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Crystals, Promotion, Evacuation and Cactus Groups

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Erklärung zur Verfassung der Arbeit

Hiermit erkläre ich, dass ich diese Arbeit selbstständig verfasst, keine anderen als die angegebenen Quellen und Hilfsmittel verwendet und mich auch sonst keiner unerlaubten Hilfsmittel bedient habe. Ich versichere außerdem, dass ich diese Arbeit bisher weder im In- noch im Ausland in irgendeiner Form als Prüfungsarbeit vorgelegt habe.

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Abstract

Using Henriques' and Kamnitzer's cactus groups, Schützenberger's promotion and evacuation operators on standard Young tableaux can be generalised in a very natural way to operators acting on highest weight words in tensor products of crystals.

For the crystals corresponding to the vector representations of the symplectic groups, we show that Sundaram's map to perfect matchings intertwines promotion and rotation of the associated chord diagrams, and evacuation and reversal. We also exhibit a map with similar features for the crystals corresponding to the adjoint representations of the general linear groups.

We prove these results by applying van Leeuwen's generalisation of Fomin's local rules for jeu de taquin, connected to the action of the cactus groups by Lenart, and variants of Fomin's growth diagrams for the Robinson-Schensted correspondence.

This work is based on a joint research project with Martin Rubey and Bruce W. Westbury. In chapter 1 we give a general introduction and state related work. Chapter 2 connects the algebraic world of representations with combinatorics and we present our findings in chapter 3. In chapter 4 we define promotion and evacuation as actions of certain elements of a cactus group and state local rules for algorithmically calculating these actions. The local rules are strongly related to the rules of our growth diagram bijections from chapter 5. The last chapter 6 is meant for proofs only. Chapters 1, 3, 4, 5 and 6 are also published separately as a joint paper [21].

Acknowledgements

I would like to express my special appreciation and thanks to my advisor Martin Rubey for his support. Whenever I had a question, got stuck in a theorem or needed someone to share a crazy idea with, he was there to help.

As this thesis is based on a joint research project with Martin Rubey and Bruce Westbury I would also like to thank them both for their valuable input. Despite all the ups and downs during the research process, it was a great experience working on this topic with both of them. At this point I would like to thank the organisers of the 29$^{\text{th}}$ international conference on Formal Power Series and Algebraic Combinatorics for the possibility to present our findings during a poster session. This was a really inspiring and motivating event and was an important milestone for this project.

Finally, I would like to thank my family, colleagues, all my friends and especially my girlfriend Sofie Mittas. She even supported me, when I had sudden inspirations and got up in the middle of the night to rewrite some proofs completely. Although I believe she does not really know what my research is really about, she always listened when I wanted to explain my results to someone. I am very grateful for the moral and financial support of my family. My parents Waltraud and Norbert and my sister Christina Pfannerer were always there for me – not only during my studies, but my whole life. I am very thankful to know that we can work on all problems together as a family and stick together during difficult times.

Contents

Chapter 1

Introduction

Our journey begins with the discovery that Sundaram's map from oscillating tableaux to perfect matchings, regarded as chord diagrams (as in Figure 3.1), intertwines promotion and rotation, see Theorem 3.4. Oscillating tableaux are highest weight words in a tensor power of the crystal of the vector representation of the symplectic group $\mathrm{Sp}(2n)$, and promotion is a natural generalisation of Schützenberger's promotion map on standard Young tableaux.

We also exhibit a map from Stembridge's alternating tableaux, the highest weight words in the r-th tensor power of the crystal for the adjoint representation of the general linear group $\mathrm{GL}(n)$, to permutations. It turns out that this map intertwines promotion and rotation provided that $n \geq r$, see Theorem 3.8.

A convenient setting for these variants of promotion are the cactus groups, introduced by Devadoss [4, def 6.1.2] and placed into our context by Henriques and Kamnitzer [10]. These are infinite groups related to coboundary categories in a similar way as the braid groups are related to braided categories. Essentially, our goal is to make the effect of the cactus groups in the coboundary category of crystals of a complex reductive Lie algebra $\mathfrak{g}$ transparent. To do so, we use the local rules discovered by van Leeuwen [16], generalising the classical local rule for jeu de taquin by Fomin [28, app 1] to all minuscule representations of Lie groups. The relation between these local rules and the action of the cactus groups was established by Lenart [17] and was made more explicit in terms of certain growth diagrams by Westbury [31].

Let $C = C_1 \otimes \cdots \otimes C_r$ be an r-fold tensor product of crystals. Then the generators $\mathrm{s}_{p,q}$, for $1 \leq p < q \leq r$, of the cactus group map highest weight words of C bijectively to highest weight words of $C_1 \otimes \cdots \otimes C_{p-1} \otimes C_q \otimes C_{q-1} \otimes \cdots \otimes C_p \otimes C_{q+1} \otimes \cdots \otimes C_r$.

For example, when $\mathfrak{g}$ is the Lie algebra of the special linear group $\mathrm{SL}(n)$, and $C_1 = \cdots = C_r$ is the crystal of its vector representation, the highest weight words of C are standard Young tableaux of size r with at most n columns. Then, the generator $\mathrm{s}_{1,r}$ of the cactus group is precisely Schützenberger's evacuation, and $\mathrm{s}_{1,r}\,\mathrm{s}_{2,r}$ is

Schützenberger's promotion. As an aside, we remark that in this case the generators $s_{i,i+2}$ encode Assaf's dual equivalence graph.

More generally, when C_i is the crystal of the μ_i-th exterior power of the vector representation then the highest weight words of C are semistandard Young tableaux of weight μ, with at most n columns: the weight of the i-th letter specifies the columns in which the number i appears. Again, $s_{1,r}$ acts as evacuation and $s_{1,r}\, s_{2,r}$ as promotion. Using evacuation as a building block, the action of the cactus groups on semistandard Young tableaux was studied by Chmutov, Glick and Pylyavskyy [2].

By analogy we call $\operatorname{ev} w = s_{1,r}\, w$ the evacuation of a highest weight word w, and $\operatorname{pr} w = s_{1,r}\, s_{2,r}\, w$ its promotion.

The promotion operator has been studied in connection with the cyclic sieving phenomenon. Rhoades [22] established a cyclic sieving phenomenon for the promotion operator acting on rectangular standard tableaux. This was generalised to promotion on invariant words in the crystal of a minuscule representation by Fontaine and Kamnitzer [7], using the geometric Satake correspondence. A cyclic sieving phenomenon for the promotion operator acting on invariant words in any crystal was given by Westbury [32], exploiting the fact that Lusztig's canonical basis for invariant tensors is preserved by promotion.

Cyclic sieving phenomena for perfect matchings and permutations together with rotation were established by Rubey and Westbury [25, 26]. There, a basis of the space of invariant tensors in tensor powers of the vector representation of the symplectic group $\mathrm{Sp}(2n)$ was given, whose elements correspond in a natural way to $(n+1)$-noncrossing perfect matchings. In particular, this basis is invariant under rotation. Similarly, a basis of the invariant space in tensor powers of the adjoint representation of the general linear group $\mathrm{GL}(n)$ was given, whose elements correspond to permutations. This basis is invariant under rotation provided that n is large enough.

The promotion and rotation operators on invariant tensors were shown to agree by Westbury [32], in the following sense. It is true in general that, given a vector space with a linear operator of finite order and two bases each preserved by the operator then there is a bijection between the two bases which intertwines the two actions of the operator. This implies for our setting that there exists a bijection between chord diagrams and invariant words which intertwines rotation and promotion. However constructing such a bijection explicitly remained an open problem, which we solve here for the vector representation of the symplectic groups and the adjoint representation of the general linear groups.

Let us remark that the polynomials encoding the orbit structure of the promotion operator, as required in the context of cyclic sieving phenomena, can be extracted from the Frobenius character of the symmetric group action on the space of invariant tensors which permutes tensor positions. This Frobenius character was determined by Sundaram [30] for the vector representation of the symplectic groups. For the adjoint representation of the general linear groups it can be computed using Schur-Weyl duality, see [25]. A closely related project to the one discussed here is to describe the Frobenius character using the structure of the crystal graph. So far, this was accomplished only in the case of the vector representation of the symplectic groups by Rubey, Sagan and Westbury [24], and the vector representation of the odd orthogonal groups by Jagenteufel [13]. For the vector representations of the even orthogonal groups and G_2 there are conjectural descriptions.

For Lie algebras of rank 2, there is another family of bases for the space of invariant tensors, the web bases introduced by Kuperberg [15]. By definition, rotation of an invariant tensor corresponds to the rotation of the associated web. It was shown by Petersen, Pylyavskyy and Rhoades [20] that promotion of rectangular standard Young tableaux with three rows corresponds to rotation of the associated $\mathrm{SL}(3)$ webs. This result was recently translated by Patrias [19] to establish that rotation of $\mathrm{SL}(3)$ webs corresponds to promotion of the associated invariant highest weight words. However, web bases for Lie algebras of larger rank are still poorly understood, if known at all.

Chapter 2

Preliminaries

In this chapter we introduce root systems and weight lattices of complex Lie algebras and weights of their representations. The root systems are the main ingredient for the classification of simple Lie algebras. So called dominant weights will identify the finite dimensional irreducible representations. The weights and roots will also specify crystal graphs, which can be seen as combinatorial models of representations.

Our aim is to introduce highest weight words of tensor products of crystals. These are the combinatorial objects we are interested in. We will focus on highest weight words of tensor powers of crystals corresponding to the vector representation of $\mathfrak{gl}(n)$ and crystals corresponding to the adjoint representation of $\mathfrak{sp}(2n)$.

Note that in this chapter we present the results for Lie algebras. They directly translate into the world of Lie groups. In our cases it does not matter if we speak of crystals corresponding to representations of a Lie group G or the crystals corresponding representations of the Lie algebra $\mathfrak{g}$ corresponding to G.

2.1 Roots and weights

The purpose of this section is to fix notation. It is based on definitions and results in the textbooks [8, 9] and [12]. For basic definitions of Lie algebras we also refer to [5].

Recall that a *Cartan subalgebra* $\mathfrak{h}$ of a complex Lie algebra $\mathfrak{g}$ is a maximal abelian Lie subalgebra such that every element $H \in \mathfrak{h}$ is semisimple. A Cartan subalgebra is unique up to an automorphism of $\mathfrak{g}$ and always exists for semisimple Lie algebras. By definition

$$\begin{aligned} \mathrm{ad}_H : \mathfrak{g} &\to \mathfrak{g} \\ x &\mapsto [H, X], \end{aligned}$$

is diagonalisable for every $H \in \mathfrak{h}$ and since $\mathfrak{h}$ is abelian all its elements and therefore all elements of $\mathrm{ad}\,\mathfrak{h} := \{\mathrm{ad}_H : H \in \mathfrak{h}\}$ commute. Thus there exists a basis of $\mathfrak{g}$ of common eigenvectors for the elements of $\operatorname{ad}\mathfrak{h}$. For such an eigenvector X the eigenvalues are given by the *roots*. These are elements of the dual space $\alpha \in \mathfrak{h}^*$, such that the *root space* $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : \operatorname{ad}_H(X) = \alpha(H)X\}$ is not the zero vector space. We denote the set of all roots by R.

The *Killing form* is a symmetric bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined by

$$(X, Y) := \operatorname{tr}(\operatorname{ad}_X \circ \operatorname{ad}_Y).$$

Note that the Killing form is nondegenerate for semisimple Lie algebras and induces an isomorphism $\alpha \mapsto H_\alpha$ between $\mathfrak{h}^*$ and $\mathfrak{h}$ given by: H_α is the unique element in $\mathfrak{h}$, such that

$$\alpha(H) = (H, H_\alpha) \quad \text{for all } H \in \mathfrak{h}.$$

Lemma 2.1 ([8, thm. 14.22])**.** *The Killing form is positive definite on the real subspace of $\mathfrak{h}$ spanned by the vectors $\{H_\alpha : \alpha \in R\}$.*

This Lemma also allows us to define an inner product $\langle \alpha, \beta \rangle := (H_\alpha, H_\beta)$ on the real vector space spanned by the set of roots R. For each root $\alpha \in R$ we denote the *coroot* with $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ and define a linear map $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ as the reflection about the hyperplane orthogonal to α by

$$s_\alpha(H) := H - \frac{2\langle H, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha = H - \langle H, \alpha^\vee \rangle \alpha.$$

Definition 2.2 ([9, def. 7.25])**.** *We call the group W generated by all maps s_α the* Weyl group *of R.*

The root systems and their Weyl group satisfy several important properties.

Theorem 2.3 ([9, thm. 7.30])**.** *Let R denote the set of roots.*

R1 R is a finite subset of a real vector space E with an inner product. It does not contain the zero vector and it spans E.

R2 If $\alpha \in R$, then also $-\alpha \in R$. There are no other scalar multiples of α in R.

R3 If $\alpha \in R$, then s_α permutes the roots.

R4 If $\alpha, \beta \in R$, then $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$.

In general we denote a set R of vectors together with a real inner product space E satisfying the properties R1-R4 an *(abstract) root system*. We call the dimension of E the *rank* of the root system.

Definition 2.4 ([9, prop. 8.12]). *We define a subset $\Delta \subset R$ to be a* base *of the root system (R, E) if*

- *Δ is linearly independent set spanning E,*
- *each root $\alpha \in R$ can be expressed as a linear combination of elements of Δ with integer coefficients. All non-zero coefficients have the same sign.*

We call those roots with non-negative coefficients the *positive roots* and the elements of Δ the *simple roots*.

Remark 2.5 ([9, prop. 8.28]). *Note that such a base always exists and that for any two bases Δ_1 and Δ_2 there exists a unique $w \in W$ such that $w \cdot \Delta_1 = \Delta_2$. Thus, the following results and constructions are independent of the choice of a certain base $\Delta = \{\alpha_1, \dots, \alpha_r\}$.*

Definition 2.6 ([9, def. 8.34]). *We fix a root system (R, E) and a set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_r\}$ and define:*

- *An element $\mu \in E$ is called an* integral element *or* weight*, if for all $\alpha \in R$*

$$\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}.$$

- *A weight μ is* dominant *relative to Δ, if*

$$\langle \mu, \alpha \rangle \geq 0$$

  for all $\alpha \in \Delta$. We call the set of all weights the weight lattice *denoted by Λ and denote the set of all dominant weights by Λ^+.*

- *Moreover for two weights λ and μ we say μ is* higher *than λ, if $\mu - \lambda$ can be expressed as linear combination of Δ with non-negative integer coefficients. We denote this by $\mu \succ \lambda$.*
- *For each weight μ the Weyl group orbit $W \cdot \mu$ contains a unique dominant weight ν, the* dominant representative*. It is the highest weight in $W \cdot \mu$.*
- *Finally we call a set $\varpi_1, \dots, \varpi_r \in \Lambda$* fundamental weights *relative to Δ if*

$$\langle \varpi_i, \alpha_j^\vee \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

  for all $i, j \in \{1 \dots, r\}$.

We finish this section by some results for the classification of root systems. Fix a root system and a set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_r\}$. For two simple roots α, β

$$d_{\alpha\beta} := \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle$$

is an integer in $\{0, 1, 2, 3\}$. We define the *Dynkin diagram* [9, def. 8.31] as a graph with vertices $v_1, \dots, v_r$ and place exactly $d_{\alpha_i \alpha_j}$ edges between different vertices v_i and v_j. If α_i and α_j differ in their lengths and are not orthogonal, we draw an arrow pointing form the longer root to the shorter one. Note that a semisimple Lie algebra is simple if and only if its Dynkin diagram is connected.

There are only certain types of connected graphs, that are Dynkin diagrams of (abstract) root systems. The types are the *Cartan types* $A_n (n \geq 1), B_n (n \geq 2), C_n (n \geq 2), D_n (n \geq 4), E_6, E_7, E_8, F_4, G_2$. For each graph of these classes exists up to isomorphism a unique root system and a unique simple Lie algebra having that graph as its Dynkin diagram. The Cartan types A_n, B_n, C_n and D_n correspond to the classical Lie algebras. A summary of all Cartan types, their Dynkin diagrams and a standard choice of root systems and weights for the classical types can be found in Figure 2.1.

2.2 Weights of a representation

Similar to the definition of roots, we can generally define eigenvalues (elements of $\mathfrak{h}^*$) of representations (π, V) of the Lie algebra $\mathfrak{g}$ with a fixed Cartan subalgebra $\mathfrak{h}$. They are called *weight* of the representation. The roots are exactly the weights of the adjoint representation.

Definition 2.7 ([9, def. 9.1]). *An element $\lambda \in \mathfrak{h}^*$ is a weight of the representation π, if the weight space*

$$V_\lambda := \{v \in V : \pi(H)v = \lambda(H)v \text{ for all } H \in \mathfrak{h}^*\}$$

is not the zero vector space.

There is an important relation between fundamental weights and irreducible finite-dimensional representations: The theorem of the highest weight for representations.

Theorem 2.8 ([9, thm. 9.4 and 9.5]). *Fix $\mathfrak{g}$ and a set of simple weights.*

1. *Every irreducible finite-dimensional representation of $\mathfrak{g}$ has a highest weight.*

2. *Two irreducible finite-dimensional representations of $\mathfrak{g}$ with the same highest weight are isomorphic.*

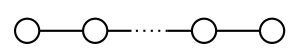
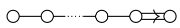
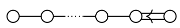
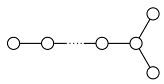
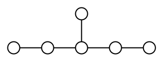
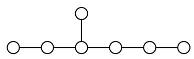
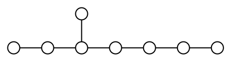
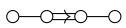
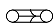
<table>
<tr><th>Cartan Type</th><th>Dynkin Diagram</th><th>Lie algebra $\mathfrak{g}$</th><th>Roots R</th><th>Simple Roots Δ</th><th>Weight Lattice Λ</th></tr>
<tr><td rowspan="2">A_n</td><td rowspan="2"></td><td>$\mathfrak{gl}(n+1)$</td><td rowspan="2">$e_i - e_j$</td><td rowspan="2">$e_i - e_{i+1}$</td><td>$\mathbb{Z}^{n+1}$</td></tr>
<tr><td>$\mathfrak{sl}(n+1)$</td><td>$\mathbb{Z}^{n+1} / \operatorname{span}\{\sum e_i\}$</td></tr>
<tr><td>B_n</td><td></td><td>$\mathfrak{so}(2n+1)$</td><td>$\pm e_i \pm e_j$
$\pm e_i$</td><td>$e_i - e_{i+1}$
e_n</td><td>$\mathbb{Z}^n$</td></tr>
<tr><td>C_n</td><td></td><td>$\mathfrak{sp}(2n)$</td><td>$\pm e_i \pm e_j$
$\pm 2e_i$</td><td>$e_i - e_{i+1}$
$2e_n$</td><td>$\mathbb{Z}^n$</td></tr>
<tr><td>D_n</td><td></td><td>$\mathfrak{so}(2n)$</td><td>$e_i \pm e_j$</td><td>$e_i - e_{i+1}$
$e_{n-1} + e_n$</td><td>$\mathbb{Z}^n$</td></tr>
<tr><td>E_6</td><td></td><td colspan="4" rowspan="5">For further details on the so called *exceptional Lie algebras* refer to [5, sec. 13.2]</td></tr>
<tr><td>E_7</td><td></td></tr>
<tr><td>E_8</td><td></td></tr>
<tr><td>F_4</td><td></td></tr>
<tr><td>G_2</td><td></td></tr>
</table>

FIGURE 2.1: The Cartan types, their Dynkin diagrams and the root systems for the classical Lie algebras. Note that $\mathfrak{gl}(n+1)$ is not a simple Lie algebra, but an important Lie algebra for our work.

