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High-Dimensional Expansion

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Abstract

Expansion properties of graphs and, in particular, bounded degree expander graphs, are a central research topic in combinatorics and theoretical computer science, with many applications and connections to other areas of mathematics. In recent years, there has been a concerted effort to generalize and extend this rich and fruitful theory to higher dimensions. In this thesis, some of these recent developments and results are surveyed. Several notions of higher-dimensional expansion, in particular coboundary expansion (introduced in the work of Linial–Meshulam [LM06] and Gromov [Gro10]) on the one hand and spectral expansion formulated in terms of the eigenvalues of higher-dimensional Laplacians (going back to the work of Eckmann [Eck45] and Garland [Gar73]) on the other hand, are described and connections and differences between these are discussed. Moreover, several applications of higher-dimensional expansion, e.g., Gromov's topological overlap theorem are presented.

Kurzfassung

Expansion als Eigenschaft von Graphen und im Besonderen Expander-Graphen mit beschränktem Knotengrad sind ein zentrales Forschungsthema in der Kombinatorik und in den theoretischen Computerwissenschaften mit vielen Anwendungen und Verbindungen zu anderen Gebieten der Mathematik. In den letzten Jahren gab es gemeinsame Bemühungen, diese reichhaltige und fruchtbare Theorie auf höherdimensionale Fragestellungen zu erweitern und zu verallgemeinern. In dieser Arbeit werden einige der aktuellen Entwicklungen und Ergebnisse beschrieben. Verschiedene Begriffe von höherdimensionaler Expansion, speziell Coboundary Expansion (zurückgehend auf die Arbeiten von Linial–Meshulam [LM06] und Gromov [Gro10]) einerseits und spektrale Expansion, definiert mithilfe der Eigenwerte höherdimensionaler Laplace-Operatoren (eingeführt in den Arbeiten von Eckmann [Eck45] und Garland [Gar73]), andererseits, werden näher beschrieben und die Unterschiede zwischen den einzelnen Ansätzen werden diskutiert. Zusätzlich wird eine Auswahl aus der Vielzahl an Anwendungen von höherdimensionaler Expansion, beispielsweise das Topological Overlap Theorem von Gromov, präsentiert.

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1 Introduction

High-dimensional expansion is a generalization of edge expansion of graphs. It was introduced in the works of Mikhail Gromov ([Gro10]) and independently in the work of Nathan Linial and Roy Meshulam ([LM06]) and has inspired a great amount of research since then.

The original notion of edge expansion describes a property of connectivity of a graph. By partitioning the graph into two subsets, a graph can be said to be highly connected, if there are many edges between the subsets. Highly connected graphs are called expander graphs. The technical definition of expansion (for details see definition 3.1.1 on page 25) leads to many interesting properties, which are very helpful for applications. Graph expansion, and in particular bounded-degree expander graphs, are also of central importance in the theory of error-correcting codes and in theoretical computer science, both for the design of algorithms and for complexity-theoretic lower bounds (e.g., the PCP theorem, probabilistically checkable proofs).

In recent years, it was felt that the concept of graph expansion should be generalized to higher dimensional analogues of graphs, simplicial complexes, in the hope that high-dimensional expansion yields as many possibilities of applications as graph expansion and that it is helpful to increase our general knowledge about discrete models like simplicial complexes and graphs. The first to define high-dimensional (combinatorial) expansion were Mikhail Gromov ([Gro10]) and independently Nathan Linial and Roy Meshulam ([LM06]). Gromov was inspired by isoperimetric inequalities that appear in geometry as well as in discrete mathematics, whereas Linial and Meshulam introduced expansion as a tool to prove vanishing cohomology of random subcomplexes of the complete simplicial complex. In both cases, the definition of edge expansion is rewritten in terms of simplicial cohomology, which allows an easy and direct generalization to higher dimensions.

Starting from these definitions, the three main questions are:

- Are there simplicial complexes that are high-dimensional expanders?
- How can we prove that a simplicial complex is an expander?
- What can we deduce from high-dimensional expansion?

As a step towards answering the first question, Gromov and Linial and Meshulam proved that the complete simplicial complex is expanding. However, the main question is whether there are infinite families of high-dimensional expanders that are *locally bounded* analogously to bounded-degree expander graphs that play such an important role in applications. The search for such examples is still going on with

high effort. The second question is tightly linked with the first one as there are many examples of simplicial complexes which may be prototypes for expanders. Research in this direction is mainly inspired by the huge reservoir of known results from graph theory. The third question, though, is the one that is inspiring a lot of research. There are already numerous applications of expansion known, see Chapter 4. Different applications led to a vast number of variations of the original definitions, each of them geared towards applications to a different mathematical problem.

It is in the nature of things that all this research leads to the field of high-dimensional expansion being very complex and confusing. As the field is very young, research is still in the phase of developing new ideas and concepts and it is unclear which of all these concepts is fruitful and usable and which is not. The author of this thesis knows of at least six different definitions of high-dimensional expansion, in a phase of consolidation of the field many of these concepts will be replaced by better ones.

In the course of this thesis, the author will try to give an overview and discuss some of the main concepts of high-dimensional expansion. The thesis is structured to fit the questions asked above. Namely, after giving some preliminaries for readers who are not familiar with simplicial complexes and cohomology in Chapter 2, we give the definition of two main notions of high-dimensional expansion in Chapter 3: coboundary expansion and spectral expansion. Then we want to survey some of the applications of expansion in Chapter 4, among them the famous topological overlap theorem by Mikhail Gromov, the appearance of expansion in the field of property testing, and how expander complexes can be used to construct error correcting codes. After motivating high-dimensional expansion we turn towards examples of high-dimensional expanders in Chapter 5. As presenting examples of expanders includes the proof of their expansion properties, this chapter also includes three methods to prove expansion. Starting with the method of "random co-filling" (Section 5.1), we show that standard complexes like the complete complex and the complete multipartite complexes are expanding. Although these proofs only work for these special complexes, they inspire methods using the local structure (the links) of the complex in consideration. The next section continues this train of thought to present so-called "local to global" methods (Section 5.2), which are described in detail for the case of 2-dimensional simplicial complexes. In the tradition of edge expansion for graphs the last section of this chapter addresses the field of random methods (Section 5.3). Using tools from probability theory, it is possible to prove expansion for a large variety of simplicial complexes. This method is also used in the original paper of Nathan Linial and Roy Meshulam ([LM06]), which brings the discussion full circle. Lastly, in the final Chapter 6, we discuss the latest open questions in research and give an overview of notions that were beyond the scope of this thesis.

2 Preliminaries

In the course of this thesis, several prerequisites from different fields are needed. For the sake of being self-contained, the most important (and maybe not so well-known) definitions and concepts will be introduced in this chapter. However, it is impossible to include all necessary definitions. In these cases, references are given and only the notation is defined.

2.1 Graphs

The first and easiest concept is that of a "graph". Graphs and graph expansion are the prototypes for the theory of high-dimensional expansion and expanders, which evolves as a generalization of these concepts. As a consequence, many ideas from graph expansion can be adopted for the high-dimensional case. Although graph expansion cannot be considered an "easier" field, it is older and therefore better understood. Moreover, graphs may appear as links (to be defined later in Section 2.2.3) of high-dimensional complexes, so the theory for graphs can be directly applied, too.

In this thesis, we will consider only finite, simple, undirected graphs (i.e., there are no loops or multiple edges, and the edges do not carry an orientation). Such a graph is given as $G = (V, E)$, where V is a finite set whose elements are called *vertices*, and $E \subseteq \binom{V}{2}$ is a set of pairs of vertices, called *edges*. Two vertices $u \neq v \in V$ are called *adjacent* or *neighbours* if they form an edge, i.e. $\{u, v\} \in E$. The set of neighbours of v is denoted by $\Gamma(v)$, and the *degree* of v is defined as $\deg(v) := |\Gamma(v)|$.

A graph is called *d-regular*, if every vertex has the same degree d. It is called *bipartite* or, more generally, *k-partite*, if the set of vertices V can be partitioned into two (or k, respectively) distinct sets and there are no edges between vertices of the same subset. We refer to [Die10] for further background on graphs.

The following definition is explicitly needed:

Definition 2.1.1. *Let S and T be disjoint subsets of the set of vertices V of a graph $G = (V, E)$. Then the set of edges between S and T, denoted $E(S, T)$, is defined to be:*

$$E(S, T) := \{e = \{u, v\} \in E \mid u \in S, v \in T\} \tag{2.1.1}$$

Moreover, the adjacency matrix is defined as:

Definition 2.1.2 (Adjacency Matrix)**.** *The* adjacency matrix $A = A(G)$ *of a graph* $G = (V, E)$ *is the* $|V| \times |V|$*-matrix indexed by the vertices of* G *with the entries:*

$$(A(G))_{u,v} = \begin{cases} 1 & \text{if } \{u, v\} \in E \\ 0 & \text{else} \end{cases} \tag{2.1.2}$$

The adjacency matrix is symmetric as G is undirected and thus possesses a real spectrum with orthogonal eigenspaces (spectral theorem for self-adjoint operators, two eigenvectors to different eigenvalues are orthogonal with respect to the standard scalar product in $\mathbb{R}^n$).

If the underlying graph is d-regular, all the sums of the columns and the rows are equal to d. So, if we apply $A(G)$ to the vector containing only ones, we get the vector containing only d's. Hence, for d-regular graphs, d is always an eigenvalue of $A(G)$ with the all-one-vector as eigenvector.

2.1.1 Laplacians of Graphs

Apart from the adjacency matrix, there is another matrix that can be associated to a given graph G, called the *Laplacian matrix* or, shorter, *Laplacian*. The definition and motivation presented here is taken from [HLW06].

The Laplacian matrix of a graph picks up the idea of the Laplacian operator from differential calculus and transfers it to a discrete setting. In differential calculus the Laplacian operator is defined as the composition of the gradient and the divergence operator. The gradient operator applied to a scalar function on $\mathbb{R}^n$ is defined via its derivative and maps the function to a vector field that represents the best linear approximation of the function and points in the direction of its largest ascent. By taking the scalar product of the gradient and a vector the directional derivatives of the function can be calculated. The divergence, on the other hand, can be applied to vector fields and is linked to the sources and sinks of vector fields (cf. Gauß' integral theorem).

Back in the setting of graphs, scalar functions are functions on the vertices that take real or complex values. The change of such a function can only be calculated with respect to two vertices that are adjacent. Inspired by the difference quotient, we have the definition of the gradient of a scalar function on the vertices:

Definition 2.1.3 (Discrete Gradient)**.** *Let* $G = (V, E)$ *be a graph and choose an (arbitrary) orientation of the edges. Let* $\vec{E}$ *be the set of oriented edges (with the chosen orientation). Let* $f : V \to \mathbb{R}$ *be a scalar function on the vertices.*

The (discrete) gradient $\operatorname{grad} f$ *of* f *(with respect to this orientation) is defined as the function:*

$$\operatorname{grad} f : \begin{cases} \vec{E} & \to \mathbb{R} \\ e = (u, v) & \mapsto (\operatorname{grad} f)(e) := f(u) - f(v) \end{cases} \tag{2.1.3}$$

Thus, the discrete gradient of a function is just the function evaluating as the differences of the original values on the endpoints of the edges. The discrete gradient takes a positive value on the edge (u, v) directed from u to v if the value of f at u is larger than the value at v and hence "points" along the edge to the lower value. Unlike the continuous gradient, the discrete gradient defined here "points" in the direction of descent and not ascent, but this is just a matter of sign.

The definition of a discrete divergence is yet more straightforward. We consider scalar functions on the edges that are interpreted as "flows along the edges". The source strength or divergence of a vertex v is defined as the net flow coming from this vertex, that is the difference of the cumulated out-flow and the cumulated in-flow:

Definition 2.1.4 (Discrete Divergence)**.** *Let $G = (V, E)$ be a graph and choose an (arbitrary) orientation of the edges. Let $\vec{E}$ be the set of oriented edges. Let $g : \vec{E} \to \mathbb{R}$ be a scalar function on the edges.*

The (discrete) divergence $\operatorname{div} g$ *of g (with respect to this orientation) is defined as the function:*

$$\operatorname{div} g : \begin{cases} V & \to \mathbb{R} \\ u & \mapsto (\operatorname{div} g)(u) := \sum\limits_{\substack{e \in \vec{E} \\ e=(u,v)}} g(e) - \sum\limits_{\substack{e \in \vec{E} \\ e=(v,u)}} g(e) \end{cases} \tag{2.1.4}$$

As the discrete gradient returns functions on the edges and the discrete divergence takes functions on the edges, gradient and divergence can be composed (just as for the continuous case) to get the discrete Laplacian:

Definition 2.1.5 (Discrete Laplacian)**.** *Let $G = (V, E)$ be a graph and choose an (here indeed arbitrary) orientation of the edges. Let $f : V \to \mathbb{R}$ be a scalar function on the vertices.*

The (discrete) Laplacian Δf *of f is defined as the function:*

$$\Delta f : \begin{cases} V & \to \mathbb{R} \\ u & \mapsto (\Delta f)(u) := (\operatorname{div}(\operatorname{grad} f))(u), \end{cases} \tag{2.1.5}$$

where divergence and gradient are taken with respect to the same orientation of the edges.

This definition is independent of the chosen orientation, as the following calcula-

tion shows:

$$(\Delta f)(u) = (\operatorname{div}\operatorname{grad} f)(u) = \sum_{\substack{e\in\vec{E}\\ e=(u,v)}} (\operatorname{grad} f)(e) - \sum_{\substack{e\in\vec{E}\\ e=(v,u)}} (\operatorname{grad} f)(e) \tag{2.1.6}$$

$$= \sum_{\substack{e\in\vec{E}\\ e=(u,v)}} (f(u)-f(v)) - \sum_{\substack{e\in\vec{E}\\ e=(v,u)}} (f(v)-f(u)) \tag{2.1.7}$$

$$= \underbrace{\sum_{u\sim v} f(u)}_{=\deg(u)f(u)} - \sum_{u\sim v} f(v) \tag{2.1.8}$$

$$= \deg(u)f(u) - \sum_{u\sim v} f(v) \tag{2.1.9}$$

As linear operators between the (finite dimensional) linear spaces of scalar functions on edges or vertices, respectively, the gradient, divergence and Laplace operators possess matrix representations. We consider the bases $(\mathbb{1}_u)_{u\in V}$ and $(\mathbb{1}_e)_{e\in\vec{E}}$ of indicator functions, respectively. The matrix representations with respect to these bases make use of the so-called incidence matrix:

Definition 2.1.6 (Incidence Matrix)**.** *The* incidence matrix $K = K(G)$ *of a graph* $G = (V, E)$ *with respect to a given orientation is the* $|V| \times |\vec{E}|$*-matrix indexed by the vertices and edges of* G *with the entries:*

$$(K(G))_{u,e} = \begin{cases} +1 & \text{if } e = (u,v) \\ -1 & \text{if } e = (v,u) \\ 0 & \text{else} \end{cases} \tag{2.1.10}$$

A short calculation, interpreting f and g as functions and simultaneously as vectors written in the above bases, shows:

$$(\operatorname{grad} f)(e) = (Kf)_e \quad \text{and} \quad (\operatorname{div} g)(u) = \left(K^T g\right)_u \tag{2.1.11}$$

Hence, K and K^T are the matrix representations of grad and div, respectively. As a consequence, grad and div are adjoint operators, if we equip the spaces with the standard scalar product which makes the bases orthogonal. This is in accordance with the continuous case, where $(-\operatorname{grad})$ and div are adjoint via the L^2-scalar product (and appropriate spaces to apply Gauß' integral theorem, because this is a corollary of it)[1].

[1] In the definition of the discrete gradient the sign is reversed. Thus, in comparison with the continuous case we consider $-\operatorname{grad}$ and $-\Delta$ instead of grad and Δ. However, in the continuous case often $-\Delta$ appears as it is positive semidefinite.

The matrix representation of the Laplacian can be calculated as the product of the representations of divergence and gradient:

$$\Delta \cong K^T K \tag{2.1.12}$$

Alternatively, equation 2.1.9 on the preceding page can be used to determine the matrix representation:

$$\Delta \cong D - A, \tag{2.1.13}$$

where D is the diagonal matrix containing the degree of the vertices in the diagonal, that is:

$$D_{u,v} = \begin{cases} \deg(u) & \text{if } u = v \\ 0 & \text{else} \end{cases} \tag{2.1.14}$$

Using the first representation, it is obvious, that Δ is symmetric and positive semidefinite, while the second one is useful for calculating the spectrum of Δ, especially when G is a d-regular graph ($D = d{\cdot}\mathrm{id}$, id denotes the identity matrix).

The following lemma summarizes some properties of the Laplacian:

Lemma 2.1.1. *Let $G = (V, E)$ be a graph with adjacency matrix A and Laplacian (matrix) Δ. Then the following holds:*

(i) *Δ is symmetric and positive semidefinite.*

(ii) *The spectrum $\sigma(\Delta)$ of Δ lies in the following interval*

$$\sigma(\Delta) \subseteq [0, \max_{v \in V} \deg(v) + \rho(A)], \tag{2.1.15}$$

where $\rho(A)$ denotes the spectral radius of A, that is, $\rho(A)$ is the maximum of the absolute values of the eigenvalues of A:

$$\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda| \tag{2.1.16}$$

(iii) $\rho(A) \leq \max_{v \in V} \deg(v)$

(iv) *0 is an eigenvalue of Δ.*

If G additionally is d-regular:

(v) $\sigma(\Delta) = d - \sigma(A)$

(vi) $\rho(A) = d$

(vii) *The spectrum of Δ is contained in the interval $[0, 2d]$.*

Proof. As mentioned above, (i) follows from the form $K^T K$ from equation 2.1.12 on the preceding page. By the symmetry and positive semidefiniteness, the spectrum of the Laplacian is contained in the non-negative real numbers. (ii) follows from equation 2.1.13 on the previous page by using the induced matrix norm $\|\cdot\|_2$ and the fact, that the spectrum of an operator T is contained in a ball with radius $\|T\|_2$:

$$\|\Delta\|_2 = \|D - A\|_2 \leq \|D\|_2 + \|A\|_2 = \max_{v \in V} \deg(v) + \rho(A) \tag{2.1.17}$$

$$\implies \sigma(\Delta) \subseteq [0, \max_{v \in V} \deg(v) + \rho(A)] \tag{2.1.18}$$

The first inequality uses the triangle inequality of the norm, while the last equality uses the fact that the spectral radius and the $\|\cdot\|_2$-norm coincide for symmetric matrices.

The same trick can be played with any norm that generates the Euclidean topology on $\mathbb{R}^n$. If we pick for example the matrix norm induced by the maximum norm $\|\cdot\|_\infty$ we obtain that the spectrum of A is contained in the $\|\cdot\|_\infty$-ball with radius $\|A\|_\infty$ intersected with the real line. We thus need to bound:

$$\rho(A) \leq \|A\|_\infty \tag{2.1.19}$$

But the matrix norm induced by the maximum norm has the easy interpretation as:

$$\|A\|_\infty = \max_{u \in V} \sum_{v \in V} |A_{u,v}|, \tag{2.1.20}$$

which is less than $\max \deg(u)$, since for every u only $\deg u$ of the summands on the right side are non-zero (and thus 1). This gives (iii).

(iv) can be seen from the representation $\Delta \cong D - A$ by applying it to the vector containing only ones. The component corresponding to the vertex u of the resulting vector can then be calculated as the difference of the degree of u (from the multiplication with D) and the sum over the row of A corresponding to u (from the multiplication with A), which evaluates to zero, as the sum over a row in the adjacency matrix gives exactly the degree. Hence, 0 is an eigenvalue of Δ with the all-one-vector as eigenvector.

If G is d-regular, the matrix representation of the Laplacian simplifies to:

$$\Delta \cong d \cdot \mathrm{id} - A, \tag{2.1.21}$$

where id denotes the identity matrix (of appropriate dimensions). Hence, the spectrum transforms likewise, giving point (v). (vi) follows from point (iii) (Using the remark below the definition of the adjacency matrix, we know that d is indeed an eigenvalue, hence $\rho(A) = d$.) and (vii) follows from (vi) and (ii). □

Laplacians of graphs play an important role in analyzing random walks and expansion for graphs (cf. Section 3.2) as they allow to use tools from linear algebra in

graph theory. Moreover, the Laplacians can be analyzed probabilistically for random graphs, giving results of concentration of eigenvalues (cf. [Fri91, Fri07, FK14]).

Sometimes Laplacians are defined slightly different, giving "normalized Laplacians" (cf. for example [GW16]). These definitions are qualitatively equivalent, however, the spectrum changes, so speaking of "eigenvalues of the Laplacian" becomes ambiguous. In this thesis, we will not use normalized Laplacians for the sake of simplicity of notation and presentation.

2.2 Simplicial Complexes and Cohomology

Graphs can be naturally generalized to *simplicial complexes*:

Definition 2.2.1 (Simplicial Complex)**.** *A (finite, abstract)* simplicial complex X *is a set of subsets of a given (finite) set V with the property:*

$$\forall F \in X, G \subseteq V: \quad G \subseteq F \implies G \in X, \tag{2.2.1}$$

i.e., that it is closed under taking subsets. The elements of V are called vertices.

The elements of X are called faces *or* simplices *of X. The* dimension *of a face F is defined to be:*

$$\dim F := |F| - 1, \tag{2.2.2}$$

where $|F|$ denotes the cardinality of F as a set. F is then called a $\dim F$-dimensional face *or just* $\dim F$-face. *The empty set $\emptyset$ is the unique (-1)-dimensional face. The maximum dimension of a face in the complex defines the dimension of the complex:*

$$\dim X := \max\{\dim F | \, F \in X\} \tag{2.2.3}$$

The set of d-faces $X(d)$ *and the d-*skeleton $X^{(d)}$ *are defined as:*

$$X(d) := \{F \in X | \, \dim F = d\} \qquad d = -1, \ldots, \dim X \tag{2.2.4}$$

$$X^{(d)} := \bigcup_{k=-1}^{d} X(k) \qquad d = -1, \ldots, \dim X \tag{2.2.5}$$

The set of 0-faces $X(0)$ (containing singletons) is identified with the vertex set V in a natural way and hence, 0-simplices $\{v\}$ are abbreviated by v.

The faces of X are naturally partially ordered by inclusion. If $F \subseteq G$, we say that F is a subface *of G.*

Among all subfaces of a given face F, the facets *(with relation symbol $\Subset$) are defined to be the subfaces of co-dimension 1 (that is, the subface has dimension 1 less than the surrounding face):*

$$G \Subset F \quad \iff \quad G \subseteq F \text{ and } \dim G = \dim F - 1 \tag{2.2.6}$$

Moreover, if all the faces that are maximal with respect to this partial order have the same dimension d, the complex X is called pure of dimension d.

Graphs are examples for 1-dimensional simplicial complexes. If there are no isolated points, the graph is a pure complex and the maximal faces correspond to the edges of the graph.

It is sometimes necessary to define orientations on the simplices. To this end, the set of vertices $X(0) \cong V$ of the simplicial complex is equipped with a linear ordering $X(0) = \{v_1, v_2, \ldots, v_n\}$ and $v_1 < v_2 < \cdots < v_n$. Every k-simplex F is a subset of $X(0)$, hence $F = \{v_{i_0}, v_{i_1}, \ldots, v_{i_k}\}$, $i_0 < i_1 < \cdots < i_k \in \{1, \ldots, n\}$. Suppose now that G is a facet of F. Then (as sets) $F \backslash G$ consists of a single element v_{i_j}. Using the index j, the *oriented incidence number* $[F : G]$ can be defined:

Definition 2.2.2 (Oriented Incidence Number)**.** *Let $F = \{v_{i_0}, v_{i_1}, \ldots, v_{i_k}\}$ ($i_0 < i_1 < \cdots < i_k \in \{0, 1, \ldots, n\}$) be a k-simplex of a simplicial complex X and let G be a $(k-1)$-simplex of X. Then the* oriented incidence number $[F : G] \in \{-1, +1\}$ *is defined as:*

$$[F : G] := \begin{cases} (-1)^j & \text{if } F \supseteq G \text{ and } F \backslash G = \{v_{i_j}\} \\ 0 & \text{else} \end{cases} \tag{2.2.7}$$

If X is a graph, these incidence numbers are just the entries of the incidence matrix from definition 2.1.6 on page 12 (with respect to the orientation coming from the chosen ordering of the vertices). Due to the connection (via the Laplacian) to the adjacency matrix, the incidence numbers play an important role in defining higher-dimensional adjacency matrices.

The aim of the incidence numbers is to describe the connection between a simplex and its facets. Another idea is to count how many (higher-/top-dimensional) faces in the complex contain a fixed simplex. This leads to the notion of the degree of a face:

Definition 2.2.3 (Degree of a Face)**.** *Let F be a k-dimensional face of the simplicial complex X, $-1 \leq k \leq \dim X$, and $k \leq l \leq \dim X$. The l-*degree $\deg_l(F)$ *of F is defined as:*

$$\deg_l(F) := |\{G \in X(l) \mid F \subseteq G\}| \tag{2.2.8}$$

The degree $\deg(F)$ *of F is defined as the top-dimensional degree of F:*

$$\deg(F) := \deg_{\dim X}(F) \tag{2.2.9}$$

The most important degrees are the top-dimensional degree and the $(k+1)$-degree.

For an infinite family $(X_i)_{i \in I}$ of simplicial complexes (with I an infinite index set) we can distinguish whether the maximum degrees (in a specified dimension) of all simplicial complexes X_i are bounded uniformly or not. If yes, we are speaking of a family of *bounded degree*. The two most important cases are that of families,

where the maximum degree of $(d-1)$-dimensional faces ($d = \dim X_i$, all complexes are assumed to have the same dimension) is bounded uniformly by a constant not depending on $i \in I$, and families of totally bounded degree, where every possible degree is bounded uniformly by a constant not depending on $i \in I$ and the dimension of the face.

2.2.1 (Reduced) Cohomology of a Simplicial Complex

Associated to a simplicial complex there is its (augmented) chain complex inducing the (reduced) cohomology (with coefficients in a group $\mathbb{G}$) of the complex.

Definition 2.2.4 (Cochains)**.** *Let $\mathbb{G}$ be an (additively written) Abelian group and let X be a simplicial complex.*

In every dimension k, the (Abelian) group of cochains *(with $\mathbb{G}$-coefficients) of the complex X is defined as:*

$$\mathrm{C}^k(X;\mathbb{G}) := \{f : X(k) \to \mathbb{G}\} \qquad k = -1, \dots, \dim X, \tag{2.2.10}$$

together with the group operation of pointwise addition in $\mathbb{G}$:

$$(f+g)(F) := f(F) + g(F) \qquad \forall F \in X(k) \tag{2.2.11}$$

Using the convention $\mathrm{C}^k(X;\mathbb{G}) = \{0\}$ for other choices of k, this definition can be extended to all $k \in \mathbb{Z}$, giving an assignment of a simplicial complex X to graded Abelian groups $\big(\mathrm{C}^k(X;\mathbb{G})\big)_{k\in\mathbb{Z}}$.

In this thesis, we will mostly take the group $\mathbb{G}$ to be $\mathbb{Z}_2$ or $\mathbb{R}$. For the sake of simplifying notation, the group of k-dimensional cochains $\mathrm{C}^k(X;\mathbb{G})$ is often denoted as $\mathrm{C}^k(X)$, if the choice of the coefficients is clear from the context.

For the case $\mathbb{G} = \mathbb{Z}_2$, the cochains have a simpler representation: Every cochain f takes only values in $\mathbb{Z}_2 = \{0,1\}$ (as a function). Hence, it can be identified with a subset of $X(k)$, namely its support $\operatorname{supp} f$. On the other hand, a subset $A \subseteq X(k)$ can be identified with its indicator function $\mathbb{1}_A$:

$$f : X(k) \to \mathbb{Z}_2 \quad \longrightarrow \quad \operatorname{supp} f := \{F \in X(k) |\, f(F) \neq 0\} \tag{2.2.12}$$

$$A \subseteq X(k) \quad \longrightarrow \quad f_A := \mathbb{1}_A \tag{2.2.13}$$

The indicator function $\mathbb{1}_A$ is defined as:

$$\mathbb{1}_A(F) := \begin{cases} 1 & \text{if} \quad F \in A \\ 0 & \text{else} \end{cases} \tag{2.2.14}$$

It is therefore equivalent to speak of cochains (with $\mathbb{Z}_2$-coefficients) as functions or as subsets. Sometimes, it will be more convenient to use one representation, sometimes the other.

The cochain groups of different dimension are connected by the coboundary map δ_k^X:

Definition 2.2.5 (Coboundary Map)**.** *The* coboundary map δ_k^X *of a simplicial complex* X *in dimension* $k \in \mathbb{Z}$ *is defined as:*

$$\delta_k^X : \begin{cases} \mathrm{C}^k(X) & \to \mathrm{C}^{k+1}(X) \\ f & \mapsto \delta_k^X f, \end{cases} \tag{2.2.15}$$

with:

$$\left(\delta_k^X f\right)(F) := \sum_{G \Subset F} [F : G] \cdot f(G) \qquad \forall F \in X(k+1) \tag{2.2.16}$$

The product $[F : G] \cdot f(G)$ *is defined using the natural group action of* $\mathbb{Z}$ *on every Abelian group, i.e.,* $1 \cdot g = g$ *and* $(-1) \cdot g = -g$ *(the inverse of* g *in the additively written Abelian group* $\mathbb{G}$*), for every* $g \in \mathbb{G}$*.*

The coboundary map obviously is a homomorphism between Abelian groups. A short computation also shows that $\delta_{k+1}^X \circ \delta_k^X = 0$. If it is clear from the context, the indices denoting the simplicial complex and/or the dimension will be omitted.

For $\mathbb{G} = \mathbb{Z}_2$ the definition of δ_k^X can be simplified, because the orientations of the simplices are irrelevant as "$-1 = 1$" in $\mathbb{Z}_2$.

The cochain groups together with the coboundary map define the following chain complex (0 denotes the Abelian group $\{0\}$ consisting only of the neutral element):

$$0 \xrightarrow{\delta_{-2}} \underbrace{\mathrm{C}^{-1}(X)}_{\cong \mathbb{G}} \xrightarrow{\delta_{-1}} \mathrm{C}^0(X) \xrightarrow{\delta_0} \mathrm{C}^1(X) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{d-2}} \mathrm{C}^{d-1}(X) \xrightarrow{\delta_{d-1}} \mathrm{C}^d(X) \xrightarrow{\delta_d} 0 \tag{2.2.17}$$

Now the groups of cocycles and coboundaries can be defined:

Definition 2.2.6 (Cocycles and Coboundaries)**.** *The* cocycles $\mathrm{Z}^k(X;\mathbb{G})$ *are defined as the kernel of the* k^{th} *coboundary map* δ_k*, whereas the* coboundaries $\mathrm{B}^k(X;\mathbb{G})$ *are defined as the image of the* $(k-1)^{st}$ *coboundary map* δ_{k-1}*:*

$$\mathrm{Z}^k(X;\mathbb{G}) := \ker \delta_k^X \tag{2.2.18}$$

$$\mathrm{B}^k(X;\mathbb{G}) := \operatorname{im} \delta_{k-1}^X \tag{2.2.19}$$

Both $\mathrm{Z}^k(X;\mathbb{G})$ *and* $\mathrm{B}^k(X;\mathbb{G})$ *are subgroups.*

Since $\delta_{k+1}^X \circ \delta_k^X = 0$, it is clear that $\mathrm{B}^k(X;\mathbb{G}) \subseteq \mathrm{Z}^k(X;\mathbb{G})$. Both groups are Abelian groups, therefore it is well-defined to consider their quotient:

Definition 2.2.7 (Cohomology Groups)**.** *The* k^{th} cohomology group $\mathrm{H}^k(X;\mathbb{G})$ *(with* $\mathbb{G}$*-coefficients) of the simplicial complex* X *is defined as the quotient of the cocycle group by the coboundary group:*

$$\mathrm{H}^k(X;\mathbb{G}) := {}^{\mathrm{Z}^k(X;\mathbb{G})}\!/_{\mathrm{B}^k(X;\mathbb{G})} \tag{2.2.20}$$

In particular, the cohomology group vanishes ($\mathrm{H}^k(X;\mathbb{G}) = 0$) if and only if the cocycles and the coboundaries coincide ($\mathrm{Z}^k(X;\mathbb{G}) = \mathrm{B}^k(X;\mathbb{G})$). Again, if there is no possibility for misinterpretation, the cohomology group will be abbreviated as $\mathrm{H}^k(X)$.

We are working with reduced cohomology, where we include the empty set as (-1)-dimensional face to obtain (-1)-dimensional cochains. In comparison with unreduced cohomology, where the empty set is not included, this only makes a difference in dimension 0.

The elements of $\mathrm{H}^k(X)$ will be denoted as $f+\mathrm{B}^k(X)$, $f \in \mathrm{Z}^k(X)$. The coboundary subgroup defines a projection [.] to the cosets (modulo coboundaries):

$$[.] : \begin{cases} \mathrm{C}^k(X) & \to {\mathrm{C}^k(X)}/{\mathrm{B}^k(X)} \\ f & \mapsto [f] := f + \mathrm{B}^k(X) \end{cases} \tag{2.2.21}$$

In the lowest dimension $k = 0$ this projection is very simple. There is only one face of dimension -1, namely the empty set $\emptyset$ and a cochain can take only one value in $\mathbb{G}$. Hence, the (-1)-cochains can be identified with the group elements of $\mathbb{G}$. As every 0-dimensional face contains the empty set as its only facet, the coboundary cochains are precisely the cochains that are constant, that is:

$$\mathrm{B}^0(X;\mathbb{G}) = \left\{ g \cdot \mathbb{1}_{X(0)} \,|\, g \in \mathbb{G} \right\} \tag{2.2.22}$$

Hence, the projection yields:

$$[f] = \left\{ f + g \cdot \mathbb{1}_{X(0)} \,|\, g \in \mathbb{G} \right\}, \tag{2.2.23}$$

$$\tag{2.2.24}$$

and for the special case $\mathbb{G} = \mathbb{Z}_2$:

$$[f] = \left\{ f, f + \mathbb{1}_{X(0)} \right\} \tag{2.2.25}$$

2.2.2 Norms on Cochains

The last ingredient to be able to define expansion for simplicial complexes is the notion of a norm on the set of cochains $\mathrm{C}^k(X;\mathbb{G})$.

Definition 2.2.8 (Norm of Cochains)**.** *A* norm $\|.\|$ *of cochains for a given simplicial complex X is a function*

$$\|.\| : \bigsqcup_{k=-1}^{\dim X} \mathrm{C}^k(X;\mathbb{G}) \to [0,\infty), \tag{2.2.26}$$

that assigns to a cochain on X of arbitrary dimension a non-negative real number and that satisfies the following conditions:

- *Positive Definiteness:* $\|f\| = 0 \iff f = 0 \quad \forall f \in \mathrm{C}^k(X;\mathbb{G})$
- *Triangle Inequality:* $\|f+g\| \leq \|f\| + \|g\| \quad \forall f,g \in \mathrm{C}^k(X;\mathbb{G})$

For the case $\mathbb{G} = \mathbb{Z}_2$ it is sometimes useful to additionally assume:

- *Monotonicity*: $\operatorname{supp} f \subseteq \operatorname{supp} g \implies \|f\| \leq \|g\| \quad \forall f,g \in \mathrm{C}^k(X;\mathbb{Z}_2)$

Actually, all $\mathbb{Z}_2$-norms that are used in this thesis are monotone since this is a quite natural property.

For some applications it is not necessary to include positive definiteness in the definition (e.g. for topological overlap, see Section 4.1), but sometimes this property is very useful (see for example the proofs in Section 5.2.1).

One may ask, however, whether the term "norm" is appropriate for this concept. Indeed, for the $\mathbb{Z}_2$-case the group of cochains can be viewed as a linear space over the field $\mathbb{Z}_2$ with (obviously) trivial scalar multiplication. Hence, the here defined "norm" satisfies the properties of a (general) norm on linear spaces over $\mathbb{Z}_2$ (positive definiteness, absolute homogeneity under scalar multiplication and triangle inequality), because the second condition is trivially fulfilled and the other two conditions are part of the definition of a norm on cochains.

Over $\mathbb{R}$ or $\mathbb{C}$, we also want to assume absolute homogeneity ($\|\alpha f\| = |\alpha|\,\|f\|$, where α is a number and f a cochain). Indeed, for these choices of coefficients $\mathbb{G}$, the group of cochains forms a (finite dimensional) linear space over $\mathbb{G}$ (with pointwise operations) and a norm on cochains then defines a norm on this linear space. The restriction to the real or complex numbers is necessary to give the term "absolute" a meaning, that is, because $\mathbb{C}$ and $\mathbb{R}$ possess a norm for themselves.

In analogy to linear spaces, norms on cochains also induce norms on quotient spaces (for appropriate equivalence relations). The relevant application is the quotient space of cochains modulo coboundaries, where the resulting norm reads:

$$\|[f]\|_X := \inf_{b \in \mathrm{B}^k(X)} \|f+b\|_X \quad \forall f \in \mathrm{C}^k(X) \tag{2.2.27}$$

In the following the coefficient group $\mathbb{G}$ will be either $\mathbb{R}$ or $\mathbb{Z}_2$.

As if to emphasize the relations to vector space norms, the most important norms for cochains (with $\mathbb{R}$-coefficients) are the ℓ^p norms, which are some of the "standard norms" for function spaces:

Definition 2.2.9 (Weighted ℓ^p-Norms for Cochains with Real Coefficients)**.** *Let p be a real number, $p \in [1,\infty]$, X a simplicial complex and let $w : X \to (0,\infty)$ be a positive weight-function.*

Then the (weighted) ℓ^p-norm *of a cochain $f \in \mathrm{C}^k(X;\mathbb{R})$ is defined as:*

$$\|f\|_p := \begin{cases} \left(\sum_{F \in X(k)} w(F)|f(F)|^p\right)^{\frac{1}{p}} & \text{if } p < \infty \\ \max_{F \in X(k)} w(F)|f(F)| & \text{if } p = \infty \end{cases} \tag{2.2.28}$$

The proof that $\|.\|_p$ really is a norm follows from the fact that the original ℓ^p-norms are norms.

As for function spaces, the case $p = 2$ is special, because the ℓ^2-norm is induced by a scalar product:

$$\|f\|_2 = \sqrt{(f,f)_2}, \tag{2.2.29}$$

where

$$(.,.)_2 : \begin{cases} \bigsqcup_{k=-1}^{\dim X} \mathrm{C}^k(X;\mathbb{R}) \times \mathrm{C}^k(X;\mathbb{R}) & \to \mathbb{R} \\ (f,g) & \mapsto (f,g)_2 := \sum_{F \in X(k)} w(F) f(F) g(F) \end{cases} \tag{2.2.30}$$

denotes the ℓ^2-scalar product. Sometimes it is convenient to extend this to pairs of cochains of different dimensions by just setting the scalar product of different-dimensional cochains to zero. Using this scalar product it will be possible to define adjoint operators on the cochains. This is necessary to define a generalization of Laplacians for simplicial complexes.

