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Definitizability of Normal Operators on Krein Spaces and Their Functional Calculus

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Abstract. We discuss a new concept of definitizability of a normal operator on Krein spaces. For this new concept we develop a functional calculus $\phi \mapsto \phi(N)$ which is the proper analogue of $\phi \mapsto \int \phi \, dE$ in the Hilbert space situation.

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1. Introduction

A bounded linear operator N on a Krein space $(\mathcal{K}, [.,.])$ is normal, if N commutes with its Krein space adjoint N^+. If we write a bounded linear N as $A + iB$ with the selfadjoint real part $A := \operatorname{Re} N := \frac{N+N^+}{2}$ and the selfadjoint imaginary part $B := \operatorname{Im} N := \frac{N-N^+}{2i}$, then N is normal if and only if $AB = BA$. In [4] we called a normal N definitizable whenever A and B were both definitizable in the classical sense, i.e. there exist so-called definitizing polynomials $p(z), q(z) \in \mathbb{R}[z]\backslash\{0\}$ such that $[p(A)x, x] \geq 0$ and $[q(B)x, x] \geq 0$ for all $x \in \mathcal{K}$.

For such definitizable operators in [4] we could build a functional calculus in analogy to the functional calculus $\phi \mapsto \int \phi \, dE$ mapping the $*$-algebra of bounded and measurable functions on $\sigma(N)$ to $B(\mathcal{H})$ in the Hilbert space case. The functional calculus in [4] can also be seen as a generalization of Heinz Langers spectral theorem on definitizable selfadjoint operators on Krein spaces; see [5,6]. Unfortunately, there are unsatisfactory phenomenons with this concept of definitizability in [4]. For example, it is not clear, whether for a bijective, normal definitizable N also N^{-1} definitizable.

In the present paper we choose a more general concept of definitizability. We shall say that a normal N on a Krein space $\mathcal{K}$ is definitizable if $[p(A, B)u, u] \geq 0$ for all $u \in \mathcal{K}$ for some, so-called definitizing, $p \in \mathbb{C}[x, y]\backslash\{0\}$

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with real coefficients. Then we study the ideal $\mathcal{I}$ generated by all definitizing polynomials with real coefficients in $\mathbb{C}[x,y]$, and assume that $\mathcal{I}$ is large in the sense that it is zero-dimensional, i.e. $\dim \mathbb{C}[x,y]/\mathcal{I} < \infty$. By the way, if N is definitizable in the sense of [4], then $\mathcal{I}$ is always zero-dimensional.

Using results from algebraic geometry, under the assumption that $\mathcal{I}$ is zero-dimensional, the variety $V(\mathcal{I}) = \{a \in \mathbb{C}^2 : f(a) = 0 \;\text{ for all }\; f \in \mathcal{I}\}$ is a finite set. We split this subset of $\mathbb{C}^2$ up as

$$V(\mathcal{I}) = (V(\mathcal{I}) \cap \mathbb{R}^2)\dot{\cup}(V(\mathcal{I})\backslash\mathbb{R}^2),$$

and interpret $V_{\mathbb{R}}(\mathcal{I}) := V(\mathcal{I}) \cap \mathbb{R}^2$ in the following as a subset of $\mathbb{C}$ by considering the first entry of an element of $\mathbb{R}^2$ as the real and the second entry as the imaginary part.

Due to the ascending chain condition the ideal $\mathcal{I}$ is generated by finitely many real definitizing polynomials $p_1, \dots, p_m$. With the help of the positive semidefinite scalar products $[p_j(A,B).,.]$, $j=1,\dots,m$, and $\sum_{k=1}^m [p_k(A,B).,.]$ we construct Hilbert spaces $\mathcal{H}_j$, $j = 1, \dots, m$, and $\mathcal{H}$ together with bounded and injective $T_j : \mathcal{H}_j \to \mathcal{K}$ and $T : \mathcal{H} \to \mathcal{K}$. We consider the $*$-algebra homomorphisms $\Theta_j : (T_jT_j^+)' \to (T_j^+T_j)'$, $C \mapsto (T_j \times T_j)^{-1}(C)$ and $\Theta : (TT^+)' \to (T^+T)'$, $C \mapsto (T \times T)^{-1}(C)$ as studied in [5],

Here $T_j \times T_j : \mathcal{H}_j \times \mathcal{H}_j \to \mathcal{K} \times \mathcal{K}$ maps the pair $(x;y)$ to the pair $(T_jx;T_jy)$ and $T \times T : \mathcal{H} \times \mathcal{H} \to \mathcal{K} \times \mathcal{K}$ maps $(x;y)$ to $(Tx;Ty)$. By $(T_jT_j^+)', (TT^+)' \subseteq B(\mathcal{K})$ and $(T_j^+T_j)' \subseteq B(\mathcal{H}_j), (T^+T)' \subseteq B(\mathcal{H})$ we denote the commutant of the respective operators.

The proper family $\mathcal{F}_N$ of functions suitable for the aimed functional calculus are functions defined on

$$(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I}))\,\dot{\cup}(V(\mathcal{I})\backslash\mathbb{R}^2).$$

Moreover, the functions $\phi \in \mathcal{F}_N$ assume values in $\mathbb{C}$ on $\sigma(\Theta(N))\backslash V_{\mathbb{R}}(\mathcal{I})$ and values in certain finite dimensional $*$-algebras $\mathcal{A}(z)$ at $z \in V_{\mathbb{R}}(\mathcal{I})$ and $\mathcal{B}((\xi,\eta))$ at $(\xi,\eta) \in V(\mathcal{I})\backslash\mathbb{R}^2$. On $\sigma(\Theta(N))\backslash V_{\mathbb{R}}(\mathcal{I})$ we assume ϕ to be bounded and measurable. Finally, $\phi \in \mathcal{F}_N$ satisfies a growth regularity condition at all w points from $V_{\mathbb{R}}(\mathcal{I})$ which are not isolated in $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})$. Vaguely speaking, this growth regularity condition means that around w the function ϕ admits an approximation by a Taylor polynomial, which is determined by $\phi(w) \in \mathcal{A}(w)$. Any polynomial $s \in \mathbb{C}[x,y]$ can be seen as a function $s_N \in \mathcal{F}_N$ in a natural way.

For each $\phi \in \mathcal{F}_N$ we will see that there exist $p \in \mathbb{C}[x,y]$ and bounded, measurable $f_1, \dots, f_m : \sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I}) \to \mathbb{C}$ with $f_j(z) = 0$ for $z \in V_{\mathbb{R}}(\mathcal{I})$ such that

$$\phi(z) = p_N(z) + \sum_j f_j(z)\,(p_j)_N(z) \tag{1.1}$$

for all $z \in \sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})$, and that $\phi((\xi,\eta)) = p_N((\xi,\eta))$ for all $(\xi,\eta) \in V(\mathcal{I})\backslash\mathbb{R}^2$. Defining ($E$ denotes the spectral measure of $\Theta(N)$)

$$\phi(N) := p(A,B) + \sum_{k=1}^m T_k \left(\int_{\sigma(\Theta_k(N))} f_k\, dE_k \right) T_k^+,$$

we show that this operator does not depend on the actual decomposition (1.1) and that $\phi \mapsto \phi(N)$ is indeed a $*$-homomorphism satisfying $\phi(N) = s(A, B)$ for $\phi = s_N$.

2. Multiple Embeddings

In the present section $(\mathcal{K}, [.,.])$ will be a Krein space and $(\mathcal{H}, (.,.)), (\mathcal{H}_j, (.,.))$, $j = 1, \dots, m$, will denote Hilbert spaces. Moreover, let $T : \mathcal{H} \to \mathcal{K}, T_j : \mathcal{H}_j \to \mathcal{K}$ and $R_j : \mathcal{H}_j \to \mathcal{H}$ bounded, linear and injective mappings such that $TR_j = T_j$. By $T^+ : \mathcal{K} \to \mathcal{H}$ and $T_j^+ : \mathcal{K} \to \mathcal{H}_j$ we denote the respective Krein space adjoints.

If D is an operator on a Krein space, then we shall denote by D' the commutant of D, i.e. the algebra of all operators commuting with D. For a selfadjoint D this commutant is a $*$-algebra with respect to forming adjoint operators.

For $j = 1, \dots, m$ we shall denote by $\Theta_j : (T_jT_j^+)'$ $(\subseteq B(\mathcal{K})) \to (T_j^+T_j)'$ $(\subseteq B(\mathcal{H}_j))$, and by $\Theta : (TT^+)'$ $(\subseteq B(\mathcal{K})) \to (T^+T)'$ $(\subseteq B(\mathcal{H}))$ the $*$-algebra homomorphisms mapping the identity operator to the identity operator as in Theorem 5.8 from [5] corresponding to the mappings T_j and T:

$$\begin{aligned} \Theta_j(C_j) &= (T_j \times T_j)^{-1}(C_j) = T_j^{-1} C_j T_j, \quad C_j \in (T_jT_j^+)', \\ \Theta(C) &= (T \times T)^{-1}(C) = T^{-1} C T, \quad C \in (TT^+)'. \end{aligned} \tag{2.1}$$

We can apply Theorem 5.8 in [5] also to the bounded linear, injective $R_j : \mathcal{H}_j \to \mathcal{H}$, and denote the corresponding $*$-algebra homomorphisms by $\Gamma_j : (R_jR_j^*)'$ $(\subseteq B(\mathcal{H})) \to (R_j^*R_j)'$ $(\subseteq B(\mathcal{H}_j))$:

$$\Gamma_j(D) = (R_j \times R_j)^{-1}(D) = R_j^{-1} D R_j, \quad D \in (R_jR_j^*)'.$$

For the following note that due to $(\operatorname{ran} T^+)^{[\perp]} = \ker T = \{0\}$ the range of T^+ is dense in $\mathcal{H}$.

Lemma 2.1. *For $j = 1, \dots, m$ we have $\Theta((T_jT_j^+)' \cap (TT^+)') \subseteq (R_jR_j^*)' \cap (T^+T)'$, where in fact*

$$\Theta(C)R_jR_j^* = R_j\Theta_j(C)R_j^* = R_jR_j^*\Theta(C), \quad C \in (T_jT_j^+)' \cap (TT^+)'. \tag{2.2}$$

Moreover,

$$\Theta_j(C) = (\Gamma_j \circ \Theta)(C), \quad C \in (T_jT_j^+)' \cap (TT^+)'. \tag{2.3}$$

Proof. According to Theorem 5.8 in [5] we have $\Theta_j(C)T_j^+ = T_j^+C$ and $\Theta(C)T^+ = T^+C$ for $C \in (T_jT_j^+)' \cap (TT^+)'$. Therefore,

$$\begin{aligned} T(\, R_j\Theta_j(C)R_j^* \,)T^+ &= T_j\Theta_j(C)T_j^+ = T_jT_j^+C \\ &= TR_jR_j^*T^+C = T(\, R_jR_j^*\Theta(C) \,)T^+. \end{aligned}$$

Because of $\ker T = \{0\}$ and by the density of $\operatorname{ran} T^+$ we have $R_j\Theta_j(C)R_j^* = R_jR_j^*\Theta(C)$. Applying this equation to C^+ and taking adjoints yields $R_j\Theta_j(C) R_j^* = \Theta(C)R_jR_j^*$. In particular, $\Theta(C) \in (R_jR_j^*)'$. Therefore, we can apply Γ_j to $\Theta(C)$ and get

$$(\Gamma_j \circ \Theta)(C) = R_j^{-1} T^{-1} C T R_j = T_j^{-1} C T_j = \Theta_j(C). \qquad \square$$

For the following Corollary 2.3 note that by (2.3) and by the fact that Γ_j is a $*$-algebra homomorphism mapping the identity operator to the identity operator, for $j = 1, \dots, m$ we have

$$\sigma(\Theta(C)) \supseteq \sigma(\Theta_j(C)) \quad \text{for all} \quad C \in (T_j T_j^+)' \cap (TT^+)'. \tag{2.4}$$

Remark 2.2. We would also like to make some clarifications regarding to the integrals over spectral measures. If E is a spectral measure on a Hilbert space $\mathcal{H}$ defined on the Borel subsets of $\mathbb{C}$ such that $E(\mathbb{C} \backslash K) = 0$ for some measurable subset $K \subseteq \mathbb{C}$ and if $h : \operatorname{dom} h \to \mathbb{C}$ is a Borel measurable function with a Borel measurable $\operatorname{dom} h \subseteq \mathbb{C}$ such that $K \subseteq \operatorname{dom} h$ and such that h is bounded on K, then $(x; y) \mapsto \int_{\operatorname{dom} h} h \, d(Ex, y)$ is bounded sesquilinear form on $\mathcal{H}$. Hence,

$$\int h \, dE := \int_{\operatorname{dom} h} h \, dE$$

is a well defined bounded operator on $\mathcal{H}$. Clearly, $\int h \, dE = \int_K h \, dE$. If E is the spectral measure for a bounded normal operator L on $\mathcal{H}$, then this considerations apply for each measurable superset K of $\sigma(L)$. $\lozenge$

Corollary 2.3. *For $j \in \{1, \dots, m\}$ let $N \in B(\mathcal{K})$ be normal, i.e. $NN^+ = N^+N$, such that $N \in (T_j T_j^+)' \cap (TT^+)'$. Then $\Theta(N)$ is a normal operator on the Hilbert space $\mathcal{H}$, and $\Theta_j(N)$ is a normal operator on the Hilbert space $\mathcal{H}_j$. Denoting by E (E_j) the spectral measure of $\Theta(N)$ ($\Theta_j(N)$), we have $E(\Delta) \in (R_j R_j^*)' \cap (T^+T)'$ and*

$$\Gamma_j(E(\Delta)) = E_j(\Delta),$$

for all Borel subsets Δ of $\mathbb{C}$, and $E_j(\Delta) \in (R_j^* R_j)' \cap (T_j^+ T_j)'$.

Moreover, $\int h \, dE \in (R_j R_j^*)' \cap (T^+T)'$ and

$$\Gamma_j \left(\int h \, dE \right) = \int h \, dE_j$$

for any bounded and measurable $h : \sigma(\Theta(N)) \to \mathbb{C}$, and $\int h \, dE_j \in (R_j^* R_j)' \cap (T_j^+ T_j)'$.

Proof. The normality of $\Theta(N)$ and $\Theta_j(N)$ is clear, since Θ and Θ_j are $*$-homomorphisms. From Lemma 2.1 we know that $\Theta(N) \in (R_j R_j^*)' \cap (T^+T)'$. According to the well known properties of $\Theta(N)$'s spectral measure we obtain $E(\Delta) \in (R_j R_j^*)' \cap (T^+T)'$ and, in turn, $\int h \, dE \in (R_j R_j^*)' \cap (T^+T)'$. In particular, Γ_j can be applied to $E(\Delta)$ and $\int h \, dE$. Similarly, $\Theta_j(N) \in (T_j^+ T_j)'$ implies $E_j(\Delta), \int h \, dE_j \in (T_j^+ T_j)'$ for a bounded and measurable h.

