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DISSERTATION

Valuations on Convex and Log-Concave Functions

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unter der Leitung von

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Kurzfassung

Eine Funktion Z, welche auf einem Funktionenraum $\mathcal{S}$ definiert ist und Werte in einer abelschen Halbgruppe annimmt, ist eine Bewertung, wenn

$$\mathrm{Z}(u \vee v) + \mathrm{Z}(u \wedge v) = \mathrm{Z}(u) + \mathrm{Z}(v)$$

für alle $u, v \in \mathcal{S}$ erfüllt ist, für welche auch $u \vee v, u \wedge v \in \mathcal{S}$. Hierbei stellen $u \vee v$ und $u \wedge v$ das punktweise Maximum und Minimum von $u, v \in \mathcal{S}$ dar.

In dieser Arbeit werden Bewertungen auf dem Raum $\mathrm{Conv}(\mathbb{R}^n)$ aller unterhalbstetigen, koerziven, konvexen Funktionen $u : \mathbb{R}^n \to (-\infty, +\infty]$, sodass $u \not\equiv +\infty$, studiert und klassifiziert. Weiters werden Bewertungen auf dem dazugehörigen Raum logarithmisch konkaver Funktionen, $\mathrm{LC}(\mathbb{R}^n)$, betrachtet. Dabei werden aus dem Gebiet der Konvexgeometrie bekannte Operatoren von den konvexen Körpern auf $\mathrm{Conv}(\mathbb{R}^n)$ beziehungsweise $\mathrm{LC}(\mathbb{R}^n)$ verallgemeinert. Insbesondere liegt der Fokus nicht nur auf reellwertigen Bewertungen, sondern auch auf Bewertungen, welche einer Funktion ein Maß oder einen konvexen Körper zuordnen.

Einige Resultate dieser Arbeit stammen aus gemeinsamen Arbeiten mit Andrea Colesanti und Monika Ludwig.

Abstract

A function Z defined on a function space $\mathcal{S}$ and taking values in an abelian semigroup is called a valuation if

$$\mathrm{Z}(u \vee v) + \mathrm{Z}(u \wedge v) = \mathrm{Z}(u) + \mathrm{Z}(v)$$

for all $u, v \in \mathcal{S}$ such that $u \vee v, u \wedge v \in \mathcal{S}$. Here, $u \vee v$ and $u \wedge v$ denote the pointwise maximum and minimum of $u, v \in \mathcal{S}$.

In this thesis, valuations on the space $\mathrm{Conv}(\mathbb{R}^n)$ of all lower semi-continuous, coercive, convex functions $u : \mathbb{R}^n \to (-\infty, +\infty]$ such that $u \not\equiv +\infty$ are studied and classified. Furthermore, valuations on the corresponding space of log-concave functions, $\mathrm{LC}(\mathbb{R}^n)$, are considered. Thereby, well-known operators from convex geometry are generalized from convex bodies to $\mathrm{Conv}(\mathbb{R}^n)$ and $\mathrm{LC}(\mathbb{R}^n)$, respectively. In particular, the focus is not only on real-valued valuations, but also on valuations that assign to a function a measure or a convex body.

Some results of this thesis are joint work with Andrea Colesanti and Monika Ludwig.

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"I have a very bad feeling about this."

Luke Skywalker

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Introduction

> *"I find your lack of faith disturbing."*
>
> Darth Vader

A function Z defined on a lattice $(\mathcal{L}, \vee, \wedge)$ and taking values in an abelian semigroup is called a *valuation* if

$$\mathrm{Z}(u \vee v) + \mathrm{Z}(u \wedge v) = \mathrm{Z}(u) + \mathrm{Z}(v), \tag{$\star$}$$

for all $u, v \in \mathcal{L}$. A function Z defined on a set $\mathcal{S} \subset \mathcal{L}$ is called a valuation if ($\star$) holds whenever $u, v, u \vee v, u \wedge v \in \mathcal{S}$. In the classical theory, valuations on the set of convex bodies (non-empty, compact, convex sets), $\mathcal{K}^n$, in $\mathbb{R}^n$ are studied, where $\vee$ and $\wedge$ denote union and intersection, respectively. Valuations played a critical role in Dehn's solution of Hilbert's Third Problem in 1901 and have been a central focus in convex geometric analysis. In many cases, well known functions in geometry could be characterized as valuations. For example, a first classification of the Euler characteristic and volume as continuous, $\mathrm{SL}(n)$ and translation invariant valuations on $\mathcal{K}^n$ was established by Blaschke [10] and the celebrated Hadwiger classification theorem [25] provides a characterization of intrinsic volumes as continuous, rotation and translation invariant valuations on $\mathcal{K}^n$.

In addition to the ongoing research on real-valued valuations on convex bodies, valuations with values in $\mathcal{K}^n$ have attracted interest. Such a map is called a *Minkowski valuation* if the addition in ($\star$) is given by Minkowski addition, that is $K + L = \{x + y : x \in K, y \in L\}$ for $K, L \in \mathcal{K}^n$. The first results in this direction were established by Ludwig [32, 33].

More recently, valuations were defined on function spaces. For a space $\mathcal{S}$ of real-valued functions we denote by $u \vee v$ the pointwise maximum of u and v while $u \wedge v$ denotes their pointwise minimum. For Sobolev spaces [34,36,41] and L^p spaces [37,49,56,57] complete classifications of valuations intertwining the $\mathrm{SL}(n)$ were established. For definable functions, an analog to Hadwiger's theorem was proven [8].

In the following, let $\mathrm{Conv}(\mathbb{R}^n)$ denote the space of all lower semi-continuous, convex functions $u : \mathbb{R}^n \to (-\infty, +\infty]$ such that $u \not\equiv +\infty$ and

$$\lim_{|x| \to +\infty} u(x) = +\infty.$$

Valuations on $\mathrm{Conv}(\mathbb{R}^n)$ and similar function spaces were already considered in [12, 13, 18]. For every $K \in \mathcal{K}^n$ the convex indicator function

$$\mathrm{I}_K(x) = \begin{cases} 0, & \text{if } x \in K \\ +\infty, & \text{if } x \notin K \end{cases}$$

is an element of $\mathrm{Conv}(\mathbb{R}^n)$. If $K, L \in \mathcal{K}^n$ are such that $K \cup L, K \cap L \in \mathcal{K}^n$, then

$$\mathrm{I}_{K \cup L} = \mathrm{I}_K \wedge \mathrm{I}_L, \qquad \mathrm{I}_{K \cap L} = \mathrm{I}_K \vee \mathrm{I}_L.$$

Hence, valuations on $\mathrm{Conv}(\mathbb{R}^n)$ can be understood as generalizations of valuations on $\mathcal{K}^n$.

The aim of this thesis is to find and classify valuations on $\mathrm{Conv}(\mathbb{R}^n)$ and the corresponding space of log-concave functions, $\mathrm{LC}(\mathbb{R}^n)$. Thereby, analogs of classical characterization results for valuations on $\mathcal{K}^n$ are extended to valuations on $\mathrm{Conv}(\mathbb{R}^n)$ and $\mathrm{LC}(\mathbb{R}^n)$, respectively. In particular, not only real-valued valuations, but also Minkowski valuations and measure-valued valuations are studied.

In Chapter 1, some classical results from convex geometry are gathered and already established classification results for valuations on $\mathcal{K}^n$ are recited.

The function spaces $\mathrm{Conv}(\mathbb{R}^n)$ and $\mathrm{LC}(\mathbb{R}^n)$ are studied in Chapter 2. In particular, $\mathrm{Conv}(\mathbb{R}^n)$ is equipped with the topology associated to epi-convergence and some simple properties are proven. Similarly, $\mathrm{LC}(\mathbb{R}^n)$ is equipped with a corresponding topology. Furthermore, intrinsic volumes and projections of quasi-concave functions are considered.

In Chapter 3, valuations on $\mathrm{Conv}(\mathbb{R}^n)$ and $\mathrm{LC}(\mathbb{R}^n)$ are introduced. They can be seen as functional analogs of the Euler characteristic, volume, surface area measure, projection body, identity, reflection, difference body and moment vector, respectively. Apart from sufficing the valuation property, it is shown that these operators are continuous and furthermore their behavior with respect to group actions is considered.

In Chapter 4, the valuations from Chapter 3 are characterized. The proofs of all the main results are based on two main ideas. First, every valuation on $\mathrm{Conv}(\mathbb{R}^n)$ is uniquely determined by its behavior on so-called cone functions, that are introduced in Chapter 2. Based on classical results for valuations on $\mathcal{K}^n$, which are stated in Chapter 1, the valuations on $\mathrm{Conv}(\mathbb{R}^n)$ and $\mathrm{LC}(\mathbb{R}^n)$ can therefore be described by a number of continuous functions on the reals. Second, the relation between the values of a valuation on cone functions and its values on indicator functions is investigated.

The results on real-valued valuations on $\mathrm{Conv}(\mathbb{R}^n)$ can be found in [17], whereas the results on measure-valued valuations and Minkowski valuations on $\mathrm{Conv}(\mathbb{R}^n)$ are to appear in [16]. For the classification of real-valued valuations and translation covariant Minkowski valuations on $\mathrm{LC}(\mathbb{R}^n)$ see [47].

Chapter 1

Background on Convex Bodies

"Your weapons, you will not need them."

Yoda

In this chapter we will introduce the basic notation and collect some results on convex bodies. In particular, valuations on convex bodies will be introduced and classification results will be recalled.

1.1 Basic Notation

We work in n-dimensional Euclidean space, $\mathbb{R}^n$, and denote the canonical basis vectors by $e_1, \dots, e_n$. For $x \in \mathbb{R}^n$ we will write $|x|$ for the *Euclidean norm* of x and τ_x for the *translation* $y \mapsto y + x$ on $\mathbb{R}^n$. Furthermore, for two elements $x, y \in \mathbb{R}^n$, let $x \cdot y$ denote the usual *inner product* of x and y. We will write $\mathbb{S}^{n-1}$ for the *unit sphere* in $\mathbb{R}^n$, that is

$$\mathbb{S}^{n-1} = \{z \in \mathbb{R}^n : |z| = 1\}.$$

Furthermore, for $x \in \mathbb{R}^n$ and $r > 0$ let

$$B(x,r) = \{y \in \mathbb{R}^n : |y - x| \leq r\},$$

and $B^n = B(0,1)$. For the *volume* in $\mathbb{R}^n$ we write V_n. The k-dimensional *Hausdorff measure* will be denoted by $\mathcal{H}^k$ and we write $v_k := \mathcal{H}^k(B^k)$ for the *volume of the unit ball* in $\mathbb{R}^k$. Moreover, for a k-dimensional linear subspace $E \subseteq \mathbb{R}^n$, we will denote by $\operatorname{proj}_E : \mathbb{R}^n \to E$ the *orthogonal projection* onto E and we write

$$E^\perp = \{y \in \mathbb{R}^n : x \cdot y = 0, \forall x \in E\}$$

for the *orthogonal complement* of E in $\mathbb{R}^n$. Furthermore, for a vector $z \in \mathbb{S}^{n-1}$ we denote by $z^\perp = \{x \in \mathbb{R}^n : x \cdot z = 0\}$ the *hyperplane orthogonal to* z.

For a set $A \subset \mathbb{R}^n$ we denote by

$$\operatorname{lin} A = \left\{ \sum_{i=1}^m \lambda_i x_i : \lambda_i \in \mathbb{R}, x_i \in A, m \in \mathbb{N} \right\}$$

its *linear hull* and by

$$\operatorname{aff} A = \left\{ \sum_{i=1}^m \lambda_i x_i : \lambda_i \in \mathbb{R}, \sum_{i=1}^m \lambda_i = 1, x_i \in A, m \in \mathbb{N} \right\}$$

its *affine hull*. The *dimension* of A, $\dim A$, is now defined as the dimension of aff A. Moreover, we write $\operatorname{int} A$ for the *interior* and $\operatorname{relint} A$ for the *relative interior* of A, that is the interior of A with respect to its affine hull. The *topological boundary* of A will be denoted by ∂A. Furthermore,

$$\operatorname{conv} A = \left\{ \sum_{i=1}^{m} \lambda_i\, x_i \,:\, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1, x_i \in A, m \in \mathbb{N} \right\}$$

and

$$\operatorname{pos} A = \left\{ \sum_{i=1}^{m} \lambda_i\, x_i \,:\, \lambda_i \geq 0, x_i \in A, m \in \mathbb{N} \right\}$$

denote the *convex hull* and *positive hull* of A, respectively. Using this notation, we write $T^n = \operatorname{conv}\{0, e_1, \ldots, e_n\}$ for the *standard simplex* in $\mathbb{R}^n$.

We will need certain groups of linear transforms. Denoting by $\det \phi$ the *determinant* of $\phi \in \mathbb{R}^{n \times n}$, we write

$$\begin{aligned} \mathrm{GL}(n) &= \{\phi \in \mathbb{R}^{n\times n} : \det \phi \neq 0\} \\ \mathrm{SL}(n) &= \{\phi \in \mathbb{R}^{n\times n} : \det \phi = 1\} \\ \mathrm{SO}(n) &= \{\phi \in \mathbb{R}^{n\times n} : \det \phi = 1,\, \phi^{-1} = \phi^t\}, \end{aligned}$$

for the *general linear group*, *special linear group* and *special orthogonal group*, respectively. Here, ϕ^{-1} denotes the *inverse* and ϕ^t denotes the *transpose* of an $n \times n$-matrix ϕ. Furthermore, we will sometimes write ϕ^{-t} instead of $(\phi^t)^{-1}$.

Lastly, we denote by $C(\mathbb{R})$ the space of continuous real-valued functions on $\mathbb{R}$ and we write $C^k(\mathbb{R})$ for the subset of the k times continuously differentiable functions. The usual space of Lebesgue integrable functions on $\mathbb{R}^n$ will be denoted by $L^1(\mathbb{R}^n)$.

1.2 The Space of Convex Bodies

We will now collect some results from convex geometry. Standard references are the books by Schneider [54] and Gruber [20].

In the following, we will denote by $\mathcal{K}^n$ the set of all non-empty, compact, convex subsets of $\mathbb{R}^n$, which are also called *convex bodies*. Furthermore, $\mathcal{K}^n_o$ denotes the subset of convex bodies that contain the origin and $\mathcal{K}^n_{(o)}$ denotes the subset of convex bodies that contain the origin in their interiors. Accordingly, we will use $\mathcal{P}^n$, $\mathcal{P}^n_o$ and $\mathcal{P}^n_{(o)}$ for the corresponding sets of convex polytopes. Here, a *convex polytope* is the convex hull of finitely many points in $\mathbb{R}^n$. Clearly,

$$\mathcal{P}^n \subset \mathcal{K}^n, \qquad \mathcal{P}^n_o \subset \mathcal{K}^n_o, \qquad \mathcal{P}^n_{(o)} \subset \mathcal{K}^n_{(o)},$$

as well as

$$\mathcal{P}^n_{(o)} \subset \mathcal{P}^n_o \subset \mathcal{P}^n, \qquad \mathcal{K}^n_{(o)} \subset \mathcal{K}^n_o \subset \mathcal{K}^n.$$

Each convex body $K \in \mathcal{K}^n$ is uniquely described by its *support function*

$$h(K, x) = \max\{y \cdot x : \; y \in K\}$$

for $x \in \mathbb{R}^n$. It is easy to see that $h(K, \cdot) \geq 0$ for every $K \in \mathcal{K}^n_o$ and furthermore $h(K, \cdot) > 0$ for every $K \in \mathcal{K}^n_{(o)}$. Moreover, it will be convenient to use $h(\emptyset, \cdot) \equiv 0$. For $z \in \mathbb{S}^{n-1}$, the convex body $K \in \mathcal{K}^n$

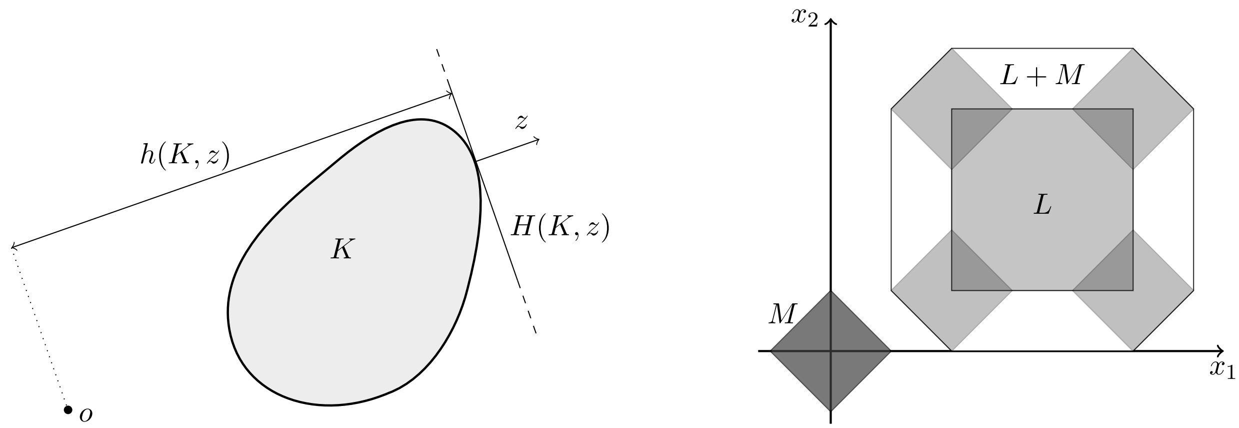

Figure 1.1: The support function and supporting hyperplane of $K \in \mathcal{K}^n$ and the Minkowski sum of $L, M \in \mathcal{K}^n$.

is contained in the closed half space $\{x \in \mathbb{R}^n : \ x \cdot z \leq h(K, z)\}$ which is bounded by the *supporting hyperplane*

$$H(K, z) = \{x \in \mathbb{R}^n : \ x \cdot z = h(K, z)\}.$$

Observe, that $H(K, z)$ has non-empty intersection with K for every $z \in \mathbb{S}^{n-1}$ and that the support function $h(K, z)$ gives the signed distance of $H(K, z)$ from the origin. Furthermore, for $x \in H(K, z) \cap K$ we say that z is an *outer unit normal* of K at x. See also Figure 1.1.

For $p \geq 0$, a function $h : \mathbb{R}^n \to \mathbb{R}$ is *p-homogeneous* if $h(t\,x) = t^p h(x)$ for $t \geq 0$ and $x \in \mathbb{R}^n$ and it is *sublinear* if it is 1-homogeneous and $h(x + y) \leq h(x) + h(y)$ for $x, y \in \mathbb{R}^n$. It is well known, that a function $h : \mathbb{R}^n \to \mathbb{R}$ is sublinear if and only if it is the support function of a unique convex body, that is $h = h(K, \cdot)$ for some $K \in \mathcal{K}^n$. Also, since $h(K, \cdot)$ is homogeneous of degree 1, it is uniquely described by its values on $\mathbb{S}^{n-1}$.

For elements of $\mathcal{K}^n$ the operation $+$ will always denote the *Minkowski sum*, that is

$$K + L := \{x + y \ : \ x \in K, y \in L\},$$

for every $K, L \in \mathcal{K}^n$, which is depicted in Figure 1.1. It is easy to see, that for $K, L \in \mathcal{K}^n$, also $K + L \in \mathcal{K}^n$. Furthermore, for the corresponding support functions we have

$$h(K + L, x) = h(K, x) + h(L, x), \tag{1.1}$$

for all $K, L \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$. For any translation τ_x with $x \in \mathbb{R}^n$ and any linear transform $\phi \in \mathrm{GL}(n)$, we write

$$\tau_x K = K + x = \{y + x \ : \ y \in K\} \qquad \text{and} \qquad \phi K = \{\phi y \ : \ y \in K\}.$$

Moreover, we denote by $-K \in \mathcal{K}^n$ the *reflection* of K, which is defined via

$$h(-K, x) = h(K, -x),$$

for every $x \in \mathbb{R}^n$.

The natural topology on $\mathcal{K}^n$ and its subspaces is induced by the *Hausdorff metric*, which is given by

$$\delta(K,L) = \sup_{z\in\mathbb{S}^{n-1}} |h(K,z) - h(L,z)|,$$

for all $K, L \in \mathcal{K}^n$. Throughout this thesis, continuity on $\mathcal{K}^n$ will always be understood with respect to this metric. The next result gives another description of Hausdorff convergence.

Theorem 1.1 ([54], Theorem 1.8.8)**.** *The convergence* $\lim_{i\to\infty} K_i = K$ *in* $\mathcal{K}^n$ *is equivalent to the following conditions taken together:*

(i) each point in K *is the limit of a sequence* $(x_i)_{i\in\mathbb{N}}$ *with* $x_i \in K_i$ *for* $i \in \mathbb{N}$*.*

(ii) the limit of any convergent sequence $(x_{i_j})_{j\in\mathbb{N}}$ *with* $x_{i_j} \in K_{i_j}$ *for* $j \in \mathbb{N}$ *belongs to* K*.*

Remark 1.2*.* It is easy to see that the first condition in Theorem 1.1 can be replaced by

(i^*) each point in $\operatorname{relint} K$ *is the limit of a sequence* $(x_i)_{i\in\mathbb{N}}$ *with* $x_i \in \mathcal{K}_i$ *for* $i \in \mathbb{N}$*.*

1.3 Valuations on Convex Bodies

In this section we introduce valuations on convex bodies. They played a critical role in Dehn's solution of Hilbert's Third Problem and have been a central focus in convex geometric analysis. Several well known operators in geometry not only have the valuation property, but can also be characterized as valuations with certain additional properties.

Definition 1.3. A function Z defined on a subset $\mathcal{Q}^n \subset \mathcal{K}^n$ and taking values in an abelian semigroup $\langle A, +\rangle$ is called a *valuation* if

$$\mathrm{Z}(K\cup L) + \mathrm{Z}(K\cap L) = \mathrm{Z}(K) + \mathrm{Z}(L),$$

whenever $K, L, K\cup L, K\cap L \in \mathcal{Q}^n$.

We will see that in many cases valuations can be characterized by their behavior with respect to group actions, especially translations. We say that a map $\mathrm{Z} : \mathcal{Q}^n \to \langle A, +\rangle$, defined on some subset $\mathcal{Q}^n \subseteq \mathcal{K}^n$ is *translation invariant* if $\mathrm{Z}(\tau_x K) = \mathrm{Z}(K)$ for every $x \in \mathbb{R}^n$ and $K \in \mathcal{Q}^n$ with $\tau_x K \in \mathcal{Q}^n$.

1.3.1 Real-Valued Valuations

Let $K \in \mathcal{K}^n$. By the well known *Steiner formula* (see, for example, [54, Section 4.2]) there exist coefficients $V_i(K)$ such that

$$V_n(K + rB^n) = \sum_{i=0}^{n} r^{n-i} v_{n-i} V_i(K),$$

for every $r > 0$. For $0 \le k \le n$, the number $V_k(K)$ is called the *k-th intrinsic volume* of K and coincides with the $(n-k)$-th quermassintegral, up to a renormalization. It is easy to see, that for $\dim K \le k$, $V_k(K)$ is just the k-dimensional volume of K. The 0-th intrinsic volume, V_0, is also called the *Euler characteristic* and $V_0(K) = 1$ for every $K \in \mathcal{K}^n$. Furthermore, we set $V_k(\emptyset) = 0$ for every $0 \le k \le n$.

We say that a valuation $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ is *rigid motion invariant* if it is translation invariant and *rotation invariant*, that is $\mathrm{Z}(\phi K) = \mathrm{Z}(K)$ for every $\phi \in \mathrm{SO}(n)$ and $K \in \mathcal{K}^n$. We are now able to state one of the most important results in the theory of valuations.

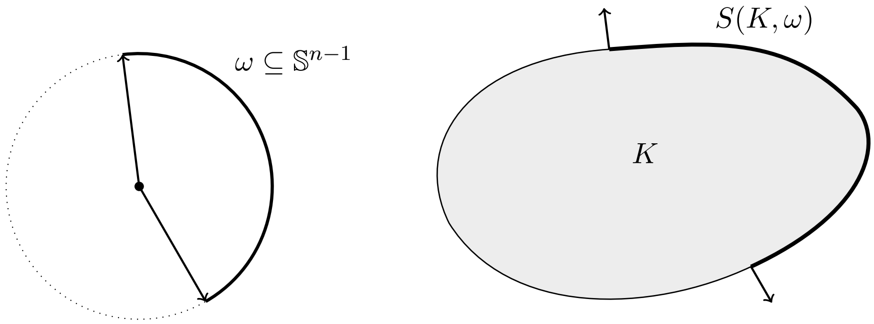

Figure 1.2: The surface area measure of $K \in \mathcal{K}^n$.

Theorem 1.4 (Hadwiger's characterization theorem, [25]). *A map $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ is a continuous, rigid motion invariant valuation if and only if there exist constants $c_0, \dots, c_n \in \mathbb{R}$ such that*

$$\mathrm{Z}(K) = \sum_{i=0}^{n} c_i V_i(K),$$

for every $K \in \mathcal{K}^n$.

Remark 1.5. A newer and shorter proof of Theorem 1.4 is given in [27].

We say that a map Z defined on some subset $\mathcal{Q}^n \subseteq \mathcal{K}^n$ is $\mathrm{SL}(n)$ *invariant* if $\mathrm{Z}(\phi K) = \mathrm{Z}(K)$ for every $\phi \in \mathrm{SL}(n)$ and $K \in \mathcal{Q}^n$ with $\phi K \in \mathcal{Q}^n$. A first classification of volume and the Euler characteristic as continuous, $\mathrm{SL}(n)$ and translation invariant valuations on $\mathcal{K}^3$ was obtained by Blaschke [10]. We remark, that this also follows from Theorem 1.4. We will need the following more general result, see for example [39, Corollary 1.2].

Theorem 1.6. *For $n \geq 2$, a functional $\mathrm{Z} : \mathcal{P}_o^n \to \mathbb{R}$ is an upper semicontinuous and $\mathrm{SL}(n)$ invariant valuation if and only if there are constants $c_0, c_n \in \mathbb{R}$ such that*

$$\mathrm{Z}(P) = c_0 V_0(P) + c_n V_n(P),$$

for every $P \in \mathcal{P}_o^n$.

For more information on the classical theory of (real-valued) valuations we refer to [25, 28] and for some recent results see [1–3, 22, 38].

1.3.2 Measure-Valued Valuations

Denote by $\mathcal{M}(\mathbb{S}^{n-1})$ the space of finite positive Borel measures on the sphere. An important operator assigned to a convex body $K \in \mathcal{K}^n$ is its *surface area measure*, $S(K, \cdot) \in \mathcal{M}(\mathbb{S}^{n-1})$. For a Borel set $\omega \subseteq \mathbb{S}^{n-1}$ and $K \in \mathcal{K}^n$, the surface area measure $S(K,\omega)$ is the $\mathcal{H}^{n-1}$ measure of all points $x \in \partial K$ at which there exists an outer unit normal vector belonging to ω, see also Figure 1.2, and furthermore

$$\int_{\mathbb{S}^{n-1}} z \, \mathrm{d}S(K,z) = 0,$$

for every $K \in \mathcal{K}^n$. By the solution of the classical Minkowski problem, a full-dimensional convex body K is - up to translations - uniquely described by its surface area measure. More precisely, a measure $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ is the surface area of an n-dimensional convex body K if and only if μ is not concentrated on a great subsphere and $\int_{\mathbb{S}^{n-1}} z \,\mathrm{d}\mu(z) = 0$. In this case, K is unique up to translation. See also [54, Section 8.2].

We say that a valuation $\mu : \mathcal{Q}^n \to \mathcal{M}(\mathbb{S}^{n-1})$ defined on some subset $\mathcal{Q}^n \subset \mathcal{K}^n$ is $\mathrm{SL}(n)$ *contravariant of degree* $p \in \mathbb{R}$ if

$$\int_{\mathbb{S}^{n-1}} b(z) \,\mathrm{d}\mu(\phi P, z) = \int_{\mathbb{S}^{n-1}} b(\phi^{-t} z) \,\mathrm{d}\mu(P, z),$$

for every map $\phi \in \mathrm{SL}(n)$, every $P \in \mathcal{Q}^n$ with $\phi P \in \mathcal{Q}^n$ and every continuous, p-homogeneous function $b : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$. The following result is due to Haberl and Parapatits.

Theorem 1.7 ([23], Theorem 1)**.** *For $n \geq 3$, a map $\mu : \mathcal{P}_o^n \to \mathcal{M}(\mathbb{S}^{n-1})$ is a valuation that is $\mathrm{SL}(n)$ contravariant of degree 1 if and only if there exist constants $c_1, c_2, c_3, c_4 \in \mathbb{R}$ with $c_1, c_2 \geq 0$ and $c_1 + c_3 \geq 0$, $c_2 + c_4 \geq 0$ such that*

$$\mu(P, \cdot) = c_1 S(P, \cdot) + c_2 S(-P, \cdot) + c_3 S^*(P, \cdot) + c_4 S^*(-P, \cdot),$$

for every $P \in \mathcal{P}_o^n$.

Remark 1.8. For details on the measures $S^*(P, \cdot)$, see [23]. They will not be of further interest for this thesis.

A sequence μ_k in $\mathcal{M}(\mathbb{S}^{n-1})$ is said to *converge weakly* to a measure $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$, if

$$\int_{\mathbb{S}^{n-1}} b(z) \,\mathrm{d}\mu_k(z) \longrightarrow \int_{\mathbb{S}^{n-1}} b(z) \,\mathrm{d}\mu(z),$$

for every continuous function $b : \mathbb{S}^{n-1} \to \mathbb{R}$. If in Theorem 1.7 one additionally assumes weak continuity (that is, if a sequence of convex bodies converges then their images converge weakly), then $c_3 = c_4 = 0$, i.e. the measures $S^*(P, \cdot)$ and $S^*(-P, \cdot)$ do not appear anymore. Hence, we have the following corollary of Theorem 1.7.

Corollary 1.9. *For $n \geq 3$, a map $\mu : \mathcal{K}_o^n \to \mathcal{M}(\mathbb{S}^{n-1})$ is a weakly continuous valuation that is $\mathrm{SL}(n)$ contravariant of degree 1 if and only if there exist constants $c_1, c_2 \geq 0$ such that*

$$\mu(K) = c_1 S(K, \cdot) + c_2 S(-K, \cdot),$$

for every $K \in \mathcal{K}_o^n$.

We say that a measure $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ is *even*, if $\mu(-\omega) = \mu(\omega)$ for every Borel set $\omega \subseteq \mathbb{S}^{n-1}$ and denote the set of all such measures by $\mathcal{M}_e(\mathbb{S}^{n-1})$. Since $\frac{1}{\sqrt{n}}(1, \ldots, 1)^t$ is always an outer unit normal of the standard simplex T^n but never of $-T^n$, the following holds true.

Corollary 1.10. *For $n \geq 3$, a map $\mu : \mathcal{K}_o^n \to \mathcal{M}_e(\mathbb{S}^{n-1})$ is a weakly continuous valuation that is $\mathrm{SL}(n)$ contravariant of degree 1 if and only if there exists a constant $c \geq 0$ such that*

$$\mu(K, \cdot) = c(S(K, \cdot) + S(-K, \cdot)),$$

for every $K \in \mathcal{K}_o^n$.

1.3.3 Minkowski Valuations

A valuation $\mathrm{Z} : \mathcal{Q}^n \to \langle \mathcal{K}^n, + \rangle$ defined on some subset $\mathcal{Q}^n \subset \mathcal{K}^n$ is also called *Minkowski valuation*. In the following we will distinguish between the behavior of Minkowski valuations with respect to special linear transforms.

Contravariant Minkowski Valuations

Definition 1.11. The *projection body* ΠK of a convex body $K \in \mathcal{K}^n$ is given by

$$h(\Pi K, z) := V_{n-1}(\operatorname{proj}_{z^\perp} K) = \tfrac{1}{2} \int_{\mathbb{S}^{n-1}} |y \cdot z| \,\mathrm{d}S(K, y),$$

for every $z \in \mathbb{S}^{n-1}$.

More generally, for a finite Borel measure μ on $\mathbb{S}^{n-1}$, we define its *cosine transform* $\mathscr{C}\mu : \mathbb{R}^n \to \mathbb{R}$ by

$$\mathscr{C}\mu(x) = \int_{\mathbb{S}^{n-1}} |z \cdot x| \,\mathrm{d}\mu(z) \tag{1.2}$$

for $x \in \mathbb{R}^n$. Since $x \mapsto \mathscr{C}\mu(x)$ is easily seen to be sublinear and non-negative on $\mathbb{R}^n$, the cosine transform $\mathscr{C}\mu$ is the support function of a convex body that contains the origin.

We require the following result where the support function of certain projection bodies is calculated for specific vectors. Let $n \geq 2$.

Lemma 1.12. *For the polytopes $P = \operatorname{conv}\{0, \frac{1}{2}(e_1 + e_2), e_2, \dots, e_n\}$ and $Q = \operatorname{conv}\{0, e_2, \dots, e_n\}$ we have*

$$\begin{aligned} h(\Pi P, e_1) &= \tfrac{1}{(n-1)!} & \qquad h(\Pi Q, e_1) &= \tfrac{1}{(n-1)!} \\ h(\Pi P, e_2) &= \tfrac{1}{2(n-1)!} & h(\Pi Q, e_2) &= 0 \\ h(\Pi P, e_1 + e_2) &= \tfrac{1}{(n-1)!} & h(\Pi Q, e_1 + e_2) &= \tfrac{1}{(n-1)!}. \end{aligned}$$

Proof. We use induction on the dimension and start with $n = 2$. In this case, P is a triangle in the plane with vertices $0, \frac{1}{2}(e_1 + e_2)$ and e_2 and Q is just the line segment connecting the origin with e_2. It is easy to see that $h(\Pi P, e_2) = V_1(\operatorname{proj}_{e_2^\perp} P) = \frac{1}{2}$ and $h(\Pi Q, e_2) = 0$ while $h(\Pi P, e_1) = h(\Pi Q, e_1) = 1$. It is also easy to see that

$$h(\Pi P, e_1 + e_2) = h(\Pi Q, e_1 + e_2) = \sqrt{2}\tfrac{\sqrt{2}}{2} = 1.$$

Assume now that the statement holds for $(n-1)$. All the projections to be considered are simplices that are the convex hull of e_n and a base in $e_n^\perp$ which is just the projection as in the $(n-1)$-dimensional case. Therefore, the corresponding $(n-1)$-dimensional volumes are just $\frac{1}{n-1}$ multiplied with the $(n-2)$-dimensional volumes from the previous case. To illustrate this, we will calculate $h(\Pi P, e_1 + e_2)$ and remark that the other cases are similar. Note that $\operatorname{proj}_{(e_1+e_2)^\perp} P = \operatorname{conv}\{e_n, \operatorname{proj}_{(e_1+e_2)^\perp} P^{(n-1)}\}$, where $P^{(n-1)}$ is the set in $\mathbb{R}^{n-1}$ from the $(n-1)$-dimensional case embedded via the identification of $\mathbb{R}^{n-1}$ and $e_n^\perp \subset \mathbb{R}^n$. Using the induction hypothesis and $|e_1 + e_2| = \sqrt{2}$, we obtain

$$V_{n-1}(\operatorname{proj}_{(e_1+e_2)^\perp} P) = \tfrac{1}{n-1} V_{n-2}(\operatorname{proj}_{(e_1+e_2)^\perp} P^{(n-1)}) = \tfrac{1}{\sqrt{2}(n-1)!},$$

and therefore $h(\Pi P, e_1 + e_2) = \frac{1}{(n-1)!}$. ■

The projection body has some useful properties concerning $\mathrm{SL}(n)$ transforms and translations. For every $\phi \in \mathrm{SL}(n)$ and $x \in \mathbb{R}^n$ we have

$$\Pi(\phi K) = \phi^{-t}\,\Pi\, K \qquad \text{and} \qquad \Pi(K + x) = \Pi(K), \tag{1.3}$$

for all $K \in \mathcal{K}^n$. Furthermore, the map $K \mapsto \Pi\, K$ is continuous and the origin is an interior point of $\Pi\, K$, if K is n-dimensional. See also [54, Section 10.9].

We say that a Minkowski valuation $\mathrm{Z} : \mathcal{Q}^n \to \mathcal{K}^n$ defined on some subset $\mathcal{Q}^n \subseteq \mathcal{K}^n$ is $\mathrm{SL}(n)$ *contravariant* if $\mathrm{Z}(\phi K) = \phi^{-t}\,\mathrm{Z}(K)$ for every $\phi \in \mathrm{SL}(n)$ and $K \in \mathcal{Q}^n$ with $\phi K \in \mathcal{Q}^n$. The following result is due to Haberl.

Theorem 1.13 ([21], Theorem 4)**.** *For $n \geq 3$, a map $\mathrm{Z} : \mathcal{K}^n_o \to \mathcal{K}^n$ is a continuous, $\mathrm{SL}(n)$ contravariant Minkowski valuation if and only if there exists a constant $c \geq 0$ such that*

$$\mathrm{Z}\, K = c\,\Pi\, K,$$

for every $K \in \mathcal{K}^n_o$.

Remark 1.14. The first characterization of the projection body, due to Ludwig, can be found in [32].