Interpreting $\mathbb{Z}_2 = \{0,1\}$ as a subset of $\mathbb{R}$, the ℓ^p norms ($p < \infty$) can be extended to $\mathbb{Z}_2$-cochains. However, these norms are independent of p and just reduce to the (weighted) counting or Hamming norm:

Definition 2.2.10 (Weighted Counting or Hamming Norm)**.** *Let $w : X \to (0,\infty)$ be a positive weight-function. The* weighted counting *or* Hamming norm *is defined as:*

$$\|f\| := \sum_{F \in X(k), f(F)=1} w(F) \quad \forall f \in \mathrm{C}^k(X;\mathbb{Z}_2) \tag{2.2.31}$$

As one easily checks, the weighted counting norm is indeed a norm on cochains that is monotone due to the condition that $w(F) > 0$ for all faces F.

For the trivial weight function $w \equiv 1$, the weighted counting norm reduces to the cardinality of the support, denoted by $|.|$:

$$|f| := |\operatorname{supp} f| \tag{2.2.32}$$

Following [Gar73], [KKL14a] and [EK15], the following weight function turns out to be very useful for proving expansion ($d = \dim X$):

$$wt(F) := \frac{|\{G \in X(d) \mid F \subseteq G\}|}{\binom{d+1}{k+1}|X(k)|} = \frac{\deg(F)}{\binom{d+1}{k+1}|X(k)|} \quad \text{for } F \in X(k), k \in \{-1,0,\dots,d\} \tag{2.2.33}$$

In the top dimension ($k = d$), this is just the uniformly weighted Hamming norm, while on lower dimensions it depends on the complex and especially the top-dimensional degrees of the faces. The constants in the denominator ensure that the cochain $f \equiv 1$ always has norm 1.

2.2.3 Links of a Complex

Just as the neighbours of a vertex in a graph, it is often useful to consider a local view of a simplicial complex. Indeed, the notion of neighbours can be generalized to the notion of the link of a simplex in a simplicial complex. The link is a local view from a simplex:

Definition 2.2.11 (Link of a Simplex)**.** *Let X be a simplicial complex with vertex set $V \cong X(0)$ and let $\sigma \in X$ be an arbitrary simplex.*

The link X_σ *of σ in X is defined as the following simplicial complex:*

$$X_\sigma := \{\tau \subseteq V \backslash \sigma \mid \tau \cup \sigma \in X\} \tag{2.2.34}$$

The set of neighbours $\Gamma(v)$ of a vertex v in the graph case is the link of v in the graph viewed as 1-complex. By definition, any k-simplex $\tilde{\tau}$ that contains σ corresponds bijectively to a $(k - |\sigma|)$-simplex in the link X_σ. The link with respect to the empty set is just the complex itself, while the link with respect to a maximal face is empty. It is also obvious from the definition that the link is again a simplicial complex, which is contained in the original complex (simplicial complexes are closed under taking subsets). Thus, it is possible to define norms on the link, usually denoted by $\|\cdot\|_\sigma$.

Moreover, as cochains are just functions on the simplices, they can be "localized" to links as well:

Definition 2.2.12 (Localization of Cochains)**.** *Let X be a simplicial complex and X_σ the link of a simplex $\sigma \in X$, $0 \leq k \leq \dim X$.*

The localization *with respect to σ is defined as the map:*

$$(\cdot)_\sigma : \begin{cases} \mathrm{C}^k(X) & \to \mathrm{C}^{k-|\sigma|}(X_\sigma) \\ f & \mapsto f_\sigma, \end{cases} \tag{2.2.35}$$

where

$$f_\sigma(\tau) := f(\sigma \cup \tau) \tag{2.2.36}$$

As with $X_\sigma \subseteq X$, the cochains of X_σ can be embedded into the cochains of X in the same dimension by setting the functions to zero for all simplices in $X \backslash X_\sigma$. This embedding is often suppressed if it is clear from the context.

2.3 Rayleigh Quotients and Eigenvalues

Rayleigh quotients are a concept from linear algebra that is very useful for calculating or estimating eigenvalues of matrices and they are used for this purpose in this thesis. They allow connections between the eigenvalues of, for example, the Laplacian matrix or the adjacency matrix of a graph and "combinatorial" properties of the

graph. Moreover, Rayleigh quotients use a variational approach to eigenvalues, that fits well together with the combinatorial definitions. An introduction to Rayleigh quotients can be found, for example, in [HJ12, section 4.2].

We consider here only real, symmetric $n \times n$-matrices. By the spectral theorem they possess n real eigenvalues with eigenvectors that form an orthonormal basis (with respect to the standard scalar product on $\mathbb{R}^n$). For such a matrix, the Rayleigh quotient is defined as:

Definition 2.3.1 (Rayleigh Quotient)**.** *Let A be a real, symmetric $n \times n$-matrix and let $x \in \mathbb{R}^n$ be an arbitrary vector. The* Rayleigh Quotient *$R_A(x)$ is defined (using the standard Euclidean scalar product $(.,.)_2$ and norm $\|\cdot\|_2$) as:*

$$R_A(x) := \frac{(Ax, x)_2}{\|x\|_2^2} \tag{2.3.1}$$

Using the Rayleigh quotients, the eigenvalues of A can be characterized as minima or maxima, respectively, of certain expressions, written down in the theorem by Courant and Fischer:

Theorem 2.3.1 (Courant-Fischer)**.** *Let A be a real, symmetric $n \times n$-matrix with real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Let $1 \leq k \leq n$ and let S denote a subspace of $\mathbb{R}^n$.*

Then the eigenvalues can be calculated by:

$$\lambda_k = \min_{S:\dim S=k} \max_{0\neq x\in S} R_A(x) = \min_{S:\dim S=k} \max_{0\neq x\in S} \frac{(Ax, x)_2}{\|x\|_2^2}, \tag{2.3.2}$$

or

$$\lambda_k = \max_{S:\dim S=n-k+1} \min_{0\neq x\in S} R_A(x) = \max_{S:\dim S=n-k+1} \min_{0\neq x\in S} \frac{(Ax, x)_2}{\|x\|_2^2}. \tag{2.3.3}$$

Proof. See [HJ12, Theorem 4.2.6] for a proof. □

The theorem by Courant and Fischer contains the following characterization of minimum and maximum eigenvalues as a special case, called the theorem of Rayleigh. However, the proof of Courant and Fischer uses Rayleigh's theorem in the proof.

Theorem 2.3.2 (Rayleigh)**.** *Let A be a real, symmetric $n \times n$-matrix and let λ_{min} and λ_{max} be its smallest and largest eigenvalue, respectively.*

Then:

$$\lambda_{min} = \min_{x\neq 0} R_A(x) = \min_{x\neq 0} \frac{(Ax, x)_2}{\|x\|_2^2} \tag{2.3.4}$$

$$\lambda_{max} = \max_{x\neq 0} R_A(x) = \max_{x\neq 0} \frac{(Ax, x)_2}{\|x\|_2^2} \tag{2.3.5}$$

$$\tag{2.3.6}$$

Proof. See [HJ12, Theorem 4.2.2] for a proof. □

For the sake of being self-contained we state an easy corollary of Rayleigh's theorem. It is used to calculate the second smallest eigenvalue and therefore can be (and is) used to calculate the eigenvalue gap (defined in Section 3.2).

Corollary 2.3.1. *Let A be a real, symmetric $n \times n$-matrix with real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Let x_1 be the eigenvector associated with the smallest eigenvalue λ_1.*

Then:

$$\lambda_2 = \min_{0 \neq x \perp x_1} R_A(x) = \min_{0 \neq x \perp x_1} \frac{(Ax, x)_2}{\|x\|_2^2} \tag{2.3.7}$$

$$\tag{2.3.8}$$

Proof. By restricting to the subspace orthogonal to x_1, A can be interpreted as a matrix on $\mathbb{R}^{n-1}$ with smallest eigenvalue λ_2. The characterization now follows from Rayleigh's theorem for this restricted matrix. □

Rem. 2.3.1. We can prove an analogous result for the eigenvalue λ_k, $2 \leq k \leq n$, by considering only vectors x orthogonal to the first $k-1$ eigenvectors $x_1, \ldots, x_{k-1}$ in the minimum.

3 Different Notions of Expansion and their Interplay

Expansion was defined originally as a property of graphs. There are two (at least qualitatively) equivalent approaches to this property: purely graph-theoretical (combinatorial) and via spectral theory. Sometimes a third approach is mentioned, namely the approach via the probabilistic properties of a random walk on the graph. However, this approach is linked tightly to the spectral theory.

As graphs can easily be generalized to high-dimensional (simplicial) complexes, the question arises whether the notion of expansion can be generalized as well. Indeed, this is possible – and in no way unique. A particularly important generalization goes back to Nathan Linial and Roy Meshulam ([LM06]) and Mikhail Gromov ([Gro10]). This generalization is very strong, which motivates alternative definitions that are considerably weaker but sufficient for many purposes. To make things more complex (and also more interesting) high-dimensional expansion also yields geometric properties that are trivial in the lowest dimension. These geometric expansion properties are defined and discussed in Chapter 4.1.

There are several slightly different definitions of expansion for simplicial complexes, that are specially tailored to the known theorems and mirror the various viewpoints. In the following, some of the most important concepts are presented and compared.

3.1 Combinatorial Expansion

Starting with the most intuitive concept, formulated in a purely graph-theoretical notation, we define:

Definition 3.1.1 (Edge Expansion for Graphs)**.** *A graph $G = (V, E)$ is said to be* ϵ-edge expanding*, $\epsilon \geq 0$, if:*

$$\forall S \subseteq V : \quad \epsilon \frac{\min\{|S|, |V \backslash S|\}}{|V|} \leq \frac{|E(S, V \backslash S)|}{|E|} \tag{3.1.1}$$

The edge expansion *of G is defined as the maximum ϵ for which the inequality from above is fulfilled.*

In the definition the factors $|V|$ and $|E|$ are just for scaling and denote a choice of "norm", if we view the graph as a simplicial complex.

Rem. 3.1.1 (Cheeger Constant). Edge expansion as defined here corresponds – if we neglect the scaling factors – to the *Cheeger constant* or *edge-isoperimetric constant* $h(G)$ of the graph G:

$$h(G) := \min \left\{ \frac{|E(S, V \backslash S)|}{|S|} \,\middle|\, S \subseteq V, 0 < |S| \leq \frac{|V|}{2} \right\} \tag{3.1.2}$$

The concept of the Cheeger constant originated in Riemannian geometry (see for example [Bus82], where it is introduced as an isoperimetric constant) and was translated to graph theory. See Section 3.1.3 for further explanations.

What is happening here intuitively is that we pick a subset of the vertices and separate it from the others (cf. Fig. 3.1). Then we look at the edges that run between these two groups of vertices, count them and compare the result with the size of the subset. Hence, a good (meaning high) edge expansion says that there are "many edges", where "many" depends on the size of the subset. Conversely, the expansion constant is limited by the existence of so-called "bottlenecks", that is, two relatively big subsets of the vertices that are connected by relatively few edges.

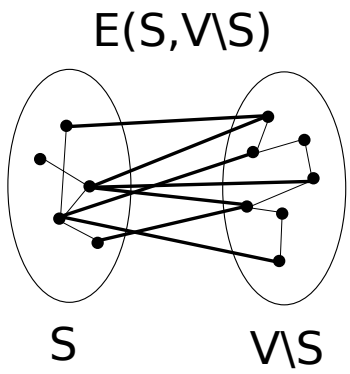

Fig. 3.1: Illustration of Edge Expansion

From the inequality it is clear, that – for a fixed vertex set V – we get good expansion, if the graph is highly connected (i.e. $|E(S, V \backslash S)|$ is big compared to $|E|$).

If the edge expansion of a graph is zero, $|E(S, V \backslash S)|$ must be zero for some $S \subseteq V$, since there are only finitely many choices for S. Keeping the above picture in mind, this means that the graph consists of at least two disconnected components, namely S and $V \backslash S$. On the other hand, if the graph is disconnected, there exists a nonempty proper subset $\emptyset \neq S \subset V$ such that $|E(S, V \backslash S)|$ is equal to zero. Hence, ϵ must be zero as well.

The connection still goes deeper: If the edge expansion of a graph is greater than zero, it takes at least a share of $\epsilon \frac{|S|}{|V|}$ ($|S| \leq \frac{|V|}{2}$) of all edges to be removed to disconnect a given subset S from the rest of the graph. Thus, a good edge expansion already implies "classical" graph-theoretical notions like edge-connectedness (that

is, it answers questions like: How many edges do we have to remove to make the graph disconnected?).

Now let us think about G as a 1-dimensional simplicial complex X. Choosing a subset of the vertices $S \subseteq V$ translates (by the comment in equation 2.2.12 on page 17) to choosing a 0-dimensional cochain $f \in \mathrm{C}^0(X;\mathbb{Z}_2)$. The edges between S and its complement $V\backslash S$ in V are precisely those that are incident to one vertex in $S = \operatorname{supp} f$ and one vertex in $V\backslash S = X(0)\backslash \operatorname{supp} f$. As the coboundary map δ_0^X maps the 0-dimensional chain f to a 1-dimensional chain that evaluates on the edges as $\mathbb{Z}_2$-sum of the values of f at the incident vertices, we have:

$$E(S, V\backslash S) = \operatorname{supp}\left(\delta_0^X f\right) \tag{3.1.3}$$

The defining inequality now translates to:

$$\epsilon \cdot \min\left\{\|f\|_X, \left\|f + \mathbb{1}_{X(0)}\right\|_X\right\} \leq \left\|\delta_0^X f\right\|_X, \tag{3.1.4}$$

where $\|\cdot\|_X$ denotes the Hamming norm of definition 2.2.10 on page 21 with weights $\frac{1}{|X(0)|}$ and $\frac{1}{|X(1)|}$, respectively. The second term $f + \mathbb{1}_{X(0)}$ is just an abbreviation for the cochain that is complementary to f, that is, the cochain that takes the value 0 whenever f takes the value 1 and vice versa. Using the characterization of 0-dimensional coboundaries from equation 2.2.22 on page 19 and the factor norm from equation 2.2.27 on page 20, this yields:

$$\epsilon \, \|[f]\|_X \leq \left\|\delta_0^X f\right\|_X \tag{3.1.5}$$

We can now reformulate definition 3.1.1 on page 25:

Definition 3.1.2 (Edge Expansion for Graphs – Reformulated)**.** *A 1-dimensional simplicial complex X is said to be ϵ-*edge expanding *in dimension 0, $\epsilon \geq 0$, if:*

$$\forall f \in \mathrm{C}^0(X;\mathbb{Z}_2): \quad \epsilon \, \|[f]\|_X \leq \left\|\delta_0^X f\right\|_X, \tag{3.1.6}$$

or, equivalently:

$$\epsilon \leq \min_{f \in \mathrm{C}^0(X;\mathbb{Z}_2)\backslash \mathrm{B}^0(X;\mathbb{Z}_2)} \frac{\left\|\delta_0^X f\right\|_X}{\|[f]\|_X} \tag{3.1.7}$$

This definition motivates a generalization to high-dimensional complexes: coboundary expansion.

3.1.1 Coboundary Expansion

Definition 3.1.3 (Coboundary Expansion)**.** *Let X be a d-dimensional simplicial complex. The* coboundary expansion *parameters $h^k(X)$ of X with respect to a norm $\|\cdot\|_X$ on X are defined as:*

$$h^k(X) := \min_{f \in \mathrm{C}^k(X)\backslash \mathrm{B}^k(X)} \frac{\left\|\delta_k^X f\right\|_X}{\|[f]\|_X} \qquad , k = 0, \ldots, d-1, \tag{3.1.8}$$

or, equivalently, $h^k(X)$ is the maximum constant such that:

$$\forall f \in \mathrm{C}^k(X): \quad h^k(X) \, \|[f]\|_X \leq \left\|\delta_k^X f\right\|_X \tag{3.1.9}$$

This definition depends mainly on the coboundary map and does not use any geometrical or combinatorial property of the simplicial complex. Hence, this definition is also valid for any cochain complex. The choice of coefficients (and of the norm) is relevant for the application, but not for the statement of the definition.

If we forget about the underlying structure of a simplicial complex for some time, the coboundary map just becomes a linear map between two "normed" linear spaces, namely $\mathrm{C}^k(X)$ and $\mathrm{C}^{k+1}(X)$. As there appears the quotient norm $\|[\cdot]\|_X$ in the definition, it is reasonable to consider the induced map (denoted by the same symbol):

$$\delta_k^X : {}^{\mathrm{C}^k(X)}\!/_{\mathrm{B}^k(X)} \to \mathrm{C}^{k+1}(X) \tag{3.1.10}$$

This map is well-defined, because the coboundaries are a subspace of the cocycles which is defined as the kernel of the coboundary map.

For positive $h^k(X) > 0$ equation 3.1.9 immediately implies the injectivity of the induced map. Since the induced map is injective if and only if the quotient is taken by a subspace which contains the kernel, the coboundary group and the cocycle group must coincide. If, on the other hand, these two groups coincide, the minimum in equation 3.1.8 on the preceding page is taken among finitely many, strictly positive values. Hence, the coboundary expansion parameter is positive as well. We have proved:

Lemma 3.1.1. *The coboundary expansion parameter $h^k(X)$ and the kth cohomology group $\mathrm{H}^k(X)$ satisfy, $k = 0, \ldots, d-1$:*

$$h^k(X) > 0 \quad \Longleftrightarrow \quad \mathrm{B}^k(X) = \mathrm{Z}^k(X) \quad \Longleftrightarrow \quad \mathrm{H}^k(X) = 0 \tag{3.1.11}$$

This is the direct analogue of the fact, that the expansion parameter of a graph is positive if and only if the graph is connected. Just as for the graph case, the kth coboundary expansion also gives a lower bound on the number of $(k+1)$-simplices that one has to remove from the complex to get non-vanishing cohomology. However, this bound depends critically on the choice of norm. In higher dimensions vanishing cohomology is quite a strong requirement – a complex with non-zero coboundary expansion necessarily has vanishing cohomology. This leads to the notion of cocycle expansion, which will be described later on.

But first, we will give another interpretation of coboundary expansion. As we have seen, a positive expansion parameter implies that the induced map of the coboundary map is injective. If we restrict its co-domain to the image of the map, that is, the coboundary group of one dimension higher, we get a bijective map δ:

$$\delta : {}^{\mathrm{C}^k(X)}\!/_{\mathrm{B}^k(X)} \to \mathrm{B}^{k+1}(X), \tag{3.1.12}$$

for which we can look at the inverse map δ^{-1}:

$$\delta^{-1} : \mathrm{B}^{k+1}(X) \to {}^{\mathrm{C}^k(X)}\!/_{\mathrm{B}^k(X)}. \tag{3.1.13}$$

As a linear map between finite-dimensional, "normed" linear spaces δ^{-1} is bounded in the following sense[1]:

$$C := \max_{0 \neq g \in \mathrm{B}^{k+1}(X)} \frac{\|\delta^{-1} g\|_X}{\|g\|_X} < \infty \tag{3.1.14}$$

The constant C, called the operator norm of δ^{-1}, is positive, since δ^{-1} is not zero and well-defined. Using the fact that every $g \in \mathrm{B}^{k+1}(X)$ corresponds (via δ) to an equivalence class $[f] \in \mathrm{C}^k(X)/\mathrm{B}^k(X)$, we can compute:

$$\begin{aligned}
C &= \max_{0 \neq g \in \mathrm{B}^{k+1}(X)} \frac{\|\delta^{-1} g\|_X} & (3.1.15)\\
&= \max_{0 \neq [f] \in \mathrm{C}^k(X)/\mathrm{B}^k(X)} \frac{\|[f]\|_X}{\|\delta\ [f]\|_X} \quad (\neq 0) & (3.1.16)\\
&= \left(\min_{0 \neq [f] \in \mathrm{C}^k(X)/\mathrm{B}^k(X)} \frac{\|\delta\ [f]\|_X}{\|[f]\|_X} \right)^{-1} & (3.1.17)\\
&= \frac{1}{h^k(X)} & (3.1.18)
\end{aligned}$$

Hence, the coboundary expansion parameter is nothing else than the inverse of the operator norm of the map δ^{-1}.

3.1.2 Cocycle Expansion

As mentioned above, the (necessary) condition $\mathrm{B}^k(X) = \mathrm{Z}^k(X)$ often is too strong, especially if we want to construct complexes with good expansion properties. The easiest way to weaken this condition is to consider the quotient space by the cocycle group instead of the coboundary group and the induced coboundary map on this quotient space. By definition, the induced map is now injective (fundamental theorem of homomorphisms) and we cannot argue with the injectivity anymore. This leads to the concept of cocycle expansion. Unfortunately, the notation becomes a bit more complex:

Definition 3.1.4 (Cocycle Expansion)**.** *Let X be a d-dimensional simplicial complex. The* cocycle expansion *parameters $h_z^k(X)$ of X with respect to a norm $\|\cdot\|_X$ on X are defined as:*

$$h_z^k(X) := \min_{f \in \mathrm{C}^k(X) \setminus \mathrm{Z}^k(X)} \frac{\left\|\delta_k^X f\right\|_X}{\min_{z \in \mathrm{Z}^k(X)} \|f + z\|_X} \qquad , k = 0, \ldots, d-1 \tag{3.1.19}$$

[1] This definition of boundedness of a linear map normally appears in the theory of infinite-dimensional normed linear spaces, where it is not trivial that this expression is bounded. However, as we are interested in quantitative statements, this value can be interesting.

From the definition it is clear, that $h_z^k(X)$ is always strictly positive. Moreover, the notions of coboundary expansion and cocycle expansion coincide, if the cohomology group vanishes:

Lemma 3.1.2. *If* $\mathrm{H}^k(X) = 0$, *then* $h_z^k(X) = h^k(X) > 0$.

The converse is not true already for the simplest case of graphs and $\mathbb{Z}_2$-coefficients, as the following example shows:

Example 1. Let X be the 1-dimensional simplicial complex defined by, see Fig. 3.2:

$$X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{c,d\}\} \subseteq 2^{\{a,b,c,d\}} \tag{3.1.20}$$

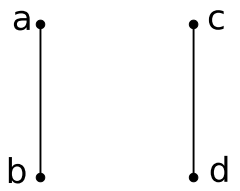

Fig. 3.2: Illustration to example 1

X consists of two connected components, namely $\{a,b\}$ and $\{c,d\}$. A short calculation (and the remarks to equation 2.2.22 on page 19) shows, that the homology does not vanish and there are three non-zero types of cochains modulo cocycles:

$$\mathrm{H}^0(X;\mathbb{Z}_2) = \left\{[0], [\mathbb{1}_{\{a,b\}}]\right\} \cong \mathbb{Z}_2 \neq 0 \tag{3.1.21}$$

$$\mathrm{C}^0(X;\mathbb{Z}_2)\big/\mathrm{Z}^0(X;\mathbb{Z}_2) = \left\{[0], [\mathbb{1}_{\{a\}}], [\mathbb{1}_{\{a,c\}}], [\mathbb{1}_{\{d\}}]\right\} \tag{3.1.22}$$

We choose the cochains with minimal Hamming norm from the three non-zero cosets:

$$f_1 := \mathbb{1}_{\{a\}} \tag{3.1.23}$$

$$f_2 := \mathbb{1}_{\{a,c\}} \tag{3.1.24}$$

$$f_3 := \mathbb{1}_{\{d\}} \tag{3.1.25}$$

The cocycle expansion parameter with respect to the Hamming norm with trivial weights can then be computed by:

$$h_z^0(X) = \min\left\{\frac{\left|\delta_0^X f_1\right|}{|f_1|}, \frac{\left|\delta_0^X f_2\right|}{|f_2|}, \frac{\left|\delta_0^X f_3\right|}{|f_3|}\right\} \tag{3.1.26}$$

$$= \min\left\{\frac{1}{1}, \frac{2}{2}, \frac{1}{1}\right\} = 1 \neq 0 \tag{3.1.27}$$

But the coboundary expansion parameter $h^0(X)$ is zero, because the cohomology group does not vanish.

In the graph case, cocycle expansion is linked (up to some constant factors) to the coboundary expansion of the single connected components (viewed separately as independent graphs), because these are exactly the supports of the different cocycles.

3.1.3 Co-filling or Co-isoperimetric Inequalities

Although it is a weaker property than coboundary expansion, cocycle expansion permits a relatively intuitive interpretation, which reveals astonishing analogies to geometry. To see this connection, it is useful to define the notion of a co-filling or co-isoperimetric inequality in the setting of simplicial complexes (which turns out to be an equivalent concept to cocycle expansion, c.f. lemma 3.1.3 on the next page):

Definition 3.1.5 (Co-filling or Co-isoperimetric Inequality)**.** *Let X be a d-dimensional simplicial complex. X is said to satisfy a* co-filling *or* co-isoperimetric inequality *in dimension k, $0 < k \leq d$, with constant $L_k > 0$, if:*

$$\forall \beta \in \mathrm{B}^k(X)\, \exists \alpha \in \mathrm{C}^{k-1}(X): \quad \delta^X_{k-1}\alpha = \beta \quad \textit{and} \quad \|\alpha\|_X \leq L_k \|\beta\|_X \tag{3.1.28}$$

The first condition (existence of α) is trivial, since this is exactly the definition of the coboundary group. Thus, the second condition is the relevant one, which needs to be checked.

So, what does this definition mean? We take an arbitrary coboundary β and search for a cochain α that (co-)"fills" it, that is, the first is the coboundary of the later. Then we compare the "size" (that is, the norm) of the cochain and its coboundary. The maximal ratio of the size of a cochain (chosen minimal) to the size of its coboundary is the co-isoperimetric constant L_k.

This concept of co-filling or co-isoperimetric inequalities is dual to the concept of filling or isoperimetric inequalities, which is going back to ancient times. Instead of co-boundaries, (geometric) boundaries are considered for filling inequalities. If we take, for example, a closed curve β (or more generally a k-dimensional manifold) in some surrounding Euclidean space, then the "filling" α would correspond to a surface (or a (k+1)-dimensional manifold, respectively) that has the curve β as its (geometric) boundary. Thus, comparing the "sizes" of α and β has the intuitive interpretation of comparing the area of α to the length of β (or the high-dimensional intrinsic volumes). To find the optimal constant for this similar problem is equivalent to finding an answer to the question:

> How do the closed curves look, where the minimal (in terms of area) filling surfaces have the biggest area relative to the length of the boundary curve (i.e. to its perimeter), that is, what are the closed curves that make a high isoperimetric constant in the inequality necessary?

This is the classical isoperimetric problem, also called "Dido's problem", named after the founder of antique Carthage.

In the abstract setting of simplicial complexes this geometric interpretation is somehow lost. However, the structure remains the same, giving a reason why such questions are asked altogether. Sometimes, there are methods of proof which use duality results to switch from cohomology to homology, replacing "coboundary" by "boundary". In homology, the boundary map has the same interpretation as the

geometric boundary. Hence, the intuition and the real structure coincide to some extent. The classical isoperimetric problem is a "continuous" problem, though, meaning that we have to deal with surfaces and area definitions, while in the simplicial complex setting only discrete analogies appear and we just have to "count".

The last link between the (co-)isoperimetric problem and (cocycle) expansion is established with the next lemma, stating that co-isoperimetric inequalities and cocycle expansion are the same, qualitatively and quantitatively.

Lemma 3.1.3. *Let X be a simplicial complex with cocycle expansion parameter $h_z^k(X)$, which satisfies a co-filling inequality in dimension $k+1$, with optimal (meaning minimal) constant $L_{k+1} > 0$, $0 \leq k < d$. Then we have:*

$$h_z^k(X) = \frac{1}{L_{k+1}} \tag{3.1.29}$$

Proof. The proof splits into two parts:

1. "Cocycle expansion yields a co-filling inequality."

   Assume that X has cocycle expansion parameter $h_z^k(X) > 0$ and let $\beta \in \mathrm{B}^{k+1}(X)$ be a given, arbitrary coboundary.

   Thus, there exists a cochain $c \in \mathrm{C}^k(X)$ which "co-fills" β, that is, $\delta_k^X c = \beta$. Now we use the definition of cocycle expansion for the cochain c to get:

$$h_z^k(X) \leq \frac{\left\|\delta_k^X c\right\|_X}{\min_{z \in \mathrm{Z}^k(X)} \|c+z\|_X} = \frac{\|\beta\|_X}{\min_{z \in \mathrm{Z}^k(X)} \|c+z\|_X} \tag{3.1.30}$$

   Let $\alpha = c + z \in \mathrm{C}^k(X)$ be the cochain where the minimum in the denominator is attained. Then this inequality is equivalent to:

$$\|\alpha\|_X \leq \frac{1}{h_z^k(X)} \|\beta\|_X \quad \text{and} \quad \delta_k^X \alpha = \beta, \tag{3.1.31}$$

   which is a co-filling inequality in dimension $k+1$ with constant $\frac{1}{h_z^k(X)}$, because the constant is independent of the chosen β. L_{k+1} is – by assumption – the optimal constant for the inequality to be satisfied, hence:

$$L_{k+1} \leq \frac{1}{h_z^k(X)} \tag{3.1.32}$$

2. "A co-filling inequality implies cocycle expansion."

   We now want to find a lower bound on $h_z^k(X)$ in terms of L_{k+1}. For this reason, let $f \in \mathrm{C}^k(X) \backslash \mathrm{Z}^k(X)$ be a possible candidate for the minimum in the definition of cocycle expansion. $\delta_k^X f$ is a (k+1)-dimensional coboundary

and as such valid for the co-filling inequality. Hence, there exists a cochain $\alpha \in \mathrm{C}^k(X)$, such that:

$$\delta_k^X \alpha = \delta_k^X f \quad \text{and} \quad \|\alpha\|_X \leq L_{k+1} \left\|\delta_k^X f\right\|_X \tag{3.1.33}$$

As the coboundary map is linear, the first equality translates to:

$$\delta_k^X (\alpha - f) = 0, \tag{3.1.34}$$

which is equivalent to α and f being in the same equivalence class modulo cocycles. As a result, we get:

$$\min_{z \in \mathrm{Z}^k(X)} \|f + z\|_X \leq \|f + (\alpha - f)\|_X = \|\alpha\|_X \tag{3.1.35}$$

Putting together equations 3.1.33 and 3.1.35 yields:

$$\min_{z \in \mathrm{Z}^k(X)} \|f + z\|_X \leq L_{k+1} \left\|\delta_k^X f\right\|_X, \tag{3.1.36}$$

or, equivalently,

$$\frac{1}{L_{k+1}} \leq \frac{\left\|\delta_k^X f\right\|_X}{\min_{z \in \mathrm{Z}^k(X)} \|f + z\|_X}. \tag{3.1.37}$$

The left side is independent of the choice of f, so we can replace the right side by the minimum over the choices of f, which is exactly $h_z^k(X)$, hence:

$$\frac{1}{L_{k+1}} \leq h_z^k(X) \tag{3.1.38}$$

The inequalities of both steps together give the claimed equality. □

Due to this lemma, co-filling inequalities do not imply coboundary expansion. This is not surprising, if we look at the proof. Relative to the situation for coboundary expansion, co-filling inequalities "live one level higher", that is we need to first apply the coboundary map to "get to the higher level" and then apply the inequality there. In the first step, we loose any information about cocycles, because they are all mapped to zero, and we cannot restore it.

However, using co-filling inequalities (in combination with vanishing cohomology) is a feasible method to prove coboundary expansion, since they are very intuitive and suggest a relatively straight-forward way to prove. The method of random co-filling described in Section 5.1 uses this idea.

3.1.4 Cosystoles and Cosystolic Expansion

As mentioned above, there is a gap between coboundary and cocycle expansion, if the cohomology does not vanish. Sometimes vanishing cohomology cannot be guaranteed (and is not necessary for the results), but cocycle expansion alone is too weak for some theorems. What seems to be missing here, is some "control" over the discrepancy between cocycles and coboundaries, that is, control over the cocycles that are not coboundaries. This gives rise to the definition of cosystoles:

Definition 3.1.6 (Cosystoles)**.** *Let X be a d-dimensional simplicial complex with non-vanishing cohomology in dimension k, $0 \leq k \leq d-1$. The* cosystole *parameter $syst^k(X)$ with respect to a norm $\|\cdot\|_X$ in dimension k is defined as:*

$$syst^k(X) := \min \left\{ \|f\|_X \mid f \in \mathrm{Z}^k(X) \backslash \mathrm{B}^k(X) \right\} < \infty \tag{3.1.39}$$

*X is said to have ϑ-*large cosystoles *in dimension k, with $\vartheta > 0$ a constant if*

$$syst^k(X) \geq \vartheta, \tag{3.1.40}$$

that is,

$$\forall f \in \mathrm{Z}^k(X) \backslash \mathrm{B}^k(X): \quad \|f\|_X \geq \vartheta \tag{3.1.41}$$

The direct motivation for the name comes from the mathematical field of systolic geometry, where "systole" denotes the minimal length of a closed curve, which is not contractible (cf. [Kat07]).[2] Thus, the definition presented here is the same definition with "length" replaced by an arbitrary norm on cochains.

Bounds on cosystoles are needed for example in the proof of theorem 4.1.3 on page 48 (Gromov's topological overlap theorem) to show that a cocycle is indeed a coboundary by bounding the norm of the cocycle to be smaller than the cosystole parameter.

The combination of cocycle expansion and large cosystoles appears very often, thus, it gets its own name:

Definition 3.1.7 (Cosystolic Expansion)**.** *Let X be a d-dimensional simplicial complex with non-vanishing cohomology in dimension k for $k \in \{0, \ldots, d-1\}$, and cocycle parameter $h_z^k(X)$, which has ϑ-large cosystoles in dimension k, $\vartheta > 0$. Then X is said to possess* cosystolic expansion *or said to be a* cosystolic expander *in dimension k.*

[2] The name "cosystoles" is a combination of "co" (for cohomology) and "systoles", which is a term from medicine describing a part of the cardiac cycle and appears in the name systolic blood pressure (in contrast to diastolic blood pressure). Systolic blood pressure denotes the maximal value of blood pressure during a cardiac cycle and is linked to the difference between maximal and minimal/average blood pressure. Likewise, the cosystole parameter is linked to the difference of norms of cocycles (corresponds to maximal blood pressure) and of coboundaries (corresponds to minimal/average blood pressure), which explains the name "(co)systole".

3.1.5 Families of Expanders

For a given simplicial complex X, there are now several constants regarding and describing expansion, which have to be determined. In some applications the exact quantitative values are important, but sometimes only qualitative statements (e.g. that the constants are strictly positive) are needed. However, all these constants are strictly positive (for the coboundary expansion we need to additionally assume vanishing cohomology). Thus, qualitative statements can only be made, if we consider infinite families of simplicial complexes and want to regard their expansion constants as a whole:

Definition 3.1.8 (Families of Expanders)**.** *An infinite family $(X_i)_{i\in I}$ of d-dimensional simplicial complexes, where I is an arbitrary infinite index set, is called a* family of (coboundary, cocycle, etc.) expanders *in dimension k if there exists a constant $\epsilon > 0$ that uniformly lower-bounds the (coboundary, cocycle, etc.) expansion parameters of the complexes in dimension k.*

Hence, families of expanders consist of simplicial complexes that possess "comparably good" expansion properties. In recent years, the construction of such families was an important topic in research, especially regarding families with additional conditions on the simplicial complexes such as bounded degree (cf. definition 2.2.3 on page 16, asked in [Gro10, DK10]).

3.2 Spectral Expansion

Apart from the combinatorial definition of expansion presented in the last section, there is a definition using "spectral properties" of the graph or simplicial complex in consideration. Again, we want to start with the definition for graphs and then show a way to generalize it to any simplicial complex.

As introduced in Section 2.1 there are (among others) two matrices associated with a graph: the adjacency matrix and the Laplacian. Speaking of matrices it is a natural question to ask for their eigenvalues and what properties of the graph can be deduced from them. Indeed, the graph's expansion is qualitatively equivalent to the so-called eigenvalue gap, as is shown in lemma 3.2.1 on the following page. The eigenvalue gap of the Laplacian and associated with it the spectral expansion of a graph is defined as:

Definition 3.2.1 (Eigenvalue Gap and Spectral Expansion for Graphs)**.** *Let $G = (V, E)$ be a graph on n vertices with Laplacian Δ and let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of Δ.*

The eigenvalue gap *(or* spectral gap*) $\lambda = \lambda(G)$ of G is defined as the distance between the smallest eigenvalue $\lambda_1 = 0$ and the second-smallest eigenvalue λ_2, that is:*

$$\lambda(G) = \lambda_2 - \lambda_1 = \lambda_2 \tag{3.2.1}$$

The graph G is said to be spectrally expanding*, if $\lambda(G) > 0$, and $\lambda(G)$ is called its* spectral expansion *parameter.*

For a d-regular graph, the eigenvalue gap can also be calculated from the spectrum of the adjacency matrix using the connection of the spectra as observed in lemma 2.1.1 on page 13. If we denote the eigenvalues of the adjacency matrix A as $d = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, the eigenvalue gap can be calculated as:

$$\lambda(G) = \mu_1 - \mu_2 = d - \mu_2 \tag{3.2.2}$$

The connection between spectral expansion of a graph and its combinatorial $\mathbb{Z}_2$coboundary expansion parameter $h^0(G)$ is established by the following lemma:

Lemma 3.2.1 (Discrete Cheeger Inequality)**.** *Let $G = (V, E)$ be a d-regular graph, let $\lambda(G)$ be its eigenvalue gap and denote by $h^0(G)$ its coboundary expansion parameter (with respect to the normalized Hamming norm).*

Then the following inequality holds:

$$\frac{\lambda(G)}{d} \leq h^0(G) \leq \sqrt{8\frac{\lambda(G)}{d}} \tag{3.2.3}$$

The name "Cheeger inequality" is originated in Riemannian geometry, where Jeff Cheeger ([Che70], right inequality) and Peter Buser ([Bus82], version of the left inequality) proved inequalities connecting the Cheeger or isoperimetric constant (cf. remark 3.1.1 on page 26) and the eigenvalue gap of the continuous Laplace operator. Following the same analogy as in Section 2.1.1 this result was proved in the discrete setting by Dodziuk ([Dod84, Theorem 2.3]) and by Alon and Milman ([AM85, Alo86]). The lemma above can be generalized to non-regular graphs, cf. [Chu97].

A complete proof of the discrete Cheeger inequality can be found for example in [HLW06, Theorem 4.11, p. 474ff] or in [Lub94, Propositions 4.2.4 and 4.2.5]. In the proof, the eigenvalues of the adjacency matrix are estimated using Rayleigh quotients (cf. Section 2.3), which then can be bounded by the expansion parameters. This utilizes that multiplying the adjacency matrix with 0-1-vectors from both sides can be interpreted as counting the edges between two vertex sets with these "indicator functions" (that is: a vertex is in the set, if the corresponding vector has a one at this position). The lower bound on $h^0(G)$ is given by lemma 3.2.2 on the next page.