Recall from Theorem 5.8 in [5] that $\Gamma_j(D) R_j^* x = R_j^* D$ for $D \in (R_j R_j^*)'$. Hence, for $x \in \mathcal{H}$ and $y \in \mathcal{H}_j$ we have

$$(\Gamma_j(E(\Delta)) R_j^* x, y) = (R_j^* E(\Delta) x, y) = (E(\Delta) x, R_j y)$$

and, in turn,

$$\begin{aligned}\int_{\mathbb{C}} s(z,\bar{z})\, d(\Gamma_j(E)R_j^* x, y) &= \int_{\mathbb{C}} s(z,\bar{z})\, d(Ex, R_j y)\\ &= (s(\Theta(N), \Theta(N)^*)x, R_j y)\\ &= (R_j^* s(\Theta(N), \Theta(N)^*)x, y)\\ &= (\Gamma_j\,(s(\Theta(N), \Theta(N)^*))\, R_j^* x, y)\end{aligned}$$

for any $s \in \mathbb{C}[z,w]$. By (2.3) and the fact, that Γ_j is a $*$-homomorphism, we have $\Gamma_j(s(\Theta(N), \Theta(N)^*)) = s(\Theta_j(N), \Theta_j(N)^*)$. Consequently,

$$\int_{\mathbb{C}} s(z,\bar{z})\, d(\Gamma_j(E)R_j^* x, y) = \int_{\mathbb{C}} s(z,\bar{z})\, d(E_j R_j^* x, y).$$

Since $E(\mathbb{C}\backslash K) = 0$ and $E_j(\mathbb{C}\backslash K) = 0$ for a certain compact $K \subseteq \mathbb{C}$ and since the set of all $s(z,\bar{z}),\ s \in \mathbb{C}[z,w]$, is densely contained in $C(K)$, we obtain from the uniqueness assertion in the Riesz Representation Theorem

$$(\Gamma_j(E(\Delta))R_j^* x, y) = (E_j(\Delta)R_j^* x, y) \quad \text{for all} \quad x \in \mathcal{H},\, y \in \mathcal{H}_j,$$

for all Borel subsets Δ of $\mathbb{C}$. Due to the density of $\operatorname{ran} R_j^*$ in $\mathcal{H}_j$ we even have $(\Gamma_j(E(\Delta))v, y) = (E_j(\Delta)v, y)$ for all $y, v \in \mathcal{H}_j$, and in turn $\Gamma_j(E(\Delta)) = E_j(\Delta)$. Since Γ_j maps into $(R_j^* R_j)'$, we have $E_j(\Delta) \in (R_j^* R_j)'$. This yields $\int h\, dE_j \in (R_j^* R_j)'$ for any bounded and measurable h.

If $h : \sigma(\Theta(N)) \to \mathbb{C}$ is bounded and measurable, then by (2.4) also its restriction to $\sigma(\Theta_j(N)) = \sigma((\Gamma_j \circ \Theta)(N))$ is bounded and measurable. Due to $E_j(\Delta)R_j^* = \Gamma_j(E(\Delta))R_j^* = R_j^* E(\Delta)$, for $x \in \mathcal{H}$ and $y \in \mathcal{H}_j$ we have

$$\begin{aligned}\left(\Gamma_j \left(\int h\, dE\right) R_j^* x, y\right) &= \left(R_j^* \left(\int h\, dE\right) x, y\right)\\ &= \left(\left(\int h\, dE\right) x, R_j y\right)\\ &= \int h\, d(Ex, R_j y) = \int h\, d(E_j R_j^* x, y)\\ &= \left(\left(\int h\, dE_j\right) R_j^* x, y\right).\end{aligned}$$

The density of $\operatorname{ran} R_j^*$ yields $\Gamma_j\left(\int h\, dE\right) = \int h\, dE_j$. □

Recall from Lemma 5.11 in [5] the mappings $(j = 1, \ldots, m)$

$$\Xi_j : B(\mathcal{H}_j) \to B(\mathcal{K}), \quad \Xi_j(D_j) = T_j D_j T_j^+,$$

$$\Xi : B(\mathcal{H}) \to B(\mathcal{K}), \quad \Xi(D) = TDT^+. \tag{2.5}$$

By $(j = 1, \ldots, m)$

$$\Lambda_j : B(\mathcal{H}_j) \to B(\mathcal{H}), \quad \Lambda_j(D_j) = R_j D_j R_j^*,$$

we shall denote the corresponding mappings outgoing from the mappings $R_j : \mathcal{H}_j \to \mathcal{H}$. Due to $T_j = TR_j$ we have $\Xi_j = \Xi \circ \Lambda_j$.

According to Lemma 5.11 in [5], $(\Lambda_j \circ \Gamma_j)(D) = DR_jR_j^*$ for $D \in (R_jR_j^*)'$. Hence, using the notation from Corollary 2.3,

$$\Xi_j\left(\int h\,dE_j\right) = \Xi\left((\Lambda_j \circ \Gamma_j)\left(\int h\,dE\right)\right) = \Xi\left(R_jR_j^* \int h\,dE\right). \tag{2.6}$$

Lemma 2.4. *Assume that for $j \in \{1, \dots, m\}$ the operator $T_jT_j^+$ commutes with TT^+ on $\mathcal{K}$. Then the operators $R_jR_j^*, T^+T$ commute on $\mathcal{H}$ and $R_j^*R_j$, $T_j^+T_j$ commute on $\mathcal{H}_j$. Moreover,*

$$\Theta(T_jT_j^+) = R_jR_j^*T^+T = T^+TR_jR_j^*. \tag{2.7}$$

Proof. If $T_jT_j^+$ and TT^+ commute on $\mathcal{K}$, then

$$T(\,T^+TR_jR_j^*\,)T^+ = TT^+T_jT_j^+ = T_jT_j^+TT^+ = T(\,R_jR_j^*T^+T\,)T^+.$$

Employing T's injectivity and the density of $\operatorname{ran} T^+$, we see that $R_jR_j^*$ and T^+T commute. From this we derive

$$T_j^+T_jR_j^*R_j = R_j^*(T^+TR_jR_j^*)R_j = R_j^*(R_jR_j^*T^+T)R_j = R_j^*R_jT_j^+T_j.$$

(2.7) follows from

$$T^{-1}T_jT_j^+T = T^{-1}TR_jR_j^*T^+T = R_jR_j^*T^+T. \qquad \square$$

3. Definitizability

In [4] we said that a normal $N \in B(\mathcal{K})$ is definitizable, if its real part $A := \frac{N+N^+}{2}$ and its imaginary part $B := \frac{N-N^*}{2i}$ are definitizable in the sense that there exist polynomials $p, q \in \mathbb{R}[z]\setminus\{0\}$ such that $[p(A)v, v] \geq 0$ and $[q(B)v, v] \geq 0$ for all $v \in \mathcal{K}$. In the present note we will relax this condition.

Definition 3.1. For a normal $N \in B(\mathcal{K})$ we call $p \in \mathbb{C}[x, y]\setminus\{0\}$ a definitizing polynomial for N, if

$$[p(A, B)v, v] \geq 0 \quad \text{for all} \quad v \in \mathcal{K}. \tag{3.1}$$

where $A = \frac{N+N^+}{2}$ and $B = \frac{N-N^+}{2i}$. If such a definitizing $p \in \mathbb{C}[x, y]\setminus\{0\}$ exists, then we call N definitizable normal. $\Diamond$

Clearly, we could also write p as a polynomial of the variables N and N^+. Because of $A = A^+$ and $B = B^+$, writing p as a polynomial of the variables A and B, has some notational advantages.

Remark 3.2. According to (3.1) the operator $p(A, B) \in B(\mathcal{K})$ must be selfadjoint; i.e. $p(A, B)^+ = p^\#(A, B)$, where $p^\#(x, y) = \overline{p(\overline{x}, \overline{y})}$. Hence, $q := \frac{p+p^\#}{2}$ is real, i.e. $q \in \mathbb{R}[x, y]\setminus\{0\}$, and satisfies $q(A, B) = p(A, B)$. Thus, we can assume that a definitizing polynomial is real. $\Diamond$

In the present section we assume that $p_j(x, y) \in \mathbb{R}[x, y]\setminus\{0\}, j = 1, \dots, m$, are real, definitizing polynomial for N.

Proposition 3.3. *With the above assumptions and notation there exist Hilbert spaces* $(\mathcal{H}, (.,.)), (\mathcal{H}_j, (.,.)), j = 1, \ldots, m,$ *and bounded linear and injective operators* $T : \mathcal{H} \to \mathcal{K}, T_j : \mathcal{H}_j \to \mathcal{K}$, *such that*

$$T_j T_j^+ = p_j(A, B), \quad \textit{and} \quad TT^+ = \sum_{k=1}^{m} T_k T_k^+ = \sum_{k=1}^{m} p_k(A, B). \tag{3.2}$$

Proof. Let $(\mathcal{H}_j, (.,.))$ be the Hilbert space completion of $\mathcal{K}/\ker p_j(A, B)$ with respect to $[p_j(A, B).,.]$ and let $T_j : \mathcal{H}_j \to \mathcal{K}$ be the adjoint of the factor mapping $x \mapsto x + \ker p_j(A, B)$ of $\mathcal{K}$ into $\mathcal{H}_j$. Since T_j^+ has dense range, T_j must be injective. Similarly, let $(\mathcal{H}, (.,.))$ be the Hilbert space completion of $\mathcal{K}/(\ker \sum_{k=1}^m p_k(A, B))$ with respect to $[\big(\sum_{k=1}^m p_k(A, B)\big).,.]$ and let $T : \mathcal{H} \to \mathcal{K}$ be the injective adjoint of the factor mapping of $\mathcal{K}$ into $\mathcal{H}$.

Finally, (3.2) follows from $[TT^+x, y] = (T^+x, T^+y) = (x, y) = [\big(\sum_{k=1}^m p_k(A, B)\big)x, y]$ and $[T_jT_j^+x, y] = (T_j^+x, T_j^+y) = (x, y) = [p_j(A, B)x, y]$ for all $x, y \in \mathcal{K}$. □

Since for $x \in \mathcal{K}$ and $j \in \{1, \ldots, m\}$ we have

$$(T^+x, T^+x) = [TT^+x, x] = \sum_{k=1}^{m} [T_kT_k^+x, x] = \sum_{k=1}^{m} (T_k^+x, T_k^+x) \geq (T_j^+x, T_j^+x),$$

one easily concludes that $T^+x \mapsto T_j^+x$ constitutes a well-defined, contractive linear mapping from $\operatorname{ran} T^+$ onto $\operatorname{ran} T_j^+$. By $(\operatorname{ran} T^+)^\perp = \ker T = \{0\}$ and $(\operatorname{ran} T_j^+)^\perp = \ker T_j = \{0\}$ these ranges are dense in the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_j$. Hence, there is a unique bounded linear continuation of $T^+x \mapsto T_j^+x$ to $\mathcal{H}$, which has dense range in $\mathcal{H}_j$.

Denoting by R_j the adjoint mapping of this continuation, we clearly have $T_j = TR_j$ and $\ker R_j \subseteq \ker T_j = \{0\}$. From (3.2) we conclude

$$T(\, I_{\mathcal{H}}\,)T^+ = TT^+ = \sum_{k=1}^{m} TR_kR_k^+T^+ = T\left(\sum_{k=1}^{m} R_kR_k^+\right)T^+.$$

$\ker T = \{0\}$ and the density of $\operatorname{ran} T^+$ yield $\sum_{k=1}^m R_kR_k^* = I_{\mathcal{H}}$.

Lemma 3.4. *With the above notations and assumptions for* $j = 1, \ldots, m$ *there exist injective contractions* $R_j : \mathcal{H}_j \to \mathcal{H}$ *such that* $T_j = TR_j$ *and* $\sum_{k=1}^m R_kR_k^* = I_{\mathcal{H}}$. *Moreover, we have*

$$\{N, N^+\}' = \{A, B\}' \subseteq \bigcap_{k=1,\ldots,m} (T_kT_k^+)' \subseteq (TT^+)' \tag{3.3}$$

for all $j \in \{1, \ldots, m\}$. *Finally,*

$$\begin{aligned} p_j(\Theta(A), \Theta(B)) &= R_jR_j^* \left(\sum_{k=1}^{m} p_k(\Theta(A), \Theta(B))\right) \\ &= \left(\sum_{k=1}^{m} p_k(\Theta(A), \Theta(B))\right) R_jR_j^*, \end{aligned} \tag{3.4}$$

and for any $u \in \mathbb{C}[x,y]$

$$\begin{aligned} p_j(A,B)\,u(A,B) &= \Xi_j\left(u(\Theta_j(A),\Theta_j(B))\right)\\ &= \Xi\left(R_jR_j^*\,u(\Theta(A),\Theta(B))\right), \end{aligned} \tag{3.5}$$

where $\Theta : (TT^+)' \ (\subseteq B(\mathcal{K})) \to (T^+T)' \ (\subseteq B(\mathcal{H}))$ *is as in* (2.1) *and* $\Xi : B(\mathcal{H}) \to B(\mathcal{K})$ *as in* (2.5).

Proof. The first part was shown above, and (3.3) is clear from Proposition 3.3.

From (2.7)—Lemma 2.4 can be applied since by (3.2) the operators $T_jT_j^+$ all commute with TT^+—and Theorem 5.8 in [5] we get

$$\begin{aligned} p_j(\Theta(A),\Theta(B)) &= \Theta(p_j(A,B)) = \Theta(T_jT_j^+) = R_jR_j^*\,T^+T = R_jR_j^*\,\Theta(TT^+)\\ &= R_jR_j^*\,\Theta\left(\sum_{k=1}^m p_k(A,B)\right) = R_jR_j^*\left(\sum_{k=1}^m p_k(\Theta(A),\Theta(B))\right), \end{aligned}$$

where $R_jR_j^*$ commutes with $T^+T = \sum_{k=1}^m p_k(\Theta(A),\Theta(B))$ by Lemma 2.4. Finally, (3.5) follows from (see Lemma 5.11 in [5])

$$\begin{aligned} p_j(A,B)\,u(A,B) &= \Xi_j\left(\Theta_j(u(A,B))\right) = (\Xi\circ\Lambda_j\circ\Gamma_j)\left(\Theta(u(A,B))\right)\\ &= \Xi\left(R_jR_j^*\,u(\Theta(A),\Theta(B))\right). \end{aligned}$$ □

By (3.3) we can apply Corollary 2.3 in the present situation. In particular, $\Theta(N)$ is a normal operator on the Hilbert space $\mathcal{H}$. Property (3.1) for $p = p_j,\ j = 1,\dots,m$, imply certain spectral properties of $\Theta(N)$.

Lemma 3.5. *With the above assumptions and notation for* $j \in \{1,\dots,m\}$ *we have*

$$\{z \in \mathbb{C} : |p_j(\operatorname{Re} z, \operatorname{Im} z)| > \|R_jR_j^*\| \cdot |\sum_{k=1}^m p_k(\operatorname{Re} z, \operatorname{Im} z)|\} \subseteq \rho(\Theta(N)).$$

In particular, the zeros of $z \mapsto \sum_{k=1}^m p_k(\operatorname{Re} z, \operatorname{Im} z)$ *in* $\mathbb{C}$ *are contained in* $\rho(\Theta(N)) \cup \{z \in \mathbb{C} : p_j(\operatorname{Re} z, \operatorname{Im} z) = 0 \ \text{ for all } \ j = 1,\dots,m\}$.

