Diplomarbeit

The stochastic heat equation in two dimensions

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THE STOCHASTIC HEAT EQUATION IN TWO DIMENSIONS

Abstract. For the equation $dv(t) = \left(\frac{1}{2}\Delta v + F_t\right)\, dt + \nabla v \cdot dW_t$ in 2 dimensions with $F_t\phi := \int_0^t \phi(-W_r - \mu)\, dr$, we will show the existence and uniqueness of a solution in the sense of tempered distributions. Further, a connection between this solution and the self-intersection local time of a planar Brownian motion will be establised. We will also show that the first and second moment of the solution satisfy, in the sense of tempered distributions, certain PDEs and the moment generating function satisfies a certain PDE in the sense of distributions. A byproduct of this result is the existence of the moment generating function of the self-intersection local time $\mathbb{E}[\exp(\theta\beta_2(x,t))]$ for points $x \neq 0$ and certain values of θ .

Contents

1. Introduction 2
2. A Crash course in SPDEs 3
3. Self Intersection Local Times (SILTs) of Brownian Motions 23
4. An Interesting SPDE 43
4.1. The First Moment 51
4.2. The Second Moment 55
4.3. The Moment Generating Function 57
5. Appendix 67
5.1. A weak form of Ito's Lemma 67
5.2. Some results regarding a "Fundamental solution" 69
5.3. A list of useful integrals 70
References 72

1. Introduction

In this work, we will try to look at the stochastic heat equation in two dimensions. As this is a rather broad topic, I already have to disappoint the reader by narrowing it down to a special case, which was (to my knowledge) first introduced in their studies of the limit order book by Hubalek, Krühner and Rheinländer. In [22] and [23], they considered the equation

$$dv(t) = \left(\frac{1}{2}D^2 v(t) + f\right)\, dt + Dv(t)\, dW_t,$$

$$v(0) = 0,$$

where $f = \delta_\mu$ for some $\mu \in \mathbb{R}_+$ in order to describe the accumulation of orders at specific levels. Here W_t is a one-dimensional Brownian motion. Their study revealed that a (in a certain sense) weak solution is directly connected to the Brownian local time and the moments of the solution satisfy certain PDEs.

The initial goal was to find a similar representation when we consider the same equation in the two-dimensional case ($\mathbb{R}^2$, planar Brownian motion). Unfortunately this task proved rather difficult and the obtained results will therefore only be mentioned in the Appendix. The assumption that such a L^2-representation of a solution, if it existed, could be connected to the self-intersection local time (SILT) of a planar Brownian motion lead then to a slightly different SPDE, which turned out to be slightly easier to handle than the originally proposed equation.

The equation studied was derived by the attempt to find a possible connection between the "fundamental solution" ($f = \delta.$) and the SILT. We also take a look at the first two moments of the solution, but first we will introduce the concept of SILT and collect results from different approaches, which will prove to be rather useful later on. Another chapter will be devoted to the theory of SPDEs. The approach taken in this chapter is rather old for this subject and doesn't use the more "hip" semigroup approach which is known from [28, 13], but provides results about the existence and uniqueness of distributional solutions. Nonetheless, the Ansatz used by Hubalek, Krühner and Rheinländer to find an explicit representation of the solution which we will follow too, is rooted in Duhamel's principle and thereby closely related to the semigroup-approach.

2. A Crash course in SPDEs

In this part we want to consider SPDEs on $\mathbb{R}^d$ of the following form:

$$du(t) = \sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}\frac{\partial u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu + f dt + \left(\sum_{k=0}^{N}\left(\sum_{i=1}^{d}\sigma_{ik}\frac{\partial u}{\partial x_i} + \nu_k u + g_k\right) dW_t^k\right) \tag{2.1}$$

Remark 1. We will try to find solutions in the Sobolev-Space $W^{2,p}(\mathbb{R}^d)$. An inconvenience which arises is that $W^{2,n}(\mathbb{R}^d) \hookrightarrow C^{n-\frac{d}{2}}(\mathbb{R}^d)$ if and only if $2n > d$. It can be shown that the solution belongs to $W^{2,n}(\mathbb{R}^d)$ only if the coefficients are $n-2$ times continuously differentiable with respect to $x \in \mathbb{R}^d$, so we have to suppose that our coefficients are more than $m + \frac{d}{2} - 2$ times continuously differentiable even if the free terms belong to $C_0^\infty(\mathbb{R}^d)$. At the same time, $W^{n,p}(\mathbb{R}^d) \hookrightarrow C^{n-\frac{d}{p}}(\mathbb{R}^d)$ if $pn > d$. By taking p sufficiently large, we see that the solutions have almost as many usual derivatives as weak ones.

Let us rewrite equation (2.1) into a more compact form:

$$du(t) = (Lu + f)\ dt + (\Lambda_k u + g_k)\ dW_t^k \qquad t > 0,$$

where

$$Lu := \sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}\frac{\partial u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu,$$

$$\Lambda_k u = \sum_{k=0}^{N}\left(\sum_{i=1}^{d}\sigma_{ik}\frac{\partial u}{\partial x_i} + \nu_k u\right).$$

We assume W_t^k to be independent Brownian motions.

Considering the (deterministic) case, where all σ_k, ν_k and g_k vanish, it would be adequate to recall some aspects, coming from the theory of parabolic PDEs.

With $\mathcal{D}$, we will denote the space of real valued Schwarz distributions on $\mathbb{R}^d$, defined on $C_0^\infty(\mathbb{R}^d)$. For a given $p \in (1,\infty)$ and $n \in (-\infty,\infty)$, the space $H^{n,p}(\mathbb{R}^d)$ is defined as the space of (generalized) functions, such that $(1-\Delta)^{\frac{n}{2}} u \in L^p(\mathbb{R}^d)$. To give a proper meaning to this definition, let us introduce the term $(1-\Delta)^{\frac{n}{2}}$ in a slightly different way. Let $\alpha \in (0,1)$, then, for a constant c_α, and all $z < 0$

$$(1-z)^\alpha = c_\alpha \int_0^\infty \frac{\exp(-t)\exp(tz) - 1}{t^\alpha}\frac{1}{t}\, dt.$$

By a formal substitution of Δ instead of z, we get the following definition

$$(1-\Delta)^\alpha u = c_\alpha \int_0^\infty \frac{\exp(t)T_t u - u}{t^\alpha}\frac{1}{t}\, dt, \tag{2.2}$$

where T_t denotes the semigroup generated by Δ. We formally substituted e^{zt} which is the solution of the ODE $f' = zf$ by the solution of $T_t' = \Delta T_t$.

As a quick reminder, T_t is given by

$$T_t u(x) := \frac{1}{(4\pi t)^{\frac{d}{2}}}\int_{\mathbb{R}^d} u(y)\exp\left(-\frac{1}{4t}|x-y|^2\right)\, dy. \tag{2.3}$$

Similarly, we define for any $\alpha > 0$

$$(1-\Delta)^{-\alpha} u = d_\alpha \int_0^\infty t^\alpha \exp(-t) T_t u \frac{1}{t}\, dt, \tag{2.4}$$

with an appropriate constant d_α. It turns out that these formulas are sufficient to consistently define $(1-\Delta)^{\frac{n}{2}}$ for any $n \in (-\infty, \infty)$.

The application of $(1-\Delta)^{\frac{n}{2}}$ to an $f \in L^p$ is defined as a limit of the respective truncated integral in (2.2) or (2.4). We say that a distribution $u \in H^{n,p}$, if there exists an $f \in L^p$, such that $u = (1-\Delta)^{\frac{n}{2}} f$ in the sense of distributions.

For $u \in H^{n,p}$, we introduce the following norm

$$\|u\|_{n,p} := \left\|(1-\Delta)^{\frac{n}{2}} u\right\|_p ,$$

where $\|.\|_p$ denotes the usual L^p norm.

It can be shown ([39]) that $H^{n,p}$ as defined above is a Banach space and C_0^∞ lies dense.

For fixed $T > 0$, we introduce the space $H_p^{1,2}(T) = H^{1,p}((0,T), H^{2,p}(\mathbb{R}^d))$ as

$$\left\{ u(t,x) \,:\, \|u\|_{1,2,p}^p := \int_0^T \left\| \frac{\partial u}{\partial t}(t,\cdot) \right\|_p^p dt + \int_0^T \|u(t,\cdot)\|_{2,p}^p \, dt < \infty \right\}.$$

Proposition 2. *For any $f \in L^p((0,T)\times\mathbb{R}^d)$ and $u_0 \in H_p^{2-\frac{2}{p}}$ there exists a unique solution $u \in H_p^{1,2}(T)$ of the (deterministic) heat equation*

$$\frac{\partial u}{\partial t} = \Delta u + f,$$

on $(0,T)\times\mathbb{R}^d$ with initial data $u(0) = u_0$.

In addition,

$$\left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^p((0,T)\times\mathbb{R}^d)} + \left\| \frac{\partial u}{\partial t} \right\|_{L^p((0,T)\times\mathbb{R}^d)} \le N(d,p)(\|f\|_{L^p((0,T)\times\mathbb{R}^d)} + \|u_0\|_{2-\frac{2}{p},p}), \tag{2.5}$$

$$\|u\|_{1,2,p} \le N(d,p,T)(\|f\|_{L^p((0,T)\times\mathbb{R}^d)} + \|u_0\|_{2-\frac{2}{p},p}).$$

Remark 3. For integers $n \ge 0$ the space $H^{n,p}$ coincides with the Sobolev space $W^{n,p}$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $(\mathcal{F}_t, t \ge 0)_t$ an increasing filtration of σ fields $\mathcal{F}_t \subset \mathcal{F}$ containing all $\mathbb{P}$ null subsets of Ω and $\mathcal{P}$ the predictable σ field generated by by $(\mathcal{F}_t,\, t \ge 0)_t$. Let $\{W_t^k;\, k = 1, 2, \ldots\}$ be a family of independent one dimensional $\mathcal{F}_t$ adapted Brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We fix a $p \ge 2$ and an integer $d \ge 1$ and consider a distribution u and a function $\phi \in C_0^\infty$. We observe that, for $u \in H^{n,p}$ and $\phi \in C_0^\infty$, by definition we get

$$(u,\phi) = \left\langle (1-\Delta)^{\frac{n}{2}} u, (1-\Delta)^{-\frac{n}{2}} \phi \right\rangle = \int_{\mathbb{R}^d} \left((1-\Delta)^{\frac{n}{2}} u \right)(x) (1-\Delta)^{-\frac{n}{2}} \phi \, dx.$$

Since $(1-\Delta)^{\frac{n}{2}} u \in L^p$, (u,ϕ) can be defined for any ϕ whose derivatives vanish sufficiently fast at infinity.

We can apply the same definition to l^2 valued functions h (l^2 being the real valued sequence space of square summable sequences).

For this section, let us define the following norm

$$|||h|||_p = \|\, \|h\|_{l^2}\, \|_p, \qquad |||h|||_{n,p} := |||(1-\Delta)^{\frac{n}{2}} h\|_{l^2}\|_p.$$

For stopping times τ, we denote $(0,\tau]\!] := \{(\omega,t)\,:\, 0<\mathrm{t}\leq \tau(\omega)\}$and

$$\mathbb{H}^{n,p}(\tau) := L^p((0,\tau]\!], \mathcal{P}, H^{n,p}),$$

$$\mathbb{H}^{n,p} := \mathbb{H}^{n,p}(\infty),$$

$$\mathbb{H}^{n,p}(\tau, l^2) = L^p((0,\tau]\!], \mathcal{P}, H^{n,p}(\mathbb{R}^d, l^2)),$$

$$\mathbb{L}(\dots) := \mathbb{H}^{0,2}(\dots).$$

These spaces are equipped with the natural/obvious norms. Out of convenience, we treat the elements of these spaces as functions and, if for a given element, there exists a modification with "better" properties, we will always immediately consider this modification.

Although the spaces $\mathbb{H}^{n,p}$ carry some familiarity, elements of the space $\mathbb{H}^{n,p}(\tau, l^2)$ need not be defined on, or belong to $H^{n,p}$ for all $(\omega, t) \in (0,\tau]\!]$. These properties are, as usual, only needed for almost all (ω, t).

For $n \in \mathbb{R}$ and

$$(f,g) \in \mathcal{F}^{n,p}(\tau) := \mathbb{H}^{n,p}(\tau) \times \mathbb{H}^{n+1,p}(\tau, l^2),$$

we define

$$\|(f,g)\|_{\mathcal{F}^{n,p}(\tau)} := \|f\|_{\mathbb{H}^{n,p}(\tau)} + \|g\|_{\mathbb{H}^{n+1,p}(\tau,l^2)}.$$

Definition 4. For a $\mathcal{D}$ valued function $u \in \bigcap_{T>0} \mathbb{H}^{n,p}(\tau\wedge T)$, we write $u \in \mathcal{H}^{n,p}(\tau)$, if $u_{xx} \in \mathbb{H}^{n-2,p}(\tau)$, $u(0,\cdot) \in L^p(\Omega, \mathcal{F}_0, H^{n-\frac{2}{p},p})$ and there exists a pair $(f,g) \in \mathcal{F}^{n-2,p}(\tau)$ such that for any $\phi \in C_0^\infty$, the equality

$$\langle u(t,\cdot),\phi\rangle = \langle u(0,\cdot),\phi\rangle + \int_0^t \langle f(s,\cdot),\phi\rangle\, ds + \sum_{k=1}^{\infty}\int_0^t \langle g_k(s,\cdot),\phi\rangle\, dW_s^k \tag{2.6}$$

holds for all $t \leq \tau$ with probability 1. We also define

$$\mathcal{H}_0^{n,p}(\tau) := \mathcal{H}^{n,p}(\tau) \bigcap \{u\,:\, u(0,\cdot) = 0\},$$

$$\|u\|_{\mathcal{H}^{n,p}(\tau)} := \|u_{xx}\|_{\mathbb{H}^{n-2,p}(\tau)} + \|(f,g)\|_{\mathcal{F}^{n-2,p}(\tau)} + \left(\mathbb{E}\left[\|u(0,\cdot)\|^p_{n-\frac{2}{p},p}\right]\right)^{\frac{1}{p}}.$$

As always, if $\tau = \infty$, we drop it in $\mathcal{H}^{n,p}(\tau)$ and $\mathcal{F}^{n,p}(\tau)$.

Remark 5. The elements of $\mathcal{H}^{n,p}(\tau)$, which is obviously a linear space, can be assumed to be defined for all (ω, t) and to take values in $\mathcal{D}$. Two elements of $\mathcal{H}^{n,p}(\tau)$ are, as usual, identified with each other, if $\|u_1 - u_2\|_{\mathcal{H}^{n,p}(\tau)} = 0$. It is also worth noting that the series of stochastic integrals $\sum_{k=1}^{\infty}\int_0^t \langle g_k(s,\cdot),\phi\rangle\, dW_s^k$

converges uniformly in t in probability on $[0,\tau\wedge T]$ for any finite T, since its quadratic variation satisfies

$$\sum_{k=1}^{\infty}\int_0^{\tau\wedge T}\langle g_k(s,\cdot),\phi\rangle^2\,ds=\sum_{k=1}^{\infty}\int_0^{\tau\wedge T}\left\langle(1-\Delta)^{\frac{n-1}{2}}g_k(s,\cdot),(1-\Delta)^{\frac{1-n}{2}}\phi\right\rangle^2\,ds$$

$$\le\left\|(1-\Delta)^{\frac{1-n}{2}}\phi\right\|_1\int_0^{\tau\wedge T}\sum_{k=1}^{\infty}\left\langle\left|(1-\Delta)^{\frac{n-1}{2}}g_k(s,\cdot)\right|^2,\left|(1-\Delta)^{\frac{1-n}{2}}\phi\right|\right\rangle\,ds$$

$$\le N\int_0^{\tau\wedge T}\left\|\left(\sum_{k=1}^{\infty}\left|(1-\Delta)^{\frac{n-1}{2}}g_k(s,\cdot)\right|^2\right)^{\frac12}\right\|_p^2\,ds<\infty\qquad\text{a.s.},$$

with $N:=\left\|(1-\Delta)^{\frac{1-n}{2}}\phi\right\|_1\left\|(1-\Delta)^{\frac{1-n}{2}}\phi\right\|_q$, $q:=\frac{p}{p-2}$. We have also used that $p\ge 2$.

$\langle u(t,\cdot),\phi\rangle$ is continuous in t on $[0,\tau\wedge T]$, as a consequence of the uniform convergence for any finite T (a.s.).

Remark 6. The pair (f,g) is unique, as otherwise 0 could be written as the sum of a continuous process of finite variation and a continuous local martingale, which is only possible, if both processes vanish.

Remark 7. The operator $(1-\Delta)^{\frac{m}{2}}$ maps $H^{n,p}$ isometrically onto $H^{n-m,p}$ for any n and m. The previous remarks also show, that the same relation holds true for $\mathcal{H}^{n,p}(\tau)$, since for any given $u\in\mathcal{H}^{n,p}(\tau)$, we can take functions ϕ whose derivatives vanish exponentially fast at infinity and substitute ϕ with $(1-\Delta)^{\frac{m}{2}}\phi$, which gives us this result. We also have the same result for $\mathbb{H}^{n,p}(\tau)$.

Definition 8. If (2.6) holds for $u\in\mathcal{H}^{n,p}(\tau)$, we write $f=Au$, $g=Bu$ and also

$$u(t)=u(0)+\int_0^t Au(s)\,ds+\sum_{k=1}^{\infty}\int_0^t B^k u(s)\,dW_s^k\qquad t\le\tau,$$

$$du=f\,dt+\sum_{k=1}^{\infty}g^k\,dW_s^k\qquad t\le\tau.$$

Remark 9. A is a continuous operator from $\mathcal{H}^{n,p}(\tau)$ to $\mathbb{H}^{n-2,p}(\tau)$ and B is a continuous operator from $\mathcal{H}^{n,p}(\tau)$ to $\mathbb{H}^{n-2,p}(\tau,l_2)$ (which follows directly from the definitions). Even though we don't know that much about $\mathcal{H}^{n,p}(\tau)$, it is obvious, that $H^{12,p}(\tau)\subset\mathcal{H}^{2,p}(\tau)$.

Theorem 10. *The spaces $\mathcal{H}^{n,p}(\tau)$ and $\mathcal{H}_0^{n,p}(\tau)$, equipped with the norm*

$$\|u\|_{\mathcal{H}^{n,p}(\tau)}:=\|u_{xx}\|_{\mathbb{H}^{n-2,p}(\tau)}+\|(f,g)\|_{\mathcal{F}^{n-2,p}(\tau)}+\left(\mathbb{E}\left[\|u(0,\cdot)\|^p_{n-\frac{2}{p},p}\right]\right)^{\frac1p},$$

are Banach spaces. If $\tau\le T$ for a finite T, then for $u\in\mathcal{H}^{n,p}(\tau)$ the following holds

$$\|u\|_{\mathbb{H}^{n,p}(\tau)}\le N(d,T)\|u\|_{\mathcal{H}^{n,p}(\tau)},\tag{2.7}$$

$$\mathbb{E}\left[\sup_{t\le\tau}\|u(t,\cdot)\|^p_{n-2,p}\right]\le N(d,T)\|u\|^2_{\mathcal{H}^{n,p}(\tau)}. \tag{2.8}$$

Proof. Obviously, $\|u\|_{\mathbb{H}^{n,p}(\tau)}=\|(1-\Delta)u\|_{\mathbb{H}^{n-2,p}(\tau)}\le\|u\|_{\mathbb{H}^{n-2,p}(\tau)}+\|u\|_{\mathcal{H}^{n,p}(\tau)}$.

We will remind ourselves of the previous remarks and assume $n=2$. Let us take a nonnegative $\rho\in C_c^\infty$ with integral equal to 1 and define $\rho_\epsilon(x)=\frac{\rho(\frac{x}{\epsilon})}{\epsilon}$, and for functions u, let $u^{(\epsilon)}(x):=u*\rho_\epsilon(x)$. $u^{(\epsilon)}$ is still a continuous, infinitely differentiable function for any distribution u. If we plug $\rho_\epsilon(\cdot-x)$ into (2.6) instead of ϕ, we get for any x that the following equality holds almost surely

$$u^{(\epsilon)}(t)=u^{(\epsilon)}(0)+\int_0^t f^{(\epsilon)}(s,x)\,ds+\sum_{k=1}^\infty\int_0^t g^{(\epsilon)k}(s,x)\,dW^k_s\qquad t\le\tau. \tag{2.9}$$

If necessary, we redefine the stochastic integral in such a way, that (2.9) holds for all ω, t and x, such that $t\le\tau$.

$$\mathbb{E}\left[\|u^{(\epsilon)}(0,\cdot)\|^p_p\right]\le\mathbb{E}\left[\|u(0,\cdot)\|^p_p\right]\le\mathbb{E}\left[\|u(0,\cdot)\|^p_{\frac{n-2}{p},p}\right]\le\|u\|_{\mathcal{H}^{n,p}(\tau)},$$

where we used that, by Minkowski's inequality, $\|h^{(\epsilon)}\|_p\le\|\rho_\epsilon\|_1\|h\|_p=\|h\|_p$. Similarly

$$\left|\int_0^t f^{(\epsilon)}(s,x)\,ds\right|^p\le T^{p-1}\int_0^\tau|f^{(\epsilon)}(s,x)|^p\,ds,$$

$$\mathbb{E}\left[\sup_{t\le\tau}\left\|\int_0^t f^{(\epsilon)}(s,x)\,ds\right\|_p^p\right]\le T^{p-1}\mathbb{E}\left[\int_0^t\left\|f^{(\epsilon)}(s,x)\right\|_p^p\,ds\right]\le T^{p-1}\|u\|^p_{\mathcal{H}^{n,p}(\tau)}.$$

By Burkholder-Davis-Grundy inequalities,

$$\mathbb{E}\left[\sup_{t\le\tau}\left|\sum_{k=1}^\infty\int_0^t g^{(\epsilon)k}(s,x)\,dW^k_s\right|^p\right]\le N\mathbb{E}\left[\left|\int_0^\tau\sum_{k=1}^\infty|g^{(\epsilon)k}(s,x)|^2\,ds\right|^{\frac{p}{2}}\right]$$
$$=N\mathbb{E}\left[\left|\int_0^\tau\|g^{(\epsilon)}(s,x)\|^2_{l_2}\,ds\right|^{\frac{p}{2}}\right].$$

As above ((2.9) gives again sense to the first term below)

$$\mathbb{E}\left[\sup_{t\le\tau}\left\|\sum_{k=1}^\infty\int_0^t g^{(\epsilon)k}(s,x)\,dW^k_s\right\|_p^p\right]|\le\int_{\mathbb{R}^d}\mathbb{E}\left[\sup_{t\le\tau}\left|\sum_{k=1}^\infty\int_0^\tau g^{(\epsilon)k}(s,x)\,ds\right|^p\right]dx$$

$$\le N\mathbb{E}\left[\int_{\mathbb{R}^d}\left|\int_0^\tau\|g^{(\epsilon)}(s,x)\|^2_{l_2}\,ds\right|^{\frac{p}{2}}dx\right]\le N\mathbb{E}\left[\left(\int_0^\tau\left\|\|g^{(\epsilon)}(s,\cdot)\|^2_{l_2}\right\|_{\frac{p}{2}}ds\right)^{\frac{p}{2}}\right]$$

$$=N\mathbb{E}\left[\left(\int_0^\tau\left\|\|g^{(\epsilon)}(s,\cdot)\|_{l_2}\right\|_p^2ds\right)^{\frac{p}{2}}\right]\le N\mathbb{E}\left[\int_0^\tau\left\|\|g^{(\epsilon)}(s,\cdot)\|_{l_2}\right\|_p^p ds\right]$$

$$\leq N\|g\|^p_{\mathbb{L}_p(\tau,l_2)} \leq N\|u\|_{\mathcal{H}^{n,p}(\tau)}.$$

Along with (2.9), this leads to

$$\mathbb{E}\left[\sup_{t\leq\tau}\|u^{(\epsilon)}(t,\cdot)\|^p_p\right] \leq N\|u\|^p_{\mathcal{H}^{n,p}(\tau)}. \tag{2.10}$$

Using the fact, that $\|h^{(\epsilon)} - h^{(\gamma)}\|_p \to 0$ for $h \in L^p$, when ϵ, $\gamma \to 0$, considering $u^{(\frac{1}{m})} - u^{(\frac{1}{k})}$, we see that $u^{(\frac{1}{m})}(t\wedge\tau, x)$ is a Cauchy sequence in $L^p(\Omega, B([0,T], L^p))$. Let us denote the limit in this space by v. For a subsequence m', we have $u^{(\frac{1}{m'})}(t,\cdot) \to v(t,\cdot)$ in L^p for $t \leq \tau$, with probability 1. On the other hand $u^{(\frac{1}{m})}(t,\cdot) \to u(t,\cdot)$, in the sense of distributions for all ω, t, such that $t \leq \tau(\omega)$. Therefore, it holds for $t \leq \tau$, with probability 1, that $u(t,\cdot) \in L^p$. For $n = 2$, (2.10) and Fatou's lemma give us the second inequality of the Theorem.

Let us check now the completeness of $\mathcal{H}^{n,p}(\tau)$. If we take a Cauchy sequence u_i in $\mathcal{H}^{n,p}(\tau)$, then it is also a Cauchy sequence in $\mathbb{H}^{n,p}(\tau\wedge T)$ for any T and there exists a $u \in \cap_{T\geq 0}\mathbb{H}^{n,p}(\tau\wedge T)$, such that $\|u - u_i\|_{\mathbb{H}^{n,p}(\tau\wedge T)} \to 0$. Additionally, $\frac{\partial^2}{\partial x^2}u_i$ form a Cauchy sequence and therefore converge in $\mathbb{H}^{n-2,p}(\tau)$, from which follows, that $\left\|\frac{\partial^2}{\partial x^2}u_i - \frac{\partial^2}{\partial x^2}u\right\|_{\mathbb{H}^{n-2,p}(\tau)} \to 0$.

For $u_i(0)$, f_i , g_i, corresponding to u_i, there exist $u(0) \in L^p(\Omega, \mathcal{F}_0, H^{n-\frac{2}{p},p})$ and $(f, g) \in \mathcal{F}^{n-2,p}(\tau)$, such that

$$\mathbb{E}\left[\|u(0) - u_i(0)\|^p_{n-\frac{2}{p},p}\right] \to 0,$$

$$\|f - f_i\|_{\mathbb{H}^{n-2,p}(\tau)} \to 0,$$

$$\|g - g_i\|_{\mathbb{H}^{n-1,p}(\tau,l_2)} \to 0.$$

By using the remark directly after (2.6), one can show that for any $\phi \in C_0^\infty$, (2.6) holds in $(0,\tau\|$ almost everywhere.

On the other hand, the previously proven inequalities imply that (at least for a modification of) u

$$\mathbb{E}\left[\sup_{t\leq\tau\wedge T}\|u(t,\cdot) - u_i(t,\cdot)\|^p_{n-2,p}\right] \to 0,$$

for any constant $t < \infty$. Remarking, that the processes $\langle u_i(t,\cdot), \phi\rangle$ are a.s. continuous, we can conclude, that $\langle u(t,\cdot), \phi\rangle$ is also a.s. continuous. Thus, (2.6) holds not only in $(0,\tau]\!]$ almost everywhere, but also for all $t \leq \tau$ almost surely. Hence $u \in \mathcal{H}^{n,p}(\tau)$ and $u_i \to u$ in $\mathcal{H}^{n,p}(\tau)$. □

Theorem 11. *Let $g \in \mathbb{H}^{n,p}(l_2)$, then there exists a sequence $g_i \in \mathbb{H}^{n,p}(l_2)$, such that $\|g - g_i\|_{\mathbb{H}^{n,p}(l_2)}$ and*

$$g_i^k = \begin{cases} \sum_{j=1}^i \chi_{(\tau^i_{j-1},\tau^i_j]}(t)g_i^{jk} & k \leq i \\ 0 & k > i \end{cases},$$

where $\tau^i_{j-1} \leq \tau^i_j$ are bounded stopping times and $g_i^{jk} \in C_c^\infty$.

