TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

Master thesis

Meucci Black-Litterman Portfolio:

An Analysis concerning Performance and Stability

carried out for the purpose of obtaining the degree of Master of Science (MSc or Dipl.-Ing. or DI), submitted at TU Wien, Faculty of Mechanical and Industrial Engineering, by

Peter KNOBL

under the supervision of

Univ.-Prof. Dr. Walter S.A. Schwaiger, MBA

Institute of Management Science, Financial Enterprise Management

Vienna, June 2018

Abstract

In an asset allocation task an investor seeks the optimal combination of assets that best suits his needs in an uncertain environment. The most popular approach to asset allocation is the mean variance model by Markowitz. However, using the Markowitz optimization will lead to portfolio weights that tend to be extreme, instable and poorly diversified. This is because the traditional Markowitz approach treats the inputs as if they were known with 100% certainty.

An alternative approach is the Black-Litterman model. The Black-Litterman model uses the market portfolio as a neutral starting point that initially requires zero certainty about the inputs. The market returns implied in the market portfolio can then be tilted in accordance with the investor's views by using a Bayesian approach.

The Black-Litterman model extended by Meucci's weighting approach gives the opportunity to valuate this certainty/uncertainty by using confidence in subjective views. In this context, the following questions are particularly of interest:

a. In how far affect different levels of confidence in subjective views the asset allocation?
b. How do different levels of confidence in subjective views concern the portfolio's performance?
c. And how do different levels of confidence in subjective views affect the portfolio's stability and risk?

A detailed analysis of these questions unveils interesting asymmetries in the risk-return profile that an investor may take advantage of.

In addition, the finalizing outlook suggests new fields of application for the Meucci Black-Litterman model and Bayesian approaches in general, such as a decision making/investment tool in corporate finance as well as a new method to bring in subjective views on the second moment of the return distribution with regard to stress events.

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1. Introduction

“Prediction,” goes an old Danish proverb, “is hazardous, especially about the future.”[1] As far as finance and asset allocation is concerned, we know that there is no model that works well all the time. However, most investors seek a portfolio that performs well (at least) *most* of the time. The keywords here are risk and stability. One model that provides some stability is the Black-Litterman model. It bases upon two established tenets of modern portfolio theory - the mean-variance optimization framework of Markowitz and the Capital Asset Pricing Model (CAPM) of Sharpe et al. The market portfolio as defined by the CAPM serves as a stable foundation that can be updated by subjective views.

The Black-Litterman model is a useful framework that gives reasonable flexibility to adjust the returns and weights given by different views with the ebbs and flows of (new) information. This is the basic idea of this approach: the opportunity to update and adjust one's beliefs in the light of new information.

The thesis is structured as follows:

The ensuing chapter demonstrates the main motivation of this treatise. It treats practical issues with the standard model of modern portfolio theory, Markowitz's mean variance approach, and how the (Meucci) Black-Litterman model tries to solve it.

The "problem statement" explains the intention of this thesis and why the confidence in subjective views is the critical element of the Meucci Black-Litterman model.

The chapter "Methodology" gives an overview of the methods that are used throughout this work.

"The mean variance approach of Markowitz" deals with assumptions and insights of Markowitz's modern portfolio theory, which is in the broader sense the foundation of the Black-Litterman model and of portfolio theory in general.

The next chapter explains the Capital Asset Pricing Model (CAPM) along general lines. The Black-Litterman model assumes that the market is in equilibrium on average and uses the expected equilibrium returns given by the equilibrium asset pricing model CAPM as a neutral reference point for expected returns.

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[1] https://quoteinvestigator.com/2013/10/20/no-predict/, 4.3.2018

Chapter 6 presents the Black-Litterman model from a Bayesian point of view. It shows how the two sources of information - the implied market and the subjective information - are combined in a Bayesian manner.

The ensuing chapter 7 treats questions that concern the practical use of the Black-Litterman model:

a. How to implement the Black-Litterman model from a practical point of view,
b. What kind of problems may occur by adjusting the input parameters (τ and ω) and
c. How to weight conveniently between the market and the subjective views (i.e. the introduction of Meucci's approach).

The actual performance and stability analysis of a Black-Litterman portfolio takes place in chapter 8. Over a 3-months period of time the performance of different calibrations of the Black-Litterman model is compared to two benchmarks: the market portfolio and a mean variance portfolio that is calculated with historical estimates of expected returns. In this context the issues by estimating expected returns that are derived from historical data are examined as well.

Founded on the results of the analysis, the following chapter addresses drawbacks of the Black-Litterman model in general.

"Outlook" proposes two new possible areas for the application of the Black-Litterman model (in a "naive" asset allocation and in product portfolios) as well as a method to take stress events in Portfolio Theory more into account.

The final chapter summarizes the most important proposals made in the thesis.

2. Problem statement

2.1. Motivation

In the early 1900s portfolio management was rather straightforward. Well acknowledged works like the one from Williams (1938) published by the Harvard University Press suggested to rely *exclusively* on the investment with the highest expected return. So, putting all eggs in one basket was kind of state of the art. [2]

In the 1950s Markowitz's article "portfolio selection" changed portfolio theory for good.[3] His framework is a quantification of the two basic objectives of investing: maximizing expected return and minimizing risk. In other words: maximizing the investor's *utility*. The quintessence of his insights about the risk-return trade-off is diversification. Even after more than half a century of dissection by bright minds, Markowitz's work is still the foundation of modern portfolio theory.

In the practical world of asset allocation however, the Markowitz framework has had little impact.[4] There are a few reasons why. Probably the most important one is its instability of portfolio weights. The weights received by Markowitz's model (also referred as the mean variance approach) are extremely sensitive to (small) changes of input parameters. For example, a 0,1% upward change in the expected return may easily increase the weight of an asset in an optimized mean variance portfolio to, say, 17% (compare chapter 4.2.2.). As expected returns, i.e. *future* returns, are very difficult to estimate (and impossible to know for sure, unless the investor is a clairvoyant), mean variance portfolios often lead to poor results. Using historical estimates only helps to a certain extent since they base on history that is not going to repeat.[5]

If already small changes may give rise to large opportunity costs,[6] things get even worse, considering that the classical mean variance optimization requires (historical estimates of) expected returns and (co)variances of *all* securities in the investment universe considered. Investors are very unlikely to have reliable forecasts in all securities, companies and sectors they have at their disposal.[7] Usually they have detailed understanding in only few assets and markets. But focusing on a small amount of assets contradicts diversification. Therefore, investors often try to augment their views by using auxiliary assumptions. However, vague assumptions concerning expected returns are considered to

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[2] Rebonato and Denev (2013), p.6
[3] Markowitz (1952)
[4] He and Litterman (1999), p. 2
[5] https://www.forbes.com/sites/johnmauldin/2017/06/01/modern-portfolio-theory-2-0-the-best-investment-strategy-today/2/#6ee9ceba70db, 7.3.2018
[6] Meucci (2005)
[7] Fabozzi (2006), p. 285

be "the natural enemies" of the mean variance framework and may turn it into the "error maximization machine", as Scherer describes it.[8]

Furthermore, optimal portfolio weights of the standard asset allocation model tend not only to be instable and sensitive, but also extreme, not intuitive and, if not ruled out, highly leveraged. This is because the classical mean variance approach overweights assets with large expected returns and low standard deviations and underweights those with low expected returns and high standard deviations. This usually results in large short positions in many assets if an investor does not use constraints concerning short selling. On the other hand, when constraints rule out short positions, the model prescribes corner solutions in which only a few assets are assigned while many assets receive zero weights.

Over the last years, many approaches have been made to tackle these issues. One of the better-known solutions is given by Michaud (1989), also known as the resampling approach. The key point is to introduce a simulated-sample vector of expected returns and to average the weights obtained in the simulations in order to get "resampled" better behaving weights. This procedure of combining - individually bad behaving - asset allocations indeed reduces the instability and leads to more diversified, less sensitive and (surprisingly) well performing[9] portfolios. However, as each set of weights has been obtained under conditions of parameters certainty, the resampling approach determines the "expectation of weights" instead of the expectation of the utility over uncertain weights.[10] From this perspective, its theoretical justification might seem a bit opaque.

In the early 1990s, Black and Litterman developed an alternative approach, the so called Black Litterman model (BL model). This model differs significantly from other proposed solutions to stabilize asset allocations. Unlike mean variance optimizations, the BL model does not start from scratch and does not require (historical) estimates of expected returns, since it uses the *equilibrium market portfolio* based on the Capital Asset Pricing Model (CAPM) of Sharpe[11], Lintner[12] and Mossin as a neutral starting point for further calibration. By Bayesian updating this neutral portfolio may then be adjusted in accordance with the investor’s subjective views.

The underlying of the BL model may be explained by “reverse optimization": Since the market equilibrium returns are already implied in equilibrium asset pricing models like the CAPM, no historical estimates of expected returns are needed. The equilibrium values are furthermore the

[8] Scherer (2002), p. 452
[9] Rebonato and Denev (2013), p.76 and Scherer (2002)
[10] Rebonato and Denev (2013), p.75
[11] Sharpe (1964)
[12] Lintner (1965)

reason why BL portfolios are generally well diversified, because the market equilibrium portfolio consists of a broad variety of positively weighted assets.

On top of this CAPM foundation, the BL model allows to bring up subjective views. If an investor has one or more views, he may adjust the weights according to his views by using *Bayesian updating*. On the other hand, if an investor does not have any views, he simply holds the market portfolio. In addition, the Meucci Black-Litterman model (compare chapter 2.2.) provides the opportunity to specify the *confidence* in his views in line with the market model.

The Bayesian combination of statistical information that pertains to the market conditions with subjective expertise seems promising to model future returns, particularly because of the possibility to weight freely between these two sources of information. The fact that the Black-Litterman model obtains its inputs from two forward-looking sources - the implied market information and the subjective views - is another benefit compared to the mean variance approach.

Its biggest advantage though is that it (usually) results in stable, intuitive and well diversified portfolios. According to He and Litterman (1999) this was actually the main intention of Black and Litterman to develop their model:

When managers try to optimize using the Markowitz approach, they usually find that the portfolio weights returned by the optimizer […] tend to appear extreme and no particularly intuitive. In practice most managers find that the effort required to specify expected returns and constraints that lead to reasonable answers does not lead to a commensurate benefit. Indeed this was the original motivation [...].

Its Bayesian nature is an important aspect of the BL model. Modern asset allocation models are supposed to provide not only statistically sophisticated asset allocations but also a coherent way to organize the subjective intentions of an asset manager. Embodying Bayesian views into a model allows one to "rationalize" subjectivity within a quantitative framework.[13] As Markowitz stated (1987): "The rational investor is a Bayesian."[14]

Researches over the last decades showed that no model, neither "pure" historical/statistical data based models nor "naive" asset allocation models perform well all the time. As for future forecasts in general and economic forecasts in detail, it is difficult to estimate future developments ex ante. It is therefore beneficial to exploit *any* useful piece of information available, statistical information as well as subjective expertise.

[13] Fabozzi (2006), p. 285
[14] Markowitz (1987), p. 57

From this perspective asset allocation has similarities to other social sciences that have to forecast the future. An interesting analogy is given by Lewis (2017):[15] the NBA draft. For decades, players were picked only because of a (subjective) professional expertise. Then Morey introduced a model that relied on historic-statistical data and "outperformed" (made better picks) most experts. Today every information, however limited, is cherished and squeezed as much as possible by combining statistical data with subjective expertise, just like in the Black-Litterman model.

The Black-Litterman approach has certainly some downsides as well. They are discussed in detail in chapter 9. Beforehand we can state that its biggest drawback occurs if it is calibrated rather extremely, say, the investor has views that differ a lot from the implied market returns and has very high confidence in his views.

2.2. Performance and Stability of a Meucci Black-Litterman portfolio

Although the Black-Litterman model provides a sophisticated model to combine two different sources of information, the question of how to weight conveniently between them remained an open question in the early nineties. In 1999 He and Litterman suggested an approach to equally weight the confidence in the market and in the views.

Six years later Meucci proposed a method which allows to weight freely between the (implied) market information and the subjective information. This is done by introducing a confidence parameter $c \in (0,1)$ that valuates one's belief in the subjective views in an intuitive and comprehensive manner.

Note that the possibility to bring in subjective views implicates that the Black-Litterman model expects its user to have superior information that enables to generate abnormal returns compared to an uninformed market observer. Without this assumption we would withdraw the Black-Litterman model its right to exist - a Black-Litterman portfolio would then only describe the market portfolio. The subjective views - and the confidence we have in them - are therefore *the* crucial elements of the Black-Litterman model.

While many (if not all) performance analyses of the Black-Litterman model examine whether certain subjective views lead to a better performance than the market portfolio or not, we take the implied assumption that the subjective views are more accurate than the market returns, as granted. Thus this treatise does not focus on how to select views that may outperform the market. It rather

[15] Lewis (2017), p.23ff

concentrates on how different levels of confidence affect a Meucci Black-Litterman portfolio's performance and stability, if subjective views are more accurate than the implied market assumptions. In particular we investigate the following questions:

a. In how far affect different levels of confidence in subjective views the asset allocation?
b. How do different levels of confidence in subjective views concern the portfolio's performance?
c. And how do different levels of confidence in subjective views affect the portfolio's stability and risk?

A detailed analysis of these questions unveils interesting asymmetries in the risk-return profile that an investor may take advantage of.

3. Methodology

To answer the questions raised in the problem statement we come back to methods that base in the broader sense on the core foundation of modern portfolio theory: *Markowitz*'s portfolio selection. His insights on diversification and the risk return trade-off paved the way for any modern portfolio management as well as the method we use to obtain the weights of the portfolios.

Market equilibrium returns implied in the Capital Asset Pricing Model by *Sharp*, *Lintner* and *Mossin* are used as the center of gravity in the asset allocation model of *Black* and *Litterman*. The combination of their model with the entropy pooling approach by *Meucci* finally leads to a weighting method between the implied market information and the subjective views, which will be used throughout this work. Chapter 7 illustrates how an asset allocation of the Meucci Black-Litterman model takes place and explains the influence of different levels of confidence in subjective views on the asset allocation.

There exists a broad range of different methods to evaluate the performance of a portfolio. They are generally classified into two categories: conventional methods and risk-adjusted methods. [16] Risk-adjusted methods have the advantage that the returns are adjusted in order to take different levels of risk into account. This makes sense since higher risk portfolios are expected to have ceteris paribus a higher return (in the long term).

In our case the conventional as well as the risk-adjusted method suffer from the subjective element of the Meucci Black-Litterman model. The disadvantage is that they may (and will) be easily manipulated due to the subjective views. However we are not interested in the question whether the views we apply to the Black-Litterman model lead to a better performing portfolio *at all* (which we assume that they will), but rather on *how* different levels of confidence in subjective views affect the performance and the stability. For this purpose the following methods are used:

To examine this impact of subjective views on the *pure* performance we use a so-called conventional *benchmark comparison* based on empirical data.[17] A portfolio is said to have beaten a benchmark once the return of the portfolio exceeds that of the benchmark, measured during the same period of time.[18] The results in chapter 8 below highlights why the conventional benchmark comparison is therefore the method of choice (keyword: tension field between two benchmarks).

[16] Samarakoon and Hasan (2006), p. 617
[17] Samarakoon and Hasan (2006), p. 617
[18] Samarakoon and Hasan (2006), p. 618

The total risk of a portfolio is defined as the standard deviation of the returns of the portfolio.[19] Thus for the stability and risk analysis, we will examine how the portfolio's standard deviation is affected by different levels of confidence in the subjective views.

As far as risk-adjusted performance measurements are concerned, we focus on the method proposed by Sharpe 1966. This method evaluates the excess return, i.e. the risk premium, per unit of risk[20] via the so called Sharpe ratio. We do so because other risk-adjusted performance measurements such as the methods proposed by Treynor or Jensen do not contribute much for our purpose since, a) their measurements aim on the systematic risk (i.e. market risk) which is only a part of the overall risk and b) Treynor's or Jensen's method work well if the portfolios of interest are well diversified, since then a part of the overall risk may be diversified away, and the systematic risk is the predominate risk. But as the confidence in the subjective views increases, the Meucci Black-Litterman portfolios may become not well diversified which may cause biased results.

Thus, for eventually not well diversified portfolios is the appropriate measure of risk and stability the portfolio's standard deviation and therefore the appropriate risk-adjusted performance measurement the Sharpe ratio.[21]

The empirical dataset is composed of market-daily observations of the Austrian Traded Index ATX covering the period between 01.08.2017 and 31.10.2017. In order to evaluate the performance of a Meucci Black-Litterman portfolio we apply two benchmarks: on the one hand the ATX itself serving as a proxy for the market portfolio and on the other hand a mean variance portfolio calibrated with historical estimates of expected returns.

The thoughts proposed in the chapter "Outlook" about subjective views on the covariance matrix concerning stress events have been inspired by *Rebonato*'s Bayesian-net approach. The suggested method of varying volatilities and correlations among assets (embodied in the covariance matrix) is a rather straightforward solution to take tail events more into account.

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[19] Samarakoon and Hasan (2006), p. 618
[20] Sharpe (1966), p. 123
[21] Samarakoon and Hasan (2006), p. 620

4. The mean variance approach of Markowitz

As already mentioned, the origin of modern portfolio theory is the portfolio selection model of Markowitz (1952). Till then, asset allocation and stock picking were basically equivalent (Williams 1938). Markowitz set the foundation of his work by modeling the rate of returns on assets as random variables. The first central moment of the random variable, i.e. the expected value, is used as the expected return, and the second central moment, i.e. the variance of the expected return, is applied as the measure of risk. So, expected return and risk are the basis for any investment decision.

The key to Markowitz's portfolio selection lies in the words diversification and correlation. In general, the expected return of a portfolio, which consists of individual assets, is a linear combination of the expected returns of the individual components of that portfolio:

$$E[R_P] = \sum_{i=1}^{n} w_i R_i$$

where $E[R_P]$= expected return of the portfolio

w_i= weight of asset i

R_i= return of asset i

However the return standard deviation (and the variance), i.e. the risk, of a portfolio is *not* a linear combination of the standard deviations of the individual assets. Only in the very special and unrealistic case that the correlation among those assets is exactly +1 it is a linear combination. For any correlation <+1 a subadditivity effect comes into play. The following example explains why.

Assume a portfolio consisting of two assets only, asset A and asset B. Then the expected return for this portfolio is:[22]

$$E[R_P] = w_A R_A + w_B R_B$$

with

$$w_A + w_B = 1$$

The portfolio return variance σ_P^2 is

$$\sigma_P^2 = E[R_P - E(R_P)]^2 = E[w_A \cdot R_A + w_B \cdot R_B - (w_A \cdot E(R_A) + w_B \cdot E(R_B))]^2$$

$$\sigma_P^2 = w_A^2 \cdot E[R_A - E(R_A)]^2 + w_B^2 \cdot E[R_B - E(R_B)]^2 + 2 \cdot w_A \cdot w_B \cdot E[(R_A - E(R_A)) \cdot (R_B - E(R_B))]$$

$$\sigma_P^2 = w_A^2 \cdot \sigma_A^2 + w_B^2 \cdot \sigma_B^2 + 2 \cdot w_A \cdot w_B \cdot Cov(R_A, R_B)$$

[22] Schwaiger (2012), p.134, Aussenegg (2016), p.98

The covariance $cov(R_A, R_B)$ of the returns R_A, R_B is defined as

$$cov(R_A, R_B) = \rho_{A,B} \cdot \sigma_A \cdot \sigma_B$$

where $\rho_{A,B}$represents the correlation between A and B.

The risk or volatility of the portfolio, i.e. the standard deviation, is the square root of the portfolio variance:

$$\sigma_P = \sqrt{w_A^2 \cdot \sigma_A^2 + w_B^2 \cdot \sigma_B^2 + 2 \cdot w_A \cdot w_B \cdot \rho_{A,B} \cdot \sigma_A \cdot \sigma_B}$$

The three following cases show where the subadditivity comes from:[23]

$$\rho_{A,B} = +1:$$

$$\sigma_P = \sqrt{w_A^2 \cdot \sigma_A^2 + w_B^2 \cdot \sigma_B^2 + 2 \cdot w_A \cdot w_B \cdot \sigma_A \cdot \sigma_B}$$

$$= w_A \cdot \sigma_A + w_B \cdot \sigma_B$$

$$\rho_{A,B} = -1:$$

$$\sigma_P = \sqrt{w_A^2 \cdot \sigma_A^2 + w_B^2 \cdot \sigma_B^2 - 2 \cdot w_A \cdot w_B \cdot \sigma_A \cdot \sigma_B}$$

$$= w_A \cdot \sigma_A - w_B \cdot \sigma_B$$

$$\rho_{A,B} < +1:$$

$$\sigma_P = \sqrt{w_A^2 \cdot \sigma_A^2 + w_B^2 \cdot \sigma_B^2 - 2 \cdot w_A \cdot w_B \cdot \rho_{A,B} \cdot \sigma_A \cdot \sigma_B}$$

$$\overset{!}{<} w_A \cdot \sigma_A + w_B \cdot \sigma_B$$

As for the case $\rho_{A,B} = +1$, it is straightforward to see that no subaddivity effect takes place - the portfolio's volatility adds up just like the expected returns. For $\rho_{A,B} = -1$, the overall risk is $\sigma_P = w_A \cdot \sigma_A - w_B \cdot \sigma_B$, which means that there exists a combination of the assets A and B (i.e. the

[23] Aussenegg (2016), p.100

weights w_A and w_B) that eliminates the portfolio risk $\sigma_P = 0$. The most common case however is $\rho_{A,B} < +1$. As the equation above shows, the portfolio risk σ_P is lower for *any* correlation $< +1$ than the linear combination of the individual asset risks $w_A \cdot \sigma_A + w_B \cdot \sigma_B$.

This subadditivity caused by noncorrelation is known as the diversification effect. It means that an investor can reduce his exposure to an individual asset risk simply by building a diversified portfolio of assets that have a correlation $< +1$. A diversified portfolio consisting of at least two assets may have less risk than the least risky individual asset, according to their correlation. This diversification is the very heart of modern asset allocation and portfolio management.

Furthermore it is assumed that investors are risk averse. This means that an investor prefers the investment with the lowest risk, if all investments have the same expected return. On the other hand if all investments are equally risky, he will choose the one with the highest return. Thus, more risk must be taken in order to get more return. The risk-return trade-off of a certain investment is for all investors per se the same. However not all investors have the same level of risk aversion and will therefore differently evaluate the risk-return trade-off based on the individual risk aversion.

Before advancing to the portfolio weights of an efficient portfolio, it might be useful to sum up the assumptions of Markowitz's portfolio selection:[24]

- Investors are rational and risk averse
- Investors make decisions based on expected return and risk
- Investors estimate risk on basis of the volatility of expected returns
- Investors try to maximize their utility

4.1. Portfolio weights

A common way to describe the risk aversion is via a quadratic utility function U:[25]

$$U = \boldsymbol{w}'\boldsymbol{\mu} - \frac{\lambda}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}$$

where $\boldsymbol{\mu}$= return vector

λ= risk-aversion coefficent (compare chapter 4.2.3.)

$\boldsymbol{\Sigma}$= covariance matrix

[24] Fabozzi (2006), p. 208 and https://cfacuecards.wordpress.com/2012/06/10/5-assumptions-to-markowitz-portfolio-theory, 5.4.2018

[25] See e.g. Fabozzi (2006), p. 45, Idzorek (2005), p.5, Rebonato and Denev (2013), p. 58, Aussenegg (2016), p. 90

Other important utility functions incorporate linear utility function, exponential utility function, power utility function and logarithmic utility function. However, as several studies and Fabozzi (2006) state: "The quadratic utiltiy function provides a good approximation for many other standard utility functions such as exponential, power, and logarithmic utility."[26]

Differentiation with respect to the weights and setting the derivative equal to zero leads to

$$\frac{\partial U}{\partial w}|_{w=w} = \boldsymbol{\mu} - \lambda \boldsymbol{\Sigma} \boldsymbol{w}^* = 0$$

4.1.1. Unconstrained solution

The unconstrained solution for $\boldsymbol{w}^*$ is

$$\boldsymbol{w}^* = \frac{1}{\lambda} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$$

As this solution does not include any constraints, it is not particularly useful in practice. Besides the fact that short selling is not ruled out, the solution just obtained does not reflect a budget constraint, meaning that the weights do not sum up to one. Throughout this work, the only constraint we use is the budget constraint. There are two reasons why: while the budget constraint maintains a linear-algebra solution,[27] short selling constraints no longer allow such a solution. In addition the following applies primarily to the Black-Litterman approach, which usually has none or only a few and small short positions (compare chapter 7 and 8). Therefore the theoretically "cleanest" solution possible is of interest. So besides the fact that short selling constraints are reasonable when using the pure mean variance approach, further calculation will be done with the budget constraint only.

4.1.2. Budget constraint

Carrying out the maximization problem of the quadratic utility function with the budget constraint

$$\sum_{i=1}^{n} \boldsymbol{w}_i = 1$$

requires the introduction of a vector of Lagrangian multipliers δ and the creation of a Lagrangian function $\mathcal{L}$.

[26] Fabozzi (2006), p. 47
[27] Rebonato and Denev (2013), p.58

$$\mathcal{L} = \boldsymbol{w}'\boldsymbol{\mu} - \frac{1}{2}\lambda\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} + \delta(\mathbf{1} - \boldsymbol{w}'\mathbf{1})$$

$$\frac{\partial\mathcal{L}}{\partial\boldsymbol{w}} = \boldsymbol{\mu} - \lambda\boldsymbol{\Sigma}\boldsymbol{w} - \delta\mathbf{1} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\delta} = \mathbf{1} - \boldsymbol{w}'\mathbf{1} = 0$$

The differentiations with respect to the portfolio weights w and to the Lagrangian multipliers are set to zero to obtain the portfolio weights.

$$\boldsymbol{w}^* = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-\mathbf{1}}\left(\boldsymbol{\mu} - \frac{\boldsymbol{B} - \lambda}{\boldsymbol{A}}\mathbf{1}\right)$$

with

$$\boldsymbol{A} = \mathbf{1}'\boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{1}$$

$$\boldsymbol{B} = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{1}$$

4.2. Disadvantages

Although Markowitz's work is still one of the most widely used quantitative asset allocation models[28] and resulted in the Nobel Prize, it has some severe downsides that are worth considering.