Proof. Let $n \in \mathbb{N}$ and set

$$\Delta_n := \left\{ z \in \mathbb{C} : |p_j(\operatorname{Re} z, \operatorname{Im} z)|^2 > \frac{1}{n} + \|R_jR_j^*\|^2 \cdot |\sum_{k=1}^m p_k(\operatorname{Re} z, \operatorname{Im} z)|^2 \right\}.$$

For $x \in E(\Delta_n)(\mathcal{H})$, where E denotes $\Theta(N)$'s spectral measure, we then have

$$\begin{aligned} \|p_j(\Theta(A),\Theta(B))x\|^2 &= \int_{\Delta_n} |p_j(\operatorname{Re}\zeta, \operatorname{Im}\zeta)|^2\, d(E(\zeta)x,x)\\ &\geq \int_{\Delta_n} \frac{1}{n}\, d(E(\zeta)x,x) + \|R_jR_j^*\|^2 \int_{\Delta_n}\\ &\quad \left|\sum_{k=1}^m p_k(\operatorname{Re}\zeta, \operatorname{Im}\zeta)\right|^2 \ d(E(\zeta)x,x)\\ &\geq \frac{1}{n}\|x\|^2 + \|R_jR_j^*\left(\sum_{k=1}^m p_k(\Theta(A),\Theta(B))\right) x\|^2. \end{aligned}$$

By (3.4) this inequality can only hold for $x = 0$. Since Δ_n is open, by the Spectral Theorem for normal operators on Hilbert spaces we have $\Delta_n \subseteq \rho(\Theta(N))$. The asserted inclusion now follows from

$$\left\{z \in \mathbb{C} : |p_j(\operatorname{Re} z, \operatorname{Im} z)| > \|R_j R_j^*\| \cdot \left|\sum_{k=1}^{m} p_k(\operatorname{Re} z, \operatorname{Im} z)\right|\right\} = \bigcup_{n\in\mathbb{N}} \Delta_n. \qquad \square$$

In the following let $\mathcal{I}$ be the ideal $\langle p_1, \ldots, p_m\rangle$ generated by the real definitizing polynomials $p_1, \ldots, p_m$ in the ring $\mathbb{C}[x, y]$. The variety $V(\mathcal{I})$ is the set of all common zeros $a = (a_1, a_2) \in \mathbb{C}^2$ of all $p \in \mathcal{I}$. Clearly, $V(\mathcal{I})$ coincides with the set of all $a \in \mathbb{C}^2$ such that $p_1(a_1, a_2) = \cdots = p_m(a_1, a_2) = 0$. Denote by $V_{\mathbb{R}}(\mathcal{I})$ the set of all $a \in \mathbb{R}^2$, which belong to $V(\mathcal{I})$. It is convenient for our purposes, to consider $V_{\mathbb{R}}(\mathcal{I})$ as a subset of $\mathbb{C}$:

$$\begin{aligned} V_{\mathbb{R}}(\mathcal{I}) &:= \{z \in \mathbb{C} : f(\operatorname{Re} z, \operatorname{Im} z) = 0 \quad \text{for all} \quad f \in \mathcal{I}\} \\ &= \{z \in \mathbb{C} : p_k(\operatorname{Re} z, \operatorname{Im} z) = 0 \quad \text{for all} \quad k \in \{1, \ldots, m\}\}. \end{aligned} \tag{3.6}$$

Corollary 3.6. *Let E denote the spectral measure of $\Theta(N)$. Then we have*

$$\begin{aligned} R_j R_j^*\, E(\mathbb{C}\backslash V_{\mathbb{R}}(\mathcal{I})) &= E(\mathbb{C}\backslash V_{\mathbb{R}}(\mathcal{I}))\, R_j R_j^* \\ &= \int_{\mathbb{C}\backslash V_{\mathbb{R}}(\mathcal{I})} \frac{p_j(\operatorname{Re} z, \operatorname{Im} z)}{\sum_{k=1}^m p_k(\operatorname{Re} z, \operatorname{Im} z)}\, dE(z). \end{aligned}$$

Proof. First note that the integral on the right hand side exists as a bounded operator, because by Lemma 3.5 we have $|p_j(\operatorname{Re} z, \operatorname{Im} z)| \leq \|R_j R_j^*\| \cdot |\sum_{k=1}^m p_k(\operatorname{Re} z, \operatorname{Im} z)|$ for $z \in \sigma(\Theta(N))$. The first equality is known from Corollary 2.3.

Concerning the second equality, note that both sides vanish on the range of $E(V_{\mathbb{R}}(\mathcal{I}))$. Its orthogonal complement $\mathcal{Q} := \operatorname{ran} E(\mathbb{C}\backslash V_{\mathbb{R}}(\mathcal{I}))$ is invariant under

$$\int \left(\sum_{k=1}^{m} p_k(\operatorname{Re} z, \operatorname{Im} z)\right) dE(z) = \sum_{k=1}^{m} p_k(\Theta(A), \Theta(B)).$$

By Lemma 3.5 the restriction of this operator to $\mathcal{Q}$ is injective, and hence, has dense range in $\mathcal{Q}$. If x belongs to this dense range, i.e. $x = \big(\sum_{k=1}^m p_k(\Theta(A), \Theta(B))\big)y$ with $y \in \mathcal{Q}$, then

$$\begin{aligned} &\int_{\mathbb{C}\backslash V_{\mathbb{R}}(\mathcal{I})} \frac{p_j(\operatorname{Re} z, \operatorname{Im} z)}{\sum_{k=1}^m p_k(\operatorname{Re} z, \operatorname{Im} z)}\, dE(z)x \\ &\quad = \int_{\mathbb{C}\backslash V_{\mathbb{R}}(\mathcal{I})} p_j(\operatorname{Re} z, \operatorname{Im} z)\, dE(z)y \\ &\quad = p_j(\Theta(A), \Theta(B))y = R_j R_j^* \left(\sum_{k=1}^{m} p_k(\Theta(A), \Theta(B))\right) y \\ &\quad = R_j R_j^* x. \end{aligned}$$

By a density argument the second asserted equality of the present corollary holds true on $\mathcal{Q}$ and in turn on $\mathcal{H}$. $\square$

Remark 3.7. In Proposition 3.3 the case that $p_j(A,B) = 0$ for some j, or even for all j, is not excluded, and yields $\mathcal{H}_j = \{0\}, T_j = 0$ and $R_j = 0$ (in Lemma 3.4), or even $\mathcal{H} = \{0\}$ and $T = 0$. Also the remaining results hold true, if we interpret $\rho(R)$ as $\mathbb{C}$ and $\sigma(R)$ as $\emptyset$ for the only possible linear operator $R = (0 \mapsto 0)$ on the vector space $\{0\}$. ♢

4. An Abstract Functional Calculus

In this section let $\mathcal{K}$ be again a Krein space and let $N \in B(\mathcal{K})$ be a definitizable normal operator. Let $\mathcal{I}$ be the ideal in $\mathbb{C}[x,y]$, which is generated by all real definitizing polynomials. In order to increase readability, from now on we often write $p(z)$ short for $p(\operatorname{Re} z, \operatorname{Im} z)$ if $p \in \mathbb{C}[x,y]$ and $z \in \mathbb{C}$.

By the ascending chain condition for the ring $\mathbb{C}[x,y]$ (see for example [2], Theorem 7, Chap. 2, Sect. 5) $\mathcal{I}$ is generated by finitely many real definitizing polynomials $p_1,\dots,p_m$, i.e. $\mathcal{I} = \langle p_1,\dots,p_m\rangle$. In fact, if $\mathcal{I}$ would not be generated by finitely many real definitizing polynomials, then, in contrast to the ascending chain condition, we could find a sequence $(p_n)_{n\in\mathbb{N}}$ of such polynomials with $p_{n+1} \notin \langle p_1,\dots,p_n\rangle$ for all $n \in \mathbb{N}$.

Using these polynomials $p_1,\dots,p_m$, for $j = 1,\dots,m$ we define the spaces $\mathcal{H}_j, \mathcal{H}$, the operators T_j, R_j, T, and the spectral measures E_j and E of $\Theta_j(N)$ and $\Theta(N)$, respectively, as in the previous sections, where Θ_j, Θ is defined in (2.1). Accordingly we define Ξ_j and Ξ as in (2.5).

Lemma 4.1. *For any bounded and measurable $f : \sigma(\Theta(N)) \to \mathbb{C}$ and $j \in \{1,\dots,m\}$ we have*

$$\Xi_j\left(\int f\,dE_j\right) = \Xi\left(\int_{\sigma(\Theta(N))\setminus V_{\mathbb{R}}(\mathcal{I})} f\,\frac{p_j}{\sum_{l=1}^m p_l}\,dE \right.$$
$$\left. + R_j R_j^* \int_{\sigma(\Theta(N))\cap V_{\mathbb{R}}(\mathcal{I})} f\,dE\right).$$

Proof. By (2.6) the left hand side coincides with

$$\Xi\left(R_j R_j^* \int_{\sigma(\Theta(N))\setminus V_{\mathbb{R}}(\mathcal{I})} f\,dE + R_j R_j^* \int_{\sigma(\Theta(N))\cap V_{\mathbb{R}}(\mathcal{I})} f\,dE\right).$$

As $\int_{\sigma(\Theta(N))\setminus V_{\mathbb{R}}(\mathcal{I})} f\,dE = E(\mathbb{C}\setminus V_{\mathbb{R}}(\mathcal{I})) \int_{\sigma(\Theta(N))\setminus V_{\mathbb{R}}(\mathcal{I})} f\,dE$ Corollary 3.6 proves the asserted equality. □

Lemma 4.2. *Let $f, g : \sigma(\Theta(N)) \to \mathbb{C}$ be bounded and measurable, and let $r \in \mathbb{C}[x,y]$. For $j, k \in \{1,\dots,m\}$ we then have*

$$r(A,B)\,\Xi_j\left(\int f\,dE_j\right) = \Xi_j\left(\int f\,dE_j\right) r(A,B) = \Xi_j\left(\int r f\,dE_j\right), \tag{4.1}$$

and

$$\Xi_j\left(\int f\,dE_j\right)\Xi_k\left(\int g\,dE_k\right) = \Xi\left(\int fg\,\frac{p_j p_k}{\sum_{l=1}^m p_l}\,dE\right) \tag{4.2}$$
$$= \Xi_j\left(\int fg\,p_k\,dE_j\right) = \Xi_k\left(\int fg\,p_j\,dE_k\right).$$

Proof. By Lemma 5.11 in [5] we have

$$\begin{aligned} r(A,B)\,\Xi_j(D) &= \Xi_j(\Theta(r(A,B))D) = \Xi_j(r(\Theta_j(A),\Theta_j(B))D),\\ \Xi_j(D)r(A,B) &= \Xi_j\left(D\,\Theta_j(r(A,B))\right) = \Xi_j\left(D\,r(\Theta_j(A),\Theta_j(B))\right) \end{aligned}$$

for $D \in (T^+T)'$. For $D = \int f\,dE_j$ this implies (4.1).

According to (2.6) the expression in (4.2) coincides with

$$\Xi\left(R_jR_j^*\int f\,dE\right)\Xi\left(R_kR_k^*\int g\,dE\right).$$

By Lemma 5.11 and Theorem 5.8 in [5], we also know that $\Xi(D_1)\Xi(D_2) = \Xi(T^+TD_1D_2) = \Xi(\Theta(TT^+)D_1D_2)$, where (see Propositions 3.3 and (3.6))

$$\Theta(TT^+) = \sum_{l=1}^{m} p_l(\Theta(A),\Theta(B)) = \int\sum_{l=1}^{m} p_l\,dE = \left(\left(\int\sum_{l=1}^{m} p_l\,dE\right)\,E(\mathbb{C}\backslash V_{\mathbb{R}}(\mathcal{I}))\right).$$

Therefore, by Corollary 3.6 and by the fact, that $E(\mathbb{C}\backslash V_{\mathbb{R}}(\mathcal{I}))$ commutes with $\int_{\sigma(\Theta(N))} f\,dE$, (4.2) can be written as

$$\begin{aligned} &\Xi\left(\left(\int\sum_{l=1}^{m} p_l\,dE\right)\left(\int\frac{p_j}{\sum_{l=1}^{m}p_l}\,dE\right)\left(\int f\,dE\right)\left(\int\frac{p_k}{\sum_{l=1}^{m}p_l}\,dE\right)\left(\int g\,dE\right)\right)\\ &\quad = \Xi\left(\int fg\,\frac{p_jp_k}{\sum_{l=1}^{m}p_l}\,dE\right). \end{aligned}$$

The remaining equalities follow from Lemma 4.1 since the respective integrands vanish on $V_{\mathbb{R}}(\mathcal{I})$. $\square$

Lemma 4.3. *For any bounded and measurable* $f : \sigma(\Theta(N)) \to \mathbb{C}$ *and* $j \in \{1,\dots,m\}$ *the operator* $\Xi_j\left(\int f\,dE_j\right)$ *belongs to* $\{N,N^+\}''$.

Proof. Take $C \in \{N,N^+\}' = \{A,B\}' \subseteq \bigcap_{j=1,\dots,m}(T_jT_j^+)'$; see (3.3). From Lemma 5.11 in [5] we conclude

$$C\,\Xi_j\left(\int f\,dE_j\right) = \Xi_j\left(\Theta_j(C)(\int f\,dE_j)\right).$$

Since Θ_j is a homomorphism, $\Theta_j(C)$ commutes with $\Theta_j(N)$ and, in turn, with $\int_{\sigma(\Theta_j(N))} f\,dE_j$. Hence, employing Lemma 5.11 in [5] once more, the above expression coincides with

$$\Xi_j\left(\left(\int f\,dE_j\right)\,\Theta_j(C)\right) = \Xi_j\left(\int f\,dE_j\right)C.$$

$\square$

In order to have a better picture of what is going on, the tupel $(r, f_1,\dots, f_m)$ appearing in the subsequent definition should be imagined as the function $r+p_1\cdot f_1+\dots p_m\cdot f_m$ with a special behaviour at the points $\sigma(\Theta(N))\cap V_{\mathbb{R}}(\mathcal{I})$.

Definition 4.4. Denoting by $\mathfrak{B}\big(\sigma(\Theta(N))\big)$ the $*$-algebra of complex valued, bounded and measurable functions on $\sigma(\Theta(N))$, for $(r, f_1,\dots,f_m) \in \mathcal{R} := \mathbb{C}[x,y]\times\mathfrak{B}\big(\sigma(\Theta(N))\big)\times\cdots\times\mathfrak{B}\big(\sigma(\Theta(N))\big)$ we set

$$\Psi(r, f_1,\dots,f_m) := r(A,B) + \sum_{k=1}^{m}\Xi_k\left(\int f_k\,dE_k\right).$$

By $\mathcal{N}$ we denote the set of all $(r, f_1, \dots, f_m) \in \mathcal{R}$ such that

$$r + \sum_{k=1}^{m} f_k p_k = 0 \quad \text{on} \quad \sigma(\Theta(N)) \backslash V_{\mathbb{R}}(\mathcal{I})$$

and such that there exist $u_1, \dots, u_m \in \mathbb{C}[x, y]$ with $r = \sum_{k=1}^{m} u_k p_k$ and $(f_j + u_j)(z) = 0$ for $j = 1, \dots, m, z \in V_{\mathbb{R}}(\mathcal{I}) \cap \sigma(\Theta(N))$. ♢

Remark 4.5. Obviously, Ψ is linear. From $\Xi_j(D^*) = \Xi_j(D)^+$ we easily deduce $\Psi(r^\#, \overline{f_1}, \dots, \overline{f_m}) = \Psi(r, f_1, \dots, f_m)^*$. Moreover, $\mathcal{N}$ constitutes a linear subspace of $\mathcal{R}$ invariant under $.^\# : (r, f_1, \dots, f_m) \mapsto (r^\#, \overline{f_1}, \dots, \overline{f_m})$. ♢

Lemma 4.6. *If* $(r, f_1, \dots, f_m) \in \mathcal{N}$, *then* $\Psi(r, f_1, \dots, f_m) = 0$.

Proof. Due to (3.5) $r = \sum_{k=1}^{m} u_k p_k$ implies

$$r(A, B) = \sum_{k=1}^{m} p_k(A, B) u_k(A, B) = \sum_{k=1}^{m} \Xi_k \left(u_k(\Theta_k(A), \Theta_k(B)) \right).$$

From this and Lemma 4.1 we obtain

$$\Psi(r, f_1, \dots, f_m) = \sum_{k=1}^{m} \Xi_k \left(\int (f_k + u_k)\, dE_k \right) =$$

$$\Xi \left(\int\limits_{\sigma(\Theta(N)) \backslash V_{\mathbb{R}}(\mathcal{I})} \sum_{k=1}^{m} \frac{f_k p_k + u_k p_k}{\sum_{l=1}^{m} p_l}\, dE + \sum_{k=1}^{m} R_k R_k^* \int\limits_{\sigma(\Theta(N)) \cap V_{\mathbb{R}}(\mathcal{I})} (f_k + u_k)\, dE \right),$$

which by the definition of $\mathcal{N}$ equals to 0. □

Lemma 4.7. *For* $(r, f_1, \dots, f_m), (s, g_1, \dots, g_m) \in \mathcal{R}$ *have*

$$\begin{aligned}
&\Psi(r, f_1, \dots, f_m)\, \Psi(s, g_1, \dots, g_m) \\
&\quad = \Psi \left(rs, rg_1 + sf_1 + f_1 \sum_{k=1}^{m} g_k p_k, \dots, rg_m + sf_m + f_m \sum_{k=1}^{m} g_k p_k \right) \\
&\quad = \Psi \left(rs, rg_1 + sf_1 + g_1 \sum_{k=1}^{m} f_k p_k, \dots, rg_m + sf_m + g_m \sum_{k=1}^{m} f_k p_k \right).
\end{aligned}$$

Proof. By Lemma 4.2 we have

$$\begin{aligned}
\Psi(r, f_1, \dots, f_m)\, \Psi(s, g_1, \dots, g_m) &= r(A, B)\, s(A, B) \\
&+ \sum_{k=1}^{m} r(A, B)\, \Xi_k \left(\int g_k\, dE_k \right) + \sum_{j=1}^{m} \Xi_j \left(\int f_j\, dE_j \right) s(A, B) \\
&+ \sum_{j,k=1}^{m} \Xi_j \left(\int f_j\, dE_j \right) \Xi_k \left(\int g_k\, dE_k \right) \\
= (rs)(A, B) &+ \sum_{k=1}^{m} \Xi_k \left(\int r g_k\, dE_k \right) + \sum_{j=1}^{m} \Xi_j \left(\int s f_j\, dE_j \right) \\
&+ \sum_{j=1}^{m} \Xi_j \left(\sum_{k=1}^{m} \int f_j g_k\, p_k\, dE_j \right),
\end{aligned}$$

where this last term can also be written as

$$\sum_{j=1}^{m} \Xi_j \left(\sum_{k=1}^{m} \int f_k g_j \, p_k \, dE_j \right). \qquad \square$$

We provide $\mathcal{R}$ with a multiplication:

$$(r, f_1, \ldots, f_m) \cdot (s, g_1, \ldots, g_m) :=$$
$$\left(rs, rg_1 + sf_1 + f_1 \sum_{j=1}^{m} g_j p_j, \cdots, rg_m + sf_m + f_m \sum_{j=1}^{m} g_j p_j \right). \tag{4.3}$$

Remark 4.8. Obviously, $\cdot$ is bilinear and compatible with $.^{\#}$ as defined in Remark 4.5. It is elementary to check its associativity.

