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DISSERTATION

Tensor Valuations
on Lattice Polytopes

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Kurzfassung

In dieser Dissertation wird ein Überblick über Tensorbewertungen auf Gitterpolytopen gegeben, der auf zwei Arbeiten basiert, in denen mit der Entwicklung der Theorie dieser Bewertungen begonnen wurde. Dabei wird einerseits eine Klassifikation von Tensorbewertungen hergeleitet und andererseits werden die Basiselemente des Vektorraums dieser Bewertungen untersucht.

Basierend auf der gemeinsamen Arbeit [43] mit Monika Ludwig wird für symmetrische Tensorbewertungen bis zum Rang 8, die kovariant bezüglich Translationen und der speziellen linearen Gruppe über den ganzen Zahlen sind, eine vollständige Klassifikation hergeleitet. Der Spezialfall der skalaren Bewertungen stammt von Betke und Kneser, die zeigten, dass alle solche Bewertungen Linearkombinationen der Koeffizienten des Ehrhart-Polynoms sind. Als Verallgemeinerung dieses Resultats wird gezeigt, dass für Rang kleiner gleich 8 alle solchen Tensorbewertungen Linearkombinationen der entsprechenden Ehrhart-Tensoren sind und, dass dies für Rang 9 nicht mehr gilt. Für Rang 9 wird eine neue Bewertung beschrieben und für Tensoren höheren Ranges werden ebenfalls Kandidaten für solche Bewertungen angegeben. Weiter werden der Begriff des Ehrhart-Polynoms und die Reziprozitätssätze von Ehrhart & Macdonald auf Tensorbewertungen verallgemeinert.

Das Ehrhart-Tensorpolynom ist eine natürliche Verallgemeinerung des Ehrhart-Polynoms. Basierend auf der gemeinsamen Arbeit [10] mit Sören Berg und Katharina Jochemko werden diese Ehrhart-Tensorpolynome untersucht. Verallgemeinerungen der klassischen Formel von Pick werden im Fall von vektor- und matrixwertigen Bewertungen hergeleitet, wobei Triangulierungen des gegebenen Gitterpolygons verwendet werden. Der Begriff des h^*-Polynoms wird auf den Begriff des h^r-Tensorpolynoms erweitert und dessen Koeffizienten werden für Matrizen auf positive Semidefinitheit untersucht. Im Unterschied zum klassischen h^*-Polynom sind die Koeffizienten nicht notwendigerweise monoton. Trotzdem wird positive Semidefinitheit im planaren Fall bewiesen. Basierend auf Rechnungen wird positive Semidefinitheit auch für höhere Dimensionen vermutet. Darüber hinaus wird Hibi's Palindromsatz für reflexive Polytope auf h^r-Tensorpolynome verallgemeinert.

Abstract

An overview of tensor valuations on lattice polytopes is provided composed of two contributions that began the development of the theory of these valuations; a characterization result preceded by a thorough study of the basis elements of the vector space of valuations.

A complete classification, based on a joint paper with Monika Ludwig [43], is established of symmetric tensor valuations of rank up to eight that are translation covariant and intertwine the special linear group over the integers. The real-valued case was established by Betke & Kneser where it was shown that the only such valuations are the coefficients of the Ehrhart polynomial. The Ehrhart polynomial is generalized to the Ehrhart tensor polynomial with coefficients Ehrhart tensors. Extending the result of Betke & Kneser, it is shown that every tensor valuation with these properties is a combination of the Ehrhart tensors, for rank at most eight, which is shown to no longer hold true for rank nine. A new valuation that emerges in rank nine is described along with candidates for tensors of higher rank. Furthermore, the reciprocity theorems by Ehrhart & Macdonald are extended to tensor valuations.

Based on a joint paper with Sören Berg and Katharina Jochemko [10], the Ehrhart tensors are investigated. Pick-type formulas are given, for the vector and matrix cases, in terms of triangulations of the given lattice polygon. The notion of the Ehrhart h^*-polynomial is extended to h^r-tensor polynomials and, for matrices, their coefficients are studied for positive semidefiniteness. In contrast to the classic h^*-polynomial, the coefficients are not necessarily monotone with respect to inclusion. Nevertheless, positive semidefiniteness is proven in the planar case. Based on computational results, positive semidefiniteness of the coefficients in higher dimensions is conjectured. Furthermore, Hibi's palindromic theorem for reflexive polytopes is generalized to h^r-tensor polynomials.

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Contents

Chapter 1

Introduction

Tensor valuations on convex bodies have attracted increasing attention in recent years (see, e.g., [11, 33, 35]). They were introduced by McMullen in [49] and Alesker subsequently obtained a complete classification of continuous and isometry equivariant tensor valuations on convex bodies (based on [3] but completed in [4]). Tensor valuations have found applications in different fields and subjects; in particular, in Stochastic Geometry and Imaging (see [35]). This thesis is a compilation of an article jointly done with Monika Ludwig [43] and an article jointly done with Sören Berg and Katharina Jochemko [10] that together help develop the theory of tensor valuations on lattice polytopes.

Let $\mathcal{P}(\mathbb{Z}^n)$ denote the set of lattice polytopes in $\mathbb{R}^n$; that is, the set of convex polytopes with vertices in the integer lattice $\mathbb{Z}^n$. In general, a full-dimensional lattice in $\mathbb{R}^n$ is an image of $\mathbb{Z}^n$ by an invertible linear transformation and, therefore, all results can easily be translated to the general situation of polytopes with vertices in an arbitrary lattice. A function Z defined on $\mathcal{P}(\mathbb{Z}^n)$ with values in an abelian semigroup is a *valuation* if

$$\mathrm{Z}(P) + \mathrm{Z}(Q) = \mathrm{Z}(P \cup Q) + \mathrm{Z}(P \cap Q)$$

whenever $P, Q, P \cup Q, P \cap Q \in \mathcal{P}(\mathbb{Z}^n)$ and $\mathrm{Z}(\varnothing) = 0$.
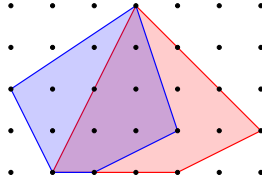
For $P \subset \mathbb{R}^n$, the lattice point enumerator, $\mathrm{L}(P)$, is defined as

$$\mathrm{L}(P) = \sum_{x \in P \cap \mathbb{Z}^n} 1. \qquad (1.1)$$

Hence, $\mathrm{L}(P)$ is the number of lattice points in P and $P \mapsto \mathrm{L}(P)$ is a valuation on $\mathcal{P}(\mathbb{Z}^n)$. A function Z defined on $\mathcal{P}(\mathbb{Z}^n)$ is $\mathrm{SL}_n(\mathbb{Z})$ *invariant* if $\mathrm{Z}(\phi P) = \mathrm{Z}(P)$ for all $\phi \in \mathrm{SL}_n(\mathbb{Z})$ and $P \in \mathcal{P}(\mathbb{Z}^n)$ where $\mathrm{SL}_n(\mathbb{Z})$ is the special linear group over the integers; that is, the group

of transformations that can be described by $n \times n$ matrices of determinant 1 with integer coefficients. A function Z is *translation invariant* on $\mathcal{P}(\mathbb{Z}^n)$ if $Z(P + y) = Z(P)$ for all $y \in \mathbb{Z}^n$ and $P \in \mathcal{P}(\mathbb{Z}^n)$. It is *i-homogeneous* if $Z(k P) = k^i Z(P)$ for all $k \in \mathbb{N}$ and $P \in \mathcal{P}(\mathbb{Z}^n)$ where $\mathbb{N}$ is the set of non-negative integers.

A fundamental result on lattice polytopes by Ehrhart [19] introduces the so-called Ehrhart polynomial and was the beginning of what is now known as Ehrhart Theory (see [8, 9]). The Ehrhart polynomial of a lattice polytope counts the number of lattice points in its integer dilates and is arguably the most fundamental arithmetic invariant of a lattice polytope.

Theorem (Ehrhart). *There exist* $L_i : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ *for* $i = 0, \ldots, n$ *such that*

$$L(kP) = \sum_{i=0}^{n} L_i(P)k^i$$

for every $k \in \mathbb{N}$ *and* $P \in \mathcal{P}(\mathbb{Z}^n)$. *For each* i, *the functional* L_i *is an* $\mathrm{SL}_n(\mathbb{Z})$ *and translation invariant valuation that is homogeneous of degree* i.

Note that $L_n(P)$ is the n-dimensional volume, $V_n(P)$, and $L_0(P)$ the Euler characteristic of P, that is, $L_0(P) = 1$ for $P \in \mathcal{P}(\mathbb{Z}^n)$ non-empty and $L_0(\varnothing) = 0$. Also note that $L_i(P) = 0$ for $P \in \mathcal{P}(\mathbb{Z}^n)$ with $\dim(P) < i$, where $\dim(P)$ is the dimension of the affine hull of P.

Extending the definition of the lattice point enumerator (1.1), for $P \in \mathcal{P}(\mathbb{Z}^n)$ and a non-negative integer r, we define the *discrete moment tensor of rank* r by

$$L^r(P) = \frac{1}{r!} \sum_{x \in P \cap \mathbb{Z}^n} x^r$$

where x^r denotes the r-fold symmetric tensor product of $x \in \mathbb{R}^n$. Let $\mathbb{T}^r$ denote the vector space of symmetric tensors of rank r on $\mathbb{R}^n$. We then have $\mathbb{T}^0 = \mathbb{R}$ and $L^0 = L$. For $r = 1$, we obtain the *discrete moment vector*, which was introduced in [16]. For $r \geq 2$, discrete moment tensors were introduced in [15]. The discrete moment tensor is a natural discretization of the *moment tensor of rank* r of $P \in \mathcal{P}(\mathbb{Z}^n)$ which is defined to be

$$M^r(P) = \frac{1}{r!} \int_P x^r \, dx. \tag{1.2}$$

For $r = 0$ and $r = 1$, respectively, this is the n-dimensional volume, V_n, and the moment vector. See [55, Section 5.4] for more information on moment tensors and [25, 26, 42, 41, 24, 45, 44] for some recent results.

Corresponding to the theorem of Ehrhart, we establish the existence of a homogeneous decomposition for the discrete moment tensors for integers $r \geq 1$.

Theorem 1. *There exist* $L_i^r : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *for* $i = 1, \ldots, n + r$ *such that*

$$L^r(kP) = \sum_{i=1}^{n+r} L_i^r(P)k^i$$

for every $k \in \mathbb{N}$ *and* $P \in \mathcal{P}(\mathbb{Z}^n)$. *For each* i, *the function* L_i^r *is an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant, translation covariant and i-homogeneous valuation.*

For the definition of $\mathrm{SL}_n(\mathbb{Z})$ equivariance and translation covariance, see Section 2. The coefficients yield new valuations that we introduce here as *Ehrhart tensors.* Note that $\mathrm{L}^r_{n+r}(P)$ is the moment tensor of P and that $\mathrm{L}^r_{i+r}(P) = 0$ for $i > \dim(P)$ (see Section 4.1). The existence of the homogeneous decomposition is proved for general tensor valuations in Section 3.1. The proof is based on results by Khovanskiĭ & Pukhlikov [52].

A second fundamental result on lattice polytopes is the reciprocity theorem of Ehrhart [19] and Macdonald [46].

Theorem (Ehrhart & Macdonald)**.** *For $P \in \mathcal{P}(\mathbb{Z}^n)$, the relation*

$$\mathrm{L}(P^\circ) = (-1)^m \sum_{i=0}^{m} (-1)^i \, \mathrm{L}_i(P)$$

holds where $m = \dim(P)$.

Here, we write P° for the relative interior of P with respect to the affine hull of P. We establish a reciprocity result corresponding to the Ehrhart-Macdonald Theorem for the discrete moment tensor.

Theorem 2. *For $P \in \mathcal{P}(\mathbb{Z}^n)$, the relation*

$$\mathrm{L}^r(P^\circ) = (-1)^{m+r} \sum_{i=1}^{m+r} (-1)^i \, \mathrm{L}^r_i(P)$$

holds where $m = \dim(P)$.

In Section 3.2, we establish reciprocity theorems for general tensor valuations; the above theorem is a special case. We follow the approach of McMullen [47].

A fundamental and intensively studied question in Ehrhart theory is the characterization of Ehrhart polynomials and their coefficients. The only known coefficients that are known to have explicit geometric descriptions are the leading, second-highest, and constant coefficients for the classic Ehrhart polynomial (see, e.g., [9]). For the Ehrhart tensor polynomial, we obtain that the leading coefficient is equal to the moment tensor, we give an interpretation for the second-highest coefficient (Proposition 22) as the weighted sum of moment tensors over the facets of the polytope, and the constant tensor coefficient, for $r \geq 1$, is given to be identically zero; the descriptions of all are given in Section 4.1.

Conversely, for lattice polygons, the coefficients of the Ehrhart polynomial are positive and well-understood. They are given by Pick's Formula [51]. Let ∂P denote the boundary of the polytope P.

Theorem 3 (Pick's Formula)**.** *For any lattice polygon P and $k \in \mathbb{N}$, we have*

$$\mathrm{L}(kP) \;=\; \mathrm{L}_0(P) + \mathrm{L}_1(P)k + \mathrm{L}_2(P)k^2$$

where $\mathrm{L}_0(P) = 1$, $\mathrm{L}_1(P) = \frac{1}{2}\,\mathrm{L}(\partial P)$, and $\mathrm{L}_2(P)$ equals the area of P.

In Section 4.3, we determine Pick-type formulas for the discrete moment vector and matrix. Our interpretation of the coefficients is given with respect to a triangulation of the respective polygon. Our principal tool to study Ehrhart tensor polynomials are h^r-**tensor polynomials** which encode the Ehrhart tensor polynomial in a certain binomial basis. Extending the notion of the usual Ehrhart h^*-polynomial, we consider

$$\mathrm{L}^r(kP) \;=\; h_0^r(P)\binom{k+n+r}{n+r} + h_1^r(P)\binom{k+n+r-1}{n+r} + \;\cdots\; + h_{n+r}^r(P)\binom{k}{n+r} \quad (1.3)$$

for an n-dimensional lattice polytope P and define the h^r-*tensor polynomial* of P to be

$$h_P^r(t) \;=\; \sum_{i=0}^{n+r} h_i^r(P)t^i.$$

We determine a formula for the h^r-tensor polynomial of half-open simplices as Theorem 26 by using *half-open decompositions of polytopes*; an important tool which was introduced by Köppe and Verdoolaege [37]. From this formula and the existence of a unimodular triangulation, we deduce an interpretation of all Ehrhart vectors and matrices of lattice polygons.

Stanley's Nonnegativity Theorem [58] is a foundational result which states that all coefficients of the h^*-polynomial of a lattice polytope are nonnegative. Stanley moreover proved that the coefficients are monotone with respect to inclusion, i.e., for all lattice polytopes $Q \subseteq P$ and all $0 \leq i \leq n$ it holds that $h_i^*(Q) \leq h_i^*(P)$. Using half-open decompositions, it was proven in [34] that, with regard to translation invariant valuations, monotonicity and nonnegativity are equivalent. In Section 4.4, we discuss notions of positivity for Ehrhart tensors and investigate Ehrhart tensor polynomials and h^2-tensor polynomials with respect to *positive semidefiniteness*. In contrast to the usual Ehrhart polynomial, Ehrhart tensors can even be indefinite for lattice polygons (Example 31). Moreover, the coefficients of h^2-tensor polynomials are not monotone which is demonstrated by Example 33. Nevertheless, considering an intricate decomposition of lattice points inside a polygon, we are able to prove positive semi-definiteness in dimension two (Theorem 32). Based on computational results, we further conjecture positive semidefiniteness of the coefficients for higher dimensions (Conjecture 38).

In Section 4.5, we prove a generalization of Hibi's Palindromic Theorem [31] characterizing reflexive polytopes as having palindromic h^r-tensor polynomials for $r \in \mathbb{N}$ of even rank and discuss possible future research directions for Ehrhart tensor polynomials.

Another fundamental result on lattice polytopes is the Betke & Kneser Theorem [14]. It provides a complete classification of $\mathrm{SL}_n(\mathbb{Z})$ and translation invariant real-valued valuations on $\mathcal{P}(\mathbb{Z}^n)$ and a characterization of the Ehrhart coefficients.

Theorem (Betke & Kneser)**.** *A functional* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *and translation invariant valuation if and only if there are* $c_0, \ldots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{Z}(P) = c_0 \, \mathrm{L}_0(P) + \cdots + c_n \, \mathrm{L}_n(P)$$

for every $P \in \mathcal{P}(\mathbb{Z}^n)$.

The above result was established by Betke [13] and first published in [14]. In both papers, it was assumed that the functional is invariant with respect to unimodular transformations where these are defined to be a combination of translations by integral vectors and $\mathrm{GL}_n(\mathbb{Z})$ transformations; that is, linear transformations with integer coefficients and determinant ± 1. The proofs remain unchanged for the $\mathrm{SL}_n(\mathbb{Z})$ case (see [16]).

The Betke & Kneser Theorem is a discrete analogue of what is presumably the most celebrated result in the geometric theory of valuations, Hadwiger's Characterization Theorem [27]. Let $\mathcal{K}^n$ denote the space of convex bodies (that is, compact convex sets) on $\mathbb{R}^n$ equipped with the topology coming from the Hausdorff metric.

Theorem (Hadwiger). *A functional* $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous and rigid motion invariant valuation if and only if there are* $c_0, \dots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{Z}(K) = c_0\, V_0(K) + \cdots + c_n\, V_n(K)$$

for every $K \in \mathcal{K}^n$.

Here $V_0(K), \dots, V_n(K)$ are the *intrinsic volumes* of $K \in \mathcal{K}^n$, which are classically defined through the Steiner polynomial. That is, for $s \geq 0$,

$$V_n(K + s\, B^n) = \sum_{j=0}^{n} s^{n-j} v_{n-j}\, V_j(K),$$

where B^n is the n-dimensional Euclidean unit ball with volume v_n and

$$K + s\, B^n = \{x + s\, y : x \in K, y \in B^n\}.$$

The Hadwiger Theorem has powerful applications within Integral Geometry and Geometric Probability (see [27, 36]).

Hadwiger's theorem was extended to vector valuations by Hadwiger & Schneider [28].

Theorem (Hadwiger & Schneider). *A function* $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}^n$ *is a continuous, rotation equivariant, and translation covariant valuation if and only if there are* $c_1, \dots, c_{n+1} \in \mathbb{R}$ *such that*

$$\mathrm{Z}(K) = c_1\, \mathrm{M}_1^1(K) + \cdots + c_{n+1}\, \mathrm{M}_{n+1}^1(K)$$

for every $K \in \mathcal{K}^n$.

Here $\mathrm{M}_i^1(K) = \Phi_i^{1,0}(K)$ are the *intrinsic vectors* of K (see (1.4) below). The key ingredient in the proof is a characterization of the Steiner point by Schneider [54].

Correspondingly, we obtain the following classification theorem for $n \geq 2$.

Theorem 4. *A function* $\mathrm{Z} \colon \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant and translation covariant valuation if and only if there are* $c_1, \dots, c_{n+1} \in \mathbb{R}$ *such that*

$$\mathrm{Z}(P) = c_1\, \mathrm{L}_1^1(P) + \cdots + c_{n+r}\, \mathrm{L}_{n+1}^1(P)$$

for every $P \in \mathcal{P}(\mathbb{Z}^n)$.

The proof is given in Section 6.1 and is based on a characterization [16] of the discrete Steiner point. For $n = 1$, the characterization follows from the Betke & Kneser Theorem as only translation covariance has to be considered.

The theorems by Hadwiger and Hadwiger & Schneider were extended by Alesker [4, 2] (based on [3]) to a classification of continuous, rotation equivariant, and translation covariant tensor valuations on $\mathcal{K}^n$ involving extensions of the intrinsic volumes. Just as the intrinsic volumes can be obtained from the Steiner polynomial, the moment tensor M^r satisfies the Steiner formula

$$\mathrm{M}^r(K + s\,B^n) = \sum_{j=0}^{n+r} s^{n+r-j} v_{n+r-j} \sum_{k \geq 0} \Phi_{j-r+k}^{r-k,k} \tag{1.4}$$

for $K \in \mathcal{K}^n$ and $s \geq 0$. The coefficients $\Phi_k^{r,s}$ are called the *Minkowski tensors* (see [55, Section 5.4]). Let $Q \in \mathbb{T}^2$ be the metric tensor, that is, $Q(x,y) = x \cdot y$ for $x,y \in \mathbb{R}^n$.

Theorem (Alesker)**.** *A function* $\mathrm{Z} : \mathcal{K}^n \to \mathbb{T}^r$ *is a continuous, rotation equivariant, and translation covariant valuation if and only if* Z *can be written as linear combination of the symmetric tensor products* $Q^l \, \Phi_k^{m,s}$ *with* $2l + m + s = r$.

We remark that there are linear relations, called syzygies, between the tensors described above and that the dimension of the space of continuous, rotation equivariant and translation covariant matrix valuations $\mathrm{Z} : \mathcal{K}^n \to \mathbb{T}^2$ is $3n + 1$ for $n \geq 2$ (see [3]).

For tensor valuations of rank up to eight, we obtain the following complete classification. For $r \geq 3$, symbolic computation is used in the proof to show that certain matrices are non-singular.

Theorem 5. *For* $2 \leq r \leq 8$, *a function* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant and translation covariant valuation if and only if there are* $c_1, \ldots, c_{n+r} \in \mathbb{R}$ *such that*

$$\mathrm{Z}(P) = c_1 \, \mathrm{L}_1^r(P) + \cdots + c_{n+r} \, \mathrm{L}_{n+r}^r(P)$$

for every $P \in \mathcal{P}(\mathbb{Z}^n)$.

While the Betke & Kneser Theorem looks similar to the Hadwiger Theorem and Theorem 4 looks similar to the Hadwiger & Schneider Theorem, the similarity between the discrete and continuous cases breaks down for rank $r = 2$, as corresponding spaces have even different dimensions. For $n = 2$ and $r = 9$, there exists a new $\mathrm{SL}_2(\mathbb{Z})$ equivariant and translation invariant valuation which is not a linear combination of the Ehrhart tensors; it is described in Section 6.3. Hence, we do not expect that a classification similar to Theorem 5 continues to hold for $r \geq 9$.

Additionally, we obtain a classification of translation covariant and $(n + r)$-homogeneous tensor valuations on $\mathcal{P}(\mathbb{Z}^n)$ for $r \geq 1$ in Theorem 23 which provides a characterization of the moment tensor. The scalar case of this result corresponds to Hadwiger's classification [27, Satz XIV] of translation invariant and n-homogeneous valuations on convex polytopes while the case of tensors of general rank r corresponds to McMullen's classification [49] of continuous, translation covariant, and $(n + r)$-homogeneous tensor valuations on convex bodies.

Chapter 2

Preliminaries

For quick later reference, we aggregate most of the basics into this section and refer the reader, for more general reference, to [9, 8, 23, 61].

2.1 Symmetric Tensors

Our setting will be the n-dimensional Euclidean space, $\mathbb{R}^n$, equipped with the scalar product $x \cdot y$, for $x, y \in \mathbb{R}^n$, to identify $\mathbb{R}^n$ with its dual space. The identification of $\mathbb{R}^n$ with its dual space allows us to regard each symmetric r-tensor as a symmetric multi-linear functional on $(\mathbb{R}^n)^r$. Let $\mathbb{T}^r$ denote the vector space of symmetric tensors of rank r on $\mathbb{R}^n$. We will also write this as $\mathbb{T}^r(\mathbb{R}^n)$ if we want to stress the vector space in which we are working.