3. *If $\mu \in \Lambda$ is the highest weight of an irreducible finite-dimensional representations of $\mathfrak{g}$, then μ is a dominant weight.*

4. *If $\mu \in \Lambda^+$ is a dominant weight, then there exists irreducible finite-dimensional representations of $\mathfrak{g}$ with highest weight μ denoted by $V(\mu)$.*

To illustrate the definitions and terms of this chapter, we now consider the general linear Lie algebra $\mathfrak{gl}(n+1, \mathbb{C})$ and the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$ and analyse their vector representations.

Example 2.9 (General linear Lie algebra)**.** Consider $\mathfrak{g} = \mathfrak{gl}(n+1, \mathbb{C})$. This is the Lie algebra of the linear maps $\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, or equivalently, the Lie algebra of complex $(n+1) \times (n+1)$ matrices. Note that this Lie algebra is not semisimple but reductive (its adjoint representation is completely reducible), nevertheless we can calculate its roots and the highest weight of the vector representation (its defining representation).

A Cartan subalgebra is given by the set of diagonal matrices $\mathfrak{d}$. Let $e_{ij} \in \mathfrak{g}$ be the matrix with a one in row i and column j and zeros elsewhere. For $D = \operatorname{diag}(d_1, \dots, d_{n+1})$ we obtain $\operatorname{ad}_D(e_{ij}) = D \cdot e_{ij} - e_{ij} \cdot D = (d_i - d_j)e_{ij}$.

Thus the maps $\operatorname{diag}(d_1, \dots, d_n) \mapsto d_i - d_j$ are roots, which we will identify with $e_i - e_j$, where e_k denotes the k^{th} standard basis vector of $\mathbb{R}^{n+1}$. It turns out, that the inner product on the real vector space spanned by the roots defined via the Killing form is compatible with the usual inner product on $\mathbb{R}^{n+1}$ under this identification. (They only differ by a positive factor.) A basis of the root system is given by $\alpha_i := e_i - e_{i+1}$ for $1 \leq i \leq n$. The coroots and roots are coincident. The fundamental weights are $\varpi_i = e_1 + e_2 + \cdots + e_i$ for $1 \leq i \leq n$. A weight $(\lambda_1, \dots, \lambda_{n+1})$ is dominant if and only if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1}$.

The Weyl group is given by the symmetric group $\mathfrak{S}_{n+1}$ and therefore the dominant representative of a weight is obtained by sorting its components into weakly decreasing order.

Next we calculate the weights of the vector representation $\pi(X) = (v \mapsto Xv)$. For $D = \operatorname{diag}(d_1, \dots, d_{n+1}) \in \mathfrak{h}$ we get $De_i = d_i \cdot e_i$ for $i = 1, \dots, n+1$. With the identification above we get the weights $e_1, \dots, e_{n+1}$. We observe $e_{i+1} = e_i - \alpha_i$ and obtain an order on the weights, which we represent in the following diagram:

$$e_1 \overset{-\alpha_1}{\succ} e_2 \overset{-\alpha_2}{\succ} \cdots \overset{-\alpha_{n-1}}{\succ} e_n \overset{-\alpha_n}{\succ} \bar{e}_{n+1}. \tag{2.1}$$

The highest weight is e_1, which is also the first fundamental weight.

Example 2.10 (Symplectic Lie algebra)**.** For an integer n we denote by E_n the $n \times n$ identity matrix and define the block matrix $J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$. Then the symplectic Lie algebra is $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C}) := \{X \in \mathfrak{gl}(2n, \mathbb{C}) : X^t J + JX = 0\}$.

A Cartan subalgebra is given by the subalgebra $\mathfrak{h}$ of diagonal matrices in $\mathfrak{g}$, that are the diagonal matrices of the form $D = \operatorname{diag}(d_1, \dots, d_n, d_{n+1}, \dots, d_{2n})$ that satisfy $d_i + d_{i+n} = 0$ for all $1 \leq i \leq n$.

As before we obtain $\operatorname{ad}_D(e_{ij}) = (d_i - d_j)e_{ij}$ and depending on the cases $\max(i,j) \leq n$, $i \leq n < j$, $j \leq n < i$ and $n < \min(i,j)$ and using the property $d_i + d_{i+n} = 0$ we get the roots $D \mapsto \pm d_i \pm d_j$ with i and j both running from 1 up to n. Again we can identify them with linear combinations of the standard basis vectors of $\mathbb{R}^n$ and use the usual inner product as inner product on the roots.

Thus $R = \{\pm e_i \pm e_j : i \neq j\} \cup \{\pm 2e_i\}$ and we choose positive roots $R^+ = \{e_i \pm e_j : i < j\} \cup \{2e_i\}$. A basis Δ of simple roots is given by $\alpha_i = e_i - e_{i+1}$ for $1 \leq i < n$ and $\alpha_n = 2e_n$. The coroots are $\alpha_i^\vee = \begin{cases} \alpha_i & i < n \\ \alpha_i/2 & i = n \end{cases}$ and the fundamental weights are $\varpi_i = e_1 + e_2 + \cdots + e_i$ for $1 \leq i \leq n$. A weight $(\lambda_1, \ldots, \lambda_n)$ is dominant if and only if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.

The Weyl group is the hyperoctahedral group, also known as the group of signed permutations of $\{\pm 1, \ldots, \pm n\}$. Therefore, the dominant representative of a vector is obtained by sorting the absolute values of its components into weakly decreasing order.

Again we consider the vector representation $\pi(X) = (v \mapsto Xv)$ and calculate its weights. For $D = \operatorname{diag}(d_1, \ldots, d_n, -d_1, \ldots, -d_n) \in \mathfrak{h}$ we get $De_i = d_i \cdot e_i$ and $De_{i+n} = -d_i \cdot e_{i+n}$ for $1 \leq i \leq n$. Thus, using the identification from above, we get the weights $\pm e_1, \ldots, \pm e_n$. We write $\bar{e}_i$ in place of $-e_i$. Moreover we observe $e_{i+1} = e_i - \alpha_i$, $\bar{e}_n = e_n - \alpha_n$ and $\bar{e}_i = \bar{e}_{i+1} - \alpha_i$ for $1 \leq i < n$, which is visualised in the diagram

$$e_1 \overset{-\alpha_1}{\succ} e_2 \overset{-\alpha_2}{\succ} \cdots \overset{-\alpha_{n-1}}{\succ} e_n \overset{-\alpha_n}{\succ} \bar{e}_n \overset{-\alpha_{n-1}}{\succ} \bar{e}_{n-1} \overset{-\alpha_{n-2}}{\succ} \cdots \overset{-\alpha_1}{\succ} \bar{e}_1. \tag{2.2}$$

The highest weight of the representation is e_1, it is also the first fundamental weight of the Lie algebra.

2.3 Crystals

For the rest of the chapter let $\mathfrak{g}$ be a reductive complex Lie algebra with a fixed set of simple roots $\{\alpha_i\}_{i \in I}$, its corresponding weight lattice Λ and set of dominant weights Λ^+. A crystal can be seen as a combinatorial model for a representation stripped of its linear structure. Crystal bases were first developed as certain limits of quantum groups by Kashiwara [14]. We will follow an axiomatic approach for defining crystals similar to [17, sec. 2.1] and [10, sec. 2].

Definition 2.11. *A* $\mathfrak{g}$-crystal *is a finite set* B *not containing* 0 *together with maps*

$$\begin{aligned} \mathrm{wt} : B &\mapsto \Lambda \\ \varepsilon_i, \varphi_i : B &\to \mathbb{Z} \\ \tilde{e}_i, \tilde{f}_i : B &\to B \sqcup \{0\} \end{aligned}$$

for each $i \in I$ satisfying:

1. *If $b, b' \in B$ then $\tilde{e}_i(b) = b'$ if and only if $b = \tilde{f}_i(b')$.*
2. *If $\tilde{e}_i(b) \neq 0$ then $\mathrm{wt}(\tilde{e}_i(b)) = \mathrm{wt}(b) + \alpha_i$ and similarly if $\tilde{f}_i(b) \neq 0$ then $\mathrm{wt}(\tilde{f}_i(b)) = \mathrm{wt}(b) - \alpha_i$.*
3. *For all $b \in B$ and $i \in I$ we have*
$$\begin{aligned} \varepsilon_i &= \max\{n \geq 0 : \tilde{e}_i^n(b) \neq 0\} \quad \textit{and} \\ \varphi_i &= \max\{n \geq 0 : \tilde{f}_i^n(b) \neq 0\}. \end{aligned}$$
4. *For all $b \in B$ and $i \in I$ we have*
$$\varphi_i(b) - \varepsilon_i(b) = \langle \mathrm{wt}(b), \alpha_i^\vee \rangle. \tag{2.3}$$

The functions $\tilde{e}_i$ and $\tilde{f}_i$ are called the Kashiwara operators*.*

Remark 2.12. *Note that parts of the definition of a crystal are redundant. The minimal ingredients for a crystal are the set B and the functions $\tilde{e}_i$.*

With a crystal B we associate a certain directed coloured graph with vertices B and we draw edges from b to b' with colour (label) i, if $\tilde{f}_i(b) = b'$. There some are vertices, with no incoming edge or equivalently $\varepsilon_i(b) = 0$ for all i. We call such elements of B the *highest weight elements* or *highest weight words*. We call their weights the *highest weights* of the crystals B. By equation (2.3) we obtain that all highest weights of a crystal are dominant weights in Λ^+.

For each crystal we define its *dual* by, roughly speaking, interchanging φ_i with ε_i and $\tilde{e}_i$ with $\tilde{f}_i$.

Definition 2.13 ([1, def. 2.20])**.** *Let B be a crystal. Let $B^\vee$ be a set that is in bijection with B and bijection $b \mapsto b^\vee$. Then we define the dual crystal of B to be the crystal with elements $B^\vee$,*

$$\begin{aligned} \mathrm{wt}(b^\vee) &:= \mathrm{wt}(b) & \quad \textit{and} \quad && \tilde{e}_i(b^\vee) &:= \tilde{f}_i(b)^\vee \\ \varepsilon_i(b^\vee) &:= \varphi_i(b) & && \tilde{f}_i(b^\vee) &:= \tilde{e}_i(b)^\vee. \\ \varphi_i(b^\vee) &:= \varepsilon_i(b) & && & \end{aligned}$$

We now give some examples for crystal graphs.

Example 2.14. The following crystals consist of elements of the form $\boxed{i}$, $\boxed{\bar{i}}$ and $\boxed{0}$ with weights

$$\mathrm{wt}\,\boxed{i} = e_i, \quad \mathrm{wt}\,\boxed{\bar{i}} = -e_i \quad \text{and} \quad \mathrm{wt}\,\boxed{0} = 0.$$

We will present a standard crystal graph for each classical Cartan type.

Type A_n The standard crystal is given by the graph

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{n+1}.$$

Its highest weight element is $\boxed{1}$ with weight e_1. We also denote this as the crystal corresponding to the vector representation of $\mathfrak{gl}(n+1)$. Compare this graph to the diagram (2.1) in example 2.9. For later use we also present the dual crystal graph and for better readability we are using $\boxed{\bar{i}}$ instead of $\boxed{i}^{\vee}$.

$$\boxed{\overline{n+1}} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$$

It corresponds to the dual of the vector representation.

Type B_n The standard crystal graph is

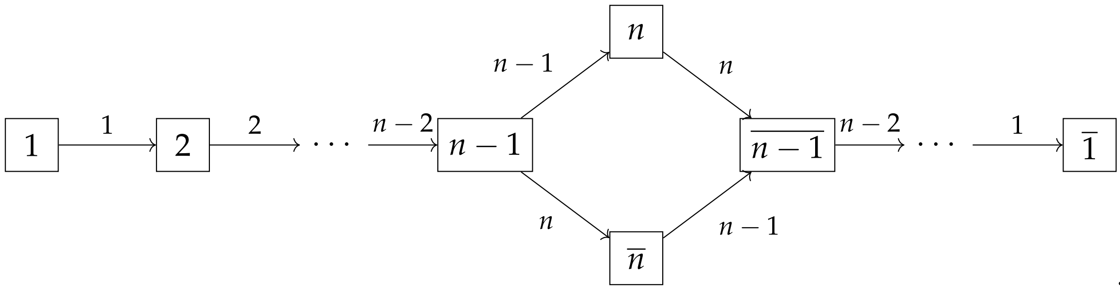
$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{1} \boxed{\bar{1}}.$$

Type C_n As in type A_n we can see in example 2.10 and diagram (2.2) how the standard crystal

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{1} \boxed{\bar{1}}$$

relates to the vector representation.

Type D_n The standard crystal graph is



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The standard crystals for the types B_n, C_n, D_n are self dual.

Crystals have a *direct sum*, given by their disjoint union and a *tensor product*, which we define now.

Definition 2.15 ([14, sec. 4.3]). *Let B_1, B_2 be crystals, then the tensor product $B_1 \otimes B_2$ is defined by:*

- *The underlying set is $B_1 \times B_2$. Instead of pairs (b_1, b_2) we also write $b_1 \otimes b_2$.*
- *The weight is given by $\mathrm{wt}(b_1 \otimes b_2) := \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$.*
- *The Kashiwara operators are*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i(b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i(b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

This implies

$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \mathrm{wt}(b_1), \alpha_i^\vee \rangle)$$

and therefore $b_1 \otimes b_2$ is a highest weight word in $B_1 \otimes B_2$ if and only if $\varepsilon_i(b_1) = 0$ and $\varepsilon_i(b_2) - \langle \mathrm{wt}(b_1), \alpha_i^\vee \rangle \leq 0$ for all $i \in I$.

We are in particular interested in the highest weight words and their weights in tensor products of more parts. By iterating the construction of tensor products, we obtain for the crystal $B_1 \otimes B_2 \otimes \cdots \otimes B_r$ the results:

- The underlying set is $B_1 \times B_2 \times \cdots \times B_r$.

- The weight is given by the sum of the weights of the letters b_i.

$$\mathrm{wt}(b_1 \otimes b_2 \otimes \cdots \otimes b_r) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2) + \cdots + \mathrm{wt}(b_r))$$

- $b := b_1 \otimes b_2 \otimes \cdots \otimes b_r$ is a highest weight word, if and only if

$$\varepsilon_i(b) := \max_{1 \leq j \leq n} (\varepsilon_i(b_j) - \langle \mathrm{wt}(b_1), \alpha_i^\vee \rangle - \langle \mathrm{wt}(b_2), \alpha_i^\vee \rangle - \cdots - \langle \mathrm{wt}(b_{j-1}), \alpha_i^\vee \rangle) = 0.$$

- Note that it also possible to describe the Kashiwara operators, but for obtaining highest weight words this is not needed.

Remark 2.16. *It is convenient to define a function $\varepsilon : B \to \Lambda$ satisfying $\varepsilon_i(b) = \langle \varepsilon(b), \alpha_i^\vee \rangle$. One can obtain such a function using the fundamental weights. For the word $b_1 \otimes b_2 \otimes \cdots \otimes b_r$ we define a finite sequence of weights by $\mu^0 := 0$ and $\mu^i := \mu^{i-1} + \mathrm{wt}(b_i)$ for $i = 1, \ldots, r$. Then the condition for highest weight words reads as follows.*

Lemma 2.17. *The word $b_1 \otimes b_2 \otimes \cdots \otimes b_r$ is a highest weight word in $B_1 \otimes B_2 \otimes \cdots \otimes B_r$ if and only if*

$$\mu^{j-1} - \varepsilon(b_j) \in \Lambda^+ \quad \text{for all} \quad j = 1, \ldots, r. \tag{2.4}$$

We denote the set of all $b_1 \otimes b_2 \otimes \cdots \otimes b_r$ satisfying this condition as $\max B_1 \otimes B_2 \otimes \cdots \otimes B_r$.

A direct consequence of this is, that for a highest weight word all weights μ_i in the sequence above are dominant.

We now follow Lenart's approach [17, sec. 2.1] and axiomatically introduce the category of $\mathfrak{g}$-crystals, which is according to a result of Joseph, uniquely defined.

A1 For each dominant weight $\lambda \in \Lambda^+$, the category contains a crystal B_λ with a unique highest weight element with weight λ.

A2 The category consists of all crystals isomorphic to a direct sum of B_λ. We denote isomorphism of crystals by $\cong$.

A3 The category is closed under tensor products. For all dominant weights λ, μ there exists an inclusion of crystals $\iota_{\lambda,\mu} : B_{\lambda+\mu} \hookrightarrow B_\lambda \otimes B_\mu$.

By axiom A3 we obtain a decomposition of the tensor product of crystals into connected components

$$B_{\lambda_1} \otimes B_{\lambda_2} \otimes \cdots \otimes B_{\lambda_r} \cong \bigoplus_{(b_1 \otimes b_2 \otimes \cdots \otimes b_r) \in \max B_{\lambda_1} \otimes B_{\lambda_2} \otimes \cdots \otimes B_{\lambda_r}} B_{\mathrm{wt}(b_1) + \mathrm{wt}(b_2) + \cdots + \mathrm{wt}(b_r)}. \tag{2.5}$$

Example 2.18. To illustrate, consider the standard crystal for C_2 denoted by C as shown in Example 2.14. In figure 2.2 we see the crystal $C^{\otimes 3}$. It consists of six components:

- one component with highest weight $3e_1$,
- three isomorphic components with highest weight e_1 and
- two isomorphic components with highest weight $2e_1 + e_2$.

The decomposition (2.5) is closely related to the decomposition of tensor products of highest weight modules, which follows from [11, thm. 5.2.1].

Lemma 2.19. *For dominant weights $\lambda, \mu \in \Lambda^+$ denote $V(\lambda)$ and $V(\mu)$ the corresponding irreducible highest weight representations and B_λ and B_μ corresponding crystals. Then the crystal $B_\lambda \otimes B_\mu$ corresponds to the representation $V(\lambda) \otimes V(\mu)$. That is,*

- *the connected components of $B_\lambda \otimes B_\mu$ are precisely the crystals of the irreducible representations in the decomposition of $V(\lambda) \otimes V(\mu)$*
- *for $\nu \in \max B_\lambda \otimes B_\mu$ we get as many copies of B_ν as the multiplicity of $V(\lambda)$ in $V(\lambda) \otimes V(\mu)$.*

For a crystal B we may write the decomposition (2.5) as isomorphism of crystals

$$\otimes^r B \cong \bigoplus_\lambda B_\lambda \times U_\lambda$$

where B_λ is a connected crystal and U_λ is the set of highest weight words of weight λ. (Compare to [31, sec. 5.2].) When B is the crystal of the vector representation of $\mathfrak{gl}(n)$, an explicit isomorphism is given by the well known Robinson-Schensted correspondence. In this case U_λ is the set of standard tableaux of shape λ and B_λ can be identified with the set of semistandard tableaux of shape λ and largest entry n.

As final results of this section we continue the examples 2.9 and 2.10 and make the highest weight words of the tensor powers of the crystals explicit. We already know that for a highest weight word $b_1 \otimes b_2 \otimes \cdots \otimes b_r$ all the weights μ^i defined as in Remark 2.16 are dominant. We prove in two cases that we already get all highest words from this weaker condition.