Proof. Due to the argument in Remark 7 and the density of C_c^∞ in any $H^{n,p}$, we only need to consider the case $n = 0$. Further, we can easily see, that the set of $g \in \mathbb{L}^p(l_2)$ for which this statement holds is a linear, closed subspace $\mathbb{L}$ of $\mathbb{L}^p(l_2)$. What remains to show is that $\mathbb{L} = \mathbb{L}^p(l_2)$. If this was not true then there exists, by Riesz's theorem, a nonzero $h \in \mathbb{L}^q(l_2)$ (with $q = \frac{p}{p-1}$) such that

$$\mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d} \langle h, g\rangle_{l_2} \, dx \, dt\right] = 0$$

for any $g \in \mathbb{L}$. In particular

$$\mathbb{E}\left[\int_0^\infty \chi_{(0,\tau]} \left(\int_{\mathbb{R}^d} h^k, g \, dx\right) dt\right] = 0$$

for any bounded stopping time τ, $k \geq 1$ and $g \in C_c^\infty$. Since $\int_{\mathbb{R}^d} h^k, g \, dx$ is (almost everywhere equal to) a predictable function, it follows that $\int_{\mathbb{R}^d} h^k, g \, dx = 0$ on $(0, \infty]\!](\text{a.e.})$.

Taking g from a countable subset $\mathcal{G} \subset C_c^\infty$ that is dense in L^p, we get that on a subset of $(0, \infty]\!]$ of full measure

$$\int_{\mathbb{R}^d} h^k, g \, dx = 0 \qquad \forall \, g \in \mathcal{G}, \; k \geq 1.$$

But then $h^k = 0$ (a.e.) on $(0, \infty]\!] \times \mathbb{R}^d$, which contradicts $h \neq 0$. ☐

Theorem 12. *Let $T \in (0, \infty)$. If $u_i \in \mathcal{H}^{n,p}(T)$, $i = 1, 2, \ldots$, and $\|u_i\|_{\mathcal{H}^{n,p}(T)} \leq K$ for a finite constant K, then there exists a subsequence i' and a function $u \in \mathcal{H}^{n,p}(T)$, such that*

(i)

$$u_{i'} \rightharpoonup u \qquad in \, \mathbb{H}^{n,p}(T),$$

$$u_{i'}(0, \cdot) \rightharpoonup u(0, \cdot) \qquad in \, L^p(\Omega, H^{n-\frac{p}{2},p}),$$

$$Au_{i'} \rightharpoonup Au \qquad in \, \mathbb{H}^{n-2,p}(T),$$

$$Bu_{i'} \rightharpoonup Bu \qquad in \, \mathbb{H}^{n-1,p}(T, l_2).$$

(ii)

$$\|u\|_{\mathcal{H}^{n,p}(T)} \leq K.$$

(iii)

For any $\phi \in C_c^\infty$ and any $t \in [0, T]$, we have $\langle \phi, u_{i'}(t, \cdot)\rangle \rightharpoonup \langle \phi, u(t, \cdot)\rangle$ in $L^p(\Omega)$.

Proof. From the properties of the L^p spaces, the existence of a subsequence and i' and the weak convergence to some u, $u(0, \cdot)$, Au, Bu in the respective spaces. For any $\phi \in C_c^\infty$, the expressions in

$$\langle u_{i'}(t, \cdot), \phi\rangle = \langle u_{i'}(0, \cdot), \phi\rangle + \int_0^t \langle Au_{i'}(s, \cdot), \phi\rangle \, ds + \sum_{k=1}^\infty \int_0^t \left\langle B^k u_{i'}(s, \cdot), \phi\right\rangle dW_s^k,$$

converge in the corresponding spaces. Since integration and stochastic integration can be consider as continuous linear operators (which means that they are also weakly continuous operators), we have that for any $\phi \in C_c^\infty$,

$$\langle u(t,\cdot),\phi\rangle = \langle u(0,\cdot),\phi\rangle + \int_0^t \langle Au(s,\cdot),\phi\rangle\, ds + \sum_{k=1}^\infty \int_0^t \left\langle B^k u(s,\cdot),\phi\right\rangle\, dW_s^k, \tag{2.11}$$

for almost all $(\omega,t)\in\Omega\times[0,T]$.

By the Banach-Saks theorem, there exists a sequence $(v_{i'},Av_{i'},Bv_{i'})$ of convex combinations of $(u_{i'},Au_{i'},Bu_{i'})$, which converges strongly to (u,f,g) in $\mathbb{H}^{n,p}(T)\times\mathbb{H}^{n-2,p}(T)\times\mathbb{H}^{n-1,p}(T,l_2)$. From (2.8), it follows that

$$\mathbb{E}\left[\sup_{t\le T}\|v_i-v_j\|_{n-2,p}^p\right]\to 0$$

as $i,\ j\to\infty$. Therefore, there exists a $H^{n-2,p}$ valued function v on $\Omega\times[0,T]$, such that

$$\mathbb{E}\left[\sup_{t\le T}\|v_i-v\|_{n-2,p}^p\right]\to 0.$$

In particular, we have that for any $\phi\in C_c^\infty$ $\langle v_i(t,\cdot),\phi\rangle\to\langle v(t,\cdot),\phi\rangle$ uniformly on $[0,T]$ in probability. On the other hand, the strong convergence of v_i to u in $\mathbb{H}^{n,p}(T)$ implies that $\langle v_i(t,\cdot),\phi\rangle\to\langle u(t,\cdot),\phi\rangle$ on $\Omega\times[0,T]$ in measure. From this, we can conclude that $\langle v_i(t,\cdot),\phi\rangle\to\langle u(t,\cdot),\phi\rangle$ a.e.. Because ϕ was arbitrary and by the density of C_c^∞ in the spaces conjugate to $\mathbb{H}^{n,p}(T)$, $u=v$ a.e. on $\Omega\times[0,T]$ (as generalized functions).

Thus, we have $v\in\mathbb{H}^{n,p}(T)$ and as $\langle v_i(t),\phi\rangle$ are given by equations, similar to (2.11), implies that $\langle v_i(t),\phi\rangle$ is (a.s.) continuous in t. The uniform convergence of $\langle v_i(t),\phi\rangle$ to $\langle v(t),\phi\rangle$ yields the a.s. continuity of $\langle v(t),\phi\rangle$. By the above, (2.11) still holds for almost all $(\omega,t)\in\Omega\times[0,T]$, if $\langle u(t),\phi\rangle$ is replaced by $\langle v(t),\phi\rangle$. Since the latter is continuous and the right hand side of (2.11) is continuous as well, $\langle v(t),\phi\rangle$ equals the right hand side of (2.11) for all $t\in[0,T]$ (a.s.). Hence, $v\in\mathcal{H}^{n,p}(T)$, which shows (i) for v instead of u, but this is irrelevant.

(ii) follows from the a.e. equality of $u=v$ on $\Omega\times[0,T]$ and from the fact that the norm of the weak limit is less or equal to the liminf of the norms of the sequence (Banach-Steinhaus).

For (iii), we take a $\phi\in C_c^\infty$ and a $\psi\in L^q(\Omega)$ with $q=\frac{p}{p-1}$ and write

$$\mathbb{E}\left[\psi\left(\langle u_i(t,\cdot),\phi\rangle\right)\right]$$

$$=\mathbb{E}\left[\psi\left(\langle u_i(0,\cdot),\phi\rangle\right)\right]+\mathbb{E}\left[\psi\left(\int_0^t\langle Au_i(s,\cdot),\phi\rangle\, ds\right)\right]+\mathbb{E}\left[\psi\left(\sum_{k=1}^\infty\int_0^t\langle B^k u(s,\cdot),\phi\rangle\, dW_s^k\right)\right].$$

By the previously stated properties of the operators and (i),

$$\lim_{i'\to\infty}\mathbb{E}\left[\psi\left(\langle u_{i'}(t,\cdot),\phi\rangle\right)\right]$$

$$= \lim_{i'\to\infty} \left(\mathbb{E}\left[\psi\left(\langle u_{i'}(0,\cdot),\phi\rangle\right)\right] + \mathbb{E}\left[\psi\left(\int_0^t \langle Au_{i'}(s,\cdot),\phi\rangle\, ds\right)\right] + \mathbb{E}\left[\psi\left(\sum_{k=1}^{\infty}\int_0^t \langle B^k u_{i'}(s,\cdot),\phi\rangle\, dW_s^k\right)\right]\right)$$

$$= \mathbb{E}\left[\psi\left(\langle u(0,\cdot),\phi\rangle\right)\right] + \mathbb{E}\left[\psi\left(\int_0^t \langle Au(s,\cdot),\phi\rangle\, ds\right)\right] + \mathbb{E}\left[\psi\left(\sum_{k=1}^{\infty}\int_0^t \langle B^k u(s,\cdot),\phi\rangle\, dW_s^k\right)\right]$$

$$= \mathbb{E}\left[\psi\left(\langle u(t,\cdot),\phi\rangle\right)\right].$$

This proves (iii). □

Now, we want to look for functions $u \in \mathcal{H}_0^{n,p}(\tau)$, such that A, B are of the form
$Au = Lu + f$,
$Bu = Cu + g$.
So (2.1) will basically be of the form

(2.12)
$$du(t) = \sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(t)\frac{\partial^2 u}{\partial x_i \partial x_j} + cu + f\, dt + \left(\sum_{k=0}^{N}\left(\sum_{i=1}^{d}\sigma_{ik}(t)\frac{\partial u}{\partial x_i} + g_k\right) dW_t^k\right).$$

In order to make things a bit easier, we consider the special case

$$du(t,x) = \Delta u(t,x) + f(t,x)\, dt + \left(\sum_{k=0}^{N} g_k\, dW_t^k\right) \qquad t > 0. \tag{2.13}$$

In the following, the operators T_t are defined by (2.3) and $p \geq 2$.

Lemma 13. *Let $-\infty \leq a < b \leq \infty$, $g \in L^p((a,b)\times\mathbb{R}^d, l_2)$, then*
$\int_{\mathbb{R}^d}\int_a^b \left(\int_a^t \|\nabla T_{t-s} g(s,\cdot)(x)\|_{l_2}^2\, ds\right)^{\frac{p}{2}} dt\, dx \leq N(d,p)\int_{\mathbb{R}^d}\int_a^b \|g(t,x)\|_{l^2}^p\, dt\, dx.$

Proof. [?]. □

Theorem 14. *For $f \in \mathbb{H}^{-1,p}$, $g \in \mathbb{L}^p(l_2)$,*
(i) (2.13) with zero initial condition has a unique solution $u \in \mathcal{H}^{1,p}(\tau)$.
(ii) For this equation we have

$$\left\|\frac{\partial^2 u}{\partial x \partial x}\right\|_{\mathbb{H}^{-1,p}} \leq N(d,p)(\|f\|_{\mathbb{H}^{-1,p}} + \|g\|_{\mathbb{L}(l_2)}). \tag{2.14}$$

(iii) For this solution we have $u \in C_{loc}([0,\infty), L^p)$ almost surely, and for any λ, $T > 0$,

$$\mathbb{E}\left[\sup_{t\leq T}(\exp(-p\lambda t)\|u(t,\cdot)\|_p^p)\right] + \mathbb{E}\left[\int_0^T \exp(-p\lambda t)\left\| |u|^{\frac{p-2}{p}}\left|\frac{\partial u}{\partial x}\right|^{\frac{2}{p}}(t,\cdot)\right\|_p^p\right]$$

$$\leq N(d,p,\lambda)\left(\|\exp(-\lambda t)f\|_{\mathbb{H}^{-1,p}}^p + \|\exp(-\lambda t)g\|_{\mathbb{L}(T,l_2)}^p\right). \tag{2.15}$$

Proof. There exists a linear operator
$P\ :\ H^{-1,p}\to (L^p)^{d+1}$,
such that if $h\in H^{-1,p}$ and $Ph=(h_0,\tilde h^1,\dots,\tilde h^d)$, then $h=h_0+\operatorname{div}(\tilde h)$ and

$$\|\tilde h\|^p+\|h_0\|^p\le N(d,p)\|h\|_{-1,p},\quad \|h\|_{-1,p}\le N(d,p)(\|\tilde h\|^p+\|h_0\|^p). \tag{2.16}$$

Actually, we can choose $\tilde h=-\nabla((1-\Delta)^{-1}h)$ and $h_0=h-\operatorname{div}(\tilde h)=(1-\Delta)^{-1}h$.

Indeed, $\|h_0\|_p=\|h\|_{-2,p}\le\|h\|_{-1,p}$. Also, the fact, that $\frac{\partial}{\partial x_i}$ is a bounded operator from $H^{n,p}$ to $H^{n+1,p}$ for any n ([39]) means that $\frac{\partial}{\partial x_i}(1-\Delta)^{-\frac12}$ is a bounded operator from $H^{n,p}$ to $H^{n,p}$ and $\frac{\partial}{\partial x_i}(1-\Delta)^{-1}$ is a bounded operator from $H^{n,p}$ to $H^{n-1,p}$. This is the reason, why $\|\tilde h\|_p\le N(d,p)\|h\|_{-1,p}$. This results in the first estimate of (2.16). On the other hand $(1-\Delta)^{-\frac12}h=(1-\Delta)^{-\frac12}h_0+\operatorname{div}\left(\frac{\partial}{\partial x_i}(1-\Delta)^{-\frac12}\right)\tilde h$ and both operators on the right hand side are bounded on L^p.

Define $(f_0,\tilde f)=Pf$. Equation (2.13) takes the form

$$du=(\Delta u+f_0+\operatorname{div}(\tilde f))\,dt+\sum_k g^k\,dW_t^k, \tag{2.17}$$

and we supply it with zero initial condition. Now we will prove, that for arbitrary f_0, $\tilde f\in\mathbb{L}^p$, our assertions hold for (2.17) instead of (2.13). Obviously, in (2.14) and (2.15), $f=f_0+\operatorname{div}(\tilde f)$.

First we consider the (very particular) case, in which

$$f_0(t,x)=\sum_{i=1}^m\chi_{(\tau_{i-1},\tau_i]}(t)f_{0i}(x), \tag{2.18}$$

$$\tilde f(t,x)=\sum_{i=1}^m\chi_{(\tau_{i-1},\tau_i]}(t)\tilde f_i(x),$$

$$g(t,x)=\sum_{k=1}^m g^k(t,x)h_k,$$

$$g^k(t,x)=\sum_{i=1}^m\chi_{(\tau_{i-1},\tau_i]}(t)g^{ik}(x),$$

where $\{h_k\}$ is the standard orthonormal basis in l_2, $m<\infty$, τ_i are bounded stopping times with $\tau_{i-1}\le\tau_i$ and f_{0i}, $\tilde f$, $g^{ik}\in C_c^\infty$.

Set

$$v(t,x)=\sum_k\int_0^t g^k(s,x)\,dW_s^k=\sum_{i=1}^m\sum_{k=1}^m g^{ik}(x)\left(W^k_{t\wedge\tau_i}-W^k_{t\wedge\tau_{i-1}}\right),$$

$$u(t,x))v(t,x)+\int_0^t T_{t-s}(\Delta v+f)(s,\cdot)(x)\,ds,\quad \forall t\ge 0. \tag{2.19}$$

It is easy to see that, by definition, the function $u-v$ is infinitely often differentiable in (t,x) and satisfies the equation

$$\frac{\partial z}{\partial t} = \Delta z + \Delta v + f.$$

For any x, it follows that, the function $u(t,x)$ satisfies almost surely

$$u(t,x) = \int_0^t \left(\Delta u(s,x) + f(s,x)\right)\, ds + \sum_{k=1}^m \int_0^t g^k(s,x)\, dW_s^k. \tag{2.20}$$

Now, we want to obtain some bounds on the norms of u, for that we define

$$u_1(t,x) = \int_0^t T_{t-s} f(s,x)\, ds.$$

By Proposition (2), dealing with the deterministic case, for any ω,

$$\left\|\frac{\partial^2 u_1}{\partial x^2}\right\|_{L^p(\mathbb{R}_+, H^{-1,p})} \leq N\|f\|_{L^p(\mathbb{R}_+, H^{-1,p})}. \tag{2.21}$$

Using once again, that the operators $\frac{\partial}{\partial x_i}(1-\Delta)^{-\frac{1}{2}}$ are bounded in L^p for any $p > 1$.

(2.22)

$$\left\|\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_1}{\partial x^2}\right\|^p_{\mathbb{H}^{-1,p}} \leq N\left\|\frac{\partial u}{\partial x} - \frac{\partial u_1}{\partial x}\right\|^p_{\mathbb{L}^p} = N\int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left[\left|\frac{\partial u}{\partial x} - \frac{\partial u_1}{\partial x}\right|^p\right](t,x)\, dx\, dt.$$

We will make further transformations to this formula. If $z^k = z^k(x)$ are bounded Borel functions, then by Ito's formula, applied to the increment over $[0,t]$ of

$$\left(\int_r^t T_{t-s} z^k\, ds\right)\left(W^k_{r\wedge\tau_2} - W^k_{r\wedge\tau_1}\right)$$

as a function of r, we obtain a.s.

$$0 = -\int_0^t \left(W^k_{r\wedge\tau_2} - W^k_{r\wedge\tau_1}\right) T_{t-r} z^k\, dr + \int_0^t \chi_{(\tau_1,\tau_2]}(r)\left(\int_r^t T_{t-s} z^k\, ds\right)\, dW_r^k.$$

Either by using this for our particular g, or by using the stochastic version of the Fubini theorem and coming back to (2.19), for any $t \geq 0$ and $x \in \mathbb{R}^d$, we get (almost surely)

$$\frac{\partial u}{\partial x}(t,x) - \frac{\partial u_1}{\partial x}(t,x) = \frac{\partial v}{\partial x}(t,x) + \int_0^t T_{t-s}\sum_{k=1}^m \int_0^t \Delta\frac{\partial g^k}{\partial x}(r,x)\, dW_r^k\, ds$$

$$= \frac{\partial v}{\partial x}(t,x) + \sum_{k=1}^m \int_0^t\int_0^t \frac{d}{ds}T_{t-s}\frac{\partial g^k}{\partial x}(r,x)\, ds\, dW_r^k = \sum_{k=1}^m \int_0^t T_{t-r}\frac{\partial g^k}{\partial x}(r,x)\, dW_r^k.$$

By the Burkholder-Davis-Grundy inequality

$$\mathbb{E}\left[\left|\frac{\partial u}{\partial x}(t,x) - \frac{\partial u_1}{\partial x}(t,x)\right|^p\right](t,x) \leq N\mathbb{E}\left[\left(\int_0^t \sum_{k=1}^m \left|T_{t-s}\frac{\partial g^k}{\partial x}(r,x)\right|^2\, dr\right)^{\frac{p}{2}}\right]$$

$$= N\mathbb{E}\left[\left(\int_0^t \left\|T_{t-s}\frac{\partial g^k}{\partial x}(r,x)\right\|_{l_2}^2 \, dr\right)^{\frac{p}{2}}\right].$$

Applying this to (2.22) and applying Lemma (13),

$$\left\|\frac{\partial u}{\partial x}(t,x) - \frac{\partial u_1}{\partial x}(t,x)\right\|_{\mathbb{L}^p}^p \le N\mathbb{E}\left[\int_0^\infty \int_{\mathbb{R}^d}\left(\int_0^t \sum_{k=1}^m \left|T_{t-s}\frac{\partial g^k}{\partial x}(r,x)\right|^2 dr\right)^{\frac{p}{2}} dx\, dt\right]$$

$$\le N\|g\|_{\mathbb{L}^p(l_2)}^p.$$

Along with (2.21), this gives us (2.14). We do not know yet, if $u \in \mathcal{H}^{1,p}$. We want to prove (2.15) for sufficiently large λ.

From (2.20) and Ito's formula, we get

$$|u(t,x)|^p \exp(-\lambda t) = \int_0^t \exp(-\lambda t)(p|u|^{p-2}u\Delta u + p|u|^{p-2}uf$$
$$+\frac{1}{2}p(p-1)|u|^{p-2}\|g\|_{l_2}^2 - \lambda|u|^p)(s,x)\, ds$$
$$+p\sum_{k\le m}\int_0^t \exp(-\lambda t)|u|^{p-2}ug^k(s,x)\, dW_s^k.$$

We integrate with respect to x, use the stochastic Fubini theorem and the fact that $u(t,x)$, $g(t,x)$ and their derivatives decrease very fast, when $|x|\to\infty$. Then we integrate by parts in $\int |u|^{p-2}u\Delta u\, dx$ and notice that for $q = \frac{p}{p-1}$

$$\int_{\mathbb{R}^d}|u|^{p-2}uf(s,x)dx = -(p-1)\int_{\mathbb{R}^d}|u|^{p-2}\frac{\partial u}{\partial x}(t,x)\cdot\tilde{f}(s,x)dx + \int_{\mathbb{R}^d}|u|^{p-2}uf_0(s,x)dx,$$

$$\left|\int_{\mathbb{R}^d}|u|^{p-2}\frac{\partial u}{\partial x}(t,x)\cdot\tilde{f}(s,x)\, dx\right|$$
$$\le \int_{\mathbb{R}^d}\left(|u|^{\frac{p-2}{2}}\left|\frac{\partial u}{\partial x}(t,x)\right|\right)^q |u|^{q(\frac{p-2}{2})}\, dx + \|\tilde{f}(s,\cdot)\|_p^p$$
$$\le N\|f(s,\cdot)\|_{-1,p}^p + N_1\|u(s,\cdot)\|_p^p + \frac{1}{2}\left\||u|^{\frac{p-2}{p}}\left|\frac{\partial u}{\partial x}(t,x)\right|^{\frac{2}{p}}(s,\cdot)\right\|_p^p,$$

$$\int_{\mathbb{R}^d}|u(s,x)|^{p-2}u(s,x)f_0(s,x)\, dx \le \|f_0(s,\cdot)\|_p^p + \|u(s,\cdot)\|_p^p$$
$$\le N\|f(s,\cdot)\|_{-1,p}^p + \|u(s,\cdot)\|_p^p.$$

For

$$\lambda \ge p(p-1)N_1 + p + \frac{p(p-1)}{2},$$

we get

$$\|u(s,\cdot)\|_p^p \exp(-\lambda t) + \frac{p(p-1)}{2}\int_0^t \left\||u|^{\frac{p-2}{p}}\left|\frac{\partial u}{\partial x}\right|^{\frac{2}{p}}(s,\cdot)\right\|_p^p \exp(-\lambda s)\, ds$$

$$\leq N\int_0^t \left(\|f(s,\cdot)\|_{-1,p}^p + \|g(s,\cdot)\|_p^p\right)\exp(-\lambda s)\,ds$$
$$+p\sum_{k\leq m}\int_0^t \exp(-\lambda s)\left(\int_{\mathbb{R}^d}|u|^{p-2}ug^k(s,x)\,dx\right)\,dW_s^k,$$

where $N=N(p)$. After this, we basically just have to take the expectation and apply certain transformations based on the Burkholder-Davis-Grundy inequalities. More can be found in [26].

The assumption about the arbitrariness of λ in (2.15) can be justified by a rescaling argument, when instead of f,g and ω, we take (c^2f,cg), (c^2t,cx) and $\frac{1}{c}\omega_{c^2t}$ and get $u(c^2t,cx)$ instead of $u(t,x)$.

From our explicit formulas and from the particular choices of f and g, it follows that $u\in C_{\mathrm{loc}}([0,\infty),H^{n,p})$ for any n (and any ω). This proves (iii).

From (2.14) and (2.15), it follows that $u\in\cap_{T>0}\mathbb{H}^{1,p}(T)$. Furthermore, by the stochastic Fubini theorem, we get from (2.20), that u solves (2.13) in the sense of Definition 4. Hence $u\in\mathcal{H}^{1,p}$, which proves (i). The uniqueness is a consequence from setting $f=g=0$ and arriving at the heat equation for which the uniqueness of the solution in our class of functions is a standard fact. This completes the proof if f, g are step functions.

In the general case, the uniqueness is proven in the exact same way. Concerning the other points, we will use Theorem 11 and Remark 9.

If we consider all functions f_0, $\tilde{f}^j$, g^k as one sequence, then by Theorem 11, we can approximate the by functions f_{0i}, $\tilde{f}_i^j$, g_i^k of type (2.18). Let u_i be the corresponding solutions of (2.17). By the result for the particular case, u_i is a Cauchy sequence in $\mathcal{H}^{1,p}$ and by Theorem 10, there exists a $u\in\mathcal{H}^{1,p}$ to which u_i converges in $\mathcal{H}^{1,p}$. Remark 9 and the convergence $\left\|\frac{\partial^2u}{\partial x^2}-\frac{\partial^2u_1}{\partial x^2}\right\|_{\mathbb{H}^{-1,p}}^p\to 0$ show that $Au=\Delta u+f$ and $Bu=g$. In particular, this proves (i).

(ii) follows from the construction of u. From (iii) in the particular case, we get that u_i is a Cauchy sequence in $L^p(\Omega,C([0,T],L^p))$ for any T. Therefore, it converges in this space to a function v. It follows, that for any $\phi\in C_c^\infty$,

$$\langle v(t,\cdot),\phi\rangle=\int_0^t\left(\langle v(s,\cdot),\Delta\phi\rangle+\langle f(s,\cdot),\phi\rangle\right)\,ds+\sum_{k=1}^\infty\int_0^t\langle g^k(s,\cdot),\phi\rangle\,dW_s^k,$$

for all t (a.s.). Therefore$u-v$ is a generalized solution to the heat equation with zero initial condition and with bounded L^p-norm (a.s.). This implies that $\|(u-v)(t,\cdot)\|_p=0$ for all t (a.s.), so that $u\in C([0,T],L^p)$ for all T (a.s.). Finally, we get (2.15) by Fatou's lemma, taking into account that

$$\int_0^T\int_{\mathbb{R}^d}|\nabla(u-u_i)|^p\,dx\,dt=\int_0^T\int_{\mathbb{R}^d}|\nabla(1-\Delta)^{-\frac{1}{2}}(1-\Delta)^{\frac{1}{2}}(u-u_i)|^p\,dx\,dt$$
$$\leq N\int_0^T\int_{\mathbb{R}^d}|(1-\Delta)^{\frac{1}{2}}(u-u_i)|^p\,dx\,dt\to 0$$

in probability for any T. This proves the theorem. □

Remark 15. Although (2.5) holds for $p\in(1,\infty)$, Lemma 13 is false for $p<2$.

Let us look at the task of dealing with equations with constant coefficients formally. We will try to make these observations more rigorous further on.

Assume, we have

$$du(t,x) = f(t,x)\,dt + \sum_k g_k(t,x)\,dW_t^k, \tag{2.23}$$

and we define a process x_t and a function v by

$$x_t^i := \sum_k \int_0^t \sigma_{i,k}(s)\,dW_s^k, \qquad i = 1,\dots,d, \tag{2.24}$$

$$v(t,x) := u(t, x - x_t).$$

Applying Ito's formula to v, we get

(2.25)
$$dv(t,x) = \left(f(t,x-x_t) + \sum_i \left(\sum_j a_{i,j}(t) \frac{\partial^2 v}{\partial x_i \partial x_j}(t,x) - \left\langle \frac{\partial g}{\partial x_i}(t, x - x_t), \sigma_i(t) \right\rangle_{l_2} \right) \right) dt$$
$$+ \sum_k \left(g_k(t, x-x_t) - \sum_i \left(\frac{\partial v}{\partial x_i}(t,x)\sigma_{i,k}(t) \right) \right) dW_t^k.$$

This shows, ho to introduce the terms $\frac{\partial v}{\partial x_i}$ and $\sigma_{i,k}$ in equation (2.23) and also illustrates the necessity of g having a first derivative in x. If we had $\Delta u + f$ in (2.23) instead of f, then we would get the second order differential operator $\sum_i \sum_j (\delta_{i,j} + \alpha_{i,j}) \frac{\partial^2}{\partial x_i \partial x_j}$ which coefficients strongly relate to the coefficients of $\frac{\partial v}{\partial x_i}$ and $\sigma_{i,k}(t)$. We could get around this problem, if we managed to start off with an equation with more general linear operators L instead of Δ.

If, instead of (2.23), we consider

$$du(t,x) = (\Delta u + \bar f)\,dt + \sum_k g_k(t,x)\,dW_t^k,$$

and take expectations in the (2.25) counterpart, then, assuming σ to be nonrandom, we get an equation for $\mathbb{E}[v(t,x)]$ with operator L different from Δ.

