MSc Economics

The effect of information on ambiguous portfolio choices

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MSc Economics

Affidavit

I, Árpád Illyés

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Abstract

This paper analyzes an extension of the portfolio choice model under ambiguity of Gollier (2011) which uses the smooth model of ambiguity. It introduces an additional information setting the investor can acquire that can reduce the exposure to ambiguity. Within this framework it is shown that acquiring information about the plausible return distributions increases the ex ante welfare of the investor. The relation of the average ex ante investment and the original investment level under no information is also examined. Throughout various examples it is demonstrated that this relation is not so clear. When the case of constant absolute ambiguity aversion is considered, however it can be established that under certain conditions information increases the average investment, as absolute ambiguity aversion tends to infinity.

1 Introduction

Ambiguity has been in the main focus of the standard decision theory since the striking realization of the famous experiment of Ellsberg (1961). It revealed that even in a really simple setting decision makers tend to violate the subjective expected utility maximization paradigm of Savage (1954). In other words there might not be a unique personal probability assessment about the states of nature because of uncertainty and as a result the reduction of compound lotteries can fail. This problem has led to the development of various utility representations incorporating ambiguity and attitude towards ambiguity. A common feature of these models is that the decision makers have multiple priors about the possible outcomes.

The conceptual difference of ambiguity models lies in the evaluation of these prior beliefs. For example in the celebrated maxmin expected utility representation of Gilboa and Schmeidler (1989) the outcomes are evaluated using the prior which gives the lowest expected utility. A similar model by Ghirardato et al. (2004) is the so-called α-maxmin expected utility representation. It also uses the worst possible prior to compute expected utility, but only with weight α. With weight $1-\alpha$ however they take the expected utility under the best prior into account, hence the weight α can be interpreted as a parameter of pessimism.

Although these models are well-founded there are two main problems with them. The first is that they ignore the information contained in the rest of the priors and only focus on the best or worst ones. The second problem is that the attitude towards ambiguity is fixed by the set of priors and the model itself. These problems are solved by the so-called *smooth model* of ambiguity by Klibanoff et al. (2005). Under the smooth model all the priors are evaluated by taking some kind of transformed average of them using subjective second-order beliefs. The other attractive feature of the smooth model is that it separates tastes from beliefs and therefore attitudes towards ambiguity can be analyzed independently of the actual degree of ambiguity.

The extensive applicability of the smooth model provides a way to generalize economic models built around the expected utility framework. These generalizations allow the decision makers to possess multiple prior distributions about the outcomes instead of a single one as in subjective expected utility theory. One such an extension is by Gollier (2011), who uses the smooth model to investigate a standard portfolio choice problem under ambiguity. Ambiguity is introduced in a way that the investor does not know with certainty which of the multiple plausible return distributions of the risky asset will be the *true* one. In this model

Gollier finds sufficient conditions under which an increasing aversion towards ambiguity reduces the demand for the asset with uncertain returns. As intuitive as this result might sound, it does not hold in general. Gollier (2011) also shows in a cleverly constructed counterexample that higher ambiguity aversion can in fact lead to an increased demand for the uncertain asset, a phenomena similar to the existence of Giffen goods.

This paper expands the model of Gollier (2011) by introducing an informational setting to the model. The key idea is that the investor could get the help of a financial advisor who can narrow down the possible return distributions of the ambiguous asset. This allows for decreasing the exposure to ambiguity while keeping the ambiguity attitude fixed. Under this modified setting it can be examined how the ex ante welfare of an investment plan changes as a result of more information. Furthermore the ex ante average demand for the ambiguous asset can also be compared to that of the original case where no information is available.

The structure of the paper is as follows. Section 2 presents the smooth model of ambiguity in details. Section 3 introduces the standard portfolio choice model under ambiguity of Gollier (2011). Section 4 extends this model by introducing information structures which can reduce exposure to ambiguity. Section 5 concludes.

2 The smooth model of ambiguity

The *smooth model* of ambiguity refers to the representation established by Klibanoff et al. (2005). Building on the idea of Segal (1987) that is to relax the reduction of compound lotteries assumption they axiomatize a utility representation of the double expectation form

$$V(f) = \int_{\Delta} \phi \left(\int_{S} u(f) d\pi \right) d\mu = \mathbb{E}_{\mu}\phi\mathbb{E}_{\pi}(u \circ f) \tag{1}$$

where f is a Savage act defined over a state space S and u is a von-Neumann–Morgenstern utility function. It represents preferences $\succeq$ in the sense that for any two acts f and g it holds that $f \succeq g \iff V(f) \geq V(g)$.

More formally the functional V evaluates the act $f : S \to C$, a mapping from states to consequences, while the standard utility function $u : C \to \mathbb{R}$ evaluates the consequences by assigning real numbers to them. The interpretation of this form is done in a recursive flavor. First for a given probability measure π on S the decision maker computes her expected utility. However the decision maker might be subjectively uncertain about which is the *true* probability distribution according to which the nature will decide among the states. This subjective uncertainty is represented by μ, a personal belief of the decision maker about the "right" probability distribution π. Using this subjective probability measure the decision maker computes in the second step the expected value of a ϕ – transformation of the expected utility levels for every plausible priors. The transforming function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and increasing and captures the attitude towards ambiguity.