Covariant Minkowski Valuations

Definition 1.15. The difference body $\mathrm{D}\, K$ of a convex body $K \in \mathcal{K}^n$ is given by

$$h(\mathrm{D}\, K, z) := V_1(\mathrm{proj}_{\mathrm{lin}\{z\}}\, K) = h(K, z) + h(-K, z)$$

for every $z \in \mathbb{S}^{n-1}$. Equivalently, one writes

$$\mathrm{D}\, K = K + (-K).$$

Furthermore, the *moment body* $\mathrm{M}\, K$ of K is defined by

$$h(\mathrm{M}\, K, z) := \int_K |x \cdot z| \,\mathrm{d}x$$

for every $z \in \mathbb{S}^{n-1}$. Moreover, we define the moment vector $\mathrm{m}(K)$ of K as

$$h(\mathrm{m}(K), z) := \int_K x \cdot z \,\mathrm{d}x$$

for every $z \in \mathbb{S}^{n-1}$. Note, that the moment vector is an element of $\mathbb{R}^n$.

Observe, that for $\phi \in \mathrm{GL}(n)$, we have

$$\begin{aligned} h(\mathrm{M}\, \phi K, y) &= \int_{\phi K} |x \cdot y| \,\mathrm{d}x \\ &= |\det \phi| \int_K |\phi x \cdot y| \,\mathrm{d}x \\ &= |\det \phi| \int_K |x \cdot \phi^t y| \,\mathrm{d}x \\ &= |\det \phi| h(\mathrm{M}\, K, \phi^t y) = h(|\det \phi| \phi\, \mathrm{M}\, K, y) \end{aligned}$$

and similarly $\mathrm{m}(\phi K) = |\det \phi| \phi\, \mathrm{m}(K)$ for all $K \in \mathcal{K}^n$ and $y \in \mathbb{R}^n$. See also [54, Sections 5.4 & 10.1].

We require the following result where the support functions of certain moment bodies and moment vectors are calculated for specific vectors. Let $n \geq 2$.

Lemma 1.16. *For $r > 0$ and $T_r = \operatorname{conv}\{0, r\, e_1, e_2, \dots, e_n\}$,*

$$\begin{aligned} h(T_r, e_1) &= r & h(-T_r, e_1) &= 0 \\ h(\mathrm{m}(T_r), e_1) &= \tfrac{r^2}{(n+1)!} & h(\mathrm{M}\, T_r, e_1) &= \tfrac{r^2}{(n+1)!}. \end{aligned}$$

Proof. It is easy to see that $h(T_r, e_1) = r$ and $h(-T_r, e_1) = 0$. Let $\phi_r \in \mathrm{GL}(n)$ be such that $e_1 \mapsto r\, e_1$ and $e_i \mapsto e_i$ for $i = 2, \dots, n$. Then $T_r = \phi_r T^n$ and therefore

$$h(\mathrm{m}(T_r), e_1) = h(\mathrm{m}(\phi_r T^n), e_1) = |\det \phi_r|\, h(\mathrm{m}(T^n), (\phi_r)^t e_1) = r^2\, h(\mathrm{m}(T^n), e_1) = \tfrac{r^2}{(n+1)!}.$$

Finally, since $e_1 \cdot x \geq 0$ for every $x \in T_r$, we have $h(\mathrm{M}\, T_r, e_1) = h(\mathrm{m}(T_r), e_1)$. ■

We say that a Minkowski valuation $\mathrm{Z} : \mathcal{Q}^n \to \mathcal{K}^n$ defined on some subset $\mathcal{Q}^n \subseteq \mathcal{K}^n$ is $\mathrm{SL}(n)$ *covariant* if $\mathrm{Z}(\phi K) = \phi\, \mathrm{Z}(K)$ for every $\phi \in \mathrm{SL}(n)$ and $K \in \mathcal{Q}^n$ with $\phi K \in \mathcal{Q}^n$. The following result is due to Haberl.

Theorem 1.17 ([21], Theorem 6)**.** *For $n \geq 3$, a map $\mathrm{Z} : \mathcal{K}_o^n \to \mathcal{K}^n$ is a continuous, $\mathrm{SL}(n)$ covariant Minkowski valuation if and only if there exist constants $c_1, c_2, c_3 \geq 0$ and $c_4 \in \mathbb{R}$ such that*

$$\mathrm{Z}\, K = c_1 K + c_2(-K) + c_3\, \mathrm{M}\, K + c_4\, \mathrm{m}(K),$$

for every $K \in \mathcal{K}_o^n$.

Next, we state a characterization of the difference body by Ludwig.

Theorem 1.18 ([33], Corollary 1.2)**.** *For $n \geq 2$, a map $\mathrm{Z} : \mathcal{P}^n \to \mathcal{K}^n$ is a translation invariant and $\mathrm{SL}(n)$ covariant Minkowski valuation if and only if there exists a constant $c \geq 0$ such that*

$$\mathrm{Z}\, P = c\, \mathrm{D}\, P,$$

for every $P \in \mathcal{P}^n$.

We say that a Minkowski valuation $\mathrm{Z} : \mathcal{K}^n \to \mathcal{K}^n$ is *translation covariant* if there exists a function $\mathrm{Z}^0 : \mathcal{K}^n \to \mathbb{R}$ associated with Z such that

$$\mathrm{Z}(K + x) = \mathrm{Z}(K) + \mathrm{Z}^0(K) x,$$

for every $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$. Since several important geometric operators have this property, translation covariant valuations have attracted considerable interest. For example, the identity on $\mathcal{K}^n$ and the reflection $K \mapsto -K$ are translation covariant. Furthermore, for $z \in \mathbb{S}^{n-1}$ we have

$$h(\mathrm{m}(K + x), z) = \int_{K+x} z \cdot y \,\mathrm{d}y = \int_K z \cdot (y + x)\,\mathrm{d}y = \int_K z \cdot y\,\mathrm{d}y + V_n(K)\, z \cdot x. \tag{1.4}$$

Hence, $\mathrm{m}(K+x) = \mathrm{m}(K) + V_n(K)x$ for every $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$. Based on Schneider's characterization of the Steiner point [53], Hadwiger & Schneider [26] proved that the quermassvectors form a basis of the space of continuous, rotation and translation covariant vector-valued valuations. In [42], McMullen characterized weakly continuous and translation covariant vector-valued valuations on convex polytopes, extending a previous result by Hadwiger [24]. In his result the intrinsic moment vectors of the faces of a polytope appear. For further results on translation covariant valuations see [43, 44].

We derive the following simple consequence of Theorem 1.17.

Corollary 1.19. *For $n \geq 3$, a map $\mathrm{Z} : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous, $\mathrm{SL}(n)$ and translation covariant Minkowski valuation if and only if there exist constants $c_1, c_2 \geq 0$ and $c_3 \in \mathbb{R}$ such that*

$$\mathrm{Z}\, K = c_1 K + c_2(-K) + c_3\, \mathrm{m}(K), \tag{1.5}$$

for every $K \in \mathcal{K}^n$.

Proof. We have already seen in Theorem 1.17 and (1.4) that (1.5) defines a continuous, $\mathrm{SL}(n)$ and translation covariant valuation. Conversely, let Z be a continuous, $\mathrm{SL}(n)$ and translation covariant valuation on $\mathcal{K}^n$. Obviously, the restriction of Z to $\mathcal{K}^n_o$ is a continuous, $\mathrm{SL}(n)$ covariant valuation. Hence, by Theorem 1.17 there exist constants $c_1, c_2, c_4 \geq 0$ and $c_3 \in \mathbb{R}$ such that

$$\mathrm{Z}\, K = c_1 K + c_2(-K) + c_3\, \mathrm{m}(K) + c_4\, \mathrm{M}\, K, \tag{1.6}$$

for every $K \in \mathcal{K}^n_o$. Define the polytope P as $P := [-e_1, 2e_1] + [0, e_2] + \cdots [0, e_n]$ and observe that $P, P + e_1, P - e_1 \in \mathcal{K}^n_o$. By the translation covariance of Z we obtain

$$\begin{aligned}
\mathrm{Z}(P) + \mathrm{Z}^0(P)e_1 = \mathrm{Z}(P+e_1) = c_1 P + c_1 e_1 + c_2(-P) - c_2 e_1 + c_3\, \mathrm{m}(P) + V_n(P)e_1 + c_4\, \mathrm{M}(P+e_1),\\
\mathrm{Z}(P) - \mathrm{Z}^0(P)e_1 = \mathrm{Z}(P-e_1) = c_1 P - c_1 e_1 + c_2(-P) + c_2 e_1 + c_3\, \mathrm{m}(P) - V_n(P)e_1 + c_4\, \mathrm{M}(P-e_1).
\end{aligned}$$

Adding these equations shows that

$$2\,\mathrm{Z}(P) = \mathrm{Z}(P+e_1) + \mathrm{Z}(P-e_1) = 2c_1 P + 2c_2(-P) + 2c_3\, \mathrm{m}(P) + c_4(\mathrm{M}(P+e_1) + \mathrm{M}(P-e_1)).$$

On the other hand by (1.6)

$$2\,\mathrm{Z}(P) = 2c_1 P + 2c_2(-P) + 2c_3\, \mathrm{m}(P) + 2c_4\, \mathrm{M}\, P.$$

Evaluating and comparing the support function of $2\,\mathrm{Z}(P)$ at e_1

$$2c_4 \tfrac{5}{2} = c_4(\tfrac{9}{2} + \tfrac{5}{2}),$$

and therefore $c_4 = 0$. Furthermore, this shows that $\mathrm{Z}^0(K) = c_1 - c_2 + c_3 V_n(K)$ for every $K \in \mathcal{K}^n_o$. Now, fix an arbitrary $K \in \mathcal{K}^n$. Then, there exist $K_o \in \mathcal{K}^n_o$ and $x \in \mathbb{R}^n$ such that $K = K_o + x$. By the properties of Z this gives

$$\begin{aligned}
\mathrm{Z}(K) &= \mathrm{Z}(K_o + x)\\
&= \mathrm{Z}(K_o) + \mathrm{Z}^0(K_o)x\\
&= c_1 K_o + c_2(-K_o) + c_3\, \mathrm{m}(K_o) + (c_1 - c_2 + V_n(K_o))x\\
&= c_1 K + c_2(-K) + c_3\, \mathrm{m}(K).
\end{aligned}$$

■

Remark 1.20. For the valuation Z in Corollary 1.19 we see that Z^0 is a linear combination of the Euler characteristic and volume. Indeed, it is easy to show, that for a continuous, $\mathrm{SL}(n)$ and translation covariant valuation Z on $\mathcal{K}^n$, its associated function Z^0 has to be a continuous, $\mathrm{SL}(n)$ and translation invariant real-valued valuation. See also Lemma 4.27.

Chapter 2

Convex, Log-Concave and Quasi-Concave Functions

"Help me, Obi-Wan Kenobi.
You're my only hope."

Princess Leia

In order to extend valuations from convex bodies to convex functions and related function spaces, we need to define a suitable space of convex functions first. Furthermore, we will equip said function space with the topology associated to epi-convergence and discuss some important properties. Moreover, we will see that geometric constructions, such as intrinsic volumes and projections, have already been established for the related class of quasi-concave functions. The results of this chapter can be found in [17].

2.1 A Suitable Space of Convex Functions

To every convex function $u : \mathbb{R}^n \to (-\infty, +\infty]$ there can be assigned several convex sets. For any $t \in (-\infty, +\infty]$ we can consider the *sublevel sets*

$$\{u < t\} := \{x \in \mathbb{R}^n : u(x) < t\}, \quad \{u \leq t\} := \{x \in \mathbb{R}^n : u(x) \leq t\},$$

which are convex sets. Then, the *(effective) domain* of u is defined as the set

$$\operatorname{dom} u := \{u < +\infty\}.$$

Furthermore, the *epigraph* of u

$$\operatorname{epi} u := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : u(x) \leq y\}$$

is a convex subset of $\mathbb{R}^n \times \mathbb{R}$. See also [54, Section 1.5].

In the following, let $\operatorname{Conv}(\mathbb{R}^n)$ denote the set of all convex functions $u : \mathbb{R}^n \to (-\infty, +\infty]$ that are lower semi-continuous, proper and coercive. Here we say that u is *proper* if $\operatorname{dom} u$ is not empty or equivalently $u \not\equiv +\infty$. Furthermore, u is called *coercive* if

$$\lim_{|x| \to +\infty} u(x) = +\infty. \tag{2.1}$$

The function space $\operatorname{Conv}(\mathbb{R}^n)$ was already discussed by Cavallina & Colesanti in [12] with the slight difference that they also included the function that is $+\infty$ everywhere. We will see that the inclusion of this function conflicts with epi-convergence and it is therefore not an element of the space $\operatorname{Conv}(\mathbb{R}^n)$, that is discussed in this thesis. See also Remark 2.17. Furthermore, we remark that Colesanti & Fragalà as well as Cordero-Erausquin & Klartag used similar spaces in [13] and [18], respectively.

Lower semi-continuity of a convex function $u : \mathbb{R}^n \to (-\infty, +\infty]$ is equivalent to its epigraph $\operatorname{epi} u$ being closed and to the closure of all sub-level sets $\{u \leq t\}$ for $t \in \mathbb{R}$. Therefore, such a function is also called *closed*. Furthermore, the growth condition (2.1) is equivalent to the boundedness of all sublevel sets $\{u \leq t\}$. Hence, for $u \in \operatorname{Conv}(\mathbb{R}^n)$ we have

$$\{u \leq t\} \in \mathcal{K}^n, \tag{2.2}$$

for all $t \geq \min_{x\in\mathbb{R}^n} u(x)$. Note, that every function $u \in \operatorname{Conv}(\mathbb{R}^n)$ attains its minimum.

It is easy to see, that for two convex functions $u, v \in \operatorname{Conv}(\mathbb{R}^n)$ the pointwise minimum $u \wedge v$ corresponds to the union of their epigraphs and therefore to the union of their sublevel sets. Similarly, the pointwise maximum $u \vee v$ corresponds to the intersection of the epigraphs and sublevel sets. Hence, for all $t \in \mathbb{R}$

$$\{u \wedge v \leq t\} = \{u \leq t\} \cup \{v \leq t\} \qquad \text{and} \qquad \{u \vee v \leq t\} = \{u \leq t\} \cap \{v \leq t\}, \tag{2.3}$$

where for $u \wedge v \in \operatorname{Conv}(\mathbb{R}^n)$ all occurring sublevel sets are either empty or convex bodies.

We collect some basic results.

Lemma 2.1 ([12], Lemma 3.2)**.** *If* $u \in \operatorname{Conv}(\mathbb{R}^n)$*, then*

$$\operatorname{relint}\{u \leq t\} \subseteq \{u < t\},$$

for every $t > \min_{x\in\mathbb{R}^n} u(x)$.

The following result is also known as the *cone property*.

Lemma 2.2 ([13], Lemma 2.5)**.** *For* $u \in \operatorname{Conv}(\mathbb{R}^n)$ *there exist constants* $a, b \in \mathbb{R}$ *with* $a > 0$ *such that*

$$u(x) > a|x| + b \quad \forall x \in \mathbb{R}^n.$$

Next, we define $\operatorname{LC}(\mathbb{R}^n)$ as the set of all log-concave functions f that can be written as

$$f = e^{-u},$$

for some $u \in \operatorname{Conv}(\mathbb{R}^n)$. Thus, $\operatorname{LC}(\mathbb{R}^n)$ is the set of all upper semi-continuous, log-concave functions $f : \mathbb{R}^n \to [0, +\infty)$ that have non-empty *support* $\operatorname{supp} f := \{x \in \mathbb{R}^n : f(x) \neq 0\}$ and that *vanish at infinity*,

$$\lim_{|x|\to+\infty} f(x) = 0.$$

Observe, that for $f = e^{-u} \in \operatorname{LC}(\mathbb{R}^n)$ the support of f is equal to the domain of u, $\operatorname{supp} f = \operatorname{dom} u$, and that the function that is 0 for every $x \in \mathbb{R}^n$ is not an element of $\operatorname{LC}(\mathbb{R}^n)$.

2.2 Piecewise Affine, Cone and Indicator Functions

In this section we will restrict our attention to special elements of $\operatorname{Conv}(\mathbb{R}^n)$. A function $\ell \in \operatorname{Conv}(\mathbb{R}^n)$ is called *piecewise affine*, if there exist finitely many n-dimensional convex polyhedra $C_1, \dots, C_m$ with pairwise disjoint interiors, such that $\bigcup_{i=1}^m C_i = \mathbb{R}^n$ and the restriction of ℓ to each C_i is affine. Here, a

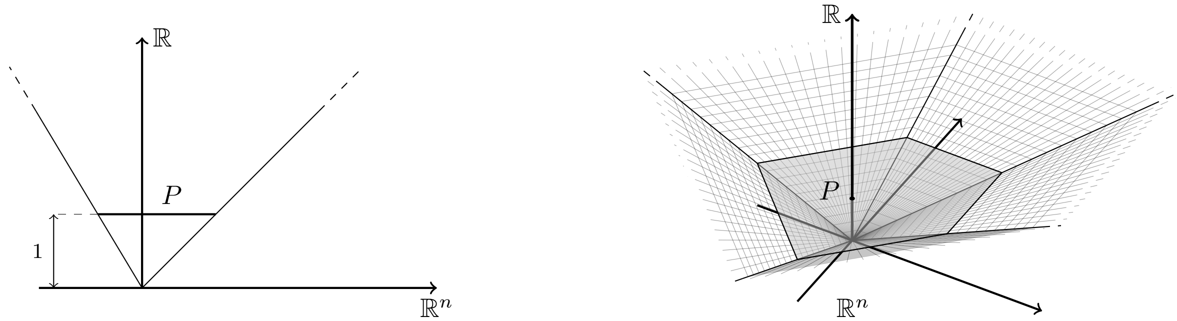

Figure 2.1: The function ℓ_P for $P \in \mathcal{P}^n_{(o)}$.

convex polyhedron is the intersection of finitely many half-spaces. The set of piecewise affine elements in $\mathrm{Conv}(\mathbb{R}^n)$ will be denoted by $\mathrm{Conv}_{\text{p.a.}}(\mathbb{R}^n)$. Furthermore, we call $u \in \mathrm{Conv}(\mathbb{R}^n)$ a *finite element* of $\mathrm{Conv}(\mathbb{R}^n)$ if $u(x) < +\infty$ for every $x \in \mathbb{R}^n$. Note, that convex piecewise affine functions are finite elements of $\mathrm{Conv}(\mathbb{R}^n)$.

Next, for $K \in \mathcal{K}^n_o$ we define the convex function ℓ_K via

$$\operatorname{epi} \ell_K := \operatorname{pos}(K \times \{1\}).$$

This means that the epigraph of ℓ_K is a cone in $\mathbb{R}^n \times \mathbb{R}$ with apex at the origin and $\{\ell_K \leq t\} = t\,K$ for every $t \geq 0$. A function of type $\ell_K + t$ with $K \in \mathcal{K}^n_o$ and $t \in \mathbb{R}$ is also called *cone function*. It is easy to see that every $K \in \mathcal{K}^n_o$ the function ℓ_K is an element of $\mathrm{Conv}(\mathbb{R}^n)$. Furthermore, $\operatorname{dom} \ell_K = \mathbb{R}^n$ if and only if K contains the origin in its interior. Lastly, for $P \in \mathcal{P}^n_{(o)}$ we can see that $\ell_P \in \mathrm{Conv}_{\text{p.a.}}(\mathbb{R}^n)$, see also Figure 2.1. For $P \in \mathcal{P}^n_{(o)}$ we can also describe ℓ_P as follows: Let P have facets $F_1, \dots, F_m$ and denote by z_i the outer unit normal vectors of P at F_i. Furthermore, let C_i denote the positive hull of F_i. Now $C_1, \dots, C_m$ have pairwise disjoint interiors with $\bigcup_{i=1}^m C_i = \mathbb{R}^n$ and we have

$$\ell_P(x) = \frac{z_i \cdot x}{h(P, z_i)},$$

for every $x \in C_i$.

Another important class of functions in $\mathrm{Conv}(\mathbb{R}^n)$, that can be seen as generalizations of convex bodies, are *indicator functions*. The (convex) indicator function of $K \in \mathcal{K}^n$ is given by

$$\mathrm{I}_K(x) = \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{if } x \notin K, \end{cases}$$

for every $x \in \mathbb{R}^n$. Clearly, $\mathrm{I}_K \in \mathrm{Conv}(\mathbb{R}^n)$ for every $K \in \mathcal{K}^n$. Furthermore, it is easy to see that I_K corresponds to the *characteristic function* of $K \in \mathcal{K}^n$, $\chi_K \in \mathrm{LC}(\mathbb{R}^n)$, via the relation

$$\chi_K(x) = e^{-\mathrm{I}_K(x)} = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } x \notin K, \end{cases}$$

for every $x \in \mathbb{R}^n$.

We will see in Chapter 4 that the relation between cone and indicator functions plays a key role in the classification of valuations on $\mathrm{Conv}(\mathbb{R}^n)$ and $\mathrm{LC}(\mathbb{R}^n)$, respectively.

2.3 Duality

In this section we discuss the convex conjugate and how it interacts with operations on the epigraph of a convex function. For further details we refer to [54, Section 1.6.2].

The *conjugate function* u^* of a convex function $u : \mathbb{R}^n \to (-\infty, +\infty]$ is defined as

$$u^*(y) := \sup_{x \in \mathbb{R}^n} \big(y \cdot x - u(x)\big),$$

for every $y \in \mathbb{R}^n$. If u is a closed convex function, then also u^* is a closed convex function and $u^{**} = u$. Furthermore, we will consider the *infimal convolution* $u_1 \,\square\, u_2$ of two closed convex functions $u_1, u_2 : \mathbb{R}^n \to (-\infty, +\infty]$, which is given by

$$(u_1 \,\square\, u_2)(x) := \inf_{x = x_1 + x_2} \big(u_1(x_1) + u_2(x_2)\big),$$

for every $x \in \mathbb{R}^n$. We remark, that infimal convolution corresponds to Minkowski addition of epigraphs, that is

$$\operatorname{epi}(u_1 \,\square\, u_2) = \operatorname{epi} u_1 + \operatorname{epi} u_2. \tag{2.4}$$

Hence, another name for infimal convolution is also *epi-addition*. Furthermore, if $u_1 \,\square\, u_2 > -\infty$ pointwise, then

$$(u_1 \,\square\, u_2)^* = u_1^* + u_2^*. \tag{2.5}$$

Another operation is the so-called *epi-multiplication*. For $\lambda > 0$ and a closed convex function $u : \mathbb{R}^n \to (-\infty, +\infty]$ consider the convex function u_λ which is defined by

$$u_\lambda(x) := \lambda u\big(\tfrac{x}{\lambda}\big),$$

for every $x \in \mathbb{R}^n$. Observe, that for the convex conjugate of u_λ one has

$$u_\lambda^*(y) = \sup_{x \in \mathbb{R}^n} \big(y \cdot x - \lambda u\big(\tfrac{x}{\lambda}\big)\big) = \sup_{x \in \mathbb{R}^n} \big(y \cdot \lambda x - \lambda u(x)\big) = \lambda u^*(y), \tag{2.6}$$

for every $y \in \mathbb{R}^n$ and $\lambda > 0$.

Remark 2.3. For two closed convex functions u_1, u_2 the infimal convolution $u_1 \,\square\, u_2$ need not be closed, even when it is convex.

2.4 Epi-Convergence

We will now introduce the topology on $\mathrm{Conv}(\mathbb{R}^n)$ and collect some results.

Definition 2.4. A sequence $u_k : \mathbb{R}^n \to (-\infty, +\infty]$ is said to be *epi-convergent* to $u : \mathbb{R}^n \to (-\infty, +\infty]$ if for all $x \in \mathbb{R}^n$ the following conditions hold:

(i) For every sequence x_k that converges to x

$$u(x) \le \liminf_{k \to \infty} u_k(x_k). \tag{2.7}$$

(ii) There exists a sequence x_k that converges to x such that

$$u(x) = \lim_{k \to \infty} u_k(x_k). \tag{2.8}$$

In this case we write $u = \text{epi-lim}_{k\to+\infty} u_k$ and $u_k \xrightarrow{epi} u$. Correspondingly, we say that a sequence f_k in $\mathrm{LC}(\mathbb{R}^n)$ is *hypo-convergent* to $f \in \mathrm{LC}(\mathbb{R}^n)$ if there exist $u_k, u \in \mathrm{Conv}(\mathbb{R}^n)$ such that $f_k = e^{-u_k}$ for every $k \in \mathbb{N}$, $f = e^{-u}$ and $u_k \xrightarrow{epi} u$. In this case we write $f = \text{hypo-lim}_{k\to\infty} f_k$ and $f_k \xrightarrow{hypo} f$.

Remark 2.5. If $u_k \xrightarrow{epi} u$, then by equation (2.7) the function u is an asymptotic common lower bound for the sequence u_k. Consequently, (2.8) states that this bound is optimal.

Remark 2.6. The name epi-convergence is due to the fact, that this convergence is equivalent to the convergence of the corresponding epigraphs in the Painlevé-Kuratowski sense. Another name for epi-convergence is also Γ-convergence. See also [19, Theorem 4.16] and [51, Proposition 7.2] and especially the commentary section of the same chapter.

Immediately from Definition 2.4 we obtain the following result.

Lemma 2.7 ([19], Proposition 6.1)**.** *If $u_k : \mathbb{R}^n \to (-\infty, +\infty]$ is a sequence that epi-converges to $u : \mathbb{R}^n \to (-\infty, +\infty]$, then also every subsequence u_{k_i} of u_k epi-converges to u.*

The next result shows some of the strong implications that follow from epi-convergence of convex functions.

Theorem 2.8 ([51], Theorem 7.17)**.** *If u_k is a sequence of convex functions that epi-converges to a function u, then u is convex. Moreover, if $\operatorname{dom} u$ has non-empty interior, then u_k converges uniformly to u on every compact set that does not contain a boundary point of $\operatorname{dom} u$.*

From Theorem 2.8 we derive the following result that connects epi-convergence with pointwise convergence, see also [19, Example 5.13].

Lemma 2.9. *Let $u_k : \mathbb{R}^n \to \mathbb{R}$ be a sequence of finite convex functions and let $u : \mathbb{R}^n \to \mathbb{R}$ be a finite convex function. If u_k is epi-convergent to u, then u_k also converges pointwise to u.*

Remark 2.10. The last statement is no longer true if the functions may attain the value $+\infty$. In that case

$$\text{epi-lim}_{k\to+\infty} u_k(x) \leq \lim_{k\to+\infty} u_k(x),$$

for all $x \in \mathbb{R}^n$ such that these limits exist. See also [19, Example 5.13].

Remark 2.11. In fact, also the reverse of Lemma 2.9 is true. Hence, for finite convex functions epi-convergence and pointwise convergence coincide.

Each sublevel set of a function from $\mathrm{Conv}(\mathbb{R}^n)$ is either empty or in $\mathcal{K}^n$. We say that $\{u_k \leq t\} \to \emptyset$ as $k \to +\infty$ if there exists $k_0 \in \mathbb{N}$ such that $\{u_k \leq t\} = \emptyset$ for $k \geq k_0$. The following simple result describes one of the consequences of epi-convergence on $\mathrm{Conv}(\mathbb{R}^n)$. See also [46, Lemma 3.1].

Lemma 2.12. *Let $u_k, u \in \mathrm{Conv}(\mathbb{R}^n)$. If $u_k \xrightarrow{epi} u$, then $\{u_k \leq t\} \to \{u \leq t\}$ as $k \to +\infty$ for every $t \in \mathbb{R}$ with $t \neq \min_{x\in\mathbb{R}^n} u(x)$.*

Proof. First, let $t > u_{\min} := \min_{x\in\mathbb{R}^n} u(x)$. For $x \in \operatorname{relint}\{u \leq t\}$, it follows from Lemma 2.1 that $s := u(x) < t$. Since $u_k \xrightarrow{epi} u$ there exists a sequence x_k that converges to x such that $u_k(x_k)$ converges to $u(x)$. Therefore, there exist $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$u_k(x_k) \leq s + \varepsilon \leq t.$$

Thus, $x_k \in \{u_k \leq t\}$, which shows that x is a limit of a sequence of points from $\{u_k \leq t\}$. Obviously, this implies (i*) of Remark 1.2 and therefore (i) of Theorem 1.1.

Now, let $(x_{i_j})_{j\in\mathbb{N}}$ be a convergent sequence in $\{u_{i_j} \leq t\}$ with limit $x \in \mathbb{R}^n$. By Lemma 2.7 the subsequence u_{i_j} epi-converges to u. Therefore

$$u(x) \leq \liminf_{j\to\infty} u_{i_j}(x_{i_j}) \leq t$$

which gives (ii) of Theorem 1.1.

Second, let $t < u_{\min}$. Since $\{u \leq t\} = \emptyset$, we have to show that there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ and $x \in \mathbb{R}^n$,

$$u_k(x) > t.$$

Assume that there does not exist such an index k_0. Then there are infinitely many points x_{i_j} such that $u_{i_j}(x_{i_j}) \leq t$. Note, that

$$x_{i_j} \in \{u_{i_j} \leq t\} \subseteq \{u_{i_j} \leq u_{\min} + 1\}.$$

By Lemma 2.7, we know that $u_{i_j} \xrightarrow{epi} u$ and therefore we can apply the previous argument to obtain that $\{u_{i_j} \leq u_{\min} + 1\} \to \{u \leq u_{\min} + 1\}$, which shows that x_{i_j} is bounded. Hence, there exists a convergent subsequence $x_{i_{j_k}}$ with limit $x \in \mathbb{R}^n$. Applying Lemma 2.7 again, we obtain that $u_{i_{j_k}} \xrightarrow{epi} u$ and therefore

$$u(x) \leq \liminf_{k\to\infty} u_{i_{j_k}}(x_{i_{j_k}}) \leq t,$$

which is a contradiction. Hence $\{u_k \leq t\}$ must be empty eventually. ■

For the next result we use a characterization of epi-convergence, due to Beer, Rockafellar and Wets. It uses Painlevé-Kuratowski convergence (PK-lim), which we are not going to define here. The only important fact for our purposes is that Hausdorff convergence implies Painlevé-Kuratowski convergence.

Theorem 2.13 ([9], Theorem 3.1)**.** *Let X be a separable metrizable space and let $u, u_1, u_2, \dots$ be extended real valued lower semi-continuous functions on X.*

1. *If $u = \text{epi-lim}_{k\to+\infty} u_k$, then for each $t \in \mathbb{R}$ there exists a sequence t_k of reals convergent to t such that $\{u \leq t\} = \text{PK-lim}\{u_k \leq t_k\}$.*
2. *If for each $t \in \mathbb{R}$ there exists a sequence t_k of reals convergent to t such that $\{u \leq t\} = \text{PK-lim}\{u_k \leq t_k\}$, then $u = \text{epi-lim}_{k\to+\infty} u_k$.*

Lemma 2.14. *Let $K_i, K \in \mathcal{K}^n_o$. The sequence K_i converges to K in the Hausdorff metric, if and only if ℓ_{K_i} epi-converges to ℓ_K. Furthermore, for $L_i, L \in \mathcal{K}^n$, Hausdorff convergence of L_i to L is equivalent to epi-convergence of I_{L_i} to I_L.*

Proof. Let $K_i, K \in \mathcal{K}^n_o$ be such that $K_i \to K$. For $t < 0$ we have $\{\ell_{K_i} \leq t\} = \{\ell_K \leq t\} = \emptyset$, for all $i \in \mathbb{N}$ and for $t \geq 0$ we have

$$\{\ell_K \leq t\} = tK \qquad \text{and} \qquad \{\ell_{K_i} \leq t\} = tK_i,$$

for every $i \in \mathbb{N}$. Since K_i converges to K in the Hausdorff metric, also tK_i converges to tK in the same metric and furthermore in the Painlevé-Kuratowski sense. Therefore all sublevel sets are convergent and $\ell_{K_i} \xrightarrow{epi} \ell_K$ by Theorem 2.13.

Conversely, if $K_i, K \in \mathcal{K}_o^n$ are such that $\ell_{K_i} \xrightarrow{epi} \ell_K$, then Lemma 2.12 shows that

$$K_i = \{\ell_{K_i} \leq 1\} \to \{\ell_K \leq 1\} = K.$$

The proof for the second statement is analog. ■

A fundamental relationship between convex functions and their conjugates in terms of epi-convergence was established by Wijsman, see, for example, [51, Theorem 11.34]. In fact, it is one of the reasons why epi-convergence was introduced.

Theorem 2.15 (Epi-continuity of the convex conjugate)**.** *If $u_k, u : \mathbb{R}^n \to (-\infty, +\infty]$ are closed, proper and convex, then*

$$u_k \xrightarrow{epi} u \qquad \textit{if and only if} \qquad u_k^* \xrightarrow{epi} u^*.$$

Next, we extend Lemma 2.2 to an epi-convergent sequence of functions in $\mathrm{Conv}(\mathbb{R}^n)$ and obtain a *uniform cone property.*

Lemma 2.16. *Let $u_k, u \in \mathrm{Conv}(\mathbb{R}^n)$. If $u_k \xrightarrow{epi} u$, then there exist constants $a, b \in \mathbb{R}$ with $a > 0$ such that*

$$u_k(x) > a|x| + b \qquad \textit{and} \qquad u(x) > a|x| + b$$

for every $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$.

Proof. By Lemma 2.2, there exist constants $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$u(x) > \alpha|x| + \beta := \ell(x).$$

Switching to conjugates gives $u^* < \ell^*$. Observe, that

$$\ell^*(y) = \sup_{x \in \mathbb{R}^n} \big(y \cdot x - (\alpha|x| + \beta)\big) = \left(\sup_{x \in \mathbb{R}^n} \big(y \cdot x - \alpha|x|\big)\right) - \beta,$$

for every $y \in \mathbb{R}^n$. Since

$$\sup_{x \in \mathbb{R}^n} \big(y \cdot x - \alpha|x|\big) = \begin{cases} 0 & \text{if } |y| \leq \alpha \\ +\infty & \text{if } |y| > \alpha, \end{cases}$$

we have $\ell^* = \mathrm{I}_{\alpha B^n} - \beta$. Setting $a := \alpha/2 > 0$, we see that aB^n is a compact subset of $\operatorname{int} \operatorname{dom} u^*$. Therefore, Theorems 2.8 & 2.15 imply that that u_k^* converges uniformly to u^* on aB^n. Since $u^* < -\beta$ on aB^n, there exists a constant b such that $u_k^*(y) < -b$ for every $y \in aB^n$ and $k \in \mathbb{N}$ and therefore

$$u_k^* < \mathrm{I}_{aB^n} - b,$$

for every $k \in \mathbb{N}$. Consequently

$$u_k(x) > a|x| + b,$$

for every $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$. ■

Remark 2.17. Note, that Lemmas 2.12 and 2.16 are no longer true if $u \equiv +\infty$. For example, consider $u_k(x) = \mathrm{I}_{B(k^2 x_0, kR)}$ for some $R > 0$ and $x_0 \in \mathbb{R}^n \backslash \{0\}$. Then $\text{epi-lim}_{k \to +\infty} u_k = u$ but every set $\{u_k \leq t\}$ is a ball of radius kR for $t \geq 0$. In this case, the sublevel sets are not even bounded. Furthermore, it is clear that there does not exist a uniform pointed cone that contains all the sets $\operatorname{epi} u_k$.

2.5 Moreau-Yosida Envelopes

The aim of this section is to establish Lemma 2.21, which states that $\mathrm{Conv}_{\mathrm{p.a.}}(\mathbb{R}^n)$ is a dense subset of $\mathrm{Conv}(\mathbb{R}^n)$. We remark, that this can also be deduced from more general results (see for example [7, Corollary 3.42]), however we will give a self-contained proof using Moreau-Yosida envelopes. See also [51, Chapter 1, Section G].

Definition 2.18. Let $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $\lambda > 0$. Set $q(x) = \frac{1}{2}|x|^2$ and recall that $q_\lambda(x) = \lambda q\big(\frac{x}{\lambda}\big)$ for $x \in \mathbb{R}^n$. The *Moreau-Yosida envelope* or *Moreau-Yosida approximation* $e_\lambda u$ of u is defined as

$$e_\lambda u := u \,\square\, q_\lambda,$$

or equivalently

$$e_\lambda u(x) = \inf_{y \in \mathbb{R}^n} \big(u(y) + \tfrac{1}{2\lambda}|x-y|^2\big) = \inf_{x_1+x_2=x} \big(u(x_1) + \tfrac{1}{2\lambda}|x_2|^2\big),$$

for every $x \in \mathbb{R}^n$.