Example 2.20. Let $\mathfrak{g} = \mathfrak{gl}(n+1)$, $\Lambda = \mathbb{Z}^{n+1}$ and Λ^+ the corresponding dominant weights. Moreover denote A the crystal of the vector representation V and $A^\vee$ the

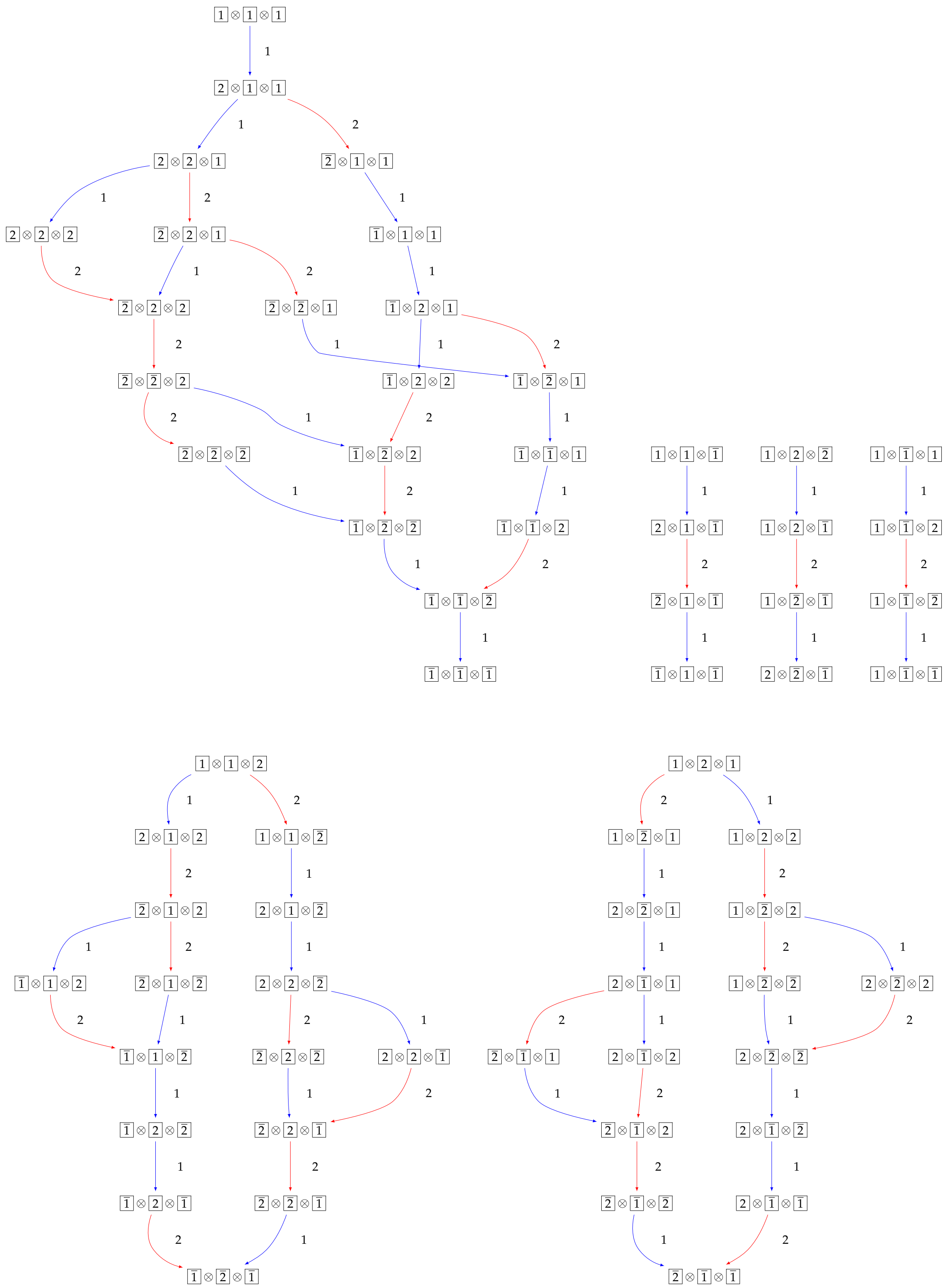


FIGURE 2.2: The crystal $C^{\otimes 3}$, where C denotes the standard crystal for C_2.

crystal of the dual of the vector representation V^*. A is the standard A_n crystal and $A^\vee$ its dual (compare to Example 2.14). Consider a sequence $(0 = \mu^0, \mu^1, \ldots, \mu^r)$ of dominant weights, such that two consecutive weights differ exactly by a unit vector. We define a word $b_1 \otimes b_2 \otimes \cdots \otimes b_r$ and a crystal $B := B_1 \otimes B_2 \otimes \cdots \otimes B_r$ by

$$b_i := \begin{cases} \boxed{\ell} & \text{if } \mu^i - \mu^{i-1} = e_\ell \\ \boxed{\overline{\ell}} & \text{if } \mu^i - \mu^{i-1} = -e_\ell \end{cases} \quad \text{and} \quad B_i := \begin{cases} A & \text{if } \mu^i - \mu^{i-1} = e_\ell \\ A^\vee & \text{if } \mu^i - \mu^{i-1} = -e_\ell \end{cases}.$$

Then the word is a highest weight word in B.

Proof. We show that the condition (2.4) for highest weight words in Lemma 2.17 is satisfied. For $b \in A \sqcup A^\vee$ we obtain

$$\varepsilon(b) = \sum_{i=1}^{n} \varepsilon_i \varpi_i = \begin{cases} 0 & \text{if } b = \boxed{1} \text{ or } b = \boxed{\overline{n+1}} \\ \varpi_{\ell-1} & \text{if } b = \boxed{\ell}, \ell \neq 1 \\ \varpi_\ell & \text{if } b = \boxed{\overline{\ell}}, \ell \neq n \end{cases}$$

where $\varpi_i = e_1 + \cdots + e_i$ denotes the fundamental weights. Now we consider different cases.

1. Let $\mu^i - \mu^{i-1} = e_1$ or $\mu^i - \mu^{i-1} = -e_{n+1}$, then $\varepsilon(b_i) = 0$ and thus $\mu^{i-1} - \varepsilon(b_i) = \mu^{i-1} \in \Lambda^+$.

2. Let $\mu^i - \mu^{i-1} = e_\ell$ with $\ell \neq 1$. As $\mu^i, \mu^{i-1} \in \Lambda^+$ we have

$$\mu^{i-1}_{\ell-1} = \mu^i_{\ell-1} \geq \mu^i_\ell = \mu^{i-1}_\ell + 1$$

and get $\mu^{i-1}_{\ell-1} - 1 \geq \mu^{i-1}_\ell$. This yields

$$\mu^{i-1}_1 - 1 \geq \mu^{i-1}_2 - 1 \geq \cdots \geq \mu^{i-1}_{\ell-1} - 1 \geq \mu^{i-1}_\ell \geq \mu^{i-1}_{\ell+1} \geq \cdots \geq \mu^{i-1}_{n+1}$$

and thus $\mu^{i-1} - \varpi_{\ell-1} = \mu^{i-1} - \varepsilon(b_i) \in \Lambda^+$.

3. Let $\mu^i - \mu^{i-1} = -e_\ell$ with $\ell \neq n+1$. Similar to the case above we have

$$\mu^{i-1}_\ell - 1 = \mu^i_\ell \geq \mu^i_{\ell+1} = \mu^{i-1}_{\ell+1}.$$

As before this yields

$$\mu^{i-1}_1 - 1 \geq \mu^{i-1}_2 - 1 \geq \cdots \geq \mu^{i-1}_\ell - 1 \geq \mu^{i-1}_{\ell+1} \geq \mu^{i-1}_{\ell+2} \geq \cdots \geq \mu^{i-1}_{n+1}$$

and thus $\mu^{i-1} - \varpi_\ell = \mu^{i-1} - \varepsilon(b_i) \in \Lambda^+$.

□

Later we focus on those sequences of weights $(0 = \mu^0, \mu^1, \ldots, \mu^{2r})$ that correspond to highest weights in $\otimes^r(A \otimes A^\vee)$ and call them *alternating tableaux*. These are precisely the highest weight words corresponding to tensor powers of the crystal corresponding to the adjoint representation regarded as $V \otimes V^*$.

Finally we consider the symplectic case again. As the standard C_n crystal differs from $A \sqcup A^\vee$ from the former example only by an additional edge $\boxed{n} \xrightarrow{n} \boxed{\overline{n}}$ we expect to get a similar result and a similar proof.

Example 2.21. Let $\mathfrak{g} = \mathfrak{sp}(2n)$, $\Lambda = \mathbb{Z}^n$ and Λ^+ the corresponding dominant weights. Moreover denote C the crystal of the vector representation V. C is the standard C_n crystal (compare to Example 2.14). Consider a sequence $(0 = \mu^0, \mu^1, \ldots, \mu^r)$ of dominant weights, such that two consecutive weights differ exactly by a unit vector. We define a word $b_1 \otimes b_2 \otimes \cdots \otimes b_r \in \otimes^r C$ by

$$b_i := \begin{cases} \boxed{\ell} & \text{if } \mu^i - \mu^{i-1} = e_\ell \\ \boxed{\overline{\ell}} & \text{if } \mu^i - \mu^{i-1} = -e_\ell \end{cases} .$$

Then the word is a highest weight word in C.

Proof. Again we show that the highest weight word condition (2.4) is satisfied and this time we obtain

$$\varepsilon(b) = \begin{cases} 0 & \text{if } b = \boxed{1} \\ \varpi_{\ell-1} & \text{if } b = \boxed{\ell}, \ell \neq 1 \\ \varpi_\ell & \text{if } b = \boxed{\overline{\ell}} \end{cases} .$$

Note that the dominant weights also have the condition that all its components are non-negative. We consider similar cases as before, the first three are solved exactly as before.

1. Let $\mu^i - \mu^{i-1} = e_1$. Solve this exactly as case 1. from before.

2. Let $\mu^i - \mu^{i-1} = e_\ell$ with $\ell \neq 1$. Solve this exactly as case 2. from before.

3. Let $\mu^i - \mu^{i-1} = -e_\ell$ with $\ell \neq n$. Solve this exactly as case 3. from before.

4. Let $\mu^i - \mu^{i-1} = -e_n$, then we obtain $\mu_n^{i-1} - 1 = \mu_n^i \geq 0$. This gives

$$\mu_1^{i-1} - 1 \geq \mu_2^{i-1} - 1 \geq \cdots \geq \mu_n^{i-1} - 1 \geq 0$$

and thus $\mu^{i-1} - \varpi_n = \mu^{i-1} - \varepsilon(b_i) \in \Lambda^+$. $\square$

We call sequences of dominant weights corresponding to highest weight words in $\otimes^r C$ *oscillating tableaux*.

Chapter 3

Results

We aim at making the action of the cactus group on the highest weight words of a tensor power of certain representations transparent. Our approach works best for tensor products of minuscule representations of a Lie group. A representation is *minuscule* if the Weyl group W of the Lie group acts transitively on the weights of the representation: the set of weights forms a single orbit under the action of W. The non-trivial minuscule representations are:

Type A_n All exterior powers of the vector representation.

Type B_n The spin representation.

Type C_n The vector representation.

Type D_n The vector representation and the two half-spin representations.

Type E_6 The two fundamental representations of dimension 27.

Type E_7 The fundamental representation of dimension 56.

There are no nontrivial minuscule representations in types G_2, F_4 or E_8. For all other types, any crystal can be embedded into a tensor product of minuscule crystals.

For tensor products of exterior powers of the vector representation of $\mathrm{GL}(n)$, the action of the cactus group is known, as already mentioned in the introduction.

FIGURE 3.1: A 3-noncrossing perfect matching and a permutation as chord diagrams.

Highest weight words of weight zero of $\otimes^r S$, where S is the the spin representation of the spin group $\mathrm{Spin}(2n+1)$ can be identified directly with fans of n Dyck paths of length r. One can show that ev acts on these as reversal.

The vector representation of the odd orthogonal group $\mathrm{SO}(2n+1)$ is not minuscule, but appears as a direct summand in $S \otimes S$. In particular, highest weight words of weight zero of a tensor power of the vector representation of $\mathrm{SO}(3)$ can be identified with noncrossing set partitions without singletons, and pr acts on these as rotation. Because the vector representation of the Lie algebras of $\mathrm{SO}(3)$ coincides with the the adjoint representation of the Lie algebra of $\mathrm{SL}(2)$, this description can also be deduced from the results in Section 3.2 below.

However, our main contributions concern the vector representation of $\mathrm{Sp}(2n)$ and the adjoint representation of $\mathrm{GL}(n)$ - regarded as the tensor product of the vector representation and its dual.

3.1 The vector representation of the symplectic groups

Definition 3.1 (Sundaram [30])**.** *An n-symplectic oscillating tableau of length r and (final) shape μ is a sequences of partitions*

$$\emptyset = \mu^0, \mu^1, \dots, \mu^r = \mu$$

such that the Ferrers diagrams of two consecutive partitions differ by exactly one box, and each partition μ^i has at most n non-zero parts.

Proposition 3.2. *Let C be the crystal corresponding to $\otimes^r V$, where V is the vector representation of the symplectic group $\mathrm{Sp}(2n)$. Then the highest weight words of C are obtained from n-symplectic oscillating tableaux by considering each partition as a vector in $\mathbb{Z}^n$ and taking successive differences. Explicitly, the highest weight word corresponding to $\mathcal{O}$ is*

$$\mu^1 - \mu^0, \mu^2 - \mu^1, \dots, \mu^r - \mu^{r-1}.$$

A now classic bijection due to Sundaram [30] maps an oscillating tableau $\mathcal{O}$ of length r to a pair $(\mathcal{M}(\mathcal{O}), \mathcal{M}_T(\mathcal{O}))$, consisting of a matching of a subset of $\{1, \dots, r\}$ and a partial standard Young tableau on the complementary subset. We describe this bijection in Section 5.

Theorem 3.3. *Let $\mathcal{O}$ be an n-symplectic oscillating tableau of length r, not necessarily of empty shape. Then $\mathcal{M}(\operatorname{ev}\mathcal{O})$ is the reversal of $\mathcal{M}(\mathcal{O})$ and $\mathcal{M}_T(\operatorname{ev}\mathcal{O})$ is the Schützenberger evacuation of $\mathcal{M}_T(\mathcal{O})$.*

There is a remarkable geometric description of perfect matchings corresponding to n-symplectic oscillating tableaux of empty shape under Sundaram's bijection: visualise a perfect matching by drawing its pairs as (straight) diagonals connecting the vertices of a labelled regular r-gon. Then a perfect matching is $(n+1)$-noncrossing, and the image of an n-symplectic oscillating tableau, if it contains at most n pairs that mutually cross in this picture.

Theorem 3.4. *The bijection $\mathcal{M}$ between n-symplectic oscillating tableaux of empty shape and $(n+1)$-noncrossing perfect matchings intertwines promotion and rotation, and evacuation and reversal:*

$$\operatorname{rot}\mathcal{M}(\mathcal{O}) = \mathcal{M}(\operatorname{pr}\mathcal{O}) \text{ and } \operatorname{rev}\mathcal{M}(\mathcal{O}) = \mathcal{M}(\operatorname{ev}\mathcal{O}).$$

3.2 The adjoint representation of the general linear groups

Definition 3.5 (Stembridge [29])**.** *A vector in $\mathbb{Z}^n$ with weakly decreasing entries is called a staircase.*

A $\mathrm{GL}(n)$-alternating tableau of length r and weight μ is a sequence of staircases

$$\emptyset = \mu^0, \mu^1, \ldots, \mu^{2r} = \mu$$

such that

for even i, μ^{i+1} is obtained from μ^i by adding 1 to an entry, and

for odd i, μ^{i+1} is obtained from μ^i by subtracting 1 from an entry.

When μ is the zero weight, slightly abusing language, we say that the alternating tableau is of empty shape.

Proposition 3.6. *Let C be the crystal corresponding to $\otimes^r\mathrm{GL}(n)$, where $\mathrm{GL}(n)$ is the adjoint representation of the general linear group $\mathrm{GL}(n)$. Then the highest weight words of C are obtained from $\mathrm{GL}(n)$-alternating tableaux by taking successive differences. Explicitly, the highest weight word corresponding to $\mathcal{A}$ is the sequence of r pairs*

$$(\mu^1 - \mu^0, \mu^2 - \mu^1), \ldots, (\mu^{2r-1} - \mu^{2r-2}, \mu^{2r} - \mu^{2r-1}).$$

It is tempting to regard each vector in an alternating tableau as a pair of partitions by separating the positive and negative terms. Indeed, this is what we will do below. However, for $n > 2$ promotion does not preserve the maximal number of non-zero entries in a vector of an alternating tableau. In fact, it is not clear whether there is an embedding ι of the set of $\mathrm{GL}(n)$-alternating tableaux for into the set of $\mathrm{GL}(n+1)$-alternating tableaux such that $\mathrm{pr}\,\iota(\mathcal{A}) = \mathrm{pr}\,\mathcal{A}$. In spite of this, we prove a stability phenomenon for promotion of alternating tableaux in Section 6.4.

In Section 5, we introduce a bijection similar in spirit to Sundaram's, that maps an alternating tableau $\mathcal{A}$ of length r to a triple $\big(\mathcal{P}(\mathcal{A}), \mathcal{P}_P(\mathcal{A}), \mathcal{P}_Q(\mathcal{A})\big)$, consisting of a bijection between two subsets, R and S, of $\{1, \dots, r\}$, and two partial standard Young tableaux. The shapes of these tableaux are obtained by separating the positive and negative terms in the weight of the alternating tableaux. The entries of the first tableau then form the complementary subset of R, the entries of the second form the complementary subset of S.

Theorem 3.7. *Let $\mathcal{A}$ be a $\mathrm{GL}(n)$-alternating tableau of length $r \leq \lfloor \frac{n+1}{2} \rfloor$, not necessarily of empty shape. Then $\mathcal{P}(\mathrm{ev}\,\mathcal{A})$ is the reversal of the complement of $\mathcal{P}(\mathcal{A})$ and*

$$\big(\mathcal{P}_P(\mathrm{ev}\,\mathcal{A}), \mathcal{P}_Q(\mathrm{ev}\,\mathcal{A})\big) = \big(\,\mathrm{ev}\,\mathcal{P}_P(\mathcal{A}), \mathrm{ev}\,\mathcal{P}_Q(\mathcal{A})\big).$$

Theorem 3.8. *For $n \geq r-1$ and also for $n \leq 2$ the bijection $\mathcal{P}$ between $\mathrm{GL}(n)$-alternating tableaux of empty shape and permutations intertwines promotion and rotation:*

$$\mathrm{rot}\,\mathcal{P}(\mathcal{A}) = \mathcal{P}(\mathrm{pr}\,\mathcal{A}).$$

For even $n \geq r$ and for odd $n \geq r-1$, it intertwines evacuation and reverse-complement:

$$\mathrm{rc}\,\mathcal{P}(\mathcal{A}) = \mathcal{P}(\mathrm{ev}\,\mathcal{A}).$$

For $n \leq 2$ it intertwines evacuation and inverse-reverse-complement:

$$\mathrm{rc}\,\mathcal{P}(\mathcal{A})^{-1} = \mathcal{P}(\mathrm{ev}\,\mathcal{A}).$$

We remark that the case $n \leq 2$ is special, because our bijection identifies $\mathrm{GL}(2)$-alternating tableaux of empty weight in a natural way with noncrossing partitions, which form an invariant set under rotation. In fact, this set coincides with the web basis for $\mathrm{GL}(2)$. Moreover, in this case the evacuation of an alternating tableau is its reversal. In terms of noncrossing partitions, the inverse-reverse-complement of the corresponding permutation is the mirror image of the partition.

As indicated in the introduction, Patrias [19] demonstrated that the growth algorithm of Khovanov and Kuperperg also intertwines promotion of GL(3)-alternating tableaux of empty shape and rotation of webs.

