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Pricing Financial Derivatives using Brownian Motion and a Gaussian Markov Alternative to Fractional Brownian Motion

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Abstract

This thesis examines different models for pricing financial options. Instead of using Brownian motion as the underlying process, as is done in the Black–Scholes model, fractional Brownian motion is introduced and discussed. Then the Dobrić–Ojeda process, a Gaussian Markov alternative, and a modified version of it will be presented as an alternative to fractional Brownian motion, based on the analysis of Conus and Wildman. In contrast to Brownian motion, fractional Brownian motion and its alternatives incorporate past dependencies, using the Hurst index. The Black–Scholes and the Conus–Wildman model will be tested on options of the S&P 500 index, where the implied volatility and the implied Hurst index are estimated. The pricing accuracy of the two models will be compared using the obtained estimators. We find that the Conus–Wildman model estimates option prices better than the Black–Scholes model, concluding that past dependencies matter and should be incorporated when pricing options.

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Chapter 1

Introduction

The Black–Scholes model, sometimes called Black–Scholes–Merton model, is probably the most famous model for financial option pricing. The Nobel price winning paper *"The pricing of Options and Corporate Liabilities"*, published in the Journal of Political Economy in 1973 [3], has since facilitated the calculation of option prices. The Black–Scholes price frequently overprices options, due to several assumptions made in the model, especially assuming a constant volatility. Black calls the simplicity of the model its greatest weakness and strength, as it is easy for people to understand. It is a good 'first approximation' and knowing its holes, can be extended [2], as will be done in this paper.

An option is a security that gives the holder the right, but not the obligation, to *exercise* it, that is to buy or sell an asset. While an *American option* can be exercised at any time during a specified period of time, a *European option* can only be exercised at the end of that time period, on the *maturity date*. The price that is paid when the option is exercised is called the *strike price*. Options that give the option holder the right to buy the underlying asset are referred to as *call options*, while *put options* give the holder the right to sell the underlying.

In 1900 L. Bachelier was the first one to develop a model describing the evolution of stock prices $S = (S_t)_{t \geq 0}$,

$$S_t = S_0 + \mu t + \sigma W_t,$$

where $W = (W_t)_{t \geq 0}$ is a standard Brownian motion process and t indicates time, $S_0 \in \mathbb{R}_+$ is the initial condition, today's stock price, $\mu \in \mathbb{R}$ represents the drift of the stock price and $\sigma \in \mathbb{R}_+^*$ its volatility. The main criticism of Bachelier's model is that stock prices can take on negative values, which let P. Samuelson to describe the logarithms of the stock prices S_t as a linear model instead of the prices itself,

$$\ln \frac{S_t}{S_0} = \mu t + \sigma W_t, \tag{1.1}$$

describing stock prices using a geometric Brownian motion,

$$S_t = S_0 e^{\mu t} e^{\sigma W_t - \frac{\sigma^2}{2} t}.$$

Solving (1.1) with Itô's formula (4.1) the stochastic differential equation (SDE) can be obtained straightforward

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t). \tag{1.2}$$

Adding a bank account $B = (B_t)_{t \geq 0}$ with $dB_t = rB_t\, dt$ for a (fixed) risk-free interest rate $r > 0$, the *standard diffusion (B,S)-model.* On the basis of this model F. Black, M. Scholes and R. Merton obtained the famous *Black–Scholes formula* [3], [26].

F. Black and M. Scholes based their model on the assumption that when options are priced correctly, it should not be possible to make an immediate profit through buying *(long position)* and selling *(short position)* options [3]. This concept known as *arbitrage* is central in financial markets. An arbitrage free market is a realistic assumption; if there would exist an arbitrage opportunity traders would immediately act upon it, driving prices towards an arbitrage free equilibrium and eliminating the opportunity. The following other assumptions that were made in [3] are considering an *'ideal' market* and are common for economic models in general: the short term interest rate is known and constant, stock prices follow a random walk, the distribution of stock prices is log-normal, the variance rate is constant, the stock pays no dividends, there are no transaction costs, it is possible to borrow any fraction of a security at any time and there are no penalties for short selling.

A major weakness of the (B,S)-model is the assumption of a known and constant variance. Financial prices have shown that the volatility changes for different maturity dates T with a fixed strike price K, and also for different strike prices with a fixed maturity; in this case the volatility takes the shape of a convex function, which is known as the *(volatility) smile effect* [26]. There have been several attempts to allow for a stochastic volatility as a more realistic model. In chapter 3 Heston's stochastic volatility model will be introduced.

In comparison to Brownian motion which is the underlying process of the Black–Scholes model, fractional Brownian motion has been shown to model stock prices more accurately, as it incorporates long-range dependencies through including Hurst index H. Previous models usually assume independent increments (as is the case for Brownian motion), which means that the stock price process would be independent of its past. But fractional Brownian motion is not a semimartingale (except for $H = 0.5$, which is the classical Brownian motion) and therefore allows for arbitrage. For this reason fractional Brownian motion was discarded as an underlying process for financial models for a while, but has picked up on popularity in recent years as the amount of literature shows. It is not a Markov process either, so all the stochastic calculus has to be derived from Gaussian properties. The motivation for using fractional Brownian motion is that market data has shown, that stock prices do have some dependence of the past. As shown in section 8.3 we estimate the Hurst index of the historical S&P 500 data to be $H = 0.58$.

Dobrić and Ojeda proposed a Gaussian Markov alternative to fractional Brownian motion. Gaussian Markov processes are commonly used to model dynamic systems. For financial stochastic models the majority of processes are Markovian and many are Gaussian. That is, because Gaussian Markov processes have many nice properties like long-range dependence, fat tails and non-stationary volatility, corresponding to the real market and allowing a more realistic approach [23].

Throughout this paper we will assume to be on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and if not stated otherwise all stochastic processes will be real valued. Further, the underlying stock price process $(S_t)_{t\geq 0}$ satisfies $S_0 = 1$.

Chapter 2

A Discourse on Chaos and Fractal Geometry

Chaos theory has fascinated all kinds of scientists, cutting across traditional scientific disciplines; explaining irregularities, randomness and chaos through patterns and order. In the 1970s chaos theory began evolving with the help of the computer, which facilitated computing repeated iterations of simple mathematical formulas, the basis of chaos. Physiologists found an order in the chaos developing in human hearts, Ecologists in the rise and fall of moth populations and Economists in stock price data.

The best known principle of chaos theory is probably the *Butterfly Effect* or in more technical terms the 'sensitive dependence on initial conditions'. The name 'Butterfly Effect' arises from the graph as shown in figure 2.2. In 1961 E. Lorenz discovered that the weather's unpredictability had some consistency. He found that insignificant changes in input had significant changes in output. For a more detailed history and overview we will refer to [11].

Houthakker, a Harvard economics professor, who collected cotton prices could not make his data fit under the in statistics popular Gaussian normal distribution, the bell shaped curve. Up until then it was believed that all sorts of data would be best described by the normal distribution, having only a few big outliers.

Benoit Mandelbrot discovered that he had found similar data as Houthakker had, but on a completely different topic, the distribution of small and large incomes. Mandelbrot started looking for patterns in cotton price data, because those prices had been documented regularly. When studying the (ir)regularities of transmission noise in telephone lines, Mandelbrot found that the transmission errors described a Cantor set over time. Other interesting examples are the Koch snowflake, the Sierpinski triangle and carpet

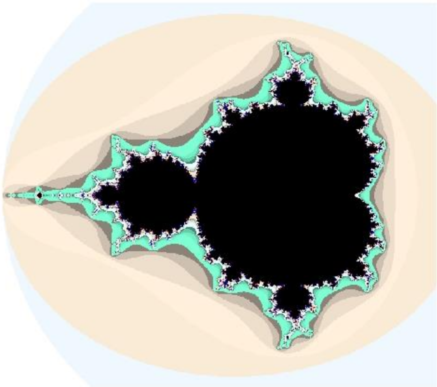

Figure 2.1: Mandelbrot set

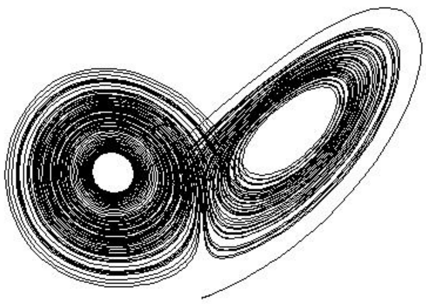

Figure 2.2: Lorenz attractor

and the Menger sponge. Then Mandelbrot discovered the data Hurst had collected, which will be described in the following section, and introduced the *Noah Effect* and the *Joseph Effect*. The Noah Effect describes the phenomenon that large changes can happen very fast and not as previously assumed slowly, for example a stock price can drop suddenly without having gone through all prices in between. The Joseph Effect describes persistence, for example a lot of transmission errors happen in one period of time, while in another one there are none, meaning that once you have some errors you are more likely to have more following. Another famous example of Mandelbrot is the question of how long Britain‘s coastline is. The answer is, that it depends on the size of the ruler being used; the shorter the ruler, the longer the coastline, as one starts to measure more details of the coastline. Mandelbrot was the one who introduced the word *fractal* as he was searching for a name to describe his findings. "[A] fractal is a way of seeing infinity"[11]. Fractal means *self-similarity*, which is a form of invariance across time and space. The famous Mandelbrot set is shown in figure 2.1, displaying self-similarity on all scales.

Physicists then used chaos to approach turbulence, a topic previously left to the force of nature. Kolmogorov [15], examining turbulent flow, described smaller scale turbulent motions on a random self-similar field with stationary increments.

2.1 Hurst Discoveries and Long-Range Dependece

The British hydrologist H. E. Hurst observed a surprising phenomenon while studying the fluctuations of yearly run-offs of rivers, especially the Nile. Following [26] let $x_1, x_2, \ldots, x_n$ be the values of n successive water run-offs of the Nile, with expected value $\frac{1}{n}X_n$, where $X_n = \sum_{k=1}^{n} x_k$. The deviation of the cumulative value X_k to k successive years from the empirical mean is $X_k - \frac{k}{n}X_n$ with range

$$R_n = \max_{k \leq n}\left(X_k - \frac{k}{n}X_n\right) - \min_{k \leq n}\left(X_k - \frac{k}{n}X_n\right).$$

Hurst considered the normalized values $Q_n = \frac{R_n}{S_n}$, with

$$S_n = \sqrt{\frac{1}{n}\sum_{k=1}^{n} x_k^2 - (\frac{1}{n}\sum_{k=1}^{n} x_k)^2}$$

and discovered for large n

$$\frac{R_n}{S_n} \sim cn^H,$$

that H, which is now known as the *Hurst index*, is approximately 0.7, with c being some constant. This came as a surprise as he would have expected the index to be $H = 0.5$, indicating that the run offs were independent of the past. Mandelbrot was the one who introduced the name Hurst index, when he discovered Hurst's findings [1]. Note that H is also referred to as *self-similarity parameter* in some literature as in [22], the definition of self-similarity is given in section 5.2. In [23] the Hurst index of the S&P 500 index was calculated to be 0.61, while we have found it to be 0.58 using the ergodic ratio of second moments method described in section 8.2. Further, Scansaroli [23] made three main observations about the Hurst index:

1. The Hurst index is not constant over time.
2. The Hurst index is greater than $H = 0.5$ on a 95% confidence interval for most of the time.
3. Capitalization, volume and liquidity may influence the index.

The Hurst index allows for *long-range dependence*, meaning that the increments are positively correlated, this is the case for $H > 0.5$, and the closer H is to 1, the more long memory the process has. For $H = 0.5$, we have classical Brownian motion and increments that are not correlated. The case $H < 0.5$ indicates a negative correlation between increments. Most authors who have studied fractional Brownian motion in the field of financial mathematics focus on the case $H > 0.5$ as the data suggests being realistic.

Gaussian white noise $\epsilon = (\epsilon_t)_{t\geq 1}$ can be obtained with Brownian motion $B = (B_t)_{t\geq 0}$, representing the randomness of the process,

$$\epsilon_t = B_t - B_{t-1},$$

for $t \geq 1$ and with $\mathbb{E}[\epsilon_t] = 0$ and $\mathbb{E}[\epsilon_t^2] = 1$, where ϵ_t are independent identically distributed random Gaussian variables. Accordingly, *fractional Gaussian white noise* is obtained through the increments of fractional Brownian motion,

$$\epsilon_t = B_H(t) - B_H(t-1),$$

with Hurst index $H \in (0, 1)$.

Now we can define *long-range dependence*, see [10], [26]:

Definition 2.1.1. *If $\frac{1}{2} < H < 1$ a process is said to have long-range dependence if*

$$\sum_{n=0}^{\infty} |\rho(n)| = \infty,$$

where $\rho(n)$ is the covariance function

$$\rho_H(n) = Cov(\epsilon_t, \epsilon_{t+n}) = \frac{1}{2}\left\{|n+1|^{2H} - 2|n|^{2H} + |n-1|^{2H}\right\}.$$

Note that for $H < \frac{1}{2}$, we have $\sum_{n=0}^{\infty} |\rho(n)| < \infty$.

Chapter 3

Stochastic Volatility Models

In the Black–Scholes model a constant and fixed volatility was assumed in order to facilitate calculations. As is known from financial markets, this is not a very realistic assumption. There have been several attempts since to incorporate a stochastic volatility into financial option pricing models. In stochastic volatility models the stock price process is accompanied by another stochastic differential equation (SDE) for the volatility σ, these models are of the form

$$\begin{aligned} dS_t &= S_t\sqrt{\sigma_t}\left(\rho\, dB_t^{(1)} + \sqrt{1-\rho^2}\, dB_t^{(2)}\right), \\ d\sigma_t &= a(t,\sigma_t)\, dt + b(t,\sigma_t)\, dB(t), \quad \sigma_0 > 0, \end{aligned}$$

where $(B_t^{(1)}, B_t^{(2)})_{t\geq 0}$ is a two dimensional standard Brownian motion, correlation $\rho \in [-1,1]$ and with a and b satisfying some regularity conditions so that the solution of the SDEs exists. In the following section Heston's stochastic volatility model will be introduced. For further models, see [14] and [19].

3.1 Heston

In [12] Heston assumes that the volatility follows an Ornstein–Uhlenbeck process,

$$d\sqrt{\sigma_t} = -\beta\sqrt{\sigma_t}\, dt + \delta\, dB(t),$$

where $B(t)$ is Brownian motion. Then using Itô's formula (4.1) the Cox–Ingersoll–Ross process can be obtained,

$$\begin{aligned} dS_t &= S_t\sqrt{\sigma_t}\left(\rho\, dB_t^{(1)} + \sqrt{1-\rho^2}\, dB_t^{(2)}\right), \\ d\sigma_t &= \kappa(\theta-\sigma_t)\, dt + \xi\sqrt{\sigma_t}\, dB(t), \quad \sigma_0 > 0, \end{aligned}$$

where the constants $\kappa, \theta, \xi > 0$ satisfy the *Feller condition* $2\kappa\theta > \xi^2$.

As shown in [25] calibration results are pretty accurate for a wide range of maturities, but for short time maturities the generated volatility smile is not steep enough, which is known as the *small-time explosion* feature.

Chapter 4

Preliminaries

As a short recap, the main processes and theorems that are used throughout this paper will be defined in this section. As main reference for the stochastic analysis, we refer to [24].

Definition 4.0.1. *A* Gaussian process *is a real valued stochastic process* $(X_t)_{t\in[0,T]}$*, if for any* $t_1, \dots, t_k$ *in* T *the random variables* $X_{t_1}, \dots, X_{t_k}$ *are jointly normal.*

A Gaussian process X_t is called *centered* if $\mathbb{E}[X_t] = 0$ for all $t \in [0, T]$. It is completely characterized by its covariance, i.e. $\mathbb{E}[X_t X_s]$ for all $s, t \in [0, T]$.

Definition 4.0.2. *A random process* $(X_t)_{t\in[0,T]}$ *is called a* Markov process *with respect to filtration* $\mathcal{F}$*, if for all* $s \leq t$ *the random variable* X_t *is conditionally independent of* $\mathcal{F}_s$ *given* X_s*.*

4.1 Short Review on Brownian Motion

Brownian motion originally observed in 1827 in the physical field by the Scottish botanist Robert Brown, who under a microscope followed the jittery motion of tiny particles in water, introduced as a mathematical concept in 1900 by L. Bachelier, who presented a stochastic analysis of the stock and option markets, and A. Einstein (1905), who explained the motion as a result of numerous collisions with even smaller particles, is also called a *Wiener process* after Norbert Wiener, who proved and expanded the theory in 1923.

Definition 4.1.1. *An adapted* $\mathbb{R}^d$*-valued stochastic process* $B = (B_t)_{t\geq 0}$ *is called* d-dimensional Brownian motion *with respect to filtration* $\mathbb{F}$ *if the following properties hold:*

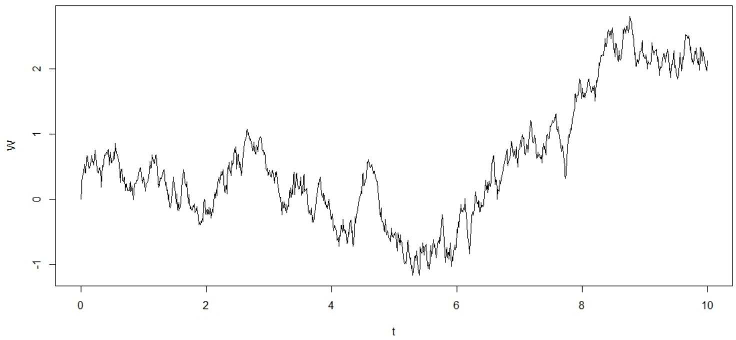

Figure 4.1: Brownian motion

1. $\mathbb{P}[B_0 = X] = 1$.

2. *Independence of increments, i.e.* $B_t - B_s$ *is independent of* $\mathcal{F}_s$ *for all* $s < t$ *in* $\mathbb{R}_+$.

3. *Stationarity of increments, i.e.* $(B_t - B_s) \stackrel{d}{=} (B_{t-s} - B_0)$ *for all* $s < t$ *in* $\mathbb{R}_+$.

4. *Normal distribution of increments, i.e.* $(B_t - B_0) \stackrel{d}{=} \mathcal{N}(0, tI_d)$ *for all* t *in* $\mathbb{R}_+$, *where* $I_d \in \mathbb{R}^{d\times d}$ *is the identity matrix.*

5. B *has continuous paths, i.e.* $t \to B_t(\omega)$ *is continuous for every* $\omega \in \Omega$ *and every* t *in* $\mathbb{R}_+$.

Brownian motion is called *standard Brownian motion* if the process starts at the origin, i.e (1) is replaced with $\mathbb{P}[B_0 = 0] = 1$. (Standard) Brownian motion is a continuous Gaussian process with homogeneous independent increments. It is one of the wideliest used stochastic processes in financial mathematics, being the underlying process of several financial models, including the Black–Scholes model. It involves a multi-dimensional normal distribution. It is a Gaussian process, a Lévy process, a Markov process, a diffusion process, a martingale and a self-similar process [10].

Definition 4.1.2. *A* d-dimensional Lévy process *belongs to a larger class of processes, where requirement (4) is dropped in the definition of Brownian motion and (5) is replaced with the condition of càdlàg paths.*

4.2 Itô's Lemma and Girsanov's Theorem

Itô's lemma, the most important result of stochastic analysis, named after the Japanese mathematician Kiyosi Itô, who proved the lemma in 1944. The version for continuous semimartingales will be given here.