Lemma 2.19. *For $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $\lambda > 0$, the Moreau-Yosida envelope $e_\lambda u$ is a finite element of $\mathrm{Conv}(\mathbb{R}^n)$. Moreover, $e_\lambda u(x) \le u(x)$ for every $x \in \mathbb{R}^n$.*

Proof. Throughout the proof fix an arbitrary $\lambda > 0$. Since

$$\inf_{x_1+x_2=x} \big(u(x_1) + \tfrac{1}{2\lambda}|x_2|^2\big) \le u(x) + \tfrac{1}{2\lambda}|0|^2,$$

we have $e_\lambda u(x) \le u(x)$ for every $x \in \mathbb{R}^n$. Since u is proper, there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) < +\infty$. This shows that

$$e_\lambda u(x) = \inf_{x_1+x_2=x} \big(u(x_1) + \tfrac{1}{2\lambda}|x_2|^2\big) \le u(x_0) + \tfrac{1}{2\lambda}|x-x_0|^2 < +\infty,$$

for every $x \in \mathbb{R}^n$, which shows that $e_\lambda u$ is finite. Using (2.4) we obtain that

$$\mathrm{epi}\, e_\lambda u = \mathrm{epi}\, u + \mathrm{epi}\, q_\lambda.$$

It is therefore easy to see, that $e_\lambda u$ is a convex function such that $\lim_{|x|\to+\infty} e_\lambda u(x) = +\infty$. ■

Lemma 2.20. *For every $u \in \mathrm{Conv}(\mathbb{R}^n)$, $\operatorname{epi-lim}_{\lambda\to 0^+} e_\lambda u = u$.*

Proof. By Theorem 2.15, $e_\lambda u \xrightarrow{epi} u$ if and only if $(e_\lambda u)^* \xrightarrow{epi} u^*$. By the definition of e_λ, (2.5) and (2.6) we have

$$(e_\lambda u)^* = (u \,\square\, q_\lambda)^* = u^* + \lambda q^*.$$

Therefore, we need to show that $u^* + \lambda q^* \xrightarrow{epi} u^*$. Observe, that for $q(x) = \frac{1}{2}|x|^2$ we have $q = q^*$. Since epi-convergence is equivalent to pointwise convergent if the functions are finite, it follows that $\operatorname{epi-lim}_{\lambda\to0^+} \lambda q^* = 0$. It is now easy to see that $\operatorname{epi-lim}_{\lambda\to0^+}(u^* + \lambda q^*) = u^*$ and therefore $\operatorname{epi-lim}_{\lambda\to0^+}(e_\lambda u)^* = u^*$. ■

Lemma 2.21. *The piecewise affine functions $\mathrm{Conv}_{\mathrm{p.a.}}(\mathbb{R}^n)$ are dense in $\mathrm{Conv}(\mathbb{R}^n)$, equipped with the topology associated to epi-convergence.*

Proof. By Lemma 2.9, epi-convergence coincides with pointwise convergence on finite functions in $\mathrm{Conv}(\mathbb{R}^n)$. Therefore, it is easy to see that $\mathrm{Conv}_{\mathrm{p.a.}}(\mathbb{R}^n)$ is epi-dense in the finite elements of $\mathrm{Conv}(\mathbb{R}^n)$. Now for arbitrary $u \in \mathrm{Conv}(\mathbb{R}^n)$ if follows from Lemma 2.19 that $e_\lambda u$ is a finite element of $\mathrm{Conv}(\mathbb{R}^n)$. Since Lemma 2.20 shows that $\operatorname{epi-lim}_{\lambda\to0^+} e_\lambda u = u$, the finite elements of $\mathrm{Conv}(\mathbb{R}^n)$ are a dense subset of $\mathrm{Conv}(\mathbb{R}^n)$. Finally, since denseness is transitive, the piecewise affine functions are an epi-dense subset of $\mathrm{Conv}(\mathbb{R}^n)$. ■

2.6 Intrinsic Volumes and Projections

In this section we describe how log-concave and more generally quasi-concave functions can be seen as direct extensions of $\mathcal{K}^n$. Furthermore, extensions of intrinsic volumes and orthogonal projections to these function spaces are discussed.

For $K \in \mathcal{K}^n$, consider the characteristic function $\chi_K \in \mathrm{LC}(\mathbb{R}^n)$, which we will view as a natural representative of K in $\mathrm{LC}(\mathbb{R}^n)$. An intuitive way to generalize the volume on $\mathbb{R}^n$ to $\mathrm{LC}(\mathbb{R}^n)$ is the integral with respect to the Lebesgue measure,

$$I(f) := \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x,$$

for $f \in \mathrm{LC}(\mathbb{R}^n)$. Clearly $I(\chi_K) = V_n(K)$ for every $K \in \mathcal{K}^n$. Furthermore, by Lemma 2.2 it is easy to see, that $I(f) < \infty$ for every $f \in \mathrm{LC}(\mathbb{R}^n)$. Sometimes $I(f)$ is also called the *total mass* of f, see for example [13]. This notion for the volume of a (log-concave) function is commonly accepted and there are several examples of functional counterparts of geometric inequalities, in which the volume $V_n(K)$ of a convex body K is replaced by the integral $\int f$ of a function f. For example, the Prékopa-Leindler inequality is the functional analog of the Brunn-Minkowski inequality [31, 50].

Since the layer-cake principle yields

$$I(f) = \int_0^{+\infty} V_n(\{f \geq t\})\,\mathrm{d}t,$$

one can analogously extend the intrinsic volumes V_i to $\mathrm{LC}(\mathbb{R}^n)$ via

$$V_i(f) := \int_0^{+\infty} V_i(\{f \geq t\})\,\mathrm{d}t, \tag{2.9}$$

for every $f \in \mathrm{LC}(\mathbb{R}^n)$ and $0 \leq i \leq n$. Again, this gives $V_i(\chi_K) = V_i(K)$ for every $K \in \mathcal{K}^n$. These functional versions of the intrinsic volumes where recently introduced for a more general class of quasi-concave functions [11, 45]. A function $f : \mathbb{R}^n \to [0, +\infty)$ is said to be *quasi-concave* if the level sets $\{f \geq t\}$ are convex for very $t \geq 0$, which clearly holds for elements in $\mathrm{LC}(\mathbb{R}^n)$. Furthermore, if $\zeta : \mathbb{R} \to [0, +\infty)$ is non-increasing and continuous, then $\zeta \circ u$ is quasi-concave for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ and the level sets $\{\zeta \circ u \geq t\}$ are convex bodies for every $0 < t \leq \max_{x\in\mathbb{R}^n} \zeta(u(x))$. Similar to (2.9) we can therefore consider

$$V_i(\zeta \circ u) := \int_0^{+\infty} V_i(\{\zeta \circ u \geq t\})\,\mathrm{d}t, \tag{2.10}$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $0 \leq i \leq n$. Note, that this expression might not be finite for every choice of ζ and u. However, observe that for $i = 0$ we obtain

$$V_0(\zeta \circ u) = \max\nolimits_{x\in\mathbb{R}^n} \zeta(u(x)) = \zeta(\min\nolimits_{x\in\mathbb{R}^n} u(x)) \tag{2.11}$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$.

Likewise, the projection of a convex set onto a subspace can be extended to quasi-concave functions, see for example [11, 29, 55]. For this, let E be a linear subspace of $\mathbb{R}^n$. The projection $\mathrm{proj}_E f : E \to [0, +\infty)$ of a quasi-concave function $f : \mathbb{R}^n \to [0, +\infty)$ onto E is now defined as

$$\mathrm{proj}_E f(x) := \sup\nolimits_{y\in E^\perp} f(x + y), \tag{2.12}$$

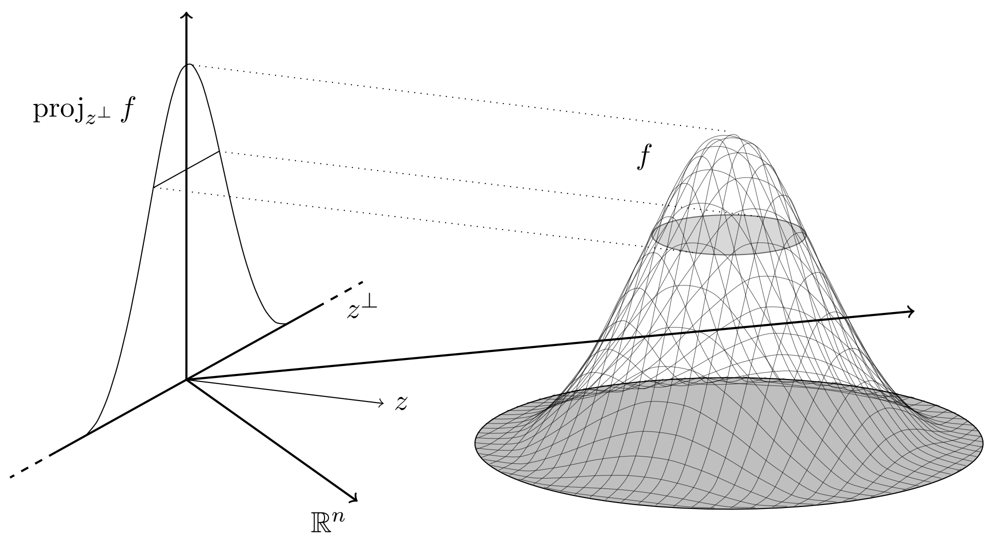

Figure 2.2: Projection of a log-concave function $f \in \mathrm{LC}(\mathbb{R}^n)$ onto the hyperplane $z^\perp$.

for every $x \in E$. If we furthermore assume that f is upper semi-continuous and $\lim_{|x|\to+\infty} f(x) = 0$ we can rewrite this as

$$\mathrm{proj}_E f(x) = \max_{y \in E^\perp} f(x+y).$$

For such a function f and $t \geq 0$, we have $\max_{y\in E^\perp} f(x+y) \geq t$ if and only if there exists $y \in E^\perp$ such that $f(x+y) \geq t$. Hence,

$$\{\mathrm{proj}_E f \geq t\} = \mathrm{proj}_E\{f \geq t\}, \tag{2.13}$$

for every $t \geq 0$, where proj_E on the right side denotes the usual orthogonal projection onto E in $\mathbb{R}^n$. Consequently $\mathrm{proj}_E \chi_K = \chi_{\mathrm{proj}_E K}$ and moreover

$$\mathrm{hypo}\,\mathrm{proj}_E f = \mathrm{proj}_{E\times\mathbb{R}}\,\mathrm{hypo}\, f,$$

where $\mathrm{hypo}\, f = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq y \leq f(x)\}$ denotes the *hypograph* of f. See also [11, Lemma 5.2] and Figure 2.2.

Again, let $\zeta : \mathbb{R} \to [0,+\infty)$ be a non-increasing, continuous function. For $f = \zeta \circ u$ with $u \in \mathrm{Conv}(\mathbb{R}^n)$ we can rewrite (2.12) as

$$\mathrm{proj}_E(\zeta \circ u)(x) = \zeta\big(\min_{y\in E^\perp} u(x+y)\big),$$

for every $x \in E$. Hence, it makes sense to also define the projection of a convex function $u \in \mathrm{Conv}(\mathbb{R}^n)$ as

$$\mathrm{proj}_E u(x) := \min_{y\in E^\perp} u(x+y),$$

for every $x \in E$. Similarly, we obtain

$$\{\mathrm{proj}_E u \leq t\} = \mathrm{proj}_E\{u \leq t\},$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $t \in \mathbb{R}$ and furthermore

$$\mathrm{epi}\,\mathrm{proj}_E u = \mathrm{proj}_{E\times\mathbb{R}}\,\mathrm{epi}\, u.$$

We remark that since the orthogonal projection of a convex body is again a convex body, it is easy to see that for $\dim E = k$, the function $\mathrm{proj}_E u$ is an element of $\mathrm{Conv}(E)$.

Chapter 3

Valuations on Convex and Log-Concave Functions

"These aren't the droids you're looking for."

Obi-Wan Kenobi

In this chapter, we study valuations on convex and log-concave functions. In addition to introducing some real-valued, measure-valued and Minkowski valuations, we will prove important properties such as continuity and consider the behavior with respect to group actions. Furthermore, we will see that studying valuations on $\mathrm{Conv}(\mathbb{R}^n)$ is equivalent to studying valuations on $\mathrm{LC}(\mathbb{R}^n)$.

3.1 Basic Observations

Let $K, L \in \mathcal{K}^n$ be such that $K \cup L \in \mathcal{K}^n$. It is easy to see that

$$h(K \cup L, \cdot) = h(K, \cdot) \vee h(L, \cdot) \qquad \text{and} \qquad h(K \cap L, \cdot) = h(K, \cdot) \wedge h(L, \cdot),$$

where $\vee$ and $\wedge$ denote the pointwise maximum and minimum, respectively. Moreover, we have for the convex indicator functions

$$\mathrm{I}_{K \cap L} = \mathrm{I}_K \vee \mathrm{I}_L \qquad \text{and} \qquad \mathrm{I}_{K \cup L} = \mathrm{I}_K \wedge \mathrm{I}_L, \tag{3.1}$$

and similarly for the characteristic functions

$$\chi_{K \cap L} = \chi_K \wedge \chi_L \qquad \text{and} \qquad \chi_{K \cup L} = \chi_K \vee \chi_L\,. \tag{3.2}$$

Hence, the following definition can be seen as a generalization of Definition 1.3.

Definition 3.1. A map Z defined on a space of real-valued functions $\mathcal{S}$ and taking values in an abelian semigroup $\langle A, + \rangle$ is called a *valuation* if

$$\mathrm{Z}(u \vee v) + \mathrm{Z}(u \wedge v) = \mathrm{Z}(u) + \mathrm{Z}(v),$$

whenever $u, v, u \vee v, u \wedge v \in \mathcal{S}$.

For Sobolev spaces [34,36,41] and L^p spaces [37,49,56,57] complete classifications of valuations intertwining the $\mathrm{SL}(n)$ were established. For definable functions, an analog to Hadwiger's theorem was proven [8] and for quasi-concave functions, valuations were introduced and classified in [14, 15]. A first classification of valuations on $\mathrm{Conv}(\mathbb{R}^n)$ was obtained by Cavallina & Colesanti [12]. See also [4, 30, 35, 59].

If Z is a valuation on $\mathrm{Conv}(\mathbb{R}^n)$, then (3.1) shows that

$$K \mapsto \mathrm{Z}(\mathrm{I}_K) \tag{3.3}$$

defines a valuation on $\mathcal{K}^n$ and using (3.2), an analog statement can be made for valuations on $\mathrm{LC}(\mathbb{R}^n)$. Note, that for $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ we always have $u \vee v \in \mathrm{Conv}(\mathbb{R}^n)$ but not necessarily $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$. In the same way $f \wedge g \in \mathrm{LC}(\mathbb{R}^n)$ for every $f, g \in \mathrm{LC}(\mathbb{R}^n)$ but we need not have $f \vee g \in \mathrm{LC}(\mathbb{R}^n)$. Furthermore, if $K, L \in \mathcal{K}_o^n$ are such that $K \cup L \in \mathcal{K}_o^n$, then (2.3) shows that

$$\ell_{K \cap L} = \ell_K \vee \ell_L \qquad \text{and} \qquad \ell_{K \cup L} = \ell_K \wedge \ell_L.$$

Consequently,

$$K \mapsto \mathrm{Z}(\ell_K) \tag{3.4}$$

defines a valuation on $\mathcal{K}_o^n$. In Chapter 4 we will see that studying these connections between valuations on $\mathrm{Conv}(\mathbb{R}^n)$ and valuations on $\mathcal{K}^n$ and $\mathcal{K}_o^n$, respectively, will be crucial in order to classify the former. Moreover, we make the following observations on actions of affine transformations. For every $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$ we have

$$\mathrm{I}_{\tau_x K} = \mathrm{I}_K \circ \tau_x^{-1} \qquad \text{and} \qquad \chi_{\tau_x K} = \chi_K \circ \tau_x^{-1} \tag{3.5}$$

and for every $\phi \in \mathrm{GL}(n)$

$$\mathrm{I}_{\phi K} = \mathrm{I}_K \circ \phi^{-1} \qquad \text{and} \qquad \chi_{\phi K} = \chi_K \circ \phi^{-1}. \tag{3.6}$$

If $K \in \mathcal{K}_o^n$, then

$$\ell_{\phi K} = \ell_K \circ \phi^{-1}. \tag{3.7}$$

Hence, we have the following analogs of the properties that were studied in Section 1.3. Let $\mathcal{S}$ denote a space of real-valued functions defined on $\mathbb{R}^n$. A map Z defined on $\mathcal{S}$ is called *translation invariant* if $\mathrm{Z}(u \circ \tau_x^{-1}) = \mathrm{Z}(u)$ for every $x \in \mathbb{R}^n$ and $u \in \mathcal{S}$ with $u \circ \tau_x^{-1} \in \mathcal{S}$. Furthermore, Z is said to be $\mathrm{SL}(n)$ *invariant* if $\mathrm{Z}(u \circ \phi^{-1}) = \mathrm{Z}(u)$ for every $\phi \in \mathrm{SL}(n)$ and $u \in \mathcal{S}$ with $u \circ \phi^{-1} \in \mathcal{S}$.

We say that a measure-valued map $\mu : \mathcal{S} \to \mathcal{M}(\mathbb{S}^{n-1})$ is $\mathrm{SL}(n)$ *contravariant of degree* $p \in \mathbb{R}$ if

$$\int_{\mathbb{S}^{n-1}} b(z) \,\mathrm{d}\mu(u \circ \phi^{-1}, z) = \int_{\mathbb{S}^{n-1}} b(\phi^{-t} z) \,\mathrm{d}\mu(u, z),$$

for every $\phi \in \mathrm{SL}(n)$, every $u \in \mathcal{S}$ with $u \circ \phi^{-1} \in \mathcal{S}$ and every continuous, p-homogeneous function $b : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$.

Moreover, an operator $\mathrm{Z} : \mathcal{S} \to \mathcal{K}^n$ is said to be $\mathrm{SL}(n)$ *covariant* if $\mathrm{Z}(u \circ \phi^{-1}) = \phi\, \mathrm{Z}(u)$ for every $\phi \in \mathrm{SL}(n)$ and $u \in \mathcal{S}$ with $u \circ \phi^{-1} \in \mathcal{S}$ and it is called $\mathrm{SL}(n)$ *contravariant* if $\mathrm{Z}(u \circ \phi^{-1}) = \phi^{-t}\, \mathrm{Z}(u)$. Furthermore, Z is *translation covariant* if there exists a function $\mathrm{Z}^0 : \mathcal{S} \to \mathbb{R}$ associated with Z such that

$$\mathrm{Z}(u \circ \tau_x^{-1}) = \mathrm{Z}(u) + \mathrm{Z}^0(u) x,$$

for every $x \in \mathbb{R}^n$ and $u \in \mathcal{S}$ with $u \circ \tau_x^{-1} \in \mathcal{S}$.

Lastly, for a map $\mathrm{Y} : \mathrm{LC}(\mathbb{R}^n) \to \mathbb{R}$ we say that Y is *homogeneous of degree* $q \in \mathbb{R}$ if $\mathrm{Y}(sf) = s^q\, \mathrm{Y}(f)$ for every $s > 0$ and $f \in \mathrm{LC}(\mathbb{R}^n)$. Similarly, one defines homogeneity for an operator defined on $\mathrm{LC}(\mathbb{R}^n)$ and taking values in $\mathcal{M}(\mathbb{S}^{n-1})$ or $\mathcal{K}^n$.

Remark 3.2. Let $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \langle A, + \rangle$ be a valuation where $\langle A, + \rangle$ is an abelian semigroup. Let $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ be such that $u \vee v \in \mathrm{Conv}(\mathbb{R}^n)$ and let $f, g \in \mathrm{LC}(\mathbb{R}^n)$ be such that $f = e^{-u}$ and $g = e^{-v}$. Since

$$f \vee g = e^{-(u \wedge v)} \qquad \text{and} \qquad f \wedge g = e^{-(u \vee v)},$$

the map Z is a valuation if and only if $\mathrm{Y} : \mathrm{LC}(\mathbb{R}^n) \to \langle A, + \rangle$ is a valuation, where

$$\mathrm{Y}(f) = \mathrm{Z}(-\log f),$$

for every $f \in \mathrm{LC}(\mathbb{R}^n)$. Furthermore, for $u_k, u \in \mathrm{Conv}(\mathbb{R}^n)$ we have $u_k \xrightarrow{epi} u$ if and only if $e^{-u_k} \xrightarrow{hypo} e^{-u}$. Hence, if $\langle A, + \rangle$ is a topological semigroup, then Z is continuous if and only if Y is continuous. Moreover, for $x \in \mathbb{R}^n$ we have $f \circ \tau_x^{-1} = e^{-u \circ \tau_x^{-1}}$ and therefore Z is translation invariant if and only if Y is translation invariant. Similarly, translation covariance, $\mathrm{SL}(n)$ invariance and $\mathrm{SL}(n)$ covariance are equivalent for valuations on $\mathrm{LC}(\mathbb{R}^n)$ and their counterparts on $\mathrm{Conv}(\mathbb{R}^n)$. Therefore, studying valuations on $\mathrm{LC}(\mathbb{R}^n)$ is equivalent to studying valuations on $\mathrm{Conv}(\mathbb{R}^n)$ and it will be convenient for us to switch between these points of view. Lastly, we want to point out that $u \mapsto qu$ is continuous for all $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $q > 0$, and therefore $f \mapsto f^q$ is a continuous map from $\mathrm{LC}(\mathbb{R}^n)$ to $\mathrm{LC}(\mathbb{R}^n)$.

For $k \geq 0$, we say that a non-negative function $\zeta \in C(\mathbb{R})$ has *finite k-th moment* if

$$\int_0^\infty t^k \zeta(t)\,\mathrm{d}t < \infty.$$

Furthermore, we define

$$D^k(\mathbb{R}) := \big\{ \zeta \in C(\mathbb{R}) \,:\, \zeta \geq 0,\, \zeta \text{ is decreasing and has finite } k\text{-th moment} \big\}.$$

We conclude this section with two lemmas that will be needed in order to define measure-valued valuations and Minkowski valuations on $\mathrm{Conv}(\mathbb{R}^n)$ and $\mathrm{LC}(\mathbb{R}^n)$ respectively.

Lemma 3.3. *For $k \geq 0$ and $\zeta \in D^k(\mathbb{R})$ there exists $\xi \in D^k(\mathbb{R})$ such that ξ is smooth, strictly decreasing and $\xi(t) > \zeta(t)$ for every $t \in \mathbb{R}$.*

Proof. Fix $\varepsilon > 0$ and let $\rho_\varepsilon \in C^\infty(\mathbb{R})$ denote a standard mollifying kernel such that $\int_{\mathbb{R}^n} \rho_\varepsilon \,\mathrm{d}x = 1$ and $\rho_\varepsilon(x) \geq 0$ for all $x \in \mathbb{R}^n$ while the support of ρ_ε is contained in $[-\varepsilon, \varepsilon]$. Write τ_ε for the translation $t \mapsto t + \varepsilon$ on $\mathbb{R}$ and define $\xi(t)$ for $t \in \mathbb{R}$ by

$$\xi(t) = (\rho_\varepsilon \star (\zeta \circ \tau_\varepsilon^{-1}))(t) + e^{-t} = \int_{-\varepsilon}^{+\varepsilon} \zeta(t - \varepsilon - s)\rho_\varepsilon(s)\,\mathrm{d}s + e^{-t}.$$

It is easy to see, that ξ is non-negative and smooth. Since $t \mapsto \int_{-\varepsilon}^{+\varepsilon} \zeta(t - \varepsilon - s)\rho_\varepsilon(s)\,\mathrm{d}s$ is decreasing, ξ is strictly decreasing. Since

$$\int_{-\varepsilon}^{+\varepsilon} \zeta(t - \varepsilon - s)\rho_\varepsilon(s)\,\mathrm{d}s \geq \int_{-\varepsilon}^{+\varepsilon} \zeta(t)\rho_\varepsilon(s)\,\mathrm{d}s = \zeta(t),$$

we get $\xi(t) > \zeta(t)$ for every $t \in \mathbb{R}$. Finally, ξ has finite k-th moment, since $t \mapsto e^{-t}$ has finite k-th moment and

$$\begin{aligned}
\int_0^{+\infty} t^k \int_{-\varepsilon}^{+\varepsilon} \zeta(t - \varepsilon - s)\rho_\varepsilon(s)\,\mathrm{d}s\,\mathrm{d}t &= \int_{-\varepsilon}^{+\varepsilon} \rho_\varepsilon(s) \int_0^{+\infty} t^k \zeta(t - \varepsilon - s)\,\mathrm{d}t\,\mathrm{d}s \\
&\leq \int_{-\varepsilon}^{+\varepsilon} \rho_\varepsilon(s)\,\mathrm{d}s \int_0^{+\infty} t^k \zeta(t - 2\varepsilon)\,\mathrm{d}t < +\infty.
\end{aligned}$$

■

Lemma 3.4. *For $k \geq 0$ let $\xi \in D^k(\mathbb{R})$. If ξ is smooth and strictly decreasing, then*

$$\int_0^{\xi(b)} (\xi^{-1}(t) - b)^{k+1} \,\mathrm{d}t < +\infty,$$

for every $b \in \mathbb{R}$.

Proof. Using the substitution $t = \xi(s)$ and integration by parts, we obtain

$$\begin{aligned}\int_0^{\xi(b)} (\xi^{-1}(t) - b)^{k+1} \,\mathrm{d}t &= - \int_b^{+\infty} \underbrace{(s-b)^{k+1}\xi'(s)}_{<0} \,\mathrm{d}s \\ &\leq - \underbrace{\liminf_{s \to +\infty} (s-b)^{k+1}\xi(s)}_{\in[0,+\infty]} + (k+1) \underbrace{\int_b^{+\infty} (s-b)^k \xi(s) \,\mathrm{d}s}_{<+\infty} < +\infty.\end{aligned}$$

■

3.2 Real-Valued Valuations

The aim of this section is to find analogs of the Euler characteristic and volume on $\mathrm{Conv}(\mathbb{R}^n)$ and $\mathrm{LC}(\mathbb{R}^n)$, respectively. Furthermore, we study their properties, such as their behavior with respect to the $\mathrm{SL}(n)$ and translations.

Section 3.2.1 is original to this thesis. The results of Sections 3.2.2 and 3.2.3 for valuations on $\mathrm{Conv}(\mathbb{R}^n)$ can be found in [17] whereas the corresponding results on $\mathrm{LC}(\mathbb{R}^n)$ are proved in [47].

3.2.1 A Simple Approach

Before we consider general dimensions, we start with some basic observations in the 1-dimensional case. Let $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}) \to \mathbb{R}$ be a continuous and translation invariant valuation. By Lemma 2.14, (3.3) and (3.5), the map $K \mapsto \mathrm{Z}(\mathrm{I}_K + t)$ defines a continuous and translation invariant valuation on K^1 for every $t \in \mathbb{R}$ and therefore has to be a linear combination of the Euler characteristic V_0 and volume V_1 (see, for example, [28, p. 39]). Similarly, by (3.4), the map $K \mapsto \mathrm{Z}(\ell_K + t)$ defines a continuous valuation on $\mathcal{K}_o^1$ for every $t \in \mathbb{R}$. However, there exist infinitely many valuations of this type, e.g. if $K = [-a, b] \in \mathcal{K}_o^1$ with $a, b \geq 0$, then $[-a, b] \mapsto \alpha(a) + \beta(b)$ defines a continuous valuation for every continuous $\alpha, \beta : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Having said this, we will see in Section 4.2 that in the higher-dimensional setting with the additional assumption of $\mathrm{SL}(n)$ invariance only linear combinations of the Euler characteristic and volume will remain. Hence, in the following we consider the two cases that $K \mapsto \mathrm{Z}(\ell_K + t)$ is either a multiple of the Euler characteristic or volume and in both cases we want to find an explicit representation of Z.

We begin with the case of the Euler characteristic, that is we assume that there exists a constant c_t such that $\mathrm{Z}(\ell_K + t) = c_t V_0(K) = c_t$ for every $K \in \mathcal{K}_o^1$. Obviously, by the continuity of Z the map $t \mapsto c_t$ is continuous. Hence, there exists a continuous function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$\mathrm{Z}(\ell_K + t) = \psi(t), \tag{3.8}$$

for every $K \in \mathcal{K}_o^1$ and $t \in \mathbb{R}$.

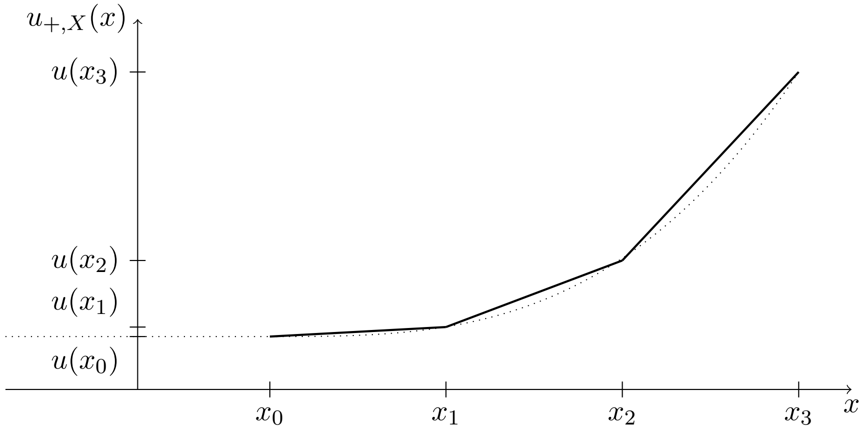

Figure 3.1: Illustration of of $u_{+,X}$.

Fix an arbitrary $u \in \mathrm{Conv}(\mathbb{R})$. Without loss of generality let $u(0) = \min_{x\in\mathbb{R}^n} u(x)$ and let $u_+ := u + \mathrm{I}_{[0,+\infty)}$. Observe, that u_+ is increasing on its domain. Similarly, $u_- := u + \mathrm{I}_{(-\infty,0]}$ is decreasing on its domain and by the valuation property of Z we have

$$\mathrm{Z}(u_+) + \mathrm{Z}(u_-) = \mathrm{Z}(u) + \mathrm{Z}(u + \mathrm{I}_{\{0\}})$$

and therefore

$$\mathrm{Z}(u) = \mathrm{Z}(u_+) + \mathrm{Z}(u_-) - \psi(u(0)). \tag{3.9}$$

We will now calculate $\mathrm{Z}(u_+)$, provided that $\mathrm{dom}\, u_+ \neq \emptyset$. First, assume that $\mathrm{dom}\, u_+$ is bounded, that is $\mathrm{dom}\, u_+ = [0, b]$ with $b \in \mathbb{R}$ and furthermore that $u(b) < +\infty$. Let

$$X := \{0 = x_0 < x_1 < \cdots < x_n = b\}$$

denote a *partition* of $[0, b]$ and let $u_{+,X}$ denote the piecewise affine approximation of u_+ that arises from X,

$$u_{+,X}(x) = \begin{cases} u(x_i) + \frac{x-x_i}{x_{i+1}-x_i}(u(x_{i+1}) - u(x_i)), & x_i \le x \le x_{i+1} \\ +\infty, & x \notin \mathrm{dom}\, u_+. \end{cases}$$

See also Figure 3.1. It is easy to see, that $u_{+,X} \xrightarrow{epi} u_+$ as the *norm* $|X| = \max_{1\le i\le n}(x_i - x_{i-1})$ of X approaches zero and by the valuation property

$$\mathrm{Z}(u_{+,X}) = \sum_{i=0}^{n-1} \mathrm{Z}(u_{+,X} + \mathrm{I}_{[x_i,x_{i+1}]}) - \sum_{i=1}^{n-1} \mathrm{Z}(u_{+,X} + \mathrm{I}_{\{x_i\}}).$$

Note, that for $0 \le i \le n-1$ we have

$$\begin{aligned}\mathrm{Z}(u_{+,X} + \mathrm{I}_{[x_i,x_{i+1}]}) =& \mathrm{Z}\left(\ell_{\left[0,\frac{x_{i+1}-x_i}{u(x_{i+1})-u(x_i)}\right]} \circ \tau_{x_i}^{-1} + u(x_i)\right) - \mathrm{Z}\left(\ell_{\left[0,\frac{x_{i+1}-x_i}{u(x_{i+1})-u(x_i)}\right]} \circ \tau_{x_{i+1}}^{-1} + u(x_{i+1})\right) \\ &+ \mathrm{Z}(u + \mathrm{I}_{\{x_{i+1}\}}),\end{aligned}$$

see also Figure 3.2. By (3.8) and the translation invariance of Z this reduces to

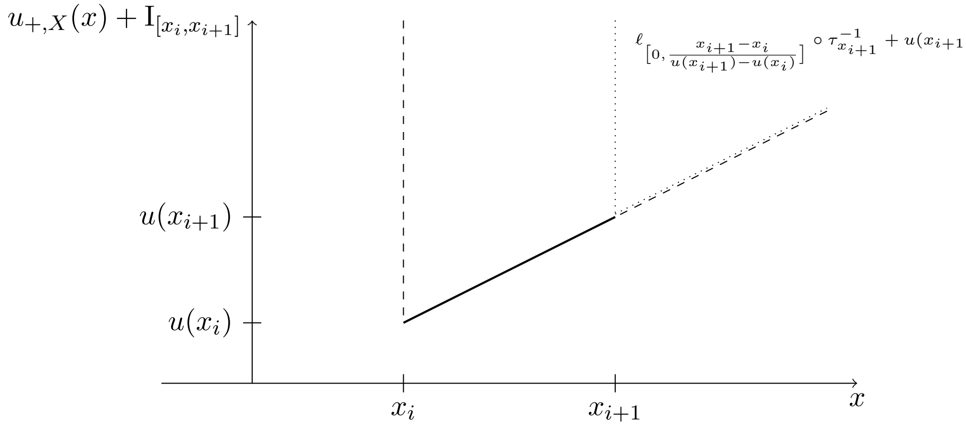

Figure 3.2: Inclusion-exclusion principle for $u_{+,X} + \mathrm{I}_{[x_i,x_{i+1}]}$.

$$\mathrm{Z}(u_{+,X} + \mathrm{I}_{[x_i,x_{i+1}]}) = \psi(u(x_i)) - \psi(u(x_{i+1})) + \psi(u(x_{i+1})) = \psi(u(x_i)),$$

for every $0 \leq i \leq n-1$. Therefore,

$$\mathrm{Z}(u_{+,X}) = \sum_{i=0}^{n-1} \psi(u(x_i)) - \sum_{i=1}^{n-1} \psi(u(x_i)) = \psi(u(x_0)) = \psi(u(0)),$$

and furthermore by continuity

$$\mathrm{Z}(u_+) = \psi(u(0)).$$

In the case $\lim_{x\to b^-} u(x) = +\infty$, consider the sequence $u_{+,k} := u_+ + \mathrm{I}_{[0,b-1/k]}$, $k \in \mathbb{N}$. Since $u_{+,k} \xrightarrow{epi} u_+$ and by the continuity of Z, we obtain again $\mathrm{Z}(u_+) = \psi(u(0))$. Similarly, if $\operatorname{dom} u_+$ is unbounded, consider the sequence $u_+ + \mathrm{I}_{[0,k]}$ to obtain $\mathrm{Z}(u_+) = \psi(u(0))$.

In the same way, one shows that $\mathrm{Z}(u_-) = \psi(u(0))$. Therefore, it follows from (3.9) that

$$\mathrm{Z}(u) = \psi(u(0)) + \psi(u(0)) - \psi(u(0)) = \psi(u(0)) = \psi(\min_{x\in\mathbb{R}^n} u(x)).$$

We will see in Lemma 3.5 that for general dimensions any continuous function composed with the minimum of a convex function is a continuous, $\mathrm{SL}(n)$ and translation invariant valuation that can be understood as a functional analog of the Euler characteristic.

Next, we want to find an analog for the volume (or length) in the 1-dimensional case. Therefore, let $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}) \to \mathbb{R}$ be again a continuous and translation invariant valuation but this time there exists a continuous function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$\mathrm{Z}(\ell_K + t) = \psi(t) V_1(K), \tag{3.10}$$

for every $K \in \mathcal{K}_o^1$ and $t \in \mathbb{R}$. Consider the function $u_h = \ell_{[0,1/h]} + \mathrm{I}_{[0,1]}$ for $h > 0$. Note, that by (3.10) and translation invariance, Z vanishes on functions with 0-dimensional domain. Thus, by translation invariance and the valuation property,

$$\mathrm{Z}(u_h + t) = \mathrm{Z}(\ell_{[0,1/h]} + t) - \mathrm{Z}(\ell_{[0,1/h]} \circ \tau_1^{-1} + t + h) = \frac{\psi(t) - \psi(t+h)}{h}$$

for every $t \in \mathbb{R}$. Since $u_h + t \xrightarrow{epi} \mathrm{I}_{[0,1]} + t$ as $h \searrow 0$, it follows that also $u_h + t + k \xrightarrow{epi} \mathrm{I}_{[0,1]} + t$ whenever $h \searrow 0$, $k \to 0$ and therefore by the continuity of Z

$$\mathrm{Z}(\mathrm{I}_{[0,1]} + t) = \lim_{h\to 0^+, k\to 0} \frac{\psi(t+k) - \psi(t+h+k)}{h} = -\psi'(t).$$

In particular, ψ is continuously differentiable. Moreover, consider the function $u_b = \ell_{[0,1]} + \mathrm{I}_{[0,b]}$ for any $b > 0$. Again, since Z is translation invariant and vanishes on functions with 0-dimensional domain, we have

$$\mathrm{Z}(u_b + t) = \mathrm{Z}(\ell_{[0,1]} + t) - \mathrm{Z}(\ell_{[0,1]} \circ \tau_b^{-1} + t + b) = \psi(t) - \psi(t+b),$$

for every $t \in \mathbb{R}$. Since $u_b \xrightarrow{epi} \ell_{[0,1]}$ as $b \to +\infty$, it follows that $\psi(t) \to 0$ as $t \to +\infty$. Therefore

$$\int_0^{+\infty} \psi'(t)\,\mathrm{d}t = \lim_{t\to+\infty} \psi(t) - \psi(0) = -\psi(0).$$

Fix an arbitrary $u \in \mathrm{Conv}(\mathbb{R})$ and let u_+ and u_- be defined as before. Assume first that $\operatorname{dom} u_+ = [0,b]$ with $b \in \mathbb{R}$ and $u(b) < +\infty$. Denote the piecewise affine approximation of u_+ that arises from the partition X by $u_{+,X}$. Since Z vanishes on functions with 0-dimensional domain and by translation invariance we have

$$\mathrm{Z}(u_{+,X}) = \sum_{i=0}^{n-1} \mathrm{Z}(u_{+,X} + \mathrm{I}_{[x_i,x_{i+1}]}),$$

with

$$\mathrm{Z}(u_{+,X} + \mathrm{I}_{[x_i,x_{i+1}]}) = \mathrm{Z}\left(\ell_{\left[0,\frac{x_{i+1}-x_i}{u(x_{i+1})-u(x_i)}\right]} \circ \tau_{x_i}^{-1} + u(x_i)\right) - \mathrm{Z}\left(\ell_{\left[0,\frac{x_{i+1}-x_i}{u(x_{i+1})-u(x_i)}\right]} \circ \tau_{x_{i+1}}^{-1} + u(x_{i+1})\right).$$

By (3.10) and the translation invariance of Z this reduces to

$$\mathrm{Z}(u_{+,X} + \mathrm{I}_{[x_i,x_{i+1}]}) = \tfrac{x_{i+1}-x_i}{u(x_{i+1})-u(x_i)}(\psi(u(x_i)) - \psi(u(x_{i+1}))).$$

Hence,

$$\mathrm{Z}(u_{+,X}) = \sum_{i=0}^{n-1} -\frac{\psi(u(x_{i+1})) - \psi(u(x_i))}{u(x_{i+1}) - u(x_i)}(x_{i+1} - x_i) \longrightarrow -\int_{\operatorname{dom} u_+} \psi'(u(x))\,\mathrm{d}x = \mathrm{Z}(u_+),$$

as $|X|$ approaches zero. Using continuity arguments, one can show that this representation also holds for unbounded domains and unbounded functions u. Moreover, a similar representation can be obtained for u_-. Thus,

$$\mathrm{Z}(u) = -\int_{\operatorname{dom} u} \psi'(u(x))\,\mathrm{d}x.$$

In Section 2.6 we saw that the integral of a quasi-concave function can be interpreted as a generalization of the n-dimensional volume. In Section 3.2.3 we will introduce the integral of a non-negative function composed with a convex function as a functional analog of the n-dimensional volume on $\mathrm{Conv}(\mathbb{R}^n)$. Furthermore, these examples show that under certain circumstances a valuation Z on $\mathrm{Conv}(\mathbb{R})$ is uniquely determined by its values on the functions $\ell_K + t$ with $K \in \mathcal{K}_o^1$ and $t \in \mathbb{R}$. We will prove a more general version of this observation in Lemma 4.1.

