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DISSERTATION

The Lifting Problem for Category on Uncountable Cardinals and Boolean Ultrapowers and Cichoń's Diagram

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Kurzfassung der Dissertation

Diese Arbeit besteht aus zwei Teile.

Der erste Teil behandelt die Anwendung Boolescher Ultrapotenzen auf Cichoń's Diagramm, und basiert auf zwei Forschungsarbeiten:

1. ([KTT18], gemeinsam mit J. Kellner und F. Tonti) Wir beginnen mit einer finite support ccc Iteration $\tilde{P}^4$ aus [Mej13b], die erzwingt, dass

   $$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{d} = 2^{\aleph_0}.$$

   Ausgehend von drei stark kompakten Kardinalzahlen zeigen wir dann, dass eine Boolesche Ultrapotenz dieser Iteration acht verschiedene Werte erzwingt:

   $$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

   Dieses Resultat wurde zur Veröffentlichung im Journal of Symbolic Logic angenommen und ist als preprint verfügbar (arXiv:1706.09638).

2. ([KST17], gemeinsam mit J. Kellner und S. Shelah.) Aufbauend auf [She00] erhalten wir eine andere (als in in [GMS16]) Reihenfolge der Einträge auf der linken Seite des Diagramms: Wir vertauschen $\mathrm{cov}(\mathcal{N})$ und $\mathfrak{b}$ und erhalten:

   $$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}.$$

   Diese Konstruktion ist eine deutlich kompliziertere Variante der Konstruktion in [GKS17].

   Ausgehend von vier stark kompakten Kardinalzahlen und mit Hilfe von Booleschen Ultrapotenzen (ähnlich wie in [GKS17]) können wir das Resultat auf die rechte Seite erweitern, wobei die Duale von $\mathfrak{b}$ und $\mathrm{cov}(\mathcal{N})$ auch vertauscht werden, und erhalten:

   $$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \\ < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{M}) = \mathfrak{d} < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

   Dieses Resultat is zur Veröffentlichung in den "Commentationes Mathematicae Universitatis Carolinae" (special issue in honor of Bohuslav Balcar) eingereicht und als preprint verfügbar (arXiv:1712.00778).

Der zweite Teil (grösstenteils mit J. Kellner und S. Shelah) ist durch folgende Frage motiviert: Können wir Shelahs oracle-cc Konstruktion auf überabzählbare λ (mit $\lambda^{<\lambda} = \lambda$) verallgemeinern?

Wir untersuchen zwei Probleme, die beide für $\lambda = \omega$ von Shelah durch eine oracle-cc Konstruktion gelöst wurden:

1. Die Existenz eines lifting Homomorphismus' für $\mathrm{Bor}(\lambda)/\mathcal{M}(\lambda)$.

   Die klassischen Beweise von Neumann, Stone und von Carlson lassen sich verallgemeinern: Die *Existenz* folgt aus $2^\lambda = \lambda^+$, und ist konsistent mit $2^\lambda = \lambda^{++}$ (bezeugt durch das λ-Cohen Modell).

   Wir beschreieben in dieser Arbeit zwei unserer Versuche (keine davon von Erfolg gekrönt), um ein Modell *ohne* lifting zu konstruieren:

   - (Gemeinsam mit S. Friedman) Wir versuchen, die oracle-cc Methode aus [She82] direkt zu verallgemeinern. Viele Aspekte davon funktionieren auf die offensichtliche Art, aber die Limiten kleiner Kofinalität stellen ein Problem dar.
   - Wir definieren eine Iteration die "essentially Cohen"ist (ein Begriff der Ahnlichkeit zu einer Iteration von λ-Cohens ausdrückt). Damit kommen wir ziemlich weit in einem möglichen Beweis für die Nichtexistenz, aber den Beweis abzuschließen ist ein Ziel für künftige Arbeiten.

   Wir erwähnen auch, dass die Existenz eines $<\lambda$-vollständigen Booleschen Algebra lifting Homomorphismus impliziert, dass λ messbar ist.

2. Die Existenz eines nichttrivialen Automorphismus' von $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$.

   Für unerreichbares λ wurde in [SS15] gezeigt, dass $2^\lambda = \lambda^+$ die Existenz impliziert. Hier geben wir einen vereinfachten Beweis für den messbaren Fall.

   Unter den Voraussetzungen $\lambda < 2^{\aleph_0}$ und MA(σ-centered) zeigen wir, dass jeder Automorphismus trivial ist.

   (Die Konsistenz von "jeder Automorphismus ist trivial" für nicht unerreichbare λ oder für $2^\lambda > \lambda^+$ ist ein Ziel für künftige Forschung.)

Abstract

The thesis consists of two parts.

The first part is concerned with applications of Boolean ultrapowers to Cichoń's diagram and is based on two research papers:

1. ([KTT18], joint work with J. Kellner and F. Tonti) We started with a finite support ccc iteration $\tilde{P}^4$ from [Mej13b] forcing that
$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{d} = 2^{\aleph_0}.$$
Assuming three strongly compact cardinals, we then showed that a Boolean ultrapower of this forcing iteration (again a finite support ccc iteration) forces eight different values to the characteristics in Cichoń's diagram:
$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$
This result has been accepted for publication in the Journal of Symbolic Logic and available as a preprint (arXiv:1706.09638).

2. ([KST17], joint work with J. Kellner and S. Shelah) Building on [She00], we give a construction to get a different order of the characteristics in the left hand side of Cichoń's diagram than the one in [GMS16]. We swap $\mathrm{cov}(\mathcal{N})$ and $\mathfrak{b}$:
$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}.$$
This construction is a modification of the one in [GKS17], however, considerably more complicated.
Assuming four strongly compact cardinals, using a Boolean ultrapower (in a similar way to [GKS17]) we can then expand our result to the right hand side, where also the characteristics dual to $\mathfrak{b}$ and $\mathrm{cov}(\mathcal{N})$ are swapped, resulting in:
$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \\ < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{M}) = \mathfrak{d} < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$
This result is submitted for publication in the special issue of "Commentationes Mathematicae Universitatis Carolinae" in honor of Bohuslav Balcar and available as a preprint (arXiv:1712.00778).

The second part (mostly joint work with J. Kellner and S. Shelah) is motivated by the search for generalizations of Shelah's oracle-cc construction to cardinals other than $\aleph_0$, more specifically, to uncountable λ with $\lambda^{<\lambda} = \lambda$.

We investigate two problems (which were solved by Shelah for the case $\lambda = \aleph_0$ using oracle-cc):

1. The existence of lifting homomorphisms for $\mathrm{Bor}(\lambda)/\mathcal{M}(\lambda)$.

   The classical proofs of Neumann, Stone and Carlson generalize: The *existence* is implied by $2^\lambda = \lambda^+$, and consistent with $2^\lambda = \lambda^{++}$ (witnessed by the λ-Cohen model).

   From the various tries (none of which successful) for obtaining a model with *no lifting* homomorphisms, we present two in this thesis:

   - (Joint work with S. Friedman.) We tried to generalize the oracle-cc machinery as presented in [She82]. While most of the results generalize in the obvious way, the limit steps of small cofinality pose a problem.
   - We defined a forcing iteration which is "essentially Cohen" (a notion which describes similarity to an iteration of λ-Cohens). This gets us quite far in a hopeful proof for the nonexistence of a lifting; but to complete the proof remains as a goal for future research.

   We also mention that the existence of a $<\lambda$-complete Boolean algebra lifting homomorphism implies that λ has to be a measurable cardinal.

2. The existence of nontrivial automorphism of $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$.

   For λ inaccessible, [SS15] shows that $2^\lambda = \lambda^+$ implies that there is a nontrivial automorphism. Here, we give a simplified proof for the measurable case.

   Assuming $\lambda < 2^{\aleph_0}$ and MA(σ-centered), we show that every automorphism is trivial.

   The consistency of "every automorphism is trivial" for λ not inaccessible and/or $2^\lambda > \lambda^+$ is in early progress stages and remains a goal for future research.

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Contents

Part I

Boolean ultrapowers and Cichoń's Diagram

Chapter 1

Introduction

Cantor's result (from 1874) that the cardinality $\mathfrak{c} = 2^{\aleph_0}$ of the real line is strictly bigger than the cardinality $\aleph_0$ of a countable infinite set, was the first theorem about cardinal characteristics of the continuum. This is a central result for, e.g., real analysis: We often study notions of "smallness", such as Lebesgue measure zero or meager, with the property that countable sets are small. (So if the real line was countable, these notions would not make sense.) A **cardinal characteristic** (also called cardinal invariant), is, roughly speaking, the minimal cardinal number for which such a smallness property (which holds for all countable sets) fails.

To recall: A subset A of the Cantor space 2^ω (or of any other Polish space) is nowhere dense if its closure has an empty interior and *meager* (or: of first category) if it is contained in the countable union of (closed) nowhere dense sets. The collection $\mathcal{M}$ of meager sets forms a σ-ideal (i.e., $\mathcal{M}$ is closed under taking countable unions) and has a basis consisting of Borel (and even F_σ) sets. We can also see the Cantor space as a probability space, equipped with the standard product measure (each basic clopen set $[s]$ has measure $2^{-|s|}$, and this measure can be extended to all Lebesgue measurable sets). The collection $\mathcal{N}$ of (Lebesgue) measure zero sets forms a σ-ideal as well, its basis consisting of Borel (and even G_δ) sets.

The ideal generated by σ-compact sets $\mathcal{K}_\sigma$ contains those subsets of the Baire space ω^ω (this space is homeomorphic to the irrational numbers) which can be covered by a countable union of compact sets.

We can now define some cardinal characteristics associated with these σ-ideals:

Definition 1.1. Let $\mathcal{I}$ be a σ-ideal on a set X (in particular, $\mathcal{I}$ can be $\mathcal{M}$, $\mathcal{N}$ or $\mathcal{K}_\sigma$). The additivity, covering, uniformity and cofinality numbers are defined respectively as follows:

- $\mathrm{add}(\mathcal{I}) := \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I} \text{ with } \bigcup \mathcal{J} \notin \mathcal{I}\}$,
- $\mathrm{cov}(\mathcal{I}) := \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I} \text{ with } \bigcup \mathcal{J} = X\}$,
- $\mathrm{non}(\mathcal{I}) := \min\{|I| : I \subseteq X,\ I \notin \mathcal{I}\}$, and
- $\mathrm{cof}(\mathcal{I}) := \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I} \text{ such that } (\forall I \in \mathcal{I})(\exists J \in \mathcal{J}) : I \subseteq J\}$.

Definition 1.2. For $f, g \in \omega^\omega$, f is eventually dominated by g, or: $f \leq^* g$, if $(\forall^* n \in \omega)\ f(n) \leq g(n)$ (where $\forall^*$ means "all but finitely many"). A family $\mathcal{F} \subseteq \omega^\omega$ is **dominating** if every $f \in \omega^\omega$ is eventually dominated by some $g \in \mathcal{F}$ and **unbounded** if no single $f \in \omega^\omega$ eventually dominates all members of $\mathcal{F}$. The **dominating number** $\mathfrak{d}$ is the minimal cardinality of a dominating family and the **bounding number** $\mathfrak{b}$ is the minimal cardinality of an unbounded family.

Note that for any $f \in \omega^\omega$, $\{g \in \omega^\omega : g \leq^* f\} \in \mathcal{K}_\sigma$ and every element of $\mathcal{K}_\sigma$ can be covered by such a set, and that $\operatorname{add}(\mathcal{K}_\sigma) = \operatorname{non}(\mathcal{K}_\sigma) = \mathfrak{b}$ and $\operatorname{cov}(\mathcal{K}_\sigma) = \operatorname{cof}(\mathcal{K}_\sigma) = \mathfrak{d}$.

1.1 Cichoń's diagram and previous results

The ZFC-provable inequalities between the cardinal characteristics defined above (for $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{K}_\sigma$) are summarized in Cichoń's diagram:

$$\begin{tikzcd}
 & \operatorname{cov}(\mathcal{N}) \arrow[r] & \operatorname{non}(\mathcal{M}) \arrow[r] & \operatorname{cof}(\mathcal{M}) \arrow[r] & \operatorname{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \operatorname{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \operatorname{add}(\mathcal{M}) \arrow[u] \arrow[r] & \operatorname{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \operatorname{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

An arrow between $\mathfrak{x}$ and $\mathfrak{y}$ indicates a ZFC-provable inequality $\mathfrak{x} \leq \mathfrak{y}$. Moreover,

$$\max(\mathfrak{d}, \operatorname{non}(\mathcal{M})) = \operatorname{cof}(\mathcal{M}) \text{ and } \min(\mathfrak{b}, \operatorname{cov}(\mathcal{M})) = \operatorname{add}(\mathcal{M}).$$

Every assignment of $\aleph_1$ and $\aleph_2$ to the entries of Cichoń's diagram that honors these restrictions can be shown to be consistent. These facts have been proven by various authors, cf. [BJ95; Bar84; BS10; CKP85; JS90; Kam89; Mil81; Mil84; RS83], a complete proof can be found in [BJ95, ch. 7]. It is even more challenging and requires more involved techniques to get show the consistency of many simultaneously different values.

For example, matrix iterations, which are a are special kinds of FS ccc iterations (introduced by Blass and Shelah in [BS89]) were used in the context of Cichon's diagram by D. Mejia in [Mej13b] and [Mej13a] for obtaining at most six different values. In [FFMM18] and [Mej18], the technique was extended to arbitrary coherent systems of finite support iterations, resulting in seven different values. The limitation is that a finite support ccc iteration of uncountable cofinality δ always results in $\operatorname{non}(\mathcal{M}) \leq \delta \leq \operatorname{cov}(\mathcal{M})$.

A different approach can be seen in [FGKS17], where five different cardinal characteristics on the right side of Cichoń's diagram were separated using a creature forcing construction (between a product and an iteration), as in Kellner and

Shelah's [KS09; KS12]. This construction forces $\mathfrak{d} = \aleph_1$, it is ω^ω-bounding, so it cannot be used to separate the cardinal below $\mathfrak{d}$.

In our research, we use a Boolean ultrapower construction to control characteristics in Cichoń's diagram (this idea is due to Shelah).

1.2 An overview of the results

It was a longstanding question whether *all* cardinal characteristics in Cichoń's Diagram (other than $\operatorname{add}(\mathcal{M})$ and $\operatorname{cof}(\mathcal{M})$, of course) can be simultaneously different.

In the **second chapter** of the thesis, which is basically the paper [KTT18] (joint work with J. Kellner and F. Tonti), we assume three strongly compact cardinals, and start with a well known finite support ccc iteration $\tilde{P}^4$ (introduced in Mejia's [Mej13b]) for the "left hand side", forcing that $\aleph_1 < \operatorname{add}(\mathcal{N}) < \operatorname{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{d} = 2^{\aleph_0}$ (and actually something stronger). We than apply Boolean ultrapowers to $\tilde{P}^4$, resulting in another finite support ccc iteration, which also controls the "right hand side", forcing

$$\aleph_1 < \operatorname{add}(\mathcal{N}) < \operatorname{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{d} < \operatorname{non}(\mathcal{N}) < \operatorname{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

I.e., we get the following values in the diagram (for some increasing cardinals λ_i):

$$\begin{tikzcd}
 & \lambda_2 \arrow[r] & \boxed{} \arrow[r] & \boxed{} \arrow[r] & \lambda_6 \arrow[r] & \lambda_7 \\
 & & \lambda_3 \arrow[u] \arrow[r] & \lambda_4 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[uu] \arrow[r] & \boxed{} \arrow[u] \arrow[r] & \boxed{} \arrow[u] \arrow[r] & \lambda_5 \arrow[uu] &
\end{tikzcd}$$

The kind of Boolean ultrapower construction we use was introduced in Mansfield's [Man71], similar methods have recently been applied, e.g., by Malliaris and Shelah [MS16] and Raghavan ans Shelah [RS]. The idea to apply Boolean ultrapowers to control characteristics in Cichoń's diagram is due to Shelah.

More recently, Goldstern, Kellner and Shelah [GKS17] solved the question for the whole diagram, assuming four strongly compact cardinals. The construction is similar, but, of course, more complicated: For the left hand side, the construction of [GMS16] (Goldstern, Mejia, Shelah) is used, which gives

$$\aleph_1 < \operatorname{add}(\mathcal{N}) < \operatorname{cov}(\mathcal{N}) < \operatorname{add}(\mathcal{M}) = \mathfrak{b} < \operatorname{non}(\mathcal{M}) < \operatorname{cov}(\mathcal{M}) = 2^{\aleph_0}.$$

However, the construction has to be modified to be compatible with GCH. Then a similar Boolean ultrapower construction is used to get the following order:

$$\begin{tikzcd}
 & \lambda_2 \arrow[r] & \lambda_4 \arrow[r] & \boxed{} \arrow[r] & \lambda_8 \arrow[r] & \lambda_9 \\
 & & \lambda_3 \arrow[u] \arrow[r] & \lambda_6 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[uu] \arrow[r] & \boxed{} \arrow[u] \arrow[r] & \lambda_5 \arrow[u] \arrow[r] & \lambda_7 \arrow[uu] &
\end{tikzcd}$$

Recently Mejia [Mej18] managed to use matrix iterations and ultrafilter limits to obtain seven different values in Cichoń's diagram. His iteration requires weaker assumptions and it is simpler to construct than the Goldstern-Mejía-Shelah's version presented in [GMS16] and used (as the initial iteration) in [GKS17]. The Boolean ultrapower construction applied to this iteration in exactly the same way as in [GKS17], assuming only three compact cardinals, gives the same model, with weaker hypotheses. So maybe it might be possible to construct ever more sophisticated FS-ccc methods that could replace the use of Boolean ultrapowers in the specific application altogether, but this remains a goal for future research.

In the **third chapter**, which is basically the paper [KST17] (joint work with J. Kellner and S. Shelah), we give a construction to get a different order for these characteristics. This time we build on a finitely additive measure (FAM) method originating from Shelah's [She00], which results in a left hand configuration with swapped $\mathrm{cov}(\mathcal{N})$ and $\mathfrak{b}$:

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}.$$

Once we get this, we can very similarly apply Boolean ultrapowers to get the following order:

$$\begin{tikzcd}
 & \lambda_3 \arrow[r] & \lambda_4 \arrow[r] & \boxed{} \arrow[r] & \lambda_8 \arrow[r] & \lambda_9 \\
 & & \lambda_2 \arrow[u] \arrow[r] & \lambda_7 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[uu] \arrow[r] & \boxed{} \arrow[u] \arrow[r] & \lambda_5 \arrow[u] \arrow[r] & \lambda_6 \arrow[uu] &
\end{tikzcd}$$

Getting this order is considerably more complicated, and we briefly describe the difference.

In both constructions, we assign to each of the cardinal characteristics of the left hand side a relation R (e.g., $R \subseteq \omega^\omega \times \omega^\omega$ is "eventually different" in case of the characteristic $\mathrm{non}(\mathcal{M})$). We can then show that the characteristic remains "small" (i.e., is at most the intended value λ in the final model), because all single forcings we use in the iterations are either small (i.e., smaller than λ) or are "R-good". However, $\mathfrak{b}$ is an exception: We do not know any variant of an eventually different forcing (which we need to increase $\mathrm{non}(\mathcal{M})$) which satisfies that all of its subforcings are "eventually dominating"-good. To show that $\mathfrak{b}$ "remains small" is therefore the main difficulty (in both constructions).

In the old construction, each non-small forcing is a subforcing of the eventually different forcing $\mathbb{E}$. To deal with such forcings, ultrafilter limits of sequences of $\mathbb{E}$-conditions are introduced and used (and it is required that all $\mathbb{E}$-subforcings are basically $\mathbb{E}$ intersected with some small elementary model, and thus closed under limits of sequences in the model). In the new construction, we have to deal with an additional kind of "large" forcing: (subforcings of) random forcing. Ultrafilter limits do not work any more, but, similarly to [She00], we can use finite additive measures

(FAMs) and interval-FAM-limits of random conditions. But now $\mathbb{E}$ doesn't seem to work with interval-FAM-limits any more, so we replace it with a creature forcing notion $\tilde{\mathbb{E}}$.

We also have to show that $\mathrm{cov}(\mathcal{N})$ remains small. In the old construction, we could use a very simple relation R and use the fact that all σ-centered forcings are R-good: All large forcings are subforcings of either $\mathbb{E}$ or of Hechler, both σ-centered. In the new construction, the large forcings we have to deal with are subforcings of $\tilde{\mathbb{E}}$. But $\tilde{\mathbb{E}}$ is not σ-centered, just (ρ, π)-linked. So we use a different (and more cumbersome) relation R'; and use the fact of [OK14] that (ρ, π)-linked forcings are R'-good.

Chapter 2

Eight values in Cichoń’s Diagram

This chapter is based on the paper “Compact cardinals and eight values in Cichoń’s Diagram” (arXiv:1706.09638, http://dx.doi.org/10.1017/jsl.2018.17.), joint work with J. Kellner and F. Tonti. The notation is slightly changed, namely the iteration will be $\tilde{P}$ and we use LCU and COB, instead of $\circledcirc$ and $\boxplus$, in order to have a uniform notation.

Introduction

The result consists of two parts: In the first one, Section 2.1, we present a finite support ccc iteration $\tilde{P}^4$ forcing that $\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{d} = 2^{\aleph_0}$ (and actually something stronger, cf. Lemmas 2.18 and 2.20). This is nothing new: The forcing and all required properties were presented in [Mej13b]. We recall all the facts that are required for our result, in a form convenient for our purposes.

In the second part 2.2, we investigate the (iterated) Boolean ultrapower $\tilde{P}^7$ of $\tilde{P}^4$. Assuming three strongly compact cardinals, this ultrapower (again a finite support ccc iteration) forces

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0},$$

i.e., we get the following values in the diagram (for some increasing cardinals λ_i):

$$\begin{tikzcd}
 & \lambda_2 \arrow[r] & \square \arrow[r] & \square \arrow[r] & \lambda_6 \arrow[r] & \lambda_7 \\
 & & \lambda_3 \arrow[r] \arrow[u] & \lambda_4 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[r] \arrow[uu] & \square \arrow[r] \arrow[u] & \square \arrow[r] \arrow[u] & \lambda_5 \arrow[uu] &
\end{tikzcd}$$

It seems unlikely that the large cardinals assumption is actually needed, but we would expect a proof without it to be considerably more complicated.

The kind of Boolean ultrapower that we use was investigated in [Man71], and recently applied, e.g., in [MS16] and [RS] (where a Boolean ultrapower of a forcing notion is applied to cardinal characteristics of the reals). Recently Shelah developed a method of using Boolean ultrapowers to control characteristics in Cichoń's diagram. The current chapter is a relatively simple application of these methods. A more complicated one, in a later paper of Goldstern, Kellner and Shelah [GKS17], shows that all possible invariants in Cichoń's diagram can be pairwise different.

2.1 The initial iteration $\tilde{P}^4$

The goal for this section is to obtain the following constellation:

$$\begin{tikzcd}
 & \lambda_2 \arrow[r] & \boxed{} \arrow[r] & \boxed{} \arrow[r] & \lambda_4 \arrow[r] & \lambda_4 \\
 & & \lambda_3 \arrow[r] \arrow[u] & \lambda_4 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[r] \arrow[uu] & \boxed{} \arrow[r] \arrow[u] & \boxed{} \arrow[r] \arrow[u] & \lambda_4 \arrow[uu] &
\end{tikzcd}$$

Figure 2.1: The initial iteration for eight values.

We want to show that some forcing $\tilde{P}^4$ results in $\mathfrak{x}_i = \lambda_i$ (for $i = 1, 2, 3$). So we have to show two "directions", $\mathfrak{x}_i \leq \lambda_i$ and $\mathfrak{x}_i \geq \lambda_i$. The direction $\mathfrak{x}_i \leq \lambda_i$ will be given by the fact that $\tilde{P}^4$ is (R_i, λ_i)-good for a suitable relation R_i.

As mentioned before, this iteration is nothing new, it is just a suitable rewrite of properties and results which were presented in [Mej13b].

2.1.1 Good iterations and the LCU property

The notion of "goodness" was first explored in by Judah and Shelah [JS90] and Brendle [Bre91], later expanded by Brendle and Mejía [BM14]. In this section, we will recall the facts of good iterations, and specify the instances of the relations we use.

Assumption 2.1. We will consider binary relations R on $X = \omega^\omega$ (or on $X = 2^\omega$) that satisfy the following: There are relations R^n such that $\mathrm{R} = \bigcup_{n\in\omega} \mathrm{R}^n$, each R^n is a closed subset (and in fact absolutely defined) of $X \times X$, and for $g \in X$ and $n \in \omega$, the set $\{f \in X : f \,\mathrm{R}^n\, g\}$ is nowhere dense. Also, for all $g \in X$ there is some $f \in X$ with $f \,\mathrm{R}\, g$.

We will actually use another space as well, the space C of strictly positive rational sequences $(q_n)_{n\in\omega}$ such that $\sum_{n\in\omega} q_n \leq 1$. It is easy to see that C is homeomorphic to ω^ω, when we equip the rationals with the discrete topology and use the product topology.

We use the following instances of relations R on X; it is easy to see that they all satisfy the assumption (in case of $X = \mathcal{C}$ we use the homeomorphism mentioned above):

Definition 2.2.
1. $X = \mathcal{C}$: $f \mathrel{\mathrm{R}_1} g$ if $(\forall^* n \in \omega)\, f(n) \le g(n)$.
   (We use "$\forall^* n \in \omega$" for "$(\exists n_0 \in \omega)\,(\forall n > n_0)$".)
2. $X = 2^\omega$: $f \mathrel{\mathrm{R}_2} g$ if $(\forall^* n \in \omega)\, f \restriction I_n \ne g \restriction I_n$,
   where $(I_n)_{n\in\omega}$ is the increasing interval partition of ω with $|I_n| = 2^{n+1}$.
3. $X = \omega^\omega$: $f \mathrel{\mathrm{R}_3} g$ if $(\forall^* n \in \omega)\, f(n) \le g(n)$.

We say "f is bounded by g" if $f \mathrel{\mathrm{R}} g$; and, for $\mathcal{Y} \subseteq \omega^\omega$, "$f$ is bounded by $\mathcal{Y}$" if $(\exists y \in \mathcal{Y})\, f \mathrel{\mathrm{R}} y$. We say "unbounded" for "not bounded". (I.e., f is unbounded by $\mathcal{Y}$ if $(\forall y \in \mathcal{Y})\, \neg f \mathrel{\mathrm{R}} y$.) We call $\mathcal{X}$ an R-unbounded family, if $\neg(\exists g)\,(\forall x \in \mathcal{X})\, x \mathrel{\mathrm{R}} g$, and an R-dominating family if $(\forall f)\,(\exists x \in \mathcal{X})\, f \mathrel{\mathrm{R}} x$. Let $\mathfrak{b}_i$ be the minimal size of an R_i-unbounded family, and $\mathfrak{d}_i$ of an R_i-dominating family.

We only need the following connection between R_i and the cardinal characteristics:

Lemma 2.3.
1. $\mathrm{add}(\mathcal{N}) = \mathfrak{b}_1$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}_1$.
2. $\mathrm{cov}(\mathcal{N}) \le \mathfrak{b}_2$ *and* $\mathrm{non}(\mathcal{N}) \ge \mathfrak{d}_2$.
3. $\mathfrak{b} = \mathfrak{b}_3$ *and* $\mathfrak{d} = \mathfrak{d}_3$.

Proof. (3) holds by definition.

(1) can be found in [BJ95, 6.5.B].

To prove (2), note that for fixed $g \in 2^\omega$ the set $\{f \in 2^\omega : \neg g \mathrel{\mathrm{R}_2} f\}$ is a null set, call it N_g. Let $\mathcal{G}$ be an R_2-unbounded family. Then $\{N_g : g \in \mathcal{G}\}$ covers 2^ω: Fix $f \in 2^\omega$. As f does not bound $\mathcal{G}$, there is some $g \in \mathcal{G}$ unbounded by f, i.e., $f \in N_g$. Let X be a non-null set. Then X is R_2-dominating: For any $g \in 2^\omega$ there is some $x \in X \setminus N_g$, i.e., $g \mathrel{\mathrm{R}_2} x$. □

Definition 2.4. [JS90] Let P be a ccc forcing, λ an uncountable regular cardinal, and R as above. P is (R, λ)**-good**, if for each P-name $r \in \omega^\omega$ there is (in V) a nonempty set $\mathcal{Y} \subseteq \omega^\omega$ of size $<\lambda$ such that every f (in V) that is R-unbounded by $\mathcal{Y}$ is forced to be R-unbounded by r as well.

Note that λ-good trivially implies μ-good if $\mu \ge \lambda$ are regular.

How do we get good forcings? Let us just quote the following results:

Lemma 2.5. *A FS iteration of Cohen forcing is good for any* (R, λ)*, and the composition of two* (R, λ)*-good forcings is* (R, λ)*-good.*
Assume that $(P_\alpha, Q_\alpha)_{\alpha<\delta}$ *is a FS ccc iteration. Then* P_δ *is* (R, λ)*-good, if each* Q_α *is forced to satisfy the following:*

1. For $\mathrm{R} = \mathrm{R}_1$*:* $|Q_\alpha| < \lambda$*, or* Q_α *is* σ*-centered, or* Q_α *is a sub-Boolean-algebra of the random algebra.*

2. *For* $\mathrm{R} = \mathrm{R}_2$*:* $|Q_\alpha| < \lambda$*, or* Q_α *is* σ*-centered.*

3. *For* $\mathrm{R} = \mathrm{R}_3$*:* $|Q_\alpha| < \lambda$.

Proof. (R, λ)-goodness is preserved by FS ccc iterations (in particular compositions), as proved in [JS90], cf. [BJ95, pp. 6.4.11–12]. Also, ccc forcings of size $<\lambda$ are (R, λ)-good [BJ95, p. 6.4.7], which takes care of the case of Cohens and of $|Q_\alpha| < \lambda$.

So it remains to show that (for $i = 1, 2$) the "large" iterands in the list are (R_i, λ)-good. For R_1 this follows from [JS90] and [Kam89], cf. [BJ95, pp. 6.5.17–18]. For R_2 this is proven in [Bre91]. □

Lemma 2.6. *Let* $\lambda \leq \kappa \leq \mu$ *be uncountable regular cardinals. After forcing with* μ *many Cohen reals* $(c_\alpha)_{\alpha \in \mu}$*, followed by an* (R, λ)*-good forcing, we get: For every real* r *in the final extension, the set* $\{\beta \in \kappa : c_\beta$ *is unbounded by* $r\}$ *is cobounded in* κ*. I.e.,* $(\exists \alpha \in \kappa)\,(\forall \beta \in \kappa \setminus \alpha)\,\neg c_\beta \,\mathrm{R}\, r$.

(The Cohen real c_β can be interpreted both as Cohen generic element of 2^ω and as Cohen generic element of ω^ω; we use the interpretation suitable for the relation R.)

Proof. Work in the intermediate extension after κ many Cohen reals, let us call it V_κ. The remaining forcing (i.e., $\mu \setminus \kappa$ many Cohens composed with the good forcing) is good; so applying Definition 2.4 we get (in V_κ) a set $\mathcal{Y}$ of size $<\lambda$.

As the initial Cohen extension is ccc, and $\kappa \geq \lambda$ is regular, we get some $\alpha \in \kappa$ such that each element y of $\mathcal{Y}$ already exists in the extension by the first α many Cohens, call it V_α. The set of reals M_y bounded by y is meager (and absolute). Any c_β for $\beta \in \kappa \setminus \alpha$ is Cohen over V_α, and therefore not in M_y, i.e., not bounded by y. As this holds for all y, c_β is unbounded by $\mathcal{Y}$, and thus, according to the definition of good, unbounded by r as well. □

In the light of this result, let us revisit Lemma 2.3 with some new notation:

Definition 2.7. For $i = 1, 2, 3$, $\lambda > \aleph_0$ regular, and P a ccc forcing notion, let $\mathsf{LCU}_i(P, \lambda)$ stand for: "There is a sequence $(x_\alpha)_{\alpha \in \lambda}$ of P-names such that for every P-name y we have $(\exists \alpha \in \lambda)\,(\forall \beta \in \lambda \setminus \alpha)\, P \Vdash \neg x_\beta \,\mathrm{R}_i\, y$."

Lemma 2.8. $\mathsf{LCU}_i(P, \lambda)$ *implies* $\mathfrak{b}_i \leq \lambda$ *and* $\mathfrak{d}_i \geq \lambda$*. In particular:*

1. $\mathsf{LCU}_1(P, \lambda)$ *implies* $P \Vdash (\,\mathrm{add}(\mathcal{N}) \leq \lambda \,\&\, \mathrm{cof}(\mathcal{N}) \geq \lambda\,)$.

2. $\mathsf{LCU}_2(P, \lambda)$ *implies* $P \Vdash (\,\mathrm{cov}(\mathcal{N}) \leq \lambda \,\&\, \mathrm{non}(\mathcal{N}) \geq \lambda\,)$.

3. $\mathsf{LCU}_3(P, \lambda)$ *implies* $P \Vdash (\,\mathfrak{b} \leq \lambda \,\&\, \mathfrak{d} \geq \lambda\,)$.

Proof. The set $\{x_\alpha : \alpha \in \lambda\}$ is certainly forced to be R_i-unbounded; and given a set $Y = \{y_j : j < \theta\}$ of $\theta < \lambda$ many P-names, each has a bound α_j, so for any $\beta \in \lambda$ above all α_j we get $P \Vdash \neg x_\beta \,\mathrm{R}_i\, y_j$ for all j; i.e., Y cannot be dominating. □

2.1.2 Ground model Borel functions, partial random forcing

The following lemma seems to be well known (but we are not aware of a good reference or an established notation):

Definition 2.9. Let Q be a forcing notion, and let η be a Q-name for a real. We say that Q is "generically Borel determined (by η, via B)", if

- Q consists of reals,
- the Q-generic filter is determined by the real η, and moreover:
- $B \subseteq \mathbb{R}^2$ is a Borel relation such that for all $q \in Q$, $Q \Vdash (B_Q(q, \eta) \leftrightarrow q \in G)$.

We investigate iterations of such forcings:

Lemma 2.10. *Assume that $(P_\beta, Q_\beta)_{\beta<\alpha}$ is a FS ccc iteration such that each Q_β is generically Borel determined (in an absolute way already fixed in V). Then for each P_α-name r for a real, there is (in the ground model) a Borel function $F : \mathbb{R}^\omega \to \mathbb{R}$ and a sequence $(\alpha_i)_{i\in\omega}$ of ordinals in α such that P_α forces $r = F((\eta_{\alpha_i})_{i\in\omega})$.*

Proof. We prove by induction on $\gamma \leq \alpha$:

- For all $p \in P_\gamma$ there is a Borel relation $B^p \subseteq \mathbb{R}^\omega$ and a sequence $(\alpha_i^p)_{i\in\omega}$ of elements of γ such that $P_\gamma \Vdash B^p((\eta_{\alpha_i^p})_{i\in\omega}) \leftrightarrow p \in G_\gamma$.
- For each P_γ-name r for a real, there is a Borel function F^r and a sequence $(\alpha_i^r)_{i\in\omega}$ of elements of γ such that $P_\gamma \Vdash r = F^r((\eta_{\alpha_i^p})_{i\in\omega})$.

The second item follows from the first, as we can use the countable maximal antichains that decide $r(n) = m$.

If γ is a limit ordinal, then P_γ has no new elements, so there is nothing to do.

So assume $\gamma = \zeta + 1$. By our assumption, Q_ζ is generically Borel determined from η_ζ via a Borel relation B_ζ. Consider $(p, q) \in P_\zeta * Q_\zeta$. This is in G_γ iff $p \in G_\zeta$ (which, by induction, is Borel) and $q \in G(\zeta)$. As q is a real, it is forced that $q = B^q((\alpha_i^q)_{i\in\omega})$. Moreover, P_ζ forces that Q_ζ forces that $q \in G(\zeta)$ iff $B_\zeta(\eta_\zeta, q)$ iff $B_\zeta(\eta_\zeta, B^q((\alpha_i^q)_{i\in\omega}))$. □

Definition 2.11. Given $(P_\beta, Q_\beta)_{\beta<\alpha}$ as above, and some $w \subseteq \alpha$, we define the P_α-name $\mathbb{R}^w$ to consist of all reals r such that in the ground model there are a Borel function F and a sequence $(\alpha_i)_{i\in\omega}$ of elements of w such that $r = F((\eta_{\alpha_i})_{i\in\omega})$.

The following is straightforward:

Fact 2.12.

- Set (in V) $\mu = (|w|+2)^{\aleph_0}$. Then it is forced that $\mathbb{R}^w$ has cardinality $\leq \mu$.
- If $w' \supseteq w$, then (it is forced that) $\mathbb{R}^{w'} \supseteq \mathbb{R}^w$.
- If w is the increasing union of $(w_\alpha)_{\alpha\in\gamma}$ with $\mathrm{cf}(\gamma) \geq \omega_1$, then (it is forced that) $\mathbb{R}^w = \bigcup_{\alpha\in\gamma} \mathbb{R}^{w_\alpha}$.

- For every P_α-name r for a real, there is a countable w such that (it is forced that) $r \in \mathbb{R}^w$.

Definition 2.13. Let $\mathbb{B}$ be the definition of random forcing, i.e., positive pruned trees T, ordered by inclusion. Given $(P_\beta, Q_\beta)_{\beta<\alpha}$ as above, $w \subseteq \alpha$, we define the P_α-name $\mathbb{B}^w := \mathbb{B} \cap \mathbb{R}^w$ and call it "partial random forcing defined from w".

Clearly $\mathbb{B}^w$ is a subforcing (not necessarily a complete one) of $\mathbb{B}$, and if p, q in $\mathbb{B}^w$ are incompatible in $\mathbb{B}^w$, then they are incompatible in random forcing. In particular $\mathbb{B}^w$ is ccc.

Note that Q^w is again generically Borel determined (by the generic real η defined by $\{\eta\} = \bigcap\{[s] \in G : s \in 2^{<\omega}\}$, and by the Borel relation "$\eta \in [T]$").

Remark 2.14. In this section, we have provided a very explicit notion of "partial random", using Borel functions. The use of Borel functions is not essential, we could use any other method of calculating reals from generic reals at certain restricted positions, provided this method satisfies Fact 2.12. One such alternative definition has been used in [GMS16]: We can define the sub-forcing $P_\alpha \restriction w$ of P_α in a natural way, and require that it is a complete subforcing (which is a closure property of w). Then we can define Q_α to be the random forcing, as evaluated in the $P_\alpha \restriction w$-extension.

While this approach is basically equivalent (and may seem slightly more natural than the artificial use of Borel functions), it has the disadvantage that we have to take care of the closure property of w.

Definition 2.15. Analogously to "partial random", we define the "partial Hechler" and "partial amoeba" forcings.

These forcings are generically Borel determined as well.

2.1.3 The initial forcing $\tilde{P}^4$

Assume that λ is regular uncountable and $\mu < \lambda$ implies $\mu^{\aleph_0} < \lambda$. Then $|w| < \lambda$ implies that the size of a partial forcing defined by w us $<\lambda$.