Definition 16. Denote by $\mathbb{D}$ the set of all $\mathcal{D}$-valued functions u $(u(t,x))$ $\Omega \times [0,\infty)$, such that for any $\phi \in C_c^\infty$,

(i) the function $\langle u, \phi \rangle$ is $\mathcal{P}$ - measurable,

(ii) for any $\omega \in \Omega$ and $T \in (0,\infty)$, we have

$$\int_0^T \sup_{x \in \mathbb{R}^d} |\langle u(t,\cdot), \phi(\cdot - x)\rangle|^2\,dt < \infty. \tag{2.26}$$

In the same way, we define $\mathbb{D}(l_2)$ by considering l_2-valued linear functionals on C_c^∞ and replacing $|\cdot|$ by $\|\cdot\|_{l_2}$.

Remark 17. We note that $\langle u(t,\cdot), \phi(\cdot - x)\rangle$ is continuous in x and Borel in t, so (2.26) makes sense. Also, for $p \geq 2$, $q = \frac{p}{p-1}$, and any n,

$$\int_0^T \sup_{x\in\mathbb{R}^d} |\langle u(t,\cdot), \phi(\cdot - x)\rangle|^2 \, dt \le \int_0^T \sup_{x\in\mathbb{R}^d} \|u(t,\cdot)\|_{n,p}^2 \|\phi(\cdot - x)\|_{-n,q}^2 \, dt$$

$$\le \|\phi\|_{-n,q}^2 T^{\frac{p-2}{p}} \left(\int_0^T \|u(t,\cdot)\|_{n,p}^p \, dt \right)^{\frac{2}{p}}. \tag{2.27}$$

This shows, that if $u \in \mathcal{H}^{n,p}$, condition (2.26) is satisfied (at least for almost all ω). If $u \in \mathcal{H}^{n,p}$, then (2.6) holds true, which in turn shows, that $\langle u(t,\cdot), \phi\rangle$ is indistinguishable from a predictable process, which holds true for any $\phi \in C_c^\infty$. From the separability of $H^{-n,q}$, it follows that we can modify u on a set of measure zero and get a function belonging to $\mathbb{D}$. In this sense, we write

$$\mathcal{H}^{n,p} \subset \mathbb{D}. \tag{2.28}$$

Definition 18. Let f, $u \in \mathbb{D}$ and $g \in \mathbb{D}(l_2)$. We say that the equality

$$du(t,x) = f(t,x)\, dt + g(x,t)\, dW_t \qquad t > 0, \tag{2.29}$$

holds in the sense of distributions, if for any $\phi \in C_c^\infty$, with probability 1 for all $t \ge 0$, we have

$$\langle u(t,\cdot), \phi\rangle = \langle u(0,\cdot), \phi\rangle + \int_0^t \langle f(s,\cdot), \phi\rangle \, ds + \sum_{k=1}^\infty \int_0^t \langle g^k(s,\cdot), \phi\rangle \, dW_s^k. \tag{2.30}$$

Since $\|\langle g, \phi\rangle\|_{l_2}^2$ is locally summable in t, the last series in (2.30) converges uniformly in t in probability over every finite interval in time.

Note that, if $u \in \mathcal{H}^{n,p}$ and u satisfies (2.30) in the sense of distributions, then by (2.28), $u \in \mathbb{D}$ and (2.29) holds in the sense of distributions.

Lemma 19. *Let f, $u \in \mathbb{D}$ and $g \in \mathbb{D}(l_2)$. Assume the definitions in (2.24), then (2.23) holds (in the sense of distributions), if (2.25) holds (in the sense of distributions).*

Proof. Remember that for a distribution $\psi(x)$ and $y \in \mathbb{R}^d$, we interpret $\psi(x-y)$ as the distribution defined by $\langle \psi, \phi(\cdot + y)\rangle$. From

$$\int_0^T \sup_{y\in\mathbb{R}^d} \left| \left\langle \frac{\partial^2 v}{\partial x^2}(t,\cdot), \phi(\cdot - y) \right\rangle \right|^2 dt = \int_0^T \sup_{y\in\mathbb{R}^d} \left| \left\langle v(t,\cdot), \frac{\partial^2 \phi}{\partial x^2}(\cdot - x) \right\rangle \right|^2 dt$$

$$= \int_0^T \sup_{y\in\mathbb{R}^d} \left| \left\langle u(t,\cdot), \frac{\partial^2 \phi}{\partial x^2}(\cdot + x_t - y) \right\rangle \right|^2 dt = \int_0^T \sup_{y\in\mathbb{R}^d} \left| \left\langle u(t,\cdot), \frac{\partial^2 \phi}{\partial x^2}(\cdot - y) \right\rangle \right|^2 dt < \infty$$

$$\int_0^T \sup_{y\in\mathbb{R}^d} \left| \left\langle \left\langle \sum_i \frac{\partial g}{\partial x_i}(t, \cdot - x_t), \sum_i \sigma^i(t) \right\rangle_{l_2}, \phi(\cdot - y) \right\rangle \right|^2 dt$$

$$= \int_0^T \sup_{y\in\mathbb{R}^d} \left| \left\langle \left\langle \sum_i \frac{\partial g}{\partial x_i}(t, \cdot - x_t), \phi(\cdot - y) \right\rangle, \sum \sigma_i^i(t) \right\rangle_{l_2} \right|^2 dt$$

$$\leq \int_0^T \|\sigma^i(t)\|_{l_2}^2 \, dt \int_0^T \sup_{y\in\mathbb{R}^d} \left\| \left\langle \sum_i \frac{\partial g}{\partial x_i}(t, \cdot - x_t), \phi(\cdot - y) \right\rangle \right\|_{l_2}^2 dt < \infty,$$

it follows that $v(t,x)$, $f(t, x-x_t)$ and $\left\langle \sum_i \frac{\partial g}{\partial x_i}(t, \cdot - x_t), \phi(\cdot - y) \right\rangle_{l_2}$ belong to $\mathbb{D}$, $g(t, x-x_t)$ and $\sum_i \frac{\partial v}{\partial x_i}(t, \cdot)\sigma^i$ belong to $\mathbb{D}(l_2)$. Furthermore, for any $\phi \in C_c^\infty$, the function $F(t,x) := \langle u(t, \cdot - x)\phi \rangle$ has a stochastic differential in t for any x and is infinitely often differentiable with respect to x. The assertion we made now follows immediately from Ito's formula applied to $F(t, x_t)$. □

Remark 20. If, instead of (2.23), u satisfies the equation

$$u(t) = \left(\sum_{i=1}^d \sum_{j=1}^d a_{ij}(t) \frac{\partial^2 u}{\partial x_i \partial x_j} + h(t,x) \right) dt + \left(\sum_{k=0}^N \left(\sum_{i=1}^d \sigma_{ik}(t) \frac{\partial u}{\partial x_i}) \, dW_t^k \right) \right),$$

then (2.25) takes the form

$$\frac{\partial}{\partial t} v(t,x) = \sum_{i=1}^d \sum_{j=1}^d \left(a_{ij}(t) - \alpha_{ij}(t) \right) \frac{\partial^2 v}{\partial x_i \partial x_j}(t,x) + h(t, x - x_t) \qquad t > 0, \tag{2.31}$$

and can be considered on each ω separately. If $a(t) < \alpha(t)$, then the initial value problem $v(0) = v_0$ is ill-posed.

This shows, that the operators appearing in the stochastic part should be, in a certain sense, subordinated to the operators appearing in the deterministic part of the equation. This is essential, when constructing an L^p-theory.

In spite of what we just said, if we take $d = 1$ and a one dimensional Brownian motion W_t, and consider the following equation

$$du(t,x) = iu_x(t,x) \, dW_t,$$

then this equation has a somewhat nice solution for initial data $u_0 \in L^2$. We use the Fourier transform and it turns out that $\hat{u}(t,\xi) = u_0(\xi)\exp(\xi W_t - \frac{1}{2}|\xi|^2 t)$ is the Fourier transform of the solution. We see that it decays very fast for $|\xi| \to \infty$, showing us that $u(t,x)$ is infinitely differentiable in x. Taking expectations, we also see, that $\mathbb{E}[u(t,x)] = u_0(x)$, if u_0 is non random, and in this case, we get a representation of any L^2 function as an integral over Ω of functions $u(\omega, 1, x)$ which are infinitely often differentiable in x. However, a major drawback from such equations is, that $\mathbb{E}[|u(t,0)|^p] = \infty$ for any $p > 1$ if, for example, $\hat{u}_0(\xi) \geq \exp(-\lambda\xi)$, where λ is a constant.

Lemma 21. *Let $f \in \mathbb{D}$, $g \in \mathbb{D}(l_2)$ and u_0 be a $\mathcal{D}$-valued function on Ω, then the following assertions hold true*

(i) There can only exist one solution to (2.12) in $\mathbb{D}$ with initial condition $u(0,\cdot) = u_0$.

(ii) Let $\mathcal{F}_t = \mathcal{W}_t \vee \mathcal{B}_t$ for $t \geq 0$, and assume that the σ-fields $\mathcal{W}_t$ and $\mathcal{B}_t$ form independent increasing filtrations. Let W and B be sets, such that $W \cup B = \{1, 2, \dots\}$. Assume that $(W_t^k, \mathcal{W}_t)$ and $(W_t^r, \mathcal{B}_t)$ are Wiener processes for $k \in W$ and $r \in B$. Let $u \in \mathbb{D}$ satisfy equation (2.12) in the sense of distributions and let a, f, σ, g be $\mathcal{W}_t$-adapted. Finally, assume that there exists an $n \in (-\infty, \infty)$ such that $f \in \mathbb{H}^{n,2}(T)$ and $g \in \mathbb{H}^{n,2}(T, l_2)$ for any $t \in (0, \infty)$ and $u(0, \cdot)$ is W_0-measurable and

$$\mathbb{E}\left[\|u(0,\cdot)\|_{n,2}^2\right] < \infty.$$

Then there exists a unique solution $\tilde{u}$, in $\mathbb{D}$, of the equation

(2.32)
$$d\tilde{u} = \left(\sum_i \sum_j a_{ij} \frac{\partial^2 \tilde{u}}{\partial x_i \partial x_j} + f\right) dt + \sum_{k \in W} \left(\sum_i \sigma_{ik} \frac{\partial \tilde{u}}{\partial x_i} + g_k\right) dW_t^k, \qquad t > 0.$$

In addition, for any $\phi \in C_c^\infty$ and $t \geq 0$,

$$\langle \tilde{u}(t,\cdot), \phi\rangle = \mathbb{E}\left[\langle u(t,\cdot), \phi\rangle | \mathcal{W}_t\right] \qquad \text{(a.s.)}. \tag{2.33}$$

Proof. (i) As usual, we will set $f = g = 0$ and $u_0 = 0$ and use Lemma 19, it suffices to consider only the case where $\sigma = 0$. For any given $\phi \in C_c^\infty$ we have

$$\langle u(t,\cdot), \phi\rangle = \int_0^t \langle u(s,\cdot), L(s)\phi\rangle\, ds, \qquad t \geq 0,$$

almost surely. Substituting ϕ with $\phi(\cdot - x)$ and noting that both sides are continuous and bounded in (t, x) on $[0, T] \times \mathbb{R}^d$ for any $T < \infty$, we get that the function $F(t, x) := \langle u(t,\cdot), \phi(\cdot - x)\rangle$ is bounded in (t, x) on $[0, T] \times \mathbb{R}^d$ for any $T < \infty$, infinitely often differentiable in x, and almost surely satisfies the equation

$$F(t, x) = \int_0^t L(s) F(s, x)\, ds \qquad \forall\, t,\ x.$$

From the theory of parabolic equations, $F(t, x) = 0$, $\forall\, t,\ x$ (a.s.), follows. This means, that $\langle u(t,\cdot), \phi\rangle = 0$ for all t almost surely. Let us now take ϕ with integral 1, then for any x and n with probability 1, we have $\langle u(t,\cdot), n^d\phi(n(\cdot - x))\rangle = 0$ for all t. By the continuity of this function in x, we get that it is 0 for all t and x with probability 1. Finally, $\langle u(t,\cdot), n^d\phi(n(\cdot - x))\rangle \to u(t, x)$ as $n \to \infty$ for all (ω, t, x) in the sense of distributions, implying that with probability 1, we have $u(t,\cdot) = 0$ for all t, as stated.

(ii) We first notice that according to [26] equation (2.12) has a unique solution v in the space $\mathbb{H}^{n-1,2}(T)$, for any T. The definition of solutions in this space in [26] is slightly continuous, but v is continuous (a.s.) as an $H^{n,2}$-valued process and

$$\mathbb{E}\left[\sup_{t \leq T} \|v(t,\cdot)\|_{n,2}^2\right] < \infty \qquad \forall\, T < \infty, \tag{2.34}$$

so that v is a $\mathbb{D}$ solution of (2.12). It follows from (i) that our function u coincides with v and therefore belongs to $\mathbb{H}^{n-1,2}(T)$, for any T and (2.34) holds for u. Furthermore, with probability 1 for all t at once,

$$u(t) = u(0)+\int_0^t \left(\sum_i \sum_j a_{ij}(s)\frac{\partial^2 u}{\partial x_i \partial x_j}(s) + f(s)\right) ds+\int_0^t \sum_{k\in W} \left(\sum_i \sigma_{ik}(s)\frac{\partial u}{\partial x_i}(s) + g(s)\right) dW_s^k,$$

where all integrals are taken in the sense of the Hilbert space $H^{n-1,2}$. By the Hilbert-space counterpart of Theorem 1.4.7 in [26], there exists an $H^{n+1,2}$- valued, $\mathcal{W}_t$-predictable function $\bar{u}(t)$, such that for almost all t, we have (a.s.)

$$\bar{u}(t) = \mathbb{E}\left[u(t)|\mathcal{W}_t\right],$$

$$\frac{\partial \bar{u}}{\partial x}(t) = \mathbb{E}\left[\frac{\partial u}{\partial x}(t)|\mathcal{W}_t\right],$$

$$\frac{\partial^2 \bar{u}}{\partial x^2}(t) = \mathbb{E}\left[\frac{\partial^2 u}{\partial x^2}|\mathcal{W}_t\right]$$

and

(2.35)

$$\bar{u}(t) = u(0)+\int_0^t \left(\sum_i \sum_j a_{ij}(s)\frac{\partial^2 \bar{u}}{\partial x_i \partial x_j}(s) + f(s)\right) ds+\int_0^t \sum_{k\in W} \left(\sum_i \sigma_{ik}(s)\frac{\partial \bar{u}}{\partial x_i}(s) + g(s)\right) dW_s^k,$$

for almost all t and ω. The right hand side is a continuous $H^{n-1,2}$-valued process, which we will denote by $\bar{u}$ and we will show that $\tilde{u}$ is indeed the function we were looking for.

By definition and the equality $\bar{u} = \tilde{u}$ (a.e.), $\tilde{u}$ satisfies (2.35) for all t with probability 1 and is also a continuous process in $H^{n-1,2}$. This implies that $\tilde{u} \in \mathbb{D}$ and $\tilde{u}$ is a solution of (2.32). To prove (2.33) for any t, it remains to observe that again by Theorem 1.4.7 in [26], the conditional expectation $\mathbb{E}[u(t)|\mathcal{W}_t]$ is equal to the right hand side of (2.35), almost surely. □

Theorem 22. *Take $n \in \mathbb{R}$, $f \in \mathbb{H}^{n-1,p}$ and $g \in \mathbb{H}^{n,p}(l_2)$, then*
(i) equation (2.12) with zero initial condition has a unique solution $u \in \mathcal{H}^{n+1,p}$,
(ii) for this solution, we have

$$\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{\mathbb{H}^{n-1,p}} \leq N(\|f\|_{\mathbb{H}^{n-1,p}} + \|g\|_{\mathbb{H}^{n,p}(l_2)}), \tag{2.36}$$

$$\|u\|_{\mathbb{H}^{n+1,p}} \quad \leq \quad N\|\langle f, g\rangle\|_{\mathcal{F}^{n-1,p}},$$

where $N = N(d, p, \sigma, K)$,
(iii) we have $u \in C_{loc}([0,\infty), H^{n,p})$ almost surely and for any λ, $T > 0$,

(2.37)

$$\mathbb{E}\left[\sup_{t\leq T} \exp(-p\lambda t)\|u(t,\cdot)\|_{n,p}^p\right] \leq N(\|\exp(-\lambda t)f\|^p_{\mathbb{H}^{n-1,p}(T)}+\|\exp(-\lambda t)g\|^p_{\mathbb{H}^{n,p}(T,l_2)}),$$

where $N = N(d, p, \delta, K, \lambda)$.

Proof. Since one can apply the operator $(I-\Delta)^{\frac{n}{2}}$ to both sides of (2.12), it suffices to prove the theorem only for $n=0$. As we have already noticed, any function $u \in \mathcal{H}^{1,p}$ also belongs to $\mathbb{D}$. This and Lemma 21 prove the uniqueness in (i). The translation invariance of our norms, combined with Lemma 19, shows that in order to prove existence in (i) and all the other assertions, we only need to consider the case $\sigma=0$. As in the proof of Theorem 14, we can assume f, g as in (2.18).

In this case, we know from [26], equation (2.12) has a unique $\mathbb{D}$-valued solution u that belongs to $C_b([0,T]\times\mathbb{R}^d)$ and $C((0,T],L^2)$ almost surely for any $T<\infty$. It follows, that $u \in C([0,T],L^p)$ almost surely for any $T<\infty$. Estimate (2.37) also follows from [26], as in the proof of theorem 14). Now, it only remains to prove, that $u \in \mathcal{H}^{1,p}$ and (2.36) holds. Since we already know that u is a $\mathbb{D}$ solution, it suffices to show that $u \in \mathbb{H}^{1,p}(T)$ for any $T<\infty$, in order for it to be an element of $\mathcal{H}^{1,p}$.

Since the matrix a is uniformly non-degenerate, by making a nonrandom time change, we can reduce the general case to the case $a \geq I$. On this case, define the matrix-valued function $\bar\sigma(t)=\bar\sigma^*(t)\geq 0$ as the solution of he equation $\bar\sigma^2(t)+2I=2a(t)$. Without loss of generality, we can assume that on our probability space we are also given a d-dimensional Wiener process B_t, which is independent of $\mathcal{F}_t$.

Now we consider the equation

$$dv(t,x)=\left(\Delta v(t,x)+f\left(t,x-\int_0^t\bar\sigma(s)\,dB_s\right)\right)dt+\sum_k g_k\left(t,x-\int_0^t\bar\sigma(s)\,dB_s\right)dW_t^k \tag{2.38}$$

with zero initial condition. We replace the predictable σ-field $\mathcal{P}$ with the predictable σ-field generated by $\mathcal{F}\vee\sigma(B_s\,:\,s\leq t)$. The space $\mathcal{H}^{n,p}$ becomes larger. By Theorem 14, there exists a solution v of (2.38) possessing properties (i)-(iii) listed in the theorem. We use again that, after changing, if necessary, v on a set of probability zero, the function v becomes a $\mathbb{D}$-solution of (2.38). By Lemma 19, the function $z(t,x):=v\left(t,x+\int_0^t\bar\sigma(s)\,dB_s\right)$ is a $\mathbb{D}$-solution of

$$dz(t,x)=\left(\sum_i\sum_j a_{ij}(t)\frac{\partial^2 z}{\partial x_i\partial x_j}(t,x)+f(t,x)\right)dt+\sum_k g_k(t)dW_t^k+\sum_i\sum_j\sigma_{ij}(s)\frac{\partial z}{\partial x_i}(t,x)dB_t^j,$$

and by Lemma 21, there exists a solution $\tilde u \in \mathbb{D}$ of

$$d\tilde u(t,x)=\left(\sum_i\sum_j a_{ij}(t)\frac{\partial^2\tilde u}{\partial x_i\partial x_j}(t,x)+f(t,x)\right)\,dt+\sum_k g_k(t,x)\,dW_t^k,$$

which is (2.12) in our case. In addition, for any $\phi\in C_c^\infty$ and $t\geq 0$, (a.s.)

$$\langle\tilde u(t,\cdot),\phi\rangle=\mathbb{E}\left[\langle z(t,\cdot),\phi\rangle|\mathcal{F}_t\right]=\mathbb{E}\left[\left\langle v\left(t,\cdot+\int_0^t\bar\sigma(s)\,dB_s\right),\phi\right\rangle|\mathcal{F}_t\right].$$

In particular, it follows from this equality that $\tilde u$ is a $\mathbb{D}$-solution with respect to the initial predictable σ-field $\mathcal{P}$, and from the uniqueness, we get $\tilde u=u$. Therefore, (a.s.)

$$\langle u(t,\cdot),\phi\rangle = \mathbb{E}\left[\left\langle v\left(t,\cdot+\int_0^t \bar\sigma(s)\,dB_s\right),\phi\right\rangle |\mathcal{F}_t\right]. \tag{2.39}$$

Further, it follows that

$$|\langle u(t,\cdot),\phi\rangle|^p \le \mathbb{E}\left[\|v(t,\cdot)\|_{1,p}^p|\mathcal{F}_t\right]\|\phi\|_{-1,p}^p \tag{2.40}$$

(a.s.) for any $\phi\in C_c^\infty$ and $t\ge 0$, where $q=\frac{p}{p-1}$.

Next, we take a countable family $\Phi\subset C_c^\infty$, which is dense in C_c^∞. We observe that, given a distribution ψ, we have $\psi\in H^{1,p}$ if, and only if, for any $\phi\in\Phi$ we have $|\langle\psi,\phi\rangle|\le N\|\phi\|_{-1,q}$ with a constant N independent of ϕ. In this case exists a bounded linear functional l on $H^{-1,q}$, such that $l(\phi)=\langle\psi,\phi\rangle$ for any $\phi\in\Phi$. Since $l(\phi)=\langle(1-\Delta)^{-\frac12}h,\phi\rangle$ with an $h\in L^p$ and Φ dense in C_c^∞, we have $\psi=(1-\Delta)^{-\frac12}h\in H^{1,p}$. This also implies, that the set $\{(\omega,t) : w(\omega,t,\cdot)\in H^{1,p}\}$ is measurable (even predictable) for any $w\in\mathbb{D}$,say $w=u$.

We also know that $v\in\mathcal{H}^{1,p}$, which implies that $\mathbb{E}\left[\|v(t,\cdot)\|_{1,p}^p\right]<\infty$ for almost all t. We now fix such a t. Then there exists a set Ω' of probability 1 such that $\mathbb{E}\left[\|v(t,\cdot)\|_{1,p}^p|\mathcal{F}_t\right]<\infty$ on Ω' and (2.40) holds for all $\omega\in\Omega'$ and $\phi\in\Phi$. Hence $u(t,\cdot)\in H^{1,p}$ for the chosen t and all $\omega\in\Omega'$. In particular, $u(t,\cdot)\in H^{1,p}$ for almost all (ω,t) and from (2.40) it follows that

$$\|u(t,\cdot)\|_{1,p}^p\le\mathbb{E}\left[\|v(t,\cdot)\|_{1,p}^p|\mathcal{F}_t\right] \qquad \text{(a.s.)},$$

$$\|u\|_{\mathbb{H}^{1,p}(T)}\le\|v\|_{\mathbb{H}^{1,p}(T)}<\infty.$$

Thus, $u\in\mathbb{H}^{1,p}(T)$ for any $T<\infty$ and $u\in\mathcal{H}^{1,p}$.

Similarly, from the equality

$$\left\langle\frac{\partial^2u}{\partial x^2}(t,\cdot),\phi\right\rangle=\mathbb{E}\left[\left\langle\frac{\partial^2v}{\partial x^2}(t,\cdot)\left(t,\cdot+\int_0^t\bar\sigma(s)\,dB_s\right),\phi\right\rangle|\mathcal{F}_t\right] \qquad \text{(a.s.)},$$

we get

$$\left\|\frac{\partial^2u}{\partial x^2}(t,\cdot)\right\|_{-1,p}^p\le\mathbb{E}\left[\left\|\frac{\partial^2v}{\partial x^2}(t,\cdot)\right\|_{-1,p}^p|\mathcal{F}_t\right] \qquad \text{(a.s.)}.$$

This and the properties of v, immediately yield (2.36). □

3. Self Intersection Local Times (SILTs) of Brownian Motions

In the first part of this section we will follow the more "traditional" way of defining the notion of SILTs as it is presented in [18, 4, 32].

Formally, the self intersection local time of a planar Brownian motion on the Borel set $B \subset \mathbb{R}_+^2$, is defined as

$$\beta_2(x,B) := \int\int_B \delta_0(W_s - W_r - x)\, dr\, ds \quad x \in \mathbb{R}^2,\ t \in \mathbb{R}_+.$$

More precisely, it is defined as

$$\beta_2(x,B) := \lim_{\epsilon \to 0} \int\int_B \rho_\epsilon(W_s - W_r - x)\, dr\, ds,$$

where

$$\rho_\epsilon(x) := \frac{\exp\left(-\frac{|x|^2}{2\epsilon}\right)}{2\pi\epsilon}.$$

Rosen [32] showed that $\beta_2(x,B)$, where B is a bounded Borel set in $\mathbb{R}_+^2(\epsilon) := \{(s,t) \,:\, s,t \geq 0,\ |s-t| > \epsilon\}$ for an $\epsilon > 0$, is a continuous function in x.

We will not deal with general Borel sets and remain on the set $\{t,\ s \in \mathbb{R}_+ \mid 0 \leq s \leq t\}$ and introduce the following notation

$$\beta_2(x,t) := \lim_{\epsilon \to 0} \int_0^t \int_0^s \rho_\epsilon(X_s - X_r - x)\, dr\, ds \quad x \in \mathbb{R}^2,\ t \in \mathbb{R}_+.$$

Remark 23.

$$\mathbb{E}\left[\int_0^t \int_0^s \rho_\epsilon(X_s - X_r - x)\, dr\, ds\right] = \int_0^t \int_0^s \mathbb{E}\left[\rho_\epsilon(W_s - W_r - x)\right]\, dr\, ds$$

$$= \int_0^t \int_0^s \int_{\mathbb{R}^2} \rho_\epsilon(z-x) \frac{1}{2\pi(s-r)} \exp\left(-\frac{|z|^2}{2(s-r)}\right)\, dz\, dr\, ds.$$

Letting $\epsilon \to 0$, we get

$$\int_0^t \int_0^s \frac{1}{2\pi(s-r)} \exp\left(-\frac{|x|^2}{2(s-r)}\right)\, dr\, ds.$$

We already see, that for $x = 0$, the points on the diagonal are problematic, as the integral will not be finite any longer.

To deal with those problematic points one can introduce, like it was first done by [4], the so-called regularized version of the SILT,

$$\gamma(x,t) := \lim_{\epsilon \to 0} \left(\int_0^t \int_0^s \rho_\epsilon(W_s - W_r - x)\, dr\, ds - \mathbb{E}\left[\int_0^t \int_0^s \rho_\epsilon(W_s - W_r - x)\, dr\, ds\right]\right).$$

Sometimes, slightly different normalizations are used, which differ from this one by, at most, a constant times t.

Through this regularization/normalization, we obtain convergence a.s. and continuity of the limit in t.

Remark 24. A rather trivial but, for our purposes, important result, is that $\beta_2(0,t)$ is, in law, equal to $t\beta_2(0,1)$.

$$\beta_2^\epsilon(at,x) := \int_0^{at}\int_0^s \rho_\epsilon(W_s - W_r - x)\,dr\,ds$$
$$= \int_0^{at} h(s)\,ds = \int_0^t h(sa)a\,ds$$
$$= a\int_0^t\int_0^{as} \rho_\epsilon(W_{sa} - W_r - x)\,dr\,ds = a^2\int_0^t\int_0^s \rho_\epsilon(W_{sa} - W_{ra} - x)\,dr\,ds$$
$$=^d a^2\int_0^t\int_0^s \rho_\epsilon(W_{a(s-r)} - x)\,dr\,ds = a^2\int_0^t\int_0^s \rho_\epsilon(\sqrt{a}W_{s-r} - x)\,dr\,ds$$
$$= a^2 a^{-\frac{1}{2}}\int_0^t\int_0^s \rho_{\frac{\epsilon}{\sqrt{a}}}\left(W_{s-r} - \frac{x}{\sqrt{a}}\right)dr\,ds = a^{\frac{3}{2}}\int_0^t\int_0^s \rho_{\frac{\epsilon}{\sqrt{a}}}\left(W_{s-r} - \frac{x}{\sqrt{a}}\right)dr\,ds$$
$$= a^{\frac{3}{2}}\beta_2^{\frac{\epsilon}{\sqrt{a}}}\left(t, \frac{x}{\sqrt{a}}\right).$$

More general results regarding scaling can be found in [11, 10].