3.2.2 The Euler Characteristic

We show that the previously discussed functional analog of the Euler characteristic for quasi-concave functions, (2.11), can be generalized to a more general class of functions.

Lemma 3.5. *For $\zeta \in C(\mathbb{R})$, the map*

$$u \mapsto \zeta\big(\min_{x\in\mathbb{R}^n} u(x)\big) \tag{3.11}$$

is a continuous, $\mathrm{SL}(n)$ *and translation invariant valuation on* $\mathrm{Conv}(\mathbb{R}^n)$*.*

Proof. Let $u \in \mathrm{Conv}(\mathbb{R}^n)$. Since

$$\min_{x\in\mathbb{R}^n} u(x) = \min_{x\in\mathbb{R}^n} u(\tau x) = \min_{x\in\mathbb{R}^n} u(\phi^{-1}x),$$

for every $\phi \in \mathrm{SL}(n)$ and translation $\tau : \mathbb{R}^n \to \mathbb{R}^n$, (3.11) defines an $\mathrm{SL}(n)$ and translation invariant map. It is easy to see that

$$e^{-\min_{x\in\mathbb{R}^n} u(x)} = \int_{\mathbb{R}} V_0(\{u \le t\})e^{-t}\,\mathrm{d}t.$$

If $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ are such that $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$, then the valuation property of V_0 implies that

$$\begin{aligned} e^{-\min_{x\in\mathbb{R}^n}(u\vee v)(x)} + e^{-\min_{x\in\mathbb{R}^n}(u\wedge v)(x)} &= \int_{\mathbb{R}} \big(V_0(\{u\vee v \le t\}) + V_0(\{u\wedge v \le t\})\big)e^{-t}\,\mathrm{d}t \\ &= \int_{\mathbb{R}} \big(V_0(\{u \le t\}\cap\{v \le t\}) + V_0(\{u \le t\}\cup\{v \le t\})\big)e^{-t}\,\mathrm{d}t \\ &= \int_{\mathbb{R}} \big(V_0(\{u \le t\}) + V_0(\{v \le t\})\big)e^{-t}\,\mathrm{d}t \\ &= e^{-\min_{x\in\mathbb{R}^n} u(x)} + e^{-\min_{x\in\mathbb{R}^n} v(x)}. \end{aligned}$$

Therefore, since

$$\min_{x\in\mathbb{R}^n}(u\wedge v)(x) = \min\{\min_{x\in\mathbb{R}^n} u(x), \min_{x\in\mathbb{R}^n} v(x)\},$$

we obtain that

$$\min_{x\in\mathbb{R}^n}(u\vee v)(x) = \max\{\min_{x\in\mathbb{R}^n} u(x), \min_{x\in\mathbb{R}^n} v(x)\}.$$

Hence, a function $\zeta \in C(\mathbb{R})$ composed with the minimum of a function $u \in \mathrm{Conv}(\mathbb{R}^n)$ defines a valuation on $\mathrm{Conv}(\mathbb{R}^n)$. The continuity of (3.11) follows from Lemma 2.12. ■

Remark 3.6. For an alternative proof of the valuation property see [12, Lemma 3.7].

Lemma 3.7. *For every $q \in \mathbb{R}$, the map*

$$f \mapsto V_0(f)^q = \big(\max_{x\in\mathbb{R}^n} f(x)\big)^q$$

is a continuous, $\mathrm{SL}(n)$ *and translation invariant valuation on* $\mathrm{LC}(\mathbb{R}^n)$ *that is homogeneous of degree q.*

Proof. Since

$$\big(\max_{x\in\mathbb{R}^n} s\, f(x)\big)^q = s^q\big(\max_{x\in\mathbb{R}^n} f(x)\big)^q,$$

for every $f \in \mathrm{LC}(\mathbb{R}^n)$ and $s > 0$, the map $f \mapsto V_0(f)^q$ is homogeneous of degree q. By Remark 3.2 and Lemma 3.5 it is a continuous, $\mathrm{SL}(n)$ and translation invariant valuation. ■

3.2.3 Volume

Let $\zeta \in C(\mathbb{R})$ be non-negative. For $u \in \operatorname{Conv}(\mathbb{R}^n)$, define

$$\mathrm{Z}_\zeta(u) := \int_{\operatorname{dom} u} \zeta(u(x))\,\mathrm{d}x.$$

We want to investigate conditions on ζ such that Z_ζ defines a continuous valuation on $\operatorname{Conv}(\mathbb{R}^n)$.

It is easy to see, that in order for $\mathrm{Z}_\zeta(u)$ to be finite for every $u \in \operatorname{Conv}(\mathbb{R}^n)$, it is necessary for ζ to have finite $(n-1)$-st moment. Indeed, if $u(x) = |x|$, then

$$\mathrm{Z}_\zeta(u) = \int_{\mathbb{R}^n} \zeta(|x|)\,\mathrm{d}x = n\,v_n \int_0^{+\infty} t^{n-1}\zeta(t)\,\mathrm{d}t. \tag{3.12}$$

We will see in Lemma 3.9 that this condition is also sufficient. First, we require the following result.

Lemma 3.8. *Let u_k be a sequence in $\operatorname{Conv}(\mathbb{R}^n)$ with epi-limit $u \in \operatorname{Conv}(\mathbb{R}^n)$. If $\zeta \in C(\mathbb{R})$ is non-negative with finite $(n-1)$-st moment, then, for every $\varepsilon > 0$, there exist $t_0 \in \mathbb{R}$ and $k_0 \in \mathbb{N}$ such that*

$$\int_{\operatorname{dom} u \cap \{u > t\}} \zeta(u(x))\,\mathrm{d}x < \varepsilon \qquad \text{and} \qquad \int_{\operatorname{dom} u_k \cap \{u_k > t\}} \zeta(u_k(x))\,\mathrm{d}x < \varepsilon$$

for every $t \geq t_0$ and $k \geq k_0$.

Proof. Without loss of generality, let $\min_{x\in\mathbb{R}^n} u(x) = u(0)$. By the definition of epi-convergence, there exists a sequence x_k in $\mathbb{R}^n$ such that $x_k \to 0$ and $u_k(x_k) \to u(0)$. Therefore, there exists $k_0 \in \mathbb{N}$ such that $|x_k| < 1$ and $u_k(x_k) < u(0) + 1$ for every $k \geq k_0$. By Lemma 2.16, there exist constants $a > 0$ and $\bar{b} \in \mathbb{R}$, such that

$$u(x), u_k(x) > a|x| + \bar{b},$$

for every $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$. Setting $\widetilde{u}_k(x) = u_k(x - x_k)$, we have

$$\widetilde{u}_k(x) > a|x - x_k| + \bar{b} \geq a|x| - a|x_k| + \bar{b} \geq a|x| + (\bar{b} - a),$$

for every $k \geq k_0$. Hence, with $b = \bar{b} - a$, we have

$$u(x), \widetilde{u}_k(x) > a|x| + b, \tag{3.13}$$

for every $x \in \mathbb{R}^n$ and $k \geq k_0$.

For $x \in \mathbb{R}^n$ we use polar coordinates, that is $x = r\omega$ with $r \in [0, +\infty)$ and $\omega \in \mathbb{S}^{n-1}$. For $u(r\omega) \geq 1$, we obtain from (3.13) that

$$r^{n-1} < \left(\frac{u(r\omega)}{a} - \frac{b}{a}\right)^{n-1} \leq c\,u(r\omega)^{n-1} \qquad \text{and similarly} \qquad r^{n-1} < c\,\widetilde{u}_k(r\omega)^{n-1}, \tag{3.14}$$

for every $r \in [0, +\infty)$, $\omega \in \mathbb{S}^{n-1}$ and $k \geq k_0$, where c only depends on a, b and the dimension n. Now choose $\bar{t}_0 \geq \max\{1, 2(u(0)+1) - b\}$. Then for all $t \geq \bar{t}_0$

$$\frac{t - u(0)}{t - b} \geq \frac{1}{2} \qquad \text{and} \qquad \frac{t - (u(0)+1)}{t - b} \geq \frac{1}{2}. \tag{3.15}$$

For $\omega \in \mathbb{S}^{n-1}$, let $v_\omega(r) := u(r\omega)$. The function v_ω is non-decreasing and convex on $[0,+\infty)$. In particular, the left and right derivatives, $v'_{\omega,l}, v'_{\omega,r}$ of v_ω exist and for the subgradient $\partial v_\omega(r) = [v'_{\omega,l}, v'_{\omega,r}]$. Furthermore, it follows from $r < \bar{r}$ that $\eta \le \bar{\eta}$ for $\eta \in \partial v_\omega(r)$ and $\bar{\eta} \in \partial v_\omega(\bar{r})$.

For $t \ge \bar{t}_0$, set

$$D_\omega(t) := \{r \in [0,+\infty) \,:\, t < u(r\omega) < +\infty\}.$$

For every $\omega \in \mathbb{S}^{n-1}$, the set $D_\omega(t)$ is either empty or there exists

$$r_\omega(t) = \inf D_\omega(t) \le \tfrac{t-b}{a} \tag{3.16}$$

and $v_\omega(r_\omega(t)) = t$. Therefore, if $D_\omega(t)$ is non-empty, we have

$$t - u(0) \le \xi\, r_\omega(t)$$

for $\xi \in \partial v_\omega(r_\omega(t))$. Hence, it follows from (3.16) and (3.15) that

$$\vartheta \ge \xi \ge \frac{t-u(0)}{r_\omega(t)} \ge \frac{a(t-u(0))}{t-b} \ge \frac{a}{2}, \tag{3.17}$$

for all $r \in D_\omega(t)$, $\vartheta \in \partial v_\omega(r)$ and $\xi \in \partial v_\omega(r_\omega(t))$. Similarly, setting $\widetilde{v}_{k,\omega}(r) = \widetilde{u}_k(r\omega)$ and

$$\widetilde{D}_{k,\omega}(t) = \{r \in [0,+\infty) \,:\, t < u_k(r\omega) < +\infty\},$$

it is easy to see that $\widetilde{v}_{k,\omega}$ is convex on $[0,+\infty)$ and monotone increasing on $\widetilde{D}_{k,\omega}(t)$ for all $k \ge k_0$. By the choice of $\bar{t}_0$ and (3.15) we have

$$\vartheta \ge \frac{a}{2},$$

for every $t \ge \bar{t}_0$, $r \in D_{k,\omega}(t)$, $k \ge k_0$ and $\vartheta \in \partial \widetilde{v}_{k,\omega}(r)$. Recall, that as a convex function v_ω is locally Lipschitz and differentiable almost everywhere on the interior of its domain. Using polar coordinates, (3.14) and the substitution $v_\omega(r) = s$, we obtain from (3.17) that

$$\begin{aligned}\int_{\operatorname{dom} u \cap \{u>t\}} \zeta(u(x))\,\mathrm{d}x &= \int_{\mathbb{S}^{n-1}} \int_{D_\omega(t)} r^{n-1}\zeta(v_\omega(r))\,\mathrm{d}r\,\mathrm{d}\omega \\ &\le c\int_{\mathbb{S}^{n-1}} \int_{D_\omega(t)} v_\omega(r)^{n-1}\zeta(v_\omega(r))\,\mathrm{d}r\,\mathrm{d}\omega \\ &\le \frac{2\,n\,v_n\,c}{a}\int_t^{+\infty} s^{n-1}\zeta(s)\,\mathrm{d}s\end{aligned} \tag{3.18}$$

for every $t \ge \bar{t}_0$. In the same way,

$$\int_{\operatorname{dom} u_k \cap \{u_k>t\}} \zeta(u_k(x))\,\mathrm{d}x = \int_{\operatorname{dom} \widetilde{u}_k \cap \{\widetilde{u}_k>t\}} \zeta(\widetilde{u}_k(x))\,\mathrm{d}x \le \frac{2\,n\,v_n\,c}{a}\int_t^{+\infty} s^{n-1}\zeta(s)\,\mathrm{d}s,$$

for every $t \ge \bar{t}_0$ and $k \ge k_0$ with the same constant c as in (3.18). The statement now follows, since ζ is non-negative and has finite $(n-1)$-st moment. ■

Lemma 3.9. *Let $\zeta \in C(\mathbb{R})$ be non-negative. Then $\mathrm{Z}_\zeta(u) < +\infty$ for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ if and only if ζ has finite $(n-1)$-st moment.*

Proof. As already pointed out in (3.12), it is necessary for ζ to have finite $(n-1)$-st moment in order for Z_ζ to be finite.

Now let $u \in \mathrm{Conv}(\mathbb{R}^n)$ be arbitrary, let ζ have finite $(n-1)$-st moment and let $u_{\min} = \min_{x\in\mathbb{R}^n} u(x)$. By Lemma 3.8, there exists $t \in \mathbb{R}$ such that

$$\int_{\mathrm{dom}\, u \cap \{u>t\}} \zeta(u(x))\,\mathrm{d}x \le 1.$$

It follows that

$$\mathrm{Z}_\zeta(u) = \int_{\mathrm{dom}\, u} \zeta(u(x))\,\mathrm{d}x = \int_{\{u\le t\}} \zeta(u(x))\,\mathrm{d}x + \int_{\mathrm{dom}\, u\cap\{u>t\}} \zeta(u(x))\,\mathrm{d}x \le \max_{s\in[u_{\min},t]} \zeta(s)\, V_n(\{u\le t\}) + 1$$

and hence $\mathrm{Z}_\zeta(u) < \infty$. ■

Lemma 3.10. *For $\zeta \in C(\mathbb{R})$ non-negative and with finite $(n-1)$-st moment, the functional Z_ζ is continuous on $\mathrm{Conv}(\mathbb{R}^n)$.*

Proof. Let $u \in \mathrm{Conv}(\mathbb{R}^n)$ and let u_k be a sequence in $\mathrm{Conv}(\mathbb{R}^n)$ such that $u_k \xrightarrow{epi} u$. Set $u_{\min} = \min_{x\in\mathbb{R}^n} u(x)$. By Lemma 3.8, it is enough to show that

$$\int_{\{u_k\le t\}} \zeta(u_k(x))\,\mathrm{d}x \to \int_{\{u\le t\}} \zeta(u(x))\,\mathrm{d}x$$

for every fixed $t > u_{\min}$. Lemma 2.12 implies that $\{u_k \le t\} \to \{u \le t\}$ in the Hausdorff metric. Furthermore, by Lemma 2.16, there exists $b \in \mathbb{R}$ such that $u(x), u_k(x) > b$ for $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$. Set $c = \max_{s\in[b,t]} \zeta(s) \ge 0$. We distinguish the following cases.

- $\dim(\mathrm{dom}\, u) < n$:
  In this case $V_n(\{u \le t\}) = 0$ and since volume is continuous on convex sets, $V_n(\{u_k \le t\}) \to 0$. Hence,
  $$0 \le \int_{\{u_k\le t\}} \zeta(u_k(x))\,\mathrm{d}x \le c\, V_n(\{u_k \le t\}) \to 0.$$

- $\dim(\mathrm{dom}\, u) = n$:
  In this case, $\{u \le t\}$ is a set in $\mathcal{K}^n$ with non-empty interior. Therefore, for $\varepsilon > 0$ there exist $k_0 \in \mathbb{N}$ and $C \in \mathcal{K}^n$ such that for every $k \ge k_0$ the following hold:
  $$C \subset \mathrm{int}(\{u \le t\}) \cap \{u_k \le t\},$$
  $$V_n(\{u \le t\} \cap C^c) \le \frac{\varepsilon}{3c},$$
  $$V_n(\{u_k \le t\} \cap C^c) \le \frac{\varepsilon}{3c},$$
  where C^c is the complement of C. Note, that $u(x), u_k(x) \in [b,t]$ for $x \in C$ and $k \ge k_0$. Since $C \subset \mathrm{int}\,\mathrm{dom}\, u$, Theorem 2.8 implies that u_k converges to u uniformly on C. Since ζ is continuous,

the restriction of ζ to $[b,t]$ is uniformly continuous. Hence, $\zeta \circ u_k$ converges uniformly to $\zeta \circ u$ on C. Therefore, there exists $k_1 \geq k_0$ such that

$$|\zeta(u(x)) - \zeta(u_k(x))| \leq \frac{\varepsilon}{3V_n(C)},$$

for all $x \in C$ and $k \geq k_1$. This gives

$$\begin{aligned}
\Big| \int_{\{u \leq t\}} & \zeta(u(x))\,\mathrm{d}x - \int_{\{u_k \leq t\}} \zeta(u_k(x))\,\mathrm{d}x \Big| \\
&\leq \int_C |\zeta(u(x)) - \zeta(u_k(x))|\,\mathrm{d}x + \int_{\{u \leq t\} \cap C^c} \zeta(u(x))\,\mathrm{d}x + \int_{\{u_k \leq t\} \cap C^c} \zeta(u_k(x))\,\mathrm{d}x \\
&\leq V_n(C) \frac{\varepsilon}{3V_n(C)} + c\frac{\varepsilon}{3c} + c\frac{\varepsilon}{3c} = \varepsilon,
\end{aligned}$$

for $k \geq k_1$. The statement now follows, since $\varepsilon > 0$ was arbitrary. ∎

Lemma 3.11. *For $\zeta \in C(\mathbb{R})$ non-negative and with finite $(n-1)$-st moment, the functional Z_ζ is a continuous, $\mathrm{SL}(n)$ and translation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n)$.*

Proof. By Lemma 3.10, the map Z_ζ is continuous. Furthermore, it is easy to see that Z_ζ is $\mathrm{SL}(n)$ and translation invariant. It remains to show the valuation property. Therefore, let $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ be such that $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$. We have

$$\begin{aligned}
\mathrm{Z}_\zeta(u \wedge v) &= \int_{\mathrm{dom}\, v \cap \{v<u\}} \zeta(v(x))\,\mathrm{d}x + \int_{\mathrm{dom}\, v \cap \{u=v\}} \zeta(v(x))\,\mathrm{d}x + \int_{\mathrm{dom}\, u \cap \{u<v\}} \zeta(u(x))\,\mathrm{d}x, \\
\mathrm{Z}_\zeta(u \vee v) &= \int_{\mathrm{dom}\, u \cap \{v<u\}} \zeta(u(x))\,\mathrm{d}x + \int_{\mathrm{dom}\, u \cap \{u=v\}} \zeta(u(x))\,\mathrm{d}x + \int_{\mathrm{dom}\, v \cap \{u<v\}} \zeta(v(x))\,\mathrm{d}x.
\end{aligned}$$

Hence,

$$\mathrm{Z}_\zeta(u \wedge v) + \mathrm{Z}_\zeta(u \vee v) = \mathrm{Z}_\zeta(u) + \mathrm{Z}_\zeta(v)$$

which proves the valuation property. ∎

In the following we consider the special case $\zeta(t) = e^{-qt}$ with $q > 0$, to obtain a valuation on log-concave functions.

Lemma 3.12. *For every $q > 0$, the map*

$$f \mapsto V_n(f^q) = \int_{\mathbb{R}^n} f^q(x)\,\mathrm{d}x,$$

is a continuous, $\mathrm{SL}(n)$ and translation invariant valuation on $\mathrm{LC}(\mathbb{R}^n)$ that is homogeneous of degree q.

Proof. By Remark 3.2 and Lemma 3.11, the map $f \mapsto V_n(f^q)$ is a well-defined, continuous, $\mathrm{SL}(n)$ and translation invariant valuation on $\mathrm{LC}(\mathbb{R}^n)$. Since

$$\int_{\mathbb{R}^n} (sf)^q(x)\,\mathrm{d}x = s^q \int_{\mathbb{R}^n} f^q(x)\,\mathrm{d}x,$$

for every $f \in \mathrm{LC}(\mathbb{R}^n)$ and $s > 0$, it is homogeneous of degree q. ∎

We need the following calculation of the volume of a specific function.

Lemma 3.13. *For $r>0$, $q>0$ and $T_r = \operatorname{conv}\{0, r\, e_1, e_2, \dots, e_n\}$,*

$$V_n(e^{-q\ell_{T_r}}) = \tfrac{r}{q^n}.$$

Proof. By definition we have

$$V_n(e^{-q\ell_{T_r}}) = \int_0^1 V_n(\{e^{-q\ell_{T_r}} \leq t\})\,\mathrm{d}t = \int_0^1 V_n(\{\ell_{T_r} \leq -\tfrac{\log t}{q}\})\,\mathrm{d}t.$$

Using the substitution $s = -\frac{\log t}{q}$ we have $\mathrm{d}t = -qe^{-qs}\,\mathrm{d}s$ and therefore

$$V_n(e^{-q\ell_{T_r}}) = q\int_0^{+\infty} V_n(\{\ell_{T_r} \leq s\})\, e^{-qs}\,\mathrm{d}s.$$

By definition, $\{\ell_{T_r} \leq s\} = s\, T_r$ for every $s \geq 0$. Hence,

$$V_n(\{\ell_{T_r} \leq s\}) = s^n V_n(T_r) = s^n \tfrac{r}{n!}.$$

This gives

$$V_n(e^{-q\ell_{T_r}}) = \tfrac{r}{n!}\int_0^{+\infty} s^n e^{-qs} q\,\mathrm{d}s = \tfrac{1}{q^n}\tfrac{r}{n!}\int_0^{+\infty} (qs)^n e^{-qs} q\,\mathrm{d}s = \tfrac{1}{q^n}\tfrac{r}{n!}\int_0^{+\infty} p^n e^{-p}\,\mathrm{d}p = \tfrac{r}{q^n}.$$

■

3.3 Measure-Valued Valuations

On the Sobolev space $W^{1,1}(\mathbb{R}^n)$ (that is, the space of functions $f \in L^1(\mathbb{R}^n)$ with weak gradient $\nabla f \in L^1(\mathbb{R}^n)$), Gaoyong Zhang [60] defined the *projection body* $\Pi\,\langle f\rangle$, given by

$$h(\Pi\,\langle f\rangle, y) = \int_{\mathbb{R}^n} |y \cdot \nabla f(x)|\,\mathrm{d}x,$$

for $y \in \mathbb{R}^n$. The operator that associates to f the convex body $\Pi\,\langle f\rangle$ is easily seen to be $\mathrm{SL}(n)$ contravariant. The projection body of f turned out to be critical in Zhang's affine Sobolev inequality [60], which is a sharp affine isoperimetric inequality essentially stronger than the L^1 Sobolev inequality. The convex body $\Pi\,\langle f\rangle$ is the classical projection body of another convex body $\langle f\rangle$, which is the unit ball of the so-called optimal Sobolev norm of f and was introduced by Lutwak, Yang and Zhang [40]. The operator $f \mapsto \langle f\rangle$ is called the *LYZ operator* and it is $\mathrm{SL}(n)$ covariant. Furthermore, in [36], a characterization of the operators $f \mapsto \Pi\,\langle f\rangle$ and $f \mapsto \langle f\rangle$ as $\mathrm{SL}(n)$ contravariant and $\mathrm{SL}(n)$ covariant valuations was established.

In this section, we extend the *LYZ measure*, that is, the surface area measure of the image of the LYZ operator, to functions $\zeta \circ u$, where $\zeta \in D^{n-2}(\mathbb{R})$ and $u \in \mathrm{Conv}(\mathbb{R}^n)$. Following [40], for $W^{1,1}(\mathbb{R}^n)$ not vanishing a.e., we define the Borel measure $S(\langle f\rangle, \cdot)$ on $\mathbb{S}^{n-1}$ (using the Riesz-Markov-Kakutani representation theorem) by the condition that

$$\int_{\mathbb{S}^{n-1}} b(z)\,\mathrm{d}S(\langle f\rangle, z) = \int_{\mathbb{R}^n} b(-\nabla f(x))\,\mathrm{d}x \tag{3.19}$$

for every $b : \mathbb{R}^n \to \mathbb{R}$ that is continuous and positively 1-homogeneous. Since the LYZ measure $S(\langle f \rangle, \cdot)$ is not concentrated on a great subsphere of $\mathbb{S}^{n-1}$ (see [40]), the solution to the Minkowski problem implies that there is a unique convex body $\langle f \rangle$ whose surface area measure is $S(\langle f \rangle, \cdot)$. See also Section 1.3.2 and [54, Section 8.2].

By the co-area formula, we may rewrite (3.19) if, in addition, $f = \zeta \circ u$ with $\zeta \in D^{n-2}(\mathbb{R})$ and $u \in \mathrm{Conv}(\mathbb{R}^n)$, as

$$\int_{\mathbb{S}^{n-1}} b(z) \,\mathrm{d}S(\langle f \rangle, z) = \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} b(z) \,\mathrm{d}S(\{f \geq t\}, z) \,\mathrm{d}t.$$

This formula provides the motivation of our extension. The results of this section will appear in [16].

Lemma 3.14. *If $\zeta \in D^{n-2}(\mathbb{R})$, then*

$$\int_0^{+\infty} \mathcal{H}^{n-1}(\partial\{\zeta \circ u \geq t\}) \,\mathrm{d}t < +\infty$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$.

Proof. Fix $u \in \mathrm{Conv}(\mathbb{R}^n)$. By Lemma 3.3 there exists $\xi \in D^{n-2}(\mathbb{R})$ such that ξ is smooth, strictly decreasing and $\xi > \zeta$ pointwise. Thus, $\{\zeta \circ u \geq t\} \subseteq \{\xi \circ u \geq t\}$ for every $t \in \mathbb{R}$. Since those are compact convex sets for every $t \geq 0$, we obtain $\mathcal{H}^{n-1}(\partial\{\zeta \circ u \geq t\}) \leq \mathcal{H}^{n-1}(\partial\{\xi \circ u \geq t\})$ for every $t \in \mathbb{R}$. Hence, it is enough to show that

$$\int_0^{+\infty} \mathcal{H}^{n-1}(\partial\{\xi \circ u \geq t\}) \,\mathrm{d}t < +\infty.$$

By Lemma 2.2, there exist constants $a, b \in \mathbb{R}$ with $a > 0$ such that $u(x) > v(x) = a|x| + b$ for all $x \in \mathbb{R}^n$. Therefore $\xi \circ u < \xi \circ v$, which implies that $\{\xi \circ u \geq t\} \subset \{\xi \circ v \geq t\}$ for every $t > 0$. Hence,

$$\int_0^{+\infty} \mathcal{H}^{n-1}(\partial\{\xi \circ u \geq t\}) \,\mathrm{d}t < \int_0^{+\infty} \mathcal{H}^{n-1}(\partial\{\xi \circ v \geq t\}) \,\mathrm{d}t = \tfrac{n\, v_n}{a^{n-1}} \int_0^{\xi(b)} (\xi^{-1}(t) - b)^{n-1} \,\mathrm{d}t,$$

which is finite by Lemma 3.4. ■

Lemma and Definition 3.15. *For $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $\zeta \in D^{n-2}(\mathbb{R})$, a finite Borel measure $S(\langle \zeta \circ u \rangle, \cdot)$ on $\mathbb{S}^{n-1}$ is defined by the condition that*

$$\int_{\mathbb{S}^{n-1}} b(z) \,\mathrm{d}S(\langle \zeta \circ u \rangle, z) = \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} b(z) \,\mathrm{d}S(\{\zeta \circ u \geq t\}, z) \,\mathrm{d}t \tag{3.20}$$

for every continuous function $b : \mathbb{S}^{n-1} \to \mathbb{R}$. Moreover, if $u_k, u \in \mathrm{Conv}(\mathbb{R}^n)$ are such that $u_k \xrightarrow{epi} u$, then the measures $S(\langle \zeta \circ u_k \rangle, \cdot)$ converge weakly to $S(\langle \zeta \circ u \rangle, \cdot)$.

Proof. For fixed $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $\zeta \in D^{n-2}(\mathbb{R})$, we have

$$\left| \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} b(z) \,\mathrm{d}S(\{\zeta \circ u \geq t\}, z) \,\mathrm{d}t \right| \leq \max_{z \in \mathbb{S}^{n-1}} |b(z)| \int_0^{+\infty} \mathcal{H}^{n-1}(\partial\{\zeta \circ u \geq t\}) \,\mathrm{d}t$$

for every continuous function $b : \mathbb{S}^{n-1} \to \mathbb{R}$. Hence Lemma 3.14 shows that

$$b \mapsto \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} b(z) \,\mathrm{d}S(\{\zeta \circ u \geq t\}, z) \,\mathrm{d}t$$

defines a non-negative, bounded, linear functional on the space of continuous functions on $\mathbb{S}^{n-1}$. It follows from the Riesz-Markov-Kakutani representation theorem (see, for example, [52]), that there exists a unique Borel measure $S(\langle\zeta \circ u\rangle, \cdot)$ on $\mathbb{S}^{n-1}$ such that (3.20) holds. Moreover, the measure is finite.

Next, let $u_k, u \in \mathrm{Conv}(\mathbb{R}^n)$ with $u_k \xrightarrow{epi} u$ and fix a continuous function $b : \mathbb{S}^{n-1} \to \mathbb{R}$. By Lemma 2.14, the convex sets $\{u_k \leq t\}$ converge in the Hausdorff metric to $\{u \leq t\}$ for every $t \neq \min_{x\in\mathbb{R}^n} u(x)$, which implies the convergence of $\{\zeta \circ u_k \geq t\} \to \{\zeta \circ u \geq t\}$ for every $t \neq \max_{x\in\mathbb{R}^n} \zeta(u(x))$. Since the map $K \mapsto S(K, \cdot)$ is weakly continuous on the space of convex bodies, we obtain

$$\int_{\mathbb{S}^{n-1}} b(z)\,\mathrm{d}S(\{\zeta \circ u_k \geq t\}, z) \to \int_{\mathbb{S}^{n-1}} b(z)\,\mathrm{d}S(\{\zeta \circ u \geq t\}, z),$$

for a.e. $t \geq 0$. By Lemma 2.16, there exist $a, d \in \mathbb{R}$ with $a > 0$ such that $u_k(x) > v(x) = a|x| + d$ and therefore $\zeta \circ u_k(x) < \zeta \circ v(x)$ for $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$. By convexity,

$$\mathcal{H}^{n-1}(\partial\{\zeta \circ u_k \geq t\}) < \mathcal{H}^{n-1}(\partial\{\zeta \circ v \geq t\})$$

for every $k \in \mathbb{N}$ and $t > 0$ and therefore

$$\begin{aligned}\Big|\int_{\mathbb{S}^{n-1}} b(z)\,\mathrm{d}S(\{\zeta \circ u_k \geq t\}, z)\Big| &\leq \max_{z\in\mathbb{S}^{n-1}} |b(z)|\ \mathcal{H}^{n-1}(\partial\{\zeta \circ u_k \geq t\})\\ &< \max_{z\in\mathbb{S}^{n-1}} |b(z)|\ \mathcal{H}^{n-1}(\partial\{\zeta \circ v \geq t\}).\end{aligned}$$

By Lemma 3.14, the function $t \mapsto \int_{\mathbb{S}^{n-1}} |b(z)|\,\mathrm{d}S(\{\zeta \circ v \geq t\}, z)$ is integrable. Hence, we can apply the dominated convergence theorem to conclude the proof. ■

We say that an operator $\mu : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{M}(\mathbb{S}^{n-1})$ is *decreasing*, if the real-valued function $u \mapsto \mu(u, \mathbb{S}^{n-1})$ is decreasing on $\mathrm{Conv}(\mathbb{R}^n)$, that is, if $u \geq v$, then

$$\mu(u, \mathbb{S}^{n-1}) \leq \mu(v, \mathbb{S}^{n-1}). \tag{3.21}$$

Similarly, we define *increasing* and we say that μ is *monotone* if it is decreasing or increasing.

Remark 3.16. Another attempt to define a decreasing valuation $\mu : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{M}(\mathbb{S}^{n-1})$ would be

$$\mu(u, \omega) \leq \mu(v, \omega), \tag{3.22}$$

whenever $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ such that $u \geq v$ and for all Borel sets $\omega \subset \mathbb{S}^{n-1}$. Clearly, every μ that satisfies (3.22) also satisfies (3.21). However, such a property would fail for any non-trivial measure-valued valuation μ that attempts to generalize the surface area measure of a convex body, e.g. that there exists $c \in \mathbb{R}$ such that $\mu(\mathrm{I}_K, \cdot) = cS(K, \cdot)$ for every $K \in \mathcal{K}^n$. Since $\mathrm{I}_{\sqrt{n}B^n} \leq \mathrm{I}_{[-1,1]^n}$ and

$$0 = S(\sqrt{n}B^n, \{e_1\}) \leq S([-1,1]^n, \{e_1\}) = 2^{n-1},$$

c can not be positive. On the other hand, $\mathrm{I}_{[-1,1]^n} \leq \mathrm{I}_{[0,1]^n}$ but

$$2^{n-1} = S([-1,1]^n, \{e_1\}) \geq S([0,1]^n, \{e_1\}) = 1.$$

Hence, μ must vanish on indicator functions. Similarly, if there exists $\widetilde{c} \in \mathbb{R}$ such that $\mu(\ell_K, \cdot) = \widetilde{c}S(K, \cdot)$ for every $K \in \mathcal{K}^n_o$, then $\widetilde{c} = 0$. We will see in Lemma 4.1 that under the additional assumptions of translation invariance and continuity, this implies that $\mu(u, \omega) = 0$ for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $\omega \subset \mathbb{S}^{n-1}$.

For $u \in \mathrm{Conv}(\mathbb{R}^n)$, we write $u^- \in \mathrm{Conv}(\mathbb{R}^n)$ for the function that is defined via $u^-(x) = u(-x)$ for every $x \in \mathbb{R}^n$. Similarly, for $f \in \mathrm{LC}(\mathbb{R}^n)$ we write $f^-(x) = f(-x)$ for every $x \in \mathbb{R}^n$. Observe, that

$$\{u^- \leq t\} = -\{u \leq t\} \qquad \text{and} \qquad \{f^- \geq s\} = -\{f \geq s\}$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$, $f \in \mathrm{LC}(\mathbb{R}^n)$, $t \in \mathbb{R}$ and $s > 0$.

Lemma 3.17. *For $\zeta_1, \zeta_2 \in D^{n-2}(\mathbb{R})$, the map*

$$u \mapsto S(\langle \zeta_1 \circ u\rangle, \cdot) + S(\langle \zeta_2 \circ u^-\rangle, \cdot)$$

defines a weakly continuous, decreasing, translation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n)$ that is $\mathrm{SL}(n)$ contravariant of degree 1.