4.2.1. Leverage

The first is that unless constraints rule out short selling, Markowitz's mean variance approach usually results in highly leveraged portfolios with large long positions financed by large short positions. If positivity constraints are assigned to fix this problem, the model provides corner solutions in which only a few assets are assigned - mostly those with the highest Sharpe ratio, i.e. the expected (excess) return to risk ratio - while many assets receive zero weights. This contradicts the common-sense notion of diversification. In other words, the portfolio returned by the mean variance optimizer tends to be extreme, rather poorly diversified and not very intuitive.

4.2.2. Instability

The second and practically even more important issue is the instability of weights received by the mean variance model. Small changes in inputs may give rise to large changes of portfolio weights.

[28] Idzorek (2005), p.2

Considering that the inputs can only be estimated and often are based on history that is not going to repeat, the high sensitivity of the mean variance approach leads to (very) large opportunity costs.

The behavior of the Markowitz portfolio weights bears analogy to the chaos theory: Although fully deterministic, small differences of initial conditions (i.e. expected returns) yield to diverging outcomes that are almost impossible to predict.

But where does this instability come from? And is there a way to fix it? To answer these questions, it is worth to have a closer look at the (unconstrained) solution

$$\boldsymbol{w}^* = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$$

We obtain the sensitivity of the weights to the returns by differentiating the equation with respect to the return vector $\boldsymbol{\mu}$:

$$\frac{\partial \boldsymbol{w}^*}{\partial \boldsymbol{\mu}} = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}$$

The sensitivity of the weights to the returns is inversely related to λ, the risk aversion coefficient and directly related to $\boldsymbol{\Sigma}^{-1}$, the inverse of the covariance matrix. As the covariance matrix consists of very "small numbers" (products of *squares of* volatilities times correlations), the inverse of the covariance matrix is made up of "large numbers" (compared to unity).[29] This implies that the sensitivity of the "pure" mean variance approach is high too. Although the risk aversion coefficient has a stabilizing effect on the sensitivity, for reasonable values of λ (say between 1 and 4) the large inverse covariance matrix $\boldsymbol{\Sigma}^{-1}$ dominates the equation $\frac{\partial \boldsymbol{w}^*}{\partial \boldsymbol{\mu}} = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}$. For understanding reasons it is helpful to go back to a two asset portfolio (consisting of asset A and B only).

Assume that asset A has a volatility of 14% and asset B one of 16% and that the investor has a risk aversion factor of $\lambda = 1$, to keep the example simple. For a correlation of 0.4, the sensitivities are $\frac{\partial w_A}{\partial \mu_A} = 60.74$, $\frac{\partial w_A}{\partial \mu_B} = \frac{\partial w_B}{\partial \mu_A} = -21.26$ and $\frac{\partial w_B}{\partial \mu_B} = 46.5$. If the correlation is 0.75, then the sensitivities increase and the first weight to a change in the expected return of the first asset becomes 116.62.[30] That means that the small change of 0,1% of the expected return of asset A leads to a change of 11.66% of asset's A weight in the portfolio.

This explains (at least mathematically) where the high sensitivity and instability comes from. However, as the expected return is not the only parameter that has to be estimated, the uncertainty

[29] Rebonato (2013), p. 450
[30] compare Rebonato (2013), p.72

brought by the estimation of the covariance matrix can also be a source of portfolio estimation errors. Actually Palczewski and Palczewski[31] show that the portfolio estimation errors consist not only of two parts (the expected return and the covariance matrix) but of three: the third one is a non-linear component due to the superposition of errors in the two estimates.

4.2.3. Risk aversion coefficient

The unconstrained solution of the maximization problem highlights that the risk-aversion coefficient plays an essential role in asset allocation - not only in the mean variance model but also in asset valuation in general and in many other allocation models that build on Markowitz's risk-return trade-off foundation, including the Black-Litterman model. Although the risk-aversion coefficient is such a fundamental element, research has provided comparatively little help as how risk aversion should be modeled.[32] Risk aversion is ultimately an empirical challenge[33] and rather subjective. Amos Tversky and Nobel laureate Daniel Kahneman addressed already 1979 in their paper[34] the issue of modeling risk aversion. (Low) laboratory incentives do not reflect the behavior in actual "real-world" situations of choice.

As the proposed results of this treatise are supposed to appeal as many investors as possible (each with a different level of risk aversion), we will use the *implied* market risk-aversion coefficient, which is a convenient choice for this purpose (compare chapter 7.2.).

[31] Palczewski (2010), p.1
[32] Hault and Laury (2002), p. 2
[33] Hault and Laury (2002), p. 3
[34] Kahneman and Tversky (1979)

5. The capital asset pricing model (CAPM)

The following chapter describes the motivation, intuition and assumptions of the CAPM model and why the risk-return trade-off introduced by Markowitz plays an important role not only in portfolio optimization but also in security valuation.

5.1. Capital Market Line (CML)

So far, all calculations base on the assumption that no risk free asset is available. However, if one assumes that borrowing and lending at the risk free rate is possible, the weights of the efficient portfolio are different. This is because the presence of a risk free asset allows obtaining a better risk-return trade-off than without a risk free asset.[35] Thus, adding a risk free asset with the risk free return R_f changes the efficient portfolio weighting as follows:

$$\max\left[\boldsymbol{w}'(\boldsymbol{\mu} - R_f\mathbf{1}) \ -\frac{1}{2}\lambda\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} + R_f)\right]$$

$$f = \ \boldsymbol{w}'(\boldsymbol{\mu} - R_f\mathbf{1}) \ -\frac{1}{2}\lambda\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w} + R_f)$$

$$\frac{\partial f}{\partial \boldsymbol{w}} = \ (\boldsymbol{\mu} - R_f\mathbf{1}) - \ \lambda\boldsymbol{\Sigma}\boldsymbol{w} = 0$$

$$\boldsymbol{w^t} = \frac{1}{\lambda}\boldsymbol{\Sigma^{-1}}\,(\boldsymbol{\mu} - R_f\mathbf{1})$$

Comparing the tangency weights w^t with the weights in absence of a riskless asset w^* leads to:

$$R_f = \ \frac{\boldsymbol{B} - \lambda}{\boldsymbol{A}}$$

$$\lambda = \ \boldsymbol{B} - R_f\boldsymbol{A}$$

$$\boldsymbol{w^t} = \frac{1}{\boldsymbol{B} - R_f\boldsymbol{A}}\boldsymbol{\Sigma^{-1}}\,(\boldsymbol{\mu} - R_f\mathbf{1})$$

Without a risk free asset, investors select a portfolio with the highest expected return for a given level of risk according to their risk aversion, which is represented by the risk aversion coefficient λ. Any portfolio with the highest expected return for a given risk level is efficient. All of these portfolios maximize the respective investor's utility. The combination of all efficient portfolios over different levels of risk represents the efficient frontier.

[35] Aussenegg (2016), p. 161

However, the introduction of a risk free asset with a standard deviation of zero $\sigma_F = 0$ shifts the efficient frontier from a combination of portfolios only consisting of risky assets to a new proportional line consisting of *one* risky portfolio and a riskless asset. Since all but one efficient portfolio combinations the investor obtains in the absence of a riskless asset are dominated by this new linear combination of riskless and risky assets all but one portfolio combinations of the risky portfolio become redundant.

The one and only portfolio whose combination with the risk free asset dominates all other portfolios is called the *market portfolio* or tangency portfolio, because it corresponds to the point of tangency between the line through the risk free asset and the efficient frontier of risky assets. This new efficient frontier line is called the Capital Market Line (CML). The portfolio of all rational investors, no matter how risk averse they are, lies on the CML. The slope of the CML is equal to:[36]

$$\frac{E(R_M) - R_F}{\sigma_M} = Market\ price\ of\ risk$$

The market price of risk represents the premium for one unit of risk. The expected return of the CML portfolio P can be obtained by considering that the minimum expected return for zero risk $\sigma_P = 0$ is given by R_F:

$$E(R_P) = R_F + \frac{E(R_M) - R_F}{\sigma_M}\sigma_P$$

The equation above shows that the expected return of the portfolio P is a linear function of the expected return of the market portfolio.

5.2. Capital Asset Pricing Model (CAPM) and Security Market Line (SML)

The Capital Asset Pricing Model is an equilibrium asset pricing model based on Markowitz's mean variance portfolio selection. It is an abstraction of the real world capital markets using the following assumptions:[37]

- Investors subscribe to Markowitz's method of portfolio diversification (compare Markowitz portfolio assumptions)
- Investors have the same time horizon
- Investors have the same expectations about the expected return and variance of all assets

[36] Aussenegg (2016), p. 166
[37] Aussenegg (2016), p. 179 and Fabozzi (2006), p. 208

- There is a risk free asset with the expected return R_F and zero risk $\sigma_F = 0$: Investors can borrow or lend any amount at the risk free rate
- Capital markets are (perfectly) competitive and frictionless
- There are no taxes, transaction costs or short sale restrictions

Sharpe developed an asset pricing model that explains how a risky asset should be priced by using the Capital Market Line (compare *market portfolio* or tangency portfolio) as a starting point. The CML sheds light on the expected return of the efficient portfolio; however it does not give any information about the valuation and pricing of an individual asset. To do so, the introduction of the notion of *systematic* and *unsystematic* risk is helpful.[38]

Assume a portfolio that consists of N assets, where R_i represents the expected return of the asset i and w_i the weight of the asset i of the portfolio P. The variance of the expected return of portfolio P is

$$\sigma_P^2(R_P) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j cov(R_i, R_j)$$

For the market portfolio M holds therefore

$$\sigma_M^2(R_M) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_{iM} w_{jM} cov(R_i, R_j)$$

where w_{iM} and w_{iM} represent the percentage of asset i and asset j in the market portfolio. By collecting the terms, the equation can be rewritten as

$$\sigma_M^2(R_M) = w_{1M}\sum_{j=1}^{N} w_{jM} cov\left(R_1, R_j\right) + w_{2M}\sum_{j=1}^{N} w_{jM} cov\left(R_2, R_j\right) + \ \dots + w_{NM}\sum_{j=1}^{N} w_{NM} cov\left(R_N, R_j\right)$$

With the covariance of asset i and the market portfolio M

$$cov(R_j, R_M) = \sum_{j=1}^{N} w_{jM} cov\left(R_i, R_j\right)$$

follows

$$\sigma_M^2(R_M) = w_{1M} cov(R_1, R_M) + w_{2M} cov(R_2, R_M) + \ \dots + w_{NM}\sum_{j=1}^{N} w_{jM} cov\left(R_N, R_j\right)$$

[38] Fabozzi (2006), p. 209

The equation above unveils that the variance of the market portfolio is only a function of the covariance of each asset with the market portfolio. The degree to which an asset covaries with the market portfolio is called the systematic risk. Sharpe defines systematic risk as the portion of an asset's variability that can be attributed to a common factor.[39] The systematic risk results from economy-wide sources of risk, mainly based on general economic conditions, that affects the overall general market. Therefore it is also called the market risk. This risk cannot be reduced by building a portfolio. In other words, this risk is not diversifiable.

On the other hand, the diversifiable asset's risk is called the non- or unsystematic risk. It represents the asset's specific risk, that only affects asset i. With growing N the asset's specific risk will cancel out. Thus, the unsystematic risk can be diversified. This means that a highly diversified portfolio has ceteris paribus a lower unsystematic risk than a poorly diversified portfolio.

As already mentioned the Capital Market Line (CML) gives information about the expected return of the efficient portfolio lying on the CML, but not about the expected return of *individual* assets. The expected return of individual assets $E(R_i)$ is:[40]

$$E(R_i) = R_F + \frac{E(R_M) - R_F}{\sigma_M^2} cov(R_i, R_M)$$

This is called the Security Market Line (SML). In equilibrium all individual securities are on the SML and not on the CML, because of the nonsystematic risk that is incorporated in individual assets. In well diversified portfolios like the CML's market portfolio however, the nonsystematic risk has (almost) no impact.

With regard to the equation of the SML, it is obvious that the expected return of an individual asset $E(R_i)$ does not depend on its total risk, but rather on how it covaries with the market (portfolio). The degree to which the asset's covariance depends on the market is defined as

$$\beta_i = \frac{cov(R_i, R_M)}{\sigma_M^2} = \rho_{i,M} \frac{\sigma_i}{\sigma_M}$$

The covariance prescribes the direction and intensity of the asset's correlation with the market, while the division through the market's variance σ_M^2 normalizes β_i in order to make it comparable with other assets.[41]

[39] Sharpe (1964), Fabozzi (2006), p. 210
[40] Fabozzi (2006), p. 211
[41] Külcür (2008), p.9

Substituting the ratio

$$\frac{cov(R_i, R_M)}{\sigma_M^2}$$

in the SML equation gives the beta version of the SML:[42]

$$E[R_i] = \mu_i = R_f + \beta_i(E[R_m] - R_f)$$

This is called the Capital Asset Pricing Model (CAPM). The CAPM is a theory which states that *in equilibrium* all securities are on the SML.[43] If the security is temporarily not on the SML, an adjustment process takes places to bring it back "on track", i.e. back on the Security Market Line.

According to the CAPM, the expected return on an individual asset is a linear function of the asset's *systematic* risk as measured by beta. Thus, a higher beta leads ceteris paribus to a higher expected return.

The CAPM serves as the starting point for the Black-Litterman approach is content of the next chapter.

[42] Sharpe (1964)
[43] Aussenegg (2016), p.174

6. The Black-Litterman model

In the early 1990s, Fischer Black and Robert Litterman proposed a model for portfolio selection. This model, known as the *Black-Litterman model*, uses a Bayesian approach to combine the market equilibrium with additional market views of an investor.

Since its publication the Black-Litterman asset allocation model has gained wide application in many financial institutions.

"At Goldman Sachs, the Black-Litterman model is a key tool in the Investment Management Division's asset allocation process", Litterman stated in 2003[44] (both, Litterman and Black were working for Goldman Sachs when they developed their asset allocation model).

6.1. The motivation behind the Black-Litterman model

As already mentioned, more than half a century ago, Markowitz set the foundation of modern portfolio theory with the formulation of the two basic objectives of investing: maximizing expected return and minimizing risk.[45] In the practical world of investment management however, the mean-variance optimization has had (surprisingly) little impact. There are 2 main reasons why (compare 2.1. Motivation):

First, classical mean-variance optimization requires (historical estimates of) expected returns and (co)variances of *all* securities of the relevant investment universe. But investors typically have knowledge and expertise to provide reliable forecasts of the returns of only a few assets and markets. Therefore, investors often try to augment their views based on auxiliary assumptions, which lead to poor results.[46] Thus, the Markowitz approach makes it difficult to focus on small segments of the investment universe, since it (unrealistically) requires expected returns for *every* security of the investment universe considered.

Second, optimal portfolio weights of standard asset allocation models tend to be extreme and not very intuitive. This is because the classical mean-variance approach is very sensitive to the return assumptions used, since it overweights assets with large expected returns and low standard deviations and underweights those with low expected returns and high standard deviations. This usually results in large short positions in many assets if an investor does not use constraints concerning short selling. On the other hand, when constraints rule out short positions, the models

[44] Litterman (2003)
[45] He and Litterman (1999), p.2
[46] Black and Litterman (1992), p.28

prescribe corner solutions in which only a few assets are assigned while many assets receive zero weights.[47]

Actually, this was the main motivation for Black and Litterman to develop their approach. Their asset allocation model provides an intuitive solution for the two problems. The key is combining the mean-variance optimization framework of Markowitz[48] and the capital asset pricing model (CAPM) of Sharpe[49] and Lintner.[50]

6.2. Equilibrium returns

The starting point of the Black-Litterman model is the market equilibrium. The market equilibrium is the condition in which expected returns equilibrate the demand for assets with the outstanding supply. One of the basic assumptions of the Black-Litterman model is that unless an investor has specific views on assets, the assets' expected returns are consistent with the market equilibrium returns. Therefore, an investor without any views holds the market portfolio.[51]

The expected equilibrium returns given by an equilibrium asset pricing model such as the CAPM provide a neutral reference point for expected returns. This allows the Black-Litterman model to generate optimal portfolio weights that are much better behaved than the unreasonable portfolios that standard models typically suggest. The CAPM serves as the "center of gravity" for expected returns.

The equilibrium returns are the set of returns that clear the market[52] if all investors had identical views. Supposing the asset universe consists of *N* assets, the implied excess equilibrium return vector $\boldsymbol{\Pi}$ may be derived via the CAPM as follows:

$$\boldsymbol{\Pi} = \boldsymbol{R} - R_f \mathbf{1}$$

where $\boldsymbol{R}$ is the vector of asset returns, $\mathbf{1}$ is a vector of ones and R_f is the risk-free rate. Assuming that the CAPM holds, $\boldsymbol{\Pi}$ is given by:[53]

$$\boldsymbol{\Pi} = \boldsymbol{\beta}(R_m - R_f)$$

[47] Black and Litterman (1992), p. 28
[48] Markowitz (1952)
[49] Sharpe (1964)
[50] Lintner (1965)
[51] Fabozzi et al (2008)
[52] Idzorek (2005), p.3
[53] Fabozzi (2006), p. 287

where:

- $R_m - R_f$ is the market risk premium.
- $\beta = cov(\boldsymbol{R}, \boldsymbol{R'}\boldsymbol{\omega}_{mkt})/\sigma_m^2$ is the vector of asset betas, where $\boldsymbol{R'}\boldsymbol{\omega}_{mkt}$ is the market return.
- $\boldsymbol{R}$ is the vector of asset returns.
- $\boldsymbol{\omega}_{mkt}$ is the vector of market capitalization weights
- σ_m^2 is the variance of the market return, i.e. $\sigma_m^2 = \boldsymbol{\omega}'_{mkt}\boldsymbol{\Sigma}\boldsymbol{\omega}_{mkt}$, where $\boldsymbol{\Sigma}$ is the asset return covariance matrix.

Denote by λ the expression $(R_m - R_f)/\sigma_m^2$. The vector of equilibrium risk premiums, i.e. the implied excess equilibrium return vector $\boldsymbol{\Pi}$, may be written as

$$\boldsymbol{\Pi} = \lambda\boldsymbol{\Sigma}\boldsymbol{\omega}_{mkt}$$

Rearranging leads to

$$\boldsymbol{\omega}_{mkt} = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Pi}$$

This represents the vector of market capitalization positions and λ is the risk-aversion coefficient. The risk-aversion coefficient reflects the expected risk-return tradeoff. It is the rate at which an investor will forego expected return for less variance.[54]

The estimation of the risk-aversion coefficient is a critical issue, since

a. the risk-aversion coefficient is different for each investor depending on the investor's risk aversion level and
b. is in principle difficult to estimate.

As already explained in chapter "4.2.2. Instability” and again subject of chapter “10.3. Covariance matrix during stress events”, the risk-aversion coefficient plays an important role for the stability and the composition of the asset allocation.

6.3. Distributional assumptions

Within the framework of Black-Litterman it is assumed that the asset returns $\boldsymbol{R}$ follow a multivariate normal distribution with an expected return vector $\boldsymbol{\mu}$ and a covariance matrix $\boldsymbol{\Sigma}$.[55] This is because the

[54] Idzorek (2005), p.3

Black-Litterman model does not assume that the world is *always* at CAPM equilibrium, but rather that the market is in equilibrium *on average*. At any given point in time the equilibrium could be perturbed by shocks. Therefore,

$$\boldsymbol{\mu} = \boldsymbol{\Pi} + \boldsymbol{\epsilon}$$

where the *N x 1* vector $\boldsymbol{\epsilon}$ represents the perturbations to the equilibrium. The vector $\boldsymbol{\epsilon}$ is assumed to have a multivariate normal distribution so that the prior distribution on $\boldsymbol{\mu}$ is given by

$$\boldsymbol{\mu} \sim N(\boldsymbol{\Pi}, \tau\boldsymbol{\Sigma})$$

The parameter τ is a scalar. It may be interpreted as the remaining uncertainty in the estimation of $\boldsymbol{\Pi}$. Blamont and Firoozy[56] interpret $\tau\boldsymbol{\Sigma}$ as the standard error of the estimation of the implied equilibrium return vector $\boldsymbol{\Pi}$. Alternatively τ represents the investor's uncertainty that the CAPM holds.

So τ is a parameter to adjust the variance of the distribution of $\boldsymbol{\mu}$. The smaller the value of τ, the smaller the variance of the distribution and the larger the confidence in the estimation of the equilibrium return vector $\boldsymbol{\Pi}$. Thus, a small value of τ corresponds to a high confidence in the equilibrium return estimates. There are many different approaches to set the value of τ. It is discussed in greater detail in chapter "7.4. The choice of τ and ω" below.

The mean (i.e. the equilibrium return vector $\boldsymbol{\Pi}$) and the variance (i.e. $\tau\boldsymbol{\Sigma}$) of the normally distributed $\boldsymbol{\mu}$ vector directly enter the Black-Litterman Formula, representing a neutral starting point for further calibration.

6.4. Investor views

As already mentioned, the Black-Litterman model provides the flexibility to combine the market equilibrium with additional views of an investor. Investors' views are expressed as deviations from the equilibrium returns. If an investor does not have any views, the optimal portfolio of the Black-Litterman model matches the market equilibrium portfolio. If an investor has one or more views about the market, the Black-Litterman approach tilts the optimal portfolio away from the market portfolio in the direction of the investor's views.[57]

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[55] Fabozzi et al (2008)
[56] Blamont and Firoozy (2003)
[57] He and Litterman (1999), p. 6

There are 2 types of views within the framework of Black-Litterman: Absolute views and relative views. An absolute view could be expressed as “next period’s expected return of Apple is 8%”, or as “General Motors will have an absolute return by 4%”. A relative view on the other hand could be expressed by “Apple will outperform General Motors by 4%”. Among portfolio managers, relative views are the predominant type,[58] since many portfolio strategies produce relative rankings of assets (assets are expected to underperform/outperform other assets) rather than absolute expected returns.

In the Black-Litterman model investors’ views are expressed by view portfolios (i.e. sub-portfolios) composed of the assets involved in the respective views. For instance, the two absolute views above correspond to two view portfolios, one long in Apple and the other one long in General Motors.

The assets involved in relative views form two separate sub-portfolios, one long portfolio for the assets that are expected to outperform and one short portfolio for the assets expected to underperform. The number of outperforming assets need not match the number of underperforming assets. However, the net long positions less the net short positions must be equal to 0.[59]

Implementing the stated views into the Black-Litterman model is done via 3 variables:

- $\boldsymbol{P}$ … is a matrix that identifies the assets involved in the views *(K x N matrix or 1 x N row vector in the special case of 1 view);*
- $\boldsymbol{Q}$ … is the view vector *(K x 1 column vector);*
- $\boldsymbol{\Omega}$… is a diagonal covariance matrix of error terms from the expressed views representing the uncertainty in each view (*K x K matrix);*

where *K* represents the number of views and *N* is the number of assets.

$\boldsymbol{P}$, the matrix that identifies the assets involved in the views, is the “pick” matrix: each of its *K* rows is an *N*-dimensional vector that corresponds to one view. Thus, each row of $\boldsymbol{P}$ represents a view portfolio, where an element of $\boldsymbol{P}$ is nonzero if the asset is involved in the view and zero if it is not. In the case of relative views, the elements of a row sum up to zero. In the case of absolute views, the corresponding row consists of only one 1 in the place of the asset involved and zeros everywhere else. Therefore the sum of the elements in a row is 1 in absolute views.

58 Fabozzi et al (2008)
59 Idzorek (2005), p.11

The *K x 1* vector of expected returns on the view portfolios is given by $\boldsymbol{P\mu}$. It is assumed that it is normally distributed which leads to the distributional assumption of the investor's views:[60]

$$\boldsymbol{P\mu} \sim N(\boldsymbol{Q}, \boldsymbol{\Omega})$$

The mean of this distribution is the view vector $\boldsymbol{Q}$. In the case of e.g. 4 views, $\boldsymbol{Q}$ is a *4 x 1* column vector. The vector $\boldsymbol{Q}$ includes the investor's views on the assets' expected returns.

The degree of confidence an investor has in his views is reflected by the covariance matrix $\boldsymbol{\Omega}$. The covariance matrix $\boldsymbol{\Omega}$ consists only of diagonal elements ω_{kk}. Its off-diagonal elements are set equal to zero, since the model assumes that the views are independent of each other. The investor's confidence in the kth view is inversely proportional to the value of ω_{kk} – the larger the value of ω_{kk}, the smaller the confidence in the kth view. Just like for τ, there are different approaches to specify the value of $\boldsymbol{\Omega}$. It is discussed in greater detail in chapter "7.4. The choice of τ and ω" as well.

6.5. The Black-Litterman formula

A possibility to combine the two sources of information represented by the "market information" embodied in

$$\boldsymbol{\mu} \sim N(\boldsymbol{\Pi}, \tau\boldsymbol{\Sigma})$$

and the subjective information embodied in

$$\boldsymbol{P\mu} \sim N(\boldsymbol{Q}, \boldsymbol{\Omega})$$

is given by the Bayes' theorem. The Bayes' theorem is a rule that may be used to update the beliefs that one holds in the light of new information. It states that after observing the data *D*, the belief in *E* (expressed as the probability $P(E)$) is adjusted according to the following ratio:[61]

$$P(E|D) = \frac{P(D|E) \cdot P(E)}{P(D)}$$

where

- $P(E|D)$ is the conditional probability of the data given that the prior evidence *E* is true.

[60] Fabozzi et al (2008)
[61] Fabozzi (2006), p.566

- $P(D)$ is the unconditional probability of the data. That is the probability of *D* irrespective of *E* that may also be expressed as $P(D) = P(D|E) \cdot P(E) + P(D|E^c) \cdot P(E^c)$, where the subscript *c* describes the complementary event.

The probability $P(E)$ is called the prior probability (before having seen the data) and $P(E|D)$ is called the posterior probability (after having seen the data, i.e. the updated probability).

If we apply the Bayes' theorem to asset allocation modeling, it is expressed in terms of distributions not probabilities.[62]

In the context of the Black-Litterman model it is assumed that an investor is aware of the CAPM equilibrium returns. If an investor wants to form his views based on this knowledge, he may – according to the Bayes' theorem – use the expected equilibrium returns as the prior returns and update them with his own views. This results in the posterior returns that combine the equilibrium returns and the views. Figure 1 illustrates the process of combining the two sources of information – the market information and the subjective information.