The main statement of the Cheeger inequality is that spectral and combinatorial expansion for graphs are qualitatively the same. However, the values of the different expansion parameters need not be equal, which can be problematic in estimates using them. To understand the difference between the two definitions it is useful to look at the estimates on graph properties that arise "naturally".

On the one hand, this is already clear from the definition of the combinatorial expansion parameter: $h^0(G)$ is the biggest constant such that the number of edges between two complementary sets $|E(S, V\backslash S)|$ can be lower-bounded by a *linear* function in $|S|$ with slope $h^0(G)$ (up to some normalizing constants).

The eigenvalue gap, on the other hand, turns out to establish a lower bound by a *quadratic* function in $|S|$! This can be seen by – following [Tre11] – defining the notion of the minimal sparsity of a cut:

Definition 3.2.2 (Minimal Sparsity of a Cut)**.** *Let $G = (V, E)$ be a d-regular graph. The* minimal sparsity (of a cut) *$\phi(G)$ is defined as:*

$$\phi(G) = \min_{\emptyset \neq S \subsetneq V} \frac{|E(S, V \backslash S)|}{\frac{d}{|V|} \cdot |S| \cdot |V \backslash S|} \tag{3.2.4}$$

If G is not regular, then d can be replaced by the average degree (cf. handshake lemma):

$$d = \frac{2|E|}{|V|} \tag{3.2.5}$$

By applying the handshake lemma for d-regular graphs ($2|E| = d|V|$) and by rewriting the eigenvalue gap using Rayleigh quotients, a connection between minimal sparsity, combinatorial expansion and the eigenvalue gap can be drawn. This already gives one direction of the Cheeger inequality, called the "easy one".

Lemma 3.2.2. *Let $G = (V, E)$ be a d-regular graph with minimal sparsity $\phi(G)$, combinatorial expansion parameter $h^0(G)$ and eigenvalue gap $\lambda(G)$. Then we have:*

$$\frac{\lambda(G)}{d} \leq \phi(G) \leq h^0(G) \tag{3.2.6}$$

For a non-regular graph, this chain of inequalities holds, if we replace d by the average degree:

$$d = \frac{2|E|}{|V|} \tag{3.2.7}$$

Indeed, $\phi(G)$ and $\lambda(G)$ can be calculated by minimizing the Rayleigh quotients for the adjacency matrix (up to some normalization, or for the Laplacian matrix) among all 0-1-vectors (indicator functions of subsets of V) or all real vectors, respectively. Hence, the eigenvalue gap is a "continuous relaxation" of the minimal sparsity and the first inequality follows directly.[3]

The first inequality immediately gives the claimed quadratic lower bound on the number of edges:

$$\forall S \subseteq V: \quad |E(S, V \backslash S)| \geq \phi(G) \cdot d \cdot \frac{|S| \cdot |V \backslash S|}{|V|} \geq \lambda(G) \cdot \frac{|S| \cdot |V \backslash S|}{|V|} \tag{3.2.8}$$

Neither of the two lower bounds (linear or quadratic) is better than the other and both have their applications. The following two examples visualize these lower bounds in the case of the complete graph and for some arbitrary graph. In one case, the spectral lower bound holds with equality, in the other case, the combinatorial lower bound is better.

[3] This gives the hint, that the eigenvalue gap is linked to the expansion parameters for cochains with coefficients from $\mathbb{R}$ instead of $\mathbb{Z}$.

Example 2 (Complete Graph)**.** Let $K_n = (V, E)$ be the complete graph on n vertices, that is, the graph containing all possible edges. We have:

$$|V| = n, \quad |E| = \binom{n}{2} = \frac{n(n-1)}{2} \tag{3.2.9}$$

As the graph is complete, every vertex set S of size $|S| = s$ has the same amount of edges to its complement:

$$|E(S, V\backslash S)| = |S| \cdot |V\backslash S| = |S|(|V| - |S|) = s(n-s) \tag{3.2.10}$$

The combinatorial expansion parameter hence can be calculated by:

$$h^0(K_n) = \frac{|V|}{|E|} \min_{\substack{\emptyset \neq S \subsetneq V \\ |S| \leq \frac{|V|}{2}}} \frac{|E(S, V\backslash S)|}{|S|} \tag{3.2.11}$$

$$= \frac{2}{n-1} \min_{0 < s \leq \frac{n}{2}} \frac{s(n-s)}{s} \tag{3.2.12}$$

$$= \begin{cases} \frac{n}{n-1} & \text{if } n \in 2\mathbb{N} \\ \frac{n+1}{n-1} & \text{if } n \in 2\mathbb{N}+1 \end{cases} \tag{3.2.13}$$

A short calculation shows that $\lambda(K_n) = n$, hence the spectral lower bound turns out to be exact ($|S| = s$):

$$s(n-s) = |E(S, V\backslash S)| \geq n\frac{s(n-s)}{n} = s(n-s) \tag{3.2.14}$$

Fig. 3.3 on the facing page illustrates the combinatorial lower bound.

Example 3. This example shows the lower bounds for a randomly generated graph. Fig. 3.4 on the next page shows the graph and illustrates the expansion properties. As in this case the explicit choice of the subset S is relevant for the amount of edges, here $|E(S, V\backslash S)|$ is minimized over all subsets of a given size. On the x-axis the size of the subset is plotted. The subset of vertices, where the minimum for the calculation of $h^0(G)$ is taken, is marked red in the illustration of the graph.

As we can see here, the combinatorial lower bound is better than the spectral one for all choices of subsets.

3.2.1 High-Dimensional Laplacians

To generalize the definition of spectral expansion or eigenvalue gap to higher dimensions it is first necessary to define "high-dimensional Laplacians". In this thesis we will follow the construction sketched in [GW16].

In the case of graphs the eigenvalues of the Laplacian could be linked to $\mathbb{Z}_2$-expansion using Rayleigh quotients. As remarked at that point, in the calculation of the Rayleigh quotients all real vectors are considered, while for $\mathbb{Z}_2$-expansion only

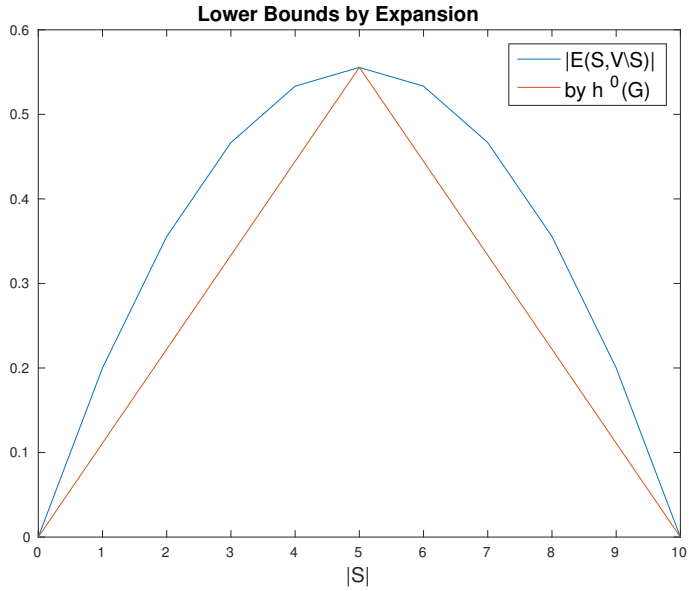

Fig. 3.3: Illustration of the combinatorial lower bound for the complete graph K_n ($n = 10$). The size of the subset S is plotted on the x-axis, the relative size of the set of edges between the set and its complement and the lower bounds are plotted on the y-axis. The spectral lower bound coincides with the number of edges here.

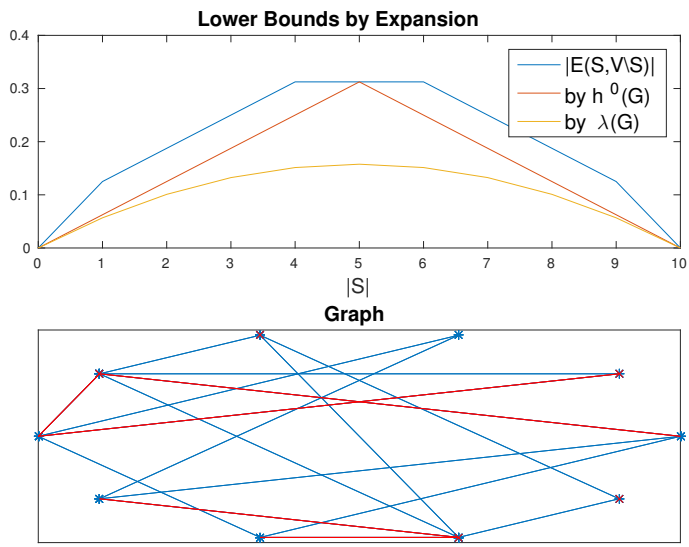

Fig. 3.4: Illustration of the lower bounds for a specific randomly generated graph. The size of the subset S is plotted on the x-axis, the relative size of the set of edges between the set and its complement and the lower bounds are plotted on the y-axis.

cochains with $\mathbb{Z}_2$-coefficients (corresponding to subsets of vertices) are important. It is therefore reasonable to work with coefficients in $\mathbb{R}$ to define high-dimensional eigenvalue gaps. Moreover, the Rayleigh quotients need a scalar product, thus the (weighted) ℓ^2-norm is the preferable choice. Hence, in this chapter, we will work with the Hilbert spaces $\left(\mathrm{C}^k(X;\mathbb{R}), (.,.)_2\right)$.

Graph Laplacians were introduced using discrete analogies of gradient and divergence. However, it is not obvious, how these should look like in higher dimensions. Here the approach for the matrix representation is easier. Recall, that the matrix representation of the graph Laplacian is $K^T K$ with the incidence matrix K. The incidence matrix can now be generalized using the oriented incidence numbers:

Definition 3.2.3 (High-Dimensional Incidence Matrix)**.** *Let X be a d-dimensional simplicial complex and let $0 \leq k < d$ be a natural number. The* k-dimensional incidence matrix *K_k is defined as the $|X(k+1)| \times |X(k)|$-matrix indexed by the k- and $(k+1)$-faces of X with the entries:*

$$(K_k)_{F,G} := [F : G] \tag{3.2.15}$$

This definition depends on a choice of orientation/ordering of the vertices (as the incidence numbers do). K_k can be interpreted as the representation matrix of a linear map between $\mathrm{C}^k(X;\mathbb{R})$ and $\mathrm{C}^{k+1}(X;\mathbb{R})$ with respect to the (standard) basis $\left(\mathbb{1}_{\{F\}}\right)_{F\in X(k)}$ of indicator functions. Equation 2.2.16 on page 18 shows that this map is exactly the kth coboundary map δ_k^X.

Using the incidence matrix the Laplacian could be defined just as for graphs as $K_k^T K_k$. However, as we allow the weighted ℓ^2-scalar product we have to be more general here. In the unweighted Hilbert space, K_k^T is just the matrix representation of the Hilbert space adjoint of K_k with respect to the standard ℓ^2-scalar product, but in the weighted case we have to take the adjoint with respect to the weighted scalar product. That is, we need to find a linear map δ_k^* that satisfies:

$$(\delta_k f, g)_2 = (f, \delta_k^* g)_2 \qquad \forall f \in \mathrm{C}^k(X;\mathbb{R}), g \in \mathrm{C}^{k+1}(X;\mathbb{R}) \tag{3.2.16}$$

δ_k^* can be calculated evaluating this equation for example by substituting f and g by indicator functions to get, for any $f \in \mathrm{C}^{k+1}(X;\mathbb{R})$ and $G \in X(k)$:

$$\left(\delta_k^* f\right)(G) = \sum_{F\in X(k+1)} \frac{w(F)}{w(G)} [F : G]\, f(F) \tag{3.2.17}$$

In the graph case with trivial weights this is just the definition of divergence, as expected. In the homology theory of simplicial complexes this map is also known as the boundary map ∂_{k+1} (with trivial weights, after identifying chains and cochains in a natural way).

The high-dimensional Laplacian can now be defined in the same manner as for graphs:

Definition 3.2.4 (High-Dimensional Laplacians)**.** *Let X be a d-dimensional simplicial complex and let $0 \leq k < d$ be a natural number.*

The up-Laplacian Δ_k^{up}*, the* down-Laplacian Δ_k^{down} *and the* Laplacian Δ_k *of X are defined as:*

$$\Delta_k^{up} := \delta_k^* \delta_k \tag{3.2.18}$$

$$\Delta_k^{down} := \delta_{k-1} \delta_{k-1}^* \tag{3.2.19}$$

$$\Delta_k := \Delta_k^{up} + \Delta_k^{down} \tag{3.2.20}$$

All Laplacians are linear maps from $\mathrm{C}^k(X;\mathbb{R})$ to itself, where the up-Laplacian first goes "up" in dimension and then "down" and the down-Laplacian first goes "down" and then "up". The up-Laplacian is the most important Laplacian for our purposes and generalizes the definition of Laplacians for graphs. In this thesis, mainly the up-Laplacian will be used.

From the definition and the properties of adjoint maps we get the following properties of the Laplacians:

Lemma 3.2.3 (Properties of the Laplacians)**.** *Let X be a simplicial complex with up-Laplacian Δ_k^{up}. Then:*

(i) Δ_k^{up} *is self-adjoint (i.e.* $(\Delta_k^{up})^* = \Delta_k^{up}$*).*

(ii) Δ_k^{up} *is positive semidefinite (i.e.* $(\Delta_k^{up} f, f)_2 \geq 0$*, for all f).*

(iii) $\mathrm{B}^k(X;\mathbb{R}) \subseteq \mathrm{Z}^k(X;\mathbb{R}) \subseteq \ker \Delta_k^{up}$*, hence 0 is an eigenvalue of* Δ_k^{up}*,* $0 \in \sigma(\Delta_k^{up})$*.*

Proof. (i) and (ii) follow directly from the definition and (iii) follows from the fact that already $\delta_k f = 0$, hence $\Delta_k^{\mathrm{up}} f = \delta_k^* \delta_k f = 0$ for all $f \in \mathrm{Z}^k(X;\mathbb{R}) = \ker \delta_k$. □

The up-Laplacian (and similarly the down-Laplacian) for trivial weights can be calculated directly from the definition, which yields a representation $\Delta_k^{\mathrm{up}} \cong D_k - A_k$ with a degree matrix D_k and a high-dimensional analogue of the adjacency matrix A_k. Starting with an arbitrary cochain $f \in \mathrm{C}^k(X;\mathbb{R})$ and a face $G \in X(k)$, we compute:

$$(\Delta_k^{\mathrm{up}} f)(G) = \sum_{F \in X(k+1)} [F : G] \, (\delta_k f)(F) \tag{3.2.21}$$

$$= \sum_{F \in X(k+1)} [F : G] \left(\sum_{H \in X(k)} [F : H] \, f(H) \right) \tag{3.2.22}$$

$$= \left(\sum_{F \in X(k+1)} [F : G] \, [F : G] \right) f(G) \tag{3.2.23}$$

$$+ \sum_{G \neq H \in X(k)} \sum_{F \in X(k+1)} [F : G] \, [F : H] \, f(H) \tag{3.2.24}$$

The incidence number $[F:G]$ is zero if and only if the face G is no facet of F. Thus, the first sum counts just the faces F that contain G and is therefore equal to $\deg_{k+1}(G)$. The second term looks more complicated, but the double sum reduces to a single sum, because:

$$[F:G]\,[F:H] \neq 0 \implies F = G \cup \{u\} = H \cup \{v\}, u \neq v \in X(0) \tag{3.2.25}$$

$$\implies G \cap H = F \backslash \{u,v\}, F = G \cup H \tag{3.2.26}$$

As a consequence, the only term in the sum over all F that does not vanish is the term $F = G \cup H$. The set $G \cup H$ need not be a face in X, so we need to condition on that:

$$(\Delta_k^{\mathrm{up}} f)(G) = \deg_{k+1}(G) f(G) + \sum_{\substack{G \neq H \in X(k) \\ G \cap H \in X(k-1) \\ G \cup H \in X(k+1)}} [G \cup H : G]\,[G \cup H : H]\, f(H) \tag{3.2.27}$$

From this equation it is clear how a degree matrix D_k and an adjacency matrix A_k has to look like to guarantee $\Delta_k^{\mathrm{up}} \cong D_k - A_k$:

Definition 3.2.5 (High-Dimensional Degree and Adjacency Matrix)**.** *Let X be a d-dimensional simplicial complex and let $0 \leq k < d$ be a natural number.*

The degree matrix *D_k of dimension k is defined as the $|X(k)| \times |X(k)|$-diagonal matrix indexed by the k-faces of the complex with the entries ($F \in X(k)$):*

$$(D_k)_{F,F} = \deg_{k+1}(F) \tag{3.2.28}$$

The adjacency matrix *A_k of dimension k is defined as the $|X(k)| \times |X(k)|$-matrix indexed by the k-faces of the complex with the entries ($G, H \in X(k)$):*

$$(A_k)_{G,H} = \begin{cases} -[G \cup H : G]\,[G \cup H : H] & \text{if } G \cap H \in X(k-1), G \cup H \in X(k+1) \\ 0 & \text{else} \end{cases} \tag{3.2.29}$$

To get back to expansion, we now want to define the spectral gap for the up-Laplacian:

Definition 3.2.6 (Trivial Eigenvalues and Spectral Gap of up-Laplacian and Spectral Expansion)**.** *Let X be a d-dimensional simplicial complex with up-Laplacian Δ_k^{up}, $0 \leq k < d$, and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{|X(k)|}$ be the eigenvalues of Δ_k^{up}.*

The eigenvectors of Δ_k^{up} to the eigenvalue 0 that belong to $\mathrm{B}^k(X;\mathbb{R})$ are called trivial eigenvectors*.*

The eigenvalue gap *(or* spectral gap*) $\lambda^k = \lambda^k(X)$ of X is defined as the smallest non-trivial eigenvalue $\lambda_{\dim \mathrm{B}^k(X;\mathbb{R})+1}$:*

$$\lambda^k(X) := \lambda_{\dim \mathrm{B}^k(X;\mathbb{R})+1} \tag{3.2.30}$$

If $\lambda^k(X) > 0$, the complex X is said to be spectrally expanding *in dimension k with* spectral expansion *parameter $\lambda^k(X)$.*

The spectral gap $\lambda^k(X)$ can be calculated using the Rayleigh quotient:

$$\lambda^k(X) = \min_{f \perp \mathrm{B}^k(X;\mathbb{R})} \frac{(\Delta_k^{\mathrm{up}} f, f)_2}{(f, f)_2}, \tag{3.2.31}$$

where the minimum is taken among all cochains that are orthogonal (with respect to the weighted ℓ^2-scalar product) to the coboundaries. From this equation the connection to $\mathbb{R}$-coboundary expansion can be seen. Using the property of an adjoint map we have:

$$(\Delta_k^{\mathrm{up}} f, f)_2 = (\delta_k^* \delta_k f, f)_2 = (\delta_k f, \delta_k f)_2 = \|\delta_k f\|_2^2 \tag{3.2.32}$$

Moreover, the condition $f \perp \mathrm{B}^k(x;\mathbb{R})$ immediately implies that $\|f\|_2 = \|[f]\|_2$, because in the Hilbert space case the projection to equivalence classes modulo coboundaries becomes the orthogonal projection to the orthogonal complement of the subspace of coboundaries. As a result, we have the equation:

$$\lambda^k(X) = \min_{f \perp \mathrm{B}^k(X;\mathbb{R})} \frac{\|\delta_k f\|_2^2}{\|f\|_2^2} \tag{3.2.33}$$

$$= \min_{f \in \mathrm{C}^k(X;\mathbb{R}) \setminus \mathrm{B}^k(X;\mathbb{R})} \frac{\|\delta_k f\|_2^2}{\|[f]\|_2^2} \tag{3.2.34}$$

$$= h_{\mathbb{R}}^k(X)^2, \tag{3.2.35}$$

where $h_{\mathbb{R}}^k(X)$ denotes the coboundary expansion parameter with respect to $\mathbb{R}$-coefficients and the weighted ℓ^2-norm.

This also shines a different light on spectral expansion for graphs and the Cheeger inequality there: The Cheeger inequality shows the (qualitative) equivalence of coboundary expansion with $\mathbb{Z}$-coefficients and Hamming norm to coboundary expansion with $\mathbb{R}$-coefficients and (weighted) ℓ^2-norm.

4 Applications of Expansion

High-dimensional expansion allows many fields of applications, which makes it very interesting for research. Already the historical ancestor "graph expansion" could be applied in many fields like algorithms and communication networks (cf. [HLW06] for a survey). High-dimensional expansion extends the applications further, leading to a rich field for research. In this thesis we will only survey some of the applications to give the reader a feeling of what is possible. We will give an introduction of the geometric properties of expansion established by Gromov's theorem, show the connection between expansion and property testing and give a hint how expansion can be used in quantum computing as well as to construct good error correcting codes.

For further information we refer to the literature cited in the corresponding subsections.

4.1 Topological Overlapping – Gromov's Theorem

As already remarked in the last chapter 3, high-dimensional expansion yields geometric properties of the simplicial complex. Thus, the first application of expansion that we want to discuss is the connection between combinatorial expansion and the geometry of the complex.

We start with an arbitrary simplicial (or polyhedral cell) complex X. Although it is defined as an abstract combinatorial structure, a simplicial complex can be identified with a topological space, called its geometric realization. Indeed, a k-dimensional face can be viewed as the convex hull of $(k+1)$ points in an Euclidean space $\mathbb{R}^N$, that is, the vertex set of the simplicial complex is identified with points in the Euclidean space and subsets of the vertex set correspond to the convex hulls of these subsets. If we take for example a 2-dimensional simplex σ, then we have a subset of three points $\sigma = \{v_0, v_1, v_2\}$ that lie for example in $\mathbb{R}^2$. This corresponds to a triangle like in Fig. 4.1 on the next page. The facets of σ are the three subsets consisting of two of the vertices, corresponding to the edges on the boundary of the triangle, and likewise for the vertices of the triangle.

If we pick the vertices from an Euclidean space of sufficiently large dimension (one dimension for every vertex in $X(0)$ is enough), we can guarantee that there are no intersections between different disjoint simplices. As a subset of $\mathbb{R}^N$, the geometrical realization of the simplicial complex is equipped with the subspace topology of the Euclidean topology and hence a topological space on its own. An *embedding* of the simplicial complex into an Euclidean space $\mathbb{R}^d$ is a continuous map from the

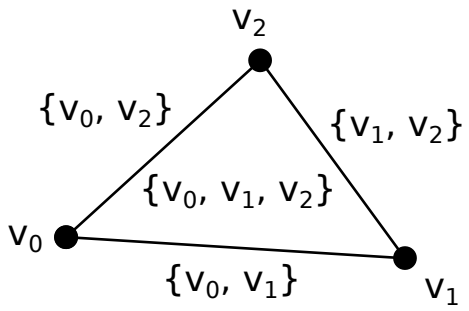

Fig. 4.1: Illustration of a 2-dimensional simplex

geometric realization (all geometric realizations are homeomorphic) to $\mathbb{R}^d$ that is a homeomorphism (i.e. the map is bijective and its inverse map is also continuous) onto its image. As a finite simplicial complex always has a compact geometric realization, the image under a continuous map is also compact. Moreover the Euclidean space is a Hausdorff space, hence if we assume injectivity of the map, then it is already a homeomorphism.

The natural question in this context now is the following:

> For a given simplicial complex X and a given Euclidean space $\mathbb{R}^d$, is there an embedding of X into $\mathbb{R}^d$? If not, what are the obstructions?

This general question can be specialized to:

> Is there an embedding of a given d-dimensional simplicial complex into $\mathbb{R}^d$?

As the question before, this question can be very difficult and has inspired a lot of research. In this setting, it is helpful to consider the following, slightly different question:

> For a given continuous map from the geometric realization of a d-dimensional simplicial complex X to R^d, is the map injective? If it is not injective, how far is it from being injective?

This leads to the notion of *intersection, heavily covered or overlap points*. These are points that have a preimage that consists of more than one point and thus the existence of such points can be used for quantitative non-embeddability results.

Simplifying the problem further, we only consider affine maps at first, that is, we consider only functions that map the vertices of the simplicial complex into $\mathbb{R}^d$ and interpret the convex hulls of the images of a subset of vertices as the image of the corresponding face. For the complete simplicial complex (consisting of all possible subsets of the vertex set) we can forget about the simplicial complex and only consider points in $\mathbb{R}^d$ and their convex hulls.

The following theorem of Endre Boros and Zoltan Füredi solves this problem for points in the plane:

Theorem 4.1.1 (Boros, Füredi, [BF84]). *Let $P = \{P_1, P_2, \ldots, P_n\}$ be a family of n points in the plane $\mathbb{R}^2$ such that no three of them lie on a line (hence, every three of them form a triangle). These points form $\binom{n}{3}$ triangles in the plane.*

Then there exists a point $X \in \mathbb{R}^2$ that lies in the interior of at least

$$\frac{n^3}{27} + \mathcal{O}(n^2) = \frac{2}{9}\binom{n}{3} + \mathcal{O}(n^2) \tag{4.1.1}$$

of the triangles. We say that X is covered *by these triangles.*

The proof uses elementary geometric arguments. It can be found in [BF84].

Formulated in terms of simplicial complexes, the theorem implies that the 2-skeleton $X^{(2)}$ of the complete complex on n vertices cannot be affinely embedded into $\mathbb{R}^2$.

The theorem of Boros and Füredi was generalized to higher dimensions by Imre Bárány:

Theorem 4.1.2 (Bárány, [Bá82]). *Let $P = \{P_1, P_2, \ldots, P_n\}$ be a family of n points in $\mathbb{R}^d$. These points form $\binom{n}{d+1}$ (maybe degenerated) d-simplices in $\mathbb{R}^d$.*

Then there exists a point $X \in \mathbb{R}^d$ that lies in at least

$$\frac{1}{(d+1)^{d+1}}\binom{n}{d+1} + \mathcal{O}(n^d) \tag{4.1.2}$$

of the d-simplices.

The proof uses the theorems of Carathéodory and of Tverberg about convex hulls of points in $\mathbb{R}^d$. It can be found in [Bá82].

Again, there is a point that is covered by a fixed fraction of all d-simplices. Thus, the d-skeleton of the complete complex cannot be embedded into $\mathbb{R}^d$. But what about other complexes that are sparser? The answer to the question depends critically on the complex, of course. A single triangle, for example, can certainly be embedded into the plane, whereas the skeletons of complete complexes definitely cannot. To distinguish between these cases, the notion of geometric expansion was introduced:

Definition 4.1.1 (Geometric Expansion). *Let X be a d-dimensional simplicial complex.*

X is said to be geometrically ϵ-expanding *or* geometrically ϵ-overlapping*, if for every* ***affine*** *map f from the geometric realization of X into $\mathbb{R}^d$ there is a point $x \in \mathbb{R}^d$ that is in the image of at least an ϵ-fraction of all d-simplices of X, that is:*

$$|\{\sigma \in X(d) \,|\, x \in f(\sigma)\}| \geq \epsilon\,|X(d)| \tag{4.1.3}$$

($f(\sigma)$ denotes the image of the subset of the geometric realization that corresponds to the face σ.)

Following this definition, the skeletons of complete complexes possess "good" geometric expansion properties, while the triangle for example does not. Logically, the name "geometric expansion" seems quite arbitrary as the connection to high-dimensional expansion as introduced in Chapter 3 is far from obvious. Indeed, the link is established by Gromov's topological overlap theorem. In the proof of this theorem a new method is used that utilizes coboundary expansion to prove the stronger property of topological expansion:

Definition 4.1.2 (Topological Expansion)**.** *Let X be a d-dimensional simplicial complex.*

X is said to be topologically ϵ-expanding *or* topologically ϵ-overlapping*, if for every* ***continuous*** *map f from the geometric realization of X into $\mathbb{R}^d$ there is a point $x \in \mathbb{R}^d$ that is in the image of at least an ϵ-fraction of all d-simplices of X, that is:*

$$|\{\sigma \in X(d) \,|\, x \in f(\sigma)\}| \geq \epsilon \, |X(d)| \tag{4.1.4}$$

($f(\sigma)$ denotes the image of the subset of the geometric realization that corresponds to the face σ.)

Of course, every affine map is continuous but not vice versa. Hence, to be topologically expanding is a much stronger condition. By the following theorem it is a consequence of high-dimensional expansion! The next theorem is the reason for the name "topological expansion" and a big motivation to study coboundary/cosystolic expansion:

Theorem 4.1.3 (Gromov, [Gro10], [DKW15])**.** *Let X be a d-dimensional simplicial complex with normalized Hamming norm $\|\cdot\|_X$ on the cochains.*

Let $L, \vartheta > 0$ be constants. Then there exists a constant $\epsilon_0 = \epsilon_0(L, \vartheta, d)$ used in the statement of the theorem for technical purposes.

Assume X fulfills the following properties:

- *X is a cosystolic expander with respect to $\|\cdot\|_X$ in every dimension $0 \leq k \leq d$, that is, it has ϑ-large cosystoles in every dimension and fulfills a co-filling inequality with constant L.*
- *X is* locally ϵ-sparse *with $\epsilon \leq \epsilon_0$, that is, for every face $\tau \in X$ and every $0 \leq k \leq d$ we have:*

$$|\{\sigma \in X(k) \,|\, \tau \cap \sigma \neq \emptyset\}| \leq \epsilon \, |X(k)| \tag{4.1.5}$$

Under this conditions there exists a constant $\mu = \mu(L, \vartheta, d, \epsilon) > 0$ such that for every continuous map f from the geometric realization of X to $\mathbb{R}^d$ there is a point $p \in \mathbb{R}^d$ that is "overlapped" by at least a μ-fraction of the d-faces, that is:

$$|\{\sigma \in X(d) \,|\, p \in f(\sigma)\}| \geq \mu \, |X(d)| \tag{4.1.6}$$

(As before, $f(\sigma)$ denotes the image of the subset of the geometric realization that corresponds to the face σ.)

A proof of this theorem would need several new notions to be introduced and leads too far from the core of this thesis. It will thus be omitted. A proof can be found in [DKW15].

The theorem stated here is not the most general form of Gromov's overlap theorem. Indeed, X need only be a general cell complex with an arbitrary (normalized) norm on the cochains and the co-domain of the map under consideration may be any compact connected d-dimensional piecewise-linear manifold. The theorem then yields a result in terms of the chosen norm (in our case the norm is the normalized Hamming norm). If the chosen norm is an ℓ^p-norm, the result may not give estimates on the number of simplices overlapping the point p, but it still gives the qualitative statement that the complex cannot be embedded into $\mathbb{R}^d$.

Gromov's theorem is insofar surprising as it establishes a connection between the (purely) combinatorial concept of coboundary/cosystolic expansion and a geometric property like topological overlapping. However, it is plausible that the structure of the complex has a strong influence on its embeddability and coboundary expansion seems to be a good/the right measure to quantify this behaviour.

Indeed, Gromov's theorem can be checked (and is trivial) for the lowest dimension, namely for the embeddability of a 1-dimensional complex (a graph) into the real line:

Example 4 (Topological Overlapping in the Graph Case)**.** Let $G = (V, E)$ be an arbitrary connected graph and f an arbitrary map from the geometric realization of G into the real line as illustrated in Fig. 4.2 on the following page. Now consider the images of the vertices under f, $\{f(v_i) \,|\, v_i \in V\}$, and choose an arbitrary point $p \in \mathbb{R}\backslash\{f(v_i) \,|\, v_i \in V\}$ such that there is at least one point $f(v_i)$ left from (smaller than) p and at least one point right from (larger than) p. The point p now partitions $f(V) = \{f(v_i) \,|\, v_i \in V\}$ into two subsets $f(S)$ and $f(\bar{S})$, $S, \bar{S} \subseteq V$, namely the set of points smaller than p and that larger than p, respectively. We thus have:

$$V = S \sqcup \bar{S} \quad \text{and} \quad f(V) = f(S) \sqcup f(\bar{S}) \tag{4.1.7}$$

The number of (images of) edges that overlap p is now – due to the intermediate value theorem – greater or equal to the number of edges $|E(S, \bar{S})|$ between S and $\bar{S}$. Indeed, any edge that is incident to a vertex from S and one from $\bar{S}$ gives rise to a path (the image of the edge under f) in $\mathbb{R}$ that starts left from p and ends right from p. By the intermediate value theorem the point p must lie on the path and thus in the image of the edge. Moreover, there may be other edges inside S or $\bar{S}$ that "hit" p, thus greater or equal. Hence, we have:

$$|\{e \in E \,|\, p \in f(e)\}| \geq |E(S, \bar{S})| \tag{4.1.8}$$

The right-hand side of the inequality can now be estimated by the coboundary expansion $h^0(G)$ (with unnormalized Hamming norm) of the graph and the minimal size of S and $\bar{S}$:

$$|\{e \in E \,|\, p \in f(e)\}| \geq |E(S, \bar{S})| \geq h^0(G) \min\{|S|, |\bar{S}|\} \tag{4.1.9}$$

If we now pick a point $p \in \mathbb{R}$ such that $|S| \approx \frac{|V|}{2} \approx |\bar{S}|$, we get one of the points promised by Gromov's theorem with the constant:

$$\mu \approx \frac{h^0(G)}{2} \frac{|V|}{|E|} \tag{4.1.10}$$

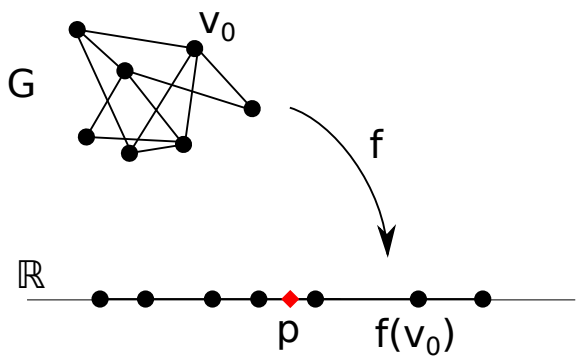

Fig. 4.2: Illustration of topological overlapping in the case of a graph (example 4 on the preceding page)

Due to Gromov's theorem, there are great efforts to find families of bounded degree cosystolic expanders, because these families are automatically topologically expanding. The question whether there exist infinite families of topological expanders was posed by Gromov in [Gro10] and (partially) answered in the work of Tali Kaufman, David Kazhdan, Alexander Lubotzky and Shai Evra ([KKL14a, KKL14b, EK15]). In [LM13] and [LLR15] other examples are given by Alexander Lubotzky, Roy Meshulam, Zur Luria and Ron Rosenthal.

4.2 Property Testing

Beside the geometric/topological application, high-dimensional expansion also appears in the theory of property testing (as observed in [KL13]). Indeed, it can be shown that $\mathbb{Z}_2$-coboundary expansion is just a reformulation of testability of the subspace of coboundaries inside the space of cochains. In this section we follow the discussion in [KL13] on this topic.

In property testing we have given an element from a certain space and want to develop a (random) test that asserts whether the element is in a given subspace or not. If the element is contained in the subspace, then we want to give the correct answer "yes", while in the other case we want to give the answer "no" with a certain probability. This probability should be bounded or estimated in terms of the "distance" of the element to the subspace, that is, we allow more errors if we are close to the subspace and less if we are far away. In this thesis we only want to consider linear bounds as these are the cases linked to expansion.

The easiest test fulfilling this property is to take every element of the subspace and compare it with the given element. However, this will not be very efficient especially for big subspaces. Thus, a very important quality attribute of a test is its complexity, that is, how expensive it is to perform the test. This leads to a compromise, as low complexity can sometimes only be obtained by allowing bad error probabilities.

Altogether, this leads to the following definition of testability (cf. [KL13]):

Definition 4.2.1 (Testability)**.** *Let A be a finite set of "values", let $W_n \subseteq A^n$ be a subset of "tuples of values" and let $P_n \subseteq W_n$ be the subset of "tuples possessing a property". We want to test the property, that is, whether an element $x \in W_n$ is element of P_n or not. Let $q \in \mathbb{N}$ and $\epsilon > 0$ be constants independent of n.*

This property is said to be (q, ϵ)-testable*, if there exists a (randomized) algorithm (called the* tester*) that – given $x \in W_n$ – queries at most q of the n coordinates and then answers "yes" or "no" with the following two conditions :*

$$\mathbb{P}(\textit{algorithm answers "yes"} | x \in P_n) = 1 \tag{4.2.1}$$

$$\mathbb{P}(\textit{algorithm answers "no"} | x \notin P_n) \geq \epsilon \frac{\operatorname{dist}(x, P_n)}{n} \tag{4.2.2}$$

Here, $\mathbb{P}(\cdot)$ denotes the probability measure for the randomized algorithm and by $\operatorname{dist}(x, P_n)$ *we denote the Hamming distance of x to the set P_n, i.e., for the coordinates x_i and p_i of x and p, respectively:*

$$\operatorname{dist}(x, P_n) = \min_{p \in P_n} |\{1 \leq i \leq n \,|\, p_i \neq x_i\}| \tag{4.2.3}$$

A popular example of a testable property is the linearity of functions (interpreting elements from A^{B^n} as functions from B^n to A). We want to show another example, namely the testability of the property "to be a coboundary" in the set of cochains of a simplicial complex, which turns out to be equivalent to coboundary expansion.

4.2.1 Testability and Coboundary Expansion

Proving testability requires the definition of a testing algorithm. Thus, we introduce the so-called cocycle tester:

Definition 4.2.2 (Cocycle Tester)**.** *Let X be a d-dimensional simplicial complex and let $0 \leq k \leq d-1$. We consider values from $\mathbb{Z}_2$, that is, $A = \mathbb{Z}_2$, and the space $\mathrm{C}^k(X; \mathbb{Z}_2)$ of cochains with $\mathbb{Z}_2$-coefficients, interpreted as $\mathbb{Z}_2^{|X(k)|}$ in a natural way.*

The cocycle tester *works in the following way when given an arbitrary cochain $f \in \mathrm{C}^k(X; \mathbb{Z}_2)$:*

1. *Choose a $(k+1)$-simplex τ uniformly at random from X.*
2. *Evaluate $\delta_k f(\tau)$.*

3. If $\delta_k f(\tau) = 0$ answer "Yes, f is a cocycle", else answer "No, f is no cocycle".

In the second step, the cocycle tester queries exactly $k+2$ simplices in $X(k)$, namely the facets of τ, to calculate the value of $\delta_k f$. If the tested cochain f is indeed a cocycle, then the cocycle tester will always answer "yes". The next lemma gives an answer for the other case:

Lemma 4.2.1. *Let X be a d-dimensional simplicial complex and let $0 \leq k \leq d-1$. Let $h^k(X)$ be the coboundary expansion parameter in dimension k with respect to the normalized Hamming norm $\|\cdot\|_X$.*

Then we have that $h^k(X) > 0$ is equivalent to the property of being a coboundary (that is, $P_n \cong \mathrm{B}^k(X;\mathbb{Z}_2)$ and $W_n \cong \mathrm{C}^k(X;\mathbb{Z}_2)$) is $(k+2, h^k(X))$-testable using the cocycle tester.