Proof. It is easy to see, that it suffices to proof the desired properties for the map

$$u \mapsto S(\langle \zeta \circ u\rangle, \cdot), \tag{3.23}$$

with $\zeta \in D^{n-2}(\mathbb{R})$. As $K \mapsto S(K, \cdot)$ is translation invariant, it follows from the definition that also $S(\langle \zeta \circ u\rangle, \cdot)$ is translation invariant. Lemma and Definition 3.15 gives weak continuity. If $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ are such that $u \geq v$, then

$$\{u \leq s\} \subseteq \{v \leq s\}, \qquad \{\zeta \circ u \geq t\} \subseteq \{\zeta \circ v \geq t\}$$

and consequently by convexity

$$S(\{\zeta \circ u \geq t\}, \mathbb{S}^{n-1}) \leq S(\{\zeta \circ v \geq t\}, \mathbb{S}^{n-1}),$$

for all $s \in \mathbb{R}$ and $t \geq 0$. For $\phi \in \mathrm{SL}(n)$,

$$\{\zeta \circ u \circ \phi^{-1} \geq t\} = \phi\{\zeta \circ u \geq t\},$$

and hence by the properties of surface area measure, we obtain

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} b(z)\,\mathrm{d}S(\langle \zeta \circ u \circ \phi^{-1}\rangle, z) &= \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} b(z)\,\mathrm{d}S(\phi\{\zeta \circ u \geq t\}, z)\,\mathrm{d}t \\ &= \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} b(\phi^{-t} z)\,\mathrm{d}S(\{\zeta \circ u \geq t\}, z)\,\mathrm{d}t \\ &= \int_{\mathbb{S}^{n-1}} b(\phi^{-t} z)\,\mathrm{d}S(\langle \zeta \circ u\rangle, z)\end{aligned}$$

for every continuous, 1-homogeneous function $b : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$. Finally, let $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ be such that $u \wedge v \in \mathrm{Conv}(\mathbb{R}^n)$. Since $\zeta \in D^{n-2}(\mathbb{R})$ is decreasing, we obtain by (2.3) and the valuation property of surface area measure that

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}} b(z)\,\mathrm{d}\big(S(\langle \zeta \circ (u \vee v)\rangle, z) + S(\langle \zeta \circ (u \wedge v)\rangle, z)\big) \\ &\quad= \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} b(z)\,\mathrm{d}\big(S(\{\zeta \circ u \wedge \zeta \circ v \geq t\}, z) + S(\{\zeta \circ u \vee \zeta \circ v \geq t\}, z)\big)\,\mathrm{d}t \\ &\quad= \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} b(z)\,\mathrm{d}\big(S(\{\zeta \circ u \geq t\} \cap \{\zeta \circ v \geq t\}, z) + S(\{\zeta \circ u \geq t\} \cup \{\zeta \circ v \geq t\}, z)\big)\,\mathrm{d}t \\ &\quad= \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} b(z)\,\mathrm{d}\big(S(\{\zeta \circ u \geq t\}, z) + S(\{\zeta \circ v \geq t\}, z)\big)\,\mathrm{d}t \\ &\quad= \int_{\mathbb{S}^{n-1}} b(z)\,\mathrm{d}\big(S(\langle \zeta \circ u\rangle, z) + \,\mathrm{d}S(\langle \zeta \circ v\rangle, z)\big).\end{aligned}$$

Hence (3.23) defines a valuation. ■

Remark 3.18. Since for every $u \in \operatorname{Conv}(\mathbb{R}^n)$ and $\zeta \in D^{n-2}(\mathbb{R})$ the measure $S(\langle \zeta \circ u \rangle, \cdot)$ is the surface area measure of a (not necessarily unique) convex body, a simple calculation shows that $S(\langle \zeta \circ u^- \rangle, \cdot) = S(-\langle \zeta \circ u \rangle, \cdot)$.

Lemma 3.19. *For every $q > 0$ and $c_1, c_2 \geq 0$, the map*

$$f \mapsto c_1 S(\langle f^q \rangle, \cdot) + c_2 S(\langle (f^-)^q \rangle, \cdot),$$

is a weakly continuous, translation invariant valuation on $\mathrm{LC}(\mathbb{R}^n)$ that is $\mathrm{SL}(n)$ contravariant of degree 1 and homogeneous of degree q.

Proof. By Remark 3.2 and Lemma 3.17, the map $f \mapsto c_1 S(\langle f^q \rangle, \cdot) + c_2 S(\langle (f^-)^q \rangle, \cdot)$ is a weakly continuous, translation invariant valuation that is $\mathrm{SL}(n)$ contravariant of degree 1. It remains to show homogeneity. Therefore, let $s > 0$ and $b : \mathbb{S}^{n-1} \to \mathbb{R}$ be continuous. Using the substitution $r = t/s^q$ we have

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} b(z) \, \mathrm{d}S(\langle (sf)^q \rangle, z) &= \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} b(z) \, \mathrm{d}S(\{s^q f^q \geq t\}, z) \, \mathrm{d}t \\
&= \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} b(z) \, \mathrm{d}S(\{f^q \geq \tfrac{t}{s^q}\}, z) \, \mathrm{d}t \\
&= s^q \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} b(z) \, \mathrm{d}S(\{f^q \geq r\}, z) \, \mathrm{d}r \\
&= s^q \int_{\mathbb{S}^{n-1}} b(z) \, \mathrm{d}S(\langle f^q \rangle, z),
\end{aligned}$$

for every $f \in \mathrm{LC}(\mathbb{R}^n)$. Similarly, one shows that $f \mapsto S(\langle (f^-)^q \rangle, \cdot)$ is homogeneous of degree q. ■

We need the following result where the LYZ measure of a specific function is calculated.

Lemma 3.20. *For $q > 0$, $t \in \mathbb{R}$ and $K \in \mathcal{K}_o^n$,*

$$S(\langle e^{-q(\ell_K + t)} \rangle, \cdot) = \tfrac{(n-1)!}{q^{n-1}} e^{-qt} S(K, \cdot).$$

Proof. Observe, that by Lemma 3.19 it is enough to show that $S(\langle e^{-q\,\ell_K} \rangle, \cdot) = \frac{(n-1)!}{q^{n-1}} S(K, \cdot)$. By Lemma and Definition 3.15 we have

$$S(\langle e^{-q\,\ell_K} \rangle, \cdot) = \int_0^1 S(\{e^{-q\,\ell_K} \geq s\}, \cdot) \, \mathrm{d}s) = \int_0^1 S(\{\ell_K \leq -\tfrac{\log s}{q}\}, \cdot) \, \mathrm{d}s.$$

Using the substitution $r = -\log s/q$ we have $\mathrm{d}s = -qe^{-qr} \, \mathrm{d}r$ and therefore

$$S(\langle e^{-q\,\ell_K} \rangle, \cdot) = q \int_0^{+\infty} S(\{\ell_K \leq r\}, \cdot) e^{-qr} \, \mathrm{d}r.$$

By definition, $\{\ell_K \leq r\} = rK$ for every $r \geq 0$. Hence,

$$S(\{\ell_K \leq r\}, \cdot) = r^{n-1} S(K, \cdot).$$

This gives

$$S(\langle e^{-q\,\ell_K} \rangle, \cdot) = S(K, \cdot) \tfrac{1}{q^{n-1}} \int_0^{+\infty} (qr)^{n-1} e^{-qr} q \, \mathrm{d}r = S(K, \cdot) \tfrac{1}{q^{n-1}} \int_0^{+\infty} s^{n-1} e^{-s} \, \mathrm{d}s = \tfrac{(n-1)!}{q^{n-1}} S(K, \cdot).$$

■

In [58], Tuo Wang extended the definition of the LYZ operator and the LYZ measure from $W^{1,1}(\mathbb{R}^n)$ to the space of functions of bounded variation, $\mathrm{BV}(\mathbb{R}^n)$. For a function $f \in L^1(\mathbb{R}^n)$, the *total variation* of f is given by

$$V(f,\mathbb{R}^n) = \sup\left\{\int_{\mathbb{R}^n} f \text{ div } \psi \,:\, \psi \in C_c^\infty(\mathbb{R}^n,\mathbb{R}^n), \|\psi\|_{L^\infty(\mathbb{R}^n)} \leq 1\right\},$$

where $C_c^\infty(\mathbb{R}^n,\mathbb{R}^n)$ denotes the set of smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ with compact support, $\operatorname{div}\psi$ is the *divergence* of $\psi \in C_c^\infty(\mathbb{R}^n,\mathbb{R}^n)$ and $\|\cdot\|_{L^\infty(\mathbb{R}^n)}$ is the *essential supremum norm.* We say that f is of *bounded variation*, $f \in \mathrm{BV}(\mathbb{R}^n)$, if $V(f,\mathbb{R}^n) < \infty$. For such a f, there exists a vector valued Radon measure $Df = (D_1 f, \dots, D_n f)$ on $\mathbb{R}^n$ with

$$\int_{\mathbb{R}^n} f \frac{\partial \phi}{\partial x_i} \,\mathrm{d}x = -\int_{\mathbb{R}^n} \phi \,\mathrm{d}D_i f,$$

for every continuously differentiable function $\phi : \mathbb{R}^n \to \mathbb{R}$ with compact support. Furthermore, we write $|Df|$ for the total variation of Df and σ_f for the corresponding Radon-Nikodým derivative. We can now state Wang's definition.

Definition 3.21. For $f \in BV(\mathbb{R}^n)$ which is not 0 a.e. with respect to the n-dimensional Lebesgue measure, the *LYZ body* is defined to be the unique convex body $\langle f\rangle$ with centroid at the origin, such that

$$\int_{\mathbb{S}^{n-1}} b(z) \,\mathrm{d}S(\langle f\rangle, z) = \int_{\mathbb{R}^n} b(-\sigma_f) \,\mathrm{d}|Df|,$$

for every $b : \mathbb{R}^n \to \mathbb{R}$ that is continuous and 1-homogeneous.

Remark 3.22. In [58], it is furthermore assumed that b is an even function. Hence, $S(\langle f\rangle, \cdot)$ is considered to be an even measure which determines an origin-symmetric convex body $\langle f\rangle$. The so obtained body is a symmetrization of the body given in Definition 3.21.

By the co-area formula [6, Theorem 3.40]

$$V(f,\mathbb{R}^n) = \int_{-\infty}^{+\infty} \operatorname{Per}(\{f > t\},\mathbb{R}^n) \,\mathrm{d}t = |Df|(\mathbb{R}^n),$$

where $\operatorname{Per}(A,\mathbb{R}^n)$ denotes the *perimeter* of $A \subseteq \mathbb{R}^n$. Since $\operatorname{Per}(K,\mathbb{R}^n) = \mathcal{H}^{n-1}(K)$ for every $K \in \mathcal{K}^n$ and $\operatorname{Per}(\emptyset,\mathbb{R}^n) = 0$, Lemma 3.14 shows that $V(\zeta \circ u,\mathbb{R}^n) < +\infty$ for for every $\zeta \in D^{n-2}(\mathbb{R})$ and $u \in \operatorname{Conv}(\mathbb{R}^n)$. Together with Lemma 3.9 this gives $\zeta \circ u \in \mathrm{BV}(\mathbb{R}^n)$ for every $\zeta \in D^{n-1}(\mathbb{R})$. Thus, Definition 3.21 gives a different definition of $S(\langle \zeta \circ u\rangle, \cdot)$ with $\zeta \in D^{n-1}(\mathbb{R})$ and $u \in \operatorname{Conv}(\mathbb{R}^n)$ with $\dim \operatorname{dom} u = n$ and it can be shown that in this case both definitions coincide. However, we also assign a non-trivial measure to functions whose support is $(n-1)$-dimensional and furthermore to functions $\zeta \circ u$ with $\zeta \in D^{n-2}(\mathbb{R})$, which are not necessarily in $\mathrm{BV}(\mathbb{R}^n)$.

Remark 3.23. Wang's definition allows to extend the LYZ operator to $\mathrm{BV}(\mathbb{R}^n)$ with values in the space of n-dimensional convex bodies. However, Wang's extended operators $f \mapsto S(\langle f\rangle, \cdot)$ and $f \mapsto \langle f\rangle$ are only semi-valuations (see [59] for the definition) but no longer valuations on $\mathrm{BV}(\mathbb{R}^n)$ and Wang [59] characterizes $f \mapsto \langle f\rangle$ as a Blaschke semi-valuation.

3.4 Minkowski Valuations

In this section we study functional analogs the projection body, the difference body and the moment vector on $\operatorname{Conv}(\mathbb{R}^n)$. Furthermore we find functional representations of the identity and the reflection on $\mathcal{K}^n$.

In the following, an operator $\mathrm{Z} : \operatorname{Conv}(\mathbb{R}^n) \to \mathcal{K}^n$ is called *decreasing* if $\mathrm{Z}(u) \subseteq \mathrm{Z}(v)$ for all $u, v \in \operatorname{Conv}(\mathbb{R}^n)$ such that $u \geq v$ and it is said to be *increasing* if $\mathrm{Z}(v) \subseteq \mathrm{Z}(u)$ for all $u, v \in \operatorname{Conv}(\mathbb{R}^n)$ such that $u \geq v$. Moreover, Z is *monotone* if it is decreasing or increasing.

The results for the projection body and the level set body of a convex function $u \in \operatorname{Conv}(\mathbb{R}^n)$ can be found in [16]. Most of the results for log-concave functions, especially the moment vector of a function $f \in \operatorname{LC}(\mathbb{R}^n)$, are to appear in [47].

3.4.1 Projection Body

By Definition 1.11 and the definition of the cosine transform in (1.2), the support function of the classical projection body is the cosine transform of the surface area measure. Since the measure $S(\langle \zeta \circ u \rangle, \cdot)$, defined in Lemma and Definition 3.15, is finite for all $\zeta \in D^{n-2}(\mathbb{R})$ and $u \in \operatorname{Conv}(\mathbb{R}^n)$, its cosine transform is finite as well. Hence, setting

$$h(\Pi \langle \zeta \circ u \rangle, z) = \tfrac{1}{2} \mathscr{C} S(\langle \zeta \circ u \rangle, \cdot)(z)$$

for $z \in \mathbb{S}^{n-1}$, defines a convex body $\Pi \langle \zeta \circ u \rangle$ for $\zeta \in D^{n-2}(\mathbb{R})$ and $u \in \operatorname{Conv}(\mathbb{R}^n)$. Here we use that the cosine transform of a measure gives a non-negative and sublinear function, which also shows that $\Pi \langle \zeta \circ u \rangle$ contains the origin. By the definition of the cosine transform and the definition of the LYZ measure $S(\langle \zeta \circ u \rangle, \cdot)$, we have

$$\begin{aligned} h(\Pi \langle \zeta \circ u \rangle, z) &= \tfrac{1}{2} \int_{\mathbb{S}^{n-1}} |y \cdot z| \, \mathrm{d}S(\langle \zeta \circ u \rangle, y) \\ &= \tfrac{1}{2} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} |y \cdot z| \, \mathrm{d}S(\{\zeta \circ u \geq t\}, y) \, \mathrm{d}t \\ &= \int_0^{+\infty} h(\Pi \{\zeta \circ u \geq t\}, z) \, \mathrm{d}t \end{aligned} \tag{3.24}$$

for $\zeta \in D^{n-2}(\mathbb{R})$, $u \in \operatorname{Conv}(\mathbb{R}^n)$ and $z \in \mathbb{S}^{n-1}$. Hence the projection body of $\zeta \circ u$ is a Minkowski average of the classical projection bodies of the level sets of $\zeta \circ u$.

Using the definition of the classical projection body (Definition 1.11), the definitions (2.10) and (2.12) of intrinsic volumes and projections of quasi-concave functions respectively and (2.13), we also obtain for $z \in \mathbb{S}^{n-1}$,

$$\begin{aligned} h(\Pi \langle \zeta \circ u \rangle, z) &= \int_0^{+\infty} h(\Pi \{\zeta \circ u \geq t\}, z) \, \mathrm{d}t \\ &= \int_0^{+\infty} V_{n-1}(\operatorname{proj}_{z^\perp} \{\zeta \circ u \geq t\}) \, \mathrm{d}t \\ &= \int_0^{+\infty} V_{n-1}(\{\operatorname{proj}_{z^\perp}(\zeta \circ u) \geq t\}) \, \mathrm{d}t \\ &= V_{n-1}(\operatorname{proj}_{z^\perp}(\zeta \circ u)). \end{aligned} \tag{3.25}$$

Thus, the definition of the projection body of the function $\zeta \circ u$ is analog to the definition of the projection body of a convex body. In [5], this connection was established for functions that are log-concave and in $W^{1,1}(\mathbb{R}^n)$.

Lemma 3.24. *For $\zeta \in D^{n-2}(\mathbb{R})$, the map*

$$u \mapsto \Pi \langle \zeta \circ u \rangle \tag{3.26}$$

defines a continuous, decreasing, $\mathrm{SL}(n)$ *contravariant and translation invariant Minkowski valuation on* $\mathrm{Conv}(\mathbb{R}^n)$.

Proof. Let $\zeta \in D^{n-2}(\mathbb{R})$ and $u \in \mathrm{Conv}(\mathbb{R}^n)$. By (1.3) and (3.24), we get for every $\phi \in \mathrm{SL}(n)$ and $z \in \mathbb{S}^{n-1}$,

$$\begin{aligned} h(\Pi \langle \zeta \circ u \circ \phi^{-1} \rangle, z) &= \int_0^\infty h(\Pi\{\zeta \circ u \circ \phi^{-1} \geq t\}, z) \,\mathrm{d}t \\ &= \int_0^\infty h(\Pi\, \phi\{\zeta \circ u \geq t\}, z) \,\mathrm{d}t \\ &= \int_0^\infty h(\phi^{-t}\, \Pi\{\zeta \circ u \geq t\}, z) \,\mathrm{d}t \\ &= \int_0^\infty h(\Pi\{\zeta \circ u \geq t\}, \phi^{-1} z) \,\mathrm{d}t \;=\; h(\Pi \langle \zeta \circ u, \rangle\, \phi^{-1} z). \end{aligned}$$

Similarly, we get for every $x \in \mathbb{R}^n$ and $z \in \mathbb{S}^{n-1}$,

$$h(\Pi \langle \zeta \circ u \circ \tau_x^{-1} \rangle, z) = h(\Pi \langle \zeta \circ u \rangle, z).$$

Thus for every $\phi \in \mathrm{SL}(n)$ and every $x \in \mathbb{R}^n$,

$$\Pi \langle \zeta \circ u \circ \phi^{-1} \rangle = \phi^{-t}\, \Pi \langle \zeta \circ u \rangle \quad \text{and} \quad \Pi \langle \zeta \circ u \circ \tau_x^{-1} \rangle = \Pi \langle \zeta \circ u \rangle$$

and the map defined in (3.26) is translation invariant and $\mathrm{SL}(n)$ contravariant. By Lemma 3.17, the map $u \mapsto S(\langle \zeta \circ u \rangle, \cdot)$ is a weakly continuous valuation. Hence, the definition of $\Pi \langle \zeta \circ u \rangle$ via the cosine transform and (1.1) imply that (3.26) is a continuous Minkowski valuation. Finally, let $\zeta \in D^{n-2}(\mathbb{R})$ and $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ be such that $u \geq v$. Then $\{\zeta \circ u \geq t\} \subseteq \{\zeta \circ v \geq t\}$ for every $t \geq 0$ and consequently, $h(\Pi\{\zeta \circ u \geq t\}, z) \leq h(\Pi\{\zeta \circ v \geq t\}, z)$ for every $z \in \mathbb{S}^{n-1}$ and $t \geq 0$. Hence, for every $z \in \mathbb{S}^{n-1}$,

$$h(\Pi \langle \zeta \circ u \rangle, z) = \int_0^{+\infty} h(\Pi\{\zeta \circ u \geq t\}, z) \,\mathrm{d}t \leq \int_0^{+\infty} h(\Pi\{\zeta \circ v \geq t\}, z) \,\mathrm{d}t = h(\Pi \langle \zeta \circ v \rangle, z),$$

or equivalently $\Pi \langle \zeta \circ u \rangle \subseteq \Pi \langle \zeta \circ v \rangle$. Thus, the map defined in (3.26) is decreasing. ∎

In the following we consider the homogeneous case for log-concave functions.

Lemma 3.25. *For every $q > 0$, the map*

$$f \mapsto \Pi \langle f^q \rangle \tag{3.27}$$

is a continuous, $\mathrm{SL}(n)$ *contravariant and translation invariant Minkowski valuation on* $\mathrm{LC}(\mathbb{R}^n)$ *that is homogeneous of degree q.*

Proof. Remark 3.2 and Lemma 3.24 imply that (3.27) defines a continuous, $\mathrm{SL}(n)$ contravariant and translation invariant Minkowski valuation on $\mathrm{LC}(\mathbb{R}^n)$. By Lemma 3.19 and (3.24) we have homogeneity of degree q. ∎

3.4.2 Identity, Reflection and Difference Body

Similar to the definition of the projection body of a convex function in (3.24) we want to define the difference body of a convex function. In order to do so, we will assign to each convex function a body that can be interpreted as a weighted Minkowski average of the level sets.

Lemma 3.26. *For $\zeta \in D^0(\mathbb{R})$,*

$$\left| \int_0^{+\infty} h(\{\zeta \circ u \geq t\}, z)\, \mathrm{d}t \right| < +\infty$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $z \in \mathbb{S}^{n-1}$.

Proof. Fix $u \in \mathrm{Conv}(\mathbb{R}^n)$. By Lemma 3.3 there exists $\xi \in D^0(\mathbb{R})$ such that ξ is smooth, strictly decreasing and $\xi > \zeta$ pointwise. Hence, $\{\zeta \circ u \geq t\} \subseteq \{\xi \circ u \geq t\}$ for every $t \geq 0$ and therefore it suffices to show that

$$\left| \int_0^{+\infty} h(\{\xi \circ u \geq t\}, z)\, \mathrm{d}t \right| < +\infty$$

for every $z \in \mathbb{S}^{n-1}$. By Lemma 2.2, there exist constants $a, b \in \mathbb{R}$ with $a > 0$ such that $u(x) > v(x) = a|x| + b$ for all $x \in \mathbb{R}^n$. Hence,

$$\left| \int_0^{+\infty} h(\{\xi \circ u \geq t\}, z)\, \mathrm{d}t \right| \leq \int_0^{+\infty} h(\{\xi \circ v \geq t\}, z)\, \mathrm{d}t = \frac{1}{a} \int_0^{\xi(b)} (\xi^{-1}(t) - b)\, \mathrm{d}t,$$

which is finite by Lemma 3.4. ∎

Lemma and Definition 3.27. *For $\zeta \in D^0(\mathbb{R})$ and $u \in \mathrm{Conv}(\mathbb{R}^n)$, the level set body $[\zeta \circ u]$ is defined by*

$$h([\zeta \circ u], z) = \int_0^{+\infty} h(\{\zeta \circ u \geq t\}, z)\, \mathrm{d}t,$$

for every $z \in \mathbb{S}^{n-1}$. Furthermore, the map $u \mapsto [\zeta \circ u]$ from $\mathrm{Conv}(\mathbb{R}^n)$ to $\mathcal{K}^n$ is a continuous, decreasing, $\mathrm{SL}(n)$ and translation covariant Minkowski valuation.

Proof. Let $u, v \in \mathrm{Conv}(\mathbb{R}^n)$ be such that $u \geq v$. Then

$$\{\zeta \circ u \geq t\} \subseteq \{\zeta \circ v \geq t\}$$

for every $t \geq 0$ and consequently,

$$h(\{\zeta \circ u \geq t\}, z) \leq h(\{\zeta \circ v \geq t\}, z)$$

for every $z \in \mathbb{S}^{n-1}$. Since the integral in the definition of $[\zeta \circ u]$ converges by Lemma 3.26, this shows that $u \mapsto [\zeta \circ u]$ is well-defined and decreasing on $\mathrm{Conv}(\mathbb{R}^n)$.

Now, let $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $u_k \in \mathrm{Conv}(\mathbb{R}^n)$ be such that $\text{epi-lim}_{k\to\infty} u_k = u$. By Lemma 2.14, the sets $\{u_k \leq t\}$ converge in the Hausdorff metric to $\{u \leq t\}$ for every $t \neq \min_{x\in\mathbb{R}^n} u(x)$, which is equivalent to the convergence $\{\zeta \circ u_k \geq t\} \to \{\zeta \circ u \geq t\}$ for every $t \neq \max_{x\in\mathbb{R}^n} \zeta(u(x))$. By Lemma 2.16, there exist constants $a, b \in \mathbb{R}$ with $a > 0$ such that for every $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$

$$u_k(x) > v(x) = a|x| + b$$

and therefore $\zeta(u_k(x)) < \zeta(v(x))$ for every $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$ and hence also

$$|h(\{\zeta \circ u_k \geq t\}, z)| \leq h(\{\zeta \circ v \geq t\}, z)$$

for every $t \geq 0, k \in \mathbb{N}$ and $z \in \mathbb{S}^{n-1}$ where we have used the symmetry of v. By Lemma 3.26, we can apply the dominated convergence theorem, which shows that $u \mapsto [\zeta \circ u]$ is continuous.

Finally, since

$$u \mapsto \{\zeta \circ u \geq t\}$$

defines an $\mathrm{SL}(n)$ and translation covariant Minkowski valuation for every $t \geq 0$, it is easy to see that also $u \mapsto [\zeta \circ u]$ has these properties. ■

Let $f = \zeta \circ u$ with $\zeta \in D^0(\mathbb{R})$ and $u \in \mathrm{Conv}(\mathbb{R}^n)$. By the definition of the level set body it is easy to see that $[f^-] = -[f]$. Furthermore, by the definition of the difference body, the projection of a quasi-concave function (2.12) and (2.13) we have

$$\begin{aligned} h(\mathrm{D}\,[f], z) &= h([f], z) + h(-[f], z) \\ &= \int_0^{+\infty} h(\{f \geq t\}, z) + h(-\{f \geq t\}, z)\,\mathrm{d}t \\ &= \int_0^{+\infty} h(\mathrm{D}\{f \geq t\}, z)\,\mathrm{d}t \\ &= \int_0^{+\infty} V_1(\mathrm{proj}_{\mathrm{lin}\{z\}}\{f \geq t\})\,\mathrm{d}t \\ &= V_1(\mathrm{proj}_{\mathrm{lin}\{z\}}\, f), \end{aligned}$$

for every $z \in \mathbb{S}^{n-1}$. This corresponds to the geometric interpretation of the projection body from (3.25).

Lemma 3.28. *For $\zeta \in D^0(\mathbb{R})$, the map $u \mapsto \mathrm{D}\,[\zeta \circ u]$ from $\mathrm{Conv}(\mathbb{R}^n)$ to $\mathcal{K}^n$ is a continuous, decreasing, $\mathrm{SL}(n)$ covariant and translation invariant Minkowski valuation.*

Proof. For every $x \in \mathbb{R}^n$, $z \in \mathbb{S}^{n-1}$ and $u \in \mathrm{Conv}(\mathbb{R}^n)$, we have

$$h(\mathrm{D}\,[\zeta \circ u \circ \tau_x^{-1}], z) = \int_0^{+\infty} h(\mathrm{D}\{\zeta \circ u \circ \tau_x^{-1} \geq t\}, z)\,\mathrm{d}t = \int_0^{+\infty} h(\mathrm{D}\{\zeta \circ u \geq t\}, z)\,\mathrm{d}t = h(\mathrm{D}\,[\zeta \circ u], z),$$

since the difference body operator is translation invariant. The further properties follow immediately from the properties of the level set body proved in Lemma and Definition 3.27. ■

Lemma 3.29. *For every $q > 0$, the map*

$$f \mapsto [f^q] \tag{3.28}$$

is a continuous, $\mathrm{SL}(n)$ and translation covariant Minkowski valuation on $\mathrm{LC}(\mathbb{R}^n)$ that is homogeneous of degree q. Moreover, the map

$$f \mapsto \mathrm{D}\,[f^q]$$

is a continuous, $\mathrm{SL}(n)$ covariant and translation invariant Minkowski valuation on $\mathrm{LC}(\mathbb{R}^n)$ that is homogeneous of degree q.

Proof. By Remark 3.2 and Lemma and Definition 3.27, the map $f \mapsto [f^q]$ is a well-defined, continuous, $\mathrm{SL}(n)$ and translation covariant Minkowski valuation on $\mathrm{LC}(\mathbb{R}^n)$. Furthermore, for $s > 0$, $z \in \mathbb{S}^{n-1}$ and $f \in \mathrm{LC}(\mathbb{R}^n)$, we have

$$\begin{aligned} h([(s\,f)^q], z) &= \int_0^{+\infty} h(\{(s\,f)^q \geq t\}, z)\,\mathrm{d}t \\ &= \int_0^{+\infty} h(\{f^q \geq \tfrac{t}{s^q}\}, z)\,\mathrm{d}t \\ &= s^q \int_0^{+\infty} h(\{f^q \geq r\}, z)\,\mathrm{d}r = s^q\, h([f^q], z), \end{aligned}$$

where we used the substitution $r = t/s^q$. Hence, (3.28) is homogeneous of degree q. The properties for the second map follow immediately from the properties of the difference body. ∎

3.4.3 Moment Vectors

The following lemma will allow us to give a definition for the moment vector of functions $\zeta \circ u$, where $\zeta \in D^n(\mathbb{R})$ and $u \in \mathrm{Conv}(\mathbb{R}^n)$.

Lemma 3.30. *For $\zeta \in D^n(\mathbb{R})$,*

$$\int_0^{+\infty} |h(\mathrm{m}(\{\zeta \circ u \geq t\}, z)|\,\mathrm{d}t < +\infty$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $z \in \mathbb{S}^{n-1}$.

Proof. Fix $u \in \mathrm{Conv}(\mathbb{R}^n)$ and observe, that for $K \in \mathcal{K}^n$ and $z \in \mathbb{S}^{n-1}$,

$$|h(\mathrm{m}(K), z)| = \left| \int_K x \cdot z\,\mathrm{d}x \right| \leq V_n(K) \max_{y \in \mathbb{S}^{n-1}} |h(K, y)|.$$

By Lemma 3.3 there exists $\xi \in D^n(\mathbb{R})$ such that ξ is smooth, strictly decreasing and $\xi > \zeta$ pointwise. Moreover, Lemma 2.2 shows that there exist constants $a, b \in \mathbb{R}$ with $a > 0$ such that

$$u(x) > v(x) = a|x| + b,$$

for every $x \in \mathbb{R}^n$. Hence, $\zeta \circ u < \xi \circ u < \xi \circ v$ pointwise and therefore

$$\{\zeta \circ u \geq t\} \subset \{\xi \circ v \geq t\} = \big\{x \,:\, |x| \leq \tfrac{\xi^{-1}(t) - b}{a}\big\}$$

for every $0 < t \leq \xi(b)$. This gives

$$|h(\mathrm{m}(\{\zeta \circ u \geq t\}), z)| \leq V_n(\{\xi \circ v \geq t\}) \max_{y \in \mathbb{S}^{n-1}} |h(\{\xi \circ v \geq t\}, y)| = \tfrac{v_n}{a^{n+1}} \big(\xi^{-1}(t) - b\big)^{n+1},$$

for every $0 < t \leq \xi(b)$ and $z \in \mathbb{S}^{n-1}$. Thus,

$$\int_0^{+\infty} |h(\mathrm{m}(\{\zeta \circ u \geq t\}), z)|\,\mathrm{d}t \leq \frac{v_n}{a^{n+1}} \int_0^{\xi(b)} (\xi^{-1}(t) - b)^{n+1}\,\mathrm{d}t,$$

for every $z \in \mathbb{S}^{n-1}$, which is finite by Lemma 3.4. ∎

By Lemma 3.30, the integral
$$\int_0^{+\infty} h(\mathrm{m}(\{\zeta\circ u\geq t\}),y)\,\mathrm{d}t \tag{3.29}$$
converges for every $\zeta\in D^n(\mathbb{R})$, $u\in\mathrm{Conv}(\mathbb{R}^n)$ and $y\in\mathbb{R}^n$. Since each of the support functions
$$y\mapsto h(\mathrm{m}(\{\zeta\circ u\geq t\}),y)$$
is sublinear, it is easy to see that (3.29) defines a sublinear function in y and thus is the support function of a convex body $\mathrm{m}(\zeta\circ u)\in\mathcal{K}^n$. Using the definition of the moment vector and the layer-cake principle, we obtain
$$h(\mathrm{m}(\zeta\circ u),y)=\int_0^{+\infty}\int_{\{\zeta\circ u\geq t\}} x\cdot y\,\mathrm{d}x\,\mathrm{d}t=\int_0^{+\infty}\int_{\mathbb{R}^n}\chi_{\{\zeta\circ u\geq t\}}(x)\,(x\cdot y)\,\mathrm{d}x\,\mathrm{d}t=\int_{\mathbb{R}^n}\zeta(u(x))\,(x\cdot y)\,\mathrm{d}x.$$
Hence,
$$\mathrm{m}(\zeta\circ u)=\int_{\mathbb{R}^n}\zeta(u(x))\,x\,\mathrm{d}x$$
is an element of $\mathbb{R}^n$ and will be called the moment vector of $\zeta\circ u$.

Lemma 3.31. *For $\zeta\in D^n(\mathbb{R})$, the map $u\mapsto\mathrm{m}(\zeta\circ u)$ from $\mathrm{Conv}(\mathbb{R}^n)$ to $\mathbb{R}^n$ is a continuous, $\mathrm{SL}(n)$ and translation covariant valuation.*

Proof. Fix $\zeta\in D^n(\mathbb{R})$. For $\phi\in\mathrm{SL}(n)$ we have
$$\mathrm{m}(\zeta\circ u\circ\phi^{-1})=\int_{\mathbb{R}^n}\zeta(u(\phi^{-1}x))\,x\,\mathrm{d}x=\int_{\mathbb{R}^n}\zeta(u(x))\,\phi x\,\mathrm{d}x=\phi\,\mathrm{m}(\zeta\circ u),$$
for every $u\in\mathrm{Conv}(\mathbb{R}^n)$, which shows $\mathrm{SL}(n)$ covariance. Furthermore, for $x\in\mathbb{R}^n$ we obtain
$$\mathrm{m}(\zeta\circ u\circ\tau_x^{-1})=\int_{\mathbb{R}^n}\zeta(u(y-x))\,y\,\mathrm{d}y=\int_{\mathbb{R}^n}\zeta(u(y))\,y\,\mathrm{d}y+x\int_{\mathbb{R}^n}\zeta(u(y))\,\mathrm{d}y=\mathrm{m}(\zeta\circ u)+V_n(\zeta\circ u)\,x,$$
which proves translation covariance. In order to show the valuation property, let $u,v\in\mathrm{Conv}(\mathbb{R}^n)$ such that $u\wedge v\in\mathrm{Conv}(\mathbb{R}^n)$. We have
$$\begin{aligned}\mathrm{m}(\zeta(u\vee v)) &= \int_{\{u\geq v\}}\zeta(u(x))\,x\,\mathrm{d}x+\int_{\{u<v\}}\zeta(v(x))\,x\,\mathrm{d}x\\ \mathrm{m}(\zeta(u\wedge v)) &= \int_{\{u\geq v\}}\zeta(v(x))\,x\,\mathrm{d}x+\int_{\{u<v\}}\zeta(u(x))\,x\,\mathrm{d}x.\end{aligned}$$
Hence,
$$\mathrm{m}(\zeta(u\vee v))+\mathrm{m}(\zeta(u\wedge v))=\mathrm{m}(\zeta(u))+\mathrm{m}(\zeta(v)).$$
It remains to show continuity. Let $u_k,u\in\mathrm{Conv}(\mathbb{R}^n)$ such that $u_k\xrightarrow{epi}u$. By Lemma 2.16, there exist $a>0$ and $b\in\mathbb{R}$ such that $u_k(x)>a|x|+b$ for every $k\in\mathbb{N}$ and $x\in\mathbb{R}^n$. Similar as in the proof of Lemma 3.30, this gives
$$|h(\mathrm{m}(\{\zeta\circ u_k\geq t\}),\cdot)|\leq V_n(\{\zeta\circ v\geq t\})\max_{y\in\mathbb{S}^{n-1}}|h(\{\zeta\circ v\geq t\},y)|,$$

which shows that these functions are dominated by an integrable function. Furthermore, Lemma 2.12 and the continuity of the moment vector on $\mathcal{K}^n$ imply that $h(\mathrm{m}(\{\zeta \circ u_k \geq t\}), \cdot) \to h(\mathrm{m}(\{\zeta \circ u \geq t\}), \cdot)$ pointwise for every $t \neq \max_{x\in\mathbb{R}^n} \zeta(u(x))$. Hence, by the dominated convergence theorem, we have

$$h(\mathrm{m}(\zeta \circ u_k), \cdot) = \int_0^{+\infty} h(\mathrm{m}(\{\zeta \circ u_k \geq t\}, \cdot)\,\mathrm{d}t \longrightarrow \int_0^{+\infty} h(\mathrm{m}(\{\zeta \circ u \geq t\}, \cdot)\,\mathrm{d}t = h(\mathrm{m}(\zeta \circ u), \cdot),$$

pointwise, which implies Hausdorff convergence of $\mathrm{m}(\zeta \circ u_k)$ to $\mathrm{m}(\zeta \circ u)$. ■

Lemma 3.32. *For every $q > 0$, the map*

$$f \mapsto \mathrm{m}(f^q)$$

is a continuous, $\mathrm{SL}(n)$ *and translation covariant Minkowski valuation on* $\mathrm{LC}(\mathbb{R}^n)$ *that is homogeneous of degree q.*

Proof. By Remark 3.2 and Lemma 3.31, the map $f \mapsto \mathrm{m}(f^q)$ is a continuous, $\mathrm{SL}(n)$ and translation covariant Minkowski valuation. Furthermore, it is homogeneous of degree q since

$$\mathrm{m}((sf)^q) = \int_{\mathbb{R}^n} (sf)^q(x)\,x\,\mathrm{d}x = s^q \int_{\mathbb{R}^n} f^q(x)\,x\,\mathrm{d}x = s^q\,\mathrm{m}(f^q),$$

for every $s > 0$ and $f \in \mathrm{LC}(\mathbb{R}^n)$. ■

Chapter 4

Classification of Valuations on Convex and Log-Concave Functions

"Never tell me the odds."