Chapter 4

The cactus groups, local rules, promotion and evacuation

4.1 Promotion and evacuation

In this section, following Henriques and Kamnitzer [10], we define promotion and evacuation of highest weight words as an action of certain elements of the cactus group on r-fold tensor products of crystals. Then, following van Leeuwen [16] and Lenart [17] we encode the action of the cactus group by certain local rules, generalising Fomin's.

Definition 4.1. *The r-fruit cactus group, $\mathfrak{C}_r$, has generators $\mathrm{s}_{p,q}$ for $1 \leq p < q \leq r$ and defining relations*

- $\mathrm{s}_{p,q}^2 = 1$
- $\mathrm{s}_{p,q}\,\mathrm{s}_{k,l} = \mathrm{s}_{k,l}\,\mathrm{s}_{p,q}$ *if* $q < k$ *or* $l < p$
- $\mathrm{s}_{p,q}\,\mathrm{s}_{k,l} = \mathrm{s}_{p+q-l,p+q-k}\,\mathrm{s}_{p,q}$ *if* $p \leq k < l \leq q$

For convenience we additionally define $\mathrm{s}_{p,p} = 1$.

The following lemma shows that it is sufficient to define the action of the composites $\mathrm{s}_{1,q}\,\mathrm{s}_{2,q}$ for $2 \leq q \leq r$. The first relation was observed by White [33, lem. 2.3], the second is in analogy to Schützenberger's original definition of evacuation of standard Young tableaux in [27, sec. 5].

Lemma 4.2. *We have*

$$\mathrm{s}_{p,q} = \mathrm{s}_{1,q}\,\mathrm{s}_{1,q-p+1}\,\mathrm{s}_{1,q} \quad \text{and} \quad \mathrm{s}_{1,q} = \mathrm{s}_{1,2}\,\mathrm{s}_{2,2}\;\;\mathrm{s}_{1,3}\,\mathrm{s}_{2,3}\;\ldots\;\mathrm{s}_{1,q}\,\mathrm{s}_{2,q}\,.$$

Proof. The first equality is obtained from the third defining relation by replacing p, q, k and ℓ with $1, q, p$ and q respectively. The second equality follows from $\mathrm{s}_{1,\ell}\,\mathrm{s}_{2,\ell} = \mathrm{s}_{1,\ell-1}\,\mathrm{s}_{1,\ell}$, which is also an instance of the third defining relation. □

Henriques and Kamnitzer [10] defined an action of the cactus group on r-fold tensor products of crystals in terms of the commutor, which in turn is defined using Lusztig's involution. Let us first briefly recall the latter, as introduced in [18]:

Definition 4.3. *Let B be any highest weight crystal. Lusztig's involution η maps the highest weight element of B to its lowest weight element, and the Kashiwara operator f_i to e_{i^*}, where $i \mapsto i^*$ is the Dynkin diagram automorphism specified by $\alpha_{i^*} = -w_0(\alpha_i)$, and w_0 is the longest element of the Weyl group. This definition is extended to arbitrary crystals by applying the involution to each component separately.*

Note that Lusztig's involution is not a morphism of crystals. For a crystal of semistandard Young tableaux, Lusztig's involution is precisely Schützenberger's evacuation of semistandard Young tableaux, not to be confused with the same operation on standard Young tableaux.

Definition 4.4. *For two crystals A and B, the commutor is the crystal morphism*

$$\begin{aligned} \sigma_{A,B} : A \otimes B &\to B \otimes A \\ (a, b) &\mapsto \eta\big(\eta(b), \eta(a)\big). \end{aligned}$$

We can now define the action of the cactus group.

Definition 4.5. *The action of $\mathfrak{C}_r$ on words in $C_1 \otimes \cdots \otimes C_r$ is defined inductively by letting $\mathrm{s}_{p,p+1}$ act as $1 \otimes \sigma_{C_p,C_{p+1}} \otimes 1$ and $\mathrm{s}_{p,q}$ as $1 \otimes \sigma_{C_p,C_{p+1}\otimes\cdots\otimes C_q} \otimes 1 \circ \mathrm{s}_{p+1,q}$.*

The action can be expressed more explicitly in terms of Lusztig's involution:

Proposition 4.6.

$$\mathrm{s}_{p,q}\, w_1 \dots w_r = w_1 \dots w_{p-1} \eta\big(\eta(w_q)\eta(w_{q-1}) \dots \eta(w_p)\big) w_{q+1} \dots w_r.$$

Proof. Induction on $q - p$. □

Definition 4.7. *The promotion $\mathrm{pr}\, w$ of w is $\mathrm{s}_{1,r}\, \mathrm{s}_{2,r}\, w$, and the evacuation $\mathrm{ev}\, w$ of w is $\mathrm{s}_{1,r}\, w$.*

Proposition 4.8. $\mathrm{s}_{1,q}\, \mathrm{s}_{2,q}(w) = \sigma_{C_1,C_2\otimes\cdots\otimes C_q}(w)$.

Proof. Immediate from the definition of the action of $\mathrm{s}_{1,q}$ and the fact that $\mathrm{s}_{2,q}^2 = 1$. □

4.2 Local rules

We now follow Lenart's approach [17] and realise the action of the cactus group using van Leeuwen's local rules [16, Rule 4.1.1], which generalise Fomin's [28, A 1.2.7].

Definition 4.9. *Let λ be a weight of a minuscule representation of a Lie group with Weyl group W. Then $\mathrm{dom}_W(\lambda)$ is the dominant representative of the W-orbit $W\lambda$.*

Let A be a crystal and B and C be crystals of minuscule representations. Then the local rule

$$\tau^A_{B,C}\colon A \otimes B \otimes C \to A \otimes C \otimes B$$

is a weight preserving bijection defined for highest weight words $a \otimes b \otimes c$ as follows: let κ be the weight of a, let λ be the weight of $a \otimes b$ and let ν be the weight of $a \otimes b \otimes c$. Then

$$\tau^A_{B,C}(a \otimes b \otimes c) = a \otimes \hat{c} \otimes \hat{b},$$

where, regarding κ, λ, μ and ν as vectors,

$$\mu = \mathrm{dom}_W(\kappa + \nu - \lambda), \quad \hat{c} = \mu - \kappa \quad \text{and} \quad \hat{b} = \nu - \mu.$$

We represent this by the following diagram:

$$\begin{array}{ccc} \lambda & \xrightarrow{c} & \nu \\ b\uparrow & & \uparrow\hat{b} \\ \kappa & \xrightarrow[\hat{c}]{} & \mu \end{array} \tag{4.1}$$

From now on we omit the labels on the edges, because they are determined by the weights.

Since any isomorphism between crystals is determined by specifying a bijection between the corresponding highest weight words, the definition of the local rule can now be extended to the whole crystal by applying the lowering operators.

For example, the Weyl group of $\mathrm{Sp}(2n)$ is the hyperoctahedral group of signed permutations of $\{\pm 1, \ldots, \pm n\}$. Therefore, the dominant representative of a vector is obtained by sorting the absolute values of its components into weakly decreasing order.

Remark 4.10. *As in the classical case the local rule is symmetric in the sense that $\mu = \mathrm{dom}_W(\kappa + \nu - \lambda)$ if and only if $\lambda = \mathrm{dom}_W(\kappa + \nu - \mu)$.*

Theorem 4.11 ([17, thm. 4.4])**.** *Let A and B be crystals, embedded into tensor products $A_1 \otimes \cdots \otimes A_k$ and $B_1 \otimes \cdots \otimes B_\ell$ of crystals of minuscule representations. Let w be a highest weight* *word in $A \otimes B$. Then $\sigma_{A,B}(w)$ can be computed as follows. Create a $k \times \ell$ grid of squares as in* (4.1)*, labelling the edges along the left border with $w_1, \dots, w_k$ and the edges along the top border with $w_{k+1}, \dots, w_{k+\ell}$:*

$$
\begin{tikzcd}
{} \arrow[r, "w_{k+1}"] & {} \arrow[r, dotted, dash] & {} \arrow[r, "w_{k+\ell}"] & {} \\
{} \arrow[u, "w_k"] \arrow[r] & {} \arrow[u] \arrow[r, dotted, dash] & {} \arrow[u] \arrow[r] & {} \arrow[u, "\hat{w}_{k+\ell}"'] \\
{} \arrow[u, dotted, dash] \arrow[r] & {} \arrow[u, dotted, dash] \arrow[r, dotted, dash] & {} \arrow[u, dotted, dash] \arrow[r] & {} \arrow[u, dotted, dash] \\
0 \arrow[u, "w_1"] \arrow[r, "\hat{w}_1"'] & {} \arrow[u] \arrow[r, dotted, dash] & {} \arrow[u] \arrow[r, "\hat{w}_\ell"'] & {} \arrow[u, "\hat{w}_{1+\ell}"']
\end{tikzcd} \tag{4.2}
$$

For each square whose left and top edges are already labelled use the local rule to compute the labels on the square's bottom and right edges. The labels $\hat{w}_1 \dots \hat{w}_{k+\ell}$ of the edges along the bottom and the right border of the grid then form $\sigma_{A,B}(w)$.

Example 4.12. Let $C_1 = C_2 = C_4$ be the crystal corresponding to the exterior square of $\mathrm{SL}(3)$ and $C_3 = C_5$ the crystal corresponding to its vector representation. Then $w = 110 \otimes 101 \otimes 100 \otimes 110 \otimes 010$ is the highest weight word corresponding to the semistandard tableau

$$
\begin{array}{|c|c|c|}
\hline
1 & 1 & 2 \\
\hline
2 & 4 & \multicolumn{1}{c}{} \\
\cline{1-2}
3 & 5 & \multicolumn{1}{c}{} \\
\cline{1-2}
4 & \multicolumn{2}{c}{} \\
\cline{1-1}
\end{array}\;.
$$

The promotion $\mathrm{pr}\, w$ can now be computed by writing down the sequence of cumulative weights in one line, writing the zero weight just below the second element of this line, and then successively applying the local rule (4.1). Finally, append the weight of w to the second line.

Because the Weyl group of $\mathrm{SL}(n)$ is the symmetric group $\mathfrak{S}_n$, $\mathrm{dom}_{\mathfrak{S}_n}$ is just returning its argument sorted into weakly decreasing order:

$$
\begin{array}{ccccccc}
000 & 110 & 211 & 311 & 421 & 431 & \\
 & 000 & 110 & 210 & 320 & 330 & 431.
\end{array}
$$

Thus, the promotion of the semistandard tableau above is

1	1	5
2	3	
3	4	
5		

.

As mentioned in the introduction, this definition of promotion coincides with the classical definition of promotion in terms of Bender-Knuth moves on tableaux when the crystals correspond to exterior powers of the vector representation of $\mathrm{SL}(n)$.

Moreover, we obtain a formulation of the commutor, and therefore also of promotion of highest weight words in crystals of minuscule representations, analogous to the definition in terms of slides in tableaux:

Corollary 4.13. *Let A and B be crystals, embedded into tensor products of crystals of minuscule representations. Let $a \in A$ and $b \in B$ such that ab is a highest weight word in $A \otimes B$, and let $\hat{b}\hat{a} = \sigma_{A,B}(ab)$ with $\hat{b} \in B$ and $\hat{a} \in A$.*

Then

$\hat{b}$ is the highest weight element in the same component of B as b, and

$\hat{a}$ is an element of A such that the weight of $\hat{b}\hat{a}$ equals the weight of ab.

In particular, when A is a crystal of a minuscule representation, $\hat{a}$ is determined uniquely by its weight.

Proof. Let B_λ be the component of B containing b. Because of the naturality of the commutor in B (see property $(C1)$ in [17]), $\sigma_{A,B}(ab)$ equals $\sigma_{A,B_\lambda}(ab)$.

Since ab is a highest weight element and the commutor is an isomorphism of crystals, $\sigma_{A,B_\lambda}(ab) = \hat{b}\hat{a}$ is also a highest weight element. It follows that $\hat{b}$ is of highest weight, and therefore equals the highest weight element of B_λ. □

Let us now make promotion and evacuation of $\mathrm{GL}(n)$-alternating tableaux explicit. We regard the adjoint representation as the tensor product $V \otimes V^*$, where V is the vector representation of $\mathrm{GL}(n)$ and V^* is its dual. Both of these are minuscule, so we can apply Theorem 4.11.

As before, we begin by writing down the sequence of cumulative weights in the first line. Following Theorem 4.11, we begin the second line just below the third element of the first with the second element of the word. Below this we begin a third line with the zero weight. We then successively apply the local rule (4.1). Finally, we append the final element of the second line and the weight of the original word to the third line. We call the resulting diagram the *promotion diagram* of an alternating tableau:

$$
\begin{array}{cccccc}
\mu^0{=}\varnothing & \mu^1{=}1 & \mu^2 & \dots\dots & \mu^{2r} & \\
 & & \check{\mu}^1{=}\mu^1 & \dots\dots & \check{\mu}^{2r-1} & \\
 & & \check{\check{\mu}}^0{=}\mu^0 & \dots\dots & \check{\check{\mu}}^{2r-2} & \check{\check{\mu}}^{2r-1}{=}\check{\mu}^{2r-1} \quad \check{\check{\mu}}^{2r}{=}\mu^{2r}
\end{array}
\tag{4.3}
$$

Example 4.14. To illustrate, let us compute the promotion of the GL(3)-alternating tableau $(e_1, e_3), (e_1, e_2), (e_2, e_2), (e_2, e_1), (e_3, e_1)$, where e_i is the i-th unit vector. The Weyl group of $\mathrm{GL}(n)$ is the symmetric group $\mathfrak{S}_n$, so $\mathrm{dom}_{\mathfrak{S}_n}$ is just returning its argument sorted into decreasing order. The first row is the original alternating tableau. For better readability we write $\bar{1}$ in place of -1.

$$
\begin{array}{ccccc|cc|cccccc}
000 & 100 & 10\bar{1} & 20\bar{1} & 2\bar{1}\bar{1} & 20\bar{1} & 2\bar{1}\bar{1} & 20\bar{1} & 10\bar{1} & 100 & 000 & & \\
 & & 100 & 200 & 20\bar{1} & 21\bar{1} & 20\bar{1} & 21\bar{1} & 11\bar{1} & 110 & 100 & & \\
 & & 000 & 100 & 10\bar{1} & 11\bar{1} & 10\bar{1} & 11\bar{1} & 10\bar{1} & 100 & 10\bar{1} & 100 & 000
\end{array}
\tag{4.4}
$$

The six vectors in the rectangle demonstrate that the naive embedding of $\mathrm{GL}(n)$-alternating tableaux into the set of $\mathrm{GL}(n+1)$-alternating tableaux is not compatible with promotion, as already mentioned in Section 3.2: padding the vectors of the original word with zeros, and applying the local rules, we obtain the rectangle

$$
\begin{array}{cc}
200\bar{1} & 20\bar{1}\bar{1} \\
210\bar{1} & 21\bar{1}\bar{1} \\
110\bar{1} & 11\bar{1}\bar{1},
\end{array}
$$

with bottom right vector $11\bar{1}\bar{1}$, rather than $100\bar{1}$ as one might expect.

To obtain the evacuation of an alternating tableau we use the second identity of Lemma 4.2. We start by computing the promotion of the initial alternating tableau as above, except that we do not append anything to the third line. We then repeat this process a total of r times, creating a (roughly) triangular array of weights, which we call the *evacuation diagram* of an alternating tableau. The sequence of cumulative weights of the evacuation can then be read off the vertical line on the right hand side, from bottom to top. An example can be found in Figure 4.1. The symbols $\oplus$, $\ominus$ and $\otimes$ occurring in the figure should be ignored for the moment.

$$
\begin{array}{rccccccccccccccc}
\mathcal{A} = & 000 & 100 & 10\bar{1} & 20\bar{1} & 20\bar{2} & 20\bar{1} & 2\bar{1}\bar{1} & 3\bar{1}\bar{1} & 2\bar{1}\bar{1} & 20\bar{1} & 20\bar{2} & 30\bar{2} & 30\bar{3} & 31\bar{3} & 21\bar{3} \\
 & & & & & & & & & & \ominus & & & & & \\
 & & & 100 & 200 & 20\bar{1} & 200 & 20\bar{1} & 30\bar{1} & 20\bar{1} & 21\bar{1} & 21\bar{2} & 31\bar{2} & 31\bar{3} & 32\bar{3} & 22\bar{3} \\
 & & & 000 & 100 & 10\bar{1} & 100 & 10\bar{1} & 20\bar{1} & 10\bar{1} & 11\bar{1} & 11\bar{2} & 21\bar{2} & 21\bar{3} & 22\bar{3} & 21\bar{3} \\
 & & & & & & \ominus & & & & & & & & & \\
 & & & & & 100 & 110 & 11\bar{1} & 21\bar{1} & 11\bar{1} & 21\bar{1} & 21\bar{2} & 31\bar{2} & 31\bar{3} & 32\bar{3} & 31\bar{3} \\
 & & & & & 000 & 100 & 10\bar{1} & 20\bar{1} & 10\bar{1} & 20\bar{1} & 20\bar{2} & 30\bar{2} & 30\bar{3} & 31\bar{3} & 30\bar{3} \\
 & & & & & & & 100 & 200 & 100 & 200 & 20\bar{1} & 30\bar{1} & 30\bar{2} & 31\bar{2} & 30\bar{2} \\
 & & & & & & & 000 & 100 & 000 & 100 & 10\bar{1} & 20\bar{1} & 20\bar{2} & 21\bar{2} & 20\bar{2} \\
 & & & & & & & & & \otimes & & & & & & \\
 & & & & & & & & & 100 & 200 & 20\bar{1} & 30\bar{1} & 30\bar{2} & 31\bar{2} & 30\bar{2} \\
 & & & & & & & & & 000 & 100 & 10\bar{1} & 20\bar{1} & 20\bar{2} & 21\bar{2} & 20\bar{2} \\
 & & & & & & & & & & & 100 & 200 & 20\bar{1} & 21\bar{1} & 20\bar{1} \\
 & & & & & & & & & & & 000 & 100 & 10\bar{1} & 11\bar{1} & 10\bar{1} \\
 & & & & & & & & & & & & & 100 & 110 & 100 \\
 & & & & & & & & & & & & & & & \oplus \\
 & & & & & & & & & & & & & 000 & 100 & 10\bar{1} \\
 & & & & & & & & & & & & & & & 100 \\
 & & & & & & & & & & & & & & & 000 \\
 & & & & & & & & & & & & & & & \mathrm{ev}\,\mathcal{A} =
\end{array}
$$

FIGURE 4.1: The evacuation of an alternating tableau.

Finally, we would like to point out that for alternating tableaux of empty shape there is a second way to compute the promotion, exploiting the fact that the next-to-last element is forced to be $10\dots0$. Let $w = w_1 \dots w_r$ be the highest weight word corresponding to the alternating tableau. We consider w as an element of $A \otimes A^* \otimes B_\lambda$, where A is the crystal corresponding to V, A^* is the crystal corresponding to V^* and, similar to what was done in the proof of Corollary 4.13, B_λ is the component of

$\bigotimes^{r-1}(A \otimes A^*)$ containing $w_3 \dots w_r$. Then we first compute $\hat{w} = \sigma_{A,A^*\otimes B_\lambda}(w)$, followed by computing $\hat{\hat{w}} = \sigma_{A^*,B_\lambda\otimes A}(\hat{w})$:

$$\begin{array}{ccccccccccccc}
000 & 100 & 10\bar{1} & 20\bar{1} & 2\bar{1}\bar{1} & 20\bar{1} & 2\bar{1}\bar{1} & 20\bar{1} & 10\bar{1} & 100 & 000 & & \\
 & 000 & 00\bar{1} & 10\bar{1} & 1\bar{1}\bar{1} & 10\bar{1} & 1\bar{1}\bar{1} & 10\bar{1} & 00\bar{1} & 000 & 00\bar{1} & 000 & \\
 & & 000 & 100 & 10\bar{1} & 11\bar{1} & 10\bar{1} & 11\bar{1} & 10\bar{1} & 100 & 10\bar{1} & 100 & 000
\end{array} \tag{4.5}$$

Because the initial segment of $\hat{\hat{w}}$ is an element of B_λ and is of highest weight, it must coincide with the initial segment of the promotion of w.