Moreover, for $(r, f_1, \ldots, f_m) \in \mathcal{N}$ and $(s, g_1, \ldots, g_m) \in \mathcal{R}$ we have

$$rs + \sum_{j=1}^{m} p_j \left(rg_j + sf_j + f_j \sum_{k=1}^{m} g_k p_k \right) = \left(r + \sum_{j=1}^{m} f_j p_j \right) \left(s + \sum_{k=1}^{m} g_k p_k \right) = 0$$

on $\mathbb{C} \backslash V_{\mathbb{R}}(\mathcal{I})$. For the corresponding $u_1, \ldots, u_m \in \mathbb{C}[x, y]$ with $r = \sum_{j=1}^{m} u_j p_j$ and $(f_j + u_j)(z) = 0$ for all $z \in V_{\mathbb{R}}(\mathcal{I})$ we have $rs = \sum_{j=1}^{m} (u_j s) p_j$ and

$$rg_j + sf_j + f_j \sum_{k=1}^{m} g_k p_k + u_j s = rg_j + f_j \sum_{k=1}^{m} g_k p_k = 0$$

on $V_{\mathbb{R}}(\mathcal{I})$ since r and the p_j vanish there. Hence, $\mathcal{N}$ is a right ideal. Similarly, one shows that it is also a left ideal. Finally, the commutator

$$(r, f_1, \ldots, f_m) \cdot (s, g_1, \ldots, g_m) - (s, g_1, \ldots, g_m) \cdot (r, f_1, \ldots, f_m) =$$
$$\left(0, \sum_{j=1}^{m} (f_1 g_j - g_1 f_j) p_j, \ldots, \sum_{j=1}^{m} (f_m g_j - g_m f_j) p_j \right)$$

belongs to $\mathcal{N}$. Consequently, $\mathcal{R}/\mathcal{N}$ is a commutative $*$-algebra. $\lozenge$

Gathering the previous results we obtain the final result of the present section.

Theorem 4.9. *$\Psi/\mathcal{N} : (r, f_1, \ldots, f_m) + \mathcal{N} \mapsto \Psi(r, f_1, \ldots, f_m)$ is a well-defined $*$-homomorphism from $\mathcal{R}/\mathcal{N}$ into $\{N, N^+\}'' \subseteq B(\mathcal{K})$.*

5. Algebra of Zero-Dimensional Ideals

By the Noether–Lasker Theorem (see for example [2], Theorem 7, Chap. 4, Sect. 7) any ideal $\mathcal{I}$ in $\mathbb{C}[x, y]$ admits a minimal primary decomposition

$$\mathcal{I} = Q_1 \cap \cdots \cap Q_l, \tag{5.1}$$

Q_j being a primary ideal means that $fg \in Q_j$ implies $f \in Q_j$ or $g^k \in Q_j$ for some $k \in \mathbb{N}$, and minimal means that $Q_j \not\supseteq \bigcap_{i \neq j} Q_i$ for all $j = 1, \ldots, l$, and $P_j \neq P_i$ for $i \neq j$, where P_j denotes the radical

$$\sqrt{Q_j} := \{ f \in \mathbb{C}[x, y] : f^k \in Q_j \ \text{ for some } \ k \in \mathbb{N} \}.$$

For an ideal $\mathcal{I}$ in $\mathbb{C}[x,y]$ such a decomposition is in general not unique. Nevertheless, the First Uniqueness Theorem on minimal primary decompositions states that the number $l \in \mathbb{N}$ and the radicals $P_1, \dots, P_l$ are uniquely determined by $\mathcal{I}$; see for example [1], Theorem 8.55 on page 362. Moreover, the Second Uniqueness Theorem on minimal primary decompositions says that if $Q_1' \cap \dots \cap Q_l' = \mathcal{I} = Q_1 \cap \dots \cap Q_l$ are minimal primary decompositions ordered such that $P_j = \sqrt{Q_j} = \sqrt{Q_j'}$ for $j = 1, \dots, l$ and if P_k is minimal in $\{P_1, \dots, P_l\}$ with respect to $\subseteq$, then $Q_k' = Q_k$; see for example [1], Theorem 8.56 on page 364.

Assume now that $\mathcal{I}$ is a zero-dimensional ideal in $\mathbb{C}[x,y]$, i.e.

$$\dim \mathbb{C}[x,y]/\mathcal{I} < \infty.$$

For necessary and sufficient conditions see for example [1], Theorem 6.54 and Corollary 6.56 on pages 274 and 275 and [3], pages 39 and 40. Let (5.1) be a minimal primary decomposition. Then any Q_j, and in turn $P_j \supseteq Q_j$, is also zero-dimensional. In particular, $\mathbb{C}[x,y]/P_j$ is a finite integral domain, and hence, a field. In turn, the radicals $P_1, \dots, P_l$ of $Q_1, \dots, Q_l$ are maximal ideals. By [2], Theorem 11, Chap. 4, Sect. 5, this means that the P_j are generated by $x - a_{x,j}, y - a_{y,j}$, i.e. $P_j = \langle x - a_{x,j}, y - a_{y,j} \rangle$, for pairwise distinct $a_j = (a_{x,j}, a_{y,j}) \in \mathbb{C}^2$. Consequently, any P_k is minimal in $\{P_1, \dots, P_l\}$, and by what was said above, (5.1) is the unique minimal primary decomposition of $\mathcal{I}$.

By Hilbert's Nullstellensatz (see for example [2], Theorem 2, Chap. 4, Sect. 1) the set $V(Q_j)$ of common zeros in $\mathbb{C}^2$ of all $f \in Q_j$ coincides with $V(P_j) = \{a_j\}$. By [2], Theorem 7, Chap. 4, Sect. 3, we also have

$$V(\mathcal{I}) = V(Q_1) \cup \dots \cup V(Q_l) = \{a_1, \dots, a_l\}.$$

Since $V(Q_j + Q_i) = V(Q_j) \cap V(Q_i) = \{a_j\} \cap \{a_i\} = \emptyset$ (see [2], Theorem 4, Chap. 4, Sect. 3) for $i \neq j$, the weak Nullstellensatz (see for example [2], Theorem 1, Chap. 4, Sect. 1) yields $Q_j + Q_i = \mathbb{C}[x,y]$. By the Chinese Remainder Theorem the mapping

$$\theta : \begin{cases} \mathbb{C}[x,y]/\mathcal{I} \to (\mathbb{C}[x,y]/Q_1) \times \dots \times (\mathbb{C}[x,y]/Q_l), \\ \\ x + \mathcal{I} \qquad \mapsto (x + Q_1, \dots, x + Q_l) \end{cases} \tag{5.2}$$

constitutes an isomorphism. Moreover,

$$\mathcal{I} = Q_1 \cap \dots \cap Q_l = Q_1 \cdot \ldots \cdot Q_l. \tag{5.3}$$

Remark 5.1.

1. Since the ring $\mathbb{C}[x,y]/Q_j$ is finite dimensional, its invertible elements $f + Q_j$ are exactly those, for which $fg \in Q_j$ implies $g \in Q_j$. Q_j being primary this is equivalent to $f \notin P_j$. Hence, $f + Q_j$ is invertible in $\mathbb{C}[x,y]/Q_j$ if and only if $f(a_j) \neq 0$.
2. As $\sqrt{Q_j} = P_j$ we have $(x - a_{x,j})^m, (y - a_{y,j})^n \in Q_j$ for sufficiently large $m, n \in \mathbb{N}$. Therefore, the ideal $P_j \cdot Q_j$ contains $(x - a_{x,j})^{m+1}$ and $(y - a_{y,j})^{n+1}$. Thus, $P_j \cdot Q_j$ is also zero-dimensional and $\sqrt{P_j \cdot Q_j} = P_j$.

$\Diamond$

Definition 5.2. For $a \in V(\mathcal{I})$ we set by $Q(a) := Q_j$ and $P(a) := P_j$, where j is such that $a = a_j$. By $d_x(a)$ ($d_y(a)$) we denote the smallest natural number m (n) such that $(x - a_x)^m \in Q(a)$ ($(y - a_y)^n \in Q(a)$). Moreover, we set

$$\mathcal{A}(a) := \mathbb{C}[x, y]/(P(a) \cdot Q(a)) \quad \text{and} \quad \mathcal{B}(a) := \mathbb{C}[x, y]/Q(a).$$

for $a \in V(\mathcal{I})$. $\Diamond$

Since $P(a) \cdot Q(a)$ and $Q(a)$ are ideals with finite codimension satisfying $P(a) \cdot Q(a) \subseteq Q(a), \mathcal{A}(a)$ and $\mathcal{B}(a)$ are finite dimensional algebras with $\dim \mathcal{A}(a) \geq \dim \mathcal{B}(a)$.

Remark 5.3. Assume that $\mathcal{I}$ is invariant under $.^{\#}$, where $f^{\#}(x, y) := \overline{f(\bar{x}, \bar{y})}$. This is for sure the case if $\mathcal{I}$ is generated by real polynomial $p_1, \ldots, p_m$. Then $V(\mathcal{I}) \subseteq \mathbb{C}^2$ is invariant under $(z, w) \mapsto (z, w)^{\#} := (\bar{z}, \bar{w})$. Moreover, it is elementary to check that with Q also $Q^{\#}$ is a primary ideal. Hence, with $\mathcal{I} = Q_1 \cap \cdots \cap Q_l$ also $\mathcal{I} = \mathcal{I}^{\#} = Q_1^{\#} \cap \cdots \cap Q_l^{\#}$ is a minimal primary decomposition. By the uniqueness of the minimal primary decomposition for our zero dimensional ideal $\mathcal{I}$ one has $Q(a)^{\#} = Q(a^{\#})$ for all $a \in V(\mathcal{I})$.

Consequently, $f \mapsto f^{\#}$ induces a conjugate linear bijection from $\mathcal{A}(a)$ ($\mathcal{B}(a)$) onto $\mathcal{A}(a^{\#})$ ($\mathcal{B}(a^{\#})$). $\Diamond$

For the following note that if we conversely start with primary and zero-dimensional ideals $Q_1, \ldots, Q_l$ with $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $i \neq j$, then $\mathcal{I} := Q_1 \cap \cdots \cap Q_l$ is also zero-dimensional, and by the above mentioned uniqueness statement, $Q_1 \cap \cdots \cap Q_l$ is indeed the unique minimal primary decomposition of $\mathcal{I}$.

Proposition 5.4. *Let $\mathcal{I}$ be a zero-dimensional ideal in $\mathbb{C}[x, y]$ which is generated by $p_1, \ldots, p_m$, and let $\mathcal{I} = \bigcap_{a \in V(\mathcal{I})} Q(a)$ be its unique primary decomposition. Assume that W is a subset of $V(\mathcal{I})$. Then*

$$\mathcal{J} := \bigcap_{a \in V(\mathcal{I}) \setminus W} Q(a) \cap \bigcap_{a \in W} (P(a) \cdot Q(a))$$

is also a zero-dimensional ideal satisfying $\mathcal{J} \subseteq \mathcal{I}$. The mapping

$$\psi : \begin{cases} \mathbb{C}[x, y]/\mathcal{J} \to \bigtimes_{a \in V(\mathcal{I}) \setminus W} (\mathbb{C}[x, y]/Q(a)) \times \bigtimes_{a \in W} (\mathbb{C}[x, y]/(P(a) \cdot Q(a))), \\ \\ x + \mathcal{J} \qquad \mapsto \Big((x + Q(a))_{a \in V(\mathcal{I}) \setminus W}, \big(x + (P(a) \cdot Q(a))\big)_{a \in W} \Big) \end{cases}$$

is an isomorphism, and any $p \in \mathcal{J}$ can be written in the form $p = \sum_j u_j p_j$, where $u_j(a) = 0$ for all $a \in W$.

Proof. We already mentioned that $P(a) \cdot Q(a)$ is zero-dimensional with $\sqrt{P(a) \cdot Q(a)} = P(a)$ and that the intersection $\mathcal{J} = \bigcap_{a \in V(\mathcal{I}) \setminus W} Q(a) \cap \bigcap_{a \in W} P(a) \cdot Q(a)$ is the unique primary decomposition of the zero-dimensional $\mathcal{J}$. The isomorphism property of ψ is a special case of the corresponding fact concerning θ; see (5.2). We also have

$$\mathcal{J} = \prod_{a \in V(\mathcal{I}) \setminus W} Q(a) \cdot \prod_{a \in W} P(a) \cdot Q(a) = \prod_{a \in V(\mathcal{I})} Q(a) \cdot \prod_{a \in W} P(a)$$
$$= \mathcal{I} \cdot \prod_{a \in W} P(a) = \left\langle p_1 \cdot \prod_{a \in W} P(a), \ldots, p_m \cdot \prod_{a \in W} P(a) \right\rangle.$$

This means that any $p \in \mathcal{J}$ has a representation $p = \sum_j u_j p_j$ with $u_j \in \prod_{a \in W} P(a) = \bigcap_{a \in W} P(a)$. Hence, $u_j(a) = 0$ for all $a \in W$. $\square$

Example 5.5. Assume that $\mathcal{I}$ is generated by two polynomial $p_1, p_2 \in \mathbb{C}[x, y]$ such that p_1 only depend on x and p_2 only depends on y. The set $V(\mathcal{I})$ of common zeros of $\mathcal{I}$, or equivalently of p_1 and p_2, in $\mathbb{C}^2$ then consists of all points of the form (z, w), where $z \in \mathbb{C}$ is a zero of p_1 and $w \in \mathbb{C}$ is a zero of p_2, i.e. $V(\mathcal{I}) = p_1^{-1}\{0\} \times p_2^{-1}\{0\}$. For $z \in p_1^{-1}\{0\}$ by $\mathfrak{d}_1(z)$ we denote p_1's multiplicity of the zero z, and for $w \in p_2^{-1}\{0\}$ by $\mathfrak{d}_2(w)$ we denote p_2's multiplicity of the zero w.