Observe that we use a cocycle tester to test the property of being a coboundary. Testability thus already implies vanishing cohomology. Indeed, if there were a cocycle that is no coboundary, then the tester would give the wrong answer with probability one. Hence, the algorithm would not qualify as a tester.

Proof. We start by reformulating the definition of testability. When does the cocycle tester fail? – If we got a cochain f, that is no cocycle, and we evaluate its coboundary on a simplex that is not in the support of the coboundary. On the other hand, the cocycle tester gives the correct (negative) answer if – by coincidence – we sample a simplex that is in the support of $\delta_k f$. Among all $(k+1)$-simplices these simplices make the tester work correctly. Hence, we have:

$$\mathbb{P}(\text{algorithm answers "no"} | f \notin \mathrm{B}^k(X;\mathbb{Z}_2)) = \frac{|\delta_k f|}{|X(k+1)|} = \|\delta_k f\|_X \tag{4.2.4}$$

Moreover, the Hamming distance of a cochain f to the subset of coboundaries is nothing than the Hamming norm on the quotient space modulo coboundaries:

$$\mathrm{dist}(x, \mathrm{B}^k(X;\mathbb{Z}_2)) = |[x]| \tag{4.2.5}$$

Putting this together, we obtain that the property of being a coboundary is $(k+2, \epsilon)$-testable if and only if:

$$\|\delta_k f\|_X \geq \epsilon \frac{|[f]|}{|X(k)|} = \epsilon \, \|[f]\|_X \tag{4.2.6}$$

But this is exactly the defining inequality for the coboundary expansion parameter and thus the two concepts are equivalent. □

We see that testing this property and high-dimensional expansion are just different viewpoints on the same problem. However, there is also a connection between (graph) expansion and more general property testing as the next section shows.

4.2.2 Testability and Constraints

Expansion also appears if we test properties that are defined by constraints. Take for example the cocycle tester from the last section: A cochain f is a cocycle if its coboundary δf evaluates to zero on every face in one dimension higher. Every such face τ poses the constraint that $\delta f(\tau) = 0$. If f meets all these constraints then it is indeed a cocycle.

A *constraint-defined property* now corresponds to a subset P_n that is defined likewise by a set of constraints. Every constraint can be seen as a function depending on a subset of coordinates that evaluates to one if an element $x \in A^n$ meets the constraint and zero else.

For a given constraint-defined property it is simple to define a candidate for a tester. Namely, we could define a tester that makes the following steps to test an element $x \in W_n$:

1. Choose a constraint uniformly at random (or distributed according to a chosen probability distribution) from the set of all constraints.

2. Check if x meets the constraint and answer accordingly.

The cocycle tester from above is such a *"constraint tester"* where the constraints are indexed by the $(k+1)$-dimensional faces.

Besides the simplicity of defining a tester, constraint-defined properties allow quite a nice illustration/description in terms of graphs. Namely, we can consider a graph G (called the *constraint graph*) that contains a vertex for every coordinate in A_n. As edges we choose the sets of two coordinates such that there exists a constraint which uses both of the coordinates.[1]

Back in the example of the cocycle tester, we would get a graph which has the k-simplices of X as vertices and two k-simplices are connected by an edge if and only if there is a $(k+1)$-simplex in X that contains both of them.

For the easiest case that all constraints depend on exactly two coordinates, the constraint graph G allows to interpret the probability of rejection by this tester in graph-theoretical terms, namely the fraction of edges rejecting the element.

If we assume that the property is testable by this tester, we can already deduce (under some technical preconditions) graph-theoretical properties of the constraint graph (following [DK12]):

Lemma 4.2.2. *Let P_n be a constraint-defined property such that every constraint depends on exactly two coordinates and let $G = (V, E)$ be its constraint graph. Denote by $d = \frac{2|E|}{|V|}$ its average degree and let Δ be the* relative distance *of P_n, that is (using the Hamming distance* dist*):*

$$\Delta := \frac{\min_{p_1 \neq p_2 \in P_n} \operatorname{dist}(p_1, p_2)}{n} > 0 \tag{4.2.7}$$

[1] One could also define a hypergraph on the same vertex set with hyperedges for all constraints.

Moreover, let P_n be non-degenerate*, that is (using the projection π_A onto the coordinates defined by A):*

$$\forall A \subseteq \{1, \dots, n\} : \quad \exists x \neq y \in \pi_A(P_n) : \operatorname{dist}(x, y) > \frac{|A|}{3} \tag{4.2.8}$$

If P_n is (q, ϵ)-testable by the tester described above the lemma (and arbitrary, unimportant q), then the graph G is edge expanding on small sets:

$$\forall S \subseteq V, |S| < \frac{3\Delta}{4}|V| : \quad |E(S, V \backslash S)| \geq \frac{\epsilon}{3}\frac{d}{2}|S| \tag{4.2.9}$$

Proof. We want to prove the lemma by contraposition, i.e. we have to show for any subset $S \subseteq V$:

$$|E(S, V \backslash S)| < \frac{\epsilon}{3}\frac{d}{2}|S| \quad \Longrightarrow \quad |S| \geq \frac{3\Delta}{4}|V| \tag{4.2.10}$$

Let S be an arbitrary subset of V and define $\gamma := \frac{|S|}{3|V|}$. Since P_n is non-degenerate we can find two elements $a, a' \in \pi_S(P_n)$ that have a large distance:

$$\operatorname{dist}(a, a') > \frac{|S|}{3} = \gamma|V| \tag{4.2.11}$$

The element a is in the projection of P_n to the subset S of coordinates. Hence, there exists an element $b \in \pi_{V \backslash S}(P_n)$ such that $w := (a, b) \in P_n$ (just pick any point in the preimage of a under the projection and project it onto the other coordinates). Let $w' \in P_n$ be the point that is closest to (a', b) with respect to the Hamming distance. We have $w \neq w'$. Indeed, if we would have $w = w'$, then:

$$\operatorname{dist}((a', b), P_n) = \operatorname{dist}((a', b), w) = \operatorname{dist}(a', a) > \gamma|V| \tag{4.2.12}$$

As P_n is testable we thus obtain:

$$\mathbb{P}(\text{algorithm rejects } (a', b) | (a', b) \notin P_n) \geq \epsilon \frac{\operatorname{dist}((a', b), P_n)}{n} > \epsilon\gamma \tag{4.2.13}$$

This leads to a contradiction if we think about what it means that the algorithm rejects (a', b): We know that $a' \in \pi_S(P_n)$ and $b \in \pi_{V \backslash S}(P_n)$. They are coordinates of points that already lie in P_n. Hence, the constraints that depend only on coordinates in S and only on coordinates in $V \backslash S$ are fulfilled. The only constraints that may reject (a', b) are those that depend on coordinates both from S and $V \backslash S$, corresponding to edges between S and $V \backslash S$:

$$\mathbb{P}(\text{algorithm rejects } (a', b) | (a', b) \notin P_n) = \frac{|E(S, V \backslash S)|}{|E|} \tag{4.2.14}$$

This is the point, where the connection to expansion is established. Assuming that there are few edges between the two parts (as we do) we get a contradiction on $w = w'$:

$$\mathbb{P}(\text{algorithm rejects } (a', b) | (a', b) \notin P_n) = \frac{|E(S, V \backslash S)|}{|E|} < \frac{\epsilon}{3} \frac{d}{2} \frac{|S|}{|E|} = \epsilon\gamma \tag{4.2.15}$$

Using the definition of the testability property we further obtain:

$$\begin{aligned} \text{dist}((a', b), w') &= \text{dist}((a', b), P_n) && (4.2.16) \\ &\leq \frac{\mathbb{P}(\text{algorithm rejects}(a', b) | (a', b) \notin P_n)|V|}{\epsilon} < \gamma|V| && (4.2.17) \end{aligned}$$

From this we can deduce a lower bound on the size of S. By the inverse triangle inequality for the Hamming distance, we get:

$$|S| \geq \text{dist}(a, a') = \text{dist}((a, b), (a', b)) \geq \underbrace{\text{dist}(w, w')}_{\geq \Delta|V|} - \underbrace{\text{dist}(w', (a', b))}_{< \gamma|V|} \geq (\Delta - \gamma)|V| \tag{4.2.18}$$

Altogether, using $\gamma = \frac{|S|}{3|V|}$:

$$|S| \geq \frac{3\Delta}{4}|V|, \tag{4.2.19}$$

and that proves expansion for small sets. □

This lemma is not the last connection between expansion and testing. Indeed, it can be shown that (another) generalization of expansion to higher dimensions (linked with the convergence of high-dimensional random walks) implies a property called *agreement expansion* that is defined in a quite similar way and is used for so-called *agreement tests*. These tests work with functions that are partially defined on subsets and check whether these functions can be extended to a global function or not. For more information see [DK17].

4.3 Error Correcting Codes

Another application of expanders is the construction of codes. Technically speaking, a code is a subset $C \subseteq \mathbb{Z}_2^n$ of binary words that is used to transmit information. In the trivial code we have $C = \mathbb{Z}_2^n$, where every word is used as code word. However, if there is any noise or other sources of errors, codes also allow to implement error correcting algorithms. For this reason, bits are added which leads to a code $C \subsetneq \mathbb{Z}_2^n$. Two code words have a Hamming distance, that is, we can count the number of coordinates where they differ. By choosing a code where the minimal Hamming distance of two code words is big, we can correct small errors by taking the code

word that is next to the perturbed word. This leads to the definition of the *distance* d of a code C, namely the minimal Hamming distance of two code words.

As adding more bits (and thus getting a higher distance) leads to more bits that have to be transferred, a quality attribute of codes is the so-called *rate*. If the code is a k-dimensional linear subspace of $\mathbb{Z}_2^n$, the rate is defined as $R = \frac{k}{n}$, that is, the ratio of information bits to transferred bits.

To obtain good codes we thus have to search for codes that have a high distance (and thus a high ability to suppress errors) and also a high rate. Moreover, we want to construct codes of any arbitrary size such that the relative distance $\frac{d}{n}$ and the rate R is bounded away from zero by a constant independent on n. This is the point where expanders come into play.

Moreover, in the construction in Section 4.2.2 the property that is tested can be seen as a code itself, that is, $P_n = C$. Indeed, the discussion in [DK12] uses the notion of "locally testable codes" instead of that of property testing.

4.3.1 Expander Codes

The first kind of codes that uses/leads to expanders are the so-called *expander codes*. These are codes that work with parity checks, that is, we add bits that are calculated from the information bits by linear operations (addition modulo 2). The easiest example is the parity bit, which is added at the end of the word and is zero if there is an even number of ones in the information bits and one else. We now consider the bipartite graph which has a vertex for every bit in the code word in one block and one vertex for every parity bit in the other block. There is an edge between a code bit vertex and a parity bit vertex if and only if the code bit is used to calculate the parity bit. Every parity check code has thus a bipartite graph associated to it. This graph is called the *factor graph* of the code.

If the factor graph of a code is some kind of expander, we call the code expander code:

Definition 4.3.1 (Expander Code)**.** *Let $C \subseteq \mathbb{Z}_2^n$ be a parity check code and $G = (L \cup R, E)$ its factor graph (with code bits L and parity bits R).*

C is called an expander code*, if G is a one-sided expander graph, that is:*

$$\exists \epsilon > 0 : \forall S \subseteq L : \quad |\Gamma(S)| \geq \epsilon |S| \tag{4.3.1}$$

($\Gamma(S)$ denotes the set of neighbours of vertices in S, that is, all vertices in R that are adjacent to a vertex in S.)

ϵ is called the one-sided expansion constant*.*

The following lemma (cf. [Gur10]) shows that good one-sided expander graphs give codes with good distance:

Lemma 4.3.1. *Let $G = (L \cup R, E)$ be a one-sided expander graph that is D-left-regular, that is the degree of any vertex in the left set L is equal to D, and let*

$C = C(G)$ be the corresponding parity check code. Let $D(1-\epsilon)$ be the one-sided expansion constant of G.

Then the distance d of C fulfills:

$$d \geq 2(1-\epsilon)|L| \tag{4.3.2}$$

A proof and deeper going discussion can be found in [Gur10].

By this lemma, a good code can be constructed from a good expander.

Further constructions of codes using expansion are presented in [SS06]. [Lub11, Ch. 3.1] includes an introduction on codes as well.

4.3.2 Quantum Codes

The second type of codes where expansion appears are the so-called *quantum codes*. Quantum codes are codes that are designed to be used in quantum computing. The principle of quantum computers is quite different from "normal" computers, because they are following quantum laws. However, quantum computers are also very vulnerable to the effects of noise. Quantum codes may help to stabilize quantum computers by reducing the influence of noise to the result. In [NC11, Ch. 10.1.1] Michael Nielsen and Isaac Chuang survey some of the critical differences between quantum coding and normal coding that influence the design of quantum codes:

- Quantum states may not be cloned or duplicated. Hence, the easiest type of code (just repeating the bits several times) does not work.
- There is a continuum of errors that may affect a single quantum bit. It may not be possible to determine the error that occurred.
- By measuring a quantum state it is destroyed irreproducably. This makes decoding difficult.

One of the first and most important quantum codes are the *CSS-Codes*, named after their inventors Arthur Calderbank, Peter Shor and Andrew Steane. CSS-Codes are given by two classical codes that lie in orthogonal spaces of $\mathbb{Z}_2^n$ with respect to the following bilinear form (similar to the ℓ^2-scalar product restricted on 0-1-valued vectors, but taken modulo 2), x and y are 0-1-valued vectors with components x_i and y_i:

$$(x, y) := \sum_i x_i y_i \mod 2 \tag{4.3.3}$$

Definition 4.3.2 (CSS-Code)**.** *A* quantum CSS-code *C consists of a pair (C_1, C_2) of (linear) subspaces of $\mathbb{Z}_2^n$ that are orthogonal, that is ($(\cdot)^\perp$ denotes the orthogonal space):*

$$C_1 \subseteq C_2^\perp \quad \textit{and} \quad C_2 \subseteq C_1^\perp \tag{4.3.4}$$

The rate *of C is defined by $\frac{k}{n}$, using the dimension k:*

$$k = \dim {C_2^\perp}/{C_1} = \dim {C_1^\perp}/{C_2} \tag{4.3.5}$$

The distance *d of C is defined as:*

$$d = \min \left\{ |x| \mid x \in (C_1^\perp \backslash C_2) \cup (C_2^\perp \backslash C_1) \right\} \tag{4.3.6}$$

($|\cdot|$ denotes the Hamming norm.)

As in the classical case we want to construct a code that possesses a high rate and high distance. It turns out that topology and expansion can be used for that, leading to the so-called *Homological codes*.

Let X be a given simplicial complex that may be obtained by a triangulation of a high-dimensional manifold, covers or likewise. The cohomology of X yields the associated CSS-code. Indeed, in every dimension we have the subgroup of coboundaries $\mathrm{B}^i(X;\mathbb{Z}_2)$, which is the candidate for one of the subspaces in the definition. For the other subspace we have to consider the adjoint map δ^* ($= \partial$ in the terms of homology that was not introduced in this thesis) of the coboundary map with respect to the restricted ℓ^2-scalar product. Denote by $\mathrm{Z}_i(X;\mathbb{Z}_2)$ the kernel of δ^*_{i-1} (the set of cycles) and by $\mathrm{B}_i(X;\mathbb{Z}_2)$ the image of δ^*_i. As a consequence of the definition of the adjoint map, we can conclude that:

$$(\mathrm{B}_i(X;\mathbb{Z}_2))^\perp = \mathrm{Z}^i(X;\mathbb{Z}_2) \quad \text{and} \quad \left(\mathrm{B}^i(X;\mathbb{Z}_2)\right)^\perp = \mathrm{Z}_i(X;\mathbb{Z}_2) \tag{4.3.7}$$

Thus, the pair $(\mathrm{B}_i(X;\mathbb{Z}_2), \mathrm{B}^i(X;\mathbb{Z}_2))$ defines a CSS-code. The rate of this code has the following nice description:

$$\frac{k}{n} = \frac{1}{n} \dim {\mathrm{Z}^i(X;\mathbb{Z}_2)}/{\mathrm{B}^i(X;\mathbb{Z}_2)} = \frac{1}{n} \dim \mathrm{H}^i(X;\mathbb{Z}_2) \tag{4.3.8}$$

Moreover, the distance of the code is the cochain in ${\mathrm{Z}^i(X;\mathbb{Z}_2)}/{\mathrm{B}^i(X;\mathbb{Z}_2)}$ or in ${\mathrm{Z}_i(X;\mathbb{Z}_2)}/{\mathrm{B}_i(X;\mathbb{Z}_2)}$ of minimal Hamming weight. This corresponds to the minimum of the cosystole parameter $\mathrm{syst}^i(X)$ and the so-called systole parameter (defined accordingly for the sets of cycles and boundaries). To find cosystolic expanders on the one hand is a way to find CSS-codes with high distance. On the other hand, many of the results that are known for high-dimensional expanders (the specific definitions are varying as the need varies) can be translated to the theory of quantum codes or may help as suggestions.

A deeper-going discussion of the connection of high-dimensional expansion and CSS-codes and important results can be found in [EOT16]. [Zém09] presents several constructions of homological codes and discusses some limits. In [GL14] and [Fet12] a good introduction to homological codes can be found as well as an explanation of their connections to manifolds and systoles. Finally, [BMD07] gives an overview and compares classical and quantum codes.

5 Methods to Prove Expansion and Examples

After introducing the definition of high-dimensional expansion in Chapter 3, we now want to give examples for expanders and show methods to prove expansion. The field of high-dimensional expansion is still very young and there are only a few methods known. Many of these methods are designed to fit just the complex in consideration, while other methods may be generalized.

In the following sections we want to present several strategies to prove expansion and try to extract the ideas behind them. We will determine bounds on the expansion parameters of some "standard" complexes which can be found in Table 5.1 and develop methods to "lift" expansion from low dimensions to high dimensions by so-called local to global methods. Finally, some random methods are introduced.

	Hamming Norm	Normalized Hamming Norm
$h^0(K_n)$	$\lceil \frac{n}{2} \rceil$	$1 + \mathcal{O}\left(\frac{1}{n}\right)$
$h^0(K_{n,n})$	$\frac{n}{2} + \mathcal{O}\left(\frac{1}{n}\right)$	$1 + \mathcal{O}\left(\frac{1}{n^2}\right)$
$h^k(\Delta_{n-1})$	$\geq \frac{n}{k+2}$	$\geq \frac{n}{n-k-1}$
	$\leq \lceil \frac{n}{k+2} \rceil$	$\leq \frac{k+2}{n-k-1} \lceil \frac{n}{k+2} \rceil$
$h^{k-1}(\Lambda_n^k)$	$\geq \frac{n^k(n-2)}{(3n-4)(2n-2)^{k-1}-n^k}$	$\frac{kn^{k-1}(n-2)}{(3n-4)(2n-2)^{k-1}-n^k}$
	$\leq n$	$\leq k$

Table 5.1: Expansion parameters for selected complexes

5.1 Random Co-filling

The first method to prove expansion that we will discuss is called "random co-filling", although there is no "real randomness" involved. As the name is saying, the aim is to prove a co-filling inequality and thus cocycle expansion. If in addition the complex under investigation has vanishing cohomology, then we already proved coboundary expansion with this construction.

The method of random co-filling appeared in [MW07] and independently in [Gro10] where it was used to prove the expansion of complete complexes.

In this section we first want to explain the general idea of random co-filling and then apply it to prove expansion for standard complexes like simplices and k-partite complexes.

5.1.1 General Idea

We consider the following problem: Given a simplicial complex X we want to show a co-filling inequality (in dimension $k+1$) and thus cocycle expansion (in dimension k). Together with vanishing cohomology this would already imply coboundary expansion.

Hence, we start with an arbitrary coboundary $\beta \in \mathrm{B}^{k+1}(X)$ and need to find a small cochain $\alpha \in \mathrm{C}^k(X)$ that co-fills β. Every choice of α gives an upper bound on the co-filling constant. To obtain the optimal constant we must therefore find the smallest of these cochains, that is, with the minimal upper bound. However, it is not easy to explicitly write down the minimal co-filling cochain for all different choices of β. This problem may be avoided if we can write down and analyze "enough" cochains, that is, if we are able to write down a large family of co-fillings for the given β. Next, we try to analyze all these co-fillings together, for example by calculating the mean of all upper bounds on the co-filling constant. Using this calculation, we deduce a bound for the optimal co-filling, for example by the fact that the minimum of a set of values is less than the mean of these values.

This is where the name "random co-filling" comes from. The process of choosing the optimal cochain can be seen as a random process where we pick a cochain from the family uniformly at random. Calculating the mean translates to calculating the expected upper bound from this random variable. From this we can deduce that the probability for the upper bound to be low is non-zero, hence there exists a cochain which gives rise to this upper bound.

Summarizing, for a given cochain β we make the following steps:

1. Find a family of co-fillings α that we know or hope includes the optimal or a very good co-filling. For every α in this family denote the co-filling constant by $c_\alpha = \frac{\|\alpha\|_X}{\|\beta\|_X}$. We want to find the α with minimal c_α.

2. Calculate the mean of the family $(c_\alpha)_\alpha$, e.g. by double counting arguments. This yields an upper bound on $\min_\alpha c_\alpha$ and hence on the co-filling constant.

In this method there are two critical points. First, how can we find such a family? The more cochains there are in the family, the higher is the probability to include the right one. However, if the percentage of "really bad cochains" is too high, the estimates using the mean do not give a good bound. Hence, this method gives good results and is easy to apply if we can find many co-fillings that usually do not vary much in size. It is often used when there are already good co-fillings known for special cases of coboundaries, but these co-fillings depend on the form of the coboundary. By the "random process" the best fitting co-filling is chosen. In the next sections some examples of such families are given.

The second critical point is the calculation of the mean. Here, the main tool is double counting of the right quantities. For example many families of co-fillings use local views of the coboundary. Counting all simplices in the support of a coboundary and counting it locally gives the same result (up to a factor because some simplices may be counted more than once).

Finally, it has to be pointed out, that random co-filling just gives an upper bound on the co-filling constant (and hence a lower bound on the cocycle expansion parameter). This bound need not be sharp in any way. Actually, the sharpness depends on the chosen family of co-fillings. Sharpness of the result can be proven by giving examples of critical coboundaries.

5.1.2 The Simplex Δ_{n-1}

The first example, where random co-filling works pretty well, is the complete complex or simplex on n vertices, that is, the complex that contains all possible faces. We denote by Δ_{n-1} the $(n-1)$-dimensional complete simplicial complex on n vertices. For the sake of simplicity we choose the vertex set by $V = [n] = \{1, 2, \dots n\}$.

To keep the notation simple and to make the idea clear we use coefficients from $\mathbb{Z}_2$, hence, we do not have to deal with orientations. However, the idea of proof does not change for other coefficients. Moreover, we calculate the expansion parameters for the counting norm. The results for the normalized Hamming norm follow easily by multiplying the norms with the number of k-simplices:

$$|\Delta_{n-1}(k)| = \binom{n}{k+1} \tag{5.1.1}$$

The 0-dimensional expansion parameter in the 1-dimensional case was already given in example 2 on page 38 (up to the normalizing constant $\frac{n-1}{2}$):

$$h^0(\Delta_{n-1}) = \begin{cases} \frac{n}{2} & \text{if } n \in 2\mathbb{N} \\ \frac{n+1}{2} & \text{if } n \in 2\mathbb{N}+1 \end{cases} = \left\lceil \frac{n}{2} \right\rceil \tag{5.1.2}$$

We start now with a detailed discussion of the proof of expansion in the 2-dimensional case, as all the important ideas already appear here, while sketches and illustrations are still possible and it is quite easy to imagine, what is happening. The results for higher dimensions follow in a similar way. The proof discussed here appeared in [LM06] and [MW07]. Another proof can be found in [Gro10] using a somewhat different notation.

Lemma 5.1.1. *We have:*

$$\frac{n}{3} \leq h^1(\Delta_{n-1}) \leq \left\lceil \frac{n}{3} \right\rceil \tag{5.1.3}$$

Proof. We follow the steps from Section 5.1.1 for the random co-filling method to prove the lower bound on the expansion parameter. Using the same idea (just one

dimension lower), we show that the first cohomology group vanishes, which is needed to obtain coboundary expansion. Finally, we give an example that shows the upper bound on $h^1(\Delta_{n-1})$.

Let $\beta \in \mathrm{Z}^2(\Delta_{n-1}; \mathbb{Z}_2)$ be an arbitrary cocycle[1].

1. Construction of a family of co-fillings

   We need to find a family of 1-cochains that have β as its coboundary. Assume $\alpha \in \mathrm{C}^1(\Delta_{n-1}; \mathbb{Z}_2)$ is such a cochain: $\delta_1\alpha = \beta$. This means that α evaluates to one for an odd number of edges, that is, one or three edges, in each triangle in the support of β. We are looking for a co-filling with minimal Hamming norm, hence, we want that there is only one non-zero edge in most of these triangles. One strategy may be to choose/mark for every triangle in $\operatorname{supp}\beta$ an edge and define α by adding all marked edges to $\operatorname{supp}\alpha$. Then try to correct any "errors" in the coboundary by "skillfully" adding further edges.

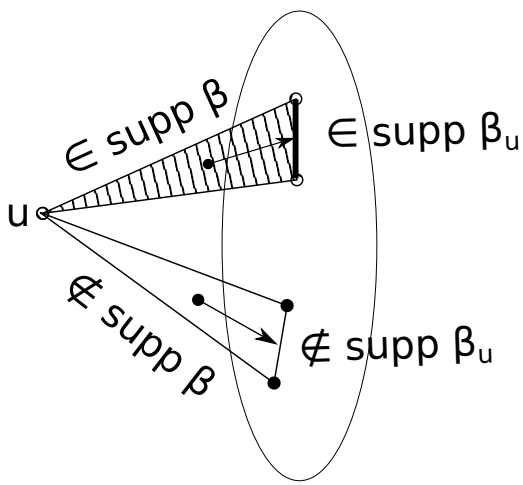

Fig. 5.1: Illustration of β_u

   However, there is an easier strategy. Choose an arbitrary vertex $u \in [n] \cong \Delta_{n-1}(0)$. Now for any triangle that contains u there is a natural choice of an edge to mark, namely the edge that is opposite to u. The value of β associated with the triangle is "pushed" to the edge in the link of u (cf. Fig. 5.1). For all the other triangles that do not contain u we do nothing – it turns out (using that β is a cocycle) that this is already enough. Going through the definitions, the cochain $\alpha^{(u)}$ that is defined in this way is just the localization β_u (cf. definition 2.2.12 on page 22) of β to the link of u in Δ_{n-1}, which we denote by $(\Delta_{n-1})_u$ (cf. definition 2.2.11 on page 22), embedded into the space

[1]For the (general) random co-filling method we would start with an arbitrary coboundary, but we may as well start with a cocycle as every coboundary is a cocycle. In the course of the proof we only use that β is a cocycle, so we prove more than needed.

of cochains on the whole complex, that is, $\alpha^{(u)} := \beta_u$:

$$\alpha^{(u)}(\{v,w\}) = \begin{cases} 0 & \text{if } u \in \{v,w\}, \text{ i.e., } \{v,w\} \notin (\Delta_{n-1})_u \\ \beta(\{u\} \cup \{v,w\}) & \text{if } \{v,w\} \in (\Delta_{n-1})_u \end{cases} \tag{5.1.4}$$

A short calculation evaluating both sides for any $\{a,b,c\} \in \Delta_{n-1}(2)$ with $b,c \neq u$ shows that $\delta_1\alpha^{(u)} = \beta$:

$$\delta_1\alpha^{(u)}(\{a,b,c\}) = \alpha^{(u)}(\{a,b\}) + \alpha^{(u)}(\{b,c\}) + \alpha^{(u)}(\{a,c\}) \tag{5.1.5}$$

$$= \begin{cases} \alpha^{(u)}(\{b,c\}) = \beta(\{a,b,c\}) & \text{if } a = u \\ \beta(\{u,a,b\}) + \beta(\{u,b,c\}) + \beta(\{u,a,c\}) & \text{else} \end{cases} \tag{5.1.6}$$

$$= \beta(\{a,b,c\}) \tag{5.1.7}$$

In the last equality, we use that β is a cocycle, namely (addition is mod 2):

$$\delta_2\beta(\{u,a,b,c\}) = \beta(\{u,a,b\}) + \beta(\{u,b,c\}) + \beta(\{u,a,c\}) + \beta(\{a,b,c\}) \tag{5.1.8}$$

$$\implies \beta(\{u,a,b\}) + \beta(\{u,b,c\}) + \beta(\{u,a,c\}) = \beta(\{a,b,c\}) \tag{5.1.9}$$

This calculation is illustrated in Fig. 5.2.

The family we use for the random co-filling method is $\left(\alpha^{(u)}\right)_{u\in\Delta_{n-1}(0)}$.

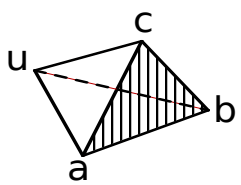

Fig. 5.2: Illustration of $\delta_2\beta(\{u,a,b,c\})$

The construction of this family of co-fillings is independent of the specific dimension. Indeed, we can consider the localization of an arbitrary-dimensional cocycle with respect to any vertex and obtain a co-filling. The last ingredient to show coboundary expansion is that the first cohomology group $\mathrm{H}^1(\Delta_{n-1};\mathbb{Z}_2)$ is vanishing. Hence, we use the construction in this dimension. In the high-dimensional case (cf. lemma 5.1.2 on page 67) this step is not necessary/already included, but for dimension 1, formally, we have not proved it yet. The proof is included for the sake of completeness, but it has no further benefit:

2. The first cohomology group is vanishing.

   Let $\alpha \in \mathrm{Z}^1(\Delta_{n-1};\mathbb{Z}_2)$ be a 1-dimensional cocycle, that is, a set of edges in the complete graph with the property that in every triangle there are either zero or two edges. Let $u \in \Delta_{n-1}(0)$ be an arbitrary vertex and consider the localization α_u of α to the link of u. For any edge $\{a,b\} \in \Delta_{n-1}(1), b \neq u$, we calculate:

$$\delta_0\alpha_u(\{a,b\}) = \alpha_u(\{a\}) + \alpha_u(\{b\}) \tag{5.1.10}$$

$$= \begin{cases} \alpha(\{u,b\}) & \text{if } a = u \\ \alpha(\{u,a\}) + \alpha(\{u,b\}) & \text{if } a \neq u \end{cases} \tag{5.1.11}$$

$$= \alpha(\{a,b\}) \tag{5.1.12}$$

   In the last equality it is again used that α is a cocycle. Thus, we have shown, that α is indeed a coboundary, $\mathrm{Z}^1(\Delta_{n-1};\mathbb{Z}_2) = \mathrm{B}^1(\Delta_{n-1};\mathbb{Z}_2)$ and that $\mathrm{H}^1(\Delta_{n-1};\mathbb{Z}_2)$ vanishes.

3. Calculation of the mean co-filling constants

   As the family of co-fillings consists of "local views" of the cochain β, it is reasonable to use double counting. Namely, we have that the Hamming norm of the support of each $\alpha^{(u)}$ is equal to the count of triangles in $\operatorname{supp}\beta$ that contain u. Every triangle contains three vertices, hence, it is counted three times:

$$\sum_{u\in\Delta_{n-1}(0)} \left|\alpha^{(u)}\right| = 3\,|\beta| \tag{5.1.13}$$

4. Lower Bound on the expansion parameter

   Using the calculation above, we get the following inequality for any co-filling α of β, that is $\delta_1\alpha = \beta$:

$$\min_{z\in\mathrm{Z}^1(\Delta_{n-1};\mathbb{Z}_2)} |\alpha + z| \leq \min_{u\in\Delta_{n-1}(0)} \left|\alpha^{(u)}\right| \leq \frac{1}{n}\sum_{u\in\Delta_{n-1}(0)} \left|\alpha^{(u)}\right| = \frac{3}{n}\,|\beta| \tag{5.1.14}$$

   This yields an upper bound on the optimal co-filling constant L_2 and hence a lower bound on the cocycle expansion parameter $h_z^1(\Delta_{n-1})$:

$$L_2 \leq \frac{3}{n} \quad \text{and} \quad h_z^1(\Delta_{n-1}) \geq \frac{n}{3} \tag{5.1.15}$$

   As we have already shown that $\mathrm{H}^1(\Delta_{n-1};\mathbb{Z}_2) = \{0\}$, we have:

$$h^1(\Delta_{n-1}) = h_z^1(\Delta_{n-1}) \geq \frac{n}{3} \tag{5.1.16}$$

5. The lower bound on the expansion parameter is almost sharp.

   To show the approximate sharpness of the lower bound we give an explicit example of a critical cochain. This example is taken from [MW07].

   First suppose that n is divisible by 3. Then we can partition the vertex set $\Delta_{n-1}(0)$ into three equal-sized sets:

$$\Delta_{n-1}(0) = V_0 \sqcup V_1 \sqcup V_2 \quad \text{and} \quad |V_i| = \frac{n}{3} \tag{5.1.17}$$

   Now we define a cochain $\phi \in \mathrm{C}^1(\Delta_{n-1}; \mathbb{Z}_2)$ that has a "small" coboundary (illustrated in Fig. 5.3):

$$\phi(F) := \begin{cases} 1 & \text{if } |F \cap V_i| = 1,\ \forall i = 0, 1 \\ 0 & \text{else} \end{cases} \qquad \forall F \in \Delta_{n-1}(1) \tag{5.1.18}$$

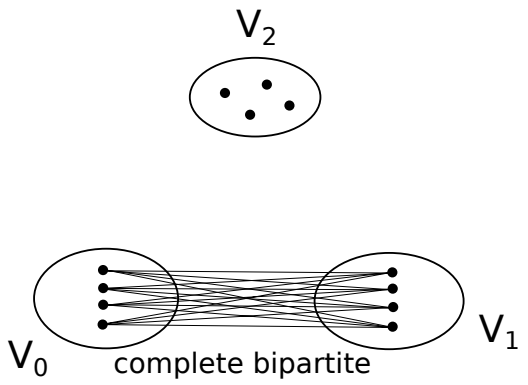

Fig. 5.3: Illustration of ϕ

   Thus, the support of ϕ consists of all edges that run between V_0 and V_1, but no other edges. Each of these edges induces $\frac{n}{3}$ triangles in the coboundary of ϕ, namely the triangles that consist of the edge and a vertex from V_2. All the other triangles contain either none of the edges in the support of ϕ or they contain two of them. It is not possible for a triangle to contain three edges, as this means that two vertices of the triangle have to be either in V_0 or in V_1, but there are no connections inside V_0 or V_1 in the support of ϕ. Hence, the coboundary $\delta_1\phi$ calculates to:

$$\delta_1\phi(F) = \begin{cases} 1 & \text{if } |F \cap V_i| = 1,\ \forall i = 0, 1, 2 \\ 0 & \text{else} \end{cases} \tag{5.1.19}$$

   The Hamming norms of ϕ and $\delta_1\phi$ can easily be calculated using this representation:

$$|\phi| = \left(\frac{n}{3}\right)^2 \quad |\delta_1\phi| = \left(\frac{n}{3}\right)^3 \tag{5.1.20}$$

To calculate an upper bound on the coboundary expansion, it is necessary to find the cochain that has minimal norm in the equivalence class of ϕ modulo coboundaries. Actually, we have $|\phi| = |[\phi]|$. Indeed, adding an arbitrary coboundary to ϕ does not increase the Hamming norm. A 1-dimensional coboundary consists of the edges of a cut $(S, \Delta_{n-1}(0)\backslash S)$ of the 1-skeleton. That is, we partition the vertex set into two subsets S and $\Delta_{n-1}(0)\backslash S$ to get a 0-dimensional cochain which has all the edges $E(S, \Delta_{n-1}(0)\backslash S)$ between S and $\Delta_{n-1}(0)\backslash S$ in its coboundary. Minimal norm modulo coboundaries thus means, that for every cut $(S, \Delta_{n-1}(0)\backslash S)$ there are at most half of the edges in $\operatorname{supp}\phi$. As all the edges can be counted immediately and ϕ contains less than half of them (this can be shown in a short elementary calculation), ϕ is a cochain with minimal Hamming norm modulo coboundaries and we can finally deduce our result for the special case:

$$\forall n \in \mathbb{N},\ 3|n: \quad h^1(\Delta_{n-1}) \leq \frac{|\delta_1\phi|}{|[\phi]|} = \frac{\left(\frac{n}{3}\right)^3}{\left(\frac{n}{3}\right)^2} = \frac{n}{3} \tag{5.1.21}$$

If n is not divisible by 3, however, this construction can be adapted, but the analysis is not very beautiful. We need to partition the vertex set into three approximately equal-sized parts V_0, V_1 and V_2 (without loss of generality we assume that $\left\lfloor \frac{n}{3} \right\rfloor \leq |V_i| \leq |V_{i+1}| \leq \left\lceil \frac{n}{3} \right\rceil$, $i = 0, 1$). The definition of ϕ and the calculation of $\delta_1\phi$ carries over, the proof that ϕ has minimal Hamming norm becomes nasty. Indeed, if we took the edges between V_1 and V_2 instead of V_0 and V_1, we could add the edges of the cut $(V_0 \cup V_2, V_1)$ to decrease the Hamming norm (if this would not be the case, we would get a contradiction to the claim we proved in the first part). We now have:

$$|\delta_1\phi| = |V_0| \cdot |V_1| \cdot |V_2| \quad \text{and} \quad |[\phi]| = |V_0| \cdot |V_1|, \tag{5.1.22}$$

and hence:

$$h^1(\Delta_{n-1}) \leq \frac{|\delta_1\phi|}{|[\phi]|} = \frac{|V_0| \cdot |V_1| \cdot |V_2|}{|V_0| \cdot |V_1|} = |V_2| = \left\lceil \frac{n}{3} \right\rceil, \tag{5.1.23}$$

and that is, what we wanted to show.

□

Why does this example give a critical cochain? If we look deeper into the proof, we see that for this special choice ϕ of a cochain and its coboundary $\delta_1\phi$, the link of each vertex $u \in V_0$ can be restricted (as if the complex were multipartite) to the link inside $V_1 \sqcup V_2$, as all other vertices in V_0 are already adjacent to two edges. This restricted link is a complete graph on $\frac{2n}{3}$ vertices. The localization ϕ_u with respect to the same vertex restricted to this subset then has half of the vertices in its support. Considering the restricted link as its own complex (as we will later for

the complete k-partite complexes) we can calculate the size of the coboundary of ϕ_u in the restricted link. The size of this coboundary depends on the coboundary expansion of the complete graph, which is limited by the 0-cochain that contains half of the vertices (cf. example 2 on page 38). The cochain ϕ hence is already "critical" in the restricted link of each vertex in V_0, by construction, making it a "natural" candidate for a critical cochain of the whole complex. Moreover, for this special coboundary the co-fillings given by $\alpha^{(u)}$ all have the same size $\left(\frac{n}{3}\right)^2$ and there is nothing lost when estimating the minimum by the mean. All co-fillings are equally good.