Lemma 4.2.1. *(Itô's Lemma.) Let U be an open non-empty subset of $\mathbb{R}^d$ and $X = (X^1, \dots, X^d)$ be a U-valued continuous semimartingale. Let $f \in C^2(U, \mathbb{R}^n)$, with $f = f_1, \dots, f_n$. Then, f is an $\mathbb{R}^n$-valued continuous semimartingale and*

$$f_k(X) = f_k(X_0) + \sum_{i=1}^{d} \int_0^{\cdot} \partial_i f_k(X_s)\, dX_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^{\cdot} \partial_i \partial_j f_k(X_s) d[X^i, X^j]_s, \tag{4.1}$$

for every $k \in \{1, \dots, n\}$, up to indistinguishability, with partial derivatives ∂_i, ∂_j.

Girsanov's theorem tells us how to convert the probabilty measure $\mathbb{P}$ to a risk-neutral measure $\mathbb{Q}$, which is an essential tool when pricing financial derivatives.

Theorem 4.2.2. *(Girsanov's theorem.) Let X be a $\mathbb{R}^d$-valued continuous semimartingale with canonical decomposition $X = A + M$ and let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F})$ and Z an adapted continuous process such that $d\mathbb{Q} = Z_t\, d\mathbb{P}$ on $\mathcal{F}_t, t \in \mathbb{R}_+$. Then, $X = \tilde{A} + \tilde{M}$ is a continuous $\mathbb{Q}$-semimartingale with*

$$\tilde{A} = A + \int_0^{\cdot} \frac{1}{Z_s}\, d[M, Z]_s \quad \text{and} \quad \tilde{M} = M - \int_0^{\cdot} \frac{1}{Z_s}\, d[M, Z]_s,$$

where $\int_0^{\cdot} \frac{1}{Z_s}\, d[M, Z]_s$ is of locally bounded variation.

Chapter 5

Fractional Brownian Motion

The motivation of using fractional Brownian motion originates from empirical study, where a past dependency on stock price returns has been discovered. As shown in figure 8.1 the estimated Hurst parameter obtained through end of day closing values of the S&P 500 index, which is a good proxy for the United States stock market, but also worldwide, using $7,986$ values is around $H = 0.58$. Figures 5.1 and 5.2 demonstrate the influence Hurst index H has on fractional Brownian motion, for the negatively correlated case $H = 0.2$ and the positively correlated one $H = 0.8$. Note that the uncorrelated case

Graphs of fractional Brownian motion

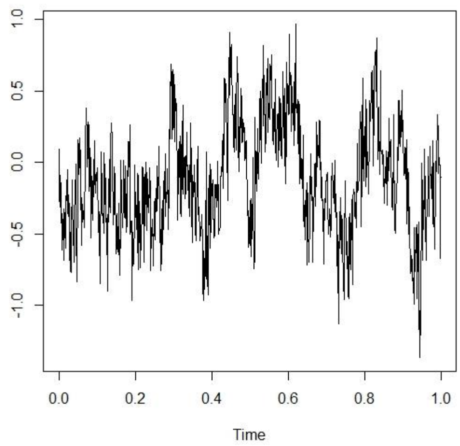

Figure 5.1: Negatively correlated increments, $H = 0.2$.

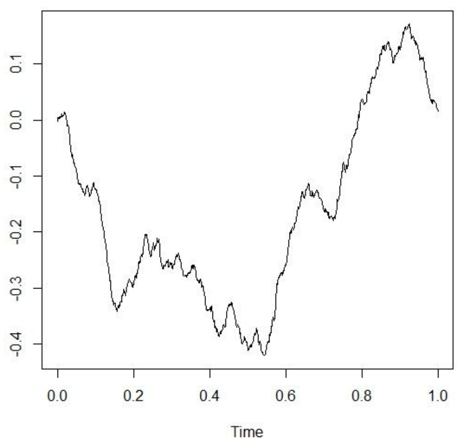

Figure 5.2: Positively correlated increments, $H = 0.8$.

$H = 0.5$, which is classical Brownian motion is shown in figure 4.1. The starting value of the fractional Brownian motion is 0 and we have chosen the number of time points, for example trading days, between starting time 0 and end time 1 to be $1,000$ in both graphs.

5.1 Background of Fractional Brownian Motion

Before fractional Brownian motion, Lévy [17] introduced another moving average process, the Holmgren–Riemann–Liouville integral

$$B_H(t) = \frac{1}{\Gamma(H+\frac{1}{2})} \int_0^t (t-s)^{H-\frac{1}{2}}\, dB(s), \tag{5.1}$$

where $B(s)$ is white noise and Γ the gamma function, but Mandelbrot found that this integral puts too great importance on the origin [18]. Kolmogorov [16] was the first one to introduce fractional Brownian motion on a Hilbert space in 1940, calling it *Wiener Helix*. The name *fractional Brownian motion (fBm)* was first used by Mandelbrot and van Ness in [18] in 1968, where they presented a stochastic integral representation of fBm:

$$\begin{aligned} B_H(t) - B_H(0) &= \frac{1}{\Gamma(H+\frac{1}{2})} \int_{\mathbb{R}} \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right) dZ(s) \\ &= \frac{1}{\Gamma(H+\frac{1}{2})} \\ &\times \left(\int_{-\infty}^0 ((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}})\, dB(s) + \int_0^t (t-s)^{H-\frac{1}{2}}\, dB(s) \right), \end{aligned}$$

where $B = (B_H(t))_{t\geq 0}$ is a real valued stochastic process with Hurst exponent $H \in (0,1)$ and starting value $B_H(0)$. $B_H(t)$ has independent increments if and only if $H = 0.5$, hence if $B_H(t)$ is Brownian motion. Note that $B_H(t)$ is a continuous centered Gaussian process.

5.2 Concepts of Fractional Brownian Motion

Fractional Brownian motion has dependent increments. In order to incorporate past dependencies of stock prices, the Hurst index H, which measures the intensity of long-range dependencies as described in section 2.1, will be used as an additional parameter.

Definition 5.2.1. *A real valued stochastic process $(X_t)_{t\geq 0}$ is called* self-similar*, if for any $a > 0$ there exists $b > 0$ such that*

$$X_{at} \stackrel{d}{=} bX_t.$$

If $b = a^H$ for each $a > 0$ and H unique, $(X_t)_{t\geq 0}$ is self-similar with Hurst exponent H.

The following example shows that Brownian motion is a self-similar process, compare to [10], [26].

Example 5.2.2. Brownian motion $B = (B_t)_{t\geq 0}$ with $\mathbb{E}\,[B_t] = 0$ and $\mathbb{E}\,[B_s, B_t] = \min(s,t)$ is a self-similar process. It follows that

$$\mathbb{E}\,[B_{as}, B_{at}] = \min(as, at) = a\min(s,t) = \mathbb{E}\,[(a^{\frac{1}{2}}B_s)(a^{\frac{1}{2}}B_t)].$$

Hence, $(B_{as}, B_{at}) \stackrel{d}{=} (a^{\frac{1}{2}}B_s, a^{\frac{1}{2}}B_t)$, which is a self-similar process according to Definition 5.2.1 with Hurst index $H = \frac{1}{2}$.

In the following theorem the covariance of fractional Brownian motion is determined.

Theorem 5.2.3. *Let $X = (X_H(t))_{t\geq 0}$ be a self-similar Gaussian process with zero mean and Hurst exponent $H \in (0,1)$. Then*

$$\mathbb{E}\,[X_H(t)X_H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad s,t \in \mathbb{R}_+, \tag{5.2}$$

is the covariance function.

Proof.

$$\begin{aligned}\mathbb{E}\,[X_H(t)X_H(s)] &= \frac{1}{2}(\mathbb{E}\,[X_H^2(t)] + \mathbb{E}\,[X_H^2(s)] - \mathbb{E}\,[(X_H(t) - X_H(s))^2] \\ &\quad -(\mathbb{E}\,[X_H^2(t)] + \mathbb{E}\,[X_H^2(s)] - \mathbb{E}\,[X_H(|t-s|)^2])) \\ &= \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).\end{aligned}$$

In order to prove that it is indeed a covariance function, which is stated without proof in most literature, it needs to be shown that the matrix is positive semi-definite, see [24]. Note that for $r \geq 0$,

$$r^{2H} = \frac{1}{C}\int_0^\infty \frac{1 - e^{-r^2u^2}}{u^{1+2H}}\,du,$$

with $C \in \mathbb{R}_+^*$ and substituting $v = ru$,

$$C := \int_0^\infty \frac{1 - e^{-v^2}}{v^{1+2H}}\, dv.$$

Then, expanding e^{2stu^2} into the power series $\sum_{k=1}^\infty \frac{1}{k!}(2stu^2)^k$,

$$\begin{aligned} -|s-t|^{2H} &= \frac{1}{C}\int_0^\infty \frac{e^{-s^2u^2-t^2u^2}(e^{2stu^2}-1)}{u^{1+2H}}\, du \\ &= \frac{1}{C}\sum_{k=1}^\infty \frac{2^k}{k!}\int_0^\infty \frac{s^k e^{-s^2u^2} t^k e^{-t^2u^2}}{u^{1-2k+2H}}\, du, \end{aligned}$$

where monotone convergence is applied to interchange the integral with the power series. Hence,

$$\begin{aligned} \sum_{i,j=1}^n \langle v_i, \mathrm{Cov}(X_H(t_i), X_H(t_j)) v_j\rangle &= \frac{1}{2}\sum_{i,j=1}^n \langle v_i (t_i^{2H} + t_j^{2H} - |t_i - t_j|^{2H}) v_j\rangle \\ &= \frac{1}{2C}\int_0^\infty \frac{|\sum_{i=1}^n (1 - e^{-t_i^2u^2}) v_i|^2}{u^{1+2H}}\, du \\ &\quad + \sum_{k=1}^\infty \frac{2^k}{k!}\int_0^\infty \frac{|\sum_{i=1}^n t_i^k e^{-t_i^2u^2 v_i}|^2}{u^{1-2k+2H}}\, du \geq 0, \end{aligned}$$

for all $n \in \mathbb{N}$, all vectors $v_1, \ldots, v_n \in \mathbb{R}^d$ and all times $t_1, \ldots, t_n \in \mathbb{R}^d$, which proves that the covariance matrix is positive semi-definite. □

Definition 5.2.4. A continuous Gaussian process $B = (B_H(t))_{t\geq 0}$ with Hurst index $H \in (0,1)$ with zero mean and covariance as in (5.2) is called *fractional Brownian motion (fBm).*

The notation $Z = (Z_H(t))_{t\geq 0}$ will be used in this paper as an indication for fBm as was done in [6], [7] and [8]. While Brownian motion has independent increments, fractional Brownian motion has negative correlated increments for $H \in (0, \frac{1}{2})$ and positive correlated increments for $H \in (\frac{1}{2}, 1)$, which is the main difference between those processes. Recall that $H = \frac{1}{2}$ is Brownian motion. As the distribution of a zero mean Gaussian process is entirely determined by its covariance function, due to its characteristic function, it follows from theorem 5.2.3 that fBm is H-self-similar and has stationary increments. Conversely, an H-self-similar process with stationary increments is fBm.

Theorem 5.2.5. *Fractional Brownian motion $Z = (Z_H(t))_{t\geq 0}$ has the following properties [8], [22], [26]:*

1. $\mathbb{P}[Z_0 = 0] = 1$.

2. $\mathbb{E}[Z_t] = 0$ *for all* $t \in \mathbb{R}_+$.

3. Z *has stationary increments, i.e.* $(Z_{t+s} - Z_s) \overset{d}{=} Z_t$, *for all* $s \le t \in \mathbb{R}_+$.

4. Z *has finite second moments, i.e.* $\mathbb{E}[Z_t^2] = |t|^{2H}$, *for all* $t \in \mathbb{R}_+$.

5. Z *has continuous sample paths.*

If $\mathbb{E}[Z_1^2] = 1$ *the process is called* standard *fractional Brownian motion.*

The following theorem states that fractional Brownian motion has almost surely Hölder continuous paths, see [24].

Theorem 5.2.6. *Fractional Brownian motion with Hurst index* $H \in (0, 1)$ *has almost surely Hölder continuous paths with* $\alpha \in (0, H)$.

Proof. Let $s, t \in \mathbb{R}_+$, using theorems 5.2.3 and 5.2.5,

$$\mathrm{Cov}(Z_s - Z_t) = \mathrm{Cov}(Z_s) - 2\mathrm{Cov}(Z_s, Z_t) + \mathrm{Cov}(Z_t) = |s-t|^{2H} I_d.$$

Hence,

$$\mathbb{E}[|Z_s - Z_t|^{2n}] = \mathbb{E}[|X|^{2n}]|s-t|^{2Hn},$$

for $X \sim \mathcal{N}(0, I_d)$ and $\mathbb{E}[|X|^{2n}] < \infty$ for all $n \ge 0$. With the Kolmogorov-Chentsov criterion, we have

$$\mathbb{E}[(Z_s, Z_t)^{2n}] \le \mathbb{E}[|X|^{2n}]|s-t|^{2Hn-1},$$

hence, the paths are locally Hölder continuous for all exponents $\alpha \in (0, \tilde{\alpha})$ with $\tilde{\alpha} = \sup_{n \in \mathbb{N}} \frac{2Hn-1}{2n}$. □

5.3 Replacing Brownian Motion with Fractional Brownian Motion in the Black–Scholes Model

Following [13] and [27] the Black–Scholes SDE (1.2) with fBm as its underlying process can be easily transformed to

$$dS_t = S_t(\mu\, dt + \sigma\, dZ_H(t)) \tag{5.3}$$

with solution, that is obtained through the use of *Wick calculus* instead of Itô calculus,

$$S_t = S_0 \exp\left\{\sigma Z_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right\},$$

with $\mathbb{E}\left[S_t\right] = S_0 e^{\mu t}$. Note that Brownian motion can theoretically be replaced by fractional Brownian motion, but as fBm is not a semimartingale, it admits arbitrage and fails to admit a hedging strategy, therefore cannot be used as a realistic financial pricing model. Y. Hu and B. Øksendal [13] argue that the arbitrage problem is not only a problem of having a martingale or non-martingale, but also of the type of integral, having an Itô type integral or Stratonovich type integral.

5.3.1 Stochastic Calculus for Fractional Brownian Motion

As fractional Brownian motion is not a semimartingale and does not allow for Itô calculus, a stochastic Itô type integral that uses the Wick product or a Stratonovich type integral must be used. Note that for stochastic calculus we need to distinguish between the two cases $H > \frac{1}{2}$ and $H < \frac{1}{2}$. Recall that the covariance of fBm is

$$\mathbb{E}\left[Z_H(t), Z_H(s)\right] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}) := R_H(t,s),$$

where $t, s \geq 0$, which can be written as

$$R_H(t,s) = \alpha_H(t,s) \int_0^t \int_0^s |r-u|^{2H-2}\, du\, dr,$$

with $\alpha_H = H(2H-1)$. We do not need stochastic calculus for fractional Brownian motion in this thesis, as we use another process, the modified Dobrić–Ojeda or the Conus–Wildman process, which allows for Itô calculus. For further reading, see [1], [21].

Chapter 6

The Dobrić–Ojeda Process

The Dobrić–Ojeda process, proposed as an alternative to fractional Brownian motion, is a Gaussian Markov process with dependent increments in time. Its main advantage is its semimartingale property, which allows for the use of Itô calculus.

6.1 Fractional Brownian Fields

Dobrić and Ojeda constructed a Gaussian field encompassing all fractional Brownian motion while searching for natural wavelets for fBm. In [8] the existence of a fractional Brownian field and its covariance has been derived and proven. Every fractional Brownian field is a sum of the odd and even part of two independent Gaussian fields, which are called *dependent fractional Brownian fields (dfBf)*,

$$B_H^o(t) = \frac{B_H(t) - B_H(-t)}{2} \quad \text{and} \quad B_H^e(t) = \frac{B_H(t) + B_H(-t)}{2}.$$

The odd and even fields, $(B_H^o(t))_{(t,H)\in[0,\infty)\times(0,1)}$ and $(B_H^e(t))_{(t,H)\in[0,\infty)\times(0,1)}$, respectively, are not independent of each other, see [8], therefore they are called *dependent*.

Definition 6.1.1. *Let the real-valued centered Gaussian process $Z = (Z_H(t))_{t\geq 0}$ be fractional Brownian motion, $B_H^i(t)$ and $W_H^i(t)$ with i=o and i=e be the odd and even part of two dfBf $B = (B_H(t))_{t\in[0,\infty)}$ and $W = (W_H(t))_{t\in[0,\infty)}$, respectively, with $H \in (0,1)$ and $Z_H(0) = 0$ almost surely, embedded in the* fractional Gaussian field $Z = (Z_H(t))_{(t,H)\in[0,\infty)\times(0,1)}$ *defined by*

$$Z_H(t) = \begin{cases} B_H^e(t) + W_H^o(t) & \textit{for } t \geq 0, \\ B_H^e(-t) + W_H^o(-t) & \textit{for } t < 0, \end{cases}$$

with covariance

$$\mathbb{E}\left[Z_H(t)Z_{H'}(s)\right] = a_{H,H'}\left(\frac{|t|^{H+H'} + |s|^{H+H'} - |t-s|^{H+H'}}{2}\right), \tag{6.1}$$

with

$$a_{H,H'} = \begin{cases} \sqrt{\Gamma(2H+1)\Gamma(3-2H)}\sin^2(\pi H) & \textit{for} \quad H+H'=1, \\ -\frac{2}{\pi}\sqrt{\Gamma(2H+1)\sin(\pi H)}\sqrt{\Gamma(2H'+1)\sin(\pi H')} & \\ \times\Gamma(-(H+H'))\cos((H'-H)\frac{\pi}{2}\cos(H'+H)\frac{\pi}{2}) & \textit{for} \quad H+H'\neq 1. \end{cases}$$

In the case where $H + H' = 1$, (H, H') is called a *dual pair*, which generates two martingales, one driving B_H and the other driving $B_{H'}$. If $H = H'$, Z_H is fractional Brownian motion and if $H = H' = \frac{1}{2}$, Z_H is standard Brownian motion. In this paper the focus shall lie on the dual pair case where $H + H' = 1$ and to simplify notation, the subscript H' shall be dropped from $a_{H,H'} =: a_H$.

6.2 Martingale Properties

We will show that a martingale can be generated from fractional Brownian motion, which then can be used to derive the variance and covariance.

Theorem 6.2.1. *The process $M_H = (M_H(t))_{t\geq 0}$ defined by*

$$M_H(t) = \mathbb{E}\left[Z_{H'}(t)|\mathcal{F}_t^H\right] \tag{6.2}$$

is a martingale with respect to $\mathcal{F}_t^H$, where $H + H' = 1$ and

$$\mathcal{F}_t^H := \sigma(Z_H(s) : 0 \leq s \leq t)$$

is the filtration generated by $M_H(t)$.

