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Radicalization and Terrorism: An Optimal Control Approach

Autor: Thomas Lechinger
Matrikelnummer: 01125290
Studienkennzahl: 066 395

Betreuer: Ao.Univ.Prof. Dipl.-Ing. Dr.techn. Gernot Tragler

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Abstract

The aim of this thesis is to construct and analyze an optimal control model that deals with radicalization in the context of a terrorist organization. Two subgroups of the population are considered, namely a group of terrorists and a group of people susceptible to the radical thoughts of the first one. There are two different reasons modeled why a person susceptible to terrorism actually joins the terrorist organization. On the one hand, recruiting happens, which is carried out by members of the terrorist organization and therefore depends on its size. On the other hand, radicalization also happens because of negative side effects of law enforcement. By modeling intelligence explicitly, the efficiency of law enforcement can be influenced. Law enforcement can be applied more efficiently by the decision maker, the least negative its effects are. In this thesis, these undesirable effects are referred to as collateral damage. It simply models mistakes made in the process of applying law enforcement measures, which also leads to radicalization. Furthermore, prevention is applied as a possibility to decrease radicalization.

After creating the model and motivating the underlying dynamics, several investigations are carried out in order to understand the dynamic behavior. The optimal control model is then solved by applying Pontryagin's Maximum Principle. Due to the complexity of the evolving system, the calculation of the equilibrium and stable saddle paths is done numerically. Therefore, the MATLAB toolbox OCMAT is used, which was created for that very purpose.

Kurzfassung

Das Ziel dieser Arbeit ist die Konstruktion und Analyse eines optimalen Kontrollmodells, um Radikalisierung im Kontext einer terroristischen Organisation abzubilden. Für diesen Zweck werden zwei unterschiedliche Teile der Gesamtbevölkerung betrachtet, nämlich Terroristen sowie eine weitere Gruppen von Menschen, welche potentiell anfällig für terroristische Gedanken und somit radikalisierbar sind. Eine Form der Radikalisierung, in dieser Arbeit, ist aktive Rekrutierung, durchgeführt durch Mitglieder der terroristischen Organisation. Um die wirtschaftlichen Auswirkungen von Terrorismus bis zu einem gewissen Grad steuern zu können, stehen unterschiedliche Kontrollinstrumente zur Verfügung. Beispielsweise besteht die Möglichkeit, die Ausgaben für präventive Maßnahmen festzulegen. Außerdem kann mittels Strafverfolgung gegen die Gruppe der Terroristen vorgegangen werden. Durch die Modellierung von Intelligenz in Form einer weitere Kontrollvariable, besteht die Möglichkeit, dass es während der Strafverfolgung zu Irtümern kommt, welche unter dem Begriff des Kollateralschadens zusammengefasst werden. Diese Beeinträchtigung Unschuldiger führt ebenfalls zu Radikalisierung und stellt somit eine weitere Möglichkeit derselbigen dar.

Nach der Motivation und Konstruktion des zugrundeliegenden Modells werden im Zuge unterschiedlicher Herangehensweisen Einblicke in dessen dynamisches Verhalten gewonnen. Das optimale Kontrollmodell wird dann durch Anwendung des Pontryagin'schen Maximumprinzips gelöst. Aufgrund der Komplexität des betrachteten Systems werden das Gleichgewicht, sowie mehrere stabile Sattelpunktpfade, numerisch berechnet. Zu diesem Zweck wird die MATLAB Toolbox OCMAT eingesetzt, welche für exakt diesen Verwendungszweck geschrieben worden ist.

Contents

Chapter 1

Introduction

Although there is a large number of papers dealing with the topic of radicalization and terrorism, there are not that many considering this topic in the context of a dynamic optimization problem. The aim of this thesis is not to question the different reasons for terrorism nor to discuss its moral point of view, but to model radicalization and its effects from an economical perspective. It is assumed that terror attacks, purely economically speaking, produce social costs. Therefore it is an understandable goal to minimize these costs, which is the main focus of this thesis. This includes the search for a dynamic optimal allocation of resources in form of money, to minimize the arising social costs. Another focus of this thesis is the comparison between dynamic and static optimization approaches.

In order to simplify the naming, the use of some words in this thesis are explained in the following sentences. Although the definition of the term "terrorist" is debatable and the demarcation between political resistance and activism is controversial, the term "terrorist" is used in this thesis only to describe radical people who commit terror attacks. It is assumed that every terrorist contributes to attacks, which is why the number of terrorist attacks increases with the number of terrorists. People who are susceptible to radicalization are simply called "susceptibles". Radicalization is modeled in two ways in this thesis. On the one hand, terrorists try to actively recruit susceptible people. On the other hand, collateral damage, induced by law enforcement, turns a certain number of susceptibles into terrorists. One important assumption about susceptible people is their reaction to recruitment performed by terrorists. If during the recruiting process a susceptible is addressed, this individual may become radical. The decision maker, who decides on the usage and the amount of the available controls will be referred to as "government". One of the three decision variables is law enforcement. In order to apply law enforcement, human resources are needed. Therefore, the words "police" and "military" will be used in order to describe the application of law enforcement. As it will be discussed in detail in Chapter 2, applying law enforcement also has a negative side effect, which will be distinctly addressed by the term "collateral damage". Although law enforcement in general induces collateral damage, speaking of the effects of law enforcement only considers the positive effects, namely a decrease in the number of terrorists.

In order to enable a roughly realistic behavior of the model, a lot of effort was put into the estimation of parameter values, especially because no previous estimations could be found for most of the considered parameters. The estimations provided in this thesis are certainly not perfect, but they are sufficient to the extent of allowing the model to behave in the desired way.

The model's dynamics are based on a proposal by the research team ORCOS (Operations Research and Control Systems) at TU Wien (Technische Univerität Wien / Vienna University of Technology). . At this point I want to thank in particular my supervisor Gernot Tragler and Dieter Grass for the effort and help they provided me during the writing process of this thesis.

The structure of this thesis is organized in the following way. In Chapter 2, the model will be formulated. This includes the dynamic behavior of the state variables, the influence of the control variables, the objective function, and the parameterization. By doing so, several motivations will be stated. In chapter 3, the uncontrolled system will be analyzed and discussed by considering the steady states and their stability. A phase portrait provides a first impression of the dynamic behavior. Chapter 4 deals with constant strategies and their effects. Again the steady states will be calculated and discussed. In the course of an example, the optimal constant strategy will be calculated and compared to the uncontrolled system. Another phase portrait will be generated by the use of a representative constant strategy. In Chapter 5, the Hamiltonian and Lagrangian will be formulated in order to solve the dynamic optimization problem. Due to its complexity, the calculations will be done numerically using the MATLAB toolbox OCMAT. Some optimal paths will be presented and compared to the corresponding optimal constant solution. Finally, two emergent questions which arose during the analysis will be discussed. Chapter 6 then contains a conclusion of the insights found in the previous chapters. Part of the code used for calculation and visualization will be presented in the Appendix A.

Chapter 2

Model Construction

In order to construct a dynamic optimization problem, also the dynamics connecting the state variables as well as the influence of the control variables must be determined. Furthermore, an objective function is needed. The purpose of this function is to represent the overall social costs for any number of terrorists and any chosen strategy. Finally all parameter values must be specified by the use of empirical observations.

The aim of the dynamic optimization problem then is to search for an optimal strategy, which minimizes the social costs and therefore the objective function for a given starting value of terrorists and susceptibles.

2.1 The System Dynamics

In this model, two different groups are modeled, namely the class of terrorists $T(t)$ and the class of susceptibles $S(t)$, respectively, at time t. The values of both state variables $T(t)$ and $S(t)$ represent the numbers of terrorists susceptibles, respectively. The overall population size P is assumed to be constant and so of course it must hold that $S(t) + T(t) \leq P \quad \forall t \geq 0$. In this model, it is only possible for a susceptible to become a terrorist and only terrorists produce social costs due to attacks. The goal for the decision maker (government) is to minimize these social costs. This can be done by means of three different control instruments, namely law enforcement, prevention, and intelligence. The decision variable $v(t)$ describes the current effort of government intervention against terrorism. Under perfect conditions the decision variable $v(t)$ determines the percentage decline in $T(t)$ due to law enforcement. It is assumed that the decrease of $T(t)$ in absolute numbers is more difficult for a smaller value of $T(t)$. Law enforcement needs time, so a limitation of $v(t)$ is assumed which is chosen to be one. The corresponding costs include for example the wage for soldiers, the costs of military equipment, and so on. Of course, such government interventions have the intention to decrease the amount of terrorists $T(t)$, but they also have the contrary effect due to collateral damage, which will be discussed in detail later in this section. The control variable $\omega(t)$ is measured in USD and describes the spendings on prevention. In general it is less efficient than government intervention but conversely it has no negative side effects. The third decision variable $\mu(t) \in [0, 1)$ describes the effect of education on the analysts working for the government. To run interventions it is assumed that analysts choose the targets. The quality of their reports vary in a sense that a report with a high quality is almost certainly correct while a report with a bad quality has a random character. Sometimes the bad quality reports predict the right target and sometimes not. A wrongly chosen target, namely one that involves innocent civilians, results in collateral damage in this model. $\mu(t)$ describes the percentage of correct reports and therefore influences the effectiveness of government intervention as well as the extent of collateral damage. It is assumed that the correctness of all reports is impossible, which is reflected by the assumption of $\mu(t)$ being less than one. This assumption ensures the existence of collateral damage during the application of law enforcement.

To describe the dependencies and influences between the two states as well as the influence of the three control variables, the following differential equation system is used, which will be discussed in detail in this chapter. Figure 2.1 provides a graphical representation of the dynamic model.

$$\begin{aligned} \dot{S}(t) &= k - \delta S(t) - f(\upsilon(t), \omega(t), \mu(t), T(t))S(t) && (2.1) \\ \dot{T}(t) &= f(\upsilon(t), \omega(t), \mu(t), T(t))S(t) - \\ &\qquad (\sigma + g(\upsilon(t), \mu(t), T(t)))T(t) && (2.2) \end{aligned}$$

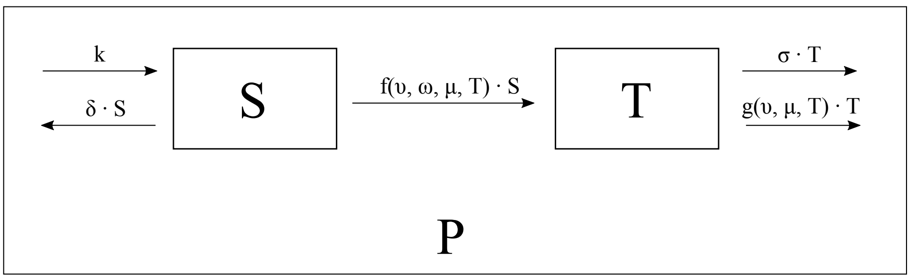


Figure 2.1: This flow chart displays the dynamic behavior between the number of susceptibles $S(t)$, the number of terrorists $T(t)$, and the whole population P.

The Equations (2.1) and (2.2) describe the dynamic changes of susceptibles and terrorists, respectively. Temporarily ignoring $T(t)$ and all its dynamics in relation to $S(t)$ and P, the remaining $S(t)$ is left as a limited growth model. There exists a constant inflow k into $S(t)$ and a dynamic outflow $\delta S(t)$. This models the assumption that under all circumstances there will be a part of the population which is not susceptible to radicalization. The upper bound of $S(t)$ and hence the maximum amount of susceptibles possible in this model is therefore given by k/δ.

The initiation function $f(\upsilon(t), \omega(t), \mu(t), T(t))$ describes the share of people currently switching from the class of susceptible people to the group of terrorists. The size of f depends on all three control variables and on the amount of terrorists, and its definition is as follows:

$$f(\upsilon(t), \omega(t), \mu(t), T(t)) = \varphi(\omega(t)) \left(\Theta(\upsilon(t), \mu(t), T(t)) + \beta \frac{T(t)}{P} \right) \qquad (2.3)$$

The first factor of this function, $\varphi(\omega(t))$, is the same as in [1] and called the "prevention function".

$$\varphi(\omega(t)) \quad = \quad h + (1-h)e^{-m\omega(t)} \qquad (2.4)$$

In this function, the parameter h represents a percentage of people that can not be retuned. This means that from all susceptibles who are inclined to become terrorists, $h \cdot 100\,\%$ cannot be influenced by prevention, no matter how big the corresponding spendings are. The parameter m represents the effectiveness of prevention. The higher m, the more effective any spendings on prevention are and therefore the lower φ will be, while holding $\omega(t)$ constant. Prevention always reduces the amount of terrorists and never increases it, so φ is always between h and 1. Because $\varphi(\omega(t))$ is multiplied by the remaining part of the initiation function $f(\upsilon(t), \omega(t), \mu(t), T)$, the amount of susceptibles that actually turn into terrorists can be reduced by $1 - \varphi(\omega(t))$.

Simmilar to [2], the term $\Theta(\upsilon(t), \mu(t), T(t))$ inside the brackets of equation (2.3) represents the collateral damage.

$$\Theta(\upsilon(t), \mu(t), T(t)) \quad = \quad \theta\upsilon(t)(1-\mu(t)) \left(1 - \frac{T(t)}{P}\right) \qquad (2.5)$$

The collateral damage results from the probability of getting a report with bad quality, $(1 - \mu(t))$, multiplied by the probability of hitting a person who is not a terrorist $\left(1 - \frac{T(t)}{P}\right)$ times the effort the government puts into the fight against terrorists $\upsilon(t)$ times a parameter θ. This parameter θ describes how much collateral damage leads to further radicalization.

The term $\beta \frac{T(t)}{P}$ in Equation (2.3) represents the influence of terrorists on the group of susceptibles for the purpose of recruitment. The term gets multiplied by $S(T)$ and $\varphi(\omega(t))$. The prevention function $\varphi(\omega(t))$ has already been discussed, so considering the term

$$\beta \frac{S(t)}{P} T(t)$$

will deepen the understanding of these dynamics. The more terrorists $T(t)$, the higher their overall influence and therefore the more people become radical. By definition, it is only possible for susceptibles to become terrorists and the smaller the fraction $\frac{S(t)}{P}$, the harder it is for a recruiting terrorist to find susceptibles, because terrorists do not know who is susceptible and who is not. The parameter β represents the average reach of each terrorist. This means that an average terrorist tries to recruit β people per year. $\frac{S(t)}{P}$ represents the possibility to randomly choose a susceptible and the multiplication by $T(t)$ represents the fact that a larger group of terrorists is able to influence more people. The more terrorists, the more recruitment, which implies more future terrorists.

Considering Equation (2.2), which describes the change in the amount of terrorists $T(t)$, it is obvious that all inflow comes from $S(t)$, namely $f(\upsilon(t), \omega(t), \mu(t), T(t))S(t)$, which has been discussed above. The outflow from the group of terrorists on the other hand can be distinguished into two parts. The first part $\sigma \cdot T(t)$ describes the part of the outflow that cannot be manipulated by any kind of control variable. One can interpret it as natural outflow out of the terrorists $T(t)$ caused by death or a change in ideology.

The second part of the outflow $g(\upsilon(t), \mu(t), T(t)) \cdot T(t)$ represents the outflow caused by law enforcement. Unlike in [2], the aim of this thesis is not to investigate the engagement between the government and the terrorists in a way that both groups try to decimate the size of their counterpart. Therefore, the size of the government military will not be considered in form of a variable which can be decimated by terrorist attacks. In this model, the effects caused by military are defined by the law enforcement $\upsilon(t)$, which in turn is determined by the decision maker, i.e., the government. The function $g(\upsilon(t), \mu(t), T(t))$ determines the current change of $T(t)$ caused by law enforcement and is defined as:

$$g(\upsilon(t), \mu(t), T(t)) = \upsilon(t) \left(\mu(t) + (1 - \mu(t)) \frac{T(t)}{P} \right) \tag{2.6}$$

Like in [2], the control variable $\mu(t) \in [0, 1)$ determines the percentage of correct reports in the process of searching for terrorists. In this way it can be called level of intelligence or level of education. In this thesis, $\mu(t) < 1$ is assumed, otherwise the model structure would change due to the possible absence of collateral damage. The military does not know which reports have a high quality and therefore every report leads to an intervention. In the case of application it is to assume that several analysts try to find terrorists. Only the fraction of analysts with high intelligence / education deliver good results. The other part, namely $(1 - \mu(t)) \cdot 100$ % of all analysts deliver reports with a random quality. In these cases it depends on the number of terrorists, whether or not the report finds a terrorist by some sort of luck. As P constitutes the size of the whole population and $T(t)$ the size of terrorists, the fraction $T(t)/P$ can be interpreted as the probability of a randomly selected individual being a terrorist.

The idea behind definition (2.6) is that in absence of any intelligence ($\mu(t) = 0$), meaning that every report of the government is random in quality, every military intervention of the government is a shot into the dark. Sometimes they hit a terrorist, sometimes they miss. Theoretically, the case $\mu(t) = 1$ and $\sigma = 0$ implies that every report is correct and no natural outflow out of $T(t)$ exists. Then the law enforcement $\upsilon(t)$ determines the current outflow rate of $T(t)$. Therefore, $\upsilon(t) \cdot T(t)$ is the current outflow in numbers caused by military actions. However, since $\mu(t) \in [0, 1)$ is defined as a measurement of intelligence, representing the fraction of reports that are not random but always correct, the change in $T(t)$ caused by government intervention is $\upsilon(t)\left(\mu(t) + (1-\mu)\frac{T(t)}{P}\right)T(t)$, which is exactly the definition of g in (2.6).

Figure 2.2 shows the impact of law enforcement as well as the impact of collateral damage for all combinations of $\mu(t)$ and $\upsilon(t)$ while holding $T(t)$ constant. The impact of law enforcement is highly dependent on $\upsilon(t)$ and $\mu(t)$. For any level of intelligence $\mu(t) < 1$, collateral damage exists, and the lower the intelligence level, the higher the collateral damage if a certain level of law enforcement $\upsilon(t)$ is applied. This in turn increases the amount of terrorists $T(t)$. Therefore, $\upsilon(t)$ has two contrary effects and as shown in Chapter 5, a necessary condition for the application of law enforcement is that its decreasing effect is more impactful than the effect of collateral damage.