The symmetric tensor product of tensors $A_i \in \mathbb{T}^{r_i}$ for $i = 1, \ldots, k$ is

$$A_1 \odot \cdots \odot A_k(v_1, \ldots, v_r) = \frac{1}{r!} \sum_\sigma A_1 \otimes \cdots \otimes A_k(v_{\sigma(1)}, \ldots, v_{\sigma(r)})$$

for $v_1, \ldots, v_r \in \mathbb{R}^n$ where $r = r_1 + \cdots + r_k$, the ordinary tensor product is denoted by $\otimes$, and we sum over all of the permutations of $1, \ldots, r$. Given the standard orthonormal basis $e_1, \ldots, e_n$, any tensor $A \in \mathbb{T}^r$ can be written uniquely as

$$A = \sum_{1 \le i_j \le n} A_{i_1, \ldots, i_r} e_{i_1} \odot \cdots \odot e_{i_r}.$$

For $r = 2$, the bilinear form $A \in \mathbb{T}^2$ can then be identified with a symmetric $n \times n$ matrix $A = (A_{ij})$. To that end, we will call the discrete moment tensor of ranks 1 and 2 the *discrete moment vector* and *discrete moment matrix*, respectively. We will also regard their associated coefficients, their Ehrhart tensors, as *Ehrhart vectors* and *Ehrhart matrices*.

For simplicity, we will use the abbreviated notation $AB = A \odot B$. Specifically, the r-fold symmetric tensor product of $x \in \mathbb{R}^n$ will be written as

$$x^r = x \odot \cdots \odot x.$$

Symmetry is inherent here; so this is equal to the r-fold tensor product. Note that, for $x \in \mathbb{R}^n$, its r-fold symmetric tensor product is

$$x^r(v_1, \ldots, v_r) = (x \cdot v_1) \cdots (x \cdot v_r)$$

for $v_1, \ldots v_r \in \mathbb{R}^n$. We also define $x^0 = 1$ whenever $x \neq 0$.

Applying this to the discrete moment tensor, in particular, gives us

$$\mathrm{L}^r(P)(v_1, \ldots, v_r) = \frac{1}{r!} \sum_{x \in P \cap \mathbb{Z}^n} (x \cdot v_1) \cdots (x \cdot v_r)$$

for $v_1, \ldots, v_r \in \mathbb{R}^n$. For the discrete moment tensor $\mathrm{L}^r : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$, the action of $\mathrm{GL}_n(\mathbb{Z})$ is observed to be

$$\mathrm{L}^r(\phi P)(v_1, \ldots, v_r) = \frac{1}{r!} \sum_{x \in \phi P \cap \mathbb{Z}^n} (x \cdot v_1) \cdots (x \cdot v_r) = \frac{1}{r!} \sum_{x \in P \cap \mathbb{Z}^n} (x \cdot \phi^t v_1) \ldots (x \cdot \phi^t v_r)$$

$$= \mathrm{L}^r(P)(\phi^t v_1, \ldots, \phi^t v_r)$$

for $P \in \mathcal{P}(\mathbb{Z}^n)$ and $\phi \in \mathrm{SL}_n(\mathbb{Z})$ where ϕ^t is the transpose of ϕ. In general, a function $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ is said to be $\mathrm{SL}_n(\mathbb{Z})$ *equivariant* if

$$\mathrm{Z}(\phi P)(v_1, \ldots, v_r) = \mathrm{Z}(P)(\phi^t v_1, \ldots, \phi^t v_r)$$

for $v_1, \ldots, v_r \in \mathbb{R}^n$, $\phi \in \mathrm{SL}_n(\mathbb{Z})$, and $P \in \mathcal{P}(\mathbb{Z}^n)$. We will write this as $\mathrm{Z}(\phi P) = \mathrm{Z}(P) \circ \phi^t$. We use the term $\mathrm{SL}_n(\mathbb{Z})$ equivariance in order to stay consistent with the vector-valued case and note that for $x_1, \ldots, x_r \in \mathbb{R}^n$, we have

$$\phi(x_1 \odot \cdots \odot x_r) = \phi x_1 \odot \cdots \odot \phi x_r$$

for $\phi \in \mathrm{SL}_n(\mathbb{Z})$.

We now show that an $\mathrm{SL}_n(\mathbb{Z})$ equivariant tensor valuation defined on a lower dimensional lattice polytope is completely determined by its lower dimensional coordinates. The precise statement is given as the following lemma. For $A \in \mathbb{T}^r$ and $r_j \in \mathbb{N}$ with $r_1 + \cdots + r_m = r$, we write $A(e_1[r_1], \ldots, e_m[r_m])$ for $A(e_1, \ldots, e_1, \ldots, e_m, \ldots, e_m)$ with e_j appearing r_j times for $j = 1, \ldots, m$. We identify the subspace of lattice polytopes lying in the span of $e_1, \ldots, e_{m-1}$ with $\mathcal{P}(\mathbb{Z}^{m-1})$ and set $\mathbb{Z}^0 = \{0\}$.

Lemma 6. *If* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant, then*

$$\mathrm{Z}(P)(e_1[r_1], \ldots, e_{m-1}[r_{m-1}], e_m[r_m]) = 0$$

for every $P \in \mathcal{P}(\mathbb{Z}^{m-1})$ *whenever* $r_m > 0$ *and* $r_1 + \cdots + r_m = r$.

Proof. If $m = 1$, then we consider $\phi \in \mathrm{SL}_n(\mathbb{Z})$ that maps e_1 to $e_1 + e_2$ and e_j to e_j for $j > 1$. For $P = \{0\}$, we have

$$
\begin{aligned}
\mathrm{Z}(P)(e_1[r-1], e_2) &= \mathrm{Z}(\phi P)(e_1[r-1], e_2) \\
&= \mathrm{Z}(P)(e_1[r-1], e_1 + e_2) \\
&= \mathrm{Z}(P)(e_1[r]) + \mathrm{Z}(P)(e_1[r-1], e_2)
\end{aligned}
$$

yielding the result.

So, let $m \geq 2$. The proof is by induction on $r_1 \geq 0$. Consider the linear transformation $\phi \in \mathrm{SL}_n(\mathbb{Z})$ that maps e_m to $e_1 + e_m$ and maps e_j to e_j for all $j \neq m$. Any lattice polytope $P \in \mathcal{P}(\mathbb{Z}^{m-1})$ is invariant with respect to the map ϕ yielding

$$
\begin{aligned}
\mathrm{Z}(P)(e_1, e_2[r_2], \ldots, e_m[r_m - 1]) &= \mathrm{Z}(\phi P)(e_1, e_2[r_2], \ldots, e_m[r_m - 1]) \\
&= \mathrm{Z}(P)(e_1 + e_m, e_2[r_2], \ldots, e_m[r_m - 1]) \\
&= \mathrm{Z}(P)(e_1, e_2[r_2], \ldots, e_m[r_m - 1]) + \mathrm{Z}(P)(e_2[r_2], \ldots, e_m[r_m])
\end{aligned}
$$

for any integers $r_2, \ldots, r_m \geq 0$ with $r_2 + \cdots + r_m = r$. Hence we have proved the statement for $r_1 = 0$.

Let $r_1 > 0$ and suppose the statement holds for $r_1 - 1$. Then the equation

$$
\begin{aligned}
\mathrm{Z}(P)&(e_1[r_1 + 1], e_2[r_2], \ldots, e_m[r_m]) \\
&= \mathrm{Z}(\phi P)(e_1[r_1 + 1], e_2[r_2], \ldots, e_m[r_m]) \\
&= \mathrm{Z}(P)(e_1 + e_m[r_1 + 1], e_2[r_2], \ldots, e_m[r_m]) \\
&= \sum_{l=0}^{r_1+1} \binom{r_1 + 1}{l} \mathrm{Z}(P)(e_1[r_1 + 1 - l], e_2[r_2], \ldots, e_m[r_m + l]) \\
&= \mathrm{Z}(P)(e_1[r_1 + 1], e_2[r_2], \ldots, e_m[r_m]) + (r_1 + 1)\, \mathrm{Z}(P)(e_1[r_1], e_2[r_2], \ldots, e_m[r_m + 1])
\end{aligned}
$$

shows that $\mathrm{Z}(P)(e_1[r_1], e_2[r_2], \ldots, e_m[r_m + 1]) = 0$, which completes the proof by induction. $\square$

2.2 Translation Covariance

Next, we look at the behavior of the discrete moment tensor L^r with respect to translations. Geometrically, see Figure 2.1 for the unit square $[0, 1]^2$, it is easy to see that the discrete moment vector of a translated polytope is

$$
\mathrm{L}^1(P + y) = \mathrm{L}^1(P) + \mathrm{L}(P)y.
$$
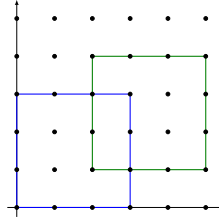
Figure 2.1: $L^1(3[0,1]^2 + (2,1)^t) = L^1([0,1]^2) + 16(2,1)^t$

For $y \in \mathbb{Z}^n$, we have

$$L^r(P+y) = \frac{1}{r!} \sum_{x \in (P+y) \cap \mathbb{Z}^n} x^r = \frac{1}{r!} \sum_{x \in P \cap \mathbb{Z}^n} (x+y)^r$$

$$= \sum_{j=0}^{r} L^{r-j}(P) \frac{y^j}{j!},$$

where on the right side we sum over symmetric tensor products. In accordance with McMullen [49], a valuation $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ is called *translation covariant* if there exist associated functions $Z^j : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^j$ for $j = 0, \ldots, r$ such that

$$Z(P+y) = \sum_{j=0}^{r} Z^{r-j}(P) \frac{y^j}{j!}$$

for all $y \in \mathbb{Z}^n$ and $P \in \mathcal{P}(\mathbb{Z}^n)$.

Certain essential properties of Z are inherited by its associated functions. These can be seen by a comparison of the coefficients in the polynomial expansion of Z evaluated at a translated lattice polytope. The following proposition gives the first of these properties. It was proven in [49] for tensor valuations on convex bodies and is included here for completeness.

Proposition 7. *If* $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is a translation covariant valuation with associated functions* $Z^0, \ldots, Z^r$, *then, for* $j = 0, \ldots, r$, *the associated function* Z^{r-j} *is a translation covariant valuation with the same associated functions as* Z, *that is,*

$$Z^{r-j}(P+y) = Z^{r-j}(P) + \cdots + Z^0(P) \frac{y^{r-j}}{(r-j)!}$$

for all $y \in \mathbb{Z}^n$ *and* $P \in \mathcal{P}(\mathbb{Z}^n)$.

Proof. We compare coefficients in the polynomial expansion in the translation vector $y \in \mathbb{Z}^n$.

Since Z is a valuation, we have

$$\sum_{j=0}^{r} Z^{r-j}(P \cup Q)\frac{y^j}{j!} = Z((P \cup Q) + y) = Z((P + y) \cup (Q + y))$$

$$= Z(P + y) + Z(Q + y) - Z((P + y) \cap (Q + y))$$

$$= Z(P + y) + Z(Q + y) - Z((P \cap Q) + y)$$

$$= \sum_{j=0}^{r} Z^{r-j}(P)\frac{y^j}{j!} + \sum_{j=0}^{r} Z^{r-j}(Q)\frac{y^j}{j!} - \sum_{j=0}^{r} Z^{r-j}(P \cap Q)\frac{y^j}{j!}.$$

Hence the associated functions of Z are valuations.

For $y, z \in \mathbb{Z}^n$, observe that

$$Z(P + y + z) = \sum_{j=0}^{r} Z^{r-j}(P + y)\frac{z^j}{j!} = \sum_{k=0}^{r} Z^{r-k}(P)\frac{(y + z)^k}{k!}$$

$$= \sum_{k=0}^{r} Z^{r-k}(P) \sum_{j=0}^{k} \frac{y^{k-j}z^j}{j!(k-j)!} = \sum_{j=0}^{r}\sum_{k=j}^{r} Z^{r-k}(P)\frac{y^{k-j}z^j}{j!(k-j)!}.$$

Therefore

$$Z^{r-j}(P + y) = \sum_{k=j}^{r} Z^{r-k}(P)\frac{y^{k-j}}{(k-j)!} = Z^{r-j}(P) + \cdots + Z^0(P)\frac{y^{r-j}}{(r-j)!},$$

that is, we obtain the same associated functions as before.   □

We require further results on the associated functions.

Proposition 8. *Let* $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *be a translation covariant valuation. If* Z *is* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant, then its associated functions are also* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant. If* Z *is* i-*homogeneous, then its associated function* Z^j *vanishes for* $j < r - i$ *and otherwise is* $(i+j-r)$-*homogeneous.*

Proof. If Z is $\mathrm{SL}_n(\mathbb{Z})$ equivariant, then, for any $\phi \in \mathrm{SL}_n(\mathbb{Z})$, we can deduce that

$$\sum_{j=0}^{r} Z^{r-j}(\phi P)\frac{y^j}{j!} = Z(\phi P + y) = Z(\phi(P + \phi^{-1}y)) = Z(P + \phi^{-1}y) \circ \phi^t$$

$$= \sum_{j=0}^{r} \left(Z^{r-j}(P) \circ \phi^t \right)\left(\frac{(\phi^{-1}y)^j}{j!} \circ \phi^t \right) = \sum_{j=0}^{r} \left(Z^{r-j}(P) \circ \phi^t \right)\frac{y^j}{j!}.$$

It follows that the associated functions are also $\mathrm{SL}_n(\mathbb{Z})$ equivariant.

Now suppose Z is i-homogeneous and let $P \in \mathcal{P}(\mathbb{Z}^n)$. For $k \in \mathbb{N}$ and $y \in \mathbb{Z}^n$, we have

$$\mathrm{Z}^r\big(k(P+y)\big) = \sum_{j=0}^{r} \mathrm{Z}^{r-j}(kP)\frac{(ky)^j}{j!}.$$

Furthermore, if we first consider the homogeneity of the valuation, we obtain

$$\mathrm{Z}^r\big(k(P+y)\big) = k^i\,\mathrm{Z}^r(P+y) = \sum_{j=0}^{r} k^i\,\mathrm{Z}^{r-j}(P)\frac{y^j}{j!}.$$

As these equations hold for any $y \in \mathbb{Z}^n$, a comparison of the two shows that for $k \in \mathbb{N}$

$$k^j\,\mathrm{Z}^{r-j}(kP) = k^i\,\mathrm{Z}^{r-j}(P).$$

Hence, if the valuation Z is i-homogeneous, then Z^{r-j} is $(i-j)$-homogeneous for $j \leq i$ and vanishes for $j > i$. □

2.3 Further Properties

The inclusion-exclusion principle is a fundamental property of valuations on lattice polytopes that was first established by Betke but left unpublished. The first published proof was given by McMullen in [50] where the following more general extension property was also established. For $m \geq 1$, we write $P_J = \bigcap_{j \in J} P_j$ for $\varnothing \neq J \subset \{1,\ldots,m\}$ and given lattice polytopes $P_1,\ldots,P_m$. Let $|J|$ denote the number of elements in J and let $\mathbb{G}$ be an abelian group.

Theorem 9 (McMullen [50]). *If* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{G}$ *is a valuation, then there exists an extension of* Z*, also denoted by* Z*, to finite unions of lattice polytopes such that*

$$\mathrm{Z}(P_1 \cup \cdots \cup P_m) = \sum_{\varnothing \neq J \subset \{1,\ldots,m\}} (-1)^{|J|-1}\,\mathrm{Z}(P_J)$$

whenever $P_J \in \mathcal{P}(\mathbb{Z}^n)$ *for all* $\varnothing \neq J \subset \{1,\ldots,m\}$.

In particular, Theorem 9 can be used to define valuations on the relative interior of lattice polytopes. We set $\mathrm{Z}(P^\circ) = \mathrm{Z}(P) - \mathrm{Z}(\partial P)$. Expressing ∂P as the union of its faces, we obtain

$$\mathrm{Z}(P^\circ) = (-1)^{\dim(P)} \sum_F (-1)^{\dim(F)}\,\mathrm{Z}(F) \tag{2.1}$$

for $P \in \mathcal{P}(\mathbb{Z}^n)$ where we sum over all non-empty faces of P.

Betke & Kneser [14] proved their classification theorem by using suitable dissections and complementations of lattice polytopes by lattice simplices. Let $T_k \in \mathcal{P}(\mathbb{Z}^n)$ be the standard k-dimensional simplex, that is, the convex hull of the origin and the vectors $e_1,\ldots,e_k$. We call a k-dimensional simplex S *unimodular* if there are $\phi \in \mathrm{SL}_n(\mathbb{Z})$ and $x \in \mathbb{Z}^n$ such that $S = \phi(T_k + x)$. We require the following results.

Proposition 10 (Betke & Kneser [14]). *For every $P \in \mathcal{P}(\mathbb{Z}^n)$, there exist unimodular simplices $S_1, \ldots, S_m$ and integers $k_1, \ldots, k_m$ such that*

$$\mathrm{Z}(P) = \sum_{i=1}^{m} k_i \, \mathrm{Z}(S_i)$$

for all valuations Z on $\mathcal{P}(\mathbb{Z}^n)$ with values in an abelian group.

The following statement is a direct consequence of this proposition.

Corollary 11. *If $\mathrm{Z}, \mathrm{Z}' : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ are $\mathrm{SL}_n(\mathbb{Z})$ equivariant and translation invariant valuations such that*

$$\mathrm{Z}(T_i) = \mathrm{Z}'(T_i) \ \text{ for } \ i = 0, \ldots, n,$$

then $\mathrm{Z} = \mathrm{Z}'$ on $\mathcal{P}(\mathbb{Z}^n)$.

A function Z is *Minkowski additive* if $\mathrm{Z}(P + Q) = \mathrm{Z}(P) + \mathrm{Z}(Q)$ for any $P, Q \in \mathcal{P}(\mathbb{Z}^n)$. The following proof is done as in [55, Remark 6.3.3] but also given here for completeness.

Proposition 12. *Every 1-homogeneous, translation invariant valuation $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ is Minkowski additive.*

Proof. Given $P, Q \in \mathcal{P}(\mathbb{Z}^n)$, the 1-homogeneous valuation Z defined on $kP + lQ$, for any $k, l \in \mathbb{N}$, is

$$\mathrm{Z}(kP + lQ) = k \, \mathrm{Z}_{1,0}(P, Q) + l \, \mathrm{Z}_{0,1}(P, Q)$$

by Theorem 13. Yet, the same theorem yields

$$k \, \mathrm{Z}_{1,0}(P, Q) = \mathrm{Z}(kP + 0Q) = k \, \mathrm{Z}_1(P)$$

and

$$l \, \mathrm{Z}_{0,1}(P, Q) = \mathrm{Z}(0P + lQ) = l \, \mathrm{Z}_1(P).$$

Hence

$$\mathrm{Z}(kP + lQ) = k \, \mathrm{Z}(P) + l \, \mathrm{Z}(P).$$

where $\mathrm{Z} = \mathrm{Z}_1$, its 1-homogeneous part. $\qquad\square$

Chapter 3

Translation Covariant Valuations

We now apply results on translative polynomial valuations to show that the evaluation of the discrete moment tensor on dilated lattice polytopes yields a homogeneous decomposition in which the coefficients themselves are new tensor valuations. In analogy to Ehrhart's celebrated result, we call this expansion the *Ehrhart tensor polynomial* of P. In Section 3.1, we collect results that demonstrate that any translation covariant valuation yields a homogeneous decomposition. In Section 3.2, we extend the reciprocity theorem of Ehrhart & Macdonald.

3.1 Ehrhart Tensor Polynomials

For the dilated unit square, the polynomiality of the discrete moment tensor can be observed geometrically (see Figure 3.1). Observe that

$$\mathrm{L}(k[0,1]^2) = (k+1)^2$$

for the discrete volume and

$$\mathrm{L}^1(k[0,1]^2) = \frac{k(k+1)^2}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

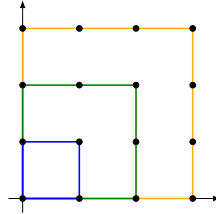for the discrete moment vector of the dilated unit square.

Figure 3.1: $\mathrm{L}^r(k[0,1]^2)$

We now consider valuations that take values in a rational vector space which we denote by $\mathbb{V}$. A valuation $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{V}$ is *translative polynomial* of degree at most d if, for every $P \in \mathcal{P}(\mathbb{Z}^n)$, the function defined on $\mathbb{Z}^n$ by $x \mapsto Z(P + x)$ is a polynomial of degree at most d. McMullen [47] considered translative polynomial valuations of degree at most one and Khovanskiĭ & Pukhlikov [52] proved Theorem 14 in the general case. Another proof, following the approach of [47], is due to Alesker [5]. These papers assume that the valuation on $\mathcal{P}(\mathbb{Z}^n)$ satisfies the inclusion-exclusion principle, which holds by Theorem 9.

Theorem 13 (Khovanskiĭ & Pukhlikov [52]). *Let $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{V}$ be a valuation which is translative polynomial of degree at most d and let $P_1, \ldots, P_m \in \mathcal{P}(\mathbb{Z}^n)$ be given. For any $k_1, \ldots, k_m \in \mathbb{N}$, the function $Z(k_1 P_1 + \cdots + k_m P_m)$ is a polynomial in $k_1, \ldots, k_m$ of total degree at most $d + n$. Moreover, the coefficient of $k_1^{r_1} \cdots k_m^{r_m}$ is an r_i-homogeneous valuation in P_i which is translative polynomial of degree at most d.*

Here we only require a special case of the result by Khovanskiĭ & Pukhlikov.

Theorem 14. *If $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{V}$ is a valuation that is translative polynomial of degree at most d, then there exist $Z_i : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{V}$ for $i = 0, \ldots, n + d$ such that*

$$Z(kP) = \sum_{i=0}^{n+d} Z_i(P) k^i$$

for every $k \in \mathbb{N}$ and $P \in \mathcal{P}(\mathbb{Z}^n)$. For each i, the function Z_i is a translative polynomial and i-homogeneous valuation.

Since Z_i is i-homogeneous, the function $x \mapsto Z_i(P + x)$ is an i-homogeneous polynomial. As a consequence, the function Z_i is translative polynomial of degree i. Note that this result contains the translation invariant case by setting $d = 0$.

Let $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ be a translation covariant tensor valuation. For given $v_1, \ldots, v_r \in \mathbb{R}^n$, we associate the real-valued valuation $P \mapsto Z(P)(v_1, \ldots, v_r)$ with the tensor valuation Z. Since

$$Z(P + y)(v_1, \ldots, v_r) = \sum_{j=0}^{r} \left(Z^{r-j}(P) \frac{y^j}{j!} \right)(v_1, \ldots, v_r),$$

the real-valued valuation $P \mapsto Z(P)(v_1, \ldots, v_r)$ is translative polynomial of degree at most r. Therefore, we immediately obtain the following consequence of Theorem 14.

Theorem 15. *If $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ is a translation covariant valuation, then there exist $Z_i : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ for $i = 0, \ldots, n + r$ such that*

$$Z(kP) = \sum_{i=0}^{n+r} Z_i(P) k^i$$

for every $k \in \mathbb{N}$ and $P \in \mathcal{P}(\mathbb{Z}^n)$. For each i, the function Z_i is a translation covariant and i-homogeneous valuation.

Note that if the tensor valuation Z is $SL_n(\mathbb{Z})$ equivariant, then so are the homogeneous components $Z_0, \ldots, Z_{n+r}$.

The homogeneous components have a translation property that agrees with the covariance of Z. The translation covariance of $\mathrm{Z} = \mathrm{Z}^r$ together with its decomposition from Theorem 15 yields

$$\mathrm{Z}^r(k(P+y)) = \sum_{j=0}^{r} \mathrm{Z}^{r-j}(kP)\frac{(ky)^j}{j!}$$

$$= \sum_{j=0}^{r} \sum_{l=0}^{n+r-j} \mathrm{Z}_l^{r-j}(P)\frac{k^{j+l}y^j}{j!}$$

$$= \sum_{j=0}^{r} \sum_{l=j}^{n+r} \mathrm{Z}_{l-j}^{r-j}(P)\frac{k^l y^j}{j!}.$$

By the homogeneous decomposition of Theorem 15, we also have

$$\mathrm{Z}^r(k(P+y)) = \sum_{l=0}^{n+r} \mathrm{Z}_l^r(P+y)k^l.$$

A comparison of the coefficients of these polynomials in k gives

$$\mathrm{Z}_l^r(P+y) = \sum_{j=0}^{l} \mathrm{Z}_{l-j}^{r-j}(P)\frac{y^j}{j!} \tag{3.1}$$

where we set $\mathrm{Z}_l^s = 0$ for $s < 0$. Furthermore, if Z^r is $\mathrm{SL}_n(\mathbb{Z})$ equivariant, then Lemma 6 implies that

$$\mathrm{Z}_0^r(P) = \mathrm{Z}^r(0P) = \begin{cases} c & \text{if } r = 0, \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

for $n \geq 2$ with $c \in \mathbb{R}$, and hence, Z_1^r is translation invariant for $r \geq 2$.