[62] Fabozzi (2006), p. 566f

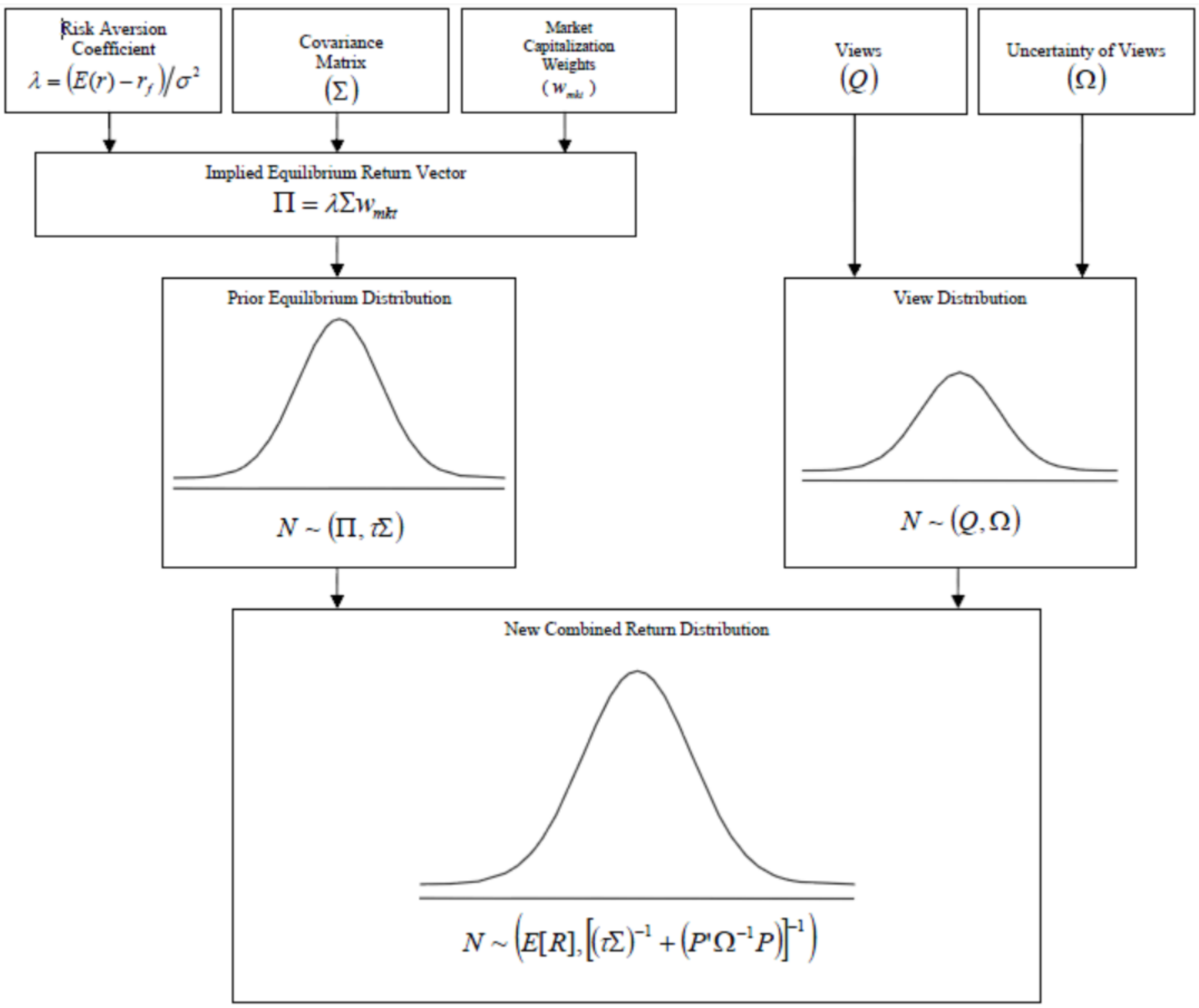

Figure 1: Combining the market information and the subjective information[63]

Applying the Bayes' theorem to the two sources[64]

$$P(E) \sim N(\boldsymbol{\Pi}, \tau\boldsymbol{\Sigma})$$

$$P(D|E) \sim N(\boldsymbol{P^{-1}Q}, [\boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{P}]^{-1})$$

leads to the posterior distribution of the expected returns $\boldsymbol{\mu}$:

$$P(E|D) \sim N([(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{Q}][(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{P}]^{-1}, ((\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{P})^{-1})$$

The posterior distribution is normal with the mean and the covariance given by the elements referred to above. The mean of the posterior expected return, i.e. $E[\boldsymbol{R}]$, represents the *Black-Litterman formula* and is given by

$$E[\boldsymbol{R}] = [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{P}]^{-1}[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{Q}]$$

[63] Idzorek (2005), p. 16
[64] Walters (2008)

- $E[\boldsymbol{R}]$ is the posterior expected return vector *(N x 1 column vector);*
- τ is a scalar;
- $\boldsymbol{\Sigma}$ is the covariance matrix of excess returns (*N x N matrix);*
- $\boldsymbol{P}$ is the a matrix that identifies the assets involved in the views *(K x N matrix or 1 x N row vector in the special case of 1 view);*
- $\boldsymbol{\Omega}$ is a diagonal covariance matrix of error terms from the expressed views representing the uncertainty in each view (*K x K matrix);*
- $\boldsymbol{\Pi}$ is the implied equilibrium return vector (*N x 1 column vector);* and,
- $\boldsymbol{Q}$ is the view vector *(K x 1 column vector).*

When no views are expressed, the posterior estimation of the expected return becomes $E[\boldsymbol{R}] = \boldsymbol{\Pi}$. When the uncertainty in the views is large, then $E[\boldsymbol{R}]$ is dominated by $\boldsymbol{\Pi}$.

Keeping the two sources where the Black-Litterman formula is derived from (the market equilibrium and the views) in mind might help to understand and interpret the results of the posterior expected return. Depending on the investor's confidence in his views, the expected returns will either tilt in the direction of the expected equilibrium returns $\boldsymbol{\Pi}$ (for a low level of confidence in the views) or will go in the direction of the "view returns" $\boldsymbol{Q}$ (for a high level of confidence). Therefore, $\boldsymbol{Q}$ can be seen as the views' equivalent to the market's $\boldsymbol{\Pi}$.

7. How to implement the Black-Litterman model

The following chapters 7 and 8 examine the practical implementation and the performance and stability analysis of the Black-Litterman model. This is done by using empirical data. However as chapter 7 and 8 focus on different issues, it might be helpful to use data that fit best the respective needs: Chapter 7 emphasizes the understandings of the Black-Litterman model. Thus, we need a database that many portfolio managers are already familiar with and have (at least) a vague idea of the weightings and dependencies among the assets included. We will therefore use the S&P 500.

Chapter eight is not about asset allocation with the Black-Litterman model in the first place, but rather how Black-Litterman portfolios actually perform compared to the market portfolio and the mean variance optimized portfolio. The S&P 500 would meet those requirements as well however we believe that the world is not in need of *another* performance comparison about the S&P 500. For this reason a less commonly used dataset appears to bring more interesting insights. Therefore we will use the ATX, the Austrian Traded Index which may be seen as the Austrian's equivalent to the S&P 500.

The starting point of the calibration of a Black-Litterman portfolio is the market portfolio. As the market portfolio is as such a theoretical device, it is necessary to use approximations for the market portfolio. The S&P 500 serves as a proxy for the US market portfolio.

An index composed of 500 companies is a benefit concerning market coverage, but in turn reduces the traceability and comprehensibility of the results. Thus, the companies of the S&P 500 index are bundled into 10 "value weighted industry portfolios", consisting of:

- Basic Materials (1000)
- Industrials (2000)
- Consumer Goods (3000)
- Health Care (4000)
- Consumer Service (5000)
- Telecommunications (6000)
- Utilities (7000)
- Financials (8000)
- Technology (9000)
- Oil & Gas (0001)

In order to cover many different market conditions, we decide to use a broad period of time: from the 6.10.1989 to the 26.08.2016. The companies' returns will be considered on a weekly basis. We receive the returns of the value weighted industry portfolios by summing up the company returns of one industry at time t multiplied by the company market values at $t-1$, and divided by the sum of the company market values at $t-1$.

$$R_{vw,t} = \frac{\sum_{i=1}^{SP}(R_{i,t} \cdot MV_{i,t-1})}{\sum_{i=1}^{SP} MV_{i,t-1}}$$

Since an individual company i generates its return R_i with the market value at $t-1$, we weight the return at t with the market value at $t-1$.

To get an idea of the returns of the value weighted industry portfolios it might be helpful to depict the returns for two examples, like e.g. the Technology and Financials value weighted industry portfolios. The return over time equals their Buy-and-Hold returns:

$$BHR_{i,T} = [\prod_{t=1}^{T}(1 + R_{vw,t})] - 1$$

Figure 2 shows the generated Buy-and-Hold returns for the industries Technology and Financials. The x-axis represents the number of weeks from 6.10.1989 by 26.08.2016. The y-axis shows the return from the starting point till the last date in percent.

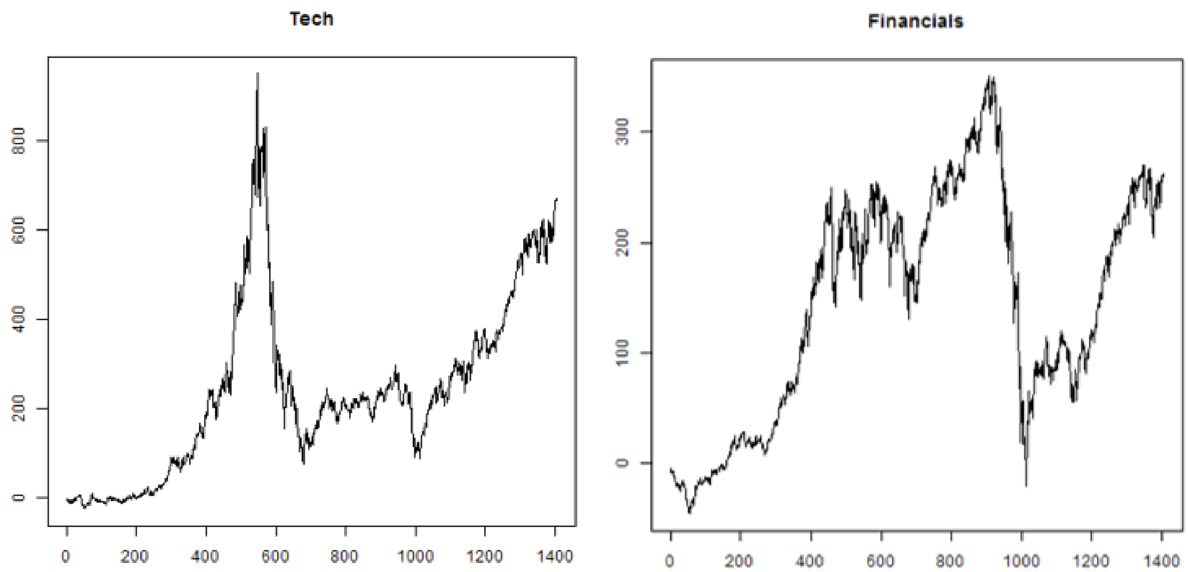

Figure 2: Buy-and-Hold returns of Tech and Financials

Chapter eight goes into detail concerning the evaluation of the relevant historical returns and the valuation of (negative) risk free returns (see chapters 8.1. and 8.2.). For now, we show how to calibrate adequate inputs in a superficial way.

To obtain the relevant excess returns, the risk free return $R_{f,t}$ (compare chapter "8.2. Risk free rate") of each week is subtracted from the return of the respective weekly value weighted returns:

$$R_{excess,vw,t} = R_{vw,t} - R_{f,t}$$

The historical estimates of expected returns as well as the covariance matrix base on these excess returns. For the expected excess returns we may use the arithmetic means of the value weighted excess returns over the whole time span.

$$AM_{\tau 1,\tau 2} = \frac{1}{n} \cdot \sum_{t=\tau 1+1}^{\tau 2} R_t$$

The choice of the historical arithmetic mean for future expected returns is certainly controversial. Chapter "8.1. Historical estimates of expected returns" discusses the question whether this makes sense or not in greater detail. As we focus in the following on the asset allocation only (and not on the actual performance), the arithmetic mean appears to be a convenient choice for the input parameters of the mean variance portfolio. The evaluated excess returns are listed in Table 1. All figures are shown as absolute values.

Industry	Expected excess return	Variance	Standard deviation
1000	0,00119549	0,00095969	0,03097883
2000	0,00153876	0,00069223	0,02631024
3000	0,00159888	0,00040258	0,02006443
4000	0,00186508	0,00058608	0,02420905
5000	0,00153398	0,00063808	0,02526032
6000	0,00097387	0,00071758	0,0267877
7000	0,00129897	0,00050572	0,02248825
8000	0,00152814	0,00123313	0,03511593
9000	0,00207168	0,00122525	0,03500352
0001	0,00166392	0,00088144	0,02968905

Table 1: Expected excess returns

Table 2 shows the covariance matrix of the ten industry portfolios, representing the variances and the dependencies among the industries.

	1000	2000	3000	4000	5000	6000	7000	8000	9000	0001
1000	0,00095969	0,00065225	0,00039749	0,00035732	0,00050569	0,00033077	0,00029456	0,00070178	0,00054666	0,00063332
2000	0,00065225	0,00069223	0,00039462	0,00039428	0,00055102	0,00038601	0,00027833	0,00073996	0,00062631	0,00048227
3000	0,00039749	0,00039462	0,00040258	0,0003436	0,00036965	0,00027158	0,00024824	0,00048139	0,00031005	0,00032099
4000	0,00035732	0,00039428	0,0003436	0,00058608	0,00038558	0,00028508	0,00023586	0,00050102	0,00035969	0,00032615
5000	0,00050569	0,00055102	0,00036965	0,00038558	0,00063808	0,0003895	0,00023403	0,00067876	0,00061218	0,00036563
6000	0,00033077	0,00038601	0,00027158	0,00028508	0,0003895	0,00071758	0,00026681	0,00050138	0,0004545	0,00032114
7000	0,00029456	0,00027833	0,00024824	0,00023586	0,00023403	0,00026681	0,00050572	0,00035578	0,00021021	0,00035726
8000	0,00070178	0,00073996	0,00048139	0,00050102	0,00067876	0,00050138	0,00035578	0,00123313	0,00067402	0,00054052
9000	0,00054666	0,00062631	0,00031005	0,00035969	0,00061218	0,0004545	0,00021021	0,00067402	0,00122525	0,00039938
0001	0,00063332	0,00048227	0,00032099	0,00032615	0,00036563	0,00032114	0,00035726	0,00054052	0,00039938	0,00088144

Table 2: Covariance matrix

7.1. Mean variance portfolio

The tangency portfolio weights $\boldsymbol{w}^{t}$, calculated via $\boldsymbol{w}^{t} = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}\left(\mu - R_f\mathbf{1}\right)$, are shown in Figure 4. Table 3 lists the precise values of $\boldsymbol{w}^{t}$. There are also negative weights, as short positions are not ruled out. The calculated weights tend to be extreme and not very intuitive. By comparing the weights to the values for the expected return and the variance in Table 1 it is evident that industries with a low Sharpe ratio[65] $S = \frac{R_{a,excess}}{\sigma_a}$ have low or negative weights. Conversely, industries with a high reward-to-variability ratio are overweighted.

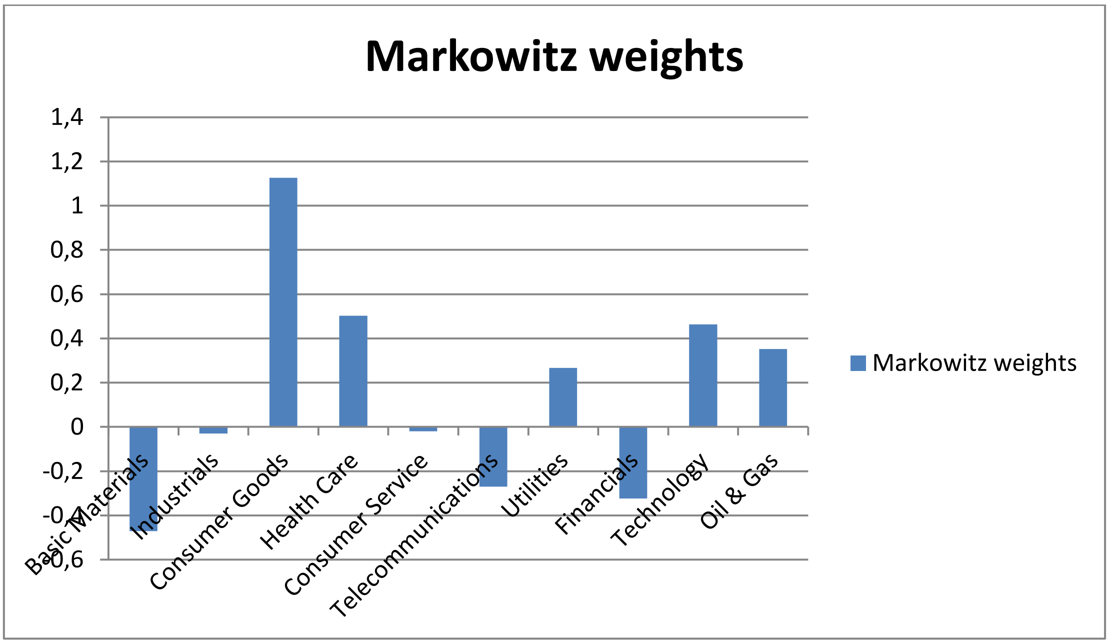

Figure 3: Portfolio weights calculated with the mean-variance approach of Markowitz

[65] Sharpe (1966)

Industry	w^t
1000	-0,47116706
2000	-0,03011255
3000	1,12648318
4000	0,50209868
5000	-0,02045573
6000	-0,27025012
7000	0,26614105
8000	-0,32328101
9000	0,46372503
0001	0,35203339

Table 3: Values of the tangency portfolio weights

7.2. Market portfolio

With $\lambda = B - R_f A$ (compare chapter 5.1.) we obtain the implied risk-aversion coefficient $\lambda = 3{,}114716$. This represents the risk-aversion coefficient that the market "expects" investors to have.

For $\lambda = 3{,}1$, the excess equilibrium returns of the 10 industry portfolios are shown in Figure 4 and Figure 5, respectively.

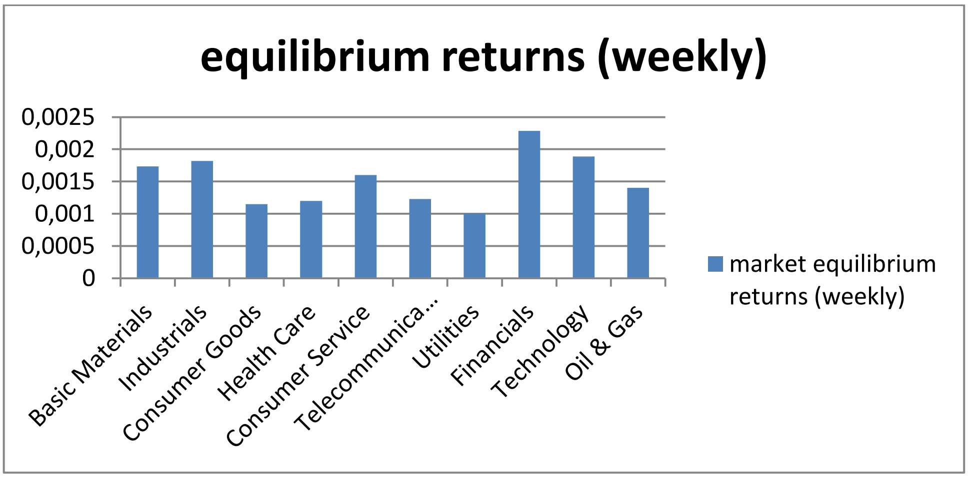

Figure 4: market equilibrium returns (weekly)

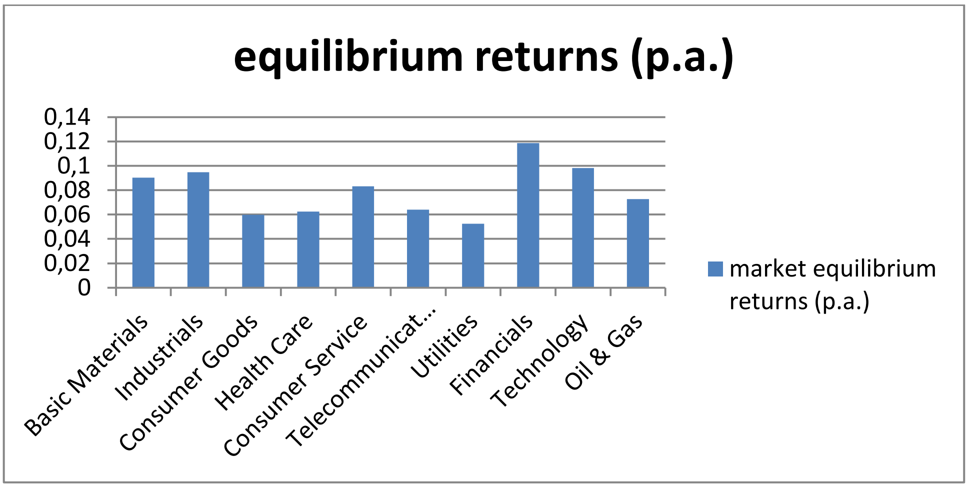

Figure 5: market equilibrium returns (p.a.)

The corresponding market weights are illustrated in Figure 6.

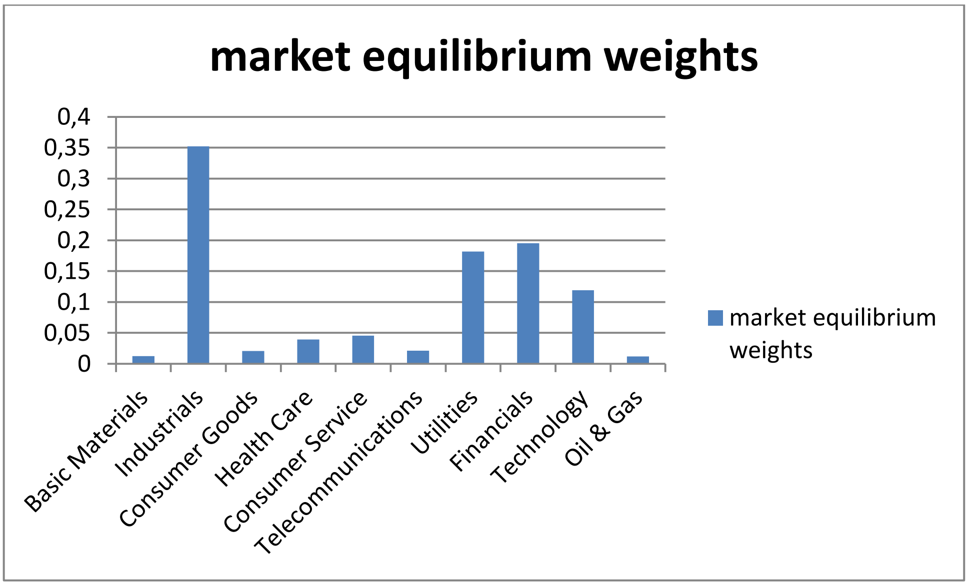

Figure 6: market equilibrium weights

Figure 6 reveals that there are no short positions or corner solutions – in contrast to the weights, that we received using the Markowitz approach. This plausible weighting of the industry portfolios provides a neutral framework that the investor can adjust according to his own views, optimization objectives and constraints.

7.3. Subjective views

In our case of 10 industry portfolios the number of assets involved is $N = 10$. Below are four sample views (i.e. $K = 4$):

> **View 1:** the Oil & Gas portfolio (0001) will have an absolute excess return of 8% (equilibrium return: 7%)

View 2: the Financials portfolio (8000) will outperform the Industrials portfolio (2000) by 2%

View 3: the Technology portfolio (9000) will outperform the Utilities portfolio (7000) by only 125 basis points (compared to the implied market assumption of 4,6%)

View 4: the Consumer Goods portfolio (3000) will have an absolute excess return by only 4% (equilibrium return: 5,75%)

View 1 and View 4 are absolute views and View 2 and 3 are relative views.

In our example is $\boldsymbol{P}$ the following 4 x 10 matrix:

$$\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Corresponding to our views, $\boldsymbol{Q}$ (on an annualized basis) is

$$\boldsymbol{Q} = \begin{pmatrix} 0.08 \\ 0.02 \\ 0.0125 \\ 0.04 \end{pmatrix}$$

and

$$\boldsymbol{Q} \cong \begin{pmatrix} 0.08/52 \\ 0.02/52 \\ 0.0125/52 \\ 0.04/52 \end{pmatrix}$$

in the case of weekly returns.

7.4. The choice of $\boldsymbol{\tau}$ and $\boldsymbol{\omega}$

The choices of τ and ω are the most difficult part in the practical application of the Black-Litterman model.[66] As already mentioned, there are many different approaches to specify the value of τ and ω.

7.4.1. He and Litterman

One approach is given by *He and Litterman* in 1999.[67] They calibrate the confidence of a view by setting the ratio of $\frac{\omega}{\tau}$ equal to the variance of the view portfolio:

$$\frac{\omega}{\tau} = p_k \Sigma p_k'$$

[66]compare e.g. Fabozzi et al (2008) and Idzorek (2005)

[67] He and Litterman (1999)

Where p_k is a single *1 x N* row vector from Matrix *P* that corresponds to the kth view. Thus there is no need to separately specify the value of τ since only the ratio of $\frac{\omega}{\tau}$ enters the Black-Litterman formula. The actual value of τ becomes irrelevant.