Given $p \in \mathbb{C}[x, y]$ we can apply polynomial division in one variable twice, once with respect to x and once with respect to y, on order to see that

$$p(x, y) = p_1(x) \cdot u(x, y) + p_2(y) \cdot v(x, y) + q(x, y)$$

with $u, v, q \in \mathbb{C}[x, y]$ such that the degree of q, seen as a polynomial on x, is less then the degree of p_1, and such that the degree of q, seen as a polynomial on y, is less then the degree of p_2; see Lemma 4.8 in [4]. Hence, $\mathcal{I}$ is zero-dimensional. Moreover, writing $p_1(x)$ and $p_2(y)$ as products of linear factors, it follows that $p \in \mathcal{I}$ if and only if

$$p \in \langle (x - z)^{\mathfrak{d}_1(z)}, (y - w)^{\mathfrak{d}_2(w)} \rangle =: Q((z, w)), \tag{5.4}$$

for all $z \in {p_1}^{-1}\{0\}, w \in {p_2}^{-1}\{0\}$. Since $Q((z, w))$ is a primary ideal in $\mathbb{C}[x, y]$,

$$\mathcal{I} = \bigcap_{(z,w) \in p_1^{-1}\{0\} \times p_2^{-1}\{0\}} Q((z, w))$$

is the minimal primary decomposition of $\mathcal{I}$. For the respective radicals we have $P((z, w)) = \langle x - z, y - w \rangle$. Moreover, $P((z, w)) \cdot Q((z, w))$ coincides with

$$\langle (x - z)^{\mathfrak{d}_1(z)+1}, (x - z)^{\mathfrak{d}_1(z)}(y - w), (x - z)(y - w)^{\mathfrak{d}_2(w)}, (y - w)^{\mathfrak{d}_2(w)+1} \rangle.$$

Therefore, $\mathcal{A}((z, w)) = \mathbb{C}[x, y]/\big(P((z, w)) \cdot Q((z, w))\big)$ is isomorphic to $\mathcal{A}_{\mathfrak{d}_1(z), \mathfrak{d}_2(w)}$ and $\mathcal{B}((z, w)) = \mathbb{C}[x, y]/Q((z, w))$ is isomorphic to $\mathcal{B}_{\mathfrak{d}_1(z), \mathfrak{d}_2(w)}$ as introduced in Definition 4.1, [4]. $\Diamond$

6. Function Classes

In the present section we make the same assumptions and use the same notation as in Sect. 4. In addition, we assume that the ideal $\mathcal{I}$ generated by all real definitizing polynomials is zero-dimensional. We fix real, definitizing polynomials $p_1, \ldots, p_m$ which generate $\mathcal{I}$. For the zero-dimensional $\mathcal{I}$ we apply the same notation as in the previous section. The variety $V(\mathcal{I}) = \{a_1, \dots, a_l\} \subseteq \mathbb{C}^2$ of common zeros of all $f \in \mathcal{I}$ will be split up as

$$V(\mathcal{I}) = \underbrace{(V(\mathcal{I}) \cap \mathbb{R}^2)}_{=V_{\mathbb{R}}(\mathcal{I})} \dot{\cup} (V(\mathcal{I}) \backslash \mathbb{R}^2),$$

where we consider $V_{\mathbb{R}}(\mathcal{I})$ as a subset of $\mathbb{C}$; see (3.6).

Definition 6.1. By $\mathcal{M}_N$ we denote the set of functions ϕ defined on

$$\underbrace{(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I}))}_{\subseteq \mathbb{C}} \dot{\cup} \underbrace{(V(\mathcal{I}) \backslash \mathbb{R}^2)}_{\subseteq \mathbb{C}^2}$$

with $\phi(z) \in \mathbb{C}$ for $z \in \sigma(\Theta(N)) \backslash V_{\mathbb{R}}(\mathcal{I}), \phi(z) \in \mathcal{A}(z)$ for $z \in V_{\mathbb{R}}(\mathcal{I})$, and $\phi(z) \in \mathcal{B}(z)$ for $z \in V(\mathcal{I}) \backslash \mathbb{R}^2$.

We provide $\mathcal{M}_N$ pointwise with scalar multiplication, addition and multiplication. We also define a conjugate linear involution $.^{\#}$ on $\mathcal{M}_N$ by

$$\begin{array}{lll} \phi^{\#}(z) := \overline{\phi(z)} & \text{for} & z \in \sigma(\Theta(N)) \backslash V_{\mathbb{R}}(\mathcal{I}), \\ \phi^{\#}(z) := \phi(z)^{\#} & \text{for} & z \in V_{\mathbb{R}}(\mathcal{I}) \\ \phi^{\#}(\xi, \eta) := \phi(\bar{\xi}, \bar{\eta})^{\#} & \text{for} & (\xi, \eta) \in V(\mathcal{I}) \backslash \mathbb{R}^2. \end{array} \qquad \Diamond$$

With the operations introduced above $\mathcal{M}_N$ is a commutative $*$-algebra as can be verified in a straight forward manner; see Remark 5.3.

Definition 6.2. Let $f : \operatorname{dom} f \to \mathbb{C}$ be a function with $\operatorname{dom} f \subseteq \mathbb{C}^2$ such that $\tau\big(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})\big) \subseteq \operatorname{dom} f$, where $\tau : \mathbb{C} \to \mathbb{C}^2$, $(x+iy) \mapsto (x, y)$, such that $f \circ \tau$ is sufficiently smooth – more exactly, at least $d_x(z) + d_y(z) - 1$ times continuously differentiable – on a sufficiently small open neighbourhood z for each $z \in V_{\mathbb{R}}(\mathcal{I})$, and such that f is holomorphic on an open neighbourhood of $V(\mathcal{I}) \backslash \mathbb{R}^2$ $(\subseteq \mathbb{C}^2)$.

Then f can be considered as an element f_N of $\mathcal{M}_N$ by setting $f_N(z) := f \circ \tau(z)$ for $z \in \sigma(\Theta(N)) \backslash V_{\mathbb{R}}(\mathcal{I})$, by

$$\begin{aligned} f_N(z) := & \sum_{(k,l) \in J(z)} \frac{1}{k!l!} \frac{\partial^{k+l}}{\partial a^k \partial b^l} f \circ \tau(a+ib)|_{a+ib=z} \\ & \cdot (x - \operatorname{Re} z)^k (y - \operatorname{Im} z)^l + (P(z) \cdot Q(z)) \in \mathcal{A}(z) \end{aligned}$$

for $z \in V_{\mathbb{R}}(\mathcal{I})$, where

$$J(z) = (\{0, \dots, d_x(z) - 1\} \times \{0, \dots, d_y(z) - 1\}) \cup \{(d_x(z), 0), (0, d_y(z))\},$$

and by

$$f_N(\xi,\eta) := \sum_{k=0}^{d_x(\xi,\eta)-1} \sum_{l=0}^{d_y(\xi,\eta)-1} \frac{1}{k!l!} \frac{\partial^{k+l}}{\partial z^k \partial w^l} f(z,w)|_{(z,w)=(\xi,\eta)}$$
$$\cdot (x-\xi)^k (y-\eta)^l + Q((\xi,\eta)) \in \mathcal{B}((\xi,\eta)),$$

for $(\xi,\eta) \in V(\mathcal{I})\backslash\mathbb{R}^2$. $\Diamond$

Remark 6.3. By the Leibniz rule $f \mapsto f_N$ is compatible with multiplication. Obviously, it is also compatible with addition and scalar multiplication. If we define for a function f as in Definition 6.2 the function $f^\#$ by $f^\#(z,w) = \overline{f(\bar{z},\bar{w})}$, $(z,w) \in \operatorname{dom} f$, then we also have $(f^\#)_N = (f_N)^\#$. $\Diamond$

Remark 6.4. A special type of functions f as in Definition 6.2 are polynomials in two variables, i.e. $f \in \mathbb{C}[x,y]$. Since for $z \in V_{\mathbb{R}}(\mathcal{I})$ and $(k,l) \notin J(z)$ we have $(x - \operatorname{Re} z)^k (y - \operatorname{Im} z)^l \in P(z) \cdot Q(z)$,

$$f_N(z) = f + (P(z) \cdot Q(z)) \in \mathcal{A}(z).$$

Similarly, $f_N(\xi,\eta) = f + Q((\xi,\eta)) \in \mathcal{B}((\xi,\eta))$ for $(\xi,\eta) \in V(\mathcal{I})\backslash\mathbb{R}^2$.

In particular, for $f = \mathbb{1}$ the element $f_N(z)$ is the multiplicative unite in $\mathcal{A}(z)$ or $\mathcal{B}(z)$ for all $z \in \big(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})\big) \dot\cup (V(\mathcal{I})\backslash\mathbb{R}^2)$. $\Diamond$

For the following recall for example from [2], Theorem 4, Chap. 2, Sect. 5, that any ideal in $\mathbb{C}[x,y]$ always has a finite number of generators.

Definition 6.5. For any $w \in \sigma(\Theta(N)) \cap V_{\mathbb{R}}(\mathcal{I})$ such that w is not isolated in $\sigma(\Theta(N))$ let $h_1,\dots,h_n$ be generators of the ideal $Q(w)$. For a sufficiently small neighbourhood $U(w)$ of w let $\chi_{Q(w)} : U(w)\backslash\{w\} \to [0,+\infty)$ be

$$\chi_{Q(w)}(z) := \max_{j=1,\dots,n} |h_j(z)|,$$

where $h_j(z)$, as usually, stands for $h_j(\operatorname{Re} z, \operatorname{Im} z)$. $\Diamond$

Remark 6.6. Since w is a common zero of all $h \in Q(w)$, we have $\chi_{Q(w)}(z) \to 0$ for $z \to w$. Moreover, for any $h \in Q(w)$ the fact, that $h_1,\dots,h_n$ are generators of $Q(w)$, yields $h(z) = O(\chi_{Q(w)}(z))$ as $z \to w$. This is particularly true for $h \in \mathcal{I}$.

Moreover, if $\chi'_{Q(w)}$ is defined in a similar manner starting with generators $h'_1,\dots,h'_{n'}$, then $\chi'_{Q(w)}(z) = O(\chi_{Q(w)})(z)$ and $\chi_{Q(w)}(z) = O(\chi'_{Q(w)})(z)$ as $z \to w$. Hence, as far as it concerns the order of growth towards w, the expression $\chi_{Q(w)}$ does not depend on the actually chosen generators.

Finally, for $h \in Q(w)$ the polynomial

$$g_h(x,y) := h(x,y) \cdot \prod_{a \in V(\mathcal{I}), a \neq (\operatorname{Re} w, \operatorname{Im} w)} (x-a_x)^{\epsilon_x(a)} (y-a_y)^{\epsilon_y(a)}$$

belongs to $\mathcal{I}$, where $\epsilon_x(a) = d_x(a), \epsilon_y(a) = 0$ or $\epsilon_x(a) = 0, \epsilon_y(a) = d_y(a)$ depending on whether $a_y = \operatorname{Im} w$ or $a_y \neq \operatorname{Im} w$; see Definition 5.2. Since $\mathcal{I}$ is generated by $p_1,\dots,p_m$, we have $g_h(z) = O(\max_{j=1,\dots,m} |p_j(z)|)$ and, in turn, $h(z) = O(\max_{j=1,\dots,m} |p_j(z)|)$ as $z \to w$. Applying this to $h = h_j$, we obtain $\chi_{Q(w)}(z) = O(\max_{j=1,\dots,m} |p_j(z)|)$. Hence, as far as it concerns the order of growth towards w, the expression $\chi_{Q(w)}$ could also be defined as $\max_{j=1,\dots,m} |p_j(z)|$. $\lozenge$

Definition 6.7. We denote by $\mathcal{F}_N$ the set of all elements $\phi \in \mathcal{M}_N$ such that $z \mapsto \phi(z)$ is Borel measurable and bounded on $\sigma(\Theta(N))\backslash V_{\mathbb{R}}(\mathcal{I})$, and such that for each $w \in \sigma(\Theta(N)) \cap V_{\mathbb{R}}(\mathcal{I})$, which is not isolated in $\sigma(\Theta(N))$,

$$\phi(z) - \phi(w)|_{x=\operatorname{Re} z, y=\operatorname{Im} z} = O(\chi_{Q(w)}(z)) \tag{6.1}$$

$$\text{as } \sigma(\Theta(N))\backslash V_{\mathbb{R}}(\mathcal{I}) \ni z \to w. \qquad \lozenge$$

Note that in (6.1) $\phi(w) \in \mathcal{A}(w)$ is a coset $p(x,y) + (P(w) \cdot Q(w))$ from $\mathbb{C}[x,y]/(P(w) \cdot Q(w))$, and $\phi(w)|_{x=\operatorname{Re} z, y=\operatorname{Im} z}$ stands for any representative of this coset $\phi(w)$ considered as a function of z. In (6.1) it does not matter what representative we take since $q = O(\chi_{Q(w)})$ as $z \to w$ for any $q \in Q(w)$, and hence, for any $q \in (P(w) \cdot Q(w))$.

Remark 6.8. Assume that our zero-dimensional ideal $\mathcal{I}$ is generated by two definitizing polynomials $p_1 \in \mathbb{R}[x], p_2 \in \mathbb{R}[y]$ as in Example 5.5. For $w \in V_{\mathbb{R}}(\mathcal{I})$, i.e. $(\operatorname{Re} w, \operatorname{Im} w) \in V(\mathcal{I})$, we conclude from (5.4) in Example 5.5 that

$$\chi_{Q(w)}(z) := \max(|(\operatorname{Re} z - \operatorname{Re} w)^{\mathfrak{d}_1(\operatorname{Re} w)}|, |(\operatorname{Im} z - \operatorname{Im} w)^{\mathfrak{d}_2(\operatorname{Im} w)}|).$$

Therefore, in this case the function class $\mathcal{F}_N$ here coincides exactly with the function class $\mathcal{F}_N$ introduced in Definition 4.11, [4]. $\lozenge$

Example 6.9. For $(\xi, \eta) \in V(\mathcal{I})\backslash\mathbb{R}^2$ and $a \in \mathcal{B}((\xi,\eta))$ the function $a\delta_{(\xi,\eta)} \in \mathcal{M}_N$, which assumes the value a at (ξ,η) and the value zero on the rest of $\big(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})\big)\dot{\cup}(V(\mathcal{I})\backslash\mathbb{R}^2)$, trivially belongs to $\mathcal{F}_N$.

Correspondingly, $a\delta_w \in \mathcal{F}_N$ for $a \in \mathcal{A}(w)$ with a $w \in V_{\mathbb{R}}(\mathcal{I})$, which is an isolated point of $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})$. $\lozenge$

Remark 6.10. Let h be defined on an open subset D of $\mathbb{R}^2$ with values in $\mathbb{C}$. Moreover, assume that for given $m, n \in \mathbb{N}$ the function h is $m+n-1$ times continuously differentiable. Finally, fix $w \in D$.