We apply the homogeneous decomposition of Theorem 15 to the discrete moment tensor to obtain the following corollary.

Corollary 16. *There exist* $\mathrm{L}_i^r : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *for* $i = 0, \ldots, n+r$ *such that*

$$\mathrm{L}^r(kP) = \sum_{i=0}^{n+r} \mathrm{L}_i^r(P)k^i$$

for every $k \in \mathbb{N}$ *and* $P \in \mathcal{P}(\mathbb{Z}^n)$. *For each* i, *the function* L_i^r *is an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant, translation covariant, and* i-*homogeneous valuation.*

Theorem 1 is then an implication of Corollary 16 and (3.2). We remark that, within Ehrhart Theory, further bases for the space of real-valued valuations on $\mathcal{P}(\mathbb{Z}^n)$ are also important (see [15, 34] for more information).

3.2 Reciprocity

The reciprocity theorem of Ehrhart and Macdonald [19, 46] is a widely used tool in combinatorics. We provide an extension of their result.

Given a function $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$, we define the function $Z^\circ : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ as

$$Z^\circ(P) = \sum_F (-1)^{\dim(F)} Z(F) \tag{3.3}$$

where the sum extends over all non-empty faces F of the lattice polytope P. Sallee [53] showed that Z° is a valuation and that $Z^{\circ\circ} = Z$. Furthermore, if Z is translation invariant or translation covariant, then Z° has the same translation property. The latter case can be seen from the equation

$$Z^\circ(P + y) = \sum_F (-1)^{\dim(F)} Z(F + y) = \sum_F (-1)^{\dim(F)} \sum_{j=0}^r Z^{r-j}(F) \frac{y^j}{j!}$$

$$= \sum_{j=0}^r \sum_F (-1)^{\dim(F)} Z^{r-j}(F) \frac{y^j}{j!} = \sum_{j=0}^r Z^{(r-j)\circ}(P) \frac{y^j}{j!}$$

for any $y \in \mathbb{Z}^n$, where we have also shown that the associated tensor $(Z^\circ)^{r-j}$ is equal to $(Z^{r-j})^\circ$ for every applicable j.

The following reciprocity theorem was established by McMullen [47]; see [34] for a different proof. Let $\mathbb{V}$ be a rational vector space.

Theorem 17 (McMullen [47]). *If* $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{V}$ *is an i-homogeneous and translation invariant valuation, then*

$$Z^\circ(P) = (-1)^i Z(-P)$$

for $P \in \mathcal{P}(\mathbb{Z}^n)$.

This result applies to any rational vector space and, therefore, includes tensor valuations with the aforementioned properties. We now use this result to prove an analogous reciprocity theorem for translation covariant tensor valuations.

Theorem 18. *If* $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is an i-homogeneous and translation covariant valuation, then*

$$Z^\circ(P) = (-1)^i Z(-P)$$

for $P \in \mathcal{P}(\mathbb{Z}^n)$.

Proof. We prove this by induction on $r \in \mathbb{N}$ where the case $r = 0$ is covered by Theorem 17. By the translation behavior of Z and Z° and by the induction hypothesis, for $P \in \mathcal{P}(\mathbb{Z}^n)$ and

$y \in \mathbb{Z}^n$, we have

$$Z^{\circ}(P + y) - (-1)^i Z(-(P + y))$$

$$= \sum_{j=0}^{r} Z^{\circ j}(P) \frac{y^{r-j}}{(r-j)!} - (-1)^i \sum_{j=0}^{r} Z^j(-P) \frac{(-y)^{r-j}}{(r-j)!}$$

$$= Z^{\circ r}(P) - (-1)^i Z^r(-P) + \sum_{j=0}^{r-1} \left((-1)^{i+j-r} Z^j(-P) \frac{y^{r-j}}{(r-j)!} - (-1)^i Z^j(-P) \frac{(-y)^{r-j}}{(r-j)!} \right)$$

$$= Z^{\circ}(P) - (-1)^i Z(-P).$$

Recall here that by Proposition 8 the associated tensor Z^j is $(i + j - r)$-homogeneous as Z is i-homogeneous and that $(Z^{\circ})^j = (Z^j)^{\circ}$.

Let $\widetilde{Z}(P) = Z^{\circ}(P) - (-1)^i Z(-P)$. Then $\widetilde{Z}$ is an i-homogeneous and translation invariant valuation. From Theorem 17, we obtain

$$\widetilde{Z}^{\circ}(P) = (-1)^i \widetilde{Z}(-P).$$

Thus

$$\widetilde{Z}(P) = (-1)^i \widetilde{Z}^{\circ}(-P) = (-1)^i \left(Z^{\circ \circ}(-P) - (-1)^i Z^{\circ}(P) \right)$$
$$= - \left(Z^{\circ}(P) - (-1)^i Z(-P) \right) = -\widetilde{Z}(P)$$

yielding $\widetilde{Z} = 0$ which proves the theorem. $\qquad\square$

The homogeneous decomposition of tensor valuations from Theorem 15 allows to consider reciprocity without the assumption of homogeneity. Since Z° is also a translation covariant valuation if Z is, the following result is a simple consequence of Theorem 18.

Corollary 19. *If* $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is a translation covariant valuation, then*

$$Z^{\circ}(P) = \sum_{i=0}^{n+r} (-1)^i Z_i(-P)$$

for $P \in \mathcal{P}(\mathbb{Z}^n)$.

Combined with (2.1), this gives the following result.

Corollary 20. *If* $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is a translation covariant valuation, then*

$$Z(P^{\circ}) = (-1)^{\dim(P)} \sum_{i=0}^{n+r} (-1)^i Z_i(-P)$$

for $P \in \mathcal{P}(\mathbb{Z}^n)$.

So, in particular, using that $L^r(-P) = (-1)^r L^r(P)$ and that $L_{i+r}^r(P) = 0$ for $\dim(P) < i \le n$, which is shown in Lemma 21 below, we obtain Theorem 2. Note that the results in this section for translation covariant tensor valuations also hold for translative polynomial valuations on lattice polytopes taking values in a rational vector space. The proofs remain the same.

Chapter 4

Ehrhart Tensor Polynomials

We aggregate the results on the Ehrhart tensor polynomial and its coefficients under two different bases into this chapter. The article done jointly with Sören Berg and Katharina Jochemko [10] is combined into Sections 4.2 - 4.5 along with the characterization of the second highest Ehrhart coefficient in Section 4.1. Other steps taken towards classifying the Ehrhart tensors in Section 4.1 will be helpful in the characterization of tensor valuations in Chapter 6.

4.1   Ehrhart Tensors

The Ehrhart tensors L_i^r include, for $r = 0$, and expand upon the Ehrhart coefficients. They are also the discrete analogues of the Minkowski tensors. By Propositions 7 and 8, the Ehrhart tensors $L_i^r : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ are $SL_n(\mathbb{Z})$ equivariant and translation covariant valuations. In this section, we derive further properties of these tensors and give a characterization for the leading Ehrhart tensor, the second highest Ehrhart tensor, and the constant Ehrhart tensor.

A complete characterization of the Ehrhart coefficients has been inaccessible up to this point. The coefficients can even be negative and, therefore, are difficult to describe combinatorially. However, it is known that the leading coefficient equals the volume, the second highest coefficient is related to the normalized surface area, and the constant coefficient is always 1.

Recall that the discrete moment tensor is the discrete analogue of the moment tensor defined in (1.2). On lattice polytopes, the moment tensor coincides with the leading Ehrhart tensor. The following result is well-known for $r = 0$ (where $M^0 = V_n$).

Lemma 21. *For $P \in \mathcal{P}(\mathbb{Z}^n)$,*
$$L_{n+r}^r(P) = M^r(P)$$
and $L_{i+r}^r(P) = 0$ for $\dim(P) < i \leq n$. Moreover, L_{i+r}^r is not simple for $0 \leq i < n$.

Proof. By the definition of the Riemann integral, we have

$$L_{n+r}^r(P) = \lim_{k\to\infty} \frac{L^r(kP)}{k^{n+r}} = \frac{1}{r!} \lim_{k\to\infty} \frac{1}{k^n} \sum_{x\in kP\cap\mathbb{Z}^n} \frac{1}{k^r} x^r$$

$$= \frac{1}{r!} \lim_{k\to\infty} \frac{1}{k^n} \sum_{x\in P\cap\frac{1}{k}\mathbb{Z}^n} x^r = \frac{1}{r!} \int_P x^r\, dx.$$

This proves the statement for $\dim(P) = n$ and shows that $L_{n+r}(P) = 0$ for $\dim(P) < n$. The statement for $\dim(P) < i$ follows by considering the affine hull of P since L_{i+r}^r is again proportional (with a positive factor) to the moment tensor calculated in this subspace. This also implies that L_{i+r}^r is not simple for $0 \le i < n$. $\qquad\square$

We give an interpretation for the second coefficient (Proposition 22) as the weighted sum of moment tensors over the facets of the polytope. The coefficient L_{n-1}, specifically, was shown to be equal to one half of the sum over the normalized volumes of the facets of P by Ehrhart [20]. We extend this statement to Ehrhart tensor polynomials by proving the following. We introduce the notation $L_P^r(k) = \sum_{i=0}^{n+r} L_i(P)k^i$ for the homogeneous decomposition of $L^r(P)$ allowing k to be any integer. Therefore, Theorem 2 can be rewritten as

$$L_P^r(-k) = (-1)^{\dim(P)+r} L^r(kP^\circ)$$

for any $P \in \mathcal{P}(\mathbb{Z}^n)$.

Proposition 22. *Let P be a lattice polytope. Then*

$$L_{\dim(P)+r-1}^r(P) \;=\; \sum_F \frac{1}{|\det(\mathrm{aff}(F)\cap\mathbb{Z}^d)|} \int_F x^r\, d_{n-1}x\,,$$

where the sum is over all facets $F \subset P$.

Proof. Theorem 2, on the one hand, implies

$$\frac{1}{r!} \sum_{x\in\partial kP} x^r \;=\; \frac{1}{r!} \sum_{F\subsetneq P}\sum_{x\in kF^\circ} x^r \;=\; \sum_{F\subsetneq P}(-1)^{\dim(F)+r} L_F^r(-k)\,,$$

where the sum is taken over all proper faces $F \subsetneq P$. On the other hand, we have

$$\frac{1}{r!} \sum_{x\in\partial kP} x^r \;=\; L^r(kP) - L^r(kP^\circ) \;=\; L^r(kP) - (-1)^{\dim(P)+r} L_P^r(-k)$$

$$= 2\sum_{i\ge 0} L_{\dim(P)+r-1-2i}^r(kP)$$

where we set $L_i^r = 0$ for all $i < 0$.

Using both equations, we obtain

$$
\begin{aligned}
\mathrm{L}^r_{\dim(P)+r-1}(P) &= \lim_{k\to\infty} \frac{1}{k^{\dim(P)+r-1}} \sum_{i\geq 0} \mathrm{L}^r_{\dim(P)+r-1-2i}(kP) \\
&= \frac{1}{2} \sum_{F\subsetneq P} (-1)^{\dim(F)+r} \lim_{k\to\infty} \frac{1}{k^{\dim(P)+r-1}} \, \mathrm{L}^r_F(-k) \\
&= \frac{1}{2r!} \sum_{F \text{ facet}} \frac{1}{|\det(\mathrm{aff}(F)\cap\mathbb{Z}^d)|} \int_F x^r \, \mathrm{d}_{n-1}x\,,
\end{aligned}
$$

where the last equality follows from Lemma 21. $\square$

For the Ehrhart tensor polynomial, for $r \geq 1$, it is clear that the constant coefficient vanishes identically by (3.2); that is, $\mathrm{L}^r_0(P) = \mathrm{L}^r(0P) = 0$ for any $P \in \mathcal{P}(\mathbb{Z}^n)$.

Although our main interest here is the classification of $\mathrm{SL}_n(\mathbb{Z})$ equivariant and translation covariant tensor valuations, we also obtain a characterization of translation covariant and $(n+r)$-homogeneous tensor valuations. In fact, by Lemma 21, they are equal to the moment tensor up to a scalar. In this simple result, which is analogous to Alesker's result on tensor valuations on convex bodies in [4], no $\mathrm{SL}_n(\mathbb{Z})$ equivariance is assumed.

Theorem 23. *If* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is a translation covariant,* $(n+r)$-*homogeneous valuation, then there is* $c \in \mathbb{R}$ *such that*

$$
\mathrm{Z}(P) = c\,\mathrm{L}^r_{n+r}(P)
$$

for every $P \in \mathcal{P}(\mathbb{Z}^n)$.

Proof. For $r = 0$, the statement is the same as Theorem 48. Suppose the assumption is true for all translation covariant and $(n+i)$-homogeneous valuations that take values in tensors of rank $i < r$. Let Z^{r-j} for $j = 1, \ldots, r$ be the associated functions of Z. By Proposition 8, the associated function Z^{r-1} is homogeneous of degree $n + r - 1$. Hence, by the induction assumption, there is $c \in \mathbb{R}$ such that $\mathrm{Z}^{r-1} = c\,\mathrm{L}^{r-1}_{n+r-1}$. Note that it follows from Proposition 7 that

$$
\mathrm{Z}^{r-j} = c\,\mathrm{L}^{r-j}_{n+r-j} \tag{4.1}
$$

for $j = 1, \ldots, r-1$. Consider the translation covariant and $(n+r)$-homogeneous valuation $\widetilde{\mathrm{Z}} = \mathrm{Z} - c\,\mathrm{L}^r_{n+r}$. For $y \in \mathbb{Z}^n$, by the translation covariance of $\widetilde{\mathrm{Z}}$ and L^r_{n+r} and (4.1), we obtain

$$
\begin{aligned}
\widetilde{\mathrm{Z}}(P + y) &= \mathrm{Z}(P + y) - c\,\mathrm{L}^r_{n+r}(P + y) \\
&= \sum_{j=0}^r \mathrm{Z}^{r-j}(P)\frac{y^j}{j!} - c\sum_{j=0}^r \mathrm{L}^{r-j}_{n+r-j}(P)\frac{y^j}{j!} \\
&= \mathrm{Z}(P) - c\,\mathrm{L}^r_{n+r}(P) \\
&= \widetilde{\mathrm{Z}}(P).
\end{aligned}
$$

Therefore, the valuation $\widetilde{\mathrm{Z}}$ is translation invariant. Theorem 14 implies that $\widetilde{\mathrm{Z}} = 0$ as nontrivial translation invariant valuations cannot be homogeneous of degree greater than n. $\square$

The characterization of the first Ehrhart tensor is the key element in the classification of tensor valuations. We show, in Lemma 24, that it can only be simple in the planar case. Faulhaber's formula oftentimes appears in the calculation of the discrete moment tensor of a lattice polytope as it does in Lemma 24. The formula was given by Bernoulli in *Ars Conjectandi* which was translated in [12] although he fully attributed it to Faulhaber due to his formulas for sums of integral powers up to the 17th power [21]. With the convention that $B_1 = -\frac{1}{2}$ and that $B_{2i+1} = 0$ for $i > 0$, the formula is stated as

$$\sum_{i=1}^{k} i^r = \frac{1}{r+1} \sum_{l=0}^{r} (-1)^l \binom{r+1}{l} B_l k^{r+1-l} \tag{4.2}$$

where B_l are the Bernoulli numbers. We will use the following convolution identity for Bernoulli polynomials (see, e.g., [1]) which, interestingly enough, is usually attributed to Leonhard Euler. For $n \geq 1$, the identity is

$$\sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i} = -nB_{n-1} - (n-1)B_n. \tag{4.3}$$

Lemma 24. *For $n \geq 2$, the valuation L_1^r is non-trivial. For $n = 2$ and $r \geq 3$ odd, it is simple.*

Proof. It suffices to prove that $\mathrm{L}_1^r(T_2)(e_1[r]) \neq 0$ and, by Lemma 6 for $n = 2$ and $r \geq 3$ odd, that $\mathrm{L}_1^r(T_1)(e_1[r]) = 0$.

For any $k \in \mathbb{N}$, we have

$$\mathrm{L}^r(kT_1)(e_1[r]) = \sum_{x \in kT_1 \cap \mathbb{Z}^n} (x \cdot e_1) \cdots (x \cdot e_1) = \sum_{i=1}^{k} i^r \tag{4.4}$$

where the sum of the first k powers of r can be expressed through Faulhaber's formula (4.2). By its homogeneous decomposition, Corollary 16, the first Ehrhart tensor is the coefficient of k, where $l = r$ in (4.2), implying that $\mathrm{L}_1^r(T_1)(e_1[r]) = (-1)^r B_r$. As $B_r = 0$ for $r = 2m+1$ where $m \neq 0 \in \mathbb{N}$, we obtain the second statement of the lemma.

Similarly to (4.4), for any $k \in \mathbb{N}$, we have

$$\mathrm{L}^r(kT_2)(e_1[r]) = \sum_{x \in kT_2 \cap \mathbb{Z}^n} (x \cdot e_1) \cdots (x \cdot e_1) = \sum_{j=1}^{k} \sum_{i=1}^{j} i^r.$$

Applying Faulhaber's formula (4.2) twice, the discrete moment tensor of kT_2 is

$$\begin{aligned}
\mathrm{L}^r(kT_2)(e_1[r]) &= \sum_{j=1}^{k} \sum_{i=1}^{j} i^r \\
&= \frac{1}{r+1} \sum_{l=0}^{r} (-1)^l \binom{r+1}{l} B_l \sum_{j=1}^{k} j^{r+1-l} \\
&= \frac{1}{r+1} \sum_{l=0}^{r} \frac{(-1)^l B_l}{r+2-l} \binom{r+1}{l} \sum_{m=0}^{r+1-l} (-1)^m \binom{r+2-l}{m} B_m k^{r+2-l-m}.
\end{aligned} \tag{4.5}$$

The value of $L_1^r(T_2)(e_1[r])$ is equal to the coefficient of k in (4.5); precisely the value when we set $m = r + 1 - l$. Hence

$$L_1^r(T_2)(e_1[r]) = \frac{(-1)^{r+1}}{r+1} \sum_{l=0}^{r} \binom{r+1}{l} B_l B_{r+1-l}. \tag{4.6}$$

Euler's identity (4.3) together with equation (4.6) and $B_0 = 1$ then gives

$$L_1^r(T_2)(e_1[r]) = (-1)^r (B_r + B_{r+1}) \neq 0$$

as $-(B_1 + B_2) = \frac{1}{3}$ and, for any $m \neq 0 \in \mathbb{N}$, $B_{2m} \neq 0$ and $B_{2m+1} = 0$. $\qquad\square$

4.2 Ehrhart h^r-tensor Polynomials

Let P be an n-dimensional lattice polytope. Since, by Theorem 1, $L^r(kP)$ is a polynomial of degree at most $n + r$, it can be written as a linear combination of the polynomials $\binom{k+n+r}{n+r}, \binom{k+n+r-1}{n+r}, \ldots, \binom{k}{n+r}$, that is,

$$L^r(kP) = h_0^r(P)\binom{k+n+r}{n+r} + h_1^r(P)\binom{k+n+r-1}{n+r} + \cdots + h_{n+r}^r(P)\binom{k}{n+r} \tag{4.7}$$

for some $h_0^r(P), \ldots, h_{n+r}^r(P) \in \mathbb{T}^r$. Equivalently, in terms of generating functions,

$$\sum_{k\geq 0} L^r(kP)t^k = \frac{h_0^r(P) + h_1^r(P)t + \cdots + h_{n+r}^r(P)t^{n+r}}{(1-t)^{n+r+1}}. \tag{4.8}$$

We call $h^r(P) = (h_0^r(P), h_1^r(P), \ldots, h_{n+r}^r(P))$ the h^r-vector of P, its entries the h^r-tensors of P, and

$$h_P^r(t) = \sum_{i=0}^{n+r} h_i^r t^i$$

the h^r-tensor polynomial of P. Observe that for $r = 0$ we obtain the usual h^*-polynomial and h^*-vector of an Ehrhart polynomial. By evaluating equation (4.7) at $k = 0, 1$, we obtain $h_0^r(P) = 0$ for $r \geq 1$ and $h_1^r(P) = L^r(P)$ for $r \geq 0$. Inspecting the leading coefficient, we obtain

$$h_1^r(P) + h_2^r(P) + \ldots + h_{n+r}^r(P) = \frac{(n+r)!}{r!} \int_P x^r \mathrm{d}x.$$

Applying Theorem 2 and evaluating at $k = 1$, we obtain

$$h_{n+r}^r(P) = L^r(P^o).$$

4.2.1   Half-open Polytopes

We will not only consider relatively open polytopes, but also *half-open polytopes*. Let P be a polytope with facets $F_1, \ldots, F_m$ and let q be a generic point in its affine span $\mathrm{aff}(P)$. Then a facet F_i is *visible* from q if $(p, q] \cap P = \varnothing$ for all $p \in F$. If $I_q(P) = \{i \in [m] : F_i \text{ is visible from } q\}$ then the point set

$$H_q(P) \;=\; P \setminus \bigcup_{i \in I_q(P)} F_i$$

defines a half-open polytope. In particular, $H_q(P) = P$ for all $q \in P$. The following result by Köppe and Verdoolaege [37] shows that every polytope can be decomposed into half-open polytopes, and is implicitely also contained in works by Stanley and Ehrhart (see [57]).

Theorem 25 ([37]). *Let P be a polytope and let $P_1, \ldots, P_m$ be the maximal cells of a triangulation of P. Let $q \in \mathrm{aff}(P)$ be a generic point. Then*

$$H_q(P) \;=\; H_q(P_1) \sqcup H_q(P_2) \sqcup \cdots \sqcup H_q(P_m)$$

is a partition.

The discrete moment tensor naturally can be defined for half-open polytopes by setting

$$\mathrm{L}^r(H_q(P)) \;:=\; \mathrm{L}^r(P) - \sum_{J \subseteq I_q(P)} (-1)^{\dim P - \dim F_J} \, \mathrm{L}^r(F_J)$$

where $F_J := \bigcap_{i \in J} F_i$. Then, from Theorem 25 and the inclusion-exclusion principle, we obtain that

$$\mathrm{L}^r(P) \;=\; \mathrm{L}^r(H_q(P_1)) \;+\; \mathrm{L}^r(H_q(P_2)) \;+\; \cdots \;+\; \mathrm{L}^r(H_q(P_m)) \tag{4.9}$$

(Compare also [34, Corollary 3.2]).

4.2.2   Half-open Simplices

Let S be an n-dimensional lattice simplex with vertices $v_1, \ldots, v_{n+1}$. Let $F_1, \ldots, F_{n+1}$ denote the facets of S such that $v_i \notin F_i$. Let $S^* = H_q(S)$ be an n-dimensional half-open simplex and let $I = I_q(S)$. We define the half-open polyhedral cone

$$C_{S^*} \;=\; \left\{ \sum_{i=1}^{n+1} \lambda_i \bar{v}_i : \lambda_i \geq 0 \text{ for } i \in [n+1], \lambda_i \neq 0 \text{ if } i \in I \right\} \subseteq \mathbb{R}^{n+1}$$

where $\bar{v}_i := (v_i, 1) \in \mathbb{R}^{n+1}$ for all $1 \leq i \leq n+1$. Then, by identifying hyperplanes of the form $\{x \in \mathbb{R}^{n+1} : x_{n+1} = k\}$ with $\mathbb{R}^n$ via $p \colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ which maps $x \mapsto (x_1, \ldots, x_n)$, we have $C_{S^*} \cap \{x_{n+1} = k\} = k S^*$. We consider the half-open parallelepiped

$$\Pi_{S^*} \;=\; \left\{ \sum_{i=1}^{n+1} \lambda_i \bar{v}_i : 0 < \lambda_i \leq 1 \text{ if } i \in I, 0 \leq \lambda_i < 1 \text{ if } i \notin I \right\} .$$

Hence
$$C_{S^*} = \bigsqcup_{u \in \mathbb{Z}^{n+1}} \Pi_{S^*} + u_1 \bar{v}_1 + \cdots + u_{n+1} \bar{v}_{n+1}.$$

Let $S_i = \Pi_{S^*} \cap \{x_{n+1} = i\}$. Then S_i is a *partially open hypersimplex*; a hypersimplex with certain facets removed.