This martingale is called a *fundamental martingale*, discovered by Molchan in 1969 in [20] as a stochastic integral with respect to a time dependent kernel. Norros et. al [22] obtained fundamental martingales in their study of Girsanov's formula for fractional Brownian motion in 1999. Dobrić and Ojeda [7] built a kernel that when integrated with respect to fBm, retrieves, up to a constant, Molchan's fundamental martingale, and is the basis of the martingale theory used throughout this paper. Our martingale M_H generates the same filtration as the non-semimartingale Z. For the case where (H, H') is not a dual pair, the process is a martingale plus an additional process, and the more $H + H'$ approaches 1, the more does the additional process approach zero and hence the closer we get to a martingale [7]. Following [6] for the proof:

Proof. Let $t \in \mathbb{R}_+$. The stochastic process $M = M_H(t)$ is *adapted* to $\mathcal{F}_t^H$, i.e. it is $\mathcal{F}_t^H$-measurable for all $t > 0$. As $M_H(t)$ is defined as the conditional expectation of the Gaussian process $Z_{H'}(t)$, it follows that $M_H(t)$ is also a Gaussian process and $\mathbb{E}\ [M_H(t)] = 0$. Using the definition of $M_H(t)$, the Gaussian property of $Z_{H'}(t) = Z_{1-H}(t)$ and Jensen's inequality the integrability of M can be obtained straightforward,

$$\begin{aligned}\mathbb{E}\ [|M_H(t)|] &= \mathbb{E}\ [|\mathbb{E}\ [Z_{H'}(t)|\mathcal{F}_t^H]|] = \mathbb{E}\ [|\mathbb{E}\ [Z_{1-H}(t)|\mathcal{F}_t^H]|] \\ &\leq \mathbb{E}\ [\mathbb{E}\ [|Z_{1-H}(t)||\mathcal{F}_t^H]] = \mathbb{E}\ [|Z_{1-H}(t)|] < \infty.\end{aligned}$$

The *martingale property* $\mathbb{E}\ [M_H(t)|\mathcal{F}_s^H] = M_H(s)$ for $0 \leq s < t$ remains to be shown. By the definition of $M_H(t)$ and the tower property we obtain

$$\begin{aligned}\mathbb{E}\ [M_H(t)|\mathcal{F}_s^H] &= \mathbb{E}\ [\mathbb{E}\ [Z_{H'}(t)|\mathcal{F}_t^H]|\mathcal{F}_s^H] = \mathbb{E}\ [Z_{H'}(t)|\mathcal{F}_s^H] \\ &= \mathbb{E}\ [Z_{H'}(t) - Z_{H'}(s)|\mathcal{F}_s^H] + \mathbb{E}\ [Z_{H'}(s)|\mathcal{F}_s^H] \\ &= \mathbb{E}\ [Z_{H'}(t) - Z_{H'}(s)|\mathcal{F}_s^H] + M_H(s).\end{aligned}$$

For the martingale property to hold we need to show that $\mathbb{E}\ [Z_{H'}(t) - Z_{H'}(s)|\mathcal{F}_s^H] = 0$. Fix $V \in \mathcal{F}_s^H$ and without loss of generality, let $V = \mathbb{1}_{\{Z_H(u)\in B\}}$ for some $u \leq s$ and B denoting a Borel set. Then

$$\mathbb{E}\ [V(Z_{H'}(t) - Z_{H'}(s))] = \mathbb{E}\ [\mathbb{1}_{\{Z_H(u)\in B\}} Z_{H'}(t)] - [\mathbb{1}_{\{Z_H(u)\in B\}} Z_{H'}(s)].$$

Note that for any Borel set B and any centered jointly Gaussian random variables X and Y with variance σ_X^2 and σ_Y^2, respectively, and covariance ρ, $\mathbb{E}\ [\mathbb{1}_{\{X\in B\}} Y] = \frac{\rho}{\sigma_X^2}\mathbb{E}\ [\mathbb{1}_{\{X\in B\}} X]$, which gives us

$$\mathbb{E}\ [\mathbb{1}_{\{Z_H(u)\in B\}} Z_{H'}(t)] = \frac{a_H u}{\mathbb{E}\ [Z_H^2(u)]}\mathbb{E}\ [\mathbb{1}_{\{Z_H(u)\in B\}} Z_H(u)] = \mathbb{E}\ [\mathbb{1}_{\{Z_H(u)\in B\}} Z_{H'}(s)].$$

It follows that $\mathbb{E}\ [V(Z_{H'}(t) - Z_{H'}(s))] = 0$ for all $V \in \mathcal{F}_s^H$. Hence, $\mathbb{E}\ [Z_{H'}(t) - Z_{H'}(s)|\mathcal{F}_s^H] = 0$, which completes the proof. $\square$

Dobrić and Ojeda proved the following stochastic integral representation of $M_H(t)$ using a hypergeometric identity,

$$M_H(t) = \mathbb{E}\ [Z_{H'}(t)|\mathcal{F}_t^H] = c_H \int_0^t (1-u)^{\frac{1}{2}-H} u^{\frac{1}{2}-H} dZ_H(u), \tag{6.3}$$

where

$$c_H = \frac{a_H}{2H\Gamma(\frac{3}{2}-H)\Gamma(H+\frac{1}{2})} = \frac{\sqrt{\Gamma(3-2H)\Gamma(2H+1)}\sin^2(\pi H)}{2H\Gamma(\frac{3}{2}-H)\Gamma(H+\frac{1}{2})},$$

for $H+H'=1$, see [7] for the proof.

By projecting fBm on to the fractional Gaussian field, our martingale M_H is used to capture information of the fractional Brownian motion process [6]. As M_H is Gaussian centered, we are searching for a process of the form $G_H M_H$, with $G_H = (G_H(t))_{t\geq 0}$ being some deterministic coefficient, to approximate fractional Brownian motion. This is done by minimizing the least-squares difference from Z_H, i.e.

$$\min\ \mathbb{E}\ [Z_H(t) - G_H(t)M_H(t)]^2,$$

where the minimizing G_H is given by

$$G_H(t) = \frac{\mathbb{E}\ [Z_H(t)M_H(t)]}{\mathbb{E}\ [M_H^2(t)]}.$$

By the definition of M_H and by the covariance of Z_H and $Z_{H'}$, given in equation 6.1, the following covariance of Z_H and M_H and the second moment of M_H can be recovered as

$$\mathbb{E}\ [Z_H(t)M_H(t)] = \mathbb{E}\ [Z_H(t)\mathbb{E}\,[Z_{H'}(t)|\mathcal{F}_t^H)]] = \mathbb{E}\,[Z_H(t)Z_{H'}(t)] = a_H t \quad (6.4)$$

and

$$\mathbb{E}\ [M_H^2(t)] = \mathbb{E}\ [M_H(t)\mathbb{E}\,[Z_{H'}(t)|\mathcal{F}_t^H] = \mathbb{E}\,[M_H(t)Z_{H'}(t)]. \quad (6.5)$$

Recall that $\mathbb{E}\,[Z_H^2(t)] = |t|^{2H}$ and note that $\mathbb{E}\,[Z_{H'}^2(t)] = |t|^{2H'} = |t|^{2(1-H)} = t^{2-2H}$, with $t \in \mathbb{R}_+$.

In [8] the following closed form solution was found for G_H

$$G_H(t) = \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{a_H\Gamma(\frac{3}{2}-H)} t^{2H-1} := c_G t^{2H-1}.$$

The Dobrić–Ojeda Gaussian Markov process $V_H = (V_H(t))_{t\in[0,\infty)}$ can now be defined as originally in [7] by V. Dobrić and F. M. Ojeda as

$$V_H(t) = G_H(t)M_H(t). \quad (6.6)$$

Proposition 6.2.2. *The second moment of M_H is given by*

$$\mathbb{E}\ [M_H^2(t)] = c_M t^{2-2H},$$

with

$$c_M = \frac{a_H^2\Gamma(\frac{3}{2}-H)}{2H\Gamma(H+\frac{1}{2})\Gamma(3-2H)} \quad \text{and} \quad a_H = 2Hc_H B\left(\frac{1}{2}+H, \frac{3}{2}-H\right),$$

where B denotes the Beta function.

Note that this cannot be seen immediately and was stated without proof in [6] and [7].

Proof. Using (6.3), (6.4) and (6.5) and substituting $u = tv$, we have

$$\begin{aligned}
\mathbb{E}[M_H^2(t)] &= \mathbb{E}\left[c_H \int_0^t (1-u)^{\frac{1}{2}-H} u^{\frac{1}{2}-H} \, dZ_H(u) Z_{H'}(t)\right] \\
&= c_H \int_0^t (1-u)^{\frac{1}{2}-H} u^{\frac{1}{2}-H} d\mathbb{E}\left[Z_H(u) Z_{H'}(t)\right] \\
&= a_H c_H \int_0^t (1-u)^{\frac{1}{2}-H} u^{\frac{1}{2}-H} \, du \\
&= a_H c_H t \int_0^1 (t-tu)^{\frac{1}{2}-H} t u^{\frac{1}{2}-H} \, du \\
&= a_H c_H t^{2-2H} \int_0^1 (1-u)^{\frac{1}{2}-H} u^{\frac{1}{2}-H} \, du \\
&= a_H c_H t^{2-2H} B\left(\frac{3}{2}-H, \frac{3}{2}-H\right).
\end{aligned}$$

□

Therefore, by proposition 6.2.2 and equation 6.4, G_H can be depict as

$$\begin{aligned}
G_H(t) &= \frac{\mathbb{E}\left[Z_H(t) M_H(t)\right]}{\mathbb{E}\left[M_H^2(t)\right]} \\
&= \frac{a_H t}{c_H a_H B(\frac{3}{2}-H, \frac{3}{2}-H) t^{2-2H}} \\
&= t^{2H-1} \frac{\Gamma(3-2H)}{c_H \Gamma^2(\frac{3}{2}-H)}.
\end{aligned}$$

Proposition 6.2.3. *The martingale process M_H has independent increments and covariance $\mathbb{E}\left[M_H(t) M_H(s)\right] = c_M (s \wedge t)$.*

Proof. Assume without loss of generality $s < t$. Then, by the martingale property and proposition 6.2.2,

$$\begin{aligned}
\mathbb{E}\left[M_H(t) M_H(s)\right] &= \mathbb{E}\left[(M_H(t) - M_H(s)) M_H(s)\right] + \mathbb{E}\left[(M_H(s))^2\right] \\
&= c_M (s \wedge t)^{2-2H} - c_M s^{2-2H} + c_M s^{2-2H} = c_M s^{2-2H},
\end{aligned}$$

which proves the covariance. In order to prove the independence of increments assume $h > 0$ small, such that $s + h < t$. Then,

$$\begin{aligned}
\mathbb{E}\ &[(M_H(t+h) - M_H(t))(M_H(s+h) - M_H(s))] \\
&= \mathbb{E}\ [M_H(t+h)M_H(s+h)] - \mathbb{E}\ [M_H(t)M_H(s+h)] \\
&-\mathbb{E}\ [M_H(t+h)M_H(s)] + \mathbb{E}\ [M_H(t)M_H(s)] \\
&= c_M(s+h)^{2-2H} - c_M(s+h)^{2-2H} - c_M s^{2-2H} + c_M s^{2-2H} = 0.
\end{aligned}$$

Since M_H is a Gaussian process, this concludes the proof. $\square$

Following [6] we will prove that the quadratic variation of M_H is equal to $c_M t^{2-2H}$, using the following lemma.

Lemma 6.2.4. *The following approximation holds for all even moments of martingale process M_t as defined in (6.2):*

$$\mathbb{E}\ [(\Delta M_{t_i})^{2k}] \leq (2k-1)!!(c_M(2-2H)(t_i \wedge t_{i-1})^{1-2H}\Delta t_i)^k,$$

where $k \geq 1$, $\Delta M_{t_i} = M_{t_i} - M_{t_{i-1}}$ and for $n > 0, t_i = \frac{it}{n}, i = 0, \dots, n$ be a partition sequence of $[0, t]$.

Proof. Let $k = 1$. By 6.2.2 and the Mean Value Theorem we have

$$\begin{aligned}
\mathbb{E}\ [(\Delta M_{t_i})^2] &= \mathbb{E}\ [M_{t_i}^2] + 2\mathbb{E}\ [M_{t_i}M_{t_{i-1}}] + \mathbb{E}\ [M_{t_{i-1}}^2] \\
&= c_M t_i^{2-2H} - 2\mathbb{E}\ [(M_{t_i} - M_{t_{i-1}} + M_{t_{i-1}})M_{t_{i-1}}] + c_M t_{i-1}^{2-2H} \\
&= c_M t_i^{2-2H} - 2\mathbb{E}\ [(\Delta M_{t_i} + M_{t_{i-1}})M_{t_{i-1}}] + c_M t_{i-1}^{2-2H} \\
&= c_M t_i^{2-2H} - 2\mathbb{E}\ [(\Delta M_{t_i} M_{t_{i-1}}] - 2\mathbb{E}\ [M_{t_{i-1}}^2] + c_M t_{i-1}^{2-2H} \\
&= c_M t_i^{2-2H} - 2\mathbb{E}\ [c_M(t_i \wedge t_{i-1})^{2-2H} - c_M t_{i-1}^{2-2H}] \\
&-2c_M t_{i-1}^{2-2H} + c_M t_{i-1}^{2-2H} \\
&= c_M(t_i^{2-2H} - t_{i-1}^{2-2H}) \\
&\leq c_M(2-2H)(t_i \wedge t_{i-1})^{1-2H}\Delta t_i.
\end{aligned}$$

As $M_H(t)$ is a Gaussian process, this holds true for all $k \geq 1$. $\square$

Proposition 6.2.5. *Let $t_i = \frac{it}{n}$ be a partition sequence of $[0, t]$ with $i = 0, \dots, n$ and $n > 0$ and martingale process M_t, then (6.2)*

$$\lim_{n\to\infty} \left\| \sum_{i=1}^{n} (\Delta M_{t_i})^2 - c_M t^{2-2H} \right\|_2 = 0$$

and

$$\lim_{n\to\infty} \sum_{i=1}^{n} (\Delta M_{t_i})^2 = c_M t^{2-2H} \quad \textit{almost surely,}$$

where $\Delta M_{t_i} = M_{t_i} - M_{t_{i-1}}$.

Proof. Since $f(t) = t^{2-2H}$ is integrable,

$$c_M(2-2H)\sum_{i=1}^{n} t_i^{1-2H}\Delta t \to c_M t^{2-2H} \quad \text{as} \quad n \to \infty,$$

in L^2 and almost surely. Therefore, by the Triangle Inequality, it suffices to show

$$\lim_{n\to\infty} \left\| \sum_{i=1}^{n} (\Delta M_{t_i})^2 - c_M(2-2H)\sum_{j=1}^{n} t_j^{1-2H}\Delta t \right\|_2 = 0.$$

With the independent increments of $M_H(t)$, Proposition 6.2.3, and Lemma 6.2.4 it follows

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=1}^{n} (\Delta M_{t_i})^2 - c_M(2-2H)\sum_{j=1}^{n} t_j^{1-2H}\Delta t\right)^2\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\left[(\Delta M_{t_i})^2\right]\mathbb{E}\left[(\Delta M_{t_j})^2\right] \\
&-2c_M(2-2H)\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\left[(\Delta M_{t_i})^2\right] t_j^{1-2H}\Delta t \\
&+c_M^2(2-2H)^2\sum_{i=1}^{n}\sum_{j=1}^{n} t_i^{1-2H} t_j^{1-2H}(\Delta t)^2 \le 0.
\end{aligned}$$

The inequality holds for both $H \in (0, 0.5)$ and $H \in [0.5, 1)$, which proves L^2-convergence and with Borel–Cantelli almost sure convergence follows. □

Define the difference equation between fractional Brownian motion and the Dobrić–Ojeda process as

$$N_H(t) = Z_H(t) - G_H(t)M_H(t) = Z_H(t) - V_H(t)$$

with

$$
\begin{aligned}
\mathbb{E}\left[N_H^2(t)\right] &= \mathbb{E}\left[Z_H^2(t)\right] - 2\mathbb{E}\left[Z_H(t)G_H(t)M_H(t)\right] + \mathbb{E}\left[G_H^2(t)M_H^2(t)\right] \\
&= t^{2H} - 2\mathbb{E}\left[Z_H(t)\frac{\mathbb{E}\left[Z_H(t)M_H(t)\right]}{\mathbb{E}\left[M_H^2(t)\right]}M_H(t)\right] \\
&\quad + \mathbb{E}\left[\left(\frac{\mathbb{E}\left[Z_H(t)M_H(t)\right]}{\mathbb{E}\left[M_H^2(t)\right]}\right)^2 M_H^2(t)\right] \\
&= t^{2H} - \frac{\left(\mathbb{E}\left[Z_H(t)M_H(t)\right]\right)^2}{\mathbb{E}\left[M_H^2(t)\right]} \\
&= t^{2H} - \frac{a_H^2 t^2 \Gamma(3-2H)}{a_H c_H \Gamma^2(\frac{3}{2}-H)t^{2-2H}} \\
&= t^{2H}\left(1 - \frac{a_H \Gamma(3-2H)}{c_H \Gamma^2(\frac{3}{2}-H)}\right) \\
&= t^{2H} b_H^2.
\end{aligned}
$$

Through substitution of $a_H = 2Hc_H B(\frac{1}{2}+H, \frac{3}{2}-H)$ the relative L^2-error is

$$
\begin{aligned}
b_H^2 &= 1 - \frac{2Hc_H \Gamma(\frac{1}{2}+H)\Gamma(\frac{3}{2}-H)\Gamma(3-2H)}{\Gamma^2(\frac{3}{2}-H)} \\
&= 1 - 2H\frac{\Gamma(\frac{1}{2}+H)\Gamma(3-2H)}{\Gamma(\frac{3}{2}-H)}.
\end{aligned}
\tag{6.7}
$$

The graph of b_H is shown in Figure 6.1. For $H \in (0.4, 1)$ the Dobrić–Ojeda

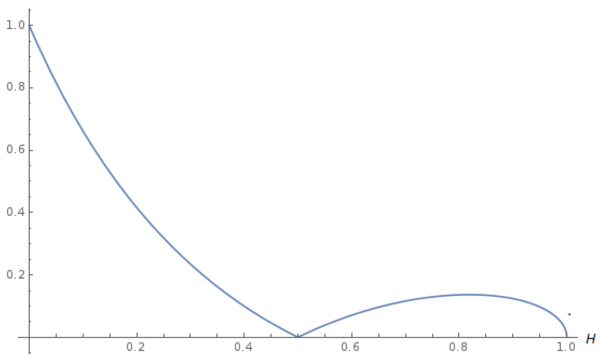

Figure 6.1: Graph of b_H

process V_H approximates Z_H with a relative L^2-error of at most 12%. Note that most literature considers $H \in (0.5, 1)$, which is the case for dependent increments and is a reasonable assumption. As H decreases below 0.4 the approximation worsens and as H approaches 0 the error increases up to 1, which can be seen straightforward from equation 6.7. In [6] and [7] the L^2-error of 12% is not questioned, although it seems to be a rather large error for pricing options.

6.3 Properties of the Dobrić–Ojeda Process

Following [6], we show that the Dobrić–Ojeda process satisfies some stochastic differential equation and find its quadratic variation.

Proposition 6.3.1. *There exists a Brownian motion process $W_t = (W_H(t))_{t\geq 0}$ adapted to the filtration $(\mathcal{F}_t^H)_{t\geq 0}$ such that the Dobrić–Ojeda process*

$$V_H(t) = G_H(t)M_H(t), \quad t \in [0,\infty),$$

is an Itô diffusion process satisfying the SDE

$$dV_H(t) = \frac{2H-1}{t}V_H(t)\,dt + D_H t^{H-\frac{1}{2}}\,dW_t$$

where $D_H = c_G\sqrt{c_M(2-2H)}$.