We will now make a small excursion into the potential theory of Brownian motions, in order to remind ourselves of some results, which will be of importance in the next section.

Let W_t denote a Brownian motion in $\mathbb{R}^d$.

Definition 25. A point $x \in \Omega$ is called regular for the closed set $\Omega \subset \mathbb{R}^d$ if the first hitting time $T_\Omega := \inf\{t \geq 0 \,:\, W_t \in \Omega\}$ satisfies $\mathbb{P}_x(T_\Omega = 0) = 1$. A point which is not regular is called irregular.

Theorem 26. *Suppose $\Omega \subset \mathbb{R}^d$ is a bounded domain and ϕ is a continuous function on $\partial\Omega$. Define $\tau := \inf\{t \geq 0 \,:\, W_t \in \partial\Omega\}$ and define $u : \bar{\Omega} \to \mathbb{R}$ by*

$$u(x) := \mathbb{E}_x[\phi(W_\tau)].$$

u satisfies the following three points:

(a) A solution to the Dirichlet problem exists if and only if the function u is a solution to the Dirichlet problem with boundary condition ϕ.

(b) u is a harmonic function on Ω with $u(x) = \phi(x)$ for all $x \in \partial\Omega$ and is continuous at every point $x \in \partial\Omega$ that is regular for the complement of Ω.

(c) If every $x \in \partial\Omega$ is regular for the complement of Ω, then u is the unique continuous function $u : \bar{\Omega} \to \mathbb{R}$ which is harmonic on Ω such that $u(x) = \phi(x)$ for all $x \in \partial\Omega$.

Proof. [29]. □

Theorem 27. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $u : \bar{\Omega} \to \mathbb{R}$ be a continuous function, which is twice continuously differentiable on Ω. Let $g : \bar{\Omega} \to \mathbb{R}$ be continuous. Then u is said to be the solution of Poisson's problem for g if $u(x) = 0$ for all $x \in \partial\Omega$ and*

$$-\frac{1}{2}\Delta u(x) = g(x) \qquad \forall\, x \in \Omega.$$

Proof. [29]. □

Remark 28. For bounded g, the solution u of the Poisson problem, if it exists, equals

$$u(x) = \mathbb{E}_x \left[\int_0^A g(W_t)\, dt \right] \qquad \text{for } x \in \Omega,$$

where $A = \inf\{t \geq 0 \,:\, W_t \notin \Omega\}$. Conversely, if g is Hölder continuous and every $x \in \partial\Omega$ is regular for the complement of Ω, then u solves the Poisson problem for g.

If u solves Poisson's problem for $g = 1$ in a domain $\Omega \subset \mathbb{R}^d$, then $u(x) = \mathbb{E}_x[A]$ is the average time it takes a Brownian motion started in x to leave the set Ω.

Remark 29.
(3.1)

$$g_R(x,y) := \begin{cases} -\frac{1}{2\pi}\log\left(\left|\frac{x}{R}-\frac{y}{R}\right|\right) + \frac{1}{2\pi}\log\left(\left|\frac{x}{|x|}-\frac{|x|y}{R^2}\right|\right) & x \neq 0,\ x,y \in B(0,R) \\ -\frac{1}{\pi}\log\left(\left|\frac{y}{R}\right|\right) & x = 0,\ y \in B(0,R) \end{cases}.$$

Which can, in the first case, be rewritten as

$$G(x) := -\frac{1}{2\pi}\log(|x|)$$

$$g(x,y) = G(y-x) - G\left(|x|\left(y - \frac{x}{|x|^2}\right)\right)$$

$$g_R(x,y) = -\frac{1}{2\pi}\log\left(\left|\frac{x}{R}-\frac{y}{R}\right|\right) - \frac{1}{2\pi}\log\left(\left|\frac{x}{R}\right|\left|\frac{y}{R}-\frac{\frac{x}{R}}{\left|\frac{x}{R}\right|^2}\right|\right)$$

$$= -\frac{1}{2\pi}\log\left(\frac{\left|\frac{x}{R}-\frac{y}{R}\right|}{\left|\frac{x}{R}\right|\left|\frac{y}{R}-\frac{\frac{x}{R}}{\left|\frac{x}{R}\right|^2}\right|}\right) = \frac{1}{4\pi}\log\left(\frac{\left|\frac{x}{R}\right|^2\left|\frac{y}{R}-\frac{\frac{x}{R}}{\left|\frac{x}{R}\right|^2}\right|^2}{\left|\frac{x}{R}-\frac{y}{R}\right|^2}\right).$$

$$\frac{\left|\frac{x}{R}\right|^2\left|\frac{y}{R}-\frac{\frac{x}{R}}{\left|\frac{x}{R}\right|^2}\right|^2}{\left|\frac{x}{R}-\frac{y}{R}\right|^2} = \frac{\left|\frac{x}{R}\right|^2\left(\left|\frac{y}{R}\right|^2 + \frac{1}{\left|\frac{x}{R}\right|^2} - 2\frac{x}{R}\frac{\frac{y}{R}}{\left|\frac{x}{R}\right|^2}\right)}{\left|\frac{x}{R}-\frac{y}{R}\right|^2}$$

$$= \frac{\left|\frac{x}{R}\right|^2\left|\frac{y}{R}\right|^2 + 1 - 2\frac{x}{R}\frac{y}{R}}{\left|\frac{x}{R}-\frac{y}{R}\right|^2} = 1 + \frac{(1+\left|\frac{x}{R}\right|)^2(1+\left|\frac{y}{R}\right|)^2}{\left|\frac{x}{R}-\frac{y}{R}\right|^2}.$$

We see that the logarithm is always positive. Consider

$$\int_{|y|\leq R} \frac{1}{4\pi}\log\left(\frac{\left|\frac{x}{R}\right|^2\left|\frac{y}{R}-\frac{\frac{x}{R}}{\left|\frac{x}{R}\right|^2}\right|^2}{\left|\frac{x}{R}-\frac{y}{R}\right|^2}\right)\, dy.$$

We change the scaling of the coordinates ($\bar{x} := \frac{x}{R}$) in order to arrive at

$$R\int_{|y|\le 1}\frac{1}{4\pi}\log\left(\frac{|\bar{x}|^2\left|y-\frac{\bar{x}}{|\bar{x}|^2}\right|^2}{|\bar{x}-y|^2}\right)\,dy.$$

Now we choose coordinates, so $\bar{x}=(S,0)$ and $y=r(\cos(\phi),\sin(\phi))$.
The integral becomes

$$\begin{aligned}R\int_0^1\int_0^{2\pi}\frac{1}{4\pi}\log\left(\frac{1+r^2S^2-2rS\cos(\phi)}{S^2+r^2-2rS\cos(\phi)}\right)r\,d\phi\,dr\\ =c(R)\frac{1}{4\pi}\int_0^1 r(I(1,rS)-I(r,R))\,dr,\end{aligned}$$

where $I(x,y)$ is defined as

$$I(x,y)=\int_0^{2\pi}\log(x^2+y^2-2xy\cos(\phi))\,d\phi=4\pi\max\{\log(x),\log(y)\}.$$

With $0\le r$ and $S\le 1$, it follows that $I(1,rS)=0$. Therefore the integral becomes

$$-\int_0^1 r\max\{\log(r),\log(S)\}\,dr=-\log(S)\int_0^1 r\,dr-\int_S^1 r\log(r)\,dr=\frac{1}{4}(1-S^2).$$

This tells us that

$$\sup_{|\bar{x}|\le 1}\int_{|y|\le 1}|g(\bar{x},y)|\,dy\le R.$$

Although we already defined the notion of SILTs, we want to use an alternative (potentially more elegant) way to approach them and derive some important results.

Let X_t,Y_t be two independent Brownian motions in $\mathbb{R}^2$ and $g_R(x,y)$ the Green function (3.1) of a Brownian motion killed on exiting the ball $B(0,R)$.
Set

$$T_R=T_R(X):=\inf\{t\,:\,|X_t|\ge R\}.$$

For each $x\in\mathbb{R}^2$ and $u\le 1$, we define the random measure

$$\mu_{x,u}(A):=\int_0^u\chi_A(X_r+x)\,dr\quad A\subset\mathbb{R}^2. \tag{3.2}$$

Lemma 30. *For each $\epsilon\in(0,1]$ and almost every ω, there exists a $K_\epsilon(\omega)$ such that*

$$\mu_{x,u}(B(y,s))(\omega)\le K_\epsilon(\omega)(s^{2-\epsilon}\wedge 1), \tag{3.3}$$

for all $y\in\mathbb{R}^2$.

Proof. We have that $\mu_{x,u}(\mathbb{R}^2)\le u$, so we will assume $s\le\frac{1}{2}$. Let $R\ge 2+2|x|$ and let

$$A_t:=\int_0^{t\wedge T_R}\chi_{B(y,s)}(X_r+x)\,dr.$$

(Proposition 5.6 in [30] tells us the following)

$\mathbb{E}_w\left[A_{T_R}\right] = \mathbb{E}_w\left[\int_0^{T_R} \chi_{B(y,s)}(X_r+x)\,dr\right] = u(w-x)$, where $u(x)$ is the solution of the Poisson problem

$$-\frac{1}{2}\Delta u = \chi_{B(y,s)}(x) \qquad x \in B(0,R),$$

$$u(x) = 0 \qquad x \in \partial B(0,R).$$

$$\mathbb{E}_w\left[A_{T_R}\right] = \mathbb{E}_w\left[\int_0^{T_R} \chi_{B(y,s)}(X_r+x)\,dr\right] = \mathbb{E}_w\left[\int_0^{T_R} \chi_{\{|X_r+x-y|\le s\}}\,dr\right]$$
$$= \int_{B(y-x,s)} \log\left(\frac{1}{|w-z|}\right)\,dz.$$

As $g_R(w,z) \le c\left(1 \vee \log\left(\frac{1}{|w-z|}\right)\right)$, if $w \in B(0,R)$,

$$\mathbb{E}_w[A_{T_R}] \le c\int_{B(y-x,s)} \left(1 \vee \log\left(\frac{1}{|w-z|}\right)\right)\,dz$$
$$\le c\int_{B(y-x,s)} \left(1 \vee \log\left(\frac{1}{|w-z|}\right)\right)\,dz \le c\int_{B(0,s)} \log\left(\frac{1}{|z|}\right)\,dx$$
$$\le cs^2 \log\left(\frac{1}{s}\right).$$

Since A_t is an additive functional, the above implies

$$\mathbb{E}_0[A_{T_R} - A_t|\mathcal{F}_t] \le \mathbb{E}_{X_t}[A_{T_R}] \le \sup_w \mathbb{E}_w[A_{T_R}] \le cs^{2-\frac{\epsilon}{2}}$$

.

By [16, 15], we have that $\mathbb{E}_0[\exp(\lambda A_{T_R})] \le 2$, if $\lambda \le \frac{1}{8}\sup_w \mathbb{E}_w[A_{T_R}]$. By Chebychev's inequality, we get

$$\mathbb{P}^0(A_{T_R} > c_1 s^{2-\epsilon}) \le 2\exp(-c_2 s^{-\frac{\epsilon}{2}}). \tag{3.4}$$

Looking at $B(0,3R)$, then we can cover this set with $N = cs^{-d}$ balls of radius $2s$ and we denote them by $B_1,\ldots,B_N$. Every ball $B(y,s)$, $y \in B(0,2R)$ is contained in one of the B_i's.

Defining

$$D_R := \{\sup_{t\le 1}|X_t| \le R\},$$

(3.4) yields (for some $y \in B(0,2R)$)

$$\mathbb{P}^0(\mu_{x,u}(B(y,s)) \ge c_1 s^{2-\epsilon}; D_R)$$

$$\le \mathbb{P}^0(\mu_{x,u}(B_i) \ge c_1 s^{2-\epsilon}; D_R)$$

$$\le c_2 s^{-d}\exp(-c_3 s^{-\frac{\epsilon}{2}}).$$

By the Borel-Cantelli lemma with $s = 2^{-i}\ i = 0,1,2,\ldots,$

$$\mathbb{P}^0\left(\frac{\mu_{x,u}B(y,2^{-i})}{(2^{-i})^{2-\epsilon}} \geq c; D_R\right) = 0.$$

Hence, for $\omega \in D_R$, exists some $K_{\epsilon R}(\omega)$, such that

$$\mu_{x,u}(B(y,2^{-i})) \leq K_{\epsilon R}(\omega)(2^{-i})^{2-\epsilon},$$

for all $y \in B(0,2R)$ and $i = 0,1,2,\ldots$. If $s \in (0,1]$, then obviously $s \in (2^{-(i+1)}, 2^{-i}]$ for some i. This leads to (provided $\omega \in D_R$)

$$\mu_{x,u}(B(y,s)) \leq K_{\epsilon R}(\omega)(2^{-i})^{2-\epsilon} \leq K_{\epsilon R}(\omega)s^{2-\epsilon}. \tag{3.5}$$

If $\omega \in D_R$ and $y \notin B(0,2R)$, then $\mu_{x,u}(B(y,s)) = 0$. Noting that each ω is in a D_R, for sufficiently large R, together with our estimates (3.5), yields the desired result (3.3). □

Define

$$\mathcal{L} = \left\{\psi \,:\, \psi : \mathbb{R}^d \to [-1,1],\ \|\psi\|_\infty \leq 1, \text{ and Lipschitz with Lipschitz constant } 1\right\}$$

and

$$d_L(\mu,\nu) := \sup_{\psi\in\mathcal{L}}\left\{\left|\int \psi\, d\mu - \int \psi\, d\nu\right|\right\}.$$

Lemma 31.

$$d_L(\mu_{x,u},\mu_{x,v}) \leq |u-v|,$$

$$d_L(\mu_{x,u},\mu_{y,u}) \leq u|x-y|.$$

Proof. The first inequality is obvious by the definition of d_L.

The second inequality follows from

$$\left|\int \psi\, d(\mu_{x,u}-\mu_{y,u})\right| = \int_0^u |\psi(X_t+x)-\psi(X_t+y)|\, dt \leq u|x-y|.$$

□

Remark 32. Lemma 30 implies that for ω not in the exceptional set, $g_R\mu_{x,y}(z)$ is continuous and bounded ([5]). Let $\alpha_2(x,\cdot,u)$ be the continuous additive functional of Y_t associated with $\mu_{x,u}$. That is the continuous additive functional, such that $\mathbb{E}_0[\alpha_2(x,T_R(Y),u)] = g_R\mu_{x,u}(z)$,

for all z and R ([8]). The stochastic interpretation of this functional is the following: $\alpha_2(x,\cdot,u)$ is the increasing part of the supermartingale $g_R\mu_{x,u}(Y_{t\wedge T_R(Y)})$.

We will state one result regarding the joint Hölder continuity of α_2 in each variable here, further results can be found in [6].

Claim 33. Assume $c\gamma > 0$ and μ being a positive measure, satisfying $\mu_{x,u}(B(y,s)) \leq c(s^{d-2+\gamma}\wedge 1)$ for all $s \in (0,\infty), y \in \mathbb{R}^d$. Let L^μ_t be the associated continuous additive functional, then L^μ_t is Hölder continuous in t a.s..

Proof. [6]. □

Proposition 34. *There exists a null set N, such that if $\omega \notin N$,*

$$\int_{\mathbb{R}^2} f(x)\alpha_2(x,r,u)(\omega)\,dx = \int_0^u \int_0^r f(Y_s(\omega) - X_t(\omega))\,ds\,dt, \tag{3.6}$$

for all bounded and measurable f.

Proof. Let f, h be continuous and compactly supported and define

$$B_u^{x,h} := \int_0^u h(X_t - x)\,dt.$$

The potential of $B_u^{x,h}$ on the ball with radius R is

$$\mathbb{E}_z[B_{T_R}^{x,h}] = \int g_R(z,y)h(y-x)\,dy.$$

So the potential of $\int f(x)B_u^{x,h}\,dx$ is

$$\int\int g_R(z,y)f(x)h(y-x)\,dy\,dx = \int\int g_R(z,y)h(x)f(y-x)\,dy\,dx,$$

which is the potential of $\int h(x)B_u^{x,f}\,dx$. Referring to [8], if two additive functionals of a Brownian motion have the same potential, they are already equal. Hence

$$\int h(x)B_u^{x,f}\,dx = \int f(x)B_u^{x,h}\,dx \quad \text{a.s.},$$

or

$$\int f(x)\left(\int h(y)\mu_{-x,u}(dy)\right)\,dx = \int h(x)\left(\int f(y)\mu_{-x,u}(dy)\right)\,dx. \tag{3.7}$$

The right hand side of (3.6) is equal to $\int_0^r(\int f(-y)\mu_{-Y_s,u}(dy))ds$. So its potential in $B(0,R)$, considered as a continuous additive functional of Y, is

$$\int g_R(z,y)\left(\int f(-\omega)\mu_{-y,u}(d\omega)\right)\,dy.$$

By (3.7), this equals

$$\int f(-x)\left(\int g_R(z,y)\mu_{-x,u}(dy)\right)\,dx = \int f(x)g_R\mu_{x,u}(z)\,dz,$$

which is the potential on the left hand side of (3.6). Since R was arbitrary, this proves the claim. □

We will now show a (Yor-Rosen-) Tanaka formula for the the ILT of two Brownian motions in $\mathbb{R}^2$.

Define

$$G(x) := \frac{1}{\pi}\log\left(\frac{1}{|x|}\right). \tag{3.8}$$

Obviously, $G(x)$ is symmetric in x.

By [9],

$$g_R\mu_{x,u}(Y_{t\wedge T_R}) - g_R\mu_{x,u}(Y_0) \tag{3.9}$$

$$= \int_0^{t\wedge T_R} \nabla g_R\mu_{x,u}(Y_s)\cdot dY_s - \alpha_2(x, t\wedge T_R, u).$$

Since $G(\cdot - y) - g_R(\cdot, y)$ is harmonic in $B(0,R)$ for each y, so is $G\mu_{x,u}(\cdot) - g_R\mu_{x,u}(\cdot)$ and we have ([9])

$$(G\mu_{x,u} - g_R\mu_{x,u})(Y_{t\wedge T_R}) - (G\mu_{x,u} - g_R\mu_{x,u})(Y_0) \tag{3.10}$$

$$= \int_0^{t\wedge T_R} \nabla(G\mu_{x,u} - g_R\mu_{x,u})(Y_s)\cdot dY_s.$$

$$G\mu_{x,u}(y) := \int G(y-z)\mu_{x,u}(dz).$$

Adding (3.9) and (3.10) and letting $R\to\infty$,

$$G\mu_{x,u}(Y_t) - G\mu_{x,u}(Y_0) = \int_0^t \nabla G\mu_{x,u}(Y_s)\cdot dY_s - \alpha_2(x,t,u).$$

Recalling the definition of $\mu_{x,u}$, yields

$$\int_0^u G(Y_t - X_r - x)\,dr - \int_0^u G(Y_0 - X_r - x)\,dr \tag{3.11}$$

$$= \int_0^t \left(\int_0^u \nabla G(Y_s - X_r - x)\,dr\right)\cdot dY_s - \alpha_2(x,t,u).$$

Theorem 35. *Let Y_r be a two dimensional Brownian motion. There exists a $b>0$ (independent of p) and constants $c(p)$, such that if $p\ge 1$, $x\in\mathbb{R}^2$ and $\sigma<1$, that*

$$\mathbb{P}\left[\int_0^1 \chi_{B(x,\sigma)}(Y_r)\,\zeta(dr) > \lambda\right] \le c(p)\frac{\sigma^{bp}}{\lambda^{bp}}. \tag{3.12}$$

Where we suppose, that $\zeta(t)$ is a nondecreasing continuous process with $\zeta(0)=0$, which satisfies, that for each $p\ge 1$, there exists an $a>0$ and a $K(p)\ge 1$, such that

$$\mathbb{E}[(\zeta(t)-\zeta(s))]^p \le K(p)|t-s|^{ap} \qquad s,t\le 1. \tag{3.13}$$

Proof. Assume that $\lambda > 2\sigma$, otherwise the result becomes rather trivial. Fix $x\in\mathbb{R}^2$ and define $R_t := |Y_t - x|$. Let $\epsilon = \frac{1}{16}$ and define

$$S_1 := \inf\{t\ :\ R_t\le\sigma\},\ T_1 := \inf\{t > S_1\ :\ R_t \ge \sigma^{1-\epsilon}\}$$

and

$$S_{i+1} := \inf\{t > T_i\ :\ R_t \le \sigma\},\ T_{i+1} := \inf\{t > S_{i+1}\ :\ R_t\ge\sigma^{1-\epsilon}\}.$$

Let

$$D_u := \inf\{i\ :\ S_i > u\},$$

so D_u is greater or equal to the number of upcrossings of $[\sigma, \sigma^{1-\epsilon}]$ by R_t up to time u. Since $\log(R_t)$ is a martingale, we can use the upcrossing inequality ([12])

$$\sup_z \mathbb{E}_z[D_1] = \mathbb{E}_\sigma[D_1] \le \frac{\mathbb{E}_z[|\log(R_1)| + |\log(\sigma)|]}{|\log(\sigma^{1-\epsilon}) - \log(\sigma)|} \le c_1.$$

Using Cebychev's inequality,

$$\sup_z \mathbb{P}_z(D_1 \ge 2c_1) \le \frac{1}{2}.$$

Applying the strong Markov property at $\inf\{t : D_t \ge 2nc_1\}$,

$$\sup_z \mathbb{P}_z(D_1 \ge 2c_1(n+1)) \le \frac{1}{2} \sup_z \mathbb{P}_z(D_1 \ge 2c_1 n),$$

which leads to

$$\mathbb{P}_z(D_1 \ge n) \le c_2 \exp(-c_3 n), \qquad n \ge 1.$$

Applying the strong Markov property applied at S_i and standard estimates on Brownian motion,

$$\mathbb{P}(T_i - S_i > K\sigma^{2-3\epsilon}) \le \mathbb{P}_0(T_1 > K\sigma^{2-3\epsilon}) \le c_4 \exp(-c_5 K). \tag{3.14}$$

Let $h \in [0,1]$. If $\zeta((t+h)\wedge 1) - \zeta(t) \ge Lh^{\frac{a}{2}}$ for some $t \in [0,1]$, then $\zeta(((j+2)h)\wedge 1) - \zeta(jh) \ge Lh^{\frac{a}{2}}$ for some $j \le [\frac{1}{h}] + 1$. The assumptions of this proposition imply

$$\mathbb{P}(\zeta(t) - \zeta(s) \ge L|t-s|^{\frac{a}{2}}) \le K(p)\frac{|t-s|^{\frac{ap}{2}}}{L^p}.$$

If $p > p_0 = \frac{8}{a}$, then

$$\mathbb{P}(\sup_{t\le 1}(\zeta((t+h)\wedge 1) - \zeta(t)) \ge Lh^{\frac{a}{2}}) \le K(p)\frac{2h^{\frac{ap}{2}}}{hL^p} \le K(p)\frac{2h^{\frac{ap}{4}}}{L^p}. \tag{3.15}$$

Note $R_{T_i} \ge \sigma^{1-\epsilon}$ and R_t doesn't return to the interval $[0,\sigma]$ until time S_{i+1}, so if $Y_r \in B(x,\sigma)$, then $r \in [S_i, T_i]$ for some i. Hence

$$\int_0^1 \chi_{B(x,\sigma)}(Y_r)\,\zeta(dr) \le \sum_{i=1}^{\infty}(\zeta(T_i\wedge 1) - \zeta(S_i \wedge 1)). \tag{3.16}$$

Set $n = (\frac{\lambda}{\sigma})^p$, $K = n^d$, $h = K\sigma^{2-5\epsilon}$ and $L = \frac{\lambda}{2h^{\frac{a}{2}}n}$, where we will choose d appropriately later. If the sum on the right hand side is bigger than λ, one of the following must hold:

(*a*) $D_1 > n$,

(*b*) $T_i - S_i \ge K\sigma^{2-3\epsilon}$ for some $i \le n$,

(*c*) $(T_i \wedge 1) - (S_i \wedge 1) > \frac{\lambda}{2n}$ (and $\max_{j\le n}(T_j - S_j) < K\sigma^{2-3\epsilon}$) for some $i \le n$.

As a quick reminder $\sigma < 1$, $\lambda \ge 2\sigma$ and $\epsilon = \frac{1}{16}$, so

$h := K\sigma^{2-5\epsilon} = \left[\frac{\lambda^d}{\sigma^d}\right]^d \sigma^{2-5\epsilon} > K\sigma^{2-3\epsilon}$. Also $S_j \wedge 1 = 1 \Rightarrow T_j \wedge 1 = 1$, which leads us to

$$(\zeta(T_i \wedge 1) - \zeta(S_i \wedge 1)) \leq \sup_{t \leq 1}(\zeta((t+h) \wedge 1) - \zeta(t)),$$

implying

$$\mathbb{P}(\zeta(T_i \wedge 1) - \zeta(S_i \wedge 1)) \leq \mathbb{P}\left(\sup_{t \leq 1}(\zeta((t+h) \wedge 1) - \zeta(t)) > \frac{\lambda}{2n}\right).$$

So,

$$\mathbb{P}\left[\int_0^1 \chi_{B(x,\sigma)}(Y_r)\, dr > \lambda\right] \leq \mathbb{P}(D_1 \geq n) + n \sup_i \mathbb{P}(T_i - S_i > K\sigma^{2-3\epsilon}) + \mathbb{P}\left((T_i \wedge 1) - (S_i \wedge 1) > \frac{\lambda}{2n}\right)$$

$$\leq c_2 \exp(-c_3 n) + nc_4 \exp(-c_5 K) + \mathbb{P}\left(\sup_{t \leq 1}(\zeta((t+h) \wedge 1) - \zeta(t)) > \frac{\lambda}{2n}\right)$$

$$\leq c_2 \exp(-c_3 n) + nc_4 \exp(-c_5 K) + K(p)\frac{2h^{\frac{ap}{4}}}{L^p}.$$

Substituting for n, K, h and L, we recall that $\lambda > 2\sigma$ and $\sigma < 1$. So, taking d sufficiently small, we obtain the result for all $p \geq p_0$. The result for $p \in [1, p_0)$ follows, since $\sigma < \lambda$. □

Remark 36. The previous result obviously holds true for $\zeta(t) = t$ (by choosing d appropriately small), which we will use in the following results.

For $\epsilon \in (0,1)$, define

$$G_\epsilon(x) := G(x) \wedge \frac{1}{\pi}\log\left(\frac{1}{\epsilon}\right), \tag{3.17}$$

$$H_\epsilon(x) := G(x) - G_\epsilon(x).$$

Proposition 37. *Assume $a > 0$. There exists a $d \geq 0$ and $\epsilon_0 < 1$ such that for $p \geq 1$ and $q \geq 1$*

$$\mathbb{E}\left[\left(\int_0^u |H_\epsilon(X_r - X_s - x)|^q\, dr\right)^p\right] \leq c(p,q)\epsilon^{dp},$$

if $u \in [0,1]$ and $\epsilon \leq \epsilon_0$.

Proof. Set $V = \int_0^u |H_\epsilon(X_r - X_s - x)|^q\, dr$ and $Y_r = X_u - X_r$. Let $n = [\frac{4}{b}] + 4$, where b is the constant from Theorem 35.