Our next result shows that $\mathrm{L}^r(kS^*)$ is given by a polynomial in k by determining its generating series. We follow the line of argumentation in [34, Proposition 3.3]. Observe that, together with equation (4.9), this reproves Theorem 1.

Theorem 26. *With the notation given above, the equation*

$$\sum_{k \geq 0} \mathrm{L}^r(kS^*) t^k = \sum_{\substack{\alpha_0,\ldots,\alpha_{n+1} \geq 0 \\ \sum k_j = r}} v_1^{\alpha_1} \cdots v_{n+1}^{\alpha_{n+1}} \frac{(1-t)^{\alpha_0} A_{\alpha_1}(t) \cdots A_{\alpha_{n+1}}(t)}{(1-t)^{n+r+1} \alpha_1! \cdots \alpha_{n+1}!} \sum_{i=0}^n \mathrm{L}^{\alpha_0}(S_i) t^i,$$

holds true where $A_j(t)$ is the j-th Eulerian polynomial.

Proof. The generating function of the discrete moment tensor allows us to consider the discrete moment tensor of kS^* by cutting the cone C_{S^*} with the hyperplane $\{x_{n+1} = k\}$. The geometric interpretation of the half-open parallelepipeds tiling the cone, the translation covariance of the discrete moment tensor, and the binomial theorem together yield the equation

$$\sum_{k \geq 0} \mathrm{L}^r(kS^*) t^k = \sum_{i=0}^n t^i \sum_{u_1,\ldots,u_{n+1} \geq 0} \mathrm{L}^r(S_i + u_1\bar{v}_1 + \cdots + u_{n+1}\bar{v}_{n+1}) t^{u_1 + \cdots + u_{n+1}}$$

$$= \sum_{i=0}^n t^i \sum_{u_1,\ldots,u_{n+1} \geq 0} \sum_{j=0}^r \frac{1}{j!} \mathrm{L}^{r-j}(S_i)(u_1\bar{v}_1 + \cdots + u_{n+1}\bar{v}_{n+1})^j t^{u_1 + \cdots + u_{n+1}}$$

$$= \sum_{i=0}^n t^i \sum_{u_1,\ldots,u_{n+1} \geq 0} \sum_{\substack{\alpha_0,\ldots,\alpha_{n+1} \geq 0 \\ \sum \alpha_j = r}} \frac{1}{\alpha_1! \cdots \alpha_{n+1}!} \mathrm{L}^{\alpha_0}(S_i)(u_1\bar{v}_1)^{\alpha_1} \cdots (u_{n+1}\bar{v}_{n+1})^{k_{n+1}} t^{u_1 + \cdots + u_{n+1}}$$

$$= \sum_{i=0}^n t^i \sum_{\substack{\alpha_0,\ldots,\alpha_{n+1} \geq 0 \\ \sum k_j = r}} \frac{1}{\alpha_1! \cdots \alpha_{n+1}!} \mathrm{L}^{\alpha_0}(S_i)\bar{v}_1^{\alpha_1} \cdots \bar{v}_{n+1}^{\alpha_{n+1}} \sum_{u_1,\ldots,u_{n+1} \geq 0} u_1^{\alpha_1} \cdots u_{n+1}^{\alpha_{n+1}} t^{u_1 + \cdots + u_{n+1}}$$
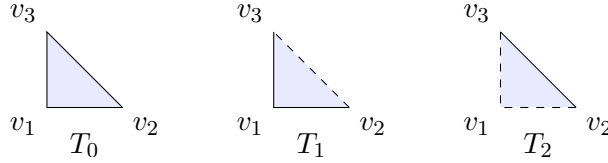
from which the result follows since

$$\sum_{k \geq 0} k^j t^k = \frac{A_j(t)}{(1-t)^{j+1}},$$

a known identity of generating functions (see, e.g., [9]). □

We remark that the results and proofs of this section immediately carry over to general translative polynomial valuations. In particular, Theorem 26 can be generalized to give a new proof of [52, Corollary 5].

Figure 4.1: Types of half-open unimodular simplices in $\mathbb{R}^2$.

4.3 Pick-type Formulas

Pick's Theorem [51] gives an interpretation for the coefficients of the Ehrhart polynomial of a lattice polygon which establishes a relationship between the area of the polygon, the number of lattice points in the polygon and on its boundary. An analogue in higher dimensions cannot exist (see, e.g., [23]) as it is crucial that every polygon in dimension two has a unimodular triangulation; that is, a triangulation into simplices of minimal possible area $1/n!$. We offer interpretations for the coefficients of the Ehrhart tensor polynomial in the vector and the matrix cases by taking the route over the h^r-tensor polynomial.

Given a polygon $P \in \mathcal{P}(\mathbb{Z}^2)$, we will consider unimodular triangulations of P where such a triangulation will always be denoted by $\mathcal{T}$. The triangulation will be described by the edge graph $G = (V, E)$ of $\mathcal{T}$ where V are the lattice points contained in P and E the edges of $\mathcal{T}$. Furthermore, the notation x will be reserved for elements of V and y, z for endpoints of the edge $\{y, z\} \in E$ for the remainder of this chapter. We define $V^o = P^o \cap \mathbb{Z}^2$, $\partial V = \partial P \cap \mathbb{Z}^2$, $E^o = \{\{y, z\} \in E : (y, z) \not\subset \partial P\}$, and $\partial E = \{\{y, z\} \in E : (y, z) \subset \partial P\}$.

Up to unimodular transformations, there are three types of half-open unimodular simplices in $\mathbb{R}^2$ that we will consider; these are T_0, T_1, and T_2 as given in Figure 4.1.

4.3.1 A Pick-type Vector Formula

To determine the h^1-tensors from Theorem 26, note that the Eulerian polynomial has closed form

$$A_j(t) \;=\; \sum_{k=0}^{j} \sum_{i=0}^{k} (-1)^i \binom{j+1}{i}(k-i)^j t^k \tag{4.10}$$

(see, e.g., [9]). We then observe that $A_0(t) = 1$, $A_1(t) = t$, and $A_2(t) = t^2 + t$.

A comparison of coefficients of the numerator of (4.8) and that in Theorem 26 yields the formula

$$h^1_{S^*}(t) \;=\; \sum_{i=0}^{2} \mathrm{L}^1(S_i)t^i(1-t) + \mathrm{L}(S_i)t^{i+1}(v_1 + v_2 + v_3)$$

implying that

$$h^1_i(S^*) \;=\; \mathrm{L}^1(S_i) - \mathrm{L}^1(S_{i-1}) + \mathrm{L}(S_{i-1})(v_1 + v_2 + v_3) \tag{4.11}$$

for a half-open simplex S^* where S_i are defined as in Section 4.2.2.

By Theorem 25, any lattice polygon can be partitioned into unimodular transformations of half-open simplices. Therefore, to calculate h^r-tensors, we will need to understand the half-open parallelepipeds Π_{T_0}, Π_{T_1}, and Π_{T_2}. For ease, we provide skeletal descriptions of these here. By setting S^* to T_0, T_1, and T_2 with the vertices given in Figure 4.1, we obtain:

$$
\begin{aligned}
T_0 &: S_0 \cap \mathbb{Z}^2 = \{0\}; \\
T_1 &: S_1 \cap \mathbb{Z}^2 = \{v_1\}; \\
T_2 &: S_2 \cap \mathbb{Z}^2 = \{v_2 + v_3\}
\end{aligned}
\tag{4.12}
$$

where $S_i \cap \mathbb{Z}^2 = \varnothing$ for any combination of S_i, T_j not given.

Proposition 27. *For any polygon $P \in \mathcal{P}(\mathbb{Z}^2)$, we have*

$$
h_P^1(t) \;=\; t \sum_V x + t^2 \left(\sum_{E^o} (y + z) - 2 \sum_{V^o} x \right) + t^3 \sum_{V^o} x.
$$

Proof. We determine the h^1-tensor polynomial of all half-open unimodular simplices, up to a unimodular transformation, with vertices v_1, v_2, v_3. Using formula (4.11) together with the values given in (4.12), we obtain the following h^1-tensor polynomials for each T_i:

$$
\begin{aligned}
h_{T_0}^1(t) &= t(v_1 + v_2 + v_3) \\
h_{T_1}^1(t) &= tv_1 + t^2(v_2 + v_3) \\
h_{T_2}^1(t) &= t^2((v_1 + v_2) + (v_1 + v_3) - 2v_1) + t^3 v_1
\end{aligned}
$$

For any interior lattice point $x \in V^o$, there must exist a simplex in $\mathcal{T}$ that is a unimodular transformation ϕ of T_0 or T_1, call it $\bar{T}$, such that $x \in \bar{T}$ and $\phi v_1 = x$. Thus, there must exist a simplex in $\mathcal{T}$ that is a unimodular transformation γ of T_2 such that $\gamma v_1 = x$. The claim follows easily now from Theorem 25. □

From Proposition 27, we can deduce formulas for the Ehrhart vectors.

Proposition 28. *For any polygon $P \in \mathcal{P}(\mathbb{Z}^2)$,*

$$
\mathrm{L}^1(kP) \;=\; \frac{k}{6} \left(2 \sum_V x + 4 \sum_{V^o} x - \sum_{E^o} (y + z) \right) + \frac{k^2}{2} \sum_{\partial V} x + \frac{k^3}{6} \left(\sum_{\partial V} x + \sum_{E^o} (y + z) \right)
$$

Proof. By definition, the Ehrhart vector polynomial equals

$$
\mathrm{L}^1(kP) \;=\; h_0^1(P) \binom{k+3}{3} + h_1^1(P) \binom{k+2}{3} + h_2^1(P) \binom{k+1}{3} + h_3^1(P) \binom{k}{3}.
$$

A substitution of values from Proposition 27 yields

$$
\mathrm{L}^1(kP) \;=\; \frac{k^3 + 3k^2 + 2k}{6} \sum_V x + \frac{k^3 - k}{6} \left(\sum_{E^o} (y + z) - 2 \sum_{V^o} x \right) + \frac{k^3 - 3k^2 + 2k}{6} \sum_{V^o} x .
$$

The result now follows from a quick comparison of coefficients. □

4.3.2 A Pick-type Matrix Formula

We now determine the h^2-tensors in order to find a Pick-type formula for the discrete moment matrix.

Similar to the vector case, by comparing coefficients of the numerator of (4.8) and that in Theorem 26, we obtain the formula

$$h_{S^*}^2(t) = \sum_{i=0}^{2} L^2(S_i) t^i (1-t)^2 + (v_1 + v_2 + v_3) L^1(S_i) t^{i+1} (1-t)$$
$$+ \frac{1}{2}(v_1^2 + v_2^2 + v_3^2) L(S_i) t^{i+1} + \frac{1}{2}(v_1 + v_2 + v_3)^2 L(S_i) t^{i+2}$$

for a half-open simplex S^* where S_i are defined as in Section 4.2.2. The h^2-tensors of a half-open simplex are then found to be

$$h_i^2(S^*) = L^2(S_i) - 2L^2(S_{i-1}) + L^2(S_{i-2}) + (v_1 + v_2 + v_3)\left(L^1(S_{i-1}) - L^1(S_{i-2})\right)$$
$$+ \frac{1}{2}(v_1^2 + v_2^2 + v_3^2) L(S_{i-1}) + \frac{1}{2}(v_1 + v_2 + v_3)^2 L(S_{i-2}). \qquad (4.13)$$

Proposition 29. *If $P \in \mathcal{P}(\mathbb{Z}^2)$ is a lattice polygon, then*

$$h_P^2(t) = \frac{t}{2}\sum_V x^2 + \frac{t^2}{2}\left(\sum_E (y+z)^2 - \sum_V x^2\right) + \frac{t^3}{2}\left(\sum_{E^o}(y+z)^2 - \sum_{V^o}x^2\right) + \frac{t^4}{2}\sum_{V^o}x^2.$$

Proof. Similar to the h^1-tensor polynomial, we determine the h^2-tensor polynomial of all half-open unimodular simplices, up to unimodular transformation. Formula (4.13) for each T_i with the values from (4.12) yields the following:

$$h_{T_0}^2(t) = \frac{t}{2}(v_1^2 + v_2^2 + v_3^2) + \frac{t^2}{2}((v_1 + v_2)^2 + (v_2 + v_3)^2 + (v_3 + v_1)^2 - v_1^2 - v_2^2 - v_3^2)$$
$$h_{T_1}^2(t) = \frac{t}{2}v_1^2 + \frac{t^2}{2}((v_1 + v_2)^2 + (v_1 + v_3)^2 - v_1^2) + \frac{t^3}{2}(v_2 + v_3)^2$$
$$h_{T_2}^2(t) = \frac{t^2}{2}(v_2 + v_3)^2 + \frac{t^3}{2}((v_1 + v_2)^2 + (v_1 + v_3)^2 - v_1^2) + \frac{t^4}{2}v_1^2$$

The claim now follows easily from Theorem 25. $\qquad \square$

From Proposition 29, we can now deduce formulas for the Ehrhart matrices.

Proposition 30. *Given a polygon $P \in \mathcal{P}(\mathbb{Z}^2)$, we have*

$$L^2(kP) = \frac{k}{24}\sum_{\partial E}(y-z)^2 + \frac{k^2}{48}\left(12\sum_V x^2 + 12\sum_{V^o}x^2 - \sum_E(y+z)^2 - \sum_{E^o}(y+z)^2\right)$$
$$+ \frac{k^3}{24}\left(2\sum_{\partial V}x^2 + \sum_{\partial E}(y+z)^2\right) + \frac{k^4}{48}\left(\sum_E(y+z)^2 + \sum_{E^o}(y+z)^2\right).$$

Proof. By definition, the Ehrhart matrix polynomial equals

$$\mathrm{L}^2(kP) \;=\; h_0^2(P)\binom{k+4}{4} + h_1^2(P)\binom{k+3}{4} + h_2^2(P)\binom{k+2}{4} + h_3^2(P)\binom{k+1}{4} + h_4^2(P)\binom{k}{4}.$$

The result follows now from Proposition 29 and comparing coefficients. For $\mathrm{L}_1^2(P)$, we further observe that

$$\mathrm{L}_1^2(P) \;=\; \frac{1}{24}\left(4\sum_{\partial V}x^2 - \sum_{\partial E}(y+z)^2\right) \;=\; \frac{1}{24}\sum_{\partial E}(y-z)^2. \qquad \square$$

4.4   Positive Semidefiniteness of h^2-tensors

A fundamental theorem in Ehrhart theory is Stanley's Nonnegativity Theorem [58] that states that the h^*-vector of every lattice polytope has nonnegative entries. While positivity of real numbers is canonically defined up to sign change, there are many different choices for higher dimensional vector spaces such as $\mathbb{T}^r$; one for every pointed cone (compare, e.g., [34]). A well-studied cone inside the vector space of symmetric matrices is the cone of positive semidefinite matrices. A matrix $M \in \mathbb{R}^{n\times n}$ is called *positive semidefinite* if $v^t M v \geq 0$ for all $v \in \mathbb{R}^n$. By the identification of $\mathbb{T}^2$ with $\mathbb{R}^{n\times n}$, we call a tensor $A \in \mathbb{T}^2$ *positive semidefinite* if its corresponding symmetric matrix (A_{ij}) is positive semidefinite. By the spectral theorem, A is a *sum of squares*; more precisely, if A has eigenvalues $\lambda_1,\ldots,\lambda_n \geq 0$ and corresponding normalized eigenvectors $u_1,\ldots,u_n$ then

$$(A_{ij}) \;=\; \sum_{i=1}^{n}\lambda_i u_i u_i^t$$

which is equivalent to $A = \sum_{i=1}^{n}\lambda_i u_i^2 \in \mathbb{T}^2$. Therefore, a tensor is positive semidefinite if and only if it is a sum of squares.

As for classic Ehrhart polynomials, the coefficients of Ehrhart tensor polynomials can be negative. However, in contrast to the usual Ehrhart polynomials, this phenomenon already appears in dimension 2 as the following example shows.

Example 31. *Let P be the triangle with vertices $v_1 = (-1,0)^t$, $v_2 = (0,-4)^t$ and $v_3 = (0,4)^t$. The Ehrhart tensor polynomial of P can be calculated as*

$$\mathrm{L}^2(kP) = \begin{pmatrix} \frac{1}{12} & 0 \\ 0 & \frac{5}{3} \end{pmatrix} k + \begin{pmatrix} -\frac{1}{12} & 0 \\ 0 & \frac{29}{3} \end{pmatrix} k^2 + \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{40}{3} \end{pmatrix} k^3 + \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{16}{3} \end{pmatrix} k^4.$$

We observe that the coefficient of k^2 is indefinite.

Our main result is the following analog to Stanley's Nonnegativity Theorem for the h^2-tensor polynomial of a lattice polygon.

Theorem 32. *The h^2-tensors of any lattice polygon are positive semidefinite.*

Before proving Theorem 32, we make a few more observations. Positive semidefiniteness of h^2-tensors is preserved under unimodular transformations since, from Equation (4.7) and comparing coefficients, we have

$$h_i^r(\phi P)(v, v) = h_i^r(\phi^t v, \phi^t v)$$

for all $P \in \mathcal{P}(\mathbb{Z}^n)$, $\phi \in \mathrm{GL}_n(\mathbb{Z})$, and $v \in \mathbb{R}^n$. However, as the next example shows, positive semidefiniteness of the h^2-vector is in general not preserved under translation. Still, as becomes apparent in the proof, Theorem 32 also holds for lattice polygons inside a higher dimensional ambient space.

Example 33. *Let $S = \mathrm{conv}\{v_1, v_2, v_3\} \setminus \mathrm{conv}\{v_2, v_3\}$ be the half-open simplex with vertices $v_1 = (2, -2)^t$, $v_2 = (3, -2)^t$, and $v_3 = (2, -1)^t$. From the formula of the h^2-vector of a half-open simplex which can be found in the proof of Proposition 29, we obtain that*

$$h_S^2(t) \;=\; \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} t + \begin{pmatrix} 37 & -28 \\ -28 & 21 \end{pmatrix} \frac{t^2}{2} + \begin{pmatrix} 25 & -15 \\ -15 & 9 \end{pmatrix} \frac{t^3}{2}.$$

That is, with a determinant of $-\frac{7}{4}$, the matrix $h_2^2(S)$ is not positive semidefinite. However, it can be seen that the positive semidefiniteness of h^2-tensors is not preserved under translations. To illustrate, consider the translate $S - v_1$. The h^2-vector of the translated simplex

$$h_{S-v_1}^2(t) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{t^2}{2} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{t^3}{2},$$

has positive semidefinite coefficients.

Example 33 moreover shows that positive semidefiniteness of h^2-tensors of half-open polytopes is not preserved under translation. This makes Theorem 32 even more suprising. In particular, it also holds true for polygons in higher dimensional ambient spaces since the results in Sections 4.2 and 4.3 are independent from the dimension of the vector space. However, positive semidefiniteness is preserved under actions of $\mathrm{GL}_n(\mathbb{Z})$ since, from Equation (4.7) and comparing coefficients, we have

$$h_i^r(\phi P)(x, x) = h_i^r(\phi^t x, \phi^t x) \geq 0$$

for all $P \in \mathcal{P}(\mathbb{Z}^n)$, $\phi \in \mathrm{GL}_n(\mathbb{Z})$ and $x \in \mathbb{R}^n$.

To prove Theorem 32, we decompose a lattice polygon into lattice polygons with few vertices for which the h^2-vectors can easily be calculated. For the remainder of this article, allow a lattice polygon to always mean a full-dimensional in $\mathbb{R}^2$ although the argument is independent from the chosen ambient space.
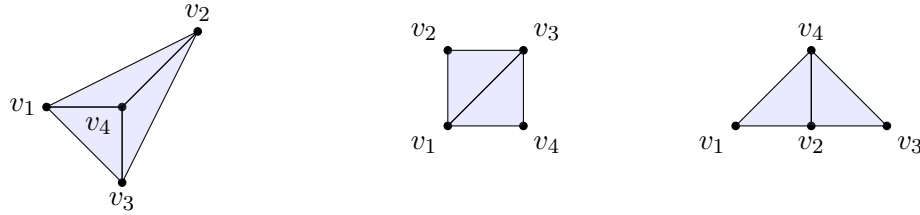
Figure 4.2: Lattice polygons containing 4 lattice points and their unimodular triangulations.

A *sparse decomposition* of P is a finite set $\mathcal{D} = \{P_1, P_2, \ldots, P_m\}$ of lattice polygons such that

i) $L(P_i) \in \{3, 4\}$ for each $i \in [m]$,

ii) $P_i \cap P_j = \varnothing$ or is a common vertex of P_i and P_j for all $i \neq j$, and

iii) $P \cap \mathbb{Z}^2 \subset \bigcup_{i=1}^m P_i$.

Lemma 34. *[40, Section 4] Up to unimodular transformation, there are exactly three different lattice polygons containing four lattice points. They are given in Figure 4.2.*

The following lemma ensures that every lattice polygon has a sparse decomposition.

Lemma 35. *Every lattice polygon has a sparse decomposition.*

Proof. We proceed by induction on $L(P)$. The statement is trivially true if $L(P) \in \{3, 4\}$. Hence, we may assume that $L(P) > 4$ and choose a vector $a \in \mathbb{R}^2 \setminus \{0\}$ such that $a^t v \neq a^t w$ for each $v, w \in P \cap \mathbb{Z}^2$ where $v \neq w$. Note that such an a exists since $L(P)$ is finite. Let $P \cap \mathbb{Z}^2 = \{v_1, \ldots, v_m\}$ be such that

$$a^t v_1 > a^t v_2 > \cdots > a^t v_m$$

and set $Q = \operatorname{conv}\{v_3, v_4, \ldots, v_m\}$. Then, by convexity, we obtain $Q \cap \mathbb{Z}^2 = P \cap \mathbb{Z}^2 \setminus \{v_1, v_2\}$.

If Q is not full-dimensional and all lattice points of Q lie on a line, then a sparse decomposition of P can easily be constructed. If u_1, u_2, and u_3 are not collinear, then we can construct a sparse decomposition which is illustrated in Figure 4.3. Let $P_1 = \operatorname{conv}\{u_1, u_2, u_3\}$. Then, by design, the triangle P_1 does not contain any other lattice point and at least one of u_1 or u_2 are visible from all points $u_4, \ldots, u_m$. Without loss of generality, assume u_1 is visible. Then for all $2 \leq i \leq \lfloor \frac{m}{2} \rfloor - 1$ define $P_i = \operatorname{conv}\{u_1, u_{2i}, u_{2i+1}\}$, $P_{\lfloor \frac{m}{2} \rfloor} = \operatorname{conv}\{u_1, u_{m-2}, u_m\}$ if m is even, and $P_{\lfloor \frac{m}{2} \rfloor} = \operatorname{conv}\{u_1, u_{m-1}, u_m\}$ if m is odd. Then $\{P_1, \ldots, P_{\lfloor \frac{m}{2} \rfloor}\}$ is a sparse decomposition. If u_1, u_2, and u_3 are collinear, then a sparse decomposition can be obtained by instead setting $P_1 = \operatorname{conv}\{u_2, u_3, u_4\}$.
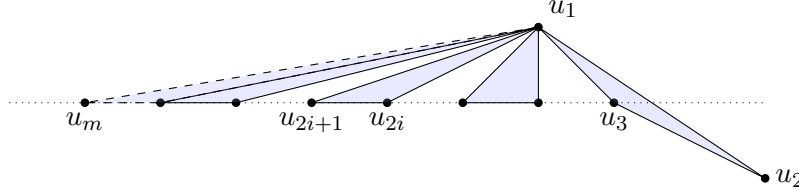
Figure 4.3: Sparse decomposition of P for the case of a collinear Q.

