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DIPLOMARBEIT

The Pseudo-Vector Glueball in the Witten-Sakai-Sugimoto Model

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Part I
Introduction

The gluon self-coupling in Quantum Chromodynamics (QCD), the theory of strong interactions, suggests the existence of bound states of gauge bosons, the so-called glueballs. In pure Yang-Mills (YM) theory, these are the only possible particle states. The non-perturbative structure of the YM-theory makes it difficult to calculate the glueball spectrum [1]. Numerical simulations of the theory on a space-time lattice have proven to be reliable means of studying glueballs. In the presence of quarks, these simulations are more difficult, because glueball states can mix with quark-antiquark states. An outstanding problem is to calculate theoretical predictions of glueball couplings and decay rates from first principles. Lattice gauge theory provides some information on euclidean correlators, but is fraught with uncertainties when extrapolating to the real-time regime. [2].

A completely different approach to strongly coupled gauge theories has been developed over the last two decades in the form of the anti-de Sitter/conformal field theory (AdS/CFT) correspondence. The motivation for this correspondence lies in superstring theory, which besides fundamental strings also contains various non-perturbative solitonic objects, known as Dirichlet branes, or D-branes for short. Those objects may be viewed from two different perspectives, the open-string and the closed-string perspective [3]. For a visualization see figure 1. Which perspective is the appropriate one depends on the value of the string coupling constant g_S controlling the interaction strength of open and closed strings. String theory moreover is invariant under the so-called S-duality, which relates the strongly coupled to the weakly coupled regime and thus also relates the two perspectives. In the open-string perspective D-branes are viewed as higher dimensional objects on which open strings can end. In the weakly coupled regime, i.e. $g_S \ll 1$, open strings might be viewed as small perturbations. By neglecting massive string excitations, i.e. for low energies, the dynamics of the open strings is described by a supersymmetric gauge theory living on the worldvolume of the D-brane. In the closed-string perspective D-branes are viewed as solitonic objects that source the gravitational field and have horizons like black holes. In the low-energy limit of superstring theory, i.e. supergravity, closed-string excitations near the horizon decouple from closed-string excitations far away. The dynamics of these closed strings is described by supergravity in the background of a near-horizon D-brane solution. Since both perspectives are equivalent descriptions of the same physics we obtain the AdS/CFT correspondence. It manifests itself as an open-closed string duality, in particular it relates a supersymmetric Yang-Mills theory to supergravity.

Witten [4] proposed a top-down construction of an AdS/CFT like duality based on non-extremal D4 black-branes in type-IIA supergravity, which breaks both supersymmetry and conformal invariance. At low energies, below a Kaluza-Klein mass scale M_{KK}, the dual field theory is a four-dimensional large-N_C Yang-Mills theory. In this duality metric fluctuations of the D4 background correspond to glueball states in the field theory [5, 6].

Quarks in the fundamental representation may be added to this duality in the form of probe flavor D-branes. Sakai and Sugimoto [7] introduced pairs of D8 and anti-D8 branes, which intersect the color D4 branes of the Witten model. The resulting Witten-Sakai-Sugimoto model has been remarkably successful in reproducing various features of low-energy QCD. It is firmly rooted in string theory and, for given N_C and N_F, has only two free parameters, i.e. the 't Hooft coupling λ and the Kaluza-Klein scale M_{KK}.

Using this model it is possible to study glueball-meson interactions and to calculate glueball decay rates from effective Lagrangians. This was first carried out by Hashimoto, Tan and Terashima in [8], which was corrected and extended by Brünner, Parganlija and Rebhan in [9]. They have considered various glueball states dual to metric fluctuations and calculated their decay rates. Using these new data it might be possible to identify glueball states in the experiment. The predicted mass spectrum alone would not be sufficient for such an identification. This work will extend these efforts by considering a pseudo-vector glueball state dual to fluctuations of the Kalb-Ramond field, which is inevitably part of the model. In the hadron spectrum, a pseudo-vector glueball would appear as a so-called h_1 meson, which is unflavored with quantum numbers $J^{PC} = 1^{+-}$.

This work is structured as follows. In Part II we briefly review the Witten model of nonsupersymmetric Yang-Mills theory. We calculate the correct supersymmetry solution and derive the corresponding linearized Einstein equations. Solutions of these equations are dual to glueballs. We present the resulting glueball modes, calculate their quantum numbers, masses, and normalizations. Then we calculate fluctuations of the Kalb-Ramond field. In Part III we extend the Witten model to the Witten-Sakai-Sugimoto model. We derive an effective action for meson fields and calculate the meson masses and normalizations. By fitting the mass of the ρ meson and the value of the pion decay constant we fix the only two free parameters of the theory. In Part IV we calculate an effective glueball-meson interaction Lagrangian with which we are able to calculate decay rates of pseudo-vector glueballs.

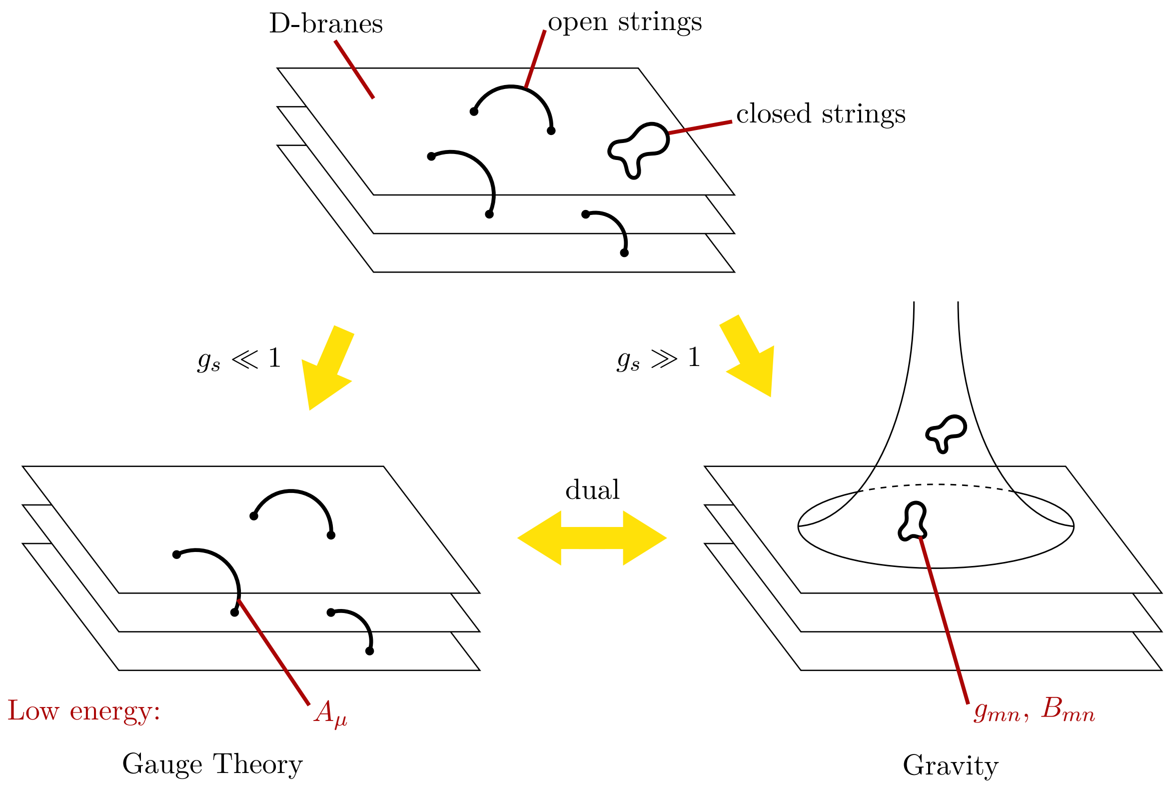

Figure 1. Motivation for the AdS/CFT correspondence.

Part II
Witten model

The Witten model of nonsupersymmetric Yang-Mills theory in 3+1 dimensions [4] is based on the AdS/CFT correspondence, which relates a 6-dimensional superconformal field theory to a large number N_C of coincident M5 branes in 11-dimensional M-theory. At low energy scales we can look at the embedding of the M5 branes in 11-dimensional supergravity. In the near-horizon limit the resulting space looks like the product space $AdS_7 \times S^4$ with line element

$$\mathrm{d}s^2 = \frac{r^2}{L^2}\left(\mathrm{d}x_4^2 + \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}x_{11}^2\right) + \frac{L^2}{r^2}\mathrm{d}r^2 + \frac{L^2}{4}\mathrm{d}\Omega_4^2. \tag{0.1}$$

The M5 brane is extended in 6 AdS_7 directions, chosen as $\mu, \nu = 0, 1, 2, 3$ and additionally 4 and 11, leaving out 10. The holographic radial coordinate coordinate is denoted by r. Dimensional reduction by

$$x_{11} \simeq x_{11} + 2\pi R_{11}, \qquad R_{11} = g_s l_s, \qquad l_s^2 = \alpha', \tag{0.2}$$

yields the the near-horizon geometry of D4 branes of type-IIA supergravity. Its dual theory is a five-dimensional super-Yang-Mills theory. Compactifying on an additional circle

$$x_4 \simeq x_4 + 2\pi R_4, \qquad R_4 \equiv \frac{1}{M_{KK}} = \frac{L^2}{3r_{KK}}, \tag{0.3}$$

breaks supersymmetry and yields the doubly Wick-rotated black hole geometry

$$\mathrm{d}s^2 = \frac{r^2}{L^2}\left(f\left(r\right)\mathrm{d}x_4^2 + \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}x_{11}^2\right) + \frac{L^2}{r^2}\frac{\mathrm{d}r^2}{f\left(r\right)} + \frac{L^2}{4}\mathrm{d}\Omega_4^2, \tag{0.4}$$

with $f\left(r\right) = 1 - \frac{r_{KK}^6}{r^6}$.

If we ignore all nontrivial harmonics on the compactification circles and on the S^4, we can interpret the bosonic normal modes of the supergravity multiplet as glueballs in the dual low-energy 3+1-dimensional Yang-Mills theory [5, 6]. There are a total of 14, coming from various fluctuations of the AdS_7 metric, which will be studied in detail below.

1 11d-SUGRA

1.1 Equations of motion

As we discussed in the introduction the geometry used in the Witten model is based on 11-dimensional supergravity. We will now derive the equations of motion of its bosonic part. They include the metric and a 4-dimensional field strength, which we will calculate in detail. The Lagrangian of 11-dimensional supergravity [10] is given by

$$\begin{aligned} 2\kappa_{11}^2\mathcal{L}_{11}^{(b)} &= \sqrt{-g}R\left(\omega\right) - \frac{\sqrt{-g}}{2\cdot 4!}F^{M_1M_2M_3M_4}F_{M_1M_2M_3M_4} \\ &\quad + \frac{1}{6\cdot 3!\cdot (4!)^2}\epsilon^{M_1\ldots M_{11}}A_{M_1M_2M_3}F_{M_4\ldots M_7}F_{M_8\ldots M_{11}}, \end{aligned} \tag{1.1}$$

where we have used the completely antisymmetric epsilon symbol $\epsilon^{M_1\ldots M_{11}}$ with $\epsilon^{0123\ldots} = 1$ and have normalized the field strength such that it satisfies

$$\begin{aligned} F_{M_1M_2M_3M_4} &:= 4\nabla_{[M_1}A_{M_2M_3M_4]} \\ &= \nabla_{M_1}A_{M_2M_3M_4} + \ldots . \end{aligned} \tag{1.2}$$

Variation of $\mathcal{L}_{11}^{(b)}$ with respect to g^{MN} yields the Einstein equations

$$\begin{aligned} \frac{\sqrt{-g}}{2\kappa_{11}^2}\left(R_{MN} - \frac{1}{2}R\,g_{MN}\right) =& \frac{\sqrt{-g}}{2\kappa_{11}^2 2\cdot 4!}\Big(4F_M{}^{M_1M_2M_3}F_{NM_1M_2M_3} \\ & - \frac{1}{2}F^{M_1M_2M_3M_4}F_{M_1M_2M_3M_4}\,g_{MN}\Big) \end{aligned} \tag{1.3}$$

On the right hand side the first term in the first brackets is obtained by varying one of the four inverse metrics which are used to raise the indices of the field strength. We thus obtain a factor of 4. The second term is obtained by using $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{MN}\delta g^{MN}$.

Variation of $\mathcal{L}_{11}^{(b)}$ with respect to A_{MNO} yields the generalized Maxwell equations

$$\begin{aligned} 0 = &\frac{1}{2\kappa_{11}^2 3!}\nabla_{M_1}\left(\sqrt{-g}F^{M_1 MNO}\right) \\ &+\frac{1}{2\kappa_{11}^2 2\cdot 3!\cdot(4!)^2}\epsilon^{MNOM_4\ldots M_{11}}F_{M_4\ldots M_7}F_{M_8\ldots M_{11}}. \end{aligned} \tag{1.4}$$

In the first term we get a factor of 4 by using $F^{M_1M_2M_3M_4}F_{M_1M_2M_3M_4} = 4F^{M_1M_2M_3M_4}\nabla_{M_1}A_{M_2M_3M_4}$ and an additional factor of 2 since the variation can hit either F. For the variation of the Chern-Simons term we used

$$\begin{aligned} &\frac{1}{2\kappa_{11}^2 6\cdot 3!\cdot(4!)^2}\epsilon^{MNOM_4\ldots M_{11}}F_{M_4\ldots M_7}F_{M_8\ldots M_{11}} \\ &-\frac{4\cdot 2}{2\kappa_{11}^2 6\cdot 3!\cdot(4!)^2}\epsilon^{M_1MNOM_5\ldots M_{11}}\nabla_{M_1}\left(A_{M_5M_6M_7}F_{M_8\ldots M_{11}}\right) \\ = &\frac{1}{2\kappa_{11}^2 6\cdot 3!\cdot(4!)^2}\epsilon^{MNOM_4\ldots M_{11}}F_{M_4\ldots M_7}F_{M_8\ldots M_{11}} \\ &+\frac{2}{2\kappa_{11}^2 6\cdot 3!\cdot(4!)^2}\epsilon^{MNOM_1M_5\ldots M_{11}}F_{M_1M_5M_6M_7}F_{M_8\ldots M_{11}} \\ &-\frac{4\cdot 2}{2\kappa_{11}^2 6\cdot 3!\cdot(4!)^2}\epsilon^{M_1MNOM_5\ldots M_{11}}A_{M_5M_6M_7}\nabla_{M_1}F_{M_8\ldots M_{11}}. \end{aligned} \tag{1.5}$$

The last term is identically 0 due to the Bianchi identity.

1.2 The Witten model geometry

To simplify further calculations we introduce the indices

$$\begin{aligned} AdS_7\times S^4 :&A,B,C\ldots,Z, \\ M_{10} :&\overline{A},\overline{B},\overline{C}\ldots,\overline{Z}, \\ AdS_7 :&a,b,c,\ldots,z, \\ S^4 :&\alpha,\beta,\gamma,\delta, \\ \text{Minkowski}_4 :&\mu,\nu,\rho,\sigma. \end{aligned} \tag{1.6}$$

M_{10} denotes the 10-dimensional space we obtain after reducing 11d SUGRA to type IIA SUGRA. The geometry of the Witten model is given by

$$\mathrm{d}s^2 = \frac{r^2}{L^2}\left(f(r)\,\mathrm{d}x_4^2 + \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\mu + \mathrm{d}x_{11}^2\right) + \frac{L^2}{r^2}\frac{\mathrm{d}r^2}{f(r)} + \frac{L^2}{4}\mathrm{d}\Omega_4^2. \tag{1.7}$$

The Einstein equations (1.3) include the metric and a 4-dimensional field strength, for which we make a Freund-Rubin ansatz

$$F_{\alpha\beta\gamma\delta} = C\sqrt{g_{S^4}}\epsilon_{\alpha\beta\gamma\delta}. \tag{1.8}$$

Here C is some constant to be determined, g_{S^4} is the determinant of the metric of the sphere and $\epsilon_{\alpha\beta\gamma\delta}$ is the 4-dimensional epsilon symbol. It is easy to see that this ansatz solves the Maxwell equations (1.4). The first line is zero, because the covariant derivative commutes with the the volume form and all other lines are trivially zero.

For the geometry (1.7) we obtain

$$R_{\alpha\beta} = \frac{12}{L^2}g_{\alpha\beta}, \tag{1.9}$$

$$R_{S^4} = \frac{48}{L^2}, \tag{1.10}$$

$$R_{ab} = -\frac{6}{L^2}g_{ab}, \tag{1.11}$$

$$R_{AdS_7} = -\frac{42}{L^2}, \tag{1.12}$$

$$R = \frac{6}{L^2}. \tag{1.13}$$

The constant C is determined by solving the Einstein equations (1.3) restricted to the sphere

$$\frac{12}{L^2}g_{\alpha\beta} - \frac{3}{L^2}g_{\alpha\beta} = \frac{C^2}{2\cdot 4!}\left(4\cdot 3!g_{\alpha\beta} - \frac{1}{2}\cdot 4!g_{\alpha\beta}\right) \qquad \Rightarrow \qquad C = \frac{6}{L} \tag{1.14}$$

Restricted to AdS_7, the Einstein equations (1.3) are also satisfied, which can be seen by calculating

$$\left(R_{ab} - \frac{1}{2}R\,g_{ab}\right) = -\frac{9}{L^2}g_{ab} \qquad \Rightarrow \qquad R_{ab} + \frac{6}{L^2}g_{ab} = 0. \tag{1.15}$$

Thus we see that the equations of motion are solved if we include the field strength $F_{\alpha\beta\gamma\delta} = \frac{6}{L}\sqrt{g_{S^4}}\epsilon_{\alpha\beta\gamma\delta}$.

1.3 Reduced Lagrangian

Since we are only interested in metric fluctuations of the AdS_7 metric it is useful to eliminate all spherical dependencies of the Lagrangian. To realize the ansatz (1.8), we can choose a gauge such that $A_{\alpha_1\ldots\alpha_3}$ depends linearly on x^α. For the field strength fluctuations we furthermore assume that there are no mixed indices terms, such as $A_{\alpha_1 a_2 a_3}$. Those are excluded because they would not be $SO(5)$ singlet states. We obtain

$$
\begin{aligned}
2\kappa_{11}^2\mathcal{L}_{11}^{(b)} &= \sqrt{-g}R\left(\omega\right) - \frac{\sqrt{-g}}{2\cdot 4!}F^{M_1M_2M_3M_4}F_{M_1M_2M_3M_4} \\
&+\frac{1}{6\cdot 3!\cdot (4!)^2}\epsilon^{M_1\ldots M_{11}}A_{M_1M_2M_3}F_{M_4\ldots M_7}F_{M_8\ldots M_{11}} \\
&= \sqrt{-g}R_{AdS_7} + \sqrt{-g}R_{S^4} - \frac{1}{2}\sqrt{-g}C^2 - \frac{\sqrt{-g}}{2\cdot 4!}F^{a_1\ldots a_4}F_{a_1\ldots a_4} \\
&-\frac{\sqrt{-g}}{2\cdot 4!}F^{\alpha_1 a_1\ldots a_3}F_{\alpha_1 a_1\ldots a_3} \\
&+\frac{2}{6\cdot 3!\cdot (4!)^2}\epsilon^{a_1\ldots a_7\alpha_1\ldots\alpha_4}A_{a_1\ldots a_3}F_{a_4\ldots a_7}F_{\alpha_1\ldots\alpha_4} \\
&+\frac{2}{6\cdot 3!\cdot (4!)^2}\epsilon^{a_1\ldots a_7\alpha_1\ldots\alpha_4}A_{\alpha_1\ldots\alpha_3}F_{\alpha_4 a_1\ldots a_3}F_{a_4\ldots a_7} \\
&= \sqrt{-g}R_{AdS_7} + \sqrt{-g}\frac{30}{L^2} - \frac{\sqrt{-g}}{2\cdot 4!}F^{a_1\ldots a_4}F_{a_1\ldots a_4} \\
&+\frac{\sqrt{g_{S^4}}}{2\cdot 4!L}\epsilon^{a_1\ldots a_7}A_{a_1\ldots a_3}F_{a_4\ldots a_7} + \text{higher spherical harmonics} \quad . \qquad (1.16)
\end{aligned}
$$

For the reduction of the Chern-Simons term we used

$$
\begin{aligned}
\epsilon^{M_1\ldots M_{11}}A_{M_1M_2M_3}F_{M_4\ldots M_7}F_{M_8\ldots M_{11}} &= \\
\epsilon^{a_1\ldots a_7\alpha_1\ldots\alpha_4}A_{a_1a_2a_3}F_{a_4\ldots a_7}F_{\alpha_1\ldots\alpha_4} & \\
+\epsilon^{a_1\ldots a_3\alpha_1\ldots\alpha_4 a_4\ldots a_7}A_{a_1a_2a_3}F_{\alpha_1\ldots\alpha_4}F_{a_4\ldots a_7} & \\
+\epsilon^{\alpha_1\ldots\alpha_4 a_1\ldots a_7}A_{\alpha_1\ldots\alpha_3}F_{\alpha_4 a_1\ldots a_3}F_{a_4\ldots a_7} & \\
+\epsilon^{\alpha_1\ldots\alpha_3 a_1\ldots a_4\alpha_4 a_5\ldots a_7}A_{\alpha_1\ldots\alpha_3}F_{a_1\ldots a_4}F_{\alpha_4 a_5\ldots a_7} &= \\
2\epsilon^{a_1\ldots a_7\alpha_1\ldots\alpha_4}A_{a_1a_2a_3}F_{a_4\ldots a_7}F_{\alpha_1\ldots\alpha_4} & \\
+2\epsilon^{\alpha_1\ldots\alpha_4 a_1\ldots a_7}A_{\alpha_1\ldots\alpha_3}F_{\alpha_4 a_1\ldots a_3}F_{a_4\ldots a_7} &= \\
3\epsilon^{a_1\ldots a_7\alpha_1\ldots\alpha_4}A_{a_1a_2a_3}F_{a_4\ldots a_7}F_{\alpha_1\ldots\alpha_4}. & \qquad (1.17)
\end{aligned}
$$

In the first line we evaluated all relevant combination of indices. In the next line we collected terms that are equivalent up to permutations and renaming of indices. In the last line we used

$$
\begin{aligned}
\epsilon^{\alpha_1\ldots\alpha_4 a_1\ldots a_7}A_{\alpha_1\ldots\alpha_3}F_{\alpha_4 a_1\ldots a_3}F_{a_4\ldots a_7} &= \\
16\epsilon^{\alpha_1\ldots\alpha_4 a_1\ldots a_7}A_{\alpha_1\ldots\alpha_3}\partial_{\alpha_4}A_{a_1\ldots a_3}\partial_{a_4}A_{a_5\ldots a_7} &= \\
-16\epsilon^{\alpha_1\ldots\alpha_4 a_1\ldots a_7}\partial_{\alpha_4}A_{\alpha_1\ldots\alpha_3}A_{a_1\ldots a_3}\partial_{a_4}A_{a_5\ldots a_7} & \\
+16\epsilon^{\alpha_1\ldots\alpha_4 a_1\ldots a_7}A_{\alpha_1\ldots\alpha_3}\partial_{a_4}A_{a_1\ldots a_3}\partial_{\alpha_4}A_{a_5\ldots a_7} &= \\
\epsilon^{a_1\ldots a_7\alpha_1\ldots\alpha_4}A_{a_1\ldots a_3}F_{a_4\ldots a_7}F_{\alpha_1\ldots\alpha_4} & \\
-\epsilon^{\alpha_1\ldots\alpha_4 a_1\ldots a_7}A_{\alpha_1\ldots\alpha_3}F_{\alpha_4 a_1\ldots a_3}F_{a_4\ldots a_7} &= \\
\frac{1}{2}\epsilon^{a_1\ldots a_7\alpha_1\ldots\alpha_4}A_{a_1\ldots a_3}F_{a_4\ldots a_7}F_{\alpha_1\ldots\alpha_4}, & \qquad (1.18)
\end{aligned}
$$

$h_{\mu\nu}$	$h_{\mu,11}$	$h_{11,11}$	$M_{n=1}[MeV]$ (Eq.)
h_{ij} 2^{++}	C_i $1^{++}_{(-)}$	ϕ 0^{++}	$1486.99(T_4)$
$h_{i\tau}$ $1^{-+}_{(-)}$	C_τ 0^{-+}		$1789.04(V_4)$
$h_{\tau\tau}$ 0^{++}			$855.174(S_4)$

Table 1. Mode classification following [5]. In the restframe μ and ν can take values among $(1, 2, 3, \tau)$ and i and j among $(1, 2, 3)$. Subscripts to J^{PC} denote odd τ-parity $P_\tau = -1$.

where in the first line we use the definition of the field strength. In the next line we partially integrate on the sphere. In the second term we also partially integrate on AdS and assume that the form field on the sphere does not depend on the AdS coordinates. In the term before the last we collect the derivative terms and note that the second term is again the term we started with. Thus we obtain the last line.