Definition 2.16. Assume GCH and let $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ be regular cardinals. Set $\delta_4 = \lambda_4 + \lambda_4$. Partition $\delta_4 \setminus \lambda_4$ into unbounded sets S^1, S^2, and S^3. Fix for each $\alpha \in \delta_4 \setminus \lambda_4$ some $w_\alpha \subseteq \alpha$ such that each $\{w_\alpha : \alpha \in S^i\}$ is cofinal in $[\delta_4]^{<\lambda_i}$.[1]

We now define $\tilde{P}^4 = (P_\alpha, Q_\alpha)_{\alpha\in\delta_4}$ to be the FS ccc iteration which first adds λ_4 many Cohen reals, and such that for each $\alpha \in \delta_4 \setminus \lambda_4$,

$$\text{if } \alpha \text{ is in} \left\{\begin{array}{c} S^1 \\ S^2 \\ S^3 \end{array}\right\}, \text{ then } Q_\alpha \text{ is the partial } \left\{\begin{array}{c} \text{amoeba} \\ \text{random} \\ \text{Hechler} \end{array}\right\} \text{ forcing defined from } w_\alpha.$$

[1] I.e., if $\alpha \in S^i$ then $|w_\alpha| < \lambda_i$, and for all $u \subseteq \delta_4$, $|u| < \lambda_i$ there is some $\alpha \in S^i$ with $w_\alpha \supseteq u$.

The forcing results in $2^{\aleph_0} = \lambda_4$, which follows from the following easy and well-known fact:

Lemma 2.17. *Let $(P_\alpha, Q_\alpha)_{\alpha<\delta}$ be a FS ccc iteration of length δ such that each Q_α is forced to consist of real numbers, and set $\lambda(\delta) := (2+\delta)^{\aleph_0}$. Then $P_\delta \Vdash 2^{\aleph_0} \leq \lambda(\delta)$.*

Proof. By induction on δ, we show that there is a dense subforcing $D_\delta \subseteq P_\delta$ of size $\leq\lambda(\delta)$. Then the continuum has size at most $\lambda(\delta)$ (as each name of a real corresponds to a countable sequence of antichains, labeled with $0, 1$, in P_δ, without loss of generality in D_δ).

For $\delta + 1$, $D_\delta \subseteq P_\delta$ is dense and has size $\leq\lambda(\delta)$, and Q_δ is forced to have size $\leq\lambda(\delta)$. Without loss of generality we can identify Q_δ with a subset of $\lambda(\delta)$. Let $D_{\delta+1}$ consist of $(p, \check{\alpha}) \in P_{\delta+1}$ such that $p \in D_\delta$ forces $\alpha \in Q_\delta$.

For δ limit, the union of D_α is dense in $P_\delta = \bigcup_{\alpha\in\delta} P_\alpha$. □

According to Lemma 2.5 $\tilde{P}^4$ is (R_i, λ_i)-good for $i = 1, 2, 3$, so Lemmas 2.6 and 2.8 gives us:

Lemma 2.18. $\mathsf{LCU}_i(\tilde{P}^4, \kappa)$ *holds for $i = 1, 2, 3$ and each regular cardinal κ in $[\lambda_i, \lambda_4]$.*

So in particular, $\tilde{P}^4$ forces $\mathrm{add}(\mathcal{N}) \leq \lambda_1$, $\mathrm{cov}(\mathcal{N}) \leq \lambda_2$, $\mathfrak{b} \leq \lambda_3$ *and* $\mathrm{cof}(\mathcal{N}) = \mathrm{non}(\mathcal{N}) = \mathfrak{d} = 2^{\aleph_0}$.

Theorem 2.19. *[Mej13b, Thm. 2] $\tilde{P}^4$ forces* $\mathrm{add}(\mathcal{N}) = \lambda_1$, $\mathrm{cov}(\mathcal{N}) = \lambda_2$, $\mathfrak{b} = \lambda_3$, *and* $\mathfrak{d} = \lambda_4 = 2^{\aleph_0}$.

Proof. It is easy to see that the partial amoebas take care of $\mathrm{add}(\mathcal{N}) \geq \lambda_1$: Let $(N_i)_{i\in\mu}$, $\aleph_1 \leq \mu < \lambda_1$ be a family of $\tilde{P}^4$-names of null sets. Each N_i is a Borel code, i.e., a real, i.e., and therefore Borel-calculated from some countable set $w^i \subseteq \delta_4$. The union of the w^i is a set w^* of size $\leq\mu$ that already Borel-decides all N_i. There is some $\beta \in S^1$ such that $w_\beta \supseteq w^*$, so the partial amoeba forcing at β sees all the null sets N_i and therefore covers their union.

Analogously one proves $\mathrm{cov}(\mathcal{N}) \geq \lambda_2$ and $\mathfrak{b} \geq \lambda_3$. □

We will reformulate the proof for $\mathrm{cov}(\mathcal{N})$ in a cumbersome manner that can be conveniently used later on, namely as the "cone of bounds" property:

Lemma 2.20. *Let $\mathsf{COB}_2(P, \lambda, \mu)$ stand for: "P is a ccc forcing notion, and there is a $<\lambda$-directed partial order $(S, \prec)$ of size μ and a sequence $(r_s)_{s\in S}$ of P-names for reals such that for each P-name N of a null set $(\exists s \in S)\,(\forall t \succ s)\; P \Vdash r_t \notin N$."*

- $\mathsf{COB}_2(P, \lambda, \mu)$ *implies* $P \Vdash (\,\mathrm{cov}(\mathcal{N}) \geq \lambda \;\&\; \mathrm{non}(\mathcal{N}) \leq \mu\,)$.
- $\mathsf{COB}_2(\tilde{P}^4, \lambda_2, \lambda_4)$ *holds.*

Proof. $\mathrm{cov}(\mathcal{N}) \geq \lambda$: Fix $<\lambda$ many P-names N_α of null sets. Each real has a "lower bound" $s_\alpha \in S$, i.e., $P \Vdash r_t \notin N_\alpha$ whenever $t \succ s_\alpha$. Let $t \succ s_\alpha$ for all α (this is possible as S is directed). So $P \Vdash r_t \notin N_\alpha$ for every α, i.e., the union doesn't cover the reals.

$\mathrm{non}(\mathcal{N}) \leq \mu$, as the set of all r_s is not null: For every name N of a null set there is some $s \in S$ such that $P \Vdash r_s \notin N$.

For $\tilde{P}^4$, we set $S = S^2$, $s \prec t$ if $w_s \subseteq w_t$, and we let r_s be the partial random real added at s. A $\tilde{P}^4$ name for a null set N depends (in a Borel way) on a countable index set $w^* \subseteq \delta_4$. Fix some $s \in S^2$ such that $w_s \supseteq w^*$, and pick any $t \succ s$. Then w_t contains all information to calculate the null set N, and therefore the partial random r_t over w_t will avoid N. □

2.2 The Boolean ultrapower of a forcing

2.2.1 Boolean ultrapowers

Boolean ultrapowers generalize regular ultrapowers by using arbitrary Boolean algebras instead of the power set algebra.

Assumption 2.21. κ is strongly compact, B is a κ-distributive, κ^+-cc, atomless complete Boolean algebra.

Lemma 2.22. *[KT64] Every κ-complete filter on B can be extended to a κ-complete ultrafilter U.*[2]

Proof. List the required properties of U as a set of propositional sentences in $\mathcal{L}_\kappa$ (a propositional language allowing conjunctions and disjunctions of any size $<\kappa$), using atomic formulas coding $b \in U$ and $b \notin U$ for $b \in B$. □

Assumption 2.23. U is a κ-complete ultrafilter on B.

Lemma 2.24. *There is a maximal antichain A_0 in B of size κ such that $A_0 \cap U = \emptyset$. In other words, U is not κ^+-complete.*

Proof. Let A_0 be a maximal antichain in the open dense set $B \setminus U$. As B is κ^+-cc A_0 has size $\leq\kappa$. It cannot have size $<\kappa$, as U is κ-complete and therefore meets every antichain of size $<\kappa$. □

The Boolean algebra B can be used as forcing notion. As usual, V denotes the universe we start with, sometimes called the ground model. In the following, we will not actually force with B (or any other p.o.); we always remain in V, but we still use forcing notation. In particular, we call the usual B-names "forcing names".

Definition 2.25. A **BUP-name** (or: labeled antichain) x is a function $A \to V$ whose domain is a maximal antichain. We may write $A(x)$ to denote A.

[2]For this, neither κ^+-cc nor atomless is required, and it is sufficient that B is κ-complete.

Each BUP-name corresponds to a forcing-name[3] for an element of V. We will identify the BUP-name and the corresponding forcing-name. In turn, every forcing name τ for an element of V has a forcing-equivalent BUP-name.

In particular, we can calculate, for two BUP-names x and y, the Boolean value $[\![x = y]\!]$.[4]

Definition 2.26.
- Two BUP-names x and y are **equivalent**, if $[\![x = y]\!] \in U$.
- For $v \in V$, let $\check{v}$ be a BUP-name-version of the standard name for v (unique up to equivalence).
- The **Boolean ultrapower** M^- consists of the equivalence classes $[x]$ of BUP-names x; and we define $[x] \in^- [y]$ by $[\![x \in y]\!] \in U$.
- $j^- : V \to M^-$ maps v to $[\check{v}]$.

We are interested in the $\in$-structure $(M^-, \in^-)$.

Given BUP-names $x_1, \dots, x_n$ and an $\in$-formula φ, the truth value $[\![\varphi^V(x_1, \dots, x_n)]\!]$ is well defined (it is the weakest element of B forcing that in the ground model $\varphi(x_1, \dots, x_n)$ holds, which makes sense as $x_1, \dots, x_n$ are guaranteed to be in the ground model).[5]

Lemma 2.27.
- *Łoś's theorem:*
$$(M^-, \in^-) \models \varphi([x_1], \dots, [x_n]) \text{ iff } [\![\varphi^V(x_1, \dots, x_n)]\!] \in U.$$
- *$j^- : (V, \in) \to (M^-, \in^-)$ is an elementary embedding.*
- *In particular, $(M^-, \in^-)$ is a ZFC model.*

Proof. Straightforward by the definition of equivalence and of $[x] \in^- [y]$, and by induction (using that U is a filter for $\varphi \wedge \psi$ and for $\exists v\, \varphi(v)$, and that it is an ultrafilter for $\neg\varphi$). For elementarity, note that $M^- \models \varphi([\check{x}_1], \dots, [\check{x}_n])$ iff $[\![\varphi^V(\check{x}_1, \dots, \check{x}_n)]\!] \in U$ iff $V \models \varphi(x_1, \dots, x_n)$. $\square$

Lemma 2.28. *$(M^-, \in^-)$ is wellfounded.*

Proof. This is the standard argument, using the fact that U is σ-complete:

Assume $[x_{n+1}] \in^- [x_n]$ for $n \in \omega$. Choose a common refinement A of the antichains $A(x_n)$. Again, let x'_n be the BUP-names with domain A equivalent to x_n.

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[3] more specifically, to the forcing-name $\{(\overset{\smile}{x(a)}, a) : a \in A(x)\}$.

[4] We can calculate $[\![x = y]\!]$ more explicitly as follows: Pick some common refinement A' of $A(x)$ and $A(y)$. This defines in an obvious way BUP-names x' and y' both with domain A': For $a \in A'$ we set $x'(a) = x(\tilde{a})$ for $\tilde{a}$ the unique element of $A(x)$ above a. Then $[\![x = y]\!]$ is $\bigvee\{a \in A' : x'(a) = y'(a)\}$ (which is independent of the refinement A').

[5] Equivalently, we can explicitly calculate $[\![\varphi^V(x_1, \dots, x_n)]\!]$ as follows: Chose a common refinement A' of $A(x_1), \dots, A(x_n)$, and set $[\![\varphi^V(x_1, \dots, x_n)]\!]$ to be $\bigvee\{a \in A' : \varphi(x'_1(a), \dots, x'_n(a))\}$; where again the BUP-names x'_i are the canonically defined BUP-names with domain A' that are equivalent to x_i.

So, by our assumption, $u_n := [\![x_{n+1} \in x_n]\!] = \bigvee\{a \in A : x'_{n+1}(a) \in x'_n(a)\}$ is in U for each n. As U is σ-complete, there is some $u \in U$ stronger than all u_n. This implies: If $a \in A$ is compatible with u, then a is compatible with u_n (for all n), and therefore $x'_{n+1}(a) \in x'_n(a)$ for all n, a contradiction $\square$

Definition 2.29. Let M be the transitive collapse of $(M^-, \in^-)$, and let $j : V \to M$ be the composition of j^- with the collapse. We denote the collapse of $[x]$ by x^U. So in particular $\check{v}^U = j(v)$.

Lemma 2.30.

- *$M \vDash \varphi(x_1^U, \dots, x_n^U)$ iff $[\![\varphi^V(x_1, \dots, x_n)]\!] \in U$. In particular, $j : V \to M$ is an elementary embedding.*
- *If $|Y| < \kappa$, then $j(Y) = j''Y$. In particular, j restricted to κ is the identity. M is closed under $<\kappa$-sequences.*
- *$j(\kappa) \neq \kappa$, i.e., $\kappa = \mathrm{cr}(j)$.*

Proof. If $[x] \in j^-(Y)$, then we can refine the antichain $A(x)$ to some A' such that each $a \in A'$ either forces $x = y$ for some $y \in Y$, or $x \notin Y$. Without loss of generality (by taking suprema), we can assume different elements a of A' giving different values $y(a)$; i.e., A' has size $|Y| + 1 < \kappa$. So U selects an element a of A', and as $[\![x \in Y]\!] \in U$, this element a proves that $[x] = j^-(y(a))$.

We have already mentioned that there is a maximal antichain $A_0 = \{a_i : i \in \kappa\}$ of size κ such that $A_0 \cap U = \emptyset$. The BUP-name x with $A(x) = A_0$ and $x(a_i) = i$ satisfies $[x] \in^- j^-(\kappa)$, but is not equivalent to any $\check{v}$; so $\kappa \leq x^U < j(\kappa)$. $\square$

As we have already mentioned, an arbitrary forcing-name for an element of V has a forcing-equivalent BUP-name, i.e., a maximal antichain labeled with elements of V. If τ is a forcing-name for an element of Y ($Y \in V$), then without loss of generality τ corresponds to a maximal antichain labeled with elements of Y. We call such an object y a "BUP-name for an element of $j(Y)$" (and not "for an element of Y", for the obvious reason: unlike in the case of a forcing extension, y^U is generally not in Y, but, by definition of $\in^-$, it is in $j(Y)$).

2.2.2 The algebra and the filter

We will now define the concrete Boolean algebra we are going to use:

Definition 2.31. Assume GCH, let κ be strongly compact, and $\theta > \kappa$ regular.

$P_{\kappa,\theta}$ is the forcing notion adding θ Cohen subsets of κ. More concretely: $P_{\kappa,\theta}$ consists of partial functions from θ to κ with domain of size $<\kappa$, ordered by extension. Let $f^* : \theta \to \kappa$ be the name of the generic function.

$\mathcal{B}_{\kappa,\theta}$ is the complete Boolean algebra generated by $P_{\kappa,\theta}$.

Clearly $\mathcal{B}_{\kappa,\theta}$ is κ^+-cc and κ-distributive, as $P_{\kappa,\theta}$ is even κ-closed.

Lemma 2.32. *There is a κ-complete ultrafilter U on $B = \mathcal{B}_{\kappa,\theta}$ such that:*

a. The Boolean ultrapower gives an elementary embedding $j : V \to M$. M is closed under $<\kappa$-sequences.

b. The elements x^U of M are exactly (the collapses of equivalence classes of) B-names x for elements of V; more concretely, a function from an antichain (of size κ) to V. We sometimes say "x^U is a mixture of κ many possibilities".

Similarly, for $Y \in V$, the elements x^U of $j(Y)$ correspond to the B-names x of elements of Y, i.e., antichains labeled with elements of Y.

c. If $|A| < \kappa$, then $j''A = j(A)$. In particular, j restricted to κ is the identity.

d. j has critical point κ, $\mathrm{cf}(j(\kappa)) = \theta$, and $\theta \leq j(\kappa) \leq \theta^+$.

e. If $\lambda > \kappa$ is regular, then $\max(\theta, \lambda) \leq j(\lambda) < \max(\theta, \lambda)^+$.

f. If S is a $<\lambda$-directed partial order, and $\kappa < \lambda$, then $j''S$ is cofinal in $j(S)$.

g. If $\mathrm{cf}(\alpha) \neq \kappa$, then $j''\alpha$ is cofinal in $j(\alpha)$, so in particular $\mathrm{cf}(j(\alpha)) = \mathrm{cf}(\alpha)$.

Proof. We have already seen (a)–(c).

(d): For each $\delta \in \theta$, $f^*(\delta)$ is a forcing-name for an element of κ, and thus a BUP-name for an element of $j(\kappa)$. Let x be some other BUP-name for an element of $j(\kappa)$, i.e., an antichain A of size κ labeled with elements of κ. Let $\delta \in \theta$ be bigger than the supremum of $\mathrm{supp}(a)$ for each $a \in A$. We call such a pair (x, δ) "suitable", and set $b_{x,\delta} \coloneqq [\![f^*(\delta) > x]\!]$. We claim that all these elements form a basis for a κ-complete filter. To see this, fix suitable pairs (x_i, δ_i) for $i < \mu$ where $\mu < \kappa$; we have to show that $\bigwedge_{i\in\mu} b_{x_i,\delta_i} \neq \mathbb{0}$. Enumerate $\{\delta_i : i \in \mu\}$ increasing (and without repetitions) as δ^j for $j \in \gamma \leq \mu$. Set $A_j = \{i : \delta_i = \delta^j\}$. Given q_j, define $q_{j+1} \in P_{\kappa,\theta}$ as follows: $q_{j+1} \leq q_j$; $\delta^j \in \mathrm{supp}(q_{j+1}) \subseteq \delta^j \cup \{\delta^j\}$; and $q_{j+1} \restriction \delta^j$ decides for all $i \in A_j$ the values of x_i to be some α_i; and $q_{j+1}(\delta^j) = \sup_{i\in A_j}(\alpha_i) + 1$. For $j \leq \gamma$ limit, let q_j be the union of $\{q_k : k < j\}$. Then q_γ is stronger than each b_{x_i,δ_i}.

As κ is strongly compact, we can extend the κ-complete filter generated by all b_{x_i,δ_i} to a κ-complete ultrafilter U. Then the sequence $(f^*(\delta)^U)_{\delta\in\theta}$ is strictly increasing (as $(f^*(\delta), \delta')$ is suitable for all $\delta < \delta'$) and cofinal in $j(\kappa)$ (as we have just seen); so $\mathrm{cf}(j(\kappa)) = \theta$.

(e): We count all BUP-names for elements of $j(\lambda)$. As we can assume that the antichains are subsets of $P_{\kappa,\theta}$, which has size θ, and as λ is regular and GCH holds, we get $|j(\lambda)| \leq [\theta]^\kappa \times \lambda^\kappa = \max(\theta, \lambda)$.

(f): An element x^U of $j(S)$ is a mixture of κ many possibilities in S. As $\kappa < \lambda$, there is some $t \in S$ above all the possibilities. Then $j(t) > x^U$.

(g): Set $\mu = \mathrm{cf}(\alpha)$, and pick an increasing cofinal sequence $\bar\beta = (\beta_i)_{i\in\mu}$ in α. $j(\bar\beta)$ is increasing cofinal in $j(\alpha)$ (as this is absolute between M and V). If $\mu < \kappa$, then $j''\bar\beta = j(\bar\beta)$, otherwise use (f). $\square$

2.2.3 The ultrapower of a forcing notion

We now investigate the relation of a forcing notion $P \in V$ and its image $j(P) \in M$, which we use as a forcing notion over V. (Think of P as being one of the forcings of Section 2.1; it has no relation with the boolean algebra B.)

Note that as $j(P) \in M$ and M is transitive, every $j(P)$-generic filter G over V is trivially generic over M as well, and we will use absoluteness between $M[G]$ and $V[G]$ to prove various properties of $j(P)$.

Lemma 2.33. *If P is κ-cc, then j gives a complete embedding from P into $j(P)$. I.e., $j''P$ is a complete subforcing of $j(P)$, and j is an isomorphism from P to $j''P$.*

Proof. It is clear that j is an isomorphism onto $j''P$: By definition the order $<_{j(P)}$ on $j(P)$ is $j(<_P)$, and by elementarity $p \leq_P q$ iff $j(q) <_{j(P)} j(p)$. Also, $p \perp q$ is preserved: $M \vDash p \perp_{j(P)} q$ by elementarity, so $p \perp_{j(P)} q$ holds in V (as $j(P) \in M$ and M is transitive).

It remains to be shown that each maximal antichain A of P is preserved, i.e., $j''A \subseteq j(P)$ is predense.

By our assumption, $|A| < \kappa$, so $j''A = j(A)$ (by Lemma 2.32(c)), which is maximal in M (by elementarity) and thus maximal in V (by absoluteness). $\square$

Accordingly, we can canonically translate P-names into $j(P)$-names, etc.

For later reference, let us make this a bit more explicit: Let g be a P-name for a real (i.e., an element of ω^ω). Each $g(n)$ is decided by a maximal antichains A_n, where $a \in A_n$ forces $g(n) = g_{n,a} \in \omega$. Then the $j(P)$-name $j(g)$ corresponds to the antichains

$$j(A_n) = j''A_n, \text{ and } j(a) \text{ forces } j(g)(n) = g_{n,a} \text{ for each } a \in A_n. \tag{2.34}$$

Lemma 2.35. *If $P = (P_\alpha, Q_\alpha)_{\alpha<\delta}$ is a finite support (FS) ccc iteration of length δ, then $j(P)$ is a FS ccc iteration of length $j(\delta)$ (more formally: it is canonically equivalent to one).*

Proof. M certainly thinks that $j(P) = (P^*_\alpha, Q^*_\alpha)_{\alpha<j(\delta)}$ is a FS iteration of length $j(\delta)$.

By induction on α we define the FS ccc iteration $(\tilde{P}_\alpha, \tilde{Q}_\alpha)_{\alpha<j(\delta)}$ and show that P^*_α is a dense subforcing of $\tilde{P}_\alpha$: Assume this is already the case for P^*_α. M thinks that Q^*_α is a P^*_α-name, so we can interpret it as a $\tilde{P}_\alpha$-name and use it as $\tilde{Q}_\alpha$. Assume that (p, q) is an element (in V) of $\tilde{P}_\alpha * \tilde{Q}_\alpha$. So p forces that q is a name in M; we can increase p to some p' that decides q to be the name $q' \in M$. By induction we can further increase p' to $p'' \in P^*_\alpha$, then $(p'', q') \in P^*_{\alpha+1}$ is stronger than (p, q). (At limits there is nothing to do, as we use FS iterations.)

$j(P)$ is ccc, as any $A \subseteq j(P)$ of size $\aleph_1$ is in M (and M thinks that $j(P)$ is ccc). $\square$

Similarly, we get:

- If $\tau = x^U$ is in M a $j(P)$-name for an element of $j(Z)$, then τ is a mixture of κ many P-names for an element of Z (i.e., the BUP-name x consists of an antichain $A \subseteq B$ labeled, without loss of generality, with P-names for elements of Z).

  (This is just the instance of "each $x^U \in j(Y)$ is a mixture of elements of Y", where we set Y to be the set[6] of P-names for elements of Z.)

- A $j(P)$-name τ for an element of $M[G]$ has an equivalent $j(P)$-name in M.

  (There is a maximal antichain A of $j(P)$ labeled with $j(P)$-names in M. As M is countably closed, this labeled antichain is in M, and gives a $j(P)$-name in M equivalent to τ.)

- In $V[G]$, $M[G]$ is closed under $<\kappa$ sequences.

  (We can assume the names to be in M and use $<\kappa$-closure.)

- In particular, every $j(P)$-name for a real, a Borel-code, a countable sequence of reals, etc., is in M (more formally: has an equivalent name in M).

- If each iterand is forced to consist of reals, then $j(P)$ forces the continuum to have size at most $|2 + j(\delta)|^{\aleph_0}$.

  (This follows from Lemma 2.17 as $j(P)$ also satisfies that each iterand consists of reals.)

2.2.4 Preservation on values of characteristics

Lemma 2.36. *Let λ be a regular uncountable cardinal and P a ccc forcing.*

a. Let $\mathfrak{x}$ be either $\mathrm{add}(\mathcal{N})$ *or* $\mathfrak{b}$. *If $P \Vdash \mathfrak{x} = \lambda$ and $\kappa \neq \lambda$, then $j(P) \Vdash \mathfrak{x} = \lambda$.*

b. Let $\mathfrak{y}$ be either $\mathrm{cof}(\mathcal{N})$ *or* $\mathfrak{d}$. *If $P \Vdash \mathfrak{y} \geq \lambda$ and $\kappa < \lambda$, then $j(P) \Vdash \mathfrak{y} \geq \lambda$.*

c. Let $(\mathfrak{x}, \mathfrak{y})$ be either $(\mathfrak{b}, \mathfrak{d})$ *or* $(\mathrm{add}(\mathcal{N}), \mathrm{cof}(\mathcal{N}))$. *Then we get:*
If $P \Vdash (\,\kappa < \mathfrak{x} \;\&\; \mathfrak{y} \leq \lambda\,)$ then $j(P) \Vdash \mathfrak{y} \leq \lambda$.

Proof. **(a)** We formulate the proof for $\mathrm{add}(\mathcal{N})$; the proof for $\mathfrak{b}$ is the same.

Let $\bar{N} = (N_i)_{i<\lambda}$ be P-names for an increasing sequence of null sets such that $\bigcup_{i<\lambda} N_i$ is not null. So in particular for every P-name N of a null set: $(\exists i_0 \in \lambda)\,(\forall i \in \lambda \setminus i_0)\, P \Vdash N_i \nsubseteq N$. (We can choose the i_0 in V due to ccc.)

Therefore M thinks that the same holds for the sequence $j(\bar{N})$ of $j(P)$-names of length $j(\lambda)$. So whenever N is a $j(P)$-name of a null set, we can assume without loss of generality that $N \in M$, so M thinks that from some i_0 on it is forced that $N_i \nsubseteq N$, which is absolute.

[6]Formally: We set Y to be some set that contains representatives of each equivalence class of P-names of elements of Z.

As $\kappa \neq \lambda$, we know that $j''\lambda$ is cofinal in $j(\lambda)$. So (since the sequence $j(\bar{N})$ is increasing) we can use $(j(N_i))_{i\in\lambda}$ and get the same property.

This shows that $j(P) \Vdash \mathrm{add}(\mathcal{N}) \leq \lambda$

For the other inequality, fix some $\chi < \lambda$, and $(N_i)_{i<\chi}$ a family of $j(P)$-names for null sets (without loss of generality each name is in M), and $p \in j(P)$.

- Case 1: $\kappa \geq \lambda$. Then the sequence $(N_i)_{i<\chi}$ (as well as p) is in M, and $M \models \big(p \Vdash \bigcup N_i \text{ null}\big)$; which is absolute.

- Case 2: $\kappa < \lambda$. Every N_i is a "mixture" of κ many P-names for null sets, so there is a single P-name N_i' such that P forces N_i' is superset of all the names involved. Therefore, $j(P)$ forces that $j(N_i') \supseteq N_i$. And P forces that $\bigcup_{i<\chi} N_i'$ is null, i.e., covered by some null set N^*. Then $j(P)$ forces that $j(N^*)$ covers $\bigcup_{i<\chi} N_i$.

(b) We show that a small set cannot be dominating: Fix a sequence $(f_i)_{i<\chi}$ of $j(P)$-names of reals, with $\chi < \lambda$. Each f_i corresponds to $\kappa < \lambda$ many possible P-names. As $\chi < \lambda$, there is a P-name g unbounded by all $\chi \times \kappa < \lambda$ many possible P-names. So if f is any of the possibilities, then P forces $g \not\leq^* f$; and thus $j(P)$ forces $j(g) \not\leq^* f_i$ for all i. So $j(P)$ forces $\mathfrak{d} \geq \lambda$.

The same proof works for $\mathrm{cof}(\mathcal{N})$ (using "the null set g is not a subset of any of the possible null sets").

(c) For $(\mathfrak{x}, \mathfrak{y}) = (\mathfrak{b}, \mathfrak{d})$: Fix a P-name of a dominating family $\bar{f} = (f_i)_{i\in\lambda}$.

We claim that $j(P)$ forces that $j''\bar{f} = (j(f_i))_{i<\lambda}$ is dominating. Let r be a $j(P)$-name of a real, i.e., a mixture of κ many possibilities (each possibility corresponding to a P-name for a real). As $P \Vdash \kappa < \mathfrak{b}$, P forces that these reals cannot be unbounded, i.e., there is a P-name $\alpha \in \lambda$ such that f_α is forced to dominate all the possibilities. By absoluteness, $j(P) \Vdash j(f_\alpha) >^* r$.

It remains to be shown that $j(P) \Vdash j(f_\alpha) \in j''\bar{f}$. (Note that α is just a P-name.) Fix a maximal antichain A in P deciding α, i.e., $a \in A$ forces $\alpha = \alpha(a)$. As j maps P completely into $j(P)$, $j''A$ is a maximal antichain in $j(P)$. So $j(P)$ forces that exactly on $j(a)$ for $a \in A$ is in the generic filter, cf. (2.34). Accordingly $j(f_\alpha) = j(f_{\alpha(a)}) \in j''\bar{f}$.

The proof for $\mathrm{cof}(\mathcal{N})$ is the same. □

For the other direction of the invariants, and the pair $(\mathrm{cov}(\mathcal{N}), \mathrm{non}(\mathcal{N}))$, we use the following two lemmas, which are reformulations of results of Shelah.[7]

Recall Definition 2.7 (which is useful because of Lemma 2.8 and satisfied for the inital forcing according to Lemma 2.18).

Lemma 2.37. *Assume* $\mathsf{LCU}_i(P, \lambda)$. *Then* $\mathsf{LCU}_i(j(P), \mathrm{cf}(j(\lambda)))$.
So if $\kappa \neq \lambda$, *then* $\mathsf{LCU}_i(j(P), \lambda)$, *and if* $\kappa = \lambda$, *then* $\mathsf{LCU}_i(j(P), \theta)$.

[7] S. Shelah, personal communication.

Proof. Let $\bar{y} = (y_\alpha)_{\alpha<\lambda}$ be the sequence of P-names witnessing $\mathsf{LCU}_i(P, \lambda)$. Note that $j(\bar{y})$ is a sequence of length $j(\lambda)$; we denote the β-th element by $(j(\bar{y}))_\beta$. So M thinks: For every $j(P)$-name r of a real $(\exists \alpha \in j(\lambda))\,(\forall \beta \in j(\lambda) \setminus \alpha)\,\neg(j(\bar{y}))_\beta \,\mathrm{R}_i\, r$. This is absolute. In particular, pick in V a cofinal subset A of $j(\lambda)$ of order type $\mathrm{cf}(j(\lambda)) =: \mu$. Then $j(\bar{y}) \restriction A$ witnesses that $\mathsf{LCU}_i(j(P), \mu)$ holds. $\square$

We have seen in Lemma 2.20 that $\mathsf{COB}_2(P^5, \lambda_2, \lambda_4)$ holds and implies that $\tilde{P}^4$ forces $\mathrm{cov}(\mathcal{N}) \geq \lambda_2$ and $\mathrm{non}(\mathcal{N}) \leq \lambda_4$ (which is trivial in the case of $\tilde{P}^4$).

Lemma 2.38. *Assume* $\mathsf{COB}_2(P, \lambda, \mu)$. *If* $\kappa > \lambda$, *then* $\mathsf{COB}_2(j(P), \lambda, |j(\mu)|)$*; if* $\kappa < \lambda$, *then* $\mathsf{COB}_2(j(P), \lambda, \mu)$.

Proof. Let $(S, \prec)$ and $\bar{r}$ witness $\mathsf{COB}_2(P, \lambda, \mu)$.

M thinks that

for each $j(P)$-name N of a null set

$$(\exists s \in j(S))\,(\forall t \in j(S))\, t \succ s \rightarrow j(P) \Vdash (j(\bar{r}))_t \notin N, \quad (*)$$

which is absolute.

If $\kappa > \lambda$, then $j(\lambda) = \lambda$, and $j(S)$ is λ-directed in M and therefore in V as well, and so we get $\mathsf{COB}_2(j(P), \lambda, |j(\mu)|)$.

So assume $\kappa < \lambda$. We claim that $j''(S)$ and $j''\bar{r}$ witness $\mathsf{COB}_2(j(P), \lambda, \mu)$. $j''S$ is isomorphic to S, so directedness is trivial. Given a $j(P)$-name N, without loss of generality in M, there is in M a bound $s \in j(S)$ as in $(*)$. As $j''S$ is cofinal in $j(S)$ (according to Lemma 2.32(f)), there is some $s' \in S$ such that $j(s') \succ s$. Then for all $t' \succ s'$, i.e., $j(t') \succ j(s')$, we get $j(P) \Vdash j(r_t) \notin N$. $\square$

2.2.5 The main theorem

We now have all everything required for the main result:

Theorem 2.39. *Assume GCH and that* $\aleph_1 < \kappa_7 < \lambda_1 < \kappa_6 < \lambda_2 < \kappa_5 < \lambda_3 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7$ *are regular,* κ_i *strongly compact for* $i = 5, 6, 7$. *Then there is a ccc order* P^7 *forcing*

$$\mathrm{add}(\mathcal{N}) = \lambda_1 < \mathrm{cov}(\mathcal{N}) = \lambda_2 < \mathfrak{b} = \lambda_3 < \mathfrak{d} = \lambda_4 < \mathrm{non}(\mathcal{N}) = \lambda_5$$
$$< \mathrm{cof}(\mathcal{N}) = \lambda_6 < 2^{\aleph_0} = \lambda_7.$$

Proof. Let $j_i : V \rightarrow M_i$ be the Boolean ultrapower embedding with $\mathrm{cf}(j(\kappa_i)) = \lambda_i$ (for $i = 5, 6, 7$). We set $\tilde{P}^5 := j_5(\tilde{P}^4)$, $\tilde{P}^6 := j_6(\tilde{P}^5)$, and $\tilde{P}^7 := j_7(\tilde{P}^6)$; and $j_6(\delta_5) =: \delta_6$ and $j_7(\delta_6) =: \delta_7$.

It is enough to show the following:

a. $\tilde{P}^i$ is a FS ccc iteration of length δ_i and forces $2^{\aleph_0} = \lambda_i$ for $i = 4, 5, 6, 7$.

b. $\tilde{P}^i \Vdash (\,\mathrm{add}(\mathcal{N}) = \lambda_1 \;\&\; \mathfrak{b} = \lambda_3 \;\&\; \mathfrak{d} = \lambda_4\,)$ for $i = 4, 5, 6, 7$.

c. $\tilde{P}^i \Vdash \mathrm{non}(\mathcal{N}) \geq \lambda_5$ for $i = 5, 6, 7$.
$\tilde{P}^i \Vdash \mathrm{cof}(\mathcal{N}) \geq \lambda_6$ for $i = 6, 7$.
$\tilde{P}^i \Vdash \mathrm{cov}(\mathcal{N}) \leq \lambda_2$ for $i = 4, 5, 6, 7$.

d. $\tilde{P}^i \Vdash \mathrm{cof}(\mathcal{N}) = \lambda_6$ for $i = 6, 7$.

e. $\tilde{P}^i \vDash (\,\mathrm{cov}(\mathcal{N}) \geq \lambda_2\ \&\ \mathrm{non}(\mathcal{N}) \leq \lambda_5\,)$ for $i = 4, 5, 6, 7$.

(a) was shown in Section 2.2.3.

(b): For $\tilde{P}^4$ this is Theorem 2.19). For $\tilde{P}^5$ use Lemma 2.36 (using for $\mathfrak{d}$ that $\kappa_5 < \lambda_3$). Using the same lemma again we get the result for $\tilde{P}^6$ and $\tilde{P}^7$ (using that $\kappa_i < \lambda_3$ for $i = 6, 7$ as well.)

(c): As $\kappa_5 > \lambda_2$, we have $\mathsf{LCU}_2(\tilde{P}^4, \kappa_5)$ (by Lemma 2.18), and thus $\mathsf{LCU}_2(\tilde{P}^5, \lambda_5)$ (by Lemma 2.37, as $\mathrm{cf}(j_5(\kappa_5)) = \lambda_5$), so $\tilde{P}^5 \Vdash \mathrm{non}(\mathcal{N}) \geq \lambda_5$ (Lemma 2.8). Repeating the same argument we get $\mathsf{LCU}_2(\tilde{P}^i, \lambda_5)$ for $i = 6, 7$ (as $\kappa_i \neq \lambda_5$ for $i = 6, 7$).

Analogously, as $\kappa_6 > \lambda_1$, we start with $\mathsf{LCU}_1(\tilde{P}^4, \kappa_6)$, get $\mathsf{LCU}_1(\tilde{P}^5, \kappa_6)$ (as $\kappa_5 \neq \kappa_6$) and then $\mathsf{LCU}_1(\tilde{P}^6, \lambda_6)$ (as $\mathrm{cf}(j_6(\kappa_6)) = \lambda_6$) and $\mathsf{LCU}_1(\tilde{P}^7, \lambda_6)$ (again as $\kappa_7 \neq \lambda_6$). So we get thus $\tilde{P}^i \Vdash \mathrm{cof}(\mathcal{N}) \geq \lambda_6$ for $i = 6, 7$.

Similarly, $\mathsf{LCU}_2(\tilde{P}^4, \lambda_2)$ holds, which is preserved by all embeddings, so we get $\mathrm{cov}(\mathcal{N}) \leq \lambda_2$.

(d): As $\tilde{P}^6$ forces the continuum to have size λ_6, the previous item implies $\tilde{P}^6 \Vdash \mathrm{cof}(\mathcal{N}) = \lambda_6$. And as in (b), this implies the same for $\tilde{P}^7$ (as $\kappa_7 < \lambda_1$, the value of $\mathrm{add}(\mathcal{N})$).

(e): $\mathsf{COB}_2(\tilde{P}^4, \lambda_2, \lambda_4)$ holds (cf. Lemma 2.20). So by Lemma 2.38 for the case $\kappa > \lambda$, and as $|j_5|(\lambda_4) = \lambda_5$, according to Lemma 2.32(e), $\mathsf{COB}_2(\tilde{P}^5, \lambda_2, \lambda_5)$ holds. I.e., $\tilde{P}^5$ forces $\mathrm{cov}(\mathcal{N}) \geq \lambda_2$ and $\mathrm{non}(\mathcal{N}) \leq \lambda_5$ (the latter being trivial as the continuum has size λ_5). For $i = 6, 7$, the same lemma, now for the case $\kappa < \lambda$, gives $\mathsf{COB}_2(\tilde{P}^i, \lambda_2, \lambda_5)$, i.e., $\tilde{P}^i$ forces $\mathrm{cov}(\mathcal{N}) \geq \lambda_2$ and $\mathrm{non}(\mathcal{N}) \leq \lambda_5$.

□

2.2.6 An alternative

In the same way we can prove the consistency of

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

(I.e., we can replace $\mathfrak{b}$ and $\mathfrak{d}$ by $\mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M})$, respectively.)