Inserting the definition of $f(\upsilon(t), \omega(t), \mu(t), T(t))$ and $g(\upsilon(t), \mu(t), T(t))$ into the system of equations (2.1) and (2.2) results in the following dynamical system:

$$\begin{aligned} \dot{S}(t) &= k - \delta S(t) - \left(h + (1-h)e^{-m\omega(t)}\right) \cdot \\ &\quad \cdot \left(\theta\upsilon(t)(1-\mu(t))\left(1 - \frac{T(t)}{P}\right) + \beta\frac{T(t)}{P}\right) S(t) \qquad (2.7) \\ \dot{T}(t) &= \left(h + (1-h)e^{-m\omega(t)}\right) \cdot \\ &\quad \cdot \left(\theta\upsilon(t)(1-\mu(t))\left(1 - \frac{T(t)}{P}\right) + \beta\frac{T(t)}{P}\right) S(t) \\ &\quad - \left(\sigma + \upsilon(t)\left(\mu(t) + (1-\mu(t))\frac{T(t)}{P}\right)\right) T(t) \qquad (2.8) \end{aligned}$$

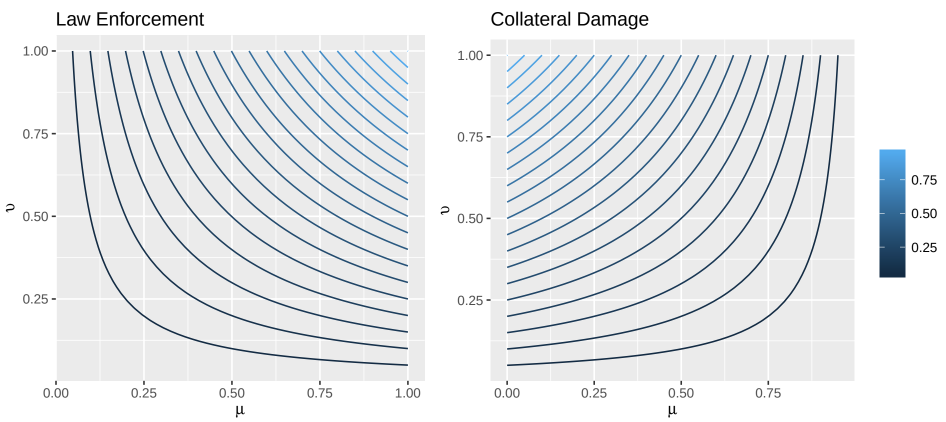


Figure 2.2: These two figures demonstrate the influence of the level of intelligence $\mu(t)$ and the level of law enforcement $\upsilon(t)$ on the impact of law enforcement and collateral damage under ceteris paribus conditions. Both illustrations show the contour lines of the underlying function. The size of terrorists was set to $30,000$ and the size of the whole population was set to $10,000,000$. The left figure shows the level of law enforcement $g(\upsilon(t), \mu(t), T(t))$ and the right figure shows the collateral damage for $\Theta(\upsilon(t), \mu(t), T(t))$ for every $(\mu(t), \upsilon(t))$ combination while holding the number of terrorists constant.

2.2 The Objective Function

The objective of the dynamic optimization problem in this thesis is to minimize the overall costs caused by terrorism and to trigger the appropriate reactions in form of controls. There are two different main components for the overall costs, namely the direct costs caused by terrorist attacks, $c_1(T(t))$, as well as the costs of the control variables against them, $c_2(\upsilon(t), \omega(t), \mu(t))$. To account for time preference, the discount factor e^{-rt} will be multiplied with the sum of the two cost terms in order to get the value of the current arising cost. Considering an infinite time horizon, the objective function is modeled by

$$\min_{\upsilon(t),\omega(t),\mu(t)} \int_0^\infty e^{-rt}(c_1(T(t)) + c_2(\upsilon(t), \omega(t), \mu(t)))dt \tag{2.9}$$

The two cost terms $c_1(T(t))$ and $c_2(\upsilon(t), \omega(t), \mu(t))$ will be discussed in detail in the following sections.

Social costs caused by terrorists

The first part of costs $c_1(T(t))$ represents the costs occurring because of the existence of terrorists. In this thesis it is assumed that all of these costs arise as a result of terrorist attacks.

Of course every attack that harms or kills humans is dreadful and it is impossible to measure the calamity inflicted upon the affected people and their relatives, but to get some kind of measurement, which is needed for this thesis, only the economic aspects of such attacks are considered.

Terrorism has various ways to produce social costs. Some of these costs arise because attacks harm tourism and the fear of instability and subsequent attacks lower the investment by companies or other states. In an economic way, every human represents a worker. Killing or wounding humans through terrorist attacks diminishes labor power and therefore the gross domestic product. Also, property destruction from incidents of terrorism damages the economy and therefore produces social costs.

In [3] an estimation based on IEP's methodology for the global economic impact of terrorism for the years 2000 to 2015 is made. Also in [3], the number of all terrorist attacks worldwide are provided for the same span of time. Figure 2.3 shows the combination of these two data series and suggests a linear connection between the economic impact and the number of attacks. The regression line is forced to go through the origin because zero attacks cause zero costs. It is important to note that the data do not distinguish between different kinds of terrorist attacks. An attack in which humans are harmed counts the same as an attack against a non-person target in this survey. The attacks on September 11 in the year 2001 and their economic impact are not considered because of their uniqueness.

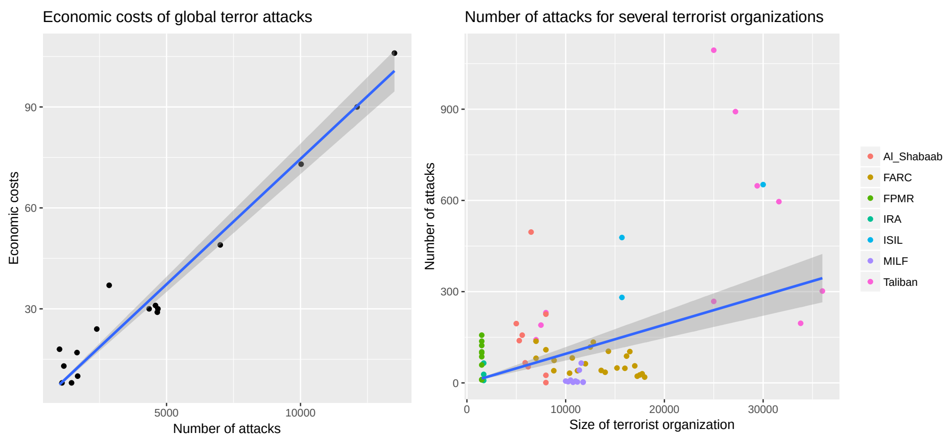


Figure 2.3: The left plot of this figure displays the global economic cost of terror attacks in billion USD in dependence of the number of global attacks. The right plot shows the size of the terrorist organization as well as the number of attacks per year for several terrorist organizations and years.

To continue the modeling of $c_1(T(t))$ a connection between the number of attacks and the size of a terrorist organization of use. Figure 2.3 also provides a visualization of the number of attacks per year compared to the size of the specific terrorist organization. The data of this plot are based on the Global Terrorism Database (GTD) [4] as well as various sources stated in [5] used for the estimation of the size of the terrorist group. The estimations for the size of the groups rely on the following procedure. If there is more than one estimation for one year, the mean is taken. If there are no estimations for a year, the surrounding values are used for a linear interpolation. Because the plot in Figure 2.3 does not suggest any specific kind of function, a linear model is assumed. The different terrorist organizations vary heavily among themselves, so the average number of attacks per member depends on various factors, e.g., the declared goal of the organization or the background of the members.

Obviously, any kind of function modeling the number of attacks must start in the origin, because if there are no terrorists, there will not be any attacks. Furthermore, it must have a positive slope, assuming that more terrorists commit more attacks.

The consideration of sleeper cells underlines the linear approach because they act mostly independently. On average, a doubling from such nearly independent cells leads to a doubling of attacks.

Overall, the sequential execution of two linear functions leads to a linear function which provides a cost function of the type $c_1(T(t)) = c_T T(t)$.

Social costs of the control instruments

The second cost-driving factor is the cost which occurs due to the fight against terrorism. As mentioned in Section 2.1 there are three possible control variables $\upsilon(t), \omega(t)$, and $\mu(t)$.

As discussed above $\mu(t) \in [0, 1)$ represents the fraction of high quality reports. Any randomly chosen report is correct with the probability $\mu(t) \cdot 100\,\%$. With the probability $(1 - \mu(t)) \cdot 100\,\%$, the report has a bad quality and will almost certainly lead to collateral damage, especially for small $T(t)$. The higher $\mu(t)$, the better, but this comes with additional costs. To increase $\mu(t)$ the government must invest into better education (e.g., training courses) for the analysts. In a heterogeneous society, the level of intelligence varies. So, in order to gain a medium level of intelligence (quality), not much investment is needed. Teaching with the goal of reaching a certain level of skills, namely to produce correct reports, the costs rise exponentially. This again is justified in the heterogeneous society. While some learn quickly and only need little training, others need more. Because of the assumption that the government is not able to distinguish between analysts who produce useful reports and those who do not, every analyst must get additional education, which explains the supposed behavior. The costs for intelligence / education therefore model training costs but also costs which arise due to the creation of reports. This includes spy reports, equipment costs, and so on. In case of $\mu(t) = 0$, no costs are produced, while a perfect intelligence $\mu(t) = 1$ is assumed to generate infinite costs. It is also assumed that the costs rise with the level of intelligence, the desired cost structure therefore has a convex appearance. One way to model the desired behavior is by using $-c_\mu \log_e(1-\mu(t))$ as a cost function for $\mu(t)$. Another approach would be to use $c_\mu \frac{\mu}{1-\mu}$, which also starts in the origin and has a pole at $\mu(t) = 1$. The favored version is the logarithmic approach, because of its favorable mathematical manageability.

The government intervention $\upsilon(t) \in [0,1]$ represents the effort the government puts into military activities to fight terrorism. Under the condition of perfect intelligence ($\mu(t) = 1$), the effect of government intervention $g(\upsilon(t), \mu(t), T(t))$ is equal to $\upsilon(t)$. This implies that a certain level of $\upsilon(t)$ reduces the amount of terrorists by $\upsilon(t)T(t)$ at time t. The costs for law enforcement are measured in USD and represent the costs the government spends on the fight against terrorism (e.g., wage for soldiers, tanks, aircraft, investigations, etc.). The structure of costs for the control variable $\upsilon(t)$ is therefore also assumed to be convex, starting in the origin and having a pole at $\upsilon(t) = 1$. For the sake of mathematical simplicity, the cost function for government intervention will also be modeled by $-c_\upsilon \log_e(1-\upsilon(t))$.

The spendings on prevention activities $\omega(t)$ are measured in USD and represent the costs the government spends on the fight against terrorism through prevention. Therefore, this variable can directly be added to the other costs.

Summing up the costs of all three control variables $\upsilon(t), \omega(t)$, and $\mu(t)$) we obtain

$$c_2(\upsilon(t), \omega(t), \mu(t)) = -c_\upsilon \log_e(1-\upsilon(t)) + \omega(t) - c_\mu \log_e(1-\mu(t))$$

This leads to the following objective function

$$\min_{\upsilon(t),\omega(t),\mu(t)} \int_0^\infty e^{-rt} \left(c_T T(t) - c_\upsilon \log_e(1-\upsilon(t)) + \omega(t) - c_\mu \log_e(1-\mu(t))\right) dt \tag{2.10}$$

2.3 The Dynamic Optimization Problem

In order to maintain the desired behavior of the model, all used parameters must be greater than zero. Of course, the values of the two state variables $S(T)$ and $T(t)$ must be non-negative and their sum has to be less than the whole population size P. All control variables are non-negative and two of them are limited. Furthermore, starting values for the number of terrorists and susceptibles, respectively, are needed and they are labeled as T_0 and S_0, respectively. The whole optimization problem can therefore be stated as

$$\min_{\upsilon(t),\omega(t),\mu(t)} \int_0^\infty e^{-rt} \left(c_T T(t) - c_\upsilon \log_e(1-\upsilon(t)) + \omega(t) - c_\mu \log_e(1-\mu(t))\right) dt \tag{2.11}$$

s.t.

$$\begin{aligned}
\dot{S}(t) &= k - \delta S(t) - \left(h + (1-h)e^{-m\omega(t)}\right) \\
&\quad \cdot \left(\theta\upsilon(t)(1-\mu(t))\left(1 - \frac{T(t)}{P}\right) + \beta\frac{T(t)}{P}\right) S(t) && (2.12) \\
\dot{T}(t) &= \left(h + (1-h)e^{-m\omega(t)}\right) \\
&\quad \cdot \left(\theta\upsilon(t)(1-\mu(t))\left(1 - \frac{T(t)}{P}\right) + \beta\frac{T(t)}{P}\right) S(t) \\
&\quad - \left(\sigma + \upsilon(t)\left(\mu(t) + (1-\mu(t))\frac{T(t)}{P}\right)\right) T(t) && (2.13) \\
P &\geq S(t) + T(t) \qquad \forall t \geq 0 && (2.14) \\
0 &\leq S(t), T(t), \omega(t) \qquad \forall t \geq 0 && (2.15) \\
0 &\leq \upsilon(t) \leq 1 \qquad \forall t \geq 0 && (2.16) \\
0 &\leq \mu(t) \leq 1 \qquad \forall t \geq 0 && (2.17) \\
0 &< k, \delta, h, m, \theta, P, \beta, \sigma, r, c_T, c_\upsilon, c_\mu && (2.18) \\
S(0) &= S_0 && (2.19) \\
T(0) &= T_0 && (2.20)
\end{aligned}$$

2.4 Parameterization

It is difficult to estimate reasonable parameter values for this model. To do so, reliable sources are needed for several different values. It is important to underline that the aim of the next section is not to model the reality in the best possible way. The goal is to get reasonable parameter values that fit into the dynamics of the stated model. For that purpose, various sources are used in the following subsections. In most cases, the IS conflict is used as a source of information. Some of the estimated values (e.g., the size of terrorists) differ greatly between different sources. In some cases, it is important to stick to one source (e.g., the official information) in order to obtain the correct proportion to other parameter values needed.

Estimation of P

Because most of the estimations refer to the IS conflict, also the total number of people P in the model will be set to a size corresponding with this conflict. As stated in [6], the total number of people living in the IS active region is estimated to be approximately 10 million. Therefore, P is assumed to be $10,000,000$.

Estimation of c_υ

The parameter c_υ calibrates the cost function for government interventions. In order to get an estimation for c_υ, a point in the $(\upsilon(t), cost)$-space is defined by research which fits the function $-c_\upsilon \log_e(1-\upsilon(t))$.

According to [7] and [8], the US Coalition killed about $26,000$ IS fighters on average over the duration of one year leaving about $17,500$ left. At the beginning of the intervention, the group size was $62,500$ according to these estimations, while ignoring all other influences like inflow or natural outflow. This corresponds to a reduction of 41.6% per year due to law enforcement.

According to [9], the spendings of the US on the war against IS are estimated to be 11 mio. USD per day, which corresponds to $4,015$ mio. per year. This provides the estimation $c_\upsilon = 7.4648 \cdot 10^9$.

Estimation of c_T

According to Section 2.2, c_T represents the average economic cost caused by one terrorist per year. To estimate c_T, the data from [3], The Global Terrorism Database (GTD) [4], and several sources collected in [5] are used. The collected data are displayed in the two plots in Figure 2.3. Both linear regressions are robust linear models, and their slopes represent the values of interest. Every attack costs on average $7,457,614$ USD, and every terrorist commits on average 0.009567419 attacks per year, which leads to average costs of $c_T = 71,350.12$ USD per terrorist per year.

Estimation of c_μ

The probability of hitting a terrorist is given by $\mu(t) + (1 - \mu(t)) \cdot T(t)/P$. The civilian casualty ratio (CCR) indicates the percentage of civilians hit by government interventions. Therefore, $1 - (\mu(t) + (1 - \mu(t)) \cdot T(t)/P)$ can be treated as equal to the civilian casualty ratio. For a given civilian casualty ratio, the current value of $\mu(t)$ can therefore be estimated by $\mu(t) = \frac{1-CCR-T(t)/P}{1-T(t)/P}$.

Combining these considerations with an estimation of costs for the analyst part of the military expenditures results in an estimated point in the $(\mu(t), cost)$-space. Again, the cost function $-c_\mu \log_e(1 - \mu(t))$ is assumed to go through this point which leads to an estimation for c_μ.

In order to estimate $\mu(t)$ consider the IS conflict with one counterpart being the US-led coalition. According to [10], the minimum number of civilians killed by the coalition is $5,637$. Several of the latest estimations of IS fighters killed in total by the coalition [11] vary heavily and are assumed to be $37,500$. Dividing the number of civilians killed by the total number of people killed, leads to a civilian death ratio of about 13%. The population size in the territory controlled by IS was about 10 millions at its peak according to [6]. This implies the estimation $\hat{\mu}_1 = 0.869$, if again we assume that the size of the terrorists group was at $62,500$ at the beginning.

In historic data, the civilian casualty ratio is typically much larger. Consider the so-called $80 - 90$ rule, which claims that about eighty to ninety percent of war victims are civilians. Although this rule is controversial, it can provide another reference point for this estimation. Assuming a civilian casualty ratio of 80% results in the estimation $\hat{\mu}_2 = 0.197$.

Taking the mean of these two estimations ($\hat{\mu} = 0.533$) is probably a promising value to start with.

According to [12], the US employs about $1,500$ analysts with an average wage of $75,000$ USD per year [13]. This implies costs of $112,500,000$ USD per year. Because of several additional costs (e.g., education, equipment, costs for informants, etc.) it is assumed that the total expenditure for reports are ten times as high. Of course not all of these resources are used in the IS conflict alone. According to [14], the share of expenditures for the IS conflict was 12.47% for the year 2015. This leads to an estimation of costs of about $140,287,500$ USD per year.

Inserting this μ - cost combination into the assumed cost structure implies the estimation $c_\mu = 1.8424 \cdot 10^8$.

Estimation of r

The parameter r controls the time preference rate. Humans prefer payments of the same height in the near future over long-term payments, especially because of inflation. According to [15], from 2000 to 2015 the global average inflation rate is about 3.97%, which is used as an estimation for r. There are several investigations about personal time preference, so it would also be possible to add these two rates, if the inflation rate is not considered in the personal time preference. Nevertheless, in this thesis only the average inflation is used as an estimation for r.

Estimation of β

β represents the influence range of each terrorist. It describes the average rate at which a terrorist tries to radicalize other people per year. Of course, this does not mean that the number of terrorists increases by $\beta T(t)$, because only susceptibles become radical if terrorists try to recruit them. In order to estimate β, consider a specific terror organization recruiting in several countries, starting at time η. In what follows, the time index (η) will sometimes be omitted for the purpose of better readability. $T(\eta)$ is equal to the number of terrorists at time η, so the term $T(\eta+1) - T(\eta)$ represents the increase in terrorists in one year and will be defined by $M(\eta)$.

All countries people are joining the terrorist organization from are divided into two groups, A and B. It is assumed, that the average structures (sympathy for terrorism) of theses two main groups are the same. The total number of terrorists recruiting can be split up into two main groups $(T(\eta) = \widetilde{T} + \widehat{T})$. Terrorists in group $\widetilde{T}$ are only recruiting in country A, while all terrorists in group $\widehat{T}$ are recruiting in country B only. Also, the total number of people joining can be split accordingly $(M(\eta) = \widetilde{M} + \widehat{M})$.

The reason for this approach is that for a sample of n countries, the susceptible rates $\frac{S_i}{P_i}$, as well as the number of people joining the terror organization M_i are known for each country $i \in \{1, \ldots, n\}$. The first group of countries (A), therefore, is set to these n countries. The total number of new recruits, coming from A, is equal to the sum of people coming from each country $i \in \{1, \ldots, n\}$, which is equal to $\widetilde{M} = \sum_i^n M_i$. Therefore, $\widehat{M}$ can be calculated by $\widehat{M} = M(\eta) - \sum_i^n M_i$. It holds that $\widehat{M}$ is equal to the number of people joining the terrorist organization in all but the n sample countries.

Because the effort of recruiting (β) is the same for each terrorist, the number of terrorists recruiting at time η can be split up into the two groups $\widetilde{T}$ and $\widehat{T}$ according to the proportional share of $\widetilde{M}$ and $\widehat{M}$. Therefore $T(\eta) = \widetilde{T} + \widehat{T}$ and

$$T(\eta) = \frac{\widetilde{M} + \widehat{M}}{M(\eta)} T(\eta) = \frac{\sum_i^n M_i}{M(\eta)} T(\eta) + \frac{\widehat{M}}{M(\eta)} T(\eta)$$

lead to

$$\widetilde{T} = \frac{\sum_i^n M_i}{T(\eta+1) - T(\eta)} T(\eta)$$

The number of people joining (M_i) is not put into relation with the total number of people in that specific country (P_i) because the size of a terror organization is usually very small compared to the population size of a country. So in the process of recruiting, the pool of potential followers is virtually endless. Therefore, a difference must not be made between the different countries for example by different weights.