To see how much D_H with $H \in (0,1)$ impacts V_H, the graph of D_H, which is defined as

$$D_H = \sqrt{\frac{2H(2-2H)\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}}, \tag{6.8}$$

is shown in figure 6.2. For the case of regular Brownian motion, $H = \frac{1}{2}$, D_H

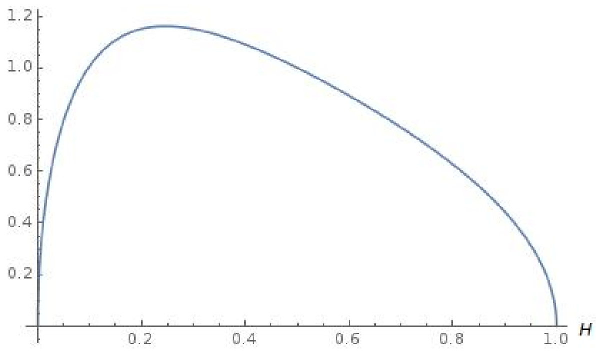

Figure 6.2: Graph of D_H

is equal to 1.

Proof. By the Martingale Representation Theorem there exists a Brownian motion process W_t adapted to the filtration $(\mathcal{F}_t^H)_{t\in[0,\infty)}$ for which

$$dM_H(t) = \sqrt{(c_M(2-2H)}t^{\frac{1}{2}-H}\, dW_t.$$

Note that $M_H(t) = (G_H(t))^{-1}V_H(t)$and recall that $G_H(t) = c_G t^{2H-1}$ and $\mathbb{E}[M_H(t)] = 0$. By the definition of V_H (6.6) and the quadratic variation of M_H, proposition 6.2.2, and using integration by parts,

$$\begin{aligned}
dV_H(t) &= d(G_H(t)M_H(t)) \\
&= G_H(t)dM_H(t) + M_H(t)dG_H(t) \\
&= G_H(t)\sqrt{c_M(2-2H)}t^{\frac{1}{2}-H}\, dW_t + (G_H(t))^{-1}V_H(t)d(c_G t^{2H-1}) \\
&= c_G\sqrt{c_M(2-2H)}t^{2H-1}t^{\frac{1}{2}-H}dW_t + c_G^{-1}t^{-2H+1}V_H(t)c_G(2H-1)t^{2H-2}\, dt \\
&= D_H t^{H-\frac{1}{2}}\, dW_t + \frac{2H-1}{t}V_H(t)\, dt.
\end{aligned}$$

□

Note, that $dV_H(t)$ is well-defined. Conus and Wildman [6] observed that the martingale part of this representation has a similar form as the Holmgren–Riemann–Liouville fractional integral (5.1), but while the fractional integral is not Itô integrable the diffusion process is non-anticipating and therefore Itô integrable. Concluding, that the drift part of the diffusion process compensates the difference and somehow imitates fractional Brownian motion, while remaining a semimartingale.

Corollary 6.3.2. *The quadratic variation of $V_H(t)$ is*

$$[V_H, V_H] = \frac{D_H^2}{2H}t^{2H},$$

where $D_H = c_G\sqrt{c_M(2-2H)}$ and without loss of generality $H \neq \frac{1}{2}$.

Proof. As the quadratic variation of Brownian motion W_t is $d[W]_t = dt$, the quadratic variation of $V_H(t)$ can be obtained straightforward,

$$[V_H, V_H] = D_H^2 \int_0^t s^{(H-\frac{1}{2})^2}\, d[W]_s = D_H^2 \int_0^t s^{2H-1}\, ds = \frac{D_H^2}{2H}t^{2H}.$$

□

Chapter 7

Option Pricing

As is shown in the previous section there exists a Brownian motion $(W_H(t))_{t\geq 0}$ adapted to the filtration $(\mathcal{F}_t^H)_{t\geq 0}$, so that the Dobrić–Ojeda process can be written as

$$dV_H(t) = \frac{2H-1}{t} V_H(t)\,dt + D_H t^{H-\frac{1}{2}}\,dW_t,$$

for D_H as in (6.2). The Dobrić–Ojeda process is a semimartingale and therefore allows for Itô calculus. For option pricing a *risk neutral measure* is needed; a measure equivalent to our original measure $\mathbb{P}$ such that the share price is equal to the discounted expectation of the share price under the measure. A risk-neutral measure exists is and only if the market is arbitrage free. Note that a risk neutral measure cannot be obtained straightforward as was the case for the Black–Scholes model, because the drift has t in the denominator and explodes for $t = 0$. First, let us replace Brownian motion with the Dobrić–Ojeda process in the Black–Scholes SDE (1.2):

$$dS_t = S_t(\mu dt + \sigma\, dV_H(t)).$$

Recall that $H = \frac{1}{2}$ is Brownian motion and assume without loss of generality $H \neq \frac{1}{2}$, then applying Itô's formula (4.1) to $Y_t = \ln S_t$,

$$\begin{aligned} dY_t &= \frac{dS_t}{S_t} - \frac{1}{2}\frac{(dS_t)^2}{(S_t)^2} \\ &= \mu\, dt + \sigma\, dV_H(t) - \frac{1}{2}\sigma^2 d[V_H, V_H]_t, \end{aligned}$$

and with the quadratic variation from corollary 6.3.2,

$$\begin{aligned} Y_t &= Y_0 + \mu t + \sigma V_H(t) - \frac{1}{2}\sigma^2 [V_H, V_H]_t \\ &= Y_0 + \mu t + \sigma V_H(t) - \frac{D_H^2 \sigma^2}{4H} t^{2H}. \end{aligned}$$

Hence, the stock price process $S_t = \exp(Y_t)$ can be expressed as

$$S_t = S_0 \exp\left\{\mu t + \sigma V_H(t) - \frac{D_H^2 \sigma^2}{4H} t^{2H}\right\}.$$

To simplify notations the subscript H will be dropped from $V_H(t), M_H(t)$ and $\mathcal{F}_t^H$ in the following.

7.1 Risk-Neutral Measure

We are searching for a risk neutral probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ on $\mathcal{F}_t$ for every $t \in [0, \infty)$, such that $d\mathbb{Q} = Z_t \, d\mathbb{P}$, where Z_t is an $\mathcal{F}_t$-adapted density, which exists after Radon–Nikodým, under which the discounted stock price process,

$$dZ_t = Z_t((\mu - r)\, dt + \sigma \, dV_t),$$

with $Z_t = \frac{S_t}{B_t}$, is a martingale. As in the Black–Scholes model, $r > 0$ is the risk-free constant interest rate and $B_t = e^{rt}$ is the bond price process. Plugging in the SDE from proposition 6.3.1, we obtain

$$dZ_t = Z_t((\mu - r)\, dt + \sigma \frac{2H-1}{t} V_t \, dt + \sigma D_H t^{H-\frac{1}{2}} \, dW_t),$$

rearranging,

$$dZ_t = \sigma D_H t^{H-\frac{1}{2}} Z_t (dW_t + \gamma_t)\, dt,$$

where γ_t is the drift correction from Girsanov's theorem, theorem 4.2.2,

$$\gamma_t = \frac{\mu - r}{\sigma D_H} t^{\frac{1}{2}-H} + \frac{2H-1}{D_H} t^{-\frac{1}{2}-H} V_t. \tag{7.1}$$

As shown in [6] the *Novikov condition* fails to hold,

Proposition 7.1.1. *For $t \in [0, T]$,*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \gamma_s^2 \, ds\right)\right] = \infty,$$

where γ is as defined in (7.1).

Note that for the Novikov condition to hold we would need $\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \gamma_s^2 \, ds\right)\right] < \infty$.

Proof. Taking the square of (7.1),

$$\gamma_s^2 = A^2 s^{1-2H} + 2ABs^{-2H} V_s + B^2 s^{-1-2H} V_s^2,$$

where A and B are constants defined as $A = \frac{\mu - r}{\sigma D_H}$ and $B = \frac{2H-1}{D_H}$. Recall that $V_s = G_s M_s$ with $\mathbb{E}\,[M_s] = 0$, $\mathbb{E}\,[M_s^2] = c_M s^{2-2H}$ and $G_s = c_G s^{2H-1}$. Then, using Jensen's inequality, as we have the convex exponential function,

$$\begin{aligned}
&\mathbb{E}\,\left[\exp\left(\frac{1}{2}\int_0^t \gamma_s^2\,ds\right)\right] \\
&\geq \exp\left(\mathbb{E}\,\left[\frac{1}{2}\int_0^t A^2 s^{1-2H} + 2ABs^{-2H}V_s + B^2 s^{-1-2H} V_s^2\,ds\right]\right) \\
&= \exp\left(\frac{A^2}{2(2-2H)} t^{2-2H}\right) \exp\left(\frac{B^2 c_G^2 c_M}{2} \int_0^t \frac{1}{s}\,ds\right) = \infty,
\end{aligned}$$

where the middle term $\mathbb{E}\,[2ABs^{-2H} G_s M_s] = 0$. □

Therefore Girsanov's theorem cannot be applied to the Dobrić–Ojeda process and without Girsanov it is rather difficult to find a risk-neutral measure as this is usually the way to proceed. Alternatively, Conus and Wildman introduced the *modified Dobrić–Ojeda process* for which the Novikov condition holds and Girsanov's theorem can be applied, that will be presented in the following section.

7.2 Modified Dobrić–Ojeda Process

The modified Dobrić–Ojeda process was introduced in [6] as a way around Girsanov's formula, replacing V_t with V_t^ϵ, a process that has zero drift for $t \in [0, \epsilon)$ for small $\epsilon > 0$ and is equal to the original Dobrić–Ojeda process, $V_t = V_t^\epsilon$, for all $t \geq \epsilon$.

Definition 7.2.1. *Let $\epsilon > 0$. The* modified Dobrić–Ojeda process $V_t^\epsilon = (V_H^\epsilon(t))_{t\in[0,\infty)}$ *is given by*

$$dV_H^\epsilon(t) = c_G(2H-1)t^{2H-2} M_t \mathbb{1}_{[\epsilon,\infty)}(t)\,dt + D_H t^{H-\frac{1}{2}}\,dW_t, \qquad (7.2)$$

with $D_H = c_G\sqrt{c_M(2-2H)}$.

Following [6] the properties of the modified Dobrić–Ojeda process will be shown.

Proposition 7.2.2. *The modified Dobrić–Ojeda process V_t^ϵ as defined in (7.2) is well-defined.*

Proof. Using Itô isometry,

$$\mathbb{E}\left[\left(D_H \int_0^t s^{H-\frac{1}{2}}\,dW_s\right)^2\right] = D_H^2 \int_0^t s^{2H-1}\,ds = \frac{D_H^2}{2H}t^{2H} < \infty,$$

as D_H is independent of s and $t \in [0,\infty)$. By assumption and without loss of generality $H \neq \frac{1}{2}$ as $H = \frac{1}{2}$ is Brownian motion. Hence, it remains to be shown that the first integral is well-defined for $t \geq \epsilon$. Note that the integral is 0 for $t < \epsilon$, and not as suggested in [6] for $t \leq \epsilon$. Using Itô isometry and the covariance of M_t, proposition 6.2.3.

$$\begin{aligned}
&\mathbb{E}\left[\left(c_G(2H-1)\int_0^t s^{2H-2}M_s\mathbb{1}_{[\epsilon,\infty)}(s)\,ds\right)^2\right] \\
&= c_G^2(2H-1)^2\int_\epsilon^t\int_\epsilon^t s_1^{2H-2}s_2^{2H-2}\mathbb{E}\left[M_{s_1}M_{s_2}\right]ds_2\,ds_1 \\
&= c_G^2(2H-1)^2\int_\epsilon^t\int_\epsilon^t s_1^{2H-2}s_2^{2H-2}c_M(s_1\wedge s_2)^{2-2H}\,ds_2\,ds_1 \\
&= c_G^2c_M(2H-1)^2\int_\epsilon^t\left(\int_\epsilon^{s_1} s_1^{2H-2}\,ds_2 + \int_{s_1}^t s_2^{2H-2}\,ds_2\right)ds_1 \\
&= c_G^2c_M(2H-1)^2\int_\epsilon^t s_1^{2H-1} - \epsilon s_1^{2H-2} + \frac{t^{2H-1}-s_1^{2H-1}}{2H-1}\,ds_1 \\
&= c_G^2c_M(2H-1)^2\frac{t^{2H-1}}{2H-1}(t-\epsilon) + \int_\epsilon^t s_1^{2H-1}\left(1-\frac{1}{2H-1}\right) - \epsilon s_1^{2H-2}\,ds_1 \\
&= c_G^2c_M(2H-1)^2\frac{t^{2H-1}}{2H-1}(t-\epsilon) + \frac{2H-2}{(2H-1)2H}\left(t^{2H}-\epsilon^{2H}\right) \\
&-\frac{\epsilon}{2H-1}\left(t^{2H-1}-\epsilon^{2H-1}\right) \\
&= 2c_G^2c_M(2H-1)^2\left(\frac{1}{2H}(t^{2H}-\epsilon^{2H}) - \frac{\epsilon}{2H-1}(t^{2H-1}-\epsilon^{2H-1})\right) < \infty.
\end{aligned}$$

Hence, both integrals are well-defined, which concludes the proof. Note that we have extended the proof here, solving the integral with the minimum $\wedge$, which is not obvious and left open in [6]. ☐

Next, the first and second moment of the modified Dobrić–Ojeda process will be derived.

Proposition 7.2.3. *The modified Dobrić–Ojeda process V_t^ϵ satisfies for all $\epsilon > 0$,*

1. $\mathbb{E}\left[V_t^\epsilon\right] = 0$ *for all* $t > 0$,

2. $$\mathbb{E}\left[(V_t^\epsilon)^2\right] = \begin{cases} \dfrac{D_H^2}{2H} t^{2H} & \textit{if } t < \epsilon, \\ \dfrac{D_H^2}{2H} t^{2H} + 2D_H^2(2H-1)\dfrac{1}{2H}(t^{2H} - \epsilon^{2H}) & \\ +2c_G^2 c_M (2H-1)^2 \left(\dfrac{1}{2H}(t^{2H} - \epsilon^{2H}) \right. & \\ \left. -\dfrac{\epsilon}{2H-1}(t^{2H-1} - \epsilon^{2H-1}) \right) & \textit{if } t \geq \epsilon. \end{cases}$$

Proof. 1. For $t < \epsilon$,

$$\mathbb{E}\left[V_t^\epsilon\right] = \mathbb{E}\left[D_H \int_0^t s^{H-\frac{1}{2}}\, dW_s\right] = 0,$$

as it is the expectation of a measurable square integrable Itô integral, i.e. $\mathbb{E}\left[D_H^2 \int_0^t s^{2H-1}\, ds\right] < \infty$ for $H \neq \frac{1}{2}$.
For $t \geq \epsilon$,

$$\begin{aligned} \mathbb{E}\left[V_t^\epsilon\right] &= \mathbb{E}\left[c_G(2H-1)\int_0^t s^{2H-2} M_s \mathbb{1}_{[\epsilon,\infty)}(s)\, ds + D_H \int_0^t s^{H-\frac{1}{2}}\, dW_s\right] \\ &= c_G(2H-1)\int_0^t s^{2H-2}\mathbb{E}\left[M_s\right]\mathbb{1}_{[\epsilon,\infty)}(s)\, ds = 0, \end{aligned}$$

as $\mathbb{E}\left[D_H \int_0^t s^{H-\frac{1}{2}}\, dW_s\right] = 0$ and $\mathbb{E}\left[M_s\right] = 0$.

2. For the second moment, we have for $t < \epsilon$,

$$\mathbb{E}\left[(V_t^\epsilon)^2\right] = \mathbb{E}\left[\left(D_H \int_0^t s^{H-\frac{1}{2}}\, dW_s\right)^2\right] = \frac{D_H^2}{2H} t^{2H}$$

and for $t \geq \epsilon$, with the proof of proposition 7.2.2,

$$\begin{aligned} &\mathbb{E}\left[(V_t^\epsilon)^2\right] \\ &= \mathbb{E}\left[\left(c_G(2H-1)\int_0^t s^{2H-2} M_s \mathbb{1}_{[\epsilon,\infty)}(s)\, ds + D_H \int_0^t s^{H-\frac{1}{2}}\, dW_s\right)^2\right] \\ &= 2c_G^2 c_M (2H-1)^2 \left(\frac{1}{2H}(t^{2H} - \epsilon^{2H}) - \frac{\epsilon}{2H-1}(t^{2H-1} - \epsilon^{2H-1})\right) \\ &\quad +2c_G D_H (2H-1)\mathbb{E}\left[\int_0^t \int_0^t s_1^{H-\frac{1}{2}} s_2^{2H-2} M_{s_2} \mathbb{1}_{[\epsilon,\infty)}(s_2)\, dW_{s_1}\, ds_2\right] \\ &\quad +\frac{D_H^2}{2H} t^{2H}. \end{aligned}$$

By the martingale representation $M_{s_2} = \sqrt{c_M(2-2H)} \int_0^{s_2} u^{\frac{1}{2}-H} \, dW_u$, we have

$$
\begin{aligned}
&2c_G D_H(2H-1)\mathbb{E}\left[\int_\epsilon^t \int_0^t s_1^{H-\frac{1}{2}} s_2^{2H-2} M_{s_2} \, dW_{s_1} \, ds_2\right] \\
&= 2c_G D_H(2H-1) \int_\epsilon^t s_2^{2H-2} \mathbb{E}\left[M_{s_2} \int_0^t s_1^{H-\frac{1}{2}} \, dW_{s_1}\right] ds_2 \\
&= 2c_G \sqrt{c_M(2-2H)} D_H(2H-1) \\
&\times \int_\epsilon^t s_2^{2H-2} \mathbb{E}\left[\int_0^{s_2} u^{\frac{1}{2}-H} \, dW_u \int_0^t s_1^{H-\frac{1}{2}} \, dW_{s_1}\right] ds_2 \\
&= 2D_H^2(2H-1) \int_\epsilon^t s_2^{2H-2} \int_0^{s_2 \wedge t} du \, ds_2 \\
&= 2D_H^2(2H-1) \frac{1}{2H}(t^{2H} - \epsilon^{2H}),
\end{aligned}
$$

which is the middle term and hence completes the proof.

□

Proposition 7.2.4. *The quadratic variation of the modified Dobrić–Ojeda process $V_t^\epsilon = (V_H^\epsilon(t))_{t \geq 0}$ is given by*

$$
[V^\epsilon, V^\epsilon]_t = \frac{D_H^2}{2H} t^{2H}
$$

where $D_H = c_G \sqrt{c_M(2-2H)}$.

Note that the modified Dobrić–Ojeda process has the same quadratic variation as the original Dobrić–Ojeda process, V_t, which follows from the fact that we have modified the drift, but not the martingale part of the process, from which the quadratic variation is made up of. Next, following [6], we will show that the modified Dobrić–Ojeda process converges to the original Dobrić–Ojeda process.