The well-known Taylor Approximation Theorem from multidimensional calculus then yields

$$h(z) = \sum_{j=0}^{m+n-2} \sum_{\substack{k,l \in \mathbb{N}_0 \\ k+l=j}} \frac{1}{k!l!} \frac{\partial^j h}{\partial x^k \partial y^l}(w) \operatorname{Re}(z-w)^k \operatorname{Im}(z-w)^l + O(|z-w|^{m+n-1})$$

as $z \to w$. Since

$$\begin{aligned} |z-w|^{m+n-1} &\leq 2^{m+n-1} \max(|\operatorname{Re}(z-w)|^{m+n-1}, |\operatorname{Im}(z-w)|^{m+n-1}) \\ &= O(\max(|\operatorname{Re}(z-w)|^m, |\operatorname{Im}(z-w)|^n)), \end{aligned}$$

and since $\operatorname{Re}(z-w)^k \operatorname{Im}(z-w)^l = O(\max(|\operatorname{Re}(z-w)|^m, |\operatorname{Im}(z-w)|^n))$ for $k \geq m$ or $l \geq n$, we also have

$$\begin{aligned} h(z) = &\sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \frac{1}{k!l!} \frac{\partial^{k+l} h}{\partial x^k \partial y^l}(w) \operatorname{Re}(z-w)^k \operatorname{Im}(z-w)^l \\ &+ O(\max(|\operatorname{Re}(z-w)|^m, |\operatorname{Im}(z-w)|^n)). \end{aligned} \qquad \lozenge$$

Lemma 6.11. *Let $f : \operatorname{dom} f \ (\subseteq \mathbb{C}^2) \to \mathbb{C}$ be a function with the properties mentioned in Definition 6.2. Then f_N belongs to $\mathcal{F}_N$.*

Proof. For a $w \in \sigma(\Theta(N)) \cap V_{\mathbb{R}}(\mathcal{I})$, which is not isolated in $\sigma(\Theta(N))$, and $z \in \sigma(\Theta(N)) \backslash V_{\mathbb{R}}(\mathcal{I})$ sufficiently near at w by Remark 6.10 the expression

$$\begin{aligned} & f_N(z) - f_N(w)|_{x=\operatorname{Re} z, y=\operatorname{Im} z} \\ & \quad = f(\operatorname{Re} z, \operatorname{Im} z) - \sum_{(k,l)\in J(w)} \frac{1}{k!l!} \frac{\partial^{k+l} f}{\partial x^k \partial y^l}(\operatorname{Re} w, \operatorname{Im} w) \\ & \qquad\qquad \cdot(\operatorname{Re} z - \operatorname{Re} w)^k (\operatorname{Im} z - \operatorname{Im} w)^l \end{aligned}$$

is a $O(\max(|\operatorname{Re}(z-w)|^{d_x(w)}, |\operatorname{Im}(z-w)|^{d_y(w)}))$, and therefore a $O(\chi_{Q(w)}(z))$ as $z \to w$. Consequently $f_N \in \mathcal{F}_N$. $\square$

Lemma 6.12. *If $\phi \in \mathcal{F}_N$ is such that $\phi(z)$ is invertible in $\mathbb{C}, \mathcal{A}(z), \mathcal{B}(z)$, respectively, for all $z \in \big(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})\big)\dot{\cup}(V(\mathcal{I})\backslash\mathbb{R}^2)$ and such that $0 \in \mathbb{C}$ does not belong to the closure of $\phi\big(\sigma(\Theta(N))\backslash V_{\mathbb{R}}(\mathcal{I})\big)$, then $\phi^{-1} : z \mapsto \phi(z)^{-1}$ also belongs to $\mathcal{F}_N$.*

Proof. By the first assumption ϕ^{-1} is a well-defined object belonging to $\mathcal{M}_N$. Clearly, with ϕ also $z \mapsto \phi(z)^{-1} = \frac{1}{\phi(z)}$ is measurable on $\sigma(\Theta(N))\backslash V_{\mathbb{R}}(\mathcal{I})$. By the second assumption of the present lemma $z \mapsto \phi(z)^{-1} = \frac{1}{\phi(z)}$ is bounded on this set.

It remains to verify (6.1) for ϕ^{-1} at each $w \in \sigma(\Theta(N)) \cap V_{\mathbb{R}}(\mathcal{I})$, which is not isolated in $\sigma(\Theta(N)$. To do so, first note that due to $\phi(w)$'s invertibility for $z \in \sigma(\Theta(N))\backslash V_{\mathbb{R}}(\mathcal{I})$ sufficiently near at w we have $\phi(w)|_{x=\operatorname{Re} z, y=\operatorname{Im} z} = p(z) \neq 0$, where $p(x, y)$ is a representative of $\phi(w)$. Now calculate

$$\phi^{-1}(z) - \phi(w)^{-1}|_{x=\operatorname{Re} z, y=\operatorname{Im} z} \tag{6.2}$$

$$= \frac{1}{\phi(z)} - \frac{1}{\phi(w)|_{x=\operatorname{Re} z, y=\operatorname{Im} z}} \tag{6.3}$$

$$+ \frac{1}{\phi(w)|_{x=\operatorname{Re} z, y=\operatorname{Im} z}} - \phi(w)^{-1}|_{x=\operatorname{Re} z, y=\operatorname{Im} z}. \tag{6.4}$$

The expression in (6.3) can be written as

$$-\frac{1}{\phi(z) \cdot \phi(w)|_{x=\operatorname{Re} z, y=\operatorname{Im} z}} \cdot \left(\phi(z) - \phi(w)|_{x=\operatorname{Re} z, y=\operatorname{Im} z}\right).$$

Here $\frac{1}{\phi(z)}$ is bounded by assumption. The assumed invertibility of $\phi(w)$ implies $\phi(w)|_{x=\operatorname{Re} w, y=\operatorname{Im} w} \neq 0$. Hence, $\frac{1}{\phi(w)|_{x=\operatorname{Re} z, y=\operatorname{Im} z}}$ is bounded for z in a certain neighbourhood of w. From $\phi \in \mathcal{F}_N$ we then conclude that (6.3) is a $O(\chi_{Q(w)}(z))$ as $z \to w$.

The expression in (6.4) can be rewritten as

$$-\frac{1}{\phi(w)|_{x=\operatorname{Re} z, y=\operatorname{Im} z}} \cdot \left(\phi(w)|_{x=\operatorname{Re} z, y=\operatorname{Im} z} \cdot \phi(w)^{-1}|_{x=\operatorname{Re} z, y=\operatorname{Im} z} - 1\right).$$

The product in the brackets is a representative of $\phi(w) \cdot \phi(w)^{-1} = 1 + (P(w) \cdot Q(w)) \in \mathcal{A}(w)$. Hence, (6.4) equals to $\frac{1}{\phi(w)|_{x=\operatorname{Re} z, y=\operatorname{Im} z}} q(\operatorname{Re} z, \operatorname{Im} z)$ for a $q \in$

$(P(w) \cdot Q(w))$, and is therefore a $O(\chi_{Q(w)}(z))$ as $z \to w$. Altogether (6.2) is a $O(\chi_{Q(w)}(z))$ as $z \to w$. Thus, $\phi^{-1} \in \mathcal{F}_N$. $\square$

7. Functional Calculus for Zero-Dimensional $\mathcal{I}$

For the following recall from Remark 6.4 that for $p \in \mathbb{C}[x, y]$ the function $p_N \in \mathcal{F}_N$ is defined in Definition 6.2.

Lemma 7.1. *For each $\phi \in \mathcal{F}_N$ there exists $p \in \mathbb{C}[x, y]$ and complex valued $f_1, \ldots, f_m \in \mathfrak{B}(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I}))$ with $f_j(z) = 0$ for $z \in V_{\mathbb{R}}(\mathcal{I})$ such that*

$$\phi(z) = p_N(z) + \sum_j f_j(z)\,(p_j)_N(z)$$

for all $z \in \sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})$, and that $\phi((\xi, \eta)) = p_N((\xi, \eta))$ for all $(\xi, \eta) \in V(\mathcal{I})\backslash\mathbb{R}^2$.

Proof. We apply Proposition 5.4 to $W = V_{\mathbb{R}}(\mathcal{I})$. The fact, that ψ is an isomorphism, then yields the existence of a polynomial $p \in \mathbb{C}[x, y]$ such that $p + (P(w) \cdot Q(w)) = \phi(w)$ for all $w \in V_{\mathbb{R}}(\mathcal{I})$ and such that $p + Q((\xi, \eta)) = \phi((\xi, \eta))$ for all $(\xi, \eta) \in V(\mathcal{I})\backslash\mathbb{R}^2$.

By Remark 6.4 we have $\phi(w) = p + (P(w) \cdot Q(w)) = p_N(w) \in \mathcal{A}(w)$ for $w \in V_{\mathbb{R}}(\mathcal{I})$. For $(\xi, \eta) \in V(\mathcal{I})\backslash\mathbb{R}^2$ we have $\phi((\xi, \eta)) = p + Q((\xi, \eta)) = p_N((\xi, \eta)) \in \mathcal{B}((\xi, \eta))$.

For $j = 1, \ldots, m$ we set $f_j(z) := \frac{\phi(z) - p(z)}{\sum_k p_k(z)}$ if $z \in \sigma(\Theta(N))\backslash V_{\mathbb{R}}(\mathcal{I})$ (see Lemma 3.5), and $f_j(z) = 0$ if $z \in V_{\mathbb{R}}(\mathcal{I})$. On $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})$ we then have

$$\phi(z) = p_N(z) + \sum_j f_j(z)\,(p_j)_N(z).$$

It remains to verify that the functions f_j are measurable and bounded on $\sigma(\Theta(N))\backslash V_{\mathbb{R}}(\mathcal{I})$. The measurability easily follows from the definition of f_j and the measurability of ϕ on this set. Since there are only finitely many points in $V_{\mathbb{R}}(\mathcal{I})$, the measurability of f_j on $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})$ follows.

Concerning boundedness, note that by Lemma 6.11 $\phi - p_N$ belongs to $\mathcal{F}_N$. Since any representative $(\phi - p_N)(w)|_{x=\operatorname{Re} z, y=\operatorname{Im} z}$ of $(\phi - p_N)(w) \in \mathcal{A}(w)$ belongs to $P(w) \cdot Q(w) \subseteq Q(w)$, we have $(\phi - p_N)(z) = O(\chi_{Q(w)}(z))$ as $z \to w$ for any $w \in \sigma(\Theta(N)) \cap V_{\mathbb{R}}(\mathcal{I})$ which is not isolated on $\sigma(\Theta(N))$. By Remark 6.6 and Lemma 3.5 we have $\chi_{Q(w)}(z) = O(\sum_k p_k(z))$ as $z \to w$ for $z \in \sigma(\Theta(N))\backslash V_{\mathbb{R}}(\mathcal{I})$. Therefore,

$$f_j(z) = \frac{\phi(z) - p(z)}{\sum_k p_k(z)} = O(1) \quad \text{as} \quad z \to w$$

for $z \in \sigma(\Theta(N))\backslash V_{\mathbb{R}}(\mathcal{I})$. $\square$

Definition 7.2. Let Δ be the set of all pairs $(\phi; (p, f_1|_{\sigma(\Theta(N))}, \ldots, f_m|_{\sigma(\Theta(N))}))$ such that all assertions from Lemma 7.1 hold true for ϕ and $(p, f_1, \ldots, f_m)$. $\lozenge$

Remark 7.3. It is straight forward to check that Δ is a linear subspace of $\mathcal{F}_N \times \Big(\mathbb{C}[x,y] \times \mathfrak{B}\big(\sigma(\Theta(N))\big) \times \cdots \times \mathfrak{B}\big(\sigma(\Theta(N))\big)\Big)$, i.e. Δ is a linear relations. Moreover, it is easy to check that with $(\phi; (p, f_1|_{\sigma(\Theta(N))}, \ldots, f_m|_{\sigma(\Theta(N))}))$ also $(\phi^{\#}; (p^{\#}, \overline{f_1|_{\sigma(\Theta(N))}}, \ldots, \overline{f_m|_{\sigma(\Theta(N))}}))$ belongs to Δ; see Remark 4.5. $\Diamond$

Δ is also compatible with multiplication as will be shown next.

Lemma 7.4. *If both,* $(\phi; (p, f_1|_{\sigma(\Theta(N))}, \ldots, f_m|_{\sigma(\Theta(N))}))$ *and* $(\psi; (q, g_1|_{\sigma(\Theta(N))}, \ldots, g_m|_{\sigma(\Theta(N))}))$*, belong to* Δ*, then also the pair* $(\phi \cdot \psi; (r, h_1|_{\sigma(\Theta(N))}, \ldots, h_m|_{\sigma(\Theta(N))}))$ *belongs to* Δ*, where (see* (4.3)*)*

$$\begin{aligned}&(r, h_1|_{\sigma(\Theta(N))}, \ldots, h_m|_{\sigma(\Theta(N))})\\ &\quad = (p, f_1|_{\sigma(\Theta(N))}, \ldots, f_m|_{\sigma(\Theta(N))})(q, g_1|_{\sigma(\Theta(N))}, \ldots, g_m|_{\sigma(\Theta(N))}).\end{aligned}$$

Proof. On $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})$ we have

$$\phi(z) = p_N(z) + \sum_j f_j(z)(p_j)_N(z) \quad \text{and} \quad \psi(z) = q_N(z) + \sum_j g_j(z)(p_j)_N(z).$$

Moreover, $f_j(z) = 0 = g_j$ for $z \in V_{\mathbb{R}}(\mathcal{I})$, and

$$\phi((\xi,\eta)) = p_N((\xi,\eta)), \psi((\xi,\eta)) = q_N((\xi,\eta)) \quad \text{for all} \quad (\xi,\eta) \in V(\mathcal{I})\backslash\mathbb{R}^2.$$

Since $p \mapsto p_N$ is compatible with multiplication, $r = p \cdot q$ satisfies $(\phi \cdot \psi)((\xi,\eta)) = r_N((\xi,\eta))$ for all $(\xi,\eta) \in V(\mathcal{I})\backslash\mathbb{R}^2$. Clearly, $h_j = pg_j + qf_j + f_j \sum_{k=1}^m g_k p_k$ vanishes on $V_{\mathbb{R}}(\mathcal{I})$. For $z \in \sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})$ we have

$$\begin{aligned}\phi(z)\,\psi(z) = p_N(z)\,q_N(z) + \sum_j \Bigg(& p_N(z)g_j(z) + q_N(z)f_j(z)\\ & + f_j(z)\sum_k g_k(z)(p_k)_N(z)\Bigg)\,(p_j)_N(z),\end{aligned}$$

which, for $z \in V_{\mathbb{R}}(\mathcal{I})$, coincides with $r_N(z) = r_N(z) + \sum_j h_j(z)(p_j)_N(z)$. For $z \in \sigma(\Theta(N))\backslash V_{\mathbb{R}}(\mathcal{I})$ the above equation can be written as

$$\begin{aligned}\phi(z)\,\psi(z) &= r(z) + \sum_j \Bigg(p(z)g_j(z) + q(z)f_j(z) + f_j(z)\sum_k g_k(z)p_k(z)\Bigg)\,p_j(z)\\ &= r_N(z) + \sum_j h_j(z)\,(p_j)_N(z).\end{aligned}$$

$\square$

We are going to determine the multivalued part $\operatorname{mul}\Delta$ of Δ.

Lemma 7.5. *Let* $p \in \mathbb{C}[x,y]$ *and* $f_1, \ldots, f_m \in \mathfrak{B}(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I}))$ *with* $f_j(z) = 0$ *for* $z \in V_{\mathbb{R}}(\mathcal{I})$ *be such that*

$$0 = p_N(z) + \sum_j f_j(z)(p_j)_N(z)$$

on $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})$ *and that* $p_N((\xi,\eta)) = 0$ *for all* $(\xi,\eta) \in V(\mathcal{I})\backslash\mathbb{R}^2$*. Then* $(p, f_1|_{\sigma(\Theta(N))}, \ldots, f_m|_{\sigma(\Theta(N))})$ *belongs to the ideal* $\mathcal{N}$ *in* $\mathcal{R}$ *as defined in Definition* 4.4*.*

Proof. Clearly, $p + \sum_{j=1}^m f_j p_j = 0$ on $\sigma(\Theta(N))\backslash V_{\mathbb{R}}(\mathcal{I})$. According to Remark 6.4 we have $p + (P(w) \cdot Q(w)) = 0 \in \mathcal{A}(w)$ for all $w \in V_{\mathbb{R}}(\mathcal{I})$ and $p + Q((\xi,\eta)) = 0 \in \mathcal{B}((\xi,\eta))$ for all $(\xi,\eta) \in V(\mathcal{I})\backslash\mathbb{R}^2$. Hence, p belongs to

$$\bigcap_{(\xi,\eta)\in V(\mathcal{I})\backslash\mathbb{R}^2} Q((\xi,\eta)) \cap \bigcap_{w\in V_{\mathbb{R}}(\mathcal{I})} (P(w)\cdot Q(w)).$$

By Proposition 5.4 we therefore have $p = \sum_j u_j p_j$ with $u_j(w) = 0$ for all $w \in V_{\mathbb{R}}(\mathcal{I})$. We see that $(f_j + u_j)(z) = 0$ for all $z \in V_{\mathbb{R}}(\mathcal{I}) \cap \sigma(\Theta(N))$. Thus, $(p, f_1|_{\sigma(\Theta(N))}, \ldots, f_m|_{\sigma(\Theta(N))}) \in \mathcal{N}$. □

Since by Lemma 4.6 $\operatorname{mul}\Delta \subseteq \mathcal{N} \subseteq \ker\Psi$, the composition $\Psi\Delta$ is a well-defined linear mapping from $\mathcal{F}_N$ into $B(\mathcal{K})$.

Definition 7.6. For $\phi \in \mathcal{F}_N$ we set $\phi(N) := (\Psi\Delta)(\phi)$. ♢

By Theorem 4.9, Lemma 7.4 and Remark 7.3 the following result can be formulated.