Suppose Q is full-dimensional. Then, by the induction hypothesis, there is a sparse decomposition $\mathcal{D}_Q$ of Q. Let i be the smallest index such that the points u_1, u_2, u_i do not lie on a common straight line. By construction, the simplex $S = \operatorname{conv}(u_1, u_2, u_i)$ contains no other lattice points and, thus, $\mathcal{D}_Q \cup \{S\}$ is a sparse decomposition of P. $\qquad\square$

Lemma 36. *If $P \in \mathcal{P}(\mathbb{Z}^2)$ is a lattice polygon containing exactly 3 or 4 lattice points, then $h_2^2(P)$ is positive semidefinite.*

Proof. If $\mathrm{L}(P) = 3$, then $P = \operatorname{conv}(v_1, v_2, v_3)$ is a unimodular lattice simplex and the statement follows from Proposition 29 as

$$2h_2^2(P) = (v_1 + v_2)^2 + (v_1 + v_3)^2 + (v_2 + v_3)^2 - v_1^2 - v_2^2 - v_3^2 = (v_1 + v_2 + v_3)^2.$$

Suppose $\mathrm{L}(P) = 4$. We have to distinguish between the three possible cases, up to unimodular transformation, given in Figure 4.2. First, if P contains one interior lattice point v_4 and vertices v_1, v_2, v_3, then we have $v_4 = \frac{1}{3}(v_1 + v_2 + v_3)$ and Proposition 29 implies that

$$\begin{aligned} 2h_2^2(P) &= (v_1 + v_2)^2 + (v_1 + v_3)^2 + (v_2 + v_3)^2 + (v_1 + v_4)^2 + (v_2 + v_4)^2 + (v_3 + v_4)^2 \\ &\quad - v_1^2 - v_2^2 - v_3^2 - v_4^2 \\ &= (v_1 + v_2)^2 + (v_1 + v_3)^2 + (v_2 + v_3)^2 + 2v_4^2 + 2v_4(v_1 + v_2 + v_3) \\ &= (v_1 + v_2)^2 + (v_1 + v_3)^2 + (v_2 + v_3)^2 + \tfrac{8}{9}(v_1 + v_2 + v_3)^2. \end{aligned}$$

Next, if P is a parallelepiped, then $v_1 + v_3 = v_2 + v_4$ and thus

$$\begin{aligned} 2h_2^2(P) &= (v_1 + v_2)^2 + (v_2 + v_3)^2 + (v_3 + v_4)^2 + (v_1 + v_4)^2 \\ &\quad + \tfrac{1}{2}(v_1 + v_3)^2 + \tfrac{1}{2}(v_2 + v_4)^2 - v_1^2 - v_2^2 - v_3^2 - v_4^2 \\ &= \tfrac{1}{2}(v_1 + v_2 + v_3 + v_4)^2 + \tfrac{1}{2}(v_1 + v_2)^2 + \tfrac{1}{2}(v_2 + v_3)^2 + \tfrac{1}{2}(v_3 + v_4)^2 + \tfrac{1}{2}(v_1 + v_4)^2. \end{aligned}$$

Finally, if P has three vertices and no interior lattice point, then one lattice point of P, say v_2 as in Figure 4.2, lies in the relative interior of the edge given by the vertices v_1 and v_3 implying that $v_2 = \frac{1}{2}(v_1 + v_3)$. In this case, we obtain

$$\begin{aligned} 2h_2^2(P) &= (v_1 + v_2)^2 + (v_2 + v_3)^2 + (v_3 + v_4)^2 + (v_1 + v_4)^2 + (v_2 + v_4)^2 - v_1^2 - v_2^2 - v_3^2 - v_4^2 \\ &= \tfrac{5}{2}v_1^2 + \tfrac{5}{2}v_3^2 + 2v_4^2 + 3v_1v_4 + 3v_3v_4 + 3v_1v_3 \\ &= \tfrac{3}{2}(v_1 + v_3 + v_4)^2 + v_1^2 + v_3^2 + \tfrac{1}{2}v_4^2. \end{aligned}\qquad\square$$

We will need the following geometric observation in our proof of Theorem 32.

Lemma 37. *Let $P \in \mathcal{P}(\mathbb{Z}^2)$ and v be a lattice point in the relative interior of P. Then at least one of the following two statements is true:*

(i) $v = \frac{1}{2}(v_1 + v_2)$ *for lattice points* $v_1, v_2 \in P$ *such that* $v_1 \neq v_2$;

(ii) $v = \frac{1}{3}(v_1 + v_2 + v_3)$ *for pairwise disjoint lattice points* $v_1, v_2, v_3 \in P$.

Proof. If v is contained in a segment formed by two lattice points in P, then v is of the form given in *(i)*.

Therefore, we may assume that v is not contained in any line segment formed by lattice points in P. By Caratheodory's Theorem (see, e.g., [55]), there are lattice points $v_1, v_2, v_3 \in P$ such that v is contained in the simplex formed by v_1, v_2, and v_3. If v, v_1, v_2, v_3 are the only lattice points in the simplex, then condition *(ii)* follows from Lemma 34. Otherwise, there is a lattice point $u \in \text{conv}\{v_1, v_2, v_3\} \setminus \{v, v_1, v_2, v_3\}$ and, consequently, v must be contained in one of the three lattice simplices

$$S_1 = \text{conv}\{v_2, v_3, u\}, \quad S_2 = \text{conv}\{v_1, v_3, u\}, \quad S_3 = \text{conv}\{v_1, v_2, u\}.$$

Without loss of generality, let $v \in S_1 \subsetneq \text{conv}\{v_1, v_2, v_3\}$. By reiteration of the above procedure, each time with a replacement of v_1 by u, we eventually find affinely independent v_1, v_2, v_3 such that $\{v, v_1, v_2, v_3\} = \text{conv}\{v_1, v_2, v_3\} \cap \mathbb{Z}^2$ and condition *(ii)* follows again from Lemma 34. $\square$

We are now equipped to give the proof of our nonnegativity theorem.

Proof of Theorem 32. From Proposition 29, it immediately follows that $h_0^2(P), h_1^2(P)$, and $h_4^2(P)$ are sums of squares.

Let $\mathcal{D} = \{P_1, P_2, \ldots, P_m\}$ be a sparse decomposition of P which exists by Lemma 35 and let $\mathcal{S}$ be some triangulation of $\cup_{i=1}^m P_i$. Observe that the closure of $P \setminus (P_1 \cup \cdots \cup P_m)$ is a union of not necessarily convex lattice polygons and any triangulation of $\cup_{i=1}^m P_i$ can be extended to a triangulation in P. Let $\mathcal{T}$ be a triangulation of P such that $\mathcal{S} \subseteq \mathcal{T}$. Let $G = (V, E)$ be the edge graph of $\mathcal{T}$ and $G' = (V', E')$ be the edge graph of $\mathcal{S}$. For every $x \in V$, we define $\alpha_x = |\{i \in [m] : x \in P_i\}|$. Note that $\alpha_x \geq 1$ for all $x \in V$ since $\mathcal{D}$ is a sparse decomposition. Proposition 29 then implies that

$$2h_2^2(P) = \sum_E (y+z)^2 - \sum_V x^2$$

$$= \sum_{E'} (y+z)^2 - \sum_V \alpha_x \, x^2 + \sum_{E \setminus E'} (y+z)^2 - \sum_V (1 - \alpha_x) \, x^2$$

$$= \sum_{i=1}^m h_2^2(P_i) + \sum_{E \setminus E'} (y+z)^2 + \sum_V (\alpha_x - 1) \, x^2 \,,$$

and therefore, by Lemma 36, $h_2^2(P)$ is a sum of squares.

We have left to show that $h_3^2(P)$ is also a sum of squares. For every $v \in V$, we define $N(v) = \{u \in V : \{u, v\} \in E\}$ to be the set of vertices adjacent to v in G. Let $E_1 \subseteq E^\circ$ be the set of edges that have exactly one endpoint on the boundary of P and $E_2 \subseteq E^\circ$ be the set of edges with both endpoints on the boundary of P but relative interior in P°. By Proposition 29, we obtain

$$2h_3^2(P) = \sum_{E^\circ}(y+z)^2 - \sum_{V^\circ}x^2$$

$$= \sum_{v \in V^\circ}\left(\sum_{u \in N(v)}\left(\frac{1}{2}(v+u)^2\right) - v^2\right) + \sum_{E_1}\frac{1}{2}(y+z)^2 + \sum_{E_2}(y+z)^2.$$

It is thus sufficient to show that

$$a(v) := \sum_{u \in N(v)}\left(\frac{1}{2}(v+u)^2 - v^2\right)$$

is a sum of squares for all $v \in V^\circ$. In view of Lemma 37, we distinguish two cases. First, suppose that there are $v_1, v_2 \in V \setminus \{v\}$ such that $v = \frac{1}{2}(v_1 + v_2)$. Then

$$a(v) = \tfrac{1}{2}(v+v_1)^2 + \tfrac{1}{2}(v+v_2)^2 - v^2 + \sum_{u \in N(v)\backslash\{v_1,v_2\}}\tfrac{1}{2}(v+u)^2$$

$$= \tfrac{1}{2}(v_1+v_2)^2 + \tfrac{1}{2}v_1^2 + \tfrac{1}{2}v_2^2 + \sum_{u \in N(v)\backslash\{v_1,v_2\}}\tfrac{1}{2}(v+u)^2.$$

In the second case, there exist pairwise disjoint $v_1, v_2, v_3 \in V \setminus \{v\}$ such that $v = \frac{1}{3}(v_1+v_2+v_3)$. Therefore

$$a(v) = \tfrac{1}{2}(v+v_1)^2 + \tfrac{1}{2}(v+v_2)^2 + \tfrac{1}{2}(v+v_3)^2 - v^2 + \sum_{u \in N(v)\backslash\{v_1,v_2,v_3\}}\tfrac{1}{2}(v+u)^2$$

$$= \tfrac{7}{18}(v_1+v_2+v_3)^2 + \tfrac{1}{2}v_1^2 + \tfrac{1}{2}v_2^2 + \tfrac{1}{2}v_3^2 + \sum_{u \in N(v)\backslash\{v_1,v_2,v_3\}}\tfrac{1}{2}(v+u)^2. \qquad \square$$

4.5 Hibi's Palindromic Theorem

It is natural to ask whether Theorem 32 holds true also in higher dimensions. Using the software package polymake [6, 22] we have calculated the h^2-tensor polynomials of several thousand randomly generated polytopes in dimension 3 and 4. Based on these computational results, we offer the following conjecture.

Conjecture 38. *For $n \geq 1$, the coefficients of the h^2-tensor polynomial of any of a lattice polytope in $\mathbb{R}^n$ are positive semidefinite.*

For our proof of Theorem 32, it was crucial that every lattice polygon has a unimodular triangulation. Since this no longer holds true in general for higher dimensional polytopes, a proof of Conjecture 38 would need to be conceptually different.

Finding inequalities among the coefficients of the h^*-polynomial of a lattice polytope, beyond Stanley's Nonnegativity Theorem, is currently of great interest in Ehrhart theory. The ultimate goal is a classification of all possible h^*-polynomials: a classification of all h^*-polynomials of degree 2 can be found in [30, Proposition 1.10]. Another fundamental inequality is due to Hibi [32] who proved that $h_i(P) - h_1(P) \geq 0$ for all $1 \leq i < n$ and full-dimensional lattice polytopes that have an interior lattice point. Calculations with polymake again suggest that there might be a version for matrices motivating the following conjecture.

Conjecture 39. *Let P be a lattice polytope containing a lattice point in its interior. Then the matrices $h_i^2(P) - h_1^2(P)$ for $1 \leq i < \dim(P) + 2$ are positive semidefinite.*

In recent years, additional inequalities for the coefficients of the h^*-polynomial have been shown (see e.g. [7, 59, 60]) which raises the question as to whether there are analogous results for Ehrhart tensors.

Question 40. *Which known inequalities among the coefficients of the h^*-polynomial of a lattice polytope can be generalized to h^r-tensor polynomials of higher rank?*

An answer would depend on the notion of positivity that is chosen. A natural choice for higher rank h^r-tensors, extending positive semidefiniteness of matrices, is to define $A \in \mathbb{T}^r$ to be positive semidefinite if and only if the coordinates $A(v[r]) \geq 0$ for all $v \in \mathbb{R}^n$. However, assuming this definition of positivity, there can not be any inequalities that are valid for all polytopes if the rank r is odd since $A(v[r]) = (-1)^r A(-v[r])$.

In the case that r is even, we are able to extend another classical result, namely Hibi's Palindromic Theorem [31] characterizing reflexive polytopes. A lattice polytope $P \in \mathcal{P}(\mathbb{Z}^n)$ is called *reflexive* if

$$P = \{x \in \mathbb{R}^n \colon Ax \leq 1\}$$

where $A \in Z^{n \times n}$ is an integral matrix.

Theorem 41 (Hibi [31]). *A polytope $P \in \mathcal{P}(\mathbb{Z}^n)$ is reflexive if and only if $h_i^*(P) = h_{n-i}^*(P)$ for all $0 \leq i \leq n$.*

A crucial step in the proof of Theorem 41 is to observe that a polytope P is reflexive if and only if

$$kP \cap \mathbb{Z}^n = (k+1)P^\circ \cap \mathbb{Z}^n$$

for all $k \in \mathbb{N}$ (see [9]). We use this fact to give the following generalization.

Proposition 42. *Let $r \in \mathbb{N}$ be even and $P \in \mathcal{P}(\mathbb{Z}^n)$ be a lattice polytope that contains the origin in its relative interior. The polytope P is reflexive if and only if $h_i^r = h_{n+r-i}^r$ for all $0 \leq i \leq n+r$.*

Proof. By Theorem 2 and comparing coefficients in equation (4.7), it follows that the assertion $h_i^r(P) = h_{n+r-i}^r(P)$ is equivalent to $\mathrm{L}^r((k-1)P) = \mathrm{L}^r(kP^\circ)$ for all integers k.

If P is a reflexive polytope, then $\mathrm{L}^r((k-1)P) = \mathrm{L}^r(kP^\circ)$ for all integers k since, as given above, we have $(k-1)P \cap \mathbb{Z}^n = kP^\circ \cap \mathbb{Z}^n$.

Now assume that P is not reflexive. Then there exists an $k \in \mathbb{N}$ such that

$$(k-1)P \cap \mathbb{Z}^n \subsetneq kP^\circ \cap \mathbb{Z}^n \, .$$

Therefore, for any $v \in \mathbb{R}^n \setminus \{0\}$, we obtain

$$\sum_{x \in (k-1)P \cap \mathbb{Z}^n} (x^t v)^r \;\; < \;\; \sum_{x \in kP^\circ \cap \mathbb{Z}^n} (x^t v)^r$$

and, in particular, $\mathrm{L}^r((k-1)P) \neq \mathrm{L}^r(kP^\circ)$ completing the proof. $\qquad\square$

Note that the proof of Proposition 42 shows that for odd rank r palindromicity of the h^r-tensor polynomial of a reflexive polynomial is still necessary, but not sufficient, since all centrally symmetric polytopes have a palindromic h^r-tensor polynomial; namely the constant zero polynomial.

Chapter 5

Translation Invariant Valuations

The classification of $\mathrm{SL}_n(\mathbb{Z})$ equivariant and translation covariant valuations follows nicely from the classification of $\mathrm{SL}_n(\mathbb{Z})$ equivariant and translation invariant tensor valuations. In Sections 5.2 and 5.3, we give part of this classification for n-homogeneous valuations. The theory of dissections of lattice polytopes will be helpful for this initial classification which we provide a short introduction to in Section 5.1.

5.1 Dissections

We begin with some definitions and adapt results from the theory of dissections on convex polytopes to lattice polytopes.

An n-dimensional lattice polytope P is said to be *dissected* into the polytopes $Q_1, \ldots, Q_m$ if $P = Q_1 \cup \cdots \cup Q_m$ where Q_i and Q_j have disjoint interiors for every $i \neq j$; this is written as $P = Q_1 \sqcup \cdots \sqcup Q_m$. We say that $P \in \mathcal{P}(\mathbb{Z}^n)$ is *equi-dissectable* by translations to $Q \in \mathcal{P}(\mathbb{Z}^n)$ if there are dissections $P = P_1 \sqcup \cdots \sqcup P_m$ and $Q = Q_1 \sqcup \cdots \sqcup Q_m$ into lattice polytopes such that P_i is a translate of Q_i for $i = 1, \ldots, m$.

As Corollary 11 shows that tensor valuations on lattice polytopes are determined by their values on unimodular simplices, we are interested in dissections of polytopes into simplices. A simplex is known to be unimodular if its vertices span the sublattice obtained from the intersection of its affine hull with $\mathbb{Z}^n$. We denote the convex hull of $v_1, \ldots, v_i \in \mathbb{Z}^n$ as $[v_1, \ldots, v_i]$. Recall that $T_0 = \{0\}$ and $T_i = [0, e_1, \ldots, e_i]$ for $i = \{1, \ldots, n\}$. The Minkowski sum of $P, Q \in \mathcal{P}(\mathbb{Z}^n)$ is $P + Q = \{x + y : x \in P, y \in Q\}$. For $j = 1, \ldots, n$, a polytope $P \in \mathcal{P}(\mathbb{Z}^n)$ is called a *j-cylinder* if there are proper independent linear subspaces $H_1, \ldots, H_j$ of $\mathbb{R}^n$ and lattice polytopes $P_i \subset H_i$ such that $P = P_1 + \cdots + P_j$. We denote by $\mathcal{Z}_j(\mathbb{Z}^n)$ the class of j-cylinders and note that $\mathcal{Z}_n(\mathbb{Z}^n) \subset \cdots \subset \mathcal{Z}_1(\mathbb{Z}^n) = \mathcal{P}(\mathbb{Z}^n)$.

Observe that an n-cylinder is an n-dimensional parallelotope. Hadwiger [27, p. 73] showed that n-dimensional parallelotopes with equal volume are equi-dissectable. We include a proof for completeness (using the Two-Tile Theorem, see [56]).

Lemma 43 (Hadwiger [27])**.** *If $P, Q \in \mathcal{Z}_n(\mathbb{Z}^n)$ have equal volume, then there exists a positive integer k such that kP and kQ are equi-dissectable by translations.*

Proof. We may assume that $V_n(P) = V_n(Q) = 1$, as the case of greater volume can be reduced to this case. Note that the polytopes P and Q tile $\mathbb{R}^n$ by translations from $\mathbb{Z}^n$. Hence

$$P = \bigsqcup_{x \in \mathbb{Z}^n} P \cap (Q + x) \text{ and } Q = \bigsqcup_{y \in \mathbb{Z}^n} (P + y) \cap Q.$$

Furthermore, each $P \cap (Q + x)$ is a rational polytope as P and Q are lattice polytopes. Therefore, there exists minimal positive integers k and l such that

$$kP = k \bigsqcup_{x \in \mathbb{Z}^n} P \cap (Q + x) \text{ and } lQ = l \bigsqcup_{y \in \mathbb{Z}^n} (P + y) \cap Q$$

where each $k\,(P \cap (Q + x))$ and $l\,((P + y) \cap Q)$ is a lattice polytope. Since each $P \cap (Q + x)$ is a translate of some $(P + y) \cap Q$, we observe that $k = l$. Hence each $k\,(P \cap (Q + x))$ is a translate of some $k\,((P + y) \cap Q)$ concluding the proof. $\qquad\square$

We require the following lemma on lattice j-cylinders. The corresponding result for general j-cylinders is due to Hadwiger [27, Hilfssatz I] and the proof in the lattice case is similar.

Lemma 44. *Let k be a positive integer and $1 \leq j \leq n - 1$. For $P \in \mathcal{Z}_j(\mathbb{Z}^n)$, there exist $Q_1, \ldots, Q_m \in \mathcal{Z}_{j+1}(\mathbb{Z}^n)$ (depending on k) such that kP is equi-dissectable to the disjoint union of k^j copies of P and $Q_1, \ldots, Q_m$.*

Proof. We denote the class of disjoint unions of j-cylinders by $\sqcup \mathcal{Z}_j(\mathbb{Z}^n)$ and by $k.Q$ any k translates of Q with disjoint interiors. Lastly, we write $P \sim Q$ for any P and Q that are equi-dissectable by translations.

We utilize Hadwiger's canonical simplex decomposition [27, p. 19] that describes a specific decomposition of the orthogonal simplex

$$T = \left\{ \sum_{i=1}^{n} s_i e_i : 0 \leq s_1 \leq \cdots \leq s_n \leq 1 \right\}.$$

For positive integers p and q, the decomposition is of the form

$$(p + q)T = \bigsqcup_{k=0}^{n} \left(p\, T_{k,\bullet} + q\,(T_{\bullet,n-k} + x_k) \right) \tag{5.1}$$

where $T_{0,\bullet} = T_{\bullet,0} = \{0\}$ and for $k = 1, \ldots, n - 1$,

$$T_{k,\bullet} = \left\{ \sum_{i=1}^{k} s_i e_i : 0 \leq s_1 \leq \cdots \leq s_k \leq 1 \right\},$$

$$T_{\bullet,n-k} = \left\{ \sum_{i=k+1}^{n} s_i e_i : 0 \leq s_{k+1} \leq \cdots \leq s_n \leq 1 \right\},$$

and $x_k = e_1 + \cdots + e_k$ while $x_0 = 0$.

The proof is given in two steps. We will first show that the statement holds for $j = 1$ and then use this for the general case. For $j = 1$, it is sufficient to prove the statement for a simplex and hence for T. The relation (5.1), for $i = 1, \ldots, k$, with $p = k - i$ and $q = 1$ yields $(k - i + 1)T \sim (k - i)T \sqcup (T + y_i) \sqcup Q_i$ where $Q_i \in \sqcup \mathcal{Z}_2(\mathbb{Z}^n)$ and $y_i \in \mathbb{Z}^n$. Combining these k relations, we obtain $k\,T \sim (T + y_1) \sqcup \cdots \sqcup (T + y_k) \sqcup (Q_1 \sqcup \cdots \sqcup Q_k)$ which implies that $k\,T \sim k.T \sqcup Q$ where $Q \in \sqcup \mathcal{Z}_2(\mathbb{Z}^n)$.

Second, we will show that the result holds for $j > 1$. Therefore, let $P = P_1 + \cdots + P_j$ with $P_1, \ldots, P_j \in \mathcal{P}(\mathbb{Z}^n)$ lying in complementary subspaces. It follows that $k\,P = k\,P_1 + \cdots + k\,P_j$. By assumption, there exists $Q_i \in \mathcal{Z}_2(\mathbb{Z}^n)$ such that $k\,P_i \sim k.P_i \sqcup Q_i$ as $P_i \in \mathcal{Z}_{l_i}(\mathbb{Z}^n)$ for $l_i < j$ and $i = 1, \ldots, j$. Recall here that $\mathcal{Z}_2(\mathbb{Z}^n) \subseteq \mathcal{Z}_{l_i + 1}(\mathbb{Z}^n)$ as $l_i \geq 1$. Hence

$$k\,P \sim (k.P_1 \sqcup Q_1) + \cdots + (k.P_j \sqcup Q_j) \tag{5.2}$$

where $k.P_i = P_{i_1} \sqcup \cdots \sqcup P_{i_k}$ and $P_{i_l} = P_i + x_{i_l}$ for some $x_{i_l} \in \mathbb{R}^n$, for $1 \leq i \leq j$ and $1 \leq l \leq k$. Distribution of the right hand side of (5.2) yields our result. Precisely, we get

$$k.P_1 + \cdots + k.P_j = \sum_{i=1}^{j} P_{i_1} \sqcup \cdots \sqcup P_{i_k} \sim k^j.P.$$

Further, for any $Q_i \neq \varnothing$, we have

$$Q_i + \sum_{\substack{1 \leq l \leq k \\ l \neq i}} (k.P_l \sqcup Q_l) \in \mathcal{Z}_{k+1}(\mathbb{Z}^n)$$

as the $k\,P_l \sim k.P_l \sqcup Q_l$ are in complementary linear subspaces by the definition of a j-cylinder and, trivially, in $\mathcal{Z}_1(\mathbb{Z}^n)$. Thus, we obtain $k\,P \sim k^j.P \sqcup Q$ for some $Q \in \sqcup \mathcal{Z}_{j+1}(\mathbb{Z}^n)$. $\square$

Every lattice polygon has a unimodular triangulation, that is, a dissection into unimodular triangles (see, e.g., [17]). If the union of two unimodular triangles in such a triangulation is a convex quadrilateral Q, then replacing the diagonal of Q given by the edges of the adjacent triangles with the opposite diagonal produces a new unimodular triangulation. This process is called a *flip*.