First, we note that

$$H_\epsilon^q(z + x) \leq c_1 \sum_{\{j\,:\,2^{-j} \leq \epsilon\}} j^q \chi_{B(x,2^{-j})}(z).$$

Also, if $\lambda \geq \epsilon^{\frac{1}{4}}$, with ϵ being sufficiently small and $2^{-j} \leq \epsilon$, then $\frac{\lambda}{40 c_1 j^{2+q}} \geq 2^{-\frac{j}{2}}$. Using Theorem 35

$$\mathbb{P}[V > \lambda] \leq \sum_{\{j\,:\,2^{-j} \leq \epsilon\}} \mathbb{P}\left[c_1 j^q \int_0^u \chi_{B(x,2^{-j})}(Y_r)\, dr > \frac{\lambda}{20j^2}\right]$$

$$\begin{aligned}
&\le c(np) \sum_{\{j\,:\,2^{-j}\le\epsilon\}} \frac{\left(2^{-j}\right)^{bnp}}{\left(\frac{\lambda}{20j^{2+q}}\right)^{bnp}} \\
&= c(p,q) \sum_{\{j\,:\,2^{-j}\le\epsilon\}} \frac{2^{-j\frac{p}{2}}}{\lambda^{bnp}} \\
&\le c(p,q)\epsilon^{d_1p}\frac{\epsilon^{d_1p}}{\lambda^{p+2}},
\end{aligned}$$

if ϵ is small enough.

Multiplying by $p\lambda^{p-1}$ and integrating from $\epsilon^{\frac{1}{4}}$ to ∞, gives

$$\mathbb{E}\left[V^p\chi_{V\le\epsilon^{\frac{1}{4}}}\right] \le c(p,q)\epsilon^{dp}.$$

Since

$$\mathbb{E}\left[V^p\chi_{V\le\epsilon^{\frac{1}{4}}}\right] \le c(p,q)\epsilon^{dp},$$

adding up the two terms, gives the result. □

Let $U_t = M_t - V_t$, where M_t is a martingale with mean 0 and V_t is a non-decreasing process with $V_0 = 0$. Furthermore, U, M and V have right-continuous paths with left limits and are adapted to a filtration satisfying the usual conditions.

Lemma 38. *Suppose that for an $a > 0$ and $p \ge 1$, there exists a $K(p)$, such that*

$$\mathbb{E}[|U_t|^p] \le K(p), \qquad t \le 1 \tag{3.18}$$

$$\mathbb{E}[|U_t - U_s|^p] \le K(p)|t-s|^{pa}, \qquad s,t \le 1 \tag{3.19}$$

.

Let $\bar{K} = K(p) \vee K(p+1)$, then there exists a $b > 0$, independent of p, and constants $c(p)$, such that for $p \ge 1$

$$\mathbb{E}[V_1{}^p] \le c(p)\bar{K}(p) \tag{3.20}$$

$$\mathbb{E}[V_t - V_s{}^p] \le c(p)\bar{K}(p)|t-s|^{pb}, \qquad s,t \le 1 \tag{3.21}$$

Proof. We focus on the case $p \ge p_0 = \frac{2}{a}$, since the result for $p < p_0$ follows from applying Jensen's inequality.

Using a chaining argument, like in the proof of Kolmogorov's theorem the first two inequalities would imply, that we find a version of U_t, such that

$$\mathbb{E}[\sup_{t\le1}|U_t|^p] \le c(p)\bar{K}(p).$$

As U_t and $-V_t$ only differ by a martingale, for $t \le 1$, we have

$$\mathbb{E}[V_1 - V_t|\mathcal{F}_t] = \mathbb{E}[U_t - U_1|\mathcal{F}_t] \le 2\mathbb{E}[\sup_s |U_s|\,|\mathcal{F}_t].$$

Using the inequality [3] (Lemma 2.3), we get

$$\mathbb{E}[V_1^p] \leq c(p)\mathbb{E}[\sup_t |U_t|^p].$$

Together with (3.18), this proves (3.20).

In a similar fashion,

$$\mathbb{E}[\sup_{s\leq r\leq t} |U_r - U_s|^p] \leq c(p)\bar{K}(p)|t-s|^{pa}.$$

If we apply the above argument to $\bar{V}_r := V_{s+r} - V_s$, $\bar{U}_r := U_{s+r} - U_s$ and $\bar{M}_r := M_{s+r} - M_r$ $r \leq t-s$, we get the second inequality. □

Remark 39. Setting $p > \frac{1}{b}$, implies, that V_t is Hölder continuous a.s. on a dense subset. As V_t was assumed to be increasing, it turns out to be continuous a.s..

Suppose now, that $U_t^i = M_t^i - V_t^i$, $i = 1, 2$ and V^i, M^i as above. Set $V_t = V_t^1 - V_t^2$ and analogously M_t and U_t.

Proposition 40. *Let $a, b, \delta \in (0,1)$. Assuming, that for each p, there exists a $K(p)$, such that*

$$\mathbb{E}[|U_t^i|^p] \leq K(p), \qquad t \leq 1,\ i = 1,2$$

$$\mathbb{E}[|U_t^i - U_s^i|^p] \leq K(p)|t-s|^{pa}, \qquad s,t \leq 1,\ i = 1,2$$

and

$$\mathbb{E}[|U_t|^p] \leq K(p)\delta^{pb}, \qquad t \leq 1. \tag{3.22}$$

Then there exists a $d > 0$, such that

$$\mathbb{E}[{V_1}^p] \leq c(p)\bar{K}(p)\delta^{dp}, \qquad t \leq 1. \tag{3.23}$$

Proof. Again, we suppose that $p \geq \frac{2}{a} + 2$. As in the preceding proof,

$$\mathbb{E}\left[\sup_{s\leq t\leq s+h} |U_t^i - U_s^i|^p\right] \leq c(p)\bar{K}(p)|h|^{pa}.$$

For $n \geq 1$,

$$\sup_{t\leq 1} |U_t| \leq \sup_{j\leq n} |U_{\frac{j}{n}}| + \sum_{i=1}^{2} \sup_{j\leq n} \sup_{\frac{j}{n}\leq t\leq \frac{j+1}{n}} |U_t^i - U_{\frac{j}{n}}^i|.$$

It follows, that

$$\mathbb{E}[\sup_t |U_t|^p]$$

$$\leq c(p)n \sup_{j\leq n} \mathbb{E}[|U_{\frac{j}{n}}|^p] + 2c(p)n \max_{1\leq i\leq 2} \sup_{j\leq n} \mathbb{E}\left[\left|\sup_{\frac{j}{n}\leq t\leq \frac{j+1}{n}} |U_t^i - U_{\frac{j}{n}}^i|\right|^p\right]$$

$$\leq c(p)n\bar{K}(p)\delta^{bp} + 2nc(p)\bar{K}(p)(\frac{1}{n})^{ap}.$$

Since $ap > 2$, we set $n = [\delta^{-\frac{b}{2}}] + 1$ in order to get

$$\mathbb{E}[\sup_t |U_t|^p] \le c(p)\bar{K}(p)\delta^{\frac{abp}{2}}. \tag{3.24}$$

Define $Z = \sup_t |U_t|$ and $W = 1 + V_1^1 + V_1^2$, then by the preceding Lemma 38, we get $W \in L^p$ for all p.

If $t \le 1$,

$$|\mathbb{E}[V_1 - V_t|\mathcal{F}_t]| = |\mathbb{E}[U_t - U_1|\mathcal{F}_t]| \le 2\mathbb{E}[Z\,|\mathcal{F}_t].$$

Referring to the proof [2] (Lemma 2.3),

$$\begin{aligned} \mathbb{E}[(V_1 - V_t)^2|\mathcal{F}_t] &= 2\mathbb{E}\left[\int_t^1 (V_1 - V_s)\,dV_s|\mathcal{F}_t\right] \\ &= 2\mathbb{E}\left[\int_t^1 \mathbb{E}[(V_1 - V_s)|\mathcal{F}_s]\,dV_s|\mathcal{F}_t\right] \\ &\le 2\mathbb{E}\left[\int_t^1 \mathbb{E}[Z|\mathcal{F}_s]\,d(V_s^1 + V_s^2)|\mathcal{F}_t\right]. \end{aligned} \tag{3.25}$$

Now set $Y_t = \mathbb{E}[(V_1 - V_s)|\mathcal{F}_s]$, $N_t = \mathbb{E}[(V_1)|\mathcal{F}_s]$, such that $Y_t = N_t - V_t$. We take the right continuous versions of Y and N. Jensen's inequality implies, that

$$Y_t^2 = (\mathbb{E}[(V_1 - V_s)|\mathcal{F}_s])^2 \le \mathbb{E}[(V_1 - V_s)^2|\mathcal{F}_s] \le 2\mathbb{E}[WZ|\mathcal{F}_s].$$

Further, if we apply Ito's lemma, we get

$$Y_1^2 - Y_t^2 = 2\int_t^1 Y_s\,dY_s + \langle N\rangle_1 - \langle N\rangle_t,$$

which in turn tells us, that

$$\begin{aligned} \mathbb{E}[\langle N\rangle_1 - \langle N\rangle_t|\mathcal{F}_t] &\le |\mathbb{E}[Y_1^2 - Y_t^2|\mathcal{F}_t]| + 2|\mathbb{E}[\int_t^1 Y_s\,dY_s|\mathcal{F}_t]| \\ &\le 4\mathbb{E}[WZ|\mathcal{F}_t] + 2\left|\mathbb{E}\left[\int_t^1 Y_s\,dV_s|\mathcal{F}_t\right]\right| \\ &\le 4\mathbb{E}[WZ|\mathcal{F}_t] + 2\left|\mathbb{E}\left[\int_t^1 \mathbb{E}[Z|\mathcal{F}_s]\,d(V_s^1 + V_s^2)|\mathcal{F}_t\right]\right| \\ &\le 8\mathbb{E}[WZ|\mathcal{F}_t]. \end{aligned}$$

By applying Lemma 2.3 from [3] and Lemma 38,

$$\begin{aligned} \mathbb{E}[\langle N\rangle_1^p] &\le c(p)\sqrt{\mathbb{E}[Z^{2p}]}\sqrt{\mathbb{E}[W^{2p}]} \\ &\le (c(2p)\bar{K}(2p)\delta^{\frac{2abp}{2}})^{\frac{1}{2}}(\bar{K}(2p))^{\frac{1}{2}} \\ &\le c(p)\bar{K}(2p)\delta^{\frac{abp}{2}}. \end{aligned}$$

Applying Jensen's inequality ones more, we get

$$\mathbb{E}[|Y_t|^{2p}] \le \mathbb{E}[2\mathbb{E}[WZ|\mathcal{F}_t]^p] \le c(p)\mathbb{E}[(WZ)^p] \le c(p)\bar{K}(2p)\delta^{\frac{abp}{2}}.$$

Therefore,

$$\mathbb{E}[|V_t|^{2p}] \leq c(p)\mathbb{E}[|N_t|^2] + c(p)\mathbb{E}[|Y_t|^{2p}]$$

$$\leq c(p)\mathbb{E}[\langle N\rangle_1^p] + c(p)\mathbb{E}[|Y_t|^{2p}] \leq c(p)\bar{K}(2p)\delta^{\frac{abp}{2}}.$$

By setting $d = \frac{ab}{4}$, the proof is completed. □

We now want to use some of the results obtained in the study of the intersection of two planar Brownian motions, in order to study the SILT for double points of a single Brownian motion.

Let t be fixed, $\Delta_n = 2^{-n}$ and set $s_i = ti\Delta_n$ for $i = 0, \ldots, 2^n$. Setting $Y_r = (X_{s_i+r} - X_{s_i}) + X_{s_i} = X_{s_i+r} \quad 0 \leq r \leq \Delta$. For an $x \in \mathbb{R}^2$, let $\mu_{x,u}(A) = \int_0^{s_i} \chi_A(X_r)\, dr$. There now exists, with the same argumentation, a continuous, additive functional of Y_r, we will call $\alpha_2^{ni}(x,\cdot)$, such that, if $A_{n,i,x} := \alpha_2^{ni}(x, \Delta_n)$, then

$$\int_0^{s_i} \big(G(X_{s_{i+1}} - X_r - x) - G(X_{s_i} - X_r - x)\big)\, dr \tag{3.26}$$
$$= \int_{s_i}^{s_{i+1}} \left(\int_0^{s_i} \nabla G(X_s - X_r - x)\, dr\right) dX_s - A_{n,i,x}.$$

Furthermore, it holds that $A_{n,i,x} \geq 0$, continuous in x and

$$\int f(x) A_{n,i,x}\, dx = \int_{s_i}^{s_{i+1}} \int_0^{s_i} f(X_r - X_s)\, ds\, dr. \tag{3.27}$$

Let

$$U_t^n = U_t^n(x) := \sum_{i=0}^{2^n-1} \int_0^{s_i} \big(G(X_{s_{i+1}} - X_r - x) - G(X_{s_i} - X_r - x)\big)\, dr,$$

$$M_t^n = M_t^n(x) := \int_{s_i}^{s_{i+1}} \int_0^{s_i} \nabla G(X_s - X_r - x)\, dr\, dX_s,$$

$$\beta_t^n(x) := \sum_{i=0}^{2^n-1} A_{n,i,x},$$

$$U_t = U_t(x) := \int_0^t (G(X_t - X_r - x) - G(-x))\, dr,$$

$$M_t = M_t(x) := \int_0^t \int_0^s \nabla G(X_s - X_r - x)\, dr\, dX_s.$$

Summing over all i in (3.26), we get

$$U_t^n = M_t^n - \beta_t^n(x). \tag{3.28}$$

Proposition 41. *If $x \neq 0$, $U_t^n \to U_t$ in L^p for $p > 1$.*

Proof.

$$\begin{aligned}
U_t^n &= \sum_{i=0}^{2^n-1} \sum_{j=0}^{i-1} \int_{s_j}^{s_{j+1}} \big(G(X_{s_{i+1}} - X_r - x) - G(X_{s_i} - X_r - x)\big)\ dr \\
&= \sum_{j=0}^{2^n-1} \sum_{i=j+1}^{2^n-1} \int_{s_j}^{s_{j+1}} \big(G(X_{s_{i+1}} - X_r - x) - G(X_{s_i} - X_r - x)\big)\ dr \\
&= \sum_{j=0}^{2^n-1} \int_{s_j}^{s_{j+1}} \big(G(X_t - X_r - x) - G(X_{s_{j+1}} - X_r - x)\big)\ dr.
\end{aligned}$$

Defining the function

$$h_r^n := \sum_{j=0}^{2^n-1} \big(G(-x) - G(X_{s_{j+1}} - X_r - x)\big)\, \chi_{(s_j, s_{j+1})}(r),$$

it suffices to prove that

$$\int_0^t h_r^n\ dr \to 0 \qquad \text{in } L^p.$$

By applying the generalized triangle inequality, as well as Hölder's inequality,

$$\mathbb{E}\left[\left|\int_0^t h_r^n\ dr\right|^p\right] \leq \left(\mathbb{E}\left[\int_0^t |h_r^n|^{2p}\ dr\right]\right)^{\frac{1}{2}} t^{2p-1}.$$

We have to show, that

$$\mathbb{E}\left[\sum_{j=0}^{2^n-1} \int_{s_j}^{s_{j+1}} |(G(-x) - G(X_{s_{j+1}} - X_r - x))|^{2p}\ dr\right] \to 0. \tag{3.29}$$

For a $z \in B(x, \frac{|x|}{2})$, choose ϵ small enough, such that $G(z) = G_\epsilon(z)$.

$$\begin{aligned}
&\mathbb{E}\left[\sum_{j=0}^{2^n-1} \int_{s_j}^{s_{j+1}} |(G_\epsilon(-x) - G_\epsilon(X_{s_{j+1}} - X_r - x))|^{2p}\ dr\right] \qquad (3.30)\\
&\leq \|\nabla G_\epsilon\|^{2p} \mathbb{E}\left[\sum_{j=0}^{2^n-1} \int_{s_j}^{s_{j+1}}\ dr\right] \qquad \sup_{u,v\leq 1, |u-v|\leq \Delta_n} |X_u - X_v|^{2p}] \\
&\leq_{\text{Cauchy-Schwarz}} c\epsilon^{-2p} \left(\mathbb{E}\left[\sup_{u,v\leq 1, |u-v|\leq \Delta_n} |X_u - X_v|^{4p}\right]\right)^{\frac{1}{2}} \\
&\leq c\epsilon^{-2p} \Delta_n^p.
\end{aligned}$$

We define the set

$$V := \left\{\sup_{u,v\leq 1, |u-v|\leq \Delta_n} |X_u - X_v| > \frac{|x|}{2}\right\}.$$

By our choice of ϵ, we get that $H_\epsilon(-x) = 0$ and

$$\mathbb{E}\left[\sum_{j=0}^{2^n-1}\int_{s_j}^{s_{j+1}} \chi_{V^C}|(H_\epsilon(-x)-H_\epsilon(X_{s_{j+1}}-X_r-x))|^{2p}\,dr\right]=0. \tag{3.31}$$

Also, by

$$\mathbb{E}\left[\left(\sum_{j=0}^{2^n-1} Z_j\right)^2\right] \le 2^n\mathbb{E}\left[\sum_{j=0}^{2^n-1} {Z_j}^2\right] \le 2^{2n}\sup_j \mathbb{E}[Z_j^2],$$

we get

$$\mathbb{E}\left[\sum_{j=0}^{2^n-1}\int_{s_j}^{s_{j+1}} \chi_V|H_\epsilon(X_{s_{j+1}}-X_r-x)|^{2p}\,dr\right] \tag{3.32}$$

$$\le \left(\mathbb{E}\left[\sum_{j=0}^{2^n-1}\int_{s_j}^{s_{j+1}} |H_\epsilon(X_{s_{j+1}}-X_r-x)|^{2p}\,dr\right]^2\right)^{\frac{1}{2}}(\mathbb{P}(V))^{\frac{1}{2}}$$

$$\le 2^n c\left(\sup_j\mathbb{E}\left[\sum_{j=0}^{2^n-1}\int_{s_j}^{s_{j+1}} |H_\epsilon(X_{s_{j+1}}-X_r-x)|^{2p}\,dr\right]^2\right)^{\frac{1}{2}}\exp\left(-\frac{|x|}{16\Delta_n}\right)$$

$$\le 2^n c\exp\left(-\frac{|x|}{16\Delta_n}\right).$$

Here, we used Proposition (37). Adding up the results from our inequalities (3.30), (3.31), (3.32) and letting $\epsilon=\epsilon(n)\to 0$, as $n\to\infty$, such that $\Delta_n^{\frac{1}{2}}\le \epsilon(n)^2$, we get the desired result. □

Proposition 42. *$\beta_t^n(x)$ is increasing as $n\to\infty$ and denote the limit by $\beta_2(x,t)$. For a function f, which is continuous and has compact support, the following equality holds almost everywhere,*

$$\int f(x)\beta_2(x,t)\,dx = \int_0^t\int_0^s f(X_r-X_s)\,dr\,ds. \tag{3.33}$$

Proof. Let ϕ_ϵ be a Dirac sequence, then by (3.27)

$$\int \phi_\epsilon(x-x_0)\beta_t^n(x)\,dx = \sum_{i=0}^{2^n-1}\int_{s_i}^{s_{i+1}}\int_0^{s_i}\phi_\epsilon(X_r-X_s-x_0)\,dr\,ds. \tag{3.34}$$

For each n, as $\epsilon\to 0$, the left hand side converges almost surely to $\beta_t^n(x_0)$, since $A_{n,i,x}$ is continuous in x. For each fixed ϵ, the right hand side of (3.34) is increasing with respect to n. This means, that for each $x_0\ne 0$, $\beta_t^n(x)$ increases, as $n\to\infty$. We call the limit $\beta_2(x,t)$. Using the monotone convergence theorem, we get

$$\int f(x)\beta_2(x,t)\,dx = \lim_{n\to\infty}\int f(x)\beta_2^n(x)\,dx$$

$$= \lim_{n\to\infty} \int_0^t \int_0^s f(X_r - X_s)\chi_{\{r\le s_i\}}\chi_{\{s_i\le s\le s_{i+1}\}}\, dr\, ds = \int_0^t \int_0^s f(X_r - X_s)\, dr\, ds.$$

□

If we define $\bar\beta_2(x,t)$ as the limit of $\beta_t^n(x)$ for each $x\in\mathbb{R}\backslash\{0\}$and rational t, by Proposition 42, it is not difficult to see, that $\bar\beta_2(x,t)\ge\bar\beta_2(x,s)$ a.s. for $t\ge s$. For $t\in[0,1]$, let

$$\beta_2(x,t) = \inf_{u\ge t,\, u\in\mathbb{Q}} \bar\beta_2(x,u).$$

Lemma 43. *For each $p\ge 1$, there exists a $v(p)$, such that*

$$\mathbb{E}[|U_t(x)|^p] \le c(p)(1\vee|G(x)|)^{v(p)}, \qquad t\le 1,$$

There exists an $a>0$, such that

$$\mathbb{E}[|U_t(x)-U_s(x)|^p] \le c(p)(1\vee|G(x)|)^{v(p)}|t-s|^{pa}, \qquad s,t\le 1.$$

Proof. Obviously, $G(x)t$ has moments of all orders. We choose ϵ small but fixed. By Proposition 37,

$\int_0^t H_\epsilon(X_t - X_r - x)\, dr$ has a pth moment. At the same time, it holds that

$$|\int_0^t G_\epsilon(X_t - X_r - x)\, dr| \le c\log(\frac{1}{\epsilon})t.$$

The first claim directly follows.

Regarding the second claim,

$$|U_t - U_s| \le |G(x)|(t-s) + |\int_0^t H_\epsilon(X_t - X_r - x)\, dr|$$

$$+|\int_0^s H_\epsilon(X_s - X_r - x)\, dr| + |\int_0^t G_\epsilon(X_t - X_r - x)\, dr|$$

$$+|\int_0^s (G_\epsilon(X_t - X_r - x) - G_\epsilon(X_s - X_r - x))\, dr|.$$

By Proposition 37,

$$\mathbb{E}[|U_t - U_s|^p] \le c(p)|G(x)|^p|t-s|^p + c(p)\epsilon^{dp}$$

$$+c(p)|\log\left(\frac{1}{\epsilon}\right)|^p|t-s|^p + \|\nabla G_\epsilon\|^p\mathbb{E}[(\int_0^s dr\, |X_t - X_s|)^p]$$

$$\le (1\vee|G(x)|)^p|t-s|^p + c(p)\epsilon^{dp} + c(p)|\log\left(\frac{1}{\epsilon}\right)|^p|t-s|^p + c|t-s|^{\frac{p}{2}}\frac{1}{\epsilon^p}.$$

We used Cauchy Schwarz in order to arrive at the last term on the right hand side. Choosing $\epsilon = |t-s|^b$ for a suitable b proves the claim. □

Theorem 44. *For $p \geq 1$, there exists a $v(p)$, such that*

$$\mathbb{E}[\beta_2(x,t)^p] \leq c(p)(1 \vee |G(x)|)^{v(p)}, \qquad t \leq 1.$$

There exists an $a > 0$, such that

$$\mathbb{E}[(\beta_2(x,t) - \beta_2(x,s))^p] \leq c(p)(1 \vee |G(x)|)^{v(p)}|t-s|^{pa}, \qquad s,t \leq 1.$$

Proof. As we have

$$\mathbb{E}[\beta_1^n(x) - \beta_t^n(x)^p|\mathcal{F}_t] = \mathbb{E}[U_t^n(x) - U_1^n(x)|\mathcal{F}_t],$$

we can make use of the monotone convergence of $\beta_t^n(x)$ to $\bar{\beta}_2(x,t)$ for rational t, the monotonicity of $\beta_2(x,t)$ and the L^p convergence of $U_t^n(x)$ to $U_t(x)$ in order to get

$$\mathbb{E}[(\beta_2(x,1) - \beta_2(x,t))|\mathcal{F}_t] = \mathbb{E}[U_t(x) - U_1(x)|\mathcal{F}_t].$$

We see, that $M_t = U_t(x) + \beta_2(x,t)$ is a martingale. The result now follows from Lemma 38. □

Theorem 45. *The following (Yor-Rosen-) Tanaka formula holds*

$$\begin{aligned}(3.35)\qquad &\int_0^t \left(G(X_t - X_r - x) - G(-x)\right)\, dr \\ &= \int_0^t \left(\int_0^s \nabla G(X_s - X_r - x)\, dr\right) dX_s - \beta_2(x,t).\end{aligned}$$

Proof. Since $\beta_t^n(x)$ converges to $\beta_2(x,t)$ and $\beta_2(x,t)$ is in L^p, the convergence happens in L^p. Since also $U_t^n(x) \to U_t(x)$ in L^p, we can conclude, that $M_t^n(x)$ converges in L^p to some N_t.

As

$$M_t^n(x) = \int_0^t h_s^n \cdot dX_s,$$

with

$$h_s^n = \int_0^s \nabla G(X_s - X_r - x)\chi_{\{r \leq s_i\}}\chi_{\{s_i \leq s \leq s_{i+1}\}}\, dr,$$

then

$$\int_0^t |h_s^n - h_s^m|^2\, ds = \langle M^n - M^m\rangle_t \to 0.$$

As h_s^n converges for each s to $h_s = \int_0^s \nabla G(X_s - X_r - x)dr$, then $\int_0^t |h_s^n - h_s^m|^2 ds \to 0$. It follows, that N_t must be equal to $M_t(x)$. We get the full formula by simply applying a limit to (3.28). □

Despite having already looked at two approaches to derive the notion of SILTs, we will, rather briefly, look at a third one. This approach is outlined in [38] and although we will not go into much detail here, there is one rather astonishing result (the occupation times formula) which we will make use of later on.

In [38] the following (Yor-Rosen-) Tanaka formula was shown,

(3.36)
$$\int_0^t \log(|W_t - W_r - x|)\, dr = \int_0^t \int_0^s \frac{W_s - W_r - x}{|W_s - W_r - x|^2}\, dr\, dW_s + t\log(x) + \pi\beta_2(t,x),$$
as the limit of embedded random walks.

Also the renormalized version of the SILT is introduced

$$\alpha(t,x) := \begin{cases} \beta_2(t,x) - \frac{1}{\pi}\log\left(\frac{1}{|x|}\right) & x \neq 0 \\ \lim_{x\to 0} \beta_2(t,x) - \frac{1}{\pi}\log\left(\frac{1}{|x|}\right) & x = 0 \end{cases}$$

It was shown by LeGall [27] that this limit exists almost everywhere and in L^2.