For this, we use the following relation as R_3:

$$f\ \mathrm{R}_3\ g, \text{ if } f, g \in \omega^\omega \text{ and } (\forall^* n \in \omega)\, f(n) \neq g(n).$$

By a result of [Mil82; Bar87] (cf. [BJ95, 2.4.1 and 2.4.7]) we have

$$\mathrm{non}(\mathcal{M}) = \mathfrak{b}_3 \text{ and } \mathrm{cov}(\mathcal{M}) = \mathfrak{d}_3.$$

As before, we use that iterations where each iterand has size $<\lambda_3$ is (λ_3, R_3)-good.

To define $\tilde{P}^4$, we use partial eventually different (instead of partial Hechler) forcings.

Unlike for $(\mathfrak{b}, \mathfrak{d})$, we do not know whether $\mathrm{non}(\mathcal{M}) = \lambda$ is generally preserved if $\kappa \neq \lambda$ and $\mathrm{cov}(\mathcal{M}) = \lambda$ is preserved if κ is small; but we can use the same argument for $(\mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}))$ that we have used for $(\mathrm{cov}(\mathcal{N}), \mathrm{non}(\mathcal{N}))$. So we can get the analogous of Lemma 2.20) that proves that $\mathrm{non}(\mathcal{M})$ is large and $\mathrm{cov}(\mathcal{M})$ small; and LCU_3 implies that $\mathrm{non}(\mathcal{M})$ is small and $\mathrm{cov}(\mathcal{M})$ large.

Chapter 3

Another ordering of ten values

This chapter is based on the paper “Another ordering of the ten cardinal characteristics in Cichoń’s diagram” ([KST17], arXiv:1712.00778), joint work with J. Kellner and S. Shelah, accepted for publication in the special issue of "Commentationes Mathematicae Universitatis Carolinae" in honor of Bohuslav Balcar.

We show the consistency of

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}.$$

Assuming four strongly compact cardinals, it is consistent that

$$\begin{aligned}\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) <& \\ < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{M}) = \mathfrak{d} < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}&.\end{aligned}$$

Introduction

We assume that the reader is familiar with basic properties of Amoeba, Hechler, random and Cohen forcing, and with the cardinal characteristics in Cichoń’s diagram, given in Figure 3.1: An arrow between $\mathfrak{x}$ and $\mathfrak{y}$ indicates that ZFC proves $\mathfrak{x} \leq \mathfrak{y}$. Moreover, $\max(\mathfrak{d}, \mathrm{non}(\mathcal{M})) = \mathrm{cof}(\mathcal{M})$ and $\min(\mathfrak{b}, \mathrm{cov}(\mathcal{M})) = \mathrm{add}(\mathcal{M})$. These (in)equalities are the only one provable. More precisely, all assignments of the

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

Figure 3.1: Cichoń’s diagram

values $\aleph_1$ and $\aleph_2$ to the characteristics in Cichoń's diagram are consistent, provided they do not contradict the above (in)equalities. (A complete proof can be found in [BJ95, ch. 7].)

In the following, we will only deal with the ten "independent" characteristics listed in Figure 3.2 (they determine $\mathrm{cof}(\mathcal{M})$ and $\mathrm{add}(\mathcal{M})$).

$$\begin{tikzcd}
& \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \square \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
& & \mathfrak{b} \arrow[r] \arrow[u] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \square \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

Figure 3.2: The ten "independent" characteristics.

$$\begin{tikzcd}
& \lambda_2 \arrow[r] & \lambda_4 \arrow[r] & \square \arrow[r] & \lambda_8 \arrow[r] & \lambda_9 \\
& & \lambda_3 \arrow[r] \arrow[u] & \lambda_6 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[r] \arrow[uu] & \square \arrow[r] \arrow[u] & \lambda_5 \arrow[r] \arrow[u] & \lambda_7 \arrow[uu] &
\end{tikzcd}$$

Figure 3.3: The old order.

$$\begin{tikzcd}
& \lambda_3 \arrow[r] & \lambda_4 \arrow[r] & \square \arrow[r] & \lambda_8 \arrow[r] & \lambda_9 \\
& & \lambda_2 \arrow[r] \arrow[u] & \lambda_7 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[r] \arrow[uu] & \square \arrow[r] \arrow[u] & \lambda_5 \arrow[r] \arrow[u] & \lambda_6 \arrow[uu] &
\end{tikzcd}$$

Figure 3.4: The new order.

Regarding the left hand side, it was shown in [GMS16] that consistently

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}. \quad (\mathrm{left}_{\mathrm{old}})$$

(This corresponds to λ_1 to λ_5 in Figure 3.3.) The proof is repeated in [GKS17], in a slightly different form which is more convenient for our purpose. Let us call this construction the "old construction".

In this paper, building on [She00], we give a construction to get a different order for these characteristics, where we swap $\mathrm{cov}(\mathcal{N})$ and $\mathfrak{b}$:

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}. \quad (\mathrm{left}_{\mathrm{new}})$$

(This corresponds to λ_1 to λ_5 in Figure 3.4.)

This construction is more complicated than the old one. Let us briefly describe the reason: In both constructions, we assign to each of the cardinal characteristics of the left hand side a relation R. E.g., we use the "eventually different" relation $\mathrm{R}_4 \subseteq \omega^\omega \times \omega^\omega$ for $\mathrm{non}(\mathcal{M})$. We can then show that the characteristic remains "small" (i.e., is at most the intended value λ in the final model), because all single forcings we use in the iterations are either small (i.e., smaller than λ) or are "R-good". However, $\mathfrak{b}$ (with the "eventually dominating" relation $\mathrm{R}_2 \subseteq \omega^\omega \times \omega^\omega$) is an exception: We do not know any variant of an eventually different forcing (which we need to increase $\mathrm{non}(\mathcal{M})$) which satisfies that all of its subalgebras are R_2-good. Accordingly, the main effort (in both constructions) is to show that $\mathfrak{b}$ remains small.

In the old construction, each non-small forcing is a (σ-centered) subalgebra of the eventually different forcing $\mathbb{E}$. To deal with such forcings, ultrafilter limits of sequences of $\mathbb{E}$-conditions are introduced and used (and we require that all $\mathbb{E}$-subforcings are basically $\mathbb{E}$ intersected with some model, and thus closed under limits of sequences in the model). In the new construction, we have to deal with an additional kind of "large" forcing: (subforcings of) random forcing. Ultrafilter limits do not work any more, but, similarly to [She00], we can use finite additive measures (FAMs) and interval-FAM-limits of random conditions. But now $\mathbb{E}$ doesn't seem to work with interval-FAM-limits any more, so we replace it with a creature forcing notion $\tilde{\mathbb{E}}$.

We also have to show that $\mathrm{cov}(\mathcal{N})$ remains small. In the old construction, we could use a rather simple (and well understood) relation $\mathrm{R}^{\mathrm{old}}$ and use the fact that all σ-centered forcings are $\mathrm{R}^{\mathrm{old}}$-good: As all large forcings are subalgebras of either eventually different forcing or of Hechler forcing, they are all σ-centered. In the new construction, the large forcings we have to deal with are subforcings of $\tilde{\mathbb{E}}$. But $\tilde{\mathbb{E}}$ is not σ-centered, just (ρ, π)-linked for a suitable pair (ρ, π) (a property between σ-centered and σ-linked, first defined in [OK14], see Def. 3.18). So we use a different (and more cumbersome) relation R_3, introduced in [OK14], where it is also shown that (ρ, π)-linked forcings are R_3-good.

Regarding the whole diagram: In [GKS17], starting with the iteration for $(\mathrm{left}_{\mathrm{old}})$, a new iteration is constructed to get simultaneously different values for all characteristics: Assuming four strongly compact cardinals, the following is consistent (cf. Figure 3.3):

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \\ < \mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

The essential ingredient is the concept of the Boolean ultrapower of a forcing notion.

In exactly the same way we can expand our new version $(\mathrm{left}_{\mathrm{new}})$ to the right hand side, where also the characteristics dual to $\mathfrak{b}$ and $\mathrm{cov}(\mathcal{N})$ are swapped. So we get: If four strongly compact cardinals are consistent, then so is the following (cf. Figure 3.4):

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \\ < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathfrak{d} < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

We closely follow the presentation of [GKS17]. Several times, we refer to [GKS17] and to [She00] for details in definitions or proofs. We thank Martin Goldstern and Diego Mejía for valuable discussions, and an anonymous referee for a very detailed and helpful report pointing out (and even fixing) several mistakes in the first version of the paper.

3.1 Finitely additive measure limits and the $\tilde{\mathbb{E}}$-forcing.

3.1.1 FAM-limits and random forcing

We briefly list some basic notation and facts around finite additive measures. (A bit more details can be found in Section 1 of [She00].)

Definition 3.1.
- A "partial FAM" (finitely additive measure) Ξ' is a finitely additive probability measure on a sub-Boolean algebra $\mathcal{B}$ of $\mathcal{P}(\omega)$, the power set of ω, such that $\{n\} \in \mathcal{B}$ and $\Xi'(\{n\}) = 0$ for all $n \in \omega$. We set $\mathrm{dom}(\Xi') = \mathcal{B}$.
- Ξ is a FAM if it is a partial FAM with $\mathrm{dom}(\Xi) = \mathcal{P}(\omega)$.
- For every FAM Ξ and bounded sequence of non-negative reals $\bar{a} = (a_n)_{n\in\omega}$ we can define in the natural way the average (or: integral) $\mathrm{Av}_\Xi(\bar{a})$, a non-negative real number.

[She00, p. 1.2] lists several results that informally say:

> There is a FAM Ξ that assigns the values a_i to the sets A_i (for all i in some index set I) iff for each $I' \subseteq I$ finite and $\epsilon > 0$ there is an arbitrary large[1] finite $u \subseteq \omega$ such that the counting measure on u for A_i approximates a_i with an error of at most ϵ, for all $i \in I'$. (∗)

For the size of such an "ϵ-good approximation" u to some FAM Ξ we can give an upper bound for $|u|$ which only depends on $|I'|$ and ϵ (and not on Ξ):

Lemma 3.2. *Given $N, k^* \in \omega$ and $\epsilon > 0$, there is an $M \in \omega$ such that: For all FAMs Ξ and $(A_n)_{n<N}$ there is a nonempty $u \subseteq \omega$ of size $\leq M$ such that* $\min(u) > k^*$ *and $\Xi(A_n) - \epsilon < \frac{|A_n\cap u|}{|u|} < \Xi(A_n) + \epsilon$ for all $n < N$.*

Proof. We can assume that $\epsilon = \frac{1}{L}$ for an integer L. $\{A_n : n \in N\}$ generates the set algebra $\mathfrak{B} \subseteq \mathcal{P}(\omega)$. Let $\mathcal{X}$ be the set of atoms of $\mathfrak{B}$. So $\mathcal{X}$ is a partition of ω of size $\leq 2^N$. Set $\mathcal{X}' = \{x \in \mathcal{X} : \Xi(x) > 0\}$. Every $x \in \mathcal{X}'$ is infinite, and $\sum_{x\in\mathcal{X}'} \Xi(x) = 1$.

Round $\Xi(x)$ to some number $\Xi^\epsilon(x) = \ell_x \cdot \frac{1}{L\cdot 2^N}$ for some integer $0 \leq \ell_x \leq L\cdot 2^N$, such that $|\Xi(x) - \Xi^\epsilon(x)| < \frac{1}{L\cdot 2^N}$ and $\sum_{x\in\mathcal{X}'} \Xi^\epsilon(x)$ is still 1. So $\sum_{x\in\mathcal{X}'} \ell_x = L \cdot 2^N$, and we construct u consisting of ℓ_x many points that are bigger than k^* and in x (for each $x \in \mathcal{X}'$). □

We will use the following variants of (∗), regarding the possibility to extend a partial FAM Ξ' to a FAM Ξ. The straightforward, if somewhat tedious, proofs are given in [She00, 1.3(G) and 1.7].

[1] Equivalently: "a finite u with arbitrary large minimum", which is the formulation actually used in most of the results.

Fact 3.3. Let Ξ' be a partial FAM, and I some index set.

(a) Fix for each $i \in I$ some $A_i \subseteq \omega$.
If $A \cap \bigcap_{i \in I'} A_i \neq \emptyset$ for all $I' \subseteq I$ finite and $A \in \mathrm{dom}(\Xi')$ with $\Xi'(A) > 0$,
then Ξ' can be extended to a FAM Ξ such that $\Xi(A_i) = 1$ for all $i \in I$.

(b) Fix for each $i \in I$ some real b^i and some bounded sequence of non-negative reals $\bar{a}^i = (a^i_k)_{k \in \omega}$.
If for each finite partition $(B_m)_{m<m^*}$ of ω into elements of $\mathrm{dom}(\Xi')$, for each $\epsilon > 0$, $k^* \in \omega$, and $I' \subseteq I$ finite there is a finite $u \subseteq \omega \setminus k^*$ such that

- for all $m < m^*$, $\Xi'(B_m) - \epsilon \leq \frac{|B_m \cap u|}{|u|} \leq \Xi'(B_m) + \epsilon$, and
- for all $i \in I'$, $\frac{1}{|u|} \sum_{k \in u} a^i_k \geq b^i - \epsilon$,

then Ξ' can be extended to a FAM Ξ such that $\mathrm{Av}_\Xi(\bar{a}^i) \geq b^i$ for all $i \in I$.

We first define what it means for a forcing Q to have FAM limits.

Remark 3.4. Intuitively, this means (in the simplest version): Fix a FAM Ξ. We can define for each sequence q_k of conditions that are all "similar" (e.g., have the same stem and measure) a limit $\lim_\Xi \bar{q}$. And we find in the Q-extension a FAM Ξ' extending Ξ, such that $\lim_\Xi(\bar{q})$ forces that the set of k satisfying $P(k) \equiv$ "$q_k \in G$" has "large" Ξ'-measure. Up to here, we get the notion used in [GMS16] and [GKS17] (but there we use ultrafilters instead of FAMs, and "large" means being in the ultrafilter). However, we need a modification: Instead of single conditions q_k we use a finite sequence $(p_\ell)_{\ell \in I_k}$ (where I_k is a fixed, finite interval); and the condition $P(k)$, which we want to satisfy on a large set, now is "$\frac{|\{\ell \in I_k : p_\ell \in G\}|}{|I_k|} > b$" for some suitable b. This is the notion used implicitly in [She00].

Notation. Let T^* be a compact subtree of $\omega^{<\omega}$, for example $T^* = 2^{<\omega}$. Let $s, t \in T^*$. Let S be a subtree of T^*.

- $t \rhd s$ means "t is immediate successor of s".
- $|s|$ is the length of s (i.e.: the height, or level, of s).
- $[t]$ is the set of nodes in T^* comparable with t.
- We set $\lim(S) = \{x \in \omega^\omega : (\forall n \in \omega)\ x \restriction n \in S\}$.
- $\mathrm{trunk}(S)$ is the smallest splitting node of S. With "$t \in S$ above the stem" we mean that $t \in S$ and $t \geq \mathrm{trunk}(S)$; or equivalently: $t \in S$ and $|t| \geq |\mathrm{trunk}(S)|$.
- Leb is the canonical measure on the Borel subsets of $\lim(T^*)$. We also write $\mathrm{Leb}(S)$ instead of $\mathrm{Leb}(\lim(S))$.[2]

[2] I.e., we define $\mathrm{Leb}([s])$ by induction on the height of $s \in T^*$ as follows: $\mathrm{Leb}(T^*) = 1$, and if s has n many immediate successors in T^*, then $\mathrm{Leb}([t]) = \frac{\mathrm{Leb}([s])}{n}$ for any such successor. This defines a measure on each basic clopen set, which in turn defines a (probability) measure on the Borel subsets of $\lim(T^*)$ (a closed subset of ω^ω).

We fix, for the rest of the paper, an interval partition $\bar{I} = (I_k)_{k\in\omega}$ of ω such that $|I_k|$ converges to infinity. We will use forcing notions Q satisfying the following setup:

Assumption 3.5.
- $Q' \subseteq Q$ is dense and the domain of functions $\mathtt{trunk}$ and loss, where $\mathtt{trunk}(q) \in H(\aleph_0)$ and loss(q) is a non-negative rational.
- For each $\epsilon > 0$ the set $\{q \in Q' : \text{loss}(q) < \epsilon\}$ is dense (in Q' and thus in Q).
- $\{p \in Q' : (\mathtt{trunk}(p), \text{loss}(p)) = (\mathtt{trunk}^*, \text{loss}^*)\}$ is $\lfloor\frac{1}{\text{loss}^*}\rfloor$-linked. I.e., each $\lfloor\frac{1}{\text{loss}^*}\rfloor$ many such conditions are compatible.[3]

In this paper, Q will be one of the following two forcing notions: random forcing, or $\tilde{\mathbb{E}}$ (as defined in Definition 3.12). We will now specify the instance of random forcing that we will use:

Definition 3.6.
- A random condition is a tree $T \subseteq 2^{<\omega}$ such that $\text{Leb}(T \cap [t]) > 0$ for all $t \in T$.
- $\mathtt{trunk}(T)$ is the stem of T (i.e., the shortest splitting node).
- If $\text{Leb}(T) = \text{Leb}([\mathtt{trunk}(T)])$, we set $\text{loss}(T) = 0$. Otherwise, let m be the maximal natural number such that
$$\text{Leb}(T) > \text{Leb}([\mathtt{trunk}(T)])(1 - \frac{1}{m})$$
and set[4] $\text{loss}(T) = \frac{1}{m}$.

Note that $\text{Leb}(T) \geq 2^{-|\mathtt{trunk}(T)|}(1 - \text{loss}(T))$ (and the inequality is strict if $\text{loss}(T) > 0$).

Note that this definition of random forcing satisfies Assumption 3.5 (with $Q' = Q$).

Definition 3.7. Fix Q and functions $(\mathtt{trunk}, \text{loss})$ as in Assumption 3.5, a FAM Ξ and a function $\lim_\Xi : Q^\omega \to Q$. Let us call the objects mentioned so far a "limit setup". Let a $(\mathtt{trunk}^*, \text{loss}^*)$-sequence be a sequence $(q_\ell)_{\ell\in\omega}$ of Q-conditions such that $\mathtt{trunk}(q_\ell) = \mathtt{trunk}^*$ and $\text{loss}(q_\ell) = \text{loss}^*$ for all $\ell \in \omega$.

We say "$\lim_\Xi$ is a strong FAM limit for intervals", if the following is satisfied: Given

- a pair $(\mathtt{trunk}^*, \text{loss}^*)$, $j^* \in \omega$, and $(\mathtt{trunk}^*, \text{loss}^*)$-sequences $\bar{q}^j$ for $j < j^*$,
- $\epsilon > 0$, $k^* \in \omega$,
- $m^* \in \omega$ and a partition of ω into sets B_m ($m \in m^*$), and

[3] In [She00, p. 2.9], $\mathtt{trunk}$ and loss are called h_2 and h_1; and instead of I_k the interval is called $[n^*_k, n^*_{k+1} - 1]$. Moreover, in [She00] the sequence $(n^*_k)_{k\in\omega}$ is one of the parameters of a "blueprint", whereas we assume that the I_k are fixed.

[4] In [She00], this is implicit in 2.11(f).

- a condition q stronger than all $\lim_\Xi(\bar{q}^j)$ for all $j < j^*$,

there is a finite $u \subseteq \omega \setminus k^*$ and a q' stronger than q such that

- $\Xi(B_m) - \epsilon < \frac{|u \cap B_m|}{|u|} < \Xi(B_m) + \epsilon$ for $m < m^*$,
- $\frac{1}{|u|} \sum_{k \in u} \frac{|\{\ell \in I_k : q' \leq q^j_\ell\}|}{|I_k|} \geq 1 - \text{loss}^* - \epsilon$ for $j < j^*$

(We are only interested in $\lim_\Xi(\bar{q})$ for $\bar{q}$ as above, so we can set $\lim_\Xi(\bar{q})$ to be undefined or some arbitrary value for other $\bar{q} \in Q^\omega$.)

The motivation for this definition is the following:

Lemma 3.8. *Assume that* $\lim_\Xi$ *is such a limit. Then there is a* Q*-name* Ξ^+ *such that for every* $(\text{trunk}^*, \text{loss}^*)$*-sequence* $\bar{q}$ *the limit* $\lim_\Xi(\bar{q})$ *forces* $\Xi^+(A_{\bar{q}}) \geq 1 - \sqrt{\text{loss}^*}$*, where*

$$A_{\bar{q}} = \{k \in \omega : |\{\ell \in I_k : q_\ell \in G\}| \geq |I_k| \cdot (1 - \sqrt{\text{loss}^*})\} \tag{3.9}$$

Proof. Work in the Q-extension. Now Ξ is a partial FAM. Let J enumerate all suitable sequences $\bar{q} \in V$ with $\lim_\Xi(\bar{q}) \in G$, and for such a sequence $\bar{q}^j$ set $a^j_k = \frac{|\{\ell \in I_k : q^j_\ell \in G\}|}{|I_k|}$, and $b^j = 1 - \text{loss}^*$. Using that Ξ satisfies Definition 3.7, we can apply Fact 3.3(b), we can extend Ξ to some FAM Ξ^+ such that $\text{Av}_{\Xi^+}(\bar{a}^j) \geq 1 - \text{loss}^*$ for $j < j^*$. So $\Xi^+(A_{\bar{q}^j}) + (1 - \Xi^+(A_{\bar{q}^j})) \cdot (1 - \sqrt{\text{loss}^*}) \geq \text{Av}_{\Xi^+}(a^j_k) \geq 1 - \text{loss}^*$, and thus $\Xi^+(A_{\bar{q}^j}) \geq 1 - \sqrt{\text{loss}^*}$. □

Definition 3.10. $(Q, \text{trunk}, \text{loss})$ as in Assumption 3.5 "has strong FAM limits for intervals", if for every FAM Ξ there is a function $\lim_\Xi$ that is a strong FAM limit for intervals.

Lemma 3.11. *[She00] Random forcing has strong FAM-limits for intervals.*

Proof. $\lim_\Xi$ is implicitly defined in [She00, p. 2.18], in the following way: Given a sequence r_ℓ with $(\text{trunk}(p_\ell), \text{loss}(p_\ell)) = (\text{trunk}^*, \text{loss}^*)$, we can set $r^* = [\text{trunk}^*]$ and $b = 1 - \text{loss}^*$; and we set n^*_k such that $I_k = [n^*_k, n^*_{k+1} - 1]$. We now use these objects to apply [She00, p. 2.18] (note that (c)($*$) is satisfied). This gives $r^\otimes$, and we define $\lim_\Xi(\bar{r})$ to be $r^\otimes$.

In [She00, p. 2.17], it is shown that this $r^\otimes$ satisfies Definition 3.7, i.e., is a limit: If r is stronger than all limits $r^{\otimes i}$, then r satisfies [She00, 2.17($*$)]. □

3.1.2 The forcing $\tilde{\mathbb{E}}$

We now define $\tilde{\mathbb{E}}$, a variant of the forcing notion Q^2 defined in [HS]:

Definition 3.12. By induction on the height $h \geq 0$, we define a compact homogeneous tree $T^* \subset \omega^{<\omega}$, and set

$$\rho(h) := \max(|T^* \cap \omega^h|, h+2) \quad \text{and} \quad \pi(h) := ((h+1)^2 \rho(h)^{h+1})^{\rho(h)^h}, \tag{3.13}$$

we set Ω_s to be the set $\{t \rhd s : t \in T^*\}$, i.e., the set of immediate successors of s, and define for each s a norm μ_s on the subsets of Ω_s. In more detail:

- The unique element of T^* of height 0 is $\langle\rangle$, i.e., $T^* \cap \omega^0 = \{\langle\rangle\}$.
- We set

  $$a(h) = \pi(h)^{h+2}, \quad M(h) = a(h)^2, \quad \text{and} \quad \mu_h(n) = \log_{a(h)}\left(\frac{M(h)}{M(h)-n}\right)$$

  for natural numbers $0 \leq n < M(h)$, and we set $\mu_h(M(h)) = \infty$.
- For any $s \in T^* \cap \omega^h$, we set $\Omega_s = \{s^\frown \ell : \ell \in M(h)\}$ (which defines $T^* \cap \omega^{h+1}$). For $A \subset \Omega_s$, we set $\mu_s(A) := \mu_h(|A|)$. So $|\Omega_s| = M(h)$, $\mu_s(\emptyset) = 0$ and $\mu_s(\Omega_s) = \infty$. Note that $|A| = |\Omega_s| \cdot \left(1 - a(h)^{-\mu_s(A)}\right)$.

We can now define $\tilde{\mathbb{E}}$:

Definition 3.14.
- For a subtree $p \subseteq T^*$, the stem of p is the smallest splitting node. For $s \in p$, we set $\mu_s(p) = \mu_s(\{t \in p : t \rhd s\})$.

  $\tilde{\mathbb{E}}$ consists of subtrees p with some stem s^* of height h^* such that $\mu_t(p) \geq 1 + \frac{1}{h^*}$ for all $t \in p$ above the stem. (So the only condition with $h^* = 0$ is the full condition, where all norms are ∞.)

  $\tilde{\mathbb{E}}$ is ordered by inclusion.
- $\mathtt{trunk}(p)$ is the stem of p.

  loss(p) is defined if there is an $m \geq 2$ satisfying the following, and in that case loss$(p) = \frac{1}{m}$ for the maximal such m:
  - p has stem s^* of height $h^* > 3m$,
  - $\mu_s(p) \geq 1 + \frac{1}{m}$ for all $s \in p$ of height $\geq h^*$.

  We set $Q' = \text{dom(loss)}$.

By simply extending the stem, we can find for any $p \in \tilde{\mathbb{E}}$ and $\epsilon > 0$ some $q \leq p$ in Q' with loss$(q) < \epsilon$; i.e., one of the assumptions in 3.5 is satisfied. (The other one is dealt with in Lemma 3.19(a).) In particular $Q' \subseteq \tilde{\mathbb{E}}$ is dense.

We list a few trivial properties of the loss function:

Facts 3.15. Assume $p \in Q'$ with $s = \mathtt{trunk}(p)$ of height h.

(a) loss$(p) < 1$, $\mu_s(p) \geq 1 + \text{loss}(p)$ for any s above the stem, and loss$(p) > \frac{3}{h}$.

(b) If q is a subtree of p such that all norms above the stem are $\geq 1 + \text{loss}(p) - \frac{2}{h}$, then q is a valid $\tilde{\mathbb{E}}$-condition.

(c) $\prod_{\ell=h}^{\infty}(1 - \frac{1}{\ell^2}) = 1 - \frac{1}{h} > 1 - \frac{\text{loss}(p)}{3}$.

Lemma 3.16. *Let $s \in T^*$ be of height h and $A \subset \Omega_s$.*

(a) If $\mu_s(A) \geq 1$, then $|A| \geq |\Omega_s| \cdot (1 - \frac{1}{h^2})$.

(b) If $A \subsetneq \Omega_s$, i.e., A is a proper subset, then $\mu_s(A \setminus \{t\}) > \mu_s(A) - \frac{1}{h}$ for $t \in A$.

(c) For $i < \pi(h)$, assume that $A_i \subseteq \Omega_s$ satisfies $\mu_s(A_i) \geq x$. Then $\mu_s(\bigcap_{i\in\pi(h)} A_i) > x - \frac{1}{h}$.

(d) For $i < I$ (an arbitrary finite index set) pick proper subsets $A_i \subsetneq \Omega_s$ such that $\mu_s(A_i) \geq x$, and assign weighs a_i to A_i such that $\sum_{i\in I} a_i = 1$. Then

$$\mu_s(B) > x - \frac{1}{h} \quad \textit{for} \quad B := \Big\{ t \in \Omega_s \, : \, \sum_{t\in A_i} a_i > 1 - \frac{1}{h^2} \Big\}. \tag{3.17}$$

Proof. (a) Trivial, as $a(h)^{-\mu_s(A)} \leq \frac{1}{a(h)} < \frac{1}{h^2}$.

(b) $\mu_s(A \setminus \{t\}) = \log_{a(h)}(|\Omega_s|) - \log_{a(h)}(|\Omega_s| - |A| + 1) \geq$
$\geq \log_{a(h)}(|\Omega_s|) - \log_{a(h)}(2(|\Omega_s| - |A|)) \geq \mu_s(A) - \log_{a(h)}(2) > \mu_s(A) - \frac{1}{h}$.

(c) $\mu_s(\bigcap_{i\in\pi(h)} A_i) = \log_{a(h)}(|\Omega_s|) - \log_{a(h)}(|\Omega_s| - |\bigcap_{i\in\pi(h)} A_i|) =$
$= \log_{a(h)}(|\Omega_s|) - \log_{a(h)}(|\bigcup_{i\in\pi(h)}(\Omega_s - A_i)|) \geq$
$\geq \log_{a(h)}(|\Omega_s|) - \log_{a(h)} \left(\pi(h) \cdot \max_{i\in\pi(h)} |\Omega_s - A_i|\right) \geq x - \log_{a(h)}(\pi(h)) > x - \frac{1}{h}$.

(d) Set $y = \sum_{i\in I} a_i \cdot |A_i|$. On the one hand, $y \geq |\Omega_s| \cdot (1 - a(h)^{-x})$. On the other hand, $y = \sum_{t\in\Omega_s} \sum_{t\in A_i} a_i \leq |B| + (|\Omega_s \setminus B|) \cdot (1 - \frac{1}{h^2})$.

So $|B| \geq |\Omega_s|(1 - h^2 a(h)^{-x}) > |\Omega_s|(1 - a(h)^{-(x-\frac{1}{h})})$, as $a(h)^{\frac{1}{h}} > \pi(h) > h^2$. □

$\tilde{\mathbb{E}}$ is not σ-centered, but it satisfied a property, first defined in [OK14], which is between σ-centered and σ-linked:

Definition 3.18. Fix f, g functions from ω to ω converging to infinity. Q is (f, g)-linked if there are $g(i)$-linked $Q^i_j \subseteq Q$ for $i < \omega$, $j < f(i)$ such that each $q \in Q$ is in every $\bigcup_{j<f(i)} Q^i_j$ for sufficiently large i.

Recall that we have defined ρ and π in (3.13).

Lemma 3.19. *(a) If $\pi(h)$ many conditions $(p_i)_{i\in\pi(h)}$ have a common node s above their stems, $|s| = h$, then there is a q stronger than each p_i.*

(b) *$\tilde{\mathbb{E}}$ is (ρ, π)-linked (In particular it is ccc).*

(c) *The $\tilde{\mathbb{E}}$-generic real η is eventually different (from every real in* $\lim(T^*)$*, and therefore from every real in ω^ω as well).*

(d) $\mathrm{Leb}(p) \geq \mathrm{Leb}\left([\mathrm{trunk}(p)]\right) \cdot \left(1 - \frac{1}{2}\mathrm{loss}(p)\right)$*; more explicitly: for any $h > |\mathrm{trunk}(p)|$,*

$$\frac{|p \cap \omega^h|}{|T^* \cap \omega^h \cap [\mathrm{trunk}(p)]|} \geq 1 - \frac{1}{2}\mathrm{loss}(p).$$

(e) *Q' (which is a dense subset of $\tilde{\mathbb{E}}$) is an incompatibility-preserving subforcing of random forcing, where we use the variant*[5] *of random forcing on* $\lim(T^*)$ *instead of 2^ω. Let B' be the the sub-Boolean-algebra of* Borel/Null *generated by* $\{\lim(q) : q \in Q'\}$*. Then Q' is dense in B'.*

(Here, Borel refers to the set of Borel subsets of $\lim(T^*)$. In the following proof, we will denote the equivalence class of a Borel set A by $[A]_{\mathcal{N}}$.)

Proof. (a) Set $S = [s] \cap \bigcap_{i<\pi(h)} p_i$. According to 3.16(c), for each $t \in S$ of height $h' \geq h$, the successor set has norm bigger than $1 + 1/h - 1/h' > 1$, so in particular there is a branch $x \in S$, and $S \cap [x \restriction 2h]$ is a valid condition stronger than all p_i.

(b) For each $h \in \omega$, enumerate $T^* \cap \omega^h$ as $\{s^h_1, \dots, s^h_{\rho(h)}\}$, and set $Q^h_i = \{p \in \tilde{\mathbb{E}} : s^h_i \in p \text{ and } |\mathrm{trunk}(p)| \leq h\}$. So for all h, Q^h_i is $\pi(h)$-linked, and $p \in \bigcup_{i<\rho(h)} Q^h_i$ for all $p \in Q$ with $|\mathrm{trunk}(p)| \leq h$.

(c) Use 3.16(b).

(d) Use 3.16(a) and the definition of loss.

(e) As in the previous item, we get that $\mathrm{Leb}(p \cap [t]) > 0$ whenever $p \in Q'$ and $t \in p$. So Q' is a subset of random forcing. As both sets are ordered by inclusion, Q' is a subforcing. If $q_1, q_2 \in Q'$ and q_1, q_2 are compatible as a random condition, then $q_1 \cap q_2$ has arbitrary high nodes, in particular a node above both stems, which implies that q_1 is compatible with q_2 in $\tilde{\mathbb{E}}$ and therefore in Q'. It remains to show that Q' is dense in B'. It is enough to show: If $x \neq 0$ in B' has the form $x = \bigwedge_{i<i^*}[\lim(q_i)]_{\mathcal{N}} \wedge \bigwedge_{j<j^*}[\lim(T^*) \setminus \lim(q_j)]_{\mathcal{N}}$ then there is some $q \in Q'$ with $[\lim(q)]_{\mathcal{N}} < x$. Note that $0 \neq x = [A]_{\mathcal{N}}$ for $A = \lim\left(\bigcap_{i<i^*} q_i\right) \setminus \bigcup_{j<j^*} \lim(q_j)$, so pick some $r \in A$ and pick $h > i^*$ large enough such that $s = r \restriction h$ is not in any q_j. Then any $q \in Q'$ stronger than all $q_i \cap [s]$ (for $i < i^*$) is as required. $\square$

Lemma 3.20. *$\tilde{\mathbb{E}}$ has strong FAM-limits for intervals.*

[5] We can use Definition 3.6, replacing 2^ω with $\lim(T^*)$.

Proof. Let $(p_\ell)_{\ell\in\omega}$ be a (s^*, loss^*)-sequence, s^* of height h^*. Set $\tilde{\zeta}^{h^*} = 0$ and

$$\tilde{\zeta}^h \coloneqq 1 - \prod_{m=h^*}^{h-1}(1 - \tfrac{1}{m^2}) \text{ for } h > h^*.$$

This is a strictly increasing sequence below $\frac{1}{3}$ loss*, cf. Fact 3.15(c). Also, all norms in all conditions of the sequence are at least $1 + \text{loss}^*$, cf. Fact 3.15(a).

We will first construct $(q_k)_{k\in\omega}$ with stem s^* and all norms $> 1 + \text{loss}^* - \frac{1}{h^*}$ such that q_k forces $\frac{|\{\ell\in I_k:\, p_\ell\in G\}|}{|I_k|} > 1 - \frac{1}{3}\text{loss}^*$. We will then use $\bar{q}$ to define $\lim_\Xi(\bar{p})$, and in the third step show that it is as required.

Step 1: So let us define q_k. Fix $k \in \omega$.

- Set

  $$X_t = \{\ell \in I_k \,:\, t \in p_\ell\} \text{ and } Y_h = \left\{t \in [s^*] \cap \omega^h \,:\, |X_t| \geq |I_k| \cdot (1 - \tilde{\zeta}^h)\right\}.$$

- We define q_k by induction on the level, such that $q_k \cap \omega^h \subseteq Y_h$. The stem is s^*. (Note that $X_{s^*} = I_k$ and so $s^* \in Y_{h^*}$.) For $s \in q_k \cap \omega^h$ (and thus, by induction hypothesis, in Y_h), we set $q_k \cap [s] \cap \omega^{h+1} = [s] \cap Y_{h+1}$, i.e., a successor t of s is in q_k iff it is Y_{h+1}. Then $\mu_s(q_k) > 1 + \text{loss}^* - \frac{1}{h}$.

  Proof: Set $I = X_s$. By induction, $|X_s| \geq |I_k| \cdot (1 - \tilde{\zeta}^h)$. For $\ell \in I$, set $A_\ell = p_\ell \cap [s] \cap \omega^{h+1}$, i.e., the immediate successors of s in p_ℓ. Obviously $\mu_s(A_\ell) \geq 1 + \text{loss}^*$. We give each A_ℓ equal weight $a_\ell = \frac{1}{|I|}$. According to (3.17), the set $B = \{t \rhd s \,:\, |\{\ell \in X_s \,:\, t \in A_\ell\}| \geq |I| \cdot (1 - \frac{1}{h^2})\}$ has norm $> 1 + \text{loss}^* - \frac{1}{h}$.

- q_k forces that $p_\ell \in G$ for $\geq |I_k| \cdot (1 - \frac{1}{2}\text{loss}^*)$ many $\ell \in I_k$.

  Proof: Let $r < q_k$ have stem s' of length h', without loss of generality $h' > |I_k| + 1$. As $s' \in Y_{h'}$, there are $> |I_k| \cdot (1 - \frac{1}{3}\text{loss}^*)$ many $\ell \in I_k$ such that $s' \in p_\ell$. So we can find a a condition r' stronger than r and all these p_ℓ (as these are at most $|I_k| + 1 \leq h'$ many conditions all containing s' above the stem).

Step 2: Now we use $(q_k)_{k\in\omega}$ to construct by induction on the height $q^* = \lim_\Xi(\bar{p})$, a condition with stem s^* and all norms $\geq 1 + \text{loss}^* - \frac{2}{h}$ such that for all $s \in q^*$ of height $h \geq h^*$,

$$\Xi(Z_s) \geq 1 - \tilde{\zeta}^h, \text{ for } \quad Z_s \coloneqq \{k \in \omega \,:\, s \in q_k\}. \quad \text{So } \Xi(Z_s) > 1 - \frac{1}{3}\text{loss}^*. \quad (*)$$

Note that $Z_{s^*} = \omega$, so $(*)$ is satisfied for s^*. Fix an $s \geq s^*$ satisfying $(*)$. Set $A(k)$ to be the s-successors in q_k for each $k \in Z_s$. Enumerate the (finitely many) $A(k)$ as $(A_i)_{i\in I}$. Clearly $\mu_s(A_i) > 1 + \text{loss}^* - \frac{1}{h}$. Assign to A_i the weight $a_i = \frac{1}{\Xi(Z_s)}\Xi(\{k \in$

$Z_s : A(k) = A_i\}$). Again using (3.17), $\mu_s(B) \geq 1 + \text{loss}^* - \frac{2}{h}$, where B consists of those successors t of s such that

$$1 - \frac{1}{h^2} < \sum_{t \in A_i} a_i = \frac{1}{\Xi(Z_s)} \Xi(\{k \in Z_s : t \in q_k\}) \leq \frac{1}{\Xi(Z_s)} \Xi(Z_t).$$

So every $t \in B$ satisfies $\Xi(Z_t) > \Xi(Z_s)(1 - \frac{1}{h^2}) \geq \tilde{\zeta}^{h+1}$, i.e., satisfies $(*)$. So we can use B as the set of s-successors in q^*.

This defines q^*, which is a valid condition by Fact 3.15(b).

Step 3: We now show that this limit works: As in Definition 3.7, fix m^*, $(B_m)_{m<m^*}$, ϵ, k^*, i^* and sequences $(p^i_\ell)_{\ell<\omega}$ for $i < i^*$, such that $(\text{trunk}(p^i_\ell), \text{loss}(p^i_\ell)) = (\text{trunk}^*, \text{loss}^*)$.