Theorem 45 (Lawson [39])**.** *Given any two unimodular triangulations $\mathcal{T}$ and $\mathcal{T}'$ of a lattice polygon $P \in \mathcal{P}(\mathbb{Z}^2)$, there exists a finite sequence of flips transforming $\mathcal{T}$ into $\mathcal{T}'$.*

5.2   Translation Invariant Valuations

We include key lemmas on general translation invariant valuations here and will apply these results to tensor valuations in the next section.

The Minkowski sum of $P, Q \in \mathcal{P}(\mathbb{Z}^n)$ is $P + Q = \{x + y : x \in P, y \in Q\}$. For $j = 1, \ldots, n$, a polytope $P \in \mathcal{P}(\mathbb{Z}^n)$ is called a *j-cylinder* if there are proper independent linear subspaces $H_1, \ldots, H_j$ of $\mathbb{R}^n$ and lattice polytopes $P_i \subset H_i$ such that $P = P_1 + \cdots + P_j$. We denote by $\mathcal{Z}_j(\mathbb{Z}^n)$ the class of j-cylinders and note that $\mathcal{Z}_n(\mathbb{Z}^n) \subset \cdots \subset \mathcal{Z}_1(\mathbb{Z}^n) = \mathcal{P}(\mathbb{Z}^n)$. Observe that an n-cylinder is an n-dimensional parallelotope.

A valuation on $\mathcal{P}(\mathbb{Z}^n)$ is said to be *simple* if it vanishes on lower dimensional sets. By Theorem 9, a simple valuation Z then has the property that

$$\mathrm{Z}(Q_1 \sqcup \cdots \sqcup Q_m) = \mathrm{Z}(Q_1) + \cdots + \mathrm{Z}(Q_m).$$

The following lemma can be found for convex polytopes in [38] and for lattice polytopes in [47, Lemma 4].

Lemma 46. *If* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ *is a simple, translation invariant, i-homogeneous valuation, then* $\mathrm{Z}(P) = 0$ *for every* $P \in \mathcal{Z}_j(\mathbb{Z}^n)$ *when* $j > i > 0$.

Proof. The statement is trivial for $j = n$. Assume that $\mathrm{Z}(P) = 0$ for all lattice polytopes $P \in \mathcal{Z}_{l+1}(\mathbb{Z}^n)$ where $l > j \geq i+1$. If $P \in \mathcal{Z}_l(\mathbb{Z}^n)$, then, by Lemma 44, there exist polytopes $Q_1, \ldots, Q_m \in \mathcal{Z}_{l+1}(\mathbb{Z}^n)$ such that kP is equi-dissectable by translations to k^l copies of P and $Q_1, \ldots, Q_m$. As Z is simple, this implies that $k^i \mathrm{Z}(P) = k^l \mathrm{Z}(P)$. Hence $\mathrm{Z}(P) = 0$. □

The following lemma can be found in [55] for valuations on convex polytopes. Here, we provide a proof for lattice polytopes. Let $\mathbb{V}$ be a rational vector space.

Lemma 47. *Let* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{V}$ *be a translation invariant valuation that is i-homogeneous for some* $1 \leq i \leq n$. *If* $P \in \mathcal{P}(\mathbb{Z}^n)$ *and* $\dim(P) < i$, *then* $\mathrm{Z}(P) = 0$.

Proof. Let H be an $(i-1)$-dimensional lattice subspace of $\mathbb{R}^n$. The restriction of Z to polytopes in $H \cap \mathcal{P}(\mathbb{Z}^n)$ is a valuation on polytopes with vertices in the lattice $H \cap \mathbb{Z}^n$ which is invariant under the translations of H into itself. The homogeneous decomposition from Theorem 14 states that this restricted Z is a sum of valuations homogeneous of degrees $0, \ldots, i-1$. However, the valuation Z is i-homogeneous implying that $\mathrm{Z}(P) = 0$ for $P \subset H$. The translation invariance of Z together with the arbitrary choice of H implies that $\mathrm{Z}(P) = 0$ for every $P \in \mathcal{P}(\mathbb{Z}^n)$ such that $\dim P < i$. □

5.3 Translation Invariant Tensor Valuations

In this section, we show that the only n-homogeneous, translation invariant tensor valuation that intertwines $\mathrm{SL}_n(\mathbb{Z})$ is the trivial tensor; that is, the tensor that vanishes identically. The classification of $\mathrm{SL}_n(\mathbb{Z})$ equivariant and translation invariant tensor valuations will be the main tool in our classification of $\mathrm{SL}_n(\mathbb{Z})$ equivariant and translation covariant tensor valuations.

Corresponding to Hadwiger's result [27, Satz XIV] on polytopes and proven similarly here, it is shown that the only translation invariant, n-homogeneous, real-valued valuation on lattice polytopes is a multiple of the n-dimensional volume. Note that this is also a direct consequence of a result by McMullen [48, Theorem 1].

Theorem 48. *If* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ *is a translation invariant and n-homogeneous valuation, then there exists* $a \in \mathbb{R}$ *such that*
$$\mathrm{Z}(P) = a\, V_n(P)$$
for every $P \in \mathcal{P}(\mathbb{Z}^n)$.

Proof. Lemma 47 implies that the valuation Z is simple. Let $P \in \mathcal{Z}_n(\mathbb{Z}^n)$. By Lemma 43, there exists a $k \in \mathbb{N}$ such that $k\,P$ is equi-dissectable by translations to $k\,[0,1]^n$. Hence $k^n\,Z(P) = k^n\,Z([0,1]^n)$ and $k^n V_n(P) = k^n V_n([0,1]^n)$ as Z and V_n are both n-homogeneous and simple. Thus Z is proportional to V_n on $\mathcal{Z}_n(\mathbb{Z}^n)$. Set $Y = Z - a\,V_n$, where $a \in \mathbb{R}$ is chosen such that $Y = 0$ on $\mathcal{Z}_n(\mathbb{Z}^n)$.

From Lemma 44, we obtain $Y(k\,P) = k^{n-1}\,Y(P)$ for $P \in \mathcal{Z}_{n-1}(\mathbb{Z}^n)$. Since Y is n-homogeneous, we conclude that $Y = 0$ on $\mathcal{Z}_{n-1}(\mathbb{Z}^n)$. Continuing it follows that $Y = 0$ on $\mathcal{Z}_i(\mathbb{Z}^n)$ for $i = 1, \ldots, n$. Hence $Z = a\,V_n$ on $\mathcal{P}(\mathbb{Z}^n)$. $\qquad\square$

The argument can easily be modified for tensor valuations by substituting a tensor $A \in \mathbb{T}^r$ for the constant $a \in \mathbb{R}$. Therefore, we immediately obtain the following corollary of Theorem 48.

Corollary 49. *If* $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is a translation invariant and n-homogeneous valuation, then there exists $A \in \mathbb{T}^r$ such that*

$$Z(P) = A\,V_n(P)$$

for every $P \in \mathcal{P}(\mathbb{Z}^n)$.

As volume is $\mathrm{SL}_n(\mathbb{Z})$ invariant, Corollary 49 makes it natural to expect that the only valuation that is translation invariant, n-homogeneous, and, additionally, $\mathrm{SL}_n(\mathbb{Z})$ equivariant is the trivial valuation. We show that this is the case.

Proposition 50. *Let $r \geq 1$ and $n \geq 2$. If* $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is an $\mathrm{SL}_n(\mathbb{Z})$ equivariant, translation invariant, and n-homogeneous valuation, then $Z(P) = 0$ for every $P \in \mathcal{P}(\mathbb{Z}^n)$.*

Proof. By Corollary 49, there exists $A \in \mathbb{T}^r$ such that $Z = V_n\,A$ on $\mathcal{P}(\mathbb{Z}^n)$. For any $\phi \in \mathrm{SL}_n(\mathbb{Z})$ and $v_1, \ldots, v_r \in \mathbb{R}^n$, this implies that

$$V_n(P)\,A(v_1, \ldots, v_r) = V_n(\phi P)\,A(v_1, \ldots, v_r) = V_n(P)\,A(\phi^t v_1, \ldots, \phi^t v_r)$$

as volume is $\mathrm{SL}_n(\mathbb{Z})$ invariant and Z is $\mathrm{SL}_n(\mathbb{Z})$ equivariant.

We are left to show that the only fixed point of the action of $\mathrm{SL}_n(\mathbb{Z})$ on the space of tensors is trivial. Let $m \in \mathbb{Z}$ and $1 \leq s \leq r$. If $e_1, \ldots, e_n$ is a basis of $\mathbb{R}^n$ and $j, k \in \{1, \ldots, n\}$, then, by setting $\phi^t e_j = e_j + m\,e_k$ and $\phi^t e_l = e_l$ for $l \neq j$, we obtain a map $\phi \in \mathrm{SL}_n(\mathbb{Z})$. As volume is invariant with respect to $\mathrm{SL}_n(\mathbb{Z})$ transformations, for $j \neq l_{s+1}, \ldots, l_r \in \{1, \ldots, n\}$, we have

$$
\begin{aligned}
A(e_j[s], e_{l_{s+1}}, \ldots, e_{l_r}) &= A(e_j + m\,e_k[s], e_{l_{s+1}}, \ldots, e_{l_r}) \\
&= \sum_{i=0}^{s} \binom{s}{i} m^i A(e_j[s-i], e_k[i], e_{l_{s+1}}, \ldots, e_{l_r}).
\end{aligned}
$$

Since m is arbitrary, this implies that

$$A(e_k[s], e_{l_{s+1}}, \ldots, e_{l_r}) = \cdots = A(e_j[s-1], e_k, e_{l_{s+1}}, \ldots, e_{l_r}) = 0$$

completing the proof. $\qquad\square$

Chapter 6

Characterization of Tensor Valuations

The main aim of this chapter is the characterization of tensor valuations that are translation covariant and equivariant with respect to the special linear group over the integers. This along with all preceding work towards this classification was done jointly with Monika Ludwig [43].

Note that the characterization for $n = 1$ follows from the Betke & Kneser Theorem as only translation covariance has to be considered. Therefore, we assume $n \geq 2$ and $r \geq 1$ in this chapter. We also obtain characterization results for the one-homogeneous component of the discrete moment tensor in Corollaries 65 and 66 and construct a new $\mathrm{SL}_2(\mathbb{Z})$ equivariant and translation invariant valuation $\mathrm{N} : \mathcal{P}(\mathbb{Z}^2) \to \mathbb{T}^9$ in Section 6.3. We start with a discussion of simple tensor valuations in the planar case.

6.1 Vector Valuations

For a lattice polytope $P \in \mathcal{P}(\mathbb{Z}^n)$, the discrete Steiner point $\mathrm{L}_1^1(P)$ was introduced in [16]. The valuation $P \mapsto \mathrm{L}_1^1(P)$ has a translation property that we refer to as translation equivariance. In general, $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ is called *translation equivariant* if

$$\mathrm{Z}(P + x) = \mathrm{Z}(P) + x$$

for $x \in \mathbb{Z}^n$ and $P \in \mathcal{P}(\mathbb{Z}^n)$.

Theorem 51 (Böröczky & Ludwig [16])**.** *A function* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *and translation equivariant valuation if and only if* $\mathrm{Z} = \mathrm{L}_1^1$.

This result is the key ingredient in the classification of $\mathrm{SL}_n(\mathbb{Z})$ equivariant and translation covariant vector valuations.

Theorem 52. *A function* $\mathrm{Z} \colon \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant and translation covariant valuation if and only if there are* $c_1, \ldots, c_{n+1} \in \mathbb{R}$ *such that*

$$\mathrm{Z}(P) = c_1 \, \mathrm{L}_1^1(P) + \cdots + c_{n+r} \, \mathrm{L}_{n+1}^1(P)$$

for every $P \in \mathcal{P}(\mathbb{Z}^n)$.

Proof. Since Z is translation covariant, there is $Z^0 : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}$ such that

$$Z(P + y) = Z(P) + Z^0(P)y$$

for $P \in \mathcal{P}(\mathbb{Z}^n)$ and $y \in \mathbb{Z}^n$. It follows that Z^0 is an $\mathrm{SL}_n(\mathbb{Z})$ and translation invariant valuation. By the Betke & Kneser Theorem, there are constants $c_0, \ldots, c_n \in \mathbb{R}$ such that

$$Z^0 = \sum_{i=0}^{n} c_i \, \mathrm{L}_i \, .$$

Set

$$\widetilde{Z} = Z - \sum_{i=1}^{n+1} c_{i-1} \, \mathrm{L}_i^1 \, .$$

Note that (3.1) applied to L_i^1 gives $\mathrm{L}_i^1(P + y) = \mathrm{L}_i^1(P) + \mathrm{L}_{i-1}(P)y$. Therefore, we obtain

$$
\begin{aligned}
\widetilde{Z}(P + y) \;&=\; Z(P + y) - \sum_{i=1}^{n+1} c_{i-1} \, \mathrm{L}_i^1(P + y) \\[2mm]
&=\; Z(P) + Z^0(P)y - \sum_{i=1}^{n+1} c_{i-1} (\mathrm{L}_i^1(P) + \mathrm{L}_{i-1}(P)y) \\[2mm]
&=\; Z(P) + \sum_{i=0}^{n} c_i \, \mathrm{L}_i(P)y - \sum_{i=1}^{n+1} c_{i-1} \, \mathrm{L}_i^1(P) - \sum_{i=0}^{n} c_i \, \mathrm{L}_i(P)y \\[2mm]
&=\; \widetilde{Z}(P).
\end{aligned}
$$

Hence $\widetilde{Z}$ is translation invariant and $\widetilde{Z} + \mathrm{L}_1^1$ is $\mathrm{SL}_n(\mathbb{Z})$ and translation equivariant. Thus, Theorem 51 shows that $\widetilde{Z}(P) = 0$ for all $P \in \mathcal{P}(\mathbb{Z}^n)$. □

6.2   Simple Tensor Valuations on $\mathcal{P}(\mathbb{Z}^2)$

We make the following elementary observation for valuations that vanish on the square $[0,1]^2$.

Lemma 53. *Let $r > 1$ be even and let $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathbb{T}^r$ be a simple, $\mathrm{SL}_2(\mathbb{Z})$ equivariant, and translation invariant valuation. If $Z([0,1]^2) = 0$, then $Z(T_2) = 0$.*

Proof. The square $[0,1]^2$ can be dissected into T_2 and a translate of $-T_2$. Therefore, we obtain

$$Z(T_2) + Z(-T_2) = (1 + (-1)^r) \, Z(T_2) = 0$$

which implies that $Z(T_2) = 0$. □

We also require the following result.

Lemma 54. *Let $1 < r < 8$ be odd and let $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathbb{T}^r$ be a simple, $SL_2(\mathbb{Z})$ equivariant, and translation invariant valuation. If $Z([0,1]^2) = 0$, then there exists $c \in \mathbb{R}$ such that $Z(T_2) = c\, L_1^r(T_2)$.*

Proof. Let $\varepsilon \in \{0,1\}$. Following [25], a valuation Z is called $GL_2(\mathbb{Z})$-ε-equivariant, if

$$Z(\phi P) = (\det \phi)^\varepsilon \, Z(P) \circ \phi^t$$

for all $\phi \in GL_2(\mathbb{Z})$ and $P \in \mathcal{P}(\mathbb{Z}^2)$, where det stands for determinant.

Let $\vartheta \in GL_2(\mathbb{Z})$ be the transform that swaps e_1 with e_2 and hence has $\det \vartheta = -1$. Defining, as in [25], the valuations Z^+ and Z^- for $P \in \mathcal{P}(\mathbb{Z}^2)$ by

$$
\begin{aligned}
Z^+(P) &= \tfrac{1}{2}\big(Z(P) + Z(\vartheta^{-1}P) \circ \vartheta^t\big), \\
Z^-(P) &= \tfrac{1}{2}\big(Z(P) - Z(\vartheta^{-1}P) \circ \vartheta^t\big),
\end{aligned}
$$

we see that Z^+ is $GL_2(\mathbb{Z})$-ε-equivariant with $\varepsilon = 0$ and that Z^- is $GL_2(\mathbb{Z})$-ε-equivariant with $\varepsilon = 1$. Indeed, if $\phi \in SL_2(\mathbb{Z})$ and $P \in \mathcal{P}(\mathbb{Z}^2)$, then

$$
\begin{aligned}
Z^+(\phi P) &= \tfrac{1}{2}\big(Z(\phi P) + Z((\vartheta^{-1}\phi\vartheta)\vartheta^{-1}P) \circ \vartheta^t\big) \\
&= \tfrac{1}{2}\big(Z(P) \circ \phi^t + Z(\vartheta^{-1}P) \circ \vartheta^t \phi^t\big) \\
&= Z^+(P) \circ \phi^t.
\end{aligned}
$$

If $\phi \in GL_2(\mathbb{Z})$ with $\det \phi = -1$ and $P \in \mathcal{P}(\mathbb{Z}^2)$, then

$$
\begin{aligned}
Z^+(\phi P) &= \tfrac{1}{2}\big(Z(\phi\vartheta\vartheta^{-1}P) + Z(\vartheta^{-1}\phi P) \circ \vartheta^t\big) \\
&= \tfrac{1}{2}\big(Z(\vartheta^{-1}P) \circ \vartheta^t \phi^t + Z(P) \circ \phi^t\big) \\
&= Z^+(P) \circ \phi^t.
\end{aligned}
$$

The proof for Z^- is similar. Moreover, note that $Z = Z^+ + Z^-$.

Let $r = 2s+1$ for $s \in \mathbb{N}$. We set $a_{r_1} = Z^+(T_2)(e_1[r_1], e_2[r_2])$ for $0 \le r_1 \le r$ and $r_1 + r_2 = r$. Then

$$Z^+(T_2)(e_1[r_1], e_2[r_2]) = Z^+(\vartheta T_2)(e_1[r_1], e_2[r_2]) = Z^+(T_2)(e_1[r_2], e_2[r_1])$$

or $a_{r_1} = a_{r-r_1}$. If we set $b_{r_1} = Z^-(T_2)(e_1[r_1], e_2[r_2])$ for $0 \le r_1 \le r$ and $r_1 + r_2 = r$, then

$$Z^-(T_2)(e_1[r_1], e_2[r_2]) = Z^-(\vartheta T_2)(e_1[r_1], e_2[r_2]) = -Z^-(T_2)(e_1[r_2], e_2[r_1])$$

or $b_{r_1} = -b_{r-r_1}$. Thus, in each case, we have to determine only $s+1$ coordinates of $Z^\pm(T_2)$.

Let $\phi \in SL_n(\mathbb{Z})$ be the map sending e_1 to $-e_2$ and e_2 to $e_1 - e_2$. We have $T_2 - e_2 = \phi T_2$. For $0 \le r_1 \le r$, the translation invariance of $Z^\pm$ implies

$$
\begin{aligned}
Z^\pm(T_2)(e_1[r_1], e_2[r_2]) &= Z^\pm(\phi T_2)(e_1[r_1], e_2[r_2]) \\
&= Z^\pm(T_2)(e_2[r_1], -e_1 - e_2[r_2]) \\
&= (-1)^{r_2} \sum_{i=0}^{r_2} \binom{r_2}{i} Z^\pm(T_2)(e_2[r-i], e_1[i]).
\end{aligned}
$$

First, we look at $Z^-(T_2)$. Note that $r_1 = 0$ gives us $b_0 = -b_r$ and that we have a system of $r + 1$ equations involving $b_0, \ldots, b_r$. That is, for r_1 odd, we have

$$b_0 + \binom{r_1}{1} b_1 + \cdots + \binom{r_1}{r_1 - 1} b_{r_1-1} + 2b_{r_1} = 0 \tag{6.1}$$

and, for $r_1 > 0$ even,

$$b_0 + \binom{r_1}{1} b_1 + \cdots + \binom{r_1}{r_1 - 1} b_{r_1-1} = 0. \tag{6.2}$$

It is easily checked that, for $1 < r < 8$ odd, this system of equations combined with $b_{r_1} = -b_{r-r_1}$ has rank $r + 1$. Hence $Z^-(T_2)$ vanishes and we have $Z(T_2) = Z^+(T_2)$. Yet, equations (6.1) and (6.2) remain the same for $Z^+(T_2)$ with the replacement of each b_i by a_i. It is easy to see that for $1 < r < 8$ odd, this system of equations combined with $a_{r_1} = a_{r-r_1}$ has rank r. As the tensor $L_1^r(T_2)$ is non-trivial by Lemma 24, any solution is a multiple of $L_1^r(T_2)$ concluding the proof. $\qquad\square$

We remark that the above lemma fails to hold for $r > 8$ odd. In particular, the system of equations that determine the $(r+1)$ coordinates $Z(T_2)(e_1[r_1], e_2[r_2])$ with $r_1 + r_2 = r$ has rank $(r - 1)$ for $r = 9, 11, 13$ and rank $(r - 2)$ for $r = 15, 17, 19$; there exist new tensor valuations in these cases. For $r = 9$, we describe the construction of this new valuation in the following section.

Proposition 55. *Let $2 \leq r \leq 8$. If $Z : \mathcal{P}(\mathbb{Z}^2) \to \mathbb{T}^r$ is a simple, $\mathrm{SL}_2(\mathbb{Z})$ equivariant, and translation invariant valuation, then $Z = 0$ for r even and there is $c \in \mathbb{R}$ such that $Z = c\,L_1^r$ for r odd.*

Proof. We only need to consider the statement for Z being in addition i-homogeneous by Theorem 14. If $i = 2$, then Z is trivial due to Proposition 50. If $i = 1$, then Lemma 46 implies that Z vanishes on $\mathcal{Z}_2(\mathbb{Z}^n)$ which gives $Z([0,1]^2) = 0$. By Lemma 53, we have $Z(T_2) = 0$ for r even. By Lemma 54, there is $c \in \mathbb{R}$ such that $Z(T_2) = c\,L_1^r$ for r odd. Since Z is simple, Corollary 11 implies in both cases the result. $\qquad\square$

6.3 A New Tensor Valuation on $\mathcal{P}(\mathbb{Z}^2)$

We now define a new simple, 1-homogeneous, $\mathrm{SL}_2(\mathbb{Z})$ equivariant, and translation invariant tensor valuation $N : \mathcal{P}(\mathbb{Z}^2) \to \mathbb{T}^9$. The basic step is to set $N(T_2) = L_1^3(T_2)^3$; that is, to use the threefold symmetric tensor product of $L_1^3(T_2)$. Note that $L_1^3 : \mathcal{P}(\mathbb{Z}^2) \to \mathbb{T}^3$ is simple, $\mathrm{SL}_2(\mathbb{Z})$ equivariant, translation invariant, and, by Lemma 24, non-trivial. So, for $\phi \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\begin{aligned}
L_1^3(\phi T_2)^3 &= L_1^3(\phi T_2) \odot L_1^3(\phi T_2) \odot L_1^3(\phi T_2) \\
&= L_1^3(T_2) \circ \phi^t \odot L_1^3(T_2) \circ \phi^t \odot L_1^3(T_2) \circ \phi^t \\
&= L_1^3(T_2)^3 \circ \phi^t.
\end{aligned} \tag{6.3}$$

Also note that $L_1^3([0,1]^2) = 0$ by Lemma 46.

More precisely, we set $N(P) = 0$ for $P \in \mathcal{P}(\mathbb{Z}^2)$ with $\dim(P) \leq 1$ and for a two-dimensional lattice polygon P we choose a dissection into translates of triangles $\phi_i T_2$ with $\phi_i \in \mathrm{SL}_2(\mathbb{Z})$ for $i = 1, \ldots, m$. Here P is said to be *dissected* into the triangles $S_1, \ldots, S_m$ if $P = S_1 \cup \cdots \cup S_m$ where S_i and S_j have disjoint interiors for every $i \neq j$; this is written as $P = S_1 \sqcup \cdots \sqcup S_m$. By Theorem 9, a simple valuation N then has the property that $N(S_1 \sqcup \cdots \sqcup S_m) = N(S_1) + \cdots + N(S_m)$. We set

$$N(P) = \sum_{i=1}^{m} L_1^3(\phi_i T_2)^3. \tag{6.4}$$

Note that (6.3) implies that N is $\mathrm{SL}_2(\mathbb{Z})$ equivariant.