For every country i the share $\frac{S_i}{P_i}$ determines how hard it is for the terror organization to recruit new members, the smaller $\frac{S_i}{P_i}$, the harder. Therefore, the effort the terror organization invests into the countries differs and can be represented by the number T_i, which is the proportion of $\widetilde{T}$ responsible for country i. The sum of all terrorists, responsible for country group A, must naturally be the same as the number of terrorists responsible for the sample of n countries, which can be written as $\sum_i^n T_i = \widetilde{T}$.

The number of people joining the terror organization from the n sample countries can be written as

$$\beta \frac{S_i}{P_i} T_i = M_i \qquad i = 1, \dots, n$$

with T_i representing the amount of terrorists trying to recruit in country i. The range of this subgroup is βT_i, and because the share of susceptibles in this country is given by $\frac{S_i}{P_i}$, the number of people actually joining is given by $\beta \frac{S_i}{P_i} T_i$. Rewriting this set of equations leads to

$$\beta T_i = \frac{M_i P_i}{S_i} \qquad i = 1, \dots, n$$

while summing up all of these n equations results in a single one

$$\beta T_1 + \dots + \beta T_n = \frac{M_1 P_1}{S_1} + \dots + \frac{M_n P_n}{S_n}.$$

Because of $\sum_i^n T_i = \widetilde{T}$, an estimation for β is given by

$$\beta = \frac{\frac{M_1 P_1}{S_1} + \dots + \frac{M_n P_n}{S_n}}{\widetilde{T}}. \qquad (2.21)$$

Of course, there are a lot of aspects influencing the number of people joining from country i (M_i) which are not taken into account by this estimation. For example, the influence of collateral damage is not considered, which leads to an overestimation of β. Another discrepancy persists because the structure of the n sample countries may differ from the rest. In the concrete example used for the actual estimation, the IS conflict is considered, and the structure of the sample countries (e.g., Tunisia, Turkey) differs highly from other countries like Germany where the number of susceptibles is most likely much lower, making it harder to recruit. In this case, this leads to an underestimation of β. Furthermore, some radicalized people are not allowed to leave their country of origin and therefore do not appear in the statistics of foreign fighters, which implies an underestimation of β. Also, the effect of returnees should be considered, and not doing so results in an overestimation of β. Furthermore, the effects of a possible prevention are not considered, which leads to an underestimation of β.

The effects of these examples differ in their sizes and directions on β and there may be a lot more influences not mentioned. The reason of not taking these effects into account is simply the impossibility of measuring all of them correctly.

To get an actual estimation, consider the IS terror group. According to [16], [17], [18], and [19], the number of terrorists approximately increased by $14,300$ between the years 2014 and 2015. Because of the information in [20], [21], [22], [23], [24], and [25], the average number of foreign fighters per year can be estimated for the countries Tunisia, Malaysia, Turkey, Jordan, and Egypt. The susceptible rates $\frac{S_i}{P_i}$ are taken from [26]. Assuming that the rest of the $14,300$ fighters are coming from countries equal in their characteristics and applying the above considerations lead to the estimation $\beta = 8.03$.

Estimation of k

For the natural inflow k, the birth rate times the population size P is assumed. This underlies the assumption that everyone, especially when being young, has a phase in their life where he or she is susceptible to terrorism. According to [27], the average worldwide birthrate for the year 2015 is given by 1.9081%, which results in the estimation $k = 190,810$.

Estimation of δ

To estimate the natural outflow from the group of susceptibles, the current percentage of susceptibles is estimated and assumed to be the upper bound. The reason for this assumption is that the number of terrorists is in general relatively small compared to the size of the whole population. Therefore, the number of susceptibles is assumed to be near its upper bound. According to Section 2.1, the upper bound for the number of susceptibles is given by k/δ. In order to estimate the susceptible rate $\frac{S}{P}$, which is needed in the following calculation, the terror organization IS is again taken as an example. Because of the fact that in the subsequent consideration only one susceptible rate for one single representative country is taken into account, the susceptible rate has to be estimated. $\beta\frac{S(\eta)}{P}T(\eta) = M$ with M being the number of new terrorists joining between η and $\eta + 1$ leads to $S(\eta) = \frac{MP}{\beta T(\eta)} = 0.113P$. Assuming that $S(\eta)$ is near its upper bound is equal to $\frac{k}{\delta} \approx 0.113P$, which therefore provides the estimation $\delta \approx 0.1688$.

Estimation of θ

The parameter θ describes how much collateral damage leads to radicalization. Recall the rate representing the influence of collateral damage:

$$\varphi(\omega(t))\theta\upsilon(t)(1-\mu(t))\left(1-\frac{T(t)}{P}\right)$$

Note that $\upsilon(t) \in [0,1]$ describes the effort the government is willing to put into the fight against terrorism, where $\upsilon(t) = 1$ is therefore the maximum effort possible. In order to estimate θ, consider the extreme case $\omega(t) = 0$ (which implies $\varphi(\omega(t)) = 1), \mu(t) = 0$, and $\upsilon(t) = 1$. Because the number of terrorists is in general relatively small compared to the size of the whole population, it holds that $(1 - T(t)/P) \approx 1$. This extreme case is the worst case scenario with the maximum collateral damage. Every person in $S(t)$ is by definition susceptible to terrorism, meaning that there are circumstances under which any given person may become a terrorist. Under the worst case scenario with the maximum collateral damage, every person in $S(t)$ should become a terrorist in the long run. If a person does not become a terrorists under these circumstances, this person should not be in $S(t)$. Again, it is assumed that this transition takes time. Similar to law enforcement, it is assumed that

$$\varphi(\omega(t))\theta\upsilon(t)(1-\mu(t))\left(1-\frac{T(t)}{P}\right) \stackrel{!}{=} 1 \tag{2.22}$$

Inserting $\omega(t) = 0, \mu(t) = 0, \upsilon(t) = 1$ and $(1 - T(t)/P) \approx 1$ leads to the estimation $\theta = 1$.

Estimation of m and h

Although an "Anti-Terror Prevention Programme" exists in the United Kingdom against radicalization and there are also some results provided, these data cannot be used for the parametrization in this thesis because the mentioned program is actively searching for susceptibles. In contrast, the prevention program modeled in this thesis is a classic passive one. The idea behind the prevention function provided in [1] is that every child receives the same prevention program while reaching a specific age. Even though the cost data estimated in [28] are for a drug prevention program, it is assumed that they can be applied to this situation as well. A prevention program with the same organizational structure applied just as often with another topic should cost approximately the same. In [28] it is estimated that a prevention program costs 150 USD per student in the year 1999. Adjusting for inflation this is equal to 218 USD in the year 2015. The effects of such programs are very hard to predict due to the lack of necessary data. The structure of φ assumes that under all circumstances a certain share of people, namely h

cannot be convinced by the prevention program. It is assumed in this theses that this share of people is the same for both topics. Therefore, it is assumed that children who are positively reacting to a drug prevention program can also be positively affected by a radicalization prevention program.

With the data provided in [28], the effectiveness of the program is assumed to be 6.5%. Like in [1] it is assumed that under the largest effort possible, the prevention program can be one and a half times as effective, which leads to the estimation $h = 0.9025$. Assuming once again the birth rate to be 0.019081 [27] leads to total expenditures of $0.019081 \cdot 218P = 41,596,580$ USD per year for prevention. Inserting this values into the prevention function is equal to

$$\varphi(41,596,580) = h + (1-h)e^{-m41,596,580} \stackrel{!}{=} 1 - 0.065 \tag{2.23}$$

Solving this equation, with respect to m, provides the estimation $m = 2.641112 \cdot 10^{-8}$.

Estimation of σ

The only ways for an individual to leave the group of terrorists without government intervention is through death or by a change in their ideology. Without government intervention, a terrorist can die either a natural death or a self-induced death through a suicide attack. According to [29], IS committed 471 suicide attacks from 2014 to 2016, which corresponds to 157 per year.

The total number of fighters for the Islamic State is estimated to be approximately 30,000 at the end of the year 2015. This implies a chance of $0.52\% = 157/30000$ to die from suicide for a terrorist. Because of the assumption of a constant population, the death rate is approximately the same as the birth rate which was about 1.9081% worldwide in the year 2015 according to [27]. Adding up these two numbers implies a lower bound for $\hat{\sigma}_1$ of about 0.0243.

With the data provided in [30], the percentage of people exhibiting a change in ideology are estimated by using the percentage of returnees. By adding this percentage to $\hat{\sigma}_1$, another estimation is provided by $\hat{\sigma}_2 = 0.1128$. This is very likely an overestimation because of two reasons. First, not all of the returnees fully turned away from their radical ideology. Second, it is very likely that some foreign fighters do not know exactly what they are getting involved in. They are counted as terrorists but they should not be. If their lack of information were to be remedied, some of them would likely not try to fight for the Islamic State.

Taking the mean of these two estimations leads to the estimation $\sigma = 0.0685$.

Base Case Parameter Values

The descriptions and base case values for the model parameter are summarized in Table 2.1.

Parameter	Value	Description
P	$10,000,000$	Population size
c_υ	$7.4648 \cdot 10^9$	Cost parameter for law enforcement
c_T	71350.12	Average social costs per terrorist per year
c_μ	$1.8424 \cdot 10^8$	Cost parameter for good quality of reports
r	0.0397	Time preference rate
β	8.03	Influence of terrorists on susceptibles
k	$190,810$	Inflow to susceptibles
δ	0.1688	Natural outflow rate of susceptibles
θ	1	Describes how much collateral damage leads to radicalization
m	$2.641112 \cdot 10^{-8}$	Effectiveness of prevention ω
h	0.9025	Share of susceptibles who cannot be prevented from becoming terrorists even by large efforts of ω
σ	0.0685	"Natural" outflow rate of terrorists

Table 2.1: Parameter values and descriptions

Chapter 3

The Uncontrolled System

The analysis of the model is initiated by assuming that there is no government interaction whatsoever. Setting $\upsilon(t) = \omega(t) = \mu(t) = 0 \quad \forall t$ provides the uncontrolled system. $\upsilon(t) = 0$ eliminates all terms containing $\mu(t)$, the level of intelligence $\mu(t)$ is therefore irrelevant, if the law enforcement $\upsilon(t)$ is equal to zero. So the case where only $\mu(t) \neq 0$ would lead to the same system, even if not all controls are zero. The uncontrolled system therefore is given by

$$\dot{S}(t) = k - \delta S(t) - \beta \frac{T(t)}{P} S(t) \tag{3.1}$$

$$\dot{T}(t) = \beta \frac{T(t)}{P} S(t) - \sigma T(t) \tag{3.2}$$

$$S(0) = S_0 > 0 \tag{3.3}$$

$$T(0) = T_0 > 0 \tag{3.4}$$

This model is very similar to the well known predator–prey model by $S(t)$ being the prey and $T(t)$ being the predator, respectively. Comparing the two models, the consumption rate of the predator must be equal to the reproduction rate of predators per prey and therefore equal to $\frac{\beta}{P}$ so as to be similar to the considered model. The main difference between the two models is that the growth rate of the uncontrolled model follows a limited growth $(k - \delta S(t))$ whereas the prey in the standard predator-prey model follows an exponential growth.

3.1 Steady States Analysis

In order to find the steady states, the condition $\dot{S}(t) = \dot{T}(t) = 0$ leads to the following equations:

$$0 = k - \delta S(t) - \beta \frac{T(t)}{P} S(t) \tag{3.5}$$

$$0 = \beta \frac{T(t)}{P} S(t) - \sigma T(t) \tag{3.6}$$

By reformulating Equation (3.5), $T(t)$ can be expressed by a function of $S(t)$:

$$T(t) = \frac{kP}{\beta S(t)} - \frac{\delta P}{\beta} \tag{3.7}$$

Rewriting (3.6) results in the following equation:

$$\left(\frac{\beta S(t)}{P} - \sigma \right) T(t) = 0 \tag{3.8}$$

This leads to two possible solutions:

$$T(t) = 0 \quad \vee \quad S(t) = \frac{\sigma P}{\beta} \tag{3.9}$$

Inserting (3.9) into (3.7) leads to the following two solutions for the steady state in the (S, T) - space:

$$(S_1^*, T_1^*) = \left(\frac{k}{\delta}, 0 \right) \tag{3.10}$$

$$(S_2^*, T_2^*) = \left(\frac{P\sigma}{\beta}, \frac{\beta k - \delta P \sigma}{\beta \sigma} \right) \tag{3.11}$$

3.2 Stability of the Steady States

To investigate the stability of the two steady states, the principles of linearized stability are applied. The resulting Jacobian matrix for the system (3.1) - (3.2) is then given by:

$$J(S, T) = \begin{pmatrix} -\delta - \frac{\beta}{P} T(t) & -\frac{\beta}{P} S(t) \\ \frac{\beta}{P} T(t) & \frac{\beta}{P} S(t) - \sigma \end{pmatrix} \tag{3.12}$$

By inserting a steady state into the Jacobian $(J(S_i^*, T_i^*))$, the eigenvalues of the resulting matrix provide important information about the asymptotic behavior of the system in the considered steady state. If the real parts of all eigenvalues are negative, the steady state is asymptotically stable.

3.2.1 The First Steady State

For the first steady state $(S_1^*, T_1^*) = (\frac{k}{\delta}, 0)$, the resulting Jacobian matrix is given by

$$J(S_1^*, T_1^*) = \begin{pmatrix} -\delta & -\frac{\beta}{P}\frac{k}{\delta} \\ 0 & \frac{\beta}{P}\frac{k}{\delta} - \sigma \end{pmatrix} \tag{3.13}$$

The eigenvalues then are the solution of setting the characteristic polynomial χ_J equal to zero.

$$\chi_J(S_1^*, T_1^*)(\lambda) = \begin{vmatrix} -\delta - \lambda & -\frac{\beta}{P}\frac{k}{\delta} \\ 0 & \frac{\beta}{P}\frac{k}{\delta} - \sigma - \lambda \end{vmatrix} \tag{3.14}$$

$$= (-\delta - \lambda)\left(\frac{\beta k}{P\delta} - \sigma - \lambda\right) \stackrel{!}{=} 0 \tag{3.15}$$

The solutions of this equation are:

$$\lambda_1 = -\delta \tag{3.16}$$

$$\lambda_2 = \frac{\beta k}{P\delta} - \sigma \tag{3.17}$$

Note that all of the occurring parameters $(k, \delta, \sigma, P, \beta)$ are positive. Because of $\delta > 0$ it follows that λ_1 is always negative. λ_2 is less than zero exactly when $\beta k < \delta P \sigma \Leftrightarrow \frac{k}{\delta} < \frac{\sigma P}{\beta}$. In this case, all eigenvalues are less than zero and the steady state is asymptotically stable.

3.2.2 The Second Steady State

Inserting the second steady state $(S_2^*, T_2^*) = \left(\frac{P\sigma}{\beta}, \frac{\beta k - \delta P\sigma}{\beta\sigma}\right)$ into the Jacobian matrix (3.12) results in

$$J(S_2^*, T_2^*) = \begin{pmatrix} -\delta - \frac{\beta k - \delta P\sigma}{P\sigma} & -\sigma \\ \frac{\beta k - \delta P\sigma}{P\sigma} & 0 \end{pmatrix} \tag{3.18}$$

$$\chi_J(S_2^*, T_2^*)(\lambda) = \begin{vmatrix} -\delta - \frac{\beta k - \delta P\sigma}{P\sigma} - \lambda & -\sigma \\ \frac{\beta k - \delta P\sigma}{P\sigma} & 0 - \lambda \end{vmatrix} \tag{3.19}$$

$$= \lambda^2 + \frac{\beta k}{P\sigma}\lambda + \frac{\beta k - \delta P\sigma}{P} \stackrel{!}{=} 0 \tag{3.20}$$

With the quadratic formula, the results for $\lambda_{1,2}$ are given by

$$\lambda_{1,2} = -\frac{\beta k}{2P\sigma} \pm \sqrt{\left(\frac{\beta k}{2P\sigma}\right)^2 - \frac{\beta k - \delta P\sigma}{P}} \tag{3.21}$$

Since $\frac{\beta k}{2P\sigma}$ is always positive, $-\frac{\beta k}{2P\sigma}$ will always be less than 0. The real part of any square root is always positive, so the only possibility for one eigenvalue λ_i to be non-negative is in the positive case.

$$-\frac{\beta k}{2P\sigma} + \sqrt{\left(\frac{\beta k}{2P\sigma}\right)^2 - \frac{\beta k - \delta P\sigma}{P}} \geq 0 \tag{3.22}$$

$$\Leftrightarrow \quad \sqrt{\left(\frac{\beta k}{2P\sigma}\right)^2 - \frac{\beta k - \delta P\sigma}{P}} \geq \frac{\beta k}{2P\sigma} \tag{3.23}$$

This is only possible if $\frac{\beta k - \delta P\sigma}{P} \leq 0 \quad \Leftrightarrow \quad \beta k - \delta P\sigma \leq 0$. So the second steady state $(S_2^*, T_2^*) = \left(\frac{P\sigma}{\beta}, \frac{\beta k - \delta P\sigma}{\beta\sigma}\right)$ is asymptotically stable if $\beta k > \delta P\sigma \quad \Leftrightarrow \quad \frac{k}{\delta} > \frac{P\sigma}{\beta}$.

The asymptotic stability of the two steady states can therefore be summed up by

$$(S_1^*, T_1^*) \text{ is asymptotic stable} \quad \Leftrightarrow \quad \frac{k}{\delta} < \frac{P\sigma}{\beta} \tag{3.24}$$

$$(S_2^*, T_2^*) \text{ is asymptotic stable} \quad \Leftrightarrow \quad \frac{k}{\delta} > \frac{P\sigma}{\beta} \tag{3.25}$$

3.3 Phase Portrait

In order to get a better idea of the dynamical system, looking at the phase portrait is advisable. The isoclines have already been explicitly calculated in Section 3.1 for the simple case of an uncontrolled system. The results and the resulting dynamics are as follows:

$$\dot{S}(t) = 0 \quad \Leftrightarrow \quad T(t) = \frac{kP}{\beta S(t)} - \frac{\delta P}{\beta} \tag{3.26}$$

$$\dot{S}(t) \gtrless 0 \quad \Leftrightarrow \quad T(t) \lessgtr \frac{kP}{\beta S(t)} - \frac{\delta P}{\beta} \tag{3.27}$$

$$\dot{T}(t) = 0 \quad \Leftrightarrow \quad T(t) = 0 \quad \vee \quad S(t) = \frac{\sigma P}{\beta} \tag{3.28}$$

$$\begin{aligned} \dot{T}(t) > 0 \quad \Leftrightarrow \quad & S(t) > \frac{\sigma P}{\beta} \quad \wedge \quad T(t) > 0 \\ \vee \quad & S(t) < \frac{\sigma P}{\beta} \quad \wedge \quad T(t) < 0 \end{aligned} \tag{3.29}$$

$$\begin{aligned} \dot{T}(t) < 0 \quad \Leftrightarrow \quad & S(t) < \frac{\sigma P}{\beta} \quad \wedge \quad T(t) > 0 \\ \vee \quad & S(t) > \frac{\sigma P}{\beta} \quad \wedge \quad T(t) < 0 \end{aligned} \tag{3.30}$$

The phase portrait has two different appearances, depending on the relative sizes of the parameters k, δ, σ, P, and β. Figure 3.1 shows the isoclines for both cases as well as the behavior of each state variable in different areas.

In case of $\frac{k}{\delta} < \frac{\sigma P}{\beta}$, the only feasible steady state is a point with no terrorists at all. If in this case, for whatever reason a group of terrorists appear, they would vanish over time without the need of intervention whatsoever. So, in an obvious manner, this case is not very compelling for this thesis.