For each $i < i^*$, $\bar{q}^i = (q^i_k)_{k\in\omega}$ is defined from $\bar{p}^i = (p^i_\ell)_{\ell\in\omega}$, and in turn defines the limit $\lim_\Xi(\bar{p}^i)$. Let q be stronger than all $\lim_\Xi(\bar{p}^i)$.

Let M be as in Lemma 3.2, for $N = m^* + i^*$. So for any N many sets there is a u of size at most M (above k^*) which approximates the measure well. We use the following N many sets:

- B_m (for $m < m^*$).
- Fix an $s \in q$ of height $h > M \cdot i^*$; and use the i^* many sets $Z^i_s \subseteq \omega$ defined in $(*)$.

Accordingly, there is a u (starting above k^*) of size $\leq M$ with

- $\Xi(B_m) - \epsilon \leq \frac{|B_m \cap u|}{|u|} \leq \Xi(B_m) + \epsilon$ for each $m < m^*$, and
- $\frac{|Z^i_s \cap u|}{|u|} \geq 1 - \frac{1}{3}\text{loss}^* - \epsilon$ for each $i < i^*$.

So for each $i \in i^*$ there are at least $|u| \cdot (1 - \frac{1}{2}\text{loss}^* - \epsilon)$ many $k \in u$ with $s \in q^i_k$. There is a condition r stronger than q and all those q^i_k (as $\leq M \cdot i^* + 1$ many conditions of height $h > M \cdot i^*$ with common node s above their stems are compatible). So r forces, for all $i < i^*$ and $k \in u \cap Z^i_s$, that $q^i_k \in G$ and therefore that $|\{\ell \in I_k : p^i_\ell \in G\}| \geq |I_k|(1 - \frac{1}{3}\text{loss}^*)$. By increasing r to some q', we can assume that r decides which p^i_ℓ are in G and that r is actually stronger than each p^i_ℓ decided to be in G. So all in all we get $q' \leq q$ such that

$$\frac{1}{|u|} \sum_{k\in u} \frac{|\{\ell \in I_k : q' \leq p^j_\ell\}|}{|I_k|} \geq \frac{1}{|u|} |\{k \in u : k \in Z^j_s\}| (1 - \frac{1}{3}\text{loss}^*) >$$
$$> 1 - \text{loss}^* - \epsilon,$$

as required. □

3.2 The left hand side of Cichoń's diagram

We write $\mathfrak{x}_1$ for add($\mathcal{N}$), $\mathfrak{x}_2$ for $\mathfrak{b}$ (which will also be add($\mathcal{M}$)), $\mathfrak{x}_3$ for cov($\mathcal{N}$) and $\mathfrak{x}_4$ for non($\mathcal{M}$).

3.2.1 Good iterations and the LCU property

We want to show that some forcing P^5 results in $\mathfrak{x}_i = \lambda_i$ (for $i = 1 \dots 4$). So we have to show two "directions", $\mathfrak{x}_i \leq \lambda_i$ and $\mathfrak{x}_i \geq \lambda_i$.

For $i = 1, 3, 4$ (i.e., for all the characteristics on the left hand side apart from $\mathfrak{b} = \text{add}(\mathcal{M})$), the direction $\mathfrak{x}_i \leq \lambda_i$ will be given by the fact that P^5 is (R_i, λ_i)-good for a suitable relation R_i. (For $i = 2$, i.e., the unbounding number, we will have to work more.)

We will use the following relations:

Definition 3.21. 1. Let C be the set of strictly positive rational sequences $(q_n)_{n\in\omega}$ such that $\sum_{n\in\omega} q_n \leq 1$.[6] Let $\mathrm{R}_1 \subseteq C^2$ be defined by: $f\,\mathrm{R}_1\,g$ if $(\forall^* n \in \omega)\, f(n) \leq g(n)$.

2. $\mathrm{R}_2 \subseteq (\omega^\omega)^2$ is defined by: $f\,\mathrm{R}_2\,g$ if $(\forall^* n \in \omega)\, f(n) \leq g(n)$.

4. $\mathrm{R}_4 \subseteq (\omega^\omega)^2$ is defined by: $f\,\mathrm{R}_4\,g$ if $(\forall^* n \in \omega)\, f(n) \neq g(n)$.

So far, these relations fit the usual framework of goodness, as introduced in [JS90] and [Bre91] and summarized, e.g., in [BJ95, p. 6.4] or [GMS16, Sec. 3] or [Mej13b, Sec. 2]. For $\mathfrak{x}_3$, i.e., cov($\mathcal{N}$), we will use a relation R_3 that does not fit this framework (as the range of the relation is not a Polish space). Nevertheless, the property "(R_3, λ)-good" behaves just as in the usual framework (e.g., finite support limits of good forcings are good, etc.). The relation R_3 was implicitly used by Kamo and Osuga [OK14], who investigated (R_3, λ)-goodness.[7] It was also used in [BM14]; a unifying notation for goodness (which works for the usual cases as well as relations such as R_3) is given in [MC, §4].

Definition 3.22. We call a set $\mathcal{E} \subset \omega^\omega$ an R_3-parameter, if for all $e \in \mathcal{E}$

- $\lim e(n) = \infty$, $e(n) \leq n$, $\lim(n - e(n)) = \infty$,
- there is some $e' \in \mathcal{E}$ such that $(\forall^* n)\, e(n) + 1 \leq e'(n)$, and
- for all countable $\mathcal{E}' \subseteq \mathcal{E}$ there is some $e \in \mathcal{E}$ such that for all $e' \in \mathcal{E}'$ $(\forall^* n)\, e(n) \geq e'(n)$.

Note that such an R_3-parameter of size $\aleph_1$ exists. This is trivial if we assume CH (which we could in this paper), but also true without this assumption, see [MC, p. 4.20]. Recall that ρ and π were defined in (3.13).

[6] It is easy to see that C is homeomorphic to ω^ω, when we equip the rationals with the discrete topology and use the product topology.

[7] They use the notation $(*^{<\lambda}_{c,h})$, cf. [OK14, Def. 6].

Definition 3.23. We fix, for the rest of the paper, an R_3-parameter $\mathcal{E}$ of size $\aleph_1$, and set

$$b(h) = (h{+}1)^2 \rho(h)^{h+1}, \quad \mathcal{S} = \left\{\psi \in \prod\nolimits_{h\in\omega} P(b(h)) \,:\, (\forall h \in \omega)\, |\psi(h)| \le \rho(h)^h\right\},$$

$$\mathcal{S}_e = \left\{\phi \in \prod\nolimits_{h\in\omega} P(b(h)) \,:\, (\forall h \in \omega)\, |\phi(h)| \le \rho(h)^{e(h)}\right\} \quad \text{and} \quad \hat{\mathcal{S}} = \bigcup_{e\in\mathcal{E}} \mathcal{S}_e.$$

We can now define the relation for $\mathrm{cov}(\mathcal{N})$:

3. $\mathrm{R}_3 \subseteq \mathcal{S} \times \hat{\mathcal{S}}$ is defined by: $\psi \,\mathrm{R}_3\, \phi$ iff $(\forall^* n \in \omega)\, \phi(n) \not\subseteq \psi(n)$.

Note that $\mathcal{S}_e \subset \hat{\mathcal{S}} \subset \mathcal{S}$ and that $\mathcal{S}_e$ and $\mathcal{S}$ are Polish spaces. Assume that M is a forcing extension of V by either a ccc forcing (or by a σ-closed forcing). Then $\mathcal{E}$ is an "R_3-parameter" in M as well, and we can evaluate in M for each $e \in \mathcal{E}$ the sets $\mathcal{S}_e^M$ and $\mathcal{S}^M$, as well as $\hat{\mathcal{S}}^M = \bigcup_{e\in\mathcal{E}} \mathcal{S}_e^M$. Absoluteness gives $\mathcal{S}_e^V = \mathcal{S}_e^M \cap V$ and $\hat{\mathcal{S}}^V = \hat{\mathcal{S}}^M \cap V$.

Definition 3.24. Fix one of these relations $R \subseteq X \times Y$.

- We say "f is bounded by g" if $f \,\mathrm{R}\, g$; and, for $\mathcal{Y} \subseteq \omega^\omega$, "$f$ is bounded by $\mathcal{Y}$" if $(\exists y \in \mathcal{Y})\, f \,\mathrm{R}\, y$. We say "unbounded" for "not bounded". (I.e., f is unbounded by $\mathcal{Y}$ if $(\forall y \in \mathcal{Y})\, \neg f \,\mathrm{R}\, y$.)
- We call $\mathcal{X}$ an R-unbounded family, if $\neg(\exists g)\,(\forall x \in \mathcal{X}) x \,\mathrm{R}\, g$, and an R-dominating family if $(\forall f)\,(\exists x \in \mathcal{X})\, f \,\mathrm{R}\, x$.
- Let $\mathfrak{b}_i$ be the minimal size of an R_i-unbounded family,
- and let $\mathfrak{d}_i$ be the minimal size of an R_i-dominating family.

We only need the following connection between R_i and the cardinal characteristics:

Lemma 3.25. *1.* $\mathrm{add}(\mathcal{N}) = \mathfrak{b}_1$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}_1$.

2. $\mathfrak{b} = \mathfrak{b}_2$ *and* $\mathfrak{d} = \mathfrak{d}_2$.

3. $\mathrm{cov}(\mathcal{N}) \le \mathfrak{b}_3$ *and* $\mathrm{non}(\mathcal{N}) \ge \mathfrak{d}_3$.

4. $\mathrm{non}(\mathcal{M}) = \mathfrak{b}_4$ *and* $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}_4$.

Proof. (2) holds by definition. (1) can be found in [BJ95, 6.5.B]. (4) is a result of [Mil82; Bar87], cf. [BJ95, 2.4.1 and 2.4.7].

To see (3), we work in the space $\Omega = \prod_{h\in\omega} b(h)$, with the b defined in Definition 3.23 and the usual (uniform) measure. It is well known that we get the same values for the characteristics $\mathrm{cov}(\mathcal{N})$ and $\mathrm{non}(\mathcal{N})$ whether we define them using Ω or, as usual, 2^ω (or $[0, 1]$ for that matter, etc). Given $\psi \in \mathcal{S}$, note that

$$N_\psi = \{\eta \in \Omega \,:\, (\exists^\infty h)\, \eta(h) \in \psi(h)\}$$

is a Null set, as $\{\eta \in \Omega :\ (\forall h > k)\, \eta(h) \notin \psi(h)\}$ has measure $\prod_{h>k}(1 - \frac{|\psi(h)|}{b(h)}) \geq \prod_{h>k}(1 - \frac{1}{(h+1)^3})$, which converges to 1 for $k \to \infty$.

Let $\mathcal{A} \subseteq \mathcal{S}$ be an R_3-unbounded family. So for every $\phi \in \hat{\mathcal{S}}$ there is some $\psi \in A$ such that $(\exists^\infty h)\, \psi(h) \supseteq \phi(h)$. In particular, for each $\eta \in \Omega$, there is a $\psi \in A$ with $\eta \in N_\psi$; i.e., $\mathrm{cov}(\mathcal{N}) \leq |\mathcal{A}|$.

Analogously, let X be a non-null set (in Ω). For each ψ there is an $x \in X \setminus N_\psi$, so $\phi_x(n) = \{x(n)\}$ satisfies $\psi\, \mathrm{R}_3\, \phi_x$. □

Remark 3.26. As shown implicitly in [OK14], and explicitly in [MC, p. 4.22], we actually get $\mathrm{cov}(\mathcal{N}) \leq c^{\exists}_{b,\rho^{\mathrm{Id}}} \leq \mathfrak{b}_3$.

Definition 3.27. Let P be a ccc forcing, λ an uncountable regular cardinal, and $\mathrm{R}_i \subseteq X \times Y$ one of the relations above (so for $i = 1, 2, 4$, $Y = X$, and for $i = 3$ $Y = \hat{\mathcal{S}}_e$). P is (R_i, λ)-good, if for each P-name r for an element of Y there is (in V) a nonempty set $\mathcal{Y} \subseteq Y$ of size $<\lambda$ such that every $f \in X$ (in V) that is R_i-unbounded by $\mathcal{Y}$ is forced to be R_i-unbounded by r as well.

Note that λ-good trivially implies μ-good if $\mu \geq \lambda$ are regular.

Lemma 3.28. *Let λ be uncountable regular.*

a. *Forcings of size $<\lambda$ are (R_i, λ)-good. In particular, Cohen forcing is $(\mathrm{R}_i, \aleph_1)$-good.*

b. *A FS ccc iteration of (R_i, λ)-good forcings (and in particular, a composition of two such forcings) is (R_i, λ)-good.*

1. *A sub-Boolean-algebra of the random algebra is $(\mathrm{R}_1, \aleph_1)$-good. Any σ-centered forcing notion is $(\mathrm{R}_1, \aleph_1)$-good.*

3. *A (ρ, π)-linked forcing is $(\mathrm{R}_3, \aleph_1)$-good (for the ρ, π of Definition 3.12).*

Proof. **(a&b):** For $i = 1, 2, 4$ this is proven in [JS90], cf. [BJ95, p. 6.4]. The same proof works for $i = 3$, as shown in [OK14, Lem. 12, 13]. The proof for the uniform framework can be found in [MC, pp. 4.10, 4.14].

(1) follows from [JS90] and [Kam89], cf. [BJ95, pp. 6.5.17–18].

(3) is shown in [OK14, Lem. 10], cf. [MC, Lem. 4.24]; as our choice of π, ρ and b (see Definition 3.23) satisfies $\pi(h) \geq b(h)^{\rho(h)^h} = ((h+1)^2 \rho(h)^{h+1})^{\rho(h)^h}$. □

Each relation R_i is a subset of some $X \times Y$, where X is either 2^ω, ω^ω (or homeomorphic to it) or $\mathcal{S}$, and Y is the range of R_i.

Lemma 3.29. *For each i and each $g \in Y$, the set $\{f \in X :\ f\, \mathrm{R}_i\, g\} \subseteq X$ is meager.*

Proof. We have explicitly defined each $f\, \mathrm{R}_i\, g$ as $\forall^* n R^n_i(f, g)$ for some R^n_i. The lemma follows easily from the fact that for each $n \in \omega$, the set $\{f \in X :\ R^n_i(f, g)\}$ is closed nowhere dense. □

Lemma 3.30. *Let $\lambda \le \kappa \le \mu$ be uncountable regular cardinals. Force with μ many Cohen reals $(c_\alpha)_{\alpha\in\mu}$, followed by an (R_i, λ)-good forcing. Note that each Cohen real c_β can be interpreted as element of the Polish space X where $\mathrm{R}_i \subseteq X \times Y$. Then we get: For every real r in the final extension's Y, the set $\{\alpha \in \kappa : c_\alpha$ is R_i-unbounded by $r\}$ is cobounded in κ. I.e., $(\exists \alpha \in \kappa)\,(\forall \beta \in \kappa \setminus \alpha)\,\neg c_\alpha\, \mathrm{R}_i\, r$.*

Proof. Work in the intermediate extension after κ many Cohen reals, let us call it V_κ. The remaining forcing (i.e., $\mu \setminus \kappa$ many Cohens composed with the good forcing) is good; so applying the definition we get (in V_κ) a set $\mathcal{Y} \subseteq Y$ of size $<\lambda$.

As the initial Cohen extension is ccc, and $\kappa \ge \lambda$ is regular, we get some $\alpha \in \kappa$ such that each element y of $\mathcal{Y}$ already exists in the extension by the first α many Cohens, call it V_α.

Fix some $\beta \in \kappa \setminus \alpha$ and $y \in Y$. As $\{x \in X : x\, \mathrm{R}_i\, y\}$ is a meager set already defined in V_α, we get $\neg c_\beta\, \mathrm{R}_i\, y$. Accordingly, c_β is unbounded by $\mathcal{Y}$; and, by the definition of good, unbounded by r as well. $\square$

In the light of this result, let us revisit Lemma 3.25 with some new notation, the "linearly cofinally unbounded" property LCU:

Definition 3.31. For $i = 1, 2, 3, 4$, γ a limit ordinal, and P a ccc forcing notion, let $\mathsf{LCU}_i(P, \gamma)$ stand for:

> There is a sequence $(x_\alpha)_{\alpha\in\gamma}$ of P-names such that for every P-name y $(\exists \alpha \in \gamma)\,(\forall \beta \in \gamma \setminus \alpha)\, P \Vdash \neg x_\beta\, \mathrm{R}_i\, y)$.

Lemma 3.32.

- *$\mathsf{LCU}_i(P, \delta)$ is equivalent to $\mathsf{LCU}_i(P, \mathrm{cf}(\delta))$.*
- *If λ is regular, then $\mathsf{LCU}_i(P, \lambda)$ implies $\mathfrak{b}_i \le \lambda$ and $\mathfrak{d}_i \ge \lambda$.*
  In particular:

1. *$\mathsf{LCU}_1(P, \lambda)$ implies $P \Vdash (\,\mathrm{add}(\mathcal{N}) \le \lambda\, \&\, \mathrm{cof}(\mathcal{N}) \ge \lambda\,)$.*
2. *$\mathsf{LCU}_2(P, \lambda)$ implies $P \Vdash (\,\mathfrak{b} \le \lambda\, \&\, \mathfrak{d} \ge \lambda\,)$.*
3. *$\mathsf{LCU}_3(P, \lambda)$ implies $P \Vdash (\,\mathrm{cov}(\mathcal{N}) \le \lambda\, \&\, \mathrm{non}(\mathcal{N}) \ge \lambda\,)$.*
4. *$\mathsf{LCU}_4(P, \lambda)$ implies $P \Vdash (\,\mathrm{non}(\mathcal{M}) \le \lambda\, \&\, \mathrm{cov}(\mathcal{M}) \ge \lambda\,)$.*

Proof. Assume that $(\alpha_\beta)_{\beta\in\mathrm{cf}(\delta)}$ is increasing continuous and cofinal in δ. If $(x_\alpha)_{\alpha\in\delta}$ witnesses $\mathsf{LCU}_i(P, \delta)$, then $(x_{\alpha_\beta})_{\beta\in\mathrm{cf}(\delta)}$ witnesses $\mathsf{LCU}_i(P, \mathrm{cf}(\delta))$. And if $(x_\beta)_{\beta\in\mathrm{cf}(\delta)}$ witnesses $\mathsf{LCU}_i(P, \mathrm{cf}(\delta))$, then $(y_\alpha)_{\alpha\in\delta}$ witnesses $\mathsf{LCU}_i(P, \mathrm{cf}(\delta))$, where $y_\alpha \coloneqq x_\beta$ for $\alpha \in [\alpha_\beta, \alpha_{\beta+1})$.

The set $\{x_\alpha : \alpha \in \lambda\}$ is certainly forced to be R_i-unbounded; and given a set $Y = \{y_j : j < \theta\}$ of $\theta < \lambda$ many P-names, each has a bound $\alpha_j \in \lambda$ so that $(\forall \beta \in \lambda \setminus \alpha_j)\, P \Vdash \neg x_\beta\, \mathrm{R}_i\, y_j)$, so for any $\beta \in \lambda$ above all α_j we get $P \Vdash \neg x_\beta\, \mathrm{R}_i\, y_j$ for all j; i.e., Y cannot be dominating. $\square$

3.2.2 The initial forcing P^5 and the COB property

We will assume the following throughout the paper:

Assumption 3.33.
- $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5$ are regular uncountable cardinals such that $\mu < \lambda_i$ implies $\mu^{\aleph_0} < \lambda_i$.
- We set $\delta_5 = \lambda_5 + \lambda_5$, and partition $\delta_5 \setminus \lambda_5$ into unbounded sets S^i for $i = 1, \dots, 4$. Fix for each $\alpha \in \delta_5 \setminus \lambda_5$ a $w_\alpha \subseteq \alpha$ such that $\{w_\alpha : \alpha \in S^i\}$ is cofinal[8] in $[\delta_5]^{<\lambda_i}$ (for each $i = 1, \dots, 4$).

The reader can assume that $(\lambda_i)_{i=1,\dots,5}$ and $(S^i)_{i=1,\dots,4}$ have been fixed once and for all (let us call them "fixed parameters"), whereas we will investigate various possibilities for $\bar{w} = (w_\alpha)_{\alpha \in \delta_5 \setminus \lambda_5}$ in the following. (We will call a $\bar{w}$ which satisfies the assumption a "cofinal parameter".)

We define by induction:

Definition 3.34. We define the FS iteration $(P_\alpha, Q_\alpha)_{\alpha \in \delta_5}$ and, for $\alpha > \lambda_5$, P'_α as follows: If $\alpha \in \lambda_5$, then Q_α is Cohen forcing. In particular, the generic at α is determined by the Cohen real η_α. For $\alpha \in \delta_5 \setminus \lambda_5$:

1. $$Q_\alpha^{\text{full}} := \left\{ \begin{array}{c} \text{Amoeba} \\ \text{Hechler} \\ \text{Random} \\ \tilde{\mathbb{E}} \end{array} \right\} \text{ for } \alpha \text{ in} \left\{ \begin{array}{c} S^1 \\ S^2 \\ S^3 \\ S^4 \end{array} \right. .$$
   So Q_α^{full} is a Borel definable subset of the reals, and the Q_α^{full}-generic is determined, in a Borel way, by the canonical generic real η_α.

2. P'_α is the set of conditions $p \in P_\alpha$ satisfying the following, for each $\beta \in \operatorname{supp}(p)$: $\beta \in w_\alpha$ and there is (in the ground model) a countable $u \subseteq w_\alpha \cap \beta$ and a Borel function $B : (\omega^\omega)^u \to Q_\beta^{\text{full}}$ such that $p \restriction \beta$ forces that $p(\beta) = B((\eta_\gamma)_{\gamma \in u})$. We *assume* that
   $$P'_\alpha \text{ is a complete subforcing of } P_\alpha. \tag{3.35}$$

3. In the P_α-extension, let M_α be the induced P'_α-extension of V. Then Q_α is the M_α-evaluation of Q_α^{full}. Or equivalently (by absoluteness): $Q_\alpha = Q_\alpha^{\text{full}} \cap M_\alpha$. We call Q_α a "partial Q_α^{full} forcing" (e.g.: a "partial random forcing").

Some notes:

- For item (3) to make sense, (3.35) is required.
- We do not require any "transitivity" of the w_α, i.e., $\beta \in w_\alpha$ does generally not imply $w_\beta \subseteq w_\alpha$.
- We do not require (and it will generally not be true) that P_α forces that Q_α is a *complete* subforcing of Q_α^{full}.

[8]i.e., if $\alpha \in S^i$ then $|w_\alpha| < \lambda_i$, and for all $u \subseteq \delta_5$, $|u| < \lambda_i$ there is some $\alpha \in S^i$ with $w_\alpha \supseteq u$.

A simple absoluteness argument (between M_α and $V[G_\alpha]$) shows:

Lemma 3.36. *P_α forces:*

(a) Q_α is an incompatibility preserving subforcing of Q_α^{full} and in particular ccc. (And so, P_α itself is ccc for all α.)

(b) For $\alpha \in S^i$, $|Q_\alpha| < \lambda_i$.

(c) Q_α forces that its generic filter $G(\alpha)$ is also generic over M_α. So from the point of view of M_α, $M_\alpha[G(\alpha)]$ is a Q_α^{full}-extention.

(2) For $\alpha \in S^2$: The partial Hechler forcing Q_α is σ-centered.

(3) For $\alpha \in S^3$: The partial random forcing Q_α equivalent to a subalgebra of the random algebra.

(4) For $\alpha \in S^4$: A partial $\tilde{\mathbb{E}}$ forcing is (ρ, π)-linked and basically equivalent to a subalgebra of the random algebra (as in Lemma 3.19(e)).

Proof. (b): $|P'_\alpha| \leq |w_\alpha|^{\aleph_0} \times 2^{\aleph_0} < \lambda_i$ by Assumption 3.33. There is a set of nice P'_α-names of size $< \lambda_i$ such that every P'_α-name for a real has an equivalent name in this set. Accordingly, the size of the reals in M_α is forced to be $< \lambda_i$.

(c) is trivial, as Q_α is element of the transitive class M_α.

(4): By Lemma 3.19(b) we know that M_α thinks that $\tilde{\mathbb{E}}$ is (ρ, π)-linked; i.e., that there is a family[9] Q^i_j as in Definition 3.18. Being ℓ-linked is obviously absolute between M_α and $V[G_\alpha]$ (for any $\ell < \omega$); and $M_\alpha \vDash \bigcup_{h\in\omega, i<\rho(h)} Q^h_i = Q^{\text{full}}_\alpha$ translates to $V[G_\alpha] \vDash \bigcup_{h\in\omega, i<\rho(h)} Q^h_i = Q_\alpha$.

Similarly, M_α thinks that $\tilde{\mathbb{E}}$ satisfies 3.19(e), i.e., that there is some dense $Q' \subseteq \tilde{\mathbb{E}}$ and a dense embedding from Q' to a subalgebra B' of the random algebra.

So from the point of view of $V[G_\alpha]$, there is a Q' dense in $\tilde{\mathbb{E}} \cap M_\alpha$ and a dense embedding of Q' into some B', which is a subalgebra of the random algebra in M_α and therefore of the random algebra in $V[G_\alpha]$. $\square$

It is easy to see that (3.35) is a "closure property" of w_α:

Lemma 3.37. *Assume we have constructed (in the ground model) $(P_\beta, Q_\beta)_{\beta<\alpha}$ and w_α according to Definition 3.34; for some $\alpha \in S^i$, $i = 1, \ldots, 4$. This determines the (limit or composition) P_α.*

(a) For every P_α-name τ of a real, there is (in V) a countable $u \subseteq \alpha$ and a Borel function $B : (\omega^\omega)^u \to \omega^\omega$ such that P_α forces $\tau = B((\eta_\gamma)_{\gamma\in u})$.

(So if $w_\alpha \supseteq u$ satisfies (3.35)*, then P_α forces that $\tau \in M_\alpha$.)*

(b) The set of w_α satisfying (3.35) *is an ω_1-club in $[\alpha]^{<\lambda_i}$ (in the ground model).*

[9] Actually there is even a Borel definable family Q^i_j, see the proof of Lemma 3.19(a), but this is not required here.

(A set $A \subseteq [\alpha]^{<\lambda_i}$ is an ω_1-club, if for each $a \in [\alpha]^{<\lambda_i}$ there is a $b \supseteq a$ in A, and if $(a^i)_{i\in\omega_1}$ is an increasing sequence of sets in A, then the limit $b \coloneqq \bigcup_{i\in\omega_1} a^i$ is in A as well.)

Proof. The first item follows easily from the fact that we are dealing with a FS ccc iteration where the generics of all iterands Q_β are Borel-determined by some generic real η_β. (See, e.g., Section 2.1.2, for more details.)

Any $w \in [\alpha]^{<\lambda_i}$ defines some P^w_α. We first define w' for such a w:

Set $X = [P^w_\alpha]^{\leq\aleph_0}$, as set of size at most $(2^{\aleph_0} \times |w|^{\aleph_0})^{\aleph_0} < \lambda_i$. For $x \in X$, pick some $p \in P_\alpha$ stronger than all conditions in x (if such a condition exists), and some $q \in P_\alpha$ incompatible to each element of x (again, if possible). There is a countable $w_x \subseteq \alpha$ such that $p, q \in P^{w_x}$. Set $w' \coloneqq w \cup \bigcup_{x\in X} w_x$.

Start with any $w_0 \in [\alpha]^{<\lambda_i}$. Construct an increasing continuous chain in $[\alpha]^{<\lambda_i}$ with $w^{k+1} = (w^k)'$. Then $w^{\omega_1} \supseteq w_0$ is in the set of w satisfying (3.35); which shows that this set is unbounded. It is equally easy to see that it is closed under increasing sequences of length ω_1. □

For later reference, we explicitly state the assumption we used (for every $\alpha \in \delta_5 \setminus \lambda_5$):

Assumption 3.38. w_α is sufficiently closed so that (3.35) is satisfied.

Let us also restate Lemma 3.37(a):

Lemma 3.39. *For each P^5-name f of a real, there is a countable set $u \subseteq \delta_5$ such that $w_\alpha \supseteq u$ implies that (P^5 forces that) $f \in M_\alpha$.*

Lemma 3.40. $\mathsf{LCU}_i(P^5, \kappa)$ *holds for $i = 1, 3, 4$ and each regular cardinal κ in $[\lambda_i, \lambda_5]$.*

Proof. This follows from Lemma 3.36:

For $i = 1$: Partial random and partial $\tilde{\mathbb{E}}$ forcings are basically equivalent to a sub-Boolean-algebra of the random algebra; and partial Hechler forcings are σ-centered. The partial amoeba forcings are small, i.e., have size $<\lambda_1$. So according to Lemma 3.28, all iterands Q_α (and therefore the limits as well) are $(\mathrm{R}_1, \lambda_1)$-good.

For $i = 3$, note that partial $\tilde{\mathbb{E}}$ forcings are (ρ, π)-linked. All other iterands have size $<\lambda_3$, so the forcing is $(\mathrm{R}_3, \lambda_3)$-good.

For $i = 4$ it is enough to note that *all* iterands are small, i.e., of size $<\lambda_4$.

We can now apply Lemma 3.30. □

So in particular, P^5 forces $\mathrm{add}(\mathcal{N}) \leq \lambda_1$, $\mathrm{cov}(\mathcal{N}) \leq \lambda_3$, $\mathrm{non}(\mathcal{M}) \leq \lambda_4$ and $\mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \lambda_5 = 2^{\aleph_0}$; i.e., the respective left hand characteristics are small. We now show that they are also large, using the "cone of bounds" property COB:

Definition 3.41. For a ccc forcing notion P, regular uncountable cardinals λ, μ and $i = 1, 2, 4$, let $\mathsf{COB}_i(P, \lambda, \mu)$ stand for:

> There is a $<\lambda$-directed partial order $(S, \prec)$ of size μ and a sequence $(g_s)_{s\in S}$ of P-names for reals such that for each P-name f of a real $(\exists s \in S)\,(\forall t \succ s)\, P \Vdash f \, \mathrm{R}_i \, g_t$.

For $i = 3$, let $\mathsf{COB}_3(P, \lambda, \mu)$ stand for:

> There is a $<\lambda$-directed partial order $(S, \prec)$ of size μ and a sequence $(g_s)_{s\in S}$ of P-names for reals such that for each P-name f of a null-set $(\exists s \in S)\,(\forall t \succ s)\, P \Vdash g_t \notin f$.

So s is the tip of a cone that consists of elements bounding f, where in case $i = 3$ we implicitly use an additional relation $N \, \mathrm{R}_3{}' \, r$ expressing that the null-set N doesn't contain the real r. Note that $\mathrm{cov}(\mathcal{N})$ is the bounding number $\mathfrak{b}_3'$ of $\mathrm{R}_3{}'$, and $\mathrm{non}(\mathcal{N})$ the dominating number $\mathfrak{d}_3'$. So $\mathrm{add}(\mathcal{N}) = \mathfrak{b}_3' \le \mathfrak{b}_3$ and $\mathrm{non}(\mathcal{N}) = \mathfrak{d}_3' \ge \mathfrak{d}_3$ (as defined in Lemma 3.25).

$\mathsf{COB}_i(P, \lambda, \mu)$ implies that P forces that $\mathfrak{b}_i \ge \lambda$ and that $\mathfrak{d}_i \le \mu$ (for $i = 1, 2, 4$, and the same for $i = 3$ and $\mathfrak{b}_3', \mathfrak{d}_3'$): Clearly P forces that $\{g_s : s \in S\}$ is dominating. And if A is set of names of size $\kappa < \lambda$, then for each $f \in A$ the definition gives a bound $s(f)$ and directedness some $t \succ s(f)$ for all f, i.e., g_t bounds all elements of A. So we get:

Lemma 3.42. *1.* $\mathsf{COB}_1(P, \lambda, \mu)$ *implies* $P \Vdash (\,\mathrm{add}(\mathcal{N}) \ge \lambda \,\&\, \mathrm{cof}(\mathcal{N}) \le \mu\,)$.

2. $\mathsf{COB}_2(P, \lambda, \mu)$ *implies* $P \Vdash (\,\mathfrak{b} \ge \lambda \,\&\, \mathfrak{d} \le \mu\,)$.

3. $\mathsf{COB}_3(P, \lambda, \mu)$ *implies* $P \Vdash (\,\mathrm{cov}(\mathcal{N}) \ge \lambda \,\&\, \mathrm{non}(\mathcal{N}) \le \mu\,)$.

4. $\mathsf{COB}_4(P, \lambda, \mu)$ *implies* $P \Vdash (\,\mathrm{non}(\mathcal{M}) \ge \lambda \,\&\, \mathrm{cov}(\mathcal{M}) \le \mu\,)$.

Lemma 3.43. $\mathsf{COB}_i(P^5, \lambda_i, \lambda_5)$ *holds (for* $i = 1, 2, 3, 4$*).*

Proof. We use the following facts (provable in ZFC, or true in the P_α-extention, respectively):

1. Amoeba forcing adds a sequence $\bar{b}$ which R_1-dominates the old elements of $\mathcal{C}$.

   (The simple proof can be found in [GKS17, Lem. 1.4], a slight variation in [BJ95].)

   Accordingly (by absoluteness), the generic real η_α for partial amoeba forcing Q_α R_1-dominates $\mathcal{C} \cap M_\alpha$.

2. Hechler forcing adds a real which R_2-dominates all old reals.

   Accordingly, the generic real η_α for partial Hechler forcing Q_α R_2-dominates all reals in M_α.

3. Random forcing adds a random real.

   Accordingly, the generic real η_α for partial random forcing Q_α is not in any nullset whose Borel-code is in M_α.

4. The generic branch $\eta \in \lim(T^*)$ added by $\tilde{\mathbb{E}}$ is eventually different to each old real, i.e., R_4-dominates the old reals.

   (This was shown in Lemma 3.19(c).)

   Accordingly, the generic branch η_α for partial $\tilde{\mathbb{E}}$ forcing Q_α R_4-dominates the reals in M_α.

Fix $i \in \{1,2,3,4\}$, and set $S = S^i$ and $s \prec t$ if $w_s \subsetneq w_t$, and let g_s be η_s, i.e., the generic added at s (e.g., the partial random real in case of $i = 3$, etc).

Fix a P^5-name f for a real. It depends (in a Borel way) on a countable index set $w^* \subseteq \delta_5$. Fix some $s \in S^i$ such that $w_s \supseteq w^*$. Pick any $t \succ s$. Then $w_t \supseteq w_s \supseteq w^*$, so ($P^5$ forces that) $f \in M_t$, so, as just argued, $P^5 \Vdash f \mathrel{\mathrm{R}_i} g_t$ (or: $P^5 \Vdash f \mathrel{\mathrm{R}_3}' g_t$ for $i = 3$). □

So to summarize what we know so far about P^5: Whenever we choose (in addition to the "fixed" λ_i, S^i) a cofinal parameter $\bar{w}$ satisfying Assumptions 3.33 and 3.38, we get

Fact 3.44.

- COB_i holds for $i = 1,2,3,4$. So the left hand side characteristics are large.
- LCU_i holds for $i = 1,3,4$. So the left hand side characteristics other than $\mathfrak{b}$ are small.

What is missing is "$\mathfrak{b}$ small". We do not claim that this will be forced for every $\bar{w}$ as above; but we will show in the rest of Section 3.2 that we can choose such a $\bar{w}$.

3.2.3 FAMs in the P_α-extension compatible with M_α, explicit conditions.

We first investigate sequences $\bar{q} = (q_\ell)_{\ell\in\omega}$ of Q_α-conditions that are in M_α, i.e., the (evaluations of) P'_α-names for ω-sequences in Q^{full}_α. For $\alpha \in S^3 \cup S^4$, M_α thinks that Q_α (i.e., Q^{full}_α) has FAM-limits. So if M_α thinks that Ξ_0 is a FAM, then for any sequence $\bar{q}$ in M_α there is a condition $\lim_{\Xi_0}(\bar{q})$ in M_α (and thus in Q_α). We can relativize Lemma 3.8 to sequences in M_α:

Lemma 3.45. *Assume that $\alpha \in S^3 \cup S^4$, that Ξ is a P_α-name for a FAM and that Ξ_0, the restriction of Ξ to M_α, is forced to be in M_α. Then there is a $P_{\alpha+1}$-name Ξ^+ for a FAM such that for all* (trunk*, loss*)*-sequences $\bar{q}$ in M_α,*

$$\lim_{\Xi_0}(\bar{q}) \in G(\alpha) \textit{ implies } \Xi^+(A_{\bar{q}}) \geq 1 - \sqrt{\text{loss}^*}.$$

$A_{\bar{q}}$ was defined in (3.9) (here we use $G(\alpha)$ instead of G, of course).

Proof. This Lemma is implicitly used in [She00]. Note that P'_α is a complete subforcing of P_α, and so there is a quotient R such that $P_\alpha = P'_\alpha * R$. We consider the following (commuting) diagram:

$$\begin{tikzcd}
V \arrow[r, "P_\alpha"] \arrow[dr, "P'_\alpha"'] & V_\alpha \arrow[r, "Q_\alpha"] & V_{\alpha+1} \\
& M_\alpha \arrow[u, "R"'] \arrow[r, "Q_\alpha"'] & \square \arrow[u]
\end{tikzcd}$$

Note that (P'_α forces that) $R * Q_\alpha = R \times Q_\alpha$. So from the point of view of M_α:

- $Q_\alpha = Q^{\text{full}}_\alpha$ has FAM limits, and Ξ_0 is a FAM. So there is a Q_α-name for a FAM Ξ^+_0 satisfying Lemma 3.8.
- R is a ccc forcing, and there is an R-name[10] Ξ for a FAM extending Ξ_0.
- So there is $R \times Q_\alpha$-name Ξ^+ for a FAM extending both Ξ^+_0 and Ξ (cf. [She00, Claim 1.6]).

Back in V, this defines the $P_{\alpha+1}$-name Ξ^+. Let $\bar{q} = (q_\ell)_{\ell\in\omega}$ be a sequence in M_α. Then $M_\alpha[G(\alpha)]$ thinks: If $\lim_{\Xi_0}(\bar{q}) \in G(\alpha)$, then $\Xi^+_0(A_{\bar{q}})$ is large enough. This is upwards absolute to $V[G_{\alpha+1}]$ (as $A_{\bar{q}}$ is absolute). □

For later reference, we will reformulate the lemma for a specific instance of "sequence in M_α". Recall that a sequence in M_α corresponds to a "P'_α-name of a sequence in Q^{full}_α". This is not equivalent to a "P_α-name for a sequence in Q_α", which would correspond to an arbitrary sequence in Q_α (of which there are $|\alpha+\aleph_0|^{\aleph_0}$ many, while there are only less than λ_i many sequences in M_α). However, we can define the following:

Definition 3.46.
- An explicit Q_α-condition (in V) is a P'_α-name for a Q^{full}_α condition.
- A condition $p \in P^5$ is explicit, if for all $\alpha \in \text{supp}(p) \cap (S^4 \cup S^5)$, $p(\alpha)$ is an explicit Q_α-condition.