Further examples of critical cochains can be found in [Koz16]. There, it is proved that $h^1(\Delta_{n-1}) = \frac{n}{3}$ for all n that are no power of 2. The complexes where n is a power of 2 do not fulfill this equation, however, the error is of order $\mathcal{O}\left(\frac{1}{n}\right)$ as $n \to \infty$.

The expansion parameters for dimensions k with $n-2 \geq k \geq 2$ can be calculated using the same method. The only thing that changes is that the notation becomes more complicated:

Lemma 5.1.2. *For any $0 \leq k \leq n-2$, we have:*

$$\frac{n}{k+2} \leq h^k(\Delta_{n-1}) \leq \left\lceil \frac{n}{k+2} \right\rceil \tag{5.1.24}$$

Proof. As for the 1-dimensional case, the random co-filling method works fine for the complete complex. Indeed, we can use a similar family of co-fillings, namely the localization of the cochain to the link of a vertex. One may ask whether the cochain has to be localized to the link of a vertex and not to the link of any other simplex, but this can be explained by considering the dimensions: By localizing a k-dimensional cochain to the link of a vertex, we automatically get a $(k-1)$-dimensional cochain after re-embedding, which is the candidate for the co-filling. If we would localize to the link of an l-dimensional simplex, the result would be a $(k-l-1)$-dimensional cochain and to get a co-filling, we would need to "lift" the cochain to higher dimensions in some sense. In any way, localizing with a vertex already yields the wanted result.

Let $\beta \in \mathrm{Z}^{k+1}(\Delta_{n-1}; \mathbb{Z}_2)$ be an arbitrary cocycle. We define the family of co-fillings $\left(\alpha^{(u)}\right)_{u \in \Delta_{n-1}(0)}$ indexed by the vertices $u \in \Delta_{n-1}(0)$ by setting the values for any k-simplex σ:

$$\alpha^{(u)}(\sigma) := \beta_u(\sigma) = \begin{cases} \beta(\{u\} \cup \sigma) & \text{if } \sigma \in (\Delta_{n-1})_u \\ 0 & \text{else} \end{cases} \tag{5.1.25}$$

Calculating the coboundary of $\alpha^{(u)}$ evaluated at a simplex $\tau \in \Delta_{n-1}(k+1)$ shows:

$$\delta_k \alpha^{(u)}(\tau) = \sum_{\sigma \Subset \tau} \alpha^{(u)}(\sigma) \tag{5.1.26}$$

$$= \begin{cases} \beta(\tau) & \text{if } u \in \tau \\ \sum_{\sigma \Subset \tau} \beta(\{u\} \cup \sigma) & \text{else} \end{cases} \tag{5.1.27}$$

The sum in the last term can be rewritten using the definition of the coboundary of β:

$$\sum_{\sigma \Subset \tau} \beta(\{u\} \cup \sigma) = \delta_{k+1}\beta(\{u\} \cup \tau) + \beta(\tau), \tag{5.1.28}$$

which yields, using $\delta_{k+1}\beta = 0$ (the addition is modulo 2):

$$\delta_k \alpha^{(u)}(\tau) + \beta(\tau) = \begin{cases} 0 & \text{if } u \in \tau \\ \delta_{k+1}\beta(\{u\} \cup \tau) & \text{if } u \notin \tau \end{cases} \tag{5.1.29}$$

$$= 0 \tag{5.1.30}$$

Hence, every $\alpha^{(u)}$ is a co-filling of β. This argument (applied to different dimensions) proves that Δ_{n-1} has vanishing cohomology in any dimension lower than $n-1$.[2]

The mean can be calculated again by double counting: Every $(k+1)$-simplex τ in the support of β is counted $k+2$ times as u runs through the vertices of τ:

$$\sum_{u \in \Delta_{n-1}(0)} \left|\alpha^{(u)}\right| = (k+2)\,|\beta| \tag{5.1.31}$$

This yields the bounds on the optimal co-filling constant and the expansion parameters:

$$L_{k+1} \leq \frac{k+2}{n} \quad \text{and} \quad h^k(\Delta_{n-1}) = h_z^k(\Delta_{n-1}) \geq \frac{n}{k+2} \tag{5.1.32}$$

To show the upper bound, a similar construction as in the case $k = 1$ is used. If n is divisible by $k+2$, we can partition the vertex set into $k+2$ equal-sized sets and define a cochain $\phi \in \mathrm{C}^k(\Delta_{n-1}; \mathbb{Z}_2)$ by adding all k-simplices to its support that contain vertices from the first $k+1$ vertex sets. Every such simplex induces $\frac{n}{k+2}$ different simplices in the coboundary of ϕ and ϕ can be shown to have minimal Hamming norm, but the argument is more complicated. Hence, the expansion parameters are bounded by $\frac{n}{k+2}$. Again, the part where n is not divisible by $k+2$ is messy, but uses the same idea, and will be omitted here. □

[2] The cohomology of Δ_{n-1} in dimension $n-1$ can be calculated directly as there is only one simplex of this dimension, i.e. $\mathrm{C}^{n-1}(\Delta_{n-1}; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

5.1.3 The Complete k-Partite Complex Λ_n^k

The next example for random co-filling we want to present is the complete k-partite complex Λ_n^k. It is a direct generalization of the concept of a complete bipartite graph and is defined as follows:

Definition 5.1.1 (Complete k-Partite Complex)**.** *Let n and k be natural numbers and let $V_0, V_1, \ldots, V_k$ be $k+1$ disjoint sets of cardinality n, that is, $V_i \cong [n] := \{1, 2, \ldots, n\}$.*

The complete (k+1)-partite complex Λ_n^k on $n \times (k+1)$ vertices *is defined as the $(k+1)$-fold join of the vertex sets $V_0, \ldots, V_k$, that is:*

$$\Lambda_n^k := V_0 * V_1 * \cdots * V_k := \left\{ F \subseteq \bigsqcup_{i=0}^{k} V_i \,\middle|\, |F \cap V_i| \leq 1, \forall i = 0, \ldots, k \right\} \tag{5.1.33}$$

Λ_n^k is a k-dimensional simplicial complex.

We aim to prove lower bounds on the $(k-1)$-dimensional $\mathbb{Z}_2$-coboundary expansion $h^{k-1}(\Lambda_n^k)$ of Λ_n^k by random co-filling. Therefore, we need two ingredients: an appropriate family of co-fillings and vanishing cohomology $\mathrm{H}^{k-1}(\Lambda_n^k; \mathbb{Z}_2)$. As the proof will use induction, we will also prove coboundary expansion separately for the easiest case Λ_n^1, namely the complete bipartite graph $K_{n,n}$.

We will start with proving that the complete $(k+1)$-partite complexes Λ_n^k have vanishing cohomology in co-dimension 1:

Lemma 5.1.3. *Let Λ_n^k be the complete $(k+1)$-partite complex and $n, k \geq 1$.*
Then:

$$\mathrm{H}^{k-1}(\Lambda_n^k; \mathbb{Z}_2) = \{0\} \tag{5.1.34}$$

Proof. The proof uses induction on the dimension k of the complexes for fixed n. Indeed, the proof method is very similar to the one used later to define the family of co-fillings, but it is in a different dimension.

1. Induction Hypothesis

   As induction hypothesis in step k we assume that:

   $$\mathrm{H}^{l-1}(\Lambda_n^l; \mathbb{Z}_2) = \{0\} \quad \forall 1 \leq l < k \tag{5.1.35}$$

2. Initial Step

   For $k = 1$, we get that $\Lambda_n^1 = K_{n,n}$, the complete bipartite graph and we want to show that the cohomology in dimension zero is vanishing. As $K_{n,n}$ is connected, this is trivial. Indeed, if β is a 0-dimensional cocycle, this means that for every edge the values of β on both sides are equal and hence the value of β does not change along any path. By choosing a path from any point to any other point, we can show that β is constant and thus a coboundary (cf. equation 2.2.22 on page 19).

3. Induction Step

   Now let $k > 1$ be an arbitrary number and assume that the induction hypothesis holds. Let $\beta \in \mathrm{Z}^{k-1}(\Lambda_n^k; \mathbb{Z}_2)$ be an arbitrary cocycle, we have to show that β is a coboundary, that is, there exists a co-filling of β.

   We choose two arbitrary vertices $u, u' \in V_0$ from the first set V_0 (the first factor of the join), if $n = 1$ we can choose $u = u'$ and if $n > 1$ we need to choose $u \neq u'$. u and u' give rise to two cochains on the link of u and u', respectively, namely the localizations β_u and $\beta_{u'}$ of β. The link of any vertex of the first set V_0 consists of all simplices containing only vertices from the last vertex sets. Hence, the link is isomorphic to the k-partite complex Λ_n^{k-1} and we can consider β_u and $\beta_{u'}$ as cochains of this complex.

   a) As cochains in $\mathrm{C}^{k-2}(\Lambda_n^{k-1}; \mathbb{Z}_2)$, the cochain $\beta_u + \beta_{u'}$ is a cocycle.

      Indeed, we can calculate the coboundary for any simplex $\sigma \in \Lambda_n^{k-1}(k-1)$:

$$\delta_{k-2}(\beta_u + \beta_{u'})(\sigma) = \sum_{\tau \Subset \sigma} (\beta_u + \beta_{u'})(\tau) \tag{5.1.36}$$

$$= \underbrace{\sum_{\tau \Subset \sigma} \beta(\{u\} \cup \tau)}_{=\delta_{k-1}\beta(\{u\}\cup\sigma)+\beta(\sigma)} + \underbrace{\sum_{\tau \Subset \sigma} \beta(\{u'\} \cup \tau)}_{=\delta_{k-1}\beta(\{u'\}\cup\sigma)+\beta(\sigma)} \tag{5.1.37}$$

$$= \beta(\sigma) + \beta(\sigma) = 0 \tag{5.1.38}$$

      For the second last equality, we used that β is a cocycle, that is, it has vanishing coboundary. This shows that $\beta_u + \beta_{u'}$ is a $(k-2)$-dimensional cocycle in Λ_n^{k-1} and we can apply the induction hypothesis to obtain that it is already a coboundary. Let $\alpha^{(u,u')} \in \mathrm{C}^{k-3}(\Lambda_n^{k-1}; \mathbb{Z}_2)$ be a co-filling of $\beta_u + \beta_{u'}$.

   b) We can define a candidate for a co-filling of β by setting its values for any simplex $\sigma \in \Lambda_n^k(k-2)$:

$$\alpha(\sigma) := \begin{cases} \beta_u(\sigma) & \text{if } \sigma \cap V_0 = \emptyset \\ \alpha^{(u,u')}(\sigma \setminus \{u'\}) & \text{if } \sigma \cap V_0 = \{u'\} \end{cases} \tag{5.1.39}$$

   c) This candidate α is indeed a co-filling for β, that is, $\delta_{k-2}\alpha = \beta$.

      We need to calculate the coboundary of α, $\sigma \in \Lambda_n^k(k-1)$. Therefore, we consider the following two cases.

      Case 1: We have $\sigma \cap V_0 = \emptyset$, hence any facet τ of σ has empty intersection, as well:

$$\delta_{k-2}\alpha(\sigma) = \sum_{\tau \Subset \sigma} \alpha(\tau) \tag{5.1.40}$$

$$= \sum_{\tau \Subset \sigma} \beta(\{u\} \cup \tau) \tag{5.1.41}$$

$$= \delta_{k-1}\beta(\{u\} \cup \sigma) + \beta(\sigma) = \beta(\sigma) \tag{5.1.42}$$

Case 2: We have $\sigma \cap V_0 = \{u'\}$:

$$\delta_{k-2}\alpha(\sigma) = \sum_{\substack{\tau \Subset \sigma \\ u' \in \tau}} \alpha(\tau) + \alpha(\sigma \backslash \{u'\}) \tag{5.1.43}$$

$$= \beta_u(\sigma \backslash \{u'\}) + \sum_{\tau \Subset \sigma \backslash \{u'\}} \alpha^{(u,u')}(\tau) \tag{5.1.44}$$

$$= \beta_u(\sigma \backslash \{u'\}) + \underbrace{\delta_{k-3}\alpha^{(u,u')}}_{=\beta_u + \beta_{u'}}(\sigma \backslash \{u'\}) \tag{5.1.45}$$

$$= \beta_{u'}(\sigma \backslash \{u'\}) = \beta(\sigma) \tag{5.1.46}$$

Altogether we have $\delta_{k-2}\alpha = \beta$ and the induction step is complete.

□

In the proof, we use that the complete $(k+1)$-partite complex Λ_n^k is a composition (join) of the complete k-partite complex Λ_n^{k-1} and another set of vertices V_0. This gives a hint, why we use induction: We can reduce the problem to the part of the complex without V_0 and then we apply the induction hypothesis.

In the next step we need to lay the basis of the induction for the coboundary expansion parameters. Thus, we prove:

Lemma 5.1.4 (Coboundary Expansion of the Complete Bipartite Graph)**.** *The (unnormalized) coboundary expansion $h^0(K_{n,n})$ of the complete bipartite graph $K_{n,n} = \Lambda_n^1$ on $2n$ vertices is:*

$$h^0(K_{n,n}) = \begin{cases} \frac{n}{2} & \text{if } n \in 2\mathbb{N} \\ \frac{n}{2} + \frac{1}{2n} & \text{if } n \in 2\mathbb{N}+1 \end{cases} \tag{5.1.47}$$

Proof. This result can be proved elementarily by calculating the number of edges $|E(S, V \backslash S)|$ (V denotes the vertex set $K_{n,n}(0)$) between any subset S of vertices and its complement.

Let S be any subset of vertices of size less than $n = \frac{|V|}{2}$ and denote $S_i := S \cap V_i, i \in \{0,1\}$, the parts of S that lie in the different components V_0 and V_1. With $s_i := |S_i|$ and $s := |S| = s_0 + s_1$ we can calculate:

$$|E(S, V \backslash S)| = s_0(n - s_1) + s_1(n - s_0) = sn - 2s_0(s - s_0) \tag{5.1.48}$$

The last term on the right-hand side is maximized when $s_0 = s_1 = \frac{s}{2}$, if s is even, and for $|s_0 - s_1| \leq 1$, if s is odd, that is, the set is "levelled out left and right" (approximately for odd s). Hence, we have:

$$|E(S, V \backslash S)| \geq sn - \frac{s^2}{2} = s(n - \underbrace{\frac{s}{2}}_{\leq \frac{n}{2}}) \geq s\frac{n}{2} = \frac{n}{2}|S| \tag{5.1.49}$$

The first inequality is sharp if n is even, the second one is always sharp. If n is odd, the critical set is partitioned into subsets of size $s_0 = \left\lfloor \frac{s}{2} \right\rfloor$ and $s_1 = \left\lceil \frac{s}{2} \right\rceil$. Substituting these values for s_0 and s_1 yields:

$$|E(S, V \backslash S)| \geq sn - \frac{s^2 - 1}{2} \geq s\left(\frac{n}{2} + \frac{1}{2n}\right) = \left(\frac{n}{2} + \frac{1}{2n}\right)|S| \tag{5.1.50}$$

The last term is of order $\mathcal{O}\left(\frac{1}{n}\right)$. Putting everything together we obtain:

$$h^0(K_{n,n}) = \begin{cases} \frac{n}{2} & \text{if } n \in 2\mathbb{N} \\ \frac{n}{2} + \frac{1}{2n} & \text{if } n \in 2\mathbb{N} + 1 \end{cases} \tag{5.1.51}$$

□

The last step to prove coboundary expansion is to find a co-filling inequality for Λ_n^k. In the proof of the following lemma we will orient ourselves by the work of Dominic Dotterrer and Matthew Kahle ([DK10, Dot12]). However, the author of this thesis was able to improve the constants. The constants given here are optimal in the sense that for this proof (with this family of co-fillings and these estimates) there are no better results. Indeed, the proof yields a linear recursion for the expansion parameters of Λ_n^k which has the solution given in this thesis. Dominic Dotterrer and Matthew Kahle did not solve this recursion but gave an estimate on the solution. By varying the family or the estimates, however, there may be space for further improvements.

Lemma 5.1.5. *For any $k \geq 1$ and any $n \geq 1$, we have:*

$$h^{k-1}(\Lambda_n^k) \geq \frac{n^k(n-2)}{(3n-4)(2n-2)^{k-1} - n^k} \tag{5.1.52}$$

$$\approx \frac{n-2}{3 \cdot 2^{k-1} - 1} \quad \text{if } n \gg 1 \tag{5.1.53}$$

In [DK10], the result is:

$$h^{k-1}(\Lambda_n^k) \geq \frac{n}{2^{k+1} - 1} \tag{5.1.54}$$

Before we give a proof of this lemma, we want to discuss two special cases. First we observe that for $k = 1$ we get the bound from lemma 5.1.4 on the previous page:

$$h^0(\Lambda_n^1) \geq \frac{n(n-2)}{(3n-4) - n} = \frac{n}{2} \tag{5.1.55}$$

Secondly, if we take $n = 1$, as one can easily check, we obtain the k-dimensional simplex as "complete $(k+1)$-partite complex", that is, $\Lambda_1^k = \Delta_k$. Comparing the

results from lemma 5.1.2 on page 67 to the result from the lemma above, we can see that in this special case the lower bound is sharp:

$$\text{from lemma 5.1.2:} \quad 1 = \left\lceil \frac{k+1}{(k-1)+2} \right\rceil \geq \; h^{k-1}(\Delta_k) \geq \frac{k+1}{(k-1)+2} = 1 \tag{5.1.56}$$

$$\text{from lemma 5.1.5:} \quad h^{k-1}(\Lambda_1^k) \geq \frac{1(-1)}{0-1} = 1 \tag{5.1.57}$$

Now we want to prove lemma 5.1.5:

Proof. As mentioned before, the proof will use induction on the dimension k with n being fixed. We only need to show a co-filling inequality, because we already showed that Λ_n^k has vanishing cohomology in the right dimension.

1. Induction Hypothesis

   We assume in step k, that we have:

$$h^{l-1}(\Lambda_n^l) \geq \frac{n^l(n-2)}{(3n-4)(2n-2)^{l-1} - n^l} \quad \forall 1 \geq l < k \tag{5.1.58}$$

   To simplify notation, we abbreviate the term on the right-hand side and the corresponding co-filling constant by:

$$c_{k-1} := \frac{n^k(n-2)}{(3n-4)(2n-2)^{k-1} - n^k} \tag{5.1.59}$$

$$L_k := \frac{1}{c_{k-1}} \tag{5.1.60}$$

2. Initial Step

   The initial step for $k = 1$ is already taken in lemma 5.1.4 on page 71, cf. the discussion above the proof.

3. Induction Step

   For the induction step we use quite a similar technique as for showing that the cohomology vanishes. However, this proof works in a different dimension and we also need to introduce some "randomness" by finding a lot of co-fillings.

   To show the co-filling inequality, let $\beta \in \mathrm{B}^k(\Lambda_n^k; \mathbb{Z}_2)$ be an arbitrary coboundary which we want to find a small co-filling for. As β is a coboundary, we also know that there exists a co-filling $\alpha \in \mathrm{C}^{k-1}(\Lambda_n^k; \mathbb{Z}_2)$.

   a) Find a family of co-fillings for β

      As in the proof for the complete simplicial complexes, we will provide a family of co-fillings indexed by a set of vertices, in this case the vertices in the first subset V_0. Let $u \in V_0$ be an arbitrary vertex and consider the localization β_u to the link of this vertex. As this link $(\Lambda_n^k)_u$ is isomorphic

to Λ_n^{k-1} it is natural to use the induction hypothesis there and try to lift the result to Λ_n^k.

However, the localization β_u is (in general) no coboundary in Λ_n^{k-1}. Thus, we consider $\beta_u + \beta_{u'} \in \mathrm{C}^{k-1}(\Lambda_n^{k-1}; \mathbb{Z}_2)$ for any (other) vertex $u' \in V_0$. To show that this sum is a coboundary, we consider the localization $\alpha_u + \alpha_{u'} \in \mathrm{C}^{k-2}(\Lambda_n^{k-1}; \mathbb{Z}_2)$ of the co-filling α of β and show that it is a co-filling. Indeed, for any $\sigma \in \Lambda_n^{k-1}(k-1)$ we have:

$$\delta_{k-2}(\alpha_u + \alpha_{u'})(\sigma) = \sum_{\tau \Subset \sigma} \alpha(\{u\} \cup \tau) + \alpha(\{u'\} \cup \tau) \tag{5.1.61}$$

$$= \underbrace{\delta_{k-1}\alpha}_{=\beta}(\{u\} \cup \sigma) + \alpha(\sigma) + \underbrace{\delta_{k-1}\alpha}_{=\beta}(\{u'\} \cup \sigma) + \alpha(\sigma) \tag{5.1.62}$$

$$= \beta_u(\sigma) + \beta_{u'}(\sigma) \tag{5.1.63}$$

Hence, we can apply the induction hypothesis to $\beta_u + \beta_{u'}$ to obtain a minimal co-filling $\alpha^{(u,u')} \in \mathrm{C}^{k-2}(\Lambda_n^{k-1}; \mathbb{Z}_2)$ with the property:

$$\left|\alpha^{(u,u')}\right| \leq L_{k-1} \left|\beta_u + \beta_{u'}\right| \tag{5.1.64}$$

As $\alpha^{(u,u')}$ is minimal, this definition includes that $\alpha^{(u,u')} = 0$ if $u = u'$ as $\beta_u + \beta_u = 0$ (modulo 2).

Using $\alpha^{(u,u')}$ we can construct a co-filling of β, depending on u. For any $\sigma \in \Lambda_n^k(k-1)$ we define:

$$\alpha^{(u)}(\sigma) := \begin{cases} \beta_u(\sigma) & \text{if } \sigma \cap V_0 = \emptyset \\ \alpha^{(u,u')}(\sigma \backslash \{u'\}) & \text{if } \sigma \cap V_0 = \{u'\} \end{cases} \tag{5.1.65}$$

This definition really gives a co-filling, because for any $\sigma \in \Lambda_n^k(k)$, $\sigma \cap V_0 = \{u'\}$, we have:

$$\delta_{k-1}\alpha^{(u)}(\sigma) = \sum_{\tau \Subset \sigma} \alpha^{(u)}(\tau) \tag{5.1.66}$$

$$= \alpha^{(u)}(\sigma \backslash \{u'\}) + \sum_{\substack{\tau \Subset \sigma \\ u' \in \tau}} \alpha^{(u)}(\tau) \tag{5.1.67}$$

$$= \beta_u(\sigma \backslash \{u'\}) + \underbrace{\sum_{\tau \Subset \sigma \backslash \{u'\}} \alpha^{(u,u')}(\tau)}_{=\delta_{k-2}\alpha^{(u,u')}(\sigma \backslash \{u'\})} \tag{5.1.68}$$

$$= \beta_u(\sigma \backslash \{u'\}) + \underbrace{\delta_{k-2}\alpha^{(u,u')}}_{=\beta_u + \beta_{u'}}(\sigma \backslash \{u'\}) \tag{5.1.69}$$

$$= \beta_{u'}(\sigma \backslash \{u'\}) = \beta(\sigma) \tag{5.1.70}$$

b) Calculate the average "co-filling parameter" for this family

To calculate the average, we first observe that the support of each $\alpha^{(u)}$ can be partitioned into disjoint subsets:

$$\operatorname{supp} \alpha^{(u)} = \operatorname{supp} \beta_u \sqcup \bigsqcup_{u \neq u' \in V_0} \operatorname{supp} \alpha^{(u,u')} \tag{5.1.71}$$

Hence, we have:

$$\left|\alpha^{(u)}\right| = |\beta_u| + \sum_{u \neq u' \in V_0} \left|\alpha^{(u,u')}\right| \tag{5.1.72}$$

Moreover, we observe:

$$|\beta| = \sum_{u \in V_0} |\beta_u| \,, \tag{5.1.73}$$

because every face in the support of β contains exactly one vertex from V_0 (since it has to contain a vertex from every subset V_i and the complex is multipartite) and $|\beta_u|$ counts the faces in the support that contain u.

We may now calculate, using the co-filling inequality for the $\alpha^{(u,u')}$'s and the triangle inequality for norms on cochains:

$$\begin{aligned}
\sum_{u \in V_0} \left|\alpha^{(u)}\right| &= \sum_{u \in V_0} \left(|\beta_u| + \sum_{u \neq u' \in V_0} \left|\alpha^{(u,u')}\right| \right) && (5.1.74)\\
&= |\beta| + \sum_{\substack{u,u' \in V_0 \\ u \neq u'}} \underbrace{\left|\alpha^{(u,u')}\right|}_{\leq L_{k-1} \left|\beta_u + \beta_{u'}\right|} && (5.1.75)\\
&\leq |\beta| + L_{k-1} \sum_{\substack{u,u' \in V_0 \\ u \neq u'}} \underbrace{|\beta_u + \beta_{u'}|}_{\leq |\beta_u| + \left|\beta_{u'}\right|} && (5.1.76)\\
&\leq |\beta| + L_{k-1} \sum_{\substack{u,u' \in V_0 \\ u \neq u'}} |\beta_u| + |\beta_{u'}| && (5.1.77)
\end{aligned}$$

Collecting all terms and using equation 5.1.73, we finally get ($|V_0| = n$):

$$\begin{aligned}
\frac{1}{|V_0|} \sum_{u \in V_0} \left|\alpha^{(u)}\right| &\leq \frac{1}{n} \left(|\beta| + L_{k-1} \left((n-1) |\beta| + n |\beta| - |\beta| \right) \right) && (5.1.78)\\
&= |\beta| \, \frac{1 + L_{k-1}(2n-2)}{n} && (5.1.79)
\end{aligned}$$

c) Estimate the co-filling constant of Λ_n^k

By random co-filling, we thus get the following co-filling inequality (using that we have already proved vanishing cohomology):

$$\min_{b\in \mathrm{B}^{k-1}(\Lambda_n^k;\mathbb{Z}_2)} |\alpha+b| \leq \min_{u\in V_0} \left|\alpha^{(u)}\right| \leq \frac{1}{|V_0|}\sum_{u\in V_0}\left|\alpha^{(u)}\right| \leq |\beta|\,\frac{1+L_{k-1}(2n-2)}{n} \tag{5.1.80}$$

The optimal co-filling constant hence is less or equal than $\frac{1+L_{k-1}(2n-2)}{n}$. However, the best constant we can get from this method fulfills the recursion:

$$L_k = \frac{1+L_{k-1}(2n-2)}{n} \quad \text{and} \quad L_1 = \frac{2}{n} \tag{5.1.81}$$

This linear recursion can be solved to obtain[3]:

$$L_k = \frac{(3n-4)(2n-2)^{k-1}}{(n-2)n^k} - \frac{1}{n-2} \tag{5.1.82}$$

By lemma 3.1.3 on page 32 we thus get a lower bound on the coboundary expansion parameter:

$$h^{k-1}(\Lambda_n^k) \geq \frac{1}{L_k} = \frac{n^k(n-2)}{(3n-4)(2n-2)^{k-1}-n^k} \tag{5.1.83}$$

Hence, the induction step is complete and the result follows.

□

The last step in the proof clarifies, why the special case $k=1$ gives a sharp estimate – it is defined that way. If we look further into the proof, the other special case $n=1$ is also not very surprising as the definition of $\alpha^{(u)}$ reduces to the co-filling from the proof for the complete complex if there is no $u' \neq u \in V_0$.

In the rest of this chapter we want to give some kind of intuition why the proof works as it does and we give an example of a cochain that gives an upper bound on the expansion parameters.

The first intuition was already hinted at beneath the proof of vanishing cohomology, namely the special structure of the complexes in consideration. The $(k+1)$-partite complex Λ_n^k consists of $k+1$ equal subsets $V_0,\dots,V_k$. If we remove the first subset V_0 and all the simplices containing a vertex from V_0, what is left is an isomorphic copy of Λ_n^{k-1}, which certainly has the same expansion properties as Λ_n^{k-1}. Thus, we want to apply induction by reducing the given coboundary to a coboundary on the last k vertex sets. If we recall the proof for the simplex (and the fact that $\Delta_k = \Lambda_1^k$), our first attempt is to take a vertex u from V_0 and for every face σ containing u we "push" the value on σ to the facet $\sigma\backslash\{u\}$ in the link of u – this

[3]This is the point were the proof varies from that in [DK10, Dot12].

is exactly the construction of the localization of the coboundary. For the simplex, this was enough as there was only one vertex in the first set and we had enough high-dimensional faces to conclude that the values are correct on all the other faces (using that we have a cocycle). For the general $(k+1)$-partite complex, however, there is neither a canonical choice of vertex from V_0 nor is there a "connection" between two vertices of V_0. Indeed, if we do this construction for two different vertices $u, u' \in V_0$, we get two values for every simplex of co-dimension 1, which need not be equal. On the other hand, we also have more freedom to define the values of the co-filling on simplices that are containing a different vertex from V_0, and we use this freedom. The values on these simplices only affect the values of the coboundary on top-dimensional simplices that contain the same vertex from V_0. From the proof for the simplex we already saw, that our definition as localization works for simplices that contain u (the multipartite complexes look locally similar to the simplex), problems only arise for the other simplices. Hence, we have to set the values there to compensate for the value from the localization and then enforce the right value, which is for example given by the localization with respect to $u' \in V_0$. As we work with $\mathbb{Z}_2$ coefficients, we add both shares: $\beta_u + \beta_{u'}$ (in the notation of the proof from above). This cochain has values on simplices in the link of u (or u', which is the same), but we want to get values on simplices containing u' in one dimension lower that produce the same behaviour as $\beta_u + \beta_{u'}$. Thus, we go one dimension down by co-filling $\beta_u + \beta_{u'}$ (in a minimal way) and then "pull it back" to simplices containing u', giving the definitions of $\alpha^{(u,u')}$ and $\alpha^{(u)}$.

The co-filling constants for $\alpha^{(u,u')}$ (from the induction hypothesis) then are used to deduce a linear recursion for the new co-filling constant. In this process, also the triangle inequality for norms of cochains is used and there are possibilities to improve this estimate for large coboundaries by a double counting argument (idea due to Kristóf Huszár, IST Austria). This yields a co-filling constant that depends on the norm of β, that is, we have a quadratic co-filling inequality.

In the light of this improvement, the question arises whether the result proved in this thesis is good. This can be checked by giving upper bounds on the expansion parameters (just as for the complete complex). Finding upper bounds is not easy, as they require a very good understanding of the complex. Nevertheless, there is a trivial upper bound:

Lemma 5.1.6 (Trivial Upper Bound)**.** *For any $k \geq 1$ and $n \geq 1$, we have:*

$$h^{k-1}(\Lambda_n^k) \leq n \tag{5.1.84}$$

Proof. Let $\sigma \in \Lambda_n^k(k-1)$ be any $(k-1)$-dimensional simplex in Λ_n^k and define the cochain $\alpha \in \mathrm{C}^{k-1}(\Lambda_n^k; \mathbb{Z}_2)$ as the indicator function $\mathbb{1}_{\{\sigma\}}$. α certainly has minimal norm modulo coboundaries, because it is no coboundary (indeed, it is no cocycle, as the following calculation shows) and the Hamming norm only takes non-negative integer values. The support of the coboundary $\delta_{k-1}\alpha$ of α consists of all k-simplices that contain σ. There are exactly n such simplices (corresponding to the n vertices

in the unique vertex subset that has empty intersection with σ). We thus have:

$$|[\alpha]| = |\alpha| = 1,\ |\delta_{k-1}\alpha| = n \quad \Longrightarrow \quad h^{k-1}(\Lambda_n^k) \leq \frac{|\delta_{k-1}\alpha|}{|\alpha|} = n \tag{5.1.85}$$

□

Rem. 5.1.1. An upper bound of $\frac{n}{4}$ can also be achieved for complexes with n divisible by 4 and $k = 2$ by giving an example for a critical cochain. Lemma 5.1.5 on page 72 yields a lower bound of $\frac{n}{5-\frac{4}{n}}$. Altogether, we can determine the expansion parameter of Λ_n^2, for n divisible by 4, to be in the interval:

$$\frac{n}{5-\frac{4}{n}} \leq h^1(\Lambda_n^2) \leq \frac{n}{4} \tag{5.1.86}$$

This example is due to Kristóf Huszár and Uli Wagner (both IST Austria).

5.2 Local to Global Methods

In the last chapter about random co-filling we had to construct a family of co-fillings which we were able to analyze with respect to its expansion properties. In the special cases of the complete complex and the complete multipartite complexes a local view of the complex proved to be very useful. Indeed, for both complexes the co-fillings were defined starting from the localization of the coboundary to the link of a vertex with some structural modifications. This leads to the question that is discussed in this chapter:

> Can we use properties of the local structure of the complex (for example of its links) to deduce expansion in higher dimensions?

However, local properties are mostly not enough, we also need to assume some global "connectedness" property as well. Together with this assumptions there are methods to prove expansion in higher dimensions, which will be presented in the following sections. The following concepts appeared in [LM13] and (more elaborate) in [KKL14a, KKL14b, EK15]. In both works similar ideas are used. However, in this thesis we are using a different norm. Thus, the core concepts are adopted, while the calculation is adjusted to fit the different norm.

The size of the co-filling of a coboundary depends very much on the special structure of the coboundary in consideration. As we want to estimate the size of the minimal co-filling for any coboundary we are not always able to use this special structure (random co-filling is a try to overcome this issue by introducing randomness). Thus, we have to use the structure of the simplicial complex as a whole, that is, from a global and local view. By concentrating on the local views of the coboundary, we might not find the optimal co-filling as this co-filling may utilize the

global structure. As a consequence, the conditions on the links have to be stronger to get a good expansion of the complex, or otherwise, we only get weaker results.

In this chapter, we usually have two differences to normal expansion. First, we only get expansion for small cochains, that is:

$$\|\delta\alpha\|_X \geq \epsilon \, \|[\alpha]\|_X \tag{5.2.1}$$

only for cochains with small norms $\|[\alpha]\|_X \leq \mu$.

Rem. 5.2.1. In [GW16] it is shown that link-based methods (for expansion with respect to the normalized Hamming norm) can only work for cochains with normalized Hamming norm $\|\alpha\|_X$ bounded away from $\frac{1}{2}$. Indeed, Anna Gundert and Uli Wagner give an example of a complex and a cochain with normalized Hamming norm tending to $\frac{1}{2}$ that has a small coboundary. This cochain bounds the coboundary expansion parameter from above by $\mathcal{O}(\log n/n)$, but the complex has good expansion properties in the links (they are random graphs). For more details, see [GW16].

Bounding the norm is an easy way to obtain some kind of "local sparsity" of the cochains which then leads to improvements in the estimates. These improvements can be seen for example in the (edge) expansion of the complete graph. Here, the critical 0-cochains have large support. Hence, if we restrict ourselves to cochains with small support we get a better constant in the estimate. Of course, we only get improvements if the critical and the large cochains coincide up to some extent. This need not be the case, however.

The second difference is that we show cocycle expansion for so-called "locally minimal cochains" (cf. definition 5.2.1). This is again a property of "local sparsity". By further assuming that the complex has bounded degree in the right dimension these restrictions can be removed and we can prove cocycle expansion for all cochains.

5.2.1 Expansion for Small Cochains

We now want to start with introducing the necessary notions and discuss the results we can deduce from them.

First we define so-called minimal and locally minimal cochains. Informally, a minimal cochains is one of minimal norm modulo coboundaries, while a locally minimal cochain is minimal in every localization:

Definition 5.2.1 (Minimal and Locally Minimal Cochains)**.** *Let X be a simplicial complex with norm $\|\cdot\|_X$ and $0 \leq k \leq \dim X$. Let $\alpha \in \mathrm{C}^k(X)$ be a cochain.*

- *α is called* minimal *if $\|\alpha\|_X = \|[\alpha]\|_X = \min_{b \in \mathrm{B}^k(X)} \|\alpha + b\|_X$.*
- *α is called* locally minimal *if for every $\emptyset \neq \sigma \in X^{(k-1)}$ the localization α_σ is minimal in X_σ with respect to a norm $\|\cdot\|_{X_\sigma}$ on the link.*[4]

[4]Usually, there is a connection between $\|\cdot\|_X$ and $\|\cdot\|_{X_\sigma}$, for example both can be the Hamming norm.

Note, that there is also a slightly different definition appearing in [KKL14b], namely that in the definition of locally minimal cochain the localization only has to be with respect to vertices and not arbitrary-dimensional simplices. In this thesis, we will not use this definition. Moreover, in 2-dimensional complexes these two definitions coincide.

Obviously, we only have to check coboundary expansion for minimal cochains. Locally minimal cochains appear, because we want to apply coboundary expansion in the links.

From now on, we only consider the Hamming norm and $\mathbb{Z}_2$-coefficients. At least for 1-dimensional cochains this allows a picturesque characterization of locally minimal cochains:

Example 5 (Locally minimal 1-cochains)**.** Let X be an arbitrary simplicial complex and $\alpha \in \mathrm{C}^1(X;\mathbb{Z}_2)$ be a locally minimal cochain in dimension 1. Hence, if we take any $v \in X(0)$, we know that α_v is a minimal 0-dimensional cochain in X_v. As observed earlier, there are only two elements in the coset modulo coboundaries, namely α_v and $\alpha_v + \mathbb{1}_{X_v(0)}$, which corresponds to the complement. We thus have:

$$|\alpha_v| \leq \left|\alpha_v + \mathbb{1}_{X_v(0)}\right|, \tag{5.2.2}$$

which implies:

$$|\operatorname{supp}\alpha_v| \leq |X_v(0)\setminus \operatorname{supp}\alpha_v| = |X_v(0)| - |\operatorname{supp}\alpha_v| \tag{5.2.3}$$

$$\implies |\alpha_v| = |\operatorname{supp}\alpha_v| \leq \frac{|X_v(0)|}{2} \tag{5.2.4}$$

So local minimality in dimension 1 means that if we count all edges that emanate from an arbitrary vertex at most half of them belong to the support of α. Fig. 5.4 illustrates this situation.