Proposition 7.2.5. *For fixed $H \in (0,1)$ the process $(V_t^\epsilon)_{t \in [0,\infty)}$ converges uniformly in $L^2(\Omega)$ and almost surely to $(V_t)_{t \in [0,\infty)}$ as ϵ converges to 0.*

Proof. For $\epsilon > 0$, define the difference process $(X_t^\epsilon)_{t \in [0,\infty)}$,

$$
X_t^\epsilon = V_t - V_t^\epsilon,
$$

for all $t \geq 0$. Then, using the definition of V_t and the SDEs from proposition

6.3.1 and equation 7.2,

$$\begin{aligned}
dX_t^\epsilon &= dV_t - dV_t^\epsilon \\
&= \frac{2H-1}{t}(V_t - V_t \mathbb{1}_{[\epsilon,\infty)}(t))\, dt \\
&= \begin{cases} \dfrac{2H-1}{t} V_t \, dt & \text{if } t < \epsilon, \\ 0 & \text{if } t \geq \epsilon. \end{cases}
\end{aligned}$$

By the Minkowski and Cauchy–Schwarz inequalities,

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t<\infty} (X_t^\epsilon)^2\right] &= \mathbb{E}\left[\sup_{0\leq t\leq \epsilon} \left|(2H-1)\int_0^t \frac{V_s}{s}\, ds\right|^2\right] \\
&\leq (2H-1)^2\, \mathbb{E}\left[\sup_{0\leq t\leq\epsilon}\left(\int_0^t \left|\frac{V_s}{s}\right| ds\right)^2\right] \\
&= (2H-1)^2 \int_0^\epsilon \int_0^\epsilon \frac{1}{su}\mathbb{E}\left[|V_s V_u|\right] ds\, du \\
&\leq (2H-1)^2 \int_0^\epsilon \int_0^\epsilon \frac{1}{su}||V_s||_2 ||V_u||_2 \, ds\, du \\
&= c_G^2 c_M (2H-1)^2 \left(\int_0^\epsilon s^{H-1}\, ds\right)^2 \\
&= \frac{c_G^2 c_M (2H-1)^2}{H^2}\epsilon^{2H} \to 0
\end{aligned}$$

as $\epsilon \to 0$ and where by the martingale representation of M_s,

$$||V_s|| = ||G_s M_s|| = c_G^2 t^{4H-2} c_M (2-2H)\int_0^s u^{1-2H}\, du = c_G^2 c_M s^{2H},$$

which proves L^2-convergence. For almost-sure convergence we use the SDE of the processes V_t and V_t^ϵ and dominated convergence, where the indicator function is bounded above by 1.

$$\begin{aligned}
\lim_{\epsilon\to 0} V_t^\epsilon &= \lim_{\epsilon\to 0}\left(c_G(2H-1)\int_0^t s^{2H-2} M_s \mathbb{1}_{[\epsilon,\infty)}(s)\, ds + D_H \int_0^t s^{H-\frac{1}{2}}\, dWs\right) \\
&= c_G(2H-1)\int_0^t s^{2H-2} M_s \lim_{\epsilon\to 0}\mathbb{1}_{[\epsilon,\infty)}(s)\, ds + D_H\int_0^t s^{H-\frac{1}{2}}\, dWs \\
&= c_G(2H-1)\int_0^t s^{2H-2} M_s\, ds + D_H \int_0^t s^{H-\frac{1}{2}}\, dWs \\
&= V_t,
\end{aligned}$$

which completes the proof. □

7.3 Modified Stock Price Process

Now, we can apply the modified Dobrić–Ojeda process to the option pricing theory developed at the beginning of this chapter. Let S_t^ϵ be the modified stock price process for $\epsilon > 0$ small, then

$$dS_t^\epsilon = S_t^\epsilon(\sigma\, dV_t^\epsilon + \mu\, dt).$$

Applying Itô's formula (4.1) to $Y_t^\epsilon = \ln S_t^\epsilon$, as before to S_t,

$$\begin{aligned} dY_t^\epsilon &= \frac{dS_t^\epsilon}{S_t^\epsilon} - \frac{1}{2}\frac{(dS_t^\epsilon)^2}{(S_t^\epsilon)^2} \\ &= \mu\, dt + \sigma V_t^\epsilon - \frac{1}{2}\sigma^2 d[V^\epsilon, V^\epsilon]_t, \end{aligned}$$

which gives us, using proposition 7.2.4,

$$\begin{aligned} Y_t^\epsilon &= Y_0 + \mu t + \sigma V_t^\epsilon - \frac{1}{2}\sigma^2 [V^\epsilon, V^\epsilon]_t \\ &= Y_0 + \mu t + \sigma V_t^\epsilon - \frac{D_H^2\sigma^2}{4H} t^{2H}. \end{aligned}$$

Hence, the modified stock price process can be written as

$$S_t^\epsilon = S_0 \exp\left\{\mu t + \sigma V_t^\epsilon - \frac{D_H^2\sigma^2}{4H} t^{2H}\right\}. \tag{7.3}$$

By proposition 7.2.5 and dominated convergence, we have

$$\begin{aligned} \lim_{\epsilon\to 0} S_t^\epsilon &= \lim_{\epsilon\to 0} S_0 \exp\left\{\mu t + \sigma V_t^\epsilon - \frac{D_H^2\sigma^2}{4H} t^{2H}\right\} \\ &= S_0 \exp\left\{\mu t + \sigma \lim_{\epsilon\to 0} V_t^\epsilon - \frac{D_H^2\sigma^2}{4H} t^{2H}\right\} \\ &= S_0 \exp\left\{\mu t + \sigma V_t - \frac{D_H^2\sigma^2}{4H} t^{2H}\right\} \\ &= S_t. \end{aligned}$$

As before the bond price process is $B_t = e^{rt}$, and with S_t^ϵ (7.3), we can define

$$Z_t^\epsilon = \frac{S_t^\epsilon}{B_t} = S_0 \exp\left\{(\mu - r)t + \sigma V_t^\epsilon - \frac{D_H^2\sigma^2}{4H} t^{2H}\right\}.$$

Then, using Itô's formula (4.1) and (7.2),

$$\begin{aligned} dZ_t^\epsilon &= Z_t^\epsilon((\mu - r)\, dt + \sigma\, dV_t^\epsilon) \\ &= Z_t^\epsilon\left((\mu - r)\, dt + \sigma\left(c_G(2H-1)t^{2H-2}M_t \mathbb{1}_{[\epsilon,\infty)}(t)\, dt + D_H t^{H-\frac{1}{2}}\, dW_t\right)\right) \\ &= \sigma D_H t^{H-\frac{1}{2}} Z_t^\epsilon (dW_t + \gamma\, dt), \end{aligned}$$

where

$$\gamma_t = At^{\frac{1}{2}-H} + Bt^{H-\frac{3}{2}} M_t \mathbb{1}_{[\epsilon,\infty)}(t), \tag{7.4}$$

with constants

$$A = \frac{\mu - r}{\sigma D_H} \quad \text{and} \quad B = \frac{c_G(2H-1)}{D_H}.$$

Our goal is to obtain a risk-neutral measure $\mathbb{Q}$ through invoking Girsanov's theorem (theorem 4.2.2). In order to use Girsanov, we will show that Novikov's condition holds for the modified stock price process. Note that it only holds for some restricted ϵ.

Proposition 7.3.1. *For γ_t as defined in (7.4) and all $t \in [0,T]$, the Novikov condition holds for $\epsilon > \exp\{-\frac{1}{2B^2 c_M}\}t$,*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \gamma_s^2 \, ds\right)\right] < \infty.$$

Proof. Using the definition of γ_t (7.4) and the Cauchy–Schwarz inequality, following [6] for the proof,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \gamma_s^2 \, ds\right)\right] \\
&= \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t (As^{\frac{1}{2}-H} + Bs^{H-\frac{3}{2}} M_s \mathbb{1}_{[\epsilon,\infty)}(s))^2 \, ds\right)\right] \\
&= \exp\left(\frac{A^2 t^{2-2H}}{2(2-2H)}\right) \mathbb{E}\left[\exp\left(AB\int_\epsilon^t \frac{M_s}{s} \, ds\right) \exp\left(\frac{B^2}{2}\int_\epsilon^t s^{2H-3} M_s^2 \, ds\right)\right] \\
&\leq \exp\left(\frac{A^2 t^{2-2H}}{2(2-2H)}\right) \left(\mathbb{E}\left[\exp\left(2AB\int_\epsilon^t \frac{M_s}{s} \, ds\right)\right]\right)^{\frac{1}{2}} \\
&\times \left(\mathbb{E}\left[\exp\left(B^2\int_\epsilon^t s^{2H-3} M_s^2 \, ds\right)\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

The first term is finite as A is some constant. The second term is finite for $\mathbb{E}\left[\exp\left\{\int_\epsilon^t \frac{M_s}{s} \, ds\right\}\right] < \infty$, which is the moment generating function of the Gaussian random variable $\int_\epsilon^t \frac{M_s}{s} \, ds$. For the last term to be finite recall that $\mathbb{E}\left[M_H^2(t)\right] = c_M t^{2-2H}$ and note that for $k \geq 1$ and Brownian motion process B_t,

$$\begin{aligned}
\int_{c_M \epsilon^{2-2H}}^{c_M t^{2-2H}} \frac{\mathbb{E}\left[B_r^{2k}\right]^{\frac{1}{k}}}{r^2} dr &= \int_{c_M \epsilon^{2-2H}}^{c_M t^{2-2H}} r^{-2} \left(\frac{2^k \Gamma(k+\frac{1}{2})}{\sqrt{\pi}} r^k\right)^{1/k} dr \\
&= \frac{2\Gamma(k+\frac{1}{2})^{1/k}(2-2H)}{\pi^{1/2k}} \ln\left(\frac{t}{\epsilon}\right).
\end{aligned} \tag{7.5}$$

Then, by the time-change for martingales and the convergence given in proposition 6.2.5, we have for any $t \geq 0$, $\quad M_t = B_{\langle M \rangle_t} = B_{c_M t^{2-2H}}$, which lets us rewrite the last term using Taylor expansion of $f(x) = e^x$,

$$\begin{aligned}
\mathbb{E}\left[\exp\left(B^2 \int_\epsilon^t s^{2H-3} M_s^2 \, ds\right)\right] &= \mathbb{E}\left[\exp\left(B^2 \int_\epsilon^t s^{2H-3} B^2_{c_M s^{2-2H}} \, ds\right)\right] \\
&= \mathbb{E}\left[\exp\left(B^2 c_M \int_\epsilon^t s^{2H-3} B^2_{s^{2-2H}} \, ds\right)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{B^2 c_M}{2-2H} \int_{\epsilon^{2-2H}}^{t^{2-2H}} r^{-2} B_r^2 \, dr\right)^k\right].
\end{aligned}$$

With the Cauchy–Schwarz inequality and (7.5), the expression can be transformed in the following way

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{B^2 c_M}{2-2H} \int_{\epsilon^{2-2H}}^{t^{2-2H}} r^{-2} B_r^2 dr\right)^k\right] \leq 1 + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \left(2B^2 c_M \ln\left(\frac{t}{\epsilon}\right)\right)^k,$$

see [6]. Hence, the series converges if

$$\left| 2B^2 c_M \ln\left(\frac{t}{\epsilon}\right) \right| < 1,$$

which is

$$t e^{-\frac{1}{2B^2 c_M}} < \epsilon < t e^{\frac{1}{2B^2 c_M}}.$$

The right-hand inequality can be dropped, as we want ϵ to be small and $t e^{\frac{1}{2B^2 c_M}} > t$. Hence, $\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \gamma_s^2 \, ds\right)\right] < \infty$ holds for all $\epsilon > t e^{-\frac{1}{2B^2 c_M}}$. This completes the proof for Novikov's condition. □

Conus and Wildman argue that Novikov's condition should hold for all $\epsilon > 0$, as $\delta_H := e^{-\frac{1}{2B^2 c_M}}$, which is shown in figure 7.1.

Having shown that Novikov's condition holds, Girsanov's theorem can be applied.

Proposition 7.3.2. *Under the modified Dobrić–Ojeda process V_t^ϵ there exists by Girsanov's theorem a risk-neutral measure $\mathbb{Q}$ that is equivalent to probability measure $\mathbb{P}$ such that*

$$\begin{aligned}
dW_t^\epsilon &= dW_t + \gamma_t \, dt \\
&= dW_t + (A t^{\frac{1}{2}-H} + B t^{H-\frac{3}{2}} M_t \mathbb{1}_{[\epsilon,\infty)}(t)) \, dt
\end{aligned}$$

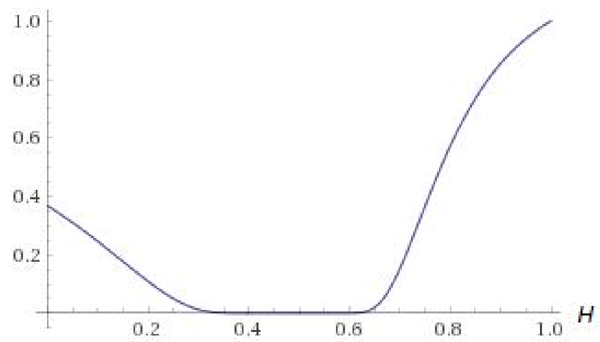

Figure 7.1: Graph of δ_H

is a Brownian motion process under $\mathbb{Q}$*. It follows that,*

$$dZ_t^\epsilon = \sigma D_H t^{H-\frac{1}{2}} Z_t^\epsilon \, dW_t^\epsilon$$

is a martingale process under $\mathbb{Q}$*.*

Proof. Under $\mathbb{Q}$, we obtain

$$Z_t^\epsilon = S_0 \exp\left\{\sigma D_H \int_0^t s^{H-\frac{1}{2}} \, dW_s^\epsilon - \frac{D_H^2 \sigma^2}{4H} t^{2H}\right\}$$

and

$$S_t^\epsilon = S_0 \exp\left\{rt + \sigma D_H \int_0^t s^{H-\frac{1}{2}} \, dW_s^\epsilon - \frac{D_H^2 \sigma^2}{4H} t^{2H}\right\}. \tag{7.6}$$

Now, taking the expectation with respect to measure $\mathbb{Q}$, we have, with the Itô formula (4.1) and the moment generating function,

$$\mathbb{E}_{\mathbb{Q}}\left[S_t^\epsilon\right] = \mathbb{E}_{\mathbb{Q}}\left[S_0 \exp\left\{rt + \sigma D_H \int_0^t s^{H-\frac{1}{2}} \, dW_s^\epsilon - \frac{D_H^2 \sigma^2}{4H} t^{2H}\right\}\right] = S_0 e^{rt}.$$

Hence, $\mathbb{Q}$ is indeed a risk-neutral measure. $\square$

With Novikov's condition and Girsanov's theorem, it follows that risk neutral measure $\mathbb{Q}$ is an equivalent local martingale measure (ELMM) and the process $(Z_t^\epsilon)_{t\in[0,T]}$ is a $\mathbb{Q}$-supermartingale, as the process Z_t^ϵ is a strictly positive local martingale, with

$$dZ_t^\epsilon = \sigma D_H t^{H-\frac{1}{2}} Z_t^\epsilon \, dW_t^\epsilon.$$

7.4 Conus–Wildman Option Pricing

We will call the underlying price process that was established in the previous section according to [6] the Conus–Wildman price process, $(S_t^\epsilon)_{t\in[0,T]}$. Let F_t be the payoff of an option with price S_t^ϵ for some $\epsilon > \delta(H)t$ at time $t > 0$ and define

$$E_t := \mathbb{E}_{\mathbb{Q}}\left[\frac{F}{B_T}\middle|\mathcal{F}_t\right].$$

By the martingale representation theorem, there exists an adapted process $(\phi_t)_{t\in[0,T]}$ such that

$$dE_t = \phi_t dZ_t^\epsilon.$$

For each $\epsilon > \delta(H)t$, we obtain a Δ-*hedging portfolio*, given by $(\phi_t, \psi_t)_{t\in[0,T]}$, where ϕ_t is the number of shares of the risky asset at time t, $\psi_t = E_t - \phi_t Z_t^\epsilon$ the number of shares of the bond at time t and Δ_t describes the sensitivity of an option's value to a change in the underlying price process, i.e. $\Delta_t = \frac{\delta V_t^\epsilon}{\delta S_t^\epsilon}$. A Δ-hedging portfolio tries to maintain the Δ as close to zero as possible. Assuming an arbitrage free market, we have

$$F_t = \phi_t S_t^\epsilon + \psi_t B_t = B_t \mathbb{E}_{\mathbb{Q}}\left[\frac{F}{B_T}\middle|\mathcal{F}_t\right], \tag{7.7}$$

at any time $t \in [0,T]$. Note that the portfolio (ϕ_t, ψ_t) is *self-financing*, no money is infused or withdrawn at any time, which means that purchasing is financed by selling, that is $dF_t = \phi_t dS_t^\epsilon + \psi_t\, dB_t$. With the no-arbitrage condition, we have a *replicating* portfolio, where the value of the portfolio is equal to the value of the option at any time $t \in [0,T]$.

The corresponding Black–Scholes partial differential equation is given in the following proposition.

Proposition 7.4.1. *The value of an option with payoff F and underlying Conus–Wildman price process, S_t^ϵ, at time $t \in [0,T]$ is given by $f(S_t^\epsilon, t)$, where $f(x,t)$ is the solution of the partial differential equation*

$$rf(x,t) = rxf_x(x,t) + f_t(x,t) + \frac{1}{2}\sigma^2 D_H^2 t^{2H-1} x^2 f_{xx}(x,t),$$

with $f(x,T) = F$ as terminal condition. Where f_t, f_x are the partial derivatives of t and x, respectively, and f_{xx} the second partial derivative of x.

Proof. By (7.2) and (7.3),

$$\begin{aligned} dS_t^\epsilon &= S_t^\epsilon(\mu\, dt + \sigma\, dV_t^\epsilon) \\ &= \left(\mu + \sigma c_G(2H-1)t^{2H-2}M_t \mathbb{1}_{[\epsilon,\infty)}(t)\right) S_t^\epsilon\, dt + \sigma D_H t^{H-\frac{1}{2}} S_t^\epsilon\, dW_t \\ &= \alpha_t S_t^\epsilon\, dt + \sigma D_H t^{H-\frac{1}{2}} S_t^\epsilon\, dW_t, \end{aligned}$$

with $\alpha_t = \mu + \sigma c_G(2H-1)t^{2H-2}M_t\mathbb{1}_{[\epsilon,\infty)}(t)$. Using Itô's formula (4.1),

$$\begin{aligned} df(S_t^\epsilon, t) &= f_x(S_t^\epsilon, t)dS_t^\epsilon + f_t(S_t^\epsilon, t) + \frac{1}{2}f_{xx}(S_t^\epsilon, t)(S_t^\epsilon)^2 \\ &= f_x(S_t^\epsilon, t)\left(\alpha_t S_t^\epsilon\, dt + \sigma D_H t^{H-\frac{1}{2}} S_t^\epsilon\, dW_t\right) + f_t(S_t^\epsilon, t) \\ &+\frac{1}{2}\sigma^2 D_H^2 t^{2H-1}(S_t^\epsilon)^2 f_{xx}(S_t^\epsilon, t)\, dt. \end{aligned} \tag{7.8}$$

As we have a self-financing hedging portfolio (ϕ, ψ) that replicates the value of the option for every $t \in [0, T]$,

$$\begin{aligned} df(S_t^\epsilon, t) &= \phi_t dS_t^\epsilon + \psi_t\, dB_t \\ &= \phi_t\left(\alpha_t S_t^\epsilon\, dt + \sigma D_H t^{H-\frac{1}{2}} S_t^\epsilon\, dW_t\right) + \psi_t r B_t\, dt. \end{aligned} \tag{7.9}$$

Equating (7.8) and (7.9),

$$\begin{aligned} &\left(\phi_t \alpha_t S_t^\epsilon + \psi_t r B_t - \alpha_t S_t^\epsilon f_x(S_t^\epsilon, t) - f_t(S_t^\epsilon, t) - \frac{1}{2}\sigma^2 D_H^2 t^{2H-1}(S_t^\epsilon)^2 f_{xx}(S_t^\epsilon, t)\right) dt \\ &= \left(\sigma D_H t^{H-\frac{1}{2}} S_t^\epsilon f_x(S_t^\epsilon, t) - \phi_t \sigma D_H t H - \frac{1}{2} S_t^\epsilon\right) dW_t. \end{aligned}$$

Since we have a martingale process with respect to Brownian motion W_t and a non martingale process on the left hand side, both sides have to be equal to zero almost surely. Hence, for the right hand side to be equal to zero, it follows that

$$\phi_t = f_x(S_t^\epsilon, t),$$

where ϕ_t is the number of shares of the underlying stock in the replicating portfolio. With $\psi_t B_t = f(S_t^\epsilon, t) - \phi_t S_t^\epsilon$, the left hand side gives us

$$rf(x, t) = rxf_x(x, t) + f_t(x, t) + \frac{1}{2}\sigma^2 D_H^2 t^{2H-1} x^2 f_{xx}(x, t),$$

which completes the proof. □

In the following sections, a Black–Scholes type call and put option price for the Conus–Wildman process will be computed.