Han Solo

We characterize the valuations that were introduced and described in Chapter 3. The principle that is used for all our results is roughly the same. In Lemma 4.1 we will see that valuations on $\operatorname{Conv}(\mathbb{R}^n)$ are uniquely described by their behavior on cone functions. By (3.4) and the classification results for valuations on $\mathcal{K}_o^n$, which were stated in Section 1.3, the valuations on $\operatorname{Conv}(\mathbb{R}^n)$ and $\operatorname{LC}(\mathbb{R}^n)$ are then described by a number of continuous functions on the reals. Furthermore, there exist continuous functions that describe the behavior on indicator functions. It will turn out, that in many cases the latter are derivatives of the former, see for example Lemma 4.2. Based on this, a classification can be established.

The results from Section 4.1 and the classification of real-valued valuations on $\operatorname{Conv}(\mathbb{R}^n)$ will be published in [17]. The classification results for Minkowski valuations and measure-valued valuations on $\operatorname{Conv}(\mathbb{R}^n)$ can be found in [16]. Lastly, the proofs of Theorems 4.10 and 4.34 are to appear in [47].

4.1 General Considerations

The next result shows that in order to classify continuous and translation invariant or covariant valuations on $\operatorname{Conv}(\mathbb{R}^n)$, it is enough to know the behavior of these valuations on cone functions. The main argument of the following lemma is due to [36, Lemma 8], where it was used for functions on Sobolev spaces. For another recent application see [41, Lemma 8].

Lemma 4.1. *Let $\langle A,+\rangle$ be a topological abelian semigroup with cancellation law and let $\mathrm{Z}_1,\mathrm{Z}_2:\operatorname{Conv}(\mathbb{R}^n)\to\langle A,+\rangle$ be continuous. If $\mathrm{Z}_1(\ell\circ\tau^{-1})=\mathrm{Z}_2(\ell\circ\tau^{-1})$ for every $\ell\in\{\ell_P+t:\,P\in\mathcal{P}_{(o)}^n,\,t\in\mathbb{R}\}$ and every translation τ on $\mathbb{R}^n$, then $\mathrm{Z}_1\equiv\mathrm{Z}_2$ on $\operatorname{Conv}(\mathbb{R}^n)$.*

Proof. By Lemma 2.21 and the continuity of Z_1 and Z_2, it suffices to show that Z_1 and Z_2 coincide on $\operatorname{Conv}_{\text{p.a.}}(\mathbb{R}^n)$. So let $u\in\operatorname{Conv}_{\text{p.a.}}(\mathbb{R}^n)$ and set $U=\operatorname{epi}u$. Note, that U is a convex polyhedron in $\mathbb{R}^{n+1}$ and that none of the facet hyperplanes of U is parallel to the x_{n+1}-axis. Here, we say that a hyperplane H in $\mathbb{R}^{n+1}$ is a *facet hyperplane* of U if its intersection with the boundary of U has positive n-dimensional Hausdorff measure. Furthermore, we call U *singular* if U has n facet hyperplanes whose intersection contains a line parallel to $\{x_{n+1}=0\}$. Since Z_1 and Z_2 are continuous, we can assume that U is not singular.

Since U is not singular and $u\in\operatorname{Conv}_{\text{p.a.}}(\mathbb{R}^n)$, there exists a unique vertex, $\bar{p}$ of U with smallest x_{n+1} coordinate. We use induction on the number m of facet hyperplanes of U that are not passing

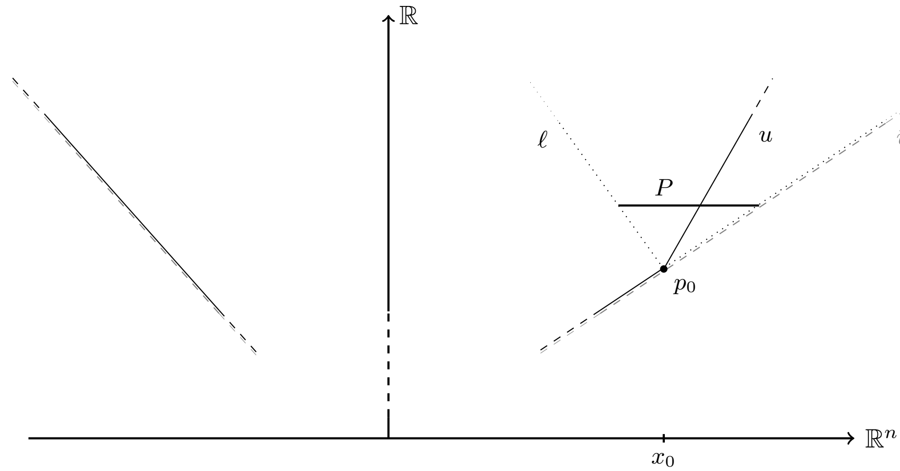

Figure 4.1: Illustration of u, $\bar{u}$ and ℓ.

through $\bar{p}$. If $m = 0$, then there exist $P \in \mathcal{P}^n_{(o)}$ and $t \in \mathbb{R}$ such that u is a translate of $\ell_P + t$ and therefore $\mathrm{Z}_1(u) = \mathrm{Z}_2(u)$.

Now let U have $m > 0$ facet hyperplanes that are not passing through $\bar{p}$ and assume that Z_1 and Z_2 coincide for all functions with at most $(m-1)$ such facet hyperplanes. Let $p_0 = (x_0, u(x_0)) \in \mathbb{R}^{n+1}$ where $x_0 \in \mathbb{R}^n$ is a vertex of U with maximal x_{n+1}-coordinate and let $H_1, \dots, H_j$ be the facet hyperplanes of U through p_0 such that the corresponding facets of U have infinite n-dimensional volume. Note, that $H_1, \dots, H_j$ do not contain $\bar{p}$ and therefore there is at least one such hyperplane. Define $\bar{U}$ as the polyhedron bounded by the intersection of all facet hyperplanes of U with the exception of $H_1, \dots, H_j$. Since U is not singular, there exists a function $\bar{u} \in \mathrm{Conv}_{\mathrm{p.a.}}(\mathbb{R}^n)$ with $\operatorname{dom} \bar{u} = \mathbb{R}^n$ such that $\bar{U} = \operatorname{epi} \bar{u}$. Note, that $\bar{U}$ has at most $(m-1)$ facet hyperplanes not containing $\bar{p}$. Hence, by the induction hypothesis

$$\mathrm{Z}_1(\bar{u}) = \mathrm{Z}_2(\bar{u}).$$

Let $\overline{H}_1, \dots, \overline{H}_i$ be the facet hyperplanes of $\bar{U}$ that contain p_0 such that the corresponding facets of $\bar{U}$ have infinite n-dimensional volume. Choose suitable hyperplanes $\overline{H}_{i+1}, \dots, \overline{H}_k$ not parallel to the x_{n+1}-axis and containing p_0 so that the hyperplanes $\overline{H}_1, \dots, \overline{H}_k$ bound a polyhedral cone with apex p_0 that is contained in $\bar{U}$, has $\overline{H}_1, \dots, \overline{H}_i$ among its facet hyperplanes and contains $\{x_0\} \times [u(x_0), +\infty)$. Define ℓ as the piecewise affine function determined by this polyhedral cone. Notice, that ℓ is a translate of $\ell_P + u(x_0)$, where $P \in \mathcal{P}^n_{(o)}$ is the projection onto the first n coordinates of the intersection of the polyhedral cone with $\{x_{n+1} = u(x_0) + 1\}$, see also Figure 4.1. Hence, Z_1 and Z_2 coincide on ℓ. Set $\bar{\ell} = u \vee \ell$. The epigraph of $\bar{\ell}$ is again a polyhedral cone with apex p_0. Hence $\bar{\ell}$ is a translate of $\ell_{\bar{P}} + u(x_0)$ with $\bar{P} \in \mathcal{P}^n_{(o)}$ since it is bounded by hyperplanes containing p_0 that are not parallel to the x_{n+1}-axis. Therefore, Z_1 and Z_2 also coincide on $\bar{\ell}$. We now have

$$u \wedge \ell = \bar{u}, \quad u \vee \ell = \bar{\ell}.$$

From the valuation property of Z_i, $i = 1, 2$, we obtain

$$\mathrm{Z}_1(u) + \mathrm{Z}_1(\ell) = \mathrm{Z}_1(\bar{u}) + \mathrm{Z}_1(\bar{\ell}) = \mathrm{Z}_2(\bar{u}) + \mathrm{Z}_2(\bar{\ell}) = \mathrm{Z}_2(u) + \mathrm{Z}_2(\ell),$$

which completes the proof. ■

The next result establishes a connection between the behavior of a valuation on cone functions and its behavior on indicator functions.

Lemma 4.2. *For $k \geq 1$, let $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^k) \to \mathbb{R}$ be a continuous, translation invariant valuation and let $\psi \in C(\mathbb{R})$. If*

$$\mathrm{Z}(\ell_P + t) = \psi(t) V_k(P) \tag{4.1}$$

for every $P \in \mathcal{P}_o^k$ and $t \in \mathbb{R}$, then

$$\mathrm{Z}(\mathrm{I}_{[0,1]^k} + t) = \frac{(-1)^k}{k!} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \psi(t)$$

for every $t \in \mathbb{R}$. In particular, ψ is k-times differentiable.

Proof. To explain the idea of the proof, we first consider the case $k = 1$. For $h > 0$, let $u_h = \ell_{[0,1/h]}$, that is, $u^h(x) = +\infty$ for $x < 0$ and $u^h(x) = h\,x$ for $x \geq 0$. Define $v^h : \mathbb{R} \to [0, +\infty]$ by $v^h = u^h + \mathrm{I}_{[0,1]}$. Since Z is a translation invariant valuation and by (4.1), we obtain

$$\mathrm{Z}(v^h + t) = \mathrm{Z}(u^h + t) - \mathrm{Z}(u^h + h + t) = \frac{1}{h}\Big(\psi(t) - \psi(t + h)\Big)$$

for $t \in \mathbb{R}$. As $h \to 0$, the epi-limit of $v^h + t$ is $\mathrm{I}_{[0,1]} + t$. Since Z is continuous, we thus obtain

$$\mathrm{Z}(\mathrm{I}_{[0,1]} + t) = \lim_{h \to 0^+} \frac{1}{h}\Big(\psi(t) - \psi(t + h)\Big)$$

for $t \in \mathbb{R}$. Hence, ψ is differentiable from the right at every $t \in \mathbb{R}$. Since $v^h + t - h \xrightarrow{epi} \mathrm{I}_{[0,1]} + t$ as $h \to 0$, we also obtain

$$\mathrm{Z}(\mathrm{I}_{[0,1]} + t) = \lim_{h \to 0^+} \big(\mathrm{Z}(u^h + t - h) - \mathrm{Z}(u^h + t)\big) = \lim_{h \to 0^+} \frac{1}{h}\Big(\psi(t - h) - \psi(t)\Big).$$

Hence, ψ is also differentiable from the left at every $t \in \mathbb{R}$ and $\mathrm{Z}(\mathrm{I}_{[0,1]} + t) = -\psi'(t)$. This concludes the proof for $k = 1$.

Next, let $\{e_1, \ldots, e_k\}$ denote the standard basis of $\mathbb{R}^k$ and set $e_0 = 0$. For $h = (h_1, \ldots, h_k)$ with $0 < h_1 \leq \cdots \leq h_k$ and $0 \leq i < k$, define the function u_i^h through its sublevel sets as

$$\{u_i^h < 0\} = \emptyset, \quad \{u_i^h \leq s\} = [0, e_0] + \cdots + [0, e_i] + \mathrm{conv}\{0, s\,e_{i+1}/h_{i+1}, \ldots, s\,e_k/h_k\},$$

for every $s \geq 0$. Let $u_k^h = \mathrm{I}_{[0,1]^k}$. Note, that u_i^h does not depend on h_j for $0 \leq j \leq i$. We use induction on i to show that $u_i^h \in \mathrm{Conv}(\mathbb{R}^k)$ and that

$$\mathrm{Z}(u_i^h + t) = \frac{(-1)^i}{k!\,h_{i+1} \cdots h_k} \psi^{(i)}(t),$$

for every $t \in \mathbb{R}$ and $0 \leq i \leq k$, where $\psi^{(i)}(t) = \frac{\mathrm{d}^i}{\mathrm{d}t^i} \psi(t)$.

For $i = 0$, set $P_h = \mathrm{conv}\{0, e_1/h_1, \ldots, e_k/h_k\} \in \mathcal{P}_o^k$ and note that $u_0^h = \ell_{P_h} \in \mathrm{Conv}(\mathbb{R}^k)$. Hence, by the assumption on Z, we have

$$\mathrm{Z}(u_0^h + t) = \mathrm{Z}(\ell_{P_h} + t) = \psi(t) V_k(P_h) = \frac{1}{k!\,h_1 \cdots h_k} \psi(t).$$

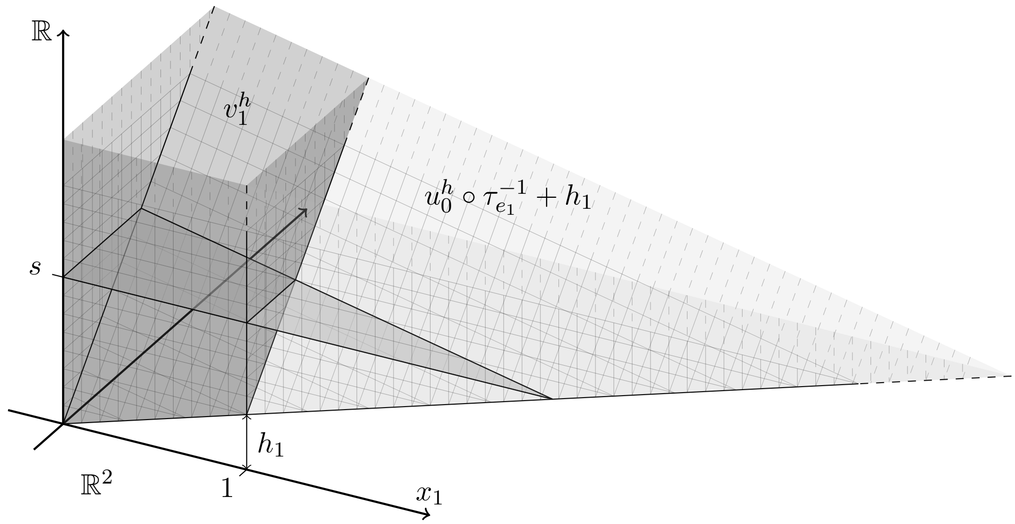

Figure 4.2: The functions v_1^h, u_0^h and $u_0^h \circ \tau_{e_1}^{-1} + h_1$ in $\mathbb{R}^2$.

Now assume that the statement holds true for $i \geq 0$. Define the function v_{i+1}^h by

$$\{v_{i+1}^h \leq s\} = \{u_i^h \leq s\} \cap \{x_{i+1} \leq 1\},$$

for every $s \in \mathbb{R}$, see also Figure 4.2. Since $\operatorname{epi} v_{i+1}^h = \operatorname{epi} u_i^h \cap \{x_{i+1} \leq 1\}$, it is easy to see that $v_{i+1}^h \in \operatorname{Conv}(\mathbb{R}^k)$. As $h_{i+1} \to 0$, we have epi-convergence of v_{i+1}^h to u_{i+1}^h. Theorem 2.8 implies that u_{i+1}^h is a convex function and hence $u_{i+1}^h \in \operatorname{Conv}(\mathbb{R}^k)$. Now, let τ_{i+1} be the translation $x \mapsto x + e_{i+1}$. Note that

$$\begin{aligned}
\{v_{i+1}^h \leq s\} \cup \{(u_i^h \circ \tau_{e_{i+1}}^{-1} + h_{i+1}) \leq s\} &= \{u_i^h \leq s\}, \\
\{v_{i+1}^h \leq s\} \cap \{(u_i^h \circ \tau_{e_{i+1}}^{-1} + h_{i+1}) \leq s\} &\subset \{x_{i+1} = 1\},
\end{aligned}$$

for every $s \in \mathbb{R}$. Since Z is a continuous, translation invariant valuation and $\mathrm{Z}(\ell_P + t) = 0$ for $P \in \mathcal{P}_o^k$ with $\dim(P) < k$, Lemma 4.1 and its proof imply that Z vanishes on all functions $u \in \operatorname{Conv}(\mathbb{R}^k)$ with $\operatorname{dom} u \subset H$, where H is a hyperplane in $\mathbb{R}^k$. Hence,

$$\mathrm{Z}(v_{i+1}^h \vee (u_i^h \circ \tau_{e_{i+1}}^{-1} + h_{i+1})) = 0.$$

Thus, by the valuation property

$$\mathrm{Z}(u_i^h + t) = \mathrm{Z}((v_{i+1}^h + t) \wedge (u_i^h \circ \tau_{e_{i+1}}^{-1} + h_{i+1} + t)) = \mathrm{Z}(v_{i+1}^h + t) + \mathrm{Z}(u_i^h \circ \tau_{e_{i+1}}^{-1} + h_{i+1} + t).$$

Using the induction assumption and the translation invariance of Z, we obtain

$$\mathrm{Z}(v_{i+1}^h + t) = \frac{(-1)^{i+1}}{k!\, h_{i+2} \cdots h_k} \frac{\psi^{(i)}(t + h_{i+1}) - \psi^{(i)}(t)}{h_{i+1}}.$$

As $h_{i+1} \to 0$, the continuity of Z shows that

$$\mathrm{Z}(u_{i+1}^h + t) = \frac{(-1)^{i+1}}{k!\, h_{i+2} \cdots h_k} \lim_{h_{i+1} \to 0^+} \frac{\psi^{(i)}(t + h_{i+1}) - \psi^{(i)}(t)}{h_{i+1}}.$$

Hence $\psi^{(i)}$ is differentiable from the right. Similarly, we have $v_{i+1}^h + t - h_{i+1} \xrightarrow{epi} u_{i+1}^h$ as $h_{i+1} \to 0$ and thus

$$\mathrm{Z}(u_{i+1}^h + t) = \lim_{h_{i+1}\to 0^+} \mathrm{Z}(v_{i+1}^h + t - h_{i+1}) = \frac{(-1)^{i+1}}{k!\, h_{i+2}\cdots h_k} \lim_{h_{i+1}\to 0^+} \frac{\psi^{(i)}(t) - \psi^{(i)}(t-h_{i+1})}{h_{i+1}},$$

which shows that $\psi^{(i)}$ is differentiable from the left and therefore,

$$\mathrm{Z}(u_{i+1}^h + t) = \frac{(-1)^{i+1}}{k!\, h_{i+2}\cdots h_k} \psi^{(i+1)}(t),$$

for every $t \in \mathbb{R}$. ∎

The next lemma gives us a sufficient condition such that a function has finite moment.

Lemma 4.3. *Let $\zeta \in C(\mathbb{R})$ have constant sign on $[t_0, \infty)$ for some $t_0 \in \mathbb{R}$. If there exist $k \in \mathbb{N}$, $c_k \in \mathbb{R}$ and $\psi \in C^k(\mathbb{R})$ with $\lim_{t\to+\infty} \psi(t) = 0$ such that*

$$\zeta(t) = c_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \psi(t)$$

for $t \geq t_0$, then

$$\left| \int_0^{+\infty} t^{k-1} \zeta(t)\, \mathrm{d}t \right| < +\infty.$$

Proof. Since we can always consider $\widetilde{\psi}(t) = \pm c_k\, \psi(t)$ instead of $\psi(t)$, we assume that $c_k = 1$ and $\zeta \geq 0$. To prove the statement, we use induction on k and start with the case $k = 1$. For $t_1 > t_0$,

$$\int_{t_0}^{t_1} \zeta(t)\, \mathrm{d}t = \int_{t_0}^{t_1} \psi'(t)\, \mathrm{d}t = \psi(t_1) - \psi(t_0).$$

Hence, it follows from the assumption for ψ that

$$\int_{t_0}^{+\infty} \zeta(t)\, \mathrm{d}t = \lim_{t_1\to+\infty} \psi(t_1) - \psi(t_0) = -\psi(t_0) < +\infty.$$

This proves the statement for $k = 1$.

Let $k \geq 2$ and assume that the statement holds true for $k-1$. Since $\zeta \geq 0$, the function $\psi^{(k-1)}$ is increasing. Therefore, the limit

$$c = \lim_{t\to+\infty} \psi^{(k-1)}(t) \in (-\infty, +\infty]$$

exists. Moreover, $\psi^{(k-1)}$ has constant sign on $[\bar{t}_0, +\infty)$ for some $\bar{t}_0 \geq t_0$. By the induction hypothesis,

$$\left| \int_0^{+\infty} t^{k-2} \psi^{(k-1)}(t)\, \mathrm{d}t \right| < +\infty,$$

which is only possible if $c = 0$. In particular, $\psi^{(k-1)}(t) \leq 0$ for all $t \geq \bar{t}_0$.

Using integration by parts, we obtain

$$\int_{t_0}^{t_1} t^{k-1} \psi^{(k)}(t)\, \mathrm{d}t = t_1^{k-1} \psi^{(k-1)}(t_1) - t_0^{k-1} \psi^{(k-1)}(t_0) - (k-1) \int_{t_0}^{t_1} t^{k-2} \psi^{(k-1)}(t)\, \mathrm{d}t. \tag{4.2}$$

Since $t^{k-1}\psi^{(k)}(t) \geq 0$ for $t \geq \max\{0, t_0\}$, we have

$$d = \int_{t_0}^{+\infty} t^{k-1}\psi^{(k)}(t)\,\mathrm{d}t \in (-\infty, +\infty].$$

Hence, (4.2) implies that $t_1^{k-1}\psi^{(k-1)}(t_1)$ converges to

$$d + t_0^{k-1}\psi^{(k-1)}(t_0) + (k-1)\int_{t_0}^{+\infty} t^{(k-2)}\psi^{(k-1)}(t)\,\mathrm{d}t.$$

Since $t_1^{k-1}\psi^{(k-1)}(t_1) \leq 0$ for $t_1 \geq \max\{\bar{t}_0, 0\}$, we conclude that d is not $+\infty$. ■

4.2 Classification of Real-Valued Valuations

The aim of this section is to give a classification of continuous, $\mathrm{SL}(n)$ and translation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n)$ and $\mathrm{LC}(\mathbb{R}^n)$ and thereby characterizing the operators studied in Section 3.2.

Lemma 4.4. *If* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous and* $\mathrm{SL}(n)$ *invariant valuation, then there exist continuous functions* $\psi_0, \psi_n, \zeta_0, \zeta_n : \mathbb{R} \to \mathbb{R}$ *such that*

$$\mathrm{Z}(\ell_P + t) = \psi_0(t) + \psi_n(t)V_n(P),$$
$$\mathrm{Z}(\mathrm{I}_P + t) = \zeta_0(t) + \zeta_n(t)V_n(P)$$

for every $P \in \mathcal{P}_o^n$ *and* $t \in \mathbb{R}$.

Proof. For $t \in \mathbb{R}$, define $\mathrm{Z}_t : \mathcal{P}_o^n \to \mathbb{R}$ as

$$\mathrm{Z}_t(P) = \mathrm{Z}(\ell_P + t).$$

By Lemma 2.14, (3.4) and (3.7), it is easy to see that Z_t defines a continuous, $\mathrm{SL}(n)$ invariant valuation on $\mathcal{P}_o^n$ for every $t \in \mathbb{R}$. Therefore, by Theorem 1.6, for every $t \in \mathbb{R}$ there exist constants $c_{0,t}, c_{n,t} \in \mathbb{R}$ such that

$$\mathrm{Z}(\ell_P + t) = \mathrm{Z}_t(P) = c_{0,t} + c_{n,t}V_n(P),$$

for every $P \in \mathcal{P}_o^n$. This defines two functions $\psi_0(t) = c_{0,t}$ and $\psi_n(t) = c_{n,t}$. Taking $P \in \mathcal{P}_o^n$ with $\dim P < n$, we have $V_n(P) = 0$. By the continuity of Z,

$$t \mapsto \mathrm{Z}(\ell_P + t) = \psi_0(t)$$

is continuous, which implies that ψ_0 is a continuous function. Similarly, taking $Q \in \mathcal{P}_o^n$ with $V_n(Q) > 0$, we see that

$$t \mapsto \psi_n(t) = \frac{\mathrm{Z}(\ell_Q + t) - \psi_0(t)}{V_n(Q)},$$

can be expressed as the difference of two continuous functions and is therefore continuous itself. Using $P \mapsto \mathrm{Z}(\mathrm{I}_P + t)$ we get the corresponding results for the functions ζ_0 and ζ_n. ■

For a continuous and $\mathrm{SL}(n)$ invariant valuation $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$, we call the functions ψ_0 and ψ_n from Lemma 4.4 the *cone growth functions* of Z. The functions ζ_0 and ζ_n are its *indicator growth functions*. By Lemma 4.1, we immediately get the following result.

Lemma 4.5. *Every continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ *is uniquely determined by its cone growth functions.*

Next, we study the relation between the cone and indicator growth functions.

Lemma 4.6. *If* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation, then the growth functions* ψ_0 *and* ζ_0 *coincide and*

$$\zeta_n(t) = \frac{(-1)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \psi_n(t),$$

for every $t \in \mathbb{R}$.

Proof. Since $\ell_{\{0\}} = \mathrm{I}_{\{0\}}$, Lemma 4.4 implies that

$$\psi_0(t) = \mathrm{Z}(\ell_{\{0\}} + t) = \mathrm{Z}(\mathrm{I}_{\{0\}} + t) = \zeta_0(t),$$

for every $t \in \mathbb{R}$.

Now define $\overline{\mathrm{Z}} : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ as

$$\overline{\mathrm{Z}}(u) = \mathrm{Z}(u) - \zeta_0\big(\min_{x\in\mathbb{R}^n} u(x)\big).$$

By Lemma 3.5, the functional $\overline{\mathrm{Z}}$ is a continuous, $\mathrm{SL}(n)$ and translation invariant valuation that satisfies

$$\overline{\mathrm{Z}}(\ell_P + t) = \psi_n(t) V_n(P)$$

and

$$\overline{\mathrm{Z}}(\mathrm{I}_P + t) = \zeta_n(t) V_n(P),$$

for every $P \in \mathcal{P}_o^n$ and $t \in \mathbb{R}$. Hence, by Lemma 4.2,

$$\zeta_n(t) = \zeta_n(t) V_n([0,1]^n) = \overline{\mathrm{Z}}(\mathrm{I}_{[0,1]^n} + t) = \frac{(-1)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \psi_n(t),$$

for every $t \in \mathbb{R}$. ■

Lemma 4.7. *If* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation, then its cone growth function* ψ_n *satisfies*

$$\lim_{t\to\infty} \psi_n(t) = 0.$$

Proof. Let

$$P = \mathrm{conv}\{0, \tfrac{e_1+e_2}{2}, e_2, e_3, \dots, e_n\} \qquad \text{and} \qquad Q = \mathrm{conv}\{0, e_2, e_3, \dots, e_n\}.$$

For $s > 0$, define $u_s \in \mathrm{Conv}(\mathbb{R}^n)$ by its epigraph as $\mathrm{epi}\, u_s = \mathrm{epi}\, \ell_P \cap \{x_1 \leq \frac{s}{2}\}$. Note, that for $t \geq 0$ this gives $\{u_s \leq t\} = tP \cap \{x_1 \leq \frac{s}{2}\}$. Define $\ell_{P,s} = \ell_P \circ \tau^{-1}_{s(e_1+e_2)/2} + s$ and similarly $\ell_{Q,s} = \ell_Q \circ \tau^{-1}_{s(e_1+e_2)/2} + s$. We will now show that

$$u_s \wedge \ell_{P,s} = \ell_P \qquad u_s \vee \ell_{P,s} = \ell_{Q,s},$$

or equivalently

$$\mathrm{epi}\, u_s \cup \mathrm{epi}\, \ell_{P,s} = \mathrm{epi}\, \ell_P \qquad \mathrm{epi}\, u_s \cap \mathrm{epi}\, \ell_{P,s} = \mathrm{epi}\, \ell_{Q,s},$$

which is the same as

$$\{u_s \le t\} \cup \{\ell_{P,s} \le t\} = \{\ell_P \le t\} \qquad \{u_s \le t\} \cap \{\ell_{P,s} \le t\} = \{\ell_{Q,s} \le t\} \tag{4.3}$$

for every $t \in \mathbb{R}$. Indeed, it is easy to see, that (4.3) holds for all $t < s$. Therefore, fix an arbitrary $t \ge s$. We have

$$\{\ell_{P,s} \le t\} = \{\ell_P + s \le t\} + s\,\tfrac{e_1+e_2}{2} = (t-s)P + s\,\tfrac{e_1+e_2}{2}.$$

This can be rewritten as

$$\{\ell_{P,s} \le t\} = tP \cap \{x_1 \ge \tfrac{s}{2}\}.$$

Hence

$$\{u_s \le t\} \cup \{\ell_{P,s} \le t\} = \big(tP \cap \{x_1 \le \tfrac{s}{2}\}\big) \cup \big(tP \cap \{x_1 \ge \tfrac{s}{2}\}\big) = tP = \{\ell_P \le t\},$$

and

$$\begin{aligned}\{u_s \le t\} \cap \{\ell_{P,s} \le t\} = tP \cap \{x_1 = \tfrac{s}{2}\} &= \big((t-s)P \cap \{x_1 = 0\}\big) + s\,\tfrac{e_1+e_2}{2} \\ &= (t-s)\,Q + s\,\tfrac{e_1+e_2}{2} = \{\ell_Q + s \le t\} + s\,\tfrac{e_1+e_2}{2} = \{\ell_{Q,s} \le t\}.\end{aligned}$$

From the valuation property of Z we now get

$$\mathrm{Z}(u_s) + \mathrm{Z}(\ell_{P,s}) = \mathrm{Z}(\ell_P) + \mathrm{Z}(\ell_{Q,s}).$$

By Lemma 4.4 and since $V_n(Q) = 0$, we have

$$\mathrm{Z}(u_s) + \psi_n(s)V_n(P) + \psi_0(s) = \psi_n(0)V_n(P) + \psi_0(0) + \psi_0(s).$$

As $s \to \infty$, we obtain $u_s \xrightarrow{epi} \ell_P$ and therefore

$$\psi_n(0)V_n(P) + \psi_0(0) - \psi_n(s)V_n(P) = \mathrm{Z}(u_s) \quad \xrightarrow{s\to\infty} \quad \mathrm{Z}(\ell_P) = \psi_n(0)V_n(P) + \psi_0(0).$$

Since $V_n(P) > 0$, this shows that $\psi_n(s) \to 0$. ■

Lemma 4.6 shows that for a continuous, $\mathrm{SL}(n)$ and translation invariant valuation Z the indicator growth functions ζ_0 and ζ_n coincide with its cone growth function ψ_0 and up to a constant factor with the n-th derivative of its cone growth function ψ_n, respectively. Since Lemma 4.7 shows that $\lim_{t\to\infty} \psi_n(t) = 0$, the cone growth functions ψ_0 and ψ_n are completely determined by the indicator growth functions of Z. Hence Lemma 4.5 immediately implies the following result.

Lemma 4.8. *Every continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathbb{R}$ *is uniquely determined by its indicator growth functions.*

Theorem 4.9. *A functional* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to [0,\infty)$ *is a continuous,* $\mathrm{SL}(n)$ *and translation invariant valuation if and only if there exist a continuous function* $\zeta_0 : \mathbb{R} \to [0,\infty)$ *and a continuous function* $\zeta_n : \mathbb{R} \to [0,\infty)$ *with finite* $(n-1)$*-st moment such that*

$$\mathrm{Z}(u) = \zeta_0\big(\min_{x\in\mathbb{R}^n} u(x)\big) + \int_{\mathrm{dom}\,u} \zeta_n\big(u(x)\big)\,\mathrm{d}x \tag{4.4}$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$.

Proof. If $\zeta_0 : \mathbb{R} \to [0,\infty)$ is continuous and $\zeta_n : \mathbb{R} \to [0,\infty)$ is continuous with finite $(n-1)$-st moment, then Lemmas 3.5 and 3.11 show that (4.4) defines a non-negative, continuous, $\mathrm{SL}(n)$ and translation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n)$.

Conversely, let $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to [0,\infty)$ be a continuous, $\mathrm{SL}(n)$ and translation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n)$ with indicator growth functions ζ_0 and ζ_n. For a polytope $P \in \mathcal{P}_o^n$ with $\dim P < n$, Lemma 4.4 implies that

$$0 \le \mathrm{Z}(\mathrm{I}_P + t) = \zeta_0(t)$$

for every $t \in \mathbb{R}$. Hence, ζ_0 is a non-negative and continuous function. Similarly, for $Q \in \mathcal{P}_o^n$ with $V_n(Q) > 0$, we have

$$0 \le \mathrm{Z}(\mathrm{I}_{sQ} + t) = \zeta_0(t) + s^n \zeta_n(t) V_n(Q),$$

for every $t \in \mathbb{R}$ and $s > 0$. Therefore, also ζ_n is a non-negative and continuous function. By Lemmas 4.3, 4.6 and 4.7, the growth function ζ_n has finite $(n-1)$-st moment. Finally, for $u = \mathrm{I}_P + t$ with $P \in \mathcal{P}_o^n$ and $t \in \mathbb{R}$, we obtain that

$$\mathrm{Z}(u) = \zeta_0(t) + \zeta_n V_n(P) = \zeta_0\big(\min\nolimits_{x\in\mathbb{R}^n} u(x)\big) + \int_{\mathrm{dom}\, u} \zeta_n(u(x))\,\mathrm{d}x.$$

By the first part of the proof,

$$u \mapsto \zeta_0\big(\min\nolimits_{x\in\mathbb{R}^n} u(x)\big) + \int_{\mathrm{dom}\, u} \zeta_n(u(x))\,\mathrm{d}x$$

defines a non-negative, continuous, $\mathrm{SL}(n)$ and translation invariant valuation on $\mathrm{Conv}(\mathbb{R}^n)$. Thus Lemma 4.8 completes the proof of the theorem. ∎

Theorem 4.10. *An operator* $\mathrm{Y} : \mathrm{LC}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, homogeneous,* $\mathrm{SL}(n)$ *and translation invariant valuation if and only if there exist constants* $c_0, c_n \in \mathbb{R}$ *and* $q \in \mathbb{R}$*, with* $q > 0$ *if* $c_n \neq 0$*, such that*

$$\mathrm{Y}(f) = c_0\big(\max\nolimits_{x\in\mathbb{R}^n} f(x)\big)^q + c_n \int_{\mathbb{R}^n} f^q(x)\,\mathrm{d}x, \tag{4.5}$$

for every $f \in \mathrm{LC}(\mathbb{R}^n)$.

Proof. Lemmas 3.7 and 3.12 show that (4.5) defines a continuous, homogeneous, $\mathrm{SL}(n)$ and translation invariant valuation on $\mathrm{LC}(\mathbb{R}^n)$ for every $c_0, c_n \in \mathbb{R}$ and $q \in \mathbb{R}$, with $q > 0$ if $c_n \neq 0$.

Conversely, let $\mathrm{Y} : \mathrm{LC}(\mathbb{R}^n) \to \mathbb{R}$ be a continuous, homogeneous, $\mathrm{SL}(n)$ and translation invariant valuation and let Z be the corresponding valuation on $\mathrm{Conv}(\mathbb{R}^n)$, that is $\mathrm{Z}(u) = \mathrm{Y}(e^{-u})$ for every $u \in \mathrm{Conv}(\mathbb{R}^n)$. Then Z is continuous, $\mathrm{SL}(n)$ and translation invariant, see also Remark 3.2. Furthermore,

$$\mathrm{Z}(u+t) = \mathrm{Y}(e^{-(u+t)}) = (e^{-t})^q\,\mathrm{Y}(e^{-u}) = e^{-qt}\,\mathrm{Z}(u),$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $t \in \mathbb{R}$, where $q \in \mathbb{R}$ denotes the degree of homogeneity of Y. Let $\psi_0, \psi_n, \zeta_0, \zeta_n$ denote the growth functions of Z. By Lemma 4.6 the functions ψ_0 and ζ_0 coincide. We have,

$$\zeta_0(t) = \mathrm{Z}(\mathrm{I}_{\{0\}} + t) = e^{-qt}\,\mathrm{Z}(\mathrm{I}_{\{0\}}),$$

for every $t \in \mathbb{R}$. Hence, there exists a constant $c_0 \in \mathbb{R}$ such that $\zeta_0(t) = c_0 e^{-qt}$ for every $t \in \mathbb{R}$. Furthermore, let $K \in \mathcal{K}_o^n$ such that $V_n(K) > 0$. Then,

$$e^{-qt}\,\mathrm{Z}(\ell_K) = \mathrm{Z}(\ell_K + t) = \zeta_0(t) + \psi_n(t) V_n(K) = c_0 e^{-qt} + \psi_n(t) V_n(K),$$

for every $t \in \mathbb{R}$. Hence, there exists a constant $\widetilde{c}_n \in \mathbb{R}$ such that $\psi_n(t) = \widetilde{c}_n\, e^{-qt}$ for every $t \in \mathbb{R}$. Lemma 4.7 shows that $\lim_{t\to+\infty} \psi_n(t) = 0$ and therefore $q > 0$ or $\widetilde{c}_n = 0$. Moreover, by Lemma 4.6,

$$\zeta_n(t) = \frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\psi_n(t) = \frac{\widetilde{c}_n\, q^n}{n!} e^{-qt} =: c_n\, e^{-qt},$$

for every $t \in \mathbb{R}$. For $t \in \mathbb{R}$, let $s = e^{-t}$. We have

$$\begin{aligned}
\mathrm{Y}(s\ \chi_K) = \mathrm{Z}(\mathrm{I}_K + t) &= c_0\, e^{-qt} + c_n\, e^{-qt} V_n(K) \\
&= c_0\, s^q + c_n\, s^q V_n(K) \\
&= c_0 \left(\int_0^{+\infty} V_0(\{s\ \chi_K \geq r\})\,\mathrm{d}r \right)^q + c_n \int_0^{+\infty} V_n(\{(s\ \chi_K)^q \geq r\})\,\mathrm{d}r \\
&= c_0 V_0(s\ \chi_K)^q + c_n V_n((s\ \chi_K)^q),
\end{aligned}$$

for every $K \in \mathcal{K}^n$. By Lemma 4.8 the valuation Z is uniquely determined by its values on indicator functions. Since

$$f \mapsto c_0 V_0(f)^q + c_n V_n(f^q),$$

defines a continuous, homogeneous, $\mathrm{SL}(n)$ and translation invariant valuation, the proof is complete. ∎

4.3 Classification of Minkowski Valuations

The Minkowski valuations that were introduced in Section 3.4 are classified. Again, we distinguish between $\mathrm{SL}(n)$ contravariance and $\mathrm{SL}(n)$ covariance.