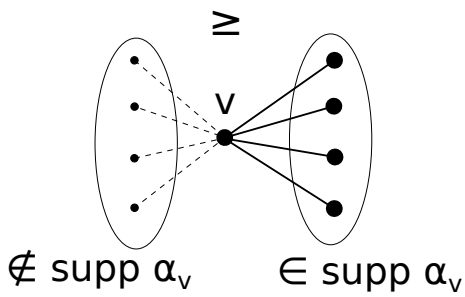

Fig. 5.4: Illustration of a locally minimal 1-cochain

Now the question arises whether "minimal" and "locally minimal" are different properties and if yes, what is their relation? Is it enough for a cochain to be locally minimal to be already minimal in the global sense? The next example shows that this is not the case:

Example 6 ("Locally minimal" does not imply "minimal"). Let X be the 1-dimensional simplicial complex illustrated in Fig. 5.5 (left picture), that is:

$$X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{b,d\}, \{c,d\}, \{a,c\}\} \subseteq 2^{\{a,b,c,d\}} \tag{5.2.5}$$

Let α be the cochain $\mathbb{1}_{\{\{a,b\},\{c,d\}\}}$. We show that α is locally minimal, but not minimal. Indeed, if we take any vertex $v \in \{a,b,c,d\}$ we always have the picture on the right in Fig. 5.5. Every vertex v is incident to two edges in X, one of these is in the support of α, the other one is in its complement:

$$|\alpha_v| = 1 \leq 1 = \left|\alpha_v + \mathbb{1}_{X_v(0)}\right| \tag{5.2.6}$$

But α is also a coboundary, since:

$$\alpha = \delta_0 \mathbb{1}_{\{a,c\}} \tag{5.2.7}$$

Hence, α is not minimal:

$$|[\alpha]| = 0 < 2 = |\alpha| \tag{5.2.8}$$

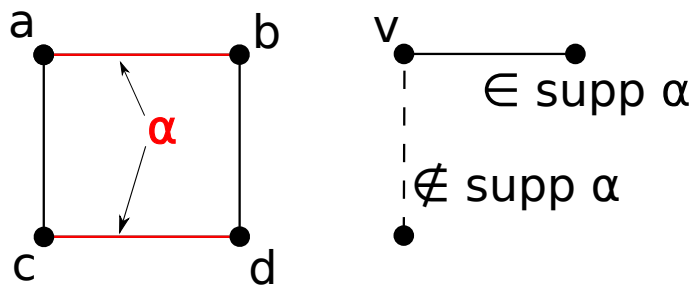

Fig. 5.5: Illustration of the complex from example 6

On the other hand, every minimal cochain is locally minimal as well, as the following lemma shows.

Lemma 5.2.1. *Let X be a simplicial complex and let $\alpha \in \mathrm{C}^k(X;\mathbb{Z}_2)$, $0 \leq k \leq \dim X$. We consider the Hamming norm on the cochains.*

Then there exists a cochain $\gamma \in \mathrm{C}^{k-1}(X;\mathbb{Z}_2)$ such that $\alpha + \delta_{k-1}\gamma$ is a locally minimal cochain and we have:

$$|\alpha + \delta_{k-1}\gamma| \leq |\alpha| \tag{5.2.9}$$

Equality (for the construction in the proof) holds if and only if α is already locally minimal.

Thus, if α is minimal, then α is also locally minimal.

Proof. Let $\alpha \in \mathrm{C}^k(X;\mathbb{Z}_2)$ be an arbitrary cochain. If α is locally minimal, then there is nothing to do, we can choose $\gamma = 0$ and the inequality holds with equality.

If α is not locally minimal, we can find a face $\emptyset \neq \sigma \in X^{(k-1)}$ such that:

$$|\alpha_\sigma| > |[\alpha_\sigma]| \tag{5.2.10}$$

Hence, we can choose a minimal cochain $c' \in \mathrm{C}^{k-|\sigma|-1}(X_\sigma;\mathbb{Z}_2)$ with the property:

$$|\alpha_\sigma + \delta c'| < |\alpha_\sigma| \tag{5.2.11}$$

We want to lift this cochain to the complex X. Therefore, we define a cochain $c \in \mathrm{C}^{k-1}(X;\mathbb{Z}_2)$ that fulfills $c_\sigma = c'$ by setting its value on every simplex $\tau \in X(k-1)$:

$$c(\tau) := \begin{cases} c'(\tau\backslash\sigma) & \text{if } \sigma \subseteq \tau \\ 0 & \text{else} \end{cases} \tag{5.2.12}$$

We can now calculate $\alpha + \delta_{k-1}c$ for any face $\tau \in X(k)$. We distinguish between two cases:
Case 1: $\sigma \subseteq \tau$:

$$(\alpha + \delta_{k-1}c)(\tau) = \alpha(\tau) + \sum_{\kappa \Subset \tau} \underbrace{c(\kappa)}_{=0,\text{ if } \sigma \not\subseteq \kappa} \tag{5.2.13}$$

$$= \alpha(\tau) + \sum_{\kappa \Subset \tau\backslash\sigma} \underbrace{c(\sigma \cup \kappa)}_{=c'(\kappa)} \tag{5.2.14}$$

$$= \alpha(\tau) + (\delta c')(\tau\backslash\sigma) \tag{5.2.15}$$

Case 2: $\sigma \not\subseteq \tau$, then every facet of τ does not contain σ as well.

$$(\alpha + \delta_{k-1}c)(\tau) = \alpha(\tau) + \sum_{\kappa \Subset \tau} \underbrace{c(\kappa)}_{=0} \tag{5.2.16}$$

$$= \alpha(\tau) \tag{5.2.17}$$

Thus, by adding this coboundary we only change α on simplices containing σ. Hence, we have:

$$\operatorname{supp}(\alpha + \delta_{k-1}c) = (\operatorname{supp}(\alpha + \delta_{k-1}c) \cap \{\tau \in X(k) \mid \sigma \subseteq \tau\}) \tag{5.2.18}$$

$$\sqcup\, (\operatorname{supp}(\alpha + \delta_{k-1}c) \cap \{\tau \in X(k) \mid \sigma \not\subseteq \tau\}) \tag{5.2.19}$$

$$= (\operatorname{supp}(\alpha + \delta_{k-1}c) \cap \{\tau \in X(k) \mid \sigma \subseteq \tau\}) \tag{5.2.20}$$

$$\sqcup\, (\operatorname{supp}\alpha \cap \{\tau \in X(k) \mid \sigma \not\subseteq \tau\}) \tag{5.2.21}$$

This can be translated to the Hamming norms on cochains since $\alpha + \delta_{k-1}c$ evaluates on simplices containing σ the same as $\alpha_\sigma + \delta c'$ does on X_σ:

$$|\alpha + \delta_{k-1}c| = |\alpha_\sigma + \delta c'| + |\operatorname{supp}\alpha \cap \{\tau \in X(k) \mid \sigma \not\subseteq \tau\}| \tag{5.2.22}$$

The first term is strictly smaller than $|\alpha_\sigma|$ by assumption, hence, we obtain by reversing the calculation:

$$|\alpha + \delta_{k-1} c| < |\alpha| \tag{5.2.23}$$

By adding a coboundary to α, we thus decreased the Hamming norm by at least one. We can now repeat this argument to decrease it further and in at most $|\alpha|$ steps we obtain a locally minimal cochain (the cochain 0 is locally minimal) and the inequality is automatically fulfilled. By summing up all added coboundaries we get the claimed γ.

Finally, if we start with a minimal cochain α, α has to be locally minimal as well, because if not, we could decrease the Hamming norm by adding a coboundary. Since α is a representative of the coset modulo coboundaries with minimal Hamming norm, this is not possible, which yields a contradiction. □

The above lemma contains more than stated. Indeed, we can also bound the norm of the "correcting" coboundary γ:

Lemma 5.2.2. *Beside the conditions of the previous lemma we further assume that the degree of each lower-dimensional simplex is bounded by a constant $Q_{k-1} > 0$, i.e.:*

$$\forall \sigma \in X^{(k-1)} : \quad \deg_{k-1} \sigma \leq Q_{k-1} \tag{5.2.24}$$

Then we can bound the Hamming norm of γ (from the previous lemma) linearly in terms of the Hamming norm of α:

$$|\gamma| \leq Q_{k-1} \, |\alpha| \tag{5.2.25}$$

Proof. We consider the same step-by-step process to construct γ as in the previous proof. Over-all, we obtain γ as a sum of cochains c_i from each step (we have to take m steps):

$$\gamma = \sum_{i=1}^{m} c_i \tag{5.2.26}$$

Every cochain c_i is constructed as a lift of a cochain from the link of a simplex $\sigma_i \in X^{(k-1)}$. The support of c_i is contained in the subset of simplices that contain σ_i as a face. By definition, there are exactly $\deg_{k-1} \sigma_i$ of them, thus:

$$\forall i = 1, \ldots, m : \quad |c_i| \leq \deg_{k-1} \sigma_i \leq Q_{k-1} \tag{5.2.27}$$

Using this estimate and the triangle inequality for the Hamming norm, we obtain:

$$|\gamma| \leq \sum_{i=1}^{m} |c_i| \leq m \cdot Q_{k-1} \tag{5.2.28}$$

Last, the number m of steps is certainly bounded by the Hamming norm of α, because in every step the Hamming norm is reduced by at least one:

$$|\gamma| \leq Q_{k-1} |\alpha| \tag{5.2.29}$$

□

However, this estimate may be far from optimal. Indeed, in every step the size of $\operatorname{supp} \alpha$ may be reduced by much more than one, which reduces the number of steps drastically. Moreover, the estimate on the size of c_i is really the worst-case estimate. A better estimate may be possible using the fact that we can choose c_i minimal. The following continuation of example 6 on page 81 illustrates this situation:

Example 7. Let X be the simplicial complex defined in example 6 on page 81 and let $\alpha = \mathbb{1}_{\{\{a,c\},\{c,d\}\}}$ be the cochain from which we want to find a locally minimal representation. As the localizations with respect to the vertices a, b and d are already minimal, we have to modify the cochain in the link of c by adding the coboundary of the cochain:

$$c = \mathbb{1}_{\{\{c\}\}} \in \mathrm{C}^0(X; \mathbb{Z}_2), \tag{5.2.30}$$

which yields the cochain:

$$\alpha + \delta_0 c = 0 \tag{5.2.31}$$

Hence, we are finished after one step in which we obtain a cochain c of size $|c| = 1$, whereas our cochain has size 2.

If we assume that for every simplex $\emptyset \neq \sigma \in X^{(k-1)}$ the link X_σ is a cocycle expander in dimension $(k - |\sigma| - 1)$ with $h_z^{k-|\sigma|-1}(X_\sigma) \geq \epsilon > 0$, we can also find a better estimate on $|c| = |c'|$ using the improvement $d_c := |\alpha| - |\alpha + \delta_{k-1} c| = |\alpha_\sigma| - |\alpha_\sigma + \delta c'|$:

$$\begin{aligned}
|c| = |c'| &\leq \frac{1}{\epsilon} |\delta c'| && (5.2.32)\\
&= \frac{1}{\epsilon} |\alpha_\sigma + (\alpha_\sigma + \delta c')| && (5.2.33)\\
&\leq \frac{1}{\epsilon} |\alpha_\sigma| + \frac{1}{\epsilon} |\alpha_\sigma + \delta c'| && (5.2.34)\\
&= \frac{1}{\epsilon} (2 |\alpha_\sigma| - d_c) && (5.2.35)
\end{aligned}$$

However, this estimate can only be useful for a large cocycle expansion parameter $\epsilon > 2 > 0$. Sadly, we cannot give a better estimate on d_c and thus we obtain the following quadratic bound ($|\alpha_\sigma| \leq |\alpha|$):

$$|\gamma| \leq |\alpha| \left(\frac{2}{\epsilon} |\alpha| - \frac{1}{\epsilon} \right) \tag{5.2.36}$$

This bound is good for small cochains α, but may be worse than the original one for larger cochains.

We now want to discuss how expansion for small and locally minimal cochains can be used to prove general expansion. This is necessary as expansion for small cochains can be proved by looking at links.

The key to the conclusion is that the complex has bounded degree.[5] Indeed, using a bound on the degree we can deduce a good cocycle expansion from cocycle expansion on small cochains in one dimension higher.

Lemma 5.2.3. *Let X be a simplicial complex with the Hamming norm $|\cdot|$ and $0 \leq k \leq \dim X - 2$. We assume that:*

- *X has degree bounded by a constant Q_k, that is:*

$$\forall \sigma \in X^{(k)} : \quad \deg_k \sigma \leq Q_k \tag{5.2.37}$$

- *X fulfills "expansion for small, locally minimal cochains" in dimension $(k+1)$, that is, there exist constants $\bar{\mu}_{k+1}, \bar{\epsilon}_{k+1} > 0$ such that for any locally minimal cochain $\alpha \in \mathrm{C}^{k+1}(X; \mathbb{Z}_2)$:*

$$|\alpha| < \bar{\mu}_{k+1} |X(k+1)| \quad \Longrightarrow \quad |\delta_{k+1}\alpha| \geq \bar{\epsilon}_{k+1} |\alpha| \tag{5.2.38}$$

Then the cocycle expansion parameter $h_z^k(X)$ of X is bounded by a constant $\epsilon_k > 0$:

$$h_z^k(X) \geq \epsilon_k > 0 \tag{5.2.39}$$

The constant ϵ_k can be chosen as:

$$\epsilon_k := \min \left\{ \bar{\mu}_{k+1} \frac{|X(k+1)|}{|X(k)|}, \frac{1}{Q_k} \right\} \tag{5.2.40}$$

The specific value of $\bar{\epsilon}_{k+1}$ is irrelevant as long as it is strictly positive.

Proof. We define ϵ_k as in the statement of the lemma. Now, let $\alpha \in \mathrm{C}^k(X; \mathbb{Z}_2)$ be an arbitrary cochain. We want to prove that:

$$|\delta_k \alpha| \geq \epsilon_k \min_{z \in \mathrm{Z}^k(X; \mathbb{Z}_2)} |\alpha + z| \tag{5.2.41}$$

To show this, we distinguish between the following two cases:

[5] This is a vacuous condition if we only consider a single complex. However, we mainly consider families of expanders and we want to uniformly bound their expansion parameters. To do this, we need that the complexes have uniformly bounded degree.

- Case 1: $|\delta_k\alpha| \geq \bar{\mu}_{k+1}\,|X(k+1)|$

  Continuing the inequality we can deduce what we want to show for this case:

$$|\delta_k\alpha| \geq \bar{\mu}_{k+1}\,|X(k+1)| \tag{5.2.42}$$
$$= \bar{\mu}_{k+1}\frac{|X(k+1)|}{|X(k)|}\underbrace{|X(k)|}_{\geq|\alpha|} \tag{5.2.43}$$
$$\geq \bar{\mu}_{k+1}\frac{|X(k+1)|}{|X(k)|}\,|\alpha| \tag{5.2.44}$$
$$\geq \epsilon_k\,|\alpha| \tag{5.2.45}$$

- Case 2: $|\delta_k\alpha| < \bar{\mu}_{k+1}\,|X(k+1)|$

  We can apply lemma 5.2.1 on page 81 and lemma 5.2.2 on page 83 to the cochain $\delta_k\alpha$ to obtain a cochain $\gamma \in \mathrm{C}^k(X;\mathbb{Z}_2)$ that makes $\delta_k\alpha$ a locally minimal cochain. Furthermore, we get:

$$|\delta_k\alpha + \delta_k\gamma| \leq |\delta_k\alpha| < \bar{\mu}_{k+1}\,|X(k+1)| \tag{5.2.46}$$
$$|\gamma| \leq Q_k\,|\delta_k\alpha| \tag{5.2.47}$$

  As $\delta_k\alpha + \delta_k\gamma$ is locally minimal, we may apply the assumption that X is expanding on small cochains to obtain ($\delta_{k+1}\delta_k = 0$):

$$0 = |\delta_{k+1}\delta_k(\alpha+\gamma)| \geq \bar{\epsilon}_{k+1}\,|\delta_k(\alpha+\gamma)| \tag{5.2.48}$$

  $\bar{\epsilon}_{k+1}$ is positive, hence the right-hand side is zero:

$$\delta_k(\alpha+\gamma) = 0, \tag{5.2.49}$$

  which means that α and γ are in the same coset modulo cocycles: $\gamma = \alpha + (\gamma+\alpha) \in \alpha + \mathrm{Z}^k(X;\mathbb{Z}_2)$. Using the estimate on the norm of γ we get cocycle expansion:

$$|\delta_k\alpha| \geq \frac{1}{Q_k}\,|\gamma| \geq \epsilon_k \min_{z\in\mathrm{Z}^k(X;\mathbb{Z}_2)} |\alpha + z|\,, \tag{5.2.50}$$

  and this is what we wanted to show.

□

The result of the lemma (that is ϵ_k) can be improved by proving expansion for bigger cochains (that is to increase $\bar{\mu}_{k+1}$) or by improving the estimate from lemma 5.2.2 on page 83. The later restriction seems to be the one where big improvements may be possible, because the estimate in the lemma seems not to be very strict. The first restriction may be loosened by making $\bar{\epsilon}_{k+1}$ very small as the specific value is irrelevant.

The content of this lemma is quite amazing as we can prove expansion from two conditions that are quite vacuous like bounded degree and positive "cocycle expansion". One could say that the second condition is trivial, because every complex has a strictly positive cocycle expansion parameter. However, we need a stricter result in the way that we do not lower-bound the norm of the coboundary in terms of the minimal norm modulo cocycles but in terms of the norm of a locally minimal cochain. As we stated earlier, a locally minimal cochain need not be minimal modulo coboundaries and hence also not minimal modulo cocycles. As a consequence, the condition of "expansion for small and locally minimal cochains" also includes that there is no non-trivial locally minimal cocycle, which is small. This is the only thing that we use in the proof. We could thus restate the lemma in the following way:

Lemma 5.2.4. *Let X be a simplicial complex with Hamming norm $|\cdot|$ and $0 \leq k \leq \dim X - 2$. Denote by Q_k the maximum degree of a face of dimension less or equal than $k-1$, that is:*

$$Q_k := \max_{\sigma \in X^{(k)}} \deg_k \sigma \tag{5.2.51}$$

Let $\bar{\mu}_{k+1}$ be the minimum of the norms of the non-trivial locally minimal cocycles in dimension $k+1$:

$$\bar{\mu}_{k+1} := \min \left\{ |\alpha| \mid 0 \neq \alpha \in \mathrm{Z}^{k+1}(X; \mathbb{Z}_2), \alpha \text{ locally minimal} \right\} \tag{5.2.52}$$

Then there exists a constant $\epsilon_k > 0$, such that:

$$h_z^k(X) \geq \epsilon_k > 0 \tag{5.2.53}$$

The constant ϵ_k can be chosen as:

$$\epsilon_k := \min \left\{ \bar{\mu}_{k+1} \frac{|X(k+1)|}{|X(k)|}, \frac{1}{Q_k} \right\} \tag{5.2.54}$$

Proof. The definition of $\bar{\mu}_{k+1}$ implies that for any $\mu < \bar{\mu}_{k+1}$ we can find a constant $\bar{\epsilon}_{k+1}$ such that the complex is expanding on locally minimal cochains which have norm smaller than $\mu\,|X(k+1)|$ (we can take the minimum over finitely many positive values). Apply lemma 5.2.3 on page 85 with this constant to obtain $h_z^k(X) \geq \epsilon_k(\mu)$. We can do this for any $\mu < \bar{\mu}_{k+1}$, hence $h_z^k(X) \geq \epsilon_k$. □

Expansion on small locally minimal cochains can also be used to obtain a lower bound on the cosystole parameter $\mathrm{syst}^k(X)$ of a complex:

Lemma 5.2.5. *Let X be a simplicial complex which fulfills expansion on small locally minimal cochains in dimension k, $0 \leq k \leq \dim X - 1$, that is, there are constants $\bar{\epsilon}_k, \bar{\mu}_k > 0$ such that for any locally minimal cochain $\alpha \in \mathrm{C}^k(X; \mathbb{Z}_2)$ we have:*

$$|\alpha| < \bar{\mu}_k \, |X(k)| \quad \Longrightarrow \quad |\delta_k \alpha| \geq \bar{\epsilon}_k \, |\alpha| \tag{5.2.55}$$

Then X has μ-large cosystoles in dimension k with $\mu \geq \bar{\mu}_k \, |X(k)|$:

$$syst^k(X) \geq \mu \geq \bar{\mu}_k \, |X(k)| \tag{5.2.56}$$

Proof. Let $\alpha \in \mathrm{Z}^k(X;\mathbb{Z}_2) \backslash \mathrm{B}^k(X;\mathbb{Z}_2)$ be a cocycle that is not a coboundary. We have to bound the Hamming norm of α from below.

If we already have that $|\alpha| \geq \bar{\mu}_k \, |X(k)|$, then there is nothing to prove. Thus, we assume that $|\alpha| < \bar{\mu}_k \, |X(k)|$ in the following.

By lemma 5.2.1 on page 81 we can find a cochain $\gamma \in \mathrm{C}^{k-1}(X;\mathbb{Z}_2)$ such that $\alpha + \delta_{k-1}\gamma$ is locally minimal and:

$$|\alpha + \delta_{k-1}\gamma| \leq |\alpha| < \bar{\mu}_k \, |X(k)| \tag{5.2.57}$$

$$\tag{5.2.58}$$

Now we can apply the condition of expansion on small locally minimal cochains to $\alpha + \delta_{k-1}\gamma$, which is still a cocycle (this is the same trick as in the previous two lemmas):

$$0 = |\delta_k(\alpha + \delta_{k-1}\gamma)| \geq \bar{\epsilon}_k \, |\alpha + \delta_{k-1}\gamma| \quad \Longrightarrow \quad \alpha + \delta_{k-1}\gamma = 0 \tag{5.2.59}$$

The last equality implies that α is equal to $\delta_{k-1}\gamma$ (modulo 2) and hence a coboundary. This is a contradiction to the assumption we started with. Hence, there cannot be such an α that has small norm and we proved that the complex has large cosystoles. □

Putting together the previous lemmas, we obtain cosystolic expansion:

Corollary 5.2.1. *Let X be a simplicial complex which fulfills the conditions of lemma 5.2.3 on page 85 and lemma 5.2.5 on the previous page.*

Then X is a cosystolic expander in dimension k.

5.2.2 Local to Global Results in Dimension 2

We are now starting with the main part of this section: the proof of "global expansion" from "local expansion". The principal idea can already be seen in the two-dimensional case, while in this case the technicalities are still not overwhelming. We thus restrict ourselves to the proof for dimension two and only give a rough idea how it can be generalized to higher dimensions (see section 5.2.3 on page 103).

In this section we mainly use two properties: coboundary expansion of the links of vertices (which are graphs) and a measure of expansion of the 1-skeleton (which is also a graph). As both are graphs, we have two (equivalent) possibilities to describe

their expansion properties: spectrally by the eigenvalue gap or combinatorially by the edge expansion. Spectral expansion is mostly easier to prove, however, edge expansion seems to be the concept that fits naturally in. In this chapter we will discuss both approaches.

Key to the proof is a double counting argument that allows to write the coboundary expansion in terms of expansion in the links.

Let X be a 2-dimensional simplicial complex where we want to prove expansion. We consider a locally minimal 1-cochain $\alpha \in \mathrm{C}^1(X;\mathbb{Z}_2)$ and we want to find a lower bound on the Hamming norm of the coboundary of α. Therefore, we define:

$$t_i := \left|\{\sigma \in X(2) \mid |\operatorname{supp}\alpha \cap \{\tau \in X(1) \mid \tau \Subset \sigma\}| = i\}\right|, \quad i = 1,2,3 \tag{5.2.60}$$

t_i denotes the number of 2-simplices in X that have exactly i of its 3 facets in the support of α. Moreover, as the localization α_v, $v \in X(0)$, is a zero-dimensional cochain, we abuse notation and use the name α_v for the support of α_v as well as for the cochain. To simplify notation we also use:

$$\bar{\alpha}_v := \alpha_v + \mathbb{1}_{X_v} \tag{5.2.61}$$

for the complementary cochain as well as for the complement of the support.

By double counting we find:

Lemma 5.2.6. *Using the notation from above:*

- $|\delta\alpha| = t_1 + t_3$
- $\sum_{v\in X(0)} \left|E_{X_v}(\alpha_v, \bar{\alpha}_v)\right| = 2t_1 + 2t_2$

($E_{X_v}(\cdot,\cdot)$ denotes the edges in the link X_v interpreted as a graph.)

If the 2-degree of each edge in X is bounded by $\hat{d}$, we also have:

$$\hat{d}\,|\alpha| \geq t_1 + 2t_2 + 3t_3 \tag{5.2.62}$$

Equality holds if all edges have degree equal to d (that is, the link of each vertex is a d-regular graph).

Proof. The first point is just the definition spelled out as the coboundary contains exactly the simplices in its support that are counted for t_1 and t_3.

For the second point consider Fig. 5.6 on the next page, which illustrates the three types of triangles that may appear. These triangles appear in the link of its vertices. Depending on the chosen vertex, the opposing edge runs between α_v and $\bar{\alpha}_v$ or not. Hence, the number of these edges counts the number of triangles with at most 2 edges from $\operatorname{supp}\alpha$ and every triangle is counted twice as there are two choices of vertices.

For the last point we use double counting as well. The right-hand side counts every triangle (2-simplex) according to the number of edges from $\operatorname{supp}\alpha$ that it

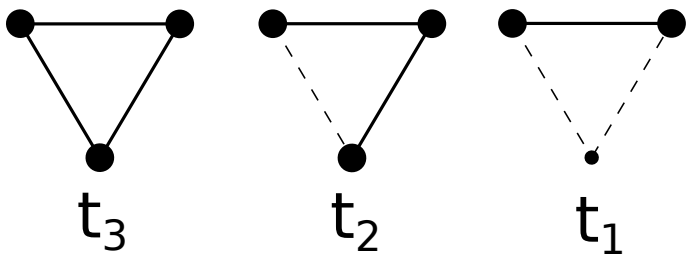

Fig. 5.6: Illustration of the double counting argument of lemma 5.2.6 on the previous page (dashed: not in $\operatorname{supp}\alpha$)

contains as facets. Every edge is contained in at most (in exactly, if all edges have equal degree) $\hat{d}$ triangles. $\hat{d}\,|\alpha|$ counts the triangles as seen from the edges in $\operatorname{supp}\alpha$. Hence, we have the inequality (equality, respectively):

$$t_1 + 2t_2 + 3t_3 = \sum_{e \in \operatorname{supp}\alpha} \deg_2 e \leq \hat{d}\,|\alpha| \tag{5.2.63}$$

□

This lemma already allows a lower bound on $|\delta\alpha|$:

$$|\delta\alpha| = t_1 + t_3 \geq t_1 - 3t_3 = (2t_1 + 2t_2) - (t_1 + 2t_2 + 3t_3) \tag{5.2.64}$$

$$\geq \sum_{v \in X(0)} \left|E_{X_v}(\alpha_v, \bar{\alpha}_v)\right| - \hat{d}\,|\alpha| \tag{5.2.65}$$

In this estimate we neglect t_3. It seems plausible that t_3 is small as we will consider only small cochains α later on. In any way, this may be a starting point for further improvements.

Now, we have to find a lower bound on the sum in terms of the Hamming norm of α. This is where we use expansion of the links X_v. Speaking in terms of cohomology we just have to lower bound the Hamming norm of $\delta\alpha_v$:

$$\left|E_{X_v}(\alpha_v, \bar{\alpha}_v)\right| = |\delta\alpha_v| \tag{5.2.66}$$

Moreover, this lower bound has to sum up to a value larger than $\hat{d}\,|\alpha|$ to get a valid result.

Assume for one moment that every link has coboundary expansion with constant $h^0(X_v) \geq \eta$, $v \in X(0)$, with respect to the unnormalized Hamming norm, that is:

$$\forall \beta \in \mathrm{C}^0(X_v; \mathbb{Z}_2): \quad |\delta\beta| \geq \eta\,|[\beta]| \tag{5.2.67}$$

We can apply this estimate to every summand and obtain ($|\alpha_v| \leq |\bar{\alpha}_v|$):

$$\sum_{v \in X(0)} |\delta\alpha_v| \geq \eta \sum_{v \in X(0)} |\alpha_v| \tag{5.2.68}$$

$$= 2\eta\,|\alpha| \tag{5.2.69}$$

The last equality holds since the sum counts every edge in $\operatorname{supp}\alpha$ twice. If η is large enough, we get coboundary expansion for any locally minimal cochain α:

$$\eta > \frac{\hat{d}}{2} \quad \Longrightarrow \quad |\delta\alpha| \geq \underbrace{(2\eta - \hat{d})}_{>0} |\alpha| \tag{5.2.70}$$

As it turns out, this condition on η is quite a restriction. We calculated earlier the expansion parameters for the complete graph K_n and the complete bipartite graph $K_{n,n}$ (cf. the results in Table 5.1 on page 59). K_n is $(n-1)$-regular and $K_{n,n}$ is n-regular. Both graphs have expansion parameters that are only slightly larger (or not larger at all) than $\frac{\hat{d}}{2}$. Hence, for these special cases our first try does not yield very good results.

Thus, the method has to be refined. In the complete graph K_n the edge expansion parameter is determined by the large sets (with size close to $\frac{n}{2}$). Indeed, if we allow only sets with size less than, for example, $(1-\epsilon)\frac{n}{2}$ we can improve the parameter from approximately $\frac{n}{2}$ to $(1+\epsilon)\frac{n}{2}$. This improvement then helps us to finish the argument from above. Inspired from the complete graph, we introduce so-called *thin* and *thick* vertices;

Definition 5.2.2 (Thin and Thick Vertices)**.** *Let X be a 2-dimensional simplicial complex and $\alpha \in \mathrm{C}^1(X;\mathbb{Z}_2)$ a locally minimal 1-cochain. Let $\epsilon > 0$ be a constant.*

A vertex $v \in X(0)$ is called ϵ-thin *with respect to α, if:*

$$|\alpha_v| \leq (1-\epsilon)\frac{|X_v(0)|}{2} \tag{5.2.71}$$

v is called ϵ-thick *otherwise.*

We denote the set of ϵ-thin and ϵ-thick by R and S, respectively:

$$R = R_\epsilon := \{v \in X(0) \,|\, \alpha_v \neq 0, v\ \epsilon\text{-thin}\} \tag{5.2.72}$$

$$S = S_\epsilon := \{v \in X(0) \,|\, \alpha_v \neq 0, v\ \epsilon\text{-thick}\} \tag{5.2.73}$$

As α is locally minimal we always have:

$$|\alpha_v| \leq \frac{|X_v(0)|}{2} \tag{5.2.74}$$

To abbreviate notation we also use:

$$r = r_\epsilon := \sum_{v \in R} |\alpha_v| \tag{5.2.75}$$

$$s = s_\epsilon := \sum_{v \in S} |\alpha_v| \tag{5.2.76}$$

$$\Longrightarrow r + s = \sum_{v \in X(0)} |\alpha_v| = 2\,|\alpha| \tag{5.2.77}$$

Furthermore, we need this technical lemma:

Lemma 5.2.7. *Let X and the other notation be as above.*

If the number of internal edges of S (in the graph $X^{(1)}$) is bounded from above, the value of r and s can be bounded as well:

$$|E_{X^{(1)}}(S)| \leq (1-\xi)\,|\alpha| \quad \Longrightarrow \quad r \geq \xi\,|\alpha| \quad \textit{and} \quad s \leq (2-\xi)\,|\alpha| \tag{5.2.78}$$

Proof. By double counting we get:

$$r = \sum_{v \in R} |\alpha_v| = |E_\alpha(S,R)| + 2\,|E_\alpha(R)| \tag{5.2.79}$$

($E_\alpha(A,B)$ denotes the the edges in $\operatorname{supp}\alpha$ between the sets A and B.)

Moreover, the support of α can be separated into edges going between thick vertices and thin vertices and edges going between vertices of the same kind:

$$|\alpha| = |E_\alpha(S,R)| + |E_\alpha(R)| + |E_\alpha(S)| \tag{5.2.80}$$

Putting this together, we obtain the desired result ($|E_{X^{(1)}}(S)| \geq |E_\alpha(S)|$):

$$r \geq |\alpha| - |E_\alpha(S)| \geq (1-(1-\xi))\,|\alpha| = \xi\,|\alpha| \tag{5.2.81}$$

The statement for s follows directly, since $r + s = 2\,|\alpha|$:

$$s = 2\,|\alpha| - r \leq (2-\xi)\,|\alpha| \tag{5.2.82}$$

□

The strategy is to find a better estimate on the thin vertices and to simultaneously bound the number of thick vertices (here we use expansion properties of the 1-skeleton of X). This is the point where it makes a difference if we use spectral or combinatorial expansion for the links.

Using Spectral Expansion in the Links

In this subsection we assume that every link is a "good" spectral expander. We prove the following theorem

Theorem 5.2.1 (Local to Global Theorem – Using Spectral Expansion)**.** *Let X be a 2-dimensional simplicial complex as in the beginning of this section and let $\alpha \in \mathrm{C}^1(X;\mathbb{Z}_2)$ be a locally minimal cochain. Let $\hat{d}$ be the maximum 2-degree of an edge in X.*

We assume that the link X_v of every vertex $v \in X(0)$ is a spectral expander with eigenvalue gap $\lambda(X_v) \geq \hat{d}(1-\epsilon_1)$.

Furthermore, we assume that the 1-skeleton $X^{(1)}$ is a k-regular graph with eigenvalue gap $\lambda(X^{(1)}) \geq k(1-\epsilon_2)$.

Under the following technical conditions:

- *Choose* $\epsilon, \epsilon' > 0, \xi \in (0,1)$*, such that:*

$$\left(\frac{\epsilon_2}{k}\frac{2}{1-\epsilon} + \frac{1}{(1-\epsilon)^2(1+\epsilon')}\frac{k-\epsilon_2}{k}\right) \leq 1-\xi \tag{5.2.83}$$

- *Using this* ξ *and* ϵ*, we need:*

$$(\epsilon(1-\epsilon_1)\xi - 2\epsilon_1) > 0 \tag{5.2.84}$$

There exists a constant $\eta > 0$ *independent of* α *(as long as* α *is locally minimal) such that:*

$$|\alpha| \leq \frac{1}{1+\epsilon'}\frac{|X(1)|}{4} \quad \Longrightarrow \quad |\delta\alpha| \geq \eta\,|\alpha| \tag{5.2.85}$$

η *can be chosen as:*

$$\eta := \frac{\hat{d}}{2}\left(\epsilon(1-\epsilon_1)\xi - 2\epsilon_1\right) \tag{5.2.86}$$

Proof. One of the two main ingredients is the quadratic lower bound from equation 3.2.8 on page 37:

$$\forall S \subseteq V: \quad |E(S, V\backslash S)| \geq \phi(G)\cdot d\cdot\frac{|S|\cdot|V\backslash S|}{|V|} \geq \lambda(G)\cdot\frac{|S|\cdot|V\backslash S|}{|V|}, \tag{5.2.87}$$

where d denotes the average degree. For subsets S with $|S| \leq \frac{|V|}{2}$ we can continue by:

$$|E(S, V\backslash S)| \geq \lambda(G)\cdot|S|\cdot\underbrace{\frac{|V\backslash S|}{|V|}}_{\geq\frac{1}{2}} \geq \frac{\lambda(G)}{2}|S| \tag{5.2.88}$$

This estimate can be improved for small subsets S with $|S| \leq (1-\epsilon)\frac{|V|}{2}$:

$$|E(S, V\backslash S)| \geq \lambda(G)\cdot|S|\cdot\underbrace{\frac{|V\backslash S|}{|V|}}_{\geq\frac{1+\epsilon}{2}} \geq (1+\epsilon)\frac{\lambda(G)}{2}|S| \tag{5.2.89}$$

Fig. 5.7 on the following page illustrates what we are doing here. The quadratic curve comes from the quadratic lower bound on the edges which we normally estimate from below by drawing a straight line (dashed in the figure). If we only consider small subsets (corresponding to thin vertices) we may draw a line with higher slope, thus getting better results. The value at the maximum of the quadratic curve is proportional to the spectral gap and thus smaller than the maximum degree $\hat{d}$. By increasing the slope we may find a linear curve that takes a value larger than the maximum degree at that point – this is what we need. Note, that this improvement is not possible for the linear lower bound

With this tool we can now start the proof:

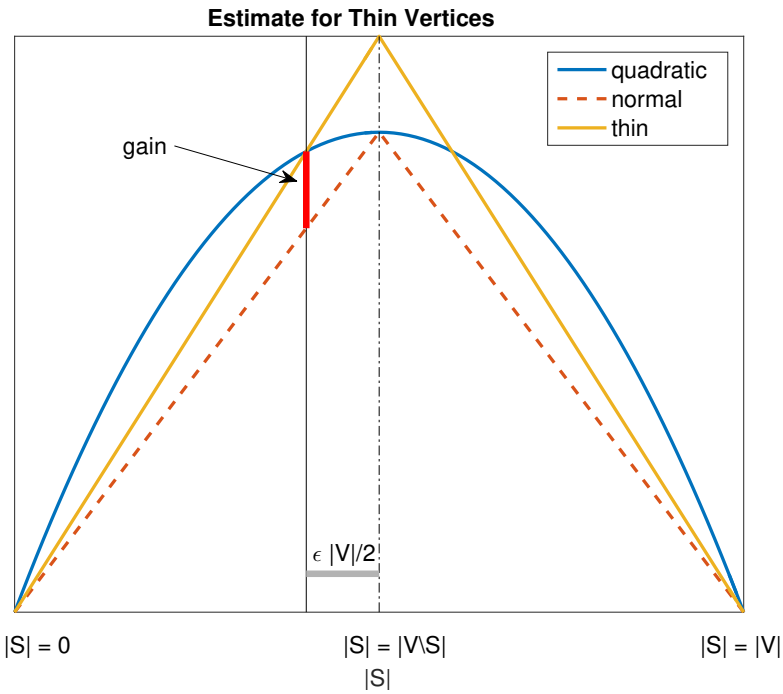

Fig. 5.7: Illustration of the improvement in the lower bound for thin vertices

1. Estimate for thin vertices

   For an ϵ-thin vertex $v \in R$ we can use the better estimate, because $|\alpha_v| \leq (1-\epsilon)\frac{|X_v(0)|}{2}$ by definition:

$$\left|E_{X_v}(\alpha_v, \bar{\alpha}_v)\right| \geq (1+\epsilon)\frac{\lambda(X_v)}{2}\,|\alpha_v| \geq (1+\epsilon)\frac{\hat{d}(1-\epsilon_1)}{2}\,|\alpha_v| \tag{5.2.90}$$

   Altogether, we obtain:

$$\sum_{v\in R}\left|E_{X_v}(\alpha_v, \bar{\alpha}_v)\right| \geq (1+\epsilon)\frac{\hat{d}(1-\epsilon_1)}{2}r \tag{5.2.91}$$

2. Estimate for thick vertices

   We can do the normal estimate for thick vertices:

$$\sum_{v\in S}\left|E_{X_v}(\alpha_v, \bar{\alpha}_v)\right| \geq \frac{\hat{d}(1-\epsilon_1)}{2}s \tag{5.2.92}$$

3. General estimate

By continuing the estimate from the beginning of this section we get:

$$|\delta\alpha| \geq \sum_{v\in X(0)} \left|E_{X_v}(\alpha_v, \bar{\alpha}_v)\right| - \hat{d}\,|\alpha| \tag{5.2.93}$$

$$\geq (1+\epsilon)\frac{\hat{d}(1-\epsilon_1)}{2}r + \frac{\hat{d}(1-\epsilon_1)}{2}s - \hat{d}\,|\alpha| \tag{5.2.94}$$

$$= \hat{d}(1-\epsilon_1)\,|\alpha| + \epsilon\frac{\hat{d}(1-\epsilon_1)}{2}r - \hat{d}\,|\alpha| \tag{5.2.95}$$

$$= \epsilon\frac{\hat{d}(1-\epsilon_1)}{2}r - \epsilon_1\hat{d}\,|\alpha| \tag{5.2.96}$$

In the second last equality we used that $r + s = 2\,|\alpha|$.