In case of $\frac{k}{\delta} \geq \frac{\sigma P}{\beta}$, two feasible steady states can exist. Again, one of them requires the number of terrorists to be zero. The analysis in Section 3.2 shows the instability of this point for the considered case. The second steady state, however, has positive values for S^* and T^*. This steady state is asymptotically stable as we have shown above.

By using the estimations for the cited parameters from Section 2.4, a phase portrait is generated in order to deepen the understanding of the dynamics. As expected, the derived parameter values imply the inequality $\frac{k}{\delta} > \frac{\sigma P}{\beta}$, which ensures the system to work in the intended way. This phase portrait of the uncontrolled system is presented in Figure 3.2. It is important to notice that the arrow length does not represent the degree of change in these figures. The changes in the system are much smaller for lower values of $S(t)$ and $T(t)$. By taking into account this information, it would be impossible to see the dynamics because of the great differences.

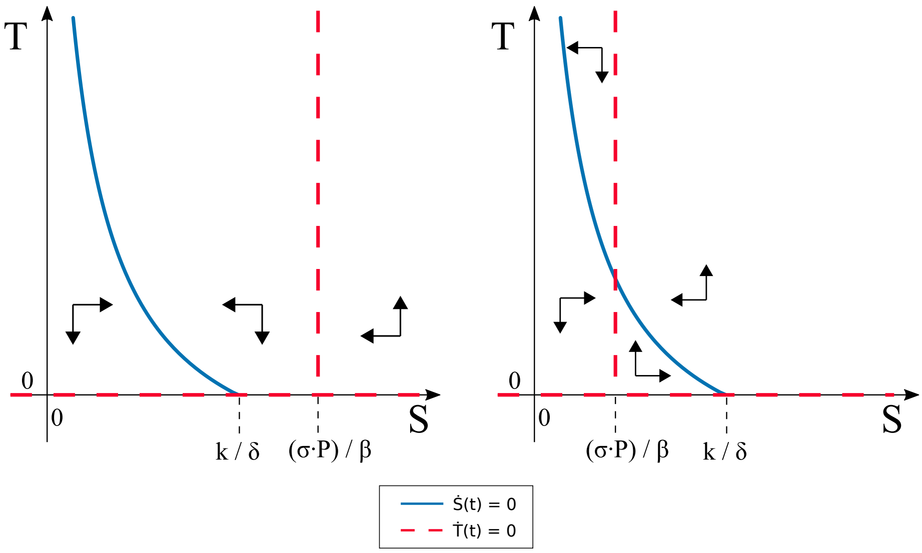
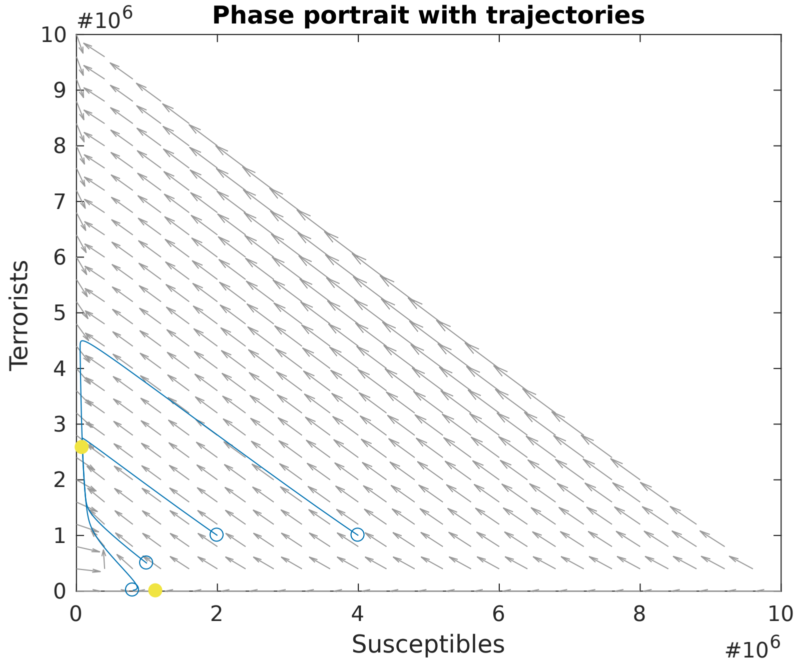
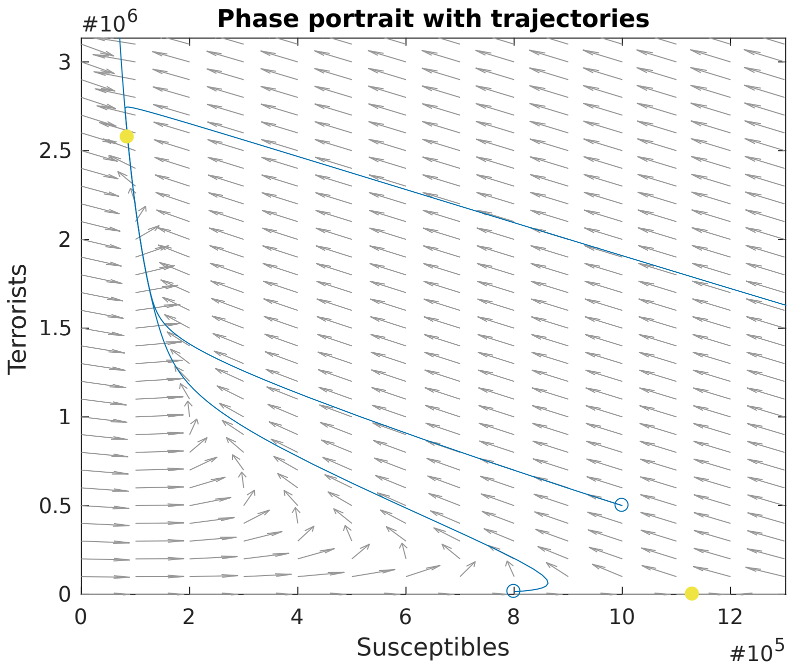


Figure 3.1: This figure indicates the behavior of the system for the two cases $\frac{k}{\delta} < \frac{\sigma P}{\beta}$ and $\frac{k}{\delta} \geq \frac{\sigma P}{\beta}$ by showing the isoclines and the directions of the changes of each state variable in different regions. An upward-pointing arrow, for example, indicates an increase in terrorists $\Leftrightarrow \dot{T}(t) > 0$.



(a) Overview



(b) Detailed view

Figure 3.2: These figures show the phase portrait of the uncontrolled system by presenting two different zoom levels. In the overview part in the upper graph (a), every feasible state in the dynamic system is displayed, whereas the detailed view in the lower graph (b) not only shows the two steady states but also a finer grid of direction arrows.

Chapter 4

The System with Constant Strategies

Studying the system under constant strategies gives a good first insight into the impact of the three different control variables. Also the effects of different parameters will be covered by means of a sensitivity analysis. Finally, the optimal constant strategy will be discussed for a specific starting value.

In order to increase readability, the function $\varphi(\omega)$ will not be displayed in its extended form in this chapter. By holding the controls constant ($\upsilon(t) = \upsilon, \mu(t) = \mu, \omega(t) = \omega \quad \forall t \geq 0$), the dynamics of the system (2.12) - (2.20) reduce to

$$\begin{aligned} \dot{S}(t) &= k - \delta S(t) \\ &\quad -\varphi(\omega) \cdot \left(\theta\upsilon(1-\mu)\left(1 - \frac{T(t)}{P}\right) + \beta\frac{T(t)}{P}\right) S(t) \end{aligned} \tag{4.1}$$

$$\begin{aligned} \dot{T}(t) &= \varphi(\omega) \cdot \left(\theta\upsilon(1-\mu)\left(1 - \frac{T(t)}{P}\right) + \beta\frac{T(t)}{P}\right) S(t) \\ &\quad - \left(\sigma + \upsilon\left(\mu + (1-\mu)\frac{T(t)}{P}\right)\right) T(t) \end{aligned} \tag{4.2}$$

$$S(0) = S_0 \tag{4.3}$$

$$T(0) = T_0 \tag{4.4}$$

4.1 Steady States Analysis

Like in Chapter 3, setting (4.1) and (4.2) equal to zero results in a system of equations whose solutions yield the desired steady states.

$$\begin{aligned} 0 &= k - \delta S(t) \\ &\quad -\varphi(\omega) \cdot \left(\theta \upsilon (1-\mu) \left(1 - \frac{T(t)}{P} \right) + \beta \frac{T(t)}{P} \right) S(t) \qquad (4.5) \\ 0 &= \varphi(\omega) \cdot \left(\theta \upsilon (1-\mu) \left(1 - \frac{T(t)}{P} \right) + \beta \frac{T(t)}{P} \right) S(t) \\ &\quad - \left(\sigma + \upsilon \left(\mu + (1-\mu) \frac{T(t)}{P} \right) \right) T(t) \qquad (4.6) \end{aligned}$$

Rewriting equation (4.5), $S(t)$ can be written as

$$S(t) = \frac{k}{\delta + \varphi(\omega) \left(\theta \upsilon (1-\mu) + (\beta - \theta \upsilon (1-\mu)) \frac{T(t)}{P} \right)} \qquad (4.7)$$

Note that under the given assumptions, the denominator is always greater than zero. This can easily be seen from (4.5), because all parameters are greater than zero, μ is less than or equal to one, and $T(t)$ is alway less than P. The term multiplied by $\varphi(\omega)$ is therefore greater than or equal to zero, and by adding δ it is definitely greater than zero. Substituting $S(t)$ from (4.7) into (4.6) and rearranging provides a cubic equation in $T(t)$:

$$\begin{aligned} &\left[2\varphi(\omega)\theta\mu\upsilon^2 - \varphi(\omega)\theta\mu^2\upsilon^2 - \beta\varphi(\omega)\mu\upsilon - \varphi(\omega)\theta\upsilon^2 \right. \\ &\quad \left. + \beta\varphi(\omega)\upsilon \right] T(t)^3 + \left[P\delta\upsilon + P\varphi(\omega)\theta\upsilon^2 + P\beta\varphi(\omega)\sigma \right. \\ &\quad - P\delta\mu\upsilon + 2P\mu^2\varphi(\omega)\theta\upsilon^2 + P\beta\mu\varphi(\omega)\upsilon \\ &\quad \left. - P\varphi(\omega)\sigma\theta\upsilon - 3P\mu\varphi(\omega)\theta\upsilon^2 + P\mu\varphi(\omega)\sigma\theta\upsilon \right] T(t)^2 \\ &\quad + \left[P^2\delta\sigma + P^2\delta\mu\upsilon - P\beta k\varphi(\omega) + P^2\mu\varphi(\omega)\theta\upsilon^2 + Pk\varphi(\omega)\theta\upsilon \right. \\ &\quad - P^2\mu^2\varphi(\omega)\theta\upsilon^2 + P^2\varphi(\omega)\sigma\theta\upsilon - P^2\mu\varphi(\omega)\sigma\theta\upsilon \\ &\quad \left. - Pk\mu\varphi(\omega)\theta\upsilon \right] T(t) + \left[P^2 k\mu\varphi(\omega)\theta\upsilon - P^2 k\varphi(\omega)\theta\upsilon \right] = 0 \end{aligned} \qquad (4.8)$$

The analytic solution exists but is way too long to present here. There are either one or three real solutions possible for $T(t)$ in (4.8). For any combination of control variables, equations (4.7) and (4.8) provide candidates for the steady states. Admissibility must be respected, for example all state variables have to be non-negative. The actual steady states will be calculated in Subsection 4.1.2.

4.1.1 Stability

Analogously to Chapter 3, the eigenvalues of the Jacobian matrix evaluated in the calculated steady states provide information about the stability of the corresponding points. For the system (4.1) - (4.2), the Jacobian can be written as

$$J(S,T) = \begin{pmatrix} -\delta - \varphi(\omega)\left(\frac{T\beta}{P} + \theta\upsilon\left(\frac{T}{P}-1\right)(\mu-1)\right) & -S\varphi(\omega)\left(\frac{\beta}{P} + \frac{\theta\upsilon(\mu-1)}{P}\right) \\ \varphi(\omega)\left(\frac{T\beta}{P} + \theta\upsilon\left(\frac{T}{P}-1\right)(\mu-1)\right) & S\varphi(\omega)\frac{\beta+\theta\upsilon(\mu-1)}{P} - \upsilon\left(\mu - \frac{T(\mu-1)}{P}\right) - \sigma + \frac{T\upsilon(\mu-1)}{P} \end{pmatrix} \quad (4.9)$$

4.1.2 Sensitivity Analysis

In order to investigate the impact of all appearing parameters, it is important to choose control variables unequal to zero. Otherwise, some terms in the system (4.1) - (4.2) containing the considered parameters vanish. The standard case in this investigation therefore contains all parameter estimations from Chapter 2.4. The control variables υ and μ are set to 0.5 whereas ω is set to $\approx 26,200,000$ because at this value the effect of $\varphi(\omega)$ is approximately half of its maximum value. Every parameter and every control variable will be changed ceteris paribus. For an estimated parameter, the considered interval is in most cases the half to the double of the estimation. Exceptions are θ and h, which are limited to 1. For the control variables, the considered intervals are $[0,1]$ for υ, $[0,1)$ for μ, and $[0;174,364,820]$ for ω. At the upper bound of the interval for ω, the effect of the prevention function $\varphi(\omega)$ is 99% of the maximum effect possible. In most cases, the equilibrium values of S and T are affected in different directions by a parameter, but as stated below, there are some exceptions.

The stability of the steady states is calculated for every considered combination by inserting the steady states into the Jacobian (4.9) and considering the eigenvalues. In all but one case there exists only one feasible steady state, which is always stable. The corresponding real parts of the eigenvalues are therefore less than zero. The single case with more than one steady state ($\upsilon = 0$) will be discussed below.

The following investigation provides a good insight into the impact of different parameters on the steady state. The results of the calculated equilibria are displayed in the Figures 4.1, 4.2, and 4.3.

As $(1-h)$ models the limit of prevention, a lower value of h increases the effectiveness of prevention and therefore causes a lower value of T^*. Prevention also influences the number of susceptibles, because fewer susceptibles become terrorists, which increases S^*. For a higher value of h, of course, the opposite effect takes place (Figure 4.1, top left).

The parameter m models the effectiveness of prevention by scaling the amount of money spent on prevention ω. A higher value of m induces a higher effectiveness of ω. As one can see in Figure 4.1 (top right), the influence of m on S^* and T^* are negligible compared to the other parameters.

Because k measures the natural inflow into the group of susceptibles, a higher value leads to a higher value of S^*. Furthermore, a higher availability of potential terrorists also increases T^* (Figure 4.1, bottom left).

δ on the other hand, models the natural outflow rate of susceptible people. A higher δ implies, of course, a lower value of S^*. But because there are fewer susceptibles who could potentially change into terrorists, T^* drops with a higher δ as well (Figure 4.1, bottom right).

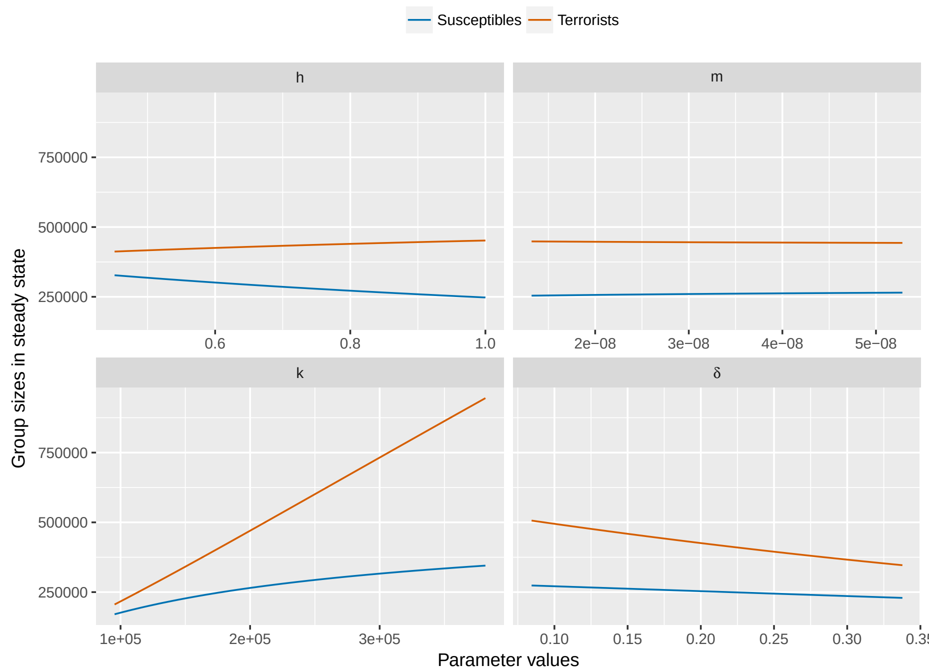


Figure 4.1: These figures show the effects of h, m, k, and δ on the steady state values under ceteris paribus conditions.

An increase in the population size P has two opposing effects. On the one hand, it decreases the influence of β and therefore the recruitment, but on the other hand it also increases the collateral damage, if $\mu < 1$. Therefore, the overall effect depends on the relative relationship between θ, υ, μ, and β.

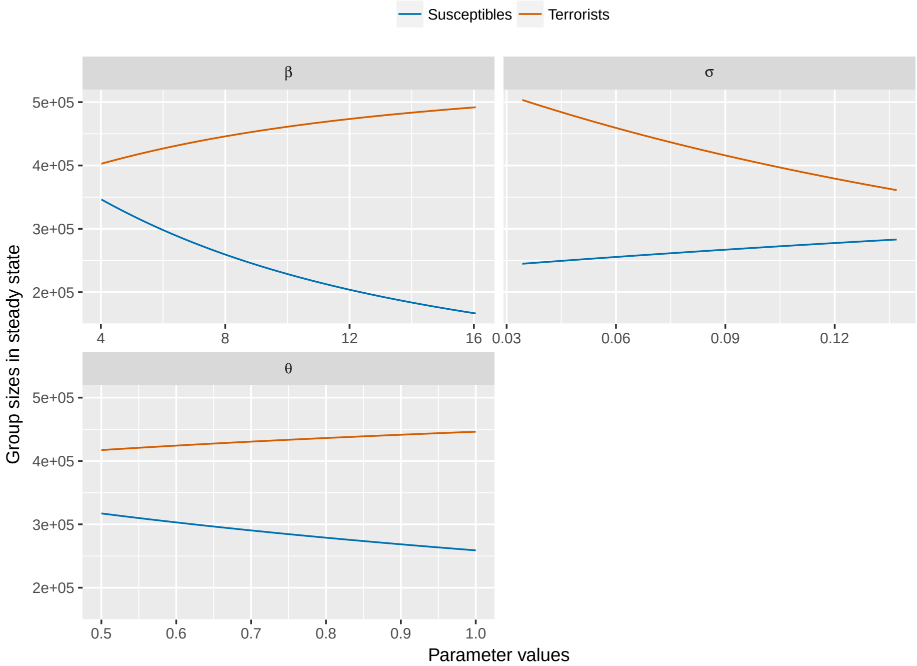


Figure 4.2: Sensitivity analysis with respect to the parameters β, σ, and θ.

Another problem with a ceteris-paribus analysis of P is that a lot of other parameters are estimated with respect to P. Visualizing the effects of P is therefore questionable, so we omit this case.

As β represents the influence of terrorists on susceptibles in terms of a recruitment rate, it is not surprising that a higher value of β causes a higher equilibrium value for T^* and a lower one for S^* (Figure 4.2, top left).