7.4.1 Call Option Pricing

Recall that the payoff of a call option is $C = (S_T - K)^+$ at maturity T, where K is the strike price and S_t the stock price at time t. The value of a Black–Scholes European call option is

$$C(S_t, t) = S_t \mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2),$$

where $\mathcal{N}$ indicates the standard normal distribution, T the expiration date, r the risk-free interest rate, with

$$d_1 = \frac{\ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Now, following [6], using the Conus–Wildman modified stock price process $(S_t^\epsilon)_{t\in[0,T]}$, the payoff of a call option is

$$C^\epsilon = (S_T^\epsilon - K)^+.$$

By (7.6) and (7.7), it follows

$$\begin{aligned}
C_t^\epsilon &= B_t \mathbb{E}_{\mathbb{Q}}\left[\frac{C^\epsilon}{B_T}\middle|\mathcal{F}_t\right] \\
&= B_t \mathbb{E}_{\mathbb{Q}}\left[\frac{(S_T^\epsilon - K)^+}{B_T}\middle|\mathcal{F}_t\right] \\
&= \frac{B_t}{B_T}\mathbb{E}_{\mathbb{Q}}\left[\left(S_t^\epsilon \frac{S_T^\epsilon}{S_t^\epsilon} - K\right)^+\middle|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\left(S_t^\epsilon e^{r(T-t)+\sigma D_H \int_t^T s^{H-\frac{1}{2}}\, dW_s^\epsilon - \frac{1}{2}\sigma^2 D_H^2 (\frac{T^{2H}-t^{2H}}{2H})} - K\right)^+\middle|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\left(x e^{r(T-t)+\sigma D_H \int_t^T s^{H-\frac{1}{2}}\, dW_s^\epsilon - \frac{1}{2}\sigma^2 D_H^2 (\frac{T^{2H}-t^{2H}}{2H})} - K\right)^+\middle|x = S_t^\epsilon\right],
\end{aligned}$$

as S_t^ϵ is $\mathcal{F}_t$ measurable and $\int_t^T s^{H-\frac{1}{2}}\, dW_s^\epsilon$ is independent of $\mathcal{F}_t$. Further, $\int_t^T s^{H-\frac{1}{2}}\, dW_s^\epsilon$ is a centered Gaussian random variable with variance $\frac{T^{2H}-t^{2H}}{2H}$, which gives us

$$\begin{aligned}
C_t^\epsilon &= e^{-r(T-t)}\frac{1}{\sqrt{2\pi}} \\
&\quad \times \int_{-\infty}^{\infty}\left(S_t^\epsilon e^{r(T-t)+\sigma D_H\sqrt{\frac{T^{2H}-t^{2H}}{2H}}z - \frac{1}{2}\sigma^2 D_H^2(\frac{T^{2H}-t^{2H}}{2H})} - K\right)^+ e^{-\frac{1}{2}z^2}dz,
\end{aligned}$$

with z as standard normal random variable. For

$$S_t^\epsilon e^{r(T-t)+\sigma D_H\sqrt{\frac{T^{2H}-t^{2H}}{2H}}z - \frac{1}{2}\sigma^2 D_H^2(\frac{T^{2H}-t^{2H}}{2H})} - K \geq 0$$

to be true,

$$z \geq \frac{\ln\left(\frac{K}{S_t^\epsilon}\right) - r(T-t) + \frac{1}{2}\sigma^2 D_H^2\left(\frac{T^{2H}-t^{2H}}{2H}\right)}{\sigma D_H\sqrt{\frac{T^{2H}-t^{2H}}{2H}}} := -d_2^\epsilon$$

needs to hold.

Then our pricing formula for a call option is,

$$\begin{aligned}
C_t^\epsilon(S_t^\epsilon, t) &= e^{-r(T-t)}\frac{1}{\sqrt{2\pi}} \\
&\times \int_{d_1}^{\infty}\left(S_t^\epsilon e^{r(T-t)+\sigma D_H\sqrt{\frac{T^{2H}-t^{2H}}{2H}}z-\frac{1}{2}\sigma^2 D_H^2(\frac{T^{2H}-t^{2H}}{2H})} - K\right)^+ e^{-\frac{1}{2}z^2}dz \\
&= S_t^\epsilon\frac{1}{\sqrt{2\pi}}\int_{d_1}^{\infty} e^{-\frac{1}{2}\left(z-\sigma D_H\sqrt{\frac{T^{2H}-t^{2H}}{2H}}\right)^2}dz - Ke^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\int_{d_1}^{\infty} e^{-\frac{1}{2}z^2}dz \\
&= S_t^\epsilon\mathcal{N}(d_1^\epsilon) - Ke^{-r(T-t)}\mathcal{N}(d_2^\epsilon),
\end{aligned}$$

with

$$d_1^\epsilon = d_2^\epsilon + \sigma D_H\sqrt{\frac{T^{2H}-t^{2H}}{2H}}.$$

Note that for $H = \frac{1}{2}$ we have the original Black–Scholes call option price.

7.4.2 Put Option Pricing

The payoff of a put option is $P = (K - S_T)^+$, where K is the strike price and S_t the stock price at time $t \in [0, T]$. The value of a Black–Scholes European put option is

$$P(S_t, t) = Ke^{-r(T-t)}\mathcal{N}(-d_2) - S_t\mathcal{N}(-d_1),$$

where d_1 and d_2 are defined as in the previous section. Similarly, as for the call option, a put option pricing formula can be obtained from the modified Dobrić–Ojeda process,

$$P_t^\epsilon(S_t^\epsilon, t) = Ke^{-r(T-t)}\mathcal{N}(-d_2^\epsilon) - S_t^\epsilon\mathcal{N}(-d_1^\epsilon),$$

with d_1^ϵ and d_2^ϵ defined as for the call option.

Chapter 8

Parameter Estimation

Thus far we have assumed to have constant parameters μ, σ and H for the stock price process. Some literature approaches this issue through estimation methods using historical stock price data, which will be introduced in this section, but estimation methods using implied volatility have been shown to estimate actual volatility better, which will be done in section 10.

8.1 Estimating the Hurst Index

As the speed and volume of trading has severely increased, facilitated through electronic trading, estimation techniques have to be fast and accurate. As Taqqu et al. examined in [28] many estimation techniques are fast and simple, but with a slow convergence rate and wide confidence intervals. They simulated 50 sample paths with $10,000$ realizations of fBm for $H = 0.5, 0.6, \ldots, 0.9$, using various estimation techniques to compare the accuracy of the estimates of the Hurst index. The following three are the main types of Hurst index estimators for fractional Brownian motion.

1. Time domain based analysis; rescaled range (R/S) method
2. Aggregated processes analysis; variance and absolute convergence methods
3. Frequency based analysis; periodogram/ spectral methods, wavelet methods and Whittle's MLE

Note that the (R/S) method is the method used in Hurst's hydrological analysis described in section 2.1. It is the best-known method, measuring the long range-dependence in a time series. Taqqu et al. find Whittle's Maximum Likelihood Estimator (MLE) approach to be the most accurate. In the following section estimators using ergodic theory, that are competitive to Whittle's approach, will be introduced, using the self-similar and stationary properties of fractional Gaussian noise.

8.2 Ratio of Second Moment Method with Ergodic Theory

"Ergodic theory says that a time average equals a space average" [9]. In order to use the ergodic theory estimation techniques introduced in [23], we assume, as in [6], that the Hurst index of the stock price following a geometric Conus–Wildman process is the same parameter as of the corresponding fractional Brownian motion, i.e. $H_{Z_t} = H_{V_t^\epsilon}$. Recall that the relative error of the processes Z_t and V_t is at most 12%, as shown in figure 6.1. Assume that the stock price follows a geometric fBm process as in the Black–Scholes SDE (1.2) and defining the log returns $y_i = \frac{S_i}{S_{i-1}}$ for $i = 1, \ldots, N$, where S_i is the observed price of the underlying stock at time $t_i = \frac{iT}{N}$, for $i = 1, \ldots, N$, for fixed time periods, $\Delta t = \frac{T}{N}$. Then,

$$y_i = \mu \Delta t + \sigma(Z_H(t_i) - Z_H(t_{i-1})) - \frac{1}{2}\sigma^2(t_i^{2H} - t_{i-1}^{2H})$$

We want to show that $y_i \to \mu\Delta t$ as $N \to \infty$. By the ergodic property of Z_H,

$$\frac{1}{N}\sum_{i=1}^{N}(Z_H(t_i) - Z_H(t_{i-1})) \to \mathbb{E}[Z_H(t_1) - Z_H(t_0)] = 0 \quad \text{a.s.} \tag{8.1}$$

The last term can be rearranged to a Riemann sum, which converges to zero as n goes to infinity, see [6]. Hence,

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N} y_i &= \mu\Delta t + \frac{\sigma}{N}\sum_{i=1}^{N}(Z_H(t_i) - Z_H(t_{i-1})) - \frac{\sigma^2}{2N}\sum_{i=1}^{N}(t_i^{2H} - t_{i-1}^{2H}) \\ &\to \mu\Delta t \quad \text{as} \quad N \to \infty,\end{aligned}$$

which gives us an estimator for μ for N sufficiently large,

$$\hat{\mu} = \frac{1}{\Delta t}\frac{1}{N}\sum_{i=1}^{N} y_i.$$

In order to estimate the volatility σ and the Hurst index H, a ratio of second moments will be used that was introduced in [23],

$$\begin{aligned}
SS_1 &:= \frac{1}{N}\sum_{i=1}^{N}(y_i - \hat{\mu}\Delta t)^2 \\
&= \frac{1}{N}\sum_{i=1}^{N}\left(\sigma(Z_H(t_i) - Z_H(t_{i-1})) - \frac{\sigma^2}{2}(t_i^{2H} - t_{i-1}^{2H})\right)^2 \\
&= \frac{\sigma^2}{N}\sum_{i=1}^{N}(Z_H(t_i) - Z_H(t_{i-1}))^2 - \frac{\sigma^3}{N}\sum_{i=1}^{N}(Z_H(t_i) - Z_H(t_{i-1}))(t_i^{2H} - t_{i-1}^{2H}) \\
&+ \frac{\sigma^4}{N}\sum_{i=1}^{N}(t_i^{2H} - t_{i-1}^{2H})^2.
\end{aligned}$$

Using the same argument as before and because we have finite variation, which means that the quadratic variation has to be zero, we have

$$\frac{\sigma^4}{N}\sum_{i=1}^{N}(t_i^{2H} - t_{i-1}^{2H})^2 \to 0 \quad \text{as} \quad N \to \infty.$$

By the Cauchy–Schwarz inequality, the second term converges to zero,

$$\begin{aligned}
&\frac{\sigma^3}{N}\sum_{i=1}^{N}(Z_H(t_i) - Z_H(t_{i-1}))(t_i^{2H} - t_{i-1}^{2H}) \\
&\leq \sqrt{\left(\frac{\sigma^2}{N}\sum_{i=1}^{N}(Z_H(t_i) - Z_H(t_{i-1}))^2\right)\left(\frac{\sigma^4}{N}\sum_{i=1}^{N}(t_i^{2H} - t_{i-1}^{2H})^2\right)} \\
&\to 0 \quad \text{as} \quad N \to \infty.
\end{aligned}$$

The first term converges to the increments of second moment, by the ergodic theorem,

$$\frac{1}{N}\sum_{i=1}^{N}\sigma(Z_H(t_i) - Z_H(t_{i-1}))^2 \to \sigma^2(\Delta t)^{2H} \quad \text{as} \quad N \to \infty,$$

and hence

$$SS_1 \to \sigma^2(\Delta t)^{2H} \quad \text{as} \quad N \to \infty. \tag{8.2}$$

As we want to obtain a ratio, we need a second estimator. In [23] two further estimates were defined, one formed from the even increments and one from the odd ones, and their average was taken. In [6], as will be done here, this step was combined to defining one second parameter with half as many points as used in SS_1.

$$SS_2 := \frac{1}{\lfloor N/2 \rfloor} \sum_{i=1}^{\lfloor N/2 \rfloor} (y_{2i} - \hat{\mu} 2\Delta t)^2 \to \sigma^2 (2\Delta t)^{2H} \quad \text{as} \quad N \to \infty, \tag{8.3}$$

where the convergence is obtained as for SS_1. We are now able to take the ratio of the two moments SS_1 (8.2) and SS_2 (8.3),

$$\frac{SS_2}{SS_1} = \frac{\sigma^2 (2\Delta t)^{2H}}{\sigma^2 (\Delta t)^{2H}} = 4^H,$$

which lets us solve for the estimator of the Hurst index H,

$$\hat{H} = \log_4 \left(\frac{SS_2}{SS_1} \right).$$

Using the ratio's method estimate of H in equation 8.2, the volatility estimator σ can be obtained,

$$\hat{\sigma}^2 = \frac{1}{(\Delta t)^{2\hat{H}}} \frac{1}{N} \sum_{i=1}^{N} (y_i - \hat{\mu}\Delta t)^2.$$

Note that outliers or large jumps shall be omitted in the data, as the ratio method is sensitive to anomalities, which will skew SS_1 and SS_2 (see [23]). Further, the estimates of H and σ are highly correlated, as is obvious from the dependent computational method.

8.3 Quadratic Variation Estimation Method

The ratio method that was introduced in the previous section cannot be applied to the Conus–Wildman process as it does not have ergodic increments. Alternatively, parameters can be estimated through quadratic variation as will be shown in this section, which then will be used for another ratio method. Note that we do not need to estimate drift μ as it is not relevant for option pricing. The estimators of the variance σ and Hurst index H remain to be calculated. Recall that the quadratic variation of both the Dobrić–Ojeda and the Conus–Wildman process are the same, corollary 6.3.2 and proposition 7.2.4,

$$I :=_{t_0} [V^\epsilon, V^\epsilon]_T = \frac{D_H^2}{2H} (T^{2H} - t_0^{2H}).$$

We want to show L^2- and almost sure convergence using the quadratic variation of V_t^ϵ for the parameter estimation, following [6].

Theorem 8.3.1. *Let* $t_i = \frac{iT}{\lfloor n^{1+\eta} \rfloor}$, $i = i_0, \ldots, \lfloor n^{1+\eta} \rfloor$, $i_0 = \frac{t_0 \lfloor n^{1+\eta} \rfloor}{T}$, *be a sequence of partitions of* $[t_0, T]$ *for some* $\eta > 0$ *and the Dobrić–Ojeda process*$V_H(t) = G_H(t) M_H(t)$*, as in (6.6). Then*

$$\lim_{n\to\infty} \left\| \sum_{i=1}^{n} (\Delta V_{t_i})^2 - I \right\|_2 = 0$$

and

$$\lim_{n\to\infty} \sum_{i=1}^{\lfloor n^{1+\eta} \rfloor} (\Delta V_{t_i})^2 = I \quad a.s.,$$

with $\Delta V_{t_i} = V_{t_i} - V_{t_{i-1}}$.

For the proof see the appendix of [6]. The term $\lfloor n^{1+\eta} \rfloor$ results from the fact that for almost sure convergence the sampling rate should be strictly greater than n, thus this holds for all $\eta > 0$. Note, that this will only marginally impact the precision of the estimator.

Corollary 8.3.2. *The sample quadratic variation of the Conus–Wildman process* V_t^ϵ *converges in* L^2 *and almost surely to* $I = \frac{D_H^2}{2H}(T^{2H} - t_0^{2H})$.

Proof. This follows immediately as there have been no changes to the quadratic variation. □

Similarly, the convergence of the quadratic variation for the log stock price process $Y_t^\epsilon = \ln(S_t^\epsilon)$ is given.

Corollary 8.3.3. *The sample quadratic variation of the log stock price process* Y_t^ϵ *converges in* L^2 *and almost surely to* $I\sigma^2 = \frac{D_H^2 \sigma^2}{2H}(T^{2H} - t_0^{2H})$.

Proof. As defined in (7.3),

$$Y_t^\epsilon = \ln(S_t^\epsilon) = \mu t + \sigma V_t^\epsilon - \frac{D_H^2 \sigma^2}{4H} t^{2H}.$$

As the drift part does not impact the quadratic variation, the quadratic variation of the log stock price process is given by $\frac{D_H^2 \sigma^2}{2H}(T^{2H} - t_0^{2H})$. □

Ratio Method with Quadratic Variation

Analogously as for the ratio method with ergodic theory (section 8.2), we want to have equally time spaced observations defined as $m := \lfloor n^{1+\eta} \rfloor$ of the stock price process $S_t^\epsilon =: s_i$ at time $t_i = \frac{iT}{m}, i = 0, \dots, m$. Assuming we have a non dividend paying stock and again let the log returns be defined by $y_i = \ln \frac{s_i}{s_{i-1}}$. Further, assume the stock price process follows a geometric Dobrić–Ojeda process, with S_t^ϵ as in (7.6):

$$S_t^\epsilon = S_0 \exp \left\{ rt + \sigma D_H \int_0^t s^{H-\frac{1}{2}} \, dW_s^\epsilon - \frac{D_H^2 \sigma^2}{4H} t^{2H} \right\}$$

Let

$$y_i = \mu \Delta t + \sigma (V_{t_i}^\epsilon - V_{t_{i-1}}^\epsilon) - \frac{D_H^2 \sigma^2}{4H} (t_i^{2H} - t_{i-1}^{2H}).$$

Note that $t_0 = 0$ according to our definition and by corollary 8.3.3 we have

$$\sum_{i=1}^{m} y_i^2 \to \frac{D_H^2 \sigma^2}{2H} T^{2H}.$$

For the sample quadratic variation for half the sample paths we have similar convergence,

$$\sum_{i=1}^{\lfloor m/2 \rfloor} y_i^2 \to \frac{D_H^2 \sigma^2}{2H} \left(\frac{T}{2} \right)^{2H}.$$

Since the convergence is almost surely we can take the ratio as in section 8.2,

$$\frac{\sum_{i=1}^{m} y_i^2}{\sum_{i=1}^{\lfloor m/2 \rfloor} y_i^2} \to \frac{\frac{D_H^2 \sigma^2}{2H} T^{2H}}{\frac{D_H^2 \sigma^2}{2H} \left(\frac{T}{2} \right)^{2H}} = 4^H.$$

Hence, the estimator of the Hurst index H for m sufficiently large is given by

$$\hat{H} = \log_4 \left(\frac{\sum_{i=1}^{m} y_i^2}{\sum_{i=1}^{\lfloor m/2 \rfloor} y_i^2} \right).$$

Note that in [6] the fraction should be reciprocal or the logarithm to base $\frac{1}{4}$. We have used the quadratic variation method to obtain a value for H using historical end-of-day closing values of the S&P 500 index with a total of $7,984$ observations from January 2^{nd}, 1986 to August 31^{st}, 2017.