The representation theorem shows that such a transforming function ϕ exist if the preference structure satisfies three assumptions. The first assumption requires that the decision maker behaves as an expected utility maximizer on the space of lotteries whose probabilities are objectively known. On a lower level this assumption translates to the usual weak order, continuity and independence axioms underlying the expected utility theory. The second assumption states that on second order acts – functions mapping distributions to consequences – the decision maker maximizes subjective expected utility. In other words there is a subjective probability assessment and a utility function on second order acts that is consistent with the second order preferences. According to this assumption the decision maker is capable of ranking bets about the right prior. Finally the third axiom establishes the connection between first and second order preferences via certainty equivalents.

These three assumptions on preferences are sufficient conditions for the existence of a strictly increasing function ϕ mapping expected utility values under different to real numbers. This function has a central role in determining attitudes towards ambiguity. If it is linear, then the decision maker is said to be *ambiguity neutral* and she simply reduces the first and second order probability distributions to a single compound distribution. In this case she behaves as a Savagian subjective expected utility maximizer.

A more interesting case arises when ϕ is concave, that is when the decision maker is *ambiguity averse*. The reduction of first and second order probabilities does not hold in this case. Intuitively, under a concave ϕ the decision maker dislikes any mean preserving spread in conditional expected utilities. The notion of aversion to mean preserving spreads is closely related to risk aversion in the expected utility framework. In fact the comparative statics of can be analogously defined for ambiguity aversion.

For a clear comparative analysis an additional assumption is necessary. Namely the separation of tastes and beliefs, which states that by varying the subjective beliefs the risk attitude embodied in u won't change and thus the same ϕ function can be used in the representation. Now it is possible to compare decision makers through the concavity of ϕ. However, it is important to mention that only decision makers who share the same von-Neumann–Morgenstern utility function u and the same subjective probabilities can be considered. Then we can say that the agent (u, ϕ_1) is *more ambiguity averse* than agent (u, ϕ_2) if

$$\phi_1 = h \circ \phi_2 \tag{2}$$

for a strictly increasing and concave h, that is when ϕ_1 is more concave than ϕ_2.

Alternatively if ϕ_1 and ϕ_2 are twice continuously differentiable then we can express the same relation between ambiguity attitudes by

$$-\frac{\phi_1''(U)}{\phi_1'(U)} \geq -\frac{\phi_2''(U)}{\phi_2'(U)}$$

for every expected utility U. Following the literature of risk theory these ratios are measures of *absolute ambiguity aversion.* Constant absolute ambiguity aversion is characterized either by the function $\phi(U) = U$ or by $\phi(U) = -\frac{1}{\eta}e^{-\eta U}$ for some $\eta \neq 0$ and for every expected utility level U. Hereafter we will say ϕ admits constant absolute ambiguity aversion (with parameter η) if it is of the second form. If $\eta \to \infty$ then we approach the maxmin expected utility model by Gilboa and Schmeidler (1989), where acts are evaluated according to the worst possible prior. In this case, the decision maker is extremely pessimistic.

Because of the crisp separation of attitude towards ambiguity and ambiguity itself and of its analytical tractability the smooth model is a popular choice when investments in ambiguous assets or insurance packages (Alary et al., 2013; Berger, 2014) is considered. Its similarity to the standard risk theory allows for an intuitive interpretation of results. The scope of the model can also be extended to multi-period decision problems involving ambiguity (Klibanoff et al., 2009).

3 Portfolio choice under ambiguity

In this section we introduce the standard portfolio choice problem under ambiguity used in Gollier (2011). The one period model considers the decision of an investor who can allocate her wealth between two assets. The first asset has a certain return, which is normalized to zero for simplicity. The second asset however is not only risky but uncertain in the Knightian sense. Its return x is governed by a distribution, but this distribution depends on some parameter θ, which the investor cannot fully observe. This parameter is commonly known to belong a parameter space Θ.

The ambiguous parameter $\theta \in \Theta = \{1, \ldots, n\}$ can simply be thought of as an index, which determines the "true" distribution of returns from a finite set $\Pi = \{F_1, \ldots, F_n\}$ of possible cumulative distribution functions for the random variable $\tilde{x}$ of the excess return. Denote by $\tilde{x}_\theta$ the random variable which is distributed according to F_θ. The supports of these priors are supposed to be bounded in the closed interval $[\underline{x}, \bar{x}]$ with $\underline{x} < 0 < \bar{x}$. If the supports were not bounded then the investor would face an unbounded loss or gain with positive probability. Since the return of the safe asset is normalized to zero, the second part of this assumption says that every prior carries the opportunity of both loss and gain.