Note that in (1.17) we must not set $F_{\alpha_4 a_1 \ldots a_3}$ to 0 yet, contrary to what one might expect from the results obtained in [5].

2 Metric fluctuations

To organize the metric fluctuations we follow [5]. On the boundary the AdS_7 metric fluctuations contain 14 independent components. The fluctuation indices can take values corresponding to $(x_1, x_2, x_3, x_{11}, \tau)$; x_0 can be excluded by imposing the transversality constraint $k^\mu h_{\mu\nu} = 0$. Furthermore the fluctuations have to be symmetric. We thus have $\frac{5 \cdot 6}{2} - 1 = 14$ components. The background geometry (1.7) is symmetric under $SO(4)$ rotations in (x_1, x_2, x_3, x_{11}). The 14 independent components thus split into 9-, 4- and 1-dimensional irreducible representations under $SO(4)$, which are denoted by T_4, V_4 and S_4 respectively. Since the x_{11}-direction is moreover compactified, the 9-, 4- and 1-dimensional irreducible representations of $SO(4)$ break into irreducible representations of $SO(3)$. The resulting states are summarized in table 1.

2.1 Tensor glueball fluctuation $h_{\mu\nu}$

2.1.1 Linearized equations of motion

To derive the linearized equations of motion for the metric fluctuations we closely follow Wald [11]. We consider fluctuations of the form

$$g_{ab}(\lambda) = g_{ab} + \lambda\, h_{ab}. \tag{2.1}$$

The covariant derivatives corresponding to $g(\lambda)$ and g are denoted by

$$\begin{aligned} \nabla_c g_{ab}(\lambda) &= 0, \\ {}^0\nabla_c g_{ab} &= 0. \end{aligned} \tag{2.2}$$

The resulting fluctuations of the Einstein tensor can be calculated by making use of the tensor field C^c_{ab}, which relates the covariant derivative of the perturbed metric and the background metric by

$$\begin{aligned} \nabla_a \omega_b &= {}^0\nabla_a \omega_b - C^c_{ab}\omega_c, \\ \nabla_a T_{bc} &= {}^0\nabla_a T_{bc} - C^d_{ab} T_{dc} - C^d_{ab} T_{cd}. \end{aligned} \tag{2.3}$$

Its unique value is

$$C^c_{ab} = \frac{1}{2} g^{cd}(\lambda) \left({}^0\nabla_a g_{bd}(\lambda) + {}^0\nabla_b g_{ad}(\lambda) - {}^0\nabla_d g_{ab}(\lambda) \right). \tag{2.4}$$

Using

$$\begin{aligned} \nabla_a \nabla_b \omega_c =& {}^0\nabla_a \nabla_b \omega_c - C^d_{ac} \nabla_b \omega_d - C^d_{ab} \nabla_d \omega_c \\ =& {}^0\nabla_a \left({}^0\nabla_b \omega_c - C^d_{bc} \omega_d \right) - C^d_{ac} \left({}^0\nabla_b \omega_d - C^e_{bd} \omega_e \right) - C^d_{ab} \left({}^0\nabla_d \omega_c - C^e_{dc} \omega_e \right) \\ =& {}^0\nabla_a {}^0\nabla_b \omega_c - {}^0\nabla_a C^d_{bc} \omega_d - \left(C^d_{bc} {}^0\nabla_a \omega_d + C^d_{ac} {}^0\nabla_b \omega_d \right) + C^d_{ac} C^e_{bd} \omega_e \\ & - \left(C^d_{ab} {}^0\nabla_d \omega_c - C^d_{ab} C^e_{dc} \omega_e \right), \end{aligned} \tag{2.5}$$

we can relate the perturbed Riemann tensor to the unperturbed one by

$$\begin{aligned} R_{abc}{}^d \omega_d =& \nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c \\ =& {}^0\nabla_a {}^0\nabla_b \omega_c - {}^0\nabla_b {}^0\nabla_a \omega_c - {}^0\nabla_a C^d_{bc} \omega_d + {}^0\nabla_b C^d_{ac} \omega_d \\ & + C^d_{ac} C^e_{bd} \omega_e - C^d_{bc} C^e_{ad} \omega_e \\ =& {}^0R_{abc}{}^d \omega_d - {}^0\nabla_a C^d_{bc} \omega_d + {}^0\nabla_b C^d_{ac} \omega_d + C^e_{ac} C^d_{be} \omega_d - C^e_{bc} C^d_{ae} \omega_d. \end{aligned} \tag{2.6}$$

Contracting the second and fourth index we obtain the Ricci tensor

$$R_{ac} = {}^0R_{ac} - {}^0\nabla_a C^b_{bc} + {}^0\nabla_b C^b_{ac} + C^e_{ac} C^b_{be} - C^e_{bc} C^b_{ae}. \tag{2.7}$$

The change of the Ricci tensor is

$$\begin{aligned} \dot{R}_{ac} =& \left(\frac{\mathrm{d}R_{ac}}{\mathrm{d}\lambda} \right) \Big|_{\lambda=0} \\ =& - {}^0\nabla_a \dot{C}^b_{bc} \omega_d + {}^0\nabla_b \dot{C}^b_{ac}, \end{aligned} \tag{2.8}$$

where we have defined

$$\begin{aligned}\dot{C}^c{}_{ab} =& \left(\frac{\mathrm{d}R_{ac}}{\mathrm{d}\lambda}\right)\Big|_{\lambda=0} \\ =& \frac{1}{2}g^{cd}\left({}^0\nabla_a h_{bd} + {}^0\nabla_b h_{ad} - {}^0\nabla_d h_{ab}\right).\end{aligned} \tag{2.9}$$

Note that all other terms vanish since they contain ${}^0\nabla_a g_{bd} = 0$.

Writing things out we get

$$\begin{aligned}\dot{R}_{ac} =& -\frac{1}{2}{}^0\nabla_a g^{bd}\left({}^0\nabla_b h_{cd} + {}^0\nabla_c h_{bd} - {}^0\nabla_d h_{bc}\right) \\ &+ \frac{1}{2}{}^0\nabla_b g^{bd}\left({}^0\nabla_a h_{cd} + {}^0\nabla_c h_{ad} - {}^0\nabla_d h_{ac}\right) \\ =& -\frac{1}{2}{}^0\nabla_a {}^0\nabla_c h + \frac{1}{2}{}^0\nabla^d\left({}^0\nabla_a h_{cd} + {}^0\nabla_c h_{ad}\right) - \frac{1}{2}{}^0\nabla^d {}^0\nabla_d h_{ac}.\end{aligned} \tag{2.10}$$

The change of the Ricci scalar is

$$\begin{aligned}\dot{R} =& \left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)\Big|_{\lambda=0} \\ =& \left(\frac{\mathrm{d}\left(g^{ac}\left(\lambda\right)R_{ac}\right)}{\mathrm{d}\lambda}\right)\Big|_{\lambda=0} \\ =& -h^{ac}\,{}^0R_{ac} + g^{ac}\dot{R}_{ac} \\ =& -h^{ac}\,{}^0R_{ac} - \frac{1}{2}{}^0\nabla^a {}^0\nabla_a h + {}^0\nabla^d {}^0\nabla^a h_{ad} - \frac{1}{2}{}^0\nabla^d {}^0\nabla_d h.\end{aligned} \tag{2.11}$$

We are now able to calculate the change of the Einstein tensor

$$G_{ab} = R_{ab} - R\, g_{ab}\left(\lambda\right), \tag{2.12}$$

which reads

$$\begin{aligned}\dot{G}_{ac} =& \left(\frac{\mathrm{d}G_{ac}}{\mathrm{d}\lambda}\right)\Big|_{\lambda=0} \\ =& \dot{R}_{ac} - \dot{R}\, g_{ac} - {}^0R\, h_{ac}.\end{aligned} \tag{2.13}$$

The equations of motion are

$$\begin{aligned}\dot{G}_{ac} =& \frac{1}{2}{}^0\nabla^d\left({}^0\nabla_a h_{cd} + {}^0\nabla_c h_{ad}\right) - \frac{1}{2}{}^0\nabla^d {}^0\nabla_d h_{ac} - {}^0R\, h_{ac} \\ =& 0.\end{aligned} \tag{2.14}$$

Comparing this with Constable and Myers [6] we find a minor mistake in their derivation in which they do not include fluctuations of R. After choosing the transverse traceless gauge this discrepancy however disappears as we can see by calculating

$$
\begin{aligned}
\dot{R}_{ac} =& -\frac{1}{2}\,{}^{0}\nabla_{a}\,{}^{0}\nabla_{c}h+\frac{1}{2}\,{}^{0}\nabla^{d}\left({}^{0}\nabla_{a}h_{cd}+{}^{0}\nabla_{c}h_{ad}\right)-\frac{1}{2}\,{}^{0}\nabla^{d}\,{}^{0}\nabla_{d}h_{ac} \\
=&\frac{1}{2}\,{}^{0}\nabla^{d}\left({}^{0}\nabla_{a}h_{cd}+{}^{0}\nabla_{c}h_{ad}\right)-\frac{1}{2}\,{}^{0}\nabla^{d}\,{}^{0}\nabla_{d}h_{ac}
\end{aligned} \tag{2.15}
$$

and thus

$$
\begin{aligned}
\dot{R} =& -h^{ac}\,{}^{0}R_{ac}-\frac{1}{2}\,{}^{0}\nabla^{a}\,{}^{0}\nabla_{a}h+{}^{0}\nabla^{d}\,{}^{0}\nabla^{a}h_{ad}-\frac{1}{2}\,{}^{0}\nabla^{d}\,{}^{0}\nabla_{d}h \qquad (2.16)\\
=&0.
\end{aligned}
$$

2.1.2 Solution

Since h_{ab} is symmetric, transverse and traceless, there are $\frac{(4\cdot 5)}{2}-1-4=5$ independent tensor fluctuations

$$
h_{\mu\nu}=q_{\mu\nu}\frac{r^{2}}{L^{2}\mathcal{N}_{T}}T_{4}\left(r\right)G_{T}\left(x^{\sigma}\right), \tag{2.17}
$$

where $q_{\mu\nu}$ is a symmetric, transverse traceless tensor, which is normalized such that $q_{\mu\nu}q^{\mu\nu}=2$. $T_4\left(r\right)$ satisfies the equation

$$
\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{7}-r\,r_{KK}^{6}\right)\frac{\mathrm{d}}{\mathrm{d}r}T_{4}\left(r\right)+L^{4}M^{2}r^{3}T_{4}\left(r\right)=0. \tag{2.18}
$$

2.1.3 Normalization

To normalize our solutions we reduce the 11-dimensional supergravity action (1.1) to 4 dimensions [8, 9]. In this way we obtain a 4-dimensional effective action, for which we demand that it has a canonically normalized kinetic term. For the fluctuation $h_{\mu\nu}$ we obtain the reduced Lagrangian

$$
\begin{aligned}
\mathcal{L}_{4}\big|_{G_{T}^{2}} =&\left(\frac{L}{2}\right)^{4}\Omega_{4}\frac{1}{2\kappa_{11}^{2}}\int\mathrm{d}r\,\mathrm{d}x^{4}\mathrm{d}x^{11}\sqrt{-g_{7}}\left(R_{AdS_{7}}+\frac{30}{L^{2}}\right)\\
=&\mathcal{C}\int\mathrm{d}r\frac{r^{3}T_{4}\left(r\right)^{2}}{2L^{3}\mathcal{N}_{T}^{2}}G_{T}\left(M^{2}-\Box\right)G_{T}\\
=&\frac{1}{2}G_{T}\left(M^{2}-\Box\right)G_{T}.
\end{aligned} \tag{2.19}
$$

Note that this 4-dimensional Lagrangian is defined with respect to the standard Minkowski line-element. Since the prefactor in the Lagrangian appears for every mode we define the constant

$$
\begin{aligned}
\mathcal{C} &= \left(\frac{L}{2}\right)^4 \Omega_4 \frac{1}{2\kappa_{11}^2} (2\pi)^2 R_4 R_{11} \\
&= \frac{L^4}{16}\frac{8\pi^2}{3}\frac{1}{(2\pi)^8 l_s^9 g_s^3} (2\pi)^2 \frac{L^2}{3r_{KK}} l_s g_s \\
&= \frac{1}{72\,(2\pi)^4}\frac{L^6}{r_{KK}}\frac{l_s g_s}{l_s^9 g_s^3} \\
&= \frac{1}{72\,(2\pi)^4}\frac{L^6}{r_{KK}}\frac{\lambda}{2\pi N_C}\frac{L^2}{3r_{KK}}\frac{4^3\,(2\pi)^3\, N_C^3}{L^9} \\
&= \frac{8}{3^3\,(2\pi)^2}\frac{\lambda N_C^2}{L\, r_{KK}^2},
\end{aligned} \tag{2.20}
$$

where we have used $R_4 = \frac{1}{M_{KK}} = \frac{L^2}{3r_{KK}}$, $R_{11} = l_s g_s$, $g_s l_s = \frac{\lambda}{2\pi M_{KK} N_C}$, $g_s l_s^3 = \frac{L^3}{8\pi N_C}$, $\Omega_4 = \frac{8\pi^2}{3}$ and $\frac{1}{2\kappa_{11}^2} = \frac{1}{(2\pi)^8 l_s^9 g_s^3}$.

We furthermore define the constant

$$
\mathcal{C}_T = \int \mathrm{d}r \frac{r^3 T_4\,(r)^2}{L^3},
$$

which depends on the specific solution to (2.18). For the lowest mode we obtain

$$
\mathcal{C}_T = 0.22547\,[T_4\,(r_{KK})]^2\,\frac{r_{KK}^4}{L^3}. \tag{2.21}
$$

The method we used to obtain the numerical value will be explained in chapter 2.1.5. From equation (2.19) we obtain the normalization condition

$$
\begin{aligned}
1 =& \frac{\mathcal{C}\mathcal{C}_T}{\mathcal{N}_T^2} \\
=& \frac{1}{\mathcal{N}_T^2}\frac{8}{3^3\,(2\pi)^2}\frac{\lambda N_C^2}{L\, r_{KK}^2} 0.22547 \frac{r_{KK}^4}{L^3} \\
=& \frac{1}{\mathcal{N}_T^2}\frac{2}{3^5\pi^2} 0.22547 \lambda N_C^2 \frac{9 r_{KK}^2}{L^4} \\
=& \frac{1}{\mathcal{N}_T^2}\frac{2}{3^5\pi^2} 0.22547 \lambda N_C^2 M_{KK}^2,
\end{aligned} \tag{2.22}
$$

which is solved by

$$
\mathcal{N}_T = 0.0137122 \lambda^{\frac{1}{2}} N_C M_{KK}. \tag{2.23}
$$

Equivalently we could rescale the function $T_4\,(r)$

$$
\begin{aligned}
h_{\mu\nu} &= q_{\mu\nu} \frac{r^2}{L^2 \mathcal{N}_T^2} T_4\,(r)\, G_T\,(x^\sigma) \\
&= q_{\mu\nu} \frac{r^2}{L^2} \widetilde{T}_4\,(r)\, G_T\,(x^\sigma)\,,
\end{aligned} \tag{2.24}
$$

where the new function satisfies

$$\widetilde{T}_4\left(r_{KK}\right)=\mathcal{N}_T. \tag{2.25}$$

We will see that for some fluctuations we have to impose the boundary conditions $T_4\left(r_{KK}\right)=0$ and $\frac{\mathrm{d}}{\mathrm{d}r}T_4\left(r\right)\Big|_{x=r_{KK}}=1$, where T_4 is then replaced by the corresponding radial mode function. For these boundary conditions the first method is more straightforward.

2.1.4 Parity and charge conjugation assignments

Following Brower, Mathur and Tan [5] we will now assign quantum numbers to our metric fluctuations. To do so we look at the dual field theory, which is governed by the low-energy action of the D-brane and resembles pure glue QCD_4. This theory is described by the DBI action of a $D4$-brane and a 5-dimensional Chern-Simons term[1]

$$S_{D4}=-\mathrm{STr}\int \mathrm{d}^5x\,\sqrt{-\det\left[G_{ab}+e^{-\frac{\phi}{2}}\left(B_{ab}+F_{ab}\right)\right]}+\int \mathrm{d}^5x\,\left(C_1\wedge F\wedge F+C_3\wedge F+C_5\right). \tag{2.26}$$

The above Lagrangian involves differently scaled fields, e.g. B_{ab}, than we will use later on. However the quantum number assignments do not change under the necessary rescalings.