A consequence is the following "occupation times formula", which is pretty similar to the one obtained before

Corollary 46. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded Borel function, then*

$$\int_0^t \int_0^s f(W_s - W_r)\, dr\, ds = \int_{\mathbb{R}^2} f(x)\beta_2(t,x)\, dx = \int_{\mathbb{R}^2} f(x)(\alpha(t,x) - \frac{t}{\pi}\log(|x|))\, dx.$$

Or, alternatively

$$\int_0^t \int_0^s (f(W_s - W_r) - \mathbb{E}[f(W_s - W_r)])\, dr\, ds = \int_{\mathbb{R}^2} f(x)(\alpha(t,x) - \mathbb{E}[\alpha(t,x)])\, dx.$$

Proof. [38]. □

Lemma 47. *$Y(t,x) := \int_0^t \int_0^s \frac{W_s - W_r - x}{|W_s - W_r - x|^2}\, dr\, dW_s$ is a continuous L^2 martingale with expectation 0 as a function of $t \in [0,K]$ for any fixed $y \in \mathbb{R}^2$.*

Proof.
$$\mathbb{E}[|Y(t,x)|^2] = \int_0^t \mathbb{E}\left[\left|\int_0^s \frac{W_s - W_r - x}{|W_s - W_r - x|^2}\, dr\right|^2\right] ds$$
$$\leq \int_0^t \mathbb{E}\left[\left(\int_0^t \frac{1}{|W_s - W_{s-r} - x|^2}\, dr\right)^2\right] ds = t\mathbb{E}\left[\left(\int_0^t \frac{1}{|W_r - x|^2}\, dr\right)^2\right].$$

By symmetry and the independence of increments of W,

$$\mathbb{E}[|Y(t,x)|^2] \leq t\mathbb{E}\left[\int_0^t \int_0^t \frac{1}{|W_{r_1} - x||W_{r_2} - x|}\, dr_2\, dr_1\right]$$
$$= 2t \int_{[0,t]\times\mathbb{R}^2} \frac{\exp\left(-\frac{|z_1|^2}{2r_1}\right)}{2\pi r_1 |z_1 - x|} \int_{[r_1,t]\times\mathbb{R}^2} \frac{\exp\left(-\frac{|z_2 - z_1|^2}{2(r_2 - r_1)}\right)}{2\pi(r_2 - r_1)|z_2 - x|}\, dr_2\, dz_2\, dr_1\, dz_1.$$

Writing $z_2 = x + \eta(\cos(\theta), \sin(\theta))$, $z_1 = x + \rho(\cos(\gamma), \sin(\gamma))$ and $r = r_2 - r_1$, we obtain, for the inner integral,

$$\int_0^{t-r_1} \int_0^{2\pi} \int_0^\infty \exp\left(-\frac{\eta^2 + \rho^2 - 2r\rho\cos(\theta - \gamma)}{2r}\right) d\eta \frac{1}{2\pi}\, d\theta \frac{1}{r}\, dr$$

$$\leq \sqrt{2\pi} \int_0^{t-r_1} \int_0^{2\pi} \int_0^{\infty} \exp\left(-\frac{(\eta-\rho)^2}{2r}\right) \frac{1}{\sqrt{2\pi r}} d\eta \frac{1}{2\pi}\, d\theta \frac{1}{\sqrt{r}}\, dr$$
$$\leq 2\sqrt{2\pi(t-r_1)}.$$

Thus,

$$\mathbb{E}[|Y(t,x)|^2] \leq 2t \int_0^t \int_0^{2\pi} \int_0^{\infty} \exp\left(-\frac{\rho^2}{2r_1}\right) 2\sqrt{2\pi(t-r_1)}\, d\rho \frac{1}{2\pi}\, d\gamma \frac{1}{r_1}\, dr_1$$
$$\leq 8\pi K^2 (0 \leq t \leq K,\ x \in \mathbb{R}^2).$$

□

Lemma 48. *$X(t,x) := \int_0^t \log(|W_t - W_r - x|)\, dr \quad (t \geq 0,\ x \in \mathbb{R}^2)$, then*

$$\mathbb{E}[X(t,x)] = t\log(|x|) - \frac{|x|^2+2t}{4} Ei\left(-\frac{|x|^2}{2t}\right) - \frac{1}{2} t \exp\left(-\frac{|x|^2}{2t}\right) \; x \neq 0,$$
$$\lim_{x\to 0} \mathbb{E}[X(t,x)] = \mathbb{E}[X(t,0)] = \frac{t}{2}(\log(2t) - C - 1),$$

where Ei denotes the exponential integral function and C is Euler's constant.

Proof. For any fixed t, $\bar{W}_r := W_t - W_{t-r}$ is a planar Brownian motion starting from 0. Thus

$$\mathbb{E}[X(t,x)] = \int_0^t \mathbb{E}\left[\log(|\bar{W}_r - x|)\right]\, dr = \int_0^t \int_{\mathbb{R}^2} \frac{\log(|y-x|)}{2\pi r} \exp\left(-\frac{|y|^2}{2r}\right) dy\, dr$$
$$= \int_0^t \int_0^{\infty} \int_0^{2\pi} \frac{\eta \log(\eta^2 + \rho^2 - 2\eta\rho\cos(\theta-\gamma))}{4\pi r} \exp\left(-\frac{\eta^2}{2r}\right) d\theta\, d\eta\, dr,$$

where we substituted $x = \eta(\cos(\theta), \sin(\theta))$, $y = \rho(\cos(\gamma), \sin(\gamma))$. As the last integral doesn't depend on γ, we can replace it by 0. Since

$$\int_0^{2\pi} \frac{1}{4\pi} \log(\eta^2 + \rho^2 - 2\eta\rho\cos(\theta-\gamma))\, d\theta = \log(\eta \vee \rho),$$

it follows, that

$$\mathbb{E}[X(t,x)] = \int_0^t \int_0^{\infty} \frac{\eta}{4\pi r} \exp\left(-\frac{\eta^2}{2r}\right) \log(\eta \vee \rho)\, d\eta\, dr.$$

This gives exactly the desired results. □

Remark 49. Although looking at all of these different approaches might not seem very educational or even confusing (regarding the notation), each offers a different insight and thereby very useful results. As we have seen, the "functional approach" results in rather nice bounds for the moments, whereas the approach used in [38] delivers a more general version of the occupation times formula. The main point, which connects all these approaches and justifies the "mixing" of these results, is the (Yor-Rosen-)Tanaka formula.

4. An Interesting SPDE

In this section, we will focus on a particularly interesting example of a 2 dimensional heat equation and a rather explicit form of its (in a certain sense) weak solution .

Before we dive into the SPDE itself, we will require some results in order to properly state the weak formulation.

We will require the following Lemma:

Lemma 50. *Let $1 \leq p < \infty$ and $k \geq 0$, then the space $C_c^\infty(\mathbb{R}^d)$ of test functions is a dense subspace of $W^{k,p}(\mathbb{R}^d)$.*

Proof. It is clear that $C_c^\infty(\mathbb{R}^d)$ is a subspace of $W^{k,p}(\mathbb{R}^d)$. First, we show that $C_{loc}^\infty(\mathbb{R}^d) \cap W^{k,p}(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$ and then, in turn that $C_c^\infty(\mathbb{R}^d)$ is dense in $C_{loc}^\infty(\mathbb{R}^d) \cap W^{k,p}(\mathbb{R}^d)$.

Let $f \in W^{k,p}(\mathbb{R}^d)$ and let ϕ_n be a sequence of smooth, compactly supported approximation of the identity with respect to the convolution (Dirac sequence). Since $f \in L^p(\mathbb{R}^d)$, $f * \phi_n$ converges to f in $L^p(\mathbb{R}^d)$. Since $\nabla^j f$ is also an element of $L^p(\mathbb{R}^d)$, we have that $(\nabla^j f) * \phi_n (= \nabla^j(f * \phi_n))$ converges to $\nabla^j f$ in $L^p(\mathbb{R}^d)$. It follows that $f * \phi_n$ converges to f in $W^{k,p}(\mathbb{R}^d)$ and as $f * \phi_n$ is smooth (due to the smoothness of ϕ_n), the first claim follows.

Let f now be a smooth function in $W^{k,p}(\mathbb{R}^d)$ ($\nabla^j f$ is also an element of $L^p(\mathbb{R}^d)$ for $0 \leq j \leq k$). Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be a compactly supported function, which equals 1 near the origin (bump function). Now, consider the functions $f_R(x) := f(x)\psi\left(\frac{x}{R}\right)$ for $R > 0$; these functions clearly are members of $C_c^\infty(\mathbb{R}^d)$. Letting $R \to \infty$, via the dominated convergence theorem, f_R converges to f in $L^p(\mathbb{R}^d)$. Applying the product rule, $\nabla f_R(x) = (\nabla f)(x)\psi\left(\frac{x}{R}\right) + \frac{1}{R} f(x)(\nabla\psi)\left(\frac{x}{R}\right)$. We see that the first term converges to ∇f in $L^p(\mathbb{R}^d)$, by the dominated convergence theorem, while the second term converges to 0. Thus, $\nabla f_R(x)$ converges to ∇f in $L^p(\mathbb{R}^d)$. An analogous argument shows the convergence of $\nabla^j f_R$ to $\nabla^j f$ in $L^p(\mathbb{R}^d)$ for all $0 \leq j \leq k$, which gives us the convergence of f_R to f in $W^{k,p}(\mathbb{R}^d)$, and the claim follows. □

Remark 51. We also see that the space $\mathcal{S}(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$.

Theorem 52. *Let $\Omega = \mathbb{R}^d$, then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous, if $1 \leq p \leq q \leq \infty$, $\frac{d}{p} - 1 \leq \frac{d}{q}$ and $(p,q) \notin \left\{(d,\infty), \left(1, \frac{d}{d-1}\right)\right\}$.*

Proof. Again, we show the claim for all test functions $f \in C_0^\infty(\mathbb{R}^d)$ and by density, it will extend to $W^{1,p}(\mathbb{R}^d)$. From

$$|f(x+s\omega) - f(x)| = \left|\int_0^s \frac{d}{dt} f(x+t\omega)\, dt\right| = \left|\int_0^s \omega \cdot \nabla f(x+t\omega)\, dt\right|,$$

follows

$$|f(x)| = \left|-\int_0^\infty \omega \cdot \nabla f(x+r\omega)\, dr\right|$$

for any $x \in \mathbb{R}^d$ and any $\omega \in S^{d-1}$. Applying the generalized triangle inequality

$$|f(x)| \leq \int_0^\infty |\nabla f(x+r\omega)|\, dr.$$

Hence, averaging over all directions ω gives us

$$\int_{S^{d-1}} |f(x)|\, dS \leq \int_0^\infty \int_{S^{d-1}} |\nabla f(x+r\omega)| \frac{r^{d-1}}{r^{d-1}}\, dS\, dr.$$

Substituting and changing coordinates gives

$$|f(x)| \leq \int_{S^{d-1}} |f(x)|\, dS \leq \int_{\mathbb{R}^d} |\nabla f(x-y)| \frac{1}{|y|^{d-1}}\, dy.$$

$$\int_{\mathbb{R}^d} |\nabla f(x-y)| \frac{1}{|y|^{d-1}}\, dy \leq C(\int_{\mathbb{R}^d} |\nabla f(y)|^p\, dy)^{\frac{1}{p}}.$$

The claim now follows.

We now handle the cases, when $\frac{d}{p} - 1 < \frac{d}{p} < \frac{d}{q}$. Here we look at the following inequality

$$f(x) = f(x+R\omega) - \int_0^R \omega \cdot \nabla f(x+r\omega)\, dr,$$

for any $R > 0$. Hence,

$$|f(x)| \leq |f(x+R\omega)| + \int_0^R |\nabla f(x+r\omega)|\, dr.$$

For any specific value of R, this corresponds to averaging f over a sphere with this radius, which will not give us the nicest expressions. Instead, we average, for instance, over a range of R's between 1 and 2. This leads to

$$|f(x)| \leq \int_1^2 |f(x+R\omega)|\, dR + \int_0^2 |\nabla f(x+r\omega)|\, dr.$$

Averaging over all directions ω and converting back to Cartesian coordinates gives us

$$|f(x)| \leq \int_{1\leq|y|\leq 2} |f(x-y)|\, dy + \int_{|y|\leq 2} |\nabla f(x-y)| \frac{1}{|y|^{d-1}}\, dy.$$

Thus we can bound f pointwise, up to a constant, by the convolution of $|f|$ with the kernel $K_1(y) := \chi_{\{1\leq|y|\leq 2\}}$ plus the convolution of $|\nabla f|$ with the kernel $K_2(y) := \chi_{\{|y|\leq 2\}} \frac{1}{|y|^{d-1}}$. It is easy to check, that K_1 and K_2 both lie in $L^r(\mathbb{R}^d)$, where r is a result of Young's inequality, $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}$ (especially $1 < r < \frac{d}{d-1}$). An application of Young's inequality gives us that

$$\|f\|_{L^q(\mathbb{R}^d)} \leq C(\|f\|_{L^p(\mathbb{R}^d)} + \|\nabla f\|_{L^p(\mathbb{R}^d)}).$$

□

Remark 53. $W^{2,p} \subset W^{1,p}$

Theorem 54. $W^{d,1}(\mathbb{R}^d) \hookrightarrow C_b(\mathbb{R}^d)$ *continuous* $\forall\, d \geq 1$

Proof. Let $f \in C_c^\infty(\mathbb{R}^d)$, then from

$$f(x_1, \dots, x_j, \dots, x_d) = \int_{-\infty}^{x_j} \partial_{x_j} f(x_1, \dots, t_j, \dots, x_d)\, dt_j$$

$$= \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_d} \partial_{x_1} \dots \partial_{x_d} f(t_1, \dots, t_d)\, dt_1 \dots dt_d,$$

which yields

$$\|f\|_\infty \le \left\| \frac{\partial}{\partial x_1 \dots \partial x_d} f \right\|_{L^1} \le \|f\|_{W^{1,d}}.$$

The result follows from the density of C_0^∞. □

Theorem 55. *The embedding $W^{k,p}(\mathbb{R}^d) \hookrightarrow W^{l,q}(\mathbb{R}^d)$ is continuous, if*
$0 \le l \le k$

$$\begin{cases} 1 < p < q \le \infty & \frac{d}{p} - k < \frac{d}{q} - l \\ 1 < p \le q < \infty & \frac{d}{p} - k \le \frac{d}{q} - l \end{cases}.$$

Proof. [19]. □

Remark 56. Throughout this section, compared to the notation used in the first section, $H^2 := H^{2,2}$.

Remark 57. Regarding the embeddings:
(Theorem 52)$d = p = 2$, $0 \le \frac{2}{1}$, so we can choose $q = 1$.
$H^2 \approx W^{2,2}(\mathbb{R}^d) \subset W^{1,2}(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d)$ continuous.
(Theorem 54, 55)$d = k = p = l = 2$, choose $q = 1$,
then $-1 = \frac{2}{2} - 2 < \frac{2}{1} - 2 = 0$.
$H^2 \approx W^{2,2}(\mathbb{R}^d) \hookrightarrow W^{2,1}(\mathbb{R}^d) \hookrightarrow C_b(\mathbb{R}^d)$ continuous.

Let us now consider the equation

$$dv(t) = \left(\frac{1}{2}\Delta v + F_t\right) dt + \nabla v \cdot dW_t, \tag{4.1}$$

$$v(0) = v_0 = 0,$$

$$F_t :\ L^2(\mathbb{R}^2) \cap C_b(\mathbb{R}^2) \to \mathbb{R},$$

$$F_t \phi := \int_0^t \phi(-W_r - \mu)\, dr. \tag{4.2}$$

Proposition 58. *Let T be arbitrary, but fixed and finite. For $t \in [0, T]$, F_t as defined above is a continuous functional on H^2.*

Proof. Let $\phi \in H^2$. Due to Remark 57, we know that we can find a continuous (and bounded) representative for any H^2 function and hence the point evaluation and translation/shift $\phi \mapsto \phi(-W_r - \mu)$ for any $r \in [0,T]$ and $\mu \in \mathbb{R}^2$ are well defined. Further, it shall be noted that the point evaluation is a linear functional. Since we are integrating a bounded function over a compact set, it is finite, which guarantees that F_t is well defined and obviously linear on H^2.

$$\left| \int_0^t \phi(-W_r - \mu)\, dr \right| \leq t\|\phi\|_\infty \leq tC\|\phi\|_{H^2}.$$

□

Remark 59. The Lemma obviously still holds if we choose $\phi \in \mathcal{S}$.

We can now state the weak formulation for $\phi \in H^2$:

$$d\langle v(t), \phi\rangle = \left(\frac{1}{2}\langle \Delta\phi, v(t)\rangle + \int_0^t \phi(-\mu - W_r)\, dr\right) dt + \langle \nabla\phi, v(t)\, dW_t\rangle \tag{4.3}$$

By, rather formally, applying Duhamel's principle, we can get a decent impression about the structure of the solution.

We define $U_{s,t} := T_{W(s)} \circ T_{-W(t)}$ and $U_{t,s} := T_{W(t)} \circ T_{-W(s)}$ for any $0 \leq s \leq t$, where $T.$ is the usual shift operator on the space L^2.

We will try to show that

$$u(t) := U_{0,t}u_0 + \int_0^t U_{s,t}F_s\, ds$$

is a weak (or rather distributional) solution of the above equation.

From now on, we will refer to the dual space of H^2 by H^{-2}.

Lemma 60. *Let g_n, f_n be sequences in H^{-2}, converging to g and f respectively, then $u_n(t) := U_{0,t}g_n + \int_0^t U_{t,s}f_n\, ds \to u(t) := U_{0,t}g + \int_0^t U_{t,s}f\, ds$ in $L^2([0,T], H^{-2})$.*

Proof. Let $t \geq s > 0$. $U_{t,s}f_n$ is still an element of H^{-2}, as H^2 is invariant with respect to translations and we have $\langle U_{t,s}f_n, \phi\rangle = \langle f_n, U^*_{s,t}\phi\rangle$ for $\phi \in H^2$.

$$\|U_{t,s}f_n - U_{t,s}f\|^2_{L^2([0,t],H^{-2})} = \int_0^t \|U_{t,s}(f_n - f)\|^2_{H^{-2}}\, ds.$$

As the mapping $(\psi_s)_{s\in[0,t]} \mapsto \int_0^t \psi_s\, ds$ from $L^2([0,t], H^{-2})$ to H^{-2} is linear and continuous, it suffices to consider the terms inside the integral:

$$\begin{aligned}\|U_{t,s}(f_n - f)\|^2_{L^2([0,t],H^{-2})} &= \int_0^t \|U_{t,s}(f_n - f)\|^2_{H^{-2}}\, ds \\ &\leq \int_0^t \|f_n - f\|^2_{H^{-2}}\, ds = t\|f_n - f\|^2_{H^{-2}} \to 0.\end{aligned}$$

In total, this gives us $u_n(t) \to u(t)$. □

Remark 61. In the case of our specific (time dependent) F, we can almost use the same proof. In order to clarify the notation: For $s > 0$, we interpret F_s as the composition of an integral operator, a translation T and a point-evaluation g ($F_t = \int_0^t \circ T \circ \delta$). Further we will denote the "actions inside the integral" by f

($f \sim T \circ g$) and the point evaluation by g. The sequence F^n is chosen rather specifically: $F_t^n = \int_0^t T_s g_n \, ds$.

$$\|U_{\cdot,t}F_\cdot^n - U_{\cdot,t}F_\cdot\|^2_{L^2([0,t],H^{-2})} = \int_0^t \|U_{t,s}(F_s^n - F_s)\|^2_{H^{-2}} \, ds$$

$$\leq \int_0^t \|F_s^n - F_s\|^2_{H^{-2}} ds = \int_0^t \left\| \int_0^s f_s^n - f_s \, dr \right\|^2_{H^{-2}} ds = \int_0^t \left\| \int_0^s U_r(g_n - g) \, dr \right\|^2_{H^{-2}} ds$$

$$\leq \int_0^t \left(\int_0^s \|U_r(g_n - g)\|_{H^{-2}} \, dr \right)^2 ds \leq \int_0^t \left(\int_0^s \|g_n - g\|_{H^{-2}} \, dr \right)^2 ds = \frac{t^3}{3} \|g_n - g\|^2_{H^{-2}}.$$

One can replace g_n by an appropriate delta sequence.

Lemma 62. *Let $\Omega \subset \mathbb{R}^d$, X, Y be Banach spaces and A a linear, closed operator $A : D(A) \subset X \to Y$. Further $g : \Omega \to D(A)$, then*

$$\int_\Omega Ag(x) \, dx = A \int_\Omega g(x) \, dx.$$

Proof. [20], Theorem 3.7.12, page 83. □

Proposition 63. *Let $\phi \in H^2$, $t \in [0,T]$ and assume F_t as defined by (4.2), then*

$$\langle U_{s,t}F_s, \phi \rangle = \int_0^t \int_0^s \phi(W_s - W_r - (\mu + W_t)) \, dr \, ds,$$

$$\langle U_{t,s}F_s, \phi \rangle = \int_0^t \int_0^s \phi(W_s - W_r - (\mu - W_t)) \, dr \, ds.$$

Proof. Following the notation from Lemma 62, $\Omega = [0,s]$, $g(\cdot) = \phi(-W_\cdot - \mu)$ and the translation operator U defined earlier replaces A. We can clearly see, due to the continuity of all the operations involved and the previously stated Sobolev embeddings, that F_t maps ϕ from $H^2(\mathbb{R}^2)$ into C_b[0,T]. By the following simple estimate, we also see that $F_t\phi$ lies in $L^p[0,T]$ for any $p \geq 1$

$$(F_t\phi)^p = \left(\int_0^t \phi(-W_s - \mu) \, ds \right)^p$$

$$\leq \left(\int_0^t \sup_{x \in \mathbb{R}^2} (\phi) \, ds \right)^p \leq t^p C^p.$$

By simply differentiating with respect to t, we see that $F_t\phi$ lies in fact in $W^{1,p}[0,T]$ for any $p \geq 1$.

As linear and bounded operators are trivially closed, we can use Lemma 62 to justify exchanging the integral and translation, interpreting each action on either the space $C[0,T]$ or $W^{1,2}[0,T]$. So

$$\langle u(t), \phi \rangle = \langle U_{0,t}\phi, g \rangle + \int_0^t \langle U_{s,t}F_s, \phi \rangle \, ds = \int_0^t U_{s,t} \int_0^t \phi(-W_r - \mu) \, dr \, ds$$

$$= \int_0^t \int_0^s U_{s,t}\phi(-W_r - \mu)\, dr\, ds = \int_0^t \int_0^s \phi(W_s - W_r - (\mu + W_t))\, dr\, ds.$$

□

At this point, referring back to Proposition 42, one could already see that a potential weak solution could have a connection with the self-intersection local time (SILT) of a planar Brownian motion:

$$\int_0^t \int_0^s \phi(-W_t + W_s - W_r - \mu)\, dr\, ds = \int_0^t \int_0^s \phi(W_s - W_r - (\mu + W_t))\, dr\, ds \tag{4.4}$$

$$= \int_{\mathbb{R}^2} \phi(x)\beta_2(x + \mu + W_t, t)\, dx,$$

if $\mu + W_t \neq 0$ $\mathbb{R}^2$.

Remark 64. For $\mu \neq 0$, the condition $\mu + W_t \neq 0$ holds with probability 1.
Let $t > 0$, $\mu \neq 0$,

$$0 \leq \mathbb{P}(W_t + \mu = 0) \leq \mathbb{P}(\mu \in W[0,t])$$

$$= \mathbb{P}\left(\mu \in \sqrt{\frac{1}{t}}W[0,1]\right) = \mathbb{P}\left(\frac{\mu}{\sqrt{\frac{1}{t}}} \in W[0,1]\right) = 0.$$

Where the last equality stems from the following Lemma.

Lemma 65. *For any* x, $y \in \mathbb{R}^2$, *we have* $\mathbb{P}_x(y \in W[0,1]) = 0$.

Proof. For any fixed $y \in \mathbb{R}^2$, by Fubini's theorem,

$$\int_{\mathbb{R}^2} \mathbb{P}_y(x \in W[0,1])\, dx = \mathbb{E}_y[\mathcal{L}_2(W[0,1])] = 0.$$

Hence, for almost every $x \in \mathbb{R}^2$ we have $\mathbb{P}_y(x \in W[0,1]) = 0$. By symmetry of the Brownian motion,

$$\mathbb{P}_y(x \in W[0,1]) = \mathbb{P}_0(x - y \in W[0,1]) = \mathbb{P}_0(y - x \in W[0,1]) = \mathbb{P}_x(y \in W[0,1]).$$

We infer that $\mathbb{P}_x(y \in W[0,1]) = 0$ for $\mathcal{L}_2$-almost every point x. For any $\epsilon > 0$ we thus have, almost surely, $\mathbb{P}_{B_\epsilon}(y \in W[0,1]) = 0$.
Hence,

$$\mathbb{P}_x(y \in W[0,1]) = \lim_{\epsilon \to 0} \mathbb{P}_x(y \in W[\epsilon,1]) = \lim_{\epsilon \to 0} \mathbb{E}_x[\mathbb{P}_x(y \in W[0,1-\epsilon])] = 0,$$

where we have used the Markov property in the second step. □

Remark 66. We encounter certain issues:

The smoothness of β_2 in order to qualify as a weak solution and of course the smoothness of ϕ, in order to even be able to apply Ito's formula. We leave the question regarding the application of Ito's formula unanswered for the case of $H^2(\mathbb{R}^2)$, but supply a short treatment of the case $f \in W^{2,p}_{\text{loc}}(\mathbb{R}^n)$ for $p > 1 + \frac{n}{2}$ in the Appendix.

A possible way out would be to consider $\phi \in C_c^\infty(\mathbb{R}^2)$ and interpret the solution as a distribution. If we followed this approach, all steps made before certainly hold, and even (4.4) holds, as shown by [6]. As $\phi \in C_c^\infty$ is quite restrictive, we will try to work with functions $\phi \in \mathcal{S}$ and interpret the solution as a tempered distribution.

Remark 67. As $\mathcal{S}$ is a dense subspace of any of our usual Sobolev spaces, the previous lemmata and propositions obviously still hold, if we replace $\phi \in H^2$ by $\phi \in \mathcal{S}$.

Remark 68. As we will turn our attention to the existence of an actual solution in a moment, we will refer to regarding the uniqueness to Theorem 22 (and the preceding Lemma).

Proposition 69. *Let $\phi \in \mathcal{S}$ and consider the set up from equation (4.3), then we have*

$$d\langle u(t), \phi\rangle = \left(\frac{1}{2}\langle \Delta\phi, u(t)\rangle + \int_0^t \phi(-\mu - W_r)\, dr\right)\, dt + \langle \nabla\phi, u(t)\, dW_t\rangle$$

for any $t \geq 0$ and $u(0) = 0$.

Proof.

$$\langle u(t), \phi\rangle = \langle U_{0,t}\phi, g\rangle + \int_0^t \langle U_{t,s}F_s, \phi\rangle\, ds = \int_0^t U_{t,s} \int_0^t \phi(-W_r - \mu)\, dr\, ds$$

$$= \int_0^t \int_0^s U_{s,t}\phi(-W_r - \mu)\, dr\, ds = \int_0^t \int_0^s \phi(W_s - W_r - (\mu + W_t))\, dr\, ds.$$

We note that

$$\frac{d}{dt}\langle u(t), \phi\rangle = \int_0^t \phi(-W_r - \mu)\, dr,$$

and

$$d\phi(W_s - W_r - (\mu + W_t)) = \frac{1}{2}\Delta\phi(W_s - W_r - (\mu + W_t))\, dt$$

$$-\frac{\partial}{\partial x_1}\phi(W_s - W_r - (\mu + W_t))\, dW_t^1 - \frac{\partial}{\partial x_2}\phi(W_s - W_r - (\mu + W_t))\, dW_t^2.$$

As we have chosen $\phi \in \mathcal{S}$, we can exchange the differentiation and integration and obtain

$$d\langle u(t),\phi\rangle = \left(\frac{1}{2}\langle \Delta\phi, u(t)\rangle + \int_0^t \phi(-\mu - W_r)\, dr\right)\, dt + \langle \nabla\phi, u(t)\, dW_t\rangle\,.$$

□

Let us define the following two quantities:
For any $\phi \in \mathcal{S}$, $\mu \in \mathbb{R}^2\backslash\{0\}$ and $t \geq 0$,

$$A_\phi(t,\mu) := \int_0^t \int_0^s \phi(W_s - W_r - \mu)\, dr\, ds,$$

$$a_\phi(t,\mu) := A_\phi(t, \mu - W_t).$$

Based on [38], formula (4.4) holds and we obtain

$$A_\phi(t,\mu) = \int_0^t \int_0^s \phi(W_s - W_r - \mu)\, dr\, ds = \int_{\mathbb{R}^2} \phi(x-\mu)\beta_2(t,x)\, dx$$

$$= \int_{\mathbb{R}^2} \phi(x)\beta_2(t, x+\mu)\, dx = \langle \phi, T_\mu \beta_2(t)\rangle.$$

Hence, we get

$$a_\phi(t,\mu) = \langle \phi, T_{\mu+W_t}\beta_2(t)\rangle.$$

Proposition 70. *For any $f \in H^2$, $\mu \in \mathbb{R}^2\backslash\{0\}$ and $t \geq 0$, $a_f(t,\cdot)$ is a version of $u(t) := \int_0^t \int_0^s U_{s,t} T_{W_r} f\, dr\, ds$.*

Proof. Let $\phi \in H^2$,

$$\langle a_f(t,\cdot), \phi\rangle = \int_{\mathbb{R}^2} a_f(t,y)\phi(y)\, dy = \int_{\mathbb{R}^2}\int_0^t \int_0^s \phi(y) U_{s,t} f\, dr\, ds\, dy$$

$$= \int_{\mathbb{R}^2}\int_0^t \int_0^s \phi(y) f(W_s - W_r - (y+W_t)) dr ds dy = \int_{\mathbb{R}^2}\int_0^t \int_0^s \phi(-W_t + W_s - W_r - z) f(z) dr ds dz$$

$$= \int_{\mathbb{R}^2} f(z) \int_0^t \int_0^s \phi(-W_t + W_s - W_r - z) dr ds dz = \int_{\mathbb{R}^2} f(z) \int_0^t \int_0^s U_{s,t}\phi(-W_r - z) dr ds dz$$

$$= \int_{\mathbb{R}^2} f(z) \int_0^t \int_0^s U_{t,s} T_{-W_r} I_- \phi(z)\, dr\, ds\, dz = \int_{\mathbb{R}^2} f(z) \int_0^t \langle U_{t,s} F_s, \phi\rangle\, ds\, dz$$

$$= \left\langle f, \int_0^t \int_0^s U_{t,s} T_{-W_r} I_- \phi\, ds \right\rangle = \left\langle \phi, \int_0^t \int_0^s U_{s,t} T_{W_r} f\, ds \right\rangle.$$

□

Corollary 71. *Let $\mu \in \mathbb{R}^2 \backslash \{0\}$ and $f = \delta_{-\mu}$, then $a(t, \cdot)$ is a version of $u(t)$.*

Proof. We can choose f_n to be a Dirac sequence of C_c^∞ functions, which are clearly in H^2. As $f_n \to \delta_{-\mu}$ in H^{-2}, we can use Lemma 60 and see that $u_n(t) \to u(t)$ in H^{-1}. The previous proposition states that $a_{f_n}(t, \cdot)$ is a version of $u_n(t)$. We obtain that $a_{f_n}(t, \cdot) \to a(t, \cdot)$ (Proposition 70) point wise and therefore, $a(t, \cdot)$ is a version of $u(t)$. □

Remark 72. Until now, we have avoided talking about the regularity of the solution $u(t, x)$. It is well known ([35]) that, away from 0, $\beta_2(x, t)$ is Hölder continuous in (x, t), but a more rigorous investigation regarding its regularity (if it could, in fact, be regular enough to qualify as a proper weak solution) is far beyond this work.