Here we mean that for $p(\alpha)$ there is a P'_α-name q_α such that $p \restriction \alpha \Vdash p(\alpha) = q_\alpha$ (and the map $\alpha \mapsto q_\alpha$ exists in the ground model, i.e., we do not just have a P_α-name for a P'_α-condition q_α).

Lemma 3.47. *The set of explicit conditions is dense.*

Proof. We show by induction that the set D_α of explicit conditions in P_α is dense in P_α. As we are dealing with FS iterations, limits are clear. Assume that $(p, q) \in P_{\alpha+1}$. Then p forces that there is a P'_α-name q' such that $q' = q$. Strengthen p to some $p' \in D_\alpha$ deciding q'. Then $(p', q') \leq (p, q)$ is explicit. □

Note that any sequence in V of explicit Q_α-conditions defines a sequence of conditions in M_α (as $V \subseteq M_\alpha$). So we get:

[10] We identify the P_α-name Ξ in V and the induced R-name in $M_\alpha = V[G'_\alpha]$.

Lemma 3.48. *Let α, Ξ, and Ξ^+ be as in Lemma 3.45, and let $(p_\ell)_{\ell\in\omega}$ be (in V) a sequence of explicit conditions in P^5 such that $\alpha \in \mathrm{supp}(p_\ell)$ for all $\ell \in \omega$. Set $q_\ell := p_\ell(\alpha)$ and $\bar q := (q_\ell)_{\ell\in\omega}$, and assume that $(\mathrm{trunk}(q_\ell), \mathrm{loss}(q_\ell))$ is forced to be equal to some constant $(\mathrm{trunk}^*, \mathrm{loss}^*)$.*

Then there is a P'_α-name for a Q^{full}_α-condition (and thus a P_α-name for a Q_α-condition) $\lim_{\Xi_0}(\bar q)$ such that $\lim_{\Xi_0}(\bar q)$ forces that $\Xi^+(A_{\bar q}) \leq 1 - \sqrt{\mathrm{loss}^*}$.

3.2.4 Dealing with $\mathfrak{b}$ (without GCH)

In this section, we follow [GKS17, p. 1.3], additionally using techniques inspired by [She00].

We assume the following (in addition to Assumption 3.33):

Assumption 3.49. (This section only.) $\chi < \lambda_3$ is regular such that $\chi^{\aleph_0} = \chi$, $\chi^+ \geq \lambda_2$ and $2^\chi = |\delta_5| = \lambda_5$.

Set $S^0 = \lambda_5 \cup S^1 \cup S^2$. So $\delta_5 = S^0 \cup S^3 \cup S^4$, and P^5 is a FS ccc iteration along δ_5 such that $\alpha \in S^0$ implies $|Q_\alpha| < \lambda_2$, i.e., $|Q_\alpha| \leq \chi$ (and Q_α is a partial random forcing for $\alpha \in S^3$ and a partial $\tilde{\mathbb{E}}$-forcing for $\alpha \in S^4$).

Let us fix, for each $\alpha \in S^0$, a P_α-name

$$i_\alpha : Q_\alpha \to \chi \text{ injective.} \tag{3.50}$$

Definition 3.51.
- A "partial guardrail" is a function h defined on a subset of δ_5 such that, for $\alpha \in \mathrm{dom}(h)$: $h(\alpha) \in \chi$ if $\alpha \in S^0$; and $h(\alpha)$ is a pair (x, y) with $x \in H(\aleph_0)$ and y a rational number otherwise. (Any $(\mathrm{trunk}, \mathrm{loss})$-pair is of this form.)
- A "countable guardrail" is a partial guardrail with countable domain. A "full guardrail" is a partial guardrail with domain δ_5.

We will use the following lemma, which is a consequence of the Engelking-Karlowicz theorem [EK65] on the density of box products (cf. [GMS16, p. 5.1]):

Lemma 3.52. *(As $|\delta_5| \leq 2^\chi$.) There is a family H^* of full guardrails of cardinality χ such that each countable guardrail is extended by some $h \in H^*$. We will fix such an H^*.*

Note that the notion of guardrail (and the density property required in Lemma 3.52) only depends on the "fixed" parameters χ, δ_5, S^0, S^3 and S^4; so we can fix an H^* that will work for all these fixed parameters and all choices of the cofinal parameter $\bar w$.

Once we have decided on $\bar w$, and thus have defined P^5, we can define the following:

Definition 3.53. $D^* \subseteq P^5$ consists of p such that there is a partial guardrail h (and we say: "p follows h") with $\mathrm{dom}(h) \supseteq \mathrm{supp}(p)$ and, for all $\alpha \in \mathrm{supp}(p)$,

- If $\alpha \in S^0$, then $p \restriction \alpha \Vdash i_\alpha(p(\alpha)) = h_\alpha$.
- If $\alpha \in S^3 \cup S^4$, the empty condition of P_α forces

$$p(\alpha) \in Q_\alpha \text{ and } (\mathrm{trunk}(p(\alpha)), \mathrm{loss}(p(\alpha))) = h(\alpha).$$

- Furthermore, $\sum_{\alpha \in \mathrm{supp}(p) \cap (S^3 \cup S^4)} \sqrt{\mathrm{loss}(p(\alpha))} < \frac{1}{2}$.
- p is explicit (as in Definition 3.46).

Lemma 3.54. *$D^* \subseteq P^5$ is dense.*

Proof. By induction we show that for any sequence $(\epsilon_i)_{i \in \omega}$ of positive numbers the following set of p is dense: If $\mathrm{supp}(p) = \{\alpha_0, \dots, \alpha_m\}$, where $\alpha_0 > \alpha_1 >, \dots$ (i.e., we enumerate downwards), $\mathrm{loss}^p_{\alpha_n} < \epsilon_n$ whenever $\alpha_n \in S^3 \cup S^4$. For the successor step, we use that the set of $q \in Q_\alpha$ such that $\mathrm{loss}(q) < \epsilon_0$ is forced to be dense. □

Remark 3.55. So the set of conditions following *some* guardrail is dense. For each *fixed* guardrail h, the set of all conditions p following h is n-linked, provided that each loss in the domain of h is $< \frac{1}{n}$ (cf. Assumption 3.5).

Definition 3.56. A "Δ-system with heart ∇ following the guardrail h" is a family $\bar{p} = (p_i)_{i \in I}$ of conditions such that

- all p_i are in D^* and follow h,
- $(\mathrm{supp}(p_i))_{i \in I}$ is a Δ system with heart ∇ in the usual sense (so $\nabla \subseteq \delta_5$ is finite)
- the following is independent of $i \in I$:
  - $|\mathrm{supp}(p_i)|$, which we call $m^{\bar{p}}$.
    Let $(\alpha_i^{\bar{p},n})_{n < m^{\bar{p}}}$ increasingly enumerate $\mathrm{supp}(p_i)$.
  - Whether $\alpha_i^{\bar{p},n}$ is less than, equal to or bigger than the k-th element of ∇.
    In particular it is independent of i whether $\alpha_i^{\bar{p},n} \in \nabla$, in which case we call n a "heart position".
  - Whether $\alpha_i^{\bar{p},n}$ is in S^0, in S^3 or in S^4.
    If $\alpha_i^{\bar{p},n} \in S^j$, we call n an "S^j-position".
  - If n is not an S^0-position:[11] The value of $h(\alpha_i^{\bar{p},n}) =: (\mathrm{trunk}^{\bar{p},n}, \mathrm{loss}^{\bar{p},n})$.
    If n is an S^0-position, we set $\mathrm{loss}^{\bar{p},n} := 0$.

A "countable Δ-system" $\bar{p} = (p_\ell : \ell \in \omega)$ is a Δ system that additionally satisfies:

[11] If n is a S^0-position, $h(\alpha_i^{\bar{p},n})$ will generally not be be independent of i; unless of course n is a heart position.

- For each non-heart position[12] $n < m^{\bar{p}}$, the sequence $(\alpha_\ell^{\bar{p},n})_{\ell\in\omega}$ is strictly increasing.

Fact 3.57.
- Each infinite Δ-system $(p_i)_{i\in I}$ contains a countable Δ-system. I.e., there is a sequence i_ℓ in I such that $(p_{i_\ell})_{\ell\in\omega}$ is a countable Δ-system..
- If $\bar{p}$ is a Δ-system (or: a countable Δ-system) following h with heart ∇, and $\beta \in \nabla \cup (\max(\nabla + 1))$, then $\bar{p} \restriction \beta \coloneqq (p_i \restriction \beta)_{i\in I}$ is again a Δ-system (or: a countable Δ-system, respectively) following h, now with heart $\nabla \cap \beta$.

Definition 3.58. Let $\bar{p}$ be a countable Δ-system, and assume that $\bar{\Xi} = (\Xi_\alpha)_{\alpha\in\nabla\cap(S^3\cup S^4)}$ is a sequence such that each Ξ_α is a P_α-name for a FAM and P_α forces that Ξ_α restricted to M_α is in M_α. Then we can define $q = \lim_{\bar{\Xi}}(\bar{p})$ to be the following P^5-condition with support ∇:

- If $\alpha \in \nabla \cap S^0$, then $q(\alpha)$ is the common value of all $p_n(\alpha)$. (Recall that this value is already determined by the guardrail h.)
- If $\alpha \in \nabla \cap (S^3 \cup S^4)$, then $q(\alpha)$ is (forced by P^5_α to be) $\lim_{\Xi_\alpha}(p_\ell(\alpha))_{\ell\in\omega}$, see Lemma 3.48.

We now give a specific way to construct such $\bar{w}$, which allows to keep $\mathfrak{b}$ small.

Lemma/Construction 3.59. *We can construct by induction on $\alpha \in \delta_5$ for each $h \in H^*$ some Ξ^h_α, and, if $\alpha > \kappa_5$, also w_α, such that:*

(a) Each Ξ^h_α is a P_α-name of a FAM extending $\bigcup_{\beta<\alpha} \Xi^h_\beta$.

(b) If α is a limit of countable cofinality: Assume $\bar{p}$ is a countable Δ-system in P_α following h, and $n < m^{\bar{p}}$ such that $(\alpha_\ell^{\bar{p},n})_{\ell\in\omega}$ has supremum α. Then $A_{\bar{p},n}$ is forced to have Ξ^h_α-measure 1, where

$$A_{\bar{p},n} \coloneqq \Big\{k \in \omega : \ \Big|\big\{\ell \in I_k : p_\ell(\alpha_\ell^{\bar{p},n}) \in G(\alpha_\ell^{\bar{p},n})\big\}\Big| \geq |I_k| \cdot \big(1 - \sqrt{\mathrm{loss}^{\bar{p},n}}\,\big)\Big\}$$

(c) For each countable Δ-system $\bar{p}$ in P_α following h, the P_α-condition $\lim_{(\Xi^h_\beta)_{\beta<\alpha}}(\bar{p})$ is well-defined and forces

$$\Xi^h_\alpha(A_{\bar{p}}) \geq 1 - \sum_{n<m^{\bar{p}}} \sqrt{\mathrm{loss}^{\bar{p},n}}, \ \textit{where}$$
$$A_{\bar{p}} \coloneqq \Big\{k \in \omega : \ \Big|\big\{\ell \in I_k : p_\ell \in G_\alpha\big\}\Big| \geq |I_k| \cdot \big(1 - \sum_{n<m^{\bar{p}}} \sqrt{\mathrm{loss}^{\bar{p},n}}\,\big)\Big\}.$$

(d) For $\alpha > \kappa_5$: w_α is "sufficiently closed". More specifically: It satisfies Assumptions 3.33 and 3.38, and if $\alpha \in S^3 \cup S^4$ then P_α forces that Ξ^h_α restricted to M_α is in M_α.

Actually, the set of w_α satisfying this is an ω_1-club set.

[12]For a heart position n, $(\alpha_\ell^{\bar{p},n})_{\ell\in\omega}$ is of course constant.

Proof. **(a&c) for cf(α) > ω:** We set $\Xi^h_\alpha = \bigcup_{\beta<\alpha} \Xi^h_\beta$. As there are no new reals at uncountable confinalities, this is a FAM. Each countable Δ-system is bounded by some $\beta < \alpha$, and, by induction, (c) holds for β; so (c) holds for α as well.

(a&b) for cf(α) = ω: Fix h. We will show that P_α forces $A \cap \bigcap_{j<j^*} A_{\bar{p}^j,n^j} \neq \emptyset$, where A is a Ξ^h_β-positive set for some $\beta < \alpha$, and each $(\bar{p}^j, n^j)$ is as in (b).

Then we can work in the P_α-extension and apply Fact 3.3(a), using $\bigcup_{\beta<\alpha} \Xi^h_\beta$ as the partial FAM Ξ'. This gives an extension of Ξ' to a FAM Ξ^h_α that assigns measure one to all $A_{\bar{p},n}$, showing that (a) and (b) are satisfied.

So assume towards a contradiction that some $p \in P_\alpha$ forces

$$A \cap \bigcap_{j<j^*} A_{\bar{p}^j,n^j} = \emptyset.$$

We can assume that p decides the β such that $A \in V_\beta$, that β is above the hearts of all Δ-sequences $\bar{p}^j$ involved, and that $\mathrm{supp}(p) \subseteq \beta$. We can extend p to some $p^* \in P_\beta$ to decide $k \in A$ for some "large" k: By large, we mean:

- Let $F(l; n, p)$ (the cumulative binomial probability distribution) be the probability that n independent experiments, each with success probability p, will have at most l successful outcomes. As $\lim_{n\to\infty} F(n \cdot p'; n, p) = 0$ for all $p' < p$, and as $\lim_{k\to\infty} |I_k| = \infty$, we can find some k such that

$$F(|I_k| p'_j; |I_k|, p_j) < \frac{1}{2 \cdot j^*} \tag{3.60}$$

  for all $j < j^*$, where we set $p'_j \coloneqq 1 - \sqrt{\mathrm{loss}^{\bar{p}^j,n^j}}$ and $p_j \coloneqq 1 - \frac{1+\sqrt{2}}{2} \cdot \mathrm{loss}^{\bar{p}^j,n^j}$. (Note that $p'_j < p_j$, as $\mathrm{loss}^{\bar{p}^j,n^j} \leq \frac{1}{2}$.)

- All elements of $Y = \{\alpha^{\bar{p}^j,n^j}_\ell : j < j^* \text{ and } \ell \in I_k\}$ are larger than β. (This is possible as each sequence $(\alpha^{\bar{p}^j,n^j}_\ell)_{\ell<\omega}$ has supremum α.) We enumerate Y by the increasing sequence $(\beta_i)_{i\in M}$, and set $\beta_{-1} = \beta$.

We will find $q \leq p^*$ forcing that $k \in \bigcap_{j<j^*} A_{\bar{p}^j,n^j}$. To this end, we define a finite tree $\mathcal{T}$ of height M, and assign to each $s \in \mathcal{T}$ of height i a condition $q_s \in P_{\beta_{i-1}+1}$ (decreasing along each branch) and a probability $\mathrm{pr}_s \in [0, 1]$, such that $\sum_{t \rhd s} \mathrm{pr}_t = 1$ for all non-terminal nodes $s \in \mathcal{T}$. For s the root of $\mathcal{T}$, i.e., for the unique s of height 0, we set $q_s = p^* \in P_{\beta_{-1}}$ and $\mathrm{pr}_s = 1$.

So assume we have already constructed $q_s \in P_{\beta_{i-1}+1}$ for some s of height $i < M$. We will now take care of index β_i and construct the set of successors of s, and for each successor t, a $q_t \leq q_s$ in P_{β_i+1}.

- If $\beta_i \in S^0$, the guardrail guarantees that $\beta_i \in \mathrm{supp}(p^j_\ell)$ implies $p^j_\ell \restriction \beta_i \Vdash i_{\beta_i}(p^j_\ell(\beta_i)) = h(\beta_i)$. In that case we use a unique $\mathcal{T}$-successor t of s, and we set $q_t = q_s^\frown(\beta_i, i^{-1}_{\beta_i} h(\beta_i))$, and $\mathrm{pr}_t = 1$.

  In the following we assume $\beta_i \notin S^0$.

- Let J_i be the set of $j < j^*$ such that there is an $\ell \in I_k$ with $\alpha_\ell^{\bar{p}^j, n^j} = \beta_i$ (there is at most one such ℓ). For $j \in J_i$, set $r_i^j = p_\ell^j(\beta_i)$ for the according ℓ. So each r_i^j is a P_{β_i}-name for an element of Q_{β_i}.

  The guardrail gives us the constant value $(\mathtt{trunk}_i^*, \mathrm{loss}_i^*) \coloneqq h(\beta_i)$ (which is equal to $(\mathtt{trunk}^{\bar{p}^j, n^j}, \mathrm{loss}^{\bar{p}^j, n^j})$ for all $j \in J_i$).

- The case $\beta_i \in S^3$, i.e., the case of random forcing, is basically [She00, p. 2.14]: For $x \subseteq [\mathtt{trunk}_i^*]$, set $\mathrm{Leb}^{\mathrm{rel}}(x) = \frac{\mathrm{Leb}(x)}{\mathrm{Leb}([\mathtt{trunk}_i^*])}$. Note that the r_i^j are closed subsets of $[\mathtt{trunk}_i^*]$ and $\mathrm{Leb}^{\mathrm{rel}}(r_i^j) \geq 1 - \mathrm{loss}_i^*$.

  Let $\mathcal{B}^*$ be the power set of $[\mathtt{trunk}_i^*]$; and let $\mathcal{B}$ be the sub-Boolean-algebra generated by by r_i^j $(j \in J_i)$, let $\mathcal{X}$ be the set of atoms and $\mathcal{X}' = \{x \in \mathcal{X} : \mathrm{Leb}^{\mathrm{rel}}(x) > 0\}$. So $|\mathcal{X}'| \leq 2^{J_i} \leq 2^{j^*}$, $\sum_{x \in \mathcal{X}'} \mathrm{Leb}^{\mathrm{rel}}(x) = 1$, and $\sum_{x \in \mathcal{X}', x \subseteq r_i^j} \mathrm{Leb}^{\mathrm{rel}}(x) = \mathrm{Leb}^{\mathrm{rel}}(r_i^j)$.

  So far, $\mathcal{X}'$ is a P_{β_i}-name. Now we increase q_s inside P_{β_i} to some q^+ deciding which of the (finitely many) Boolean combinations result in elements of $\mathcal{X}'$, and also deciding rational numbers y_x $(x \in \mathcal{X}')$ with sum 1 such that $|\mathrm{Leb}^{\mathrm{rel}}(x) - y_x| < \frac{\sqrt{2}-1}{2} \cdot \mathrm{loss}_i^* \cdot 2^{-j^*}$.

  We can now define the immediate successors of s in $\mathcal{T}$: For each $x \in \mathcal{X}'$, add an immediate successor t_x and assign to it the probability $\mathrm{pr}_{t_x} = y_x$ and the condition $q_{t_x} = q^{+\frown}(\beta_i, r_x)$, where r_x is a (name for a) partial random condition below x (such a condition exists, as the Lebesgue positive intersection of finitely many partial random condition contains a partial random condition).

  Note that when we choose a successor t randomly (according to the assigned probabilities pr_t), then for each $j \in J$ the probability of $q^+ \Vdash q_t(\beta_i) \leq r_i^j$ is at least

$$\begin{aligned}\textstyle\sum_{x \in \mathcal{X}', x \subseteq r_i^j} \mathrm{pr}_x \geq \sum_{x \in \mathcal{X}', x \subseteq r_i^j} \left(\mathrm{Leb}^{\mathrm{rel}}(x) - \frac{\sqrt{2}-1}{2} \cdot \mathrm{loss}_i^* \cdot 2^{-j^*}\right) \geq \\ \geq \left(\textstyle\sum_{x \in \mathcal{X}', x \subseteq r_i^j} \mathrm{Leb}^{\mathrm{rel}}(x)\right) - \frac{\sqrt{2}-1}{2} \cdot \mathrm{loss}_i^* = \mathrm{Leb}^{\mathrm{rel}}(r_i^j) - \frac{\sqrt{2}-1}{2} \cdot \mathrm{loss}_i^* \geq \\ \geq 1 - \mathrm{loss}_i^* - \frac{\sqrt{2}-1}{2} \cdot \mathrm{loss}_i^* = 1 - \frac{1+\sqrt{2}}{2} \cdot \mathrm{loss}_i^*.\end{aligned}$$

- The case $\beta_i \in S^4$, i.e., the case of $\tilde{\mathbb{E}}$:

  Recall that $\tilde{\mathbb{E}}$-conditions are subtrees of some basic compact tree T^*, and there is a h such that: if $\max\{|I_k|, j^*\}$ many conditions share a common node (above their stems) at height h, then they are compatible.

  All conditions r_i^j have the same stem $s^* = \mathtt{trunk}_i^*$. For each $j \in J_i$, set $d(j) = r_i^j \cap \omega^h$. Note that ($P_{\beta_i}$ forces that) $d(j)$ is a subset of $T^* \cap [s^*] \cap \omega^h$ of relative size $\geq 1 - \frac{1}{2} \mathrm{loss}_i^*$ (according to Lemma 3.19(d)). First find $q^+ \leq q_s$ in P_{β_i} deciding all $d(j)$.

We can now define the immediate successors of s in $\mathcal{T}$: For each $x \in T^* \cap [s^*] \cap \omega^h$ add an immediate successor t_x, and assign to it the uniform probability (i.e., $\mathrm{pr}_{t_x} = \frac{1}{|T^* \cap [s^*] \cap \omega^h|}$) and the condition $q_{t_x} = q^{+\frown}(\beta_i, r_x)$, where r_x is a partial $\tilde{\mathbb{E}}$-condition stronger than all r^j_i that satisfy $x \in d(j)$. (Such a condition exists, as we can intersect $\leq j^*$ many conditions of height h.)

If we chose t randomly, then for each $j \in J$ the probability of $q^+ \Vdash q_t \leq r^j_i$ is at least $1 - \frac{1}{2}\,\mathrm{loss}^*_i \geq 1 - \frac{1+\sqrt{2}}{2} \cdot \mathrm{loss}^*_i$.

In the end, we get a tree $\mathcal{T}$ of height M, and we can chose a random branch through $\mathcal{T}$, according to the assigned probabilities. We can identify the branch with its terminal node t^*, so in this notation the branch t^* has probability $\prod_{n \leq M} \mathrm{pr}_{t^* \restriction n}$.

Fix $j < j^*$. There are $|I_k|$ many levels $i < M$ such that at β_i we deal with the $(\bar{p}^j, n^j)$-case. Let M^j be the set of these levels. For each $i \in M^j$, we perform an experiment, by asking whether the next step $t \in \mathcal{T}$ (from the current s at level i) will satisfy $q_t \restriction \beta_i \Vdash q_t(\beta_i) \leq r^j_i$. While the exact probability for success will depend on which s at level i we start from, a lower bound is given by $1 - \frac{1+\sqrt{2}}{2} \cdot \mathrm{loss}^*_i$. Recall that $\mathrm{loss}^*_i = \mathrm{loss}^{\bar{p}^j, n^j}$, and that we set $p_j := 1 - \frac{1+\sqrt{2}}{2} \cdot \mathrm{loss}^*_i$ and $p'_j := 1 - \sqrt{\mathrm{loss}^{\bar{p}^j, n^j}}$ in (3.60). So the chance of our branch t^* having success fewer than $|I_k| \cdot (1 - \sqrt{\mathrm{loss}^{\bar{p}^j, n^j}})$ many times, out of the the $|I_k|$ many tries, (let us call such a t^* "bad for j") is at most $F(|I_k| p'; |I_k|, p) \leq \frac{1}{2j^*}$.

Accordingly, the measure of branches that are not bad for *any* $j < j^*$ is at least $\frac{1}{2}$. Fix such a branch t^*. Then for each $j < j^*$,

$$\left|\left\{ i \in M^j \,:\, q_{t^*} \restriction \beta_i \Vdash q_{t^*}(\beta_i) \leq r^j_i \right\}\right| \geq |I_k| \cdot \left(1 - \sqrt{\mathrm{loss}^{\bar{p}^j, n^j}}\right),$$

and thus q_{t^*} forces that

$$\left|\left\{ \ell \in I_k \,:\, p_\ell(\alpha^{\bar{p}^j, n^j}_\ell) \in G(\alpha^{\bar{p}^j, n^j}_\ell) \right\}\right| \geq |I_k| \cdot \left(1 - \sqrt{\mathrm{loss}^{\bar{p}^j, n^j}}\right).$$

(c) for $\mathrm{cf}(\alpha) = \omega$:

Fix $\bar{p}$ as in the assumption of (c). To simplify notation, let us assume that $\nabla \neq \emptyset$ and that $\sup(\nabla) < \sup(\mathrm{supp}(p_\ell))$ (for some, or equivalently: all, $\ell \in \omega$). Let $0 < n_0 < m^{\bar{p}}$ be such that $\sup(\nabla)$ is at position $n_0 - 1$ in $\mathrm{supp}(p_\ell)$, i.e., $\sup(\nabla) = \alpha^{\bar{p}, n_0 - 1}_\ell$ (independent of ℓ), and set $\beta := \sup(\nabla) + 1$.

$\bar{p} \restriction \beta$ is again a countable Δ-system following the same h, and $\lim_{(\Xi^h_\gamma)_{\gamma<\alpha}}(\bar{p})$ is by definition identical to $\lim_{(\Xi^h_\gamma)_{\gamma<\beta}}(\bar{p} \restriction \beta)$, which by induction is a valid condition and forces (c) for $\bar{p} \restriction \beta$. This gives us the set $A_{\bar{p} \restriction \beta}$ of measure at least $1 - \sum_{n < n_0} \sqrt{\mathrm{loss}^{\bar{p}, n}}$.

For the positions $n_0 \leq n < m^{\bar{p}}$, all $(\alpha^{\bar{p}, n}_\ell)_{\ell \in \omega}$ are strictly increasing sequences above β with some limit $\alpha_n \leq \alpha$. Then (b) (applied to α_n) gives us an according measure-1-set $A_{\bar{p}, n}$.

So $\lim_{(\Xi^h_\gamma)_{\gamma<\alpha}}(\bar p)$ forces that $A' = A_{\bar p\restriction\beta} \cap \bigcap_{n_0\le n<m^{\bar p}} A_{\bar p,n}$ has measure

$$\Xi^h_\alpha(A') \ge 1 - \sum_{n<n_0} \sqrt{\text{loss}^{\bar p,n}} \ge 1 - \sum_{n<m^{\bar p}} \sqrt{\text{loss}^{\bar p,n}}.$$

Note that $p_\ell \in G$ iff $p_\ell \restriction \beta \in G_\beta$ and $p_\ell(\alpha^{\bar p,n}) \in G(\alpha^{\bar p,n})$ for all $n_0 \le n < m^{\bar p}$.

Fix $k \in A'$. As $k \in A_{\bar p\restriction\beta}$, the relative frequency for $\ell \in I_k$ to *not* satisfy $p_\ell \restriction \beta \in G_\beta$ is at most $\sum_{n<n_0} \sqrt{\text{loss}^{\bar p,n}}$. For any $n_0 \le n < m^{\bar p}$, as $k \in A_{\bar p,n}$, the relative frequency for *not* $p_\ell(\alpha^{\bar p,n}) \in G(\alpha^{\bar p,n})$ is at most $\sqrt{\text{loss}^{\bar p,n}}$. So the relative frequency for $p_\ell \in G$ to fail is at most $\sum_{n<n_0} \sqrt{\text{loss}^{\bar p,n}} + \sum_{n_0\le n<m^{\bar p}} \sqrt{\text{loss}^{\bar p,n}}$, as required.

(a&c) for $\alpha = \gamma + 1$ successor:

For $\gamma \in S^0$ this is clear: Let Ξ^h_α be the name of some FAM extending Ξ^h_γ. Let $\bar p$ be as in (c), without loss of generality $\gamma \in \nabla$. Then $q^+ \coloneqq \lim_{(\Xi^h_\beta)_{\beta<\alpha}}(\bar p) = q^\frown(\gamma, r)$, where $q \coloneqq \lim_{(\Xi^h_\beta)_{\beta<\gamma}}(\bar p \restriction \gamma)$ and r is the condition determined by $h(\gamma)$, i.e., each $p_\ell \restriction \gamma$ forces $p_\ell(\gamma) = r$. In particular, q^+ forces that $p_\ell \in G_\alpha$ iff $p_\ell \restriction \gamma \in G_\alpha$. By induction, (c) holds for γ, and therefore we get (c) for α.

Assume $\gamma \in S^3 \cup S^4$. By induction we know that (d) holds for γ, i.e., that Ξ^h_γ restricted to M_γ (call it Ξ_0) is in M_γ. So the requirement in the definition 3.58 of the limit is satisfied, and thus the limit $q^+ \coloneqq \lim_{\bar\Xi^h}(\bar p)$ is well defined for any countable Δ-system $\bar p$ as in (c): q^+ has the form $q^\frown(\gamma, r)$ with $q = \lim_{(\Xi^h_\beta)_\beta<\gamma}(\bar p \restriction \gamma)$ and $r = \lim_{\Xi_0}((p_\ell(\gamma))_{\ell\in\omega})$. Now Lemma 3.48 gives us the P_α-name Ξ^+, which will be our new Ξ^h_α.

This works as required: Again without loss of generality we can assume $\gamma \in \nabla$. By induction, q forces that $\Xi^h_\gamma(A_{\bar p\restriction\gamma}) \ge 1 - \sum_{n<m^p-1} \sqrt{\text{loss}^{\bar p,n}}$. According to Lemma 3.48, r forces that $\Xi^+(A_{(p_\ell(\gamma))_{\ell\in\omega}}) \ge 1 - \sqrt{\text{loss}^{\bar p,m^p-1}}$. So $q^+ = q^\frown r$ forces that $\Xi^h_\alpha(A_{\bar p}) \ge 1 - \sum_{n<m^p} \sqrt{\text{loss}^{\bar p,n}}$.

(d):

So we have (in V) the P_α-name Ξ^h_α. We already know that there is (in V) an ω_1-club set X_0 in $[\alpha]^{<\lambda_i}$ (for the appropriate $i \in \{3, 4\}$) such that $w \in X_0$ implies that w satisfies Assumptions 3.33 and 3.38. So each such $w \in X_0$ defines a complete subforcing P_w of P_α and the P_α-mame for the according P_w-extention M_w.

Fix some $w \in X_0$. We will define $w' \supseteq w$ as follows: For a P_w-name (and thus a P_α-name) $r \in 2^\omega$, let s be the name of $\Xi_\alpha(r) \in [0, 1]$. As in Lemma 3.37(a), we can find a countable w_r determining s. (I.e., there is a Borel function that calculates the real s from the generics at w_r; moreover we know this Borel function in the ground model.) Let $w' \supseteq w$ be in X_0 and contain all these w_r, for a (small representative set of) all P_w-names for reals.

Iterating this construction ω_1 many steps gives us a suitable w_α: Note that the assignment of a name r to the Ξ_α-value s can be done in V, and thus is known to M_α. In addition, M_α sees that for each "actual real" (i.e., element of M_α), the value

s is already determined (by P'_α). So the assignment $r \mapsto s$, which is Ξ_α restricted to M_α, is in M_α. □

Note that in (c), when we deal with a countable Δ-system $\bar{p}$ following the guardrail $h \in H^*$, the condition $\lim_{\Xi^h} \bar{p}$ forces in particular that infinitely many p_ℓ are in G. So after carrying out the construction as above, we get a forcing notion P^5 satisfying the following (which is actually the only thing we need from the previous construction, in addition to the fact that we can choose each w_α in an ω_1-club):

Lemma 3.61. *For every countable Δ-system $\bar{p}$ there is some q forcing that infinitely many p_ℓ are in the generic filter.*

Proof. According to Lemma 3.52, $\bar{p}$ follows some $h \in H^*$; so $q = \lim_{\Xi^h}(\bar{p})$ will work. □

Lemma 3.62. $\mathsf{LCU}_2(P^5, \kappa)$ *for $\kappa \in [\lambda_2, \lambda_5]$ regular, witnessed by the sequence $(c_\alpha)_{\alpha<\kappa}$ of the first κ many Cohen reals.*

Proof. Fix a P^5-name $y \in \omega^\omega$. We have to show that $(\exists \alpha \in \kappa)\,(\forall \beta \in \kappa \setminus \alpha)\, P^5 \Vdash \neg c_\beta \leq^* y)$.

Assume towards a contradiction that p^* forces that there are unboundedly many $\alpha \in \kappa$ with $c_\alpha \leq^* y$, and enumerate them as $(\alpha_i)_{i\in\kappa}$. Pick $p^i \leq p^*$ deciding α_i to be some β^i, and also deciding n_i such that $(\forall m \geq n_i)\, c_{\alpha_i}(m) \leq y(m)$. We can assume that $\beta^i \in \mathrm{supp}(p^i)$. Note that β^i is a Cohen position (as $\beta^i < \kappa \leq \lambda_5$), and we can assume that $p^i(\beta^i)$ is a Cohen condition in V (and not just a P_{β_i}-name for such a condition). By strengthening and thinning out, we may assume:

- $(p^i)_{i\in\kappa}$ forms a Δ system with heart ∇.
- All n_i are equal to some n^*.
- $p^i(\beta_i)$ is always the same Cohen condition $s \in \omega^{<\omega}$, without loss of generality of length $|s| = n^{**} \geq n^*$.
- For some position $n < m^{\bar{p}}$, β^i is the n-th element of $\mathrm{supp}(p^i)$.

Note that this n cannot be a heart condition: For any $\beta \in \kappa$, at most $|\beta|$ many p^i can force $\alpha_i = \beta$, as p^i forces that $\alpha_i \geq i$ for all i.

Pick a countable subset of this Δ-system which forms a countable Δ-system $\bar{p} \coloneqq (p_\ell)_{\ell\in\omega}$. So $p_\ell = p^{i_\ell}$ for some $i_\ell \in \kappa$, and we set $\beta_\ell = \beta^{i_\ell}$. In particular all β_ℓ are distinct. Now extend each p_ℓ to p'_ℓ by extending the Cohen condition $p_\ell(\beta_\ell) = s$ to $s^\frown \ell$ (i.e., forcing $c_{\beta_\ell}(n^{**}) = \ell$). Note that $\bar{p}' \coloneqq (p'_i)_{i\in\omega}$ is still a countable Δ-system,[13] and by Lemma 3.61 some q forces that infinitely many of the p'_ℓ are in the generic filter. But each such p'_ℓ forces that $c_{\beta_\ell}(n^{**}) = \ell \leq y(n^{**})$, a contradiction. □

[13] Note that $\bar{p}'$ will not follow the same guardrail as $\bar{p}$.

3.2.5 The left hand side

We have now finished the consistency proof for the left hand side:

Theorem 3.63. *Assume GCH and let λ_i be an increasing sequence of regular cardinals, none of which is a successor of a cardinal of countable cofinality for $i = 1, \dots, 5$. Then there is a cofinalities-preserving forcing P resulting in*

$$\mathrm{add}(\mathcal{N}) = \lambda_1 < \mathrm{add}(\mathcal{M}) = \mathfrak{b} = \lambda_2 < \mathrm{cov}(\mathcal{N}) = \lambda_3 < \mathrm{non}(\mathcal{M}) = \lambda_4 < \\ < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0} = \lambda_5.$$

Proof. Set $\chi = \lambda_2$, and let R be the set of partial functions $f : \chi \times \lambda_5 \to 2$ with $|\operatorname{dom}(f)| < \chi$ (ordered by inclusion). R is $<\chi$-closed, χ^+-cc, and adds λ_5 many new elements to 2^χ. So in the R-extension, Assumption 3.49 is satisfied, and we can construct P^5 according to Assumption 3.33 and Construction 3.59. Fact 3.44 gives us all inequalities for the left hand side, apart from $\mathfrak{b} \leq \lambda_2$, which we get from 3.62.

In the R-extension, CH holds and P is a FS ccc iteration of length δ_5, $|\delta_5| = \lambda_5$, and each iterand is a set of reals; so $2^{\aleph_0} \leq \lambda_5$ is forced. Also, any FS ccc iteration of length δ (of nontrivial iterands) forces $\mathrm{cov}(\mathcal{M}) \geq \mathrm{cf}(\delta)$: Without loss of generality $\mathrm{cf}(\delta) = \lambda$ is uncountable. Any set A of (Borel codes for) meager sets that has size $<\lambda$ already appears at some stage $\alpha < \delta$, and the iteration at state $\alpha + \omega$ adds a Cohen real over the V_α, so A will not cover all reals. □

Remark 3.64. So this consistency result is reasonably general, we can, e.g., use the values $\lambda_i = \aleph_{i+1}$. This is in contrast to the result for the whole diagram, where in particular the small λ_i have to be separated by strongly compact cardinals.

3.3 Ten different values in Cichoń's diagram

We can now apply, with hardly any change, the technique of [GKS17] to get the following:

Theorem 3.65. *Assume GCH and that $\aleph_1 < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8 < \lambda_9$ are regular, λ_i is not a successor of a cardinal of countable cofinality for $i = 1, \dots, 5$, $\lambda_2 = \chi^+$ with χ regular, and κ_i strongly compact for $i = 6, 7, 8, 9$. Then there is a ccc forcing notion P^9 resulting in:*

$$\mathrm{add}(\mathcal{N}) = \lambda_1 < \mathfrak{b} = \mathrm{add}(\mathcal{M}) = \lambda_2 < \mathrm{cov}(\mathcal{N}) = \lambda_3 < \mathrm{non}(\mathcal{M}) = \lambda_4 < \\ < \mathrm{cov}(\mathcal{M}) = \lambda_5 < \mathrm{non}(\mathcal{N}) = \lambda_6 < \mathfrak{d} = \mathrm{cof}(\mathcal{M}) = \lambda_7 < \mathrm{cof}(\mathcal{N}) = \lambda_8 < 2^{\aleph_0} = \lambda_9.$$

To do this, we first have to show that we can achieve the order for the left hand side, i.e., Theorem 3.63, starting with GCH and using a FS ccc iteration P^5 alone (instead of using $P = R * P^5$, where R is not ccc). This is the only argument that requires $\lambda_2 = \chi^+$. We will just briefly sketch it here, as it can be found with all details in [GKS17, p. 1.4]:

- We already know that in the R-extension, (where R is $<\chi$-closed, χ^+-cc and forces $2^\chi = \lambda_5$) we can find by the inductive construction 3.59 suitable w_α such that $R * P^5$ works.
- We now perform a similar inductive construction in the ground model: At stage α, we know that there is an R-name for a suitable w^1_α of size $< \lambda_i$ (where i is 3 in the random and 4 in the $\tilde{\mathbb{E}}$-case). This name can be covered by some set $\tilde{w}^1_\alpha$ in V, still of size $< \lambda_i$, as R is χ^+-cc. Moreover, in the R-extension, the suitable parameters form an ω_1-club; so there is a suitable $w^2_\alpha \supseteq \tilde{w}^1_\alpha$, etc. Iterating ω_1 many times and taking the union at the end leads to w_α in V which is forced by R to be suitable.
- Not only w_α is in V, but the construction for w_α is performed in V, so we can construct the whole sequence $\bar{w} = (w_\alpha)_{\alpha\in\delta_5}$ in V.
- We now know that in the R-extension, the forcing P^5 defined from $\bar{w}$ will satisfy $\mathsf{LCU}_2(P^5, \kappa)$ in the form of Lemma 3.62.
- By an absoluteness argument, we can show that actually in V the forcing P^5 defined form $\bar{w}$ will satisfy Lemma 3.62 as well.