The parity transformation of the vector potential A_a is defined by

$$\begin{aligned} P:\ A_0\left(x^0,x^i,\tau\right)&\to A_0\left(x^0,-x^i,\tau\right),\\ P:\ A_i\left(x^0,x^i,\tau\right)&\to -A_i\left(x^0,-x^i,\tau\right),\\ P:\ A_\tau\left(x^0,x^i,\tau\right)&\to A_\tau\left(x^0,-x^i,\tau\right). \end{aligned} \tag{2.27}$$

Since our background geometry includes a compactified τ-direction it also makes sense to look at τ-parity. τ-parity transformations are defined by

$$\begin{aligned} P_\tau:\ A_0\left(x^0,x^i,\tau\right)&\to A_0\left(x^0,x^i,-\tau\right),\\ P_\tau:\ A_i\left(x^0,x^i,\tau\right)&\to A_i\left(x^0,x^i,-\tau\right),\\ P_\tau:\ A_\tau\left(x^0,x^i,\tau\right)&\to -A_\tau\left(x^0,x^i,-\tau\right). \end{aligned} \tag{2.28}$$

Since this quantum number has no analogue in 4-dimensional QCD, we will exclude fields with negative τ-parity from our physical spectrum. We can also define charge conjugation by the action

$$\begin{aligned} C:\ A_0\left(x^0,x^i,\tau\right)&\to -A_0^T\left(x^0,x^i,\tau\right),\\ C:\ A_i\left(x^0,x^i,\tau\right)&\to -A_i^T\left(x^0,x^i,\tau\right),\\ C:\ A_\tau\left(x^0,x^i,\tau\right)&\to -A_\tau^T\left(x^0,x^i,\tau\right), \end{aligned} \tag{2.29}$$

[1] The Lagrangian in [5] contains some typos.

where T denotes matrix transposition in the non-abelian case of more than one coinciding D4 branes. The Lagrangian should be invariant under the above transformations and thus we can conclude how the fields h_{ab} transform. To do so we look at the term

$$h_{ab}\mathrm{Tr}\left(F^{ac}F^{b}_{\ c}\right) = h_{ij}\mathrm{Tr}\left(F^{ic}F^{j}_{\ c}\right), \tag{2.30}$$

which is obtained by expanding (2.26) and where we have gone to the rest frame of $h_{\mu\nu}$, for which transversality implies $k^{\mu}h_{\mu\nu} = -m\,h_{0\nu} = 0$. Furthermore ∂_c and A_c have the same transformation behavior under parity transformations and thus we can read off the quantum numbers

$$h_{ij} \to 2^{++}\left(P_{\tau} = +\right). \tag{2.31}$$

2.1.5 Numerics

To solve the differential equation (2.18) we rewrite it by

$$\begin{aligned}
0 =& \frac{\mathrm{d}}{\mathrm{d}r}\left(r^7 - r\,r_{KK}^6\right)\frac{\mathrm{d}}{\mathrm{d}r}T_4\left(r\right) + L^4M^2r^3T_4\left(r\right)\\
=& r_{KK}^5\frac{\mathrm{d}}{\mathrm{d}\widetilde{r}}\left(\widetilde{r}^7 - \widetilde{r}\right)\frac{\mathrm{d}}{\mathrm{d}\widetilde{r}}T_4\left(\widetilde{r}\right) + r_{KK}^3L^4M^2\widetilde{r}^3T_4\left(\widetilde{r}\right)\\
=& r_{KK}^5\frac{\mathrm{d}}{\mathrm{d}\widetilde{r}}\left(\widetilde{r}^7 - \widetilde{r}\right)\frac{\mathrm{d}}{\mathrm{d}\widetilde{r}}T_4\left(\widetilde{r}\right) + 9r_{KK}^5M_{KK}^{-2}M^2\widetilde{r}^3T_4\left(\widetilde{r}\right)\\
=& \frac{\mathrm{d}}{\mathrm{d}\widetilde{r}}\left(\widetilde{r}^7 - \widetilde{r}\right)\frac{\mathrm{d}}{\mathrm{d}\widetilde{r}}T_4\left(\widetilde{r}\right) + \lambda\widetilde{r}^3T_4\left(\widetilde{r}\right),
\end{aligned} \tag{2.32}$$

where we have used $\widetilde{r} = \frac{r}{r_{KK}}$ and

$$\lambda = 9M_{KK}^{-2}M^2. \tag{2.33}$$

Next we map the coordinate $\widetilde{r}$ to a finite interval by

$$\widetilde{r} = \left(\cos x\right)^{-\frac{1}{3}}. \tag{2.34}$$

Using $\frac{\mathrm{d}}{\mathrm{d}\widetilde{r}}\arccos\left(\widetilde{r}\right) = \frac{-1}{\sqrt{1-\widetilde{r}^2}}$ we calculate

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}\widetilde{r}} &= \frac{\mathrm{d}}{\mathrm{d}\widetilde{r}}\arccos\left(\widetilde{r}^{-3}\right)\\
&= \frac{\mathrm{d}\left(\widetilde{r}^{-3}\right)}{\mathrm{d}\widetilde{r}}\frac{\mathrm{d}}{\mathrm{d}\left(\widetilde{r}^{-3}\right)}\arccos\left(\widetilde{r}^{-3}\right)\\
&= \frac{-3}{\widetilde{r}^4}\frac{-1}{\sqrt{1-\widetilde{r}^{-6}}}\\
&= \frac{3}{\widetilde{r}\sqrt{\widetilde{r}^6 - 1}}.
\end{aligned} \tag{2.35}$$

Furthermore we derive

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\widetilde{r}} T_4(\widetilde{r}) &= \frac{\mathrm{d}x}{\mathrm{d}\widetilde{r}} \frac{\mathrm{d}}{\mathrm{d}x} T_4(x) \\
&= \frac{3}{\widetilde{r}\sqrt{\widetilde{r}^6 - 1}} \frac{\mathrm{d}}{\mathrm{d}x} T_4(x) \\
&= \frac{3\cos^{\frac{4}{3}} x}{\sqrt{1-\cos^2 x}} \frac{\mathrm{d}}{\mathrm{d}x} T_4(x) \\
&= \frac{3\cos^{\frac{4}{3}} x}{\sin x} \frac{\mathrm{d}}{\mathrm{d}x} T_4(x) \\
&= 3\cot x\, \cos^{\frac{1}{3}} x \frac{\mathrm{d}}{\mathrm{d}x} T_4(x)
\end{aligned} \tag{2.36}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\widetilde{r}} \left(\widetilde{r}^7 - \widetilde{r}\right) \frac{\mathrm{d}}{\mathrm{d}\widetilde{r}} T_4(\widetilde{r}) &= \frac{\mathrm{d}x}{\mathrm{d}\widetilde{r}} \frac{\mathrm{d}}{\mathrm{d}x} \left(\widetilde{r}\left(\widetilde{r}^6 - 1\right) \frac{\mathrm{d}x}{\mathrm{d}\widetilde{r}} \frac{\mathrm{d}}{\mathrm{d}x} T_4(x) \right) \\
&= \frac{\mathrm{d}x}{\mathrm{d}\widetilde{r}} \frac{\mathrm{d}}{\mathrm{d}x} \left((\cos x)^{-\frac{1}{3}} \left((\cos x)^{-2} - 1\right) \frac{\mathrm{d}x}{\mathrm{d}\widetilde{r}} \frac{\mathrm{d}}{\mathrm{d}x} T_4(x) \right) \\
&= \frac{\mathrm{d}x}{\mathrm{d}\widetilde{r}} \frac{\mathrm{d}}{\mathrm{d}x} \left((\cos x)^{-\frac{7}{3}} \left(1 - \cos^2 x\right) \frac{\mathrm{d}x}{\mathrm{d}\widetilde{r}} \frac{\mathrm{d}}{\mathrm{d}x} T_4(x) \right) \\
&= \frac{\mathrm{d}x}{\mathrm{d}\widetilde{r}} \frac{\mathrm{d}}{\mathrm{d}x} \left((\cos x)^{-\frac{7}{3}} \sin^2 x \frac{\mathrm{d}x}{\mathrm{d}\widetilde{r}} \frac{\mathrm{d}}{\mathrm{d}x} T_4(x) \right) \\
&= \frac{\mathrm{d}x}{\mathrm{d}\widetilde{r}} \frac{\mathrm{d}}{\mathrm{d}x} \left(3(\cos x)^{-\frac{7}{3}} \sin^2 x\, \cot x\, \cos^{\frac{1}{3}} x \frac{\mathrm{d}}{\mathrm{d}x} T_4(x) \right) \\
&= \frac{\mathrm{d}x}{\mathrm{d}\widetilde{r}} \frac{\mathrm{d}}{\mathrm{d}x} \left(3\tan x \frac{\mathrm{d}}{\mathrm{d}x} T_4(x) \right) \\
&= \frac{\mathrm{d}x}{\mathrm{d}\widetilde{r}} \frac{\mathrm{d}}{\mathrm{d}x} \left(3(\cos x)^{-2} \frac{\mathrm{d}}{\mathrm{d}x} T_4(x) + 3\tan x \frac{\mathrm{d}^2}{\mathrm{d}x^2} T_4(x) \right) \\
&= 9\cot x\, (\cos x)^{-\frac{5}{3}} \frac{\mathrm{d}}{\mathrm{d}x} T_4(x) + 9\cos^{\frac{1}{3}} x \frac{\mathrm{d}^2}{\mathrm{d}x^2} T_4(x) ,
\end{aligned} \tag{2.37}
$$

to obtain the transformed differential equation

$$
0 = 9\cos^{\frac{1}{3}} x \frac{\mathrm{d}^2}{\mathrm{d}x^2} T_4(x) + 9\cot x\, (\cos x)^{-\frac{5}{3}} \frac{\mathrm{d}}{\mathrm{d}x} T_4(x) + \lambda (\cos x)^{-1} T_4(x) . \tag{2.38}
$$

To solve this differential equation, we make use of the so-called shooting method. In this method one chooses appropriate initial conditions with which one solves the differential equation in terms of the free parameter, which in our case is λ. The correct eigenvalues λ are then those for which the function satisfies a specified condition at the opposing boundary. First of all we note that our differential equation has singular points at 0 and $\frac{\pi}{2}$, therefore we introduce a cutoff ϵ. As initial conditions we choose $T_4(\epsilon) = 1$ and $\frac{\mathrm{d}}{\mathrm{d}x} T_4(x)\Big|_{x=\epsilon} = 0$. The first condition can be imposed without loss of generality, since we can still rescale the solution. The second condition can be understood by looking at a transformation $x \to -x$. This transformation maps any boundary point to its antipodal point, i.e. it maps τ to $\tau + \pi$. In our ansatz we assumed that the solution does not depend

T_4:	λ	$M=\sqrt{\frac{\lambda}{9}}M_{KK}\ [MeV]$
$n=1$	22.0966	1486.99
$n=2$	55.5833	2358.4
$n=3$	102.452	3201.88
$n=4$	162.699	4034.94
$n=5$	236.328	4862.99

Table 2. Mass spectrum of the tensor glueball $h_{\mu\nu}$, the vector glueball $h_{\mu,11}$ and the scalar glueball $h_{11,11}$.

on τ and thus it should also be symmetric under the transformation $x\to -x$. This however implies $\frac{\mathrm{d}}{\mathrm{d}x}T_4(x)\Big|_{x=0}=0$. To obtain the solution $T_4(x)$, with a free parameter λ, we use Mathematica and the function ParametricNDSolve. Evaluating the resulting parametric function at the boundary $x=\frac{\pi}{2}-\epsilon$ we can use the command FindRoot to find those values of λ which satisfy the condition $T_4\left(\frac{\pi}{2}-\epsilon\right)=0$. In figure 2 one can see how the boundary value $T_4\left(\frac{\pi}{2}-\epsilon\right)$ depends on λ. We obtain the spectrum in table 2, where we have used (2.33) to translate λ to the real mass M, using $M_{KK}=949$MeV as specified in chapter 5. In figure 3 we show the resulting eigenfunctions.

2.2 (P_τ-odd) axial vector mode $h_{\mu,11}/C_\mu$

In this and the following subsections we shortly present the results, obtained by a similar analysis as in chapter 2.1 applied to the other metric fluctuations. We also include P_τ-odd fluctuation modes that do not have an interpretation as glueballs in the dual Yang-Mills theory.

The fluctuation $h_{\mu,11}$ reads

$$h_{\mu,11}=q_\mu\frac{r^2}{\mathcal{N}_A L^2}T_4(r)\,G_A(x^\sigma)\,, \tag{2.39}$$

where $T_4(r)$, again, satisfies

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^7-r\,r_{KK}^6\right)\frac{\mathrm{d}}{\mathrm{d}r}T_4(r)+L^4M^2r^3T_4(r)=0, \tag{2.40}$$

with the mass spectrum in table 2. Here q_μ is a unit transverse polarization vector.

The resulting Lagrangian for a single polarization reads

$$\mathcal{L}_4|_{G_A^2}=\mathcal{C}\int \mathrm{d}r\frac{r^3T_4(r)^2}{2L^3\mathcal{N}_A^2}G_A\left(M^2-\Box\right)G_A. \tag{2.41}$$

For the lowest mass state we get

$$\begin{aligned}\mathcal{C}_A&=\int \mathrm{d}r\frac{r^3T_4(r)^2}{L^3}\\ &=0.22547\,[T_4(r_{KK})]^2\,\frac{r_{KK}^4}{L^3},\end{aligned} \tag{2.42}$$

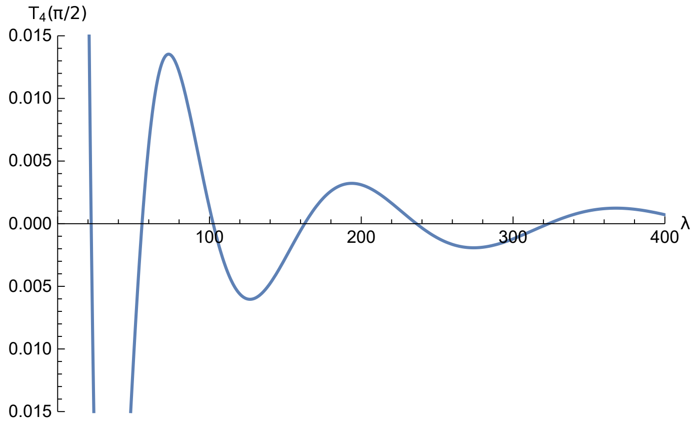

Figure 2. $T_4(x)|_{x=\frac{\pi}{2}-\epsilon}$ as a function of λ.

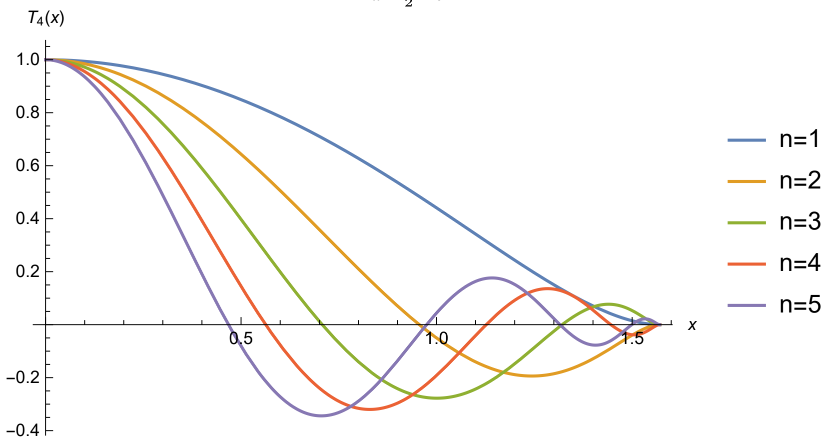

Figure 3. Solutions $T_4(x)$, where the labels n correspond to table 2.

and the normalization constant

$$\mathcal{N}_A = 0.0137122\lambda^{\frac{1}{2}} N_C M_{KK} = \mathcal{N}_T. \tag{2.43}$$

From the D4 brane action we obtain the associated quantum numbers

$$h_{ij} \to 1^{++} \, (P_\tau = -) \, . \tag{2.44}$$

The τ-parity quantum number is absent in real QCD, thus we exclude this state from our spectrum.

2.3 Scalar glueball $h_{11,11}$

The (predominantly dilatonic[2]) scalar glueball fluctuation reads

[2]Upon dimensional reduction to 10 dimensions the metric fluctuation $h_{11,11}$ will become the dilaton. See e.g. (3.49).

$$
\begin{aligned}
h_{\mu\nu} &= \frac{r^2}{\mathcal{N}_S\, L^2} T_4\left(r\right)\left(\eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box}\right) G_S\left(x^\sigma\right), \\
h_{11,11} &= -3\frac{r^2}{\mathcal{N}_S\, L^2} T_4\left(r\right) G_S\left(x^\sigma\right),
\end{aligned}
\tag{2.45}
$$

with

$$
\frac{\mathrm{d}}{\mathrm{d}r}\left(r^7 - r\, r_{KK}^6\right)\frac{\mathrm{d}}{\mathrm{d}r} T_4\left(r\right) + L^4 M^2 r^3 T_4\left(r\right) = 0 \tag{2.46}
$$

and the mass spectrum in table 2.

The fluctuation Lagrangian reads

$$
\mathcal{L}_4\big|_{G_S^2} = \mathcal{C}\int \mathrm{d}r \frac{3r^3 T_4\left(r\right)^2}{L^3 \mathcal{N}_S^2} G_S\left(M^2 - \Box\right) G_S, \tag{2.47}
$$

with

$$
\begin{aligned}
\mathcal{C}_S &= \int \mathrm{d}r \frac{r^3 T_4\left(r\right)^2}{L^3} \\
&= 0.22547\left[T_4\left(r_{KK}\right)\right]^2 \frac{r_{KK}^4}{L^3}
\end{aligned}
\tag{2.48}
$$

and

$$
\mathcal{N}_S = 0.0335879\lambda^{\frac{1}{2}} N_C M_{KK} = \sqrt{6}\mathcal{N}_T. \tag{2.49}
$$

The quantum numbers are

$$
h_{ij} \rightarrow 0^{++}\left(P_\tau = +\right). \tag{2.50}
$$

2.4 (P_τ-odd) vector mode $h_{\mu\tau}$

The corresponding fluctuation reads

$$
h_{\mu\tau} = q_\mu \frac{\sqrt{r^6 - r_{KK}^6}}{\mathcal{N}_V\, r\, L^2} V_4\left(r\right) G_V\left(x^\sigma\right), \tag{2.51}
$$

with

$$
\frac{\mathrm{d}}{\mathrm{d}r}\left(r^7 - r\, r_{KK}^6\right)\frac{\mathrm{d}}{\mathrm{d}r} V_4\left(r\right) + \left(L^4 M^2 r^3 - \frac{9 r_{KK}^{12}}{r^7 - r\, r_{KK}^6}\right) V_4\left(r\right) = 0 \tag{2.52}
$$

and the mass spectrum in table 3. Here q_μ is again a unit transverse polarization vector. To solve this differential equations we have to impose the boundary condition $V_4\left(\epsilon\right) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}x} V_4\left(x\right)\Big|_{x=\epsilon} = 1$.

The Lagrangian for a given polarization reads

V_4:	λ	$M=\sqrt{\frac{\lambda}{9}}M_{KK}\ [MeV]$
$n=1$	31.9853	1789.04
$n=2$	72.4792	2693.1
$n=3$	126.144	3552.87
$n=4$	193.133	4396.16
$n=5$	273.482	5231.3

Table 3. Mass spectrum of the pseudo-vector glueball $h_{\mu\tau}$ and the pseudo-scalar glueball $h_{\tau,11}$.

$$\begin{aligned}\mathcal{L}_4|_{G_V^2} &= \mathcal{C}\int \mathrm{d}r \frac{r^3 V_4(r)^2}{2L^3\mathcal{N}_V^2} G_V\left(M^2-\Box\right)G_V \\ &= \frac{1}{2}G_V\left(M^2-\Box\right)G_V,\end{aligned} \tag{2.53}$$

with

$$\begin{aligned}\mathcal{C}_V &= \int \mathrm{d}r \frac{r^3 V_4(r)^2}{L^3} \\ &= 0.0495616\left[V_4'(r_{KK})\right]^2 \frac{r_{KK}^4}{L^3}\end{aligned} \tag{2.54}$$

and

$$\mathcal{N}_V = 0.00642887\lambda^{\frac{1}{2}}N_C M_{KK}. \tag{2.55}$$

The quantum numbers are

$$h_{ij} \to 1^{-+}\left(P_\tau = -\right). \tag{2.56}$$

2.5 Pseudo-scalar glueball $h_{\tau,11}/C_\tau$

The pseudo-scalar glueball fluctuation reads

$$h_{\tau,11} = \frac{\sqrt{r^6 - r_{KK}^6}}{\mathcal{N}_{PS}L^2} V_4(r)\,G_{PS}(x^\sigma), \tag{2.57}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^7 - r\,r_{KK}^6\right)\frac{\mathrm{d}}{\mathrm{d}r}V_4(r) + \left(L^4M^2r^3 - \frac{9r_{KK}^{12}}{r^7 - r\,r_{KK}^6}\right)V_4(r) = 0 \tag{2.58}$$

and the mass spectrum in table 3. The Lagrangian reads

$$\begin{aligned}\mathcal{L}_4|_{G_{PS}^2} &= \mathcal{C}\int \mathrm{d}r \frac{r^3 V_4(r)^2}{2L^3\mathcal{N}_{PS}^2} G_{PS}\left(M^2-\Box\right)G_{PS} \\ &= \frac{1}{2}G_{PS}\left(M^2-\Box\right)G_{PS},\end{aligned} \tag{2.59}$$

with

$$\begin{aligned}\mathcal{C}_{PS} &= \int \mathrm{d}r \frac{r^3 V_4(r)^2}{L^3} \\ &= 0.0495616 \left[V_4(r_{KK})\right]^2 \frac{r_{KK}^4}{L^3}\end{aligned} \tag{2.60}$$

and

$$\mathcal{N}_{PS} = 0.00642887 \lambda^{\frac{1}{2}} N_C M_{KK}. \tag{2.61}$$

The quantum numbers are

$$h_{ij} \to 0^{-+} \left(P_\tau = +\right). \tag{2.62}$$

2.6 Exotic scalar glueball $h_{\tau\tau}$

The exotic scalar glueball fluctuation reads

$$\begin{aligned}h_{\tau\tau} &= -\frac{r^2}{\mathcal{N}_{ES} L^2} f(r) S_4(r) G_{ES}(x^\sigma), \\ h_{\mu\nu} &= \frac{r^2}{\mathcal{N}_{ES} L^2} S_4(r) \left[\frac{1}{4}\eta_{\mu\nu} - \left(\frac{1}{4} + \frac{3r_{KK}^6}{5r^6 - 2r_{KK}^6}\right)\frac{\partial_\mu \partial_\nu}{M^2}\right] G_{ES}(x^\sigma), \\ h_{11,11} &= \frac{r^2}{\mathcal{N}_{ES} 4L^2} f(r) S_4(r) G_{ES}(x^\sigma), \\ h_{rr} &= -\frac{L^2}{\mathcal{N}_{ES} r^2 f(r)} \frac{3r_{KK}^6}{5r^6 - 2r_{KK}^6} S_4(r) G_{ES}(x^\sigma), \\ h_{r\mu} = h_{\mu r} &= \frac{90 r^7 r_{KK}^6}{\mathcal{N}_{ES} M^2 L^2 \left(5r^6 - 2r_{KK}^6\right)} S_4(r) \partial_\mu G_{ES}(x^\sigma),\end{aligned} \tag{2.63}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^7 - r\, r_{KK}^6\right)\frac{\mathrm{d}}{\mathrm{d}r} S_4(r) + \left(L^4 M^2 r^3 + \frac{432 r^5 r_{KK}^{12}}{\left(5r^6 - 2r_0^6\right)^2}\right) S_4(r) = 0 \tag{2.64}$$

and the mass spectrum in table 4. Because this mode involves the metric component $h_{\tau\tau}$, which has no analogous in other holographic QCD models, it has been termed "exotic" in [5]. The Lagrangian reads

$$\begin{aligned}\mathcal{L}_4\big|_{G_{ES}^2} &= \mathcal{C} \int \mathrm{d}r \frac{r^3 S_4(r)^2}{2L^3 \mathcal{N}_{ES}^2} G_{ES}\left(M^2 - \Box\right) G_{ES} \\ &= \frac{1}{2} G_{ES}\left(M^2 - \Box\right) G_{ES},\end{aligned} \tag{2.65}$$

with

S_4:	λ	$M = \sqrt{\frac{\lambda}{9}} M_{KK}$ $[MeV]$
$n = 1$	7.30835	855.174
$n = 2$	46.9855	2168.34
$n = 3$	94.4816	3074.81
$n = 4$	154.963	3937.85
$n = 5$	228.709	4783.95

Table 4. Mass spectrum of the scalar glueball $h_{\tau\tau}$.

$$\begin{aligned} \mathcal{C}_{ES} &= \int \mathrm{d}r \frac{r^3 S_4(r)^2}{L^3} \\ &= 0.0918315 \left[S_4(r_{KK})\right]^2 \frac{r_{KK}^4}{L^3} \end{aligned} \tag{2.66}$$

and

$$\mathcal{N}_{ES} = 0.008751 \lambda^{\frac{1}{2}} N_C M_{KK}. \tag{2.67}$$

The quantum numbers are

$$h_{ij} \to 0^{++} (P_\tau = +). \tag{2.68}$$

The mass of this glueball mode is significantly lighter than the tensor glueball, which is in qualitative agreement with the situation in lattice gauge theory [12]. However quantitatively the mass is much too small, which together with the fact that the model has probably too many scalar glueball modes hints at the possibility that the exotic scalar mode has no counterpart in real QCD [8, 9].