4.3.1 Contravariant Minkowski Valuations

In this section, we classify the functional analogs of the projection body from Lemmas 3.24 and 3.25. In the following, let $n \geq 3$.

Lemma 4.11. *If* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{K}^n$ *is a continuous and* $\mathrm{SL}(n)$ *contravariant Minkowski valuation, then there exist continuous functions* $\psi, \zeta : \mathbb{R} \to [0, \infty)$ *such that*

$$\begin{aligned}
\mathrm{Z}(\ell_K + t) &= \psi(t)\, \Pi\, K, \\
\mathrm{Z}(\mathrm{I}_K + t) &= \zeta(t)\, \Pi\, K,
\end{aligned}$$

for every $K \in \mathcal{K}^n_o$ *and* $t \in \mathbb{R}$.

Proof. For $t \in \mathbb{R}$, define $\mathrm{Z}_t : \mathcal{K}^n_o \to \mathcal{K}^n$ as

$$\mathrm{Z}_t\, K = \mathrm{Z}(\ell_K + t).$$

As in the proof of Lemma 4.4 it follows from Lemma 2.14, (3.4) and (3.7), that Z_t defines a continuous, $\mathrm{SL}(n)$ contravariant Minkowski valuation on $\mathcal{K}^n_o$ for every $t \in \mathbb{R}$. By Theorem 1.13, there exists a non-negative constant c_t such that

$$\mathrm{Z}(\ell_K + t) = \mathrm{Z}_t\, K = c_t\, \Pi\, K$$

for all $K \in \mathcal{K}^n_o$. This defines a function $\psi(t) = c_t$, which is continuous due to the continuity of Z. Similarly, using $\mathrm{Z}_t(K) = \mathrm{Z}(\mathrm{I}_K + t)$, we obtain the function ζ. ∎

For a continuous, $\mathrm{SL}(n)$ contravariant Minkowski valuation $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{K}^n$, we call the function ψ from Lemma 4.11 the *cone growth function* of Z. The function ζ is called its *indicator growth function.* By Lemma 4.1, we immediately get the following result.

Lemma 4.12. *Every continuous,* $\mathrm{SL}(n)$ *contravariant and translation invariant Minkowski valuation* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{K}^n$ *is uniquely determined by its cone growth function.*

Next, we establish an important connection between cone and indicator growth functions.

Lemma 4.13. *Let* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{K}^n$ *be a continuous,* $\mathrm{SL}(n)$ *contravariant and translation invariant Minkowski valuation. The growth functions satisfy*

$$\zeta(t) = \frac{(-1)^{n-1}}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} \psi(t)$$

for every $t \in \mathbb{R}$.

Proof. We fix the $(n-1)$-dimensional linear subspace $E = e_n^\perp$ of $\mathbb{R}^n$. Since E is of dimension $(n-1)$, we can identify the set of functions $u \in \mathrm{Conv}(\mathbb{R}^n)$ such that $\mathrm{dom}\, u \subseteq E$ with $\mathrm{Conv}(\mathbb{R}^{n-1}) = \mathrm{Conv}(E)$. We define $\mathrm{Y} : \mathrm{Conv}(E) \to \mathbb{R}$ by

$$\mathrm{Y}(u) = h(\mathrm{Z}(u), e_n).$$

Since Z is a Minkowski valuation, Y is a real-valued valuation. Moreover, Y is continuous and translation invariant, since Z has these properties. By the definition of the growth functions we now get

$$\mathrm{Y}(\ell_P + t) = h(\mathrm{Z}(\ell_P + t), e_n) = \psi(t) h(\Pi\, P, e_n) = \psi(t) V_{n-1}(P)$$

and

$$\mathrm{Y}(\mathrm{I}_P + t) = h(\mathrm{Z}(\mathrm{I}_P + t), e_n) = \zeta(t) h(\Pi\, P, e_n) = \zeta(t) V_{n-1}(P)$$

for every $P \in \mathcal{P}_o^{n-1}(E) = \{P \in \mathcal{P}_o^n \,:\, P \subset E\}$ and $t \in \mathbb{R}$. Hence, by Lemma 4.2,

$$\zeta(t) = \zeta(t)\, V_{n-1}([0,1]^{n-1}) = \mathrm{Y}(\mathrm{I}_{[0,1]^{n-1}} + t) = \frac{(-1)^{n-1}}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} \psi(t)$$

for every $t \in \mathbb{R}$, where $[0,1]^{n-1} = [0,1]^n \cap E$. ■

Next, we establish important properties of the cone growth function.

Lemma 4.14. *If* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{K}^n$ *is a continuous,* $\mathrm{SL}(n)$ *contravariant and translation invariant Minkowski valuation, then its cone growth function* ψ *is decreasing and satisfies*

$$\lim_{t \to \infty} \psi(t) = 0. \tag{4.6}$$

Proof. In order to prove that ψ is decreasing, we have to show that $\psi(t) \geq \psi(s)$ for all $t < s$. Without loss of generality, we assume that $t = 0$, since for arbitrary t we can consider $\widetilde{\mathrm{Z}}(u) = \mathrm{Z}(u+t)$ with cone growth function $\widetilde{\psi}$ and $\widetilde{\psi}(0) = \psi(t)$. Hence, for the remainder of the proof we fix an arbitrary $s > 0$ and we have to show that $\psi(s) \leq \psi(0)$.

Let

$$P = \mathrm{conv}\{0, \tfrac{e_1+e_2}{2}, e_2, e_3, \dots, e_n\} \qquad \text{and} \qquad Q = \mathrm{conv}\{0, e_2, e_3, \dots, e_n\}.$$

For $s>0$, choose $u_s \in \mathrm{Conv}(\mathbb{R}^n)$ such that $\operatorname{epi} u_s = \operatorname{epi} \ell_P \cap \{x_1 \leq \frac{s}{2}\}$. Define $\ell_{P,s} = \ell_P \circ \tau^{-1}_{s(e_1+e_2)/2} + s$ and similarly $\ell_{Q,s} = \ell_Q \circ \tau^{-1}_{s(e_1+e_2)/2} + s$. As we have already seen in the proof of Lemma 4.7, we have

$$u_s \wedge \ell_{P,s} = \ell_P \qquad \text{and} \qquad u_s \vee \ell_{P,s} = \ell_{Q,s}.$$

Thus, the valuation property of Z gives

$$\mathrm{Z}(u_s) + \mathrm{Z}(\ell_{P,s}) = \mathrm{Z}(u_s \wedge \ell_{P,s}) + \mathrm{Z}(u_s \vee \ell_{P,s}) = \mathrm{Z}(\ell_P) + \mathrm{Z}(\ell_{Q,s}).$$

Using the translation invariance of Z and the definition of the cone growth function, this gives for the support functions

$$h(\mathrm{Z}(u_s), \cdot) = (\psi(0) - \psi(s)) h(\Pi\, P, \cdot) + \psi(s) h(\Pi\, Q, \cdot). \tag{4.7}$$

Since $\mathrm{Z}(u_s)$ is a convex body, its support function is sublinear. This yields

$$h(\mathrm{Z}(u_s), e_1 + e_2) \leq h(\mathrm{Z}(u_s), e_1) + h(\mathrm{Z}(u_s), e_2)$$

and

$$\begin{aligned}(\psi(0) - \psi(s)) h(\Pi\, P, e_1 + e_2) &+ \psi(s) h(\Pi\, Q, e_1 + e_2) \\ &\leq (\psi(0) - \psi(s))\big(h(\Pi\, P, e_1) + h(\Pi\, P, e_2)\big) + \psi(s)\big(h(\Pi\, Q, e_1) + h(\Pi\, Q, e_2)\big).\end{aligned}$$

Using Lemma 1.12, we obtain

$$\begin{aligned}(\psi(0) - \psi(s)) \tfrac{1}{(n-1)!} + \psi(s) \tfrac{1}{(n-1)!} &\leq (\psi(0) - \psi(s))\big(\tfrac{1}{(n-1)!} + \tfrac{1}{2(n-1)!}\big) + \psi(s)\big(\tfrac{1}{(n-1)!} + 0\big), \\ 0 &\leq (\psi(0) - \psi(s)) \tfrac{1}{2(n-1)!},\end{aligned}$$

which holds if and only if $\psi(s) \leq \psi(0)$.

In order to show (4.6), let s in the construction above go to $+\infty$. It is easy to see, that in this case u_s is epi-convergent to ℓ_P. Since ψ is decreasing and non-negative, $\lim_{s\to+\infty} \psi(s) = \psi_\infty$ exists. Taking limits in (4.7) therefore yields

$$\psi(0) h(\Pi\, P, \cdot) = h(\mathrm{Z}(\ell_P), \cdot) = (\psi(0) - \psi_\infty) h(\Pi\, P, \cdot) + \psi_\infty\, h(\Pi\, Q, \cdot).$$

Evaluating at e_2 now gives $\psi_\infty = 0$. ■

By Lemma 4.1, we obtain the following result as an immediate corollary from the last result. We call a Minkowski valuation on $\mathrm{Conv}(\mathbb{R}^n)$ *trivial* if $\mathrm{Z}(u) = \{0\}$ for $u \in \mathrm{Conv}(\mathbb{R}^n)$.

Lemma 4.15. *Every continuous, increasing,* $\mathrm{SL}(n)$ *contravariant and translation invariant Minkowski valuation on* $\mathrm{Conv}(\mathbb{R}^n)$ *is trivial.*

Lemma 4.13 shows that the indicator growth function ζ of a continuous, $\mathrm{SL}(n)$ contravariant and translation invariant Minkowski valuation Z determines its cone growth function ψ up to a polynomial of degree less than $n-1$. By Lemma 4.14, $\lim_{t\to\infty} \psi(t) = 0$ and hence the polynomial is also determined by ζ. Thus ψ is completely determined by the indicator growth function of Z and Lemma 4.12 immediately implies the following result.

Lemma 4.16. *Every continuous,* $\mathrm{SL}(n)$ *contravariant and translation invariant Minkowski valuation* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{K}^n$ *is uniquely determined by its indicator growth function.*

Theorem 4.17. *A function* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{K}^n$ *is a continuous, monotone,* $\mathrm{SL}(n)$ *contravariant and translation invariant Minkowski valuation if and only if there exists* $\zeta \in D^{n-2}(\mathbb{R})$ *such that*

$$\mathrm{Z}(u) = \Pi \langle \zeta \circ u \rangle \tag{4.8}$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$.

Proof. If $\zeta \in D^{n-2}(\mathbb{R})$, then Lemma 3.24 shows that (4.8) defines a continuous, decreasing, $\mathrm{SL}(n)$ contravariant and translation invariant Minkowski valuation on $\mathrm{Conv}(\mathbb{R}^n)$.

Conversely, let a continuous, monotone, $\mathrm{SL}(n)$ contravariant and translation invariant Minkowski valuation Z be given and let ζ be its indicator growth function. Lemma 4.15 implies that we may assume that Z is decreasing. It follows from the definition of ζ in Lemma 4.11 that ζ is non-negative and continuous. To see that ζ is decreasing, note that by the definition of ζ in in Lemma 4.11,

$$h(\mathrm{Z}(\mathrm{I}_{[0,1]^n} + t), e_1) = \zeta(t)\, h(\Pi[0,1]^n, e_1) = \zeta(t)$$

for every $t \in \mathbb{R}$ and that Z is decreasing. By Lemma 4.13 combined with Lemma 4.3, the function ζ has finite $(n-2)$-nd moment. Thus $\zeta \in D^{n-2}(\mathbb{R})$.

For $u = \mathrm{I}_P + t$ with $P \in \mathcal{P}_o^n$ and $t \in \mathbb{R}$, we obtain by (3.24) that

$$h(\Pi \langle \zeta \circ u \rangle, z) = \int_0^{+\infty} h(\Pi\{\zeta \circ u \geq s\}, z)\, \mathrm{d}s = \zeta(t)\, h(\Pi\, P, z)$$

for every $z \in \mathbb{S}^{n-1}$. Hence $\Pi \langle \zeta \circ (\mathrm{I}_P + t)\rangle = \zeta(t)\, \Pi\, P$ for $P \in \mathcal{P}_o^n$ and $t \in \mathbb{R}$. By Lemma 3.24,

$$u \mapsto \Pi \langle \zeta \circ u \rangle$$

defines a continuous, decreasing, $\mathrm{SL}(n)$ contravariant and translation invariant Minkowski valuation on $\mathrm{Conv}(\mathbb{R}^n)$ and ζ is its indicator growth function. Thus Lemma 4.16 completes the proof of the theorem. ∎

Theorem 4.18. *A function* $\mathrm{Y} : \mathrm{LC}(\mathbb{R}^n) \to \mathcal{K}^n$ *is a continuous, homogeneous,* $\mathrm{SL}(n)$ *contravariant and translation invariant Minkowski valuation if and only if there exist* $c \geq 0$ *and* $q > 0$ *such that*

$$\mathrm{Y}(f) = c\, \Pi \langle f^q \rangle \tag{4.9}$$

for every $f \in \mathrm{LC}(\mathbb{R}^n)$.

Proof. For $c \geq 0$ and $q > 0$ it follow from Lemma 3.25 that (4.9) defines a continuous, homogeneous, $\mathrm{SL}(n)$ contravariant and translation invariant Minkowski valuation on $\mathrm{LC}(\mathbb{R}^n)$.

Conversely, let a continuous, $\mathrm{SL}(n)$ contravariant and translation invariant Minkowski valuation Y be given that is homogeneous of degree $q \in \mathbb{R}$. Assume without loss of generality that Y is non-trivial. If Z denotes the corresponding valuation on $\mathrm{Conv}(\mathbb{R}^n)$, that is $\mathrm{Z}(u) = \mathrm{Y}(e^{-u})$ for every $u \in \mathrm{Conv}(\mathbb{R}^n)$, then Z is continuous, $\mathrm{SL}(n)$ contravariant and translation invariant, see also Remark 3.2. Furthermore,

$$\mathrm{Z}(u + t) = \mathrm{Y}(e^{-(u+t)}) = (e^{-t})^q\, \mathrm{Y}(e^{-u}) = e^{-qt}\, \mathrm{Z}(u),$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $t \in \mathbb{R}$. In particular, this gives for the cone growth function ψ of Z

$$e^{-qt}\, \mathrm{Z}(\ell_K) = \mathrm{Z}(\ell_K + t) = \psi(t)\, \Pi\, K,$$

for every $K \in \mathcal{K}_o^n$ and $t \in \mathbb{R}$. Thus, by Lemmas 4.12 and 4.14 we have $q > 0$, since we assumed that Y was non-trivial. Furthermore, Lemma 4.13 implies that there exists a constant $c \geq 0$ such that $\zeta(t) = ce^{-qt}$, where ζ is the indicator growth function of Z. For $t \in \mathbb{R}$ let $s = e^{-t}$. We have

$$\begin{aligned} h(\mathrm{Y}(s\ \chi_K), z) = h(\mathrm{Z}(\mathrm{I}_K + t), z) &= ce^{-qt} h(\Pi\, K, z) \\ &= cs^q h(\Pi\, K, z) \\ &= c \int_0^{+\infty} h(\Pi\{(s\ \chi_K)^q \geq r\}, z)\, \mathrm{d}r \\ &= h(c\, \Pi\, \langle (s\, \chi_K)^q \rangle, z), \end{aligned}$$

for every $K \in \mathcal{K}^n$ and $z \in \mathbb{S}^{n-1}$. By Lemma 4.16 the valuation Z is uniquely determined by its values on indicator functions. Since

$$f \mapsto c\, \Pi\, \langle f^q \rangle$$

defines a continuous, homogeneous, $\mathrm{SL}(n)$ contravariant and translation invariant Minkowski valuation, the proof is complete. ■

4.3.2 Covariant Minkowski Valuations

In this section we establish classification results for the operators discussed in Sections 3.4.2 and 3.4.3. Let $n \geq 3$.

Lemma 4.19. *If* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{K}^n$ *is a continuous,* $\mathrm{SL}(n)$ *covariant Minkowski valuation, then there exist continuous functions* $\psi_1, \psi_2, \psi_3 : \mathbb{R} \to [0, \infty)$ *and* $\psi_4 : \mathbb{R} \to \mathbb{R}$ *such that*

$$\mathrm{Z}(\ell_K + t) = \psi_1(t) K + \psi_2(t)(-K) + \psi_3(t)\, \mathrm{M}\, K + \psi_4(t)\, \mathrm{m}(K)$$

for every $K \in \mathcal{K}_o^n$ *and* $t \in \mathbb{R}$. *If* Z *is also translation invariant, then there exists a continuous function* $\zeta : \mathbb{R} \to [0, \infty)$ *such that*

$$\mathrm{Z}(\mathrm{I}_K + t) = \zeta(t)\, \mathrm{D}\, K$$

for every $K \in \mathcal{K}^n$ *and* $t \in \mathbb{R}$.

Proof. For $t \in \mathbb{R}$, define $\mathrm{Z}_t : \mathcal{K}_o^n \to \mathcal{K}^n$ as

$$\mathrm{Z}_t\, K = \mathrm{Z}(\ell_K + t).$$

Similar to the proof of Lemma 4.11 it follows from Lemma 2.14, (3.4) and (3.7), that Z_t defines a continuous, $\mathrm{SL}(n)$ covariant Minkowski valuation on $\mathcal{K}_o^n$ for every $t \in \mathbb{R}$. Therefore, by Theorem 1.17, for every $t \in \mathbb{R}$ there exist constants $c_{1,t}, c_{2,t}, c_{3,t} \geq 0$ and $c_{4,t} \in \mathbb{R}$ such that

$$\mathrm{Z}(\ell_K + t) = \mathrm{Z}_t\, K = c_{1,t} K + c_{2,t}(-K) + c_{3,t}\, \mathrm{M}\, K + c_{4,t}\, \mathrm{m}(K)$$

for every $K \in \mathcal{K}_o^n$. This defines functions $\psi_i(t) = c_{i,t}$ for $1 \leq i \leq 4$. By the continuity of Z,

$$t \mapsto h(\mathrm{Z}(\ell_{T_r} + t), e_1) = r\, \psi_1(t) + \frac{r^2}{(n+1)!}(\psi_3(t) + \psi_4(t))$$

is continuous for every $r > 0$, where T_r is defined as in Lemma 1.16. Setting $r = 1$ and $r = 2$ shows that

$$t \mapsto \psi_1(t) + \frac{1}{(n+1)!}(\psi_3(t) + \psi_4(t))$$

and

$$t \mapsto 2\,\psi_1(t) + \frac{4}{(n+1)!}(\psi_3(t) + \psi_4(t))$$

are continuous functions. Hence $\psi_3 + \psi_4$ and ψ_1 are continuous functions. The continuity of the map $t \mapsto h(\mathrm{Z}(\ell_{T_r} + t), -e_1)$ shows that $\psi_3 - \psi_4$ and ψ_2 are continuous. Hence, also ψ_3 and ψ_4 are continuous functions.

Similarly, if Z is also translation invariant, we consider $\mathrm{Y}_t(K) = \mathrm{Z}(\mathrm{I}_K + t)$, which defines a continuous, translation invariant and $\mathrm{SL}(n)$ covariant Minkowski valuation on $\mathcal{K}^n$ for every $t \in \mathbb{R}$, see also (3.3), (3.5) and (3.6). Therefore, by Theorem 1.18, there exists a non-negative constant d_t such that

$$\mathrm{Z}(\mathrm{I}_K + t) = \mathrm{Y}_t(K) = d_t\,\mathrm{D}\,K$$

for every $t \in \mathbb{R}$ and $K \in \mathcal{K}^n_o$. This defines a function $\zeta(t) = d_t$, which is continuous due to the continuity of Z. ■

Lemma 4.20. *If* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{K}^n$ *is a continuous,* $\mathrm{SL}(n)$ *covariant Minkowski valuation, then, for* $e \in \mathbb{S}^{n-1}$,

$$h(\mathrm{Z}(v), e) = 0$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n)$ *such that* $\mathrm{dom}\, v$ *lies in an affine subspace orthogonal to* e. *Moreover, if* ϑ *is the orthogonal reflection at* $e^\perp$, *then*

$$h(\mathrm{Z}(u), e) = h(\mathrm{Z}(u \circ \vartheta^{-1}), -e)$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$.

Proof. By Lemma 4.19, we have $h(\mathrm{Z}(\ell_K), e) = 0$ for every $K \in \mathcal{K}^n_o$ such that $K \subset e^\perp$. Hence, Lemma 4.1 implies that $h(\mathrm{Z}(v), e) = 0$ for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ such that $\mathrm{dom}\, v \subset e^\perp$. By the translation invariance of Z, this also holds for $v \in \mathrm{Conv}(\mathbb{R}^n)$ whose $\mathrm{dom}\, v$ lies in an affine subspace orthogonal to e.

Similarly, for every $K \in \mathcal{K}^n_o$, we have $h(K, e) = h(\vartheta K, -e)$ and $h(-K, e) = h(-\vartheta K, -e)$ while $h(\mathrm{m}(K), e) = h(\mathrm{m}(\vartheta K), -e)$ and $h(\mathrm{M}\, K, e) = h(\mathrm{M}(\vartheta K), -e)$. Hence Lemma 4.19 implies that $h(\mathrm{Z}(\ell_K), e) = h(\mathrm{Z}(\ell_K \circ \vartheta^{-1}), -e)$. The claim follows again from Lemma 4.1. ■

In the proof of the next lemma, we use the following classical result due to H.A. Schwarz (cf. [48, p. 37]). Suppose a real valued function ψ is defined and continuous on the closed interval I. If

$$\lim_{h \to 0} \frac{\psi(t+h) - 2\,\psi(t) + \psi(t-h)}{h^2} = 0$$

for every t in the interior of I, then ψ is an affine function.

Lemma 4.21. *Let* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{K}^n$ *be a continuous,* $\mathrm{SL}(n)$ *covariant and translation invariant Minkowski valuation and let* ψ_1, ψ_2, ψ_3 *and* ψ_4 *be the functions from Lemma 4.19. Then* ψ_1 *and* ψ_2 *are continuously differentiable,* $\psi_1' = \psi_2'$ *and both* ψ_3 *and* ψ_4 *are constant.*

Proof. For a closed interval I in the span of e_1, let the function $u_I \in \operatorname{Conv}(\mathbb{R}^n)$ be defined by

$$\{u_I < 0\} = \emptyset, \quad \{u_I \leq s\} = I + \operatorname{conv}\{0, s\,e_2, \dots, s\,e_n\}$$

for every $s \geq 0$. By the properties of Z it is easy to see that the map $I \mapsto h(\mathrm{Z}(u_I + t), e_1)$ is a real valued, continuous, translation invariant valuation on $\mathcal{K}^1$ for every $t \in \mathbb{R}$. Hence, it is easy to see that there exist functions $\zeta_0, \zeta_1 : \mathbb{R} \to \mathbb{R}$ such that

$$h(\mathrm{Z}(u_I + t), e_1) = \zeta_0(t) + \zeta_1(t) V_1(I) \tag{4.10}$$

for every $I \in \mathcal{K}^1$ and $t \in \mathbb{R}$ (see, for example, [28, p. 39]). Note, that by the continuity of Z, the functions ζ_0 and ζ_1 are continuous.

For $r, h > 0$, let $T_{r/h} = \operatorname{conv}\{0, \frac{r}{h}\,e_1, e_2, \dots, e_n\}$. Define the function u_r^h by

$$\{u_r^h \leq s\} = \{\ell_{T_{r/h}} \leq s\} \cap \{x_1 \leq r\}$$

for every $s \in \mathbb{R}$. It is easy to see that $u_r^h \in \operatorname{Conv}(\mathbb{R}^n)$ and that

$$\{u_r^h \leq s\} \cup \{\ell_{T_{r/h}} \circ \tau_{r\,e_1}^{-1} + h \leq s\} = \{\ell_{T_{r/h}} \leq s\},$$

$$\{u_r^h \leq s\} \cap \{\ell_{T_{r/h}} \circ \tau_{r\,e_1}^{-1} + h \leq s\} \subset \{x_1 = r\}$$

for every $s \in \mathbb{R}$. By translation invariance, the valuation property and Lemma 4.20, this gives

$$h(\mathrm{Z}(u_r^h + t), e_1) = h(\mathrm{Z}(\ell_{T_{r/h}} + t), e_1) - h(\mathrm{Z}(\ell_{T_{r/h}} + t + h), e_1)$$

for every $t \in \mathbb{R}$. Note, that by Theorem 2.13 we have $u_r^h \xrightarrow{epi} u_{[0,r]}$ as $h \to 0$. Hence, using the continuity of Z, Lemma 4.19 and Lemma 1.16, we obtain

$$\begin{aligned} h(\mathrm{Z}(u_{[0,r]} + t), e_1) &= \lim_{h \to 0^+} h(\mathrm{Z}(u_r^h + t), e_1) \\ &= \lim_{h \to 0^+} \left(r\,\frac{\psi_1(t) - \psi_1(t+h)}{h} + \frac{r^2}{(n+1)!}\,\frac{(\psi_3 + \psi_4)(t) - (\psi_3 + \psi_4)(t+h)}{h^2} \right) \end{aligned}$$

for every $t \in \mathbb{R}$ and $r > 0$. Comparison with (4.10) now gives

$$\zeta_1(t) = \lim_{h \to 0^+} \frac{\psi_1(t) - \psi_1(t+h)}{h}, \quad 0 = \lim_{h \to 0^+} \frac{(\psi_3 + \psi_4)(t) - (\psi_3 + \psi_4)(t+h)}{h^2}. \tag{4.11}$$

Similarly, since also $u_r^h - h \xrightarrow{epi} u_{[0,r]}$ as $h \to 0$, we obtain

$$\zeta_1(t) = \lim_{h \to 0^+} \frac{\psi_1(t-h) - \psi_1(t)}{h}, \quad 0 = \lim_{h \to 0^+} \frac{(\psi_3 + \psi_4)(t-h) - (\psi_3 + \psi_4)(t)}{h^2}.$$

Hence, ψ_1 is continuously differentiable with $-\psi_1' = \zeta_1$. In addition, by H.A. Schwarz's result, the function $\psi_3 + \psi_4$ is affine and hence by (4.11) it must be constant.

Now, let ϑ denote the reflection at $\{x_1 = 0\} = e_1^\perp$. Lemma 4.20 and the translation invariance of Z give

$$\begin{aligned} h(\mathrm{Z}(u_{[0,r]} + t), e_1) &= h(\mathrm{Z}(u_{[0,r]} \circ \vartheta^{-1} + t), -e_1) \\ &= h(\mathrm{Z}(u_{[-r,0]} + t), -e_1) = h(\mathrm{Z}(u_{[0,r]} + t), -e_1) \end{aligned}$$

for every $t \in \mathbb{R}$. Repeating the arguments from above, but evaluating at $-e_1$, shows that $-\psi_2' = \zeta_1$ and $\psi_3 - \psi_4$ is constant. Hence, both ψ_3 and ψ_4 are constant. ∎

Lemma 4.22. *If the operator* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{K}^n$ *is a continuous,* $\mathrm{SL}(n)$ *covariant and translation invariant Minkowski valuation, then there exists a non-negative* $\psi \in C^1(\mathbb{R})$ *such that*

$$\mathrm{Z}(\ell_K + t) = \psi(t)\,\mathrm{D}\,K$$

for every $t \in \mathbb{R}$ *and* $K \in \mathcal{K}_o^n$. *Moreover,* $\lim_{t\to+\infty} \psi(t) = 0$.

Proof. Let $\psi_1, \ldots, \psi_4$ be as in Lemma 4.19. By Lemma 4.21, there exist constants c_3, c_4 such that $\psi_3(t) \equiv c_3$ and $\psi_4(t) \equiv c_4$. Moreover, ψ_1 and ψ_2 are non-negative and only differ by a constant. Hence, it suffices to show that $\lim_{t\to+\infty} \psi_1(t) = \lim_{t\to+\infty} \psi_2(t) = 0$ and $c_3 = c_4 = 0$. To show this, let $r, b > 0$ and let $v_r^b \in \mathrm{Conv}(\mathbb{R}^n)$ be defined by $\operatorname{epi} v_r^b = \operatorname{epi} \ell_{T_r} \cap \{x_1 \le b\}$, where T_r is defined as in Lemma 1.16. Note, that $\text{epi-lim}_{b\to+\infty} v_r^b = \ell_{T_r}$. Set $\ell_r^b := \ell_{T_r} \circ \tau_{b\,e_1}^{-1} + \frac{b}{r}$ and observe that

$$v_r^b \wedge \ell_r^b = \ell_{T_r}, \qquad \operatorname{dom}(v_r^b \vee \ell_r^b) \subset \{x_1 = b\}.$$

Thus, by the valuation property and Lemma 4.20, we obtain

$$h(\mathrm{Z}(v_r^b), e_1) = h(\mathrm{Z}(\ell_{T_r}), e_1) - h(\mathrm{Z}(\ell_r^b), e_1).$$

Using the translation invariance and continuity of Z now gives

$$r\,\psi_1(0) + r^2 \frac{c_3 + c_4}{(n+1)!} = h(\mathrm{Z}(\ell_{T_r}), e_1) = \lim_{b\to+\infty} h(\mathrm{Z}(v_r^b), e_1) = \lim_{b+\infty} r\,(\psi_1(0) - \psi_1(\tfrac{b}{r}))$$

for every $r > 0$. Hence, $\lim_{t\to+\infty} \psi_1(t) = 0$ and $c_3 + c_4 = 0$. Similarly, evaluating the support functions at $-e_1$ gives $\lim_{t\to+\infty} \psi_2(t) = 0$ and $c_3 - c_4 = 0$. Consequently, $c_3 = c_4 = 0$. ■

By Lemma 4.1, we obtain the following result as an immediate corollary of the last result.

Lemma 4.23. *Every continuous, increasing,* $\mathrm{SL}(n)$ *covariant, translation invariant Minkowski valuation on* $\mathrm{Conv}(\mathbb{R}^n)$ *is trivial.*

For a given continuous, $\mathrm{SL}(n)$ covariant and translation invariant Minkowski valuation $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{K}^n$, we call the function ψ from Lemma 4.22 the *cone growth function* of Z.

Lemma 4.24. *If the operator* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{K}^n$ *is a continuous,* $\mathrm{SL}(n)$ *covariant and translation invariant Minkowski valuation with cone growth function* ψ*, then* ψ *is decreasing and*

$$\mathrm{Z}(\mathrm{I}_K + t) = -\psi'(t)\,\mathrm{D}\,K$$

for every $t \in \mathbb{R}$ *and* $K \in \mathcal{K}_o^n$.

Proof. Let ζ be as in Lemma 4.19. Since $\zeta \ge 0$, it suffices to show that $\zeta = -\psi'$. Therefore, for $h > 0$ let $u_h \in \mathrm{Conv}(\mathbb{R}^n)$ be defined by $\operatorname{epi} u_h = \operatorname{epi} \ell_{[0,e_1/h]} \cap \{x_1 \le 1\}$. By Theorem 2.13, we have $\text{epi-lim}_{h\to 0} u_h = \mathrm{I}_{[0,e_1]}$. Define $\ell_h = \ell_{[0,e_1/h]} \circ \tau_{e_1}^{-1} + h$ and observe that

$$u_h \wedge \ell_h = \ell_{[0,e_1/h]} \qquad \text{and} \qquad u_h \vee \ell_h = \mathrm{I}_{\{e_1\}} + h.$$

Hence, by the properties of Z and the definitions of ψ and ζ this gives

$$\zeta(t) = h(\mathrm{Z}(\mathrm{I}_{[0,e_1]} + t), e_1) = \lim_{h\to 0^+} h(\mathrm{Z}(u_h + t), e_1) = \lim_{h\to 0^+} \frac{\psi(t) - \psi(t+h)}{h}$$

for every $t \in \mathbb{R}$. The claim follows, since ψ is differentiable. ■

The function $\zeta = -\psi'$ appearing in the above Lemma is called the *indicator growth function* of Z. Lemma 4.21 shows that the indicator growth function ζ of a continuous, $\mathrm{SL}(n)$ covariant and translation invariant Minkowski valuation Z determines its cone growth function ψ up to a constant. Since $\lim_{t\to\infty}\psi(t) = 0$, the constant is also determined by ζ. Thus ψ is completely determined by the indicator growth function of Z and Lemma 4.1 implies the following result.

Lemma 4.25. *Every continuous,* $\mathrm{SL}(n)$ *covariant, translation invariant Minkowski valuation on* $\mathrm{Conv}(\mathbb{R}^n)$ *is uniquely determined by its indicator growth function.*

Theorem 4.26. *A function* $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{K}^n$ *is a continuous, monotone,* $\mathrm{SL}(n)$ *covariant and translation invariant Minkowski valuation if and only if there exists* $\zeta \in D^0(\mathbb{R})$ *such that*

$$\mathrm{Z}(u) = \mathrm{D}\,[\zeta \circ u] \tag{4.12}$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$.

Proof. If $\zeta \in D^0(\mathbb{R})$, then Lemma 3.28 shows that (4.12) defines a continuous, decreasing, $\mathrm{SL}(n)$ covariant and translation invariant Minkowski valuation on $\mathrm{Conv}(\mathbb{R}^n)$.

Conversely, let a continuous, monotone, $\mathrm{SL}(n)$ covariant and translation invariant Minkowski valuation Z be given and let ζ be its indicator growth function. Lemma 4.23 implies that we may assume that Z is decreasing. By Lemma 4.25, the valuation Z is uniquely determined by ζ. For $P = [0, e_1] \in \mathcal{P}_o^n$, we have

$$h(\mathrm{Z}(\mathrm{I}_P + t), e_1) = \zeta(t)\, h(\mathrm{D}\,P, e_1) = \zeta(t)$$

for every $t \in \mathbb{R}$. Since Z is decreasing, also ζ is decreasing. Since $\zeta = -\psi'$, it follows from Lemma 4.21 that

$$\int_0^\infty \zeta(t)\,\mathrm{d}t = \psi(0) - \lim_{t\to\infty}\psi(t) = \psi(0).$$

Thus $\zeta \in D^0(\mathbb{R})$.

For $u = \mathrm{I}_P + t$ with arbitrary $P \in \mathcal{P}_o^n$ and $t \in \mathbb{R}$, we have

$$h(\mathrm{D}\,[\zeta \circ u], z) = \int_0^{+\infty} h(\mathrm{D}\{\zeta \circ u \geq s\}, z)\,\mathrm{d}s = \zeta(t)\, h(\mathrm{D}\,P, z)$$

for every $z \in \mathbb{S}^{n-1}$. Hence $\mathrm{D}\,[\zeta \circ (\mathrm{I}_P + t)] = \zeta(t)\,\mathrm{D}\,P$ for $P \in \mathcal{P}_o^n$ and $t \in \mathbb{R}$. By Lemma 3.28,

$$u \mapsto \mathrm{D}\,[\zeta \circ u]$$

defines a continuous, decreasing, $\mathrm{SL}(n)$ covariant and translation invariant Minkowski valuation on $\mathrm{Conv}(\mathbb{R}^n)$ with indicator growth function ζ. Thus Lemma 4.25 completes the proof of the theorem. ■

In the remainder of this section we will study valuations on $\mathrm{LC}(\mathbb{R}^n)$. However, instead of translation invariance we will consider translation covariance and monotonicity will be replaced by homogeneity.

The next result extends the basic observation that the associated function $\mathrm{Z}^0 : \mathcal{K}^n \to \mathbb{R}^n$ of a translation covariant Minkowski valuation $\mathrm{Z} : \mathcal{K}^n \to \mathcal{K}^n$ is a translation invariant real-valued valuation. See for example [44, Lemma 10.5] for a corresponding result on vector-valued valuations. Similarly, $\mathrm{SL}(n)$ covariance of Z implies $\mathrm{SL}(n)$ invariance of Z^0. Hence, it is no coincidence that the associated function of the Minkowski valuation described in Corollary 1.19 is a linear combination of the Euler characteristic and volume.