σ, as the natural outflow rate of the group of terrorists, directly influences T^*. A higher outflow decreases the number of terrorists. Because T^* decreases, there are fewer terrorists which are able to recruit from the group of susceptibles. Therefore, S^* rises with an increase of σ (Figure 4.2, top right).

The parameter θ models the amount of susceptibles turning into terrorists because of collateral damage. Because this parameter only affects the dynamics between S and T and not, for example, the outflow of T, an increase in θ increases the inflow to terrorists and therefore decreases the number of susceptibles (Figure 4.2, bottom).

In Figure 4.3, the effects of the control variables on the steady states are visualized. A higher level of μ and therefore a higher percentage of high-quality reports increases the efficiency of v. It also decreases the amount of collateral damage. Therefore both effects decrease the amount of terrorists, and both effects directly or indirectly increase the amount of susceptibles (Figure 4.3, top left).

ω representing the amount of money spent on prevention decreases the flow from susceptibles to terrorists. The higher ω, the lower T^* and the higher S^*. The effects of ω on the steady state are relatively small compared to the other control variables (Figure 4.3, top right).

As υ stands for the level of law enforcement, it has two contrary effects (Figure 4.3, bottom left). The fight against terrorism decreases the number of terrorists, therefore T^* decreases with an increase of υ. But law enforcement also produces collateral damage, which in turn increases the number of terrorists. The second effect is larger than the first one if $\varphi(\omega)\theta\upsilon(1-\mu)(1-T(t)/P)S(t) > \upsilon(\mu+(1-\mu)T(t)/P)T(t)$. Of course, this could not increase the number of terrorists in the Optimal control model because υ would then best be set to zero.

In the hairline case of $\upsilon = 0$, there is a second steady state with T^{**} being zero and $S^{**} = k/\delta$. The eigenvalues of the Jacobian matrix are $\begin{pmatrix} -0.1689 \\ 0.7946 \end{pmatrix}$, which classifies this steady state as unstable. So, in all considered cases, there exists only one stable steady state.

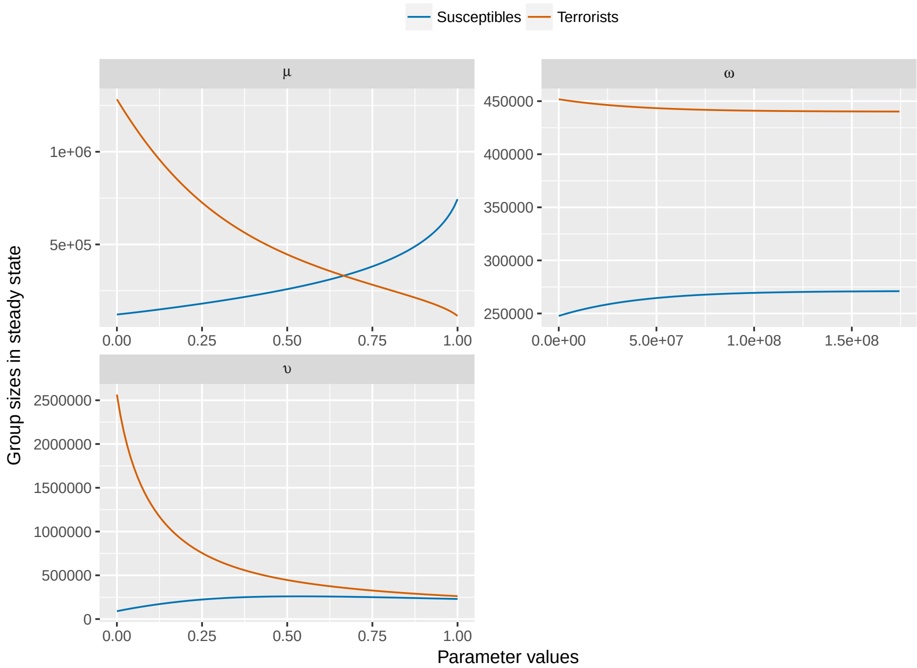


Figure 4.3: This figure shows the effects of υ, ω, and μ on the steady state.

4.2 The Optimal Constant Strategy

To find the ideal constant strategy for a given problem (starting values) a function must be defined which numerically calculates the discounted integral of the objective function for a given constant strategy. This function can then be used to find the best combination of controls by numerically searching for the minimum. The discounted structure of this problem was specifically considered to get better numeric results by enlarging the time steps exponentially. Therefore, the time steps at the beginning, where the costs are the highest, are very small.

In the following subsection the results of this procedure will be discussed for a given starting point. Theses results will be compared to the strategy of not intervening at all, hence the uncontrolled system.

A Starting Value with a Medium Terrorist Size

Considering the case that there are about $15,000$ terrorists as well as $800,000$ susceptibles in the beginning, Figure 4.4 provides an overview of the development of the costs, the number of terrorists, and the number of susceptible people. The superiority of the optimal constant strategy is obvious. In the uncontrolled case, the state variables converge to a very high equilibrium. The appropriate trajectory can be seen in Figure 3.2. The overall costs of the optimal constant strategy are about 90% less than in the uncontrolled case. In the long run, the number of susceptibles are $937,270$, and the number of terrorists are $47,397$ in the controlled case. In the uncontrolled case, the number of terrorists is $2,574,900$, and the number of susceptibles is $85,333$. The optimal constant strategy for this starting value is given by

$$\begin{pmatrix} \upsilon^* \\ \mu^* \\ \omega^* \end{pmatrix} = \begin{pmatrix} 0.61842 \\ 0.99952 \\ 135520000 \end{pmatrix} \tag{4.10}$$

The optimal strategy for μ is astonishingly high, although it is impossible for μ to become 1 due to the cost structure. During further investigations in Chapter 5, such high values for μ will appear several times. This behavior will be addressed in Section 5.4.

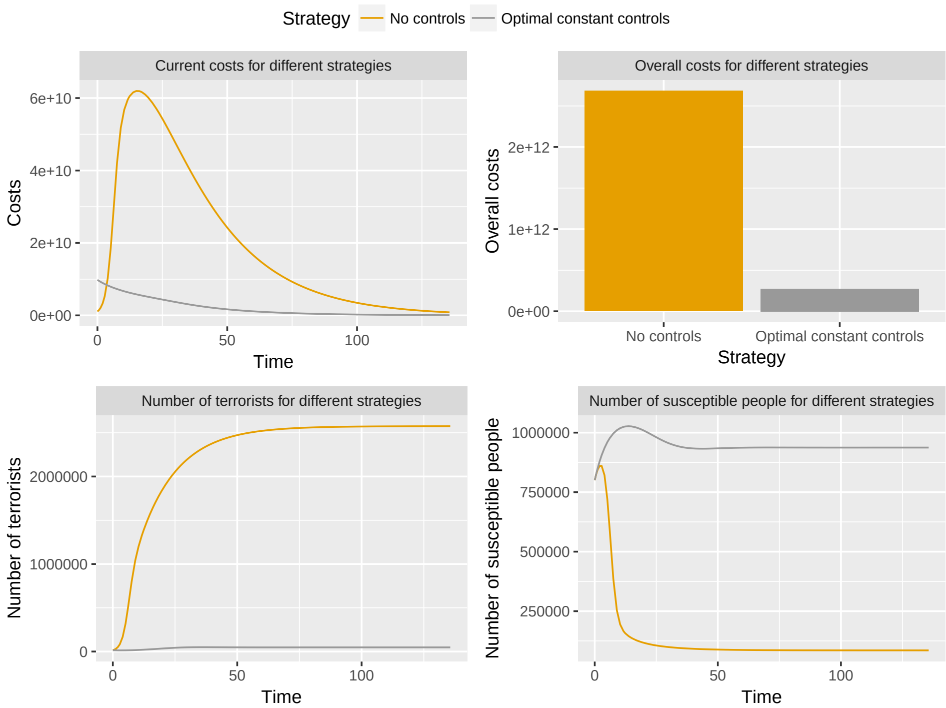


Figure 4.4: This figure shows the current costs and overall costs in USD for two different constant strategies as well as the change of the two state variables S and T over time. The starting value is given by $(S_0, T_0) = (800,000; 15,000)$.

4.3 Phase Portrait for One Constant Strategy

The results in Figure 4.4 suggest an enormous difference between the uncontrolled and the controlled system. A more suitable phase portrait could describe the dynamics in the system much better by applying a representative constant strategy. To find such a representative strategy, the optimal constant strategy was calculated for several thousand starting values inside the feasible region. The results of this process are displayed in Figure 4.5 which lead to the chosen representative strategy, the median of each group:

$$\begin{pmatrix} \bar{\upsilon} \\ \bar{\mu} \\ \bar{\omega} \end{pmatrix} = \begin{pmatrix} 0.77318 \\ 0.99939 \\ 93070000 \end{pmatrix} \tag{4.11}$$

The generated data were also used to create a cost map provided in Figure 4.6. This figure shows all predetermined starting values and the overall cost under the optimal constant strategy. In general, the lower the number of

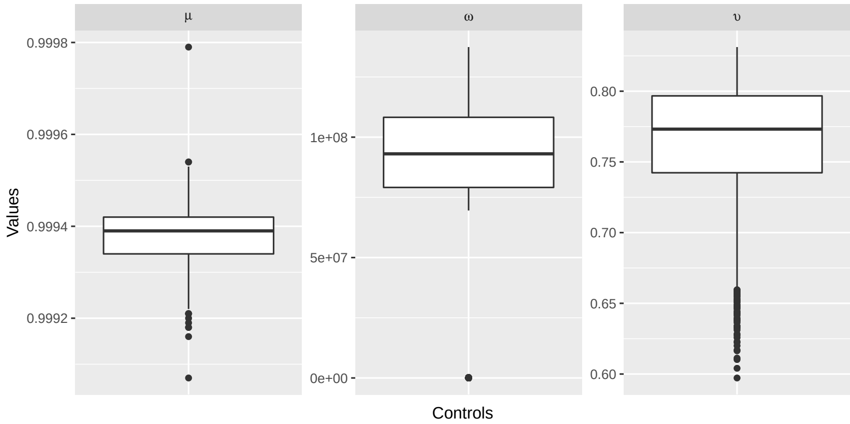


Figure 4.5: This figure shows the distribution of the optimal constant controls for several thousand starting values. Note that the values of μ are very high in all considered cases. For the representative constant strategy, the median is taken for each control.

terrorists or the number of susceptibles, the lower the total costs are. It is interesting that even for a high number of terrorists, the overall costs are relatively small, if the number of susceptibles is small enough. In this case, the size of terrorists decreases quickly because of the lack of new recruits.

Using the controls from (4.11), the corresponding phase portrait is generated and presented in Figure 4.7. Comparing this phase portrait to the one in the uncontrolled case (Figure 3.2) it stands out that the unstable equilibrium with $T(t) = 0 \quad \forall t$ does not exist anymore. This is not surprising, due to the structure of the model (4.1) - (4.2) and the analysis carried out in Subsection 4.1.2, it is only possible for this steady state to exist for $\upsilon = 0$. Since the chosen $\bar{\upsilon}$ is not equal to zero, this steady state disappeared entirely.

The stable steady state, present in both graphs in Figure 4.7 changed its position drastically. The very high number of terrorists in the uncontrolled case dropped significantly in the constant-control case to about $9,452.8$. In turn, the number of susceptibles rose, compared to the uncontrolled system to $1,082,900$ in the equilibrium. A smaller number of terrorists is not able to recruit many susceptibles, so the group of susceptibles does not get diminished like in the uncontrolled case. Moreover, the group of susceptibles is able to approach a value which is close to the equilibrium in the absence of any terrorists.

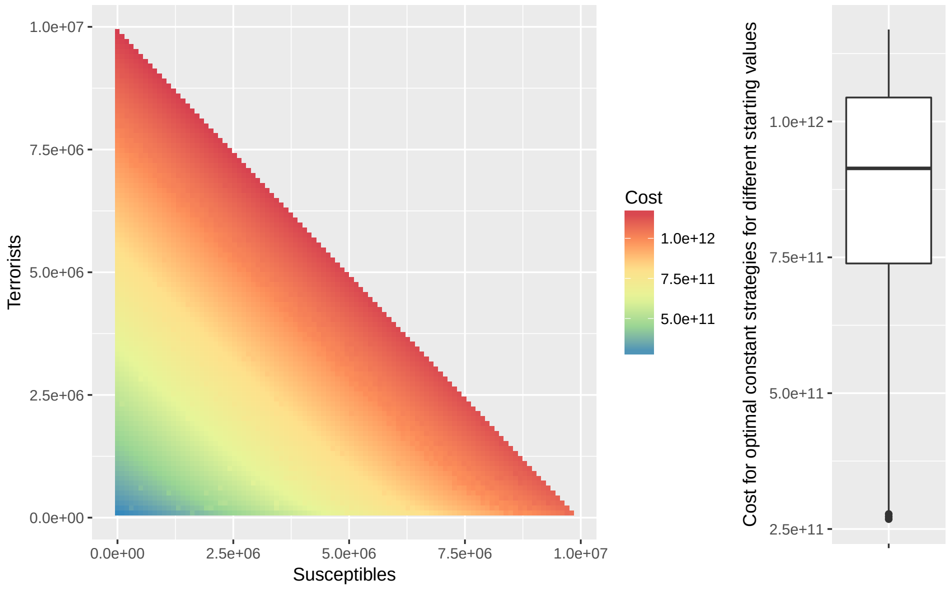
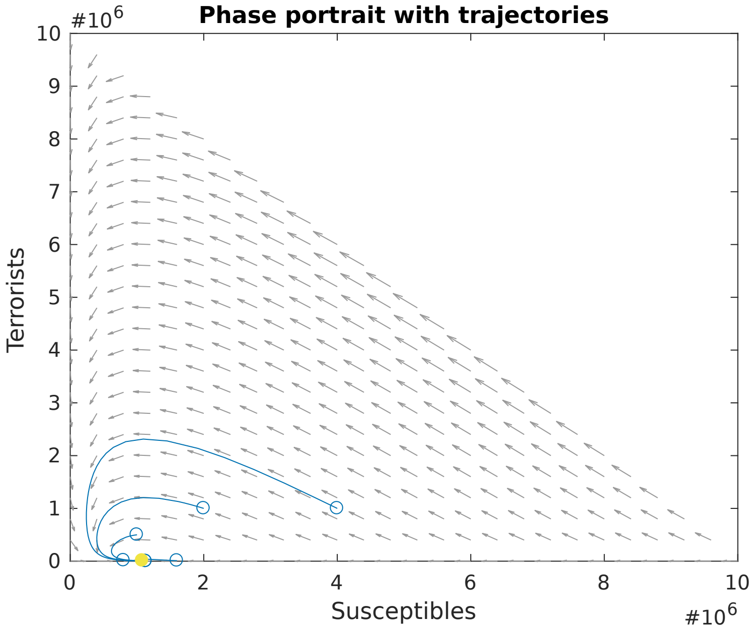
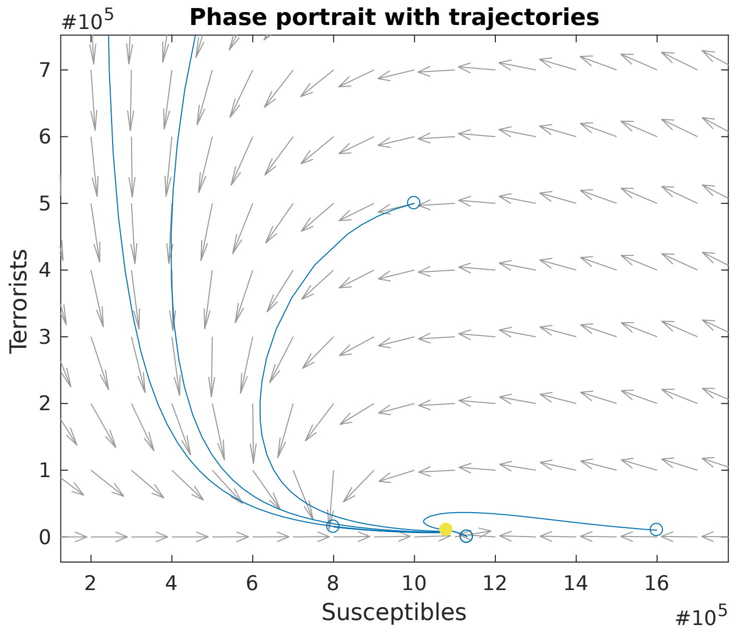


Figure 4.6: The overall costs for the optimal constant controls are calculated for several thousand starting values and displayed in two different ways. In the left figure, every position represents a starting value while the color represents the overall costs. In the right figure, the distribution of all calculated overall costs is displayed using a boxplot.



(a) Overview



(b) Detailed view

Figure 4.7: These figures show the phase portrait of the constant-control system. Again, the detailed view (b) shows a finer grid of direction-arrows. Note again that the arrow length does not represent the size of change.

Chapter 5

The Optimal Control System

In this chapter the approach will be formulated allows us to obtain dynamic optimal solutions. This will be done using the Hamiltonian and Lagrangian to get the canonical system, which describes the core of the necessary optimality conditions. The handling of implicit controls will be considered as well as the handling of constraints. For the calculated equilibrium, several optimal paths will be generated and compared to the corresponding optimal constant solution stated in Chapter 4. The calculations will be carried out by using MATLAB and the toolbox for dynamic optimization, OCMAT.

For better readability, the indication of time (t) will mostly be omitted in this chapter.

5.1 Formulating the Canonical System

Applying the Maximum Prinziple of Pontryagin according to [31] and [32] provides the foundation of the following investigations. One important step is to formulate the Hamiltonian, because the optimal control variables $\upsilon^*(t), \mu^*(t)$, and $\omega^*(t)$ jointly maximize the Hamiltonian.

In this case, the current-value form of the Hamiltonian is preferred over the present-value form in order to get an autonomous system of ODEs. Note that the objective function will be multiplied with (-1) because in the standard notation a maximization problem is considered. With $\lambda_1(t)$ and $\lambda_2(t)$ denoting the co-states, the current-value Hamiltonian can then be written as

$$
\begin{aligned}
\mathcal{H}(S,T,\upsilon,\omega,&\mu,\lambda_1,\lambda_2) = \\
&-\left(c_T T - c_\upsilon \log_e(1-\upsilon) + \omega - c_\mu \log_e(1-\mu)\right) \\
&+\lambda_1 \Bigg[k - \delta S - \left(h + (1-h)e^{-m\omega}\right) \\
&\qquad\qquad \cdot \left(\theta\upsilon(1-\mu)\left(1-\frac{T}{P}\right) + \beta\frac{T}{P}\right) S \Bigg] \\
&+\lambda_2 \Bigg[\left(h + (1-h)e^{-m\omega}\right) \\
&\qquad\qquad \cdot \left(\theta\upsilon(1-\mu)\left(1-\frac{T}{P}\right) + \beta\frac{T}{P}\right) S \\
&\qquad\qquad - \left(\sigma + \upsilon\left(\mu + (1-\mu)\frac{T}{P}\right)\right) T \Bigg]
\end{aligned}
\tag{5.1}
$$

There are several conditions on the control variables which are not considered yet, namely:

$$
0 \le \upsilon(t) \le 1, \qquad 0 \le \mu(t) < 1, \qquad 0 \le \omega(t) \tag{5.2}
$$

Because the objective function has a pole and the corresponding value is infinite for $\upsilon = 1$ and also for $\mu = 1$, any optimal solution is forced to choose values for υ and μ that are less than one. Therefore, the corresponding conditions do not have to be considered explicitly, which leaves only the three non-negativity constraints, namely

$$
0 \le \upsilon(t), \qquad 0 \le \mu(t), \qquad 0 \le \omega(t) \tag{5.3}
$$

In order to take them into account correctly, the Lagrangian is needed. The above already fits the standard form, which directly leads to the following Lagrangian function

$$
\begin{aligned}
\mathcal{L}(S,T,\upsilon,\omega,\mu,\lambda_1,\lambda_2,\xi_1,\xi_2,\xi_3) &= \\
\mathcal{H}(S,T,\upsilon,\omega,\mu,\lambda_1,\lambda_2) &+ \xi_1\upsilon + \xi_2\mu + \xi_3\omega
\end{aligned}
\tag{5.4}
$$

with the Lagrangian variables $\xi_1(t), \xi_2(t)$, and $\xi_3(t)$. These variables determine whether a constraint is active or not. In total, there are eight different combinations. An optimal solution can stay within one of these combinations, meaning that none of the constraints in (5.3) change their status from active to inactive or vice versa. On the other hand, it is also possible for an optimal solution to switch between two or more of these combinations.