This variant of the local rules for promotion was recently rediscovered, in slightly different form, by Patrias [19]. Note, however, that for an alternating tableau of non-empty shape, this procedure yields a tableau which, in general, is different from the result of promotion.

Chapter 5

Growth diagram bijections

In this chapter we recall Sundaram's bijection (using Roby's description [23] based on Fomin's growth diagrams [6]) between oscillating tableaux and matchings. We also present a new bijection, in the same spirit, between alternating tableaux and partial permutations. In both cases, the action of the cactus group on highest weight words becomes particularly transparent when using Fomin's growth diagrams and local rules for the Robinson-Schensted correspondence.

We give a slightly non-standard presentation with the benefit that these local rules can be regarded as a variation of the classical case of Definition 4.9.

$$c = \begin{tikzcd} \kappa \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] & \nu \arrow[u] \end{tikzcd} \qquad \text{or} \qquad c = \begin{tikzcd} \lambda \arrow[r] & \mu \\ \lambda \arrow[u] \arrow[r] \arrow[ur, phantom, "\times"] & \lambda \arrow[u] \end{tikzcd}$$

$$\mu' = \operatorname{sort}(\kappa' + \nu' - \lambda') \qquad \mu = \lambda + e_1$$

$$\lambda' = \operatorname{sort}(\kappa' + \nu' - \mu') \qquad \lambda = \mu - e_1$$

FIGURE 5.1: Cells of a growth diagram and corresponding local rules.

In general, a growth diagram is a finite collection of cells (as in Figure 5.1, where a prime denotes the conjugate partition and e_1 is the first unit vector), arranged in the form of a Ferrers diagram using the French convention. Thus, for each cell in the diagram all cells below and to the left are also present. The four corners of each cell c are labelled with partitions as indicated.

A difference to Definition 4.9 is that two adjacent partitions (as for example λ and κ in Figure 5.1) either coincide or the one at the head of the arrow is obtained from the

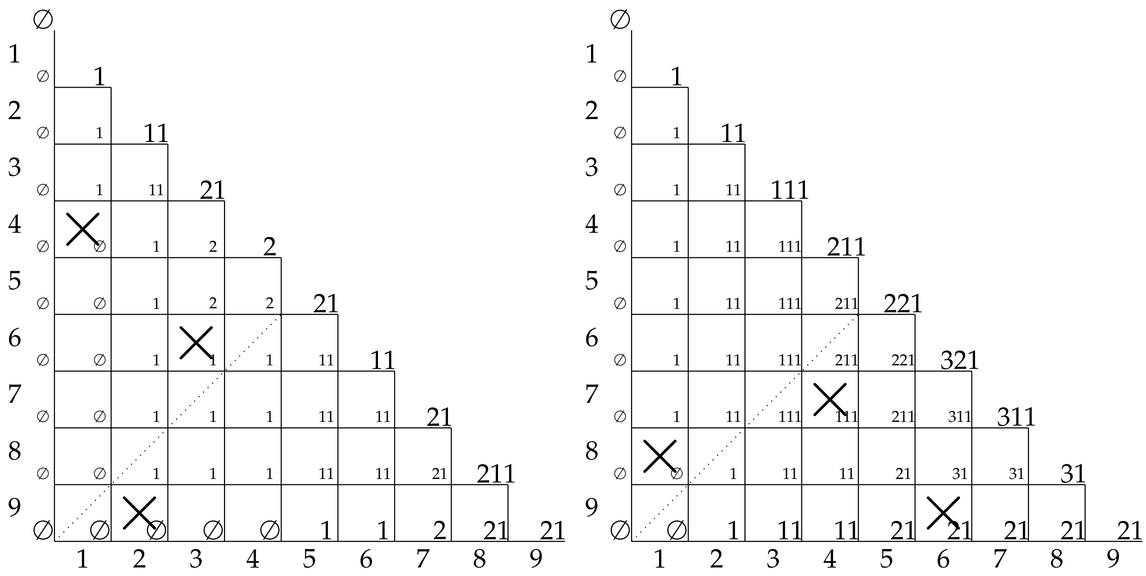


FIGURE 5.2: A pair of growth diagrams $\mathcal{G}(\mathcal{O})$ and $\mathcal{G}(s_{1,9}\,\mathcal{O})$ illustrating Theorem 3.3. The dotted line indicates the axis of reflection for the matchings $\mathcal{M}(\mathcal{O})$ and $\mathcal{M}(s_{1,9}\,\mathcal{O})$.

other by adding a single box. In the latter case, we write $\lambda \lessdot \kappa$ and, if κ is obtained by adding one to the i-th part, $\kappa = \lambda + e_i$.

The two rules in Figure 5.1 determining μ are called *forward rules*, the two rules in Figure 5.1 determining λ are called *backward rules*.

5.1 Roby's description of Sundaram's correspondence

Definition 5.1. *Let $\mathcal{O} = (\mu_0, \mu_1, \ldots, \mu_r)$ be an oscillating tableau. The associated (triangular) growth diagram $\mathcal{G}(\mathcal{O})$ consists of r left-justified rows, with $i-1$ cells in row i for $i \in \{1, \ldots, r\}$, where row 1 is the top row. Label the cells according to the following specification:*

- *R1 Label the corners of the cells along the diagonal from top-left to bottom-right with the partitions in $\mathcal{O}$.*
- *R2 Label the corners of the subdiagonal with the smaller of the two partitions labelling the two adjacent corners on the diagonal.*
- *R3 Use the backward rules to determine which cells contain a cross.*

Let $\mathcal{M}(\mathcal{O})$ be the matching containing a pair $\{i,j\}$ for every cross in column i and row j of the $\mathcal{G}(\mathcal{O})$. Furthermore, let $\mathcal{M}_T(\mathcal{O})$ be the partial standard Young tableau corresponding to the sequence of partitions along the top border of $\mathcal{G}(\mathcal{O})$.

Example 5.2. An example for this procedure, which also illustrates Theorem 3.3, can be found in Figure 5.2. Let w be the highest weight word $1,2,1,-2,2,-1,1,3,-3$. The partitions in the corresponding 3-symplectic oscillating tableau $\mathcal{O}$ label the corners of the diagonal of the first growth diagram. Applying the backward rules, we obtain the matching and the partial standard Young tableau

$$\mathcal{M}(\mathcal{O}) = \big\{\{1,4\},\{2,9\},\{3,6\}\big\} \text{ and } \mathcal{M}_T(\mathcal{O}) = \begin{array}{|c|c|}\hline 5 & 7 \\ \hline 8 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

Using Lemma 4.2 and the local rule in Definition 4.9 one can compute that $\mathrm{s}_{1,9}\, w$ is the highest weight word $1,2,3,1,2,1,-2,-3,-1$ corresponding to the 3-symplectic oscillating tableau labelling the corners of the diagonal of the second growth diagram. Applying the backward rules again, we obtain the matching and the partial standard Young tableau predicted by Theorem 3.3:

$$\mathcal{M}(\mathrm{s}_{1,9}\,\mathcal{O}) = \big\{\{1,8\},\{4,7\},\{6,9\}\big\} \text{ and } \mathcal{M}_T(\mathrm{s}_{1,9}\,\mathcal{O}) = \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

5.2 A new variant for Stembridge's alternating tableaux

Recall that a staircase is a vector with weakly decreasing integer entries. The *positive part* of the staircase is the partition obtained by removing all entries less than or equal to zero. The *negative part* of the staircase is the partition obtained by removing all entries greater than or equal to zero, removing the signs of the remaining entries and reversing the sequence.

Definition 5.3. *Let $\mathcal{A} = (\mu^0, \mu^1, \dots, \mu^{2r})$ be an alternating tableau. The associated growth diagram $\mathcal{G}(\mathcal{A})$ is an $r \times r$ square of cells, obtained as follows:*

- *P1 Label the corners of the cells along the diagonal from north-west to south-east with the staircases in $\mathcal{A}$.*
- *P2 Apply the backward rules on the positive parts of the staircases to determine which cells below the diagonal contain a cross.*
- *P3 Use the backward rules (rotated by 180°) on the negative parts of the staircases to determine which cells above the diagonal contain a cross.*

Let $\mathcal{P}(\mathcal{A})$ be the partial permutation mapping i to j for every cross in column i and row j of $\mathcal{G}(\mathcal{A})$, and let $\big(\mathcal{P}_P(\mathcal{A}), \mathcal{P}_Q(\mathcal{A})\big)$ be the pair of partial standard Young tableau corresponding to the sequence of partitions along the bottom and the right border of $\mathcal{G}(\mathcal{A})$, respectively.

An example for this procedure, which also illustrates Theorem 3.7, can be found in Figure 5.3. We render fixed points as $\otimes$, other crosses below the diagonal as $\oplus$ and crosses above the diagonal as $\ominus$. The reason for doing so is given by Corollary 6.16 in Section 6.3, where we show that the growth diagram of an alternating tableau and its evacuation diagram are very closely related.

Example 5.4. Let w be the $\mathrm{GL}(13)$ highest weight word

$$(e_1, -e_{13}), (e_1, -e_{13}), (e_{13}, -e_{12}), (e_1, -e_1), (e_{12}, -e_{13}), (e_1, -e_{13}), (e_2, -e_1)$$

where e_i is the i-th unit vector. The staircases in the corresponding alternating tableau $\mathcal{A}$ label the corners of the diagonal of the first growth diagram, where we write the negative partitions with bars. Applying the backward rules we obtain the partial permutation and the partial standard Young tableaux

$$\mathcal{P}(\mathcal{A}) = \big\{(3,2), (4,4), (5,1), (6,5)\big\}, \mathcal{P}_P(\mathcal{A}) = \boxed{3\,|\,5\,|\,6}, \text{ and } \mathcal{P}_Q(\mathcal{A}) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 7 & \\ \cline{1-1}\end{array}.$$

On the right hand side of the figure the growth diagram obtained by applying the same procedure to $\mathrm{s}_{1,7}\, w$, which yields

$$\mathcal{P}(\mathcal{A}) = \big\{(2,1), (3,7), (4,4), (5,6)\big\}, \mathcal{P}_P(\mathcal{A}) = \boxed{2\,|\,3\,|\,5}, \text{ and } \mathcal{P}_Q(\mathcal{A}) = \begin{array}{|c|c|}\hline 1 & 7 \\ \hline 6 & \\ \cline{1-1}\end{array}.$$

as predicted by Theorem 3.7.

In the example above, we could have obtained the same sequence of staircases from a $\mathrm{GL}(3)$ highest weight word. As it turns out, applying $\mathrm{s}_{1,7}$ to this word yields the same result, although for $r = 7$ Theorem 3.7 applies only when n is at least 13. The computation of the evacuated alternating tableau is carried out in Figure 4.1.

The $\mathrm{GL}(2)$ highest weight word $(e_1, -e_2), (e_2, -e_2)$ illustrates the necessity of the hypothesis in Theorem 3.7. On the one hand, it is fixed by $\mathrm{s}_{1,2}$, on the other hand, the corresponding filling of the 2×2 square has a single cross at $(2, 1)$, which is incompatible with the conclusion of the theorem.

Similarly, to justify the necessity of the hypothesis in Theorem 3.8, consider the $\mathrm{GL}(3)$-alternating tableau in the first row of Diagram (4.4), which corresponds to the

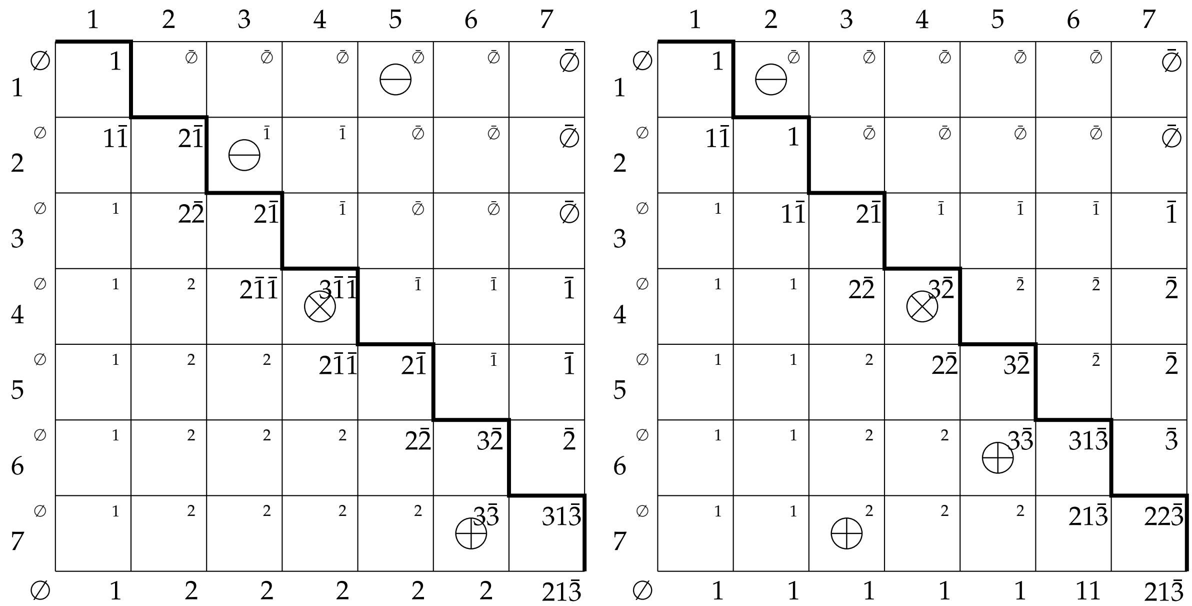


FIGURE 5.3: A pair of growth diagrams $\mathcal{G}(\mathcal{A})$ and $\mathcal{G}(s_{1,7}\,\mathcal{A})$ illustrating Theorem 3.7.

permutation depicted in Figure 3.1. Its promotion, as computed in the last row of Diagram (4.4), corresponds to the permutation 23514, which differs from the rotated permutation.

Chapter 6

Proofs

Our strategy is as follows. We first consider only $\mathrm{GL}(n)$-alternating tableaux of empty shape and length r with $n \geq r$, and show that the bijection $\mathcal{P}$ presented in Section 5.2 intertwines rotation and promotion. To do so, we demonstrate that the middle row of the promotion diagram (4.3) of an alternating tableau $\mathcal{A}$ can be interpreted as corresponding to a single-step rotation of the rows of the growth diagram $\mathcal{G}(\mathcal{A})$. Then, using a very similar argument, we find that the promotion of $\mathcal{A}$ corresponds to a single-step rotation of the columns of the growth diagram just obtained.

To prove the statements concerning evacuation, we show that the permutation $\mathcal{P}(\mathcal{A})$ can actually be read off directly from the evacuation diagram. In particular, this makes the effect of evacuation on $\mathcal{P}(\mathcal{A})$ completely transparent. The effect of the evacuation of an arbitrary alternating tableau $\mathcal{A}$ on the triple $\big(\mathcal{P}(\mathcal{A}), \mathcal{P}_P(\mathcal{A}), \mathcal{P}_Q(\mathcal{A})\big)$ is deduced from the special case of alternating tableaux of empty shape by extending $\mathcal{A}$ to an alternating tableau of empty shape.

In order to determine the exact range of validity of Theorem 3.8 we use a stability phenomenon proved in Section 6.1. The case $n = 2$ is treated completely separately in Section 6.4.

Finally, in Section 6.5, we deduce the statements for oscillating tableaux and the vector representation of the symplectic groups, Theorem 3.3 and 3.4, from the statements for alternating tableaux.

6.1 Stability

In this section we prove a stability phenomenon needed for establishing the exact bounds in Theorem 3.8, but may be interesting in its own right.

Theorem 6.1. *Let $\mathcal{A}$ be a $\mathrm{GL}(n)$-alternating tableau, not necessarily of empty shape, and suppose that each staircase in $\mathcal{A}$ and $\operatorname{pr} \mathcal{A}$ contains at most m nonzero parts.*

Then $\operatorname{pr}\tilde{\mathcal{A}} = \widetilde{\operatorname{pr}\mathcal{A}}$, *where* $\tilde{\mathcal{A}}$ *and* $\widetilde{\operatorname{pr}\mathcal{A}}$ *are the* $\mathrm{GL}(m)$*-alternating tableaux obtained from* $\mathcal{A}$ *and* $\operatorname{pr}\mathcal{A}$ *by removing* $n-m$ *zeros from each staircase.*

Before proceeding to the proof, let us remark that this is not a trivial statement: it may well be that some staircases in the intermediate row $\check{\operatorname{pr}}\,\mathcal{A}$ have more than m nonzero parts.

Proof. It suffices to consider the case $m = n-1$. We show inductively that the statement is true for every square of staircases in Diagram (4.3)

$$\begin{array}{ccccc}
\lambda = \mu^{2i-2} & + & \alpha = \mu^{2i-1} & - & \nu = \mu^{2i} \\
- & & - & & - \\
\beta = \check{\mu}^{2i-3} & + & \varepsilon = \check{\mu}^{2i-2} & - & \gamma = \check{\mu}^{2i-1} \\
+ & & + & & + \\
\kappa = \check{\check{\mu}}^{2i-4} & + & \delta = \check{\check{\mu}}^{2i-3} & - & \mu = \check{\check{\mu}}^{2i-2}
\end{array} \tag{6.1}$$

Thus, we assume that all staircases in the top and bottom line contain at least one zero entry. For such a staircase $\rho \in \mathbb{Z}^n$, let $\tilde{\rho} \in \mathbb{Z}^{n-1}$ be the staircase obtained from ρ by removing a zero entry. If ρ does not contain a zero, it must contain an entry 1 (say, at position i), followed by a negative entry. In this case, $\tilde{\rho} \in \mathbb{Z}^{n-1}$ is obtained from ρ by removing ρ_i and adding 1 to ρ_{i+1}.

With this notation, we have to show the following four equalities:

1. $\tilde{\varepsilon} = \operatorname{dom}_{\mathfrak{S}_{n-1}}(\tilde{\beta} + \tilde{\alpha} - \tilde{\lambda})$,
2. $\tilde{\gamma} = \operatorname{dom}_{\mathfrak{S}_{n-1}}(\tilde{\varepsilon} + \tilde{\nu} - \tilde{\alpha})$,
3. $\tilde{\delta} = \operatorname{dom}_{\mathfrak{S}_{n-1}}(\tilde{\kappa} + \tilde{\varepsilon} - \tilde{\beta})$, and
4. $\tilde{\mu} = \operatorname{dom}_{\mathfrak{S}_{n-1}}(\tilde{\delta} + \tilde{\gamma} - \tilde{\varepsilon})$.