The rest of the proof is the same as in [GKS17, Sec. 2], where we interchange $\mathfrak{b}$ and $\mathrm{cov}(\mathcal{N})$ as well as $\mathfrak{d}$ and $\mathrm{non}(\mathcal{N})$.

We cite the following facts from [GKS17, pp. 2.2–2.5]:

Facts 3.66. (a) If κ is a strongly compact cardinal and $\theta > \kappa$ regular, then there is an elementary embedding $j_{\kappa,\theta} : V \to M$ (in the following just called j) such that

- the critical point of j is κ, $\mathrm{cf}(j(\kappa)) = |j(\kappa)| = \theta$,
- $\max(\theta, \lambda) \leq j(\lambda) < \max(\theta, \lambda)^+$ for all $\lambda \geq \kappa$ regular, and
- $\mathrm{cf}(j(\lambda)) = \lambda$ for $\lambda \neq \kappa$ regular,

and such that the following is satisfied:

(b) If P is a FS ccc iteration along δ, then $j(P)$ is a FS ccc iteration along $j(\delta)$.

(c) $\mathsf{LCU}_i(P, \lambda)$ implies $\mathsf{LCU}_i(j(P), \mathrm{cf}(j(\lambda)))$, and thus $\mathsf{LCU}_i(j(P), \lambda)$ if $\lambda \neq \kappa$ regular.[14]

(d) If $\mathsf{COB}_i(P, \lambda, \mu)$, then $\mathsf{COB}_i(j(P), \lambda, \mu')$, for $\mu' = \begin{cases} |j(\mu)| & \text{if } \kappa > \lambda \\ \mu & \text{if } \kappa < \lambda. \end{cases}$

[14] In [GKS17], we only used "classical" relations R_3 that are defined on a Polish space in an absolute way. In this paper, we use the relation R_3 which is not of this kind. However, the proof still works without any change: The parameter $\mathcal{E}$ used to define the relation R_3, cf. Definition 3.22, is a set of reals. So $j(\mathcal{E}) = \mathcal{E}$, and we can still use the usual absoluteness arguments between M and V. (A parameter not element of $H(\kappa_9)$ might be a problem.)

Using these facts, it is easy to finish the proof:[15]

Proof of Theorem 3.65. Recall that we want to force the following values to the characteristics of Figure 3.2 (where we indicate the positions of the κ_i as well):

$$\begin{tikzcd}
 & \lambda_3 \arrow[r, "\kappa_6"] & \lambda_4 \arrow[r] & \square \arrow[r] & \lambda_8 \arrow[r] & \lambda_9 \\
 & & \lambda_2 \arrow[r] \arrow[u] \arrow[ul, dotted, dash, "\kappa_7"] & \lambda_7 \arrow[u] & & \\
\aleph_1 \arrow[r, "\kappa_9"'] & \lambda_1 \arrow[r] \arrow[uu] \arrow[ur, dotted, dash, "\kappa_8"] & \square \arrow[r] \arrow[u] & \lambda_5 \arrow[r] \arrow[u] & \lambda_6 \arrow[uu] &
\end{tikzcd}$$

Step 5: Our first step, called "Step 5" for notational reasons, just uses P^5. This is an iteration of length δ_5 with $\mathrm{cf}(\delta_5) = |\delta_5| = \lambda_5$, satisfying:

$$\text{For all } i\text{: } \mathsf{LCU}_i(P^5, \mu) \text{ for all } \mu \in [\lambda_i, \lambda_5] \text{ regular, and } \mathsf{COB}_i(P^5, \lambda_i, \lambda_5). \tag{3.67}$$

As a consequence, the characteristics are forced by P^5 to have the following values[16] (we also mark the position of κ_6, which we are going to use in the following step):

$$\begin{tikzcd}
 & \lambda_3 \arrow[r, "\kappa_6"] & \lambda_4 \arrow[r] & \square \arrow[r] & \lambda_5 \arrow[r] & \lambda_5 \\
 & & \lambda_2 \arrow[r] \arrow[u] & \lambda_5 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[r] \arrow[uu] & \square \arrow[r] \arrow[u] & \lambda_5 \arrow[r] \arrow[u] & \lambda_5 \arrow[uu] &
\end{tikzcd}$$

Step 6: Consider the embedding $j_6 \coloneqq j_{\kappa_6,\lambda_6}$. According to 3.66(b), $P^6 \coloneqq j_6(P^5)$ is a FS ccc iteration of length $\delta_6 \coloneqq j_6(\delta_5)$. As $|\delta_6| = \lambda_6$, the continuum is forced to have size λ_6.

For $i = 1$, we have $\mathsf{LCU}_1(P^5, \mu)$ for all regular $\mu \in [\lambda_1, \lambda_5]$, so using 3.66(c) we get $\mathsf{LCU}_1(P^6, \mu)$ for all regular $\mu \in [\lambda_1, \lambda_5]$ different to κ_6; as well as $\mathsf{LCU}_1(P^6, \lambda_6)$ (as $\mathrm{cf}(j(\kappa_6)) = \lambda_6$). For $\mu = \lambda_1$ the former implies $P^6 \Vdash \mathrm{add}(\mathcal{N}) \leq \lambda_1$, and the latter $P^6 \Vdash \mathrm{cof}(\mathcal{N}) \geq \lambda_6 = 2^{\aleph_0}$.

More generally, we get from (3.67) and 3.66(c)

$$\begin{gathered}\text{For all } i\text{: } \mathsf{LCU}_i(P^6, \mu) \text{ for all regular } \mu \in [\lambda_i, \lambda_5] \setminus \{\kappa_6\}. \\ \text{For } i < 4\text{: } \mathsf{LCU}_i(P^6, \lambda_6).\end{gathered} \tag{3.68}$$

[15] This is identical to the argument in [GKS17], with the roles of $\mathfrak{b}$ and $\mathrm{cov}(\mathcal{N})$, as well as their duals, switched.

[16] These values, and the ones forced by the "intermediate forcings" P^6 to P^8, are not required for the argument; they should just illustrate what is going on.

So in particular for $\mu = \lambda_i$, we see that the characteristics on the left do not increase; for $\mu = \lambda_5$ that the ones on the right are still at least λ_5; and for $i < 4$ an $\mu = \lambda_6$ that the according characteristics on the right will have size continuum. (But not for $i = 4$, as $\kappa_4 < \lambda_4$. And we will see that $\mathrm{cov}(\mathcal{M})$ is at most λ_5.)

Dually, because $\lambda_3 < \kappa_6 < \lambda_4$, we get from (3.67) and 3.66(d)

$$\text{For } i < 4\text{: } \mathsf{COB}_i(P^6, \lambda_i, \lambda_6). \text{ For } i = 4\text{: } \mathsf{COB}_4(P^6, \lambda_4, \lambda_5). \tag{3.69}$$

(The former because $|j_6(\lambda_5)| = \max(\lambda_6, \lambda_5) = \lambda_6$.) So the characteristics on the left do not decrease, and $P^6 \Vdash \mathrm{cov}(\mathcal{M}) \leq \lambda_5$.

Accordingly, P^6 forces the following values:

$$\begin{tikzcd}
 & \lambda_3 \arrow[r] \arrow[dr, dash, dotted, "\kappa_7"] & \lambda_4 \arrow[r] & \boxed{} \arrow[r] & \lambda_6 \arrow[r] & \lambda_6 \\
 & & \lambda_2 \arrow[r] \arrow[u] & \lambda_6 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[r] \arrow[uu] & \boxed{} \arrow[r] \arrow[u] & \lambda_5 \arrow[r] \arrow[u] & \lambda_6 \arrow[uu] &
\end{tikzcd}$$

Step 7: We now apply a new embedding, $j_7 := j_{\kappa_7,\lambda_7}$, to the forcing P^6 that we just constructed. (We always work in V, not in any inner model M or any forcing extention.) As before, set $P^7 := j_7(P^6)$, a FS ccc iteration of length $\delta_7 = j_7(\delta_6)$, forcing the continuum to have size λ_7.

Now $\kappa_7 \in (\lambda_2, \lambda_3)$, so arguing as before, we get from (3.68)

$$\begin{aligned}
&\text{For all } i\text{: } \mathsf{LCU}_i(P^7, \mu) \text{ for all regular } \mu \in [\lambda_i, \lambda_5] \setminus \{\kappa_6, \kappa_7\}.\\
&\text{For } i < 4\text{: } \mathsf{LCU}_i(P^7, \lambda_6). \text{ For } i < 3\text{: } \mathsf{LCU}_i(P^7, \lambda_7).
\end{aligned} \tag{3.70}$$

and from (3.69)

$$\begin{aligned}
&\text{For } i < 3\text{: } \mathsf{COB}_i(P^7, \lambda_i, \lambda_7).\\
&\text{For } i = 3\text{: } \mathsf{COB}_3(P^7, \lambda_3, \lambda_6). \text{ For } i = 4\text{: } \mathsf{COB}_4(P^7, \lambda_4, \lambda_5).
\end{aligned} \tag{3.71}$$

Accordingly, P^7 forces the following values:

$$\begin{tikzcd}
 & \lambda_3 \arrow[r] & \lambda_4 \arrow[r] & \boxed{} \arrow[r] & \lambda_7 \arrow[r] & \lambda_7 \\
 & & \lambda_2 \arrow[r] \arrow[u] & \lambda_7 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[r] \arrow[uu] \arrow[ur, dash, dotted, "\kappa_8"] & \boxed{} \arrow[r] \arrow[u] & \lambda_5 \arrow[r] \arrow[u] & \lambda_6 \arrow[uu] &
\end{tikzcd}$$

Step 8: Now we set $P^8 := j_{\kappa_8,\lambda_8}(P^7)$, a FS ccc iteration of length δ_8. Now $\kappa_8 \in (\lambda_1, \lambda_2)$, and as before, we get from (3.70)

$$\begin{aligned}
&\text{For all } i\text{: } \mathsf{LCU}_i(P^8, \mu) \text{ for all regular } \mu \in [\lambda_i, \lambda_5] \setminus \{\kappa_6, \kappa_7, \kappa_8\}.\\
&\text{For } i < 4\text{: } \mathsf{LCU}_i(P^8, \lambda_6). \text{ For } i < 3\text{: } \mathsf{LCU}_i(P^8, \lambda_7).\\
&\text{For } i < 2 \text{ (i.e., } i = 1\text{): } \mathsf{LCU}_1(P^8, \lambda_8).
\end{aligned} \tag{3.72}$$

and from (3.71)

$$\begin{aligned}&\text{For } i = 1\text{: } \mathsf{COB}_1(P^8, \lambda_1, \lambda_8). \text{ For } i = 2\text{: } \mathsf{COB}_2(P^8, \lambda_2, \lambda_7).\\ &\text{For } i = 3\text{: } \mathsf{COB}_3(P^8, \lambda_3, \lambda_6). \text{ For } i = 4\text{: } \mathsf{COB}_4(P^8, \lambda_4, \lambda_5).\end{aligned} \tag{3.73}$$

Accordingly, P^8 forces the following values:

$$\begin{tikzcd}
 & \lambda_3 \arrow[r] & \lambda_4 \arrow[r] & \boxed{} \arrow[r] & \lambda_8 \arrow[r] & \lambda_8 \\
 & & \lambda_2 \arrow[u] \arrow[r] & \lambda_7 \arrow[u] & & \\
\aleph_1 \arrow[r, "\kappa_9"'] & \lambda_1 \arrow[uu] \arrow[r] & \boxed{} \arrow[u] \arrow[r] & \lambda_5 \arrow[u] \arrow[r] & \lambda_6 \arrow[uu] &
\end{tikzcd}$$

Step 9: Finally we set $P^9 \coloneqq j_{\kappa_9,\lambda_9}(P^8)$, a FS ccc iteration of length δ_9 with $|\delta_9| = \lambda_9$, i.e., the continuum will have size λ_9. As $\kappa_9 < \lambda_1$, (3.72) and (3.73) also hold for P^9 instead of P^8. Accordingly, we get the same values for the diagram as for P^8, apart from the value for the continuum, λ_9. $\square$

Part II

On Liftings for $\mathrm{Bor}(\lambda)/\mathcal{M}(\lambda)$ and Automorphisms of $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$

Chapter 4

Introduction

In this part of the thesis we consider the generalized Cantor space 2^λ and study the lifting problem for $\mathrm{Bor}(\lambda)/\mathcal{M}(\lambda)$ and the existence of trivial automorphisms of $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$. All the results presented are joint work with J. Kellner and S. Shelah, appart from Section 6.1, which was done under the supervision of S. Friedman.

The study of the generalized Cantor space was started by Sikorski in [Sik50], where he studied compactness properties of the space 2^{ω_1}. Recently, this area of research has been progressing quickly, the generalized spaces have been studied from the topological, model theoretical and combinatorial perspective (in [KLLS16], Komskii, Laguzzi, Löwe and Sharakou collected many questions inspired by the series of workshops on this topic).

Our motivation for studying the generalized Cantor space was to find a generalization of Shelah's oracle-cc forcing method ([She83] and [She82, Ch. IV], [She98, Ch. IV]). In the "classical" case ($\lambda = \omega$), the consistency of "no lifting for $\mathrm{Bor}/\mathcal{M}$ plus $2^{\aleph_0} = \aleph_2$" and the consistency of "all automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\omega}$ are trivial" were shown using this method.

In the rest of this chapter, we will introduce well knonw basic notions and standard facts concerning the generalized space (none of which are due to the author).

4.1 Basic Notions and Notation

We always assume λ is an uncountable cardinal , with $\lambda^{<\lambda} = \lambda$ (which implies that λ is regular). Whenever we use the terms real, we mean λ-real (i.e., an element of 2^λ); and with Borel, meager or Cohen we mean λ-Borel, λ-meager and λ-Cohen (as defined in the following).

The *bounded topology*, or λ-box topology, on 2^λ is generated by the *cones*

$$[s] = \{\nu \in 2^\lambda : \nu > s\}$$

for $s \in 2^{<\lambda}$ (which are in fact clopen).

The family of *λ-Borel sets* (or Borel sets for short, denoted by Bor) is the smallest family containing the cones and which is closed under complements and $\leq\lambda$ unions. Without the assumption $2^{<\lambda} = \lambda$, controlling the Borel sets is difficult, in particular, even some open sets may not be Borel (since they might not be a λ-union of cones).

A subset of 2^λ is *λ-meager* (or just meager), if it is contained in the union of λ many closed nowhere dense sets. Let $\mathcal{M}$ be the family of meager Borel sets.

Remark 4.1. As in the classcical case,

- Borel sets satisfy the *Baire property* (i.e., for every Borel B there is an open set O with $B\Delta O$ meager).

  (Proof: The collection of sets with Baire property contains the cones and is closed under complements and λ-unions.)

- The *Baire Category theorem* holds (i.e., a nonempty open set is not meager).

  (Proof: Let $\{D_i : i < \lambda\}$ be a family of dense open sets and let $D := \bigcap_{i\in\lambda} D_i$. We argue that D is dense: Given $s_0 \in 2^{<\lambda}$, build $\langle s_i : i < \lambda\rangle$ increasing such that $[s_{i+1}] \subseteq D_i$ (possible since D_i is dense) and for limit α, define $s_\alpha := \bigcup_{i<\alpha} s_i$. Then $s_\lambda := \bigcup_{i<\lambda} s_i$ is such that $[s_\lambda] \subseteq D$, witnessing that D is dense.)

We fix an (injective) enumeration $(s_\alpha)_{\alpha<\lambda}$ of $2^{<\lambda}$. Let T be a wellfounded subtree of $\lambda^{<\omega}$. We can interpret it as *Borel code* in the following way: We calculate the Borel set $B(T,t)$ for all $t \in T$ by induction: If $t = s^\frown\alpha$ is a terminal node, then $B(T,t)$ is the cone $[s_\alpha]$. Otherwise, $B(T,t)$ is $2^\lambda \setminus \bigcup_{s\rhd t} B(T,s)$ (where $\rhd$ denotes "immediate successor"). We set $B(T) := B(T,\langle\rangle)$.

Obviously each Borel set has a code (which is not unique). Abusing notation, we will often identify codes with their resulting Borel sets.

λ-Cohen forcing (or just Cohen, for short, written as $\mathbb{C}$) is $2^{<\lambda}$ ordered by extension. $\mathbb{C}$ is λ-closed, and satisfies the λ^+-cc (cf. [Jec03, Lem. 15.4])

With $\mathbb{C}_I$ we denote the $<\lambda$-support product (with index set I) of copies of $\mathbb{C}$. This is a $<\lambda$-closed, λ^+-cc forcing for any I (cf., e.g., [Jec03, Lem. 15.17]). Note that $\mathbb{C}_\alpha$ is isomorphic to $\mathbb{C}_{|\alpha|}$ and dense in the the $<\lambda$-support iteration of length α of copies of $\mathbb{C}$. By "α many Cohens" we mean either of these two forcing notions.

In forcing notions, we write $q \leq p$ for "q is stronger than p", $p \parallel q$ for "p and q are compatible" and $p \perp q$ for "p and q are incompatible".

A *dense embedding* is a function between two forcings that preserves $\leq$ and $\perp$ which has a dense image. More generally, a *complete embedding* $F : P \to Q$ preserves $\leq$ and $\perp$ and satisfies: For every $q \in Q$ there is a $p \in P$ such that $F(p') \parallel q$ for all $p' \leq_P p$ (or equivalently: Q forces that $F^{-1}(G_Q)$ is P-generic).

Define $q \leq^* p$ by $q \Vdash p \in G$. Let $q \equiv^* p$ mean $q \leq^* p$ and $p \leq^* q$. We call P *separative*, if $q \leq^* p$ implies $q \leq p$. Note that $\mathbb{C}_\kappa$ is separative. If F is complete, then $F(q) \leq F(p)$ implies $q \leq^* p$.

Fact 4.2. Any λ-complete, atomless forcing Q of size λ is equivalent to $\mathbb{C}$ (as there is a dense embedding $f : \mathbb{C} \to \mathrm{ro}(Q)$).

Proof. Construct $f(s)$ by induction on length(s), such that $\{f(s^\frown 0), f(s^\frown 1)\}$ is maximal antichain under $f(s)$, each value either being below $a_{\mathrm{length}(s)}$ or $\neg a_{\mathrm{length}(s)}$, where a_α enumerates Q; and extend f continuously to limits. Then $\{f(s) : \mathrm{length}(s) = \alpha\}$ is a maximal antichain for each $\alpha < \lambda$. □

Absoluteness in the generalized context is a big issue, even Σ^1_1 absoluteness generally fails:

Example 1. Let $S \subseteq S^{\omega_1}_\omega$ be such that both S and $S^{\omega_1}_\omega \setminus S$ are stationary. Shoot a club through $S \cup S^{\omega_1}_\omega$, by forcing with the poset P_S, consisting of all bounded closed sets of ordinals $\subseteq S$, ordered by end-extension. This forcing even preserved $\aleph_1$, since it is ω-distributive. Looking at the Σ^1_1-formula defining the club filter, we have an example for the failure of Σ^1_1-absoluteness.

We will always work with forcing notions that are λ-complete. Note that Σ^1_1-formulas are absolute between the ground model and the extension via a λ-complete forcing notion:

Lemma 4.3. *Given P a λ-complete forcing and G a P-generic filter over V, Σ^1_1-formulas are absolute between V and $V[G]$.*

Proof. Consider Φ a Σ^1_1 formula with parameters in V, and assume $V^P \models \Phi(x)$ for some $x \in \lambda^\lambda$. In V, let T be a tree such that its projection to the first coordinate $p[T]$ is $\{x \in \lambda^\lambda : \Phi(x)\}$.

Consider $\dot{h}$ a P-name for an $h \in \lambda^\lambda$ such that $V^P \models (x, h) \in [T]$.

We can now define, by induction, an increasing sequence of condition $\{p_i : i < \lambda\} \subseteq P$ and an increasing sequence $\{t_i \in \lambda^{<\lambda}, i < \lambda\}$ such that $p_i \Vdash t_i \subseteq \dot{h}$. Successor stages are no problem, and in limit stages the $< \lambda$ closure of P comes to use: We can define, for δ limit ordinal, $t_\delta = \bigcup_{i<\delta} t_i$ and pick p_α to be a lower bound for $\{p_i : i < \delta\}$. Since for every $i < \delta$, $p_i \Vdash (\check{x}, \dot{h}) \in [T]$, we have $(x \restriction_{|t_i|}, t_i) \in T)$. Letting $g := \bigcup_{i<\lambda} t_i(\text{ in } V)$, we get $(x, g) \in [T]$, yielding that $\Phi(x)$ holds in V. □

The Cohen-generic filter is determined by the generic Cohen real $c \in 2^\lambda$ in the obvious way; and a real c is Cohen over V iff c avoids all meager sets of V (by which of course we mean that there is in V a code T for a meager set such that $c \notin B(T)$ in $V[c]$).

Here it does not matter whether "c is meager" is evaluated in V or in $V[c]$, as it turns out that many properties of Borel codes are *absolute* under forcing with λ-complete forcings,[1] in particular:

1. T is a Borel code,

2. $B(T_1) = B(T_2)$, and analogously for $\subseteq$,

[1] Note that such forcings also preserve $\lambda^{<\lambda} = \lambda$.

3. $B(T_1)\Delta B(T_2) = B(T_3)$,
4. $B(T)$ is meager.

4.2 Liftings for $\mathbf{Bor}/\mathcal{M}$

Whenever P and Q are mathematical structures and $\pi : P \to Q$ is a surjective homomorphism, we can ask if a right inverse exists, *i.e.* a homomorphism $f : Q \to P$ such that $\pi(f(x)) = x$ for all $x \in Q$. These right inverse homomorphisms are called liftings or splitting homomorphisms.

For example, any surjection between sets has a lifting, the projection $\pi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ to the second component from the Klein-4-group to the cyclic group of order 2 has a lifting, namely $(a \mapsto (0, a))$, while there is no lifting between $\mathbb{Z}_4$ and $\mathbb{Z}_2$.

In this thesis, we will use the term "lifting" for the following special case:

Definition 4.4. A *lifting* is a Borel algebra homomorphism $H : \mathrm{Bor}/\mathcal{M} \to \mathrm{Bor}$ such that $H([A])\Delta A$ is λ-meager for all λ-Borel sets A (where $[A]$ denotes the equivalence class of A modulo λ-meager).

Equivalently, we can search for the homomorphism $H' : \mathrm{Bor} \to \mathrm{Bor}$, $H' = \pi \circ H$ (where π is the canonical projection, mapping each Borel sets to its equivalence class modulo meager), or for a subalgebra C of Bor such that $\pi \restriction C$ is bijective.

Chapter 5

Liftings under GCH and in the Cohen Model

In the case $\lambda = \omega$, CH implies that there is a lifting of Borel modulo Meager (cf. [NS35]), and Carlson proved that there is still a lifting after adding $\aleph_2$ many Cohen reals[1]. Both facts are proved in [CFZ94], and the proofs there work in the case of general λ as well; we just have to replace every instance of "ω", "$\aleph_0$" and "countable" with "λ". So we get the following:

Theorem 5.1. *(Neumann, Stone) $2^\lambda = \lambda^+$ implies that there is a lifting of $Bor/\mathcal{M}$.*

Actually, we have a lifting under $2^\lambda = \lambda^+$ for any $\leq \lambda$-complete Bolean algebra $\mathcal{B}$ and any proper $\leq \lambda$-closed ideal I, with $|\mathcal{B}/\mathcal{I}| \leq 2^\lambda$.

Proof. We use Sikorski's extension lemma:

Lemma 5.2. *Let $f : \mathcal{A} \to \mathcal{B}$ be a Boolean algebra homomorphism, $x \notin \mathcal{A}$ and $y \in \mathcal{B}$. Then there is a homomorphism $g : \langle \mathcal{A} \cup \{x\} \rangle \to \mathcal{B}$ extending f and mapping x to y iff for all $a, a' \in \mathcal{A}$, if $a \leq x \leq a'$ in $\mathcal{A}$, then $f(a) \leq y \leq f(a')$ in $\mathcal{B}$.*

Enumerate $\mathcal{A} := \mathcal{B}/\mathcal{I}$ as $\{A_\alpha : \alpha < \lambda^+\}$ and denote by $\mathcal{A}_\alpha$ the Boolean algebra generated by $\{A_\beta : \beta < \alpha\}$. Note that this Boolean algebra will have size $\leq \lambda$ for $|\beta| \leq \lambda$. We will inductively construct homomorphisms $f_\alpha : \mathcal{A}_\alpha \to \mathcal{B}$ such that

1. f_α extends f_β whenever $\beta \leq \alpha \leq \lambda^+$.
2. $\pi \circ f_\alpha = id_{\mathcal{A}_\alpha}$,

At **successor steps** $\alpha+1$, define $\mathcal{A}_{\alpha+1} := \langle \mathcal{A}_\alpha \cup \{A_\alpha\} \rangle$, that is, the subalgebra of $\mathcal{A}$ generated by A_α and elements of $\mathcal{A}_\alpha$. Since $\mathcal{A}_\alpha$ has size $\leq \lambda$, $\{A \in \mathcal{A}_\alpha : A \leq A_\alpha\}$ obviously has size $\leq \lambda$. Define $D := \sup\{f_\alpha(A) : A \in \mathcal{A}_\alpha, A \leq A_\alpha\}$ and

[1] Actually the results are formulated for Lebesgue measurable modulo null, but it is obvious that they apply to meager (and similar ideals) as well.

$D' := \inf\{f_\alpha(A) : A \in \mathcal{A}_\alpha, A \leq A_\alpha\}$ (they exist in $\mathcal{B}$, and clearly $D \leq D'$: we know $\inf\{A \in \mathcal{A}_\alpha, A \leq A_\alpha\}$ and $\sup\{A \in \mathcal{A}_\alpha, A \leq A_\alpha\}$ exist in $\mathcal{A}$ and A_α is in between). Moreover, π is a $\leq \lambda$-complete Boolean algebra homomorphism, and $D' > D$ would contradict the previous statement.

By $\leq \lambda$-completeness we can find a representative in this interval: Whenever $B \in \mathcal{B}$ is such that $\pi(B) = A_\alpha$, then $B' := (B \vee D') \wedge D$ is in the interval and clearly $\pi(B') = A_\alpha$. Moreover, whenever $A, A' \in \mathcal{A}_\alpha$, if $A \leq A_\alpha \leq A'$, $f_\alpha(A) \leq D \leq B' \leq D' \leq f_\alpha(A')$ in $\mathcal{B}$. Therefore, by the above extension lemma, we know there is a unique extension $f_{\alpha+1}$ of f_α mapping A_α to B' and $\{A \in \mathcal{A}_{\alpha+1} : \pi(f_\alpha(A)) = A\}$ is a subalgebra of $\mathcal{A}_{\alpha+1}$ containing $\mathcal{A}_\alpha \cup \{A_\alpha\}$, hence equals $\mathcal{A}_{\alpha+1}$.

Obviously $\mathcal{A}_{\alpha+1}$ will also have size at most λ, so we can iterate the above.

At **limit steps** δ, $\mathcal{A}_\delta := \bigcup_{\alpha<\delta} \mathcal{A}_\alpha$ and $f_\delta := \bigcup_{\alpha<\delta} f_\alpha$'s. □

Theorem 5.3. *(Carlson) It is consistent with $2^\lambda = \lambda^{++}$ that there is a lifting. (More precisely, adding λ^{++} many λ-Cohens to a model of $2^\lambda = \lambda^+$ preserves the lifting.)*

Proof. If $\mathcal{B}$ is a subalgebra of $\mathrm{Bor}/\mathcal{M}$ and $b \in \mathrm{Bor}/\mathcal{M}$ then **the gap determined by** b **over** $\mathcal{B}$ is a pair $\langle C, D\rangle$, where $C = \{c \in \mathcal{B} : c \leq b\}$ and where $D = \{d \in \mathcal{B} : d \geq b\}$. We call a gap **$\lambda$-generated** if C is λ-generated as an ideal and D is λ-generated as a filter.

We also call a pair $\langle M, N\rangle$ of models of ZFC with $M \subseteq N$ **good** if the gap determined by any element of $(\mathrm{Bor}/\mathcal{M})^N$ over $(\mathrm{Bor}/\mathcal{M})^M$ is λ-generated.

We first show that she pair $\langle V, V[G]\rangle$ is good, where G is a V-generic filter for λ-Cohen forcing: Let $b \in (\mathrm{Bor}/\mathcal{M})^{V[G]}$ and fix a name $\dot{b}$ for it. For each $p \in G$, let $b_p := \sup\{c \in (\mathrm{Bor}/\mathcal{M})^V : p \Vdash c \leq \dot{b}\}$. Then $\{b_p : p \in G\}$ is obviously of size λ and generates the lower part of the gap. The upper part is generated by $\{b^p : p \in G\}$, where $b^p := \inf\{c \in (\mathrm{Bor}/\mathcal{M})^V : p \Vdash c \geq \dot{b}\}$.

Since every $b \in (\mathrm{Bor}/\mathcal{M})^{V[G]}$ appears in some λ-Cohen extension, we get that the pair $\langle V, V[G]\rangle$ is good, where G is a V-generic filter for $Add(\kappa, \lambda)$, the forcing adding κ many λ-Cohen reals.

Assume $\mathcal{B}$ is a subalgebra of $\mathcal{B}'$ such that the gap determined by b over $\mathcal{B}$ is λ-generated for any $b \in \mathcal{B}'$. If $\mathcal{D} := \langle \mathcal{B} \cup C\rangle$ for some set $C \subseteq \mathcal{B}'$ of size λ, then the gap determined by b over $\mathcal{D}$ is λ-generated for any $b \in \mathcal{B}'$: The lower part of this gap determined by b over $\mathcal{D}$ is generated by $\{b' \cap c : c \in \langle C\rangle, b'$ is in the lower part of the λ-generated gap determined by $b \cup (\neg c)$ over $\mathcal{B}\}$, hence λ-generated. Analogously for the upper part.

Start with $V \vDash 2^\lambda = \lambda^+$ and G an $Add(\lambda^{++}, \lambda)$-generic filter over V. We show that in $V^{Add(\lambda^{++},\lambda)}$ there is an enumeration of $\mathrm{Bor}/\mathcal{M}$ of length λ^{++} such that the gap determined by each element over the algebra generated by the previous ones is λ-generated. Whenever we have such small gaps (we are λ-complete), we can use the extension Lemma (since there is a candidate).

We will denote the $Fn(\lambda^{++} \times \lambda, 2)$-generic filter by G and for each $\alpha < \lambda^{++}$ define $G_\alpha := G \cap Fn(\alpha \times \lambda, 2)$. Note that G_α will be generic and consider the

λ-complete algebra of λ-Borel sets in $V[G_\alpha]$ (not in $V[G]$), which will be further denoted by $\mathcal{B}_\alpha$. The following hold:

1. $\alpha < \beta \leq \lambda^{++}$ implies $\mathcal{B}_\alpha \subseteq \mathcal{B}_\beta$,
2. $|\mathcal{B}_\alpha| = \lambda^+$ for each $\alpha < \lambda^{++}$.

List Bor$/\mathcal{M}$ in a sequence $\langle b_\beta : \beta < \lambda^{++}\rangle$ such that $\mathcal{B}_\alpha$ is enumerated in the α'th interval of length λ^+(maybe with repetitions).

Suppose $\beta < \lambda^{++}$ and let C be the algebra generated by $\{b_\gamma : \gamma < \beta\}$. We need to show that the gap generated by b_β over C is λ-generated.

We know there is $\alpha < \lambda^{++}$ and $\xi < \lambda^+$ such that $\beta = \lambda \times \alpha + \xi$. Let $\mathcal{B}$ be the algebra generated by $\{b_\gamma : \gamma < \lambda^+\alpha\}$. Then by the previous lemma it suffices to show that the gap determined by b over $\mathcal{B}$ is λ-generated for all $b \in$ Bor$/\mathcal{M}$ (there is some C of size λ between them).

The case $\alpha = 0$ is clear.

If α is a limit ordinal with $\mathrm{cof}(\alpha) = \lambda^+$ then $\mathcal{B} = \mathcal{B}_\alpha$ and again we are done (no new reals).

If α is a limit ordinal with $\mathrm{cof}(\alpha) < \lambda^+$, then $\mathcal{B} = \bigcup\{\mathcal{B}_\delta : \delta < \alpha\}$ and the gap determined by b over $\mathcal{B}$ is just the union of the gaps determined by b over $\mathcal{B}_\delta$ for $\delta < \alpha$, hence λ-generated.

If α_1 is a successor ordinal and $\mathcal{B} = \mathcal{B}_{\alpha+1}$ and we have some countable C below, the gap determined by b_β over everyting prevoius is λ-generated since it is just some some λ-Cohen extensions and everyting is λ-generated Cohen extensions. In the middle we won't know if it is a Cohen extension, but we know it is small generated, more precisely, there is some C of size λ. $\square$

Chapter 6

Trying to obtain a model with no liftings

In the "classical" case ($\lambda = \omega$) the consistency of no lifting plus $2^{\aleph_0} = \aleph_2$ was shown in [She83], using the oracle-cc forcing method introduced in [She82, Ch. IV] (where it is shown that there may be only trivial automorphisms of $\mathcal{P}(\omega)/\mathrm{fin}$. Both results are summarized in [She98, Ch. IV].)

We wanted to find a λ-variant of this oracle construction. Unfortunately, up to this point we did not succeed, some of the ideas looked promising for a long time. This chapter will be an exposition of these tries.

6.1 First try: A direct generalization of the oracle-c.c.

To obtain a model with no such lifting, Shelah defined "oracles", what it means for a poset to be "$\bar{M}$-cc" for an oracle $\bar{M}$ and described how these posets should be iterated. With Sy Friedman as adviser, the approach was to find appropriate generalizations (to $\lambda = \aleph_2$) of these notions and the corresponding iteration results with the scope of generalizing the involved construction of a model with no lift. While some of the notions and results generalize in an obvious way, the preservation of the oracle chain condition in limit steps of small cofinality is far from clear. We tried thinning out the inverse limit, but we did not manage to obtain the desired result.

6.1.1 ω_2-oracles

We denote by $S^{\omega_2}_{\omega_1}$ the set of ordinals less than ω_2 of cofinality ω_1.

Definition 6.1. An ω_2-oracle is a sequence $\bar{M} = \langle M_\delta : \delta \in S^{\omega_2}_{\omega_1} \rangle$, such that

- for each such δ, M_δ is a countably closed transitive model of ZFC^- containing δ and $M_\delta \models \delta < \omega_2, \mathrm{cof}(\delta) = \omega_1$

- $\forall A \subseteq \omega_2 : I_{\bar{M}}(A) := \{\alpha \in S^{\omega_2}_{\omega_1} : A \cap \alpha \in M_\alpha\}$ contains a trace of a club($C \cap S^{\omega_2}_{\omega_1}$)

The existence of an oracle is equivalent with the existence of a $\diamondsuit^*(S^{\omega_2}_{\omega_1})$-sequence, that is, a sequence $\langle S_\delta : \delta \in S^{\omega_2}_{\omega_1}\rangle$ such that $|S_\delta| \leq \omega_1$ for each δ and for every set $A \subseteq \omega_2$, $\{\delta \in S^{\omega_2}_{\omega_1} : A \cap \delta \in S_\delta\}$ contains a club.

To each ω_2-oracle $\bar{M}$ we associate a trapping filter $D_{\bar{M}}$, generated by $\{I_{\bar{M}}(A) : A \subseteq \omega_2\}$. Every restriction to $S^{\omega_2}_{\omega_1}$ of a club $C \subseteq \omega_2$ is in this filter $D_{\bar{M}}$ and hence, $D_{\bar{M}}$ is the restriction of the club filter on ω_2 to $S^{\omega_2}_{\omega_1}$ (recall that, since ω_2 regular, the club filter on ω_2 is proper and normal).

For $\bar{M}$ an oracle and P a poset with universe ω_2, we introduce the following notation:

- $P \cap \delta <_{M_\delta} P$ iff predense sets in $P \cap \delta$ which are in M_δ are predense in P
- $P \cap \delta \ll_{M_\delta} P$ iff $P \cap \delta <_{M_\delta} P$ and incompatibility is preserved (m.a.c of $P \cap \delta$ which are in M_δ remain maximal antichains in P).
- $P \cap \delta \ll_V P$ just means completely embedded.

For a poset with universe ω_2, i.e. $P = (\omega_2, <)$, we say that P is $\bar{M}$-cc for an ω_2-oracle $\bar{M}$ iff $\{\delta \in S^{\omega_2}_{\omega_1} : P \cap \delta <_{M_\delta} P\} \in D_{\bar{M}}$. If the universe is not ω_2 but the poset has size $\aleph_2$ then we just go to the isomorphic partial order with universe ω_2. Note that $\bar{M}$-cc posets of size $\aleph_2$ have the $\aleph_2$-cc.

Lemma 6.2. *Given ω_2-many oracles $\{\bar{M}^\alpha : \alpha < \omega_2\}$, we can find an oracle $\bar{N}$ encompassing all the given ones, i.e. for any partial order P if P is $\bar{N}$-cc then P is $\bar{M}^\alpha$-cc for all $\alpha < \omega_2$.*

The central idea of the oracle chain condition is the following Omitting Type-type Theorem, again a straightforward generalization of the classical case:

Theorem 6.3. *(OTT)*

Assume $\diamondsuit^*(S^{\omega_2}_{\omega_1})$. Suppose $\langle \psi_i(x) : i < \omega_2\rangle$ is given, each $\psi_i(x)$ is Π^1_1 with free variable x and possibly a generalized-real parameter. Suppose there is no solution x, neither in V nor in any ω_1-Cohen extension to

$$\bigwedge_{i<\omega_2} \psi_i(x) \tag{6.4}$$

Then there is an oracle $\bar{M}$ such that, for every $\bar{M}$-cc countaly closed poset P, there is still no solution in V^P.

Proof. Let κ be large enough so that $H(\kappa)$ reflects V. Given a countably closed forcing notion P of size $\aleph_1$ and a nice P-name for a real τ, let

- $M(P, \tau)$ be a countably closed elementary submodel of $H(\kappa)$ containing P, τ and the sequence of formulas $\langle \psi_i(x) : i < \omega_2\rangle$.

- $\mathcal{I}(P,\tau)$ the collection of predense subsets of P which are in $M(P,\tau)$

The required oracle $\bar{M}$ will be build using $\diamondsuit^*(S^{\omega_2}_{\omega_1})$ and a closing process (since we need to deal with all possible names τ) in such a way that whenever P is a countably closed forcing notion of size $\aleph_1$ and τ is a P-name, both in M_δ, we have $M_\delta \supseteq I(P,\tau)$.

We now need to argue that this ω_2-oracle satisfies the requirements, therefore, we assume towards a contradiction that for an $\bar{M}$-cc, countably closed poset Q of size $\aleph_2$ (w.l.o.g. with universe ω_2), a Q-name τ for a solution exists. But then we can find $\delta \in S^{\omega_2}_{\omega_1}$ such that

- τ is a $Q \restriction \delta$-name
- $Q \restriction \delta, \tau \in M_\delta$
- $Q \restriction \delta \subseteq_{ic} Q$
- $Q \restriction \delta <_{M_\delta} Q$

Letting $P := Q \restriction \delta$ we obtain a contradiction using the following claim. □

Claim 6.5. *If P, Q are posets, G is a Q-generic filter over V and τ is a P-name for a real with P countably closed of size $\aleph_1$, $P \subseteq_{ic} Q$ and every predense subset of P lying in $M(P,\tau)$ is predense in Q, then, for some i*

$$V^Q \models \neg \bigwedge_{i<\omega_1} \psi_i(\tau[G])$$

Proof. Every predense subset of P lying in $M(P,\tau)$ is predense in Q, therefore we get that $G \cap P$ is P-generic over $M(P,\tau)$ and that $\tau[G] = \tau[G \cap P]$.