Theorem 7.7. *$\phi \mapsto \phi(N)$ constitutes a $*$-homomorphism from $\mathcal{F}_N$ into $\{N, N^*\}'' \subseteq B(\mathcal{K})$. It satisfies $p_N(N) = p(A,B)$ for all $p \in \mathbb{C}[x,y]$.*

Proof. The final assertion is clear because of $(p_N; (p, 0, \ldots, 0)) \in \Delta$. □

8. Spectral Properties of the Functional Calculus

For $w \in V_{\mathbb{R}}(\mathcal{I})$ we will need the following notation. By $\pi_w : \mathcal{A}(w) \to \mathcal{B}(w)$ we denote the mapping

$$\pi_w\,(f + (P(w)\cdot Q(w))) = f + Q(w).$$

Lemma 8.1. *If $\phi \in \mathcal{F}_N$ vanishes everywhere except at a fixed $w \in V_{\mathbb{R}}(\mathcal{I})$ and if $\pi_w\phi(w) = 0$, then*

$$\phi(N) = \Psi(0; g_1, \ldots, g_m)$$

for $g_1, \ldots, g_m \in \mathfrak{B}\,(\sigma(\Theta(N)))$ which vanish on $(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I}))\backslash\{w\}$.

Proof. Let $p \in \mathbb{C}[x,y]$ and $f_1, \ldots, f_m \in \mathfrak{B}(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I}))$ with $f_j(z) = 0$ for $z \in V_{\mathbb{R}}(\mathcal{I})$ such that

$$\phi(z) = p_N(z) + \sum_j f_j(z)\,(p_j)_N(z)$$

for all $z \in \sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})$, and that $p_N((\xi,\eta)) = \phi((\xi,\eta)) = 0$ for all $(\xi,\eta) \in V(\mathcal{I})\backslash\mathbb{R}^2$. The latter fact just means $p \in Q((\xi,\eta))$. From $0 = \phi(z) = p_N(z) + \sum_j f_j(z)\,(p_j)_N(z)$ for $z \in V_{\mathbb{R}}(\mathcal{I})\backslash\{w\}$ we infer $p \in (P(z)\cdot Q(z))$. From $\pi_w\phi(w) = 0$ we obtain $p \in Q(w)$.

By Proposition 5.4 we have $p = \sum_j u_j p_j$, where $u_j(z) = 0$ for all $z \in V_{\mathbb{R}}(\mathcal{I})\backslash\{w\}$. We define g_j to be zero on $(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I}))\,\backslash\{w\}$ and set $g_j(w) = u_j(w)$. The difference

$$\begin{aligned}&(p; f_1|_{\sigma(\Theta(N))}, \ldots, f_m|_{\sigma(\Theta(N))}) - (0; g_1, \ldots, g_m)\\ &\quad = (p; f_1|_{\sigma(\Theta(N))} - \delta_w(.)u_1(w), \ldots, f_m|_{\sigma(\Theta(N))} - \delta_w(.)u_m(w))\end{aligned}$$

satisfies $p + \sum_j (f_j(z) - \delta_w(z)u_j(w))p_j(z) = \phi(z) = 0$ for $z \in \sigma(\Theta(N)) \backslash V_{\mathbb{R}}(\mathcal{I})$ and $f_j(z) - \delta_w(z)u_j(w) + u_j(z) = 0$ for all $z \in V_{\mathbb{R}}(\mathcal{I}) \cap \sigma(\Theta(N))$. It therefore belongs to the ideal $\mathcal{N}$ of $\mathcal{R}$. Thus,

$$\phi(N) = \Psi(p; f_1|_{\sigma(\Theta(N))}, \ldots, f_m|_{\sigma(\Theta(N))}) = \Psi(0; g_1, \ldots, g_m). \qquad \square$$

Corollary 8.2. *Assume that the spectral measure E of $\Theta(N)$ satisfies $E\{w\} = 0$ for a fixed $w \in V_{\mathbb{R}}(\mathcal{I})$, which surely happens if $w \notin \sigma(\Theta(N))$. Then $\phi(N) = \psi(N)$ for all ϕ, ψ that coincide on $\big((\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I}))\backslash\{w\}\big) \dot\cup (V(\mathcal{I})\backslash\mathbb{R}^2)$ and that satisfy $\pi_w\phi(w) = \pi_w\psi(w)$. Here $\pi_w : \mathcal{A}(w) \to \mathcal{B}(w)$ is defined by $\pi_w\big(f + (P(w) \cdot Q(w))\big) = f + Q(w)$.*

Proof. By Lemma 8.1 there exist $g_1, \ldots, g_m \in \mathfrak{B}\big(\sigma(\Theta(N))\big)$, which vanish on $(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I}))\backslash\{w\}$, such that

$$\phi(N) - \psi(N) = \Psi(0; g_1, \ldots, g_m) = \sum_{k=1}^{m} \Xi_k \left(\int_{\sigma(\Theta_k(N))} g_k \, dE_k \right)$$

According to Lemma 4.1 together with our assumption $E\{w\} = 0$, this operator vanishes. $\square$

Remark 8.3. For $\zeta \in V(\mathcal{I})\backslash\mathbb{R}^2$ or a $\zeta \in V_{\mathbb{R}}(\mathcal{I})$, which is isolated in $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})$, we saw in Example 6.9 that $a\delta_\zeta \in \mathcal{F}_N$. If a is the unite e in $\mathcal{B}(\zeta)$ or in $\mathcal{A}(\zeta)$, i.e. the coset $1 + Q(\zeta)$ for $\zeta \in V(\mathcal{I})\backslash\mathbb{R}^2$ or the coset $1 + (P(\zeta) \cdot Q(\zeta))$ for $\zeta \in V_{\mathbb{R}}(\mathcal{I})$, then $(e\delta_\zeta) \cdot (e\delta_\zeta) = (e\delta_\zeta)$ together with the multiplicativity of $\phi \mapsto \phi(N)$ show that $(e\delta_\zeta)(N)$ is a projection. It is a kind of Riesz projection corresponding to ζ.

We set $\xi := \operatorname{Re}\zeta,\ \eta := \operatorname{Im}\zeta$ if $\zeta \in V_{\mathbb{R}}(\mathcal{I})$ and $(\xi, \eta) := \zeta$ if $\zeta \in V(\mathcal{I})\backslash\mathbb{R}^2$. For $\lambda \in \mathbb{C}\backslash\{\xi + i\eta\}$ and for $s(z, w) := z + iw - \lambda$ we then have $s_N \cdot (e\delta_\zeta) = \big(s_N(\zeta)\big)\delta_\zeta$. As $s(\xi, \eta) \neq 0, s_N(\zeta)$ does not belong to $P(\zeta) \supseteq Q(\zeta)$. Therefore, it is invertible in $\mathcal{B}(\zeta)$ or in $\mathcal{A}(\zeta)$. For its inverse b we obtain

$$s_N \cdot (e\delta_\zeta) \cdot (b\delta_\zeta) = e\delta_\zeta.$$

From $s_N(N) = N - \lambda$ we derive that $(N|_{\operatorname{ran}(e\delta_\zeta)(N)} - \lambda)^{-1} = (b\delta_\zeta)(N)|_{\operatorname{ran}(e\delta_\zeta)(N)}$ on $\operatorname{ran}(e\delta_\zeta)(N)$. In particular, $\sigma(N|_{\operatorname{ran}(e\delta_\zeta)(N)}) \subseteq \{\xi + i\eta\}$. $\lozenge$

Lemma 8.4. *If $\phi \in \mathcal{F}_N$ vanishes on*

$$(\sigma(\Theta(N)) \cup (V_{\mathbb{R}}(\mathcal{I}) \cap \sigma(N))) \,\dot\cup \left\{(\alpha, \beta) \in V(\mathcal{I})\backslash\mathbb{R}^2 : \alpha + i\beta, \bar\alpha + i\bar\beta \in \sigma(N)\right\},$$

then $\phi(N) = 0$.

Proof. Since any $w \in V_{\mathbb{R}}(\mathcal{I})\backslash\sigma(N)$ is isolated in $\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})$, we saw in Remark 8.3 that for

$$\zeta \in \underbrace{(V_{\mathbb{R}}(\mathcal{I})\backslash\sigma(N))}_{=:Z_1} \dot\cup \underbrace{\{(\alpha, \beta) \in V(\mathcal{I})\backslash\mathbb{R}^2 : \alpha + i\beta \in \rho(N)\}}_{=:Z_2}$$

the expression $(e\delta_\zeta)(N)$ is a bounded projection commuting with N. Hence, $(e\delta_\zeta)(N)$ also commutes with $(N - (\xi + i\eta))^{-1}$, where $\xi := \operatorname{Re}\zeta,\ \eta := \operatorname{Im}\zeta$ if $\zeta \in Z_1$ and $(\xi, \eta) := \zeta$ if $\zeta \in Z_2$.

Consequently, $N|_{\mathrm{ran}(e\delta_\zeta)(N)} - (\xi + i\eta)$ is invertible on $\mathrm{ran}(e\delta_\zeta)(N)$, i.e. $\xi + i\eta \notin \sigma(N|_{\mathrm{ran}(e\delta_\zeta)(N)})$. In Remark 8.3 we saw $\sigma(N|_{\mathrm{ran}(e\delta_\zeta)(N)}) \subseteq \{\xi + i\eta\}$. Hence, $\sigma(N|_{\mathrm{ran}(e\delta_\zeta)(N)}) = \emptyset$, which is impossible for $\mathrm{ran}(e\delta_\zeta)(N) \neq \{0\}$. Thus, $(e\delta_\zeta)(N) = 0$.

For $(\xi, \eta) \in Z_3 := \{(\alpha, \beta) \in V(\mathcal{I})\backslash\mathbb{R}^2 : \bar{\alpha} + i\bar{\beta} \in \rho(N)\}$ one has $(\bar{\xi}, \bar{\eta}) \in Z_2$. Hence,

$$0 = (e\delta_{(\bar{\xi},\bar{\eta})})(N)^* = (e^{\#}\delta_{(\xi,\eta)})(N) = (e\delta_{(\xi,\eta)})(N).$$

Since, by our assumption, ϕ is supported on $Z_1 \cup Z_2 \cup Z_3$, we obtain

$$\phi(N) = \left(\sum_{\zeta \in Z_1 \cup Z_2 \cup Z_3} \phi(\zeta)\delta_\zeta \right)(N) = \sum_{\zeta \in Z_1 \cup Z_2 \cup Z_3} \phi(\zeta)(e\delta_\zeta)(N) = 0. \qquad \square$$

As a consequence of Lemma 8.4 for $\phi \in \mathcal{F}_N$ the operator $\phi(N)$ only depends on ϕ's values on

$$(\sigma(\Theta(N)) \cup (V_{\mathbb{R}}(\mathcal{I}) \cap \sigma(N))) \\ \dot{\cup}\ \{(\alpha, \beta) \in V(\mathcal{I})\backslash\mathbb{R}^2 : \alpha + i\beta, \bar{\alpha} + i\bar{\beta} \in \sigma(N)\}. \quad (8.1)$$

Thus, we can, and will from now on, re-define the function class $\mathcal{F}_N$ for our functional calculus so that the elements ϕ of $\mathcal{F}_N$ are functions on this set with values in $\mathbb{C}, \mathcal{A}(z)$ or $\mathcal{B}(z)$, such that $z \mapsto \phi(z)$ is measurable and bounded on $\sigma(\Theta(N))\backslash V_{\mathbb{R}}(\mathcal{I})$ and such that (6.1) holds true for every $w \in \sigma(\Theta(N)) \cap V_{\mathbb{R}}(\mathcal{I})$ which is not isolated in $\sigma(\Theta(N))$.

Lemma 8.5. *If $\phi \in \mathcal{F}_N$ is such that $\phi(z)$ is invertible in $\mathbb{C}, \mathcal{A}(z)$ or $\mathcal{B}(z)$, respectively, for all z in (8.1), and such that 0 does not belong to the closure of $\phi\big(\sigma(\Theta(N))\backslash V_{\mathbb{R}}(\mathcal{I})\big)$, then $\phi(N)$ is a boundedly invertible operator on $\mathcal{K}$ with $\phi^{-1}(N)$ as its inverse.*

Proof. We think of ϕ as a function on $\big(\sigma(\Theta(N)) \cup V_{\mathbb{R}}(\mathcal{I})\big)\dot{\cup}(V(\mathcal{I})\backslash\mathbb{R}^2)$ by setting $\phi(z) = e$ for all z not belonging to (8.1). Then all assumptions of Lemma 6.12 are satisfied. Hence $\phi^{-1} \in \mathcal{F}_N$, and we conclude from Theorem 7.7 and Remark 6.4 that

$$\phi^{-1}(N)\phi(N) = \phi(N)\phi^{-1}(N) = (\phi \cdot \phi^{-1})(N) = \mathbb{1}_N(N) = I_{\mathcal{K}}. \qquad \square$$

Corollary 8.6. *$\sigma(N)$ equals to*

$$\sigma(\Theta(N)) \cup (V_{\mathbb{R}}(\mathcal{I}) \cap \sigma(N)) \\ \cup\{\alpha + i\beta : (\alpha, \beta) \in V(\mathcal{I})\backslash\mathbb{R}^2, \alpha + i\beta, \bar{\alpha} + i\bar{\beta} \in \sigma(N)\}. \quad (8.2)$$

In particular, $\sigma(N)\backslash\sigma(\Theta(N))$ is finite.

Proof. Since Θ is a homomorphism, we have $\sigma(\Theta(N)) \subseteq \sigma(N)$. Hence, (8.2) is contained in $\sigma(N)$. For the converse, consider the polynomial $s(z, w) = z + iw - \lambda$ for a λ not belonging to (8.2). We conclude that for any

$$\zeta \in (V_{\mathbb{R}}(\mathcal{I}) \cap \sigma(N)) \cup \{(\alpha, \beta) \in V(\mathcal{I})\backslash\mathbb{R}^2 : \alpha + i\beta, \bar{\alpha} + i\bar{\beta} \in \sigma(N)\}$$

the polynomial s does not belong to $P(\zeta) \supseteq Q(\zeta)$. Hence, $s_N(\zeta)$ is invertible $\mathcal{A}(\zeta)$ or $\mathcal{B}(\zeta)$. Clearly, $s_N(\zeta) \neq 0$ for $\zeta \in \sigma(\Theta(N))\setminus V_{\mathbb{R}}(\mathcal{I})$. Finally, 0 does not belong to the closure of

$$s_N\left(\sigma(\Theta(N))\setminus V_{\mathbb{R}}(\mathcal{I})\right) = s(\sigma(\Theta(N))\setminus V_{\mathbb{R}}(\mathcal{I})) \subseteq \sigma(\Theta(N)) - \lambda.$$

Applying Lemma 8.5, we see that $s_N(N) = (N - \lambda)$ is invertible. □

Remark 8.7. We set $K_r := V_{\mathbb{R}}(\mathcal{I}) \cap \sigma(N)$,

$$Z := \left\{(\alpha, \beta) \in V(\mathcal{I})\setminus\mathbb{R}^2 : \alpha + i\beta, \bar{\alpha} + i\bar{\beta} \in \sigma(N)\right\},$$

and $K_i := \{\alpha + i\beta : (\alpha, \beta) \in Z\}$. Using Corollary 8.6 we could re-define once more the functions $\phi \in \mathcal{F}_N$ as functions ϕ on $\sigma(N)$ such that

1. ϕ is complex valued, bounded and measurable on $\sigma(N)\setminus(K_r \cup K_i)$,
2. $\phi(\zeta) \in \mathcal{A}(\zeta)$ for $\zeta \in K_r\setminus K_i$,
3. $\phi(\zeta) \in \bigtimes_{(\alpha,\beta)\in Z, \alpha+i\beta=\zeta} \mathcal{A}(\zeta)$ for $\zeta \in K_i\setminus K_r$,
4. $\phi(\zeta) \in \mathcal{A}(\zeta) \times \bigtimes_{(\alpha,\beta)\in Z, \alpha+i\beta=\zeta} \mathcal{A}(\zeta)$ for $\zeta \in K_r \cap K_i$;
5. for a $w \in K_r$, which is not isolated in $\sigma(N)$, we have

$$\phi(z) - p(\operatorname{Re} z, \operatorname{Im} z) = O(\chi_{Q(w)}(z)) \quad \text{as} \quad \sigma(N)\setminus(K_r \cup K_i) \ni z \to w,$$

   where p is a representative of $\phi(w)$ for $w \in K_r\setminus K_i$ and p is a representative of the first entry of $\phi(w)$ for $w \in K_r \cap K_i$. ♢

9. Special Cases of Definitizable Operators

Unitary and selfadjoint operators are special cases of normal operators on Hilbert spaces as well as on Krein spaces. We will show how some well-known facts on definitizable selfadjoint or unitary operators on a Krein space $\mathcal{K}$ can easily be obtain from the previously obtained results.