Let us first reduce to the case where at least one of the staircases involved does not contain a zero. Consider a square of staircases

$$\begin{array}{ll}
\beta = \alpha \pm e_i & \delta = \alpha \pm e_i \pm e_j \\
 & \\
\alpha & \gamma = \operatorname{dom}_{\mathfrak{S}_n}(\alpha \pm e_j)
\end{array}$$

where all of α, β, γ and δ contain a zero. We first show that there is an index $k \notin \{i, j\}$ such that $\alpha_k = \beta_k = \delta_k = 0$. Suppose on the contrary that $\alpha_k \neq 0$ for all $k \notin \{i, j\}$. Then, since β contains a zero, we have $i \neq j$. Furthermore, we have

$$\begin{array}{lll} \alpha_i = 0 & \text{or} & \alpha_j = 0 \\ \alpha_i = \mp 1 & \text{or} & \alpha_j = 0 \\ \alpha_i = \mp 1 & \text{or} & \alpha_j = \mp 1 \\ \alpha_i = 0 & \text{or} & \alpha_j = \mp 1 \end{array}$$

because α, β, γ and δ contain a zero, respectively. However, this set of equations admits no solution. Thus, there must be a further zero in α and therefore also in β, γ and δ. From this it follows that $\tilde{\gamma} = \mathrm{dom}_{\mathfrak{S}_{n-1}}(\tilde{\alpha} + \tilde{\delta} - \tilde{\beta})$.

Returning to the square in (6.1) we show that ε contains a zero entry if β or γ do. Suppose on the contrary that ε does not contain a zero entry. Then $\varepsilon = \beta + e_i$, where i is the position of the (only) zero in β. Moreover, we have $\alpha = \beta$, because there is only one way to obtain a zero entry in α by subtracting a unit vector. Thus,

$$\lambda = \mathrm{dom}_{\mathfrak{S}_n}(\beta + \alpha - \varepsilon) = \mathrm{dom}_{\mathfrak{S}_n}(\beta - e_i) = \beta - e_i,$$

which implies that λ does not contain a zero entry, contradicting our assumption. Similarly, if γ contains a zero at position i, we have $\varepsilon = \gamma + e_i$, $\gamma = \delta$ and $\mu = \mathrm{dom}_{\mathfrak{S}_n}(\gamma - e_i)$, a contradiction.

There remain three different cases:

β contains a zero, but γ does not.

We have to show Equations (2) and (4). Let $\alpha = \varepsilon - e_i$ and $\nu = \varepsilon - e_i - e_j$. Then $\gamma = \mathrm{dom}_{\mathfrak{S}_n}(\varepsilon - e_j)$. Since, by the foregoing, ε contains a zero, we have $\varepsilon_j = 0$. Since α also has a zero we have $i \neq j$. Since ν has a zero, $\varepsilon_i = 1$. Because γ has no zero, $\mu = \nu$. Together with the fact that δ has a zero implies that $\delta = \varepsilon - e_i$. The equations can now be checked directly.

β contains no zero, but γ does.

We have to show Equations (1) and (3). Let $\lambda = \beta - e_i$ and $\alpha = \beta - e_i + e_j$. Then $\varepsilon = \mathrm{dom}_{\mathfrak{S}_n}(\beta + e_j)$. Since β has no zero, but, by the foregoing, ε does, we have $\beta_j = \bar{1}$. Since λ has a zero, $\beta_i = 1$, and thus $i \neq j$. Because β has no zero, $\kappa = \lambda$. Again, the equations can now be checked directly.

none of β, ϵ and γ contain a zero.

In this case, $\kappa = \lambda$, $\delta = \alpha$ and $\mu = \nu$. Let $\lambda = \beta - e_i$, $\alpha = \beta - e_i + e_j$. Then $\varepsilon = \mathrm{dom}_{\mathfrak{S}_n}(\beta + e_j)$. Thus $\beta_j \neq \bar{1}$, $\beta_i = 1$, $\beta_{i+1} \leq \bar{1}$ and, because $\alpha \neq \beta$, we have $i \neq j$.

Because α and β are staircases, $\beta + e_j$ has in fact decreasing entries and $\varepsilon = \beta + e_j$. Thus, $\alpha = \varepsilon - e_i$, $\nu = \varepsilon - e_i - e_k$ and $\gamma = \mathrm{dom}_{\mathfrak{S}_n}(\varepsilon - e_k)$. Again, because ε and ν are staircases, $\varepsilon - e_k$ has decreasing entries and $\gamma = \varepsilon - e_k$. Thus, the equations can now be checked directly. $\square$

6.2 Growth diagrams for staircase tableaux

It will be convenient to slightly modify and generalise the definition of $\mathcal{G}(\mathcal{A})$ as follows.

Definition 6.2. *For a pair of partitions $\mu = (\mu_+, \mu_-)$, the partition μ_+ is the positive and the partition μ_- is the negative part. Given an integer n not smaller than the sum of the lengths of the two partitions, $[\mu_+, \mu_-]_n$ is the staircase*

$$(\mu_+^0, \mu_+^1, \dots, 0, \dots, 0, \dots, -\mu_-^1, -\mu_-^0).$$

Using this equivalence, a staircase tableau is a sequence of staircases $\mathcal{A} = (\mu^0, \mu^1, \dots, \mu^r)$ such that μ^i and μ^{i+1} differ by a unit vector for $0 \leq i < r$. When $\mu^0 = \emptyset$, the tableau is straight, otherwise skew. Unless specified otherwise, a staircase tableau is straight.

The extent[1] *$\mathcal{E}(\mu)$ of a staircase $\mu = [\mu_+, \mu_-]_n$ is the number of nonzero entries in μ. Put differently, the extent is the sum of the lengths of the partitions μ_+ and μ_-. The extent of a staircase tableau is the maximal extent of its staircases.*

Definition 6.3. *The growth diagram $\mathcal{G}(\mathcal{A})$ corresponding to a (straight) staircase tableau $\mathcal{A}$ is obtained in analogy to Definition 5.3: label the top left corner with the staircase μ^0. If μ^{i+1} is obtained from μ^i by adding (respectively subtracting) a unit vector, μ^{i+1} labels the corner to the right of (respectively below) the corner labelled μ^i.* All *the remaining corners of $\mathcal{G}(\mathcal{A})$ are then labelled with staircases as follows. The positive parts on the corners to the left and below the path defined by the staircase tableau are obtained by applying the backward rule, whereas the forward rule determines the positive parts on the remaining corners. The negative parts are computed similarly.*

$\mathcal{G}_+$ (respectively $\mathcal{G}_-$) denotes the (classical) growth diagrams obtained by ignoring the negative (respectively positive) parts of the staircases labelling the corners of a growth diagram $\mathcal{G}$.

[1]It would be more logical to use 'height' for the extent of a staircase, and 'length' for the number n. However, Stembridge defines the height of a staircase as the number n. We therefore avoid the words 'length' and 'height' in the context of staircases altogether.

A (partial) filling ϕ is a rectangular array of cells, where every row and every column contains at most one cell with a cross.

Let ϕ be a partial filling having crosses in all rows except $\mathcal{R}$ (counted from the top), and in all columns except $\mathcal{C}$ (counted from the left). Let P and Q be partial standard Young tableaux having entries $\mathcal{R}$ and $\mathcal{C}$ respectively. Then the growth diagram $\mathcal{G}(\phi, P, Q)$ is obtained as follows. The sequence of partitions corresponding to Q (respectively P) determines the positive (respectively negative) parts of the staircases on the bottom (respectively right) border. The remaining positive and negative parts are computed using the forward rule.

If ϕ contains precisely one cross in every row and every column, we abbreviate $\mathcal{G}(\phi, \emptyset, \emptyset)$ to $\mathcal{G}(\phi)$.

Remark 6.4. *The classical growth diagram associated to a (partial) filling ϕ is precisely $\mathcal{G}_+(\phi, Q)$.*

Remark 6.5. *Two horizontally adjacent shapes in $\mathcal{G}_+(\phi, Q)$ differ if and only if there is no cross above in this column. Two horizontally adjacent shapes in $\mathcal{G}_-(\phi, Q)$ differ if and only if there is a cross above in this column.*

Remark 6.6. *Transposing a filling ϕ is equivalent to interchanging $\mathcal{G}_+(\phi)$ and $\mathcal{G}_-(\phi)$.*

Finally, we introduce the operations on fillings we want to relate to promotion.

Definition 6.7. *Let ϕ be a filling of a square grid. The* column rotation $\operatorname{crot}\phi$ *(respectively,* row rotation $\operatorname{rrot}\phi$*) of the filling ϕ is obtained from ϕ by removing the first column (respectively, row) and appending it at the right (respectively, bottom).*

The rotation $\operatorname{rot}\phi$ *of a filling ϕ is* $\operatorname{crot}\operatorname{rrot}\phi$.

6.3 Promotion and evacuation of alternating tableaux

Let us first recall a classical fact concerning the effect of removing the first column of a filling on the growth diagram.

Proposition 6.8. *Consider the classical growth diagrams $\mathcal{G}$ and $\check{\mathcal{G}}$ for the partial fillings ϕ and $\check{\phi}$, where $\check{\phi}$ is obtained from ϕ by deleting its first column. Let Q and $\check{Q}$ be the partial standard Young tableaux corresponding to the sequence of partitions on the top borders of the growth diagrams $\mathcal{G}$ and $\check{\mathcal{G}}$. Then* $\operatorname{jdt} Q = \check{Q}$

The following central result connects the local rule for the symmetric group with column rotation, the operation of moving the first letter of a permutation to the end.

Theorem 6.9. *Let ϕ be a filling of an $r \times r$ square grid having exactly one cross in every row and in every column. Let λ and ν be two adjacent staircases in $\mathcal{G}(\phi)$, λ being to the left or above ν. Finally, let κ and μ be the two corresponding staircases in $\mathcal{G}(\operatorname{crot}\phi)$. Then, provided that $n \geq \max(\mathcal{E}(\kappa), \mathcal{E}(\lambda), \mathcal{E}(\mu), \mathcal{E}(\nu))$, we have $\mu = \operatorname{dom}_{\mathfrak{S}_n}(\kappa + \nu - \lambda)$.*

Conversely, suppose that the staircases in $\mathcal{A} = (1 = \mu^1, \ldots, \mu^{2r} = \emptyset)$ label a sequence of adjacent corners from the corner just to the right of the top left corner to the bottom right corner of $\mathcal{G}(\phi)$, and suppose that the staircases $\check{\mathcal{A}} = (\emptyset = \check{\mu}^0, \ldots, \check{\mu}^{2r} = \emptyset)$ satisfy $\check{\mu}^i = \operatorname{dom}_{\mathfrak{S}_n}(\check{\mu}^{i-1} + \mu^{i+1} - \mu^i)$ for $i \leq 2r - 1$. Then, provided that $n \geq \max(\mathcal{E}(\mathcal{A}), \mathcal{E}(\check{\mathcal{A}}))$, the filling of $\mathcal{G}(\check{\mathcal{A}})$ is $\operatorname{crot}\phi$.

We remark that Proposition 6.8, restricted to permutations, is a special case of this result. More precisely, it is obtained by considering the staircase tableau $(1 = \mu^1, \ldots, \mu^{2r} = \emptyset)$ consisting of the partitions labelling the corners along the top and the right border of a classical growth diagram, with the empty shape in the top left corner removed.

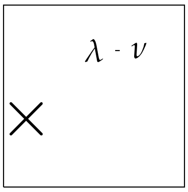
It is not hard to extend the theorem to partial fillings, the statement is completely analogous. Its proof proceeds by extending the partial filling to a permutation. However, it turns out to be more convenient to deduce the statements for staircase tableaux of non-empty shape from the corresponding statements for staircase tableaux of empty shape directly.

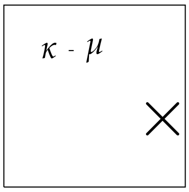
Proof. It is sufficient to prove the first statement, because the filling and the staircases of a growth diagram determine each other uniquely. Let us first determine certain local rules satisfied separately by the positive and negative parts of the staircases κ, λ, μ and ν. A summary of the various cases is displayed in Figure 6.1. In the following, addition and subtraction of integer partitions is defined by interpreting them as vectors in $\mathbb{Z}^n$.

First case, λ left of ν:

Let $Q = (\emptyset = \mu_0, \mu_1, \ldots, \mu_{s-1} = \lambda_+, \mu_s = \nu_+)$ be the partial standard Young tableau corresponding to the sequence of partitions in $\mathcal{G}_+(\phi)$ on the same line as λ and ν, beginning at the left border. Let $\check{Q} = (\emptyset = \check{\mu}_0, \check{\mu}_1, \ldots, \check{\mu}_{s-2} = \kappa_+, \check{\mu}_{s-1} = \mu_+)$ be the corresponding partial standard Young tableau in $\mathcal{G}_+(\operatorname{crot}\phi)$.

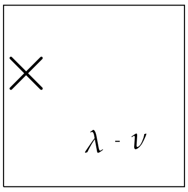
Suppose there is a cross in ϕ in the first column in a row below ν, as in Figure 6.1.a. Then, by Proposition 6.8, $\check{Q} = \operatorname{jdt} Q$, which implies that the partitions $\mu_{s-1}, \mu_s, \check{\mu}_{s-2}$ and $\check{\mu}_{s-1}$ satisfy the local (growth diagram) rule $\check{\mu}_{s-1} = \operatorname{sort}(\check{\mu}_{s-2} + \mu_s - \mu_{s-1})$. Moreover $\kappa_- = \lambda_-$ and $\nu_- = \mu_-$ because the growth for the negative parts of the staircases is from the top right to the bottom left.

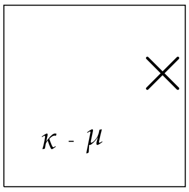
a. ϕ : 

crot ϕ : 

$$\lambda_+ = \kappa_+ + \square$$
$$\mathcal{E}(\kappa_+ + \nu_+ - \lambda_+) \leq \mathcal{E}(\nu_+)$$
$$\mathcal{E}(\kappa_- + \nu_- - \lambda_-) = \mathcal{E}(\nu_-)$$

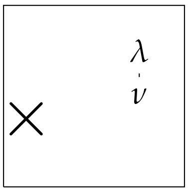
$$\mu_+ = \mathrm{sort}(\kappa_+ + \nu_+ - \lambda_+), \quad \lambda_- = \kappa_-, \quad \nu_- = \mu_-$$

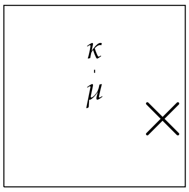
b. ϕ : 

crot ϕ : 

$$\mu_- = \nu_- + \square$$
$$\mathcal{E}(\kappa_- + \nu_- - \mu_-) \leq \mathcal{E}(\kappa_-)$$
$$\mathcal{E}(\kappa_+ + \nu_+ - \mu_+) = \mathcal{E}(\kappa_+)$$

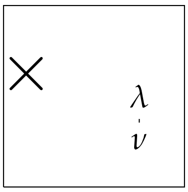
$$\lambda_- = \mathrm{sort}(\kappa_- + \nu_- - \mu_-), \quad \lambda_+ = \kappa_+, \quad \nu_+ = \mu_+$$

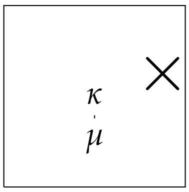
c. ϕ : 

crot ϕ : 

$$\lambda_+ = \kappa_+ + \square$$
$$\mathcal{E}(\kappa_+ + \nu_+ - \lambda_+) \leq \mathcal{E}(\nu_+)$$
$$\mathcal{E}(\kappa_- + \nu_- - \lambda_-) = \mathcal{E}(\nu_-)$$

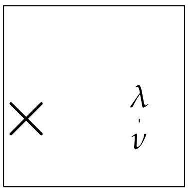
$$\mu_+ = \mathrm{sort}(\kappa_+ + \nu_+ - \lambda_+), \quad \lambda_- = \kappa_-, \quad \nu_- = \mu_-$$

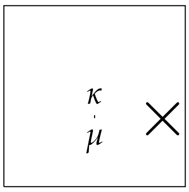
d. ϕ : 

crot ϕ : 

$$\mu_- = \nu_- + \square$$
$$\mathcal{E}(\kappa_- + \nu_- - \mu_-) \leq \mathcal{E}(\kappa_-)$$
$$\mathcal{E}(\kappa_+ + \nu_+ - \mu_+) = \mathcal{E}(\kappa_+)$$

$$\lambda_- = \mathrm{sort}(\kappa_- + \nu_- - \mu_-), \quad \lambda_+ = \kappa_+, \quad \nu_+ = \mu_+$$

e. ϕ : 

crot ϕ : 

$$\lambda'_+ - e_1 = \nu'_+ = \kappa'_+ = \mu'_+, \quad \lambda'_- = \nu'_- = \kappa'_- = \mu'_- - e_1$$

FIGURE 6.1: The cases considered in the proof of Theorem 6.9.

If there is a cross in the first column in a row above ν, as in Figure 6.1.b, we reason in a very similar way. In this case $\kappa_+ = \lambda_+$ and $\nu_+ = \mu_+$. For the negative parts of the staircases we consider the partial standard Young tableaux Q and $\check{Q}$ corresponding to the sequences of partitions beginning at the *right* border of $\mathcal{G}_-(\phi)$ and $\mathcal{G}_-(\operatorname{crot}\phi)$ respectively. We then have $Q = \operatorname{jdt}\check{Q}$ and conclude as before, using the symmetry of the local rule, see Remark 4.10.

Second case, λ above ν:

Depending on the position of the cross in the first column there are three slightly different cases, as illustrated in Figure 6.1.c, d and e.

Recall that the partitions on the right border of a (classical) growth diagram corresponding to the right to left reversal of a filling ψ are obtained by transposing the partitions on the right border of the (classical) growth diagram corresponding to ψ. Consider now the filling ψ below and to the left of λ, and let $\check{\psi}$ be the filling to the left and below κ. Note that the reversal of ψ is obtained from the reversal of $\check{\psi}$ by appending the first column of ψ to the right. We thus obtain that the transposes of the positive parts of the staircases κ, μ, λ and ν satisfy the local (growth diagram) rule.

The relation between the negative parts of the staircases κ, μ, λ and ν is obtained in a very similar way by considering the fillings above and to the right of ν and μ.

We now show $\mu = \operatorname{dom}_{\mathfrak{S}_n}(\kappa + \nu - \lambda)$, provided $n \geq \max(\mathcal{E}(\kappa), \mathcal{E}(\lambda), \mathcal{E}(\mu), \mathcal{E}(\nu))$. To do so, we extend the notion of extent to arbitrary vectors with non-negative integer entries: for such a vector $\alpha \in \mathbb{Z}^n_{\geq 0}$, the extent $\mathcal{E}(\alpha)$ is n minus the number of trailing zeros.

The case illustrated in Figure 6.1.e follows by direct inspection. We thus only consider the remaining four cases, in which $\kappa_+ + \nu_+ - \lambda_+$ and $\kappa_- + \nu_- - \lambda_-$ have all entries non-negative, because μ_+ and μ_- are obtained by sorting these vectors. Similarly, also $\kappa_+ + \nu_+ - \mu_+$ and $\kappa_- + \nu_- - \mu_-$ have all entries non-negative, because λ_+ and λ_- are obtained by sorting these vectors, by the symmetry of the local rule, see Remark 4.10.