It is interesting to check if there is a mixing term of the exotic scalar glueball and the dilatonic glueball since both have the same quantum number. A term proportional to $G_{ES} G_D$ would mean that we have not found normal modes yet and that the physical field is a superposition of G_{ES} and G_D. It turns out that such a term does not exist.

3 Field strength fluctuations

3.1 Equations of motion

To calculate the field strength fluctuations we first need to calculate the 7-dimensional equations of motion

$$
\begin{aligned}
\frac{\sqrt{-g}}{2\kappa_{11}^2}\left(R_{ab}-\frac{1}{2}R\,g_{ab}\right) &= \\
\frac{\sqrt{-g}}{\kappa_{11}^2 2\cdot 2\cdot 4!}\left(4F_a{}^{M_1M_2M_3}F_{bM_1M_2M_3}-\frac{1}{2}F^{M_1M_2M_3M_4}F_{M_1M_2M_3M_4}\,g_{ab}\right) & \\
+\frac{1}{2\cdot 6\cdot 3!\cdot (4!)^2}\epsilon^{M_1\ldots M_{11}}A_{M_1M_2M_3}F_{M_4\ldots M_7}F_{M_8\ldots M_{11}}g_{ab} &= \\
\frac{\sqrt{-g}}{\kappa_{11}^2 2\cdot 2\cdot 4!}\left(4F_a{}^{a_1a_2a_3}F_{ba_1a_2a_3}-\frac{1}{2}F^{a_1a_2a_3a_4}F_{a_1a_2a_3a_4}\,g_{ab}\right) & \\
+\frac{1}{2\cdot 6\cdot 3!\cdot (4!)^2}\epsilon^{a_1\ldots a_7\alpha_1\ldots\alpha_4}A_{a_1a_2a_3}F_{a_4\ldots a_7}F_{\alpha_1\ldots\alpha_4}g_{ab} &= 0, \\
\frac{1}{2\cdot 3!\kappa_{11}^2}\nabla_{M_1}\left(\sqrt{-g}F^{M_1abc}\right)+\frac{1}{2\kappa_{11}^2 2\cdot 3!\cdot(4!)^2}\epsilon^{abcM_4\ldots M_{11}}F_{M_4\ldots M_7}F_{M_8\ldots M_{11}} &= \\
\nabla_{a_1}\left(\sqrt{-g}F^{a_1abc}\right)+\frac{1}{(4!)^2}\epsilon^{abca_1\ldots a_4\alpha_1\ldots\alpha_4}F_{a_1\ldots a_4}F_{\alpha_1\ldots\alpha_4}+\nabla_{\alpha_1}\left(\sqrt{-g}F^{\alpha_1abc}\right) &= \\
\nabla_{a_1}\left(\sqrt{-g_7}F^{a_1abc}\right)+\frac{1}{4\cdot L}\epsilon^{abca_1\ldots a_4}F_{a_1\ldots a_4} &= 0. \quad (3.1)
\end{aligned}
$$

Note that in these particular expressions it is possible to replace the covariant derivative ∇_{a_1} with the partial derivative ∂_{a_1}.

3.2 Solutions for the pseudo-vector glueball mode

We find different kinds of solution depending on which components of A are non-vanishing, see e.g. [5]. For now we will only look at the pseudo-vector fluctuations with $J^{PC}=1^{+-}$. The reason for these quantum numbers is given in the end of this section. To find the solution corresponding to the pseudo-vector glueball we start with the ansatz

$$
\begin{aligned}
A_{\mu\nu,11} &= a(r)B_{\mu\nu}, \\
A_{\rho\tau r} &= \frac{1}{2}b(r)\epsilon^{\alpha\beta\gamma\delta}\eta_{\rho\alpha}\partial_\beta B_{\gamma\delta},
\end{aligned} \quad (3.2)
$$

where from now on we will use all Greek indices as Minkowski indices. The field strength is

$$
\begin{aligned}
F_{r\mu\nu 11} &= a'(r)B_{\mu\nu}, \\
F_{\rho\mu\nu 11} &= a(r)\left(\partial_\rho B_{\mu\nu}-\partial_\mu B_{\rho\nu}+\partial_\nu B_{\rho\mu}\right), \\
F_{\mu\rho\tau r} &= \frac{1}{2}b(r)\left(\epsilon^{\alpha\beta\gamma\delta}\eta_{\rho\alpha}\partial_\mu\partial_\beta B_{\gamma\delta}-\epsilon^{\alpha\beta\gamma\delta}\eta_{\mu\alpha}\partial_\rho\partial_\beta B_{\gamma\delta}\right).
\end{aligned} \quad (3.3)
$$

In the 7-dimensional setup indices are always raised and lowered with respect to g, whose components are

$$
\begin{aligned}
g_{\mu\nu} &= \frac{r^2}{L^2}\eta_{\mu\nu},\\
g_{\tau\tau} &= \frac{r^2 f(r)}{L^2},\\
g_{rr} &= \frac{L^2}{r^2 f(r)}.
\end{aligned}
\tag{3.4}
$$

We however define the operator $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$. Recall that $\epsilon^{\alpha\beta\gamma\delta}$ is an ϵ-symbol and does not involve the metric g.

Our equations of motion have three free indices which can take the values

$$
\{a,b,c\} = \{\tau, 11, r\},
\tag{3.5}
$$

$$
\{a,b,c\} = \{\mu, \tau, r\},
\tag{3.6}
$$

$$
\{a,b,c\} = \{\mu, \tau, 11\},
\tag{3.7}
$$

$$
\{a,b,c\} = \{\mu, 11, r\},
\tag{3.8}
$$

$$
\{a,b,c\} = \{\mu, \nu, 11\},
\tag{3.9}
$$

$$
\{a,b,c\} = \{\mu, \nu, r\},
\tag{3.10}
$$

$$
\{a,b,c\} = \{\mu, \nu, \tau\}.
\tag{3.11}
$$

To solve our ansatz we start with the equation of motion (3.1) for the indices (3.6), i.e. $\{a,b,c\} = \{\rho, \tau, r\}$:

$$
\begin{aligned}
\partial_{a_1}\left(\sqrt{-g_7}F^{a_1\rho\tau r}\right) + \frac{1}{4\cdot L}\epsilon^{\rho\tau r a_1\ldots a_4}F_{a_1\ldots a_4} &=\\
\partial_\mu\left(\frac{r^5}{L^5}F^{\mu\rho\tau r}\right) + \frac{1}{L}\epsilon^{\rho\tau r\alpha\beta\gamma 11}F_{\alpha\beta\gamma 11} &=\\
\partial_\mu\left(\frac{r^5}{L^5}g^{\mu\nu}g^{\rho\sigma}g^{rr}g^{\tau\tau}F_{\nu\sigma\tau r}\right) - \frac{1}{L}\epsilon^{\rho\alpha\beta\gamma\tau 11 r}F_{\alpha\beta\gamma 11} &=\\
\partial_\mu\left(\frac{r}{L}\eta^{\mu\nu}\eta^{\rho\sigma}F_{\nu\sigma\tau r}\right) - \frac{1}{L}\epsilon^{\rho\alpha\beta\gamma}F_{\alpha\beta\gamma 11} &=\\
\frac{r}{2L}b(r)\epsilon^{\rho\beta\gamma\delta}\Box\partial_\beta B_{\gamma\delta} - \frac{r}{2L}b(r)\epsilon^{\mu\beta\gamma\delta}\eta^{\rho\sigma}\partial_\mu\partial_\sigma\partial_\beta B_{\gamma\delta}&\\
-\frac{3}{L}\epsilon^{\rho\alpha\beta\gamma}a(r)\partial_\alpha B_{\beta\gamma} &=\\
\frac{r}{2L}b(r)\epsilon^{\rho\beta\gamma\delta}\Box\partial_\beta B_{\gamma\delta} - \frac{3}{L}\epsilon^{\rho\beta\gamma\delta}a(r)\partial_\beta B_{\gamma\delta} &= 0\\
r\, b(r)\Box B_{\gamma\delta} - 6a(r)B_{\gamma\delta} &= 0
\end{aligned}
\tag{3.12}
$$

From this we may determine $b(r)$ to be

$$b(r) = \frac{6}{r\,m^2}a(r), \tag{3.13}$$

where m is the mass of B. To determine the remaining ambiguity in our ansatz we look at (3.9), i.e. $\{a, b, c\} = \{\mu, \nu, 11\}$:

$$\begin{aligned}
\partial_{a_1}\left(\sqrt{-g_7}F^{a_1\mu\nu 11}\right) + \frac{1}{4\cdot L}\epsilon^{\mu\nu 11 a_1\ldots a_4}F_{a_1\ldots a_4} = \\
\partial_{a_1}\left(\sqrt{-g_7}g^{a_1b_1}g^{\mu\rho}g^{\nu\sigma}g^{11,11}F_{b_1\rho\sigma 11}\right) + \frac{3}{L}\epsilon^{\mu\nu 11\rho\sigma\tau r}F_{\rho\sigma\tau r} = \\
\partial_\alpha\left(\frac{L^3}{r^3}\eta^{\alpha\beta}\eta^{\mu\rho}\eta^{\nu\sigma}F_{\beta\rho\sigma 11}\right) + \partial_r\left(\frac{1}{r^5 L}\left(r^6 - r_{KK}^6\right)\eta^{\mu\rho}\eta^{\nu\sigma}F_{r\rho\sigma 11}\right) \\
-\frac{3}{L}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma\tau r} =
\end{aligned}$$

$$\eta^{\mu\rho}\eta^{\nu\sigma}\eta^{\alpha\beta}\partial_\alpha\left(\frac{L^3}{r^3}a(r)\left(\partial_\beta B_{\rho\sigma} - \partial_\rho B_{\beta\sigma} + \partial_\sigma B_{\beta\rho}\right)\right) \tag{3.14}$$

$$+\partial_r\left(\frac{1}{r^5 L}\left(r^6 - r_{KK}^6\right)\eta^{\mu\rho}\eta^{\nu\sigma}B_{\rho\sigma}a'(r)\right) \tag{3.15}$$

$$-\frac{3}{2L}b(r)\epsilon^{\mu\nu\rho\sigma}\left(\epsilon^{\alpha\beta\gamma\delta}\eta_{\alpha\sigma}\partial_\rho\partial_\beta B_{\gamma\delta} - \epsilon^{\alpha\beta\gamma\delta}\eta_{\alpha\rho}\partial_\sigma\partial_\beta B_{\gamma\delta}\right) = 0. \tag{3.16}$$

In the second line we keep the summation over ρ and thus obtain only a factor of 6 instead of 24.

As we will show now, this is solved by

$$a(r) = r^3 N_4(r). \tag{3.17}$$

The terms in the first line (3.14) simplify to

$$\eta^{\mu\rho}\eta^{\nu\sigma}\eta^{\alpha\beta}\partial_\alpha\left(L^3 N_4(r)\partial_\beta B_{\rho\sigma}\right) = m^2 L^3 N_4(r)\eta^{\mu\rho}\eta^{\nu\sigma}B_{\rho\sigma}. \tag{3.18}$$

Calculating the r-derivatives in line (3.15) we get:

$$
\begin{aligned}
&\partial_r\left(\frac{1}{Lr^5}\left(r^6-r_{KK}^6\right)a'(r)\right)=\\
&\partial_r\left(\frac{1}{Lr^5}\left(r^6-r_{KK}^6\right)\left(3r^2N_4(r)+r^3N_4'(r)\right)\right)=\\
&\partial_r\left(\frac{3}{Lr^3}\left(r^6-r_{KK}^6\right)N_4(r)\right)\\
&+\partial_r\left(\frac{1}{Lr^2}\left(r^6-r_{KK}^6\right)N_4'(r)\right)=\\
&-\frac{9}{Lr^4}\left(r^6-r_{KK}^6\right)N_4(r)+\frac{1}{L}18r^2N_4(r)+\frac{3}{Lr^3}\left(r^6-r_{KK}^6\right)N_4'(r)\\
&-\frac{2}{Lr^3}\left(r^6-r_{KK}^6\right)N_4'(r)+\frac{1}{L}6r^3N_4'(r)+\frac{1}{Lr^2}\left(r^6-r_{KK}^6\right)N_4''(r)=\\
&\frac{1}{Lr^2}\left(r^6-r_{KK}^6\right)N_4''(r)+\frac{1}{L}\left(3r^3-3\frac{r_{KK}^6}{r^3}-2r^3+2\frac{r_{KK}^6}{r^3}+6r^3\right)N_4'(r)\\
&+\frac{1}{L}\left(9\,r^2+\frac{9}{r^4}r_{KK}^6\right)N_4(r)=\\
&\frac{1}{Lr^2}\left(r^6-r_{KK}^6\right)N_4''(r)+\frac{1}{L}\left(7r^3-\frac{r_{KK}^6}{r^3}\right)N_4'(r)+\frac{1}{L}\left(9\,r^2+\frac{9}{r^4}r_{KK}^6\right)N_4(r).
\end{aligned}
\tag{3.19}
$$

Lastly for the terms in (3.16) we get

$$
\begin{aligned}
&-\frac{3}{2L}\epsilon^{\mu\nu\rho\sigma}b(r)\left(\epsilon^{\alpha\beta\gamma\delta}\eta_{\alpha\sigma}\partial_\rho\partial_\beta B_{\gamma\delta}-\epsilon^{\alpha\beta\gamma\delta}\eta_{\alpha\rho}\partial_\sigma\partial_\beta B_{\gamma\delta}\right)=\\
&-\frac{3}{L}\epsilon^{\mu\nu\rho\sigma}b(r)\epsilon^{\alpha\beta\gamma\delta}\eta_{\alpha\sigma}\partial_\rho\partial_\beta B_{\gamma\delta}=\\
&-\frac{3}{L}\delta^{\mu\nu\rho}_{\beta'\gamma'\delta'}b(r)\eta^{\beta\beta'}\eta^{\gamma\gamma'}\eta^{\delta\delta'}\partial_\rho\partial_{\beta'}B_{\gamma'\delta'}=\\
&-\frac{3}{L}\delta^{\rho}_{\beta'}\delta^{\mu\nu}_{\gamma'\delta'}b(r)\eta^{\beta\beta'}\eta^{\gamma\gamma'}\eta^{\delta\delta'}\partial_\rho\partial_{\beta'}B_{\gamma'\delta'}=\\
&-\frac{6}{L}b(r)\eta^{\beta\rho}\partial_\rho\partial_\beta B^{\mu\nu}=\\
&-\frac{36r^2}{L}N_4(r)B^{\mu\nu}.
\end{aligned}
\tag{3.20}
$$

Altogether we have

$$
\partial_r\left(r\left(r^6-r_{KK}^6\right)N_4'(r)\right)+\left(m^2L^4r^3-27r^5+\frac{9}{r}\right)N_4(r)=0,
\tag{3.21}
$$

which is exactly the mode equation for $N_4(r)$ in [5]. The corresponding eigenvalues are displayed in table 5. The mode functions are plotted in figure 4.

The solution is thus

$$
\begin{aligned}
A_{\mu\nu,11}&=r^3N_4(r)B_{\mu\nu},\\
A_{\rho\tau r}&=\frac{6r^2}{m^2}N_4(r)\epsilon^{\alpha\beta\gamma\delta}\eta_{\rho\alpha}\partial_\beta B_{\gamma\delta},
\end{aligned}
\tag{3.22}
$$

N_4:	λ	$M = \sqrt{\frac{\lambda}{9}} M_{KK}\ [MeV]$
$n=1$	53.3758	2311.09
$n=2$	109.446	3309.37
$n=3$	177.231	4211.29
$n=4$	257.959	5080.66
$n=5$	351.895	5934.05

Table 5. Mass spectrum of the pseudo-vector glueball $B_{\mu\nu}$.

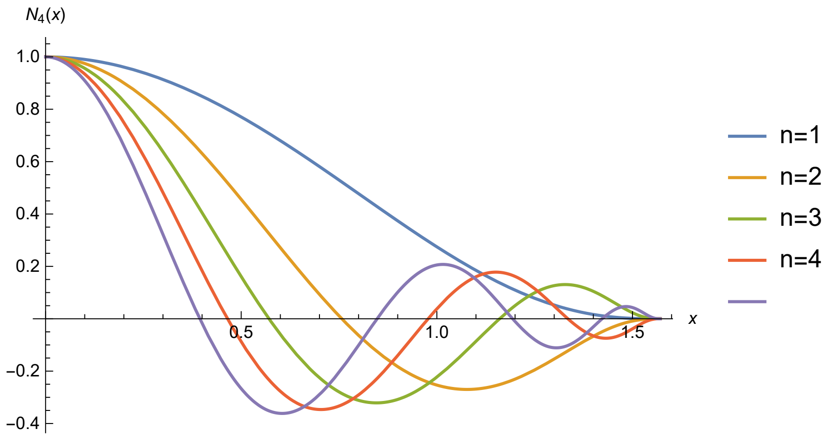

Figure 4. Solutions $N_4\,(x)$, with x defined in (2.34) and the labels n corresponding to table 5.

with field strength

$$\begin{aligned}
F_{r\mu\nu 11} &= \left(r^3 N_4(r) + 3r^2 N_4'(r)\right) B_{\mu\nu},\\
F_{\rho\mu\nu 11} &= r^3 N_4(r) \left(\partial_\rho B_{\mu\nu} - \partial_\mu B_{\rho\nu} + \partial_\nu B_{\rho\mu}\right),\\
F_{\mu\rho\tau r} &= \frac{3r^2}{m^2} N_4(r) \left(\epsilon^{\alpha\beta\gamma\delta}\eta_{\rho\alpha}\partial_\mu\partial_\beta B_{\gamma\delta} - \epsilon^{\alpha\beta\gamma\delta}\eta_{\mu\alpha}\partial_\rho\partial_\beta B_{\gamma\delta}\right).
\end{aligned} \tag{3.23}$$

A pure gauge mode of this excitation is

$$\begin{aligned}
A_{\mu\nu\tau} &= T(r) \left(q_\nu \partial_\mu G(x) - q_\mu \partial_\nu G(x)\right),\\
A_{\rho\tau r} &= T'(r) q_\rho G(x).
\end{aligned} \tag{3.24}$$

It corresponds to a gauge transformation

$$A \to A + \mathrm{d}\Lambda, \tag{3.25}$$

with

$$\Lambda_{\nu\tau} = q_\nu T(r) G(x). \tag{3.26}$$

We now have to check the remaining index combinations for consistency.
For (3.5), i.e. $\{a,b,c\} = \{\tau, 11, r\}$ we see that $F_{\mu\tau 11 r} = 0$ and $F_{\mu\nu\rho\sigma} = 0$, and thus the equations of motion are satisfied:

$$\begin{aligned}\partial_{a_1}\left(\sqrt{-g_7}F^{a_1\tau 11 r}\right) + \frac{1}{4\cdot L}\epsilon^{\tau 11 r a_1\ldots a_4}F_{a_1\ldots a_4} = \\ \partial_\mu\left(\sqrt{-g_7}F^{\mu\tau 11 r}\right) + \frac{1}{4\cdot L}\epsilon^{\tau 11 r\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma} = 0.\end{aligned} \tag{3.27}$$

For (3.7), i.e. $\{a,b,c\} = \{\mu,\tau,11\}$ we similarly use $F_{\nu\mu\tau 11} = 0$, $F_{r\mu\tau 11} = 0$ and $F_{r\nu\rho\sigma} = 0$:

$$\begin{aligned}\partial_{a_1}\left(\sqrt{-g_7}F^{a_1\mu\tau 11}\right) + \frac{1}{4\cdot L}\epsilon^{\mu\tau 11 a_1\ldots a_4}F_{a_1\ldots a_4} = \\ \partial_\nu\left(\sqrt{-g_7}F^{\nu\mu\tau 11}\right) + \partial_r\left(\sqrt{-g_7}F^{r\mu\tau 11}\right) + \frac{1}{L}\epsilon^{\mu\tau 11 r\nu\rho\sigma}F_{r\nu\rho\sigma} = 0.\end{aligned} \tag{3.28}$$

For (3.8), i.e. $\{a,b,c\} = \{\mu, 11, r\}$ we use $F_{\nu\rho\sigma\tau} = 0$ but we have to be careful since $F_{\nu\mu 11 r} \neq 0$. However we can use a gauge in which $B_{\mu\nu}$ is transverse, i.e. $\partial_\mu B^{\mu\nu} = 0$, exploiting (3.24):

$$\begin{aligned}\partial_{a_1}\left(\sqrt{-g_7}F^{a_1\mu 11 r}\right) + \frac{1}{4\cdot L}\epsilon^{\mu 11 r a_1\ldots a_4}F_{a_1\ldots a_4} = 0. \\ \partial_\nu\left(\sqrt{-g_7}F^{\nu\mu 11 r}\right) + \frac{3}{L}\epsilon^{\mu 11 r\nu\rho\sigma\tau}F_{\nu\rho\sigma\tau} = \\ \partial_\nu\left(\frac{r^5}{L^5}g^{rr}g^{11,11}g^{\nu\sigma}g^{\mu\rho}F_{\sigma\rho 11 r}\right) \\ \frac{r^5}{L^5}g^{rr}g^{11,11}g^{\nu\sigma}g^{\mu\rho}\left(r^3N_4(r) + 3r^2N_4'(r)\right)g^{\nu\sigma}\partial_\nu B_{\sigma\rho} = 0.\end{aligned} \tag{3.29}$$