We assume any value for τ, e.g. $\tau = 0.001$. We could actually ignore τ by setting it to 1, however for comprehensive reasons we keep carrying τ. The covariance matrix $\boldsymbol{\Omega}$ has the following form:

$$\boldsymbol{\Omega} = \begin{pmatrix} (p_1 \Sigma p_1') \cdot \tau & 0 & 0 & 0 \\ 0 & (p_2 \Sigma p_2') \cdot \tau & 0 & 0 \\ 0 & 0 & (p_3 \Sigma p_3') \cdot \tau & 0 \\ 0 & 0 & 0 & (p_4 \Sigma p_4') \cdot \tau \end{pmatrix}$$

$$\boldsymbol{\Omega} = \begin{pmatrix} 8.814399 \cdot 10^{-7} & 0 & 0 & 0 \\ 0 & 4.45429 \cdot 10^{-7} & 0 & 0 \\ 0 & 0 & 1.310555 \cdot 10^{-6} & 0 \\ 0 & 0 & 0 & 4.025812 \cdot 10^{-7} \end{pmatrix}$$

This approach equally weights the confidence in the market and the confidence in the views. In other words, the investor is trusted as much as the official market model, which is quite a convenient choice to calibrate the uncertainty matrix.

For understanding the mechanics of the model, it might be helpful to start with an in-depth look at one view only instead of four. If we apply only View 1 (an absolute view) to the Black-Litterman model, the pick matrix $\boldsymbol{P}$ becomes a 1 x 10 row vector

$$\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and the view vector $\boldsymbol{Q}$ is

$$\boldsymbol{Q} = (0.08/52)$$

In accordance with the approach given by He and Litterman, the variance of View 1 has the following form:

$$\boldsymbol{\Omega} = ((p_1 \Sigma p_1') \cdot \tau) = 8.814399 \cdot 10^{-7}$$

where $\tau = 0.001$.

After specifying the covariance matrix $\boldsymbol{\Omega}$, we know all inputs that enter the Black-Litterman formula and may calculate the new posterior expected return vector $E[\boldsymbol{R}]$.

	Black-Litterman $E[\boldsymbol{R}]$	Market equilibrium $\boldsymbol{\Pi}$
Basic Materials	0,0928764	0,0902841
Industrials	0,0966428	0,0946688
Consumer Goods	0,0609845	0,0596706
Health Care	0,0636346	0,0622996
Consumer Service	0,0845892	0,0830927
Telecommunications	0,0652026	0,0638881
Utilities	0,0538232	0,0523609
Financials	0,1209306	0,1187182
Technology	0,0998116	0,0981769
Oil & Gas	0,0763921	0,0727842

Table 4: Posterior expected return vector p.a. compared to market equilibrium expected return vector p.a. of View 1

Although only the Oil & Gas portfolio is directly affected by our View 1, the individual returns of the other portfolios changed from their equilibrium returns as well. This is because each individual return is linked to the other returns via the covariance matrix $\boldsymbol{\Sigma}$.[68] So the Black-Litterman model does not simply replace the expected return of one asset only, but rather adopts the expected returns of all assets in the light of the (new) subjective information given by the investor. However the return of the asset involved, i.e. the Oil & Gas portfolio, changes the most, just as expected.

As shown in Table 4, the posterior expected return vector of the Oil & Gas portfolio is higher than its prior equilibrium return. This is no surprise, since the He and Litterman approach of the Black-Litterman model equally weights the confidence of the two sources of information. If an investor adds views that have a higher expected return than the one given by the market equilibrium, the new posterior return of Black-Litterman model $E[\boldsymbol{R}]$ will be higher, ceteris paribus, than prior equilibrium returns $\boldsymbol{\Pi}$. This means that the portfolio of an optimistic investor, who believes that the assets involved in his view will outperform the market, will always have a higher expected return than the portfolio of an investor with bearish views. In other words, the investor with the highest expectations will have the portfolio with the highest *expected* returns.

With the posterior expected return vector $E[\boldsymbol{R}]$ of the Black Litterman model with one view only – View 1 – the new portfolio weights may be derived by using the mean variance approach of Markowitz with the constraint that the weights sum up to one. The Black-Litterman portfolio weights are presented in Figure 7.

[68] Idzorek (2005), p.17

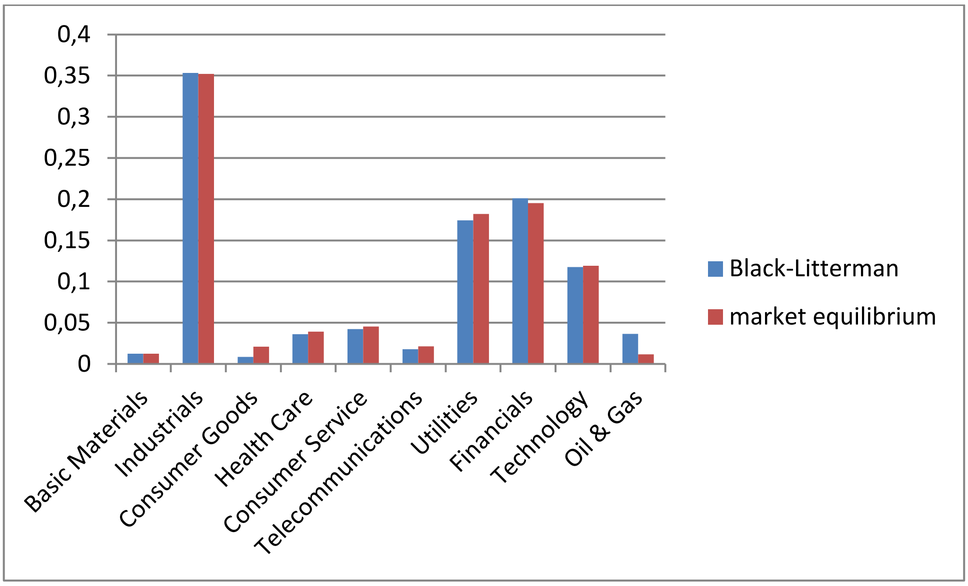

Figure 7: Black-Litterman portfolio weights with View 1 only

In accordance with the changes of the expected returns the weights of the assets change as well. Since View 1 states that the Oil & Gas portfolio will have a higher expected return than the one given by the market equilibrium, the Black-Litterman model shifts weight from other portfolios to the Oil & Gas portfolio. Thus, as far as absolute views are concerned, the behavior of the Black-Litterman model is quite intuitive.

However, among portfolio managers relative views are the predominant type. Therefore it might be useful to analyze the calibration for a relative view – View 3. View 3 expresses that the Technology portfolio will outperform the Utilities portfolio by *only* 125 basis points, which means that the difference between the expected return of the Technology portfolio and the Utilities portfolio should be smaller, ceteris paribus, than the difference at equilibrium (~4.6% p.a.).

Applying View 3 to the Black-Litterman model leads to

$$\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

In the row vector $\boldsymbol{P}$ the outperforming portfolio receives a positive weighting, i.e. one, while the underperforming portfolio receives a negative weighting, i.e. minus one. The view vector $\boldsymbol{Q}$ has the following form

$$\boldsymbol{Q} = (0.0125/52)$$

In line with the calculation of $\boldsymbol{\Omega}$ of an absolute view, the calibration of $\boldsymbol{\Omega}$ is

$$\boldsymbol{\Omega} = ((p_3 \Sigma p_3') \cdot \tau) = 1.310555 \cdot 10^{-6}$$

where $\tau = 0.001$. The actual value of τ is still irrelevant since only the ratio of $\frac{\omega}{\tau}$ enters the Black-Litterman formula.

The new posterior expected return vector $E[\boldsymbol{R}]$ is depicted in Table 5.

	Black-Litterman $E[\boldsymbol{R}]$	**Market equilibrium $\boldsymbol{\Pi}$**
Basic Materials	0,0870796	0,0902841
Industrials	0,0902457	0,0946688
Consumer Goods	0,058885	0,0596706
Health Care	0,0607256	0,0622996
Consumer Service	0,0782862	0,0830927
Telecommunications	0,0615024	0,0638881
Utilities	0,056117	0,0523609
Financials	0,1146731	0,1187182
Technology	0,0852751	0,0981769
Oil & Gas	0,0722489	0,0727842

Table 5: Posterior expected return vector p.a. compared to market equilibrium expected return vector p.a. of View 3

As shown in Table 5, the market equilibrium return of the Technology portfolio is distinctly higher than the equilibrium return of the Utility portfolio. The difference between the expected return of the Technology portfolio and the Utilities portfolio is about 4.6% p.a. at equilibrium. To reduce the difference between the expected returns in accordance with View 3, the posterior expected return of the Technology portfolio decreases, while the Utility portfolio’s posterior expected return increases. Interestingly the decrease of the Technology portfolio’s return is more than 3 times as high as the Utilities’ increase. This is due to the different variances and correlations of Technology and Utility. As already mentioned, a view does not only affect the returns of assets involved in the view, but also all the other assets’ returns via the covariance matrix. This is valid for absolute views as well as for relative ones.

Figure 8 represents the corresponding optimal portfolio weights for the new posterior expected return vector $E[\boldsymbol{R}]$ of View 3, compared to the equilibrium weights.

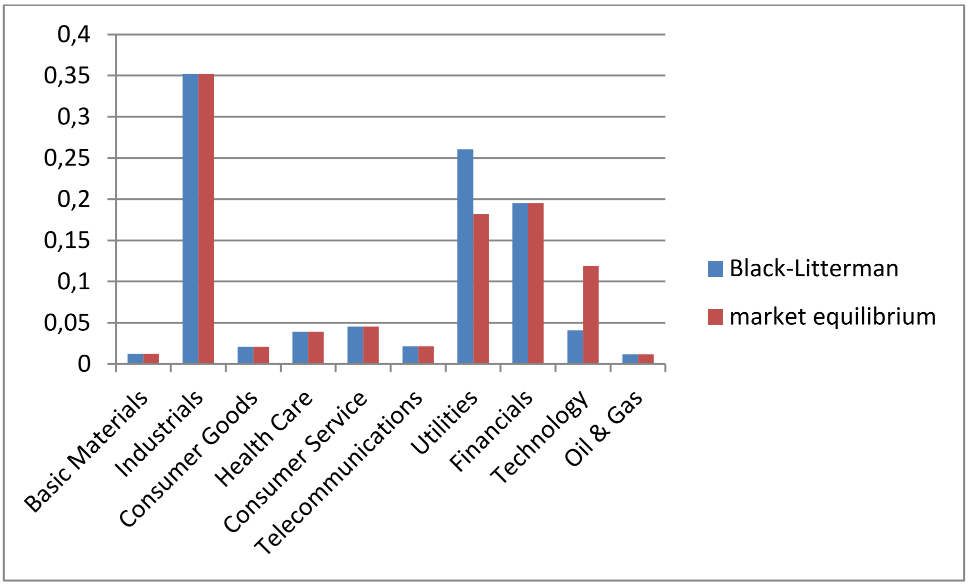

Figure 8: Black-Litterman porfolio weights with View 3 only

In line with the expected returns, the Black-Litterman model reduces weight of the Technology portfolio in favor of the Utilities portfolio, just as expected. Surprisingly the weights of the portfolios where no views are expressed remain unchanged. This is because the new portfolio can be seen as the sum of two sub portfolios, where portfolio 1 is the market equilibrium portfolio and portfolio 2 is the view portfolio consisting of the long and short positions based on the view. In a relative view, the long and short positions of portfolio 2 – the view portfolio – perfectly compensate each other and therefore have no impact on the weights of portfolio 1. On the other hand, the absolute View 1 increases the weight of the Oil & Gas portfolio without an offsetting position. However, this intuitiveness of the Black-Litterman model is a bit less apparent once investment constraints are added.[69]

All considerations so far equally weighted the confidence in the market equilibrium return and the confidence in the views given by the investor in line with the approach of He and Litterman. However, this might become an issue if the investor wants to specify different levels of confidence for the views compared to the equilibrium and among each other.

7.4.2. Meucci

Another method to specify the confidence in the market and the views is given by *Meucci.*[70] Meucci sets $\tau = 1$ and introduces c, a parameter for the view portfolio, to adjust the confidence in views. Meucci specifies the elements ω of the subjective uncertainty matrix $\boldsymbol{\Omega}$ as:

[69] Idzorek (2005), p.18
[70] Meucci (2005)

$$\omega = \left(\frac{1}{c} - 1\right) \cdot (p_k \Sigma p_k')$$

where $c \in (0,1)$.

The scalar c represents the confidence in the investor's views. The case $c \to 0$ leads to an infinitely disperse distribution of the views: this means that the investor's views have no impact. Thus, if an investor has only low confidence in his view, he sets c small and the Black-Litterman portfolio will tilt in the direction of the market portfolio.

On the other hand, the case $c \to 1$ leads to an infinitely peaked distribution of the views: this means that the investor is trusted completely over the market model.[71] In other words, for a high value of c, the view information of the Black-Litterman formula will become predominant.

The case $c = \frac{1}{2}$ equally weights the confidence in the market equilibrium model and the confidence in the investor's views. Therefore this constellation corresponds to the He and Litterman approach and leads to the same returns.

With Meucci's approach it is possible to calibrate the confidence of our views for different values of c. As View 1 concerns the Oil & Gas portfolio only, the changes of the Oil & Gas portfolio for different values of c are of interest. Figure 11 presents the expected returns for $c \to 0$ (i.e. equal the market equilibrium return), $c = 1\%$, $c = 16{,}67\%$, $c = 50\%$ (i.e. the same expected return as with the He and Litterman approach), $c = 83{,}33\%$ and $c = 99\%$, representing a very high confidence in the investor's view.

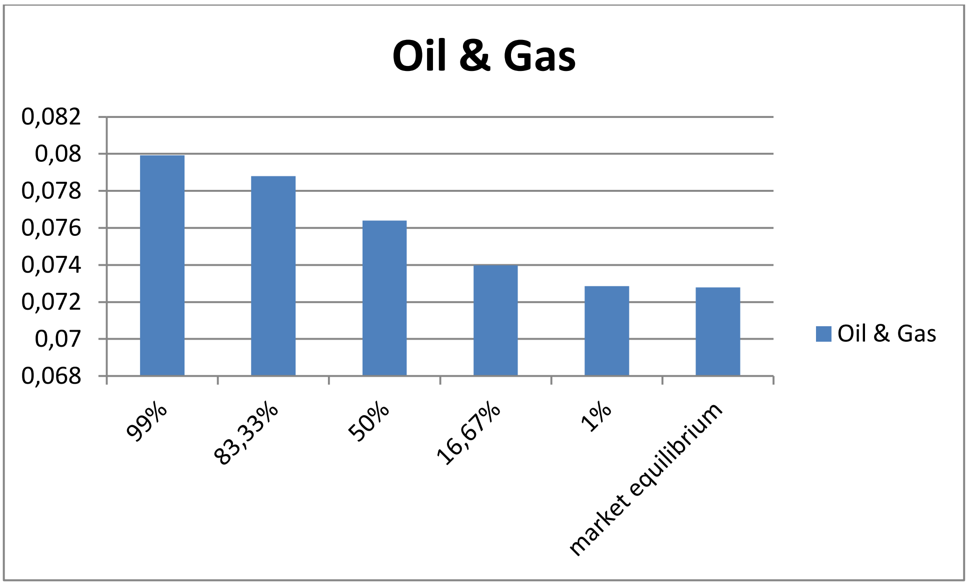

Figure 9: Expected return of the Oil & Gas portfolio for different settings of c in View 1

[71] Meucci (2005)

As Figure 9 demonstrates, in the case of a small value of c which corresponds to a low level of confidence in the view compared to the level of confidence in the market model, the expected return of the Oil & Gas portfolio is almost similar to the market equilibrium return (~7,3% p.a.). For increasing confidence, the posterior expected return rises and goes in the direction of the investor's expected return expressed by our View 1 (~8% p.a.).

The corresponding weight changes of a portfolio generated by using the mean variance approach of Markowitz for the different settings of c are depicted in Figure 10.

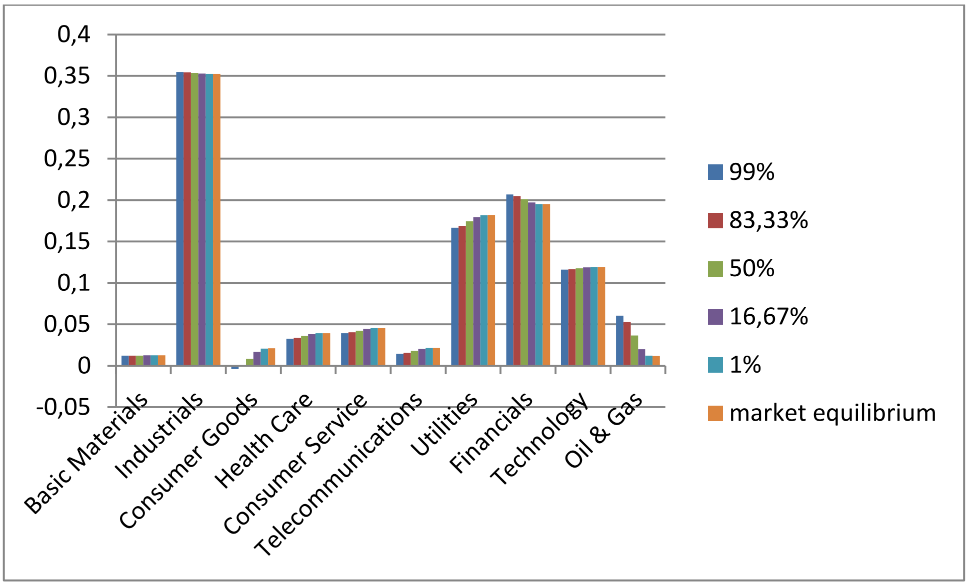

Figure 10: Portfolio weights for different settings of c in View 1

Changes in the level of confidence in View 1 lead not only to a different expected return for the Oil & Gas portfolio, but also to a different expected return of every other asset class, since each individual return is linked to the other returns via the covariance matrix (compare He and Litterman approach View 1). Thus, the weights of every industry portfolio change as well.

Figure 11 shows the weight changes for the Oil & Gas portfolio in greater detail.

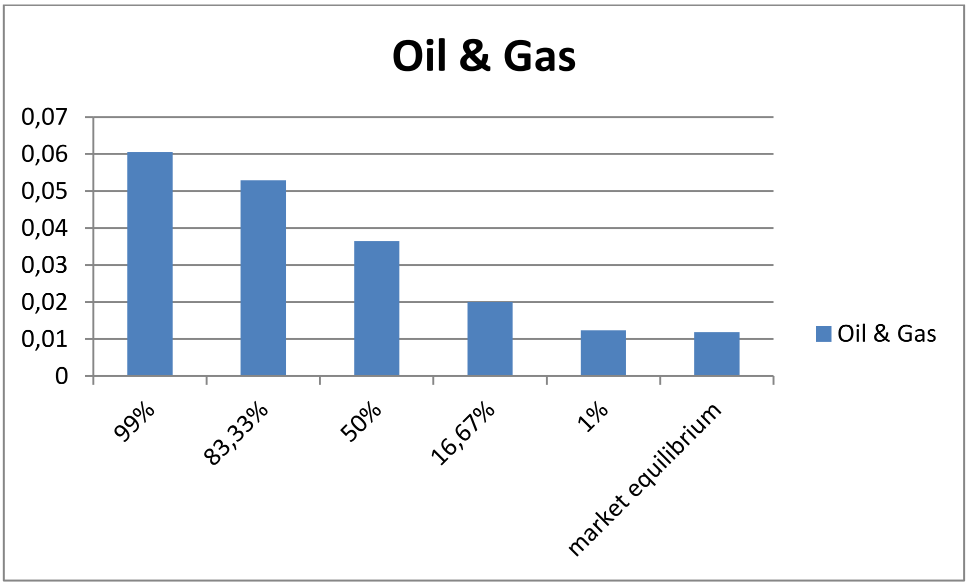

Figure 11: Weights of the Oil & Gas for different settings of c

As shown in Figure 11, different values of c lead to significant changes in the weighting of the Oil & Gas portfolio. An expected return of the Oil & Gas portfolio that is equal to the market equilibrium return (~7,3%) corresponds to a weight of about 0.012. In the case that the expected return of the Oil & Gas portfolio is almost completely dominated by the View 1 (8%), its weighting rises to almost 0.06.

Meucci's approach is not only appropriate for absolute views, but also for relative ones like View 3. As already mentioned, in View 3 the difference between the expected return of the Technology portfolio and the Utilities portfolio is of interest. The consequences of different settings of c for the posterior expected returns are illustrated in Figure 12.

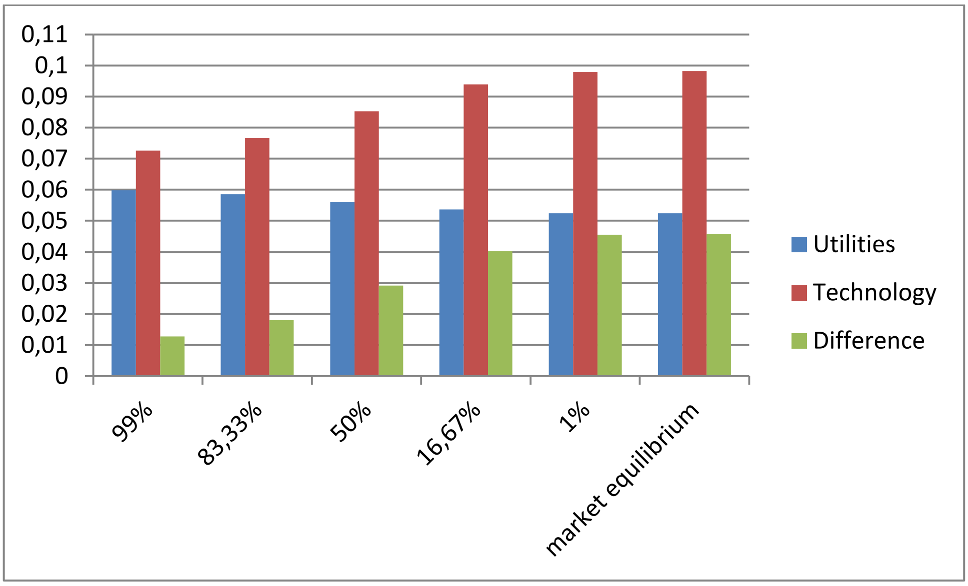

Figure 12: Expected return of the Technology portfolio and the Utilities portfolio for different settings of c

The impact of c in View 3 is in line with the effects of c in an absolute view. However the actual change of the expected return of each portfolio involved depends on the implied difference between the expected return of the Technology portfolio and the Utilities portfolio according to c. This has no consequences for $c \rightarrow 0$, but for increasing confidence in View 3 the expected returns change. This means the expected return of Technology decreases while the expected return of Utilities increases in order to reduce the difference between the expected returns to 125 basis points.

Figure 13 illustrates that the higher the confidence in the investor's view gets, the more weight is shifted from the Technology portfolio to the Utilities portfolio. For high values of c, the Technology portfolio even becomes a short position.

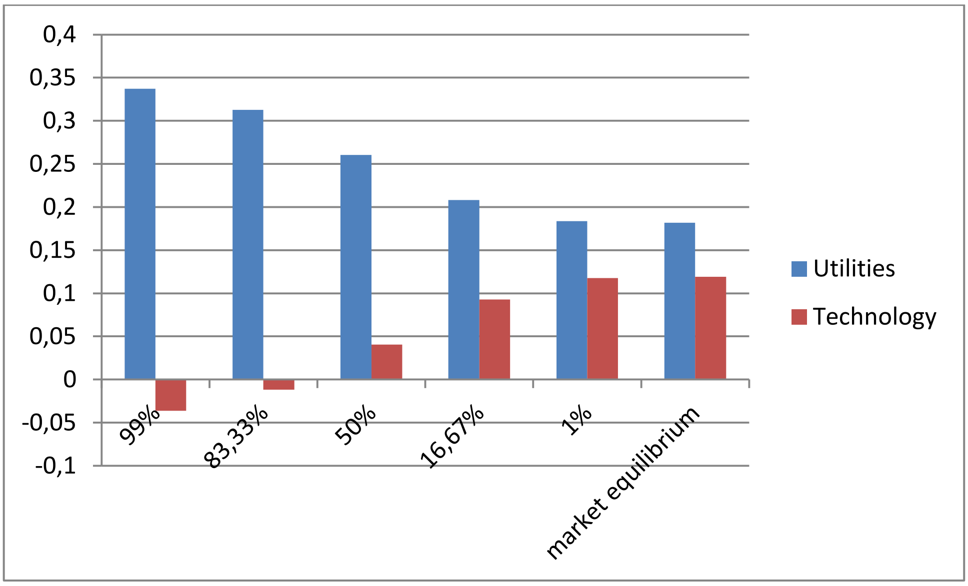

Figure 13: Weights of the Technology portfolio and the Utilities portfolio for different settings of c

8. Performance measurement of a Meucci Black-Litterman portfolio

Contrary to the previous chapter 7 that focused on asset allocation and on how the mechanics of the Black-Litterman model work, we will investigate in the following the actual performance that a Meucci Black-Litterman portfolio delivers. This performance will be evaluated by using a benchmark comparison based on empirical data. [72] The dataset for this purpose bases on the Austrian equity market. We will use the Austrian Traded Index ATX as a proxy for this market. The time period of interest is the 01.08.2017 till the 31.10.2017. For the calibration of the covariance matrix as well as the historical estimates of the expected returns, daily observations over five years are taken into account: from the 01.11.2012 until the 31.07.2017. The ATX consists of 20 assets, however due to adaptations made on the 27.10.2017, where the newly listed BAWAG GROUP AG entered the ATX, we consider for the performance test only 19 assets, as for the BAWAG GROUP AG is no statistically firm data available yet.

The benchmarks that the Black-Litterman portfolio has to compete with are: on the one hand the ATX itself, representing the market portfolio and on the other hand a mean variance portfolio that is calibrated with historical estimates of expected returns.

While the portfolio weights of the market portfolio are easily accessible via their representation in the ATX, the preparation of the mean variance portfolio requires the investigation of historical estimates of expected returns.

8.1. Historical estimates of expected returns

The determination of the expected returns for Markowitz's mean variance approach causes some issues: Not only are historical returns a questionable estimation for future returns per se (since history is not going to repeat), like Asness (2005) noted:

When it comes to forecasting the future, especially when valuations (and thus historical returns) are at extremes, the answers we get from looking at simple historical averages are bunk.[73]

But also the extraction of appropriate historical returns per se may cause some issues. There are mainly two different approaches to do so: by calculating the arithmetic average or by determining the geometric average. Both methods have their advantages and disadvantages.