In the process of finding the optimal path for a given starting value, these optimal paths will be put together resulting in the whole optimal path.

For the Lagrangian $\mathcal{L}(S, T, \upsilon, \omega, \mu, \lambda_1, \lambda_2, \xi_1, \xi_2, \xi_3)$, the necessary conditions for optimality are:

$$\frac{\partial \mathcal{L}}{\partial \upsilon} = 0, \qquad \frac{\partial \mathcal{L}}{\partial \mu} = 0, \qquad \frac{\partial \mathcal{L}}{\partial \omega} = 0 \tag{5.5}$$

$$\xi_1 \upsilon = 0, \qquad \xi_2 \mu = 0, \qquad \xi_3 \omega = 0 \tag{5.6}$$

$$\xi_i \geq 0, \quad i = 1, 2, 3 \tag{5.7}$$

$$\dot{\lambda}_1 = r\lambda_1 - \frac{\partial \mathcal{L}}{\partial S} \tag{5.8}$$

$$\dot{\lambda}_2 = r\lambda_2 - \frac{\partial \mathcal{L}}{\partial T} \tag{5.9}$$

At this point, the common way is to try to express the control variables explicitly by reformulating (5.5). However, in this case it is not possible to express all three variables but only υ and μ explicitly. This can be done by reformulating $\mathcal{L}_\upsilon = 0$ and $\mathcal{L}_\omega = 0$ from (5.5). There are two solutions for υ and μ, which are both too long to state here. The fact that there are more than one solution is addressed in this thesis by allowing MATLAB to keep both of them. The syntax for doing so within the OCMAT-toolbox can be found in the appendix.

In order to handle the implicit control variable ω, the equation $\mathcal{L}_\omega = 0$ from (5.5) is derived with respect to the time t. The result, which is of course also equal to 0, is then reformulated to express $\dot{\omega}$. The final result of this procedure is named Γ. So it is not possible to explicit express ω but $\dot{\omega}$. The idea behind this approach is to extend the canonical system by one dimension by adding $\dot{\omega}$. Therefore, a solution of the canonical system for a given starting value contains the optimal control ω in its fifth dimension.

The canonical system using only the Hamiltonian can therefore be stated as:

$$
\begin{aligned}
\dot{S} &= k - \delta S - \varphi(\omega) \cdot \left(\theta\upsilon(1-\mu)\left(1-\frac{T}{P}\right) + \beta\frac{T}{P}\right) S && (5.10)\\
\dot{T} &= \varphi(\omega) \cdot \left(\theta\upsilon(1-\mu)\left(1-\frac{T}{P}\right) + \beta\frac{T}{P}\right) S \\
&\qquad - \left(\sigma + \upsilon\left(\mu + (1-\mu)\frac{T}{P}\right)\right) T && (5.11)\\
\dot{\lambda}_1 &= \lambda_1(r+\delta) + (\lambda_1 - \lambda_2)\varphi(\omega)\left(\theta\upsilon(1-\mu)\left(1-\frac{T}{P}\right) + \beta\frac{T}{P}\right) && (5.12)\\
\dot{\lambda}_2 &= \lambda_2 r - c_T + (\lambda_1-\lambda_2)\varphi(\omega)(\beta - \theta\upsilon(1-\mu))\frac{S}{P} \\
&\qquad + \upsilon\left(2(1-\mu)\frac{T}{P} + \mu\right) + \sigma && (5.13)\\
\dot{\omega} &= \Gamma && (5.14)
\end{aligned}
$$

For a given starting value (S_0, T_0), the optimal path can then be calculated by solving the differential equation system.

Because of the complexity of the system, the further analysis will be carried out numerically using the MATLAB-toolbox OCMAT. An older version of the toolbox as well as a manual can be found in [33].

5.2 The Equilibrium of the Canonical System

After the initialization of OCMAT, an equilibrium needs to be identified. Therefore, $\dot{S} = \dot{T} = \dot{\lambda}_1 = \dot{\lambda}_2 = \dot{\omega} = 0$ must hold for the canonical system (5.10) - (5.14). Any equilibrium which is found by the toolbox must furthermore pass an admissibility test. This is necessary due to numerical errors. The only equilibrium found that satisfies all feasibility conditions is given by

$$
\begin{pmatrix} S^* \\ T^* \\ \lambda_1^* \\ \lambda_2^* \\ \omega^* \end{pmatrix} = \begin{pmatrix} 961{,}711.17 \\ 40{,}109.79 \\ -65{,}119.02 \\ -524{,}742.52 \\ 136{,}882{,}380.49 \end{pmatrix}, \qquad \begin{pmatrix} \upsilon^* \\ \mu^* \\ \omega^* \end{pmatrix} = \begin{pmatrix} 0.6404174 \\ 0.9993141 \\ 136{,}882{,}380.49 \end{pmatrix} \tag{5.15}
$$

The number of terrorists in the equilibrium is significantly smaller than in the uncontrolled case. Interestingly, the number of terrorists in the equilibrium of the representative constant strategy is $9{,}452.8$, and thus lower than in the optimal dynamic case. Also, the number of susceptibles differs. This is not a contradiction because the lower steady state values of the constant control case cause higher costs. As one can see in the examples of the following Section 5.3, the number of terrorists in the optimal dynamic solution is sometimes lower than in the optimal constant solution and sometimes higher. The reason is that in view of the overall costs it is sometimes better to accept higher numbers of terrorists.

The values of the controls in the equilibrium are also remarkable. μ^* is again very high, which will be addressed in the sensitivity analysis in Section 5.4. The expenditures for prevention in the equilibrium are easier to interpret by looking at $\varphi(\omega)$. It applies that $\varphi(\omega^*) = h + (1-h) \cdot 0.02691109$. Therefore, the effect of prevention is near its limit, leaving only about 2.7% left. Prevention is therefore used heavily in the equilibrium.

The associated Lagrange-multipliers are

$$\begin{pmatrix} \xi_1^* \\ \xi_2^* \\ \xi_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{5.16}$$

Because all ξ_i are zero, none of the constraints from (5.3) are active in the equilibrium. This implies that for any optimal solution, sooner or later all controls are used.

The eigenvalues of the Jacobian-matrix evaluated in the equilibrium are

$$\begin{pmatrix} -0.1475 + 0.1247i \\ -0.1475 - 0.1247i \\ 0.1872 + 0.1247i \\ 0.1872 - 0.1247i \\ -2.6373 \cdot 10^{-8} \end{pmatrix} \tag{5.17}$$

Because the sign of the real part varies, the equilibrium is a saddle point. In order to check whether the founded equilibrium is a maximum and not mistakenly a minimum, the Hessian matrix of the Hamiltonian with respect to the three controls can be calculated. The corresponding eigenvalues are

$$\begin{pmatrix} -3.9165 \cdot 10^{14} \\ -5.7283 \cdot 10^{10} \\ -2.6410 \cdot 10^{-8} \end{pmatrix} \tag{5.18}$$

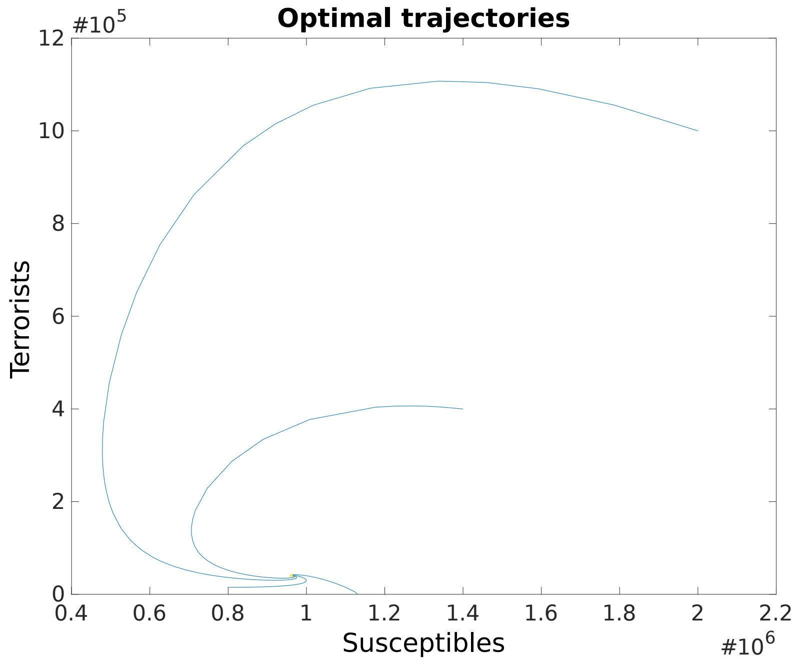
Due to the negativity of all eigenvalues, the Hessian matrix is negative definite. The equilibrium is therefore indeed a local maximum.

The investigation of the second steady state occurring in the uncontrolled case, namely the one with no terrorist at all, reveals the following. Although the numeric algorithm found equilibria when given $(k/\delta, 0)$ as a starting value, none of them are feasible. The only possibility for this point to be an equilibrium would be if all controls are set to zero, but even this equilibrium is not feasible. The corresponding Lagrangian multipliers are

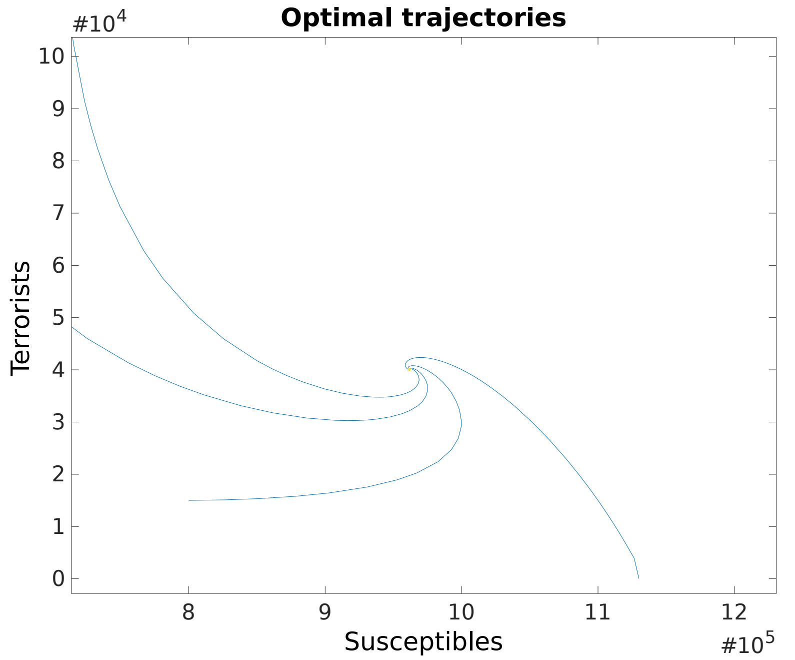
$$\begin{pmatrix} \xi_1^{**} \\ \xi_2^{**} \\ \xi_3^{**} \end{pmatrix} = 10^{10} \cdot \begin{pmatrix} -9.3419 \\ 0.0000 \\ 0.0184 \end{pmatrix} \tag{5.19}$$

The negativity of ξ_1^{**} clearly classifies this point as infeasible. The point stated in (5.15) therefore remains the only steady state.

Figure 5.1 shows the calculated equilibrium, as well as four optimal paths in the (S, T)-space. The corresponding starting values are given by $(S_0^1, T_0^1) = (800,000; 15,000)$, $(S_0^2, T_0^2) = (1,130,000; 10)$, $(S_0^3, T_0^3) = (1,400,000;$-$400,000)$, and $(S_0^4, T_0^4) = (2,000,000; 1,000,000)$. The general appearance is similar to the phase portrait with constant controls. In both cases, the trajectories approach the equilibrium on a spiral path. In the next section, some of these paths will be analyzed and discussed.



(a) Overview



(b) Detailed view

Figure 5.1: This figure shows the calculated equilibrium of the canonical system and some optimal trajectories for different starting values.

5.3 Analysis of Optimal Solutions

In this section, the the optimal dynamic solution will be analyzed and compared to the appropriate optimal constant solution for different starting values. Of special interest are of course the costs caused by the different strategies. Also the development of susceptibles and terrorists will be considered as well as the values of all three controls over time.

Although the objective value and therefore the overall cost of an optimal path are exactly given by $\frac{-1}{r}\mathcal{H}^*\Big|_{t=0}$, this value will not be used in the following comparison. Instead, the current cost will be integrated numerically, like in the case of the optimal constant strategy. The reason for this is to achieve comparability, because the numerical integration only covers a finite time horizon while the formula stated above returns the exact integral of the cost function over an infinite time horizon.

A Starting Value with a Medium Terrorist Size

The first example will use the same starting values as considered in Section 4.2. In this case the differences in terms of costs are relatively small while comparing the optimal dynamic strategy with the optimal constant one. As shown in Figure 5.2, the optimal strategy is only 0.75% better with respect to the overall cost. This suggests that in some cases the simple solution of a constant strategy already delivers very good results. Figure 5.2 also displays the number of terrorists and the number of susceptibles over time. In case of the optimal dynamic strategy, there are about $40,000$ terrorists in the long run, while in the case of the constant strategy there are about $47,000$. The optimal strategy therefore provides a reduction from approximately 15% while it costs less. Another interesting fact to observe is that for the first 25 years, the number of terrorists in the optimal dynamic solution is higher than in the constant strategy. By doing so and allowing the terrorists to grow to their equilibrium value a little bit faster, costs are mitigated. Afterwards, the controls in the optimal dynamic solution are used to an extent to ensure a lower equilibrium value for $T(t)$.

A look at the control variables in Figure 5.3 shows the following. Prevention rises a little bit over time, but the effects of $\omega(t)$ via $\varphi(\omega(t))$ do not vary that much. At the beginning, the effect of prevention is 96% of its total possible effect, and in the long run this share increases to 97.3%. More precisely, $\varphi(\omega^*(t)) \in h + (1-h) \cdot [0.0271197, 0.03923181]$. Although the overall effects of prevention on the dynamic system are relatively low as we could also see in Section 4.1, it is used to a substantial extent. Therefore it may not be very effective, but it is cost effective, otherwise it would not be used in the optimal solution.

The values of $\mu(t)$ are very high at all times, slightly dropping in the first 25 years. This control is used highly from the very beginning.

The extent of law enforcement $v(t)$ increases over time and reaches its peak approximately after 25 years. The change of this control variable is much higher over time than in the other two control variables. It seems that it rises with the number of terrorists $T(t)$.

The share of costs in Figure 5.3 shows that law enforcement costs by far the most. The rise in $v(t)$ can also be seen in the share of costs, especially because $\mu(t)$ drops a little bit at the same time. The costs for prevention are very small and take up only a fraction of the total costs.

In this example, the results yielded from constant strategies turn out to be satisfactory. There are differences, but the overall costs are not apart by that much.

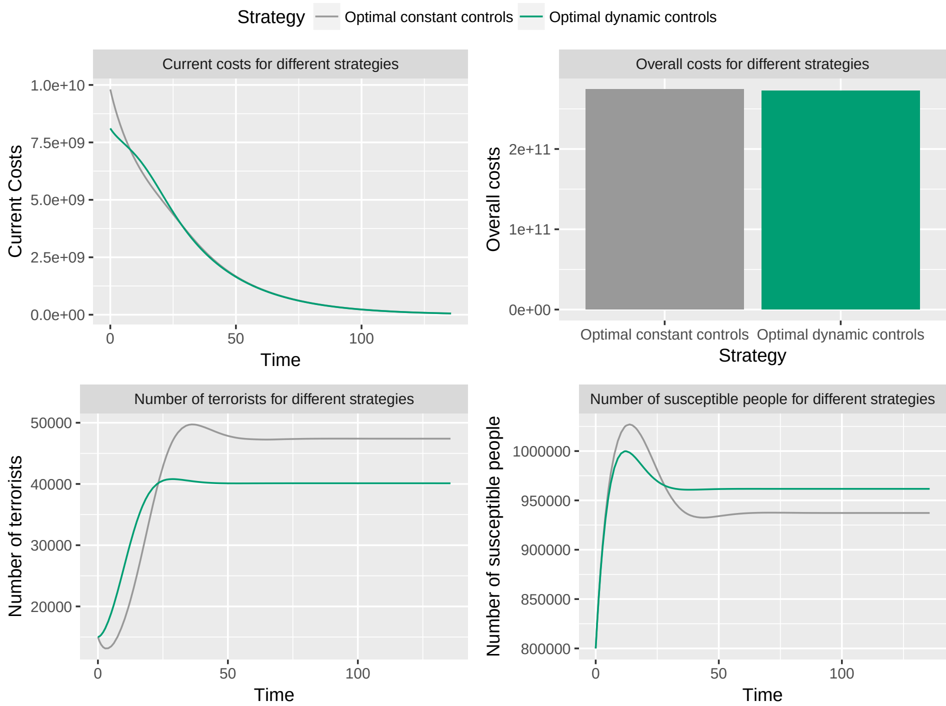


Figure 5.2: This figure shows the costs in USD as well as the changes of the two state variables S and T for the optimal constant solution and the optimal dynamic solution. The starting value is $(S_0, T_0) = (800,000; 15,000)$.

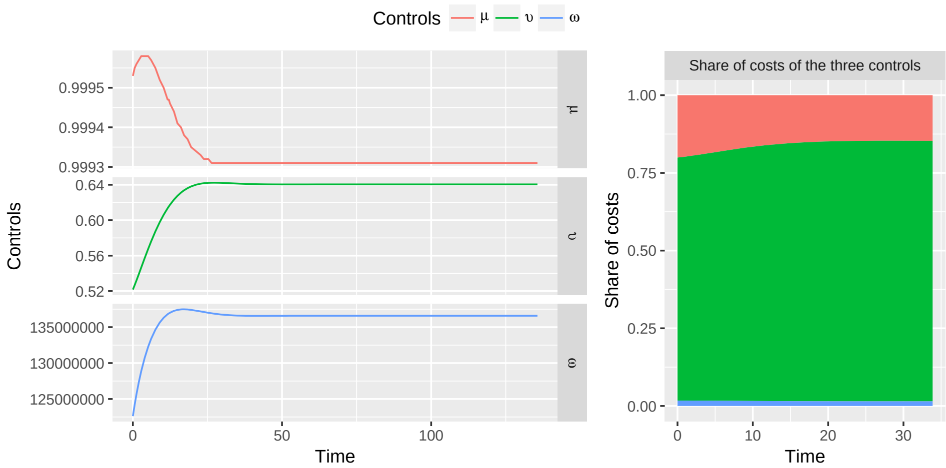


Figure 5.3: In this figure, the development of all three control variables is shown over time as well as the corresponding share of costs. The starting value is $(S_0, T_0) = (800,000; 15,000)$.