For (3.10), i.e. $\{a,b,c\} = \{\mu,\nu,r\}$ we get $F_{\rho\mu\nu r} = 0$ and $F_{\rho\sigma\tau 11} = 0$:

$$\begin{aligned}\partial_{a_1}\left(\sqrt{-g_7}F^{a_1\mu\nu r}\right) + \frac{1}{4\cdot L}\epsilon^{\mu\nu r a_1\ldots a_4}F_{a_1\ldots a_4} = \\ \partial_\rho\left(\sqrt{-g_7}F^{\rho\mu\nu r}\right) + \frac{3}{L}\epsilon^{\mu\nu r\rho\sigma\tau 11}F_{\rho\sigma\tau 11} = 0.\end{aligned} \tag{3.30}$$

For (3.11), i.e. $\{a,b,c\} = \{\mu,\nu,\tau\}$ we have to calculate:

$$
\begin{aligned}
\partial_{a_1}\left(\sqrt{-g_7}F^{a_1\mu\nu\tau}\right) + \frac{1}{4\cdot L}\epsilon^{\mu\nu\tau a_1\dots a_4}F_{a_1\dots a_4} &= \\
\partial_r\left(\sqrt{-g_7}F^{r\mu\nu\tau}\right) + \frac{3}{L}\epsilon^{\mu\nu\tau\rho\sigma 11 r}F_{\rho\sigma 11 r} &= \\
\partial_r\left(\frac{r^5}{L^5}g^{rr}g^{\mu\rho}g^{\nu\sigma}g^{\tau\tau}F_{r\rho\sigma\tau}\right) + \frac{3}{L}\epsilon^{\mu\nu\rho\sigma\tau 11 r}F_{\rho\sigma 11 r} &= \\
-\partial_r\left(\frac{r}{L}\eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma\tau r}\right) - \frac{3}{L}\epsilon^{\mu\nu\rho\sigma}F_{r\rho\sigma 11} &= \\
-\partial_r\left(\frac{r}{L}\eta^{\mu\rho}\eta^{\nu\sigma}\epsilon^{\alpha\beta\gamma\delta}\left(\frac{3r^2}{m^2}N_4(r)\left(\eta_{\sigma\alpha}\partial_\rho\partial_\beta B_{\gamma\delta} - \eta_{\rho\alpha}\partial_\sigma\partial_\beta B_{\gamma\delta}\right)\right)\right) & \\
+\frac{3}{L}\epsilon^{\mu\nu\rho\sigma}\left(r^3N_4(r) + 3r^2N_4'(r)\right)B_{\rho\sigma} &= \\
-\partial_r\left(\frac{r}{L}\left(\frac{3r^2}{m^2}N_4(r)\left(\eta^{\mu\rho}\epsilon^{\nu\beta\gamma\delta}\partial_\rho\partial_\beta B_{\gamma\delta} - \eta^{\nu\sigma}\epsilon^{\mu\beta\gamma\delta}\partial_\sigma\partial_\beta B_{\gamma\delta}\right)\right)\right) & \\
+\frac{3}{L}\epsilon^{\mu\nu\rho\sigma}\left(r^3N_4(r) + 3r^2N_4'(r)\right)B_{\rho\sigma}. & \qquad (3.31)
\end{aligned}
$$

After contracting this term with $\epsilon_{\mu'\nu'\mu\nu}$ and using the generalized Kronecker delta $\delta^{\mu_1\dots\mu_s}_{\nu_1\dots\nu_s} = \frac{-1}{(n-s)!}\epsilon^{\mu_1\dots\mu_s\rho_{s+1}\dots\rho_n}\epsilon_{\nu_1\dots\nu_s\rho_{s+1}\dots\rho_n}$, we see that it is zero:

$$
\begin{aligned}
-\epsilon_{\mu'\nu'\mu\nu}\partial_r\left(\frac{r}{L}\left(\frac{3r^2}{m^2}N_4(r)\left(\eta^{\mu\rho}\epsilon^{\nu\beta\gamma\delta}\partial_\rho\partial_\beta B_{\gamma\delta} - \eta^{\nu\sigma}\epsilon^{\mu\beta\gamma\delta}\partial_\sigma\partial_\beta B_{\gamma\delta}\right)\right)\right) & \\
+\epsilon_{\mu'\nu'\mu\nu}\frac{3}{L}\epsilon^{\mu\nu\rho\sigma}\left(r^3N_4(r) + 3r^2N_4'(r)\right)B_{\rho\sigma} &= \\
-\partial_r\left(\frac{r}{L}\left(\frac{6r^2}{m^2}N_4(r)\eta^{\mu\rho}\delta^{\beta\gamma\delta}_{\mu'\nu'\mu}\partial_\rho\partial_\beta B_{\gamma\delta}\right)\right) & \\
+\frac{6}{L}\delta^{\rho\sigma}_{\mu'\nu'}\left(r^3N_4(r) + 3r^2N_4'(r)\right)B_{\rho\sigma} &= \\
-\partial_r\left(\frac{r}{L}\left(\frac{6r^2}{m^2}N_4(r)\eta^{\mu\rho}\delta^{\gamma\delta}_{\mu'\nu'}\partial_\rho\partial_\mu B_{\gamma\delta}\right)\right) & \\
+\frac{6}{L}\delta^{\rho\sigma}_{\mu'\nu'}\left(r^3N_4(r) + 3r^2N_4'(r)\right)B_{\rho\sigma} &= \\
-\frac{12}{L}\partial_r\left(r^3N_4(r)\right)B_{\mu'\nu'} & \\
+\frac{12}{L}\left(r^3N_4(r) + 3r^2N_4'(r)\right)B_{\mu'\nu'} &= 0. \qquad (3.32)
\end{aligned}
$$

We conclude that the solution (3.22) solves all equations of motion.

3.3 Normalization

To normalize our fluctuation we plug it back into the Lagrangian

$$
\begin{aligned}
2\kappa_{11}^2\mathcal{L}_{11}^{(b)} =& \sum \mathcal{L}_{EH}^{(i)} + \sqrt{-g}\frac{30}{L^2} - \frac{\sqrt{-g}}{2\cdot 4!}F^{a_1\ldots a_4}F_{a_1\ldots a_4} \\
&+ \frac{\sqrt{g_{S^4}}}{2\cdot 4! L}\epsilon^{a_1\ldots a_7}A_{a_1\ldots a_3}F_{a_4\ldots a_7} \\
=& \sum \mathcal{L}_{EH}^{(i)} + \sqrt{-g}\frac{30}{L^2} \\
&- \frac{\sqrt{-g}}{4}F^{r\mu\nu 11}F_{r\mu\nu 11} \qquad (3.33)\\
&- \frac{\sqrt{-g}}{12}F^{\rho\mu\nu 11}F_{\rho\mu\nu 11} \qquad (3.34)\\
&- \frac{\sqrt{-g}}{4}F^{\mu\rho\tau r}F_{\mu\rho\tau r} \qquad (3.35)\\
&+ \frac{\sqrt{g_{S^4}}}{2L}\epsilon^{\rho\tau r\sigma\mu\nu 11}A_{\rho\tau r}F_{\sigma\mu\nu 11} \qquad (3.36)\\
&+ 3\frac{\sqrt{g_{S^4}}}{4L}\epsilon^{\mu\nu,11\sigma\rho\tau r}A_{\mu\nu,11}F_{\sigma\rho\tau r}. \qquad (3.37)
\end{aligned}
$$

We will calculate each line individually. The first line (3.33) yields

$$
\begin{aligned}
-\frac{\sqrt{-g}}{4}F^{r\mu\nu 11}F_{r\mu\nu 11} =& \\
-\frac{r^5\sqrt{g_{S^4}}}{L^5 4}g^{rr}g^{\rho\mu}g^{\sigma\nu}g^{11,11}F_{r\rho\sigma 11}F_{r\mu\nu 11} =& \\
-\frac{r^5\sqrt{g_{S^4}}}{L^5 4}\frac{L^4\left(r^6-r_{KK}^6\right)}{r^{10}}r^4\left(rN_4(r)+3N_4'(r)\right)^2\eta^{\rho\mu}\eta^{\sigma\nu}B_{\mu\nu}B_{\rho\sigma} =& \\
-\sqrt{g_{S^4}}\frac{r^6-r_{KK}^6}{4Lr}\left(rN_4(r)+3N_4'(r)\right)^2\eta^{\rho\mu}\eta^{\sigma\nu}B_{\mu\nu}B_{\rho\sigma} =& \\
-\sqrt{g_{S^4}}\frac{1}{4}\left(L^3m^2r^3 - \frac{36r^5}{L}\right)N_4(r)^2\eta^{\rho\mu}\eta^{\sigma\nu}B_{\mu\nu}B_{\rho\sigma}.& \qquad (3.38)
\end{aligned}
$$

The second on (3.34) simplifies to

$$
\begin{aligned}
-\frac{\sqrt{-g}}{12}F^{\rho\mu\nu 11}F_{\rho\mu\nu 11} =& \\
-\frac{r^5\sqrt{g_{S^4}}}{L^5 12}g^{\alpha\rho}g^{\beta\mu}g^{\gamma\nu}g^{11,11}F_{\alpha\beta\gamma 11}F_{\rho\mu\nu 11} =& \\
-\frac{r^5\sqrt{g_{S^4}}}{L^5 4}\frac{L^8}{r^8}r^6N_4(r)^2\eta^{\alpha\rho}\eta^{\beta\mu}\eta^{\gamma\nu}\partial_\alpha B_{\beta\gamma}\partial_\rho B_{\mu\nu} =& \\
-\sqrt{g_{S^4}}\frac{L^3r^3}{4}N_4(r)^2\eta^{\alpha\rho}\eta^{\beta\mu}\eta^{\gamma\nu}\partial_\alpha B_{\beta\gamma}\partial_\rho B_{\mu\nu}.& \qquad (3.39)
\end{aligned}
$$

Next we calculate the third line (3.35)

$$
\begin{aligned}
&-\frac{\sqrt{-g}}{4}F^{\mu\rho\tau r}F_{\mu\rho\tau r}= \\
&-\frac{r^5\sqrt{g_{S^4}}}{L^5 4}g^{\mu\nu}g^{\rho\sigma}\frac{3r^2}{m^2}N_4(r)\epsilon^{\alpha\beta\gamma\delta}\left(\eta_{\sigma\alpha}\partial_\nu\partial_\beta B_{\gamma\delta}-\eta_{\nu\alpha}\partial_\sigma\partial_\beta B_{\gamma\delta}\right) \\
&\cdot\frac{3r^2}{m^2}N_4(r)\epsilon^{\alpha'\beta'\gamma'\delta'}\left(\eta_{\rho\alpha'}\partial_\mu\partial_{\beta'}B_{\gamma'\delta'}-\eta_{\mu\alpha'}\partial_\rho\partial_{\beta'}B_{\gamma'\delta'}\right)= \\
&-\frac{9r^5\sqrt{g_{S^4}}}{4m^4L}N_4(r)^2\eta^{\mu\nu}\eta^{\rho\sigma}\epsilon^{\alpha\beta\gamma\delta}\left(\eta_{\sigma\alpha}\partial_\nu\partial_\beta B_{\gamma\delta}-\eta_{\nu\alpha}\partial_\sigma\partial_\beta B_{\gamma\delta}\right) \\
&\cdot\epsilon^{\alpha'\beta'\gamma'\delta'}\left(\eta_{\rho\alpha'}\partial_\mu\partial_{\beta'}B_{\gamma'\delta'}-\eta_{\mu\alpha'}\partial_\rho\partial_{\beta'}B_{\gamma'\delta'}\right)= \\
&-\frac{9r^5\sqrt{g_{S^4}}}{4m^4L}N_4(r)^2\eta^{\mu\nu}\eta^{\rho\sigma}\eta^{\alpha\alpha''}\eta^{\beta\beta''}\eta^{\gamma\gamma''}\eta^{\delta\delta''}\epsilon_{\alpha''\beta''\gamma''\delta''}\left(\eta_{\sigma\alpha}\partial_\nu\partial_\beta B_{\gamma\delta}-\eta_{\nu\alpha}\partial_\sigma\partial_\beta B_{\gamma\delta}\right) \\
&\cdot\epsilon^{\alpha'\beta'\gamma'\delta'}\left(\eta_{\rho\alpha'}\partial_\mu\partial_{\beta'}B_{\gamma'\delta'}-\eta_{\mu\alpha'}\partial_\rho\partial_{\beta'}B_{\gamma'\delta'}\right)= \\
&-\frac{9r^5\sqrt{g_{S^4}}}{4m^4L}N_4(r)^2\eta^{\mu\nu}\eta^{\rho\sigma}\eta^{\alpha\alpha''}\eta^{\beta\beta''}\eta^{\gamma\gamma''}\eta^{\delta\delta''}\epsilon_{\alpha''\beta''\gamma''\delta''}\epsilon^{\alpha'\beta'\gamma'\delta'}\eta_{\sigma\alpha}\partial_\nu\partial_\beta B_{\gamma\delta}\eta_{\rho\alpha'}\partial_\mu\partial_{\beta'}B_{\gamma'\delta'} \\
&\frac{9r^5\sqrt{g_{S^4}}}{4m^4L}N_4(r)^2\eta^{\mu\nu}\eta^{\rho\sigma}\eta^{\alpha\alpha''}\eta^{\beta\beta''}\eta^{\gamma\gamma''}\eta^{\delta\delta''}\epsilon_{\alpha''\beta''\gamma''\delta''}\epsilon^{\alpha'\beta'\gamma'\delta'}\eta_{\sigma\alpha}\partial_\nu\partial_\beta B_{\gamma\delta}\eta_{\mu\alpha'}\partial_\rho\partial_{\beta'}B_{\gamma'\delta'} \\
&\frac{9r^5\sqrt{g_{S^4}}}{4m^4L}N_4(r)^2\eta^{\mu\nu}\eta^{\rho\sigma}\eta^{\alpha\alpha''}\eta^{\beta\beta''}\eta^{\gamma\gamma''}\eta^{\delta\delta''}\epsilon_{\alpha''\beta''\gamma''\delta''}\epsilon^{\alpha'\beta'\gamma'\delta'}\eta_{\nu\alpha}\partial_\sigma\partial_\beta B_{\gamma\delta}\eta_{\rho\alpha'}\partial_\mu\partial_{\beta'}B_{\gamma'\delta'} \\
&-\frac{9r^5\sqrt{g_{S^4}}}{4m^4L}N_4(r)^2\eta^{\mu\nu}\eta^{\rho\sigma}\eta^{\alpha\alpha''}\eta^{\beta\beta''}\eta^{\gamma\gamma''}\eta^{\delta\delta''}\epsilon_{\alpha''\beta''\gamma''\delta''}\epsilon^{\alpha'\beta'\gamma'\delta'}\eta_{\nu\alpha}\partial_\sigma\partial_\beta B_{\gamma\delta}\eta_{\mu\alpha'}\partial_\rho\partial_{\beta'}B_{\gamma'\delta'}= \\
&-\frac{9r^5\sqrt{g_{S^4}}}{4m^4L}N_4(r)^2\eta^{\mu\nu}\eta^{\beta\beta''}\eta^{\gamma\gamma''}\eta^{\delta\delta''}\epsilon_{\alpha'\beta''\gamma''\delta''}\epsilon^{\alpha'\beta'\gamma'\delta'}\partial_\nu\partial_\beta B_{\gamma\delta}\partial_\mu\partial_{\beta'}B_{\gamma'\delta'} \\
&+\frac{9r^5\sqrt{g_{S^4}}}{4m^4L}N_4(r)^2\delta^\nu_{\alpha'}\eta^{\rho\alpha''}\eta^{\beta\beta''}\eta^{\gamma\gamma''}\eta^{\delta\delta''}\epsilon_{\alpha''\beta''\gamma''\delta''}\epsilon^{\alpha'\beta'\gamma'\delta'}\partial_\nu\partial_\beta B_{\gamma\delta}\partial_\rho\partial_{\beta'}B_{\gamma'\delta'} \\
&+\frac{9r^5\sqrt{g_{S^4}}}{4m^4L}N_4(r)\delta^\sigma_{\alpha'}\eta^{\mu\alpha''}\eta^{\beta\beta''}\eta^{\gamma\gamma''}\eta^{\delta\delta''}\epsilon_{\alpha''\beta''\gamma''\delta''}\epsilon^{\alpha'\beta'\gamma'\delta'}\partial_\sigma\partial_\beta B_{\gamma\delta}\partial_\mu\partial_{\beta'}B_{\gamma'\delta'} \\
&-\frac{9r^5\sqrt{g_{S^4}}}{4m^4L}N_4(r)^2\eta^{\rho\sigma}\eta^{\beta\beta''}\eta^{\gamma\gamma''}\eta^{\delta\delta''}\epsilon_{\alpha'\beta''\gamma''\delta''}\epsilon^{\alpha'\beta'\gamma'\delta'}\partial_\sigma\partial_\beta B_{\gamma\delta}\partial_\rho\partial_{\beta'}B_{\gamma'\delta'}= \\
&-\frac{18r^5\sqrt{g_{S^4}}}{4m^4L}N_4(r)^2\eta^{\mu\nu}\eta^{\beta\beta''}\eta^{\gamma\gamma''}\eta^{\delta\delta''}\epsilon_{\alpha'\beta''\gamma''\delta''}\epsilon^{\alpha'\beta'\gamma'\delta'}\partial_\nu\partial_\beta B_{\gamma\delta}\partial_\mu\partial_{\beta'}B_{\gamma'\delta'} \\
&+\frac{9r^5\sqrt{g_{S^4}}}{2m^4L}N_4(r)^2\eta^{\rho\alpha''}\eta^{\beta\beta''}\eta^{\gamma\gamma''}\eta^{\delta\delta''}\epsilon_{\alpha''\beta''\gamma''\delta''}\epsilon^{\nu\beta'\gamma'\delta'}\partial_\nu\partial_\beta B_{\gamma\delta}\partial_\rho\partial_{\beta'}B_{\gamma'\delta'}= \\
&\frac{9r^5\sqrt{g_{S^4}}}{2m^4L}N_4(r)^2\eta^{\mu\nu}\eta^{\beta\beta''}\eta^{\gamma\gamma''}\eta^{\delta\delta''}\delta^{\beta'\gamma'\delta'}_{\beta''\gamma''\delta''}\partial_\nu\partial_\beta B_{\gamma\delta}\partial_\mu\partial_{\beta'}B_{\gamma'\delta'}= \\
&\frac{9r^5\sqrt{g_{S^4}}}{2m^4L}N_4(r)^2\eta^{\mu\nu}\eta^{\beta\beta'}\eta^{\gamma\gamma''}\eta^{\delta\delta''}\delta^{\gamma'\delta'}_{\gamma''\delta''}\partial_\nu\partial_\beta B_{\gamma\delta}\partial_\mu\partial_{\beta'}B_{\gamma'\delta'}= \\
&\frac{9r^5\sqrt{g_{S^4}}}{m^4L}N_4(r)^2\eta^{\mu\nu}\eta^{\beta\beta'}\eta^{\gamma\gamma'}\eta^{\delta\delta'}\partial_\nu\partial_\beta B_{\gamma\delta}\partial_\mu\partial_{\beta'}B_{\gamma'\delta'}= \\
&\frac{9r^5\sqrt{g_{S^4}}}{L}N_4(r)^2\eta^{\gamma\gamma'}\eta^{\delta\delta'}B_{\gamma\delta}B_{\gamma'\delta'}.
\end{aligned}
\tag{3.40}
$$

The fourth line (3.36) reads

$$
\begin{aligned}
&\frac{\sqrt{g_{S^4}}}{2L}\epsilon^{\rho\tau r\sigma\mu\nu 11}A_{\rho\tau r}F_{\sigma\mu\nu 11} = \\
&-\frac{\sqrt{g_{S^4}}}{L}\frac{3r^2}{2m^2}N_4(r)\epsilon^{\rho\sigma\mu\nu}\epsilon^{\alpha\beta\gamma\delta}\eta_{\rho\alpha}\partial_\beta B_{\gamma\delta}r^3N_4(r)\left(\partial_\sigma B_{\mu\nu}-\partial_\mu B_{\sigma\nu}+\partial_\nu B_{\sigma\mu}\right) = \\
&\frac{\sqrt{g_{S^4}}}{L}\frac{9r^5}{2m^2}N_4(r)^2\eta^{\beta\beta'}\eta^{\gamma\gamma'}\eta^{\delta\delta'}\delta^{\sigma\mu\nu}_{\beta'\gamma'\delta'}\partial_\beta B_{\gamma\delta}\left(\partial_\sigma B_{\mu\nu}-\partial_\mu B_{\sigma\nu}+\partial_\nu B_{\sigma\mu}\right) = \\
&\frac{\sqrt{g_{S^4}}}{L}\frac{9r^5}{m^2}N_4(r)^2\eta^{\beta\beta'}\eta^{\gamma\gamma'}\eta^{\delta\delta'}\partial_\beta B_{\gamma\delta}\partial_\beta B_{\gamma'\delta'} = \\
&-\frac{\sqrt{g_{S^4}}}{L}9r^5N_4(r)^2\eta^{\gamma\gamma'}\eta^{\delta\delta'}B_{\gamma\delta}B_{\gamma'\delta'}.
\end{aligned}
\tag{3.41}
$$

And finally we calculate the last line (3.37)

$$
\begin{aligned}
&3\frac{\sqrt{g_{S^4}}}{4L}\epsilon^{\mu\nu,11\sigma\rho\tau r}A_{\mu\nu,11}F_{\sigma\rho\tau r} = \\
&-3\frac{\sqrt{g_{S^4}}}{4L}\epsilon^{\mu\nu\sigma\rho}r^3N_4(r)B_{\mu\nu}\frac{3r^2}{m^2}N_4(r)\left(\epsilon^{\alpha\beta\gamma\delta}\eta_{\rho\alpha}\partial_\sigma\partial_\beta B_{\gamma\delta}-\epsilon^{\alpha\beta\gamma\delta}\eta_{\sigma\alpha}\partial_\rho\partial_\beta B_{\gamma\delta}\right) = \\
&-\frac{\sqrt{g_{S^4}}}{L}\frac{9r^5}{2m^2}N_4(r)^2\epsilon^{\mu\nu\sigma\rho}r^3B_{\mu\nu}\epsilon^{\alpha\beta\gamma\delta}\eta_{\rho\alpha}\partial_\sigma\partial_\beta B_{\gamma\delta} = \\
&\frac{\sqrt{g_{S^4}}}{L}\frac{9r^5}{m^2}N_4(r)^2\eta^{\beta\beta'}\eta^{\gamma\gamma'}\eta^{\delta\delta'}\delta^{\mu\nu\sigma}_{\beta'\gamma'\delta'}B_{\mu\nu}\partial_\sigma\partial_\beta B_{\gamma\delta} = \\
&-\frac{\sqrt{g_{S^4}}}{L}9r^5N_4(r)^2\eta^{\gamma\gamma'}\eta^{\delta\delta'}B_{\gamma'\delta'}B_{\gamma\delta}.
\end{aligned}
\tag{3.42}
$$