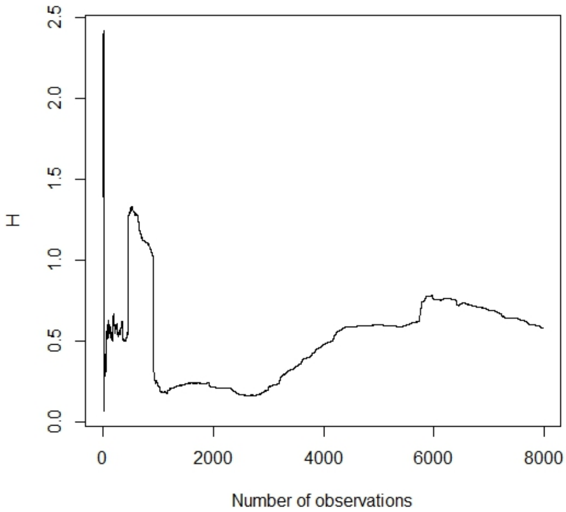

Figure 8.1: Historical H

The estimator for all 7,084 observations is $H = 0.5749$. Then, using estimator $\hat{H}$ to obtain the volatility estimator σ,

$$\hat{\sigma}^2 = \frac{2\hat{H}}{D_H T^{2\hat{H}}} \sum_{i=1}^{m} y_i^2.$$

Chapter 9

Conus–Wildman

The major goal of Conus and Wildman [6] was to apply the Dobrić–Ojeda process as noise to the Black–Scholes SDE (1.2):

$$dS_t = S_t(\mu\, dt + \sigma\, dV_t).$$

They did a simulation using the above presented underlying processes to compare the Black–Scholes price, the fractional Brownian motion price and the Dobrić–Ojeda price to the trading price. The price was computed of a European call option using historical stock price data. This was done over a period of 62 consecutive days. The case study was performed on a call option of an American Airline stock (AAL) and of a Bank of America stock (BAC) with a fixed strike price. They observe that with a lower value of H for the quadratic variation ratio method, the Dobrić–Ojeda option price is more accurate than the Black–Scholes and the fractional Brownian motion price with respect to the actual option trading price. However, when the quadratic variation ratio method estimates a higher value of H, the Dobrić–Ojeda model overestimates the option price.

While incorporating past dependencies, through their estimation method Conus and Wildman still obtained a constant volatility $\sigma > 0$, which suggests that the volatility smile is still flat.

Picking just one or two stocks makes it difficult to get a feeling for the overall market and the choice of stocks seems rather random. Therefore we have used the S&P 500 index for calibration in the following section, using different strike prices and using call and put options.

Instead of taking historical data for our estimators, we want to calculate the *implied volatility*, i.e. solving for the volatility σ using given market values. The implied volatility is believed to be superior to the historical volatility as it reflects market participants' expectations and it has been shown that it reflect the actual volatility better. Typically, the higher the implied volatility, the more expensive the option. Options that are at-the-money are usually traded more often.In the following section we have derived the implied volatility for the Conus–Wildman model.

9.1 Implied Volatility in the Conus–Wildman Process

In order to obtain the implied volatility, the derivative of our payoff function F with respect to volatility σ needs to be taken. Recall that

$$F_t = S_t^{\epsilon}\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2),$$

with

$$d_1 = \frac{\ln\left(\frac{S_t^{\epsilon}}{K}\right) + r(T-t) + \frac{1}{2}\sigma^2 D_H^2\left(\frac{T^{2H}-t^{2H}}{2H}\right)}{\sigma D_H\sqrt{\frac{T^{2H}-t^{2H}}{2H}}}$$

and

$$\begin{aligned} d_2 &= \frac{\ln\left(\frac{S_t^{\epsilon}}{K}\right) + r(T-t) - \frac{1}{2}\sigma^2 D_H^2\left(\frac{T^{2H}-t^{2H}}{2H}\right)}{\sigma D_H\sqrt{\frac{T^{2H}-t^{2H}}{2H}}} \\ &= d_1 - \sigma D_H\sqrt{\frac{T^{2H}-t^{2H}}{2H}}. \end{aligned}$$

Taking the derivative with respect to the volatility is known as the *Vega* of the *Greeks* in the Black–Scholes model.

$$\begin{aligned} \frac{\partial F_t}{\partial \sigma} &= S_t^{\epsilon}\mathcal{N}'(d_1)\left(-\frac{d_2}{\sigma}\right) - Ke^{-r(T-t)}\mathcal{N}'(d_2)\left(-\frac{d_1}{\sigma}\right) \\ &= \frac{1}{\sqrt{2\pi}}S_t^{\epsilon}e^{-\frac{d_1^2}{2}}\left(-\frac{d_2}{\sigma}\right) - \frac{1}{\sqrt{2\pi}}Ke^{-r(T-t)}e^{-\frac{d_2^2}{2}}\left(-\frac{d_1}{\sigma}\right) \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\left(-\frac{S_t^{\epsilon}d_2}{\sigma} + \frac{Ke^{-r(T-t)}d_1}{\sigma}e^{\frac{d_1^2}{2}-\frac{d_2^2}{2}}\right) \\ &= \mathcal{N}'(d_1)\left(-\frac{S_t^{\epsilon}d_2}{\sigma} + \frac{Ke^{-r(T-t)}d_1}{\sigma}e^{\frac{1}{2}(d_1-d_2)(d_1+d_2)}\right) \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{N}'(d_1)\Bigg(-\frac{S_t^\epsilon d_2}{\sigma} \\
&+ \frac{e^{-r(T-t)}d_1}{\sigma}\exp\left\{\frac{1}{2}\sigma D_H\sqrt{\frac{T^{2H}-t^{2H}}{2H}}\frac{2\ln\frac{S_t^\epsilon}{K}+2r(T-t)}{\sigma D_H\sqrt{\frac{T^{2H}-t^{2H}}{2H}}}\right\}\Bigg) \\
&= \mathcal{N}'(d_1)\left(-\frac{S_t^\epsilon d_2}{\sigma}+\frac{Ke^{-r(T-t)}d_1}{\sigma}\frac{S_t^\epsilon}{K}e^{r(T-t)}\right) \\
&= S_t^\epsilon \mathcal{N}'(d_1)D_H\sqrt{\frac{T^{2H}-t^{2H}}{2H}},
\end{aligned}
$$

where the derivative of d_1 with respect to σ is

$$
\begin{aligned}
\frac{\partial d_1}{\partial \sigma} &= \frac{\sigma^2 D_H^3\left(\frac{T^{2H}-t^{2H}}{2H}\right)^{\frac{3}{2}}}{\sigma^2 D_H^2\left(\frac{T^{2H}-t^{2H}}{2H}\right)} \\
&\quad - \frac{\left(\ln\frac{S_t^\epsilon}{K}+r(T-t)+\frac{1}{2}\sigma^2 D_H^2\left(\frac{T^{2H}-t^{2H}}{2H}\right)\right)D_H\sqrt{\frac{T^{2H}-t^{2H}}{2H}}}{\sigma^2 D_H^2\left(\frac{T^{2H}-t^{2H}}{2H}\right)} \\
&= \frac{-\ln\frac{S_t^\epsilon}{K}-r(T-t)+\frac{1}{2}\sigma^2 D_H^2\left(\frac{T^{2H}-t^{2H}}{2H}\right)}{\sigma^2 D_H\sqrt{\frac{T^{2H}-t^{2H}}{2H}}} \\
&= -\frac{d_2}{\sigma}.
\end{aligned}
$$

Similarly, the derivative of d_2 with respect to σ

$$
\frac{\partial d_2}{\partial \sigma} = -\frac{d_1}{\sigma}.
$$

In order to solve the two dimensional optimization problem with a gradient approach, the derivative with respect to H is required. First, note that,

$$
\frac{\partial}{\partial H}\frac{T^{2H}-t^{2H}}{2H} = \frac{t^{2H}-2Ht^{2H}\ln(t)-T^{2H}+2HT^{2H}\ln(T)}{2H^2} := A.
$$

Then,

$$
\begin{aligned}
\frac{\partial F_t}{\partial H} &= S_t^{\epsilon}\mathcal{N}'(d_1)\left(-\frac{d_2 AH}{T^{2H}-t^{2H}}\right) - Ke^{-r(T-t)}\mathcal{N}'(d_2)\left(-\frac{d_1 AH}{T^{2H}-t^{2H}}\right) \\
&= \frac{1}{\sqrt{2\pi}}S_t^{\epsilon}e^{-\frac{d_1^2}{2}}\left(-\frac{d_2 AH}{T^{2H}-t^{2H}}\right) - \frac{1}{\sqrt{2\pi}}Ke^{-r(T-t)}e^{-\frac{d_2^2}{2}}\left(-\frac{d_1 AH}{T^{2H}-t^{2H}}\right) \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\left(-\frac{S_t^{\epsilon}d_2 AH}{T^{2H}-t^{2H}} + \frac{Ke^{-r(T-t)}d_1 AH}{T^{2H}-t^{2H}}e^{\frac{d_1^2}{2}-\frac{d_2^2}{2}}\right) \\
&= \mathcal{N}'(d_1)\left(-\frac{S_t^{\epsilon}d_2 AH}{T^{2H}-t^{2H}} + \frac{Ke^{-r(T-t)}d_1 AH}{T^{2H}-t^{2H}}e^{\frac{1}{2}(d_1-d_2)(d_1+d_2)}\right) \\
&= \mathcal{N}'(d_1)\left(-\frac{S_t^{\epsilon}d_2 AH}{T^{2H}-t^{2H}} + \frac{Ke^{-r(T-t)}d_1 AH}{T^{2H}-t^{2H}}\frac{S_t^{\epsilon}}{K}e^{r(T-t)}\right) \\
&= S_t^{\epsilon}\mathcal{N}'(d_1)\frac{AH\sigma C}{T^{2H}-t^{2H}}
\end{aligned}
$$

with

$$\frac{\partial d_1}{\partial H} = -d_2\frac{AH}{T^{2H}-t^{2H}}$$

and

$$\frac{\partial d_2}{\partial H} = -d_1\frac{AH}{T^{2H}-t^{2H}}.$$

Chapter 10

Calibration

For the calibration we have used an iterative approach with two loops in the case of the Conus–Wildman model one for the volatility σ and one for Hurst index H. Note that with the calculations of the implied volatility of the previous section a gradient approach would be possible, but it would not simplify the estimation. We have chosen to use the S&P 500 index, as it is known to be the best representation of the United States stock market and it entails a great variety of liquid options.

10.1 S&P 500 Index

The Standard&Poor's 500 (S&P 500) index consists of 500 stocks chosen for reasons like market size and liquidity. The index differs from stocks, for example the underlying for index options is the numerical value of the index. When a S&P 500 call option is exercised, the exerciser obtains the in-the-money cash value of the option times 100$. It is an A.M. settled index option which means that the expiration day is always on the Saturday following the third Friday of the month, so the last trading day is on Thursday and the final settlement value is determined at opening Friday morning.

The Chicago board options exchange (Cboe) has created a volatility index (VIX) for the S&P 500 index (SPX). The VIX calculates the 30-day expected volatility for the U.S. stock market, derived from the SPX call and put options. It often moves in the opposite direction of the SPX. The data used in this paper has been extracted from the Cboe, as it offers a variety of options.

10.2 Estimators

We will estimate the volatility σ and the Hurst index H using different call and put option prices on the S&P 500 index across different strike prices and maturities and test the accuracy of these estimators by comparing the estimates to other given option prices. We will take the average of the bid and ask option prices. Note that we will only consider option prices where the average is strictly positive and where the ask option price is larger than the bid option price. Disproportional big outliers, where the difference between ask and bid price is disproportionately large, will be excluded. *Out-of-the money* and *in-the-money* options will be considered. Recall that while in-the-money options are worth exercising, out-of-the money options have no intrinsic value, but time value. This is done for data extracted from different days separately. After filtering the total amount of options we have is between $6,000$ and up to over $10,000$, depending on the date, that includes put and call options throughout maturities ranging from two weeks to over one and a half years and 294 different strike prices from 100 to $4,100$. All prices will be stated in US dollars. In figure 10.1 all of the call and put options data that is being used to obtain the estimates, that was extracted from Cboe on April $20^{\text{th}}, 2018$, is plotted. Observe that the data seems to be consistent,

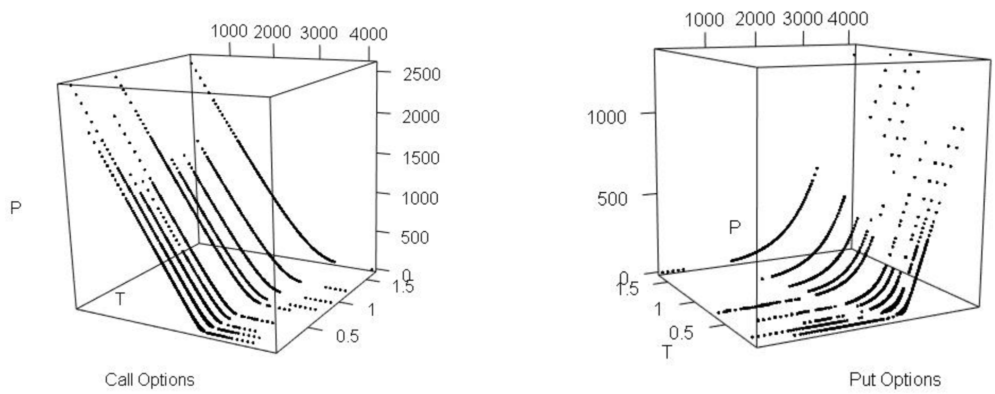

Figure 10.1: SPX call and put options across different maturities, strikes and prices

which is a good starting point for our calculations. The time to maturity T is on the z-axis, having values up to 1.67 years, while the strike price K is on the x-axis on top, strike prices range from 100 to $4,100$, and the prices P of the options are on the y-axis, where the highest price of a put option is $1,363.05$ while the highest call option price is $2,579.75$.

We will obtain a volatility estimate for the Black–Scholes model and an estimator for the volatility and for the Hurst index for the Conus–Wildman model in order to compare these two models to each other. We will do that by using the least square method with an iterative approach to the optimization problem.

Moreover, we will apply different weights to the options, i.e. more weight to expiration dates that are sooner, as they are more liquid and the pricing tends to be more accurate. Our goal is to minimize the error sum Y, defined as

$$Y_n^\epsilon = \min \sum_{i=1}^{n} w_i (F_t^\epsilon - F_t)^2,$$

where $(w_i)_{i\in[1,n]}$ is the weight vector, with $\sum_{i=1}^{n} w_i = 1$, and F is the payoff function for either put options P or call options C. F_t^ϵ indicates the price we obtain from the Conus–Wildman process and F_t is the market price. In order to test the obtained estimators every other option will be used in the minimization problem and the remaining options will be tested through calculating the sum of the weighted squared difference of the option price to the estimated option price. In order to compare the Conus–Wildman model to the Black–Scholes one, we will do the same with the Black–Scholes option price,

$$Y_n^{BS} = \min \sum_{i=1}^{n} w_i (F_t^{BS} - F_t)^2,$$

where F_t^{BS} indicates the Black–Scholes option price of either a call or a put option.

10.3 Implementation in R

First, we will show how the Black–Scholes option price can be obtained in R. We have included an indicator function I in order to differentiate between call and put option prices. The variables are defined in the usual way, where S is the stock price, K the strike price, $Tau = T - t$ the time to maturity, r the risk-free interest rate and sigma the volatility.

```
#Black-Scholes function
BS <- function(sigma, S, K, r, Tau, I) {
            d1 <- (log(S/K) + ( (r + sigma^2/2)*(Tau)))
                  /(sigma*sqrt(Tau))
            d2 <- (log(S/K) + ( (r - sigma^2/2)*(Tau)))
                  /(sigma*sqrt(Tau))
```

```
        if (I == 'C') {
                x <- pnorm(d1) * S
                        - K*exp(-r*(Tau)) * pnorm(d2) #Call
        } else {
                x <- K*exp(-r*(Tau)) * pnorm(-d2)
                        - pnorm(-d1) * S #Put
        }
        return(x)
}
```

Similarly, for the Conus–Wildman process,

```
#Conus-Wildman function
fCW <- function(sigma, S, K, r, t, T, H, I) {
  D <- D(H)
  Tau <- (T-t)
  d1 <- (log(S/K) + r*Tau + sigma^2/2
        * D^2 * (T^(2*H) - t^(2*H))/(2*H))
        /(sigma * D * sqrt((T^(2*H)-t^(2*H))/(2*H)))
  d2 <- (log(S/K) + r*Tau - sigma^2/2
        * D^2 * (T^(2*H) - t^(2*H))/(2*H))
        /(sigma * D * sqrt((T^(2*H)-t^(2*H))/(2*H)))
  if (I == 'C') {
                x <- pnorm(d1) * S
                - K*exp(-r*(Tau)) * pnorm(d2) #Call
        } else {
                x <- K*exp(-r*(Tau)) * pnorm(-d2)

                - pnorm(-d1) * S #Put
        }
  return(x)
},
```

where D(H) is D_H as defined in equation 6.2. Next, we want to sum over the weighted squared difference of our estimated option price that we obtain using function fCW defined above and the given option price P for all entries i. Note that we do the same for the Black–Scholes model, where the code is very similar and hence omitted here.

```
errorfunction <- function(sigma, S, K, r, t, T, H,
                                        P, I, w) {
i <- 1
x <- 0
```

```
for (i in 1:length(P)){
     x <- x +w[i]
          *(fCW(sigma, S, K[i], r, t, T[i], H, I)-P[i])^2
     i<-i+1
     }
return(x)
}
```

With the *findmin* function the estimators for σ and H are obtained through an iterative approach, where the values that yield the smallest sum in the *errorfunction* are saved. We restrict σ to be between 0 and 1, which is a quite generous assumption; when regarding the VIX from Cboe, the highest historical value was on October 20^{th}, 1987, the Tuesday after *Black Monday*, where the VIX reached 172.79%, but other than that it never had a value higher than 100, during the financial crisis its highest value in 2008 was 89.53%.

```
findmin <- function(S, K, r, t, T, P, I, w) {
  sigma <- 0.01
  yold <- 100000000000000
  while (sigma < 1) {
         H <- 0.01
         while (H<1) {
                 ynew <- errorfunction(sigma, S, K, r,
                                 t, T, H, P, I, w)
                 if (ynew < yold) {
                         yold <- ynew
                         sigmabest <- sigma
                         Hbest <- H
                 }
                 H <- H+0.01
         }
         sigma <- sigma+0.01
  }
  result <- c(sigmabest, Hbest, yold)
  return(result)
}
```

Note that for the Black–Scholes model we only have one iteration for the volatility σ.

Finally, we use the obtained estimators to compare the model prices that are calculated using the estimators to the ones that are given for all the options that we have not used in the calculation of the estimators, i.e. every other option.

```
compareCW <- function(S, K, r, t, T, P, I, w) {
    sigma <- findmin(S, K, r, t, T, P, I, w)[1]
    H <- findmin(S, K, r, t, T, P, I, w)[2]
    j <- 1
    x <- c()
    for (j in 1:length(P)) {
          x[j] <- w[j]
          *(fCW(sigma, S, K[j], r, t, T[j], H, I)-P[j])^2
          j <- j+1
          }
          return(x)
}
```

Notice that the *sigma* and the H are given the optimal values that have been obtained in the $findmin$ function. Then a vector is created through a loop, calculating the weighted squared difference of the calculated price through function fCW and the given price P for each of the entries.