The investor is endowed with an exogenous level of initial wealth denoted by w_0. If α amount is allocated to the ambiguous asset then the investor's wealth at the end of the period is $w_0 + \alpha x$ upon the realization of the uncertain asset. The investment problem generally should be written as $(w_0 - \alpha)(1 + r_s) + \alpha(1 + x)$, but since r_s, the return of the safe asset is zero by assumption it simplifies to the form above. To reflect for the fact that one cannot invest more money than they originally have we restrict α not to exceed w_0 basically imposing a borrowing constraint. This restriction could be mitigated but then one would need to make assumptions on the lender. But since the portfolio choice analysis focuses on the investor's behavior these additional assumptions are not made.

The decision maker has a subjective probability assessment $(q_1, \ldots, q_n)$ over the set Π, that is $\sum_{\theta=1}^{n} q_\theta = 1$ and $q_\theta > 0$. The investor believes with probability q_θ that the true excess return distribution will be F_θ. In the terminology of Klibanoff et al. (2005) the vector $(q_1, \ldots, q_n)$ is her second-order belief and it corresponds to the measure μ in the representation (1). The fact that all subjective beliefs assign strictly positive probabilities reflect the convention that if one of the q_θ was zero then the decision maker simply would not find distribution F_θ to be plausible and she could simply get rid of it.

It is assumed that the investor's preferences can be represented with the smooth model of ambiguity. In a recursive way first she computes the expected utility of investing α amount in the ambiguous asset, conditional on the true distribution being F_θ (or equivalently the parameter being θ). This conditional utility is given by

$$U(\alpha, \theta) = \mathbb{E}u(w_0 + \alpha\tilde{x}_\theta) = \int u(w_0 + \alpha\tilde{x}_\theta)dF_\theta(x))$$

The utility function u is supposed to be increasing and concave reflecting to risk aversion, a standard assumption in risk theory. The concavity of u also implies that $U(\cdot, \theta)$ is a concave function of the investment level α for every possible parameter θ.

After computing expected utilities in the first step with all plausible distribution functions $F_\theta \in \Pi$ the investor uses her second order beliefs and a transformation function ϕ to obtain the *ex ante* value of investing amount α in the ambiguous asset

$$V(\alpha) = \phi^{-1}\left(\sum_{\theta=1}^{n} q_\theta \phi(U(\alpha, \theta))\right) = \phi^{-1}\left(\sum_{\theta=1}^{n} q_\theta \phi(\mathbb{E}u(w_0 + \alpha\tilde{x}_\theta))\right) \tag{3}$$

The transformation ϕ^{-1} simply computes the certainty equivalent of the ambiguous expected utility. The objective of the investor is to maximize her value by picking an optimal investment plan α^*. Since the function ϕ^{-1} is increasing, an optimal level of investment is

$$\alpha^* \in \operatorname{argmax} \sum_{\theta=1}^{n} q_\theta \phi(U(\alpha, \theta))$$

As this function is concave in the level of investment and the all the priors are bounded in a closed interval $[\underline{x}; \bar{x}]$ the first order condition is already sufficient, therefore every optimal α^* which is an inner solution has to satisfy

$$\sum_{\theta=1}^{n} q_\theta \phi'(U(\alpha^*, \theta))\mathbb{E}\tilde{x}_\theta u'(w_0 + \alpha^*\tilde{x}_\theta) = 0 \tag{4}$$

If the agent is ambiguity neutral then her transforming function ϕ is linear. As a consequence she will simply maximize subjective expected utility using the reduced probability measures $q_\theta F_\theta$ for all parameters $\theta \in \Theta$.

A primary result from Gollier (2011) shows that the demand for the ambiguous asset is positive if the equity premium as defined by $\sum_\theta q_\theta \mathbb{E}\tilde{x}_\theta$ is positive. This result is independent of the degree of the ambiguity aversion. To keep the analysis simple we will assume in what follows that the equity premium and hence the demand for the ambiguous asset is nonnegative. Together with our previous no-borrowing assumption this means that we restrict the investment levels to belong to the compact set $[0, w_0]$.

The main proposition of Gollier (2011) is that when the equity premium is positive, then any increase in ambiguity aversion reduces the demand for the ambiguous asset when the returns $(\tilde{x}_1, \ldots, \tilde{x}_n)$ can be ranked according to *Monotone Likelihood Order* (MLR). The MLR order between two distributions with probability density functions f and g means that for every $x_1 > x_0$ in their support then $f \succeq_{\mathrm{MLR}} g$ if

$$\frac{f(x_1)}{g(x_1)} \geq \frac{f(x_0)}{g(x_0)}$$

holds. The notion of increased ambiguity used here is the one stated in (2), that is when we use a concave transformation of the function ϕ.

4 Portfolio choice with additional information

Now we turn to the question of what happens if there is additional information available about the priors. Imagine if the investor could get help from a professional she trusts to narrow down the possible distributions of the excess return. To formulate this let $(\Theta_i)_{i=1}^m$ be a partition of the original parameter space Θ, that is $\cup_{i=1}^m \Theta_i = \Theta$ and $\Theta_i \cap \Theta_j = \emptyset$ for all $i \neq j$. This means that the expert can narrow down the set of plausible priors by revealing that the true parameter θ lies in one of the Θ_i partition sets. Our setup contains an implicit assumption, namely that the expert cannot reveal a partition set which is not a subset of Θ. This assumption is not that restrictive as we are considering the ex ante welfare of the investor and thus it is enough if she *believes* that the expert cannot reveal priors that had not been thought of.