We now show that the definition (6.4) is well-defined; does not depend on the choice of the triangulation.

Lemma 56. *Let $S_1, \ldots, S_m$ and $S_1', \ldots, S_m'$ be unimodular triangles. If*

$$S_1 \sqcup \cdots \sqcup S_m = S_1' \sqcup \cdots \sqcup S_m' \in \mathcal{P}(\mathbb{Z}^n),$$

then

$$\sum_{i=1}^{m} N(S_i) = \sum_{i=1}^{m} N(S_i').$$

Proof. By Theorem 45, there is a sequence of flips that transforms any triangulation of a given polygon P to any other triangulation of P. Therefore, it suffices to check that the value of N is not changed by any flip, as N vanishes on lower dimensional polygons. So if $S_i \sqcup S_j = S_k' \sqcup S_l'$ and $S_i \sqcup S_j$ is a translate of an $\mathrm{SL}_2(\mathbb{Z})$ image of $[0, 1]^2$, we have to show that

$$N(S_i) + N(S_j) = N(S_k') + N(S_l').$$

This is easily seen. Indeed, since $S_i \sqcup S_j$ is a translate of an $\mathrm{SL}_2(\mathbb{Z})$ image of $[0, 1]^2$, we have

$$N(S_i) + N(S_j) = N(S_i \sqcup S_j) = 0,$$

by the $\mathrm{SL}_2(\mathbb{Z})$ equivariance as $L_1^3([0, 1]^2) = 0$ and the same holds for S_k', S_l'. $\square$

Lemma 56 also shows that N is a valuation. Indeed, if $P, Q \in \mathcal{P}(\mathbb{Z}^2)$ are such that $P \cup Q \in \mathcal{P}(\mathbb{Z}^2)$ and $\mathcal{T}$ is a triangulation of $P \cup Q$, then we perform a sequence of flips on $\mathcal{T}$ until the subset of the triangulation of $P \cup Q$ that minimally covers $P \cap Q$ is fully contained in $P \cap Q$. Now, the valuation property of N follows immediately from the definition. Thus, we have shown that $N : \mathcal{P}(\mathbb{Z}^2) \to \mathbb{T}^9$ is a simple, 1-homogeneous, $\mathrm{SL}_2(\mathbb{Z})$ equivariant, and translation invariant valuation. Elementary calculations show that $L_1^3(T_2)^3 \in \mathbb{T}^9$ is non-trivial and not a multiple of $L_1^9(T_2)$.

We remark that for $r > 9$ odd, we can define new valuations in a similar way using symmetric tensors products of $L_1^{s_j}(T_2)$ for $j = 1, \ldots, m$ with $s_j > 1$ odd and $s_1 + \cdots + s_m = r$. In general, there are linear dependencies among these new valuations.

6.4   Simple Tensor Valuations on $\mathcal{P}(\mathbb{Z}^n)$

Let $n \geq 3$. For the classification of simple tensor valuations, we use the following dissection of the the 2-cylinder $T_{n-1} + [0, e_n]$ into n simplices $S_1, \ldots, S_n$. Let $e_0 = 0$. We set $S_1 = T_n$ and

$$S_i = [e_0 + e_n, \ldots, e_{i-1} + e_n, e_{i-1}, \ldots, e_{n-1}] \ \ \text{for } i = 2, \ldots, n. \tag{6.5}$$

Note that each S_i is n-dimensional and unimodular (see, for example, [29, Section 2.1]).

Let $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ be a simple, $\mathrm{SL}_n(\mathbb{Z})$ equivariant, and translation invariant valuation. Applying the dissection (6.5), we make use of the translation invariance of Z and consider $\tilde{S}_i = S_i - e_n$ for all $i > 1$. Define $\phi_i \in \mathrm{SL}_n(\mathbb{Z})$, for $i \in \{2, \ldots, n\}$, by $\phi_i e_j = e_j$ for $j < i - 1$, $\phi_i e_k = e_k - e_n$ for $i - 1 \leq k \leq n - 1$, and $\phi_i e_n = e_{i-1}$. Let ϕ_1 be the identity matrix. Then $\tilde{S}_i = \phi_i T_n$ for all $i \geq 1$ and

$$\mathrm{Z}(T_{n-1} + [0, e_n]) = \mathrm{Z}(\phi_1 T_n)(e_1[r_1], \ldots, e_n[r_n]) + \cdots + \mathrm{Z}(\phi_n T_n)(e_1[r_1], \ldots, e_n[r_n])$$
$$= \mathrm{Z}(T_n)(\phi_1^t e_1[r_1], \ldots, \phi_1^t e_n[r_n]) + \cdots + \mathrm{Z}(T_n)(\phi_n^t e_1[r_1], \ldots, \phi_n^t e_n[r_n]) \tag{6.6}$$

for any $r_1, \ldots, r_n \in \{0, \ldots, r\}$ with $r_1 + \cdots + r_n = r$. For $\mathrm{Z}(T_{n-1} + [0, e_n]) = 0$, this is a system of linear and homogeneous equations for the $\binom{n+r-1}{r}$ coordinates of the tensor $\mathrm{Z}(T_n)$. In addition, if $\psi \in \mathrm{SL}_n(\mathbb{Z})$ is an even permutation of $e_1, \ldots, e_n$, then $\psi T_n = T_n$ and we can also make use of these symmetries. We checked directly that the corresponding matrix has full rank and that, therefore, all coordinates vanish by using a computer algebra system (namely, SageMath [18]) in the following cases.

Lemma 57. *Let $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ be a simple, $\mathrm{SL}_n(\mathbb{Z})$ equivariant, and translation invariant valuation such that $\mathrm{Z}(T_{n-1} + [0, e_n]) = 0$. If $3 \leq n < r \leq 8$, then $\mathrm{Z}(T_n) = 0$.*

For $n = 3$, we also require the following variants of the above lemma. The calculations were again performed with a computer algebra system.

Lemma 58. *Let $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^3) \to \mathbb{T}^r$ be a simple, $\mathrm{SL}_3(\mathbb{Z})$ equivariant, and translation invariant valuation. If*

$$\mathrm{Z}(T_2 + [0, e_3])(e_1[r_1], e_2[r_2], e_3[r_3]) = 0$$

for r_3 odd and $r \in \{3, 5, 7\}$, then $\mathrm{Z}(T_3) = 0$.

Lemma 59. *Let $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^3) \to \mathbb{T}^r$ be a simple, $\mathrm{SL}_3(\mathbb{Z})$ equivariant, and translation invariant valuation. If*

$$\mathrm{Z}(T_2 + [0, e_3])(e_1[r_1], e_2[r_2], e_3[r_3]) = 0$$

for r_3 even and $r \in \{2, 4, 6, 8\}$, then $\mathrm{Z}(T_3) = 0$.

For more information, the code can be found in the appendix.

The dissection (6.5) is also used in the proof of the following result.

Lemma 60. *Let* $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *be a simple,* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant, and translation invariant valuation such that* $Z(T_{n-1} + [0, e_n]) = 0$. *If* $n \geq 2$ *and* $n \geq r$, *then* $Z(T_n) = 0$.

Proof. As in (6.6), we have

$$0 = Z(T_{n-1} + [0, e_n]) = Z(\phi_1 T_n) + \cdots + Z(\phi_n T_n),$$

as Z is a simple, translation invariant valuation that vanishes on $T_{n-1} + [0, e_n]$. Thus, for any coordinate of $Z(T_{n-1} + [0, e_n])$ where $r_1, \ldots, r_{n-1} \in \{0, \ldots, r\}$, we have the equations

$$
\begin{aligned}
0 &= Z(T_{n-1} + [0, e_n])(e_1[r_1], \ldots, e_{n-1}[r_{n-1}]) \\
&= Z(T_n)(e_1[r_1], \ldots, e_{n-1}[r_{n-1}]) + Z(\phi_2 T_n)(e_1[r_1], \ldots, e_{n-1}[r_{n-1}]) \\
&\qquad\qquad + \cdots + Z(\phi_n T_n)(e_1[r_1], \ldots, e_{n-1}[r_{n-1}]) \\
&= Z(T_n)(e_1[r_1], \ldots, e_{n-1}[r_{n-1}]) + Z(T_n)(\phi_2^t e_1[r_1], \ldots, \phi_2^t e_{n-1}[r_{n-1}]) \qquad (6.7) \\
&\qquad\qquad + \cdots + Z(T_n)(\phi_n^t e_1[r_1], \ldots, \phi_n^t e_{n-1}[r_{n-1}]) \\
&= Z(T_n)(e_1[r_1], \ldots, e_{n-1}[r_{n-1}]) + Z(T_n)(e_1 + e_n[r_1], e_2[r_2], \ldots, e_{n-1}[r_{n-1}]) \\
&\qquad\qquad + \cdots + Z(T_n)(e_1[r_1], \ldots, e_{n-2}[r_{n-2}], e_{n-1} + e_n[r_{n-1}]).
\end{aligned}
$$

For $r_1, \ldots, r_n \in \mathbb{N}$ such that $r_1 + \cdots + r_n = r$, the corresponding coordinate of $Z(T_n)$ is $Z(T_n)(e_1[r_1], \ldots, e_n[r_n])$. As T_n is invariant under permutations, the permutations of the r_j's are irrelevant. Without loss of generality, we may then assume $r_1 \geq r_2 \geq \cdots \geq r_n$ and drop r_j when $r_j = 0$ from our notation. Set $a_{r_1, \ldots, r_m} = Z(T_n)(e_{i_1}[r_1], \ldots, e_{i_m}[r_m])$ where $r_1 + \cdots + r_m = r$ and each $r_j \geq 1$.

We define a total order $\preceq$ on the coordinates $a_{r_1, \ldots, r_m}$ by saying that $a_{r_1, \ldots, r_m} \preceq a_{s_1, \ldots, s_{m'}}$ if $r_1 < s_1$ or if $r_1 = s_1, \ldots, r_{j-1} = s_{j-1}$ and $r_j < s_j$. Therefore, the coordinates are ordered in the following way from the biggest to smallest:

$$a_r, a_{r-1,1}, a_{r-2,2}, \ldots, a_{\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor}, a_{r-2,1,1}, \ldots, a_{\lceil \frac{r-1}{2} \rceil, \lfloor \frac{r-1}{2} \rfloor, 1}, \ldots, a_{2,1,\ldots,1}, a_{1,\ldots,1},$$

where $\lfloor x \rfloor$ is the largest integer less or equal to x and $\lceil x \rceil$ is the smallest integer greater or equal to x.

We claim that the equations (6.7) imply that the coordinates of $Z(T_n)$ that involve at most $(n-1)$ of $e_1, \ldots, e_n$ all vanish. One can see this by noticing that, for given $r_1, \ldots, r_m$ with $m < n$, the linear equation (6.7) only involves $a_{r_1, \ldots, r_m}$ and coordinates that are smaller than this coordinate in the ordering defined above. Thus, for $r < n$, we have an equation for each coordinate and the system of equations can be regarded as an upper-triangular matrix that, therefore, has full-rank. Thus, each coordinate vanishes implying that $Z(T_n) = 0$ for $r < n$.

Additionally, for $n = r$, we have

$$
\begin{aligned}
0 &= Z(T_{n-1} + [0, e_n])(e_1, e_2, \ldots, e_n) \\
&= Z(T_n)(e_1, \ldots, e_n) + Z(T_n)(e_1 + e_n, e_2, \ldots, e_{n-1}, -(e_1 + \cdots + e_{n-1})) \\
&\quad + Z(T_n)(e_1, e_2 + e_n, \ldots, e_{n-1}, -(e_2 + \cdots + e_{n-1})) \\
&\quad + Z(T_n)(e_1, e_2, e_3 + e_n, \ldots, e_{n-1}, -(e_3 + \cdots + e_{n-1})) + \cdots \\
&\quad + Z(T_n)(e_1, \ldots, e_{n-2}, e_{n-1} + e_n, -e_{n-1}) \\
&= -(n-2)a_{1,\ldots,1} - (n-1)^2 a_{2,1,\ldots,1}.
\end{aligned}
$$

Thus $a_{1,\ldots,1} = 0$, as $a_{2,1,\ldots,1} = 0$ by the first step. As the first step also shows that all further coordinates vanish, this completes the proof. $\square$

We now establish the three-dimensional case first and then, using this result, the general case.

Lemma 61. *Let* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^3) \to \mathbb{T}^r$ *be a simple,* $\mathrm{SL}_3(\mathbb{Z})$ *equivariant, and translation invariant valuation. If* $2 \leq r \leq 8$*, then* $\mathrm{Z}(T_3) = 0$*.*

Proof. We only need to consider the statement for Z being in addition i-homogeneous by Theorem 14. If Z is 3-homogeneous, then it is trivial due to Proposition 50. Lemma 46 implies that $\mathrm{Z}(T_2 + [0, e_3]) = 0$ if $i = 1$. Hence, Lemma 57 and Lemma 60 imply that $\mathrm{Z}(T_3) = 0$ for $i = 1$. Therefore, let Z be 2-homogeneous.

Since Z is simple and 2-homogeneous, Theorem 17 implies that $\mathrm{Z}(Q) = -\mathrm{Z}(-Q)$, that is, Z is *odd*. Using that Z is odd, translation invariant, and $\mathrm{SL}_3(\mathbb{Z})$ equivariant, we obtain

$$
\begin{aligned}
\mathrm{Z}(T_2 + [0, e_3])(e_1[r_1], e_2[r_2], e_3[r_3]) &= -\mathrm{Z}(-T_2 - [0, e_3])(e_1[r_1], e_2[r_2], e_3[r_3]) \\
&= -\mathrm{Z}(-T_2 + [0, e_3])(e_1[r_1], e_2[r_2], e_3[r_3]) \qquad (6.8) \\
&= (-1)^{r_1 + r_2 + 1}\, \mathrm{Z}(T_2 + [0, e_3])(e_1[r_1], e_2[r_2], e_3[r_3]).
\end{aligned}
$$

First, let r be odd. Then (6.8) implies that for r_3 odd,

$$
\mathrm{Z}(T_2 + [0, e_3])(e_1[r_1], e_2[r_2], e_3[r_3]) = 0.
$$

We can therefore apply Lemma 58 and obtain that $\mathrm{Z}(T_3) = 0$. This implies the statement of the lemma for r odd.

Second, let r be even. Then (6.8) implies that for r_3 even,

$$
\mathrm{Z}(T_2 + [0, e_3])(e_1[r_1], e_2[r_2], e_3[r_3]) = 0.
$$

Applying Lemma 59 gives $\mathrm{Z}(T_3) = 0$. This completes the proof of the lemma. $\square$

Proposition 62. *Let* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *be a simple,* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant, and translation invariant valuation. If* $n \geq 3$ *and* $2 \leq r \leq 8$*, then* $\mathrm{Z} = 0$*.*

Proof. For $n = 3$, we have $\mathrm{Z}(T_3) = 0$ by Lemma 61 and the result follows from Corollary 11.

Let $n > 3$ and suppose that the statement is true in dimension $n - 1$. Let $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ be a simple, $\mathrm{SL}_n(\mathbb{Z})$ equivariant, and translation invariant valuation for $2 \leq r \leq 8$. We only need to consider the statement for Z being in addition i-homogeneous by Theorem 14. If Z is n-homogeneous, then it is trivial due to Proposition 50. So, let $1 \leq i \leq n - 1$.

Define $\mathrm{Y} : \mathcal{P}(\mathbb{Z}^{n-1}) \to \mathbb{T}^s(\mathbb{R}^{n-1})$ by setting for $P \in \mathcal{P}(\mathbb{Z}^{n-1})$

$$
\mathrm{Y}(P)(e_1[r_1], \ldots, e_{n-1}[r_{n-1}]) = \mathrm{Z}(P + [0, e_n])(e_1[r_1], \ldots, e_{n-1}[r_{n-1}], e_n[r_n])
$$

where $r_1 + \cdots + r_{n-1} = s$ and $r_1 + \cdots + r_n = r$. Then Y is a simple, $\mathrm{SL}_{n-1}(\mathbb{Z})$ equivariant, and translation invariant valuation. Furthermore, Y is $(i-1)$-homogeneous as

$$
k^i\, \mathrm{Y}(P) = k^i\, \mathrm{Z}(P + [0, e_n]) = \mathrm{Z}(k\,P + k\,[0, e_n])) = k\, \mathrm{Z}(kP + [0, e_n]) = k\, \mathrm{Y}(kP)
$$

by the simplicity and translation invariance of Z.

For $2 \leq s \leq 8$, the induction assumption implies that Y $= 0$. If $s = 0$, then Y is real-valued and $\mathrm{SL}_{n-1}(\mathbb{Z})$ and translation invariant. Since it is simple, the Betke & Kneser Theorem implies that it is a multiple of the $(n-1)$-dimensional volume as, by Lemma 21, the only simple Ehrhart coefficient is volume. Hence, Y is also $(n-1)$-homogeneous and must vanish. If $s = 1$, then Y is vector-valued and $\mathrm{SL}_{n-1}(\mathbb{Z})$ equivariant and translation invariant. Since it is simple, Theorem 52 implies that it is a multiple of the moment vector, as by Lemma 21 the only simple Ehrhart tensor of rank one is the moment tensor. Thus Y is also n-homogeneous, which implies that it vanishes.

In particular, we obtain $\mathrm{Z}(T_{n-1} + [0, e_n]) = 0$. Hence, Lemma 57 and Lemma 60 imply that $\mathrm{Z}(T_n) = 0$. The result now follows from Corollary 11. $\qquad\square$

6.5 General Tensor Valuations

Let $r \geq 2$. The translation property (3.1) combined with (3.2) gives

$$\mathrm{L}_1^r(P + x) = \mathrm{L}_1^r(P) + \mathrm{L}_0^{r-1}(P)x = \mathrm{L}_1^r(P),$$

that is, L_1^r is translation invariant. We show that every $\mathrm{SL}_n(\mathbb{Z})$ equivariant and translation invariant valuation is a multiple of L_1^r for $2 \leq r \leq 8$. We start with the case of 1-homogeneous valuations.

Proposition 63. *Let* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *be an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant and translation invariant valuation. If* Z *is 1-homogeneous and* $2 \leq r \leq 8$, *then there exists* $c \in \mathbb{R}$ *such that* $\mathrm{Z} = c\,\mathrm{L}_1^r$.

Proof. We use induction on the dimension n. The case $n = 1$ is elementary (and also follows from the Betke & Kneser Theorem) and states that, for a 1-homogeneous and translation invariant valuation $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^1) \to \mathbb{T}^r(\mathbb{R})$, we have $\mathrm{Z} = c\,\mathrm{L}_1^r$ for some $c \in \mathbb{R}$.

Assume the statement holds for $n-1$. Restrict Z to lattice polytopes with vertices in $\mathbb{Z}^{n-1}$. By Lemma 6, we may view this restricted valuation as a function $\mathrm{Z}' : \mathcal{P}(\mathbb{Z}^{n-1}) \to \mathbb{T}^r(\mathbb{R}^{n-1})$. Since Z' is an $\mathrm{SL}_{n-1}(\mathbb{Z})$ equivariant and translation invariant valuation on $\mathcal{P}(\mathbb{Z}^{n-1})$, by the induction hypothesis, there is $c \in \mathbb{R}$ such that $\mathrm{Z}'(P) = c\,\mathrm{L}_1^r(P)$ for $P \in \mathcal{P}(\mathbb{Z}^{n-1})$. By Lemma 6, for $1 \leq r_n \leq r$,

$$\mathrm{Z}(P)(e_{i_1}, \ldots, e_{i_{r-r_n}}, e_n[r_n]) = c\,\mathrm{L}_1^r(P)(e_{i_1}, \ldots, e_{i_{r-r_n}}, e_n[r_n]) = 0$$

where $P \in \mathcal{P}(\mathbb{Z}^{n-1})$ and $i_1, \ldots, i_{r-r_n} \in \{1, \ldots, n-1\}$. Hence $\mathrm{Z}(P) = c\,\mathrm{L}_1^r(P)$ for $P \in \mathcal{P}(\mathbb{Z}^{n-1})$. Set

$$\widetilde{\mathrm{Z}} = \mathrm{Z} - c\,\mathrm{L}_1^r.$$

Note that $\widetilde{\mathrm{Z}}$ vanishes on $\mathcal{P}(\mathbb{Z}^{n-1})$. By the $\mathrm{SL}_n(\mathbb{Z})$ equivariance and translation invariance of $\widetilde{\mathrm{Z}}$, this implies that $\widetilde{\mathrm{Z}}$ vanishes on lattice polytopes in any $(n-1)$-dimensional lattice hyperplane $H \subset \mathbb{R}^n$ as we have $H \cap \mathbb{Z}^n = \phi\,\mathbb{Z}^{n-1} + x$ for some $\phi \in \mathrm{SL}_n(\mathbb{Z})$ and $x \in \mathbb{Z}^n$. In other words, $\widetilde{\mathrm{Z}}$ is simple and the statement follows from Propositions 55 and 62. $\qquad\square$

We can now extend Proposition 50 to i-homogeneous valuations with $2 \le i \le n$.

Proposition 64. *Let* $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *be an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant and translation invariant valuation. If* Z *is* i-homogeneous with $2 \le i \le n$ and $2 \le r \le 8$, then $Z = 0$.

Proof. Lemma 47 implies that the valuation Z vanishes on all lattice polytopes of dimension $m < i$. We use induction on m and show that Z also vanishes on all m-dimensional lattice polytopes for $i \le m \le n$.

First, let $m = i$. Restrict Z to lattice polytopes in $\mathcal{P}(\mathbb{Z}^i)$. By Lemma 6, we may view this restricted valuation as a function $Z' : \mathcal{P}(\mathbb{Z}^i) \to \mathbb{T}^r(\mathbb{R}^i)$. Since Z' is invariant under translations of $\mathcal{P}(\mathbb{Z}^i)$ into itself and $\mathrm{SL}_i(\mathbb{Z})$ equivariant, Proposition 50 implies that Z' vanishes on $\mathcal{P}(\mathbb{Z}^i)$. Thus, by Lemma 6, we obtain that also Z vanishes on lattice polytopes with vertices in $\mathbb{Z}^i$. Now, let $P \in \mathcal{P}(\mathbb{Z}^n)$ be a general i-dimensional lattice polytope. Let H be the i-dimensional subspace of $\mathbb{R}^n$ that is parallel to the affine hull of P. There exists $x \in \mathbb{Z}^n$ such that $P + x \in H$ and there exists $\phi \in \mathrm{SL}_n(\mathbb{Z})$ such that $\phi(H \cap \mathbb{Z}^n) = \mathbb{Z}^i$. Since Z vanishes on $\mathcal{P}(\mathbb{Z}^i)$ and is $\mathrm{SL}_n(\mathbb{Z})$ equivariant and translation invariant, we obtain that $Z(P) = 0$.

Next, for $m > i$, suppose that $Z(Q) = 0$ for all $Q \in \mathcal{P}(\mathbb{Z}^n)$ with $\dim(Q) < m$. By Lemma 6, we may view the restriction of Z to lattice polytopes in $\mathcal{P}(\mathbb{Z}^m)$ as a function $Z' : \mathcal{P}(\mathbb{Z}^m) \to \mathbb{T}^r(\mathbb{R}^m)$. Since Z' is a simple, $\mathrm{SL}_m(\mathbb{Z})$ equivariant and translation invariant valuation and $m \ge 3$, Proposition 62 implies that Z' vanishes on $\mathcal{P}(\mathbb{Z}^m)$. As in the previous step, this implies that $Z(P) = 0$ for any m-dimensional lattice polytope in $\mathcal{P}(\mathbb{Z}^n)$ and, by induction, we have $Z = 0$. $\qquad\square$

The characterization of L_1^r follows immediately from the combination of Theorem 14, Proposition 63, and Proposition 64.