Lemma 4.27. *If* $\mathrm{Y} : \mathrm{LC}(\mathbb{R}^n) \to \mathcal{K}^n$ *is a continuous, homogeneous,* $\mathrm{SL}(n)$ *and translation covariant Minkowski valuation, then its associated function* $\mathrm{Y}^0 : \mathrm{LC}(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous, homogeneous,* $\mathrm{SL}(n)$ *and translation invariant valuation. Furthermore,* Y *and* Y^0 *have the same degree of homogeneity.*

Proof. Let $x \in \mathbb{R}^n \backslash \{0\}$ and $f, g \in \mathrm{LC}(\mathbb{R}^n)$ be such that $f \vee g \in \mathrm{LC}(\mathbb{R}^n)$. Since

$$(f \circ \tau_x^{-1}) \vee (g \circ \tau_x^{-1}) = (f \vee g) \circ \tau_x^{-1} \qquad \text{and} \qquad (f \circ \tau_x^{-1}) \wedge (g \circ \tau_x^{-1}) = (f \wedge g) \circ \tau_x^{-1},$$

it follows from the translation covariance and the valuation property of Y that

$$\begin{aligned} \mathrm{Y}(f \circ \tau_x^{-1}) + \mathrm{Y}(g \circ \tau_x^{-1}) &= \mathrm{Y}((f \vee g) \circ \tau_x^{-1}) + \mathrm{Y}((f \wedge g) \circ \tau_x^{-1}) \\ &= \mathrm{Y}(f \vee g) + \mathrm{Y}(f \wedge g) + \mathrm{Y}^0(f \vee g)x + \mathrm{Y}^0(f \wedge g)x. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathrm{Y}(f \circ \tau_x^{-1}) + \mathrm{Y}(g \circ \tau_x^{-1}) &= \mathrm{Y}(f) + \mathrm{Y}^0(f)x + \mathrm{Y}(g) + \mathrm{Y}^0(g)x \\ &= \mathrm{Y}(f \vee g) + \mathrm{Y}(f \wedge g) + \mathrm{Y}^0(f)x + \mathrm{Y}^0(g)x. \end{aligned}$$

Hence, Y^0 is a valuation. Now, for arbitrary $y \in \mathbb{R}^n$, we have

$$\begin{aligned} \mathrm{Y}(f) + \mathrm{Y}^0(f)x + \mathrm{Y}^0(f)y &= \mathrm{Y}(f \circ \tau_{x+y}^{-1}) \\ &= \mathrm{Y}(f \circ \tau_y^{-1} \circ \tau_x^{-1}) \\ &= \mathrm{Y}(f \circ \tau_y^{-1}) + \mathrm{Y}^0(f \circ \tau_y^{-1})x \\ &= \mathrm{Y}(f) + \mathrm{Y}^0(f)y + \mathrm{Y}^0(f \circ \tau_y^{-1})x, \end{aligned}$$

and therefore $\mathrm{Y}^0(f) = \mathrm{Y}^0(f \circ \tau_y^{-1})$. For $\phi \in \mathrm{SL}(n)$ observe that

$$(\tau_x^{-1} \circ \phi^{-1})(y) = \phi^{-1}y - x = \phi^{-1}(y - \phi x) = (\phi^{-1} \circ \tau_{\phi x}^{-1})(y)$$

for every $y \in \mathbb{R}^n$ and therefore

$$\begin{aligned} \phi\,\mathrm{Y}(f) + \mathrm{Y}^0(f)\phi x &= \phi\,\mathrm{Y}(f \circ \tau_x^{-1}) \\ &= \mathrm{Y}(f \circ \tau_x^{-1} \circ \phi^{-1}) \\ &= \mathrm{Y}(f \circ \phi^{-1} \circ \tau_{\phi x}^{-1}) \\ &= \mathrm{Y}(f \circ \phi^{-1}) + \mathrm{Y}^0(f \circ \phi^{-1})\phi x \\ &= \phi\,\mathrm{Y}(f) + \mathrm{Y}^0(f \circ \phi^{-1})\phi x. \end{aligned}$$

Hence, Y^0 is $\mathrm{SL}(n)$ invariant. Moreover, for $s > 0$ we have

$$s^q\,\mathrm{Y}(f) + s^q\,\mathrm{Y}^0(f)x = s^q\,\mathrm{Y}(f \circ \tau_x^{-1}) = \mathrm{Y}(s(f \circ \tau_x^{-1})) = \mathrm{Y}((sf) \circ \tau_x^{-1}) = s^q\,\mathrm{Y}(f) + \mathrm{Y}^0(sf)x.$$

Lastly, if $f_k, f \in \mathrm{LC}(\mathbb{R}^n)$ are such that $\text{hypo-lim}_{k\to\infty} f_k = f$, then also $\text{hypo-lim}_{k\to\infty} f_k \circ \tau_x^{-1} = f \circ \tau_x^{-1}$. Hence, by the continuity of Y,

$$\mathrm{Y}(f_k) + \mathrm{Y}^0(f_k)x = \mathrm{Y}(f_k \circ \tau_x^{-1}) \longrightarrow \mathrm{Y}(f \circ \tau_x^{-1}) = \mathrm{Y}(f) + \mathrm{Y}^0(f)x.$$

■

Lemma 4.28. *Let* $\mathrm{Y} : \mathrm{LC}(\mathbb{R}^n) \to \mathcal{K}^n$ *be a continuous,* $\mathrm{SL}(n)$ *and translation covariant Minkowski valuation that is homogeneous of degree* q. *There exist constants* $c_1, c_2, d_1, d_2, d_4 \geq 0$, *and* $c_3, d_3 \in \mathbb{R}$ *such that*

$$\mathrm{Y}(s\, e^{-\ell_K}) \;=\; s^q(d_1 K + d_2(-K) + d_4\, \mathrm{m}(K) + d_3\, \mathrm{M}(K)),$$

for every $K \in \mathcal{K}_o^n$ *and* $s > 0$ *and*

$$\mathrm{Y}(s\;\chi_K) \;=\; s^q(c_1 K + c_2(-K) + c_3\, \mathrm{m}(K)),$$

for every $K \in \mathcal{K}^n$ *and* $s > 0$. *Furthermore,* $q > 0$ *if* $c_3 \neq 0$ *and*

$$\mathrm{Y}^0(f) = (c_1 - c_2)V_0(f)^q + c_3 V_n(f^q),$$

for every $f \in \mathrm{LC}(\mathbb{R}^n)$.

Proof. Similar to the proof of Lemma 4.19 it follows from Lemma 2.14, (3.4), (3.7) and Remark 3.2, that the map

$$K \mapsto \mathrm{Y}(e^{-\ell_K})$$

defines a continuous, $\mathrm{SL}(n)$ covariant Minkowski valuation on $\mathcal{K}_o^n$. By Theorem 1.17 there exist constants $d_1, d_2, d_4 \geq 0$ and $d_3 \in \mathbb{R}$ such that

$$\mathrm{Y}(s\, e^{-\ell_K}) = s^q\, \mathrm{Y}(e^{-\ell_K}) = s^q(d_1 K + d_2(-K) + d_3\, \mathrm{m}(K) +\, \mathrm{d}_4\, \mathrm{M}(K))$$

for every $K \in \mathcal{K}_o^n$ and $s > 0$, where $q \in \mathbb{R}$ denotes the degree of homogeneity of Y. Similarly, $K \mapsto \mathrm{Y}(\chi_K)$ defines a continuous, $\mathrm{SL}(n)$ and translation covariant Minkowski valuation on $\mathcal{K}^n$. Hence, by Corollary 1.19 there exist constants $c_1, c_2 \geq 0$ and $c_3 \in \mathbb{R}$ such that

$$\mathrm{Y}(s\,\chi_K) = s^q(c_1 K + c_2(-K) + c_3\, \mathrm{m}(K)),$$

for every $K \in \mathcal{K}^n$ and $s > 0$.

For $K \in \mathcal{K}^n$, $x \in \mathbb{R}^n \backslash \{0\}$ and $s > 0$ let $f := s\,\chi_K \in \mathrm{LC}(\mathbb{R}^n)$ and observe that

$$\begin{aligned}
\mathrm{Y}(f) + \mathrm{Y}^0(f)x &= \mathrm{Y}(f \circ \tau_x^{-1}) \\
&= \mathrm{Y}(s\,\chi_{K+x}) \\
&= s^q(c_1 K + c_2(-K) + c_3\, \mathrm{m}(K) + (c_1 - c_2 + c_3 V_n(K))x) \\
&= \mathrm{Y}(f) + s^q(c_1 - c_2 + c_3 V_n(K))x.
\end{aligned}$$

On the other hand, by Theorem 4.10 and Lemma 4.27, there exist $\widetilde{c}_0, \widetilde{c}_n \in \mathbb{R}$ and $\widetilde{q} \in \mathbb{R}$, with $\widetilde{q} > 0$ if $\widetilde{c}_n \neq 0$, such that

$$\mathrm{Y}^0(g) = \widetilde{c}_0 V_0(g)^{\widetilde{q}} + \widetilde{c}_n V_n(g^{\widetilde{q}}),$$

for every $g \in \mathrm{LC}(\mathbb{R}^n)$. Noting, that $V_0(f)^q = s^q$ and $V_n(f^q) = s^q V_n(K)$, a comparison shows that

$$(c_1 - c_2)s^q V_0(K) + c_3 s^q V_n(K) = \mathrm{Y}^0(s\,\chi_K) = \widetilde{c}_0 s^{\widetilde{q}} V_0(K) + \widetilde{c}_n s^{\widetilde{q}} V_n(K),$$

for every $s > 0$ and $K \in \mathcal{K}^n$. Choosing $K = \{0\}$ and $s = 1$ gives $c_1 - c_2 = \widetilde{c}_0$. With the same K and arbitrary $s > 0$ we have $q = \widetilde{q}$ and with any full-dimensional $K \in \mathcal{K}^n$ we obtain $\widetilde{c}_n = c_3$. ∎

Lemma 4.29. *Let* $\mathrm{Y} : \mathrm{LC}(\mathbb{R}^n) \to \mathcal{K}^n$ *be a continuous,* $\mathrm{SL}(n)$ *and translation covariant Minkowski valuation that is homogeneous of degree* q. *If* c_1, c_2, d_1, d_2 *denote the constants from Lemma 4.28, then* $c_1 = q\,d_1$ *and* $c_2 = q\,d_2$.

Proof. For $h > 0$ let $u_h \in \mathrm{Conv}(\mathbb{R}^n)$ be defined via $\operatorname{epi} u_h = \operatorname{epi} \ell_{[0,e_1/h]} \cap \{x_1 \leq 1\}$. As in the proof of Lemma 4.24, we have $u_h \xrightarrow{epi} \mathrm{I}_{[0,e_1]}$ as $h \to 0$ and furthermore

$$u_h \wedge \ell_h = \ell_{[0,e_1/h]} \qquad \text{and} \qquad u_h \vee \ell_h = \mathrm{I}_{\{e_1\}} + h,$$

where $\ell_h = \ell_{[0,e_1/h]} \circ \tau_{e_1}^{-1} + h$. Let Z be the valuation on $\mathrm{Conv}(\mathbb{R}^n)$ that corresponds to Y, that is $\mathrm{Z}(u) = \mathrm{Y}(e^{-u})$ for every $u \in \mathrm{Conv}(\mathbb{R}^n)$. By Remark 3.2, the valuation Z is continuous, $\mathrm{SL}(n)$ and translation covariant and furthermore $\mathrm{Z}(u+t) = e^{-qt}\,\mathrm{Z}(u)$ for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $t \in \mathbb{R}$. We now have

$$\mathrm{Z}(\ell_h) = e^{-qh}\,\mathrm{Z}(\ell_{[0,e_1/h]}) + (c_1 - c_2)e^{-qh} e_1$$

and furthermore

$$\begin{aligned}
c_1 = h(\mathrm{Z}(\mathrm{I}_{[0,e_1]}), e_1) &= \lim_{h\to 0^+} h(\mathrm{Z}(u_h), e_1)\\
&= \lim_{h\to 0^+} \left(h(\mathrm{Z}(\ell_{[0,e_1/h]}), e_1) + h(\mathrm{Z}(\mathrm{I}_{\{e_1\}} + h), e_1) - h(\mathrm{Z}(\ell_h), e_1)\right)\\
&= \lim_{h\to 0^+} \left(\tfrac{d_1}{h} + (c_1 - c_2)e^{-qh} - e^{-qh}\tfrac{d_1}{h} - (c_1 - c_2)e^{-qh}\right)\\
&= \lim_{h\to 0^+} d_1 \tfrac{1-e^{-qh}}{h} = q\,d_1.
\end{aligned}$$

Similarly, evaluating the support functions at $-e_1$ shows that $c_2 = q\,d_2$. ∎

Lemma 4.30. *Every continuous, homogeneous,* $\mathrm{SL}(n)$ *and translation covariant Minkowski valuation* $\mathrm{Y} : \mathrm{LC}(\mathbb{R}^n) \to \mathcal{K}^n$ *is either trivial or has a positive degree of homogeneity.*

Proof. Let d_1, d_2, c_1, c_2, c_3 and q denote the constants from Lemma 4.28 and suppose that $q \leq 0$. Lemma 4.28 shows that $c_3 = 0$. Furthermore, since c_1, c_2, d_1 and d_2 are non-negative, Lemma 4.29 yields that also $c_1 = c_2 = 0$. Hence, $\mathrm{Y}^0 \equiv 0$ and Y is translation invariant. Moreover, $\mathrm{Y}(s\,\chi_K) = \{0\}$ for every $s > 0$ and $K \in \mathcal{K}^n$. Thus, Remark 3.2 and Lemma 4.25 show that Y is trivial. ∎

Lemma 4.31. *For* $a, b \in \mathbb{R}$ *and* $q > 0$ *the following holds:*

$$\lim_{h\to 0^+} \left(a\frac{1-e^{-qh}}{h^2} - b\frac{e^{-qh}}{h}\right) = \begin{cases} \frac{q}{2}b & \text{if } b = q\,a \\ +\infty & \text{else.}\end{cases}$$

Proof. Since,

$$a\frac{1-e^{-qh}}{h^2} - b\frac{e^{-qh}}{h} = \frac{a\,(1-e^{-qh}) - b\,h\,e^{-qh}}{h^2},$$

and

$$\lim_{h\to 0^+} \left(a\,(1-e^{-qh}) - b\,h\,e^{-qh}\right) = 0,$$

we can apply L'Hospital's rule to obtain

$$\lim_{h\to 0^+} \frac{a\,(1-e^{-qh}) - b\,he^{-qh}}{h^2} = \lim_{h\to 0^+} \frac{q\,a\,e^{-q\,h} - b\,e^{-qh} + q\,b\,he^{-qh}}{2h} = \lim_{h\to 0^+} \tfrac{e^{-qh}}{2h}(q\,a - b) + \tfrac{q}{2}b.$$

The claim now follows since $\frac{e^{-qh}}{2h} \to +\infty$ as $h \to 0^+$. ∎

Lemma 4.32. *Let* $\mathrm{Y} : \mathrm{LC}(\mathbb{R}^n) \to \mathcal{K}^n$ *be a continuous,* $\mathrm{SL}(n)$ *and translation covariant Minkowski valuation that is homogeneous of degree* q*. If* c_3, d_3, d_4 *denote the constants from Lemma 4.28, then* $c_3 = \frac{q^{n+1}}{(n+1)!} d_3$ *and* $d_4 = 0$*.*

Proof. By Lemma 4.30, we can assume without loss of generality that $q > 0$. Define $v \in \mathrm{Conv}(\mathbb{R}^n)$ via

$$\{v < 0\} = \emptyset, \quad \{v \le s\} = [0, e_1] + \mathrm{conv}\{0, s\, e_2, \dots, s\, e_n\},$$

for every $s \ge 0$. Now, for $h > 0$ let $T_{1/h}$ be defined as in Lemmas 1.16 and define the function u_h via

$$\{u_h \le s\} = \{\ell_{T_{1/h}} \le s\} \cap \{x_1 \le 1\},$$

for every $s \in \mathbb{R}$. Similar to the proof of Lemma 4.21 we have $u_h \in \mathrm{Conv}(\mathbb{R}^n)$ and furthermore

$$\begin{aligned} \{u_h \le s\} \cup \{\ell_{T_{1/h}} \circ \tau_{e_1}^{-1} + h \le s\} &= \{\ell_{T_{1/h}} \le s\} \\ \{u_h \le s\} \cap \{\ell_{T_{1/h}} \circ \tau_{e_1}^{-1} + h \le s\} &= \{\ell_{\mathrm{conv}\{0, e_2, \dots, e_n\}} \circ \tau_{e_1}^{-1} + h \le s\}, \end{aligned}$$

for every $s \in \mathbb{R}$. Thus, denoting $\mathrm{Z}(u) = \mathrm{Y}(e^{-u})$ for $u \in \mathrm{Conv}(\mathbb{R}^n)$, this gives

$$\mathrm{Z}(u_h) + \mathrm{Z}(\ell_{T_{1/h}} \circ \tau_{e_1}^{-1} + h) = \mathrm{Z}(\ell_{T_{1/h}}) + \mathrm{Z}(\ell_{\mathrm{conv}\{0, e_2, \dots, e_n\}} \circ \tau_{e_1}^{-1} + h). \tag{4.13}$$

By Lemmas 3.13 and 4.28 we have

$$\begin{aligned} \mathrm{Z}(\ell_{T_{1/h}} \circ \tau_{e_1}^{-1} + h) &= e^{-qh}\, \mathrm{Z}(\ell_{T_{1/h}}) + e^{-qh}\big((c_1 - c_2) + \tfrac{c_3}{h\, q^n}\big) e_1 \\ \mathrm{Z}(\ell_{\mathrm{conv}\{0, e_2, \dots, e_n\}} \circ \tau_{e_1}^{-1} + h) &= e^{-qh}\, \mathrm{Z}(\ell_{\mathrm{conv}\{0, e_2, \dots, e_n\}}) + e^{-qh}(c_1 - c_2) e_1. \end{aligned}$$

Furthermore, using Lemma 1.16 we obtain for the support functions

$$\begin{aligned} h(\mathrm{Z}(\ell_{T_{1/h}}), e_1) &= \tfrac{d_1}{h} + \tfrac{d_3 + d_4}{h^2\, (n+1)!}, \\ h(\mathrm{Z}(\ell_{T_{1/h}} \circ \tau_{e_1}^{-1} + h), e_1) &= e^{-qh}\big(\tfrac{d_1}{h} + \tfrac{d_3 + d_4}{h^2\, (n+1)!} + (c_1 - c_2) + \tfrac{c_3}{h\, q^n}\big), \\ h(\mathrm{Z}(\ell_{\mathrm{conv}\{0, e_2, \dots, e_n\}}), e_1) &= e^{-qh}(c_1 - c_2). \end{aligned}$$

Observe, that for $h \to 0^+$ we have $u_h \xrightarrow{epi} v$. Hence, by the continuity of Z and (4.13), we have

$$\begin{aligned} h(\mathrm{Z}(v), e_1) &= \lim_{h \to 0^+} h(\mathrm{Z}(u_h), e_1) \\ &= \lim_{h \to 0^+} \big(\tfrac{d_1}{h}(1 - e^{-qh}) + \tfrac{d_3 + d_4}{h^2\, (n+1)!}(1 - e^{-qh}) - \tfrac{c_3}{h\, q^n} e^{-qh}\big) \\ &= q\, d_1 + \lim_{h \to 0^+} \big(\tfrac{d_3 + d_4}{(n+1)!} \tfrac{1 - e^{-qh}}{h^2} - \tfrac{c_3}{q^n} \tfrac{e^{-qh}}{h}\big). \end{aligned}$$

Since this expression must be finite, it follows from Lemma 4.31 that

$$\frac{c_3}{q^n} = q \frac{d_3 + d_4}{(n+1)!}.$$

Similarly, repeating the calculations above but evaluating the support functions at $-e_1$ gives

$$\frac{c_3}{q^n} = q \frac{d_3 - d_4}{(n+1)!}.$$

Hence, $d_4 = 0$ and $c_3 = \frac{q^{n+1}}{(n+1)!} d_3$. ∎

By Lemma 4.1, every continuous, homogeneous, $\mathrm{SL}(n)$ and translation covariant Minkowski valuation Y on $\mathrm{LC}(\mathbb{R}^n)$ is uniquely determined by the constants $c_1, c_2, c_3, d_1, d_2, d_3, d_4$ and q from Lemma 4.28. By Lemmas 4.29 and 4.32 we have $d_1 = \frac{c_1}{q}, d_2 = \frac{c_2}{q}, d_3 = \frac{(n+1)!}{q^{n+1}} c_3$ and $d_4 = 0$. Hence, Y is completely determined by the constants c_1, c_2, c_3 and q. Thus, we have the following result.

Lemma 4.33. *Every continuous, homogeneous,* $\mathrm{SL}(n)$ *and translation covariant Minkowski valuation* $\mathrm{Y} : \mathrm{LC}(\mathbb{R}^n) \to \mathcal{K}^n$ *is uniquely determined by the values* $\mathrm{Y}(s\, \chi_K)$ *with* $s > 0$ *and* $K \in \mathcal{K}^n$*.*

Theorem 4.34. *A function* $\mathrm{Y} : \mathrm{LC}(\mathbb{R}^n) \to \mathcal{K}^n$ *is a continuous, homogeneous,* $\mathrm{SL}(n)$ *and translation covariant Minkowski valuation if and only if there exist constants* $c_1, c_2 \geq 0$*,* $c_3 \in \mathbb{R}$ *and* $q > 0$ *such that*

$$\mathrm{Y}(f) = c_1 [f^q] + c_2(-[f^q]) + c_3\, \mathrm{m}(f^q) \tag{4.14}$$

for every $f \in \mathrm{LC}(\mathbb{R}^n)$*.*

Proof. Lemmas 3.29 and 3.32 show that (4.14) defines a continuous, homogeneous, $\mathrm{SL}(n)$ and translation covariant Minkowski valuation on $\mathrm{LC}(\mathbb{R}^n)$ for every $c_1, c_2 \geq 0$, $c_3 \in \mathbb{R}$ and $q > 0$.

Conversely, let $\mathrm{Y} : \mathrm{LC}(\mathbb{R}^n) \to \mathbb{R}$ be a continuous, homogeneous, $\mathrm{SL}(n)$ and translation covariant Minkowski valuation. For arbitrary $K \in \mathcal{K}^n$ and $s > 0$, let $f = s\, \chi_K$. By Lemma 4.28, there exist constants $c_1, c_2 \geq 0$ and $c_3, q \in \mathbb{R}$ such that

$$\mathrm{Y}(f) = s^q (c_1 K + c_2(-K) + c_3\, \mathrm{m}(K))$$

and by Lemma 4.30 we may assume that $q > 0$. Since

$$\begin{aligned} h([f^q], z) &= \int_0^{+\infty} h(\{s^q\, \chi_K \geq t\}, z)\, \mathrm{d}t = s^q\, h(K, z) \\ h(\mathrm{m}(f^q), z) &= \int_{\mathbb{R}^n} s^q\, \chi_K(x)(x \cdot z)\, \mathrm{d}x = s^q\, h(\mathrm{m}(K), z) \end{aligned}$$

for every $z \in \mathbb{S}^{n-1}$, we have $\mathrm{Y}(f) = c_1[f^q] + c_2(-[f^q]) + c_3\, \mathrm{m}(f^q)$. Thus, Lemma 4.33 completes the proof of the theorem. ■

The next result is a corollary of both Theorem 4.26 and Theorem 4.34.

Corollary 4.35. *A function* $\mathrm{Y} : \mathrm{LC}(\mathbb{R}^n) \to \mathcal{K}^n$ *is a continuous, homogeneous,* $\mathrm{SL}(n)$ *covariant and translation invariant Minkowski valuation if and only if there exist* $c \geq 0$ *and* $q > 0$ *such that*

$$\mathrm{Y}(f) = c\, \mathrm{D}\, [f^q]$$

for every $f \in \mathrm{LC}(\mathbb{R}^n)$*.*

4.4 Classification of Measure-Valued Valuations

The aim of this section is to give a classification of the surface area measure that generates the operator from Theorem 4.17. Let $n \geq 3$.

Lemma 4.36. *If $\mu : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{M}(\mathbb{S}^{n-1})$ is a weakly continuous valuation that is $\mathrm{SL}(n)$ contravariant of degree 1, then there exist continuous functions $\psi_1, \psi_2, \zeta_1, \zeta_2 : \mathbb{R} \to [0, \infty)$ such that*

$$\mu(\ell_K + t, \cdot) = \psi_1(t) S(K, \cdot) + \psi_2 S(-K, \cdot),$$
$$\mu(\mathrm{I}_K + t, \cdot) = \zeta_1(t) S(K, \cdot) + \zeta_2 S(-K, \cdot),$$

for every $K \in \mathcal{K}_o^n$ and $t \in \mathbb{R}$.

Proof. For $t \in \mathbb{R}$, define $\mu_t : \mathcal{K}_o^n \to \mathcal{M}(\mathbb{S}^{n-1})$ as

$$\mu_t(K, \cdot) = \mu(\ell_K + t, \cdot).$$

As in the proof of Lemma 4.11, we see that μ_t is a weakly continuous valuation that is $\mathrm{SL}(n)$ contravariant of degree 1 for every $t \in \mathbb{R}$. By Corollary 1.9, for $t \in \mathbb{R}$, there exist $c_{1,t}, c_{2,t} \geq 0$ such that

$$\mu(\ell_K + t, \cdot) = \mu_t(K, \cdot) = c_{1,t} S(K, \cdot) + c_{2,t} S(-K, \cdot)$$

for all $K \in \mathcal{K}_o^n$. This defines non-negative functions $\psi_1(t) = c_{1,t}$ and $\psi_2(t) = c_{2,t}$. To see that those functions are continuous let $b : \mathbb{S}^{n-1} \to [0, \infty)$ be a continuous function such that

$$b\big(\{\tfrac{1}{\sqrt{n}}(1, \dots, 1)^t\}\big) > 0$$

and such that b vanishes on all other outer unit normals of T^n and $-T^n$. Since $\frac{1}{\sqrt{n}}(1, \dots, 1)^t$ is an outer unit normal of T^n but not of $-T^n$, we have

$$\begin{aligned} t \mapsto & \int_{\mathbb{S}^{n-1}} b(z) \, \mathrm{d}\mu(\ell_{T^n} + t, z) \\ & = \psi_1(t) \int_{\mathbb{S}^{n-1}} b(z) \, \mathrm{d}S(T^n, z) + \psi_2(t) \int_{\mathbb{S}^{n-1}} b(z) \, \mathrm{d}S(-T^n, z) = \psi_1(t) \, d + 0, \end{aligned}$$

with some constant $d \neq 0$. Hence, ψ_1 is continuous. Similarly, one shows that ψ_2 is a continuous function. The result for indicator functions and ζ_1, ζ_2 follows along similar lines. ■

For a weakly continuous valuation $\mu : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{M}(\mathbb{S}^{n-1})$ that is $\mathrm{SL}(n)$ contravariant of degree 1, we call the functions ψ_1 and ψ_2 from Lemma 4.36, the *cone growth functions* of μ and we call the functions ζ_1 and ζ_2 its *indicator growth functions*. By Lemma 4.1 we have the following result.

Lemma 4.37. *Every weakly continuous valuation $\mu : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{M}(\mathbb{S}^{n-1})$ that is $\mathrm{SL}(n)$ contravariant of degree 1 and translation invariant is uniquely determined by its cone growth functions.*

Lemma 4.38. *Let $\mu : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{M}(\mathbb{S}^{n-1})$ be a weakly continuous valuation that is $\mathrm{SL}(n)$ contravariant of degree 1 and translation invariant. If $\psi = \psi_1 + \psi_2$ and $\zeta = \zeta_1 + \zeta_2$, then*

$$\zeta(t) = \frac{(-1)^{n-1}}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} \psi(t).$$

Moreover, ψ is decreasing and $\lim_{t \to +\infty} \psi(t) = 0$.

Proof. Recall that the cosine transform $\mathscr{C}\mu(u,\cdot)$ is the support function of a convex body that contains the origin for every $u \in \mathrm{Conv}(\mathbb{R}^n)$. By the properties of μ, this induces a continuous, $\mathrm{SL}(n)$ contravariant and translation invariant Minkowski valuation $\mathrm{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{K}^n$ via

$$h(\mathrm{Z}(u), z) = \tfrac{1}{2}\mathscr{C}\mu(u,\cdot)(z)$$

for $z \in \mathbb{S}^{n-1}$. By Lemma 4.36, we have

$$h(\mathrm{Z}(\ell_K + t), z) = \tfrac{1}{2}\mathscr{C}\big(\psi_1(t)S(K,\cdot) + \psi_2(t)S(-K,\cdot)\big)(z) = \psi(t)h(\Pi K, z)$$

for every $K \in \mathcal{K}_o^n$, $t \in \mathbb{R}$ and $z \in \mathbb{S}^{n-1}$. Hence, by Lemma 4.11, the function ψ is the cone growth function of Z. Similarly, it can be seen, that ζ is the indicator growth function of Z. The result now follows from Lemma 4.13 and Lemma 4.14. ■

Lemma 4.39. *Every weakly continuous, increasing valuation $\mu : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{M}(\mathbb{S}^{n-1})$ that is $\mathrm{SL}(n)$ contravariant of degree 1 and translation invariant is trivial.*

Proof. Since μ is increasing, Lemma 4.36 implies that for $s < t$

$$\begin{aligned} \mu(\ell_K + s, \mathbb{S}^{n-1}) &\leq \mu(\ell_K + t, \mathbb{S}^{n-1}),\\ \psi_1(s)S(K,\mathbb{S}^{n-1}) + \psi_2(s)S(-K,\mathbb{S}^{n-1}) &\leq \psi_1(t)S(K,\mathbb{S}^{n-1}) + \psi_2(t)S(-K,\mathbb{S}^{n-1}), \end{aligned}$$

for every $K \in \mathcal{K}_o^n$. Hence, $\psi = \psi_1 + \psi_2$ is an increasing function. By Lemma 4.38, $\psi \equiv 0$ and therefore $\psi_1 \equiv 0 \equiv \psi_2$, since those are non-negative functions. Lemma 4.37 implies that μ is trivial. ■

Lemma 4.40. *Every weakly continuous valuation $\mu : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{M}_e(\mathbb{S}^{n-1})$ that is $\mathrm{SL}(n)$ covariant of degree 1 and translation invariant is uniquely determined by its indicator growth functions.*

Proof. Since μ is an even measure the cone growth functions ψ_1 and ψ_2 coincide and similarly the indicator growth functions ζ_1 and ζ_2 coincide. By Lemma 4.38, we have $\lim_{t\to+\infty}\psi(t) = 0$ and $\zeta(t) = \frac{(-1)^{n-1}}{(n-1)!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\psi(t)$, where $\psi = \psi_1 + \psi_2$ and $\zeta = \zeta_1 + \zeta_2$. This shows that ζ uniquely determines ψ and therefore the indicator growth functions uniquely determine the cone growth functions. Since Lemma 4.37 implies that μ is determined by its cone growth functions, this implies the statement of the lemma. ■

Theorem 4.41. *An operator $\mu : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{M}_e(\mathbb{S}^{n-1})$ is a weakly continuous, monotone valuation that is $\mathrm{SL}(n)$ contravariant of degree 1 and translation invariant if and only if there exists $\zeta \in D^{n-2}(\mathbb{R})$ such that*

$$\mu(u,\cdot) = S(\langle\zeta \circ u\rangle,\cdot) + S(\langle\zeta \circ u^-\rangle,\cdot) \tag{4.15}$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$.

Proof. By Lemma 3.17, the map defined in (4.15) is a weakly continuous, decreasing valuation that is $\mathrm{SL}(n)$ contravariant of degree 1 and translation invariant. Furthermore, it is easy to see $\mu(u,\cdot)$ is an even measure for every $u \in \mathrm{Conv}(\mathbb{R}^n)$.

Conversely, let $\mu : \mathrm{Conv}(\mathbb{R}^n) \to \mathcal{M}_e(\mathbb{S}^{n-1})$ be a weakly continuous, monotone valuation that is $\mathrm{SL}(n)$ contravariant of degree 1 and translation invariant. Let $\zeta_1, \zeta_2 : \mathbb{R} \to [0,\infty)$ be its indicator growth functions. If μ is increasing, then Lemma 4.39 shows that μ is trivial. Hence we may assume that μ is decreasing. Since μ is even, the cone growth functions coincide and similarly the indicator growth functions coincide. Therefore, let $\zeta = \zeta_1 = \zeta_2$. Thus, Lemma 4.38 combined with Lemma 4.3 implies that $\zeta \in D^{n-2}(\mathbb{R})$.

Now, for $u = \mathrm{I}_K + t$ with $K \in \mathcal{K}^n_o$ and $t \in \mathbb{R}$ we obtain by Lemma 4.36 and by the definition of $S(\langle \zeta \circ u\rangle, \cdot)$ in Lemma and Definition 3.15 that

$$\mu(u, \cdot) = \zeta(t)(S(K, \cdot) + S(-K, \cdot)) = S(\langle \zeta \circ u\rangle, \cdot) + S(\langle \zeta \circ u^-\rangle, \cdot).$$

By Lemma 3.17,

$$u \mapsto S(\langle \zeta \circ u\rangle, \cdot) + S(\langle \zeta \circ u^-\rangle, \cdot)$$

defines a weakly continuous, decreasing valuation on $\mathrm{Conv}(\mathbb{R}^n)$ that is $\mathrm{SL}(n)$ contravariant of degree 1, translation invariant and even and both indicator growth functions are given by ζ. Thus, Lemma 4.40 completes the proof of the theorem. ■

For a classification of the homogeneous case on $\mathrm{LC}(\mathbb{R}^n)$, the measure does not need to be even.

Theorem 4.42. *An operator* $\nu : \mathrm{LC}(\mathbb{R}^n) \to \mathcal{M}(\mathbb{S}^{n-1})$ *is a weakly continuous, homogeneous valuation that is* $\mathrm{SL}(n)$ *contravariant of degree* 1 *and translation invariant if and only if there exist* $c_1, c_2 \geq 0$ *and* $q > 0$ *such that*

$$\nu(f, \cdot) = c_1 S(\langle f^q\rangle, \cdot) + c_2 S(\langle (f^-)^q\rangle, \cdot) \tag{4.16}$$

for every $f \in \mathrm{LC}(\mathbb{R}^n)$.

Proof. If $c_1, c_2 \geq 0$ and $q > 0$, then Lemma 3.19 shows that (4.16) is a weakly continuous, homogeneous valuation that is $\mathrm{SL}(n)$ contravariant of degree 1 and translation invariant.

Conversely, let a weakly continuous, homogeneous valuation ν be given that is $\mathrm{SL}(n)$ contravariant of degree 1 and translation invariant and let μ be the corresponding valuation on $\mathrm{Conv}(\mathbb{R}^n)$, that is $\mu(u) = \nu(e^{-u})$ for every $u \in \mathrm{Conv}(\mathbb{R}^n)$. Then μ is weakly continuous, $\mathrm{SL}(n)$ contravariant of degree 1 and translation invariant, see also Remark 3.2. Furthermore,

$$\mu(u + t) = \nu(e^{-(u+t)}) = (e^{-t})^q \nu(e^{-u}) = e^{-qt}\mu(u),$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $t \in \mathbb{R}$, where $q \in \mathbb{R}$ denotes the degree of homogeneity of ν. Let ψ_1 and ψ_2 denote the cone growth functions of μ. Since $e = \frac{1}{\sqrt{n}}(1, \dots, 1)^t$ is an outer unit normal of T^n but not of $-T^n$, we have by Lemma 4.36

$$\psi_1(t) S(T^n, e) = \mu(\ell_{T^n} + t, e) = e^{-qt}\mu(\ell_{T^n}, e) = e^{-qt}\psi_1(0) S(T^n, e),$$

for every $t \in \mathbb{R}$. Since $S(T^n, e) \neq 0$, this implies that

$$\psi_1(t) = \psi_1(0) e^{-qt} = d_1\, e^{-qt}$$

for some constant $d_1 \geq 0$. Similarly, there exists a constant $d_2 \geq 0$ such that $\psi_2(t) = d_2\, e^{-qt}$. By Lemma 3.20

$$S(\langle e^{-q(\ell_K + t)}\rangle, \cdot) = \tfrac{(n-1)!}{q^{n-1}} e^{-qt} S(K, \cdot),$$

for every $t \in \mathbb{R}$ and $K \in \mathcal{K}^n_o$. Hence, it is easy to see that for $u = \ell_K + t$ with $t \in \mathbb{R}$ and $K \in \mathcal{K}^n_o$ and $f = e^{-u}$

$$\begin{aligned}\nu(f) = \mu(u, \cdot) &= d_1 e^{-qt} S(K, \cdot) + d_2 e^{-qt} S(-K, \cdot)\\ &= c_1 S(\langle e^{-qu}\rangle, \cdot) + c_2 S(\langle e^{-q(u^-)}\rangle, \cdot) = c_1 S(\langle f^q\rangle, \cdot) + c_2 S(\langle (f^-)^q\rangle, \cdot),\end{aligned}$$

with $c_i = \frac{d_i\, q^{n-1}}{(n-1)!}$, $i = 1, 2$. By Lemma 3.19,

$$f \mapsto S(\langle f^q \rangle, \cdot) + c_2 S(\langle (f^-)^q \rangle, \cdot)$$

defines a weakly continuous, homogeneous valuation on $\mathrm{LC}(\mathbb{R}^n)$ that is $\mathrm{SL}(n)$ contravariant of degree 1 and translation invariant. Thus, Lemma 4.37 completes the proof of the theorem. ■

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May the force be with you.

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