It is noteworthy that as a special case of this formulation we get back the original case with no information. When $m = 1$ then the partition is trivially the whole parameter space Θ. In this case no additional information is acquired from the expert. Another special case is when $(\Theta_i)_{i=1}^m$ is the finest partition of Θ, that is when every Θ_i is a singleton set containing only one index. This would mean that upon obtaining the information set Θ_i the investor would face no ambiguity at all. To exclude this uninteresting case we require that $|\Theta_i| \geq 2$ for $i = 1, \ldots, m$, meaning that the professional cannot fully eradicate ambiguity. For further definitions we fix one information partition $(\Theta_i)_{i=1}^m$.

Before the investor acquires the additional information she uses her subjective beliefs $(q_1, \ldots, q_n)$ to assign probabilities to hearing that the plausible parameters are either from one of the Θ_i sets. We need these beliefs because we are looking at the ex ante situation, when the investor has not obtained the actual parameter restriction Θ_i yet, but she already knows what are the possible restrictions, in other words she knows the acquirable partition $(\Theta_i)_{i=1}^m$. The beliefs need to be consistent with the original ones. So, the investor simply adds up the ones which correspond to the parameter being in the same partition set. This can be written as $p_i = \Pr(\theta \in \Theta_i) = \sum_{\theta \in \Theta_i} q_\theta$ for all i.

Upon hearing that the parameter θ is either an element of one of the Θ_i partition sets the investor updates her subjective beliefs according to Bayes rule

$$\widehat{q}_\theta = \frac{q_\theta}{p_i} = \frac{q_\theta}{\sum_{j\in\Theta_i} q_j}, \qquad \text{for } i = 1, \dots, m$$

Using these updated beliefs she computes the value of the contingent investment given that information Θ_i is revealed to her

$$V_i(\alpha) = \phi^{-1}\left(\sum_{\theta\in\Theta_i} \widehat{q}_\theta \phi\left(\mathbb{E}u(w_0 + \alpha\tilde{x}_\theta)\right)\right), \quad \text{for } i = 1, \dots, m. \tag{5}$$

The inner expression is still a concave function of the investment level therefore an optimal contingent investment plan $\alpha_i^* \in (0, w_0)$ is again pinned down by the first order condition of V_i

$$\sum_{\theta\in\Theta_i} \widehat{q}_\theta \phi'(U(\theta, \alpha_i^*))\mathbb{E}\tilde{x}_\theta u'(w_0 + \alpha_i^*\tilde{x}_\theta) = 0 \tag{6}$$

If $\boldsymbol{\alpha}^* = (\alpha_1^*, \dots, \alpha_m^*)$ is the vector containing the optimal contingent investment plans of the scenarios when the respective information set Θ_i is obtained then the ex ante value of this investment plan is

$$\begin{aligned}\widehat{V}(\boldsymbol{\alpha}^*) &= \sum_{i=1}^m p_i V_i(\alpha_i^*) \\ &= \sum_{i=1}^m p_i \phi^{-1}\left(\sum_{\theta\in\Theta_i} \widehat{q}_\theta \phi(\mathbb{E}u(w_0 + \alpha_i^*\tilde{x}_\theta))\right)\end{aligned}$$

The following proposition says that if the investor is ambiguity averse then she is always better off by obtaining additional information about the plausible priors. It is important to mention that the result is independent of the information setting.

Proposition 1. *For an ambiguity averse decision maker and for any information partition $(\Theta_i)_{i=1}^m$ with the corresponding optimal contingent investment plan $\boldsymbol{\alpha}^* = (\alpha_1^*, \dots, \alpha_m^*)$ we have*

$$\widehat{V}(\boldsymbol{\alpha}^*) \geq V(\alpha^*)$$

where V is the original value function under no additional information and α^* is the corresponding demand for the ambiguous asset.

Proof. Since the increasing function ϕ is concave under ambiguity aversion this implies that its inverse ϕ^{-1} is convex. As the probabilities $(p_1, \dots, p_m)$ define a convex combination we have

$$
\begin{aligned}
\widehat{V}(\boldsymbol{\alpha}^*) &= \sum_{i=1}^{m} p_i \phi^{-1}\left(\sum_{\theta\in\Theta_i} \widehat{q}_\theta \phi(U(\alpha_i^*,\theta))\right) \\
&\geq \phi^{-1}\left(\sum_{i=1}^{m} p_i \sum_{\theta\in\Theta_i} \widehat{q}_\theta \phi(U(\alpha_i^*,\theta))\right) \\
&\geq \phi^{-1}\left(\sum_{i=1}^{m} p_i \sum_{\theta\in\Theta_i} \widehat{q}_\theta \phi(U(\alpha^*,\theta))\right) \\
&= \phi^{-1}\left(\sum_{\theta=1}^{n} q_\theta \phi(U(\alpha^*,\theta))\right) \\
&= V(\alpha^*)
\end{aligned}
$$

where the first inequality is true because ϕ^{-1} is convex. The last inequality holds because the values of α_i^* are already optimal when α^* is available. So by changing from α_i^* to α^* the values of V_i cannot be strictly increased. □

Note that the first inequality is strict if the transformation function ϕ is strictly concave, because in this case ϕ^{-1} is strictly convex.