A Starting Value with a Small Terrorist Size

In this example, a starting value with a very small terrorist size and a relatively large number of susceptibles is chosen, namely $(S_0, T_0) = (1,130,000; 10)$. Figure 5.4 displays the comparison between the optimal dynamic strategy and the optimal constant one. In this case, the dynamic solution costs in total about 17% less than the optimal constant one. The upper left figure again shows the current costs for each time instance and it stands out that the costs in the dynamic solution are much lower at the beginning. In case of the optimal solution, the growth of $T(t)$ and the decrease of the number of susceptibles are faster at the beginning. In the long run, there are about $40,000$ terrorists in the optimal control case while there are $78,000$ in case of constant controls. This corresponds to a reduction of approximately 49% in the long run with regard to the number of terrorists.

Figure 5.5 explains this behavior by showing that $\mu(t)$ and $\upsilon(t)$ are set equal to zero for several time periods at the beginning. The share of costs of the three controls also shows that at the beginning, all invested money is used for prevention, which is not very much at all. For the overall cost it is therefore optimal not to use law enforcement at the beginning and allow the number of terrorists to grow faster to their equilibrium value. After a certain period of time, the controls $\mu(t)$ and $\upsilon(t)$ become active. The value of $\mu(t)$ instantly increases nearly to one while the level of law enforcement $\upsilon(t)$ rises to a medium high value. If active, law enforcement again constitutes the biggest part of costs for the control variables.

It is very interesting that prevention $\omega(t)$ is used at a high level from the very beginning, even if law enforcement is not used at all. It seems that the use of prevention to a certain level is always cost efficient, otherwise it would not be used.

Overall, the dynamic solution shows its capabilities in this example. By not applying law enforcement and education for some time, it is possible to reduce the emerging costs by a great amount.

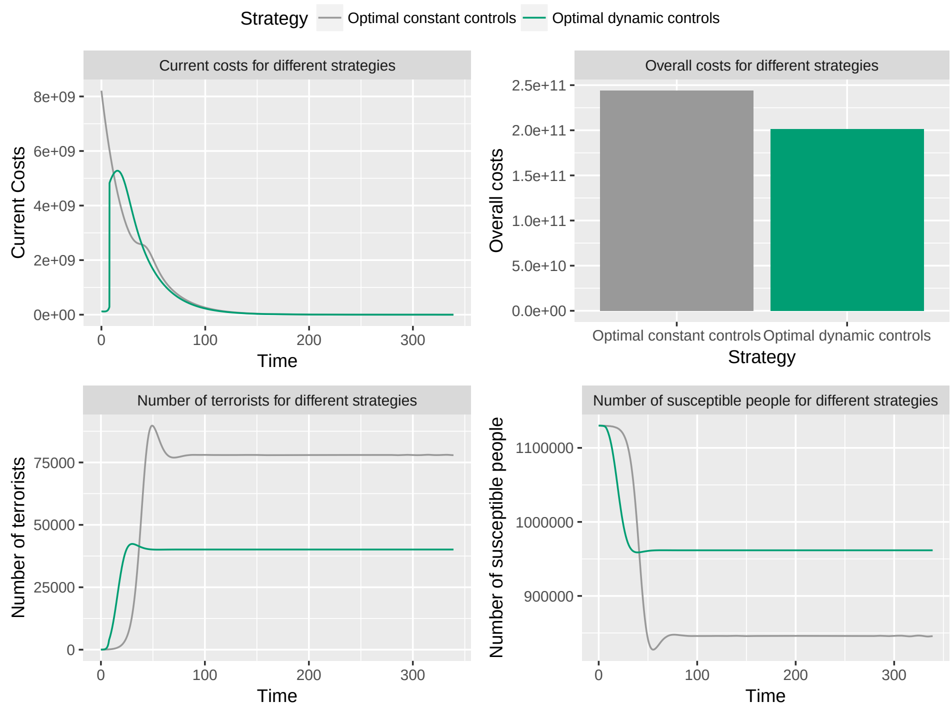


Figure 5.4: This figure shows the current costs in USD, the overall costs, as well as the changes of the two state variables S and T for two different strategies. The starting value is $(S_0, T_0) = (1,130,000; 10)$.

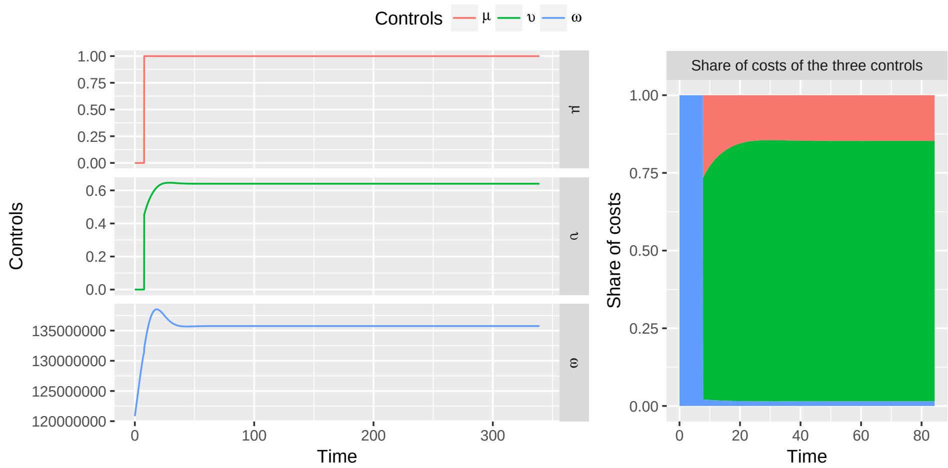


Figure 5.5: This figure shows the developement of the costs for all control variables for the optimal dynamic solution. The starting value is $(S_0, T_0) = (1, 130, 000; 10)$.

5.4 Sensitivity Analysis w.r.t. c_T, c_μ, and c_υ

In this section, the effects of the three cost parameters c_T, c_μ, and c_υ will be discussed. Since they are only appearing in the objective function, the steady state in the canonical system (5.10) - (5.14) will be investigated to determine their effects. Each parameter will be changed ceteris paribus and the equilibrium of the canonical system will be calculated. Then the effects on the state variables and the control variables in the steady state will be investigated.

The parameter c_T is the most important parameter in this model as it determines the arising costs for each terrorist. Setting this value to zero results in the obvious solution $\upsilon(t) = \omega(t) = \mu(t) = 0$. Raising this parameter increases the costs per terrorist and therefore the efforts in the fight against it. Therefore, for the sake of minimizing the overall social costs, more money is spent for all three control variables μ^*, υ^*, and ω^*. Their increase therefore lowers the number of terrorists in the equilibrium, T^*. Of course this is a foreseeable reaction to a rise in c_T. Because of a lower number of terrorists, the number of susceptibles rises in the equilibrium. These reactions to a change in c_T can be seen in Figure 5.6.

By changing the value of c_υ, the costs of law enforcement are modified. Figure 5.7 shows the conducted analysis for this parameter. A rise in this parameter will understandably decrease the value of υ^*. If a control gets more expensive, it will be used less in most of the cases in order to minimize the overall cost. At the same time, μ^* drops by a small amount, too. This

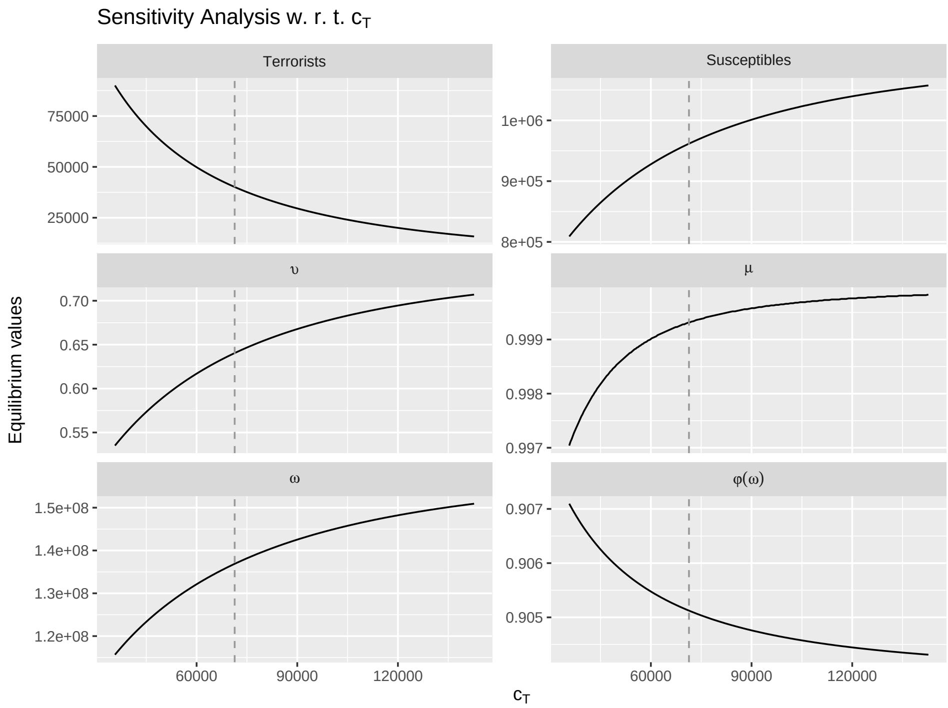


Figure 5.6: Sensitivity analysis w.r.t c_T. The range of the considered c_T compared to the original estimation $\hat{c}_T$ is chosen with $[\hat{c}_T/2, \hat{c}_T \cdot 2]$. The dashed line indicates the original estimation from Section 2.4.

development of course increases the number of terrorists T^* and at the same time decreases S^*. Although higher expenditures in prevention ω^* cannot compensate for the lower law enforcement, it is still used at a higher level. Figure 5.7 not only shows the level of ω^* but also the value of the prevention function in the equilibrium, $\varphi(\omega^*)$. Although it holds that the lower $\varphi(\omega^*)$, the higher the effect of prevention, note that $h = 0.9025$ is the infimum of φ. The increase in ω^* therefore does not have that much of an impact, because the effects of prevention are already near its maximum capacity.

The parameter c_μ controls the rise in cost for additional education. It is therefore reasonable to assume that a rise in this parameter decreases the use of μ^* in the steady state. In the first investigation, the interval was similarly chosen as in the other two parameters, namely the half of the original estimation to two times the value. However, there were almost no changes detectable for this range, so the interval for c_μ was enlarged drastically. The upper bound was set to 100 times the original value. Figure 5.8 shows the results of this sensitivity analysis. As expected, the increase in costs for a control variable increases the steady state value of T^* since

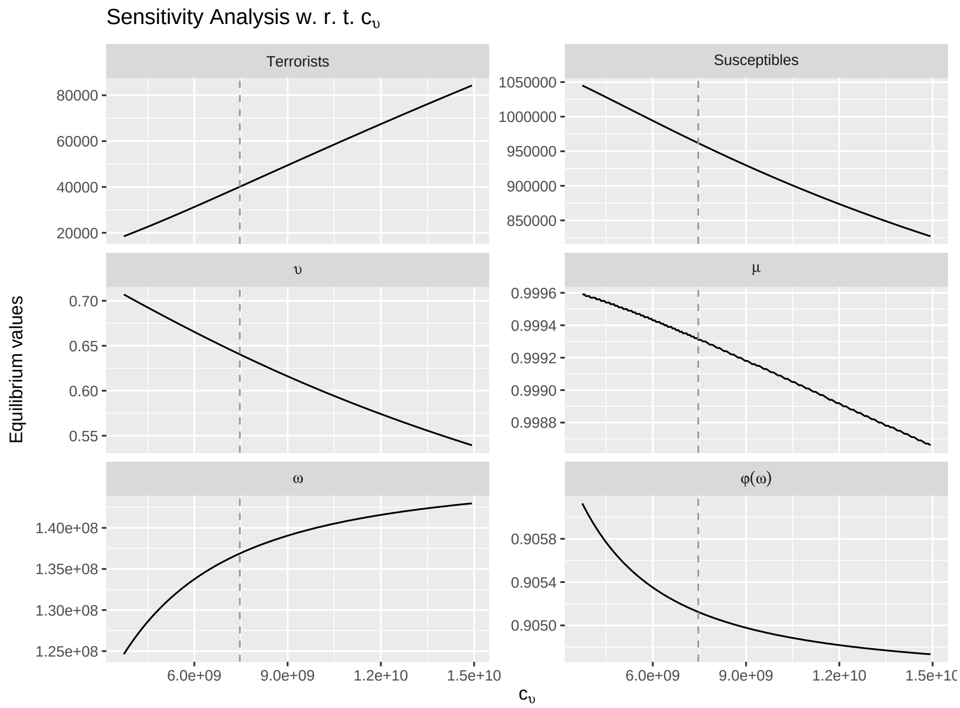


Figure 5.7: Sensitivity analysis w.r.t. c_υ. The range of the considered c_υ compared to the original estimation $\hat{c}_\upsilon$ is $[\hat{c}_\upsilon/2, \hat{c}_\upsilon \cdot 2]$. The original estimation is depicted as the dashed line.

it is optimal to tolerate a higher number of terrorists in the course of cost minimization. The reason for this behavior is simply that the costs of the fight against terrorism rise. The value of μ^* in the equilibrium stays relatively constant over a bread range despite a rise in c_μ. Only when the value of c_μ exceeds approximately 10^9, the optimal value of μ^* starts to decrease faster. Despite μ^* staying approximately at the same value for a long time, the optimal value of υ^* drops from the beginning. One explanation for this behavior is that the distribution of the money spent changes. The share of the resources used for education rises in order to maintain a high level of μ^*. After c_μ rises approximately to $5 \cdot 10^9$, the value of υ^* starts rising again. At this point, a further increase in c_μ leads to an increase of υ^* in the steady state. Unlike in the case of an increase of c_υ, a rise in c_μ decreases the equilibrium value of ω^*. Although this lowers the effect of prevention, Figure 5.8 shows that φ is still near its maximum capacity.

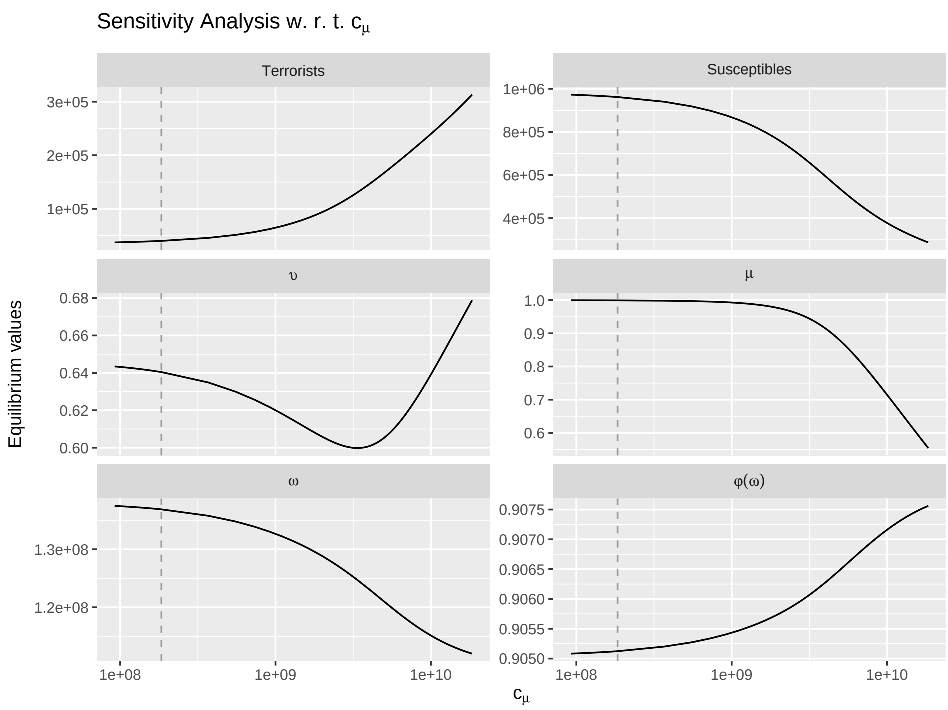


Figure 5.8: Sensitivity analysis w.r.t. c_μ. The range of the considered parameter compared to the original estimation $\hat{c}_\mu$ is $[\hat{c}_\mu/2, \hat{c}_\mu \cdot 100]$. Note that in this figure, a logarithmic scale is used for c_μ. The dashed line again indicates the original estimation.

5.5 Questions Arising During the Analysis

During the analysis of the model, several interesting behaviors of the system become visible and further questions arise. In this section, two of these questions are addressed, which hopefully deepens the understanding of the model.

5.5.1 Circumstances Causing a Large $\mu(t)$

If law enforcement is used, the optimal values for $\mu(t)$ are always very high in all previous investigations. Unlike law enforcement $\upsilon(t)$, the level of intelligence $\mu(t)$ only has positive effects inside the dynamic system. It increases the effectiveness due to law enforcement while it decreases the level of collateral damage. The only downside is of course the additional cost. Comparing the cost of $\mu(t)$ and $\upsilon(t)$, it stands out that the costs for intelligence / education c_μ estimated in Section 2.4 are much lower than the costs for law enforcement c_υ. A moderate increase in c_μ almost does not affect the equilibrium value of μ^*. The sensitivity analysis in Figure 5.8 shows that c_μ must be set to $10,000\%$ of its originally estimated value in order to have a significant impact on μ^*. This raises the question about the quality of the estimation of c_μ. It is possible that the considerations used in the process of estimating c_μ are not appropriate enough and the level of c_μ was underestimated. On the other hand, it is also possible that the effects of collateral damage were overestimated and therefore θ should decrease. These considerations result in several conclusions concerning the high values of μ^*. It is possible that c_μ has been estimated too low. In this case it is way too low and should be increased drastically to get the desired effect. Another possibility is that θ has been estimated to high and the corresponding value should decrease. It is also possible that both considerations are true and the remedies should be applied. Other possibilities are to rethink the underlying dynamics with respect to $\mu(t)$ or to stay with the original estimations, as they provide comprehensible results.

5.5.2 Circumstances Causing $\upsilon(t) = 0$

As stated several times in this thesis, the two opposing effects of $\upsilon(t)$ are on the one hand a decrease of $T(t)$ due to law enforcement and on the other hand an increase in $T(t)$ by virtue of collateral damage. The effects of collateral damage are greater than the effects of law enforcement if the following holds:

$$\varphi(\omega(t))\theta\upsilon(t)(1-\mu(t))\left(1-\frac{T(t)}{P}\right)S(t) > \upsilon(t)\left(\mu(t)+(1-\mu(t))\frac{T(t)}{P}\right)T(t)$$

This inequality can be reformulated to

$$S(t) > \frac{1}{\varphi(\omega)\theta(1-\mu(t))}\left(\mu T(t) + \frac{T(t)^2}{P-T(t)}\right) \tag{5.20}$$

For the sake of the following consideration, the control variables $\omega(t)$ and $\mu(t)$ are assumed to be constant. The effects of ω are relatively small due to the fact that $\varphi(\omega) \in (h, 1]$. μ on the other hand raises the right hand side of the inequality to infinity for $\mu \to 1$. Interpreting the right hand side of the inequality as a function of $T(t)$, it clearly has a pole at $T(t) = P$. If inequality (5.20) holds, the increase in $T(t)$ occurs because collateral damage is higher than the decrease due to law enforcement. In this case it cannot be optimal to run law enforcement, also because it causes additional cost.