Altogether we get the simple term

$$
\begin{aligned}
&-\sqrt{g_{S^4}}\frac{1}{4}L^3m^2r^3N_4(r)^2\eta^{\rho\mu}\eta^{\sigma\nu}B_{\mu\nu}B_{\rho\sigma}-\sqrt{g_{S^4}}\frac{L^3r^3}{4}N_4(r)^2\eta^{\alpha\rho}\eta^{\beta\mu}\eta^{\gamma\nu}\partial_\alpha B_{\beta\gamma}\partial_\rho B_{\mu\nu} = \\
&-\sqrt{g_{S^4}}\frac{1}{4}L^3r^3N_4(r)^2\eta^{\rho\mu}\eta^{\sigma\nu}B_{\mu\nu}\left(m^2-\Box\right)B_{\rho\sigma}.
\end{aligned}
\tag{3.43}
$$

Integrating over the sphere with radius $\frac{L}{2}$ yields

$$
\begin{aligned}
\mathcal{L}_4 &= -\int \mathrm{d}r\mathrm{d}\tau\mathrm{d}x_{11}\frac{1}{2\kappa_{11}^2}\frac{8\pi^2}{3}\frac{L^4}{16}\frac{L^3}{4}r^3N_4(r)^2\eta^{\rho\mu}\eta^{\sigma\nu}B_{\mu\nu}\left(m^2-\Box\right)B_{\rho\sigma} \\
&= -\int \mathrm{d}r\, R_{11}R_4\frac{1}{2\kappa_{11}}\frac{\pi^4}{3}\frac{L^7}{2}r^3N_4(r)^2\eta^{\rho\mu}\eta^{\sigma\nu}B_{\mu\nu}\left(m^2-\Box\right)B_{\rho\sigma} \\
&= -\frac{1}{4}\mathcal{C}_B\eta^{\rho\mu}\eta^{\sigma\nu}B_{\mu\nu}\left(m^2-\Box\right)B_{\rho\sigma}
\end{aligned}
\tag{3.44}
$$

Finally we calculate the resulting normalization

$$\begin{aligned}
\mathcal{C}_B &= R_{11}R_4\frac{2}{2\kappa_{11}^2}L^7\frac{\pi^4}{3}\int \mathrm{d}r\, r^3 N_4(r)^2 \\
N_4(r_{KK})^{-2} &= 0.116079\, R_{11}R_4\frac{2}{2\kappa_{11}^2}L^7\frac{\pi^4}{3}r_{KK}^4 \\
&= 0.116079\frac{2}{\pi^2 3^5}L^6\lambda N_C^2 M_{KK}^2 \\
&= 0.0000967936\, L^6\lambda N_C^2 M_{KK}^2 \\
N_4(r_{KK})^{-1} &= 0.00983838\, L^3\lambda^{\frac{1}{2}}N_C M_{KK}.
\end{aligned} \tag{3.45}$$

We normalize the kinetic term in B to $\frac{1}{4}$ such that each polarization mode in

$$B_{\mu\nu} = \sum_\lambda \frac{1}{\sqrt{\Box}}\epsilon_{\mu\nu\sigma\rho}q^\sigma_{(\lambda)}\partial^\rho G_{(\lambda)}(x) \tag{3.46}$$

has a canonical normalization. This ansatz automatically solves the transversality condition. The sum runs over all 3 physical polarizations with unit polarization vectors $q^\sigma_{(\lambda)}$. Suppressing the polarization label we explicitly calculate

$$\begin{aligned}
B^{\mu\nu}\left(m^2-\Box\right)B_{\mu\nu} &= \\
\frac{1}{\sqrt{\Box}}\epsilon^{\mu\nu\sigma\rho}q_\sigma\partial_\rho G(x)\left(m^2-\Box\right)\frac{1}{\sqrt{\Box}}\epsilon_{\mu\nu\sigma'\rho'}q^{\sigma'}\partial^{\rho'}G(x) &= \\
-2\delta^{\sigma\rho}_{\sigma'\rho'}\frac{1}{\sqrt{\Box}}q_\sigma\partial_\rho G(x)\left(m^2-\Box\right)\frac{1}{\sqrt{\Box}}q^{\sigma'}\partial^{\rho'}G(x) &= \\
2q_\sigma q^\sigma G(x)\left(m^2-\Box\right)G(x) - 2q^\rho\partial_\rho G(x)\frac{1}{\Box}\left(m^2-\Box\right)q_\sigma\partial^\sigma G(x), &
\end{aligned} \tag{3.47}$$

which for physical polarizations, i.e. a transverse one, yields

$$2G(x)\left(m^2-\Box\right)G(x). \tag{3.48}$$

For the unphysical longitudinal mode $q^\rho = \frac{1}{\sqrt{-p^2}}p^\rho$, where p denotes the momentum of $G(x)$, we get in momentum space $-2G(p)\left(m^2+p^2\right)G(p) + 2G(p)\left(m^2+p^2\right)G(p) = 0$.

In (3.46) we see that the pseudo-vector fluctuation is a massive vector state. It appears in the D4-brane action (2.26) in linear combination of F_{ab}. In the restframe we see that $B_{0\mu} = 0$, thus B_{ij} has to transform like F_{ij} under charge and parity transformation. Its quantum numbers are thus $J^{PC} = 1^{+-}$ as stated in the beginning of this section.

Part III
Witten-Sakai-Sugimoto model

In the Witten model it is conjectured that the geometry (1.7) is dual to the low-energy limit of pure Yang-Mills theory with $N_C \gg 1$ colors. This duality is an open-closed string

	0	1	2	3	(4)	5	6	7	8	9
N_C $D4$	○	○	○	○	○					
N_F $D8-\overline{D8}$	○	○	○	○		○	○	○	○	○

Table 6. Brane configuration in the Witten-Sakai-Sugimoto model.

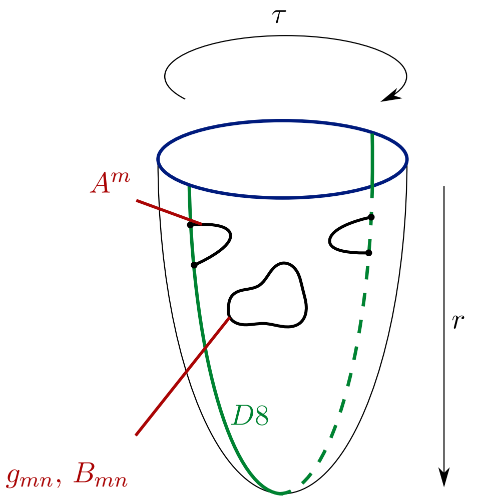

Figure 5. D8 brane embedded in the near-horizon geometry (1.7).

duality, i.e. it relates a theory of geometry to a gauge theory. To get to a QCD-like theory we also need flavor degrees of freedom. Chiral quark pairs are introduced by including probe D8 and anti-D8 branes, see figure 5. The probe approximation means that we do not include backreactions of the D8 branes to the geometry. It corresponds to the quenched case in lattice QCD, where quark loops are neglected. Sakai and Sugimoto [7] proposed the brane configuration shown in table 6. If the branes are placed at antipodal points in the compactified τ-direction the anti-D8 brane is forced to join the D8 brane at r_{KK} and thus breaks the chiral symmetry $U(N_F)_R \times U(N_F)_L \to U(N_F)_{R+L}$. This symmetry breaking gives rise to Goldstone bosons, namely pions as we will be seen below. If the branes are not placed at antipodal points, the branes join at $r > r_{KK}$.

To start the calculation we have to reduce our 11-dimensional metric to 10 dimensions. We use

$$\begin{aligned} \mathrm{d}s_{11}^2 &= \frac{r^2}{L^2}\left(f(r)\,\mathrm{d}x_4^2 + \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\mu + \mathrm{d}x_{11}^2\right) + \frac{L^2}{r^2}\frac{\mathrm{d}r^2}{f(r)} + \frac{L^2}{4}\mathrm{d}\Omega_4^2 \\ &= G_{MN}\mathrm{d}x^M\mathrm{d}x^N \\ &= e^{-\frac{2\phi}{3}}g_{\overline{MN}}\mathrm{d}x^{\overline{M}}\mathrm{d}x^{\overline{N}} + e^{\frac{4\phi}{3}}\mathrm{d}x_{11}^2 \\ &= e^{-\frac{2\phi}{3}}g_{\tilde{M}\tilde{N}}\mathrm{d}x^{\tilde{M}}\mathrm{d}x^{\tilde{N}} + \frac{r^2}{L^2}f(r)\,\mathrm{d}x_4^2 + e^{\frac{4\phi}{3}}\mathrm{d}x_{11}^2. \end{aligned} \tag{3.49}$$

With $e^\phi = \left(\frac{r}{L}\right)^{\frac{3}{2}}$ we can identify

$$
\begin{aligned}
g_{\mu\nu} &= \frac{r^3}{L^3}\eta_{\mu\nu}, \\
g_{\alpha\beta} &= \frac{rL}{4}\delta^{\alpha\beta}, \\
g_{rr} &= \frac{L}{r\,f(r)}, \\
g_{\tau\tau} &= \frac{r^3}{L^3}f(r).
\end{aligned} \tag{3.50}
$$

The volume elements read

$$
\sqrt{-\det\left(G\right)} = \frac{r^5}{16L}, \tag{3.51}
$$

$$
\sqrt{-\det\left(\overline{g}\right)} = \frac{r^9}{2^4L^5}, \tag{3.52}
$$

$$
\sqrt{-\det\left(\tilde{g}\right)} = \sqrt{\frac{r^{15}}{2^8L^7f(r)}}. \tag{3.53}
$$

For the D8 embedding it turns out to be convenient to define new coordinates Z and K by

$$
\begin{aligned}
Z^2 &= \frac{r^6}{r_{KK}^6} - 1 \\
&= K - 1 \\
&= f(r)\,K.
\end{aligned} \tag{3.54}
$$

The transformation of the line element can be calculated by

$$
\begin{aligned}
2\mathrm{d}Z\,Z &= 6\frac{r^5}{r_{KK}^6}\mathrm{d}r \\
\mathrm{d}Z^2 &= 9\frac{r^{10}}{f(r)\,K\,r_{KK}^{12}}\mathrm{d}r^2 \\
&= 9\frac{r^4}{f(r)\,r_{KK}^6}\mathrm{d}r^2,
\end{aligned} \tag{3.55}
$$

and

$$
\begin{aligned}
g_{rr}\mathrm{d}r^2 &= \frac{L}{r\,f(r)}\mathrm{d}r^2 \\
&= g_{ZZ}\mathrm{d}Z^2 \\
&= g_{ZZ}9\frac{r^4}{f(r)\,r_{KK}^6}\mathrm{d}r^2, \\
g_{ZZ} &= \frac{L\,r_{KK}^6}{9\,r^5}.
\end{aligned} \tag{3.56}
$$

The field strength tensor living on the D8 brane is

$$\begin{aligned} F_{\mu r}\mathrm{d}x^\mu \mathrm{d}r &= F_{\mu Z}\mathrm{d}x^\mu \mathrm{d}Z \\ &= 3\frac{r^2}{\sqrt{f(r)}\, r_{KK}^3} F_{\mu Z}\mathrm{d}x^\mu \mathrm{d}r \end{aligned} \tag{3.57}$$

and thus its components transform as

$$F_{\mu r} = 3\frac{r^2}{\sqrt{f(r)}\, r_{KK}^3} F_{\mu Z}. \tag{3.58}$$

The action of the joined D8 branes describes the dynamics of the $q\bar{q}$ mesons through flavor gauge fields on the branes. It is given by the DBI action

$$S_{D8} = -\mu_8 \mathrm{Tr} \int \mathrm{d}x^9 e^{-\phi} \sqrt{-\det\left(g_{\tilde{M}\tilde{N}} + 2\pi\alpha' F_{\tilde{M}\tilde{N}} + B_{\tilde{M}\tilde{N}}\right)}, \tag{3.59}$$

and a 9-dimensional Chern-Simons action

$$S_{CS} = i\mu_8 \int_{d=9} \mathrm{Tr}\left(\exp\left(2\pi\alpha' F_2 + B_2\right) \wedge C_3\right), \tag{3.60}$$

with brane tension $\mu_8 = (2\pi)^{-8}\, l_S^{-9}$. The Chern-Simons action will be important to calculate decay rates.

By using

$$\det{}^{1/2}\left(1 + M\right) = \exp\left[\frac{1}{2}\mathrm{tr}\left(M - \frac{1}{2}M^2 + \frac{1}{3}M^3 - \frac{1}{4}M^4\right)\right], \tag{3.61}$$

the DBI action may be simplified to

$$S_{D8} = -\mu_8 \mathrm{Tr} \int \mathrm{d}x^9 e^{-\phi} \sqrt{-\tilde{g}} \left(1 + \frac{1}{4}\left(2\pi\alpha'\right)^2 g^{\tilde{M}\tilde{N}} g^{\tilde{O}\tilde{P}} F_{\tilde{M}\tilde{O}} F_{\tilde{N}\tilde{P}} + \ldots\right). \tag{3.62}$$

This action contains kinetic, mass and interaction terms. As a first step let us look at the former two. The relevant second-order term in the field strength reads

$$
\begin{aligned}
S_{D8}^{\left(F^2\right)} =& -\mu_8 \mathrm{Tr} \int \mathrm{d}x^9 e^{-\phi} \sqrt{-\tilde{g}} \frac{1}{4} \left(2\pi\alpha'\right)^2 g^{\tilde{M}\tilde{N}} g^{\tilde{O}\tilde{P}} F_{\tilde{M}\tilde{O}} F_{\tilde{N}\tilde{P}} \\
=& -\mu_8 \frac{8\pi^2}{3} \frac{1}{4} \left(2\pi\alpha'\right)^2 \mathrm{Tr} \int \mathrm{d}x^4 \mathrm{d}r \, e^{-\phi} \sqrt{-\tilde{g}} g^{\tilde{M}\tilde{N}} g^{\tilde{O}\tilde{P}} F_{\tilde{M}\tilde{O}} F_{\tilde{N}\tilde{P}} \\
=& -\mu_8 \frac{2\pi^2}{3} \left(2\pi\alpha'\right)^2 \mathrm{Tr} \int \mathrm{d}x^4 \mathrm{d}Z \left(\frac{r}{L}\right)^{-\frac{3}{2}} \frac{r^{11/2} L^{-7/2} r_{KK}^3}{2^4 3} \\
& \cdot \left(\frac{L^6}{r^6} \eta^{\mu\nu} \eta^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + 2 \frac{9L^2 r^2}{r_{KK}^6} \eta^{\mu\nu} \eta^{ZZ} F_{\mu Z} F_{\nu Z}\right) \\
=& -\mu_8 \frac{4\pi^2}{3} \left(2\pi\alpha'\right)^2 \mathrm{Tr} \int \mathrm{d}x^4 \mathrm{d}Z \frac{r_{KK}^3}{16 \cdot 3L^2 r^2} \left(\frac{1}{2} \eta^{\mu\nu} \eta^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + \frac{M_{KK}^2 r^8}{r_{KK}^8} \eta^{\mu\nu} \eta^{ZZ} F_{\mu Z} F_{\nu Z}\right) \\
=& -\mu_8 \frac{\pi^2}{3} \left(2\pi\alpha'\right)^2 \frac{L^4 r_{KK}}{4 \cdot 3} \mathrm{Tr} \int \mathrm{d}x^4 \mathrm{d}Z \, K^{-1/3} \left(\frac{1}{2} \eta^{\mu\nu} \eta^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + M_{KK}^2 K^{4/3} \eta^{\mu\nu} \eta^{ZZ} F_{\mu Z} F_{\nu Z}\right) \\
=& -\kappa \mathrm{Tr} \int \mathrm{d}x^4 \mathrm{d}Z \left(\frac{1}{2} K^{-1/3} \eta^{\mu\nu} \eta^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + K M_{KK}^2 \eta^{\mu\nu} \eta^{ZZ} F_{\mu Z} F_{\nu Z}\right), \qquad (3.63)
\end{aligned}
$$

with $\kappa = \mu_8 \frac{\pi^2}{3} \left(2\pi\alpha'\right)^2 \frac{L^4 r_{KK}}{4 \cdot 3} = \frac{\lambda N_C}{216\pi^3}$.

4 Mesons

Variation of $S_{D8}^{\left(F^2\right)}$ with respect to A^α yields the equations of motion

$$
\begin{aligned}
0 =& \frac{1}{2} K^{-1/3} \eta^{\mu\nu} \eta^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + K M_{KK}^2 \eta^{\mu\nu} \eta^{ZZ} F_{\mu Z} F_{\nu Z} \\
=& 2K^{-1/3} m_A^2 A_\alpha + 2M_{KK}^2 \partial_Z \left(1 + Z^2\right) \partial_Z A_\alpha \\
\lambda_n A_\alpha =& -\left(1 + Z^2\right)^{1/3} \partial_Z \left(1 + Z^2\right) \partial_Z A_\alpha. \qquad (4.1)
\end{aligned}
$$

To solve them we make the ansatz

$$
\begin{aligned}
A_Z &= U_{KK} \phi_0 (Z) \, \pi (x^\mu), \\
A_\mu &= \psi_1 (Z) \, \rho_\mu (x^\nu), \\
F_{Z\mu} &= \psi_1' (Z) \, \rho_\mu (x^\nu) - U_{KK} \phi_0 (Z) \, \partial_\mu \pi (x^\nu), \\
F_{\mu\nu} &= \psi_1 (Z) \, \partial_\mu \rho_\nu (x^\rho) - \psi_1 (Z) \, \partial_\nu \rho_\mu (x^\rho). \qquad (4.2)
\end{aligned}
$$

To ease notation we will from now on raise indices with η instead of g. We obtain the effective action

$$
\begin{aligned}
S_{D8}^{\left(F^2\right)} &= -\kappa \mathrm{Tr} \int \mathrm{d}x^4 \mathrm{d}Z \,\left(\frac{1}{2}K^{-1/3}\eta^{\mu\nu}\eta^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} + KM_{KK}^2\eta^{\mu\nu}\eta^{ZZ}F_{\mu Z}F_{\nu Z}\right) \\
&= -\kappa \mathrm{Tr} \int \mathrm{d}x^4 \mathrm{d}Z \,\left[2K^{-1/3}\psi_1^2\left(\partial_\mu\rho_\nu\right)^2 + \frac{1}{2}2KM_{KK}^2U_{KK}^2\phi_0^2\left(\partial_\mu\pi\right)^2 + KM_{KK}^2\psi_1'^2\rho_\mu\rho^\mu + ...\right] \\
&= -\kappa \mathrm{Tr} \int \mathrm{d}x^4 \mathrm{d}Z \,\left[\left(\partial_\mu\rho_\nu\right)^2 + \frac{1}{2}2KM_{KK}^2U_{KK}^2\phi_0^2\left(\partial_\mu\pi\right)^2 + \frac{\lambda_1 M_{KK}^2}{2}\frac{2}{\lambda_1}K\psi_1'^2\rho_\mu\rho^\mu + ...\right].
\end{aligned}
\tag{4.3}
$$

Here Z runs from $-\infty$ to $+\infty$, which corresponds to integrating over a joined pair of $D8$ and $\overline{D8}$ branes. The used normalizations are explained in the following subsection.

4.1 Normalization of meson modes

In order for the meson fields in (4.3) to be canonically normalized we have to impose the conditions

$$
\begin{aligned}
2\kappa \int \mathrm{d}Z \, K^{-1/3}\psi_1^2 &= 1, \\
2\kappa M_{KK}^2 U_{KK}^2 \int \mathrm{d}Z \, K\phi_0^2 &= 1, \\
\frac{2}{\lambda_1}\kappa \int \mathrm{d}Z \, K\psi_1'^2 &= 1.
\end{aligned}
\tag{4.4}
$$

The normalization for ψ_1 is

$$
\begin{aligned}
\int \mathrm{d}Z \, K^{-1/3}\psi_1^2\left(Z\right) &= 2.80301\psi_1^2\left(0\right) \\
\psi_1^{-2}\left(0\right) &= 2\kappa \cdot 2.80301 \\
\psi_1^{-1}\left(0\right) &= 2.36771 \cdot \kappa^{\frac{1}{2}}
\end{aligned}
\tag{4.5}
$$

and the first two eigenvalues are

$$
\begin{aligned}
\lambda_1 &= 0.669314, \\
\lambda_2 &= 1.569.
\end{aligned}
\tag{4.6}
$$

For ϕ_0 we make the ansatz

$$
\phi_0 = c_1 \frac{1}{K},
\tag{4.7}
$$

which is canonically normalized if we take

$$
\begin{aligned}
2\kappa M_{KK}^2 U_{KK}^2 \int \mathrm{d}Z\, K\phi_0^2 &= 1 \\
2\kappa M_{KK}^2 U_{KK}^2 c_1^2 \int \mathrm{d}Z\, \frac{1}{K} &= 1 \\
2\kappa M_{KK}^2 U_{KK}^2 c_1^2 \pi &= 1 \\
\frac{1}{M_{KK} U_{KK} \sqrt{2\kappa\pi}} &= c_1 \\
\frac{1}{K M_{KK} U_{KK} \sqrt{2\kappa\pi}} &= \phi_0 .
\end{aligned} \tag{4.8}
$$

5 Choice of parameters

From this action (4.3) we can infer the mass of the ρ meson to be $m_\rho^2 = \lambda_1 M_{KK}$. By fitting this result to the experimental value of $m_\rho \approx 766$MeV we fix the Kaluza-Klein mass to $M_{KK} = 949$MeV [7].