[72] Samarakoon and Hasan (2006), p. 618
[73] Asness (2005), p. 37

The first common method to estimate a future return is to use the arithmetic mean of historical returns.[74] Compare to chapter 7., the arithmetic mean between the two points in time $t = \tau 1$ and $t = \tau 2$ is defined as the average of n period returns R_t:[75]

$$AM_{\tau 1,\tau 2} = \frac{1}{n} \cdot \sum_{t=\tau 1+1}^{\tau 2} R_t$$

Table 6 shows the arithmetic means of daily returns of the assets included in the ATX:

	Arithmetic means
EBS AV Equity	0,00068614
OMV AV Equity	0,0006343
VOE AV Equity	0,00066369
RBI AV Equity	0,00015739
ANDR AV Equity	0,00022289
BWO AV Equity	0,00082458
WIE AV Equity	0,00118962
IIA AV Equity	1,799E-05
CAI AV Equity	0,00080986
LNZ AV Equity	0,00083923
VER AV Equity	0,00012409
POST AV Equity	0,00041674
UQA AV Equity	0,00023192
TKA AV Equity	0,00051303
VIG AV Equity	-2,9407E-05
SBO AV Equity	7,9848E-05
SPI AV Equity	0,000905
ZAG AV Equity	0,00086774
AGR AV Equity	0,00033968

Table 6: arithmetic means

The use of arithmetic mean returns to model future returns is proposed in many finance textbooks and leads to an unbiased estimate of cumulative return if the arithmetic mean of the asset's stochastic rate of return is known.[76] However various studies prove that compounding at the arithmetic average historical return results in an upwardly biased forecast.[77] It is important to keep that in mind when using the historical arithmetic average, since this leads to overly optimistic forecasts.

[74] Campbell (2001), p. 3
[75] Aussenegg (2016), p. 52
[76] Jacquier (2003), p. 46. These books include e.g. Bodie, Kane, and Marcus (2002) pp.810-811, Brealy and Myers (2003), pp. 156-157 and Ross, Westerfield and Jaffe (2002) pp.232-233.
[77] Jacquier (2003), Hughson (2006)

An alternative approach is the use of the geometric mean. The geometric mean corresponds to the constant period based growth rate that leads to the same value as the $BHR_{\tau1,\tau2}$ over the same period.[78] The geometric mean is:

$$GM_{\tau1,\tau2} = [\prod_{t=\tau1+1}^{\tau2} (1 + R_t)]^{1/n}$$

Similarly, but expressed through the $BHR_{\tau1,\tau2}$:

$$GM_{\tau1,\tau2} = [1 + BHR_{\tau1,\tau2}]^{1/n} - 1$$

An investor should consider that the geometric average is unbiased indeed, but *only* in the special case where the sample period and the investment horizon are of equal length. In general, forecasts obtained by compounding at the geometric average will be biased downwards.[79] A convenient choice for future expected returns is therefore a return that lies between the arithmetic mean and the geometric mean.[80]

The geometric means of the daily returns of the 19 assets are depicted in table 7.

	Geometric means
EBS AV Equity	0,00046265
OMV AV Equity	0,00049656
VOE AV Equity	0,00049737
RBI AV Equity	-0,0001467
ANDR AV Equity	9,711E-05
BWO AV Equity	0,00051241
WIE AV Equity	0,00098244
IIA AV Equity	-0,00011721
CAI AV Equity	0,00071385
LNZ AV Equity	0,00067121
VER AV Equity	3,7089E-06
POST AV Equity	0,00034811
UQA AV Equity	0,00010196
TKA AV Equity	0,00040418
VIG AV Equity	-0,00015718
SBO AV Equity	-0,00014719
SPI AV Equity	0,00081287
ZAG AV Equity	0,00053749
AGR AV Equity	0,00025539

Table 7: geometric means

78 Aussenegg (2016), p. 38
79 Jacquier (2003), p.46
80 Jacquier (2003), p.46

Note that the described upwardly biased estimation errors of the arithmetic means are only pronounced over long time horizons. In our 3 month example we may use therefore arithmetic means as input parameters for the mean variance portfolio. This procedure is similar to the one presented in the previous chapter.

8.2. Risk free rate

As explained in chapter 6. the Black-Litterman requires *excess* returns as input parameters. Therefore the valuation of an appropriate risk free rate is critical.

In order to understand what makes an asset risk free, reconsider how risk is measured in investments. Investors that buy an asset have an expected return of that asset. However the actual return they make may differ a lot from this expected return. That is where the risk comes into play.[81] Based on Markowitz's portfolio theory, the variance in actual returns around the expected return represents risk in finance. Interestingly, the keynote of the risk free rate and our performance analysis is the same: the gap between expectations and actual returns.

A risk free asset must meet two requirements: no default risk and no reinvestment risk.[82]

How to identify and evaluate *in practice* such securities that have similar expected and actual returns? Frequently used proxies for such free risk rates are government bond yields.[83] Until 2008 they were characterized by low volatility and very predictable returns, just as expected and intended. However the financial crisis in 2008 and the followed Quantitative Easing programs (including government bond purchases) in the US, EU, UK and Japan significantly depressed the yields for government bonds and increased their volatility.[84] This causes some new issues: Are government bonds still a good proxy for risk free assets (due to their volatility)? How to deal with negative risk free returns?

Generally, the usage of a de facto default-free Government bonds as a practical compromise is still legitimate, however there are some adaptations and alternative options that may be considered:[85]

- the usage of an average government bond yield over a certain period (to smooth out short term volatility)
- the usage of a government bond yield from another de facto default free country with less volatility (e.g. German instead of Italian bonds)

[81] Damodaran (2008), p.3
[82] Damodaran (2008), p.6
[83] EY (2015), p.1 and Aussenegg (2016), p.153
[84] EY (2015), p.1
[85] EY (2015), p.10

Although the theoretical and practical impact of negative risk free rates is a critical question, the impact on research appears to be surprisingly little. Especially by considering that negative government bond yields give rise to a more complex issue than only increased volatility.

From a conventional point of view, nominal interest rates cannot be negative, as one has always the option to hold money as cash. However, in practice holding cash does not come for free: Storing cash requires security and therefore costs. Furthermore, using cash for transactions may be expensive and cumbersome. In effect, the lower boundary for interest rates is shifted below zero and therefore negative but it still exists (meaning that negative rates cannot be below that boundary, since at a certain point holding cash is cheaper).[86]

Negative risk free rates have some implementations and consequences one should consider:

- Negative interest rates are incompatible with a healthy and growing economy.[87]
- Negative risk free rates *increase* ceteris paribus the excess returns - which appears to be rather odd.
- The (unrealistic) CAPM assumption that an investor can borrow and lend at the risk free rate leads to the absurd constellation that lending money is more expensive than borrowing money.
- Investors that desire a fixed income have to take a higher risk.
- Not least, negative risk free rates are an opening for digital currencies.[88]

But how to deal with negative risk free rates appropriately? Damodaran (2016) suggests the following options:[89]

- Switch currencies: If the risk free rate of the currency you are using is negative, switch to a different one.
- Normalize risk free rates: Replace the negative risk free rate by a normal risk free rate (note that normalization is in the eye of the beholder and therefore always subjective).
- Stay with the negative risk free rate

In order to depict the consequences of the different options with the negative risk free rate, we will calibrate two mean variance portfolios with historical *excess* returns. The various assets' returns stay similar (i.e. not the excess returns), only the way how the risk free rate is taken into account changes,

[86] http://aswathdamodaran.blogspot.co.at/2016/03/negative-interest-rates-unreal.html, 6.5.2018
[87] Damodaran (2016), p. 15
[88] Damodaran (2016), p. 15
[89] Damodaran (2016), p. 14

resulting in different excess returns. In both cases the German government bond yields represents the proxy for risk free rates.

The first asset allocation is done in the following way: the assets' returns are reduced by the risk free rate without any adoptions - meaning that risk free rates can be negative.

$$AM_{excess,\tau1,\tau2} = \frac{1}{n} \cdot \sum_{t=\tau1+1}^{\tau2} (R_t - R_{f,t})$$

The second portfolio is calculated like the first portfolio except for the negative risk free rates - they are substituted by $R_f = 0$ if the risk free rate is negative. Thus $R_f \geq 0$.

$$AM_{excess,\tau1,\tau2,Rf\geq0} = \frac{1}{n} \cdot \sum_{t=\tau1+1}^{\tau2} (R_t - R_{f,t})$$

Both cases benefit from the usage of an average of the risk free rate, in order to smooth out the volatility:

$$AM_{excess,\tau1,\tau2} = \frac{1}{n} \cdot \sum_{t=\tau1+1}^{\tau2} (R_t - R_{f,t}) = (\frac{1}{n} \cdot \sum_{t=\tau1+1}^{\tau2} R_t) - (\frac{1}{n} \cdot \sum_{t=\tau1+1}^{\tau2} R_{f,t})$$

The obtained arithmetic mean excess returns vary significantly. The excess returns of the second portfolio are generally lower than the ones of the first portfolio, with some returns turned into negative. This is because negative risk free rates are *added* to the company's returns according to the definition of excess returns:

$$R_{excess,i,t} = R_{i,t} - R_{f,t}$$

Table 8 compares the impact of the different risk free rates on the asset's excess returns:

	$\boldsymbol{R_f = R_f}$	$\boldsymbol{R_f = \max(R_f, 0)}$
EBS AV Equity	0,00068614	0,00012151
OMV AV Equity	0,0006343	6,9667E-05
VOE AV Equity	0,00066369	9,9058E-05
RBI AV Equity	0,00015739	-0,00040724
ANDR AV Equity	0,00022289	-0,00034174
BWO AV Equity	0,00082458	0,00025922
WIE AV Equity	0,00118962	0,00062499
IIA AV Equity	1,799E-05	-0,00054664
CAI AV Equity	0,00080986	0,00024523
LNZ AV Equity	0,00083923	0,0002746

VER AV Equity	0,00012409	-0,00044054
POST AV Equity	0,00041674	-0,00014789
UQA AV Equity	0,00023192	-0,00033271
TKA AV Equity	0,00051303	-5,1595E-05
VIG AV Equity	-2,9407E-05	-0,00059404
SBO AV Equity	7,9848E-05	-0,00048478
SPI AV Equity	0,000905	0,00034037
ZAG AV Equity	0,00086774	0,00030311
AGR AV Equity	0,00033968	-0,00022495

Table 8: Comparison of excess returns

These two different excess returns that only differ in the way the risk free return is taken into account lead to different portfolio weights of an optimized unconstrained mean variance portfolio. Figure 14 and 15 illustrate the difference.

Portfolio 1:

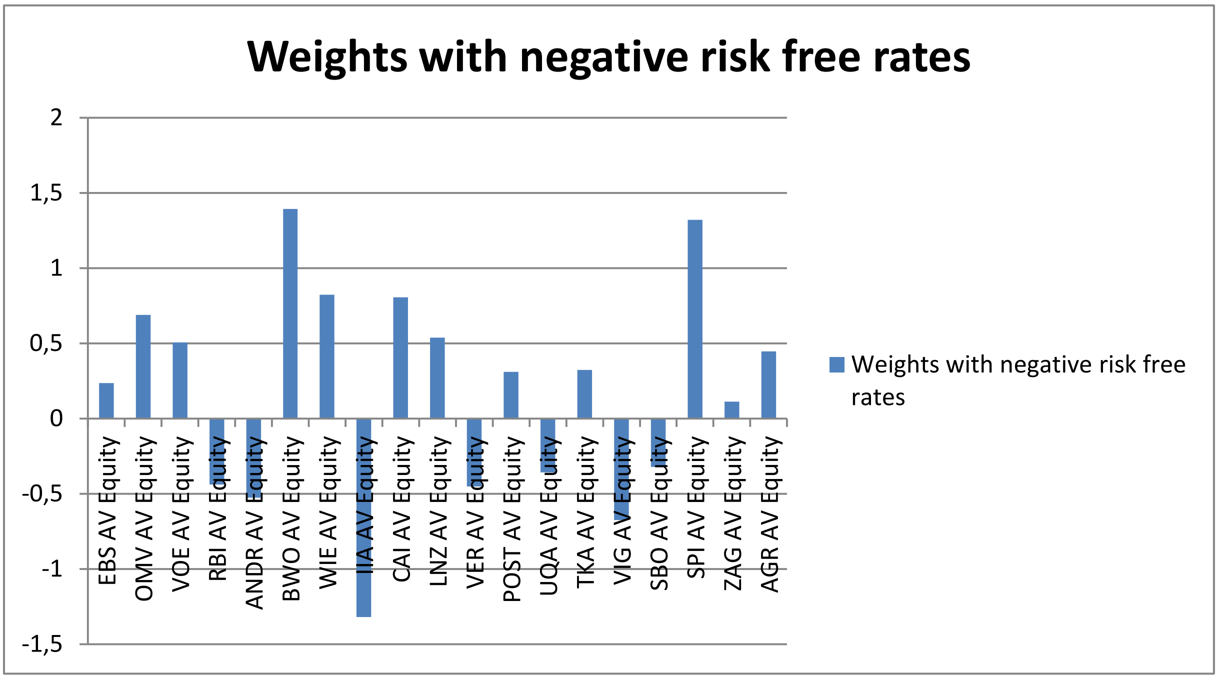

Figure 14: Portfolio weights with negative risk free rates

Portfolio 2:

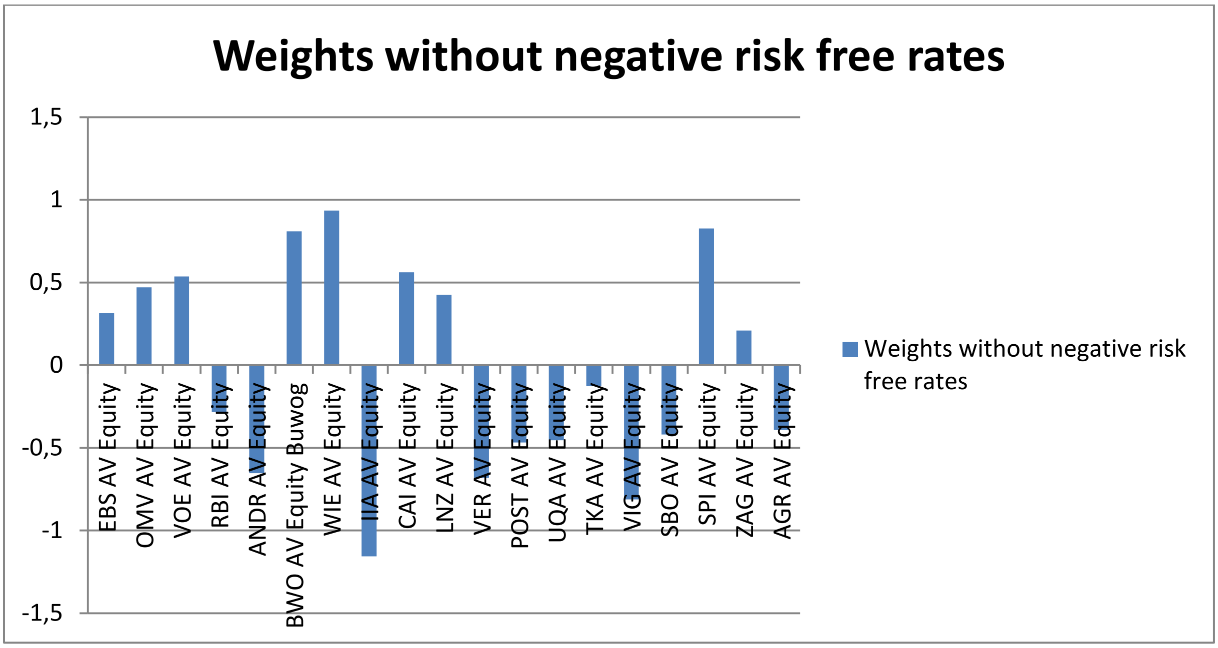

Figure 15: Portfolio weights without negative risk free rates

This is another example that highlights the high sensitivity of the mean variance approach. Even the assumed minor decision to consider *small* negative risk rates or not induces rather different portfolio weights.

8.3. The covariance matrix

Deriving a feasible covariance matrix that is real, symmetric and positive-semidefinite[90] may also cause some practical problems. Although the issue may be solved by using well-established statistical techniques, real-world covariance matrices often appear to be noisy, unavailable or inappropriate. E.g. a small numbers of outliners can easily destroy a whole sample.[91] If an investor finds himself with a bad behaving covariance matrix, several solutions have been proposed to tackle this problem (see e.g. Rebnato (1999), Kupiec (1998), Finger (1997), Brooks (1998) et al).

In our case, the covariance matrix bases on the historical (excess) returns and appears to be plausible and feasible. Table 9 shows the covariance matrix corresponding to the excess returns of the ATX assets:

[90] Rebonato (1999) p. 11, Rebonato's example refers to a correlation matrix, which must meet the same requirements, since $cov(R_A, R_B) = \rho_{A,B} \cdot \sigma_A \cdot \sigma_B$

[91] Rebonato (1999), p. 1

	EBS	OMV	VOE	RBI	ANDR	BWO	WIE	IIA	CAI	LNZ	VER	POST	UQA	TKA	VIG	SBO	SPI	ZAG	AGR
EBS	4E-04	1E-04	2E-04	3E-04	1E-04	6E-05	2E-04	1E-04	1E-04	1E-04	9E-05	7E-05	1E-04	9E-05	1E-04	8E-05	6E-05	2E-04	4E-05
OMV	1E-04	3E-04	2E-04	2E-04	9E-05	4E-05	1E-04	1E-04	7E-05	8E-05	9E-05	5E-05	9E-05	5E-05	8E-05	1E-04	5E-05	1E-04	2E-05
VOE	2E-04	2E-04	3E-04	2E-04	1E-04	5E-05	1E-04	1E-04	8E-05	1E-04	9E-05	6E-05	1E-04	7E-05	1E-04	1E-04	5E-05	1E-04	3E-05
RBI	3E-04	2E-04	2E-04	6E-04	1E-04	6E-05	2E-04	2E-04	1E-04	1E-04	1E-04	9E-05	2E-04	8E-05	2E-04	1E-04	8E-05	2E-04	4E-05
ANDR	1E-04	9E-05	1E-04	1E-04	2E-04	5E-05	1E-04	1E-04	7E-05	8E-05	8E-05	6E-05	8E-05	5E-05	9E-05	8E-05	5E-05	1E-04	3E-05
BWO	6E-05	4E-05	5E-05	6E-05	5E-05	2E-04	7E-05	8E-05	7E-05	4E-05	4E-05	4E-05	6E-05	3E-05	5E-05	4E-05	4E-05	8E-05	2E-05
WIE	2E-04	1E-04	1E-04	2E-04	1E-04	7E-05	4E-04	1E-04	9E-05	1E-04	9E-05	7E-05	1E-04	8E-05	1E-04	1E-04	7E-05	2E-04	3E-05
IIA	1E-04	1E-04	1E-04	2E-04	1E-04	8E-05	1E-04	3E-04	9E-05	8E-05	9E-05	6E-05	1E-04	6E-05	1E-04	1E-04	8E-05	1E-04	3E-05
CAI	1E-04	7E-05	8E-05	1E-04	7E-05	7E-05	9E-05	9E-05	2E-04	5E-05	5E-05	5E-05	6E-05	4E-05	7E-05	6E-05	6E-05	8E-05	2E-05
LNZ	1E-04	8E-05	1E-04	1E-04	8E-05	4E-05	1E-04	8E-05	5E-05	3E-04	6E-05	5E-05	7E-05	6E-05	8E-05	8E-05	5E-05	1E-04	3E-05
VER	9E-05	9E-05	9E-05	1E-04	8E-05	4E-05	9E-05	9E-05	5E-05	6E-05	2E-04	4E-05	6E-05	5E-05	7E-05	7E-05	4E-05	9E-05	3E-05
POST	7E-05	5E-05	6E-05	9E-05	6E-05	4E-05	7E-05	6E-05	5E-05	5E-05	4E-05	1E-04	6E-05	3E-05	6E-05	5E-05	4E-05	9E-05	2E-05
UQA	1E-04	9E-05	1E-04	2E-04	8E-05	6E-05	1E-04	1E-04	6E-05	7E-05	6E-05	6E-05	3E-04	6E-05	1E-04	9E-05	6E-05	1E-04	3E-05
TKA	9E-05	5E-05	7E-05	8E-05	5E-05	3E-05	8E-05	6E-05	4E-05	6E-05	5E-05	3E-05	6E-05	2E-04	6E-05	2E-05	3E-05	6E-05	2E-05
VIG	1E-04	8E-05	1E-04	2E-04	9E-05	5E-05	1E-04	1E-04	7E-05	8E-05	7E-05	6E-05	1E-04	6E-05	3E-04	1E-04	5E-05	1E-04	3E-05
SBO	8E-05	1E-04	1E-04	1E-04	8E-05	4E-05	1E-04	1E-04	6E-05	8E-05	7E-05	5E-05	9E-05	2E-05	1E-04	5E-04	5E-05	1E-04	3E-05
SPI	6E-05	5E-05	5E-05	8E-05	5E-05	4E-05	7E-05	8E-05	6E-05	5E-05	4E-05	4E-05	6E-05	3E-05	5E-05	5E-05	2E-04	7E-05	2E-05
ZAG	2E-04	1E-04	1E-04	2E-04	1E-04	8E-05	2E-04	1E-04	8E-05	1E-04	9E-05	9E-05	1E-04	6E-05	1E-04	1E-04	7E-05	6E-04	4E-05
AGR	4E-05	2E-05	3E-05	4E-05	3E-05	2E-05	3E-05	3E-05	2E-05	3E-05	3E-05	2E-05	3E-05	2E-05	3E-05	3E-05	2E-05	4E-05	2E-04

Table 9: Covariance matrix

8.4. Asset Allocations

With the historically derived expected returns and covariance matrix, as well as the market capitalizations of the relevant assets from the 31.07.2017, the asset allocations for the traditional mean variance approach and the market portfolio (i.e. the weightings according to the ATX) are as follows:

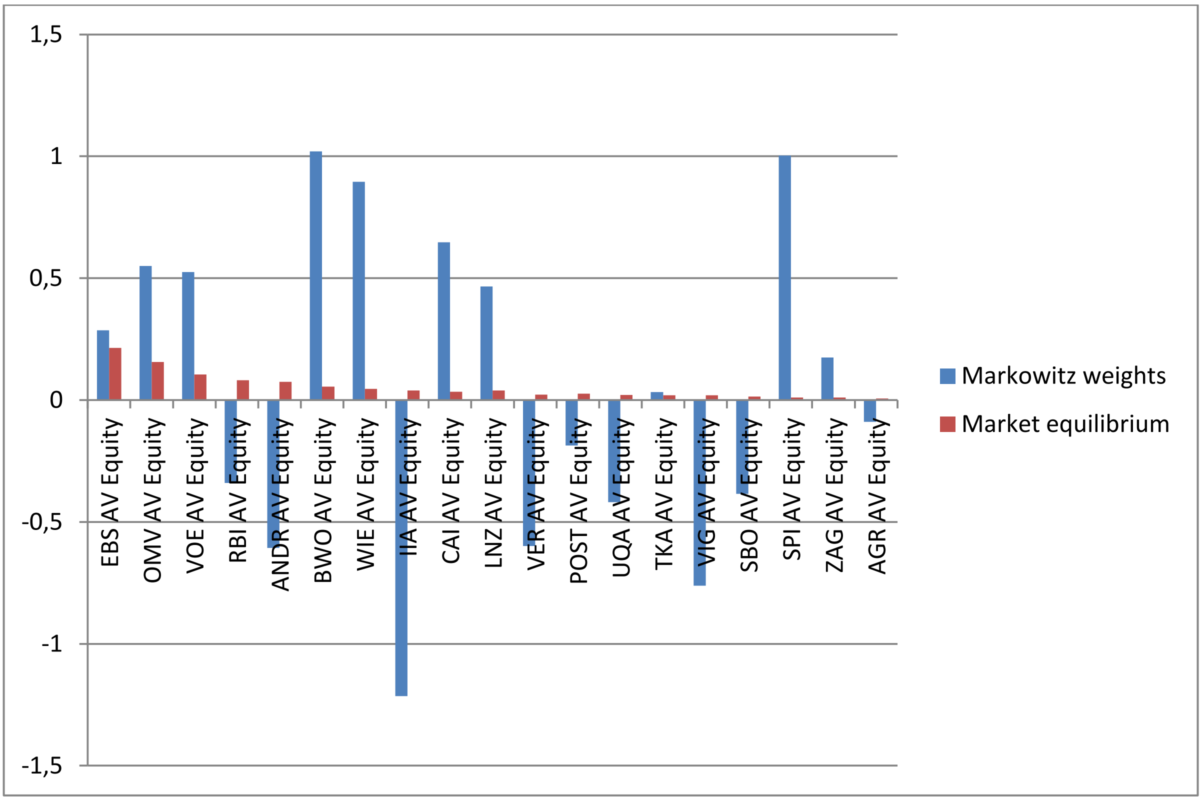

Figure 16: Comparison of portfolio weights

The allocation of the mean variance portfolio uses the arithmetic mean returns as the historical estimate of expected returns and does take negative risk free rates into account.

Figure 16 illustrates that the weights of the mean variance portfolio and the market equilibrium portfolio are completely different. This certainly has consequences on the performance: we can already predict that an extraordinary negative performance of IIA as well as positive performances of BWO, WIE and SPI will have a positive impact on the Markowitz portfolio. The market portfolio on the other hand appears to be overall robust.

As already explained, the market equilibrium portfolio is identical to a Black-Litterman portfolio if no subjective views are set. In contrast to chapter 7 that suggested that the portfolio with the highest utility was always held by the most optimistic investor (since the investor with the most optimistic views holds the portfolio with the highest expected return for a given level of risk), the performance test requires the portfolios "to prove" their promised returns. In other words: as the further performance test will demonstrate the best performing portfolio is (of course) not the one with the highest expected return ex ante, but the one with the highest return ex post.

With that in mind, the performance measure emphasizes how views impact on the portfolio performance. To highlight the importance of the accuracy of the predictions, the performance of two Black-Litterman portfolios with different views is considered:

Views Black-Litterman portfolio 1

View 1: The EBS AV (Erste Bank) will have an excess return of 0,08% instead of the implied market return of 0,06%.

View 2: LNZ AV (Lenzing) will outperform the OMV AV by 0,02%. The implied market assumption is that OMV will outperform Lenzing by 0,01%).