4. Estimate on r in terms of $|\alpha|$ – "There are many thin vertices."

   For this point we need the condition on the 1-skeleton of X. Indeed, we estimate r by bounding the amount of edges between two thick vertices. As a subset of the k-regular graph $X^{(1)}$ the edges $E(S)$ are connected with the expansion properties:

$$|E_{X^{(1)}}(S)| = \frac{1}{2}(k|S| - \underbrace{|E_{X^{(1)}}(S, X(0)\backslash S)|}_{\geq\lambda(X^{(1)})\frac{|S||X(0)\backslash S|}{|X(0)|}}) \tag{5.2.97}$$

$$\leq \frac{|S|}{2}\left(k - \underbrace{\lambda(X^{(1)})}_{\geq k-\epsilon_2}\frac{|X(0)\backslash S|}{|X(0)|}\right) \tag{5.2.98}$$

$$\leq \frac{|S|}{2}\left(k - (k-\epsilon_2)\underbrace{\frac{|X(0)\backslash S|}{|X(0)|}}_{=1-\frac{|S|}{|X(0)|}}\right) \tag{5.2.99}$$

$$= \frac{|S|}{2}\left(\epsilon_2 + (k-\epsilon_2)\frac{|S|}{|X(0)|}\right) \tag{5.2.100}$$

   This can be further estimated by bounding $|S|$ in the following way: Every vertex in S is by definition ϵ-thick, that is, we have $|\alpha_v| > (1-\epsilon)\frac{|X_v(0)|}{2}$. By counting every vertex in S we thus get ($X^{(1)}$ is k-regular, $|X_v(0)| = k$):

$$|S| = \sum_{v\in S} 1 \leq \sum_{v\in S}\frac{|\alpha_v|}{(1-\epsilon)\frac{k}{2}} \tag{5.2.101}$$

$$\leq \frac{2\,|\alpha|}{(1-\epsilon)\frac{k}{2}} = \frac{4\,|\alpha|}{(1-\epsilon)k} \tag{5.2.102}$$

Plugging this into the equation from above yields (after some calculation using also the handshake lemma: $k\,|X(0)| = 2\,|X(1)|$):

$$|E_{X^{(1)}}(S)| \leq |\alpha| \left(\frac{\epsilon_2}{k} \frac{2}{1-\epsilon} + \frac{1}{(1-\epsilon)^2} \frac{k-\epsilon_2}{k} \frac{|\alpha|}{\frac{|X(1)|}{4}} \right) \tag{5.2.103}$$

By assumption, the Hamming norm of α is smaller than $\frac{1}{1+\epsilon'} \frac{|X(1)|}{4}$ (we only consider such cochains), hence:

$$|E_{X^{(1)}}(S)| \leq |\alpha| \left(\frac{\epsilon_2}{k} \frac{2}{1-\epsilon} + \frac{1}{(1-\epsilon)^2(1+\epsilon')} \frac{k-\epsilon_2}{k} \right) \tag{5.2.104}$$

$$\leq (1-\xi)\,|\alpha| \tag{5.2.105}$$

By lemma 5.2.7 on page 92 this yields the following bound on r:

$$r \geq \xi\,|\alpha| \tag{5.2.106}$$

5. Completion of the argument

   The estimate now reads:

$$|\delta\alpha| \geq \epsilon \frac{\hat{d}(1-\epsilon_1)}{2} r - \epsilon_1 \hat{d}\,|\alpha| \tag{5.2.107}$$

$$\geq \underbrace{\frac{\hat{d}}{2} \left(\epsilon(1-\epsilon_1)\xi - 2\epsilon_1 \right)}_{=\eta>0} |\alpha| \tag{5.2.108}$$

$$\geq \eta\,|\alpha|\,, \tag{5.2.109}$$

   which finishes the proof.

□

What we did in step 4 of the proof is, that we showed a property of local sparsity of α. Indeed, for our argument we only need that there are "enough" thin vertices so the improvement of the estimate is good. Actually, the theorem states that locally sparse cochains are expanding. We can enforce local sparsity by bounding the Hamming norm of α giving only expansion for small cochains. If we could find a property that can be checked easily (unlike local sparsity), we could strengthen this theorem. However, by proving expansion for large cochains separately we get expansion for all cochains. In [LM13] expansion for large cochains is proved by a probabilistic method. It remains unclear whether it is possible to prove expansion for "locally dense" cochains in the same way.

Using Combinatorial Expansion in the Links

A similar argument can be carried through using only combinatorial properties of the complex in consideration. As the trick with thin and thick vertices uses that we have a quadratic lower bound, which we can improve by suitable linear approximations, we have to adopt the conditions of the theorem a little bit. We require again (now combinatorial) expansion of the links and some kind of expansion of the 1-skeleton, but we also need better (edge) expansion of the links for small cochains. This leads to the definition of edge expansion for small cochains:

Definition 5.2.3 (Edge Expansion for Small Cochains)**.** *Let X be a 1-dimensional simplicial complex with Hamming norm $|\cdot|$ and let $\epsilon > 0$.*

We say that X has edge expansion parameter $\eta(X)$ *for ϵ-small cochains, if for any $\alpha \in \mathrm{C}^0(X;\mathbb{Z}_2)$ we have:*

$$|\alpha| < (1-\epsilon)\frac{|X(0)|}{2} \quad \Longrightarrow \quad |E_X(\operatorname{supp}\alpha, X(0)\setminus \operatorname{supp}\alpha)| = |\delta_1\alpha| \geq \eta(X)\,|\alpha| \tag{5.2.110}$$

To replace spectral expansion of the 1-skeleton we define the notion of *skeleton expansion.* This is defined to yield a similar structure of estimate as in equation 5.2.98 on page 95 and is inspired by the concept with the same name in [EK15].

Definition 5.2.4 (Skeleton Expansion)**.** *Let X be a simplicial complex with normalized Hamming norm $\|\cdot\|_X$ and let $\beta > 0$.*

We say that X fulfills skeleton expansion *with parameter β, if:*

$$\forall A \subseteq X(0): \quad \|E_{X^{(1)}}(A)\|_X \leq \|A\|_X^2 + \beta\,\|A\|_X \tag{5.2.111}$$

(Here, we do not distinguish between subsets of vertices or edges and 0- or 1-dimensional cochains with these supports, which the norms are defined for. $E_{X^{(1)}}(A)$ denotes the set of edges in $X(1)$ with both endpoints in A.)

In terms of the Hamming norm (or cardinality of sets) this can be rewritten as:

$$|E_{X^{(1)}}(A)| \leq \frac{|X(1)|}{|X(0)|^2}\,|A|^2 + \beta\frac{|X(1)|}{|X(0)|}\,|A| \tag{5.2.112}$$

$$= \frac{\bar{k}}{2\,|X(0)|}\,|A|^2 + \beta\frac{\bar{k}}{2}\,|A| \tag{5.2.113}$$

In the last equality we used the average degree $\bar{k}$ of the graph $X^{(1)}$:

$$\bar{k} := \frac{2\,|X(1)|}{|X(0)|} \tag{5.2.114}$$

Using these two definitions we can now state and prove:

Theorem 5.2.2 (Local to Global Theorem – Using Combinatorial Expansion)**.** *Let X be a 2-dimensional simplicial complex and let $\alpha \in \mathrm{C}^1(X;\mathbb{Z}_2)$ be a locally minimal cochain.*

We assume that:

- *The link X_v of every vertex $v \in X(0)$ is an edge expander with expansion parameter $h^0(X_v)$.*
- *The link X_v of every vertex $v \in X(0)$ has edge expansion for ϵ-small cochains ($\epsilon > 0$) with parameter $\eta(X_v)$.*
- *The 1-skeleton $X^{(1)}$ is a graph that fulfills skeleton expansion with parameter $\beta < \frac{1}{2}$.*

We use the following notation:

- $\hat{d} = \max_{\sigma \in X(1)} \deg_2 \sigma$*, the maximum 2-degree of an edge in X.*
- $\bar{k} = \frac{2|X(1)|}{|X(0)|}$*, the average degree of the 1-skeleton of X.*
- $k_v = \deg_1 v = |X_v(0)|$*, the degree of a vertex $v \in X(0)$.*

The expansion parameters have to fulfill the following conditions, $\epsilon_1 > 0, \epsilon_2 \geq 0, c > 0$ are constants:

- $\eta(X_v) \geq \frac{\hat{d}}{2}(1+\epsilon_1)$*, for every $v \in X(0)$.*
- $h^0(X_v) \geq \frac{\hat{d}}{2}(1-\epsilon_2)$*, for every $v \in X(0)$.*
- $1 \leq \frac{\bar{k}}{\min_{v \in X(0)} k_v} \leq c$
- $\beta c < \frac{1}{2}$

Under the following technical conditions:

- *Choose $\epsilon, \epsilon' > 0, \xi \in (0,1)$, such that:*

$$\epsilon < 1 - 2\beta c \tag{5.2.115}$$

and

$$c^2 \frac{1}{(1-\epsilon)^2(1+\epsilon')} + \frac{2\beta c}{1-\epsilon} \leq 1 - \xi \tag{5.2.116}$$

- *Using this ξ, we need (this condition vanishes if $\epsilon_2 = 0$):*

$$\frac{\epsilon_1}{\epsilon_2} > \frac{2}{\xi} - 1 \tag{5.2.117}$$

There exists a constant $\eta > 0$ independent of α (as long as α is locally minimal) such that:

$$|\alpha| \leq \frac{1}{1+\epsilon'} \frac{|X(1)|}{4} \quad \Longrightarrow \quad |\delta\alpha| \geq \eta\,|\alpha| \tag{5.2.118}$$

η can be chosen as:

$$\eta := \frac{\hat{d}}{2}\left(\xi(\epsilon_1 + \epsilon_2) - 2\epsilon_2\right) \tag{5.2.119}$$

In principle, we require edge expansion that is only a little bit (ϵ_2) worse than $\frac{\hat{d}}{2}$, which is compensated by edge expansion for small cochains that is a little bit (ϵ_1) better than $\frac{\hat{d}}{2}$. The rest of the conditions are just technicalities that are necessary for the proof.

The theorem can also be stated for the special case of k-regular 1-skeleton and uniform 2-degree d. This makes notation a little bit less messy.

Proof. We separate the proof into the same steps as before:

1. Estimate for thin vertices

   For an ϵ-thin vertex $v \in R$ we use the improved estimate from expansion for small cochains:

$$\left|E_{X_v}(\alpha_v, \bar{\alpha}_v)\right| \geq \eta(X_v)\,|\alpha_v| \tag{5.2.120}$$

$$\geq (1+\epsilon_1)\frac{\hat{d}}{2}\,|\alpha_v| \tag{5.2.121}$$

2. Estimate for thick vertices

   For ϵ-thick vertices the only thing we can do is to use edge expansion:

$$\left|E_{X_v}(\alpha_v, \bar{\alpha}_v)\right| \geq h^0(X_v)\,|\alpha_v| \tag{5.2.122}$$

$$\geq (1-\epsilon_2)\frac{\hat{d}}{2}\,|\alpha_v| \tag{5.2.123}$$

3. General Estimate

   We again continue the estimate from the beginning of the section by applying the estimates from the previous points to obtain:

$$|\delta\alpha| \geq \sum_{v \in X(0)} \left|E_{X_v}(\alpha_v, \bar{\alpha}_v)\right| - \hat{d}\,|\alpha| \tag{5.2.124}$$

$$\geq (1+\epsilon_1)\frac{\hat{d}}{2}r + (1-\epsilon_2)\frac{\hat{d}}{2}s - \hat{d}\,|\alpha| \tag{5.2.125}$$

$$= \hat{d}\,|\alpha| + \epsilon_1\frac{\hat{d}}{2}r - \epsilon_2\frac{\hat{d}}{2}s - \hat{d}\,|\alpha| \tag{5.2.126}$$

$$= \epsilon_1\frac{\hat{d}}{2}r - \epsilon_2\frac{\hat{d}}{2}s \tag{5.2.127}$$

4. Estimate on r and s in terms of $|\alpha|$ – "There are many thin vertices."

   As in the spectral case, we have the bound on the size of S in terms of $|\alpha|$. However, there we used that we have a k-regular 1-skeleton to replace the size of the link by a constant. We cannot do this anymore, thus we estimate by the minimal size of a link:

$$|S| \leq \frac{4\,|\alpha|}{(1-\epsilon)\min_{v\in X(0)} k_v} \tag{5.2.128}$$

   Along with skeleton expansion and using the fact that α is small, this estimate gives (we omit the tiresome calculation):

$$|E_{X^{(1)}}(S)| \leq |\alpha| \left(c^2 \frac{1}{(1-\epsilon)^2} \frac{|\alpha|}{\frac{|X(1)|}{4}} + \frac{2\beta c}{1-\epsilon} \right) \tag{5.2.129}$$

$$\leq |\alpha| \left(c^2 \frac{1}{(1-\epsilon)^2(1+\epsilon')} + \frac{2\beta c}{1-\epsilon} \right) \tag{5.2.130}$$

$$\leq (1-\xi)\,|\alpha| \tag{5.2.131}$$

   By lemma 5.2.7 on page 92 this gives the desired bounds on r and s:

$$r \geq \xi\,|\alpha| \quad \text{and} \quad s \leq (2-\xi)\,|\alpha| \tag{5.2.132}$$

5. Completion of the argument

   The only thing left is the following computation:

$$|\delta\alpha| \geq \epsilon_1 \frac{\hat{d}}{2} r - \epsilon_2 \frac{\hat{d}}{2} s \tag{5.2.133}$$

$$\geq \underbrace{\frac{\hat{d}}{2}\left(\epsilon_1 \xi - \epsilon_2 (2-\xi)\right)}_{=\eta>0} |\alpha| \tag{5.2.134}$$

□

This theorem is quite nice as it shows a way to prove expansion using only (graph-theoretical) expansion in the links and 1-skeleton. However, there is one flaw: How do we prove expansion for small cochains for a complex we do not know very well? The normal way to prove expansion is via spectral expansion and the analysis of eigenvalues of matrices, which is relatively well-understood. The Cheeger inequality (in a non-normalized version) then gives a lower bound on the combinatorial expansion. This lower bound always is smaller than $\frac{\hat{d}}{2}$, because the eigenvalue gap is bounded by the maximum degree of a vertex. By taking the intermediate step with the sparsity of a cut, we can also deduce expansion for small cochains from the eigenvalues and thus make the theorem usable. This uses exactly the same trick as the spectral local-to-global theorem.

Lemma 5.2.8 (Cheeger Inequality for Small Sets)**.** *Let $G = (V, E)$ be a graph with eigenvalue gap $\lambda(G)$, minimal sparsity $\phi(G)$ of a cut and average degree $d = \frac{2|E|}{|V|}$.*

Then the expansion parameter $\eta(G)$ for ϵ-small cochains ($\epsilon > 0$) satisfies:

$$\eta(G) \geq (1+\epsilon)\frac{d}{2}\phi(G) \geq (1+\epsilon)\frac{\lambda(G)}{2} \tag{5.2.135}$$

Proof. As $|S| \leq (1-\epsilon)\frac{|V|}{2}$, then we can conclude from the definition of the minimal sparsity:

$$|E(S, V\backslash S)| \geq \phi(G)d|S| \underbrace{\frac{|V\backslash S|}{|V|}}_{\geq \frac{1+\epsilon}{2}} \geq \phi(G)(1+\epsilon)\frac{d}{2}|S| \tag{5.2.136}$$

This immediately implies the first inequality, while the second one is just a consequence of the "normal" (unnormalized) Cheeger inequality. □

As it turns out, the requirements of the local-to-global theorem can also be loosened a little bit. The skeleton expansion is only used for the special subset S of thick vertices. Since $|\alpha|$ is required to be small, we can bound the size of S as well:

$$|S| \leq |\{v \in X(0) \,|\, \alpha_v \neq 0\}| \leq 2\,|\alpha| \leq \frac{1}{1+\epsilon'}\frac{|X(1)|}{2} \tag{5.2.137}$$

Hence, skeleton expansion is only needed for smaller sets. Note, that this bound can be vacuous, if there are much more edges than vertices in X.

Application to the Complete 3-Partite 2-Complex

We want to apply the theorem to the complete 3-partite 2-complex[6] Λ_n^2 that was already analyzed in Section 5.1.3. This complex has the nice property that its 1-skeleton is a complete 3-partite graph, which is $2n$-regular, and every link is a complete bipartite graph, which is n-regular. Thus, we do not have to mess around with the degrees.

To prove skeleton expansion we take an arbitrary subset $A \subseteq \Lambda_n^2(0)$, which we split into three subsets A_0, A_1 and A_2 according to parts of the vertex set. Using Maclaurin's inequality we obtain:

$$|E(A)| = |A_0||A_1| + |A_0||A_2| + |A_1||A_2| \leq \frac{1}{3}\left(|A_0| + |A_1| + |A_2|\right)^2 = \frac{1}{3}|A|^2 \tag{5.2.138}$$

Λ_n^2 has $3n$ vertices and average degree $2n$, hence, we can choose $\beta = 0$:

$$|E(A)| \leq \frac{2n}{2 \cdot 3n}|A|^2 + 0 \cdot |A| \tag{5.2.139}$$

[6] We want to assume that n is even to make calculations easier.

We already know that the link $K_{n,n}$ is expanding with constant $\frac{n}{2}$. An easy calculation shows that for ϵ-small cochains this constant can be improved to $(1+\epsilon)\frac{n}{2}$.

Hence, the first conditions on the expansion parameters are satisfied with $\epsilon_1 = \epsilon$ and $\epsilon_2 = 0$. We may choose any $\epsilon < 1$ and $\epsilon', \xi > 0$ such that:

$$\frac{1}{(1-\epsilon)^2(1-\epsilon')} \leq 1 - \xi \tag{5.2.140}$$

The other condition is vacuous as $\epsilon_2 = 0$. For valid choices of ϵ, ϵ' and ξ by the theorem we get for a locally minimal cochain $\alpha \in \mathrm{C}^1(\Lambda_n^2; \mathbb{Z}_2)$:

$$|\alpha| \leq \frac{1}{1+\epsilon'} \frac{|\Lambda_n^2(1)|}{4} \quad \Longrightarrow \quad |\delta\alpha| \geq \frac{n}{2}\xi\epsilon\,|\alpha| \tag{5.2.141}$$

Application to a Ramanujan Complex

We consider the 2-dimensional Ramanujan complex Y that is used in [KKL14b]. As the theory of Ramanujan complexes is far beyond the scope of this thesis, we only recall the properties of Y that are used:

- The 1-skeleton $Y^{(1)}$ is a k-regular graph with $k = 2(q^2+q+1)$ (q is a sufficiently large prime power). $Y^{(1)}$ is a spectral expander with eigenvalue gap $\lambda(Y^{(1)}) \geq k - 6q = k - \mathcal{O}(\sqrt{k})$.
- The link Y_v of every vertex $v \in Y(0)$ is a d-regular graph with $d = q+1$. The eigenvalue gap of Y_v is $\lambda(Y_v) = q + 1 - \sqrt{q}$.

The construction of Ramanujan complexes can be found for example in [Lub13].

We want to apply the spectral version (or equivalently the combinatorial version and the Cheeger inequality for small cochains) of the theorem. As both eigenvalue gaps converge to the degree as $q \to \infty$, we may choose ϵ_1 and ϵ_2 arbitrarily small if q is just large enough. Thus, the technical conditions can be fulfilled by choosing ϵ small enough and ϵ' large enough. Thus, we can conclude:

Corollary 5.2.2. *There are constants $\epsilon' > 0$ and $\eta > 0$ such that the 2-dimensional Ramanujan complex Y from [KKL14b] is expanding on small locally minimal cochains $\alpha \in \mathrm{C}^1(Y; \mathbb{Z}_2)$ for q large enough:*

$$|\alpha| \leq \frac{1}{1+\epsilon'} \frac{|Y(1)|}{4} \quad \Longrightarrow \quad |\delta\alpha| \geq \eta\,|\alpha| \tag{5.2.142}$$

As it can be seen, the application of the theorem is relatively easy, one has only to take care of the constants.

5.2.3 Local to Global in High Dimension

In [EK15] the idea of thin and thick vertices was generalized to arbitrary dimensions by Shai Evra and Tali Kaufman. This needs quite a technical effort that conceals the idea behind it. In this thesis, we will cite the theorem ([EK15, Theorem 3.1]) for the sake of completeness and because it is a nice structural result. The proof, however, is quite lengthy and thus we will refer to the original work for it.

The result of Evra and Kaufman uses the norm $\|\cdot\|$ defined as the weighted Hamming norm with the weights from equation 2.2.33 on page 21.

Theorem 5.2.3 (Evra, Kaufman)**.** *Let $d \in \mathbb{N}$, $\beta > 0$ and $Q \in \mathbb{N}$ be constants. There exist $\epsilon = \epsilon(d, \beta, Q) > 0$, $\mu = \mu(d, \beta) > 0$ and $\alpha = \alpha(d, \beta) > 0$.*

Let X be a d-dimensional simplicial complex that satisfies:

- *The degree of every simplex is bounded. We have $\max_{v \in X(0)} |X_v| \leq Q$.*
- *The link X_σ of any simplex σ with $1 \leq |\sigma| \leq d-1$ has coboundary expansion parameter $h^k(X_\sigma) \geq \beta$, $0 \leq k \leq \dim X_\sigma - 1$.*
- *The link X_σ of any simplex σ with $0 \leq |\sigma| \leq d-1$ is a so-called skeleton expander with parameter α, that is:*

$$\forall A \subseteq X_\sigma(0): \quad \|E(A,A)\| \leq 4(\|A\|^2 + \alpha \|A\|) \tag{5.2.143}$$

  (We consider A and $E(A,A)$, the set of edges with both endpoints in A, as a subset as well as as a cochain with this support.)

Then the cocycle expansion and cosystole parameters of X satisfy for any $0 \leq k \leq d-2$ and any $0 \leq s \leq d-1$:

$$h_z^k(X) \geq \epsilon \quad \text{and} \quad syst^s(X) \geq \mu \tag{5.2.144}$$

Hence, the $(d-1)$-skeleton of X is a cosystolic expander.

Proof. In the proof of this theorem, the authors of [EK15] consider generalizations of thin and thick vertices, which they call "fat faces" (in every dimension). Then they look at the coboundaries of the localization of fat faces, which they estimate by the required coboundary expansion of the links. The coboundary of the cochain in consideration can then be estimated by the coboundaries of the localizations and some error term, which they show to be small. □

5.3 Random Methods

The last method to prove expansion that we are discussing is the so-called *random method*. Unlike the first two methods, we are not dealing with a given complex and are trying to prove (coboundary) expansion for this special complex, but we

consider a large variety of different complexes and want to make general statements. The random method is the historically oldest method, as well for high-dimensional expansion as for graph expansion. Indeed, the first appearance of the concept of graph expansion was in the work of Kolmogorov and Barzdin ([KB93]) and of Pinsker ([Pin73]). In these works the concept of expansion was defined and it was shown by probabilistic arguments that there exist expander graphs. Explicit constructions of expander graphs are much more difficult and involve among others deep concepts of group theory.

A random approach also was one of the starting points of high-dimensional expansion (in contrast to the approach of Gromov in [Gro10] from the viewpoint of co-filling inequalities). Nathan Linial and Roy Meshulam introduced the concept of coboundary expansion in [LM06] in order to prove that a random 2-dimensional simplicial complex (sampled according to a random distribution that is explained later on) has vanishing $\mathbb{Z}_2$-cohomology "with high probability".

An introduction to random methods for graphs can be found in [AS16] and in [JLR00].

5.3.1 General Idea

The general idea of random methods is that we are mostly unable to analyze a given complex in detail, because it is simply too large, but we can analyze a family of complexes better by concentrating on general structures or properties. If we take for example a graph and want to know whether it is connected or not, we can only check whether every vertex is connected to the rest – which is quite a complex task if there are many vertices. On the other hand, if we consider all graphs on a given vertex set, we can make a statistical statement such as "a certain fraction of all graphs is connected". So, by sampling a graph uniformly at random from these graphs, we know that we have a certain probability for it to be connected.

In general, we do not consider "all" graphs or simplicial complexes, but just a subfamily which we want to call a *random model* here. An example of such a model is presented in Section 5.3.2. A model normally is connected with a method of random sampling, that is, we want to equip the family with a probability measure. Thus, we consider the model as a random process with a certain distribution that outputs a simplicial complex of the family. Usually, this distribution is either quite easy (and hence easy to analyze) or appears in a natural way (from a natural graph-theoretical/combinatorial construction).

Introducing this "random structure" seems rather arbitrary. However, it is a useful viewpoint, because many tools from probability theory and statistics can be (and definitely were) applied. Moreover, it allows natural statements like "it is very likely that a graph is an expander".

Given the random model, we can consider properties of the simplicial complexes or graphs as random variables with expected value, variance etc. Using these properties, which are sometimes very easy to compute (this depends highly on the model, of course), we can find bounds or estimates for the probability of the random vari-

able to take specific values. If, for example, the edge expansion parameter of a graph has expected value strictly larger than zero, we may bound the probability of it to be zero and thus obtain examples for expander graphs[7].

The steps of the random method can thus be summarized as follows:

1. Define a "fitting" random model of simplicial complexes.
2. Consider the desired property of the simplicial complex as a random variable.
3. Analyze the distribution, probabilistic properties, etc. of this random variable.
4. Deduce probability estimates for the property.

Written down like this, the random method seems to be quite easy. However, the devil is in the details. First of all, what is a "fitting" model? This depends on the property that we want to analyze. The right definition of a model sometimes needs very much knowledge of the complexes in consideration and the choice of the model influences the other steps strongly. Depending on the model, the random variable may have a different distribution that is easier or harder to analyze and the probability estimates depend on this distribution.

Some of the most useful tools from probability theory are listed in the following section.

Tools from Probability Theory

Probability theory provides a great variety of estimates that concern the distribution of a random variable, namely its deviation from its expected value. These estimates are sometimes called *tail estimates*, as they estimate the tails of the probability density function. They are normally written as inequalities, named after their inventor.

The first inequality that we want to present is Chebyshev's inequality:

Lemma 5.3.1 (Chebyshev's inequality)**.** *Let X be a random variable and denote by $\mu = \mathbb{E}(X) < \infty$ and $\sigma^2 = \mathbb{E}(X-\mu)^2 \neq 0$ its expected value and variance, respectively.*

Then for any $\epsilon \in \mathbb{R}$, we have:

$$\mathbb{P}(|X-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2} \tag{5.3.1}$$

($\mathbb{P}(\cdot)$ denotes the probability measure associated with the distribution of X.)

A proof of Chebyshev's inequality can be found in any textbook on probability theory.

The Chernoff bound is also very important and will be used later on:

[7] To be exact: We only prove the existence of such examples, but the proof is not constructive.

Lemma 5.3.2 (Chernoff Bound, [Che52]). *Let $X_1, X_2, \ldots, X_n$ be n independent, identically distributed random variables and denote by S_n the sum of the random variables, that is:*

$$S_n := \sum_{i=1}^{n} X_i \tag{5.3.2}$$

Let $M(t)$ denote the moment generating function of S_n, that is ($h > 0$):

$$M(t) := \mathbb{E}(\exp(tS_n)) \quad , -h < t < h \tag{5.3.3}$$

*(*exp *denotes the exponential function.)*

Let $-\infty < \mu = \mathbb{E}(S_n) < \infty$ be the expected value of S_n.

If $a \leq \mu$, then we have:

$$\mathbb{P}(S_n \leq a) \leq \inf_{-h<t<h} \exp(-at)M(t) \tag{5.3.4}$$

If all X_i are independent Bernoulli variables, S_n is binomially distributed and we can conclude:

$$\mathbb{P}(S_n \leq (1-\epsilon)\mu) \leq \exp\left(-\frac{\epsilon^2}{2}\mu\right) \tag{5.3.5}$$

This is the form we will use. We will normally apply it to several random variables and use a so-called *union bound*, that is, we use the subadditivity of the probability measure:

$$\mathbb{P}\left(\bigcup_i A_i\right) \leq \sum_i \mathbb{P}(A_i) \tag{5.3.6}$$

Other important inequalities that deserve mentioning are the Markov inequality and the Hoeffding inequality. As we will not use these two inequalities (at least not directly) we will not state them here and refer to the classical literature about probability theory.

A second big tool of probability theory is the approximation using "standard distributions". By approximation it is possible to estimate the occurring distributions by a distribution where we can calculate the probability density function easily. A natural candidate for the approximation is the normal distribution (argued by the central limit theorem), but there are also other examples like the Poisson approximation of the binomial distribution. There are many such approximation results, one has just to find the fitting one.

5.3.2 The Linial–Meshulam Model

We now want to discuss one of the "standard" random models of simplicial complexes, the Linial–Meshulam model, named after Nathan Linial and Roy Meshulam

who introduced it in [LM06]. The *Linial–Meshulam model* is a generalization of the well-studied Erdös–Rényi model for graphs ([ER59]).

In the Erdös–Rényi model $G(n,p)$ we consider the family of all graphs on n vertices that are sampled by the following rule: Every possible edge between two points of the vertex set is added independently with probability p. Hence, every edge can be represented by an independent Bernoulli random variable and if we want to count the edges between two given subsets of vertices, we just have to add up these random variables and get a binomially distributed random variable.

The Linial–Meshulam model $X^k(n,p)$ of k-dimensional simplicial complexes on n vertices uses a similar construction. Here, we start with a complete $(k-1)$-skeleton, that is, the $(k-1)$-skeleton of the complete complex on n vertices ($k < n$), and add every k-simplex from $\Delta_{n-1}(k)$ (a subset of cardinality $(k+1)$ of the vertex set) independently with probability p. The probability for a given simplicial complex $\Delta_{n-1}^{(k-1)} \subseteq Y \subseteq \Delta_{n-1}^{(k)}$ to be sampled can be calculated by:

$$\mathbb{P}(Y) = p^{|Y(k)|}(1-p)^{\binom{n}{k+1}-|Y(k)|} \tag{5.3.7}$$

As we can see, the Erdös–Rényi model appears as the special case of the 1-dimensional Linial–Meshulam model: $G(n,p) = X^1(n,p)$. Moreover, random variables that count k-simplices are distributed binomially, which makes an analysis easy.

Simplicial complexes generated by the Linial–Meshulam model have a complete $(k-1)$-skeleton. Thus, every simplex in $X^{(k-2)}$ has unbounded degree as n tends to infinity. A $(k-1)$-dimensional simplex has a k-degree that is binomially distributed with expected value $p(n-k)$ (there are $n-k$ simplices in the complete complex that contain a given $(k-1)$-simplex). However, the maximal k-degree is unbounded as n tends to infinity. Hence, the Linial–Meshulam model need not yield families of expanders with bounded degree.

Nathan Linial, Roy Meshulam and Nolan Wallach showed in [LM06, MW07] that a sampled simplicial complex Y has vanishing cohomology in dimension $k-1$ with high probability if the parameter p is large enough and non-vanishing, if p is small, that is:

$$\lim_{n\to\infty} \mathbb{P}\left(Y \in X^k(n,p) : \mathrm{H}^{k-1}(Y;\mathbb{Z}_2) = 0\right) = \begin{cases} 0 & \text{if } p = \frac{k\log n - \omega(n)}{n} \\ 1 & \text{if } p = \frac{k\log n + \omega(n)}{n}, \end{cases} \tag{5.3.8}$$

for a function ω that satisfies $\lim_{n\to\infty}\omega(n) = \infty$.

If $p = 1$, we have the complete k-dimensional complex, which has a vanishing cohomology group (cf. Section 5.1.2). By choosing p large, we can expect to get a simplicial complex that is "dense" and thus will inherit some of the properties of Δ_{n-1}. On the other side, if we remove enough k-simplices we can generate cocycles that are no coboundaries and thus a "sparse" subcomplex is likely to have non-vanishing cohomology.

As the cohomology of a simplicial complex Y from $X^k(n,p)$ is asymptotically almost surely vanishing for large p (that is, with probability tending to one for n

tending to infinity), the complex Y also has a strictly positive coboundary parameter $h^k(Y)$. Moreover, there is an asymptotic lower bound on $h^k(Y)$ (cf. [DK10]):

Lemma 5.3.3. *Let Y be a random simplicial complex sampled from the Linial–Meshulam model $X^k(n,p)$. Let $\epsilon > 0$ and let ω be a function with $\lim_{n\to\infty} \omega(n) = \infty$.*

If the parameter p is large enough:

$$p \geq \frac{2\log(|\Delta_{n-1}(k-1)|) + \omega(n)}{\epsilon^2 h^{k-1}(\Delta_{n-1})}, \tag{5.3.9}$$

then with probability tending to 1 the coboundary expansion parameter $h^{k-1}(Y)$ is bounded by:

$$h^{k-1}(Y) \geq (1-\epsilon)p \cdot h^{k-1}(\Delta_{n-1}) \tag{5.3.10}$$

Proof. The lemma is a special case of theorem 5.3.2 on page 111 and the proof will be postponed until then. □

As we will see later, the proof is a straightforward application of the Chernoff bound and a union bound. Using methods that are technically more evolved, it is also possible to show spectral expansion (with high probability) for the Linial–Meshulam model. In [GW16] Anna Gundert and Uli Wagner prove the following theorem that is based on results of Howard Garland ([Gar73]).

Theorem 5.3.1 (Concentration of Eigenvalues for the Linial–Meshulam Model, [GW16])**.** *Let Y be a random simplicial complex sampled from the Linial–Meshulam model $X^k(n,p)$, $k \geq 2$. For every $c > 0$ and every $\gamma > c$ there exists a constant $C > 0$. Assume that:*

$$p \geq (k+\gamma)\frac{\log n}{n} \tag{5.3.11}$$

Denote by Δ^{up}_{k-1} the up-Laplacian of Y with respect to the weighted ℓ^2-scalar product with the weight-function $w(F) = \deg_k F$, and by A_{k-1} the adjacency matrix of Y in dimension $k-1$.

We use $d = p(n-k)$ for the expected k-degree of a $(k-1)$-face.

Then the following statements hold with probability greater than $1 - n^{-c}$:

- *The eigenvalues of the adjacency matrix A_{k-1} can be separated as follows (this statement even holds for $p \geq \gamma\frac{\log n}{n}$):*
  - *The largest $\binom{n-1}{k-1}$ eigenvalues lie in the interval $[d - C\sqrt{d}, d + C\sqrt{d}]$ around d.*
  - *The remaining $\binom{n-1}{k}$ eigenvalues lie in the interval $[-C\sqrt{d}, C\sqrt{d}]$ around 0.*
- *The eigenvalues of the up-Laplacian Δ^{up}_{k-1} can be separated as follows:*

- *The smallest* $\binom{n-1}{k-1}$ *eigenvalues are zero (trivial eigenvalues from the coboundary group).*
- *The remaining* $\binom{n-1}{k}$ *eigenvalues lie in the interval* $[1-\frac{C}{\sqrt{d}}, 1+\frac{C}{\sqrt{d}}]$. *(This also implies vanishing cohomology group* $\mathrm{H}^{k-1}(Y;\mathbb{R}) = 0$.*)*

Proof. A proof of this theorem can be found in [GW16]. We will omit it here, as it is very technical and beyond the scope of this thesis. The proof uses the links of $(k-2)$-faces to obtain random Erdös–Rényi model graphs and then known results for these graphs are applied to estimate the "global" eigenvalues. □

The above theorem states that the eigenvalues of the up-Laplacian or the adjacency matrix are concentrated around two values with high probability for large parameter p. Compared to the eigenvalues of the complete complex $\Delta_{n-1}^{(k)}$, the general structure of the spectra remains the same, whereas single eigenvalues are distorted by an additive error of order at most $\sqrt{d}$ (for the adjacency matrix) or $\frac{1}{\sqrt{d}}$ (for the normalized up-Laplacian):

Lemma 5.3.4 (Eigenvalues for the Complete Complex)**.** *Let* $\Delta_{n-1}^{(k)}$ *be the complete* k*-dimensional complex on* n *vertices. Denote by* Δ_{k-1}^{up} *the normalized up-Laplacian as in the previous theorem and by* A_{k-1} *the adjacency matrix in dimension* $k-1$.

Then their eigenvalues can be calculated:

- *The adjacency matrix* A_{k-1} *has the eigenvalue* $n-k$ *with multiplicity* $\binom{n-1}{k-1}$ *and the eigenvalue* $-k$ *with multiplicity* $\binom{n-1}{k}$.
- *The normalized up-Laplacian* Δ_{k-1}^{up} *has the eigenvalue* 0 *with multiplicity* $\binom{n-1}{k-1}$ *and the eigenvalue* $\frac{n}{n-k}$ *with multiplicity* $\binom{n-1}{k}$.

Proof. Every $(k-1)$-dimensional simplex σ in the complete complex $\Delta_{n-1}^{(k)}$ is contained in exactly $n-|\sigma|\,(=n-k)$ k-dimensional simplices. Using the matrix representation of the unnormalized up-Laplacian $L_{k-1}^{\mathrm{up}} \cong D_{k-1} - A_{k-1}$, we can immediately deduce one spectrum from the other:

$$L_{k-1}^{\mathrm{up}} \cong (n-k)\cdot \mathrm{id} - A_{k-1} \qquad \Longrightarrow \qquad \sigma(L_{k-1}^{\mathrm{up}}) = (n-k) - \sigma(A_{k-1}) \tag{5.3.12}$$

To obtain the spectrum of the normalized up-Laplacian $\Delta_{k-1}^{\mathrm{up}}$, we observe that the adjoint of the coboundary map (with respect to the weighted ℓ^2-scalar product) is given by ($f \in \mathrm{C}^{k-1}(\Delta_{n-1}^{(k)};\mathbb{R})$, $G \in \Delta_{n-1}(k-1)$, cf. equation 3.2.17 on page 40):

$$(\delta_k^* f)(G) = \sum_{F\in\Delta_{n-1}(k+1)} \underbrace{\frac{\deg_k F}{\deg_k G}}_{\frac{1}{n-k}} [F:G]\, f(F) \tag{5.3.13}$$

$$= \frac{1}{n-k} \sum_{F\in\Delta_{n-1}(k+1)} [F:G]\, f(F) \tag{5.3.14}$$

Here, we use that every simplex has the same degree. With this calculation, the normalized and the unnormalized up-Laplacian only differ by a constant factor:

$$\Delta_{k-1}^{\mathrm{up}} = \frac{1}{n-k} L_{k-1}^{\mathrm{up}} \qquad \Longrightarrow \qquad \sigma(\Delta_{k-1}^{\mathrm{up}}) = \frac{1}{n-k}\sigma(L_{k-1}^{\mathrm{up}}) \tag{5.3.15}$$

Thus, we only have to determine the spectrum of the unnormalized up-Laplacian and we automatically get the spectra of the normalized up-Laplacian and the adjacency matrix.

To calculate the spectrum of the unnormalized up-Laplacian, we first observe that the up-Laplacian L_{k-1}^{up} and the down-Laplacian L_{k-1}^{down} satisfy the following equality:

$$L_{k-1}^{\mathrm{up}} + L_{k-1}^{\mathrm{down}} = n \cdot \mathrm{id} \tag{5.3.16}$$

This can be seen by evaluating the left-hand side for indicator functions. The up-Laplacian gives a term $(n-k)\cdot$ id and a second term which cancels out with a part of the down-Laplacian, the remaining part gives a term $k\cdot$ id. In this calculation we use, that we consider a complete complex.