Corollary 65. *For* $2 \le r \le 8$, *a function* $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant and translation invariant valuation if and only if there exists* $c \in \mathbb{R}$ *such that* $Z = c\,\mathrm{L}_1^r$.

Together with Proposition 12, we obtain the following consequence of Corollary 65.

Corollary 66. *For* $2 \le r \le 8$, *a function* $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant, translation invariant, and Minkowski additive if and only if there exists* $c \in \mathbb{R}$ *such that* $Z = c\,\mathrm{L}_1^r$.

6.6   Characterization

The main characterization is now obtained by an inductive proof on the rank r.

Theorem 67. *For* $2 \le r \le 8$, *a function* $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant and translation covariant valuation if and only if there are* $c_1, \ldots, c_{n+r} \in \mathbb{R}$ *such that*

$$Z(P) = c_1\,\mathrm{L}_1^r(P) + \cdots + c_{n+r}\,\mathrm{L}_{n+r}^r(P)$$

for every $P \in \mathcal{P}(\mathbb{Z}^n)$.

Proof. Recall that the Betke & Kneser Theorem gives the characterization for the case $r = 0$ and Theorem 52 for $r = 1$. The induction assumption gives that

$$Z^{r-1} = \sum_{i=1}^{n+r-1} c_i \, L_i^{r-1}$$

for some constants $c_1, \ldots, c_{n+r-1} \in \mathbb{R}$. Furthermore, for any $y \in \mathbb{Z}^n$, this characterization together with Proposition 7 applied to Z^{r-1} and to L_i^{r-1} yields

$$
\begin{aligned}
Z^{r-1}(P+y) &= Z^{r-1}(P) + Z^{r-2}(P)\frac{y}{1!} + \cdots + Z^0(P)\frac{y^{r-1}}{(r-1)!} \\
&= \sum_{i=1}^{n+r-1} c_i \, L_i^{r-1}(P+y) \\
&= \sum_{i=1}^{n+r-1} c_i \left(L_i^{r-1}(P) + L_{i-1}^{r-2}(P)\frac{y}{1!} + \cdots + L_{i-r+1}^0(P)\frac{y^{r-1}}{(r-1)!} \right).
\end{aligned}
$$

A comparison of the coefficients of the polynomial expansion in y gives

$$Z^{r-j}(P) = \sum_{i=1}^{n+r-1} c_i \, L_{i-j+1}^{r-j}(P).$$

Consider the $\mathrm{SL}_n(\mathbb{Z})$ equivariant valuation

$$\widetilde{Z} = Z - \sum_{i=2}^{n+r} c_{i-1} \, L_i^r .$$

For $y \in \mathbb{Z}^n$, by Proposition 7 and the induction assumption, we obtain

$$
\begin{aligned}
\widetilde{Z}(P+y) &= Z(P+y) - \sum_{i=2}^{n+r} c_{i-1} L_i^r(P+y) \\
&= Z(P) + \sum_{j=1}^{r} Z^{r-j}(P)\frac{y^j}{j!} - \sum_{i=2}^{n+r}\sum_{j=0}^{r} c_{i-1} L_{i-j}^{r-j}(P)\frac{y^j}{j!} \\
&= Z(P) + \sum_{j=1}^{r}\sum_{i=1}^{n+r-1} c_i L_{i-j+1}^{r-j}(P)\frac{y^j}{j!} - \sum_{i=2}^{n+r} c_{i-1} L_i^r(P) - \sum_{i=2}^{n+r}\sum_{j=1}^{r} c_{i-1} L_{i-j}^{r-j}(P)\frac{y^j}{j!} \\
&= Z(P) - \sum_{i=2}^{n+r} c_{i-1} L_i^r(P) \\
&= \widetilde{Z}(P).
\end{aligned}
$$

Consequently, the function $\widetilde{Z}$ is translation invariant and Corollary 65 implies that $\widetilde{Z} = c_1 \, L_1^r$ proving the theorem.

$\square$

Appendix: SageMath code

The characterization of tensor valuations on lattice polytopes in Theorem 67 could not be done without finding the rank of several large matrices. We use SageMath to confirm that these matrices are regular. Given any simple, $\mathrm{SL}_n(\mathbb{Z})$ equivariant, and translation invariant valuation $\mathrm{Z}: \mathcal{P}(\mathbb{Z}^n) \to \mathbb{T}^r$, we consider the coordinates

$$\mathrm{Z}(T_n)(e_1[r1], e_2[r2], \ldots, e_n[rn])$$

for the given dimension $n \in \mathbb{N}$ and some $r1, \ldots, rn \in \mathbb{N}$ such that $r1 + \cdots + rn = r$. Lemma 57 is proven for dimension 3 with the following code.

#Dimension 3.

```
def coord_val3(r1,r2,r3,r):
    x = − r3 + sum(r+1−r1 for i in [0..r1])
    return x

def vars3(i,r):
    count = 0
    for r1 in [0..r]:
        for r2 in [0..r−r1]:
            r3 = r−r1−r2
            count = count + 1
            if count == i:
                return r1,r2,r3

def mat3d(r):
    n = 3
    count = 0
    num_vars = binomial(r+n−1,n−1)
    A = matrix(ZZ,2*num_vars,num_vars)
    for coord in [1..num_vars]:
        r1,r2,r3 = vars3(coord,r)
        perm = coord_val3(r2,r3,r1,r)
        A[coord+num_vars−1,coord−1] = 1
        A[coord+num_vars−1,perm−1] = A[coord+num_vars−1,perm−1] − 1
```

```
        A[ coord −1,coord −1] = 1
        for a1 in [0..r1 ]:
            for a2 in [0..r3 ]:
                cell_num = coord_val3 ( a1+a2 , r2+r3−a2 , r1−a1 , r )
                A[ coord −1,cell_num −1] = A[ coord −1,cell_num −1] +
                    (−1)^r3 ∗ binomial ( r1 , a1 ) ∗ binomial ( r3 , a2 )
        for b in [0..r2 ]:
            cell_num = coord_val3 ( r1 , r3+b , r2−b , r )
            A[ coord −1,cell_num −1] = A[ coord −1,cell_num −1] + (−1)^r3
                ∗ binomial ( r2 , b )
    return A
```

The following two functions are used to prove Lemmas 58 and 59, respectively, and call from the first two functions (namely, coord_val3 and vars3) in the preceeding code. The following functions only differ in their 'is_odd' or 'is_even" command.

```
def mat3o( r ) :
    n = 3
    count = 0
    num_vars = binomial ( r+n−1,n−1)
    A = matrix (ZZ,2∗num_vars , num_vars )
    for coord in [1.. num_vars ]:
        r1 , r2 , r3 = vars3 ( coord , r )
        perm = coord_val3 ( r2 , r3 , r1 , r )
        A[ coord+num_vars −1,coord −1] = 1
        A[ coord+num_vars −1,perm −1] = A[ coord+num_vars −1,perm −1] − 1
        A[ coord −1,coord −1] = 1
        if is_odd ( r3 ) :
            for a1 in [0.. r1 ]:
                for a2 in [0.. r3 ]:
                    cell_num = coord_val3 ( a1+a2 , r2+r3−a2 , r1−a1 , r )
                    A[ coord −1,cell_num −1] = A[ coord −1,cell_num −1] +
                        (−1)^r3 ∗ binomial ( r1 , a1 ) ∗ binomial ( r3 , a2 )
            for b in [0.. r2 ]:
                cell_num = coord_val3 ( r1 , r3+b , r2−b , r )
                A[ coord −1,cell_num −1] = A[ coord −1,cell_num −1] +
                    (−1)^r3 ∗ binomial ( r2 , b )
    return A

def mat3e( r ) :
    n = 3
    count = 0
    num_vars = binomial ( r+n−1,n−1)
    A = matrix (ZZ,2∗num_vars , num_vars )
    for coord in [1.. num_vars ]:
```

```
        r1 ,r2 ,r3 = vars3 (coord ,r )
        perm = coord_val3 (r2 ,r3 ,r1 ,r )
        A[coord+num_vars −1,coord −1] = 1
        A[coord+num_vars −1,perm −1] = A[coord+num_vars −1,perm −1] − 1
        A[coord −1,coord −1] = 1
        if is_odd (r3 ):
            for a1 in [0.. r1 ]:
                for a2 in [0.. r3 ]:
                    cell_num = coord_val3 (a1+a2 ,r2+r3 −a2 ,r1 −a1 ,r )
                    A[coord −1,cell_num −1] = A[coord −1,cell_num −1] +
                        (−1)^r3∗binomial (r1 ,a1 )∗binomial (r3 ,a2 )
            for b in [0.. r2 ]:
                cell_num = coord_val3 (r1 ,r3+b ,r2 −b ,r )
                A[coord −1,cell_num −1] = A[coord −1,cell_num −1] +
                    (−1)^r3∗binomial (r2 ,b )
    return A
```

The code for Lemma 57 is extended for dimensions $n = 4, \ldots, 7$.

```
# Dimension 4.

def coord_val4 (r1 ,r2 ,r3 ,r4 ,r ):
    x = −r4 + sum(sum(r+1−i1−i2 for i2 in [0.. r−i1 ]) for i1 in [0..
        r1 −1]) + sum(r+1−i−r1 for i in [0.. r2 ])
    return x

def vars4 (i ,r ):
    count = 0
    for r1 in [0.. r ]:
        for r2 in [0.. r−r1 ]:
            for r3 in [0.. r−r1−r2 ]:
                r4 = r−r1−r2−r3
                count = count + 1
                if count == i :
                    return r1 ,r2 ,r3 ,r4

def mat4d (r ):
    n = 4
    count = 0
    num_vars = binomial (r+n−1,n−1)
    A = matrix (ZZ,2∗num_vars ,num_vars )
    for coord in [1.. num_vars ]:
        r1 ,r2 ,r3 ,r4 = vars4 (coord ,r )
        cell_num = coord_val4 (r1 ,r2 ,r3 ,r4 ,r )
        perm = coord_val4 (r2 ,r3 ,r4 ,r1 ,r )
```

```
            A[ coord+num_vars −1,coord −1] = 1
            A[ coord+num_vars −1,perm−1] = A[ coord+num_vars −1,perm−1] − 1
            A[ coord −1,coord −1] = 1
            for a1 in [0.. r1 ]:
                for a2 in [0.. r4 ]:
                    for a3 in [0.. r4−a2 ]:
                        cell_num = coord_val4 ( a1+a2 , r2+a3 , r3+r4−a2−a3 ,
                            r1−a1 , r )
                        A[ coord −1,cell_num −1] = A[ coord −1,cell_num −1] +
                            (−1)^r4∗binomial ( r1 , a1 )∗binomial ( r4 , a2 )∗
                            binomial ( r4−a2 , a3 )
                for b1 in [0.. r2 ]:
                    for b2 in [0.. r4 ]:
                        cell_num = coord_val4 ( r1 , b1+b2 , r3+r4−b2 , r2−b1 , r )
                        A[ coord −1,cell_num −1] = A[ coord −1,cell_num −1] +
                            (−1)^r4∗binomial ( r2 , b1 )∗binomial ( r4 , b2 )
                for c in [0.. r3 ]:
                    cell_num = coord_val4 ( r1 , r2 , r4+c , r3−c , r )
                    A[ coord −1,cell_num −1] = A[ coord −1,cell_num −1] + (−1)^r4
                        ∗binomial ( r3 , c )
        return A

# Dimension 5.

def coord_val5 ( r1 , r2 , r3 , r4 , r5 , r ) :
    x = −r5 + sum(sum(sum( r+1−i1 −i2 −i3 for i3 in [0.. r+1−i1 −i2 ])
        for i2 in [0.. r+1−i1 ]) for i1 in [0.. r1 −1]) + sum(sum( r+1−r1
        −i1 −i2 for i2 in [0.. r+1−r1 −i1 ]) for i1 in [0.. r2 −1]) + sum(
        r+1−r1 −r2−i for i in [0.. r3 ])
    return x

def vars5 ( i , r ) :
    count = 0
    for r1 in [0.. r ]:
        for r2 in [0.. r−r1 ]:
            for r3 in [0.. r−r1−r2 ]:
                for r4 in [0.. r−r1−r2−r3 ]:
                    r5 = r−r1−r2−r3−r4
                    count = count + 1
                    if count == i :
                        return r1 , r2 , r3 , r4 , r5

def mat5d ( r ) :
    n = 5
```

```
    count = 0
    num_vars = binomial(r+n-1,n-1)
    A = matrix(ZZ,2*num_vars,num_vars)
    for coord in [1..num_vars]:
        r1,r2,r3,r4,r5 = vars5(coord,r)
        cell_num = coord_val5(r1,r2,r3,r4,r5,r)
        perm = coord_val5(r2,r3,r4,r5,r1,r)
        A[coord+num_vars-1,coord-1] = 1
        A[coord+num_vars-1,perm-1] = A[coord+num_vars-1,perm-1] - 1
        A[coord-1,coord-1] = 1
        for a1 in [0..r1]:
            for a2 in [0..r5]:
                for a3 in [0..r5-a2]:
                    for a4 in [0..r5-a2-a3]:
                        cell_num = coord_val5(a1+a2,r2+a3,r3+a4,r4+
                            r5-a2-a3-a4,r1-a1,r)
                        A[coord-1,cell_num-1] = A[coord-1,cell_num
                            -1] + (-1)^r5*binomial(r1,a1)*binomial(
                            r5,a2)*binomial(r5-a2,a3)*binomial(r5-a2
                            -a3,a4)
        for b1 in [0..r2]:
            for b2 in [0..r5]:
                for b3 in [0..r5-b2]:
                    cell_num = coord_val5(r1,b1+b2,r3+b3,r4+r5-b2-
                        b3,r2-b1,r)
                    A[coord-1,cell_num-1] = A[coord-1,cell_num-1] +
                        (-1)^r5*binomial(r2,b1)*binomial(r5,b2)*
                        binomial(r5-b2,b3)
        for c1 in [0..r3]:
            for c2 in [0..r5]:
                cell_num = coord_val5(r1,r2,c1+c2,r4+r5-c2,r3-c1,r)
                A[coord-1,cell_num-1] = A[coord-1,cell_num-1] +
                    (-1)^r5*binomial(r3,c1)*binomial(r5,c2)
        for d1 in [0..r4]:
            cell_num = coord_val5(r1,r2,r3,r5+d1,r4-d1,r)
            A[coord-1,cell_num-1] = A[coord-1,cell_num-1] + (-1)^r5
                *binomial(r4,d1)
    return A


# Dimension 6.

def coord_val6(r1,r2,r3,r4,r5,r6,r):
    x = -r6 + sum(sum(sum(sum(r+1-i1-i2-i3-i4 for i4 in [0..r+1-i1-
        i2-i3])for i3 in [0..r+1-i1-i2]) for i2 in [0..r+1-i1]) for
```

```
            i1 in [0..r1-1]) + sum(sum(sum(r+1-r1-i1-i2-i3 for i3 in
            [0..r+1-r1-i1-i2]) for i2 in [0..r+1-r1-i1]) for i1 in [0..
            r2-1]) + sum(sum(r+1-r1-r2-i1-i2 for i2 in [0..r+1-r1-r2-i1
            ]) for i1 in [0..r3-1]) + sum(r+1-r1-r2-r3-i1 for i1 in [0..
            r4]))
    return x


def vars6(i,r):
    count = 0
    for r1 in [0..r]:
        for r2 in [0..r-r1]:
            for r3 in [0..r-r1-r2]:
                for r4 in [0..r-r1-r2-r3]:
                    for r5 in [0..r-r1-r2-r3-r4]:
                        r6 = r-r1-r2-r3-r4-r5
                        count = count + 1
                        if count == i:
                            return r1,r2,r3,r4,r5,r6


def mat6d(r):
    n = 6
    count = 0
    num_vars = binomial(r+n-1,n-1)
    A = matrix(ZZ,2*num_vars,num_vars)
    for coord in [1..num_vars]:
        r1,r2,r3,r4,r5,r6 = vars6(coord,r)
        cell_num = coord_val6(r1,r2,r3,r4,r5,r6,r)
        perm = coord_val6(r2,r3,r4,r5,r6,r1,r)
        A[coord+num_vars-1,coord-1] = 1
        A[coord+num_vars-1,perm-1] = A[coord+num_vars-1,perm-1] - 1
        A[coord-1,coord-1] = 1
        for a1 in [0..r1]:
            for a2 in [0..r6]:
                for a3 in [0..r6-a2]:
                    for a4 in [0..r6-a2-a3]:
                        for a5 in [0..r6-a2-a3-a4]:
                            cell_num = coord_val6(a1+a2,r2+a3,r3+a4
                                ,r4+a5,r5+r6-a2-a3-a4-a5,r1-a1,r)
                            A[coord-1,cell_num-1] = A[coord-1,
                                cell_num-1] + (-1)^r6*binomial(r1,a1
                                )*binomial(r6,a2)*binomial(r6-a2,a3)
                                *binomial(r6-a2-a3,a4)*binomial(r6-
                                a2-a3-a4,a5)
        for b1 in [0..r2]:
```

```
                    for  b2  in  [0..r6]:
                        for  b3  in  [0..r6−b2]:
                            for  b4  in  [0..r6−b2−b3]:
                                cell_num  =  coord_val6(r1,b1+b2,r3+b3,r4+b4,
                                    r5+r6−b2−b3−b4,r2−b1,r)
                                A[coord−1,cell_num−1] = A[coord−1,cell_num
                                    −1] + (−1)^r6∗binomial(r2,b1)∗binomial(
                                    r6,b2)∗binomial(r6−b2,b3)∗binomial(r6−b2
                                    −b3,b4)
                for  c1  in  [0..r3]:
                    for  c2  in  [0..r6]:
                        for  c3  in  [0..r6−c2]:
                            cell_num  =  coord_val6(r1,r2,c1+c2,r4+c3,r5+r6−
                                c2−c3,r3−c1,r)
                            A[coord−1,cell_num−1] = A[coord−1,cell_num−1] +
                                (−1)^r6∗binomial(r3,c1)∗binomial(r6,c2)∗
                                binomial(r6−c2,c3)
                for  d1  in  [0..r4]:
                    for  d2  in  [0..r6]:
                        cell_num  =  coord_val6(r1,r2,r3,d1+d2,r5+r6−d2,r4−d1
                            ,r)
                        A[coord−1,cell_num−1] = A[coord−1,cell_num−1] +
                            (−1)^r6∗binomial(r4,d1)∗binomial(r6,d2)
                for  e1  in  [0..r5]:
                    cell_num  =  coord_val6(r1,r2,r3,r4,r6+e1,r5−e1,r)
                    A[coord−1,cell_num−1] = A[coord−1,cell_num−1] + (−1)^r6
                        ∗binomial(r5,e1)
        return A


# Dimension  7.

def  coord_val7(r1,r2,r3,r4,r5,r6,r7,r):
    x = −r7 + sum(sum(sum(sum(sum(r+1−i1−i2−i3−i4−i5 for i5 in [0..
        r+1−i1−i2−i3−i4]) for i4 in [0..r+1−i1−i2−i3]) for i3 in [0..
        r+1−i1−i2]) for i2 in [0..r+1−i1]) for i1 in [0..r1−1]) +
        sum(sum(sum(sum(r+1−r1−i1−i2−i3−i4 for i4 in [0..r+1−r1−i1−
        i2−i3]) for i3 in [0..r+1−r1−i1−i2]) for i2 in [0..r+1−r1−i1
        ]) for i1 in [0..r2−1]) + sum(sum(sum(r+1−r1−r2−i1−i2−i3 for
         i3 in [0..r+1−r1−r2−i1−i2]) for i2 in [0..r+1−r1−r2−i1])
        for i1 in [0..r3−1]) + sum(sum(r+1−r1−r2−r3−i1−i2 for i2 in
        [0..r+1−r1−r2−r3−i1]) for i1 in [0..r4−1]) + sum(r+1−r1−r2−
        r3−r4−i for i in [0..r5])
    return x
```

```
def vars7(i,r):
    count = 0
    for r1 in [0..r]:
        for r2 in [0..r-r1]:
            for r3 in [0..r-r1-r2]:
                for r4 in [0..r-r1-r2-r3]:
                    for r5 in [0..r-r1-r2-r3-r4]:
                        for r6 in [0..r-r1-r2-r3-r4-r5]:
                            r7 = r-r1-r2-r3-r4-r5-r6
                            count = count + 1
                            if count == i:
                                return r1,r2,r3,r4,r5,r6,r7

def mat7d(r):
    n = 7
    count = 0
    num_vars = binomial(r+n-1,n-1)
    A = matrix(ZZ,2*num_vars,num_vars)
    for coord in [1..num_vars]:
        r1,r2,r3,r4,r5,r6,r7 = vars7(coord,r)
        cell_num = coord_val7(r1,r2,r3,r4,r5,r6,r7,r)
        perm = coord_val7(r2,r3,r4,r5,r6,r7,r1,r)
        A[coord+num_vars-1,coord-1] = 1
        A[coord+num_vars-1,perm-1] = A[coord+num_vars-1,perm-1] - 1
        A[coord-1,coord-1] = 1
        for a1 in [0..r1]:
            for a2 in [0..r7]:
                for a3 in [0..r7-a2]:
                    for a4 in [0..r7-a2-a3]:
                        for a5 in [0..r7-a2-a3-a4]:
                            for a6 in [0..r7-a2-a3-a4-a5]:
                                cell_num = coord_val7(a1+a2,r2+a3,
                                    r3+a4,r4+a5,r5+a6,r6+r7-a2-a3-a4
                                    -a5-a6,r1-a1,r)
                                A[coord-1,cell_num-1] = A[coord-1,
                                    cell_num-1] + (-1)^r7*binomial(
                                    r1,a1)*binomial(r7,a2)*binomial(
                                    r7-a2,a3)*binomial(r7-a2-a3,a4)*
                                    binomial(r7-a2-a3-a4,a5)*
                                    binomial(r7-a2-a3-a4-a5,a6)
        for b1 in [0..r2]:
            for b2 in [0..r7]:
                for b3 in [0..r7-b2]:
                    for b4 in [0..r7-b2-b3]:
```

```
                    for b5 in [0..r7-b2-b3-b4]:
                        cell_num = coord_val7(r1,b1+b2,r3+b3,r4
                            +b4,r5+b5,r6+r7-b2-b3-b4-b5,r2-b1,r)
                        A[coord-1,cell_num-1] = A[coord-1,
                            cell_num-1] + (-1)^r7*binomial(r2,b1
                            )*binomial(r7,b2)*binomial(r7-b2,b3)
                            *binomial(r7-b2-b3,b4)*binomial(r7-
                            b2-b3-b4,b5)
    for c1 in [0..r3]:
        for c2 in [0..r7]:
            for c3 in [0..r7-c2]:
                for c4 in [0..r7-c2-c3]:
                    cell_num = coord_val7(r1,r2,c1+c2,r4+c3,r5+
                        c4,r6+r7-c2-c3-c4,r3-c1,r)
                    A[coord-1,cell_num-1] = A[coord-1,cell_num
                        -1] + (-1)^r7*binomial(r3,c1)*binomial(
                        r7,c2)*binomial(r7-c2,c3)*binomial(r7-c2
                        -c3,c4)
    for d1 in [0..r4]:
        for d2 in [0..r7]:
            for d3 in [0..r7-d2]:
                cell_num = coord_val7(r1,r2,r3,d1+d2,r5+d3,r6+
                    r7-d2-d3,r4-d1,r)
                A[coord-1,cell_num-1] = A[coord-1,cell_num-1] +
                    (-1)^r7*binomial(r4,d1)*binomial(r7,d2)*
                    binomial(r7-d2,d3)
    for e1 in [0..r5]:
        for e2 in [0..r7]:
            cell_num = coord_val7(r1,r2,r3,r4,e1+e2,r6+r7-e2,r5
                -e1,r)
            A[coord-1,cell_num-1] = A[coord-1,cell_num-1] +
                (-1)^r7*binomial(r5,e1)*binomial(r7,e2)
    for f1 in [0..r6]:
        cell_num = coord_val7(r1,r2,r3,r4,r5,r7+f1,r6-f1,r)
        A[coord-1,cell_num-1] = A[coord-1,cell_num-1] + (-1)^r7
            *binomial(r6,f1)
    return A
```

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