Since $M(P,\tau)$ be a countably closed elementary submodel of $H(\kappa)$, $H(\kappa)$ reflects V, and $M(P,\tau) \models$"P is countably closed of size $\aleph_1$", we know that $M(P,\tau)$ satisfies the assumptions of the Theorem, hence

$$M(P,\tau)[G \cap P] \models \}\}\neg \bigwedge_{i<\omega_2} \psi_i(\tau[G]) \text{ for some } i < \aleph_2\text{''}$$

So, for some $i < \aleph_2, i \in M(P,\tau)$, $M(P,\tau)[G \cap P] \models \neg \bigwedge_{i<\omega_2} \psi_i(\tau[G])$.

The same holds for the transitive collapse as well, since the collapse is an isomorphisms and first order properties are preserved under isomorphisms.

Since $\psi_i(x)$ is Π^1_1, we have $\neg\psi_i(x)$ is Σ^1_2, and we know Σ^1_1 statements are upwards absolute, hence $V[G] \models \neg \bigwedge_{i<\omega_1} \psi_i(\tau[G])$. □

We would like to have a way of iterating $\bar{M}$ -cc forcing notions:

Lemma 6.6. *(The two step iteration) Assuming CH and $\diamondsuit^*(S^{\omega_2}_{\omega_1})$, let $\bar{M}$ be an oracle. If P is $\bar{M}$-cc countably closed forcing notion with $|P| = \aleph_2$, then there is a P-name for an oracle $\dot{\bar{N}}$ st for each P-name $\dot{Q}$ for a partial order, if $\Vdash_P$ }}$\dot{Q}$ is $\dot{\bar{N}}$-cc" then $P * \dot{Q}$ is $\bar{M}$-cc.*

The proof is a straightforward generalization of the one in the ω-case: Without loss of generality, we restrict ourselves to forcing notions with universe ω_2. In V^P, we construct an oracle $\bar{N} = \langle N_\delta : \delta \in S^{\omega_2}_{\omega_1}\rangle$. Since P is $\bar{M}$-cc, on a filter set of δ's, predense subsets of $P \cap \delta$ which are in M_δ are predense in P. Also, $P \cap \delta$ is in M_δ. Denote the P-generic filter by G. Then for this filter set of δ's, $G \cap \delta$ is $P \cap \delta$ generic over M_δ. But, for such δ's, $P \cap \delta$ is equivalent with ω_1-Cohen forcing, since it is countably closed, of size $\aleph_1$. Thus we can take N_δ to be a countably closed transitive model, $M_\delta[G \cap \delta] \subseteq N_\delta$.

We now show that this N works. More precisely, we have to check the guessing property: working in V^P, given $A \subseteq \omega_2$ we want to argue that $\{\delta \in S^{\omega_2}_{\omega_1} : A \cap \delta \in N_\delta\}$ contains the trace of a club.

The countable support iteration does not work: Assume $\diamondsuit^*(S^{\omega_2}_{\omega_1})$, and let $\bar{M}$ be an ω_2-oracle $\alpha \leq \omega_3$ a limit ordinal, $\langle P_\beta : \beta \leq \alpha\rangle$ a countable support iteration of countably closed forcings of size $\aleph_2$ based on $\langle Q_\beta : \beta < \alpha\rangle\rangle$ such that for each $\beta < \alpha$, P_β is $\bar{M}$-cc. Then P_α is NOT $\bar{M}$-cc.

The limit steps of uncountable cofinality are no problem, everything actually appeared earlier. If we try to just take inverse limits in limit steps of countable cofinality, we do not get that $\bar{M}$-cc is preserved: Assume we look at P_ω. Then, for all n, we get that $proj_{P_n} p$ is compatible the projection to P_n of some condition in the dense set $D \subseteq P_\omega \cap \alpha$ (downwards closure of the predense set). The problem is that, as n varies, we project different elements of D and we do not have any closure for this dense set.

To solve this problem we tried to define a new support, more precisely to thin out the inverse limit, to not allow full support at stage ω (or any stage of cofinality ω).

We will need P_ω countably closed and whenever $D \subseteq P_\omega \restriction \alpha$ predense in M_α then every $p \in P_\omega$ is compatible with a condition in D. If we consider countably closed $\aleph_2$-Suslin trees which are oracle-cc for our given oracle, it seems that we can iterate those as follows:

Since P_n (with universe ω_2) is $\bar{M}$-cc for every $n \in \omega$, we have, for each n, a club C_n witnessing the $\bar{M} - cc$ of P_n. Let $C := \bigcap C_n \cap \diamondsuit^*(S^{\omega_2}_{\omega_1})$. Enumerate C in increasing order as $\alpha_0 < \alpha_1 < \alpha_2 < ..$ and define $P_\omega \restriction \alpha_i$ by induction on i.

Since M_α is countably closed for any $\alpha \in C$, it can see all ω-sequences of conditions in P_n indexed $< \alpha$. So $P_\omega \restriction \alpha \in M_\alpha$.

A condition in P_ω will be an ω sequence of nodes in $\dot{P}_n$(trees) and has to hit(since we are in the tree case) these dense sets ($\subseteq P_\omega \restriction \alpha$ which are in M_α. This condition should fall between α_i and α_{i+1}. Actually we will build a dense subset of P_ω. We will also need to close under weakening (we need to show that we still are $\bar{M}$-cc after this closure).

We define $P_\omega \restriction \alpha_0$ as the inverse limit of the $P_n \restriction \alpha_0$. Let p a condition (p is an ω-sequence of names) which falls between α_0 and α_1 which hits all the dense

$D \subseteq P_\omega \restriction \alpha_0$ which are in M_{α_0} (these are $\aleph_1$ many, so what we defined is in the ω-closed model M_{α_1}). This describes the successor steps .

When we get to α_ω, we take the conditions in this layer which respect all M_{α_n}. Our construction seems ω-closed (the only sequences that jump levels are at the ω th level). Also note that the P_ω defined this way embeds the direct limit.

We will need the following:

Lemma 6.7. *(CH and $\diamond^*(S^{\omega_2}_{\omega_1})$) Given $\bar{M} = \langle M_\delta : \delta \in S^{\omega_2}_{\omega_1}\rangle$ an ω_2-oracle and h a lifting homomorphism, there is a forcing poset P satisfying $\bar{M}$-cc and a P-name $\dot{X}$ for a Borel set such that for any generic filter $G \subseteq P * \mathbb{C}$ over V, there is no Borel set A in $V[G]$ satisfying*

- *$A = \dot{X}_G$ mod meager*
- *for each ground model Borel set B, if $B^{V[G]} \subseteq_{\mathcal{M}(\omega_1)} \dot{X}_G$ then $(h(B))^{V[G]} \subseteq A$*
- *for each ground model Borel set B, if $B^{V[G]} \cap \dot{X}_G =_{\mathcal{M}(\omega_1)} \emptyset$ then $(h(B))^{V[G]} \cap A = \emptyset$*

The poset P for destroying a lifting homomorphism is unfortunately not a tree forcing and we have not figured out a way of iterating non-arboreal oracle cc posets.

6.2 Second try: Essentially Cohen

Instead of constructing the oracle sequence with diamonds, we tried to generically generate the oracle forcing. For the $\lambda = \omega$ case, something similiar has been done in [GKSW14]; also related is [Jus92]. This is one of several approaches undertaken jointly with J. Kellner and S. Shelah.

6.2.1 Essentially Cohen and the Preparatory Forcing

Using some natural isomorphism between 2^λ and 6^λ (which is actually a homeomorphism, where in both spaces we use the bounded topology), we will often interpret a real $c \in 2^\lambda$ to code a pair (ρ, η), where $\rho \in 2^\lambda$ and $\eta \in \{-1, 0, +1\}^\lambda$.

In particular a set $\{(\rho_i, \eta_i) : i \in I\}$ is dense if for all $\xi < \lambda$, $x \in 2^\xi$ and $y \in \{-1, 0, 1\}^\xi$ there is some $i \in I$ such that $\rho_i > x$ and $\eta_i > y$. From now on we will assume $2^\lambda = \lambda^+$ and $2^{\lambda^+} = \lambda^{++}$ in the ground model V.

We intend to call a forcing iteration $\bar{P}$ "essentially Cohen", if its is, well, essentially equivalent to $\mathbb{C}_\kappa$ for some κ. We will actually use a slightly cumbersome instance of this concept, that fits the proof in this chapter.

Definition 6.8.

- P is κ-essentially Cohen (κ-e.C.), if there is a dense set $D \subseteq P$ and a dense embedding F from P to $\mathbb{C}_\kappa$ such that the image of P is closed under $<\lambda$-limits. (I.e. the union of a decreasing $<\lambda$-sequence in $F''P$ is again in $F''P$.)
- P is e.C., if it is κ-e.C. for some $\kappa \leq \lambda^+$.

Recall that $F(q) \leq F(p)$ implies $q \leq^* p$. We can select one $\equiv^*$-representative for each $p \in D$, resulting in the dense set $D' \subseteq D \subseteq P$ and an isomorphism of F from $(D', \leq^*)$ to its image. (So in the case of a separative P, we get an isomorphism from D' to the image.) The isomorphism is continuous. So in particular we get: Comparability is equivalent to compatibility in $(D', \leq^*)$; every short descending sequence p_α in $(D', \leq^*)$ has in $(D', \leq^*)$ a unique limit p (and $F(p) = \bigcup F(p_\alpha)$); and if $F(p_\alpha)$ is a descending sequence then so is p_α (with respect to $\leq^*$), etc.

The e.C. notion is not "robust" at all: For example, the dense subset of $\mathbb{C}$ consisting of sequences of successor length is not e.C., as it has no dense subset with unique limits. Let us formalize this notion:

Definition 6.9. A forcing Q "has limits", if for all decreasing $(p_i)_{i<\alpha}$ of length $\alpha < \lambda$ there is an infimum p^*, i.e.: $p^* \leq p_i$ for all α, and if $q \leq p_i$ for all α then $q \leq p^*$.

Note that such a p^* is not necessarily unique: p' is a limit as well iff $p' \leq p^* \wedge p^* \leq p'$. Nevertheless we call p^* *the* limit. If q^* is the limit of $(q_i)_{i<\alpha}$ and p^* of $(p_i)_{i<\alpha}$ and $q_i \leq p_i$ for all i, then clearly $q^* \leq p^*$.

If P is separative and e.C., then there is a dense $D \subseteq P$ which has limits. So in particular the dense subset of $\mathbb{C}$ consisting of sequences of successor length is not e.C. However, we get:

Lemma 6.10. *Assume that Q is λ-closed, atomless, has limits and has size λ.*

1. *There is a dense embedding F from $\mathbb{C}^* := (\lambda^{<\lambda}, \subseteq)$ to Q. Furthermore F is "continuous", i.e., if δ is a limit and $\eta \in \lambda^\delta$, then $F(\eta)$ is the limit of $(F(\eta \restriction \alpha))_{\alpha<\delta}$.*
2. *Q is* 1*-e.C.*

Remark 6.11. Similar (and equally simple) arguments show the following: Assume Q is λ-closed, atomless, and of size λ. (Having limits is not assumed.) Then the subset of $\mathbb{C}^*$ consisting of nodes of successor length can be densely embedded into Q; and $\mathbb{C}$ can be densely embedded into $\mathrm{ro}(Q)$.

Proof. First note that given $p, q \in Q$ there is a maximal antichain $A(q, p)$ below q of size λ such that each element of A is either incompatible with or below p.

Now enumerate Q as $(p_\alpha)_{\alpha\in\lambda}$, and set $f(\langle\rangle) = \mathbb{1}_Q$. We construct a dense embedding $F : \mathbb{C}^* \to Q$ by induction.

Assume $F(s)$ is already defined for $s \in \lambda^\alpha$. Then we define F on the successors of s such that $F(s^\frown\beta)$ enumerates $A(q, p_\alpha)$ for each $\beta \in \lambda$.

If t has limit length δ, we set $F(t)$ to be the limit of (the decreasing sequence) $(F(t \restriction \alpha))_{\alpha<\delta}$. So the constructed F will be continuous.

F clearly preserves $\leq$ and also $\perp$: Assume $s \perp t$ split at some height $\beta < \alpha$, and set $p^s = F(s \restriction (\beta + 1))$ and $p^t = F(t \restriction (\beta + 1))$. By the construction, $p^s \perp p^t$, and thus $F(s) \perp F(t)$.

For every height α, the antichain $\{f(s) : s \in \lambda^\alpha\}$ is maximal: Given $r \in Q$, set $r_0 = r$ and $\eta_0 = \langle\rangle$. and construct for $\beta < \alpha$ decreasing sequences $\eta_\beta \in \lambda^\beta$ and $r_\beta \in Q$ such that $r_\beta \leq F(\eta_\beta)$. (For successors, set $\eta_{\beta+1} = \eta_\beta{}^\frown\alpha$ for some α such that $F(\eta_{\beta+1})$ is compatible with r_β, and let $r_{\beta+1}$ be some common lower bound. At limits δ, set $\eta_\delta = \bigcup_{\beta<\delta} \eta_\beta$ and set r_δ to be the limit of $(r_\beta)_{\beta<\delta}$, which is below the limit of $(F(\eta_\beta))_{\beta<\delta}$, i.e., below $F(\eta_\delta)$.)

We now show (2). First note that, as $\mathbb{C}^*$ is seperative and $x \leq y \wedge y \leq x$ implies $x = y$, the embedding F we just constructed is actually an isomorphism onto the (dense) image, so we get an inverse, a surjective isomorphism $G : D \to \mathbb{C}^*$.

Also, we can apply (1) to $\mathbb{C}$: There is a some $F_0 : \mathbb{C}^* \to \mathbb{C}$ Then $F_0 \circ G$ gives the desired witness of e.C. □

We will use the following basic properties of e.C. forcings:

Lemma 6.12.

1. *$\mathbb{C}_\kappa$, as well as the $<\lambda$-support iteration of κ many Cohens, is κ-e.C.*
2. *If P is e.C., then it is forcing equivalent to some $\mathbb{C}_\kappa$. (The converse is not true, as already mentioned.)*
3. *In particular: e.C. implies λ^+-cc, and*

4. *if r is a P-name for an element of 2^λ, then there is a P-name c for a (single) λ-Cohen over V such that $r \in V[c]$.*

Proof. (1)–(3) are trivial. For (4): We can assume by (2) that r is a $\mathbb{C}_\kappa$-name. $r(\alpha)$ is decided by a maximal antichain A_α for all $\alpha < \lambda$. Due to λ^+-cc and $<\lambda$-support, $X_\alpha := \bigcup\{\mathrm{dom}(p) : p \in A_\alpha\}$ has size λ. Set $I = \bigcup_{\alpha\in\lambda} X_\alpha$. Then r is actually an $\mathbb{C}_I$-name, and a dense subset of $\mathbb{C}_I$ is isomorphic to $\mathbb{C}$, as $|I| = \lambda$. □

Lemma 6.13.
- *If P is κ_1-e.C. and P forces that Q is κ_2-e.C. (where $\kappa_2 \in V$), then $P * Q$ is $(\kappa_1{+}\kappa_2)$-e.C.*
- *More generally, if $(P_\alpha, Q_\alpha)_{\alpha<\delta}$ is a $<\lambda$-support iteration such that each P_α forces that Q_α is κ_α-e.C., then P_δ is $\sum \kappa_\alpha$-e.C.*

Proof.
- Let (D_1, F_1) witness e.C. for P, and let P force that (D_2, F_2) witnesses e.C. for Q.

  Set $D = \{(p, q) \in D_1 * Q : (\exists\eta(p) \in \mathbb{C}_{\kappa_2})\ p \Vdash (q \in D_2 \wedge F_2(q) = \eta(p))\}$.

  For $(p, q) \in D$, we set $F(p, q) := (F_1(p), \eta(p)) \in \mathbb{C}_{\kappa_1+\kappa_2}$.

  It is clear that F preserves $\leq$.

  The image of F is dense: given any $(x, y) \in \mathbb{C}_{\kappa_1+\kappa_2}$, let q be a name for an element of D_2 such that $F_2(q)$ extends y, then pick $p \in D_1$ deciding $F_2(q)$ such that $F_1(p)$ extends x.

  Assume $F(p, q) \parallel F(p', q')$; let (p'', q'') be such that $F(p'', q'') \leq F(p, q), F(p', q')$. In particular $F_1(p'') \leq F_1(p), F_1(p')$, and therefore $p'' \leq^* p, p'$ (here, $q \leq^* p$ denotes $q \Vdash p \in G$). So p'' decides $F_2(q)$ and $F_2(q')$ and $F_2(q'')$, and $F_2(q'') \leq F_2(q), F_2(q')$. So p'' forces $q'' \leq^* q, q'$; therefore $(p, q) \parallel (p', q')$.

- Let D_α, F_α be the P_α-names witnessing that Q_α is e.C. For each β, we set D^β to be the set of all $p \in P_\beta$ such that there is (in V) a sequence $(x_i)_{i\in\mathrm{dom}(p)}$ such that $p \restriction i$ forces $p(i) \in D_i$ and $F_i(p(i)) = x_i$ (for all $i \in \mathrm{dom}(p)$). This naturally defines $F^\beta : D^\beta \to \sum_{i<\beta} \kappa_i$.

  We show by induction on β that D^β is dense and that F^β is a dense embedding. Note that for $\alpha < \beta$ we trivially get: $p \in D^\beta$ implies $p \restriction \alpha \in D^\alpha$ and $F^\beta(p) \restriction \mathbb{C}_{\sum_{i<\alpha}\kappa_i} = F^\alpha(p \restriction \alpha)$.

  For $\beta = \alpha + 1$ a successor, we can use the previous item, setting $P := P_\alpha$, $Q := Q_\alpha$, $D_1 := D^\beta$ and $F_1 := F^\beta$.

  If the cofinality of β is $\geq \lambda$, then $P^\beta = \bigcup P^\alpha$, $D^\beta = \bigcup D^\alpha$ and $F^\beta = \bigcup F^\alpha$.

  So let let β be a limit with cofinality $\zeta < \lambda$, and pick α_i $(i < \zeta)$ cofinal.

  We start with any $p^0 \in P_\beta$; and we will construct a decreasing sequence p^i (for $i \in \zeta$) such that p^ζ in D^β.

  Given p^i, pick $q \leq p^i \restriction \alpha_i$ in D^{α_i}, and set $p^{i+1} = q \wedge p^i$ (which is the condition identical to q up to α_i and identical (or forcing-equivalent) to p^i beyond α_i).

At a limit stage $j \leq \zeta$ we can define a ("pointwise") limit condition p^j. In more detail: We set $\mathrm{dom}(p^j)$ to be $\bigcup_{i<j} \mathrm{dom}(p^i)$ (a set of size $<\lambda$). By induction on $\alpha \in \mathrm{dom}(p^j)$, we have constructed $p^j \restriction \alpha$ which is stronger than each $p^i \restriction \alpha$ (for $i < j$). So in particular $p^j \restriction \alpha$ forces that $p^i(\alpha)$ is in D_α and $F_\alpha(p^i(\alpha))$ is some x^i_α (where the sequence x^i_α exists in V), moreover this sequence is decreasing. As the image of F_α is closed, there is a $q \in D_\alpha$ mapped to $\bigcup_{i<j} x^i_\alpha$. (Pick the smallest such q in some wellorder if required.) Set $p^j(\alpha) = q$. □

Definition 6.14. $\bar{P} = (P_\alpha, Q_\alpha)_{\alpha<\beta}$ is a nice iteration, if:

- $\bar{P}$ is a $<\lambda$-support iteration.
- P_α forces Q_α to be $<\lambda$-closed.
- Each Q_α has size $\leq \lambda^+$.
- The generic object for Q_α is determined by a real $\eta_\alpha \in 2^\lambda$.
- For all $\alpha < \zeta \leq \beta$ with $\mathrm{cf}(\alpha) \neq \lambda^+$,

$$P_\alpha \Vdash P_\zeta/G_\alpha \text{ is } |\zeta \setminus \alpha|\text{-e.C.}$$

Note that for all α (including $\mathrm{cf}(\alpha) = \lambda^+$), Q_α has to be λ^+-cc (as otherwise the composition $P_\alpha * Q_\alpha$ would not be λ^+-cc and thus not e.C.).

Remark 6.15. This definition contains an essential element of the oracle notion: (The following is formally not quite correct, but morally true.) As preparatory forcing, we force with the family of nice iterations, ordered by extension. This gives us a generic iteration of length λ^{++}. Fix an $\alpha < \lambda^{++}$ and a P_α-generic filter G_α, and work in $V[G_\alpha]$. Then for any β in λ^{++} bigger than α, the forcing $Q_\alpha * (P_\beta/G_\alpha)$ is equivalent to $Q_\alpha * \mathbb{C}_\kappa$ for some κ. So if we manage to let Q_α force

$$(\forall x \in 2^\lambda)\varphi(x)$$

for some sufficiently absolute φ, and moreover not only Q_α but even $Q_\alpha * \mathbb{C}_\kappa$ forces the statement, then $P_{\lambda^{++}}$ will force it as well (as any $x \in 2^\lambda$ will appear in some stage $\beta < \lambda^{++}$). This corresponds to the omitting type property of oracle-cc.

Definition 6.16. The forcing notion AP consists of all nice iterations of length $<\lambda^{++}$, ordered by extension.

Lemma 6.17. AP *is a $<\lambda^{++}$-complete and atomless (more specifically, if $a \in$* AP *then $a^\frown\mathbb{C} \in$* AP*).*

Proof. Let $(P_\alpha, Q_\alpha)_{\alpha<\beta} \in$ AP. Set $Q_\beta = \mathbb{C}$ and $P_{\beta+1} = P_\beta * Q_\beta$. Let $\mathrm{cf}(\alpha) \neq \lambda^+$. Then P_α forces that $P_{\beta+1}/G_\alpha$ is $P_\beta/G_\alpha * Q$, and accordingly $|(\beta \setminus \alpha) + 1|$-e.C.

Let $(a_i)_{i<\delta}$ be a strictly decreasing sequence in AP (i.e., increasing as iterations) with $\delta < \lambda^{++}$. We need a lower bound. By taking a subsequence we can assume that $\delta \leq \lambda^+$.

Define b_{i+1} to be the iteration a_{i+1} restricted to length$(a_i) + 1$, and $b_j = \bigcup_{i<j} b_i$ for $j \leq \delta$ limit. We set $\beta_i :=$ length(b_i), which has cofinality $<\lambda^+$ for $i < \delta$. We have to show that the limit $b_j = (P_\alpha, Q_\alpha)_{\alpha<\beta_\delta}$ satisfies the e.C. property.

We know that $P^*_i := P_{\beta_i}$ forces that $Q^*_i = P_{\beta_j}/G_{\beta_i}$ is e.C. for all $i < j < \delta$. We can interpret $(P^*_i, Q^*_i)_{i<\delta}$ as $<\lambda$-support iteration, so the limit P^*_δ (which is isomorphic to P_{β_δ}) is e.C. according to Lemma 6.13. □

So AP does not add any new λ^+-sequences; in particular it forces $2^{\lambda^+} = \lambda^{++}$, and $2^\lambda = \lambda^+$.

Definition 6.18. For $\alpha < \lambda^{++}$, the AP-generic contains a unique $(P^\alpha_\beta, Q^\alpha_\beta)_{\beta<\alpha}$, which defines for all $\beta < \lambda^{++}$ the unique objects P^*_β and Q^*_β. Let $P^*_{\lambda^{++}}$ be the limit (i.e., the union) of the P^*_β.

Let V^+_β denote the universe we get after forcing with AP $*$ P^*_β. In this universe, we can also define $V^-_\beta := V[G_{P^*_\beta}]$.

The following lemma basically sais that we can "reflect" an AP $*$ $P^*_{\lambda^{++}}$-name for a lifting h as a P_δ name for some δ of cofinality λ^+:

Lemma 6.19. *1.* AP *forces (for $\alpha \leq \beta \leq \lambda^+$) that P^*_α is λ^+-cc, and that P^*_α is a complete subforcing of P^*_β.*

2. *No new reals appear in* AP $*$ P^*_δ *for* $\mathrm{cf}(\delta) \geq \lambda^+$. *I.e.: $2^\lambda \cap V^+_\delta = \bigcup_{\alpha<\delta} V^+_\alpha$.*

   (And of course $2^\lambda \cap V^-_\delta = \bigcup_{\alpha<\delta} V^-_\alpha$.)

3. *Fix some* AP $*$ $P^*_{\lambda^{++}}$ *name h for a function from 2^λ to 2^λ and $a = (P_\beta, Q_\beta)_{\beta<\alpha_0} \in$* AP. *Then there is a $b = (P_\beta, Q_\beta)_{\beta<\delta}$ in* AP *extending a_0 satisfying that $h \restriction V^+_\beta$ is determined by P_β; in more detail:*

   - *δ (the length of b) has cofinality λ^+.*
   - *b forces that $2^\lambda \cap V^+_\delta = 2^\lambda \cap V^-_\delta$ and that $h \restriction V^-_\delta$ is in V^-_δ.*

Proof. 1. Assumme (in $V[G_{\mathrm{AP}}]$) that $A \subset P^*_\alpha$ is an antichain.

Then A has size $\leq \lambda$: if it had size λ^+, then $A \in V$, and A is an antichain of size λ^+ in V as well, a contradiction.

If A is a maximal (in $V[G_{\mathrm{AP}}]$), then it is maximal in V as well, and thus maximal (in V and therefore also in $V[G_{\mathrm{AP}}]$) in P^*_β for any $\alpha < \beta < \lambda^{++}$.

2. This trivially follows from λ^+-cc and the fact that $P^*_\delta = \bigcup_{\alpha<\delta} P^*_\alpha$.

3. - First fix an AP $*$ $P^*_{\lambda^{++}}$-name τ for an element of $\{0, 1\}$ and work in $V[G_{\mathrm{AP}}]$. There is a maximal antichain A deciding τ. Due to λ^+-cc, $A \in P^*_\beta$ for some $\beta < \lambda^{++}$, and moreover $A \in V$.

- Now work in V and start with some $(a_0, p_0) \in AP * \bar{P}^*$. Given any τ as before, we can find $a_1 \leq a_0$ in AP deciding β as some β_1 (we can assume length$(a_1) = \beta_1$) and $A_1 \subseteq P^*_{\beta_1}$ as above. (Note that we do not have to do anything about p_0, and that assuming a_1 we can in V effectively decide τ from A_1.)
- Given a sequence $(\tau_i)_{i\in\lambda^+}$ and a_0, we can iteratively increase a_0 to a_i of length β_i and find $A_i \subseteq P_{\beta_i}$ such that (below a_i) A_i determines τ_i. Let a' be the limit of the a_i, and δ' the limit of β_i.
- Fix a^0 and $\delta^0 < \lambda^{++}$. All of the λ^+ many reals in V^+_δ, as well as their h-images, can thus be decided (below some $a^1 \leq a^0$) by a ground model sequence of antichains in some $P^*_{\delta^1}$.

  Iterating this construction gives increasing sequences a^j and δ^j for $j \in \lambda^+$; and the limit $b = \bigcup_{j\in\lambda^+} a^j$ is as required. □

6.2.2 The single forcing $Q^{\mathcal{X}}$

Definition 6.20. Fix $\rho \in 2^\lambda$ and $\eta \in \{-1, 0, 1\}^\lambda$

- The α-splitoff of ρ is the cone generated by $\rho \restriction \alpha^\frown(1 - \rho(\alpha))$.
- The set $\mathrm{In}(\rho, \eta, \zeta_0)$ is the union of the α-splitoffs of ρ with $\alpha \geq \zeta_0$ and $\eta(\alpha) = +1$.
- $\mathrm{Out}(\rho, \eta, \zeta_0)$ is defined analogously with $\eta(\alpha) = -1$.
- $\mathrm{Undec}(\rho, \eta, \zeta_0)$ is defined analogously with $\eta(\alpha) = 0$.

So In, Out, Undec are disjoint open sets, each a union of $\leq\lambda$ many cones $[s]$ with the height of s a successor; and $\mathrm{In} \cup \mathrm{Out} \cup \mathrm{Undec} = [\rho \restriction \zeta_0] \setminus \{\rho\}$.

Definition 6.21. $\mathcal{X}$ is a "suitable parameter sequence" if $\mathcal{X} = (\rho^*_\beta, \eta^*_\beta)_{\beta\in I}$ with $\lambda \leq |I| \leq \lambda^+$, the ρ^*_β are pairwise different, $\{(\rho^*_i, \eta^*_i) : i \in I\}$ is dense in $(2 \times \{-1, 0, 1\})^\lambda$, and $(\eta^*_\beta)^{-1}(i)$ is unbounded for each $\beta \in I$ and $i \in \{-1, 0, 1\}$.

For such $\mathcal{X}$, $Q^{\mathcal{X}}$ is defined as follows:

- A condition $q \in Q^{\mathcal{X}}$ consists of (A, f) such that
  - $A \subseteq I$ has size $<\lambda$,
  - $f : A \to \lambda$,
  - We set $\mathrm{In}(q) := \bigcup_{\beta\in A} \mathrm{In}(\rho^*_\beta, \eta^*_\beta, f(\beta))$; and $\mathrm{Out}(q)$ analogously;
  - We require $\mathrm{In}(q) \cap \mathrm{Out}(q) = \emptyset$.
- q' is stronger than q, if $\mathrm{In}(q') \supseteq \mathrm{In}(q)$ and $\mathrm{Out}(q') \supseteq \mathrm{Out}(q)$.

So $q' \leq q$ implies $A(q') \supseteq A(q)$, but it does not imply $f(q') \restriction A(q) = f(q)$.

Lemma 6.22.
- *$p \perp q$ iff either* $\mathrm{In}(p) \cap \mathrm{Out}(q) \neq \emptyset$ *or* $\mathrm{In}(q) \cap \mathrm{Out}(p) \neq \emptyset$.

  Two compatible elements have a greatest lower bound.
- *Q is separative, λ-closed (and even has limits), and has has size $\leq \lambda^+$.*
- *The generic is determined by the partition $I = (\mathrm{In}, \mathrm{Out}, \mathrm{Stem})$ of $2^{<\lambda}$ into* $\mathrm{In} = \{s : (\exists q \in G)\, [s] \subseteq \mathrm{In}(q)\}$, Out *(defined analogously) and* $\mathrm{Stem} = 2^{<\lambda} \setminus (\mathrm{In} \cup \mathrm{Out})$.

  $x \in \mathrm{Stem}$ iff there is a $q \in G$ and $\beta \in A(q)$ such that $\rho^*_\beta \in [x]$, and Stem *is a perfect tree without end-nodes closed under limits (i.e. $s \restriction \alpha \in \mathrm{Stem}$ for all $\alpha < \delta$ implies $s \restriction \delta \in \mathrm{Stem}$ for δ limit).*
- *Equivalently, the generic object is determined by the open set* $\mathrm{In}' = \bigcup_{q \in G} \mathrm{In}(q) = \bigcup_{s \in \mathrm{In}} [s]$.

  $[s] \subseteq \mathrm{In}$ *with s minimal implies that s has successor length.*

Lemma 6.23. *Let M be a transitive model of size λ, closed under $<\lambda$ sequences, and $\mathcal{X}^M = (\rho^*_\beta, \eta^*_\beta)_{\beta<\alpha}$ a suitable parameter sequence in M. Let $(\rho^*_\alpha, \eta^*_\alpha)$ be Cohen over M, and set $\mathcal{X} := \mathcal{X}^M {}^\frown (\rho^*_\alpha, \eta^*_\alpha)$.*

Then $Q^M := Q^{\mathcal{X}^M}$ is an M-complete[1] subforcing of $Q := Q^{\mathcal{X}}$.

6.2.3 Cohens $*$ Q

Lemma 6.24. *Let $\mathbb{C}_{\lambda^+}$ add λ^+ many Cohens $(\rho_\alpha, \eta_\alpha)$. We set $\mathcal{X} = (\rho_\alpha, \eta_\alpha)_{\alpha \in \lambda^+}$. Then $\mathbb{C}_{\lambda^+} * Q^{\mathcal{X}}$ is λ^+-e.C.*

Proof. In V, let the forcing Q^* consist of pairs (T, g) such that

- T is a subtree of $2^{<\lambda}$ of size $<\lambda$ and g a function from T to $\{-1, 0, 1\}$.
- If $g(s) \neq 0$, then $g(s)$ is a terminal node in T.
- The nodes $s \in T_\alpha$ with $g(s) = 0$ form a "closed" subtree: $g(\langle,\rangle) = 0$; if $g(s) = 0$ and s is not terminal in T then there is a $t >_T s$ with $g(t) = 0$; and if t has height δ limit and $g(t \restriction \alpha) = 0$ for all $\alpha < \delta$ then $g(t) = 0$.

Q^* is ordered by extension. (So in particular, a the tree of the stronger condition can only extend old nodes s with $g(s) = 0$.)

Given a Q^*-generic, we define Stem as the subtree $g^{-1}(0)$, and we define In as the set of nodes in $2^{<\lambda}$ extending some node in $g^{-1}(1)$, and Out analogously with -1.

Q^* has size λ, and is $<\lambda$-closed and even has limits, and is nonatomic.

In the Q^*-extension $V^* = V[G_{Q^*}]$, we define $\mathbb{C}^*$ to be the union[2] of the following two forcings $\mathbb{C}^*_{\text{ignored}}$ and $\mathbb{C}^*_{\text{chosen}}$: $\mathbb{C}^*_{\text{ignored}}$ is regular Cohen forcing (i.e., adds some

[1]Note that the definition of Q^M is absolute between M and V, so $Q^M \in M$.

[2]Formally, we let $\mathbb{C}^*$ be the disjoint union of $\mathbb{C}^*_{\text{ignored}}$ and $\mathbb{C}^*_{\text{chosen}}$ (each element of the one forcing being incompatible with every element of the other) together with a new weakest element $\mathbb{1}$.

(ρ, η) in the usual way), while $\mathbb{C}^*_{\text{chosen}}$ adds a Cohen branch ρ through the perfect tree Stem and a Cohen η which is compatible with (In, Out) above some ζ_0. In more detail: A condition p of $\mathbb{C}^*_{\text{ignored}}$ has the form (ζ_0, ρ, η), where $\zeta_0 \in \lambda$, $(\rho, \zeta) \in (2 \times \{-1, 0, 1\})^\zeta$ for some $\zeta \in [\zeta_0, \lambda)$, $\rho \in$ Stem, and η violates (In, Out) unboundedly often below ζ_0, and not anymore above ζ_0. [3] The order of $\mathbb{C}^*_{\text{ignored}}$ is defines as follows: (ζ_0', ρ', η') is stronger than (ζ_0, ρ, η) if $\zeta_0' = \zeta_0$ and (ρ', η') extends (ρ, η).

In V^*, let $\mathbb{C}^*_{\lambda^+}$ be the $<\lambda$-support product of λ^+-many copies of $\mathbb{C}^*$. We claim that $Q^* * \mathbb{C}^*_{\lambda^+}$ is equivalent to $\mathbb{C}_{\lambda^+} * Q^{\mathcal{X}}$:

Let D_0 be the dense subset of $P * Q$ consisting of conditions $(p, (A, f))$ satisfying

- (A, f) is in the ground model (not just a name).
- $\alpha \in A$ implies $\alpha \in \text{dom}(p)$ and $p(\alpha)$ has height $> f(\alpha)$.
- f is "minimal": decreasing $f(\alpha)$ for any α would lead to an inconsistency.
- If $\alpha \in \text{dom}(p) \setminus \alpha$, then α is prevented to every get into A' of a stronger $(p', (A', f'))$ (some $\beta \in A$ contradicts it).

Given such a condition, we can naturally calculate first some element of Q^*, and then map each $p(\alpha)$ to the corresponding element in the α-th copy of $\mathbb{C}^*$. □

We will actually need something more general:

Definition 6.25. Consider the following three permutations σ of $\{-1, 0, 1\}$: $\sigma_1 = (-1, 0)$, $\sigma_2 = (1, 0)$ and $\sigma_3 = (-1, 1)$.

For such a σ_i, $\zeta_0 < \lambda$, and $\eta \in \{-1, 0, +1\}^{\zeta_0}$, we call η' the "finite modification of η using σ_i above ζ_0, if $\eta'(\alpha) = \eta(\alpha)$ for $\alpha < \zeta_0$, and $\eta(\alpha) = \sigma_i \circ \eta(\alpha)$ otherwise.

In particular (ρ, η') is Cohen over some M iff (ρ, η) is Cohen over M.

Question 6.26. *After adding λ^+ many Cohens $(\rho^*_\alpha, \eta^*_\alpha)_{\alpha \in \lambda^+}$, we choose (in the extension, not the ground model) for each α a finite modification η_α of η^*_α, and set $\rho_\alpha = \rho^*_\alpha$ and $\mathcal{X} = (\rho_\alpha, \eta_\alpha)_{\alpha \in \lambda^+}$.*

Then $P_\delta * Q^{\mathcal{X}}$ is e.C.

Actually, we do not only need e.C., but nice. We have no idea how to get nice, not even in the "easy" version.

Question 6.27. *Let $\bar{P}$ be a nice iteration of length $\lambda^+ \leq \delta < \lambda^{++}$ with* $\text{cf}(\delta) = \lambda^+$. *Accordingly (as P is "essentially Cohen") we can interpret P_δ as λ^+ many Cohens $(\rho_\alpha, \eta_\alpha)$. We set $\mathcal{X} = (\rho_\alpha, \eta_\alpha)_{\alpha \in \lambda^+}$.*

Then $P_\delta * Q^{\mathcal{X}}$ is nice.

[3] More formally, let us say (η, ρ) violates (In, Out) at α if $\eta(\alpha) \in \{-1, 1\}$ and $\rho \restriction \alpha^\frown(1 - \rho(\alpha))$ has color $1 - \eta(\alpha)$. So we claim that the set of $\alpha \in \zeta$ where there is a violation is an unbounded subset of ζ_0.

The problem is that to be nice, the quotient $P_\delta * Q^{\mathcal{X}}$ by some P_α has to be e.C. But this quotient is generally not the quotient by $(\rho_\beta, \eta_\beta)_{\alpha<\beta}$, as the Cohens lie in some weird skewed way in the iteration.

Actually, we think we even need the more general version:

Question 6.28. *Let $\bar{P}$ be a nice iteration of length $\lambda^+ \leq \delta < \lambda^{++}$ with $\mathrm{cf}(\delta) = \lambda^+$. Accordingly (as P is "essentially Cohen") we can interpret P_δ as λ^+ many Cohens $(\rho_\alpha, \eta_\alpha)$. We chose some cofinal subsequence i_α of δ of order type λ^+. In the extension (not the ground model) we choose for each α a finite modification η_α of η^*_α, and set $\rho_\alpha = \rho^*_\alpha$ and $\mathcal{X} = (\rho_\alpha, \eta_\alpha)_{\alpha\in\lambda^+}$.*

Then $P_\delta * Q^{\mathcal{X}}$ is nice.