4.1. **The First Moment.** Let us, formally, take the expectations on both sides of equation (4.1), for $t \geq 0$, $x \in \mathbb{R}^2 \backslash \{0\}$, we end up with

$$\frac{\partial}{\partial t}\mathbb{E}[u(t, x)] = \frac{1}{2}\Delta\mathbb{E}[u(t, x)] + \mathbb{E}[f(t, x)], \tag{4.5}$$

$$\mathbb{E}[u(0, x)] = 0.$$

In our case, $f(t, x) \in H^{-1}$. Another aspect of this is that $\int_0^t \nabla u(x, t) \cdot dW_t$ needs to be a proper martingale and not just a strictly local one, in order for the expectation to vanish.

What we have to show is that $\mathbb{E}\left[\int_0^T (\nabla u)^2 \, dt\right] < \infty$ for every $T > 0$. As we already have a (candidate for) a weak solution, our task becomes a bit easier.

Lemma 73. *For $\phi \in \mathcal{S}$, $f \in L^2$ there exists a constant, dependent on ϕ, such that*

$$\left(\int \phi f \, dx\right)^2 \leq C \int f^2 |\phi| \, dx.$$

Proof. We use

$$\left(|\int \phi f \, dx|\right)^2 \leq \left(\int |\phi||f| \, dx\right)^2$$

$$= \left(\int \frac{\int |\phi| \, dx}{\int |\phi| \, dx} |\phi||f| \, dx\right)^2 = \left(\int |\phi| \, dx\right)^2 \left(\int \frac{|\phi|}{\int |\phi| \, dx} |f| \, dx\right)^2.$$

As $\int \frac{|\phi|}{\int |\phi| \, dx} \, dx = 1$, we can use Jensen's inequality to obtain

$$C\left(\int \frac{|\phi|}{\int |\phi| \, dx} |f| \, dx\right)^2 \leq C \int \frac{|\phi|}{\int |\phi| \, dx} |f|^2 \, dx = C \int |\phi||f|^2 \, dx.$$

□

Remark 74. In law we have

$$\int_{\mathbb{R}^2} \phi(x)\beta_2(x+\mu+W_t,t)\,dx = \langle \phi, S_{W_t}S_\mu\beta(t,\cdot)\rangle = \langle S_{-W_t}\phi, S_\mu\beta(t,\cdot)\rangle\,, \tag{4.6}$$

where $S_\cdot$ denotes the translation/shift-operator.
Further,

$$\mathbb{E}\left[\int_0^T \langle \nabla\cdot\phi, \beta_2(t,x+W_t+\mu)\rangle^2\,dt\right] = \mathbb{E}\left[\int_0^T \langle S_{-W_t}\nabla\cdot\phi, \beta_2(x+\mu,t)\rangle^2\,dt\right]$$

$$\leq C\int_0^T\int_{\mathbb{R}^2} \mathbb{E}[(\nabla\cdot\phi(x-W_t))\beta_2(x+\mu,t)^2]\,dx\,dt$$

$$\leq C\int_0^T\int_{\mathbb{R}^2} \mathbb{E}[(\nabla\cdot\phi(x-W_t))^2]^{\frac{1}{2}}\mathbb{E}[\beta_2(x+\mu,t)^4]^{\frac{1}{2}}\,dx\,dt.$$

The first inequality stems from Jensen's inequality, as shown in the previous Lemma and the second is just a routine application of the Cauchy-Schwartz inequality.

We note that for $f,\ g \in \mathcal{S}$, $fg \in \mathcal{S}$ and $f * g \in \mathcal{S}$, so $\phi^2 \in \mathcal{S}$. Further,

$$\int_{\mathbb{R}^2} \mathbb{E}[\phi(x-W_t)^2] = \int_{\mathbb{R}^2} \phi(x+y)^2 \frac{\exp\left(-\frac{|y|^2}{2t}\right)}{2\pi t}\,dy =$$

$$\left(\phi^2 * \exp\left(-\frac{|\cdot|^2}{2t}\right)\right)(x),$$

which lies again in $\mathcal{S}$.

Proposition 75. *For all $\phi \in \mathcal{S}$,*

$$\mathbb{E}\left[\int_0^T \langle \nabla\cdot\phi, \beta_2^n(x,t)\rangle^2\,dt\right] < \infty,$$

for $n = 1, 2, \dots$.

Proof. We actually just have to show, as a consequence of Fubini's theorem, Jensen's inequality and the Cauchy-Schwartz inequality, that there exists a finite constant C, such that

$$\int_{\mathbb{R}^2} \mathbb{E}[(\nabla\cdot\phi(x-W_t))^2]^{\frac{1}{2}}(\mathbb{E}[\beta_2(x+\mu,t)^{4n}])^{\frac{1}{2}}\,dx \leq Ch(t),$$

where $h \in L^1(0,T)$ for every finite T.

In the next steps we suppress the translation by μ, as we can shift this translation to the respective function $\psi \in \mathcal{S}$ and define it as $\tilde{\psi}(\cdot) := \psi(\cdot-\mu)$. Further, we will work with a generic $\psi \in \mathcal{S}$, which can obviously be replaced by the desired partial derivative of ϕ and its powers.

We now have the choice to either use the (Yor-Rosen-)Tanaka formula in order to express $\beta_2(x,t)$ and derive our estimates this way, or to use the previously stated estimates (Theorem 44), together with the scaling argument (Remark 24) from the beginning of the previous section, to obtain

$$\mathbb{E}[\beta_2(x,t)^{2n}] = t^{\frac{6n}{2}}\,\mathbb{E}\left[\beta_2\left(\frac{x}{\sqrt{t}},1\right)^{2n}\right]$$

$$\leq t^{3n}c(2n)\left(1 \vee \left|G\left(\frac{x}{\sqrt{t}}\right)\right|\right)^{v(2n)}$$

$$\leq t^{3n}c(2n)\left(1 \vee \left(\left|\frac{1}{\pi}\log(|\sqrt{t}|)\right| + |G(x)|\right)\right)^{v(2n)}.$$

In the case of $p = 2n$, $v(2n) = 2n$ and due to the positivity of all terms in the last expression,

$$\left(1 \vee \left(\left|\frac{1}{\pi}\log(|\sqrt{t}|)\right| + |G(x)|\right)\right)^{v(2n)} \leq \left(1 + \left|\frac{1}{\pi}\log(|\sqrt{t}|)\right| + |G(x)|\right)^{2n}.$$

Regarding the integral over $\mathbb{R}^2$, we only have to deal with integrals of the form $c\int_{\mathbb{R}^2}\psi(x)|G(x)|^j\,dx$, $j = 1,\dots,2n$, where $\psi \in \mathcal{S}$. These integrals can be evaluated similarly to the ones in Remark 29.

$$\left|\int_{\mathbb{R}^2}\psi(x)|G(x)|^j\,dx\right| = \left|\int_{\mathbb{R}^2}\psi(x)\left|\log\left(\frac{1}{|x|}\right)^j\right|\,dx\right|$$

$$= \left|\int_0^{2\pi}\int_0^\infty \psi(r,\rho)\left|\log\left(\frac{1}{r^2}\right)^j\right| r\,dr\,d\rho\right|$$

$$= \left|\int_0^{2\pi}\left(\int_0^1 \psi(r,\rho)\log\left(\frac{1}{r^2}\right)^j r\,dr + \int_1^\infty \psi(r,\rho)\left|\log\left(\frac{1}{r^2}\right)^j\right| r\,dr\right)\,d\rho\right|$$

$$\leq 2\pi \sup_{(r,\rho)\in B(0,1)}|\psi(r,\rho)| + \int_0^{2\pi}\int_1^\infty |\psi(r,\rho)2^j r|\log(r)|^j|\,dr\,d\rho < \infty$$

$$\leq C\frac{\Gamma(j+1)}{2} + \int_0^{2\pi}\int_1^\infty |\psi(r,\rho)|2^j r^{j+1}\,dr\,d\rho < \infty.$$

The way we conclude the existence for each of these integrals is the following: We actually just split the integral with respect to r, up into an integral over $(0,1)$ and $(1,\infty)$.

Due to the continuity of ψ, we can estimate

$$\int_0^1 |\psi(r,\rho)| r\log(r)^j \, dr \leq C \int_0^1 r\log(r)^j \, dr.$$

As $\phi \in \mathcal{S}$,

$$\left| \int_1^\infty |\psi(r,\rho) r^{j+1}| \, dr \right| \leq \int_0^\infty |\psi(r,\rho) r^{j+1}| \, dr < \infty.$$

Regarding the integration with respect to t, the only integrals we have to deal with are of the form $C\int_0^T t^m \, dt$ or $C\int_0^T t^m \log(\sqrt{t})^n \, dt$. Evaluating these integrals from 0 to T gives us

$$\int_0^T t^m \log(|\sqrt{t}|)^n \, dt$$

$$= \frac{(-1)^n(1+m)^{-n}\Gamma(1+n) + \Gamma(1+n, -(1+m)\log(T))\log(T)^n(-(1+m)\log(T))^{-n}}{2^n(1+m)}.$$

This proves our claim. $\square$

Corollary 76. *For all $\phi \in \mathcal{S}$ and $T > 0$,*

$$\mathbb{E}\left[\int_0^T \langle \nabla \cdot \phi, \beta_2^n(x,t)\rangle^2 \, dW_t\right] = 0.$$

Let us set, for a Dirac sequence ρ_ϵ and $x \neq 0$ outside the support of ρ_ϵ, $\phi(\cdot) = \rho_\epsilon(x - \cdot)$.

$$\mathbb{E}[\langle u(t,\cdot), \rho_\epsilon\rangle] = \frac{1}{2}\int_0^t \mathbb{E}[\langle \Delta\rho_\epsilon(x - \cdot), u(s,\cdot)\rangle] ds + \mathbb{E}\left[\int_0^t \int_0^s \rho_\epsilon(x - (-\mu - W(r))) \, dr \, ds\right].$$

By Lebesgue's theorem, we can interchange the expected value and integration on the left-hand side, due to the knowledge about the explicit form of u, which results in

$$\int_{\mathbb{R}^2} \rho_\epsilon(x-y)\mathbb{E}[u(s,y)] \, dy \to \mathbb{E}[u(s,x)],$$

in the sense of distributions.

On the right-hand side, we use the same argument to exchange the expectation and integration

$$\frac{1}{2}\int_0^t \langle \Delta\rho_\epsilon(x - \cdot), \mathbb{E}[u(s,\cdot)]\rangle.$$

$$\mathbb{E}\left[\int_0^s \rho_\epsilon(x - (-\mu - W_r)) \, dr\right] = \int_0^s \mathbb{E}[\rho_\epsilon(-x + (-W_r - \mu))] \, dr$$

$$= \int_0^s \int_{\mathbb{R}^2} \rho_\epsilon(z - (-\mu - x)) \frac{1}{2\pi r} \exp\left(-\frac{|z|^2}{2r}\right) dz\, dr.$$

Letting $\epsilon \to 0$, we get

$$\int_0^s \frac{1}{2\pi r} \exp\left(-\frac{|\mu + x|^2}{2r}\right) dr = \Gamma\left(0, \frac{|\mu + x|^2}{s}\right).$$

The estimate

$$\int_0^t \int_0^s \frac{1}{2\pi r} \exp\left(-\frac{|\mu + x|^2}{2r}\right) dr\, ds \leq t \int_0^t \frac{1}{2\pi r} \exp\left(-\frac{|\mu + x|^2}{2r}\right) dr,$$

ensures the following Proposition is true.

Proposition 77. *$\mathbb{E}[u(t,x)]$ is a solution of the PDE (4.5) in the sense of tempered distributions.*

4.2. **The Second Moment.** We now try to find an expression for the second moment of the solution $\mathbb{E}[u(t,x)^2]$.

If we formally apply Ito's formula to u^2,

$$du(t,x)^2 = (u\Delta u + (\nabla u)^2 + uF_t + F_t u)\, dt + 2u\nabla u \cdot dW_t.$$

We set $u^2(0,x) = 0$.

Formally taking expectations, defining $m_2(t,x) := \mathbb{E}[u^2(t,x)]$ and noticing that

$$\frac{\partial u^2}{\partial x^2} = 2u\frac{\partial u^2}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2,$$

$$\frac{\partial u^2}{\partial x} = 2u\frac{\partial u}{\partial x},$$

$$2u\nabla u \cdot dW_t = \nabla u^2 \cdot dW_t,$$

$$du(t,x)^2 = \left(\frac{1}{2}\Delta u^2 + uF_t + F_t u\right) dt + \nabla u^2 \cdot dW_t.$$

For $\phi \in \mathcal{S}$ (should also hold for $\phi \in H^2$),

$$d\langle u^2(t), \phi\rangle =$$

$$\left(\frac{1}{2}\langle \Delta\phi, u^2(t)\rangle + \int_{\mathbb{R}^2}\int_0^t \phi(-\mu - W_r)\, dr\, u(x,t)\, dx + \int_{\mathbb{R}^2}\int_0^t u(-\mu - W_r, t)\, dr\, \phi(x)\, dx\right) dt$$

$$+\langle \nabla\phi, u^2(t)\rangle\, dW_t.$$

For the "new" integral terms, it holds that

$$\left|\int_{\mathbb{R}^2}\int_0^t \phi(-\mu - W_r)\, dr\, u(x)\, dx\right| \leq \int_{\mathbb{R}^2}\int_0^t |\phi(-\mu - W_r)u(x)|\, dr\, dx$$

$$\leq \int_{\mathbb{R}^2}\int_0^t |\phi(-\mu - W_r)|^2\, dr \int_0^t |u(x)|^2\, dr\, dx \leq Cf(t)\int_{\mathbb{R}^2} |u(x)|^2\, dx \leq \infty,$$

where $f \in L^1_{\text{loc}}$, since

$$\int_0^t |\phi(-\mu - W_r)|^2\, dr \leq t \sup_{s\in[0,t]} |\phi(-\mu - W_s)|^2.$$

An analogous estimate can be made for the second term, as the explicit expression for u is continuous.

We obtain

$$\frac{\partial}{\partial t} m_2(t,x) = \frac{1}{2}\Delta m_2(t,x) + \mathbb{E}[u(t,x)F_t + F_t u(t,x)], \tag{4.7}$$

as the expectation of the stochastic integral vanishes due to the estimate

$$\int_{\mathbb{R}^2} \mathbb{E}[(\nabla\cdot\phi(x - W_t))^4]^{\frac{1}{2}}\mathbb{E}[\beta_2(x+\mu,t)^8]^{\frac{1}{2}}\, dx \leq Ch(t),$$

which we already showed in Proposition 75.

In order to proof the existence of the remaining expectations, we finally make use of 46, as we could apply it to the first double integral with the constant 1 function

$$\int_{\mathbb{R}^2} 1\cdot u(x,t)\, dx = \int_0^t\int_0^s 1\, du\, ds = \frac{t^2}{2}.$$

We are left with

$$\mathbb{E}\left[\int_0^t \rho_\epsilon(y+\mu+W_r)\, dr\right]$$

$$= \int_0^t\int_{\mathbb{R}^2} \rho_\epsilon(x - (-y-\mu))\frac{\exp\left(-\frac{|x|^2}{2r}\right)}{2\pi r}\, dx\, dr.$$

taking the limit, this is equal to

$$\int_0^t \frac{\exp\left(-\frac{|-\mu-y|^2}{2r}\right)}{2\pi r}\, dr = \Gamma\left(0, \frac{|\mu+x|^2}{t}\right).$$

In total, we get

$$\mathbb{E}\left[\int_{\mathbb{R}^2}\int_0^t \rho_\epsilon(y+\mu+W_r)\,dr\,u(x,t)\,dx\right] = \Gamma\left(0, \frac{|\mu+x|^2}{t}\right)\frac{t^2}{2}.$$

For the second integral, (as u is continuous in both components and we integrate over the compact set $[0,t]$ and by using the same estimate, as for the first moment, for the appearing double integral)

$$\mathbb{E}\left[\int_{\mathbb{R}^2}\int_0^t u(-\mu-W_r,t)\,dr\,\rho_\epsilon(y-x)\,dx\right]$$

$$\leq \mathbb{E}\left[t\sup_{s\in[0,t]}\{u(-\mu-W_s,t)\}\int_{\mathbb{R}^2}\rho_\epsilon(y-x)\,dx\right].$$

Although these estimates are not very elegant, they provide us with what we need to state the following Proposition:

Proposition 78. *$m_2(t,x) := \mathbb{E}[u^2(t,x)]$ is a solution, in the sense of tempered distributions, of the PDE (4.7).*

4.3. **The Moment Generating Function.** Let $\theta > 0$ and consider the function $M(t,x,\theta) := \mathbb{E}[\exp(\theta u(t,x))]$. We formally apply Ito's formula

$$d\exp(\theta u(t,x)) =$$

$$\frac{1}{2}\left(\exp(\theta u(t,x))(\theta\Delta u(t,x)+\theta F_t+\theta^2(\nabla u(t,x))^2)\right)dt+\theta\exp(\theta u(t,x))\nabla u(t,x)\cdot dW_t,$$

$$\exp(\theta u(0,x)) = 1.$$

Using the fact that $\Delta\exp(\theta u(t,x)) = \exp(\theta u(t,x))(\theta\Delta u(t,x)+\theta^2(\nabla u(t,x))^2)$, we should be able to obtain that

$$\frac{\partial}{\partial t}M(t,x,\theta) = \frac{1}{2}(\Delta M(t,x,\theta)+\theta F_t), \tag{4.8}$$

$$M(0,x,\theta) = 1.$$

As in the previous sections, what we need to show is that $\int_0^t \theta\exp(\theta u)\nabla u\cdot dW_s$ is a martingale or, more specifically,

$$\int_0^t \theta\exp(\theta u)\frac{\partial}{\partial x_i}u\,dW_s^i = \int_0^t \frac{\partial}{\partial x_i}\exp(\theta u)\,dW_s^i$$

is a martingale for $i = 1,2$.

First, we will state a rather useful Lemma.

Lemma 79. *Let $M = \{M_t : t \geq 0\}$ be a continuous local martingale, such that $M_0 = 0$. Suppose, that for some $\alpha > 0$ and $p \in (0, 1]$ we have $\mathbb{E}[\exp(\alpha \langle M \rangle_t^p)] < \infty$. Then,*
1.) if $p = 1$, for any $\lambda < \sqrt{\frac{\alpha}{2}}$, $\mathbb{E}[\exp(\lambda |M_t|)] < \infty$ and
2.) if $p < 1$, $\mathbb{E}[\exp(\lambda |M_t|^p)] < \infty$ for all $\lambda > 0$.

Proof. Set $X = |M_t|^p$. For any constant $c > 0$, we can write

$$\mathbb{E}[\exp(\lambda X)] = \int_0^\infty \mathbb{P}(X \geq y) \lambda \exp(\lambda y)\, dy$$

$$= \int_0^\infty (\mathbb{P}(X \geq y,\, \langle M \rangle_t^p < cy) + \mathbb{P}(X \geq y,\, \langle M \rangle_t^p \geq cy)) \lambda \exp(\lambda y)\, dy$$

$$\leq \int_0^\infty 2 \exp\left(-\frac{y^{\frac{1}{p}}}{2c^{\frac{1}{p}}}\right) \lambda \exp(\lambda y)\, dy + \int_0^\infty + \mathbb{P}\left(\frac{\langle M \rangle_t^p}{c} \geq cy\right) \lambda \exp(\lambda y)\, dy$$

$$= \int_0^\infty 2\lambda \exp\left(\lambda y - \frac{y^{\frac{1}{p}}}{2c^{\frac{1}{p}}}\right) dy + \mathbb{E}\left[\exp\left(\frac{\lambda}{c} \langle M \rangle_t^p\right)\right].$$

To complete the proof, we simply choose $c = \frac{\lambda}{\alpha}$. □

Remark 80. The constant C which will be used in the following Proposition, might change from line to line, in order to account for appearing constants.

Proposition 81. *Let $\phi \in C_c^\infty$, $T > 0$, we have, for all $0 < \theta < \theta_0$,*

$$\mathbb{E}\left[\int_0^T \left\langle \frac{\partial}{\partial x_i} \phi, \exp(\theta \beta_2(x + \mu + W_t, t)) \right\rangle^2 dt\right] < \infty.$$

Proof. Out of convenience, we will replace the partial derivatives $\frac{\partial}{\partial x_i}\phi$ again with a function $\psi \in C_c^\infty$ and, also "shift" the translation, with respect to W_t, onto our test function as in (4.6). We are left with

$$\mathbb{E}\left[\langle \psi(\cdot - W_t), \exp(\theta \beta_2(x + \mu, t)) \rangle^2\right] \leq C \int_{\mathbb{R}^2} \mathbb{E}[(\psi(x - W_t))^2]^{\frac{1}{2}} (\mathbb{E}[\exp(\theta \beta_2(x + \mu, t))^4])^{\frac{1}{2}} dx.$$

We can deal with the first expectation the same way we did in Proposition 75. Regarding the second one, we will use the Yor-Rosen-Tanaka formula (3.35, 3.36).

$$\mathbb{E}[\exp(\theta \beta_2(x + \mu, t))^4]$$

$$= \mathbb{E}\left[\exp\left(-4\theta \left(\int_0^t (G(W_t - W_r - x) - G(-x))\, dr - \int_0^t \left(\int_0^s \nabla G(W_s - W_r - x)\, dr\right) dW_s\right)\right)\right]$$

$$= \mathbb{E}\left[\exp\left(4\theta\int_0^t \log(|W_t - W_r - x - \mu|)\,dr - \int_0^t\int_0^s \frac{W_s - W_r - x - \mu}{|W_s - W_r - x - \mu|^2}\,dr\,dW_s - t\log(|x+\mu|)\right)\right]$$

$$\leq \mathbb{E}\left[\exp\left(8\theta\int_0^t \log(|W_t - W_r - x - \mu|)\,dr\right)\right]^{\frac{1}{2}}$$

$$\times\mathbb{E}\left[\exp\left(-8\theta\int_0^t\int_0^s \frac{W_s - W_r - x - \mu}{|W_s - W_r - x - \mu|^2}\,dr\,dW_s\right)\right]^{\frac{1}{2}}$$

$$\times\mathbb{E}\left[\exp\left(-16\theta t\log(|x+\mu|)\right)\right]^{\frac{1}{2}}.$$

We will treat each of these terms individually.

Regarding the first term:

As we are testing with functions from C_c^∞, our estimates are not required to be optimal, which leaves us a bit of space. Another way to deal with the first term, using an estimate from [23], will be mentioned in a following Remark.

Let $h : \mathbb{R}^n \to \mathbb{R}$ be a convex function and X_t be a process in $\mathbb{R}^n$ with continuous paths.

We consider the expression $h\left(\int_0^t X_s\,ds\right)$. Seeing that the term inside the function h exists for every ω, we apply Jensen's inequality for every sample path and arrive at

$$h\left(\frac{t}{t}\int_0^t X_s\,ds\right) \leq \frac{1}{t}\int_0^t h(tX_s)\,ds.$$

From this follows that $\mathbb{P}\left(h\left(\int_0^t X_s\,ds\right) > x\right) \leq \mathbb{P}\left(\frac{1}{t}\int_0^t h(tX_s)\,ds > x\right)$ for every $x \in \mathbb{R}$.

Hence,

$$\mathbb{E}\left[g\left(\int_0^t X_s\,ds\right)\right] \leq \mathbb{E}\left[\frac{1}{t}\int_0^t g(tX_s)\,ds\right].$$

In our case, this means

$$\mathbb{E}\left[\exp\left(8\theta\int_0^t \log(|W_t - W_r - x - \mu|)\,dr\right)\right]^{\frac{1}{2}}$$

$$\leq \mathbb{E}\left[\exp\left(8\theta t\log\left(\int_0^t \left|\frac{W_t - W_r - x - \mu}{t}\right|\,dr\right)\right)\right]^{\frac{1}{2}}$$

$$\leq \mathbb{E}\left[\left(\frac{1}{t}\right)^{8\theta t}\left(\int_0^t |W_t - W_r - x - \mu|\,dr\right)^{8\theta t}\right]^{\frac{1}{2}}.$$

We will assume that $8\theta t < 1$, which will turn out to be a reasonable assumption once we arrive at the end of the proof. Hence,

$$\mathbb{E}\left[(\frac{1}{t})^{8\theta t}\left(\int_0^t |W_t - W_r - x - \mu|\, dr\right)^{8\theta t}\right]^{\frac{1}{2}}$$

$$\leq \mathbb{E}\left[(\frac{1}{t})^{8\theta t}\left(t \sup_{s\in[0,t]} 2|W_s| + t|x| + t|\mu|\right)^{8\theta t}\right]^{\frac{1}{2}}$$

$$\leq \mathbb{E}\left[C\left(\sup_{s\in[0,t]} |W_s|\right)^{8\theta t} + C|x|^{8\theta t} + C|\mu|^{8\theta t}\right]^{\frac{1}{2}}.$$

Considering $\sup_{s\in[0,t]} |W_s| = \sup_{s\in[0,t]} \sqrt{(W_s^1)^2 + (W_s^1)^2} \leq 2\sup_{s\in[0,t]} \sqrt{(B_s)^2}$, where B denotes a one dimensional Brownian motion, we obtain the following estimate for the previous term

$$\leq \mathbb{E}\left[C\left(\left(\sup_{s\in[0,t]} |B_s|\right)^{8\theta t} + |x|^{8\theta t} + |\mu|^{8\theta t}\right)\right]^{\frac{1}{2}}.$$

To calculate the appearing moments, we can rely on the calculations for the moments of the supremum of a reflected Brownian motion, which are related to Gamma functions. We will use the estimates presented in [14], Proposition 1.1 (ii),

$$\mathbb{E}\left[\left(\sup_{s\in[0,t]} |B_s|\right)^{\nu}\right] \leq \frac{1}{\sqrt{\pi}} 2^{1+\frac{\nu}{2}} \Gamma\left(\frac{\nu+1}{2}\right) t^{\frac{\nu}{2}}.$$

Summing up,

$$\mathbb{E}\left[\exp\left(8\theta \int_0^t \log(|W_t - W_r - x - \mu|)\, dr\right)\right]^{\frac{1}{2}} \leq C\left(t^{\frac{8\theta t}{2}} + |x|^{\frac{8\theta t}{2}} + |\mu|^{\frac{8\theta t}{2}}\right).$$

When integrating with respect to x and t, we can use $0 < \frac{8\theta t}{2} < 1$ in order to dominate the respective terms to dominate the integrands based on the region of integration.