Remark. *If ϕ is strictly concave as in the case of constant absolute ambiguity aversion and not all $V_i(\alpha_i^*)$ values are equal, then*

$$\widehat{V}(\boldsymbol{\alpha}^*) > V(\alpha^*)$$

Although this result holds independently of the information obtained the same cannot be told about the average ex ante demand for the ambiguous asset, defined for a given information partition as

$$\mathbb{E}\boldsymbol{\alpha}^* = \sum_{i=1}^{m} p_i \alpha_i^* \tag{7}$$

The following example illustrates that it is not obvious whether the average ex ante investment level is always higher than the investment under no additional information.

4.1 Example 1

Let $\Theta = \{1, 2, 3, 4\}$ be the parameter space so there are $n = 4$ possible return distributions. All distributions can yield one of the following excess returns

$S = \{-0.1, -0.05, 0.1, 0.2\}$. The 4 plausible distribution assign the following probabilities to the values of returns respectively

- $f_1 = (0.35, 0.2, 0.3, 0.15)$
- $f_2 = (0.35, 0.25, 0.25, 0.15)$
- $f_3 = (0.3, 0.2, 0.25, 0.25)$
- $f_4 = (0.15, 0.15, 0.3, 0.4)$

The investor's subjective beliefs about these priors are

- $q_1 = 10\%,\ q_2 = 40\%,\ q_3 = 40\%,\ q_4 = 10\%$

The utility function admits constant absolute risk aversion with risk aversion parameter 3, and the transformation function ϕ admits constant absolute ambiguity aversion with a general parameter η

- $u(z) = -e^{-3z}$
- $\phi(U) = -\frac{1}{\eta}e^{-\eta U}$

The initial wealth endowment is $w_0 = 1.25$. Finally consider the following two information partitions

- $\Theta^a = \{\{1, 2\}, \{3, 4\}\}$
- $\Theta^b = \{\{1, 4\}, \{2, 3\}\}$

We numerically solve the original investment problem when no additional information is available as well as the case when each of the two different information is obtained. Figure 1 shows how the optimal average investment level changes when ambiguity aversion is increasing as measured by the parameter η of the transforming function ϕ. The range of η in the simulation was $[1, 500]$.

As we can observe in all 3 cases the optimal investment is decreasing as the ambiguity aversion increases. However it can be observed how different information affects the ex ante average investment. For smaller η values both partitions result in a lower average investment level than in the case when no information is obtained. Furthermore obtaining Θ^a will eventually yield a higher average investment level when ambiguity aversion is sufficiently high (approximately when η is greater than 460 in the above example). On the other hand obtaining the partition Θ^b yields a lower average demand for the ambiguous asset than under no information on the whole range of the simulation. As we will show later under

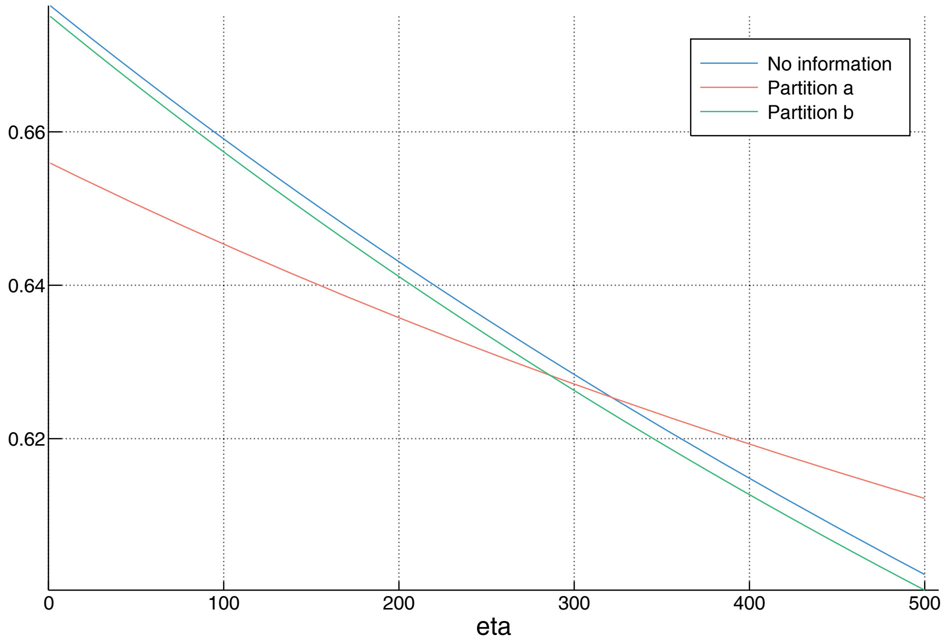


Figure 1: Average investment level under different information partitions

certain conditions on the utility function u and the priors, if ambiguity aversion is high enough the average investment level under information will always be higher than under no information *independently* of the quality of information obtained.