View 3: IIA AV (Immofinanz) will have an excess return of 0,06% instead of 0,04%.

View 4: VOE AV (Voest) will outperform ZAG AV (Zumtobel) by 0,02% instead of only 0,01% which is the corresponding implied market assumption.

Views Black-Litterman portfolio 2

In contrast to the Black-Litterman portfolio 1, the portfolio 2 uses only absolute and no relative views. This is just due to simplicity reasons: basically one could also use relative views only to obtain the same allocations.

To highlight the performance and stability as well as the possibilities and limitations of the Black-Litterman model, portfolio 2 will be calibrated rather extremely: We will have views on *all* 19 assets that differ from the implied market returns. The 19 absolute views are depicted in the following table:

	Views	**Implied market returns**	**Difference**
EBS AV Equity	0,000632467	0,00062238	1,00869E-05
OMV AV Equity	0,001036291	0,000407862	0,000628429
VOE AV Equity	0,001361233	0,00046874	0,000892493
RBI AV Equity	0,002654054	0,00064268	0,002011374
ANDR AV Equity	-0,001089418	0,000329129	-0,001418547
BWO AV Equity	0,000257342	0,000177117	8,02245E-05
WIE AV Equity	0,001804209	0,000423092	0,001381117
IIA AV Equity	0,000715306	0,000364575	0,000350731

CAI AV Equity	0,00166077	0,000248569	0,0014122
LNZ AV Equity	-0,004073601	0,000294508	-0,004368109
VER AV Equity	0,003245697	0,000256163	0,002989533
POST AV Equity	-0,000407192	0,000187203	-0,000594395
UQA AV Equity	-9,76779E-06	0,000322615	-0,000332383
TKA AV Equity	0,000698704	0,000201371	0,000497332
VIG AV Equity	-0,000303497	0,000330207	-0,000633704
SBO AV Equity	0,003302429	0,000308681	0,002993748
SPI AV Equity	0,002624089	0,000170828	0,002453261
ZAG AV Equity	-0,002082167	0,000391664	-0,00247383
AGR AV Equity	-0,001202869	9,02924E-05	-0,001293162

Table 10: Comparison of views and implied market returns

The corresponding parameters to the views expressed are:

$$\boldsymbol{P_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{P_2} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}_{19x19}$$

$$\boldsymbol{Q_1} = \begin{pmatrix} 0.08 \\ 0.02 \\ 0.06 \\ 0.02 \end{pmatrix} \cdot 10^{-2}$$

$$\boldsymbol{Q_2} = \begin{pmatrix} 0.06 \\ 0.1 \\ 0.14 \\ 0.27 \\ -0.11 \\ 0.03 \\ 0.18 \\ 0.07 \\ 0.17 \\ -0.41 \\ 0.32 \\ -0.04 \\ 0 \\ 0.07 \\ -0.03 \\ 0.33 \\ 0.26 \\ -0.21 \\ -0.12 \end{pmatrix} \cdot 10^{-2}$$

To point out the impact of the confidence on the views, the Black-Litterman portfolio 2 is split into portfolio 2a and portfolio 2b:

The overall confidence in Portfolio 2a's views is 16,67%, meaning that *all* views receive the same 16,67% of confidence, while the implied market returns are trusted with 83,33%. Why do we use such odd values like 16,67% and 83,33%? This is due to simplicity reasons: For the case $c = 16{,}67\%$, the term $\left(\frac{1}{c} - 1\right) = \frac{1}{16{,}67\%} - 1$ simplifies to 5.

Portfolio 2b on the other hand weights the confidence in the subjective views with 83,33%. Thus the confidence in the implied market returns is only 16,67%. With $c = 83{,}33\%$ follows $\frac{1}{83{,}33\%} - 1 = 0.2$.

The views of the Black-Litterman portfolio 1 are trusted as much as the implied market information, i.e. $c = 50\%$.

The corresponding $\boldsymbol{\Omega}$ matrices are:

$$\boldsymbol{\Omega_1} = \begin{pmatrix} 4{,}443 \cdot 10^{-4} & 0 & 0 & 0 \\ 0 & 4{,}56 \cdot 10^{-4} & 0 & 0 \\ 0 & 0 & 2{,}692 \cdot 10^{-4} & 0 \\ 0 & 0 & 0 & 7{,}1714 \cdot 10^{-4} \end{pmatrix}$$

$$\boldsymbol{\Omega_{2a}} = 5 \cdot \begin{pmatrix} 4{,}433 \cdot 10^{-4} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1{,}6798 \cdot 10^{-4} \end{pmatrix}_{19x19}$$

$$\boldsymbol{\Omega_{2b}} = 0{,}2 \cdot \begin{pmatrix} 4{,}433 \cdot 10^{-4} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1{,}6798 \cdot 10^{-4} \end{pmatrix}_{19x19}$$

Applying these views to the Black-Litterman formula leads to the following asset allocations of the Black-Litterman portfolios 1 and 2a:

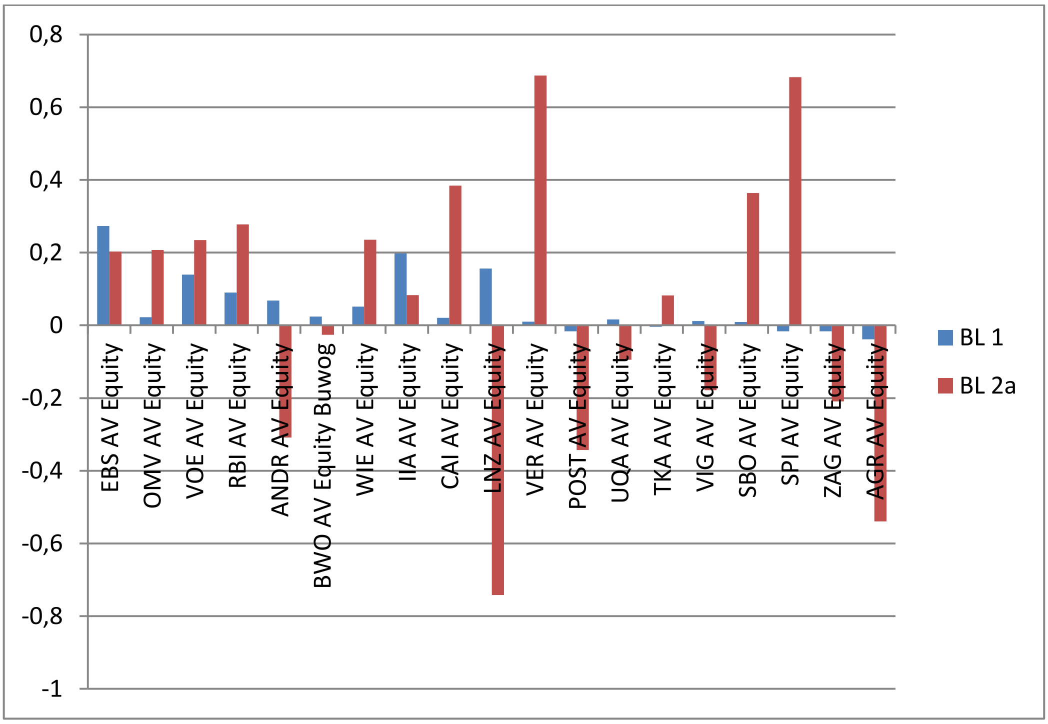

Figure 17: Comparison of BL 1 and BL 2a

Figure 17 shows that the Black-Litterman portfolio 1 (BL 1) is way more balanced than the Black-Litterman portfolio 2a (BL 2a). This is not surprising since BL 1 received only 4 views that differ from the implied market return, while in BL 2a views were expressed for all 19 assets. Although the confidence in the views of BL 2a are relatively weak (even weaker than the one of BL 1), the high difference between the implied market returns and the views causes significant changes to the portfolio weights. The underlying market equilibrium allocation is hardly determinable.

Figure 18 compares the Black-Litterman portfolio 2a (BL 2a) to the Black-Litterman portfolio 2b (BL 2b). Although BL 2a looked already extreme and highly leveraged in figure 17, one can see that the increased confidence in the views (83,33% in BL 2b compared to 16,67% in BL 2a) pushes the weights to a wholly new level. In direct comparison, BL 2a appears to be the robust allocation while BL 2b does not differ much from an ordinary mean variance portfolio.

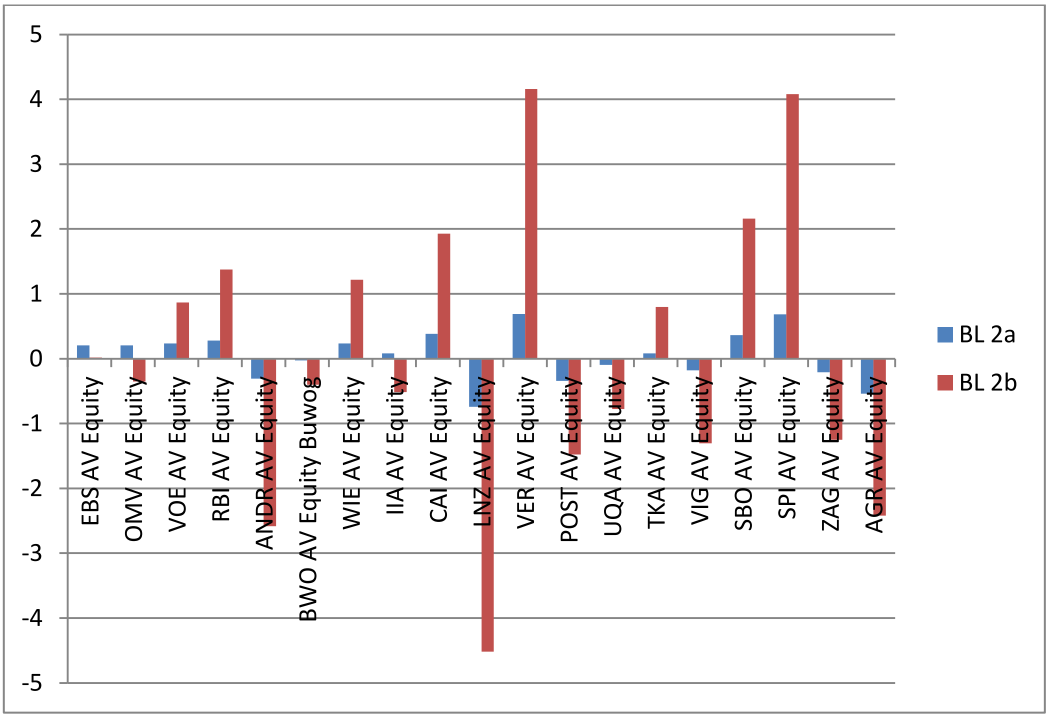

Figure 18: Comparison of BL 2a and BL 2b

Calibrating a Black-Litterman portfolio in such an extreme manner may seem strange and to miss the point of the Black-Litterman model. However, we will provide evidence that this calibration offers important insights concerning the performance and the limits of the Black-Litterman model.

8.5. Three months performance

So far, only *expected* returns have been considered. This applies irrespectively whether in the case of historical estimates for expected excess returns in the mean variance approach or in the case of the implied expected excess returns of the market portfolio and the Black-Litterman model. Now the actual excess returns come into play. The 3-month Buy and Hold returns of the 19 assets are depicted in table 11.

	BHR
EBS AV Equity	4,26%
OMV AV Equity	7,08%
VOE AV Equity	9,39%
RBI AV Equity	19,12%
ANDR AV Equity	-6,94%
BWO AV Equity	1,71%
WIE AV Equity	12,63%
IIA AV Equity	4,83%
CAI AV Equity	11,57%

LNZ AV Equity	-23,62%
VER AV Equity	23,85%
POST AV Equity	-2,65%
UQA AV Equity	-0,06%
TKA AV Equity	4,72%
VIG AV Equity	-1,98%
SBO AV Equity	24,31%
SPI AV Equity	18,88%
ZAG AV Equity	-12,85%
AGR AV Equity	-7,64%

Table 11: Buy-and-Hold returns

As one can see the 19 excess returns are rather different: the Schoeller-Bleckmann AG (SBO) had the highest BHR with more than 24% while the Lenzing AG dropped by almost 24% in 3 months. Table 12 shows the corresponding daily geometric mean returns.

EBS AV Equity	0,06%
OMV AV Equity	0,10%
VOE AV Equity	0,14%
RBI AV Equity	0,27%
ANDR AV Equity	-0,11%
BWO AV Equity	0,03%
WIE AV Equity	0,18%
IIA AV Equity	0,07%
CAI AV Equity	0,17%
LNZ AV Equity	-0,41%
VER AV Equity	0,32%
POST AV Equity	-0,04%
UQA AV Equity	0,00%
TKA AV Equity	0,07%
VIG AV Equity	-0,03%
SBO AV Equity	0,33%
SPI AV Equity	0,26%
ZAG AV Equity	-0,21%
AGR AV Equity	-0,12%

Table 12: geometric means

The geometric means are particularly of interest, since they enable to compare the *expected* daily excess returns to the daily excess returns that *actually* took place. Per definition, the geometric mean represents the constant growth rate for a certain time period (3 months in this case).

However, the actual ex-post identified GMs partially significantly differ from the historically estimated expected returns (calculated with a risk free rate that may be negative). The differences are depicted in the table 13 below.

	estimated GM	actual GM	Difference
EBS AV Equity	0,05%	0,06%	0,02%
OMV AV Equity	0,05%	0,10%	0,05%
VOE AV Equity	0,05%	0,14%	0,09%
RBI AV Equity	-0,01%	0,27%	0,28%
ANDR AV Equity	0,01%	-0,11%	-0,12%
BWO AV Equity	0,05%	0,03%	-0,03%
WIE AV Equity	0,10%	0,18%	0,08%
IIA AV Equity	-0,01%	0,07%	0,08%
CAI AV Equity	0,07%	0,17%	0,09%
LNZ AV Equity	0,07%	-0,41%	-0,47%
VER AV Equity	0,00%	0,32%	0,32%
POST AV Equity	0,03%	-0,04%	-0,08%
UQA AV Equity	0,01%	0,00%	-0,01%
TKA AV Equity	0,04%	0,07%	0,03%
VIG AV Equity	-0,02%	-0,03%	-0,01%
SBO AV Equity	-0,01%	0,33%	0,34%
SPI AV Equity	0,08%	0,26%	0,18%
ZAG AV Equity	0,05%	-0,21%	-0,26%
AGR AV Equity	0,03%	-0,12%	-0,15%

Table 13: Differences between the estimated GM and the actual GM

Eleven companies performed better than estimated and eight achieved a weaker performance than expected. The highest difference between the estimated and the actual GM occurs at Raiffeisen Bank International (RBI +0,28%), Lenzing (LNZ -0,47%), Verbund (VER +0,32), Schoeller-Bleckmann (SBO +0,34%) and Zumtobel (ZAG +0,26%). The other historical estimates are (more or less) appropriate. The high amount of appropriate historical estimates is supposed to have a positive effect on the performance of the mean variance portfolio. The next chapter investigates if this is the case.

8.6. Portfolio performance

The performance of a portfolio consisting of N assets may be calculated as the sum of the weighted (in our case daily) excess returns R_i. The weighing is effected by multiplying the asset i's portfolio weight x_i by the individual asset's excess return R_i:

$$R_P = \sum_{i=1}^{N} x_i \cdot R_i$$

with the budget constraint

$$\sum_{i=1}^{N} x_i = 1$$

Thus, the overall return of a portfolio is a linear combination of the returns of the individual components in that portfolio.[92]

Note that the asset's portfolio weights x_i may differ over time if portfolio adjustments are done. However we are interested in the Buy and Hold returns and therefore consider only static portfolios, meaning that no portfolio weight changes take place, i.e. $x_i = constant$.

The figure below shows the 3-months performance of the mean variance, the market and the Black-Litterman portfolio 1:

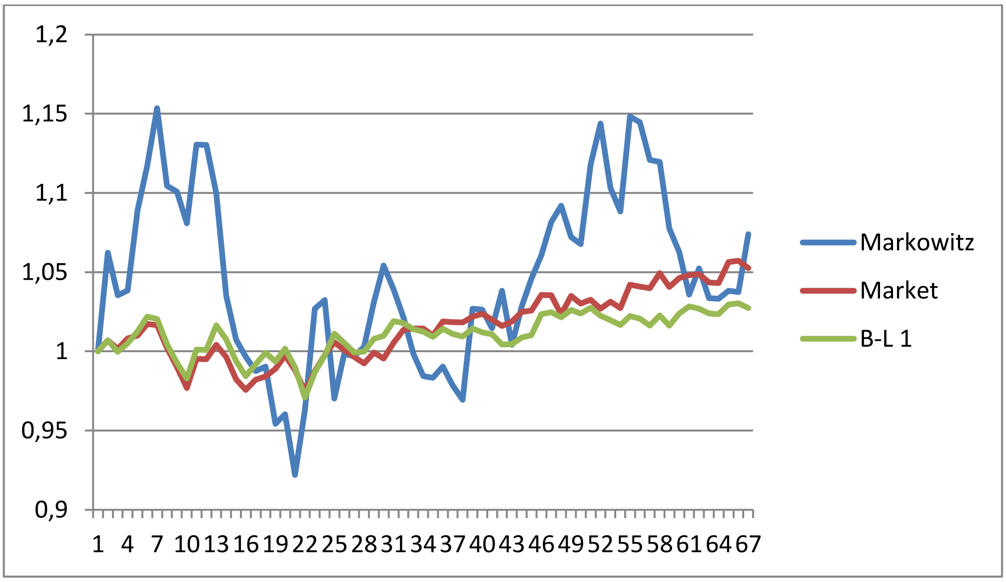

Figure 19: 3 months performance of the Markowitz, the market and the BL 1 portfolio

The extreme portfolio weights of the Markowitz approach cause pronounced ups and downs compared to the market portfolio and the Black-Litterman portfolio. The high volatility is on the one hand not surprising because of the rather extreme Markowitz weights (as depicted in figure 16). On the other hand it is still remarkable since the main motivation of the mean variance approach is to provide a portfolio with a *minimum* of risk (i.e. volatility) and a maximum of return according to the investor's risk-aversion.

The market portfolio outperformed the Black-Litterman portfolio 1 (BL 1) by about two percent. BL 1 got four moderate views, which turned out to have a worse performance than the market. Obviously the views we expressed were less appropriate than the implied market assumptions. However it may clearly be seen that the overall performance development of the market and the BL 1 portfolio is almost similar and characterized by a high correlation. If the subjective views are further away from the actual returns than the implied market returns, then the market portfolio outperforms the Black-

[92] Aussenegg (2016), p.98

Litterman portfolio. Otherwise if the subjective views are closer to the actual returns than the implied market returns, then the Black-Litterman portfolio outperforms the market. Figure 20 emphasizes this statement.

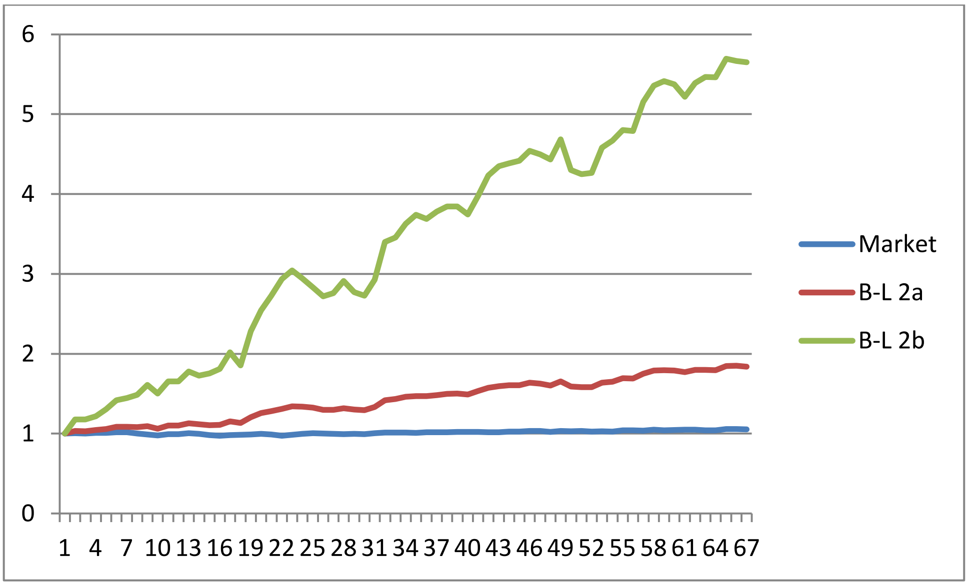

Figure 20: 3 months performance of the market, the BL 2a and the BL 2b portfolio

Figure 20 shows that both, the high and the low trusted, Black-Litterman portfolios outperform the market portfolio. In direct comparison, the market's performance appears to be almost zero.

However that did not happen by chance but on purpose: In order to show the opportunities and limitations of the Meucci Black-Litterman model, the BL portfolio 2's views are identical to the actual returns (compare geometric means of actual returns from table 12 with $\boldsymbol{Q_2}$), meaning that the views were set up with the parameters that actually took place, which in turn delivered an asset allocation that relies to 83% on the market and to 17% on the "right" predictions et vice versa (for BL 2a and BL 2b).

Just as one expects the key to the highest portfolio return lies in the accuracy of the views. The closer the predicted return of any asset is to its actual return, the better is the portfolio's performance.

The comparison of BL 2a and BL 2b illustrates two issues: The first is that an increase of the confidence in the subjective views leads to an increase in performance. Figure 21 illustrates the relationship between confidence and performance. The x-axis depicts different settings of c for the Black-Litterman portfolio 2, representing the confidence in the investor's views and the y-axis shows the associated Black-Litterman portfolio 2 performance.

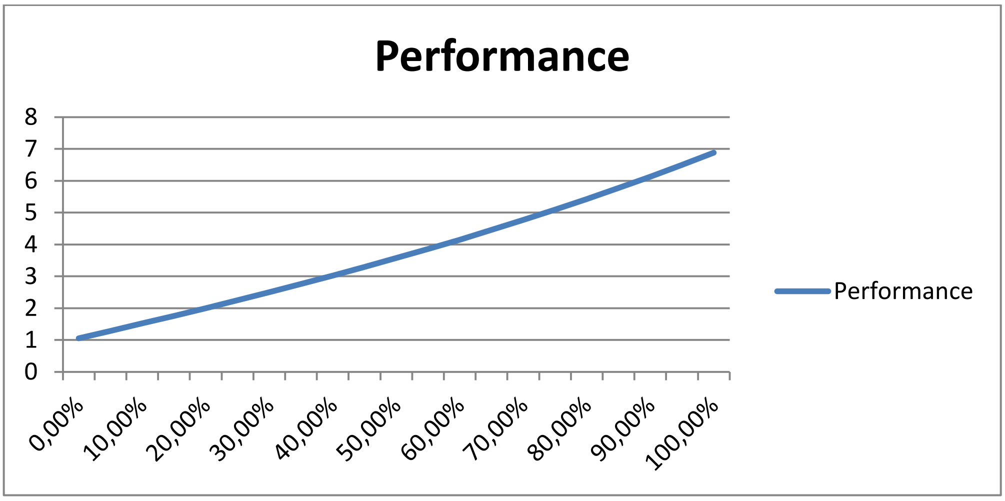

Figure 21: Performance of the BL 2 portfolio for different settings of c

This behavior corresponds to our intuition of Meucci's weighting process: The higher the confidence in the correct subjective (i.e. actual) returns, the higher the portfolio's performance. Note that the relation between confidence and performance is (almost) linear. In the following chapter "8.9. The Mahalanobis distance", we will have a look at this issue from a different angle.

The other comment we can make is that the higher the confidence in the subjective views, the more the portfolio performance reminds of an ordinary mean variance portfolio performance with a higher volatility. In order to examine this behavior we introduce another portfolio: A new mean variance (Markowitz) portfolio that uses the views of the BL portfolios 2 (i.e. $\boldsymbol{Q_2}$) as the input vector of the expected returns. The weights we receive are identical to the ones we receive if we calibrate the Black-Litterman portfolio 2 with 100% confidence in the subjective views. They are depicted in Figure 22 below.

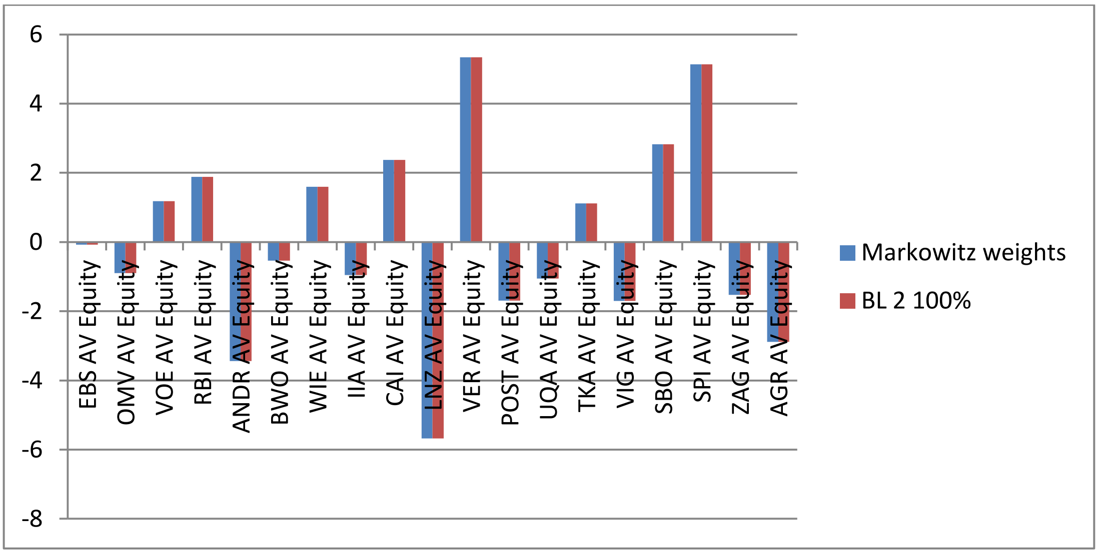

Figure 22: new Markowitz portfolio and Black-Litterman 2 portfolio with 100% confidence weights

The fact that a mean variance portfolio and a Black-Litterman portfolio with 100% confidence in the subjective views are identical is actually not surprising since the case $c \to 100\%$ leads to an infinitely peaked distribution of the views which means that the investor is trusted completely (compare chapter 7.4.2.) – just as in the mean variance approach.