Suppose first that $n \geq \max(\mathcal{E}(\kappa), \mathcal{E}(\lambda), \mathcal{E}(\nu))$. Then

$$\operatorname{dom}_{\mathfrak{S}_n}(\kappa + \nu - \lambda) = \operatorname{dom}_{\mathfrak{S}_n}([\kappa_+, \kappa_-]_n + [\nu_+, \nu_-]_n - [\lambda_+, \lambda_-]_n) = \operatorname{dom}_{\mathfrak{S}_n}(\alpha_+ + \alpha_-),$$

where $\alpha_+ = [\kappa_+, \emptyset]_n + [\nu_+, \emptyset]_n - [\lambda_+, \emptyset]_n$ and $\alpha_- = [\emptyset, \kappa_-]_n + [\emptyset, \nu_-]_n - [\emptyset, \lambda_-]_n$. It remains to show that

$$\mathcal{E}(\alpha_+) + \mathcal{E}(\alpha_-) = \mathcal{E}\left(\kappa_+ + \nu_+ - \lambda_+\right) + \mathcal{E}\left(\kappa_- + \nu_- - \lambda_-\right) \leq n,$$

because then

$$\begin{aligned}\mathrm{dom}_{\mathfrak{S}_n}(\alpha_+ + \alpha_-) &= [\mathrm{sort}(\alpha_+), \mathrm{sort}(\alpha_-)]_n \\ &= [\mathrm{sort}(\kappa_+ + \nu_+ - \lambda_+), \mathrm{sort}(\kappa_- + \nu_- - \lambda_-)]_n = [\mu_+, \mu_-]_n = \mu.\end{aligned}$$

Similarly, suppose that $n \geq \max(\mathcal{E}(\kappa), \mathcal{E}(\mu), \mathcal{E}(\nu))$. In this case, reasoning as above, we have to show that $\mathcal{E}\left(\kappa_+ + \nu_+ - \mu_+\right) + \mathcal{E}\left(\kappa_- + \nu_- - \mu_-\right) \leq n$.

The first inequality is verified by inspection of Figure 6.1.a and c, whereas the second concerns Figure 6.1.b and d. Here we write, for example, $\lambda_+ = \kappa_+ + \square$ to indicate that the partition λ_+ is obtained from the partition κ_+ by adding a single cell, which implies the inequality for the extent.

$\square$

Definition 6.10. *Let $\mathcal{A} = (\emptyset = \mu^0, \mu^1, \ldots, \mu^{2r-1}, \mu^{2r} = \mu)$ be an alternating tableau. Then $\check{\mathrm{pr}}\,\mathcal{A} = (\emptyset = \check{\mu}^0, \check{\mu}^1, \ldots, \check{\mu}^{2r-1}, \check{\mu}^{2r} = \mu)$ is the staircase tableau obtained from $\mathcal{A}$ by setting $\check{\mu}^1 = \mu^1 = 1$, and then applying the local rule* (4.1) *successively to μ^i, μ^{i+1}, and $\check{\mu}^{i-1}$ to obtain $\check{\mu}^i$ for $i \leq 2r-1$. Additionally, we set $\check{\mu}^{2r} = \mu$.*

In other words, $\check{\mathrm{pr}}\,\mathcal{A}$ can be read off from the diagram for promotion as illustrated in Diagram (4.3) beginning with the empty shape in the lower left corner, then following the second row, and terminating with the shape μ in the upper right corner.

Lemma 6.11. *Let $\mathcal{A}$ be a staircase tableau of empty shape and length r. Then the extent of $\mathcal{A}$ is at most r.*

Restricting to alternating tableaux, there is a single alternating tableau $\mathcal{A}_0$ of empty shape, length r and extent r. The filling ϕ_0 of its growth diagram $\mathcal{G}(\mathcal{A}_0)$ is invariant under rotation: $\mathrm{rot}\,\phi_0 = \phi_0$.

Restricting further to alternating tableaux of even length, the only tableau $\mathcal{A}$ such that $\check{\mathrm{pr}}\mathcal{A}$ has extent r is $\mathcal{A}_0$.

Proof. For $r = 2s+1$ odd, the only $r \times r$ filling with associated alternating tableau of extent r corresponds to the permutation $s+1, s+2, \ldots, r, 1, \ldots, s$, which is invariant under rotation.

Suppose that $r = 2s$ is even. The only $r \times r$ filling ϕ with associated alternating tableau of extent r corresponds to the permutation $s+1, s+2, \ldots, r, 1, \ldots, s$, which again is invariant under rotation.

Similarly, a staircase tableau $\check{\mathrm{pr}}\mathcal{A}$ with extent r must have filling corresponding to the permutation $s, s+1, \ldots, r, 1, \ldots, s-1$, which is the filling $\mathrm{rrot}\,\phi$, and thus $\mathcal{A} = \mathcal{A}_0$. $\square$

The first statement of Theorem 3.8, with the exception of the case $n = 2$ and the case $n = r - 1$, is a direct consequence of the following result. The special case of $\mathrm{GL}(2)$-alternating tableaux and $\mathrm{GL}(r-1)$-alternating tableaux will be considered in Section 6.4 below.

Theorem 6.12. *Let $\mathcal{A}$ be a $\mathrm{GL}(n)$-alternating tableau of length r and empty shape. Let ϕ be the filling of the growth diagram $\mathcal{G}(\mathcal{A})$. Let $\check{\phi}$ and $\check{\check{\phi}}$ be the fillings of the growth diagrams $\mathcal{G}(\check{\mathrm{pr}}\,\mathcal{A})$ respectively $\mathcal{G}(\mathrm{pr}\,\mathcal{A})$.*

Then, for $n \geq r$, $\mathrm{rrot}\,\phi = \check{\phi}$ and $\mathrm{crot}\,\check{\phi} = \check{\check{\phi}}$.

Proof. Let

$$\check{\mathrm{pr}}\,\mathcal{A} = (\emptyset = \check{\mu}^0, \check{\mu}^1, \ldots, \check{\mu}^{2r-1} = 1, \check{\mu}^{2r} = \emptyset)$$

and let

$$\mathrm{pr}\,\mathcal{A} = (\emptyset = \check{\check{\mu}}^0, \check{\check{\mu}}^1, \ldots, \check{\check{\mu}}^{2r-1} = 1, \check{\check{\mu}}^{2r} = \emptyset).$$

Furthermore, let

$$\widetilde{\mathcal{A}} = (\emptyset = \widetilde{\mu}^0, \widetilde{\mu}^1, \ldots, \widetilde{\mu}^{2r-1} = \bar{1}, \widetilde{\mu}^{2r} = \emptyset)$$

be the staircase tableau obtained by setting $\widetilde{\mu}^0 = \emptyset$ and then successively applying the local rule (4.1) to $\check{\mu}^i$, $\check{\mu}^{i+1}$, and $\widetilde{\mu}^{i-1}$ to obtain $\widetilde{\mu}^i$ for $i \leq 2r - 1$. Because of Lemma 6.11 and the assumption $n \geq r$, Theorem 6.9 is applicable and implies that the filling $\widetilde{\phi}$ of the growth diagram $\mathcal{G}(\widetilde{\mathcal{A}})$ is $\mathrm{crot}\,\check{\phi}$.

All staircases in $\widetilde{\mathcal{A}}$ except $\widetilde{\mu}^{2r-1}$ coincide with those of $\mathrm{pr}\,\mathcal{A}$. Because $\check{\mu}^{2r-1} = 1$, $\check{\mu}^{2r} = \emptyset$ and $\widetilde{\mu}^{2r-2}$ is either $\emptyset$ or $1\bar{1}$, we have $\widetilde{\mu}^{2r-1} = \bar{1}$. However, since $\widetilde{\phi}$ and $\check{\check{\phi}}$ correspond to permutations and the first $r - 1$ columns of these fillings are the same, we conclude that $\widetilde{\phi}$ equals $\check{\check{\phi}}$.

Because of the symmetry of the local rules pointed out in Remark 4.10 and because $\check{\mu}^{2r-1} = 1$ we can apply the same reasoning replacing $\check{\mathrm{pr}}\,\mathcal{A}$ and $\mathrm{pr}\,\mathcal{A}$ with the reversal of $\check{\mathrm{pr}}\,\mathcal{A}$ and the reversal of $\mathcal{A}$. Clearly, the filling corresponding to the reversal of a tableaux is obtained by flipping the original filling over the diagonal from the bottom-left to the top-right. In the process, column rotation is replaced by row rotation, which implies that $\mathrm{rrot}(\phi) = \check{\phi}$. □

Corollary 6.13. *In the setting of Theorem 6.12 if n is odd it is sufficient to require $n \geq r - 1$.*

Proof. Let $\mathcal{A}$ be a $\mathrm{GL}(n)$-alternating tableau of length $r = 2s$, with $n \geq r$. Let ϕ be the filling of $\mathcal{G}(\mathcal{A})$. Then, combining Lemma 6.11 and Theorem 6.12 we obtain

$$\mathcal{E}(\mathcal{A}) = r \Leftrightarrow \mathcal{E}(\check{\mathrm{pr}}\mathcal{A}) = r \Leftrightarrow \mathcal{E}(\mathrm{pr}\,\mathcal{A}) = r.$$

By contraposition, $\mathcal{E}(\mathcal{A}) < r \Leftrightarrow \mathcal{E}(\check{\mathrm{pr}}\mathcal{A}) < r \Leftrightarrow \mathcal{E}(\mathrm{pr}\,\mathcal{A}) < r$. Thus, the claim follows using the proof of Theorem 6.12, taking into account that Theorem 6.9 is now applicable even with $n \geq r-1$. $\square$

Finally, we can conclude one part of Theorem 3.8. Note that the case of odd n is also covered by the previous corollary.

Corollary 6.14. *Let $\mathcal{A}$ be a $\mathrm{GL}(n)$-alternating tableau of length r and empty shape. Then, for $n \geq r-1$, $\mathrm{rot}\,\mathcal{P}(\mathcal{A}) = \mathcal{P}(\mathrm{pr}\,\mathcal{A})$.*

Proof. This is a consequence of Lemma 6.11 and Theorem 6.1. $\square$

We now introduce a different way to obtain the filling of $\mathcal{G}(\mathcal{A})$. Moreover, this construction sheds some additional light on the relationship between the local rule (4.1) and those in Figure 5.1. Consider the evacuation diagram for obtaining the evacuation as illustrated in Figure 4.1. We construct a filling of the cells surrounded by three or four staircases using the symbols $\ominus$, $\oplus$ and $\otimes$ as follows:

$$\begin{tikzcd} (\alpha,\sigma) \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{\ominus}"] & (\alpha,\tau) \arrow[d] \\ (\alpha,\tau) \arrow[r] & (\beta,\tau) \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} (\beta,\tau) \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{\oplus}"] & (\alpha,\tau) \arrow[d] \\ (\alpha,\tau) \arrow[r] & (\alpha,\sigma) \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} 1 \arrow[r] \arrow[dr, phantom, "\boxed{\otimes}"] & 0 \arrow[d] \\ & 1 \end{tikzcd},$$

where β (respectively σ) is obtained from α (respectively τ) by adding a cell to the first column.

The following lemma is the main building block in establishing the connection between the filling of $\mathcal{G}(\mathcal{A})$ and the decorated evacuation diagram.

Lemma 6.15. *Let $\mathcal{A} = (\emptyset = \mu^0, \dots, \mu^{2r} = \emptyset)$ be an alternating tableau of empty shape. Let $\check{\mathrm{pr}}\,\mathcal{A} = (\emptyset = \check{\mu}^0, \check{\mu}^1, \dots, \check{\mu}^{2r-1}, \check{\mu}^{2r} = \emptyset)$ be as in Definition 6.10 and let $\mathrm{pr}\,\mathcal{A} = (\emptyset = \check{\check{\mu}}^0, \dots, \check{\check{\mu}}^{2r} = \emptyset)$ be the promotion of $\mathcal{A}$. Suppose that the filling of $\mathcal{G}(\mathcal{A})$ has a cross in column $\ell > 1$ of the first row and in row $k > 1$ of the first column. Then, for even $n \geq r$ and for odd $n \geq r-1$, we have*

- $\mu^j_+ = \check{\mu}^{j-1}_+$ *for* $2 \leq j \leq 2\ell - 2$,

- $\mu_-^j = \check{\mu}_-^{j-1}$ *for* $j > 2\ell - 2$,
- $\mu_+^{2\ell-2} = \mu_+^{2\ell-1} = \check{\mu}_+^{2\ell-3}$, *and* $\check{\mu}_+^{2\ell-2}$ *is obtained from these by adding a cell to the first column. The cell labelled with these four staircases contains a* $\ominus$.

Similarly,

- $\check{\mu}_-^j = \check{\check{\mu}}_-^{j-1}$ *for* $1 \leq j \leq 2k - 2$
- $\check{\mu}_+^j = \check{\check{\mu}}_+^{j-1}$ *for* $j > 2k - 2$
- $\check{\mu}_+^{2k-1} = \check{\check{\mu}}_+^{2k-2} = \check{\check{\mu}}_+^{2k-3}$, *and* $\check{\mu}_+^{2k-2}$ *is obtained from these by adding a cell to the first column. The cell labelled with these four staircases contains a* $\oplus$.

Finally suppose that there is a cross in the cell in the top left cell, that is $k = \ell = 1$. *Then*

- $\mu^1 = 1, \mu^2 = \emptyset$ *and* $\check{\mu}^1 = 1$. *The cell labelled with these four staircases contains a* $\otimes$.
- $\mu^j = \check{\mu}^{j-1} - e_1 = \check{\check{\mu}}^{j-2}$ *for all* $2 \leq j \leq 2r$.

Proof. Consider a square of four adjacent staircases in the diagram for computing the promotion of an an alternating tableau below:

$$
\begin{array}{ccccccccccc}
 & \mu^2 & \dots & \mu^{2\ell-2} & \mu^{2\ell-1} & \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots\dots} & \mu^{2r} = \emptyset \\
 & & & \multicolumn{2}{c}{\ominus} & & & & & \\
 & \check{\mu}^1 & \dots & \check{\mu}^{2\ell-3} & \check{\mu}^{2\ell-2} & \dots & \check{\mu}^{2k-2} & \check{\mu}^{2k-1} & \dots & \check{\mu}^{2r-1} \\
 & & & & & & \multicolumn{2}{c}{\oplus} & & \\
\emptyset = \check{\check{\mu}}^0 & \multicolumn{5}{c}{\dots\dots\dots\dots\dots\dots\dots} & \check{\check{\mu}}^{2k-3} & \check{\check{\mu}}^{2k-2} & \dots & \check{\check{\mu}}^{2r-2}
\end{array}
\tag{6.2}
$$

By definition, these satisfy the local rule, as required by Theorem 3.4. By Corollary 6.13, Theorem 6.9 is applicable with the given bounds for n. The equalities for the staircases in the second and third row are precisely the equalities listed below the illustrations in Figure 6.1: cases a and c describe the situation to the left of $\oplus$, case e describes the situation at $\oplus$ and cases b and d describe the situation to the right of $\oplus$.

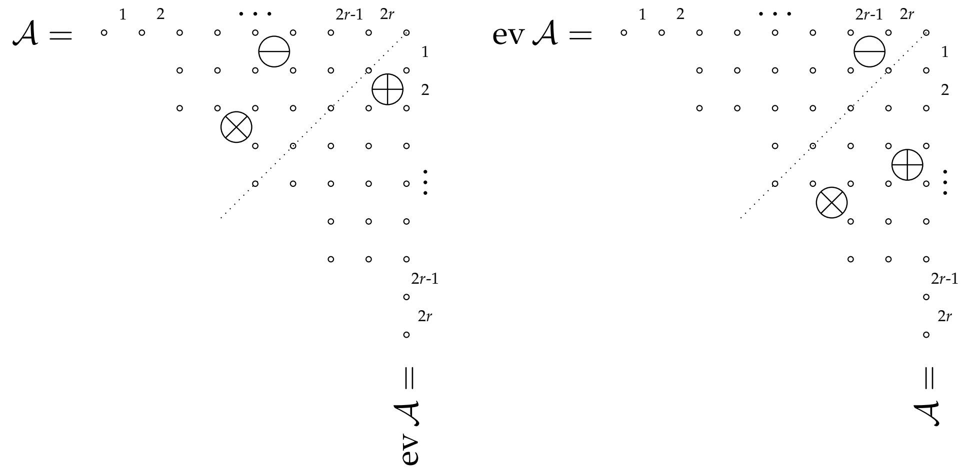
The equalities for the staircases in the first and second row can be obtained as in the last paragraph of the proof of Theorem 6.12. □

By successively applying this Lemma 6.15 we obtain the following result for the evacuation diagram:

Corollary 6.16. *Let $\mathcal{A}$ be a* $\mathrm{GL}(n)$*-alternating tableau of empty shape and length r with corresponding growth diagram $\mathcal{G}(\mathcal{A})$ and filling ϕ. Suppose that $n \geq r$ if n is even and $n \geq r-1$ if n is odd. Consider the evacuation diagram with filling obtained as above. A $\ominus$ appears only in odd columns and odd rows, a $\oplus$ appears only in even columns and even rows and a $\otimes$ appears only in even columns and odd rows. Moreover (i,j) is the position of a cell with a cross in ϕ if and only if one of the following cases holds.*

- *$i < j$ and there is a $\ominus$ in row $2i-1$ and column $2j-1$ in the evacuation diagram.*
- *$i > j$ and there is a $\oplus$ in row $2j$ and column $2i$.*
- *There is a $\otimes$ in row $2i-1$ and column $2j$. Then we also obtain $i = j$.*

By the symmetry of the local rules the evacuation diagram for $\mathrm{ev}\,\mathcal{A}$ is obtained from the evacuation diagram for $\mathcal{A}$ by mirroring it along the diagonal and interchanging $\oplus$ and $\ominus$. The cell (i,j) is interchanged with the cell $(2r+1-j, 2r+1-i)$.



This yields the part of Theorem 3.8 concerning evacuation:

Theorem 6.17. *Let $\mathcal{A} = (\emptyset = \mu^0, \mu^1, \ldots, \mu^{2r-1}, \mu^{2r} = \emptyset)$ be an alternating tableau. Suppose that $n \geq r$ if n is even and $n \geq r-1$ if n is odd. Let ϕ be the filling of the growth diagram $\mathcal{G}(\mathcal{A})$. Then the filling of $\mathcal{G}(\mathrm{ev}\,\mathcal{A})$ is obtained by rotating ϕ by $180°$.*

Proof. Let $\phi_{\mathcal{A}} = \phi$ respectively $\phi_{\mathrm{ev}\,\mathcal{A}}$ be the fillings of the growth diagrams $\mathcal{G}(\mathcal{A})$ respectively $\mathcal{G}(\mathrm{ev}\,\mathcal{A})$. Let (i,j) be the position of cell with a cross in the filling $\phi_{\mathcal{A}}$. Then, according to Corollary 6.16:

1. if $i < j$, there is a $\ominus$ in the evacuation diagram of $\mathcal{A}$ in $(2i-1, 2j-1)$. Thus there is a $\oplus$ in the evacuation diagram of $\mathrm{ev}\,\mathcal{A}$ in $(2r-2j+2, 2r-2i+2)$ and therefore there is a cross in $(r+1-i, r+1-j)$ in $\phi_{\mathrm{ev}\,\mathcal{A}}$.

2. if $j < i$, there is a $\oplus$ in the evacuation diagram of $\mathcal{A}$ in $(2j, 2i)$. Thus there is a $\ominus$ in the evacuation diagram of $\operatorname{ev}\mathcal{A}$ in $(2r+1-2i, 2r+1-2j)$ and therefore there is a cross in $(r+1-i, r+1-j)$ in $\phi_{\operatorname{ev}\mathcal{A}}$.

3. if $i = j$, then there is a $\otimes$ in the evacuation diagram of $\mathcal{A}$ in $(2i-1, 2i)$. Thus there is a $\otimes$ in the evacuation diagram of $\operatorname{ev}\mathcal{A}$ in $(2r+1-2i, 2r+2-2i)$ and therefore there is a cross in $(r+1-i, r+1-j)$ in $\phi_{\operatorname{ev}\mathcal{A}}$.