The term containing the Goldstone bosons of chiral symmetry breaking appearing in (4.3) can be matched to

$$
S_{D8} = \frac{f_\pi^2}{4} \int \mathrm{d}^4x \mathrm{Tr} \left(U^{-1} \partial U \right)^2 + ..., \quad U = \mathrm{P}exp \left\{ i \int \mathrm{d}Z\, A_Z \right\}, \tag{5.1}
$$

by identifying the so-called pion decay constant

$$
f_\pi^2 = \frac{1}{54\pi^4} \lambda N_C M_{KK}^2 . \tag{5.2}
$$

In order to calculate glueball-meson interactions, we have to extrapolate our duality to finite coupling λ and finite $N_C = 3$. To do so we match the pion decay constant to the experimental value $f_\pi \approx 92.4$MeV, which yields the coupling constant $\lambda \approx 16.63$.

Part IV
Decay of pseudo-vector glueballs

6 Chern-Simons term on D8-brane

To obtain an effective Lagrangian, which describes the decay of pseudo-vector glueballs in mesons, we look at the Chern-Simons action of a D8-brane. There are also interaction terms if we expand the DBI action to higher order terms in the field strength. We will see however that the interaction terms coming from the Chern-Simons action dominate over the DBI contributions; the latter will not change the interpretation of our results. The Chern-Simons action for a D8 brane reads (see e.g. [10])

$$
\begin{aligned}
S_{CS} =& i\mu_8 \int \mathrm{Tr}\left(exp\left(2\pi\alpha' F_2 + B_2\right) \wedge C_3\right) \\
& i\mu_8 \int \mathrm{Tr}\left(exp\left(2\pi\alpha' F_2\right) \wedge exp\left(B_2\right) \wedge C_3\right).
\end{aligned} \tag{6.1}
$$

Since we are only interested in glueball decays into mesons and not into other glueballs, it is sufficient to look at the term linear in B_2

$$
S_{CS}|_{B_2} = i\mu_8 \int \frac{(2\pi\alpha')^2}{2} \mathrm{Tr}\left(F_2 \wedge F_2 \wedge B_2 \wedge C_3\right).
$$

By partial integration

$$
\begin{aligned}
8 \cdot 6 \cdot F_2 \wedge F_2 \wedge B_2 \wedge C_3 &= \\
\epsilon^{M_1 \ldots M_9} F_{M_1 M_2} F_{M_3 M_4} B_{M_5 M_6} C_{M_7 M_8 M_9} &= \\
-2\epsilon^{M_1 \ldots M_9} A_{M_2} \partial_{M_1} F_{M_3 M_4} B_{M_5 M_6} C_{M_7 M_8 M_9} & \\
-2\epsilon^{M_1 \ldots M_9} A_{M_2} F_{M_3 M_4} \partial_{M_1} B_{M_5 M_6} C_{M_7 M_8 M_9} & \\
-2\epsilon^{M_1 \ldots M_9} A_{M_2} F_{M_3 M_4} B_{M_5 M_6} \partial_{M_1} C_{M_7 M_8 M_9} &= \\
2\epsilon^{M_1 \ldots M_9} A_{M_1} F_{M_2 M_3} B_{M_4 M_5} \partial_{M_6} C_{M_7 M_8 M_9} &= \\
\frac{1}{2}\epsilon^{M_1 \ldots M_9} A_{M_1} F_{M_2 M_3} B_{M_4 M_5} F_{M_6 M_7 M_8 M_9} &= \\
\frac{1}{2} \cdot 4 \cdot 4! \cdot A_1 \wedge F_2 \wedge B_2 \wedge F_4, &
\end{aligned} \tag{6.2}
$$

where in the second step we have used that the only non-vanishing possibility to distribute the 4 spherical indices is $\partial_{\alpha_1} C_{\alpha_2\alpha_3\alpha_4}$, we may simplify the action $S_{CS}|_{B_2}$ to

$$
\begin{aligned}
i\mu_8 \frac{(2\pi\alpha')^2}{2} \int \mathrm{Tr}\left(A_1 \wedge F_2 \wedge B_2 \wedge F_4\right) &= \\
i\mu_8 \frac{(2\pi\alpha')^2}{2} \frac{6}{L} \frac{8\pi^2}{3} \left(\frac{L}{2}\right)^4 \frac{1}{g_s} \int \mathrm{Tr}\left(A_1 \wedge F_2 \wedge B_2\right) &= \\
i\mu_8 \frac{(2\pi\alpha')^2}{2} L^3 \pi^2 \frac{1}{g_s} \int \mathrm{Tr}\left(A_1 \wedge F_2 \wedge B_2\right) &= \\
i\mu_8 \frac{(2\pi\alpha')^2}{2} \frac{1}{g_s} L^3 \pi^2 \frac{1}{4} \epsilon^{a_1 a_2 a_3 a_4 a_5} \int \mathrm{Tr} A_{a_1} F_{a_2 a_3} B_{a_4 a_5} &= \\
i\mu_8 \frac{(2\pi\alpha')^2}{2} \frac{1}{g_s} L^3 \pi^2 \frac{1}{4} \epsilon^{\mu\nu\rho\sigma Z} \int \mathrm{Tr} A_Z F_{\mu\nu} B_{\rho\sigma} & \\
-i\mu_8 \frac{(2\pi\alpha')^2}{2} \frac{1}{g_s} L^3 \pi^2 \frac{1}{2} \epsilon^{\mu\nu\rho\sigma Z} \int \mathrm{Tr} A_\mu F_{Z\nu} B_{\rho\sigma}. &
\end{aligned} \tag{6.3}
$$

For convenience we define the constant $C = \mu_8 \left(2\pi\alpha'\right)^2 \frac{1}{8g_s} L^3 \pi^2$ and use some renamed fields in our previously obtained equations

$$
\begin{aligned}
A_Z &= U_{KK}\phi_0\,(Z)\,\tilde{A}_Z\,(x^\mu)\,,\\
A_\mu &= \psi_1\,(Z)\,\tilde{A}_\mu\,(x^\nu)\,,\\
F_{Z\mu} &= \psi_1'\,(Z)\,\tilde{A}_\mu\,(x^\nu) - U_{KK}\phi_0\,(Z)\,\partial_\mu\tilde{A}\,(x^\nu)\,,\\
F_{\mu\nu} &= \psi_1\,(Z)\,\partial_\mu\tilde{A}_\nu\,(x^\rho) - \psi_1\,(Z)\,\partial_\nu\tilde{A}_\mu\,(x^\rho)\\
&= \psi_1\,(Z)\,\tilde{F}_{\mu\nu},
\end{aligned}
\tag{6.4}
$$

$$
\begin{aligned}
B_{\mu\nu} &= A_{\mu\nu,11}\\
&= r^3 N_4(r)\tilde{B}_{\mu\nu},
\end{aligned}
\tag{6.5}
$$

$$
\tilde{B}_{\mu\nu} = \frac{1}{\sqrt{\Box}}\epsilon_{\mu\nu\sigma\rho}q^\sigma\partial^\rho\tilde{G}(x),
\tag{6.6}
$$

$$
\begin{aligned}
N_4(r_{KK})^{-1} &= 0.00983838\,L^3\lambda^{\frac{1}{2}}N_C M_{KK},\\
\psi_1^{-1}\,(0) &= 2.36771\cdot\kappa^{\frac{1}{2}},\\
\phi_0 &= \frac{1}{KM_{KK}U_{KK}\sqrt{2\kappa\pi}},\\
\kappa &= \frac{\lambda N_c}{216\pi^3},\\
\lambda &= 16.63.
\end{aligned}
\tag{6.7}
$$

The lowest eigenmodes of $\tilde{A}_Z\,(x^\mu)$ and $\tilde{A}_\nu\,(x^\mu)$ correspond to $\pi\,(x^\mu)$ and $\rho_\nu\,(x^\mu)$ respectively. The components of B_2 are $B_{\mu\nu}$, and $\tilde{B}_{\mu\nu}$ is introduced to denote $B_{\mu\nu}$ of chapter 3. We accordingly renamed $G(x)$ to $\tilde{G}(x)$ and define

$$
G(x) = r^3 N_4(r)\tilde{G}(x).
\tag{6.8}
$$

We continue the calculation we stopped at (6.3) and get

$$
\begin{aligned}
C\epsilon^{\mu\nu\rho\sigma Z}\int \mathrm{Tr} A_Z F_{\mu\nu} B_{\rho\sigma} \\
-2C\epsilon^{\mu\nu\rho\sigma Z}\int \mathrm{Tr} A_\mu F_{Z\nu} B_{\rho\sigma} &= \\
\mathrm{Tr}\left(T^a T^b\right) C\epsilon^{\mu\nu\rho\sigma Z}\int A^a_Z F^b_{\mu\nu} B_{\rho\sigma} \\
-\mathrm{Tr}\left(T^a T^b\right) 2C\epsilon^{\mu\nu\rho\sigma Z}\int A^a_\mu F^b_{Z\nu} B_{\rho\sigma} \\
\mathrm{Tr}\left(T^c\left[T^a, T^b\right]\right) C\epsilon^{\mu\nu\rho\sigma Z}\int A^c_Z A^a_\mu A^b_\nu B_{\rho\sigma} \\
-\mathrm{Tr}\left(T^c\left[T^a, T^b\right]\right) 2C\epsilon^{\mu\nu\rho\sigma Z}\int A^c_\mu A^a_Z A^b_\nu B_{\rho\sigma} &= \\
-\mathrm{Tr}\left(T^a T^b\right)\frac{4C}{m_G}\int A^a_Z F^b_{\mu\nu} q^\mu r^3 N_4(r)\partial^\nu \tilde{G}(x) \\
\mathrm{Tr}\left(T^a T^b\right)\frac{4C}{m_G}\int A^a_\mu F^b_{Z\nu}\left(q^\mu\partial^\nu - q^\nu\partial^\mu\right) r^3 N_4(r)\partial^\nu \tilde{G}(x) \\
-\mathrm{Tr}\left(T^c\left[T^a, T^b\right]\right)\frac{12C}{m_G}\int A^c_Z A^a_\mu A^b_\nu q^\mu r^3 N_4(r)\partial^\nu \tilde{G}(x) &= \\
-\mathrm{Tr}\left(T^a T^b\right)\frac{4C}{m_G}\int A^a_Z F^b_{\mu\nu} q^\mu r^3 N_4(r)\partial^\nu \tilde{G}(x) \\
-\mathrm{Tr}\left(T^a T^b\right)\frac{4C}{m_G}\int A^a_\mu \partial_\nu A^b_Z r^3 N_4(r)\left(q^\mu\partial^\nu - q^\nu\partial^\mu\right)\tilde{G}(x) \\
-\mathrm{Tr}\left(T^c\left[T^a, T^b\right]\right)\frac{12C}{m_G}\int A^c_Z A^a_\mu A^b_\nu q^\mu r^3 N_4(r)\partial^\nu \tilde{G}(x) &= \\
-\mathrm{Tr}\left(T^a T^b\right)\int \frac{4C}{m_G} U_{KK}\phi_0\psi_1 r^3 N_4 \tilde{A}^a_Z \tilde{F}^b_{\mu\nu} q^\mu \partial^\nu \tilde{G}(x) \\
-\mathrm{Tr}\left(T^a T^b\right)\int \frac{4C}{m_G} U_{KK}\phi_0\psi_1 r^3 N_4 \tilde{A}^a_\mu \partial_\nu \tilde{A}^b_Z\left(q^\mu\partial^\nu - q^\nu\partial^\mu\right)\tilde{G}(x) \\
-\mathrm{Tr}\left(T^c\left[T^a, T^b\right]\right)\int \frac{12C}{m_G} U_{KK} r^3\phi_0\psi_1^2 N_4 \tilde{A}^c_Z \tilde{A}^a_\mu \tilde{A}^b_\nu q^\mu \partial^\nu \tilde{G}(x) &= \\
-i\mathrm{Tr}\left(T^a T^b\right)\frac{g_1}{m_G}\tilde{A}^a_Z \tilde{F}^b_{\mu\nu} q^\mu \partial^\nu \tilde{G}(x) & \qquad (6.9) \\
-i\mathrm{Tr}\left(T^a T^b\right)\frac{g_1}{m_G}\tilde{A}^a_\mu \partial_\nu \tilde{A}^b_Z\left(q^\mu\partial^\nu - q^\nu\partial^\mu\right)\tilde{G}(x) & \qquad (6.10) \\
-i\mathrm{Tr}\left(T^c\left[T^a, T^b\right]\right)\frac{g_2}{m_G}\tilde{A}^c_Z \tilde{A}^a_\mu \tilde{A}^b_\nu q^\mu \partial^\nu \tilde{G}(x). & \qquad (6.11)
\end{aligned}
$$

The first two lines after the last equality sign will be relevant for the 2-body and the third for the 3-body decay.

The coupling constants in this Lagrangian turn out to be

$$
\begin{aligned}
g_1 =& \int 4CU_{KK} r^3 \phi_0 \psi_1 N_4 \\
=& \int \mu_8 \frac{(2\pi\alpha')^2}{2} \frac{1}{g_s} L^3 \pi^2 U_{KK} r^3 \phi_0 \psi_1 N_4 \\
=& \int \mu_8 2\alpha'^2 \frac{1}{g_s} L^3 \pi^4 U_{KK} r^3 \phi_0 \psi_1 N_4 \\
=& \frac{1}{g_s} 2\mu_8 \pi^4 \alpha'^2 L^3 \frac{1}{M_{KK}\sqrt{2\pi\kappa}} \left[N_4\left(r_{KK}\right)\right]\left[\psi_1\left(0\right)\right] \int \mathrm{d}Z\, r^3 \frac{1}{K} N_4 \psi_1 \\
=& \frac{1}{g_s} 2\mu_8 \pi^4 \alpha'^2 \frac{r_{KK}^3}{\lambda^{\frac{1}{2}} N_C M_{KK}^2 \kappa \sqrt{2\pi}} \frac{1.63571}{0,00983838 \cdot 2,36771} \\
=& 70.2191 \frac{2\pi^4}{\sqrt{2\pi}} \left(2\pi\right)^{-8} \frac{g_s l_s}{g_s^2 l_s^6} \frac{\pi^3 216\, r_{KK}^3}{\lambda^{\frac{3}{2}} N_C^2 M_{KK}^2} \\
=& 70.2191 \frac{\left(2\pi\right)^4}{8\sqrt{2\pi}} \left(2\pi\right)^{-8} \frac{\lambda}{2\pi N_C M_{KK}} \left(\frac{8\pi N_c}{L^3}\right)^2 \frac{\left(2\pi\right)^3 27\, r_{KK}^3}{\lambda^{\frac{3}{2}} N_C^2 M_{KK}^2} \\
=& 70.2191 \left(2\pi\right)^{-5/2} \frac{\lambda}{8 N_C M_{KK}} \left(\frac{8\pi N_c}{L^3}\right)^2 \frac{27\, r_{KK}^3}{\lambda^{\frac{3}{2}} N_C^2 M_{KK}^2} \\
=& 70.2191 \left(2\pi\right)^{-1/2} 2 \frac{1}{\lambda^{\frac{1}{2}} N_C} \\
=& 56.0268 \lambda^{-\frac{1}{2}} N_C^{-1}
\end{aligned}
\tag{6.12}
$$

and

$$
\begin{aligned}
g_2 =& \int 12CU_{KK} r^3 \phi_0 \psi_1^2 N_4 \\
=& 5132.09 \lambda^{-1} N_C^{-\frac{3}{2}}.
\end{aligned}
\tag{6.13}
$$

The vector mesons corresponding to mode ψ_1 are parametrized as

$$
\tilde{A}_\mu^a T^a = \frac{\epsilon_\mu}{\sqrt{2}} \begin{pmatrix} \rho_3 + \frac{\omega_8}{\sqrt{3}} & \rho_1 - i\rho_2 & K_1^* - iK_2^* \\ \rho_1 + i\rho_2 & \frac{\omega_8}{\sqrt{3}} - \rho_3 & K_3^* - iK_4^* \\ K_1^* + iK_2^* & K_3^* + iK_4^* & -\frac{2\omega_8}{\sqrt{3}} \end{pmatrix}, \tag{6.14}
$$

the pseudoscalar Goldstone bosons as

$$
\tilde{A}_Z^a T^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_3 + \frac{\eta_8}{\sqrt{3}} & \pi_1 - i\pi_2 & K_1 - iK_2 \\ \pi_1 + i\pi_2 & \frac{\eta_8}{\sqrt{3}} - \pi_3 & K_3 - iK_4 \\ K_1 + iK_2 & K_3 + iK_4 & -\frac{2\eta_8}{\sqrt{3}} \end{pmatrix}. \tag{6.15}
$$

Furthermore there are excited modes ψ_2, ψ_3..., which however are too heavy for the considered decays. For decays of excited glueballs or if one extrapolates the glueball mass to values obtained from the lattice, those will become relevant.

7 Glueball-meson interactions

7.1 2-body decay

As already mentioned the 2-body decay is obtained from (6.9) and (6.10),

$$
\begin{aligned}
-i\mathrm{Tr}\left(T^aT^b\right)\frac{g_1}{m_G}\tilde{A}^a_Z\tilde{F}^b_{\mu\nu}q^\mu\partial^\nu\tilde{G}(x) \\
-i\mathrm{Tr}\left(T^aT^b\right)\frac{g_1}{m_G}\tilde{A}^a_\mu\partial_\nu\tilde{A}^b_Z\left(q^\mu\partial^\nu-q^\nu\partial^\mu\right)\tilde{G}(x) = \\
-i\frac{g_1}{m_G}\tilde{A}^a_Z\left(\partial_\mu\tilde{A}^a_\nu-\partial_\nu\tilde{A}^a_\mu\right)q^\mu\partial^\nu\tilde{G}(x) \\
-i\frac{g_1}{m_G}\left(\tilde{A}^a_\mu\partial_\nu\tilde{A}^a_Z-\tilde{A}^a_\nu\partial_\mu\tilde{A}^a_Z\right)q^\mu\partial^\nu\tilde{G}(x) = \\
-i\frac{g_1}{m_G}\epsilon_\nu\left(p^{\nu(G)}p^{(\omega)}_\mu q^\mu+q^\nu p^{(\eta)}_\mu p^{\mu(G)}-q^\nu p^{(\omega)}_\mu p^{\mu(G)}-p^{\nu(G)}p^{(\eta)}_\mu q^\mu\right)\left(\tilde{A}^a_{(\mu)}\tilde{A}^a_Z\tilde{G}(x)\right).
\end{aligned}
\tag{7.1}
$$

Note that to ease notation we used $p^{(\omega)}_\mu$ and $p^{(\eta)}_\mu$ as the corresponding momenta of ${\tilde{A}_\mu}^{\ a}$ and $\tilde{A}^a_Z$, the obtained equations are however valid for all mesons, not only ω and η. The polarization of the vector meson and the glueball are ϵ_ν and q^ν and the momentum of the glueball is $p^{\mu(G)}$.
In the center-of-mass frame (CMS) we can simplify the amplitude further

$$
\begin{aligned}
\mathcal{M}^\nu =& -i\frac{g_1}{m_G}\left(p^{\nu(G)}p^{(\omega)}_\mu q^\mu+q^\nu p^{(\eta')}_\mu p^{\mu(G)}-q^\nu p^{(\omega)}_\mu p^{\mu(G)}-p^{\nu(G)}p^{(\eta')}_\mu q^\mu\right) \\
=& -i\,\frac{g_1}{m_G}\left(p^{\nu(G)}p^{(\omega)}_\mu q^\mu+q^\nu p^{(\eta')}_\mu p^{\mu(G)}-q^\nu p^{(\omega)}_\mu p^{\mu(G)}-p^{\nu(G)}p^{(\eta')}_\mu q^\mu\right) \\
=& -i\,\frac{g_1}{m_G}\left(p^{\nu(G)}p^{(\omega)}_z\delta^\kappa_3-q^\nu m_G E^{(\eta')}+q^\nu m_G E^{(\omega)}-p^{\nu(G)}p^{(\eta')}_z\delta^\kappa_3\right) \\
=& -i\,\frac{g_1}{m_G}\left(2\,p^{\nu(G)}p^{(\omega)}_z\delta^\kappa_3+q^\nu m_G\left(E^{(\omega)}-E^{(\eta')}\right)\right).
\end{aligned}
\tag{7.2}
$$

To calculate the total cross section we need to take the absolute square of the amplitude

$$
\begin{aligned}
|\mathcal{M}|^2 =& \frac{1}{3}\sum_\kappa\sum_\lambda \epsilon^\mu_\lambda \epsilon^\nu_\lambda \mathcal{M}^\kappa_\mu \mathcal{M}^{\kappa *}_\nu \\
=& \frac{1}{3}\sum_\kappa \left(\eta^{\mu\nu} + \frac{p^\mu_{(\omega)} p^\nu_{(\omega)}}{m_\omega^2}\right) \mathcal{M}^\kappa_\mu \mathcal{M}^{\kappa *}_\nu \\
=& \frac{1}{3} g^2_{G\omega\eta'} \frac{1}{m_G^2} \left(-4\, m_G^2 p_z^{(\omega)2} + \sum_\kappa \left(m_G q^\nu \left(E^{(\omega)} - E^{(\eta')} \right) \right)^2 \right. \\
& \left. + \sum_\kappa 4\, \delta^\kappa_3 q_\nu p^{\nu(G)} p_z^{(\omega)} m_G \left(E^{(\omega)} - E^{(\eta')} \right) \right) \\
& + \frac{1}{3} g^2_{G\omega\eta'} \frac{1}{m_\omega^2} \frac{1}{m_G^2} \left(\left(2 p_\nu^{(\omega)} p^{\nu(G)} p_z^{(\omega)} \right)^2 + \sum_\kappa \left(p_\nu^{(\omega)} q^\nu m_G \left(E^{(\omega)} - E^{(\eta')} \right) \right)^2 \right. \\
& \left. + 4 \sum_\kappa p_\nu^{(\omega)} p^{\nu(G)} p_z^{(\omega)} \delta^\kappa_3 p_\mu^{(\omega)} q^\mu m_G \left(E^{(\omega)} - E^{(\eta')} \right) \right) \\
=& \frac{1}{3} g^2_{G\omega\eta'} \frac{1}{m_G^2} \left(-4\, m_G^2 p_z^{(\omega)2} + 3 m_G^2 \left(E^{(\omega)} - E^{(\eta')} \right)^2 \right) \\
& + \frac{1}{3} g^2_{G\omega\eta'} \frac{1}{m_\omega^2} \frac{1}{m_G^2} \left(\left(-2 E^{(\omega)} m_G p_z^{(\omega)} \right)^2 + \left(p_z^{(\omega)} m_G \left(E^{(\omega)} - E^{(\eta')} \right) \right)^2 \right. \\
& \left. - 4 E^{(\omega)} m_G^2 p_z^{(\omega)} p_z^{(\omega)} \left(E^{(\omega)} - E^{(\eta')} \right) \right) \\
=& \frac{1}{3} g^2_{G\omega\eta'} \left(-4\, p_z^{(\omega)2} + 3 \left(E^{(\omega)} - E^{(\eta')} \right)^2 + \frac{4}{m_\omega^2} E^{(\omega)2} p_z^{(\omega)2} \right. \\
& \left. \frac{1}{m_\omega^2} p_z^{(\omega)2} \left(E^{(\omega)} - E^{(\eta')} \right)^2 - \frac{4}{m_\omega^2} E^{(\omega)} p_z^{(\omega)2} \left(E^{(\omega)} - E^{(\eta')} \right) \right).
\end{aligned} \tag{7.3}
$$

Using the kinematic identities

$$
\begin{aligned}
E^{(\omega)} =& \frac{m_G^2 - m_\eta^2 + m_\omega^2}{2 m_G}, \\
E^{(\eta')} =& \frac{m_G^2 - m_\omega^2 + m_\eta^2}{2 m_G}, \\
p_z^{(\omega)} =& \frac{\left[\left(m_G^2 - (m_\omega + m_\eta)^2 \right) \left(m_G^2 - (m_\omega - m_\eta)^2 \right) \right]}{2 m_G},
\end{aligned} \tag{7.4}
$$

we get

$$
|\mathcal{M}|^2 = \frac{g^2_{G\omega\eta'} \left(m_G^6 - 2 m_G^4 \left(m_\eta^2 + 3 m_\omega^2 \right) + m_G^2 \left(m_\eta^2 + 3 m_\omega^2 \right)^2 + 8 \left(m_\omega^3 - m_\eta^2 m_\omega \right)^2 \right)}{12 m_\omega^2 m_G^2}. \tag{7.5}
$$

The total decay rate reads

$$
\Gamma = \frac{1}{8\pi} |\mathcal{M}|^2 \frac{\left| p^{(G)} \right|}{M^2}. \tag{7.6}
$$

Before calculating the 2-body decay rates we have to mention a small caveat in the calculation of decays including ω and η mesons. In the Standard model it turns that the fields η_8

Particle	π	K	η'	η	ρ	K^*	ω	ϕ	G
Mass $[MeV]$	140	497	1057.88	556.391	776	895	776	1019	2311

Table 7. Masses used in our calculation.