6.2.4 The main claim

Assuming that we can answer Question 6.28 positively, we could then show that $\mathrm{AP} * P^*_{\lambda^{++}}$ forces that there is no lifting, in the following way:

1. Assume that (a_0, p_0) forces that h is a lifting; let δ be the length of a_0. Without loss of generality $p_0 \in P^*_\delta$, $\mathrm{cf}(\delta) = \lambda^+$ and $h \restriction V^+_\delta \in V^-_\delta$, cf. Lemma 6.19.

2. We work in $V' = V^-_\delta = V[G_\delta]$, where $p_0 \in G_\delta$.

   We will construct a forcing Q and a Q-name X such that: For all Borelcodes Y in a $Q * \mathbb{C}$-extension there is in V' $\rho \in 2^\lambda$ and A open satisfying $\rho \in h(A)$, such that in the extension either ($A \subseteq X$ and $\rho \notin Y$) or ($A \cap X = \emptyset$ and $\rho \in Y$).

3. In some $\mathrm{AP} * P^*_{\lambda^{++}}$-extension over V (compatible with (a_1, p_0)), set $Y = h(X)$. Y already appears in some V^-_β, and due to niceness, V^-_β is a subuniverse of some $\mathbb{C}_{\lambda^{++}}$-extension of $V'[G_Q]$, and thus Y appears in a $\mathbb{C}$-extension of $V'[G_Q]$. So there is A, ρ as in (2): $\rho \in h(A)$, and in the $Q * \mathbb{C}$-extension of V' either $A \subseteq X$ and $\rho \notin Y$ or (which we will now assume without loss of generality)
$$A \cap X = \emptyset \text{ and } \rho \in Y. \tag{6.29}$$
   By absoluteness, (6.29) holds in the $Q * \mathbb{C}_{\lambda^{++}}$-extension of V', and therefore in V^-_β, and thus also in the final extension, implying that h is not a lifting after all (as $h(X) = Y$, $A \cap X = \emptyset$, $\rho \in h(A)$ and $\rho \in Y$).

So we work in V' and have to construct Q. We will construct for $\alpha < \lambda^+$:

- M_α, a transitive set of size λ (a model of enough of ZFC) closed under $<\lambda$-sequences, containing $\mathcal{X}_\alpha := (\rho^*_\beta, \eta^*_\beta)_{\beta<\alpha}$. Q_α is the forcing $Q^{\mathcal{X}}$ defined inside M_α (or equivalently in V').
- In M_α, a $Q_\alpha * \mathbb{C}$-name Y_α and a $Q_\alpha * \mathbb{C}$-condition (q_α, c_α).

- $(\rho^*_\alpha, \eta^*_\alpha)$, a finite modification of some $(\rho_{i_\alpha}, \eta_{i_\alpha})$ for some $i_\alpha < \delta$. (Where $(\rho_\gamma, \eta_\gamma)_{\gamma<\delta}$ is the generic Cohen sequence for P^*_δ, which we have as we assume that P^*_δ is e.C.)

In the end, we will set $Q = \bigcup Q_\alpha$ (which is $Q^{\mathcal{X}}$ for $\mathcal{X}$ the union of all $\mathcal{X}_\alpha$). We set X to be the Q name for the generic In set.

Assume we have constructed the objects listed above for all $\beta < \alpha$. This gives us $\mathcal{X} = (\rho^*_\beta, \eta^*_\beta)_{\beta<\alpha}$, which defines Q_α, and we choose M_α so that it contains $\mathcal{X}$ and $(M_\beta)_{\beta<\alpha}$ and is closed under $<\lambda$ sequences.

As the final forcing Q will be λ^+-cc, each $Q * \mathbb{C}$-name Y of an element of 2^λ can actually be captured as a Q_α-name for some α, and some bookkeeping guarantees that each such name actually appears as an Y_α, and we put Y_α into M_α as well. Also, we enumerate all $Q * \mathbb{C}$-conditions using suitable bookkeeping and deal with (q_α, c_α) at stage α. (Actually we might fail to do so; but then we will try again with the same Y and (q, c) at some later stage; and will succeed at some stage.)

We now want to find i_α and $(\rho^*_\alpha, \eta^*_\alpha)$.

1. First note that M_α is element of some $V[(\rho_\zeta, \eta_\zeta)_{\zeta<\beta}]$ for some $\beta < \delta$. By λ^+-cc we can find an upper bound i_α. So $(\rho_{i_\alpha}, \eta_{i_\alpha})$ is Cohen over M_α. Set $(\rho', \eta') = (\rho_{i_\alpha}, \eta_{i_\alpha})$.

2. According to Lemma 6.23, whenever we let (η, α) be generic over M_α, then the $Q_{\alpha+1}$ defined by extending $\mathcal{X}_\alpha$ by (η, α) will be an M_α-complete superforcing of Q_α. So we can consider Diagram 6.1. Note that Y appears in M^1, and ρ in $M_\alpha[(\rho, \eta)]$; so both exist in M^2.

3. If ρ' is not compatible with[4] q_α, then we "give up": We just use $(\rho_\alpha, \eta_\alpha) = (\rho', \eta')$. We will come back to the same Y and (q, c) cofinally often, and by density at some such stage we will have compatibility.

   So assume compatibility from now on.

4. In $M_\alpha[(\rho', \eta')]$, pick some $q' \leq_{Q_{\alpha+1}} q_\alpha$ which actually uses (ρ', η'), and find a stronger $Q_{\alpha+1}$-condition q'' and a $\mathbb{C}$-condition c'' deciding (in M^2) whether $\rho' \in Y$ or not.

5. Case 1: In $M_\alpha[(\rho', \eta')]$, (q'', c'') forces that $\rho' \in Y$.

   Then this is forced by some $(s, t) \in \mathbb{C}$ with $\rho' \in [s], \eta' \in [t]$. So whenever we modify η' above the length of (s, t) (ζ_0, say), we will still get $\rho' \in Y$.

   Recall that for each $\zeta < \lambda$,

$$\mathrm{In}(\rho', \eta', \zeta) \cup \mathrm{Out}(\rho', \eta', \zeta) \cup \mathrm{Undec}(\rho', \eta', \zeta) = [\rho' \restriction \zeta] \setminus \{\rho'\}, \text{ so}$$

$$\begin{aligned} h([\rho' \restriction \zeta]) &= h([\rho' \restriction \zeta] \setminus \{\rho'\}) = \\ &= h(\mathrm{In}(\rho', \eta', \zeta)) \cup h(\mathrm{Out}(\rho', \eta', \zeta)) \cup h(\mathrm{Undec}(\rho', \eta', \zeta)). \end{aligned}$$

[4] meaning: $\rho' \in \mathrm{In}(q_\alpha) \cup \mathrm{Out}(q_\alpha)$

$$
\begin{tikzcd}
M_{\alpha+1} \arrow[r, "Q_{\alpha+1}"'] & \tilde{M}^3 \arrow[r, "\mathbb{C}"'] & M^3 \\
M_\alpha[(\rho,\eta)] \arrow[u] \arrow[r, "Q_{\alpha+1}"'] & \tilde{M}^2 \arrow[u] \arrow[r, "\mathbb{C}"'] & M^2 \arrow[u] \\
M_\alpha \arrow[u, "\mathbb{C}"'] \arrow[r, "Q_\alpha"'] & \tilde{M}^1 \arrow[u] \arrow[r, "\mathbb{C}"'] & M^1 \arrow[u]
\end{tikzcd}
$$

Figure 6.1: Generics for the top row induce generics for the rows below (note that Q_α is an M_α-complete subforcing of $Q_{\alpha+1}$. The step from the middle to the top row is not a forcing extension.

Also, $\rho' \in h([\rho' \restriction \zeta])$, as $h([\rho' \restriction \zeta])\Delta[\rho' \restriction \zeta]$ is a meager set in M_α, and thus avoided by the Cohen ρ'.

So ρ' is in the In, Out or undecided set; we need it in the Out set. We now set ζ to be the length of (s,t), and finitely modify η' above ζ using one of the three permutations $\sigma_1, \sigma_2, \sigma_3$, resulting in an η'' so that new $A := \mathrm{Out}(\rho', \eta'', \zeta)$ contains ρ'.

We now set $(\rho^*_\alpha, \eta^*_\alpha)$ to be (ρ', η''). This is Cohen over M_α, and we can extend $(q_0, c_0) \in Q_\alpha * \mathbb{C}$ to some $(q'', c'') \in Q_\alpha + 1 * \mathbb{C}$ forcing that $\rho^*_\alpha \in Y$. Also, $\rho^* \in A$ and q'' forces that $A \cap X = \emptyset$.

6. Case 2: (q'', c'') forces that $\rho' \notin Y$. Then we do the same, but choose the permutation that results in $\mathrm{In}(\rho', \eta'', \zeta)$ containing ρ'.

Chapter 7

$<\lambda$-complete liftings

We looked at one generalization of "(finitely-complete) Boolean algebra liftings of Borel modulo meager"; another obvious generalization to λ would be $<\lambda$-complete liftings. Of these liftings, we only know the following:

Lemma 7.1. *If a $<\lambda$-complete lifting homomorphism exists, then λ has to be a measurable cardinal.*

Proof. Let h be a $<\lambda$-complete lifting homomorphism for $\mathrm{Bor}(\lambda)/\mathcal{M}(\lambda)$.

For $W \subseteq 2^{<\lambda}$ with $|W| < \lambda$, let $B_W := \bigcup\{[\eta] : \eta \in W\}$, where $[\eta] := \{\zeta \in 2^\lambda : \zeta \supseteq \eta\}$ is the basic cone with trunc η. Note that the set $E := \bigcup\{h(B_W)\Delta B_W : W \subseteq 2^{<\lambda}, |W| < \lambda\}$ is meager, as a union of $< \lambda$ many meager sets ($2^{<\lambda} = \lambda$ and h is a lifting homomorphism forBor$(\lambda)/\mathcal{M}(\lambda)$). This set will be our exception set.

Consider now $\eta^* \in 2^{<\lambda}\setminus E$. For $\alpha \in \lambda$, denote by ν_α the element of 2^α that agrees with η^* below α, but $\nu_\alpha(\alpha) \neq \eta^*(\alpha)$ and for $U \subseteq \lambda$, let $A_U := \bigcup\{[\nu_\alpha] : \alpha \in U\}$, the elements of 2^λ that split off of η^* at levels $\alpha \in U$.

Define $D := \{U \subseteq \lambda : \eta^* \in h([A_U]_{\mathcal{M}(\lambda)})\}$. We now claim that D ia a $<\lambda$-complete ultrafilter on λ, hence witnessing that λ is a measurable cardinal (it is nonprincipal because of the exception set). We know $A_U \dot\cup A_{\lambda\setminus U} = 2^\lambda \setminus \eta^*$, so either $\eta^* \in h(A_U)$ or $\eta^* \in h(A_{\lambda\setminus U})$ (its image under h must be everything), hence D is a ultrafilter. If the set U increases, so does A_U, hence D is closed under supersets.

The $<\lambda$-completeness of D follows form the $<\lambda$-completeness of h: if $\eta^* \in h(A_{U_i})$ for all $i \in \delta < \lambda$, then $\eta^* \in \bigcap h(A_{U_i}) = h(\bigcap A_{U_i}) = h(A_{\bigcap U_i})$. $\square$

Chapter 8

Trivial Automorphisms

W. Rudin [Rud56a; Rud56b] was the first to study automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\omega}$. He showed that under CH, there are $2^{\aleph_1}$ nontrivial automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\omega}$. More precisely, he showed that for any two P-points of weight $\aleph_1$, there is an automorphism sending one to the other. Parovičenko [Par16] also managed to construct non-trivial automorphisms using the countable saturation of the Boolean algebra $\mathcal{P}(\omega)/[\omega]^{<\omega}$.

In 1980, Shelah [She82] showed, using oracle-cc, that it is consistent that every automorphism of $\mathcal{P}(\omega)/[\omega]^{<\omega}$ is trivial. Shelah and Steprans [SS88] adapted the oracle-cc proof to get the same conclusion from the Proper Forcing Axiom (PFA). Velickovic showed in [Vel93] that the conjunction of the forcing axioms OCA (Open Coloring Axiom) and $\mathrm{MA}_{\aleph_1}$ implies that every automorphisms of $\mathcal{P}(\omega_1)/[\omega_1]^{<\omega}$ is induced by a function from ω_1 to ω_1 and that PFA impies that every automorphisms of $\mathcal{P}(\lambda)/[\lambda]^{<\omega}$ is induced by a function from λ to λ, for uncountable λ. (For cardinals below the first inaccessible, this follows from OCA and MA alone, see [SS16]).

In this chapter, we study automorphisms of $\mathcal{P}(\lambda)/[\lambda]^{<\mu}$ for $\mu > \aleph_0$.

8.1 Basic notions and facts

An automorphism π of $\mathcal{P}(\lambda)/[\lambda]^{<\mu}$ is called **trivial** if it is induced by an almost permutation of λ, that is, a bijection between sets in $\mathcal{P}(\lambda)/[\lambda]^{<\mu}$.

More formally:

Definition 8.1. A homeomorphisms $\pi : \mathcal{P}(\lambda)/I \to \mathcal{P}(\lambda)/J$ is **trivial** if there is a function $f : \lambda \to \lambda$ such that $\pi([X]_I) = [f^{-1}(X)]_J$ for all $X \subseteq \lambda$, where $[X]$ denotes the equivalence class of X.

We use inverse images since these are guaranteed to preserve Boolean operations, but we can often work with forward images (cf. [LM16, Lemma 2.2]), as in the case of automorphisms of $\mathcal{P}(\lambda)/[\lambda]^{<\mu}$, since f restricts to a bijection between $\lambda \setminus A$ and $\lambda \setminus B$, where $A, B \in [\lambda]^{<\mu}$ and moreover, f^{-1} witnesses π^{-1} is trivial.

8.2 The existence of nontrivial automorphisms

For inaccessible λ, S. Shelah and J. Steprans showed in [SS15] that under $2^\lambda = \lambda^+$, there is a nontrivial automorphism of $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$ (the cardinal characteristic related to the dominating number form [SS15, lem 3.1] takes values $> \lambda$ and $\leq 2^\lambda$, hence the hypotheses of the lemma hold under $2^\lambda = \lambda^+$).

We present here a simpler proof for measurable λ.

Theorem 8.2. *Assume λ is a measurable cardinal and $2^\lambda = \lambda^+$. Then $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$ has a nontrivial automorphism.*

Proof. Let $\mathcal{D}$ be a normal ultrafilter on λ (exists since λ is measurable) and denote by $\mathcal{I} := \mathcal{P}(\lambda) \setminus \mathcal{D}$ its dual (prime) ideal.

Since $2^\lambda = \lambda^+$, we can list all almost permutations of λ as $\{e_\alpha : \alpha < \lambda^+\}$. We will construct a nontrivial automorphism π of $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$ in λ^+ stages, diagonalizing over all e_α's, going along a tower $\{A_\alpha : \alpha < \lambda^+\}$ of length λ^+ that generates the ideal.

By induction on $\alpha < \lambda^+$ we define $A_\alpha \in \mathcal{I}$ and f_α an almost permutation of A_α, such that for $\alpha < \beta$:

- $A_\alpha \subset^* A_\beta$
- for almost all $x \in A_\alpha \cap A_\beta$, we have $f_\alpha(x) = f_\beta(x)$ (we say "f_β almost extends f_α")

At **successor stages** $\alpha + 1$, we will construct $A_{\alpha+1}$ and $f_{\alpha+1}$ in such a way that it is guaranteed to differ from e_α.

Fix any $X \in \mathcal{I}$ such that X is disjoint to A_α. It might happen that $e''_\alpha X \in \mathcal{D}$, but then we can split it into two parts, one of them not in $\mathcal{D}$, and take its preimage instead of X. Hence, w.l.o.g. $e''_\alpha X \in \mathcal{I}$.

First assume that $|e''_\alpha X \cap A_\alpha| = \lambda$. Set $A_{\alpha+1} = A_\alpha \cup X$ and $f_{\alpha+1} \restriction X = id$. Then $f_{\alpha+1}$ differs from e_α as witnessed by X. So we assume $|e''_\alpha X \cap A_\alpha| < \lambda$. Choose $Y \notin \mathcal{D}$, Y disjoint from $e''_\alpha X$, set $A_{\alpha+1} = A_\alpha \cup X \cup Y$ and define $f_{\alpha+1}$ to extend f_α and map X to Y bijectively. Clearly $f_{\alpha+1}$ differs from A_α as witnessed by X.

At **limit stages δ of cofinality less than** λ, let $\xi := \mathrm{cf}(\delta)$ and choose $\langle \alpha_i : i < \xi \rangle$ a cofinal increasing sequence converging to δ. The union $\bigcup_{i<\xi} A_{\alpha_i}$ is, by $< \lambda$ completeness, in $\mathcal{I}$ and f_δ defined as $f_\delta(x) = f_{\alpha_i}(x)$, where α_i is least such that $x \in A_{\alpha_i}$ is an almost permutation of A_δ and almost extends all f_{α_i}.

At **limit stages δ of cofinality** λ we choose an increasing cofinal sequence $\langle \alpha_i : i < \lambda \rangle$ converging to δ and we do some preparation:

By induction on $i \in \lambda$ we construct $A'_i =^* A_{\alpha_i}$, such that

- $A'_i \cap i = \emptyset$,
- f_{α_i}'s fully extend each other on the A'_i's,

- $f_{\alpha_i} : A'_i \to A'_i$ is a real permutation (not just almost).

At each step $i \in \lambda$, we first shrink A_{α_i} to S by taking out i, as well as the points where f_{α_i} disagrees with some f_{α_j}, for $j < i$ (these are $< \lambda$ many points). $f : S \to S$ is still an almost permutation. Then we obtain A'_i by applying the following lemma:

Lemma 8.3. Whenever we have $f : S \to S$ an almost permutation, we can find $S' \subseteq S$, $|S \setminus S'| < \lambda$, such that $f : S' \to S'$ is a permutation.

Proof. Recall that $f : S \to S$ an almost permutation if there are $A, B \subseteq S$, $|A|, |B| < \lambda$ such that $f : S \setminus A \to S \setminus B$ is bijective.

Starting with $S_0 := \mathrm{dom}(f) = S \setminus A$ and letting $S_{i+1} := S_i \cap f''S_i \cap f^{-1}S_i$, $S' := S_\omega = \bigcap_{i\in\omega} S_i$ will be as required.

If $\beta \in S_\omega$ then clearly $\beta \in \mathrm{dom}(f)$.

If $f(\beta) = \gamma \notin S_\omega$, then $\gamma \notin S_i$ for some $i < \omega$ and thus $\beta \notin S_{i+1}$, a contradiction (this works because λ has uncountable cofinality).

The injectivity is trivial since we just have the restriction of a bijective function. □

We define f_δ on $A_\delta := \bigcup_{i\in\lambda} A'_i$ (which, as $A'_i \cap i = \emptyset$, is equal to the diagonal union, thus, by normality, in the ideal) as follows: for all $x \in A_\delta$, let $f_\delta(x) := f_{\alpha_i}(x)$, for any $i < \lambda$ such that $x \in A'_i$ (we made sure that all such $f_{\alpha_i}(x)$ are identical).

Then we prove that

1. f_δ is a permutation of A_δ:

   If $x \in A_\delta$, $f_\delta(x)$ is clearly in A_δ as well. For showing the injectivity, assume $x_1 \neq x_2$ in A_δ, but $f_\delta(x_1) = f_\delta(x_2) = \gamma$. Hence there are $i, j \in \lambda$ such that $x_1 \in A'_i$, $x_2 \in A'_j$, $\gamma = f_{\alpha_i}(x_1) = f_{\alpha_j}(x_2)$. But then $\gamma \in A'_{\alpha_1} \cap A'_{\alpha_2}$ and $f^{-1}_{\alpha_i}(\gamma)$ and $f^{-1}_{\alpha_j}(\gamma)$ can't have different values.

   It remains to argue the surjectivity: take an arbitrary $\gamma \in A_\delta$. By definition, there is $i < \lambda$ such that $\gamma \in A'_i$. Then $\xi := f^{-1}_{\alpha_i}(\gamma) \in A'_i \subseteq A_\delta$ is such that $f_\delta(\xi) = \gamma$.

2. f_δ almost extends f_α, for all $\alpha < \delta$ (on A'_i we have a real extension, $A'_i =^* A_{\alpha_i}$ and for each $\alpha < \delta$ there is some $i \in \lambda$, such that f_{α_i} on A_{α_i} almost extends f_α on A_α).

On $X \subseteq \lambda$, the nontrivial automorphism π is defined as follows:

$$\pi(X) := \begin{cases} [f''_\alpha X] & \text{if } X \in \mathcal{I}, X \subseteq A_\alpha \text{ for some } \alpha < \lambda^+ \\ [\lambda \setminus f''_\alpha(\lambda \setminus X)] & \text{if } X \notin \mathcal{I}, \lambda \setminus X \subseteq A_\alpha \text{ for some } \alpha < \lambda^+ \end{cases}$$

If π were trivial, then there would be $\alpha \in \lambda$ such that e_α induces π, a contradiction, since at stage $\alpha + 1$ we made sure this does not happen. □

8.3 MA prevents nontrivial automorphisms for $\lambda < 2^{\aleph_0}$

Theorem 8.4. *Assume* $\aleph_0 < \mu \leq \lambda < 2^{\aleph_0}$, $\mathrm{cf}(\mu) > \aleph_0$, *and* $MA_{(=\lambda)}$*(σ-centered) holds. Then every automorphism of* $\mathcal{P}(\lambda)/[\lambda]^{<\mu}$ *is trivial.*

The rest of the section will contain the proof of this theorem.

Since $\lambda < 2^{\aleph_0}$, we can fix a function $\eta : \lambda \to 2^\omega$, $\alpha \mapsto \eta_\alpha$. I.e., we can see λ as a subset of 2^ω and consider an $\alpha \in \lambda$ as coded by some branch $\eta_\alpha \in 2^\omega$.

Set $A^o_{2i+j} := \{\alpha < \lambda : \eta_\alpha(i) = j\}$ and note that for all $\alpha \neq \beta$ in λ, there is an $i \in \omega$ such that $\alpha \in A^o_i$, $\beta \notin A^o_i$. Of course, $A^o_{2i} \dot\cup A^o_{2i+1} = \lambda$ for every $i \in \omega$.

Let π be an arbitrary Boolean algebra automorphism of $\mathcal{P}(\lambda)/[\lambda]^{<\mu}$.

Denote by $A^*_i \subseteq \lambda$ a representative for the image of the equivalence class of A^o_i, more precisely, A^*_i is such that $\pi(A^o_i/_{[\lambda]^{<\mu}}) = A^*_i/_{[\lambda]^{<\mu}}$. W.l.o.g. $A^*_{2i} \dot\cup A^*_{2i+1} = \lambda$ for every $i \in \omega$.

We will call Y an "image" under π of X if it is representative for the image of the equivalence class of X, i.e. such that $\pi(X/_{[\lambda]^{<\mu}}) = Y/_{[\lambda]^{<\mu}}$ and, in turn, X will be called "preimage" of Y.

For every $\beta < \lambda$ define $\nu_\beta \in 2^{\aleph_0}$ such that $\beta \in A^*_{2i+j}$ iff $\nu_\beta(i) = j$. I.e. $A^*_{2i+j} = \{\beta < \lambda : \nu_\beta(i) = j\}$.

We will prove that the mapping $\alpha \mapsto \beta$ such that $\eta_\alpha = \nu_\beta$ is an almost permutation of λ which induces the given automorphism π, hence π is trivial.

We will need the following lemma:

Lemma 8.5. *Under the assumptions of Theorem 8.4, given any two disjoint sets* $A, B \subseteq 2^\omega$ *with* $|A| \geq \mu$, $|A|, |B| \leq \lambda$, *we can find a tree* $T \subseteq 2^{<\omega}$ *such that* $|A \cap [T]| \geq \mu$, $[T] \cap B = \emptyset$.

If also $|B| \geq \mu$, *we can find* $T' \subseteq 2^{<\omega}$ *such that* $|B \cap [T']| \geq \mu$, $[T'] \cap A = \emptyset$ *and* $T \cap T' \subseteq 2^n$ *for some n.*

Note that since $[T]$ is uncountable, it has size $2^{\aleph_0} > \lambda$ and hence $[T] \nsubseteq A$.

Proof. We will prove the lemma by finding a coloring $F : 2^{<\omega} \to \{0, 1\}$, such that for $x \in 2^\omega$, if $x \in A$ then $\langle F(x \restriction n) : n \in \omega \rangle$ is eventually 0 and if $x \in B$ then $\langle F(x \restriction n) : n \in \omega \rangle$ is eventually 1.

To apply Matrin's axiom we need to define a σ-centered forcing poset Q, such that the existence of a "generic" filter (i.e. a filter that meets λ many given dense sets) is equivalent to the existence of such a function F.

A condition $q \in Q$ consists of

1. A subset S_q of 2^ω, which in turn consists of

   a. the tree 2^n for some $n \in \omega$

   b. finitely many branches to either A or B

   c. we assume that for each $s \in 2^n$ there is a branch of at most one kind (i.e., to A or to B) extending s.

2. A function $f_q : S_q \to \{0,1\}$, which can have arbitrary values below n, and above n, it is constant 0 if the branch is in A and constant 1 if the branch is in B

Q is naturally ordered by inclusion, i.e. a condition q is stronger than p if the underlying set $S_q \subseteq 2^\omega$ increases and the function f_q extends f_p.

This poset is clearly σ-centered: if p and q have the same n and f_p and f_q are identical below n, then (since (1).c. holds), their union will be a condition, stronger than both p and q.

For $a \in A$, the set D_a of conditions containing a branch to a is dense: starting with an arbitrary q in Q, the branch to a might go for a while along branches that end up in B. In this case, we choose n large enough (above all splitting points of a from these braches) and 2^n extending the finite part of q. Below n we do not care about the color of the new nodes, but above n we color the new branch to a with the color 0.

Therefore, "generically" (using $MA_{(=\lambda)}(\sigma$-centered)), if a branch is in A, then it is eventually 0. The same holds for B with eventually 1.

We know $|A| \geq \mu$, where $\mathrm{cf}(\mu) > \aleph_0$ and "eventually 0" is a countable quantifier, so there is n_0 such that $A_0 = \{x \in A : \forall n \geq n_0, F(x \restriction n) = 0\}$ of size $\geq \mu$. Now let $T_0 := F^{-1}(0) \cup 2^{n_0}$. This tree might have dying branches, so we prune it to get the tree T. We now know $[T] \cap A = [T_0] \cap A = A_0$ has size $\geq \mu$ and that $[T] \cap B = \emptyset$.

If also $|B| \geq \mu$, there is n_0 such that $B_1 = \{x \in B : \forall n \geq n_1, F(x \restriction n) = 1\}$ of size $\geq \mu$. Let $n := \max\{n_0, n_1\}$. Define $T' := F^{-1}\{1\} \cup 2^n$. Then $|B \cap [T']| \geq \mu$, $[T'] \cap A = \emptyset$. □

For a tree $T \subseteq 2^{<\omega}$, define $[T]_\eta = \{\alpha \in \lambda : \eta_\alpha \in [T]\}$ and $[T]_\nu = \{\beta \in \lambda : \nu_\beta \in [T]\}$. The set T_η is approximated by its levels, that is, $[T]_\eta$ is the intersection of the decreasing sets $[T_n]_\eta := \{\alpha \in \lambda : \eta_\alpha{\restriction}n \in T\}$ and we know what the automorphisms π does on each of these levels: $\pi([T_n]_\eta) = [T_n]_\nu$ (since we assumed $A^*_{2i} \dot\cup A^*_{2i+1} = \lambda$ for every $i \in \omega$). Thus, $\pi([T]_\eta) \subseteq \bigcap \pi([T_n]_\eta)$ (since it is contained each of the $\pi([T_n]_\eta)$'s, but $\pi([T_n]_\eta) = [T_n]_\nu$, so $\pi([T]_\eta) \subseteq \bigcap [T_n]_\nu = [T]_\nu$. Analogously we can show $\pi^{-1}([T]_\nu) \subseteq \bigcap [T_n]_\eta = [T]_\eta$

We now show that the function e defined as $e(\alpha) = \beta$ with $\eta_\alpha = \nu_\beta$ is an almost permutation of λ. We know (by definition) that there is a well defined mapping from $\alpha < \lambda$ to η_α and from $\beta < \lambda$ to ν_β. We will show that $\forall^*\alpha\ \exists!\beta\ (\eta_\alpha = \nu_\beta), \forall^*\beta\ \exists!\alpha\ (\eta_\alpha = \nu_\beta)$ (that is, e is almost injective and almost surjective). Then it remains to show that it induces the automorphism π, i.e. $\forall X \subseteq \lambda\ (\pi(X) =^* e''X)$.

Proving the following claim will thus finish the proof of the theorem:

Claim 8.6. *The following sets have cardinality $< \mu$:*

1. $\Lambda_1 := \{\alpha \in \lambda : \exists^{(\geq 2)}\beta \in \lambda\ (\nu_\beta = \eta_\alpha)\}$

2. $\Lambda_2 := \{\beta \in \lambda : \exists^{(\geq 2)}\alpha \in \lambda\ (\eta_\alpha = \nu_\beta)\}$

3. For all fixed $\nu^0 \in 2^\kappa$, $\Lambda_3 := \{\beta \in \lambda : \nu_\beta = \nu^0\}$

4. $\Lambda_4 := \{\beta \in \lambda : \nu_\beta \neq \eta_\alpha$ *for all* $\alpha \in \lambda\}$

5. $\Lambda_5 := \{\alpha \in \lambda : \eta_\alpha \neq \nu_\beta$ *for all* $\beta \in \lambda\}$

6. $\Lambda_6 := e''X \setminus \pi(X)$ *for fixed* $X \subseteq \lambda$

7. $\Lambda_7 := \pi(X) \setminus e''X$ *for fixed* $X \subseteq \lambda$

Proof. The proofs will be indirect, we always assume the set were large and use the lemma to get a contradiction.

(1) Assume that $|\Lambda_1| \geq \mu$. For every $\alpha \in \Lambda_1$, let $\beta^0_\alpha \neq \beta^1_\alpha$ in λ be such that $\eta_\alpha = \nu_{\beta^0_\alpha} = \nu_{\beta^1_\alpha}$,

For $l \in \{0, 1\}$, let $Y_l := \{\beta^l_\alpha : \alpha \in \Lambda_1\} \in [\lambda]^{\geq\mu}$ and denote by X_l its preimage under π, X_l of cardinality $\geq \mu$. As Y_0 and Y_1 are disjoint, we can assume w.l.o.g. that X_0 and X_1 are disjoint as well. For $l \in \{0, 1\}$, let $X^*_l := \{\eta_\alpha : \alpha \in X_l\}$.

We now use the Lemma to find two trees $T^0, T^1 \subseteq 2^\omega$ for X^*_0 and X^*_1. As $|[T^l]_\eta \cap X_l| \geq \mu$, it follows that $\pi([T^l]_\eta \cap X_l) \subseteq [T^l]_\nu$ (and w.l.o.g. $\subseteq Y_l$) also has size $\geq \mu$.

Let $Z_0 := \pi([T^0]_\eta \cap X_0) \subseteq [T^0]_\nu \subseteq Y_0$ and $Z_1 := \{\beta^1_\alpha : \beta^0_\alpha \in Z_0\}$. We have $Z_1 \subseteq [T^0]_\nu$, hence $\pi^{-1}(Z_1) \subseteq X_1 \cap [T^0]_\eta$, a contradiction, since $X_1 \cap [T^0]_\eta = \emptyset$.

(2) Assume towards a contradiction $|\Lambda_2| \geq \mu$. As before, for $l \in \{0, 1\}$ define $X_l := \{\alpha^l_\beta : \beta \in \Lambda_2\}$, denote its image under π by Y_l and let $Y^*_l := \{\nu_\beta : \beta \in Y_l\}$. These sets will be of cardinality $\geq \mu$ and disjoint. By the lemma we can find two disjoint trees T^l for $l \in \{0, 1\}$ such that $|[T^l] \cap Y^*_l| \geq \mu$ and $[T^l] \cap Y^*_{1-l} = \emptyset$. Letting $[T^l]_\nu = \{\beta \in \lambda : \nu_\beta \in [T^l]\}$, we know $|[T^l]_\nu \cap Y_l| \geq \mu$ and thus $|\pi^{-1}([T^l]_\nu \cap Y_l)| \geq \mu$. We also know $\pi^{-1}([T^l]_\nu \cap Y_l) \subseteq [T^l]_\eta$ (and w.l.o.g. $\subseteq X_0$).

Let $Z_0 := \pi^{-1}([T^0]_\nu \cap Y_0) \subseteq [T^0]_\eta \subseteq X_0$ and $Z_1 := \{\alpha^1_\beta : \alpha^0_\beta \in Z_0\}$. We have $Z_1 \subseteq [T^0]_\eta$, hence $\pi(Z_1) \subseteq Y_1 \cap [T^0]_\nu$, a contradiction, since $Y_1 \cap [T^0]_\nu = \emptyset$.

(3) Assume towards a contradiction $|\Lambda_3| \geq \mu$. Let Y be its preimage under the automorphism π. Hence, $|Y| \geq \mu$ as well. Let $\Lambda^*_3 := \{\nu_\beta : \beta \in \Lambda_3\}$ and $Y^* := \{\eta_\alpha : \alpha \in Y\}$, both have to be of size at least μ.

Again, using the lemma, we find a tree $T \subseteq 2^{<\omega}$, such that $|[T] \cap Y^*| \geq \mu$, $\nu^0 \notin [T]$ (we can ensure this by applying the lemma for $Y \setminus \nu^0$ and $\{\nu^0\}$. Obviously $|[T]_\eta \cap Y| \geq \mu$, and as before, $\pi([T]_\eta \cap Y) \subseteq [T]_\nu$. Therefore, if $\beta \in \pi([T]_\eta \cap Y)$, then $\nu_\beta \in [T]$. But $\beta \in \pi([T]_\eta \cap Y)$ implies $\beta \in \Lambda_3$, hence $\nu_\beta = \nu^0$, which is not in $[T]$, a contradiction.

(4) Assume that the set Λ_4 has at least μ elements. Let $Y \subseteq \lambda$ be a preimage of Λ_4. Obviously $|Y| \geq \mu$. Let $\Lambda^*_4 := \{\nu_\beta : \beta \in \Lambda_4\}$ and $Y^* := \{\eta_\alpha : \alpha \in Y\}$. By the definition of Λ_4, $\Lambda^*_4 \cap Y^* = \emptyset$.

We will use the lemma to find a tree $T \subseteq 2^{<\omega}$, such that $[T] \cap \Lambda^*_4 = \emptyset$, $|[T] \cap Y^*| \geq \mu$. Hence $|[T]_\eta \cap Y| \geq \mu$.

As before $\pi([T]_\eta \cap Y) \subseteq [T]_\nu$, and hence $\beta \in \pi([T]_\eta \cap Y)$ implies $\nu_\beta \in [T]$.

But $\beta \in \pi([T]_\eta \cap Y)$ also implies $\beta \in \Lambda_4$, thus $\nu_\beta \in \Lambda^*_4$ (by definition), a contradiction to $[T] \cap \Lambda^*_4 = \emptyset$.

(5) Assume that the set Λ_5 has at least μ elements. Let $Y \subseteq \lambda$ be an image of Λ_5. Obviously $|Y| \geq \mu$. Let $\Lambda_5^* := \{\eta_\alpha : \alpha \in \Lambda_5\}$ and $Y^* := \{\nu_\beta : \beta \in T\}$. Obviously $\Lambda_5^* \cap Y^* = \emptyset$ and Y^* also has to have size $\geq \mu$ (because of (2))

We will use the lemma to find a tree $T \subseteq 2^{<\omega}$, such that $[T] \cap {\Lambda_5}^* = \emptyset$, $|[T] \cap Y^*| \geq \mu$. Hence $|[T]_\nu \cap Y| \geq \mu$.

As before $\pi^{-1}([T]_\nu \cap Y) \subseteq [T]_\eta$, and hence $\alpha \in \pi^{-1}([T]_\nu \cap Y)$ implies $\eta_\alpha \in [T]$ and since $[T] \cap \Lambda_5^* = \emptyset$, $\alpha \notin \Lambda_5$.

But $\alpha \in \pi^{-1}(Y)$ means $\alpha \in \Lambda_5$, a contradiction.

(6) Assume $\Lambda_6 \subseteq e''X, \Lambda_6 \cap \pi(X) = \emptyset, |\Lambda_6| \geq \mu$. We know $\pi^{-1}(\Lambda_6)$ is disjoint from X and as before, define $X^* := \{\eta_\alpha : \alpha \in X\}$ and $(\pi^{-1}(\Lambda_6))^* := \{\eta_\alpha : \alpha \in \pi^{-1}(\Lambda_6)\}$

We can use the lemma to X^* and $(\pi^{-1}(\Lambda_6))^*$ to get a tree T such that $[T] \cap X^* = \emptyset$, $|[T] \cap (\pi^{-1}(\Lambda_6))^*| \geq \mu$.

Assuming $\beta \in \pi([T]_\eta \cap \pi^{-1}(\Lambda_6))$, since $\pi([T]_\eta \cap \pi^{-1}(\Lambda_6)) \subseteq [T]_\nu$, we get $\nu_\beta \in [T]$. On the other hand, it means $\beta \in \Lambda_6$ and we know $\Lambda_6 \subseteq e''X$. Since e is an almost permutation, for almost all $\beta \in \Lambda_6$, $\exists \alpha \in X$ such that $\eta_\alpha = \nu_\beta$. But since $\eta_\alpha \in X^*$, it follows $\nu_\beta \in X^*$, hence $\nu_\beta \notin [T]$, a contradiction.

(7) W.l.o.g we can assume $\pi(X) \cap e''X = \emptyset$ (otherwise just replace X with $\pi^{-1}(\pi(X) \setminus e''X)$), both of size $\geq \mu$. Hence $A := \pi^{-1}(e''X)$ is also disjoint from $\Lambda_7 = X$. Defining $A^* := \{\eta_\alpha : \alpha \in A\}$ and $X^* := \{\eta_\alpha : \alpha \in X\}$, we can apply the lemma to obtain a tree $T \subseteq 2^{\aleph_0}$ such that $|[T] \cap A^*| \geq \mu$, $[T] \cap X^* = \emptyset$. Then $\pi([T]_\eta \cap A) \subseteq [T]_\nu$, $|\pi([T]_\eta \cap A)| \geq \mu$.

If $\beta \in \pi([T]_\eta \cap A)$, then on one hand, $\nu_\beta \in [T]$. On the other hand, $\beta \in \pi(A) = e''X$, and since e is an almost permutation of λ, for most such β's there must be $\alpha \in X$ with $\nu_\beta = \eta_\alpha$. Because $\eta_\alpha \in X^*$, and $X^* \cap [T] = \emptyset$, we can conclude that ν_β can't be in $[T]$, hence the contradiction. $\square$

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Anda-Ramona Tănasie — Curriculum Vitae

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Publications

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