Figure 23 illustrates the performance of the Black-Litterman portfolio 2 with four different settings for c: the market portfolio ($c \to 0$), the BL 2a portfolio ($c = 16{,}67\%$), the BL 2b portfolio ($c = 83{,}33\%$) and the new Markowitz portfolio ($c \to 1$).

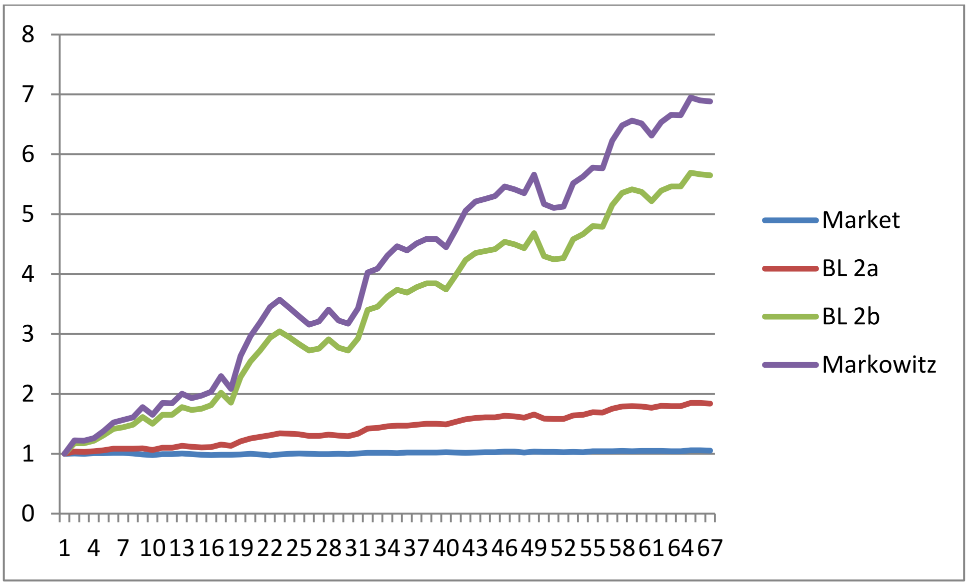

Figure 23: 3 months performance of the market, the BL 2a, the BL 2b and the new Markowitz portfolio

The performance of the BL portfolio 2b and the new Markowitz portfolio is characterized by a high correlation. The new Markowitz portfolio which bases on the “right” expected returns represents the portfolio with the *highest* return for a certain level of risk-aversion (without risk-aversion the best choice would be to go for the asset with the highest return only, like Williams suggested. Certainly this extreme method only applies if the investor is able to predict the future returns correctly.)

We know now both extremes of the Black Litterman model: With zero confidence in subjective views it results in the market portfolio while a hundred percent confidence in the investor's views leads to an efficient mean variance portfolio set up with the subjective views. This confirms that the Black-Litterman model moves in the stress field of the two Nobel prize winning ideas: the CAPM model by Sharpe et al and the Portfolio Theory by Markowitz. Black and Litterman established the missing link and Meucci proposed a convenient way to weight between them.

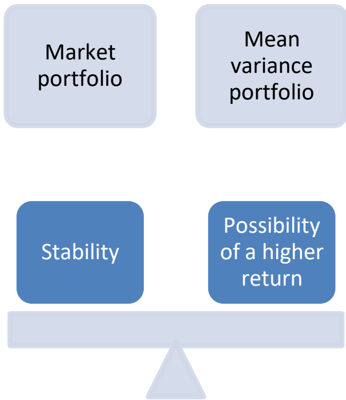

Figure 24: market portfolio - mean variance portfolio

As the CAPM model and the Portfolio Theory are both only theories, they have some drawbacks that ought to be considered. Further it should be noted that shifting between the allocations always implies a shift between the advantages and the disadvantages of the market portfolio and a mean variance portfolio, i.e. stability and the possibility of a higher performance.

With that trade-off in mind, we should also cast a glance at the Meucci Black-Litterman model from a different perspective: the traditional Markowitz approach treats the inputs as if they were known with 100% certainty[93] while holding the pure market portfolio initially requires zero certainty about the inputs. The Meucci Black-Litterman model now gives the opportunity to *valuate* this certainty/uncertainty by using confidence in subjective views.

8.7. Stability and Risk adjusted performance

We know now that the increase in performance occurs in a (more or less) linear manner: If the subjective views are more accurate than the implied market returns, the performance of a Meucci Black Litterman portfolio increases linearly with increasing confidence in the subjective views.

Can we make the same statement about stability? Beforehand remember that the Black-Litterman model nests the subjective views in a robust environment, the market portfolio. This, however, only works to a certain extent: the more confidence the investor has in his own views, the less stable his asset allocation gets. This means that stabilization is achieved by neglecting the investor's views. So if an investor considers himself as a clairvoyant who has many divergent views on returns to the market with a high level of confidence, then the Black-Litterman allocation will be rather instable, but in turn offers the possibility to significantly outperform the market (compare BL portfolio 2a, 2b).

[93] Idzorek (2006), p. 7

On the other hand, investors with almost no subjective views and forecasts enjoy the advantage of a portfolio which is in a way the most stable of portfolios: the market portfolio.[94] Or in the words of Rebonato: [95]

"[...]if the Black-Litterman is used purely as a regularization device, it [...] "sedates" the patient rather than curing it."

In portfolio theory the standard deviation of the returns of the portfolio is applied as the measure of risk and stability.[96] Thus we examine the relation between the portfolio's standard deviation σ_P, i.e. the square root of the variance of the return

$$\sigma_P^2(R_P) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j cov(R_i, R_j)$$

and the confidence parameter c in order to analyze the stability of a Meucci Black-Litterman portfolio. Figure 25 depicts this relation.

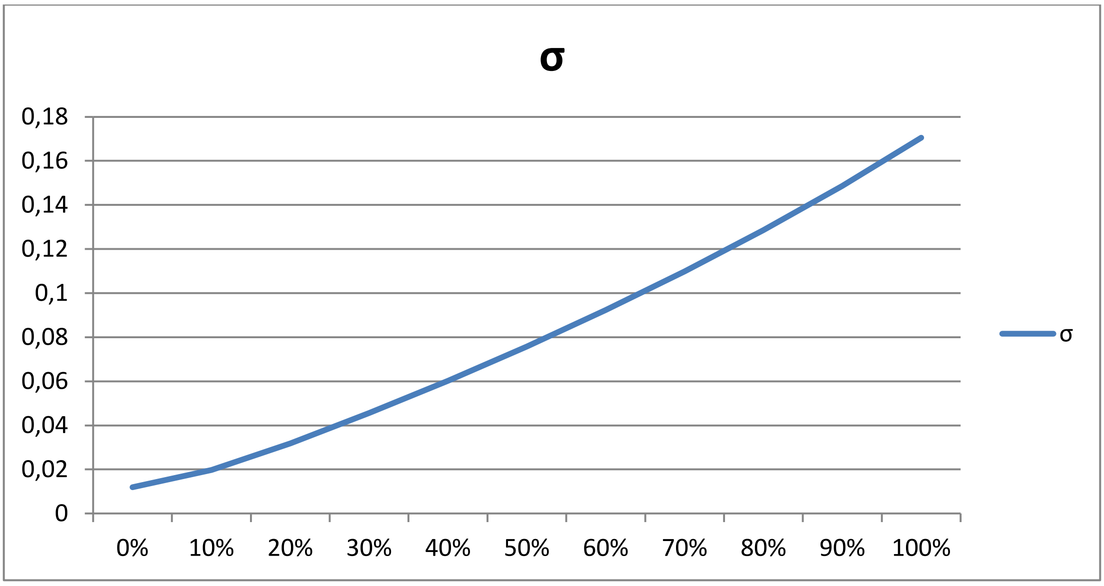

Figure 25: Portfolio's standard deviation for different settings of the confidence parameter c

At first sight figure 25 shows that the more weight we add to the subjective views, the less stable the asset allocation gets; just as expected and rather intuitive. However a closer look unveils that the relation between stability and confidence is more complex than the almost linear relation between performance and confidence: Close to the market portfolio, i.e. $c = 0\%$, the slope of σ is comparatively low. It is then continuously increasing until c is about 50% where it appears to

[94] Rebonato and Denev (2013), p.455
[95] Rebonato and Denev (2013), p.435
[96] Markowitz (1952), Samarakoon and Hasan (2006), p. 618 and compare chapter 4.

become almost linear - just like the performance-confidence relation. Obviously a positive stabilization effect brought by the market portfolio - representing an anchor of stability - comes here into play. The following risk-adjusted performance measurement proposed by Sharpe 1966 highlights this behavior.

The Sharpe ratio evaluates the excess return, i.e. the risk premium, per unit of risk of the portfolio.[97] It is defined as:

$$S = \frac{r_{p,excess}}{\sigma_p}$$

where S is the Sharpe ratio, $r_{p,excess}$ the excess return of the portfolio and σ_p the standard deviation of the returns of the portfolio. Unlike other risk-adjusted performance measurements the Sharpe ratio method also works well if the portfolios of interest are not well diversified, which might be the case for higher levels of confidence in the subjective views (compare chapter 3.). In the context of the risk-adjusted performance measurement of Sharpe a portfolio is said to outperform a benchmark when it earns a higher risk premium per unit of standard deviation than the benchmark.[98]

Figure 26 shows the Sharpe ratio for different levels of confidence in the subjective views – in analogy to figure 21 and 25:

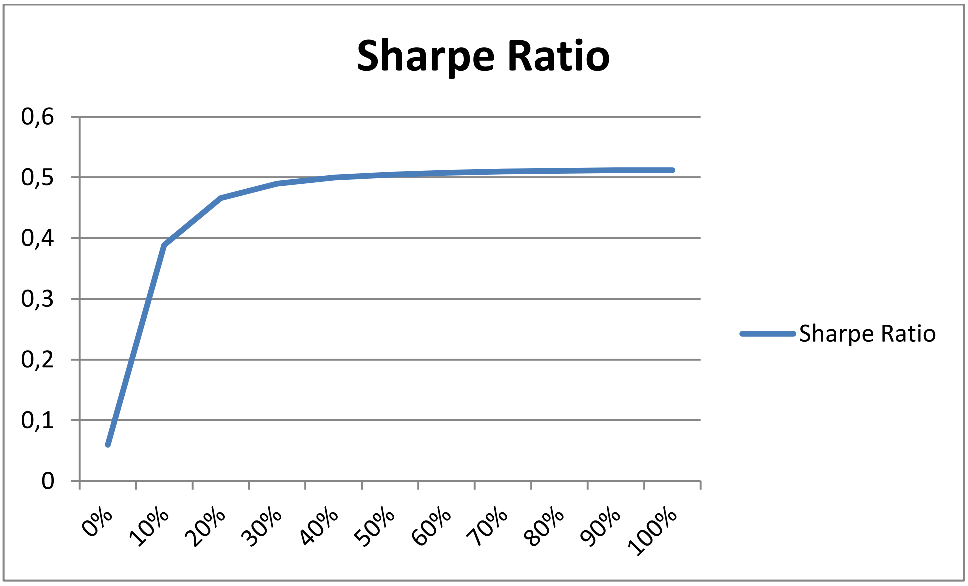

Figure 26: Sharpe Ratio for different settings of the confidence parameter c

The graph of figure 26 emphasizes the differences and asymmetries between the absolute performance (figure 21) and the risk/stability (figure 25) of the Meucci Black-Litterman portfolios:

[97] Sharpe (1966), Samarakoon and Hasan (2006), p. 618
[98] Samarakoon and Hasan (2006), p. 618

While the performance grows almost in a linear manner, the risk increases *slower* than the return for small values of c. This causes a significant increase of the Sharpe ratio for low levels of confidence with $c \sim 10\% - 30\%$. At about $c = 50\%$, meaning that the investor has the same confidence in the implied market returns as in the subjective views (compare the He and Litterman approach of chapter 7.4.1.), the risk premium per unit of standard deviation reaches almost its peak and starts to flatten out at a high level.

What are the practical implications of a concave Sharpe ratio function?

The first statement we can make is that the higher the confidence in the subjective views ceteris paribus, the higher is the risk premium we get per unit of risk, as long as the subjective views are more accurate as the implied market returns. However since a) a confidence of 100% in a view is not very realistic and causes many disadvantages of a mean variance portfolio, and b) the Sharpe ratio between $c = 50\%$ and $c = 100\%$ remains almost identical; it is fair to say that the choice for $c = 50\%$ is particularly convenient, if an investor has a relatively high level of confidence in his predictions and wants to combine the advantages of a relatively well diversified portfolio with a high risk-return profile.

The second statement may be even more relevant from a practical standpoint of view: the exploitation of the asymmetry between the risk and return relation. A study in 2016 by S&P Dow Jones Indices proved that about 90% managed funds could not outperform the market. [99] And if they do, it is often due to a higher risk that is taken. The insight that low levels of confidence lead to a linear increase of return but to a disproportionately low increase of risk might change this paradigm. Even relatively small values of c cause significantly improved returns per unit of risk. For $0 < c \leq 50\%$ the stabilization effect of the market portfolio can be conserved, however the increased return brought in by the subjective views goes directly into account.

How do the Sharpe ratios of the Meucci Black Litterman portfolios perform compared to the benchmarks? Table 14 displays them.

	Markowitz	**Market**	**BL1**	**BL2a**	**BL2b**	**Markowitz 2 = 100% Confidence**
Sharp Ratio	0,02513436	0,05954525	0,02122493	0,45028457	0,51122977	0,51192362

Table 14: Sharpe ratio of the Black-Litterman portfolios and the benchmarks

Unlike the previous conventional benchmark comparison where only the performance is considered, the Markowitz portfolio calibrated with historical returns does not outperform the market portfolio

[99] https://www.spglobal.com/our-insights/SPIVA-US-Scorecard.html, 20.06.2018

in a risk-adjusted performance measurement. This is due to its higher volatility. In fact, the risk-adjusted performance of the Markowitz portfolio is only slightly better than the one of the BL1 portfolio which had the lowest absolute portfolio performance. The Sharpe ratios of the BL2 portfolios and the Markowitz 2 portfolio correspond to the ones we determined in figure 25: This comparison highlights again that the Sharpe ratio increases significantly from the market portfolio ($c = 0\%$; $Sharp\ Ratio = 0{,}05954$) to the BL2a portfolio ($c = 17\%$; $Sharp\ Ratio = 0{,}45028$) while it remains almost identical for the BL2b and the Markowitz 2 portfolio.

8.8. Takeaways of the performance and stability analysis

In a nutshell the key takeaways of the performance and stability analysis of the Meucci Black-Litterman model are:

1. The Meucci Black-Litterman model moves in the tension field between the market portfolio and an ordinary mean variance portfolio.
2. The relation between performance and confidence is almost linear.
3. The relation between stability and confidence is split roughly into two parts: For $0 < c \leq 50\%$ the slope of σ is comparatively low, however continuously increasing until c is about 50% where it becomes almost linear as well - just like the performance relation. Obviously a positive stabilization effect brought by the anchor of stability - the market portfolio - comes here into play.
4. Point 2. and 3. cause the effect that the risk premium per unit of risk increases rapidly with increasing confidence until the confidence in the subjective views is about 50% (= the He and Litterman approach), then it starts to flatten out at a high level.
5. Thus the paradigm of holding the market portfolio might change since bringing in subjective views with a low level of confidence improves a portfolio's risk/return profile significantly due to the asymmetry between the performance-confidence and the stability-confidence relation.
6. Note however that these statements are only valid if an investor is able to determine subjective views that are closer to the actual returns than the implied market returns. But as already mentioned in the problem statement, if that is not the case, using the Black-Litterman model makes no sense at all.

8.9. The Mahalanobis distance

As already mentioned in chapter "8.6. Portfolio performance" we may also have a look at the confidence-performance linearity from a different angle.

Go back to the two sources of the Black-Litterman model: The implied market information and the subjective information. As explained in chapter 6.5., the implied market returns and the subjective views base on different distributions. The Black-Litterman model uses the Bayes' theorem to combine these two distributions to a new combined return distribution.

The corresponding weights of a portfolio are then obtained with the aid of the unconstrained or constrained solution of the same utility function that was used to receive the implied market returns. Mind that this extraction of portfolio weights occurs independently from the Black-Litterman model and bases on the mean variance approach. A common choice is the quadratic utility function when using the Black-Litterman model[100] with the unconstrained solution:

$$\boldsymbol{w}^* = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$$

Note that the usage of the same mean variance method to obtain the implied market returns $\boldsymbol{\Pi}$ (input of the Black-Litterman model) and the posterior portfolio weights, implies that the new posterior return vector $E[\boldsymbol{R}]$ (output the Black-Litterman model) and the implied market returns $\boldsymbol{\Pi}$ are two estimations of the return vector with the *same* covariance matrix $\boldsymbol{\Sigma}$. This means we substitute $\boldsymbol{\mu}$ once for $\boldsymbol{\Pi}$ and once for $E[\boldsymbol{R}]$ while the risk aversion coefficient and the covariance matrix $\boldsymbol{\Sigma}$ remain the same. This method is used for a good reason: Since the (unconstrained) Black-Litterman portfolio is the market equilibrium portfolio plus a weighted sum of the portfolios about which the investor has views,[101] the usage of different methods at the extraction of the implied market returns and the selection of the optimal portfolio weights would lead to inconsistent and unintuitive results.

Thus the only reasonable procedure of obtaining portfolio weights implies that the posterior return vector $E[\boldsymbol{R}]$ and the implied market return $\boldsymbol{\Pi}$ are two estimations of the return vector with the same covariance $\boldsymbol{\Sigma}$. This offers an interesting possibility to "measure" the impact of different levels of confidence in the subjective views.

A method to do so is given by Mahalanobis. He proposed a measure for the distance between two vectors that are linked via a covariance matrix. The Mahalanobis distance is defined as

[100] see e.g. Idzorek (2005)
[101] He and Litterman (1999), p. 8

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{(\boldsymbol{x} - \boldsymbol{y})^T \boldsymbol{S}^{-1} (\boldsymbol{x} - \boldsymbol{y})}$$

It is a dissimilarity measure between two random vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ with the same covariance matrix $\boldsymbol{S}$.[102] In our case the Mahalanobis distance of interest is

$$d(E[\boldsymbol{R}], \boldsymbol{\Pi}) = \sqrt{(E[\boldsymbol{R}] - \boldsymbol{\Pi})^T \boldsymbol{\Sigma}^{-1} (E[\boldsymbol{R}] - \boldsymbol{\Pi})}$$

It is straightforward to see that $d(E[\boldsymbol{R}], \boldsymbol{\Pi}) = 0$ if the posterior return equals the implied market return $E[\boldsymbol{R}] = \boldsymbol{\Pi}$. The more $E[\boldsymbol{R}]$ and $\boldsymbol{\Pi}$ differ, the longer the Mahalanobis distance ceteris paribus becomes. In the case that the covariance matrix is the unity matrix, the distance reduces to the usual Euclidean norm[103] of the distance vector $(E[\boldsymbol{R}] - \boldsymbol{\Pi})$.

Different levels of confidence lead to different posterior returns $E[\boldsymbol{R}]$. If we carry the assumptions of the Black-Litterman portfolio 2 (i.e. the subjective views are similar to the returns that actually took place), we receive the following Mahalanobis distance for different levels of confidence:

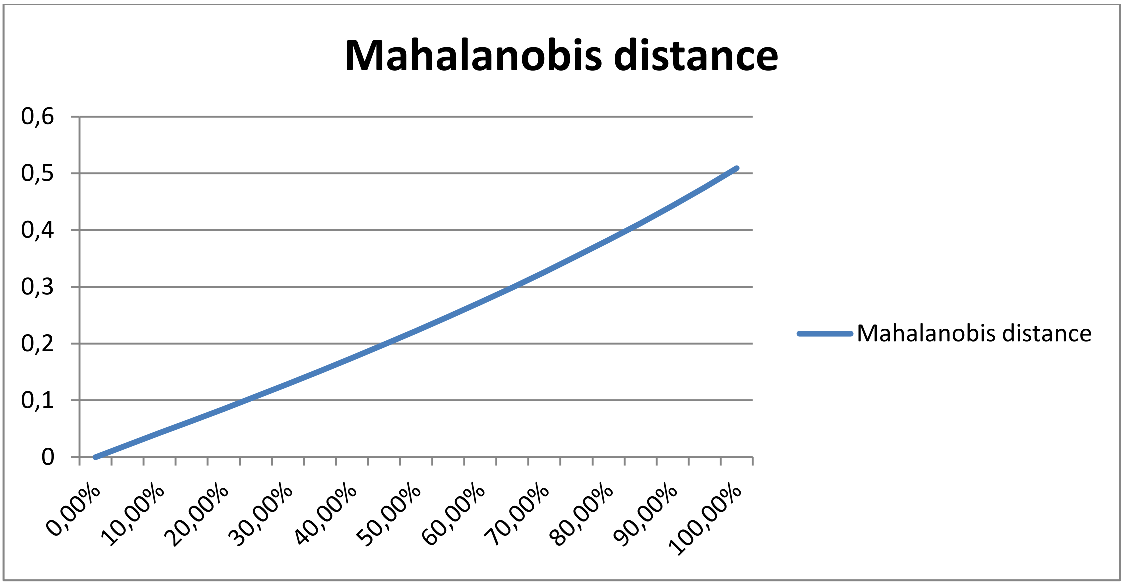

Figure 27: Mahalanobis distance for different settings of c

Figure 27 approves that the more weight we add to our subjective views, the longer the Mahalanobis distance from $\boldsymbol{\Pi}$ to $E[\boldsymbol{R}]$ becomes and thus the more do the expected returns of the market portfolio and of the Black-Litterman portfolio of interest differ. Furthermore figure 27 highlights the linearity of the (expected) return-confidence relation that has substantial similarity to Figure 21 (the performance-confidence relation). We will pick up this linearity once more in chapter 10.3. where an alternative approach of deriving a reasonable covariance matrix is proposed.

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[102] https://en.wikipedia.org/wiki/Mahalanobis_distance, 12.5.2018

[103] Rebonato and Denev (2013), p. 253

9. Drawbacks of the Black-Litterman model

We already addressed the main disadvantage of the Black-Litterman model:

In a nutshell, the stabilization is achieved by neglecting the investor's views. It "wants" its user to stay close to the market portfolio. Otherwise it results in an ordinary mean variance portfolio with all its drawbacks: instable, highly concentrated, highly leveraged if not ruled out and very sensitive to input parameters.

However there are some other issues that deserve mentioning. One is that it requires an estimation of the risk-aversion coefficient, which is difficult (see chapter 4.2.3.). Going back to the sensitivity of portfolio weights:

$$\frac{\partial \boldsymbol{w}^*}{\partial \boldsymbol{\mu}} = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}$$

Remember that the risk-aversion coefficient λ is (besides $\boldsymbol{\Sigma}^{-1}$) the critical parameter for the stability of weights. So why not use a high value of λ, e.g. 6 or 8 if the risk-aversion coefficient is an estimation anyway? And why not use even higher values like 100 if stability is the main concern in Black-Litterman portfolios with a high confidence in subjective views and mean variance portfolios in general? This leads indeed to a higher stability, however often not in a desirable way (compare e.g. Rebonato and Denev (2003) who analyzed this issue in greater detail).[104] This stabilization problem is also addressed in the outlook, where we propose a look and a method to deal with it from a different angle.

There is another drawback that deserves consideration: the implied market return of an asset *A* bases mainly on its current market capitalization. If asset *A* performed well in the recent past compared to other assets, then ceteris paribus its market capitalization increased as well compared to other assets. This implies a pro-cyclical behavior of the market and thus of the Black-Litterman model.[105]

[104] Rebonato and Denev (2013), p.451
[105] Söhnholz (2010), p.90

10. Outlook

The Bayesian nature lying underneath the Black-Litterman model works not only for the combination of the market portfolio with a portfolio that holds subjective views, but also for *any* portfolio that is supposed to be updated in the light of new information. This opens new possibilities e.g. for the naive asset allocation or product portfolios. Furthermore we suggest a method to take stress events more into account.

10.1. Naive asset allocation

Naive asset models differ from the concepts of asset allocation we proposed so far: Instead of selecting the assets of a portfolio by minimizing risk and maximizing the expected return, assets are selected from a "stand-alone" point of view.[106] This implies that the correlations among the assets are not taken into account directly.

The starting point of a naive asset allocation is the picking of stocks and other assets that investors feel are undervalued. Thus, naive asset allocations focus on the individual asset and not on the overall portfolio in the first place. The diversification effect by combining various assets is taken as a "bonus." Although its name "naive" and its rather simple approach of selecting a portfolio might suggest that naive asset allocations perform poorly or at least weaker than optimized portfolios, they can perform surprisingly well.[107]

How does the Black-Litterman model come into play at naive asset allocations? Usually naive asset allocation modeling does not end after a naive portfolio has been selected. They often get optimized in a second run. This is why these kinds of portfolios are also known as "pseudo-optimized" portfolios. After the pure naive portfolio has been selected, it gets updated by adding new assets to the original portfolio weights via analyzing if adding another asset (class) leads to a higher Sharpe ratio or not. In the very most cases it does.[108]

From this point of view, the Black-Litterman model may also be used for naive asset allocation modeling: The pure naive portfolio can be understood as the prior distribution. The identification of the implied assets' excess returns of *that* portfolio (not the market portfolio) could assist the investor as a reference point for further calibrations. Possibly he realizes that the weight (and therefore the implied return) of asset A in the pure native allocation does not correspond to his actual expected return for asset A.

[106] Söhnholz (2010), p.103
[107] Söhnholz (2010), p.19
[108] Söhnholz (2010), p.107

Therefore he could update the original pure naive allocation with subjective views concerning the expected returns. The weighting between the prior and the subjective part may then be done in the already proposed Meucci manner (see chapter 7.4.2.).

In other words, this method replaces the market portfolio by the pure native asset allocation portfolio (and its implied returns) while the underlying principles of the Black-Litterman model remain the same.

10.2. Product portfolios

The Black-Litterman model may also be used as a decision making/investment tool in corporate finance. If a company is in anticipation of changes concerning the returns of its assets, the Black-Litterman model provides information how and where to invest reasonably by considering the variance and the correlations among the different assets' returns.