9.1. Selfadjoint Definitizable Operators

An operator $N \in B(\mathcal{K})$ is by definition selfadjoint if $N = N^+$. Obviously, $N \in B(\mathcal{K})$ is selfadjoint if and only if N is normal and satisfies $p(A, B) = 0$, where $A = \frac{N+N^+}{2}, B = \frac{N-N^+}{2i}$ and $p(x, y) = y$.

Therefore, according to Definition 3.1 any selfadjoint operator on a Krein space is definitizable normal, and the ideal $\mathcal{I}$ generated by all real definitizing polynomials contains $p(x, y) = y$. Since the ideal generated by $p(x, y) = y$ is not zero-dimensional, the zero-dimensionality of $\mathcal{I}$ implies the existence of at least one real definitizing polynomial of the form

$$y \cdot s(x, y) + t(x) \quad \text{with} \quad s \in \mathbb{C}[x, y],\ t \in \mathbb{C}[x]\setminus\{0\}. \tag{9.1}$$

Proposition 9.1. *The ideal $\mathcal{I}$ is zero-dimensional if and only if there exists a $t \in \mathbb{R}[x]\setminus\{0\}$ such that $[t(A)u, u] \geq 0,\ u \in \mathcal{K}$, i.e. $N = A$ is definitizable in the classical sense; see [5].*

Proof. Any $r \in \mathbb{C}[x, y]$ can we written as $r(x, y) = y \cdot s_r(x, y) + t_r(x)$ with unique $s_r \in \mathbb{C}[x, y]$, $t_r \in \mathbb{C}[x]$. Hence, $r \in \mathcal{I}$ if and only if $t_r \in \mathcal{I}$. The set of $\mathcal{I}_x := \{t_r : r \in \mathcal{I}\}$ forms an ideal in $\mathbb{C}[x]$. If $\mathcal{I}_x$ is the zero ideal, then $\mathcal{I} = y \cdot \mathbb{C}[x, y]$ is not zero-dimensional.

If $\mathcal{I}_x \neq \{0\}$, then, applying the polynomial division, we see that dim $\mathbb{C}[x]/\mathcal{I}_x < \infty$. This also implies the zero-dimensionality of $\mathcal{I}$. If $r(x, y)$ is a real definitizing polynomial as in (9.1), then

$$[t(A)u, u] = [r(A, B)u, u] \geq 0, \ u \in \mathcal{K},$$

i.e. $t(x)$ is a definitizing polynomial. Finally, r shares the property to be real with t. $\square$

Assume that $N \in B(\mathcal{K})$ is selfadjoint and that the ideal $\mathcal{I}$ generated by all real definitizing polynomials is zero-dimensional. Consequently, we can apply the functional calculus developed in Section 7. Since $p(x, y) = y$ belongs to $\mathcal{I}$, we conclude that

$$a = (a_x, a_y) \in V(\mathcal{I}) \quad \text{implies} \quad a_y = p(a) = 0.$$

Hence, the elements of $V_{\mathbb{R}}(\mathcal{I})$ are contained in $\mathbb{R}$, and $(\xi, \eta) \in V(\mathcal{I})\backslash\mathbb{R}^2$ yields $\eta = 0$. Moreover, with N also $\Theta(N)$ is selfadjoint in the Hilbert space $\mathcal{H}$; see Proposition 3.3 and (2.1). In particular, $\sigma(\Theta(N)) \subseteq \mathbb{R}$. From Corollary 8.6 we derive that $\sigma(N)$ is contained in $\mathbb{R}$ up to finitely many points which are located in $\mathbb{C}\backslash\mathbb{R}$ symmetric with respect to $\mathbb{R}$.

9.2. Unitary Definitizable Operators

An operator $N \in B(\mathcal{K})$ is by definition unitary if $N^+N = NN^+ = I_{\mathcal{K}}$. Obviously, $N \in B(\mathcal{K})$ is unitary if and only if N is normal and satisfies $p(A, B) = 0$, where $A = \frac{N+N^+}{2}, B = \frac{N-N^+}{2i}$ and

$$p(x, y) = (x + iy)(x - iy) - 1 = x^2 + y^2 - 1.$$

Therefore, according to Definition 3.1 any unitary operator on a Krein space is definitizable normal, and the ideal $\mathcal{I}$ generated by all real definitizing polynomials always contains $p(x, y)$. Since the ideal generated by p is not zero-dimensional, the zero-dimensionality of $\mathcal{I}$ implies the existence a definitizing polynomial different from p.

Remark 9.2. If, for example, there exists a polynomial $a \in \mathbb{C}[z]\backslash\{0\}$ such that $[a(N)u, u] \geq 0, \ u \in \mathcal{K}$, then the ideal $\mathcal{J}$ generated by a (as a polynomial in $\mathbb{C}[z, w]$) and $b(z, w) = zw - 1$ in $\mathbb{C}[z, w]$ is zero-dimensional. Indeed, it is easy to see that the set $V(\mathcal{J})$ of common zeros of a and b is finite, which by [3], page 39, implies zero-dimensionality. Since $c(z, w) \mapsto c(x + iy, x - iy)$ constitutes an isomorphism from $\mathbb{C}[z, w]$ onto $\mathbb{C}[x, y]$, also the ideal generated by $a(x+iy)$ and $p(x, y)$ in $\mathbb{C}[x, y]$ is zero-dimensional. Hence, the same is true for $\mathcal{I}$, and we can apply the functional calculus developed Section 7. $\Diamond$

Assume that $N \in B(\mathcal{K})$ is unitary and that the ideal $\mathcal{I}$ generated by all real definitizing polynomials is zero-dimensional. Consequently, we can apply the functional calculus developed in Section 7. From $p \in \mathcal{I}$ we conclude that

$$a \in V(\mathcal{I}) \quad \text{implies} \quad p(a) = 0.$$

Hence, the elements of $V_{\mathbb{R}}(\mathcal{I})$ are contained in $\mathbb{T}$, and $(\xi, \eta) \in V(\mathcal{I})\backslash\mathbb{R}^2$ yields

$$(\xi + i\eta)\overline{(\bar{\xi} + i\bar{\eta})} = \xi^2 + \eta^2 = 1.$$

Moreover, with N also $\Theta(N)$ is unitary in the Hilbert space $\mathcal{H}$; see Proposition 3.3 and (2.1). In particular, $\sigma(\Theta(N)) \subseteq \mathbb{T}$. From Corollary 8.6 we derive that $\sigma(N)$ is contained in $\mathbb{T}$ up to finitely many points which are located in $\mathbb{C}\backslash\mathbb{T}$ symmetric with respect to $\mathbb{T}$.

10. Transformations of Definitizable Normal Operators

In this final section we examine, whether basic transformations, such as $\alpha N, N + \beta I_{\mathcal{K}}, N^{-1}$ with $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$, of definitizable normal operators N are again definitizable, and how the corresponding ideals $\mathcal{I}$ behave.

For $\beta \in \mathbb{C}$ it is easy to see that $p(x, y)$ is a real definitizing polynomial for N if and only if the polynomial $p(x - \operatorname{Re}\beta, y - \operatorname{Im}\beta)$ in $\mathbb{C}[x, y]$ is real definitizing for $N + \beta I_{\mathcal{K}}$. Since $r(x, y) \mapsto r(x - \operatorname{Re}\beta, y - \operatorname{Im}\beta)$ is a ring automorphism on $\mathbb{C}[x, y]$, the respective ideals $\mathcal{I}$, corresponding to N and $N + \beta I_{\mathcal{K}}$, are zero-dimensional, or not, at the same time.

Similarly, $p(x, y)$ is a real definitizing polynomial for N if and only if the polynomial $p(x \operatorname{Re} 1/\alpha - y \operatorname{Im} 1/\alpha, x \operatorname{Im} 1/\alpha + y \operatorname{Re} 1/\alpha)$ in $\mathbb{C}[x, y]$ is real definitizing for αN. Also $r(x, y) \mapsto r(x \operatorname{Re} 1/\alpha - y \operatorname{Im} 1/\alpha, x \operatorname{Im} 1/\alpha + y \operatorname{Re} 1/\alpha)$ is a ring automorphism on $\mathbb{C}[x, y]$. Hence, the ideal $\mathcal{I}$ corresponding to N is zero-dimensional if and only if the ideal $\mathcal{I}$ corresponding to αN is zero-dimensional.

For the inverse N^{-1} the situation is more complicated. We formulate two results that we will need. The first assertion is straight forward to verify. We omit its proof.

Lemma 10.1. *The mapping $\Phi : p(x, y) \mapsto p(\frac{z+w}{2}, \frac{z-w}{2i})$ from $\mathbb{C}[x, y]$ to $\mathbb{C}[z, w]$ is an isomorphism, where p is real, i.e. $p(\bar{x}, \bar{y}) = \overline{p(x, y)}$, if and only if $\overline{\Phi(p)(z, w)} = \Phi(p)(\bar{w}, \bar{z})$.*

Obviously, for a normal $N = A + iB$ and $p(x, y) \in \mathbb{C}[x, y]$ we have

$$p(A, B) = \Phi(p)(N, N^{+}). \tag{10.1}$$

For a polynomial $q \in \mathbb{C}[z, w]\backslash\{0\}$ let $d(q)$ be the maximum of the z-degree of q and the w-degree of q. Moreover, we set

$$\varpi(q)(z, w) := (zw)^{d(q)} q\left(\frac{1}{z}, \frac{1}{w}\right) \in \mathbb{C}[z, w].$$

Lemma 10.2. *If $\mathcal{I} = \langle q_1, \ldots, q_m \rangle$ is zero-dimensional with polynomials $q_1, \ldots, q_m$ such that $\overline{q_j(z, w)} = q_j(\bar{w}, \bar{z})$, then the ideal $\langle \varpi(q_1), \ldots, \varpi(r_m) \rangle$ is also zero-dimensional.*

Proof. Let $(\zeta, \eta) \in V(\varpi(q_1), \ldots, \varpi(r_m))$. For $\zeta \neq 0 \neq \eta$ we conclude $q_j(\frac{1}{\zeta}, \frac{1}{\eta}) = 0$, $j = 1, \ldots, m$, and in turn $(\zeta, \eta) \in \{(z, w) \in (\mathbb{C}\backslash\{0\})^2 : (\frac{1}{z}, \frac{1}{w}) \in V(\mathcal{I})\}$.

Assume that $\eta = 0$ and $\zeta \neq 0$. If $q_j(z, w) = \sum_{k,l=0}^{d(q_j)} b_{k,l} z^k w^l$, then $\overline{q_j(z, w)} = q_j(\bar{w}, \bar{z})$ yields $b_{k,l} = \bar{b}_{l,k}$, and we have $\varpi(q_j)(z, w) = \sum_{k,l=0}^{d(q_j)}$

$b_{d(q_j)-k,d(q_j)-l}z^k w^l$. According to the choice of $d(q_j)$ and by $b_{k,l} = \bar{b}_{l,k}$ the polynomial

$$\rho_j(z) := \varpi(q_j)(z,0) = \sum_{k=0}^{d(q_j)} b_{d(q_j)-k,d(q_j)} z^k$$

is non-zero and satisfies $\rho_j(\zeta) = 0$, i.e. $(\zeta,\eta) \in \rho_j^{-1}(\{0\}) \times \{0\}$.

From $\overline{q_j(z,w)} = q_j(\bar{w},\bar{z})$ we conclude $\rho_j(\bar{w}) = \overline{\varpi(q_j)(0,w)}$. Hence, $\zeta = 0$ and $\eta \neq 0$ yields $(\zeta,\eta) \in \{0\} \times \overline{\rho_j^{-1}(\{0\})}$.

In any case (ζ,η) is contained in

$$\{(0,0)\} \cup \left\{(z,w) \in (\mathbb{C}\backslash\{0\})^2 : \left(\frac{1}{z},\frac{1}{w}\right) \in V(\mathcal{I})\right\}$$
$$\cup \left(\bigcap_{j=1,\dots,m} \rho_j^{-1}(\{0\}) \times \{0\}\right) \cup \left(\bigcap_{j=1,\dots,m} \{0\} \times \overline{\rho_j^{-1}(\{0\})}\right).$$

Consequently, $V(\varpi(q_1),\dots,\varpi(r_m))$ is finite, and in turn $\langle\varpi(q_1),\dots,\varpi(r_m)\rangle$ is zero-dimensional; see [3], page 39. $\square$

Proposition 10.3. *Let N be normal and bijective on the Krein space $\mathcal{K}$. If $p(x,y)$ is real definitizing for N, then $\Phi^{-1}\Big(\varpi\big(\Phi(p)\big)\Big)$ is definitizing for N^{-1}. Moreover, if the ideal $\mathcal{I}$ generated by all real definitizing $p(x,y)$ for N is zero-dimensional, then also the ideal generated by all real definitizing polynomials for N^{-1} is zero-dimensional.*

Proof. Let $p(x,y)$ be real definitizing for N. By Lemma 10.1 we have $\overline{\Phi(p)(z,w)} = \Phi(p)(\bar{w},\bar{z})$, and in turn $\overline{\varpi(\Phi(p))(z,w)} = \varpi(\Phi(p))(\bar{w},\bar{z})$. Writing $\Phi(p)(z,w) = \sum_{k,l=0}^{d(\Phi(p))} b_{k,l}z^k w^l$, we obtain

$$\varpi(\Phi(p))(z,w) = \sum_{k,l=0}^{d(\Phi(p))} b_{d(\Phi(p))-k,d(\Phi(p))-l}\, z^k w^l.$$

For $u \in \mathcal{K}$ by (10.1) we have

$$\begin{aligned}
&[\Phi^{-1}\left(\varpi\left(\Phi(p)\right)\right)(\operatorname{Re} N^{-1}, \operatorname{Im} N^{-1})u,u] \\
&\quad = \left[\varpi(\Phi(p))(N^{-1},N^{-+})u,u\right] \\
&\quad = \left[\sum_{k,l=0}^{d(\Phi(p))} b_{d(\Phi(p))-k,d(\Phi(p))-l}(N^{-1})^k(N^{-+})^l u,u\right] \\
&\quad = \left[\Phi(p)(N,N^+)\,(N^{-1})^{d(\Phi(p))}u,(N^{-1})^{d(\Phi(p))}\,u\right] \\
&\quad = \left[p(A,B)\,(N^{-1})^{d(\Phi(p))}u,(N^{-1})^{d(\Phi(p))}\,u\right] \geq 0.
\end{aligned}$$

Hence, $\Phi^{-1}\Big(\varpi\left(\Phi(p)\right)\Big)$ is real definitizing for N^{-1}. Finally, if $\mathcal{I}$ is zero-dimensional and generated by real definitizing $p_1,\dots,p_m$, then $\Phi(\mathcal{I}) = \langle\Phi(p_1), \dots,\Phi(p_m)\rangle$ is zero-dimensional in $\mathbb{C}[z,w]$. According to Lemma 10.2 $\langle\varpi\big(\Phi$

$(p_1)), \dots, \varpi(\Phi(p_m))\rangle$, and hence also $\langle \Phi^{-1}\Big(\varpi\big(\Phi(p_1)\big)\Big), \dots, \Phi^{-1}\Big(\varpi\big(\Phi(p_m)\big)\Big)\rangle$ is zero-dimensional. Since its generators are real definitizing for N^{-1} also the ideal generated by all real definitizing polynomials for N^{-1} is zero-dimensional. $\square$

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