In order to get a better understanding why the average investment can be lower for certain degrees of ambiguity aversion we look at the case when the investor is ambiguity neutral. As we mentioned earlier under ambiguity neutrality the transformation function is linear $\phi(U) = U$ and we thus simply solve an expected utility maximization problem with the reduced priors $q_\theta F_\theta$. Keeping the above specification unchanged we compute the average investment levels for a linear ϕ.

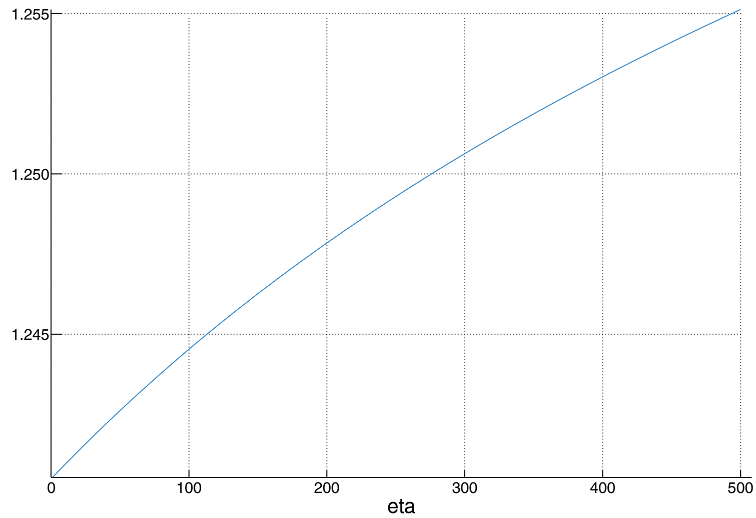
When there is no information available about the parameter θ the optimal investment level for an ambiguity neutral investor is $\alpha^* = 0.677$. But this is higher than the average investment level under both information settings

$$\mathbb{E}(\alpha_1^{a*}, \alpha_2^{a*}) = 1/2 \cdot 0.243 + 1/2 \cdot 1.07 = 0.656$$

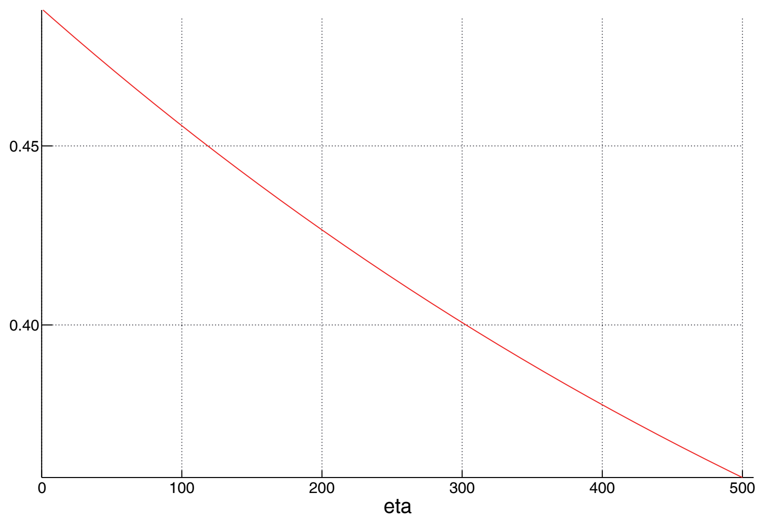
and

$$\mathbb{E}(\alpha_1^{b*}, \alpha_2^{b*}) = 1/5 \cdot 1.24 + 4/5 \cdot 0.534 = 0.675$$

where α_i^{k*} is the optimal investment level when partition i is obtained of information structure Θ^k. As it can be noticed the higher investment in the favorable parameter restriction cannot outweigh the lower investment level under the unfavorable restriction. Probably this is also the reason why we got the same result



(a) Partition a



(b) Partition b

Figure 2: Difference in values under different information

for an ambiguity averse investor for lower values of η.

This example has one more interesting feature. Let's take the differences of values under different information and that of no information, namely $\Delta V^k = \widehat{V}(\boldsymbol{\alpha}^{k*}) - V(\alpha^*)$ for $k = a, b$, where the index k refers to whether the partition Θ^a or Θ^b was obtained. Figure 2a and 2b shows how these differences evolve with increasing ambiguity aversion. The values were scaled up with a positive constant which does not affect the order of ranking. As these figures reveal acquiring different information can have the opposite welfare effects. The information Θ^a is more valuable for more ambiguity averse investors, while Θ^b is more appreciated when the ambiguity aversion is smaller.