Let us now consider the second term:

$$\mathbb{E}\left[\exp\left(-8\theta \int_0^t \int_0^s \frac{W_s - W_r - x - \mu}{|W_s - W_r - x - \mu|^2}\, dr\, dW_s\right)\right]^{\frac{1}{2}}.$$

We know that the term in the exponent is a continuous L^2 martingale (Lemma 47) with mean zero.

Referring to Lemma (79), let us consider the second variation of the term inside the exponential.

$$Y_t := -8\theta \int_0^t \int_0^s \frac{W_s - W_r - x - \mu}{|W_s - W_r - x - \mu|^2}\, dr\, dW_s,$$

then

$$\langle Y_t \rangle = 64\theta^2 \sum_{i=1}^{2} \int_0^t \left(\int_0^s \frac{(W_s - W_r - x - \mu)_i}{|W_s - W_r - x - \mu|^2}\, dr \right)^2 ds_i.$$

We will treat each term of the sum separately. Out of convenience, we omit the index i and define $y := x + \mu$.

$$0 \le 64\theta^2 \int_0^t \left(\int_0^s \frac{W_s - W_r - y}{|W_s - W_r - y|^2}\, dr \right)^2 ds \le C\theta^2 \int_0^t \left(\int_0^s \left| \frac{W_s - W_r - y}{|W_s - W_r - y|^2} \right| dr \right)^2 ds$$

$$\le C\theta^2 \int_0^t \left(\int_0^s \frac{1}{|W_s - W_r - y|}\, dr \right)^2 ds \le C\theta^2 \int_0^t \left(\int_0^t \frac{1}{|W_s - W_{s-r} - y|}\, dr \right)^2 ds$$

$$= C\theta^2 t \left(\int_0^t \frac{1}{|W_r - y|}\, dr \right)^2.$$

$$\exp(\gamma \langle Y_t \rangle) \le \exp\left(\gamma C t \left(\int_0^t \frac{1}{|W_r - y|}\, dr \right)^2 \right).$$

We use Ito's formula with $R_t := |W_t - y|$ and $h(x) := \frac{1}{x}$,

$$g(R_t) - g(R_0) = \int_0^t g'(R_s)\, dB_s + \int_0^t h(R_s)\, ds,$$

$$g(0) = 0.$$

$$g'(r) = \frac{2}{r} \int_0^r u h(u)\, du = \frac{2}{r} \int_0^r \frac{u}{u}\, du = 2.$$

$$\frac{1}{2} \left(\frac{g'(r)}{r} + g''(r) \right) = h(r).$$

We get that $g(r) = 2r$. So

$$\frac{1}{2} \int_0^t \frac{1}{|W_r - y|}\, dr = X_t - |y| - B_t.$$

X denotes a 2-dimensional Bessel process and B a standard 1-dimensional Brownian motion. We see that

$$\mathbb{E}\left[\exp\left(\gamma Ct\left(\int_0^t \frac{1}{|W_r-y|}\,dr\right)^2\right)\right] = \mathbb{E}\left[\exp\left(\gamma Ct\left(X_t-|y|-B_t\right)^2\right)\right].$$

Further,

$$\mathbb{E}\left[\exp\left(\gamma Ct\left(X_t-|y|-B_t\right)^2\right)\right] = \mathbb{E}\left[\sum_{n=1}^{\infty}\frac{\left(\gamma Ct\left(X_t-|y|-B_t\right)^2\right)^n}{n!}\right]$$

$$\leq \mathbb{E}\left[\sum_{n=1}^{\infty}\frac{(\gamma Ct)^n\left(X_t+(|y|+|B_t|)\right)^{2n}}{n!}\right].$$

By using the discrete version of Jensen's inequality, we obtain

$$\mathbb{E}\left[\sum_{n=1}^{\infty}\frac{(\gamma Ct)^n\left(X_t+(|y|+|B_t|)\right)^{2n}}{n!}\right] \leq \mathbb{E}\left[\sum_{n=1}^{\infty}\frac{(\gamma Ct)^n 3^{2n}\left(\frac{1}{3}X_t^{2n}+\frac{1}{3}|y|^{2n}+\frac{1}{3}|B_t|^{2n}\right)}{n!}\right].$$

This leaves us with the task of finding suitable bounds for the even moments of a 2-dimensional Bessel process, and a Brownian motion.

It goes without saying, that the "suitable" γ for each of the 3 terms X_t, $|B_t|$ and $|y|$ will be different, but out of convenience, we will not change the notation. It shall further be noted, that we choose the smallest value of γ obtained from the 3 following estimates, when we proceed.

Regarding the Bessel process X, we know that

$$\int_{\mathbb{R}_+^2}|x|^m\exp\left(-\frac{|x|^2}{\sigma}\right)\,dx = \frac{1}{4}\sigma^{1+\frac{m}{2}}\Gamma\left(1+\frac{m}{2}\right).$$

As we only calculate even moments, this is in turn equal to $\frac{1}{4}\sigma^{1+\frac{m}{2}}m!$.

Therefore,

$$\mathbb{E}\left[\frac{(\gamma Ct)^n 3^{2n}\frac{1}{3}X_t^{2n}}{n!}\right] \leq \frac{(\gamma Ct)^n t^{(1+n)}n!}{n!} \leq (\gamma CT^{\alpha})^n,$$

where $\alpha := \begin{cases} 1 & T<1 \\ 3 & T\geq 1\end{cases}$. For $\gamma < \frac{1}{CT^{\alpha}}$, $\gamma CT^{\alpha}<1$, which means that a summation over n converges.

Considering now $|B_t|$,

$$\mathbb{E}\left[|B_t|^{2n}\right] \leq C\int_0^{\infty}|x|^{2n}\exp\left(-\frac{x^2}{2t}\right)\,dx$$

$$\leq \frac{C}{2}(2t)^{\frac{1}{2}+n}\Gamma\left(\frac{1}{2}+n\right) = \frac{C}{2}(2t)^{\frac{1}{2}+2n}\frac{(2n)!}{n!4^n}\sqrt{\pi}.$$

From this we obtain

$$\mathbb{E}\left[\frac{(\gamma Ct)^n 3^{2n}\frac{1}{3}|B_t|^{2n}}{n!}\right] \leq \frac{(\gamma Ct)^n}{n!} t^{\frac{1}{2}+2n}\frac{(2n)!}{n!4^n} \leq (\gamma CT^{\beta})^n,$$

where $\beta := \begin{cases} \frac{1}{2} & T < 1 \\ 3 & T \geq 1 \end{cases}.$

It is easy to check that the term $\frac{(2n)!}{(n!)^2 4^n}$ is bounded by 1 for every $n = 1, 2, \dots$ and goes to 0, as n approaches infinity.

Hence, we can choose γ again small enough, such that the sum over $(\gamma CT^{\beta})^n$ converges.

The only terms we are left with, are the ones containing $|y|^{2n}$.

$$\mathbb{E}\left[\frac{(\gamma Ct)^n 3^{2n}\frac{1}{3}|y|^{2n}}{n!}\right] \leq \frac{(\gamma Ct|y|^2)^n}{n!}.$$

Combining all the three terms indicates that there exists a θ_0, which depends on T, such that the expectation of the exponential exists for every $0 < \theta < \theta_0$.

(Proceeding, we will choose our θ_0 as the minimum of the θ_0 obtained above and $\frac{1}{8T}$. The reason for this will soon become obvious.)

For the third term,

$$\mathbb{E}\left[\exp\left(-16\theta t \log(|x+\mu|)\right)\right]^{\frac{1}{2}} = \mathbb{E}\left[\exp\left(-\log(|x+\mu|^{16\theta t})\right)\right]^{\frac{1}{2}}$$

$$= \mathbb{E}\left[\exp\left(\log\left(\frac{1}{|x+\mu|^{16\theta t}}\right)\right)\right]^{\frac{1}{2}}$$

$$= \frac{1}{|x+\mu|^{8\theta t}}.$$

It should be noted that, by Young's inequality,

$$\frac{1}{|y|} = \frac{1}{\sqrt{y_1^2+y_2^2}} = \frac{1}{\sqrt{|y_1|^2+|y_2|^2}} \leq \frac{1}{\sqrt{2|y_1||y_2|}}.$$

Roughly summing up, by the previous estimates and Lemma 79

$$\mathbb{E}[\exp(\theta\beta_2(x+\mu,t))^4]$$

$$\leq C_1\left(t^{\frac{8\theta t}{2}} + |x|^{\frac{8\theta t}{2}} + |\mu|^{\frac{8\theta t}{2}}\right)\left(C_{2,\theta} + C_{3,\gamma} + \exp(\gamma C_3|x+\mu|^2)\right)\left(\frac{1}{|x+\mu|^{8\theta t}}\right).$$

When dealing with the integrability regarding x, we see that the terms which appear when we perform the previous steps (including the steps in the proof of Lemma 79 for the second term), are of the form y^{α} for $\alpha \geq 0$, $\exp(Cy^2)$ and $\frac{1}{|y|^{\beta}}$ with $\beta < 1$. As we are multiplying with a C_c^{∞} function, the first two terms are negligible and the only term requiring attention is $\frac{1}{|y|^{\beta}}$. Let $\psi \in C_c^{\infty}$ with $-\mu \in \operatorname{supp}(\psi)$,

$$\int_{\mathbb{R}^2} \psi(x) \frac{1}{|x+\mu|^{8\theta t}} \, dx = \int_{\mathbb{R}^2} \psi(x-\mu) \frac{1}{|x|^{8\theta t}} \, dx \leq C \int_{B_R(0)} \frac{1}{|x|^{8\theta t}} \, dx.$$

$B_R(0)$ denotes a ball around 0 with radius R, such that the support of $\psi(\cdot - \mu)$ lies inside it.

We see that this value is finite by either changing to polar coordinates and noting that $\frac{1}{|r|^\alpha}$ is integrable on a ball around 0 for any $\alpha < 1$, or by applying Young's inequality, as mentioned before.

Integrating with respect to t, results obviously in a finite value as, by all previous estimates and because $0 \leq 8\theta t < 1$, we are left with a polynomial in t, which we integrate over $[0, T]$. □

Proposition 82. *The moment generating function* $M(t, x, \theta) := \mathbb{E}[\exp(\theta u(t, x))]$ *for* $\theta < \theta_0$, *can be obtained as a distributional solution of (4.8) .*

Proof. Let $y \neq 0$ and ρ_ϵ again the normal approximation of the Dirac delta.

$$\mathbb{E}[\langle \exp(\theta u(s, x)), \rho_\epsilon(\cdot - y \rangle]$$

$$= \frac{1}{2} \left(\mathbb{E} \left[\int_0^t \langle \exp(\theta u(s, x)), \Delta\rho_\epsilon(\cdot - y \rangle \right] + \theta \mathbb{E} \int_0^t [\langle F_s, \rho_\epsilon(\cdot - y) \rangle] \, ds \right).$$

It only remains to justify the exchange of the integrals and the expectation in the first integral on the right hand side. The argumentation for the left hand side is the same as in the case of the first moment, where we also have already dealt with the term including F on the right hand side. By Tonelli's theorem for positive functions (as the term inside the inner product remains positive),

$$\mathbb{E} \left[\int_0^t \langle \exp(\theta u(s, x)), \Delta\rho_\epsilon(\cdot - y \rangle \, ds \right] = \int_0^t \langle \mathbb{E}[\exp(\theta u(s, x))], \Delta\rho_\epsilon(\cdot - y \rangle \, ds.$$

What is left now is using Lebesgue's theorem, letting $\epsilon \to 0$ and differentiating with respect to t. □

Remark 83.

$$\mathbb{E} \left[\exp \left(8\theta \int_0^t \log(|W_t - W_r - x - \mu|) \, dr \right) \right]^{\frac{1}{2}}$$

$$\leq \mathbb{E} \left[\exp \left(8\theta \int_0^t (1 + |W_{t-r} - x - \mu|) \, dr \right) \right]^{\frac{1}{2}}$$

$$\leq \left(\mathbb{E} \left[\exp(8\theta t) \exp \left(8\theta \int_0^t |W_{t-r}| + |x| + |\mu| \, dr \right) \right] \right)^{\frac{1}{2}}$$

$$= \left(\mathbb{E} \left[\exp(8\theta t(1 + |x| + |\mu|)) \exp \left(8\theta \int_0^t |W^1_{t-r}| + |W^2_{t-r}| \, dr \right) \right] \right)^{\frac{1}{2}},$$

as for a, $b > 0$ $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ holds.

We will replace W^i, $i = 1, 2$ by B.

$$|B_t| = |B_0| + \int_0^t \operatorname{sgn}(B_s)\, dB_s - L_t,$$

where L denotes the local time of a Brownian Motion.
As L is increasing,

$$\mathbb{E}\left[\exp\left(\int_0^t 8\theta|B_s|\, ds\right)\right] = \mathbb{E}\left[\exp\left(8\theta \int_0^t |B_0| + \int_0^s \operatorname{sgn}(B_r)\, dB_r - L_s\, ds\right)\right]$$

$$\leq \mathbb{E}\left[\exp\left(8\theta\left(\int_0^t B_s\, ds + tL_t\right)\right)\right]$$

$$\leq \mathbb{E}\left[\exp\left(16\theta \int_0^t B_s\, ds\right)\right]^{\frac{1}{2}} \mathbb{E}\left[\exp\left(16\theta t L_t\right)\right]^{\frac{1}{2}}$$

$$\leq \left(\exp\left(256\theta^2 \frac{t^3}{6}\right)\right)^{\frac{1}{2}} \left(2\exp\left(8\theta^2 t\right)\right)^{\frac{1}{2}}.$$

The estimate leading to the last inequality was taken from [23].

Remark 84. Another way to approach the previous problem, is to expand the exponent

$$\mathbb{E}[\exp(\theta\beta_2)] = \mathbb{E}[\exp(\theta\beta_2 - \theta\mathbb{E}[\beta_2])\exp(\theta\mathbb{E}[\beta_2])]$$

and use the Clark-Ocone formula to deal with the expectation involving the renormalized SILT, as in [21]. From there we get the following representation for $\tilde{L} := \beta_2(0,T) - \mathbb{E}[\beta_2(0,T)]$,

$$\tilde{L} := -\frac{1}{2\pi}\sum_{i=1}^{2}\int_0^T \left(\int_r^T \int_0^r \frac{B_r^i - B_s^i}{(t-r)^2}\exp\left(-\frac{|B_r - B_2|^2}{2(t-r)}\right) ds\, dt\right) dB_r^i.$$

The quadratic variation of this stochastic integral is

$$\langle \tilde{L}\rangle = \frac{1}{4\pi^2}\sum_{i=1}^{2}\int_0^T \left(\int_r^T \int_0^r \frac{B_r^i - B_s^i}{(t-r)^2}\exp\left(-\frac{|B_r - B_2|^2}{2(t-r)}\right) ds\, dt\right) dr$$

$$\leq \frac{1}{4\pi^2}\int_0^T \left(\int_r^T \int_0^r \frac{|B_r - B_s|}{(t-r)^2}\exp\left(-\frac{|B_r - B_s|^2}{2(t-r)}\right) ds\, dt\right) dr$$

$$= \frac{1}{\pi^2}\int_0^T \left(\int_0^r \frac{1}{|B_r - B_s|}\exp\left(-\frac{|B_r - B_s|^2}{2(T-r)}\right) ds\right) dr$$

$$\leq \frac{1}{4\pi^2}\int_0^T \left(\int_0^r \frac{1}{|B_r - B_s|}\,ds\right)\,dr.$$

A quick application of Ito's formula shows that

$$\int_0^r \frac{1}{|B_r - B_s|}\,ds = \frac{1}{d-1}(X_r - b_r),$$

where X_r has the law of the modulus of a $d-$dimensional Brownian motion at time r ($d-$dimensional Bessel process), and b_r has a normal $N(0,1)$ law. We can write

$$\exp(\lambda\langle\tilde{L}\rangle) \leq \frac{1}{T}\int_0^T \exp\left(\frac{T\lambda}{\pi^2}\left(\int_0^r \frac{1}{|B_r - B_s|}\,ds\right)^2\right)\,dr,$$

which implies the existence of some λ_0, such that $\mathbb{E}[\exp(\lambda\langle\tilde{L}\rangle)] \leq \infty$ for all $\lambda \leq \lambda_0$. From Lemma (79), we obtain the existence of a β_0, such that $\mathbb{E}[\exp(\beta|\tilde{L}|)] \leq \infty$ for all $\beta < \beta_0$. Although we are not able to obtain the critical exponent explicitly this way, we still obtain its existence.

We would proceed similarly in order to obtain our desired result.

Corollary 85. *The previous estimates also indicate that there exists a θ_0, such that for all $\theta < \theta_0$, the moment generating function of the SILT at a point $x \neq 0$ and a time $t > 0$, exists.*

5. Appendix

5.1. A weak form of Ito's Lemma.

Lemma 86. *Let*

$$\rho(t,x) = \frac{1}{(2\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{2t}\right), \qquad t>0,\ x\in\mathbb{R}^n,$$

then $\rho \in L^q((0,T)\times\mathbb{R}^n)$ *for every* $q\in(0,1+\frac{2}{n})$ *and* $T>0$.

Proof.

$$\int_0^T \int_{\mathbb{R}^n} \rho^q(t,x)\,dx\,dt = \int_0^T \int_{\mathbb{R}^n} \frac{1}{(2\pi t)^{\frac{n}{2}(q-1)}} \exp\left(-\frac{q|x|^2}{2t}\right) dx\,dt,$$

by the change of variables $y=\frac{x}{\sqrt{2t}}$, we get

$$= \frac{1}{\pi^{\frac{nq}{2}}} \int_0^T \frac{1}{(2t)^{\frac{n}{2}(q-1)}}\,dt \int_{\mathbb{R}^n} \exp(-q|y|^2)\,dy.$$

This expression is finite for $\frac{n(q-1)}{2}<1$ and $q>0$, i.e. $0<q<1+\frac{2}{n}$. □

Lemma 87. *Assume that* $f\in W^{2,p}(\mathbb{R}^n)$, *with* $p<1+\frac{n}{2}$, *then* f *is (Hölder) continuous and we have*
1.) if $p\le n$ *then* $|\nabla f|^2\in L^q(\mathbb{R}^n)$ *for some* $q>1+\frac{n}{2}$,
2.) if $p>n$ *then* $\nabla f\in C(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$.

Proof. If $p\ge n$, the claim follows from the usual Sobolev(-Morrey) embedding, [1] Theorem A.168 (as Theorem 52 is a bit too specific). If $1+\frac{n}{2}<p<n$ then, necessarily, $n>2$ and by the previously mentioned Theorem, we have $\nabla f\in L^{2q}(\mathbb{R}^n)$ with

$$2q = \frac{pn}{n-p} = \frac{n}{\frac{n}{p}-1} > \frac{n}{\frac{n}{1+\frac{n}{2}}-1} = \frac{n(n+2)}{n-2} > n+2.$$

This proves that $|\nabla f|^2\in L^q(\mathbb{R}^n)$ for some $q>1+\frac{n}{2}$ and consequently, by [1] Theorem A.168, f is Hölder continuous. □

Proposition 88. *Let* X, Y *be a.s. right-continuous stochastic processes. If* X *is a modification of* Y, *then* X, Y *are indistinguishable. In particular, we can equivalently write*
$X_t=Y_t$ *a.s. for every* t
or
$X_t=Y_t$ *for every* t *a.s..*

Proof. We refer to [1], Proposition 3.25 on page 108. □

Proposition 89. *Let* $f\in W^{2,p}_{loc}(\mathbb{R}^n)$ *with* $p>1+\frac{n}{2}$, *then we have*

$$f(W_t) = f(0) + \int_0^t \nabla f(W_s)\cdot dW_s + \frac{1}{2}\int_0^t \Delta f(W_s)\,dW_s. \tag{5.1}$$

Proof. Let us first consider the case $n > 2$. Let f_n be a regularizing sequence for f, which we obtained by a convolution with the usual mollifier. Then by Lemma 50, $f_n \in C^\infty(\mathbb{R}^n)$ and f_n converges to f, uniformly on compact sets, so that

$$\lim_{n\to\infty} f_n(W_t) = f(W_t),$$

for any $t \geq 0$. By the standard Ito formula, we have

$$f_n(W_t) = f_n(W_0) + \int_0^t \nabla f_n(W_s)\cdot dW_s + \frac{1}{2}\int_0^t \Delta f_n(W_s)\,ds.$$

Further, by the Ito isometry,

$$\mathbb{E}\left[\left(\int_0^t (\nabla f_n(W_s) - \nabla f(W_s))\cdot dW_s\right)^2\right]$$

$$= \int_0^t \mathbb{E}\left[|\nabla f_n(W_s) - \nabla f(W_s)|^2\right]\,ds$$

$$= \int_0^t \int_{\mathbb{R}^n} |\nabla f_n(x) - \nabla f(x)|^2 \rho(s,x)\,dx\,ds =: I_n.$$

If $p > n$, we have

$$\lim_{n\to\infty} I_n = 0$$

by Lebesgue's theorem, since by Lemma (87) $\nabla f \in C \cap L^\infty$, and so the integrand converges to zero pointwise and is dominated by the integrable function $\|\nabla f_n - \nabla f\|^2_{L^\infty(\mathbb{R}^n)}\rho$.

On the other hand, if $1 + \frac{n}{2} < p \leq n$, by Lemma (87) we have $|\nabla f|^2 \in L^q(\mathbb{R}^n)$ for some $q > 1 + \frac{n}{2}$. Let q' the conjugate exponent of q, then we have

$$q' = 1 + \frac{1}{p-1} < 1 + \frac{2}{n}$$

and therefore, by Lemma (86), $\rho \in L^{q'}((0,T)\times\mathbb{R}^n)$. By Hölder's inequality, we obtain

$$I_n \leq \left\||\nabla f_n - \nabla f|^2\right\|_{L^q((0,T)\times\mathbb{R}^n)} \|\rho\|_{L^{q'}((0,T)\times\mathbb{R}^n)} \to_{n\to\infty} 0.$$

Finally,

$$\mathbb{E}\left[\left|\int_0^t (\Delta f_n(W_s) - \Delta f(W_s))\,ds\right|\right]$$

$$\leq \int_0^t \mathbb{E}\left[|\Delta f_n(W_s) - \Delta f(W_s))|\right]\,ds$$

$$= \int_0^t \int_{\mathbb{R}^n} |\Delta f_n(x) - \Delta f(x)| \rho(s,x)\, dx\, ds.$$

By applying Hölder's inequality with the conjugate exponent of p, which we call p', the last term is

$$\leq \|\Delta f_n - \Delta f\|_{L^p((0,T)\times\mathbb{R}^n)} \|\rho\|_{L^{p'}((0,T)\times\mathbb{R}^n)} \to_{n\to\infty} 0,$$

since $f_n$converges to f in $W^{2,p}(\mathbb{R}^n)$ and the assumption $p > 1 + \frac{2}{n}$ implies $p' < 1 + \frac{2}{n}$. By Lemma (86), we have

$$\|\rho\|_{L^{p'}((0,T)\times\mathbb{R}^n)} < \infty.$$

In conclusion, we showed that (5.1) holds a.s. for every $t > 0$, and by 88 this is sufficient.

In the case $n \leq 2$, the hypothesis $p > 1 + \frac{n}{2}$ implies that $p > n$ and the claim can be proved as before. □

Remark 90. The previous proof can easily be adapted to functions f, which also depends on time, i.e. for functions in the parabolic Sobolev space $H^{1,p}_{\text{loc}}((0,T); H^{2,p}_{\text{loc}}(\mathbb{R}^n))$.

5.2. **Some results regarding a "Fundamental solution".**

Remark 91. Most steps in this part are rather short and formal, but can be made more rigorous by applying results from the previous section.

Consider the equation

$$du_t = \frac{1}{2}\Delta u_t + \delta_{-\mu}\, dt + \nabla u_t \cdot dW_t,$$

$$u(0) = u_0,$$

on $\mathbb{R}^2$, where W_t is again a planar Brownian motion as in the previous part. By the same argumentation as in the previous section (Remark 57, Proposition 58), the weak formulation is, again, well defined on H^2. Once again, we can use Duhamel's principle in order to obtain a general form of the solution

$$u(t) := U_{0,t} u_0 + \int_0^t U_{s,t} F_s\, ds,$$

again with $U_{s,t} := T_{W(s)} \circ T_{-W(t)}$ and $U_{t,s} := T_{W(t)} \circ T_{-W(s)}$.

Lemma 60 can be applied directly in this case.

Once again, we switch to $\phi \in \mathcal{S}$ in order to apply Ito's formula (as in Proposition 69) more easily,

$$d\langle u(t), \phi\rangle = d\left(\int_0^t \phi(W_t - W_r - \mu)\, dr\right)$$

$$= \left(\phi(-\mu) + \frac{1}{2}\int_0^t \Delta\phi(W_t - W_r - \mu)\, ds\right)\, dt$$

$$
\begin{aligned}
&+ \left(\int_0^t \nabla\phi(W_t - W_r - \mu)\, ds \right) dW_t \\
&= \left(\frac{1}{2}\langle \Delta\phi, u(t)\rangle + \phi(-\mu) \right) dt - \langle \nabla\phi, u(t)\rangle\, dW_t.
\end{aligned}
$$

Very formal, one could say that, this potential solution raises somewhat the hope, that we find a connection to the SILT,

$$
\begin{aligned}
\langle u(t), \phi\rangle &= \frac{d}{dt}\left(\int_0^t \int_0^s \phi(W_s - W_r - \mu)\, dr\, ds \right) = \frac{d}{dt}\left(\int_{\mathbb{R}^2} \phi(x)\beta_2(x-\mu, t)\, dx \right) \\
&= \int_{\mathbb{R}^2} \phi(x) \frac{d}{dt}\left(\lim_{\epsilon\to 0} \int_0^t \int_0^s \rho_\epsilon(W_s - W_r - (x-\mu))\, dr\, ds \right) dx \\
&= \int_{\mathbb{R}^2} \phi(x) \left(\lim_{\epsilon\to 0} \int_0^t \rho_\epsilon(W_t - W_s + \mu - x)\, ds \right) dx \\
&= \int_{\mathbb{R}^2} \phi(x)\eta(x-\mu, t)\, dx.
\end{aligned}
$$

Unfortunately, I have not been able to show the existence of the last limit in L^2 for all times, which is the reason this equation isn't treated as extensively. Although there are some, more or less satisfactory, results which I would like to mention (explicit proofs/calculations can be supplied at request). First, a "renormalized" version of this limit exists, namely $\lim_{\epsilon\to 0} I_\epsilon - \mathbb{E}[I_\epsilon]$. Proving this is rather tedious, but can be done following the steps in [40]. Second, for sufficiently large values of t the inequality $\int_0^t \exp(-a^2|t-s|)\, ds \le \int_0^t \int_0^s \exp(-a^2|s-r|)\, dr\, ds$ holds, which could be used after a non-determinism argument in order to show convergence in rather specific cases.

5.3. **A list of useful integrals.**

$$
\int r \log\left(\frac{1}{r^2}\right)^n dr = \frac{1}{2}\Gamma\left(n+1, \log\left(\frac{1}{r^2}\right)\right).
$$

$$
\int_0^t r \log\left(\frac{1}{r^2}\right)^n dr = \frac{1}{2}\Gamma(n+1, -2\log(t)).
$$

$$
\int_0^T t^m \log(|\sqrt{t}|)^n\, dt
$$

$$
= \frac{(-1)^n(1+m)^{-n}\Gamma(1+n) + \Gamma(1+n, -(1+m)\log(T))\log(T)^n(-(1+m)\log(T))^{-n}}{2^n(1+m)}.
$$

$$
\int t^m \log(|\sqrt{t}|)^n\, dt
$$

$$
= \frac{\Gamma(1+n, -(m+1))\log(t))\log(t)(-(1+m)\log(t))^{-n}}{2^n(1+m)}.
$$

$$
\int_{-\infty}^{\infty} \exp(-ax^2 + bx + c)\, dx = \exp\left(\frac{b^2}{4a} + c\right)\sqrt{\frac{\pi}{a}}.
$$

Let A be a positive, symmetric and invertible matrix, then

$$\int_{-\infty}^{\infty} \exp(-\langle Ax, x\rangle + \langle b, x\rangle + c)\, dx = \exp\left(\frac{1}{2}\langle b, A^{-1}b\rangle - c\right)\sqrt{\frac{\pi}{\det(A)}}.$$

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