For any $(S(t), T(t))$ combination there exists a $\mu < 1$ above which inequality (5.20) does not hold. If μ is chosen sufficiently high, the effects of collateral damage are lower than the effects of law enforcement on $T(t)$. On the other hand, if μ is sufficiently small, the effects of collateral damage are often higher than the effects of law enforcement. Therefore, for arbitrary but constant values of μ and ω, inequality (5.20) always holds if $S(t)$ is chosen high enough respectively $T(t)$ small enough. Figure 5.9 displays the right-hand-side of inequality (5.20) for three values for μ, while ω is chosen to be 0. Under the corresponding lines, it is clearly optimal to set $\upsilon(t)$ equal to zero. The lines themselves represent the points at which the effects of collateral damage and law enforcement are equal.

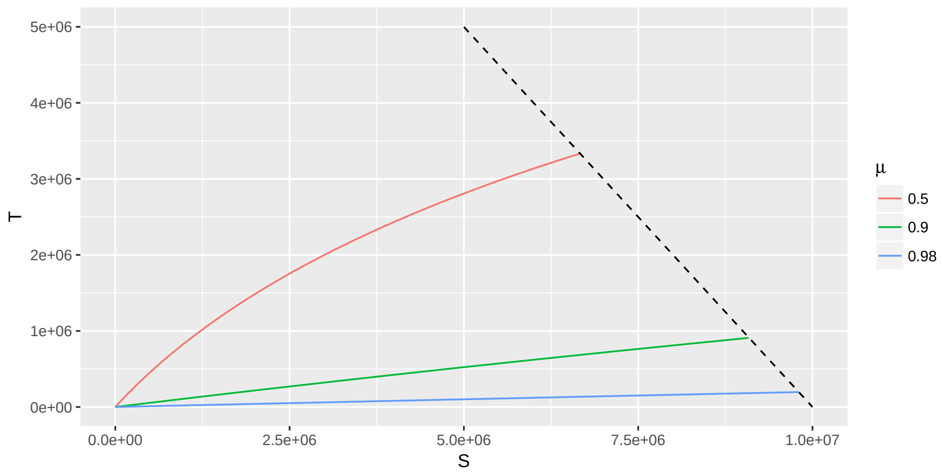


Figure 5.9: The lines in this figure display all combinations in the (S,T)-space along which the collateral damage is equal to the impact of law enforcement. Note that $\mu(t)$ is set to a constant value and $\omega(t)$ is set to 0. Underneath each line, the increase in the number of terrorists due to collateral damage is higher than the decrease because of law enforcement.

It is important to note that the corresponding costs are still completely ignored in this consideration. Furthermore dynamic effects are not considered. Note that the change from not using $\upsilon(t)$ to actually using it increases the costs abruptly. In relation to optimizing the overall cost, there is most likely a broader area in which $\upsilon(t) = 0$ is the optimal solution. The presented area defined by inequality (5.20) is just a subspace in which it is optimal to set $\upsilon(t)$ to zero for constant values of μ and ω.

However, in the dynamic optimal solution, the area at which $\upsilon(t)$ and $\mu(t)$ are set to zero differs from the areas presented in Figure 5.9. The social cost generated through terrorism changes with the value of $T(t)$. Therefore, the budget for the controls changes, too, in order to maintain cost efficiency. Furthermore, the point of using law enforcement differs from the line in figure 5.9 because of the abruptly rising costs it causes.

In order to get a more realistic picture of the area where $\upsilon(t) = \mu(t) = 0$, several optimal paths for different starting values were calculated. The parts of the paths where $\upsilon(t)$ and $\mu(t)$ are set to zero are then used to illustrate the discussed area, by surrounding a polygon. Figure 5.10 displays this polygon as well as some optimal trajectories used to generate it. The stated area is not exact due to the fact that it is approximated with a surrounding polygon. Furthermore, the optimal area is most likely larger, which could be verified by calculating additional optimal paths. Nevertheless, all points in the (S, T)-space where it is optimal to run only prevention have in common that the number of terrorists is much lower than its equilibrium value. Overall, these considerations and especially the calculations provide a reasonable answer to the question posed above.

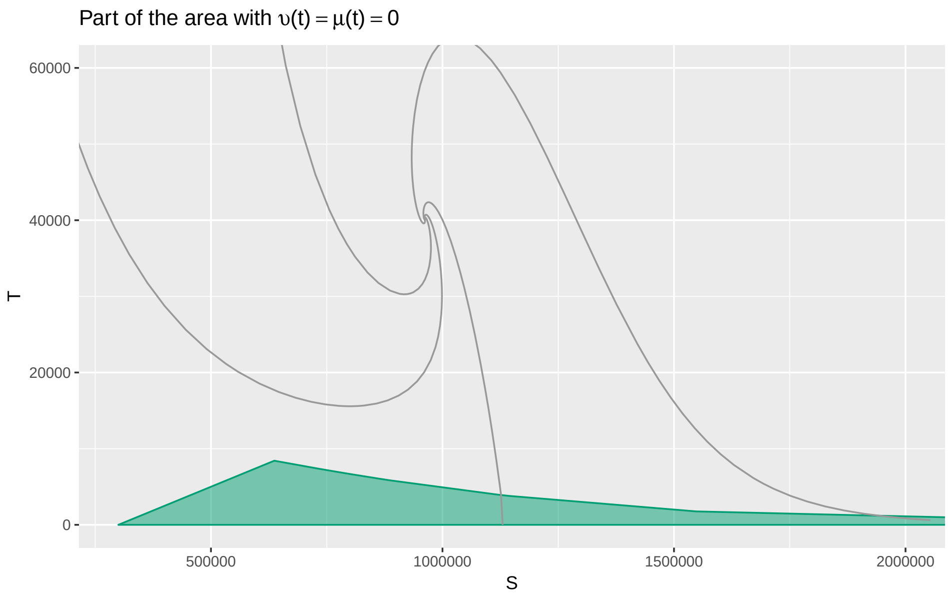


Figure 5.10: The green area in this figure approximately displays that part of the region in the (S, T)-space where it is optimal to set $\mu(t) = \upsilon(t) = 0$. A few of the trajectories used to estimate this region are also shown.

Chapter 6

Conclusion

After setting up the dynamic optimization model and considering it from different angles, the behavior of the model became visible. Under the estimated circumstances, investing in intelligence / education in order to get better reports has an enormous effect, especially because of its affordability. The sensitivity analysis for c_μ again clarifies the importance of a high level of $\mu(t)$.

The answer to the question of whether law enforcement should be applied despite the fact that it causes collateral is inconclusive. Under the assumptions of this model, there are certainly areas in the (S, T)-space at which the harm is greater than its benefits, especially in terms of additional cost. But sooner or later it is optimal to apply law enforcement. This simply results from the fact that in the only saddlepoint-stable steady state all controls are applied. In order to obtain an even better understanding, the process started in Subsection 5.5.2 could be continued in more detail.

Prevention on the other hand is always applied at a high level. Although the effects are relatively low, it is cost efficient in all investigated examples at all time.

The high values of $\mu(t)$ offer room for further investigation. It is reasonable that one of the possibilities enumerated in Subsection 5.5.1 holds or that the model, especially the efficiency of $\mu(t)$, should be reconsidered completely. With regard to further investigations, it would be very interesting to modify the model by enlarging c_μ or decreasing θ. It is not very realistic that the level of intelligence does not have any conversion cost. Another possibility to modify the stated model is therefore to introduce a new state variable which contains the level of intelligence. The corresponding dynamic must contain the education cost, respecting the upper limit of one and including some sort of forgetting rate. In either way, the modified model should be compared to the model stated in this thesis. The most important question in this context is about the desired behavior of the model. Modifying parameters simply to force the optimal values to be lower should only be applied if the changes are reasonable. One important question in either of these modifications should also address the area in which law enforcement and education are turned off.

In summary, this model considering two different ways of radicalization offers a comprehensive insight and leaves enough room for further investigations.

Appendix A

Calculation and Visualization

A.1 Applying the Matlab Toolbox OCMat

In order to investigate and solve the optimal control model numerically, the Matlab-toolbox OCMat provides the needed functions. During the procedure of solving the control model, the first step was to initialize the model. During this process, all needed functions and files are generated. The next step is to find equilibria. For this model, only one was found. The final step is to calculate the optimal path for a given starting value converging towards the unique equilibrium.

Initialization

The initialization file A.1 contains the whole optimal control model, formulated for OCMat. The control constraints are listed and labeled. The arcs are necessary because of these constraints, as they define which constraint is active. For example, the case that constraints CC1 and CC3 are active is labeled with arc number 5. As there are two solutions for the explicit expression of $v(t)$ and $\mu(t)$, both are kept. The number after the underscore defines which of these two solutions is used. Because one control cannot be expressed explicitly, the `Control` section defines for each arc, which control is given implicitly and which one explicitly.

```
Type
standardmodel

Description
terrormodel - masterthesis

Modelname
terrormodelnewa

Variable
state::S,T
```

```
control::v,mu,w

Statedynamics
ode::DS=k-delta*S-(h+(1-h)*exp(-m*w))*(theta*v*(1-mu)*(1-
    T/P)+beta*T/P)*S
ode::DT=(h+(1-h)*exp(-m*w))*(theta*v*(1-mu)*(1-T/P)+beta*
    T/P)*S-(sigma+v*(mu+(1-mu)*T/P))*T

Objective
int::-(c_T*T-c_v*log(1-v)+w-c_mu*log(1-mu))

Controlconstraint
CC1::ineq::v>=0
CC2::ineq::w>=0
CC3::ineq::mu>=0

ArcDefinition
0::[]_1
1::[]_2
2::CC2_1
3::CC2_2
4::CC3
5::CC1,CC3
6::CC2,CC3
7::CC1,CC2,CC3

Control
0::w::implicit
0::v,mu::explicit
1::w::implicit
1::v,mu::explicit
2::w::implicit
2::v,mu::explicit
3::w::implicit
3::v,mu::explicit
4::w::implicit
4::v,mu::explicit
5::w::implicit
5::v,mu::explicit
6::w::implicit
6::v,mu::explicit
7::w::implicit
7::v,mu::explicit

Parameter
P::10000000
c_v::7.4648e+10
c_T::71350.12
c_mu::1.8424e+8
```

```
r::0.0397
k::190810
beta::8.03
delta::0.168858407079646
theta::1
m::2.6411e-08
h::0.9025
sigma::0.0685
```

Code A.1: Initialization file for OCMAT

The actual commands for the initialization are displayed in A.2. The object `m` is an instance of the generated model.

```
opt=setocoptions('INIT','Jacobian','numerical','Hessian','numerical','ControlDynamics','implicit');
ocStruct=processinitfile('terrormodelnewa',opt);
modelfiles=makefile4ocmat(m,[],opt);
moveocmatfiles(m,modelfiles);
m=stdocmodel('terrormodelnewa');
save(m);
```

Code A.2: Initialization process (MATLAB)

Finding the Equilibrium

In order to find the equilibrium, Dieter Grass from the research group ORCOS at TU WIEN applied a modified version of his original function `calcep`. The admissibility will already be considered inside the function `mycalcep`. The starting values for the numeric search are values found during a constant optimization. The found equilibria are saved in `ocEP`. As stated in A.3, there are easy-to-use functions available in order to get the values of different characteristics for each equilibrium. The names of the functions are self-explanatory, a detailed description can be found inside the function or in the manual [34]. The function `state`, for example, returns the values of the state variables in the equilibrium.

```
S=759686.71;T=100981.66;v=0.552031306572620;mu=0.997613406606991;w=386212.537573212;
ocEP=mycalcep(m,1,[S;T],[v;mu;w]);

dynPrim=ocEP{1};
state(m,dynPrim)
costate(m, dynPrim)
lagrangemultiplier(m, dynPrim)
control(m,dynPrim)
eig(jacobian(dynPrim))
```

Code A.3: Search for equilibrium (MATLAB)

Finding the Optimal Path

For a given starting value (S_0, T_0), the goal is of course to find the optimal path derived from the canonical system. The code in A.4 especially addresses the calculation of paths in which it is optimal to set law enforcement and the level of intelligence to zero for some time. Inside the code, the calculation of the path is started three times in total. The first is the normal approach for finding the optimal path accounting for several options. The second uses a larger step width, and the third accounts for the change of arcs.

```
S0 = 1.1300e+006; T0 = 10;

opt=setocoptions('OCCONTARG','MaxContinuationSteps',400,'SBVPOC','FJacobian',1,'BCJacobian',0,'NMax',10000,'GENERAL','AdmissibleTolerance',1e-3,'NEWTON','MaxNewtonIters',15,'MaxProbes',10,'RelTol',5e-1,'AbsTol',5e4,'EQ','TolX',1e-10,'MaxFunEvals',5000,'MaxIter',5000);
opt0=setocoptions(opt,'OCCONTARG','MinStepWidth',1e-2,'MaxStepWidth',1e6,'InitStepWidth',1e5,'SBVPOC','MeshAdaptAbsTol',[1e-3;1e-3;1e3;1e3;1e4],'MeshAdaptRelTol',1e-5,'GENERAL','TrivialArcMeshNum',5,'NewtonSolver','newtcorr4bvp','NEWTON','CheckSingular',0);
opt00=setocoptions(opt,'GENERAL','AdmissibleTolerance',0);

epidx=1;eigval=real(eig(ocEP{epidx}));eigval(eigval>-1e-5)=[];T=50/min(abs(eigval))
sol=initocmat_AE_EP(m,ocEP{epidx},1:2,[S0;T0],[],'TruncationTime',T,'PathType','sc');
c=bvpcont('extremal2ep',sol,[],opt0);
store(m,'extremal2ep');ocEx=extremalsolution(m);n=length(ocEx)

opt0=setocoptions(opt,'OCCONTARG','MinStepWidth',1e-2,'MaxStepWidth',5e8,'InitStepWidth',1e6,'SBVPOC','MeshAdaptAbsTol',[1e-3;1e-3;1e3;1e3;1e4],'MeshAdaptRelTol',1e-5,'GENERAL','TrivialArcMeshNum',5,'NewtonSolver','newtcorr4bvp','NEWTON','CheckSingular',0);
sol=initocmat_AE_AE(m,ocEx{n},1:2,[S0;T0]);
c=bvpcont('extremal2ep',sol,[],opt0);
store(m,'extremal2ep');ocEx=extremalsolution(m);n=length(ocEx)

[b infoS newarcpos violarcarg]=testadmissibility(ocEx{n},m,opt00)
ocExN=redefinearc(ocEx{n},newarcpos(:,1),5);
```

```
opt0=setocoptions(opt,'OCCONTARG','MaxContinuationSteps'
    ,150,'MinStepWidth',1e-2,'MaxStepWidth',5e14,'
    InitStepWidth',1e6,'SBVPOC','MeshAdaptAbsTol',[1e-3;1
    e-3;1e3;1e3;1e4],'MeshAdaptRelTol',1e-5,'GENERAL','
    TrivialArcMeshNum',5,'NewtonSolver','newtcorr4bvp','
    NEWTON','CheckSingular',0);
sol=initocmat_AE_AE(m,ocExN,1:2,[S0;T0]);
c=bvpcont('extremal2ep',sol,[],opt0);
store(m,'extremal2ep');ocEx=extremalsolution(m);n=length(
    ocEx)
```

Code A.4: Optimal Path (MATLAB)

Sensitivity Analysis

Code segment A.5 displays the MATLAB-Code used to generate the needed data for the sensitivity analysis for the parameters c_T, c_μ, and c_v. Note that the considered intervals are split and always start with the original estimation. The reason for that approach is that the algorithm was not always able to find the equilibrium, if the considered parameter differs too much from the starting value. The newly calculated equilibrium was always used as a new starting value for the next iteration.

```
%% Sensitivity Analysis for c_T, c_mu and c_v
syms c_T c_mu c_v;
sensitivity_variables = [c_T c_T c_mu c_mu c_v c_v];

n = 100;
parameter_ranges = [
  linspace(71350.12, 3.5675e+004, n);    % c_T
  linspace(71350.12, 1.4270e+005, n);    % c_T
  linspace(184240000, 92120000, n);      % c_mu
  linspace(184240000, 1.8424e+010,n);    % c_mu
  linspace(7464800000, 3.7324e+009,n);   % c_v
  linspace(7464800000, 1.4930e+010,n)]; % c_v


for i = 1:length(sensitivity_variables)
    ocEP_tmp = ocEP;
    out = zeros(n, 12);

    for j = 1:length(parameter_ranges(i,:))
        switch char(sensitivity_variables(i))
            case 'c_T'
                m_tmp = changeparametervalue(m, 'c_T',
                    parameter_ranges(i,j));
            case 'c_mu'
                m_tmp = changeparametervalue(m, 'c_mu',
                    parameter_ranges(i,j));
```

```
            case 'c_v'
                m_tmp = changeparametervalue(m, 'c_v', parameter_ranges(i,j));
        end

        ocEP_tmp = calcep(m_tmp, ocEP_tmp{1}.y);
        b=isadmissible(ocEP_tmp,m_tmp); ocEP_tmp(~b)=[];

        if length(ocEP_tmp) ~= 1
            disp('Error searching for equilibrium');
            disp('Parameter'); disp(char(sensitivity_variables(i)));
            disp('Value:'); disp(parameter_ranges(i,j));
            continue;
        end

        out(j,1) = i; out(j,2) = parameter_ranges(i,j);
        out(j, 3:4) = state(m_tmp,ocEP_tmp{1})';
        out(j, 5:6) = costate(m_tmp,ocEP_tmp{1})';
        out(j, 7:9) = lagrangemultiplier(m_tmp,ocEP_tmp{1})';
        out(j, 10:12) = control(m_tmp,ocEP_tmp{1})';
    end

    filename = strcat('Sensitivity_',char(sensitivity_variables(i)),'_id_',num2str(i), '.csv');
    csvwrite_with_headers(filename, out, {'parameter_id', 'parameter_value', 'susceptible', 'terrorists', 'costate1', 'costate2', 'lagpar1', 'lagpar2', 'lagpar3', 'v', 'mu', 'w'});
end
```

Code A.5: Sensitivity Analysis (MATLAB)

A.2 Visualization

For the visualization of most figures, R [35] is used. In order to demonstrate the procedure and the packages used, the visualization of the sensitivity analysis for c_μ is demonstrated in A.6.

```
require("ggplot2")
require("dplyr")
require("tidyr")
library("stringr")
library("stringi")
library(latex2exp)
```

```
## load all data
names <- list.files(path = "data")
data_orig <- data.frame()
for(i in c(1:length(names))){
  parameter_name = stri_sub(names[i],13,-10)
  tmp <- read.csv(paste("data/",names[i], sep = ""))
  tmp$parameter_id <- parameter_name
  data_orig <- rbind(data_orig, tmp)
}
data <- data_orig %>%
  select(parameter_id, parameter_value, "Terrorists" =
      terrorists,  "Susceptibles" = susceptible, "upsilon
      " = v, "mu" = mu, "omega" = w) %>%
  mutate("varphi(omega)" = 0.9025 + (1-0.9025) *exp
      (-2.641112e-08 * omega)) %>%
  gather(group, size, -parameter_id, -parameter_value)
      %>%
  mutate(group_0 = factor(group, levels=c("Terrorists","
      Susceptibles","upsilon", "mu", "omega", "varphi(
      omega)")))


## Plot
plot_c_mu <- data %>%
  filter(parameter_id == "c_mu", parameter_value < 4e+010
       ) %>%
  ggplot(aes(x = parameter_value, y = size)) +
  geom_line() +
  geom_vline(xintercept=1.8424 * 10^8, linetype = "dashed
      ", colour = "#999999") +
  scale_x_log10() +
  facet_wrap(~group_0, ncol=2, scales = "free_y",
      labeller = label_parsed) +
  labs(title = TeX('Sensitivity Analysis for $c_\\mu$'),
      x = TeX("$c_\\mu$"), y = "Equilibrium values",
      colour = "")
```

Code A.6: Visualization of Sensitivity Analysis (R)

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