4.2 Ex ante average investment

For further results on the relation between the average ex ante investment level under information compared to the no information case, we first need to clarify the relation between the expected utilities computed using different priors and the corresponding optimal levels of investment. To answer this question, we put the question of ambiguity aside for a while and look at pure risk theory. Namely, we need conditions on when prior F_{θ_2} gives a higher expected utility than an other prior F_{θ_1}, then the optimal investment level under the first prior is bigger than under the second. Formally we seek conditions when

$$\mathbb{E}u(w_0 + \alpha\tilde{x}_{\theta_1}) < \mathbb{E}u(w_0 + \alpha\tilde{x}_{\theta_2}) \Rightarrow \alpha_1^* < \alpha_2^* \tag{8}$$

where α_i^* are the corresponding optimal investment level for the priors. Note that the expectations also depend on the level of α. It is therefore possible that for different values of investment level the inequality is reversed. Although if the priors can be ranked according to first order stochastic dominance, then one expectation will always be lower than the other for all values of α. If u is concave then a weaker condition of second order stochastic dominance is sufficient, as we shall see in the following. If these conditions are satisfied then condition (8) can be stated as

$$\mathbb{E}u(w_0 + \alpha_1^*\tilde{x}_{\theta_1}) < \mathbb{E}u(w_0 + \alpha_2^*\tilde{x}_{\theta_2}) \Rightarrow \alpha_1^* < \alpha_2^*$$

For the sake of self-containment we define here stochastic orders of the excess return distributions. *First order stochastic dominance* (FSD) between two distributions means that the dominant one yields an unambiguously higher return than the dominated one. This is equivalent to saying $F(x) \geq G(x)$ for every value of x in the support. F is said to *second order stochastic dominate* (SSD) G if for every increasing and concave utility function u we have $\mathbb{E}_F u(x) \geq \mathbb{E}_G u(x)$ where the expectations are taken with respect to the corresponding distribution functions. This concept captures the phenomenon that risk averse decision makers always dislike mean-preserving spreads in the distributions.

We finally list here the different well-established risk attitudes for a better understanding. Absolute risk aversion for a twice continuously differentiable utility function u is defined as $A(z) = -\frac{u''(z)}{u'(z)}$ and relative risk aversion is simply $R(z) = zA(z)$. If u is additionally three times continuously differentiable then absolute risk prudence can be defined as $P(z) = -\frac{u'''(z)}{u''(z)}$ and correspondingly relative prudence is $P^r(z) = zP(z)$.

Using these stochastic orders and risk attitudes we present the following auxiliary result from Gollier (2004) which was first shown by Hadar and Seo (1990)

Lemma 1. *Suppose the domain of u is $\mathbb{R}_+$. Then a shift of distribution of returns increases the demand for the risky asset if*

1. *this shift is FSD-dominant and if relative risk aversion is less than unity*
2. *this shift is SSD-dominant, relative risk aversion is less than unity and increasing, and absolute risk aversion is decreasing*
3. *this shift is SSD-dominant, relative prudence is positive and less then 2*

The problem with this result is that empirically relative risk aversion is unlikely to be smaller than unity, as Gollier (2004) notes. On the other hand, we could restrict the state space to contain only two states, in which case it is straightforward that a distribution yielding a higher expected utility results in an at least as high investment level. Note that in the case of two states, first order stochastic dominance is also a total order, thus it suffices for a distribution to first order stochastically dominate an other if it assigns higher probability to the *good state* with the higher return. The next lemma summarizes this result.

Lemma 2. *In the standard portfolio choice model with a increasing and concave utility function u if the state space consists of only two states with returns $\{\underline{x}; \bar{x}\}$, where $\underline{x} < 0 < \bar{x}$ then an FSD-dominant shift in the distribution does not decrease the optimal level of investment. Furthermore it strictly increases it if under one of the priors the optimal investment level is an inner solution*

Proof. Let F_1 and F_2 be two distributions on the state space which assign probabilities p_1 and p_2 respectively to the return $\bar{x}$, with $p_2 > p_1$. This means that $F_2 \succ_{\text{FSD}} F_1$. The expected utilities of investing α can be written as

$$(1 - p_i)u(w_0 + \alpha\underline{x}) + p_i u(w_0 + \alpha\bar{x})$$

Since we are maximizing a continuous function over a compact set $\alpha \in [0, w_0]$ the maximum exists by the extreme value theorem. We need to distinguish between the cases where the maximizing investment level is an inner solution and when it is a boundary solution.

If $\alpha_1^* \in (0, w_0)$ then the concavity of u ensures that the first order conditions are sufficient to check so α_1^* satisfies

$$(1-p_1)u'(w_0+\alpha_1^*\underline{x})\underline{x} + p_1 u'(w_0+\alpha_1^*\bar{x})\bar{x} = 0$$

or put it differently

$$\frac{p_1}{1-p_1} = -\frac{u'(w_0+\alpha_1^*\underline{x})}{u'(w_0+\alpha_1^*\bar{x})}\frac{\underline{x}}{\bar{x}}$$

It is easy to check that the fraction on the left hand side is increasing in p_1 thus if we switched to distribution F_2 that is we increased p_1 to p_2 then the left hand side would increase. By the concavity of u the right hand side is also increasing in α_1^*. To observe this take the derivative with respect to α_1^* of the right hand side

$$-\frac{\underline{x}}{\bar{x}} \cdot \frac{u''(w_0+\alpha_1^*\underline{x})u'(w_0+\alpha_1^*\bar{x})\underline{x} - u'(w_0+\alpha_1^*\underline{x})u''(w_0+\alpha_1^*\bar{x})\bar{x}}{[u'(w_0+\alpha_0^*\bar{x})]^2} > 0$$

This directly implies that $\alpha_2^* > \alpha_1^*$ needs to hold for the optimal investment levels. The same result can be shown if $\alpha_2^* \in (0, w_0)$.