From the equation we can see, that any cochain in the kernel of δ_{k-2}^* is an eigenvector to the eigenvalue n:

$$\alpha \in \ker \delta_{k-2}^* \subseteq \ker L_{k-1}^{\mathrm{down}} \qquad \Longrightarrow \qquad L_{k-1}^{\mathrm{up}}\alpha = n \cdot \alpha \tag{5.3.17}$$

The kernel of the adjoint map can be calculated using the following property of adjoint maps (orthogonal complement is taken with respect to the standard ℓ^2-scalar product):

$$\ker \delta_{k-2}^* = (\operatorname{im} \delta_{k-2})^\perp = \left(\mathrm{B}^{k-1}(\Delta_{n-1}^{(k)};\mathbb{R})\right)^\perp \tag{5.3.18}$$

In lemma 3.2.3 on page 41 we have already seen that the coboundaries are eigenvectors to the eigenvalue 0, hence we have written the space $\mathrm{C}^{k-1}(\Delta_{n-1}^{(k)};\mathbb{R})$ as the orthogonal sum of $\ker \delta_{k-2}^*$ and $\mathrm{B}^{k-1}(\Delta_{n-1}^{(k)};\mathbb{R})$, which are both eigenspaces, hence, we have determined the spectrum.

To calculate the multiplicities of the eigenvalues, we have to determine the dimension of the eigenspaces. In this course, we write down a basis of $\mathrm{B}^{k-1}(\Delta_{n-1}^{(k)};\mathbb{R})$ (we write the vertex set of Δ_{n-1} as $\{1,\dots,n\}$, as usual):

$$B := \left\{\delta_{k-2}\mathbb{1}_{\{F\}} \,|\, 1 \notin F \in \Delta_{n-1}(k-2)\right\} \tag{5.3.19}$$

As the coboundary group is generated by the images of all indicator functions of $(k-2)$-simplices, we have to show that B spans the coboundary group and that B is linearly independent in order to prove that B is a basis. For the first point, it is enough to prove that the images of indicator functions of $(k-2)$-simplices containing 1 lie in the span of B. Indeed, the image of an indicator function $\mathbb{1}_{\{F\}}$, $1 \in F \in \Delta_{n-1}(k-2)$ can be calculated by:

$$\delta_{k-2}(\underbrace{\delta_{k-3}\mathbb{1}_{\{F\setminus\{1\}\}}}_{=\mathbb{1}_{\{F\}}+\mathbb{1}_{\{F\setminus\{1\}\cup\{j\}\,|\,j\in\{1,\dots,n\}\setminus F\}}}) = 0 \tag{5.3.20}$$

$$\Longrightarrow \qquad \delta_{k-2}\mathbb{1}_{\{F\}} = -\delta_{k-2}\mathbb{1}_{\{F\setminus\{1\}\cup\{j\}\,|\,j\in\{1,\dots,n\}\setminus F\}} \in \operatorname{span} B \tag{5.3.21}$$

The linear independence of B is equally simple: Let $\sum_j c_j \delta_{k-2} \mathbb{1}_{\{F_j\}} = 0$ be a linear combination of 0, $1 \notin F_j \in \Delta_{n-1}(k-2)$. The sum has to evaluate to zero for every $G \in \Delta_{n-1}(k-1)$, so we can choose $G = F_i \cup \{1\}$ for any i and evaluate:

$$0 = \sum_j c_j \underbrace{\delta_{k-2} \mathbb{1}_{\{F_j\}}(G)}_{=0 \quad \text{if } j \neq i} = c_i \tag{5.3.22}$$

Hence, the linear combination is trivial and thus, B is a basis of $\mathrm{B}^{k-1}(\Delta_{n-1}^{(k)}; \mathbb{R})$. The cardinality of B is equal to the number of $(k-2)$-simplices that do not contain 1, that is:

$$|B| = \binom{n-1}{k-1} \tag{5.3.23}$$

As $\mathrm{B}^{k-1}(\Delta_{n-1}^{(k)}; \mathbb{R})$ and $\ker \delta_{k-2}^*$ are orthogonal complements, the dimension of the eigenspace to eigenvalue n can be calculated by subtraction, which gives the stated result. □

Rem. 5.3.1. Equation 5.3.16 on the facing page can be used likewise to calculate the spectrum of the down-Laplacian.

5.3.3 Random Subcomplexes

The Linial–Meshulam model can be generalized to the (binomial) model of random subcomplexes, as was done in [DK10]. For the Linial–Meshulam model $X^k(n,p)$, we start with a complete $(k-1)$-skeleton and add every possible k-simplex independently with probability p. Thus, we automatically obtain a random subcomplex of the simplicial complex $\Delta_{n-1}^{(k)}$. This construction is generalized to obtain a random subcomplex of any given simplicial complex X with complete $(k-1)$-skeleton. Therefore, we start again with a complete $(k-1)$-skeleton and add each k-simplex of X independently with probability p. We call the random model with this sampling algorithm the model of "random subcomplexes of X". More generally, we may allow a sequence $(X_n)_{n\in\mathbb{N}}$ of simplicial complexes to be able to consider asymptotic properties.

As in the case of the Linial–Meshulam model, a random subcomplex Y of a simplicial complex X inherits some of the properties of X. By a simple Chernoff bound, we can prove that Y is a coboundary expander with a given probability, assuming that X is already expanding:

Theorem 5.3.2 (Coboundary Expansion of Random Subcomplexes)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of k-dimensional simplicial complexes that have a complete $(k-1)$-skeleton and $\mathbb{Z}_2$-coboundary expansion parameter $h^{k-1}(X_n) > 0$. Let $\epsilon > 0$ be a constant and let ω be a function satisfying $\lim_{n\to\infty} \omega(n) = \infty$.*

Let Y_n be a random subcomplex of X_n for any $n \in \mathbb{N}$ with parameter $p = p_n$ that is large:

$$p_n \geq \frac{2 \log |X_n(k-1)| + \omega(n)}{\epsilon^2 h^{k-1}(X_n)} \tag{5.3.24}$$

Then the probability that Y_n also has positive $\mathbb{Z}_2$-coboundary expansion parameter $h^{k-1}(Y_n) > 0$ tends to 1 as n tends to infinity, that is:

$$\lim_{n\to\infty} \mathbb{P}(h^{k-1}(Y_n) \geq (1-\epsilon)p_n \cdot h^{k-1}(X_n)) = 1 \tag{5.3.25}$$

Proof. Let $\alpha \in \mathrm{C}^{k-1}(Y_n; \mathbb{Z}_2)$ be an arbitrary cochain in Y_n. Without loss of generality, we can assume that α has minimal Hamming norm modulo coboundaries, that is, $|\alpha| = |[\alpha]|$. We already know, that the complex X_n has coboundary expansion parameter $h^{k-1}(X_n)$ (as the $(k-1)$-skeletons of X_n and Y_n coincide, we can interpret α as cochain in X_n as well):

$$\left|\delta_{k-1}^{X_n}\alpha\right| \geq h^{k-1}(X_n)\,|\alpha| \tag{5.3.26}$$

Starting from this equation, we want to derive a bound (with high probability) in the complex Y_n. Therefore, we look at the differences of the coboundaries of α in the two complexes. The calculation is exactly the same, but the cochains $\delta_{k-1}^{Y_n}\alpha$ and $\delta_{k-1}^{X_n}\alpha$ have different domains. Every k-simplex in the domain of $\delta_{k-1}^{Y_n}\alpha$ has to be in Y_n and hence in X_n. On the other hand, every k-simplex in the domain of $\delta_{k-1}^{X_n}\alpha$ is contained in Y_n independently with probability p_n by the sampling process. By counting the simplices in the support of $\delta_{k-1}^{Y_n}\alpha$ we thus evaluate the sum of random Bernoulli variables corresponding to the simplices in the support of $\delta_{k-1}^{X_n}\alpha$. The Hamming norm $\left|\delta_{k-1}^{Y_n}\alpha\right|$ is a binomially distributed random variable with parameters $n = \left|\delta_{k-1}^{X_n}\alpha\right|$ and $p = p_n$. The expected value is:

$$\mathbb{E}(\left|\delta_{k-1}^{Y_n}\alpha\right|) = p_n \left|\delta_{k-1}^{X_n}\alpha\right| \tag{5.3.27}$$

By the Chernoff bound from lemma 5.3.2 on page 106 we can calculate the probability for $\left|\delta_{k-1}^{Y_n}\alpha\right|$ to be much smaller than its expected value:

$$\mathbb{P}\left(\left|\delta_{k-1}^{Y_n}\alpha\right| \leq (1-\epsilon)p_n \left|\delta_{k-1}^{X_n}\alpha\right|\right) \leq \exp\left(-\frac{\epsilon^2}{2} p_n \left|\delta_{k-1}^{X_n}\alpha\right|\right) \tag{5.3.28}$$

It turns out that this estimate is enough to prove what we want. To prove the statement, we have to show that the probability for the complex to have a small coboundary expansion parameter is tending to zero as n tends to infinity, that is:

$$\mathbb{P}\left(h^{k-1}(Y_n) < (1-\epsilon)p_n \cdot h^{k-1}(X_n)\right) \to 0 \quad (n \to \infty) \tag{5.3.29}$$

From the definition of coboundary expansion it is clear that the coboundary expansion is small if and only if there is a cochain which has a small coboundary. We can then apply a union bound to obtain:

$$\mathbb{P}\left(h^{k-1}(Y_n) < (1-\epsilon)p_n \cdot h^{k-1}(X_n)\right) \tag{5.3.30}$$

$$=\mathbb{P}\left(\exists 0 \neq \alpha \in \mathrm{C}^{k-1}(Y_n;\mathbb{Z}_2) : \left|\delta_{k-1}^{Y_n}\alpha\right| < (1-\epsilon)p_n \cdot h^{k-1}(X_n)\left|\alpha\right|\right) \tag{5.3.31}$$

$$\leq \sum_{0\neq\alpha\in\mathrm{C}^{k-1}(Y_n;\mathbb{Z}_2)} \mathbb{P}\left(\left|\delta_{k-1}^{Y_n}\alpha\right| < (1-\epsilon)p_n \cdot h^{k-1}(X_n)\left|\alpha\right|\right) \tag{5.3.32}$$

The coboundary expansion of X_n implies that:

$$\left|\delta_{k-1}^{Y_n}\alpha\right| < (1-\epsilon)p_n \cdot h^{k-1}(X_n)\left|\alpha\right| \quad \Longrightarrow \quad \left|\delta_{k-1}^{Y_n}\alpha\right| < (1-\epsilon)p_n \cdot \left|\delta_{k-1}^{X_n}\alpha\right|, \tag{5.3.33}$$

what we use to estimate further (probability measures are monotone) to obtain a form where we may apply the Chernoff bound from above:

$$\sum_{0\neq\alpha\in\mathrm{C}^{k-1}(Y_n;\mathbb{Z}_2)} \mathbb{P}\left(\left|\delta_{k-1}^{Y_n}\alpha\right| < (1-\epsilon)p_n \cdot h^{k-1}(X_n)\left|\alpha\right|\right) \tag{5.3.34}$$

$$\leq \sum_{0\neq\alpha\in\mathrm{C}^{k-1}(Y_n;\mathbb{Z}_2)} \underbrace{\mathbb{P}\left(\left|\delta_{k-1}^{Y_n}\alpha\right| < (1-\epsilon)p_n\left|\delta_{k-1}^{X_n}\alpha\right|\right)}_{\leq\exp\left(-\frac{\epsilon^2}{2}p_n\left|\delta_{k-1}^{X_n}\alpha\right|\right)} \tag{5.3.35}$$

The rest of the proof is just rewriting this estimate to show that the probability tends to zero. The bound for the parameter p_n is chosen in a way that we get the desired result. In the following we use the estimate from the coboundary expansion of X_n and the binomial formula:

$$\sum_{0\neq\alpha\in\mathrm{C}^{k-1}(Y_n;\mathbb{Z}_2)} \exp\left(-\frac{\epsilon^2}{2}p_n \underbrace{\left|\delta_{k-1}^{X_n}\alpha\right|}_{\geq h^{k-1}(X_n)|\alpha|}\right) \tag{5.3.36}$$

$$\leq \sum_{0\neq\alpha\in\mathrm{C}^{k-1}(Y_n;\mathbb{Z}_2)} \exp\left(-\frac{\epsilon^2}{2}p_n h^{k-1}(X_n)\left|\alpha\right|\right) \tag{5.3.37}$$

$$= \sum_{i=1}^{|X_n(k-1)|} \sum_{\substack{\alpha\in\mathrm{C}^{k-1}(Y_n;\mathbb{Z}_2)\\ |\alpha|=i}} \exp\left(-\frac{\epsilon^2}{2}p_n h^{k-1}(X_n)i\right) \tag{5.3.38}$$

$$= \sum_{i=1}^{|X_n(k-1)|} \binom{|X_n(k-1)|}{i} \exp\left(-\frac{\epsilon^2}{2}p_n h^{k-1}(X_n)i\right) \tag{5.3.39}$$

$$= \left(1+\exp\left(-\frac{\epsilon^2}{2}p_n h^{k-1}(X_n)\right)\right)^{|X_n(k-1)|} - 1 \tag{5.3.40}$$

Now, we can plug in the bound for p_n and use the well-known estimate $\log(1+x) \leq x$:

$$\left(1+\exp\left(-\frac{\epsilon^2}{2}p_n h^{k-1}(X_n)\right)\right)^{|X_n(k-1)|} - 1 \tag{5.3.41}$$

$$\leq \left(1+\frac{\exp\left(-\frac{\omega(n)}{2}\right)}{|X_n(k-1)|}\right)^{|X_n(k-1)|} - 1 \tag{5.3.42}$$

$$= \exp\left(|X_n(k-1)| \underbrace{\log\left(1+\frac{\exp\left(-\frac{\omega(n)}{2}\right)}{|X_n(k-1)|}\right)}_{\leq \frac{\exp\left(-\frac{\omega(n)}{2}\right)}{|X_n(k-1)|}}\right) - 1 \tag{5.3.43}$$

$$\leq \exp\left(\exp\left(-\frac{\omega(n)}{2}\right)\right) - 1 \overset{n\to\infty}{\to} 0 \tag{5.3.44}$$

We thus have proved that the probability of Y_n to have a "good" coboundary expansion parameter tends to 1 as n tends to infinity. □

Rem. 5.3.2. The above proof contains lemma 5.3.3 on page 108 as a special case by taking $X_n = \Delta_{n-1}^{(k)}$. Moreover, we observe that the size of X_n need not be unbounded (as it is for the Linial–Meshulam model). However, the statement becomes vacuous, if the value of the coboundary expansion parameter $h^{k-1}(X_n)$ grows slower than $\log|X_n(k-1)|$. In this case, there is no possible choice of ω to not obtain a bound $p_n \geq 1$.

5.3.4 Random Latin Squares

The last example for the random method that we want to discuss deals with 2-dimensional simplicial complexes that are induced by so-called random Latin squares. The construction is due to Alexander Lubotzky and Roy Meshulam ([LM13]) and yields (under some assumptions) examples[8] for 2-dimensional complexes where the 2-degree is bounded for every 1-simplex, a question that was asked by Mikhail Gromov ([Gro10]) and Dominic Dotterrer and Matthew Kahle ([DK10]).

The construction (that is, the sampling method of this random model) of random Latin squares uses special tuples of permutations. In the following, we will fix n and denote by $\mathbb{S}_n$ the symmetric group, consisting of all permutations of the set $\{1, 2, \ldots, n\}$ (a permutation is a bijective map). A *Latin square* is an n-tuple of permutations with a special property:

[8] The proof is existential and not constructive, though.

Definition 5.3.1 (Latin Square)**.** *Let $n \in \mathbb{N}$. An n-tuple $L = (\pi_1, \pi_2, \ldots, \pi_n) \in \mathbb{S}_n^n$ of permutations is called a* Latin square *(of order n), if:*

$$\forall i \neq j \in \{1, \ldots, n\} : \quad \pi_i \pi_j^{-1} \text{ has no fixed points, i.e.,} \tag{5.3.45}$$

$$\nexists k \in \{1, \ldots, n\} : \pi_i \pi_j^{-1}(k) = k \tag{5.3.46}$$

The set of all Latin squares is denoted by $\mathcal{L}_n$.

There are two possible illustrations of Latin squares. First, if we write the permutations as row vectors of its values, that is:

$$\pi \cong (\pi(1), \pi(2), \ldots, \pi(n)), \tag{5.3.47}$$

then we can write the Latin square L as a matrix consisting of all these row vectors:

$$L = (\pi_1, \pi_2, \ldots, \pi_n) \cong (\pi_i(j))_{i,j=1}^n = \begin{pmatrix} \pi_1(1) & \pi_1(2) & \cdots & \pi_1(n) \\ \pi_2(1) & \pi_2(2) & \cdots & \pi_2(n) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_n(1) & \pi_n(2) & \cdots & \pi_n(n) \end{pmatrix} \tag{5.3.48}$$

The Latin square condition now reduces to the condition that there are no two equal values in every column. This gives a hint to the origin of the name "Latin square".

The other way to visualize a Latin square is by drawing a bipartite graph. The graph consists of two vertex sets $V_1 \cong \{1, \ldots, n\}$, $V_2 \cong \{1, \ldots, n\}$ of size n and two vertices $u \in V_1$ on the left side and $v \in V_2$ on the right side are connected by an edge, if and only if there is a permutation π_i in L such that:

$$\pi_i(u) = v \tag{5.3.49}$$

The Latin square condition ensures that none of these edges coincide for different $\pi_i \neq \pi_j$.

Example 8. An example for a Latin square of order 3 can be given by:

$$L = (\pi_1, \pi_2, \pi_3) :\cong \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix} \tag{5.3.50}$$

Fig. 5.8 on the next page shows the second method of illustrating L.

We can make $\mathcal{L}_n$ a probability space by defining the probability measure to be uniform. That is, we may sample from $\mathcal{L}_n$ a random Latin square with the uniform distribution. Using this distribution, we observe that the permutations in the sampled Latin square are not independent, but one permutation alone is distributed uniformly in the space of all permutations:

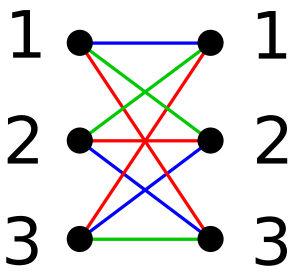

Fig. 5.8: Illustration of the Latin square L from example 8 on the previous page (blue: π_1, red: π_2, green: π_3)

Lemma 5.3.5. *Let $\mathcal{L}_n$ be the space of Latin squares with the uniform probability measure. Let $L = (\pi_1, \ldots, \pi_n)$ be a Latin square sampled randomly from $\mathcal{L}_n$.*

Then the random variable obtained by sampling L and picking π_i ($i \in \{1, \ldots, n\}$ arbitrary) is distributed uniformly (in the space $\mathbb{S}_n$ of all permutations).

Moreover, the maps:

$$i \mapsto \pi_i(j), \qquad j \in \{1, \ldots, n\} \text{ fixed, and} \tag{5.3.51}$$

$$i \mapsto (\pi_i)^{-1}(l), \qquad l \in \{1, \ldots, n\} \text{ fixed,} \tag{5.3.52}$$

are permutations by themselves and the random variables induced by these maps are distributed uniformly as well.

Proof.

- First, let $\pi \neq \pi' \in \mathbb{S}_n$ be two different permutations. Without loss of generality, we will show the first statement just for π_1 since by permuting the entries of the Latin square L we can switch π_i to the front[9]. We have to show that:

$$|\{L = (\pi_1, \ldots, \pi_n) \in \mathcal{L}_n \,|\, \pi_1 = \pi\}| = |\{L = (\pi_1, \ldots, \pi_n) \in \mathcal{L}_n \,|\, \pi_1 = \pi'\}| \tag{5.3.53}$$

  In order to prove this equality, we observe that by composing a Latin square with an arbitrary permutation $\sigma \in \mathbb{S}_n$ in the following way, we obtain another Latin square:

$$L = (\pi_1, \ldots, \pi_n) \mapsto \sigma L := (\sigma\pi_1, \ldots, \sigma\pi_n) \tag{5.3.54}$$

  σL is indeed a Latin square as can be seen easily from the second illustration of L, pictured in Fig. 5.9 on the facing page. The condition that no two edges induced by different permutations are equal is already fulfilled in the left part and it will not be perturbed by the right part. This can be seen directly from the definition, too.

[9]The proof would be the same for arbitrary i, we restrict to $i = 1$ to simplify notation.

We can now write down a bijection between the two sets, namely:

$$f : \begin{cases} \{L = (\pi_1, \dots, \pi_n) \in \mathcal{L}_n \,|\, \pi_1 = \pi\} & \to \{L = (\pi_1, \dots, \pi_n) \in \mathcal{L}_n \,|\, \pi_1 = \pi'\} \\ L & \mapsto \pi'\pi^{-1}L \end{cases} \tag{5.3.55}$$

f is well-defined and has the natural inverse $\tilde{L} \mapsto \pi(\pi')^{-1}\tilde{L}$, hence it is a bijection and both sets have the same cardinality. This shows that every permutation $\pi \in \mathbb{S}_n$ appears in an equal number of cases as π_1. π_1 is thus uniformly distributed.

- For the second and the third statement we have to show that these maps are indeed permutations. This follows directly from the Latin square condition, which implies injectivity:

$$\pi_i(j) = \pi_{i'}(j) \quad \Longrightarrow \quad \exists j' : j' = \pi_{i'}(\pi_i)^{-1}(j') \quad \Longrightarrow \quad i' = i \tag{5.3.56}$$

As we are dealing with finite and equal-sized sets, injectivity implies bijectivity and the map is a permutation. A similar argument can be applied to the second map.

We may now repeat the argument from the first point, as we get a similar bijection between the sets defined in the fitting way by composing the Latin square with the right permutations. The maps in consideration are transformed by this operation as follows:

$$(i \mapsto \pi_i(j)) \quad \mapsto \quad (i \mapsto (\sigma\pi_i)(j)) \tag{5.3.57}$$

$$\big(i \mapsto (\pi_i)^{-1}(j)\big) \quad \mapsto \quad \big(i \mapsto (\pi_i\sigma^{-1})^{-1}(j)\big) \tag{5.3.58}$$

By choosing the permutation σ correctly, we obtain the claimed bijection.

□

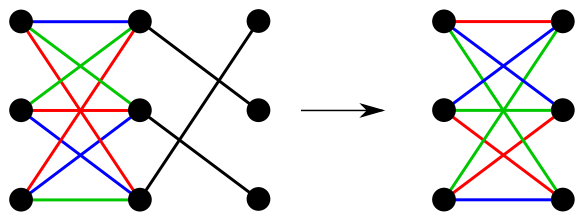

Fig. 5.9: Illustration of the composition of a Latin square L with a permutation

With this preparatory work done, we are now ready to define the simplicial complex induced by a Latin square:

Definition 5.3.2 (Latin Square Complex)**.** *Let $L = (\pi_1, \dots \pi_n) \in \mathcal{L}_n$ be an arbitrary Latin square and let Λ_n^2 be the complete 3-partite complex, where we want to denote the vertex sets by:*

$$V_0 = \{a_1, a_2, \dots, a_n\} \tag{5.3.59}$$
$$V_1 = \{b_1, b_2, \dots, b_n\} \tag{5.3.60}$$
$$V_2 = \{c_1, c_2, \dots, c_n\} \tag{5.3.61}$$

The Latin square complex *$Y(L)$ induced by L is defined as the 2-complex with $(\Lambda_n^2)^{(1)} \subseteq Y(L) \subseteq \Lambda_n^2$, which has the 2-simplices:*

$$\{a_i, b_j, c_{\pi_i(j)}\}, \quad 1 \leq i, j \leq n \tag{5.3.62}$$

For a d-tuple $\underline{L} = (L_1, \dots, L_d) \in \mathcal{L}_n^d$, we denote by $Y(\underline{L}) := \bigcup_{i=1}^d Y(L_i)$ the union of the Latin square complexes and by $\mathcal{Y}(n, d)$ we denote the space of all these complexes obtained with the probability measure induced by the product measure on $\mathcal{L}_n^d$ (that is, L_i and L_j are independent for $i \neq j$).

We directly obtain that $Y(L)$ consists of $3n$ vertices, $3n^2$ edges and n^2 triangles. Every edge is contained in exactly one triangle by the Latin square condition. Hence, in $Y(\underline{\mathrm{L}}) \in \mathcal{Y}(n, d)$, every edge has 2-degree less or equal to d.

The next aim is to show that a Latin square complex $Y(\underline{\mathrm{L}})$ is expanding on small locally minimal cochains with high probability by applying theorem 5.2.2 on page 98. Therefore, we have to discuss the structure of links of vertices. There are essentially three types of links:

- $v = a_i \in V_0$:
$$Y(\underline{\mathrm{L}})_{a_i}(1) = \Big\{\{b_j, c_{\pi_i^k(j)}\} \,|\, 1 \leq j \leq n; 1 \leq k \leq d\Big\} \tag{5.3.63}$$
- $v = b_j \in V_1$:
$$Y(\underline{\mathrm{L}})_{b_j}(1) = \Big\{\{a_i, c_{\pi_i^k(j)}\} \,|\, 1 \leq i \leq n; 1 \leq k \leq d\Big\} \tag{5.3.64}$$
- $v = c_l \in V_2$:
$$Y(\underline{\mathrm{L}})_{c_l}(1) = \Big\{\{a_i, b_{(\pi_i^k)^{-1}(l)}\} \,|\, 1 \leq i \leq n; 1 \leq k \leq d\Big\} \tag{5.3.65}$$

By lemma 5.3.5 on page 116 all of these links are bipartite graphs that are constructed by sampling d permutations independently and uniformly and adding edges from i (on the left side) to $\pi(i)$ (on the right side) for every of these permutations.[10]

If we assume that all these links are "good" spectral expanders, we can prove the necessary ingredients for theorem 5.2.2 on page 98:

[10] This random model of constructing graphs is a special case of a model called *Broder–Shamir model* and was analyzed by Joel Friedman and David-Emmanuel Kohler in [FK14] as well as by Doron Puder in [Pud12].

Lemma 5.3.6. *Let $Y(\underline{L}) \in \mathcal{Y}(n,d)$ be a Latin square complex, d large enough.*
If the link $Y(\underline{L})_v$ of every vertex $v \in Y(\underline{L})(0)$ has an eigenvalue gap that satisfies:

$$\lambda(Y(\underline{L})_v) \geq d - \mathcal{O}\left(\sqrt{d}\right), \tag{5.3.66}$$

with probability $1 - o\left(\frac{1}{n}\right)$, then the link fulfills with probability $1 - o(1)$:

- $h^0(Y(\underline{L})_v) \geq \frac{d}{2}\left(1 - \mathcal{O}\left(\frac{1}{\sqrt{d}}\right)\right)$ *and*
- $\eta(Y(\underline{L})_v) \geq \frac{d}{2}\left(1 + \epsilon - \mathcal{O}\left(\frac{1}{\sqrt{d}}\right)\right)$ *for ϵ-small cochains,*

that is:

$$\lim_{n\to\infty} \mathbb{P}(\forall v \in Y(\underline{L})(0) : \textit{the link is expanding with above estimates}) = 1 \tag{5.3.67}$$

Proof. By the normal Cheeger inequality (lemma 3.2.1 on page 36, unnormalized version) and the Cheeger inequality for small sets (lemma 5.2.8 on page 101) we obtain for an arbitrary vertex $v \in Y(\underline{\mathrm{L}})(0)$ with probability $1 - o\left(\frac{1}{n}\right)$:

$$h^0(Y(\underline{\mathrm{L}})_v) \geq \frac{1}{2}\left(d - \mathcal{O}\left(\sqrt{d}\right)\right) = \frac{d}{2}\left(1 - \mathcal{O}\left(\frac{1}{\sqrt{d}}\right)\right) \tag{5.3.68}$$

$$\eta(Y(\underline{\mathrm{L}})_v) \geq \frac{1+\epsilon}{2}\left(d - \mathcal{O}\left(\sqrt{d}\right)\right) = \frac{d}{2}\left(1 + \epsilon - \mathcal{O}\left(\frac{1}{\sqrt{d}}\right)\right) \tag{5.3.69}$$

This proves the claim for every single vertex. However, the links of the vertices are not independent (they are all dependent on the choice of the Latin square and if we take the links of all vertices from the first set together, we already can reproduce the Latin square). Nevertheless, we can estimate the probability of all links to be expanding by a union bound on the complements. Here we use that there are $3n$ vertices.

$$\mathbb{P}(\exists v \in Y(\underline{\mathrm{L}})(0) : Y(\underline{\mathrm{L}})_v \text{ is not expanding with these constants}) \tag{5.3.70}$$

$$\leq \sum_{v \in Y(\underline{\mathrm{L}})(0)} \mathbb{P}(Y(\underline{\mathrm{L}})_v \text{ is not expanding with these constants}) \tag{5.3.71}$$

$$\leq 3n \cdot o\left(\frac{1}{n}\right) = o(1) \tag{5.3.72}$$

By switching to the complements we obtain the claimed bound for the probability. □

Rem. 5.3.3. In [LM13] Alexander Lubotzky and Roy Meshulam argue that a theorem of Joel Friedman can be applied to yield the eigenvalue gap with high probability. See [Fri91, Fri07] for this theorem. Newer articles by Joel Friedman ([FK14]) and Doron Puder ([Pud12]) also deal with similar models and results.

Theorem 5.3.3 (Expansion for Small Cochains)**.** *Let $Y(\underline{L}) \in \mathcal{Y}(n,d)$ be a random Latin square complex and assume that the condition of lemma 5.3.6 on the preceding page regarding the eigenvalue gap holds.*

Then there exists a $d_0 \geq 5$ such that for $d \geq d_0$ we have that $Y(\underline{L})$ is expanding on small locally minimal cochains with probability tending to 1 as n tends to infinity. That is, there exists $\epsilon' > 0, \eta > 0$ such that for a locally minimal cochain $\alpha \in \mathrm{C}^1(Y(\underline{L}); \mathbb{Z}_2)$ with high probability:

$$|\alpha| \leq \frac{1}{1+\epsilon'} \frac{|Y(\underline{L})(1)|}{4} \quad \Longrightarrow \quad |\delta\alpha| \geq \eta\, |\alpha| \tag{5.3.73}$$

Proof. We want to apply theorem 5.2.2 on page 98. As the 1-skeleton of $Y(\underline{\mathrm{L}})$ is a complete 3-partite graph, we have already seen in Section 5.2.2 that it is a regular graph and a skeleton expander with $\beta = 0$. The previous lemma 5.3.6 on the preceding page (which is by assumption applicable) implies the needed expansion properties of the links of vertices with high probability. We can now choose $\epsilon, \epsilon' > 0$ and $\xi \in (0,1)$ such that:

$$\frac{1}{(1-\epsilon)^2(1+\epsilon')} \leq 1 - \xi \tag{5.3.74}$$

The condition on $\epsilon_1 = \epsilon - \mathcal{O}\left(\frac{1}{\sqrt{d}}\right)$ and $\epsilon_2 = \mathcal{O}\left(\frac{1}{\sqrt{d}}\right)$ can be fulfilled by choosing d large enough as:

$$\frac{\epsilon_1}{\epsilon_2} = \frac{\epsilon - \mathcal{O}\left(\frac{1}{\sqrt{d}}\right)}{\mathcal{O}\left(\frac{1}{\sqrt{d}}\right)} \to \infty, \quad (d \to \infty) \tag{5.3.75}$$

The local to global theorem in its combinatorial version now yields the claimed result. □

We thus have proved (under the condition of a good spectral gap for the links) that a majority of all Latin square complexes are expanding on small locally minimal cochains. In [LM13] it is proved that they are also expanding for large cochains. The proof there uses the special structure of Latin squares and the Chernoff bound to obtain a probability bound. As the proof is very technical we refer to the original work for the details and just state the theorem ([LM13, Theorem 1]):

Theorem 5.3.4 (Expansion for Random Latin Squares)**.** *Let $\mathcal{Y}(n,d)$ be the random space of Latin square complexes and assume that the condition of lemma 5.3.6 on the previous page regarding the eigenvalue gap holds.*

Then there exist $\epsilon > 0$ and $d < \infty$ such that:

$$\lim_{n\to\infty} \mathbb{P}(Y \in \mathcal{Y}(n,d) : h^1(Y) > \epsilon) = 1 \tag{5.3.76}$$

Random Latin squares thus yield (under this assumption) examples for 2-dimensional expanders that have bounded 2-degree for every edge. Higher-dimensional examples can be found for example by considering so-called *random Steiner systems* presented in [LLR15] by Alexander Lubotzky, Zur Luria and Ron Rosenthal.

6 Open (or just interesting) Questions – Conclusio

In the course of the previous chapters, we discussed some of the definitions, applications and examples of high-dimensional expansion. Although the field of high-dimensional expansion is still very young, we could only hope to present a small part of what is already known. Depending on the mathematical background of the researchers, there is a tremendous amount of variations of the definitions and results presented in this thesis. There are at least three more definitions of high-dimensional expansion that we were not able to include for they are far out of scope. According to the desired application, there are several examples for expanders and proofs. It is therefore hard (or simply impossible) to give a complete list of open questions. We will thus try to give an overview of interesting questions and refer to other specialized literature for more detail.

Examples for (bounded-degree) Expanders – Can we give examples of families of high-dimensional expanders beside the "standard examples" of the complete complex and the complete multipartite complexes? This is a very important question for the application of expansion, for example in (theoretical) computer sciences. For these applications, we also need special properties like bounded degree. The random Latin squares, presented in Section 5.3.4, give (under some assumptions) examples for 2-dimensional complexes with bounded 2-degree of edges. Random Steiner systems ([LLR15], linked with the theory of designs) give examples for arbitrary-dimensional complexes with bounded degree of the co-dimension 1 simplices. Both examples have complete skeletons. Thus, lower-dimensional simplices have unbounded degree. This leads to the question:

> Can we find families of expanders where *all* simplices have bounded degree?

Such complexes are needed in many applications. Moreover, both constructions give only existence of expanders, but no way of constructing them (which is necessary for the application). Similar to the development in graph theory, there are two lines of research:

Probabilistic – Here, we are searching for random models that give families of expanders with high probability. Random Latin squares and random Steiner systems are first examples which may be extended. This approach worked very well for graph-theoretical expansion and may be "lifted" to

high-dimensional expansion. There are many results in the graph case that may be used (cf. [Fri91, Fri07, FK14]).

Constructive – Beside the mathematically interesting question whether there exist families of expanders, it is also very important to find explicit examples. In the graph case, there are several constructions (Ramanujan graphs, zig-zag-product, lifts of graphs) that can be used. With Ramanujan complexes there are already approaches to generalize Ramanujan graphs to higher dimensions (cf. [Lub13, KKL14a, KKL14b, EK15]). However, some of the expansion properties of these complexes are still unclear.

This leads to the next question:

Methods to prove expansion – Given a simplicial complex, how can we prove that it has some expansion properties? There are already seemingly probabilistic methods like "random co-filling" (described in Section 5.1), that work for special complexes, and there are methods that use the local structure of the complex ("local to global" methods, described in Section 5.2, cf. [KKL14a, KKL14b, EK15]). How can these methods be refined and extended? As these methods are mainly used to prove expansion for complexes that were just constructed, can we find rules how some construction steps affect the expansion properties? If we could, for example, understand what is happening if we take the join (as simplicial complexes, see the literature for a exact definition) of two "good expanders", we may be able to construct large expanders by taking the n-fold join of some prototype expander. Another example of such a construction method is the wedge product. For spectral expansion (and Laplacians) this was studied by Danijela Horak and Jürgen Jost in [HJ11].

In general, we could also ask:

> Are there other methods, where we maybe use the knowledge of a different field of mathematics?

In the graph case, for example, there is a really powerful tool: the Cheeger inequality. By spectral expansion we are able to use linear algebra to analyze expansion. This leads to the question, whether this can be generalized:

Generalizations of the Cheeger Inequality – Is there a connection between spectral expansion and combinatorial ($\mathbb{Z}_2$-)expansion in high dimensions? Sadly, if there is a connection, it is not as easy as in the case of graphs. Anna Gundert and Uli Wagner ([GW16]) gave an example for a simplicial complex (generated by a random process) that has strong spectral properties (that is, a large spectral gap of the high-dimensional Laplacian), but there exists a cochain with small coboundary. Hence, the coboundary expansion parameter tends to zero as the size of the complex grows. This contradicts a (conjectured) linear lower-bound of the coboundary expansion parameter in terms of the spectral gap. In [SKM12], John Steenbergen, Caroline Klivans and

Sayan Mukherjee give a counterexample for the second part of the conjectured Cheeger inequality, namely an example of a complex that has non-vanishing coboundary expansion, but the eigenvalue gap tends to zero. In the same article, further results on a high-dimensional Cheeger inequality are discussed. For example, one could ask whether the structure of the inequality has to be changed or whether we need other conditions like a complete skeleton. Ori Parzanchevski, Ron Rosenthal and Ran Tessler prove Cheeger-like statements in [PRT12], using a different notion of high-dimensional expansion.

Further Generalizations of Expansion – As the field of expansion is still developing, it is not clear whether the definitions of expansion presented in this thesis are the "best ones". Indeed, there are several other approaches which have different benefits. We can only give a few examples:

Cheeger-type Expansion – As defined in [PRT12] and in [Par13], this type of expansion is a try to generalize the definition of edge expansion without using cohomology. Like in the 1-dimensional case, the vertex set is partitioned into several (depending on the dimension) subsets and instead of edges between the subsets we count the number of simplices that have one vertex in every subset. For a complete skeleton, this definition allows to prove one part of the Cheeger inequality in high dimension.

Colorful Expansion – This notion of expansion was defined in [KM16a] and [KM16b] by Tali Kaufman and David Mass. It uses a similar structure as coboundary expansion, but they replace the definition of a coboundary by another construction. Namely, for a cochain in dimension k they take a cochain $\psi(k)$ in dimension $k+1$ that has a support that consists of all simplices that contain neither no simplex of the original cochain nor all of them. In comparison to the $\mathbb{Z}_2$-coboundary, we add more simplices, because only those simplices are not taken that contain all facets or no facet in the support of the original cochain, whereas for $\mathbb{Z}_2$-coboundary expansion we do not take simplices with an even number of facets from the original cochain.

As it turns out, the notion of "colorful expansion" is linked with the convergence of high order random walks.

Agreement Expansion – The last notion of high-dimensional expansion, that we want to discuss, aims in a completely different direction. It is a continuation of the application of expansion that is described in Section 4.2 and describes the usefulness of a complex for so-called agreement tests. It was defined by Irit Dinur and Tali Kaufman in [DK17], where they also give explicit examples of bounded-degree agreement expanders using Ramanujan complexes.

The list presented here can only be a short extract of all possible trails of research. For more information we refer to the literature at the end of this thesis.

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