Assume a company that operates mainly in the automotive and the energy sector and is structured into 5 divisions:

- Automotive industries powered by internal combustion engines (ICE)
- Automotive industries powered by alternative power trains (APT)
- Fossil Energy industries (FEI)
- Sustainable Energy industries (SEI)
- Other assets (OA)

The current asset allocation of the company's 5 divisions is as follows:

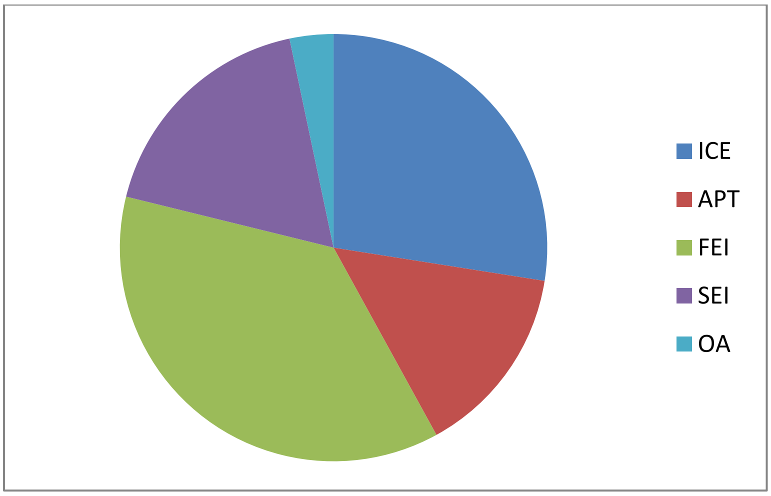

Figure 28: Company's asset allocation

ICE	27,49%
APT	14,52%
FEI	36,84%
SEI	17,85%
OA	3,31%

Table 15: Company's asset allocation

The five divisions had different returns over the last years. The historical estimates of the expected excess returns that base on the average return over the last 5 years are depicted in table 16 below:

	Historical estimates
ICE	28,12%
APT	22,31%
FEI	27,43%
SEI	19,12%
OA	15,32%

Table 14: Historical estimates of the expected excess returns

And the corresponding covariance matrix is:

	ICE	APT	FEI	SEI	OA
ICE	0,112771609	0,03227418	0,05039206	0,02145855	0,0186986
APT	0,03227418	0,12369742	0,02399253	0,01757558	0,01577877
FEI	0,050392061	0,02399253	0,09673821	0,0164831	0,01451009
SEI	0,021458551	0,01757558	0,0164831	0,07760145	0,00968203
OA	0,018698602	0,01577877	0,01451009	0,00968203	0,04587475

Table 15: Covariance matrix of the historical excess returns

The historical estimates of expected returns match the current asset allocation: obviously the current company's cash cows are the divisions "Fossil Energy" and "Automotive ICE". However due to expected changes in the Automotive as well as in the Energy industry, the company adopts its believes concerning expected returns:

	Views	Historical estimates	Difference
ICE	26%	28,12%	-2,12%
APT	24%	22,31%	+1,69%
FEI	25%	27,43%	-2,43%
SEI	22%	19,12%	+2,88%

Table 16: Views

But the company's management is not as confident about the future returns of the Automotive industry as it is about the Energy industry. Thus it wants to weight its assumptions. The following table shows the confidence for each division related to the historical estimates based on the average of the last 5 years returns:

	Confidence
ICE	83,33%
APT	83,33%
FEI	16,67%
SEI	16,67%

Table 17: Confidence in the views

Assume further that the company's two basic objectives are the same as the ones of many investors: maximizing return and minimizing risk. Certainly, the operational business underlies other constraints than ordinary security portfolios. Many operational units are quantized and cannot be divided. Think of a power plant: Suppose a new power plant costs 100 million USD and is expected to increase the overall SEI division's return by 10%. Saving 20 million USD by leaving out the turbine makes no sense, since the power plant can only run as a whole. So saving 20 million of initial investment costs does not decrease the promised 10% to 8% but to 0%. This also applies to many other operating units: new machinery, assembly lines, etc.

However if we assume the value of (new) operational units to be small compared to the existing assets, we find ourselves in a situation that is rather similar to the one many investors have: How and where to invest if we want to maximize the return and minimize the risk? And what kind of consequence do the new views have on the company's future returns and future investments?

With regard to these issues, the concept of Bayesian updating implied in the Black-Litterman model offers a potential solution: By extracting the "artificial implied" returns of the current allocation and updating them in the light of new information.

Suppose that our company subscribes to the Markowitz's assumptions introduced in chapter 4, the implied excess returns $\boldsymbol{I}$ may be calculated by:

$$\boldsymbol{I} = \lambda \boldsymbol{\Sigma} \boldsymbol{\omega}_{div}$$

With an assumed risk aversion coefficient $\lambda = 3$ follows:

ICE	17,61%
APT	11,8%
FEI	16,92%
SEI	8,61%
OA	4,81%

Table 20: implied excess returns

As we can see the implied excess returns differ a lot from the historical excess returns. In our case all 5 diversions have lower implied excess returns than its actual historical returns. Different implied returns will be the case in almost 100 percent, except for the very unrealistic scenario that a company is managed like a mean variance portfolio.

This gap between actual and implied returns requires further considerations:

The *basis* of the view for the new expected return (actual return) and of the return that the Black-Litterman model is calibrated with (i.e. the implied return) is different. The following table illustrates this issue. The blue elements base on the artificial implied returns while the greens represent the actual returns.

	ICE	APT	FEI	SEI
Implied return	17,61%	11,8%		
			16,92%	8,61%
Actual return (historically estimated)	28,12%	22,31%		
			27,43%	19,12%
View of new expected return (with a confidence of 80%)	26%	24%		
			25%	22%
View of new expected return based on the implied returns	?	?		
			?	?

Table 18: Implied and actual returns and views

The stated views refer to the actual return: E.g. the actual return of the FEI division is expected to drop from 27,43% to 19,12% while the SEI's return is expected to rise by almost 3% to 22%. Simply implementing the view that the Sustainable Energies' actual return is believed to be 22% into the Black-Litterman model would lead to biased results since further calculation in the Black-Litterman model base on the artificial implied returns and not on the actual ones. Thus we need to bridge this gap by identifying a value that bases on the implied returns and corresponds to our view.

A very straightforward solution for this problem provides the linear interpolation between the artificial implied returns on the one hand and the actual returns on the other hand:

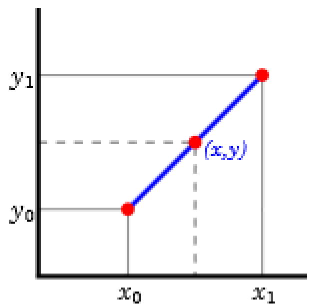

Figure 29: linear interpolation[109]

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

[109] https://en.wikipedia.org/wiki/Linear_interpolation, 20.3.2018

With $x_0 = 0$ and $y_0 = 0$ it is apparent that the value of interest y is a linear function of x with the slope $\frac{y_1}{x_1}$:

$$y = x \cdot \frac{y_1}{x_1}$$

$$Implied\ View\ Return = Implied\ Return \cdot \frac{actual\ View\ Return}{actual\ Return}$$

Applying the SEI view ($E[R] = y_1 = 22\%$) to the formula above leads to:

$$y = 8{,}61\% \cdot \frac{22\%}{19{,}12\%} = 9{,}91\%$$

According to this procedure the expressed view for Sustainable Energies corresponds to the view that the new expected return is $y = 9{,}91\%$ on an implied return basis.

Same holds for the Fossil Energies (the view $y_1 = 25\%$ leads to $y = 15{,}42\%$) as well as for the ICE and the APT.

	ICE	**APT**	**FEI**	**SEI**
Implied return	17,61%	11,8%		
			16,92%	8,61%
Actual return (historically estimated)	28,12%	22,31%		
			27,43%	19,12%
View of new expected return (with a confidence of 80%)	26%	24%		
			25%	22%
View of new expected return based on the implied returns	16,28%	12,69%		
			15,42%	9,91%

Table 19: Views based on the implied returns

The obtained views on the implied basis may then be applied to the Black-Litterman model in the usual way to receive the new expected returns and further the new weights. The only change we made compared to the original Black-Litterman formula is the exchange of the implied market excess returns $\boldsymbol{\Pi}$ for the implied company's excess returns $\boldsymbol{I}$:

$$E[\boldsymbol{R}] = [(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{P}]^{-1}[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{I} + \boldsymbol{P}'\boldsymbol{\Omega}^{-1}\boldsymbol{Q}]$$

This leads to

Prior return	Posterior return
0,17608448	0,17095727
0,11797448	0,11853935
0,16917448	0,15709548
0,08608448	0,09648353
0,04806448	0,04745848

Table 20: Comparison of the prior and the posterior return

and with

$$\boldsymbol{\omega}_{div,new} = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}(E[\boldsymbol{R}] - \frac{B-\gamma}{A}\mathbf{1})$$

$$A = \mathbf{1}'\Sigma^{-1}\mathbf{1}$$

$$B = E[\boldsymbol{R}]'\Sigma^{-1}\mathbf{1}$$

to the new allocation:

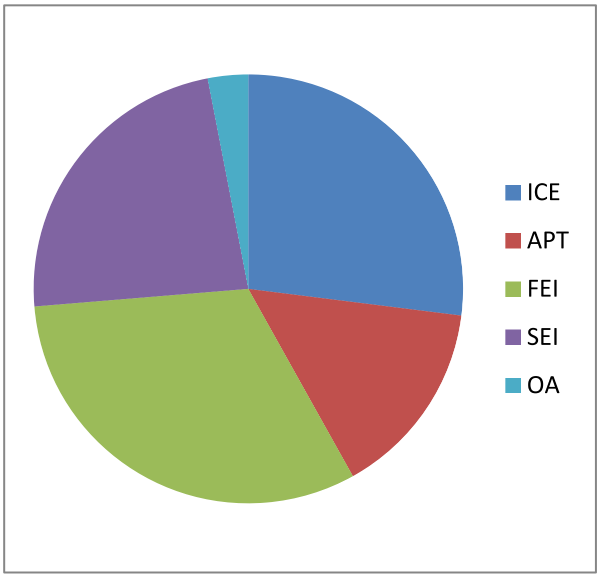

Figure 30: Posterior company's asset allocation

	Prior weights	Posterior weights	Difference
ICE	0,27487417	0,26985773	-0,50%
APT	0,1451603	0,14913984	0,04%
FEI	0,36836171	0,31789606	-5,05%
SEI	0,17851525	0,23267446	5,42%
OA	0,03308857	0,03043192	-0,27%

Table 21: Comparison prior and posterior weights

The obtained posterior weights give an idea of how new expected returns affect future investment decisions among the company's divisions. The weight changes appear rather intuitive: As the confidence in the expressed views for the ICE and the APT is low, only very moderate movements take place. However, the direction of the small shifts implies that the modified expected returns are already taken into account. The Energy Industries illustrate the added confidence in the views: The FEI's return is expected to drop while the SEI's is about to rise - which both is reflected in the suggested weight changes. The feature of this procedure is that not only the expected returns and the variance of the company's assets are taken into account, but also the often neglected correlations among the different divisions' returns.

The proposed method has similarities to how engineers deal with time dependent differential equations: they convert them via the Laplace transform from the time domain to the frequency domain where further calculations are easier. Afterwards the inverse Laplace transform takes the function of frequency and yields a function of time.

Our transformation takes place from real world returns to implied capitalization returns. Instead of the Laplace transform we use the linear interpolation to bridge the gap between actual returns that do not correspond perfectly to actual capitalizations and implied returns.

In a nutshell this is an attempt for an investment decision making tool that allows updating the current asset allocation of a company in the light of new information by considering the variance and the correlations among the assets' returns.

10.3. Covariance matrix during stress events

So far all subjective views concerned only the first moments of the return distribution: the expected returns. This is sufficient for most market conditions, since the second moments of the return distribution tend to be more stable during normal market conditions than the expected returns.[110]

However in conditions of market turmoil, there is evidence[111] that the codependence among some asset classes changes radically.

Therefore a method is proposed to receive a covariance matrix that reflects the increase of volatilities and codependences among assets, for the case that the investor is in anticipation of a market turmoil.

Generally, identifying and assessing the probability of stress events is difficult. Unfortunately, crises don't have a regular invariant signature and structure. Crises rather unfold according to their own

[110] Rebonato and Denev (2013), p. 7
[111] Söhnholz (2010), p.27ff

idiosyncratic dynamics. However there are some crises-specific aspects, like the increase of volatilities or the greater codependence among assets which are normally weakly correlated.[112] With the following proposed method the investor can make a choice: whether he believes that the next upcoming crisis will cause similar effects like any *previous* crisis (in this case he can use certain crisis-specific information) or that he is only expecting *any* crisis (in that case the model provides a more general procedure). In either case it is up to the investor to specify her *confidence*, i.e. the likelihood that the stress event occurs. This approach is therefore kind of similar to Meucci's approach to specify Ω of the Black-Litterman model, the "uncertainty matrix of subjective views".

During crises volatilities and correlations *generally* increase and *tend to* go to one.[113] Suppose that the investor is expecting a crisis with a probability of 50% but has no accurate conception of how the securities will be affected. The only assumption made is that the volatility and the correlations among the assets will increase and thus will go in the direction of one. Assume further that the feasible covariance matrix during normal market conditions is Σ_n. The corresponding inverse covariance matrix is then Σ_n^{-1}. For all feasible covariance matrices the following relation holds:

$$\Sigma_n \cdot \Sigma_n^{-1} = I$$

By the introduction of k (the confidence factor) the investor can calculate a new (crisis) covariance matrix Σ_c that takes an (unrealistic) stress scenario, where the covariance matrix equals I, with k percent into account:

$$\Sigma_c = \Sigma_n[k(\Sigma_n^{-1} - I) + I]$$

For $k = 0$ the new crisis specific covariance matrix Σ_c is equal to Σ_n meaning that no crisis is expected. In the case that the investor is in anticipation of a crisis with a probability of 50% the confidence factor k becomes $0{,}5$ and Σ_c is a linear combination of $\Sigma_n$and I (consisting of 50% respectively). For $k = 1$ the new covariance matrix Σ_c equals I. Certainly, this is absolutely unrealistic and not useful, therefore this rather straightforward way of deriving a crisis covariance matrix is supposed to be used only for $0 < k \leq 0{,}5$. However the advantage of this method is its feature not to request any crisis-specific information from the user and the fact that it can be calculated rather quickly.

With some adaptations though, the proposed method may be used in a more consistent and intuitive manner with $k \in [0{,}1]$. The drawback is that one has to make the effort to extract information from

[112] Rebonato and Denev (2013), p.7

[113] Although one has to be aware that generalizations and statements like these need strong qualifications, recent studies (compare Söhnholz (2010)) showed that during many stressful market conditions correlations *did* increase.

previous crises and that the investor is supposed to know the specific information of the upcoming crisis *ex ante*. This means that the correlations and volatilities of previous stress situations are expected to be almost similar to the ones that are likely to occur during the next crisis. For example if an investor holds the view that the next market turmoil is caused by tech companies, then the historical information of the dot-com collapse might be the one of choice. Or if the next financial crisis is expected to have the same characteristics as the lost Japanese decade, then the information during the 1990s in Japan are of interest. The new covariance matrix derived from this crisis specific information is denoted by C_s. As it is impossible to know for sure when the next stress event occurs, it might be helpful to mix and weight this crisis specific data with statistics derived during normal market conditions. The new weighted (crisis specific) covariance matrix Σ_{cs} that incorporates crisis specific data (C_s) as well as the volatilities and dependencies among assets during ordinary conditions (Σ_n) can then be obtained by:

$$\Sigma_{cs} = \Sigma_n[k((\Sigma_n^{-1} \cdot C_s) - I) + I]$$

As before, k represents the confidence factor. However this time a value close to $k = 1$ does make sense if the investor holds the view that a crisis is very likely to occur and that his crisis specific covariance matrix perfectly matches the anticipated volatilities and correlations, $\Sigma_{cs} = C_s$.

Certainly we could also create a crisis specific covariance matrix C_s without relying on historical information. E.g. Rebonato (1999) describes a consistent way of deriving a feasible covariance matrix. This gives the freedom to specify a covariance matrix C_s that corresponds to *any* crisis the investor designs. However designing or simulating a crisis can be difficult if doing so out of the blue, since crises do not have an invariant signature and structure, as already mentioned. Thus, deriving crisis specific information from history appears to be the more convenient choice.

What are the impacts of this procedure on portfolio selection? Remember the optimal portfolio weights of the unconstrained maximization problem are:

$$\boldsymbol{w}^* = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$$

Thus the term $\frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}$ defines a linear mapping between the expected returns $\boldsymbol{\mu}$ and the portfolio weights $\boldsymbol{w}^*$.[114] Figure 31 visualizes this thought:

[114] See Doust (2008) as well as Rebonato and Denev (2013), p.77

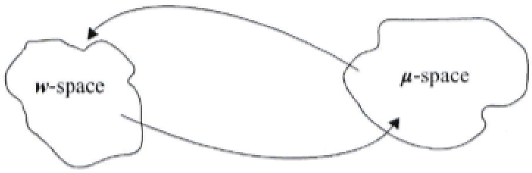

Figure 31: Linear mapping[115]

The portfolio weights are the values of interest in this transformation process. They depend of:

a. the input parameters, i.e. the expected returns and
b. the transformation mapping, i.e. the (inverse) covariance matrix multiplied by the risk-aversion coefficient.

While the Black-Litterman model approach tries to change the input parameters by substituting the expected returns $\boldsymbol{\mu}$ by $E[\boldsymbol{R}]$, the proposed method of weighting the covariance matrix aims on the mapping *itself*. Suppose that the transformation process from returns to weights resembles a physical experiment where a tennis ball machine throws tennis balls trough a plate that is pierced by holes and slits on a target behind the plate.

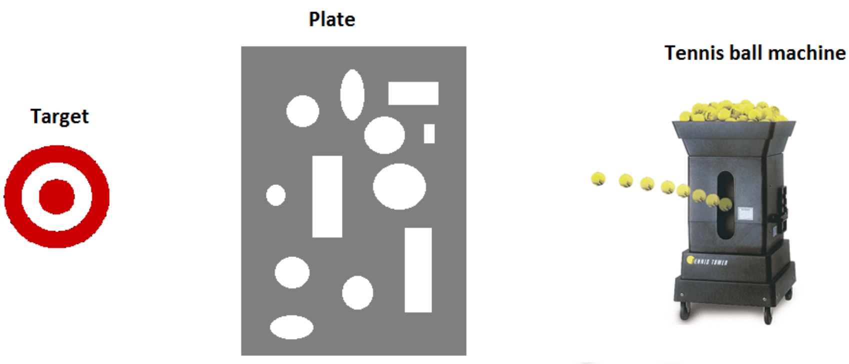

Figure 32: Experimental set-up[116]

If we assume the tennis ball machine to throw the balls perfectly stable every single time, the success of hitting the target will mainly depend on the condition of the tennis balls (are they soft or hard?) and the arrangement of the holes and slits of the plate. In this analogy the tennis balls represent the inputs, the target the output and the tennis ball machine the assignment of the inputs to the outputs.

While the Black-Litterman model focuses on the condition and "stability" of the tennis balls, the weighted covariance matrix changes (and increases) the holes and slits on the plate. Both approaches

[115] Rebonato and Denev (2013), p. 88
[116] Tennis ball machine from: http://www.tennis-aaron.de/tennisballmaschinen.php, 12.6.2018

use the "Markowitz tennis ball machine" to throw the tennis balls and both pursue the same objective: hitting the target.

Weighted covariance matrices should have a positive stabilization effect on the portfolio weights. A "stressed" covariance matrix can be an alternative to an unrealistically high risk-coefficient. Go back to sensitivity of the portfolio weights:

$$\frac{\partial \boldsymbol{w}^*}{\partial \boldsymbol{\mu}} = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}$$

Remember that the weight's stability depends only on the risk-aversion coefficient λ and the inverse covariance matrix $\boldsymbol{\Sigma}^{-1}$. Instead of increasing λ we can equivalently increase $\boldsymbol{\Sigma}$ which in turn decreases $\boldsymbol{\Sigma}^{-1}$ and therefore $\frac{\partial \boldsymbol{w}^*}{\partial \boldsymbol{\mu}}$. The advantage of this method compared to an unrealistically high value of λ is that the new crisis covariance Σ_c contains more information than λ: Σ_c describes the increased volatilities and especially codependences more precisely than (simply) multiplying every element of $\boldsymbol{\Sigma}^{-1}$ by a common factor $\frac{1}{\lambda}$. From this point of view, the new crisis covariance Σ_c is an alternative/extension to λ that embodies more information.

In the context of the Black-Litterman model, another question may come to mind: Does it make sense to substitute Ω by Σ_c or Σ_{cs} in the Black-Litterman formula if economic turbulences are expected? Unfortunately not, since a) subjective views are generally uncorrelated (however complex views may theoretically interfere with each other) and more importantly b) since the weighted covariance matrix Σ_c is way more disperse than the ordinary covariance matrix Σ_n (since crises are characterized by increased volatilities and correlations). The impact of the subjective returns will become insignificant, while the implied market returns will be overweighted in the Black-Litterman model. Adding weight to the subjective views via Meucci's approach

$$\omega = \left(\frac{1}{c} - 1\right) \cdot (p_k \Sigma_c p_k')$$

reduces de facto the crisis-specific higher covariance of Σ_{cs} to a small value of ω for $c \rightarrow 1$, corresponding to a peaked distribution of the views (compare chapter 7.4.2. Meucci). Thus, the precious crisis information embodied in Σ_{cs} gets "lost" for a high level of confidence in the subjective views.

A consistent way of using the new covariance matrix is by substituting $\Sigma_n$by Σ_{cs} throughout the whole calibration of the Black-Litterman model, including the determination of the implied market returns $\boldsymbol{\Pi} = \lambda \Sigma_{cs} \boldsymbol{\omega}_{mkt}$. Note that the use of the covariance matrix Σ_{cs} will lead ceteris paribus to different implied returns than using Σ_n.

11. Summary

The most popular asset allocation model is the mean variance approach proposed by Markowitz. This approach is a quantification of the two main objectives of investing: maximizing expected return and minimizing risk. However in the practical world of investment management, the Markowitz approach has shown its drawbacks. This is mainly because the weights received by the standard model tend to be extreme and not very intuitive. Furthermore they are extremely sensitive to the input parameters.

To overcome these problems, Black and Litterman developed an alternative approach. The Black-Litterman model is an asset allocation model that uses the Bayes' theorem to combine the market equilibrium portfolio with additional subjective views of an investor.

To unveil the impact of subjective views on the asset allocation we analyzed how an implementation of the Black-Litterman model works:

The starting point of the Black-Litterman approach is the market equilibrium. The market equilibrium provides a neutral framework that an investor can adjust according to his own views, optimization objectives and constraints.

If an investor does not have any views, he holds the market portfolio. However, if an investor does have one or more views, he may adjust the equilibrium weights according to his views. Furthermore the Black-Litterman model provides the opportunity to specify the confidence in his views in line with the market model.

A common method to calibrate the level of confidence is given be He and Litterman. Their approach equally weights the confidence in the market model and in the subjective views. A different approach is given by Meucci. Meucci's approach gives the opportunity to calibrate the Black-Litterman model for different levels of confidence in the views.

We examined during a 3-months investigation the influence of subjective views on the actual performance of a Black-Litterman portfolio compared to two benchmarks: the market portfolio and a mean variance optimized portfolio. As for a mean variance optimized portfolio the input parameters are the critical factor, we explain two different, common approaches to evaluate reasonable expected returns (arithmetic and geometric means). It turns out the current issue of negative risk free rates and the way how we consider them has a major influence on the final mean variance portfolio weights.

The performance test confirms the intuition that not the portfolio with the highest expected return ex ante, but the one with the most accurate return assumptions delivers the highest performance.

The test further highlighted that the Black-Litterman model moves in the tension field between the two Nobel Prize winning ideas: the CAPM model by Sharpe et al and the Portfolio Theory by Markowitz. Black and Litterman established the missing link and Meucci proposed a convenient way of weighting between them. The analysis of the portfolio performance proves that this weighting appears in an (almost) linear manner.

The relation between stability and confidence is split roughly into two parts: For small values of confidence the slope is comparatively low, however continuously increasing until the confidence reaches about 50%. Above 50% the steepness remains almost unchanged - it becomes almost linear just like in the performance-confidence relation. Obviously a positive stabilization effect brought by the market portfolio comes here into play.

This asymmetry between the performance-confidence and stability-confidence relation cause that the risk premium per unit of risk (represented by the Sharpe ratio) increases rapidly with increasing confidence until the confidence in the subjective views is about as high as the confidence in the market (i.e. $c = 50\%$), then it starts to flatten out at a high level.

Thus the paradigm of holding the market portfolio might change since bringing in subjective views with a low level of confidence improves a portfolio's risk/return profile significantly due to the asymmetry between the performance-confidence and the stability-confidence relation.

Note however that these statements are only valid if an investor is able to determine subjective views that are closer to the actual returns than the implied market returns.

The Bayesian nature lying underneath the Black-Litterman model works not only for the combination of the market portfolio with a portfolio that holds subjective views, but also for any portfolio that is supposed to be updated in the light of new information. This opens new possibilities:

In naive asset allocation the assets of a portfolio are selected from a stand-alone point of view. Usually the selected pure naive portfolio gets updated and optimized in a second run. If we understand the pure naive portfolio as the prior distribution that is updated in the light of new information, we may apply the presented Bayesian and "weighting" methods for this type of asset allocation models.

The Black-Litterman model may also be used as a decision making/investment tool in corporate finance. If a company is in anticipation of changes concerning the returns of its assets, the Black-Litterman model provides information how and where to invest reasonably by considering the variance and the (often neglected) correlations among the different assets' returns.

Furthermore we suggest a method to take stress events more into account and to stabilize portfolio weights. This is done by the introduction of subjective views on the second moment of the return distribution and the correlations. The suggested procedure to do so is by mixing and weighting a covariance matrix derived during normal market conditions with crisis specific information.

In a nutshell, the Meucci Black-Litterman model provides a sophisticated approach for modern asset allocation with an attractive risk-return profile.

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Affidavit

I declare in lieu of oath, that I wrote this thesis and performed the associated research myself, using only literature cited in this volume.