10.4 Results

We will show the results we have obtained by implementing the $findmin$ function, described in the previous section. This was done for call and put options separately and then combining the two. In theory the implied volatility of the put and call options should be the same under the *Put-Call Parity*, i.e. in a perfect market, in reality this does not always hold and out of curiosity it will be done here for those three cases. The estimation will be done for data extracted from three different days separately. All data has been extracted from the Cboe. The weight formula that has been applied, where more weight is put upon closer expiration dates, is consistent over put and call options and for the three different days. The options have 293 different strike prices ranging from 100 to 4,100. Further, the calculations are done with a risk free interest rate of $1\%, r = 0.01$. The results from May 9^{th} are given in appendix B.

10.4.1 Estimates for the Black–Scholes Model

For the Black–Scholes call option prices, the following estimate is obtained using 2,094 call option prices, which is half of the amount of call options we have. We used half, i.e. every other, of the available option prices to find the estimate, where the ones that are not used to obtain the estimators will be used to test the estimate using the *compare* function. The data that is used has been extracted from Cboe, we will start with the data extracted on April 20$^{\text{th}}$, 2018, where the index price was $S = 2,693.13$,

```
> findminBS(S, K3, r, Tau3, P3, 'C', w3)
[1]  0.14000 34.66505.
```

Thus, we obtain the volatility $\sigma = 0.14$ and a minimum error sum of $Y^{BS} = 34.67$.

Using the *compare* function to compare each of the other 2,094 prices that are given to the ones we estimated through the Black–Scholes formula for $\sigma = 0.14$,

```
> x <- compareBS(S, K5, r, Tau5, P5, 'C', w5)
> sum(x, na.rm=TRUE)
[1] 34.69197.
```

Hence, the error sum is very close to the one in the minimization problem, which suggests that the estimate is consistent.

Similarly, for put option prices, where we have a total of 1,887 option prices, and thus will use 944 for estimation,

```
> findminBS(S, K4, r, Tau4, P4, 'q', w4)
[1]  0.1500 53.4791.
```

Note that the minimum error sum is significantly larger than for the call options and the volatility σ has increased from 14% to 15%.

```
> x <- compareBS(S, K6, r, Tau6, P6, 'q', w6)
> sum(x, na.rm=TRUE)
[1] 53.56196.
```

Using the *compare* function again gives us a similar result for the error sum, as we obtained in the put options minimization problem with function *findmin*.

Combining put and call options in order to have one best estimate, using a total of 6,076 put and call options, where we use half, i.e. 3,038 to obtain the estimate,

```
> findminBS(S,  K3,  r,  Tau3,  P3,  I3, w3)
[1]  0.14000 44.51704.
```

The weight has been distributed so that $\sum_{i=1}^{n} w_i = 1$ holds, therefore it makes sense that our estimate lies between the one of the put option and the one of the call option. Now, using the compare function for the other put and call options,

```
> x <- compareBS(S, K5, r, Tau5, P5, I5, w5)
> sum(x, na.rm=TRUE)
[1] 44.54121,
```

which is consistent with the error sum obtained with function $findmin$.

From the data obstructed on May 3rd, where the index price was $S = 2,635.67$, and with 5,421 available call options, we use 2,710 to get the following estimate,

```
> findminBS(S, K3, r, Tau3, P3, 'C', w3)
[1]  0.13000 88.31069,

> x <- compareBS(S, K5, r, Tau5, P5, 'C', w5)
> sum(x, na.rm=TRUE)
[1] 88.14593.
```

Which again shows a consistency of the estimators. Note that the error sum is higher and the volatility has decreased to $\sigma = 13\%$. For the put option with a total of 4,556 options, 2,278 options prices are used in the estimation,

```
> findminBS(S, K4, r, Tau4, P4, 'q', w4)
[1]  0.170 50.296
```

and

```
x <- compareBS(S, K6, r, Tau6, P6, 'q', w6)
> sum(x,na.rm=TRUE)
[1] 50.56474.
```

In this case the volatility for the put options has increased to $\sigma = 17\%$ and the error sum is lower than for the call options. Combining put and call options, which sum up to 9,977 options, using 4,988 in the estimation,

```
> findminBS(S, K3, r, Tau3, P3, I3, w3)
[1]  0.15000 88.42767,

> x <- compareBS(S, K5, r, Tau5, P5, I5, w5)
> sum(x, na.rm=TRUE)
[1] 88.28723.
```

10.4.2 Estimates for the Conus–Wildman Model

Using the same data as in the previous section for the Black–Scholes model, in order to compare both models to each other. For the call options, the following results are obtained in the Conus–Wildman model, for the data extracted on April 20th,

```
> findmin(S, K3, r, t, T3, P3, 'C', w3)
[1]  0.18000  0.59000 31.80963,
```

where the volatility $\sigma^{\epsilon} = 0.18$ and Hurst index $H = 0.59$ and the minimum error sum $Y^{\epsilon} = 31.81$. Note that this is lower than in the Black–Scholes model, where the minimum error sum was $Y^{BS} = 34.67$. Then, with the *compareCW* function, we test the other options using these estimates,

```
> z <- compareCW(S, K5, r, t, T5, P5, 'C', w5)
> sum(z, na.rm=TRUE)
[1] 31.94628,
```

where we have a an error sum $Y^{\epsilon} = 31.95$, which is again lower than the one obtained from the Black–Scholes model $Y^{BS} = 34.69$.

For the put option prices, we obtain the following

```
> findmin(S, K4, r, t, T4, P4, 'q', w4)
[1]  0.18000  0.55000 49.52554.
```

Observe that while we have the same estimate for the volatility, $\sigma^{\epsilon} = 0.18$, the Hurst index is lower, $H = 0.55$.

```
> z <- compareCW(S, K6, r, t, T6, P6, 'q', w6)
> sum(z, na.rm=TRUE)
[1] 49.58949.
```

As in the Black–Scholes model, although the error sum in the Conus–Wildman model is lower, we have a larger error sum when compared to the call options.

Combining the put and call options,

```
> findmin(S, K3, r, t, T3, P3, I3, w3)
[1]  0.18000  0.57000 41.19015,
```

where $\sigma^{\epsilon} = 0.18$ is consistent with the volatility of the put and the call options and Hurst index $H = 0.57$ is the average of the two estimates obtained, the error sum $Y^{\epsilon} = 41.19$ also lies between the two error sums. Testing the estimates with the *compareCW* function again yields a smaller error sum for the Conus–Wildman model compared to the Black–Scholes model,

```
> z <- compareCW(S, K5, r, t, T5, P5, I5, w5)
> sum(z, na.rm=TRUE)
[1] 41.20687.
```

Using the data extracted on May 3rd, 2018 from [4] with index price $S = 2,635.67$ in the same way as for the Black–Scholes model,

```
> findmin(S, K3, r, t, T3, P3, 'C', w3)
[1]  0.21000  0.66000 81.97849.
```

Notice that the error sum $Y^{\epsilon} = 81.98$ is higher than the one from April 20$^{\text{th}}$ in a similar way that it has increased in the Black–Scholes model, but it is quite a bit lower, compare to $Y^{BS} = 88.31$. Further, the volatility $\sigma^{\epsilon} = 0.21$ and the Hurst index $H = 0.66$ both have increased, while in the Black–Scholes model the volatility had decreased to $\sigma = 0.13$. The *compareCW* function shows consistency,

```
> z <- compareCW(S, K5, r, t, T5, P5, 'C', w5)
> sum(z, na.rm=TRUE)
[1] 81.78778.
```

For the put options, we have

```
> findmin(S, K4, r, t, T4, P4, 'q', w4)
[1]   0.16000   0.47000  49.83244,
```

where the volatility is significantly lower and $H < 0.5$, which indicates negative past dependencies, but recall that H fluctuates and we have assumed $H > 0.4$, moreover, in the rolling H estimates in [6] it is shown that the Hurst index H can fluctuate below 0.4. In contrast, in the Black–Scholes model the volatility for the put options was higher than for the call options.

```
> z <- compareCW(S, K6, r, t, T6, P6, 'q', w6)
> sum(z, na.rm=TRUE)
[1] 49.93971,
```

the estimates are very consistent for each of the option data. Now, combining put and call options,

```
> findmin(S, K3, r, t, T3, P3, I3, w3)
[1]   0.17000   0.54000  87.55236
```

and

```
> z <- compareCW(S, K5, r, t, T5, P5, I5, w5)
> sum(z, na.rm=TRUE)
[1] 87.42714,
```

which again gives us a lower error sum than in the Black–Scholes model. In the following section the results have been summarized for a better overview.

10.4.3 Summary Statistics

The tables give a summary of the results. First, for the Black–Scholes model and then for the Conus–Wildman model. The number of options refers to the number of options used to obtain the estimators.
Call options (BS):

Date	Volatility	Error Sum	Number of Options
April 20	0.14	34.66	2,094
May 3	0.13	88.31	2,710
May 9	0.14	27.22	2,932

Put options (BS):

Date	Volatility	Error Sum	Number of Options
April 20	0.15	53.47	944
May 3	0.17	50.30	2,278
May 9	0.14	34.28	2,253

Call and put options (BS):

Date	Volatility	Error Sum	Number of Options
April 20	0.14	44.52	3,038
May 3	0.15	88.29	4,988
May 9	0.14	30.70	5,185

In the following tables the estimation results for the Conus–Wildman model are summarized. Call options (CW):

Date	Volatility	Hurst Index	Error Sum	Number of Options
April 20	0.18	0.59	31.81	2,094
May 3	0.21	0.66	81.98	2,710
May 9	0.18	0.59	25.05	2,932

Put options (CW):

Date	Volatility	Hurst Index	Error Sum	Number of Options
April 20	0.18	0.55	49.52	944
May 3	0.16	0.47	49.83	2,278
May 9	0.19	0.59	30.59	2,253

Call and put options (CW):

Date	Volatility	Hurst Index	Error Sum	Number of Options
April 20	0.18	0.57	41.19	3,038
May 3	0.17	0.54	87.55	4,988
May 9	0.18	0.58	27.90	5,185

Observe that the implied volatility is in all cases lower in the Black–Scholes model than in the Conus–Wildman model, while the error sum, which we have used to see how close the estimated prices from the models are to the actual option prices, is in all cases higher in the Black–Scholes model, which means that the Conus–Wildman model estimates option prices more accurately.

10.4.4 Error Sum Graphs

In order to see the *errorsum* function Y^ϵ on a 3D graph, figures 10.2, 10.3 and 10.4, we created a two dimensional matrix, with the volatility σ and Hurst index H, with the following R code. Note that the Conus–Wildman function fCW is defined as in section 10.3.

```
errorsum <- function(S, K, r, t, T, P, I, w) {
  k <- 1
  l <- 1
  y <- matrix(nrow=length(sigma), ncol=length(H))
  for (l in 1:length(H)) {
        H <- H[l]
        for (k in 1:length(sigma)) {
                sigma <- sigma[k]
                j <- 1
                x <- 0
                for (j in 1:length(P)){
                        x <- x + w[j]
                        *(fCW(sigma, S, K[j], r, t, T[j],
                                H, I[j])-P[j])^2
                        j <- j+1
                }
                if (x > 600) {
                y[k,l] <- 600
                } else {
                y[k,l] <- x
                }
                k <- k+1
        }
        l <- l+1
        }
        return(y)
}
```

To gain a better feeling for the shape of the *errorsum* function, we have plotted three different kind of graphs. In figure 10.2 a cap with cut-off value 600, that assumes for all error sum values Y^ϵ that are larger than 600 the value 600, has been applied, see the if condition in the code. Figure 10.3 has a cut-off value of $Y^\epsilon = 150$ and gives way to another angle, showing the curved shape of the function. Figure 10.4, has values $H \in [0.45, 0.65]$ and $\sigma \in [0.1, 0.2]$, that give a zoomed in version of the graph, enabling a better

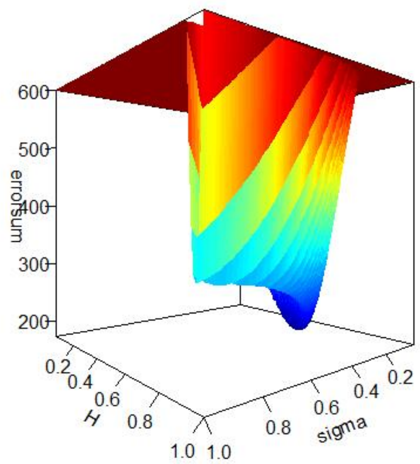

Figure 10.2: Cap of 600

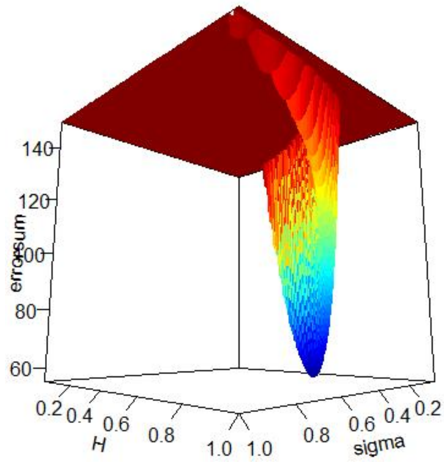

Figure 10.3: Cap of 150

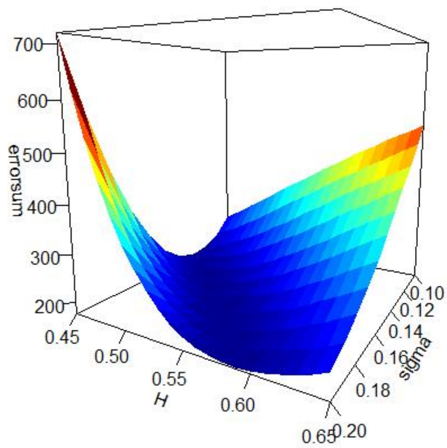

Figure 10.4: Zoomed in graph

view of the volatility and Hurst index values around the optimal value. Note that for all three days on which we have extracted the data the graphs look very similar for each of the versions. The graphs that are displayed here are from the data from May 9, 2018. From the graphs we can observe that our error sum is a convex function that has one minimum.

10.4.5 Volatility Index (VIX)

For plausibility reasons the estimated volatility is compared to the Cboe implied volatility index (VIX) of the S&P 500 on the dates where we have extracted the data. The VIX tends to move in the opposite direction of the SPX, it is an indication for market insecurities.

The data for the VIX has been extracted from Cboe [5]. The VIX on April 20$^{\text{th}}$ had a lowest value of 15.19% and a highest value of 17.50%. When we compare this to our estimated values, where we obtained a volatility of 14% in the Black–Scholes model and a volatility of 18% in the Conus–Wildman model, which do not lie in between, but just outside that range. On May 3$^{\text{rd}}$, the VIX was between 15.43% and 18.66%, where the volatility in the Black–Scholes model was estimated to be 15% and in the Conus–Wildman model 17%. The VIX had a low of 13.38% and a high of 14.63% on May 9$^{\text{th}}$, compared to the volatilities of 14% in the Black–Scholes model and 18% in the Conus–Wildman model. All volatilities seem to be in a plausible range.

Chapter 11

Conclusion

The Black–Scholes model is known to make some rather strong assumptions, as it assumes a constant volatility and past independencies, having Brownian motion as its underlying process. Our goal was to incorporate past dependencies into option pricing using the Hurst index. While fractional Brownian motion simulates stock prices quite well, it cannot be used to estimate option prices as it is not a semimartingale and admits arbitrage, which is dreaded in financial models. Therefore, Dobrić and Ojeda had proposed an alternative to fractional Brownian motion, a Gaussian Markov process, a semimartingale that incorporates past dependencies. Conus and Wildman modified this process so that Girsanov's theorem could be applied and option pricing could be derived. We used this analysis to obtain an implied volatility and an implied Hurst index and used these estimators to calculate option prices. The error sum, the minimum sum over the squared difference of the calculated option prices and the actual option prices, was compared to the error sum from the Black–Scholes model, where we also used the implied volatility as an estimator, in order to compare the pricing accuracy of the two models.

We conclude that the Conus–Wildman model outperforms the Black–Scholes model, as expected. The error sum comparing the estimated price with the given option price is in all tested cases higher in the Black–Scholes model than in the Conus–Wildman model. We find that the the Hurst index H plays an important role, as we estimate it to be around $H = 0.58$, which shows that past dependencies should be incorporated when pricing financial options. While the Black–Scholes model has Brownian motion as an underlying process and thus assumes independence of the past, $H = 0.5$, the Conus–Wildman model with the modified Dobrić–Ojeda process allows the Hurst index to assume different values, giving the model a lot more flexibility. Finally, we find that there exists a global minimum and no other local minima, which can be seen from the three dimensional graphs.

It would be interesting to deepen this analysis even further, with more data from different dates and for different types of stocks, comparing different industries and Hurst indices to each other.

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Appendices

Appendix A

Notational Conventions

B, Beta function, $B(\mu, \nu) = \int_0^1 x^{\mu-1}(1-x)^{\nu-1}\,dx = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$
$\partial_i f$, partial derivative with respect to i of function f
$\mathbb{E}$, expected value
Γ, Gamma function, $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,dx$
$\mathcal{N}$, standard normal distribution $\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt$
$\mathbb{N} = \{1, 2, \dots\}$, natural numbers
$\mathbb{R}$, real numbers
$\mathbb{R}_+$, non-negative real numbers
$\mathbb{R}_+^*$, positive real numbers
$\stackrel{d}{=}$, equal in distribution
$\wedge$, minimum

Appendix B

Results from May $9^{\mathbf{th}}$

B.1 Black–Scholes Model

Using $2,932$ call options for the estimation, from a total of $5,864$ call option prices,

```
> findminBS(S, K3, r, Tau3, P3, 'C', w3)
[1]  0.14000 27.79736
> x <- compareBS(S, K5, r, Tau5, P5, 'C', w5)
> sum(x,na.rm=TRUE)
[1] 27.21567.
```

For the put option with a total of $4,506$ options and thus $2,253$ for estimation,

```
> findminBS(S, K4, r, Tau4, P4, 'q', w4)
[1]  0.14000 34.28232
> x <- compareBS(S, K6, r, Tau6, P6, 'q', w6)
> sum(x,na.rm=TRUE)
[1] 34.56976.
```

Combining all options, a total of 10,370 put and call option,

```
> findminBS(S, K3, r, Tau3, P3, I3, w3)
[1]  0.14000 30.69687
> x <- compareBS(S, K5, r, Tau5, P5, I5, w5)
> sum(x, na.rm=TRUE)
[1] 30.39848.
```

B.2 Conus–Wildman Model

The same data and the same options are used for estimation as in the Black–Scholes model. First, the call options,

```
> findmin(S, K3, r, t, T3, P3, 'C', w3)
[1]   0.1800   0.5900  25.0508
> z <- compareCW(S, K5, r, t, T5, P5, 'C', w5)
> sum(z, na.rm=TRUE)
[1] 24.32232.
```

Then, for the put options,

```
> findmin(S, K4, r, t, T4, P4, 'q', w4)
[1]   0.19000   0.59000  30.58614
> z <- compareCW(S, K6, r, t, T6, P6, 'q', w6)
> sum(z, na.rm=TRUE)
[1] 30.57277.
```

Combining put and call options,

```
> findmin(S, K3, r, t, T3, P3, I3, w3)
[1]   0.18000   0.58000  27.89667
> z <- compareCW(S, K5, r, t, T5, P5, I5, w5)
> sum(z, na.rm=TRUE)
[1] 27.52513.
```