Channel	Γ/m_G	#
$\rho\pi$	0.120815	3
KK^*	0.0486318	4
$\eta'\omega$	0.0168338	1
$\eta'\phi$	0.00195568	1
$\eta\omega$	0.0530406	1
$\eta\phi$	0.00857957	1
Sum	0.637382	11

Table 8. 2-body decay rates.

and ω_8 in (6.14) and (6.15) and the corresponding $U(1)$ fields η_1 and ω_1 are superpositions of the physical fields η, η', ω and ω' according to

$$\begin{aligned}\eta_8 =&\eta' \sin(\theta_P) + \eta\cos(\theta_P)\,,\\ \eta_1 =&\eta' \cos(\theta_P) - \eta\sin(\theta_P)\,,\\ \omega_8 =&\omega \sin(\theta_V) + \phi\cos(\theta_V)\,,\\ \omega_{1=}&\omega\cos(\theta_V) - \phi\sin(\theta_V)\,. \end{aligned}\tag{7.7}$$

We have used the values

$$\begin{aligned}\theta_P =& -0.250503,\\ \theta_V =&0.61087,\\ =&35^\circ. \end{aligned}\tag{7.8}$$

θ_P and θ_V correspond to the standard values stated in [13].

For the calculation of the 2-body decays we use the masses displayed in table 7. The resulting 2-body decay rates are displayed in table 8. In the first column we denote the involved particles of the decay channel. The values in the second column correspond to one decay channel, e.g. $\rho_1\pi_1$. In the third column we state how many decay channels there are. In the last line we write the sum over all channels.

7.2 3-body decay

The 3-body decay is obtained from (6.11)

$$-i\mathrm{Tr}\left(T^c\left[T^a,T^b\right]\right)\frac{g_2}{m_G}\int\tilde{A}^c_Z\tilde{A}^a_\mu\tilde{A}^b_\nu q^\mu\partial^\nu\tilde{G}(x)=$$
$$f^{cab}\frac{g_2}{m_G}\int\tilde{A}^c_Z\tilde{A}^a_\mu\tilde{A}^b_\nu q^\mu\partial^\nu\tilde{G}(x),\tag{7.9}$$

which yields

$$\epsilon^{(1)\mu}_\lambda\epsilon^{(2)\nu}_{\lambda'}\mathcal{M}^\kappa_{\mu\nu}=$$
$$\epsilon^{(1)\mu}_\lambda\epsilon^{(2)\nu}_{\lambda'}f^{cab}\frac{g_2}{m_G}\left(q^\kappa_\mu p^{(G)}_\nu-p^{(G)}_\mu q^\kappa_\nu\right).\tag{7.10}$$

The notation in the last line is a little bit subtle. The indices in f^{cab} are supposed to match the cross section we want to calculate. Furthermore we fixed a and b to satisfy $a<b$. In CMS we calculate

$$\begin{aligned}
|\mathcal{M}|^2&=\frac{1}{3}\sum_\kappa\sum_{\lambda\lambda'}\epsilon^{(1)\mu}_\lambda\epsilon^{(2)\nu}_{\lambda'}\epsilon^{(1)\alpha}_\lambda\epsilon^{(2)\beta}_{\lambda'}\mathcal{M}^\kappa_{\mu\nu}\mathcal{M}^{\kappa*}_{\alpha\beta}\\
&=\frac{1}{3}\sum_\kappa\left(\eta^{\mu\alpha}+\frac{p_1^\mu p_1^\alpha}{m_1^2}\right)\left(\eta^{\nu\beta}+\frac{p_2^\nu p_2^\beta}{m_2^2}\right)\mathcal{M}^\kappa_{\mu\nu}\mathcal{M}^{\kappa*}_{\alpha\beta}\\
&=\left|f^{cab}\right|^2\frac{g_2^2}{3m_G^2}\sum_\kappa\left(\eta^{\mu\alpha}+\frac{p_1^\mu p_1^\alpha}{m_1^2}\right)\left(\eta^{\nu\beta}+\frac{p_2^\nu p_2^\beta}{m_2^2}\right)q^\kappa_\mu p^{(G)}_\nu q^\kappa_\alpha p^{(G)}_\beta\\
&\quad+\left|f^{cab}\right|^2\frac{g_2^2}{3m_G^2}\sum_\kappa\left(\eta^{\mu\alpha}+\frac{p_1^\mu p_1^\alpha}{m_1^2}\right)\left(\eta^{\nu\beta}+\frac{p_2^\nu p_2^\beta}{m_2^2}\right)p^{(G)}_\mu q^\kappa_\nu p^{(G)}_\alpha q^\kappa_\beta\\
&\quad-2\left|f^{cab}\right|^2\frac{g_2^2}{3m_G^2}\sum_\kappa\left(\eta^{\mu\alpha}+\frac{p_1^\mu p_1^\alpha}{m_1^2}\right)\left(\eta^{\nu\beta}+\frac{p_2^\nu p_2^\beta}{m_2^2}\right)q^\kappa_\mu p^{(G)}_\nu p^{(G)}_\alpha q^\kappa_\beta\\
&=\left|f^{cab}\right|^2\frac{g_2^2}{3m_G^2}\sum_\kappa\left(1+\frac{\left(p_1^\mu q^\kappa_\mu\right)^2}{m_1^2}\right)\left(-m_G^2+\frac{\left(p_2^\nu p^{(G)}_\nu\right)^2}{m_2^2}\right)\\
&\quad+\left|f^{cab}\right|^2\frac{g_2^2}{3m_G^2}\sum_\kappa\left(-m_G^2+\frac{\left(p_1^\nu p^{(G)}_\nu\right)^2}{m_1^2}\right)\left(1+\frac{\left(p_2^\mu q^\kappa_\mu\right)^2}{m_2^2}\right)\\
&\quad-2\left|f^{cab}\right|^2\frac{g_2^2}{3m_G^2}\sum_\kappa\left(0+\frac{p_1^\nu p^{(G)}_\nu p_1^\mu q^\kappa_\mu}{m_1^2}\right)\left(0+\frac{p_2^\nu p^{(G)}_\nu p_2^\mu q^\kappa_\mu}{m_2^2}\right)\\
&=\left|f^{cab}\right|^2\frac{g_2^2}{3m_G^2}\left(3+\frac{(p_1^x)^2+(p_1^y)^2+(p_1^z)^2}{m_1^2}\right)\left(-m_G^2+m_G^2\frac{\left(p_2^0\right)^2}{m_2^2}\right)\\
&\quad+\left|f^{cab}\right|^2\frac{g_2^2}{m_G^2}\left(-m_G^2+m_G^2\frac{\left(p_1^0\right)^2}{m_1^2}\right)\left(3+\frac{(p_2^x)^2+(p_2^y)^2+(p_2^z)^2}{m_2^2}\right)\\
&\quad-\left|f^{cab}\right|^2\frac{g_2^2}{3m_G^2}\left(m_G^2\frac{p_1^0p_2^0\left(p_1^xp_2^x+p_1^yp_2^y+p_1^zp_2^z\right)}{m_\rho^4}\right)=
\end{aligned}$$

$$
\begin{aligned}
=&\left|f^{cab}\right|^2 \frac{g_2^2}{3}\left(3+\frac{\left(p_1\right)^2+\left(p_1^0\right)^2}{m_\rho^2}\right)\left(-1+\frac{\left(p_2^0\right)^2}{m_\rho^2}\right) \\
&+\left|f^{cab}\right|^2 \frac{g_2^2}{3}\left(-1+\frac{\left(p_1^0\right)^2}{m_\rho^2}\right)\left(3+\frac{\left(p_2\right)^2+\left(p_2^0\right)^2}{m_\rho^2}\right) \\
&-2\left|f^{cab}\right|^2 \frac{g_2^2}{3} \frac{p_1^0 p_2^0 p_1^z p_2^z}{m_\rho^4} \\
=&\left|f^{cab}\right|^2 \frac{g_2^2}{3}\left(2+\frac{\left(p_1^0\right)^2}{m_\rho^2}\right)\left(-1+\frac{\left(p_2^0\right)^2}{m_\rho^2}\right) \\
&+\left|f^{cab}\right|^2 \frac{g_2^2}{3}\left(-1+\frac{\left(p_1^0\right)^2}{m_\rho^2}\right)\left(2+\frac{\left(p_2^0\right)^2}{m_\rho^2}\right) \\
&-2\left|f^{cab}\right|^2 \frac{g_2^2}{3} \frac{p_1^0 p_2^0 p_1^z p_2^z}{m_\rho^4} \\
=&\left|f^{cab}\right|^2 \frac{g_2^2}{3}\left(-4+\frac{\left(p_1^0\right)^2}{m_\rho^2}+\frac{\left(p_1^0\right)^2}{m_\rho^2} \frac{\left(p_2^0\right)^2}{m_\rho^2}+\frac{\left(p_2^0\right)^2}{m_\rho^2}+\frac{\left(p_1^0\right)^2}{m_\rho^2} \frac{\left(p_2^0\right)^2}{m_\rho^2}\right) \\
&-2\left|f^{cab}\right|^2 \frac{g_2^2}{3} m_G^2 \frac{p_1^0 p_2^0 p_1^z p_2^z}{m_\rho^4} \\
=&\left|f^{cab}\right|^2 \frac{g_2^2}{3}\left(-4+\frac{m_\rho^2+E_1^2}{m_\rho^2}+2 \frac{\left(m_\rho^2+E_1^2\right)}{m_\rho^2} \frac{\left(m_\rho^2+E_2^2\right)}{m_\rho^2}+\frac{m_\rho^2+E_2^2}{m_\rho^2}\right) \\
&-\left|f^{cab}\right|^2 \frac{g_2^2}{3} \frac{\sqrt{m_\rho^2+E_1^2} \sqrt{m_\rho^2+E_2^2}}{m_\rho^4} E_1 E_2 \cos (\theta) \\
=&\left|f^{cab}\right|^2 \frac{g_2^2}{3}\left(\frac{E_1^2}{m_\rho^2}+2\left(\frac{E_1^2+E_2^2}{m_\rho^2}+\frac{E_1^2 E_2^2}{m_\rho^4}\right)+\frac{E_2^2}{m_\rho^2}\right) \\
&-2\left|f^{cab}\right|^2 \frac{g_2^2}{3} \frac{\sqrt{m_\rho^2+E_1^2} \sqrt{m_\rho^2+E_2^2}}{m_\rho^4} E_1 E_2 \cos (\theta) \\
=&\left|f^{cab}\right|^2 \frac{g_2^2}{3}\left(3 \frac{E_1^2}{m_\rho^2}+3 \frac{E_2^2}{m_\rho^2}+2 \frac{E_1^2 E_2^2}{m_\rho^4}\right) \\
&-2\left|f^{cab}\right|^2 \frac{g_2^2}{3} \frac{\sqrt{m_\rho^2+E_1^2} \sqrt{m_\rho^2+E_2^2}}{m_\rho^4} E_1 E_2 \cos (\theta),
\end{aligned}
\tag{7.11}
$$

where E denotes the kinetic energy. Particles 1 and 2 are vector-mesons and θ is the angle between them. If we use the total energy e we obtain

$$
\begin{aligned}
|\mathcal{M}|^2 =& \left|f^{cab}\right|^2 \frac{g_2^2}{3}\left(3\frac{E_1^2}{m_1^2}+3\frac{E_2^2}{m_2^2}+2\frac{E_1^2E_2^2}{m_1^2m_2^2}\right)-2\left|f^{cab}\right|^2\frac{g_2^2}{3}\frac{\sqrt{m_1^2+E_1^2}\sqrt{m_2^2+E_2^2}}{m_1^2m_2^2}E_1E_2\cos(\theta)\\
=& \left|f^{cab}\right|^2 \frac{g_2^2}{3}\left(3\frac{e_1^2-m_1^2}{m_1^2}+3\frac{e_2^2-m_2^2}{m_2^2}+2\frac{\left(e_1^2-m_1^2\right)\left(e_2^2-m_2^2\right)}{m_1^2m_2^2}\right)-2\left|f^{cab}\right|^2\frac{g_2^2}{3}\frac{e_1e_2}{m_1^2m_2^2}E_1E_2\cos(\theta)\\
=& \left|f^{cab}\right|^2 \frac{g_2^2}{3}\left(-6+3\frac{e_1^2}{m_1^2}+3\frac{e_2^2}{m_2^2}+2\frac{\sqrt{e_1^2-m_1^2}\sqrt{e_2^2-m_2^2}}{m_1^2m_2^2}\right)-2\left|f^{cab}\right|^2\frac{g_2^2}{3}\frac{e_1e_2}{m_1^2m_2^2}E_1E_2\cos(\theta)\\
=& \left|f^{cab}\right|^2 g_2^2\frac{1}{2}\left(-4+2\frac{e_1^2}{m_1^2}+2\frac{e_2^2}{m_2^2}+\frac{4}{3}\frac{\sqrt{e_1^2-m_1^2}\sqrt{e_2^2-m_2^2}}{m_1^2m_2^2}\right)-2\left|f^{cab}\right|^2\frac{g_2^2}{3}\frac{e_1e_2}{m_1^2m_2^2}E_1E_2\cos(\theta)\,.
\end{aligned}
\tag{7.12}
$$

As a reminder we again state the coupling constant

$$
\begin{aligned}
g_2 =& \int 12Cr^3N_4U_{KK}\phi_0\psi_1^2\\
=&5132.095\lambda^{-1}N_C^{-\frac{3}{2}}.
\end{aligned}
\tag{7.13}
$$

The phase-space integral reads

$$
\frac{\Gamma}{m_G}=\frac{1}{(2\pi)^3}\frac{1}{32m_G^4}\int|\mathcal{M}|^2\,, \tag{7.14}
$$

where we integrate over the physical phase space. We obtain the solutions displayed in table 9.

Channel	Γ/m_G	#	Decay Products
$\pi\rho\rho$	0.0864591	3	$\pi_1\rho_2\rho_3$,$\pi_2\rho_1\rho_3$,$\pi_3\rho_1\rho_2$
πK^*K^*	0.00355496	6	$\pi_1K_1^*K_4^*$,$\pi_1K_2^*K_3^*$,$\pi_2K_1^*K_3^*$,$\pi_2K_2^*K_4^*$,$\pi_3K_1^*K_2^*$,$\pi_3K_3^*K_4^*$,
$K\rho K^*$	0.000266598	12	$K_1\rho_1K_4^*$,$K_1\rho_2K_3^*$,$K_1\rho_3K_2^*$,$K_2\rho_1K_3^*$,...
$KK^*\omega$	0.000263124	4	$K_1K_2^*\omega$,$K_2K_1^*\omega$,$K_3K_4^*\omega$,$K_4K_3^*\omega$
$KK^*\phi$	0	4	$K_1K_2^*\phi$,$K_2K_1^*\phi$,$K_3K_4^*\phi$,$K_4K_3^*\phi$
$\eta'K^*K^*$	0	2	$\eta'K_1^*K_2^*$,$\eta'K_3^*K_4^*$
ηK^*K^*	0	2	$\eta K_1^*K_2^*$,$\eta K_3^*K_4^*$
Sum	0.284959	33	

Table 9. 3-body decay rates.

Part V

Conclusion

In the first part we have briefly reviewed the Witten model, which is a predecessor of the Witten-Sakai-Sugimoto model. We have calculated metric and field strength fluctuations, which are supposed to be dual to glueballs. These fluctuations are the same in the Witten-Sakai-Sugimoto model. In part II we considered the Witten-Sakai-Sugimoto model, in which fluctuations of the D8 branes correspond to $q\bar{q}$-states in the dual field theory. We obtained an effective Lagrangian that contains glueballs, mesons and their interactions, neglecting the subdominant DBI action. We have calculated decays of the pseudo-vector glueball in 2 and 3 mesons, which we found to be relativity broad. Experimentally this means that the pseudo-vector glueball might be very hard to detect.

For our calculation we left the usual large N_C-limit and used a finite 't Hooft coupling, thus backreactions and other corrections might become important. In the Witten-Sakai-Sugimoto model quarks are massless. In [14] a holographic mechanism to include quark masses. The idea is similar to extended technicolor theories. In our calculation we however used, as a first approximation, experimentally obtained meson masses. As a future work it might be interesting to study methods to include explicit quark masses, since such mechanisms could lead to additional glueball-meson interaction terms. It could also turn out to be useful to study the D4-D6 system of [15], since it might be easier to include explicit quark masses to such a brane-setup.

Another possible research topic uses the Witten model to study dark matter. Non-abelian gauge theories are candidates for self-interacting dark matter [16]. Glueball cross-sections compatible with cosmological observations were calculated in [12], using an estimate for the leading self-interaction Lagrangian. The Witten model would allow to derive such cross-sections from first principles. It will be interesting to see how, or if, it is possible to reproduce these results.

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References

[1] C. J. Morningstar and M. J. Peardon, *The Glueball spectrum from an anisotropic lattice study*, *Phys. Rev.* **D60** (1999) 034509, [hep-lat/9901004].

[2] W. Ochs, *The Status of Glueballs*, *J. Phys.* **G40** (2013) 043001, [1301.5183].

[3] J. Erdmenger, N. Evans, I. Kirsch and E. Threlfall, *Mesons in Gauge/Gravity Duals - A Review*, *Eur. Phys. J.* **A35** (2008) 81–133, [0711.4467].

[4] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505–532, [hep-th/9803131].

[5] R. C. Brower, S. D. Mathur and C.-I. Tan, *Glueball spectrum for QCD from AdS supergravity duality*, *Nucl. Phys.* **B587** (2000) 249–276, [hep-th/0003115].

[6] N. R. Constable and R. C. Myers, *Spin two glueballs, positive energy theorems and the AdS / CFT correspondence*, *JHEP* **10** (1999) 037, [hep-th/9908175].

[7] T. Sakai and S. Sugimoto, *Low energy hadron physics in holographic QCD*, *Prog. Theor. Phys.* **113** (2005) 843–882, [hep-th/0412141].

[8] K. Hashimoto, C.-I. Tan and S. Terashima, *Glueball decay in holographic QCD*, *Phys. Rev.* **D77** (2008) 086001, [0709.2208].

[9] F. Brünner, D. Parganlija and A. Rebhan, *Glueball Decay Rates in the Witten-Sakai-Sugimoto Model*, *Phys. Rev.* **D91** (2015) 106002, [1501.07906].

[10] J. Polchinski, *String Theory: Volume 2, Superstring Theory and Beyond* -. Cambridge University Press, Cambridge, 1998.

[11] R. M. Wald, *General Relativity* -. University of Chicago Press, Chicago, 2010.

[12] L. Forestell, D. E. Morrissey and K. Sigurdson, *Non-Abelian Dark Forces and the Relic Densities of Dark Glueballs*, *Phys. Rev.* **D95** (2017) 015032, [1605.08048].

[13] Particle Data Group collaboration, C. Patrignani et al., *Review of Particle Physics*, *Chin. Phys.* **C40** (2016) 100001.

[14] K. Hashimoto, T. Hirayama, F.-L. Lin and H.-U. Yee, *Quark Mass Deformation of Holographic Massless QCD*, *JHEP* **07** (2008) 089, [0803.4192].

[15] M. Kruczenski, D. Mateos, R. C. Myers and D. J. Winters, *Towards a holographic dual of large N(c) QCD*, *JHEP* **05** (2004) 041, [hep-th/0311270].

[16] S. Tulin and H.-B. Yu, *Dark Matter Self-interactions and Small Scale Structure*, 1705.02358.