□

Proposition 6.18. *Consider the classical growth diagrams $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ for the partial fillings ϕ and $\operatorname{rc}\phi$, where $\operatorname{rc}\phi$ is obtained by rotating ϕ by $180°$. Let Q and $\widetilde{Q}$ (respectively P and $\widetilde{P}$) be the partial standard Young tableaux corresponding to the sequence of partitions on the top borders (respectively right borders) of the growth diagrams $\mathcal{G}$ and $\widetilde{\mathcal{G}}$. Then $\widetilde{Q} = \operatorname{ev} Q$ and $\widetilde{P} = \operatorname{ev} P$.*

We are now in the position to prove Theorem 3.7, which we reformulate as follows:

Theorem 6.19. *Let $\mathcal{A} = (\emptyset = \mu^0, \mu^1, \ldots, \mu^{2r-1}, \mu^{2r} = \mu)$ be an alternating tableau of length $r \leq \lfloor \frac{n+1}{2} \rfloor$. Let ϕ be the filling of the growth diagram $\mathcal{G}(\mathcal{A})$.*

Then the sequence of partitions on the bottom (respectively right) border of $\mathcal{G}_+(\operatorname{ev}\mathcal{A})$ (respectively $\mathcal{G}_-(\operatorname{ev}\mathcal{A})$) is obtained by evacuating the sequence of partitions on the bottom (respectively right) border of $\mathcal{G}_+(\mathcal{A})$ (respectively $\mathcal{G}_-(\mathcal{A})$).

Moreover, the filling of $\mathcal{G}(\operatorname{ev}\mathcal{A})$ is obtained by rotating ϕ by $180°$.

Proof. We begin by extending $\mathcal{A}$ to an alternating tableau of empty shape $\tilde{\mathcal{A}} = (\emptyset = \tilde{\mu}^0, \ldots, \tilde{\mu}^{2(r+r)} = \emptyset)$, such that $\tilde{\mu}^i = \mu^i$ for $i \leq r$, by appending the reversal of $\mathcal{A}$. Let $\tilde{\phi}$ be the filling of $\mathcal{G}(\tilde{\mathcal{A}})$, which we divide into four parts, as illustrated in the left-most diagram below. Filling A is the filling corresponding to $\mathcal{A}$, filling B is the part below and to the left of μ^{2r}, filling C is the part above and to the right of μ^{2r} and filling D is the part below and to the right of μ^{2r}.

By the symmetry of the local rules and the evacuation diagram as illustrated in Figure 4.1 we see that $\operatorname{ev}\mathcal{A}$ coincides with the first $2r+1$ staircases of $\operatorname{pr}^{(r)}(\operatorname{ev}\tilde{\mathcal{A}})$, where $\operatorname{pr}^{(r)}$ denotes $\underbrace{\operatorname{pr}\circ\operatorname{pr}\circ\cdots\circ\operatorname{pr}}_{r \text{ times}}$.

Let Q be the sequence of partitions on the bottom border of $\mathcal{G}_+(\mathcal{A})$. This sequence is also the sequence of partitions on the top border of the classical growth diagram with filling B.

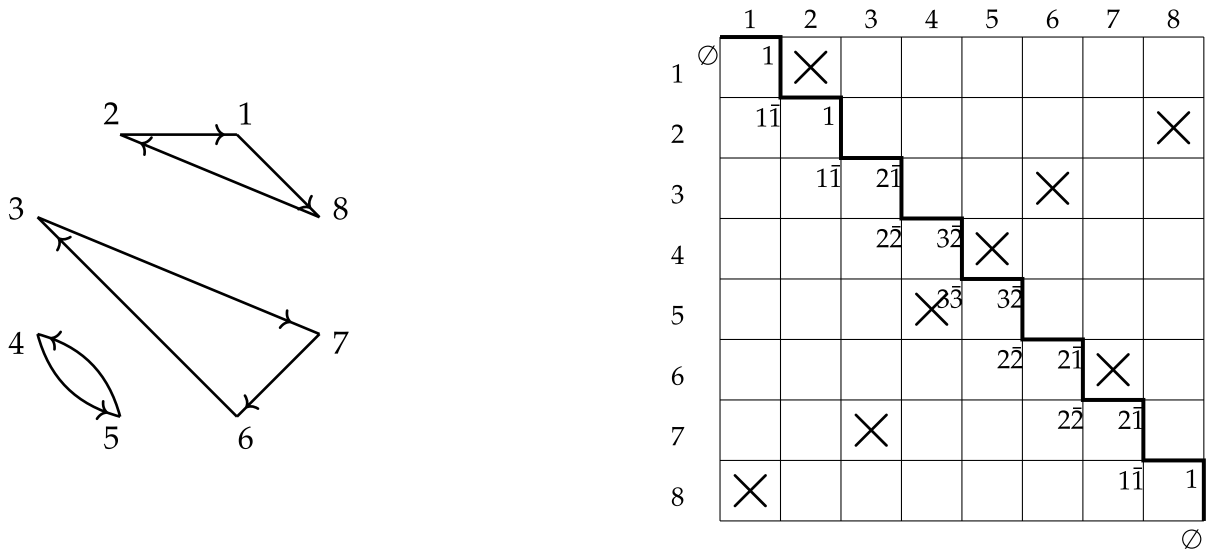
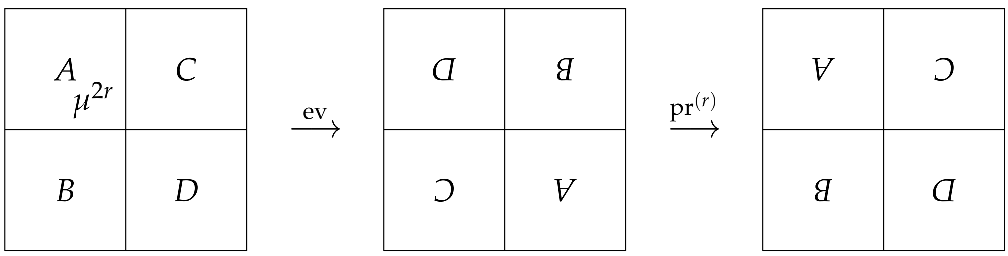


FIGURE 6.2: A noncrossing set partition corresponding to a GL(2)-alternating tableau.

The inequality $r \leq \lfloor \frac{n+1}{2} \rfloor$ implies that $n \geq 2r$ if n is even and $n \geq 2r - 1$ if n is odd. Applying Theorem 6.17 and Theorem 6.12 we obtain the following picture:



Thus the sequence of partitions on the bottom border of $\mathcal{G}_+(\operatorname{ev}\mathcal{A})$ is the same as the sequence of partitions on the top border of the regular growth diagram with filling rc B. By Proposition 6.18 we obtain the statement for the sequence of partitions on the bottom border.

The result for the right border follows using the same argument, replacing the filling A with the filling C. $\square$

6.4 GL(2)-alternating tableaux

To finish the proof of Theorem 3.8, it remains to consider the case $n = 2$.

Lemma 6.20. *The map $\mathcal{P}$ restricts to a bijection between* $\mathrm{GL}(n)$*-alternating tableaux of empty shape and length r, such that every staircase has at most two nonzero parts, and noncrossing set partitions on $\{1,\dots,r\}$.*

Proof. For simplicity, suppose that $\mathcal{A}$ is a $\mathrm{GL}(2)$-alternating tableau. Let π be the permutation corresponding to the filling associated with $\mathcal{A}$. We show that, when drawn as a chord diagram as in Figure 6.2, it is obtained from a noncrossing set partition by orienting the arcs delimiting the blocks clockwise, when the corners of the polygon are labelled counterclockwise.

We say that two arcs (i,π_i) and (j,π_j) in the chord diagram, with $i<k$, cross, if and only if the indices involved satisfy one of the following two inequalities:

$$i < j \leq \pi_i < \pi_j \qquad \text{or} \qquad \pi_i < \pi_j < i < j.$$

Let us remark that this is precisely Corteel's [3] notion of crossing in permutations.

It follows by direct inspection that the chord diagram corresponds to a noncrossing partition in the sense above if and only if no two arcs cross.

Moreover, a crossing of the first kind is the same as a pair of crosses in the rectangle below and to the left of the cell in row and column j of $\mathcal{G}(\mathcal{A})$, such that one cross is above and to the left of the other. Similarly, a crossing of the second kind is the same as a pair of crosses in the rectangle above and to the right of the cell in row and column i of $\mathcal{G}(\mathcal{A})$, such that one cross is above and to the left of the other.

By construction, a $\mathrm{GL}(2)$-alternating tableau cannot contain a vector with both entries strictly positive or both entries strictly negative. Thus, such pairs of crosses may not occur. $\square$

We can now prove another part of Theorem 3.8.

Theorem 6.21. *Let $n \leq 2$ and let $\mathcal{A}$ be a* $\mathrm{GL}(n)$*-alternating tableau of empty shape. Then* $\operatorname{rot}\mathcal{P}(\mathcal{A}) = \mathcal{P}(\operatorname{pr}\mathcal{A})$.

Proof. Let r be the length of $\mathcal{A}$ and let $\hat{\mathcal{A}}$ be the $\mathrm{GL}(r)$-alternating tableau obtained from $\mathcal{A}$ by inserting $r-n$ zeros into each staircase. Then, by Theorem 6.12, $\mathcal{P}(\operatorname{pr}\hat{\mathcal{A}}) = \operatorname{rot}\mathcal{P}(\hat{\mathcal{A}})$. By Lemma 6.20, the staircases in the alternating tableau corresponding to $\operatorname{rot}\mathcal{P}(\hat{\mathcal{A}})$ have at most two nonzero parts. Thus, the claim follows from Theorem 6.1. $\square$

To finish the proof of Theorem 3.8, we show that the evacuation of a $\mathrm{GL}(2)$-alternating tableaux of empty shape is just its reversal.

Theorem 6.22. *Let $\mathcal{A}$ be a* $\mathrm{GL}(2)$*-alternating tableau of empty shape. Then* $\operatorname{ev}\mathcal{A}$ *is the reversal of $\mathcal{A}$.*

Proof. Let $\mathcal{A} = (\emptyset = \mu^0, \dots, \mu^{2r} = \emptyset)$ and let $\operatorname{ev}\mathcal{A} = \mathcal{A} = (\emptyset = \tilde{\mu}^0, \dots, \tilde{\mu}^{2r} = \emptyset)$. Note that $\tilde{\mu}^{2i}$ is the $2i$-th (counting from zero) staircase in $\mathrm{pr}^{(r-i)}\mathcal{A}$. Thus, its negative part is the same as the negation of the positive part of $\mu^{2(r-i)}$, because the fillings in the respective regions of the corresponding growth diagrams coincide. Because the negative part and the positive part of the even labelled staircases of a $\mathrm{GL}(2)$-alternating tableau are equal, we conclude that $\mu^{2(r-i)} = \tilde{\mu}^{2i}$.

It remains to show that $\mu^{2(r-i)-1} = \tilde{\mu}^{2i+1}$. If $\tilde{\mu}^{2i} \neq \tilde{\mu}^{2(i+1)}$, the staircase $\tilde{\mu}^{2i+1}$ is uniquely determined. Otherwise, if $\tilde{\mu}^{2i} = \tilde{\mu}^{2(i+1)}$, it is obtained from $\tilde{\mu}^{2i}$ by adding the unit vector e_1 if and only if i is a fixed point of $\mathcal{P}(\tilde{\mathcal{A}})$. Equivalently, this is the case if and only if $r+1-i$ is a fixed point of $\mathcal{P}(\mathcal{A})$, as can be seen by inspecting the evacuation diagram. □

6.5 Promotion and evacuation of oscillating tableaux

We now deduce Theorem 3.3 and Theorem 3.4 from the results in the preceding section, by demonstrating that oscillating tableaux can be regarded as special alternating tableaux.

For two partitions λ, μ we define

$$\lambda \vee \mu := \max(\lambda, \mu)$$
$$\lambda \wedge \mu := \min(\lambda, \mu)$$

where max and min are defined componentwise.

Consider an oscillating tableau $\mathcal{O} = (\mu^0, \mu^1, \dots, \mu^r)$. Then

$$\mathcal{A}_{\mathcal{O}} = [\mu^0, \mu^0], [\mu^0 \vee \mu^1, \mu^0 \wedge \mu^1], [\mu^1, \mu^1], \dots, [\mu^{r-1} \vee \mu^r, \mu^{r-1} \wedge \mu^r], [\mu^r, \mu^r]$$

is an alternating tableau. Note that $[\mu^i \vee \mu^{i+1}, \mu^i \wedge \mu^{i+1}]$ is obtained by taking the larger partition as positive part and the smaller partition as negative part, because μ^i and μ^{i+1} differ by a unit vector.

If $\mathcal{O}$ is an oscillating tableau, the filling of $\mathcal{G}(\mathcal{A}_{\mathcal{O}})$ is symmetric with respect to the diagonal from the top left to the bottom right. In particular, if $\mathcal{O}$ has empty shape, the filling is precisely the permutation obtained by interpreting the perfect matching as a fixed point free involution.

Conversely, suppose that $\mathcal{A}$ is an alternating tableau, such that the filling of $\mathcal{G}(\mathcal{A})$ is symmetric with respect to the diagonal from the top left to the bottom right, and has no crosses on this diagonal. Then taking the positive part of every second staircase in $\mathcal{A}$, we obtain an oscillating tableau $\mathcal{O}_{\mathcal{A}}$. The filling of $\mathcal{G}(\mathcal{O}_{\mathcal{A}})$ is precisely the part of $\mathcal{G}(\mathcal{A})$ below and to the left of the diagonal.

It is easy to see that the rotation of a fixed point free involution corresponds to the rotation of the associated perfect matching. Also, the reversal of the complement of a symmetric filling corresponds to the reversal of the associated perfect matching. Thus, it remains to show that this correspondence between oscillating tableaux and certain alternating tableaux intertwines promotion of oscillating tableaux and alternating tableaux: $\mathcal{O}_{\operatorname{pr}\mathcal{A}} = \operatorname{pr}\mathcal{O}_{\mathcal{A}}$.

Lemma 6.23. *The promotion of an oscillating tableau equals the oscillating tableau corresponding to the promotion of the associated alternating tableau:* $\mathcal{O}_{\operatorname{pr}\mathcal{A}} = \operatorname{pr}\mathcal{O}_{\mathcal{A}}$.

Proof. Let $\mathcal{O} = (\emptyset = \mu^0, \dots, \mu^r = \emptyset)$ be an oscillating tableau. By Theorem 4.11 its promotion $\operatorname{pr}\mathcal{O} = (\emptyset = \check{\mu}^0, \dots, \check{\mu}^r = \emptyset)$ can be computed using the local rule from Definition 4.9:

$$\check{\mu}^i = \operatorname{dom}_W(\check{\mu}^{i-1} + \mu^i - \mu^{i+1}), \tag{6.3}$$

where W, the Weyl group of the symplectic group, is the hyperoctahedral group. Thus, the dominant representative of a vector is obtained by sorting the absolute values of its components into weakly decreasing order.

Let $\mathcal{A} = (\emptyset = \mu^0, \dots, \mu^{2r} = \emptyset)$ be the alternating tableau associated with the oscillating tableau $\mathcal{O}$. Let $\check{\operatorname{pr}}\,\mathcal{A} = (\emptyset = \check{\mu}^0, \check{\mu}^1, \dots, \check{\mu}^{2r-1}, \check{\mu}^{2r} = \emptyset)$ be as in Definition 6.10 and let $\operatorname{pr}\mathcal{A} = (\emptyset = \check{\check{\mu}}^0, \dots, \check{\check{\mu}}^{2r} = \emptyset)$ be the promotion of $\mathcal{A}$.

We have to show that for every square in Diagram (6.2)

$$\begin{array}{ccc} \mu^{2i-2} & \mu^{2i-1} & \mu^{2i} \\ \\ \check{\mu}^{2i-3} & \check{\mu}^{2i-2} & \check{\mu}^{2i-1} \\ \\ \check{\check{\mu}}^{2i-4} & \check{\check{\mu}}^{2i-3} & \check{\check{\mu}}^{2i-2} \end{array} \tag{6.4}$$

the positive parts of the four corners $\mu^{i-1} = \mu_+^{2i-2}$, $\mu^i = \mu_+^{2i}$, $\check{\mu}^{i-2} = \check{\check{\mu}}_+^{2i-4}$ and $\check{\mu}^{i-1} = \check{\check{\mu}}_+^{2i-2}$ satisfy Equation (6.3). Note that the positive parts and the negative parts of these staircases coincide. To avoid superscripts, we set $\mu^{2i-2} = [\lambda, \lambda]$, $\mu^{2i} = [\nu, \nu]$, $\check{\check{\mu}}^{2i-4} = [\kappa, \kappa]$ and $\check{\check{\mu}}^{2i-2} = [\mu, \mu]$.

Because the filling of $\mathcal{G}(\mathcal{A})$ is symmetric, we have $\ell = k$ in Lemma 6.15. Let us consider the case $i \neq \ell$ first. We assume that $i < \ell$, the case of $i > \ell$ is very similar. If $i < \ell$, the positive parts of the staircases in the first two lines of Diagram (6.4) coincide by Lemma 6.15, and so do the negative parts of the second two lines.

Moreover, by construction of $\mathcal{A}_{\mathcal{O}}$, the staircase in the middle of the first line either equals $[\lambda, \nu]$ or $[\nu, \lambda]$. Let us assume the latter, the former case is dealt with similarly. For the staircase in the middle we then obtain, applying the local rule to the staircases on the top left,

$$\check{\mu}^{2i-2} = \mathrm{dom}_{\mathfrak{S}_n} \left([\lambda, \kappa] + [\nu, \lambda] - [\lambda, \lambda]\right) = [\nu, \kappa].$$

Similarly, applying the local rule to the staircases on the bottom right, we find

$$[\mu, \mu] = \mathrm{dom}_{\mathfrak{S}_n} \left([\check{\check{\mu}}_+^{2i-3}, \kappa] + [\nu, \mu] - [\nu, \kappa]\right) = [\check{\check{\mu}}_+^{2i-3}, \mu].$$

Therefore, the square of staircases in Diagram (6.4) has the following form:

$$\begin{array}{ccc} [\lambda, \lambda] & [\nu, \lambda] & [\nu, \nu] \\ \\ [\lambda, \kappa] & [\nu, \kappa] & [\nu, \mu] \\ \\ [\kappa, \kappa] & [\mu, \kappa] & [\mu, \mu] \end{array} \tag{6.5}$$

Because the negative parts of the four staircases in the lower left corner are all the same, the positive parts satisfy $\mu = \mathrm{dom}_{\mathfrak{S}_n}(\kappa + \nu - \lambda)$, and therefore also Equation (6.3).

It remains to show that Equation (6.3) also holds for $i = \ell$. By Lemma 6.15 the positive part α of the staircase in the middle is obtained by adding a cell to the first column of λ, but also by adding a cell to the first column of μ. Thus, $\lambda = \mu$. Taking into account the other equalities predicted by Lemma 6.15 we see that the square of staircases in Diagram (6.4) has the form

$$\begin{array}{ccc} [\lambda,\lambda] & [\lambda,\nu] & [\nu,\nu] \\ \\ [\lambda,\nu] & [\alpha,\nu] & [\lambda,\nu] \\ \\ [\nu,\nu] & [\lambda,\nu] & [\lambda,\lambda] \end{array} \tag{6.6}$$

Considering the growth diagram $\mathcal{G}(\mathcal{A})$ we additionally find that λ is obtained from ν by adding a cell to the first column, essentially because $\ell = k$. Thus, the vector $\nu + \nu - \lambda$ is obtained from ν by subtracting 1 from the entry at position $\ell(\lambda)$, which is 0 in ν. Taking the absolute values of the entries of the vector $\nu + \nu - \lambda$ then yields λ. $\square$

List of Figures

Bibliography

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