There are two remaining cases to be checked. The first case is when $\alpha_1^* = w_0$. This means that the maximal expected utility is attained at the boundary. We need to show that $\alpha_2^* = w_0$ too. For this note that

$$\mathbb{E}U = (1-p_i)u(w_0+\alpha\underline{x}) + p_i u(w_0+\alpha\bar{x})$$

has a positive cross-derivate, that is $\frac{\partial \mathbb{E}U}{\partial p_i \partial \alpha} > 0$. This means that if we increase p_1 to p_2 then the utility increases and it could furthermore be increased by a higher α. As we are already at the boundary this yields $\alpha_2^* = w_0$. The other case is when $\alpha_2^* = 0$ but a symmetric reasoning can also show that $\alpha_1^* = 0$ in this case. We conclude that $\alpha_2^* \geq \alpha_1^*$. □

For the next proposition we need the conditions of either of the above lemmas to hold. We could also simply require condition (8) to be satisfied by the priors, but this would be a stronger assumption. We follow the first approach.

Proposition 2. *Let ϕ admit constant absolute ambiguity aversion with parameter η. If either conditions of Lemma 1 are satisfied and all priors can be ranked* *according to FSD order, or conditions of Lemma 2 are fulfilled, then for any information partition* $(\Theta_i)_{i=1}^m$ *as* $\eta \to \infty$ *we have in the limit that*

$$\mathbb{E}\boldsymbol{\alpha}^* \geq \alpha^*$$

where $\mathbb{E}\boldsymbol{\alpha}^* = \sum_{i=1}^m \alpha_i^*$ *is the average of the optimal contingent investment plan and* α^* *is the investment under no information. Moreover if the conditions of Lemma 2 hold and additionally for one of the partition sets the value*

$$\underset{\alpha \in [0,w_0]}{\operatorname{argmax}} \{ \min_{\theta \in \Theta_i} \mathbb{E}u(w_0 + \alpha \tilde{x}_\theta) \}$$

is an inner solution, then the inequality is strict.

Proof. As showed by Klibanoff et al. (2005) if $\eta \to \infty$ we approach the maxmin expected utility representation, therefore the values converge to

$$V(\alpha) \to V^{\text{MEU}}(\alpha) = \min_{\theta \in \Theta} \mathbb{E}u(w_0 + \alpha \tilde{x}_\theta)$$

and respectively

$$V_i(\alpha) \to V_i^{\text{MEU}}(\alpha) = \min_{\theta \in \Theta_i} \mathbb{E}u(w_0 + \alpha \tilde{x}_\theta)$$

for $i = 1, \ldots, m$. In the limit the optimal contingent investment plan is therefore obtained by maximizing the lowest expected utility for each partition set. Since the function u is assumed to be concave and its domain is a closed interval the problem has a solution. Denote α_i^* this optimal contingent investment level for $i = 1, \ldots, m$. Now note that the optimal α^* which is the optimal investment level when there is no additional information available has to coincide with one of the α_i^* values. This is because if $V^{\text{MEU}}(\alpha)$ is minimal at $\bar{\theta} \in \Theta$, then one of the $V_i^{\text{MEU}}(\alpha)$ is also minimal at $\bar{\theta} \in \Theta_i \subseteq \Theta$. Let assume without loss of generality that $\alpha^* = \alpha_1^*$. If $\mathbb{E}\boldsymbol{\alpha}^* < \alpha^*$ was true in the limit then

$$\mathbb{E}\boldsymbol{\alpha}^* = \sum_{i=1}^m p_i \alpha_i^* < \alpha_1^* = \alpha^*$$

would yield

$$\sum_{i=2}^m p_i \alpha_i^* < (1 - p_1)\alpha_1^* = \sum_{i=2}^m p_i \alpha_1^*$$

But from Lemma 1 and Lemma 2 we have that $\alpha_i^* \geq \alpha_1^*$ for all $i = 1, \ldots, m$, so we got a contradiction. We conclude that $\mathbb{E}\boldsymbol{\alpha}^* \geq \alpha^*$.

Now if we additionally have that in one of the partition sets the optimal investment level under the worst prior is an inner solution, that is $\alpha_i^* \in (0, w_0)$ for some $i \neq 1$ then by the second part of Lemma 2 we also have that

$$\min_{\theta\in\Theta_1} \mathbb{E}u(w_0 + \alpha\tilde{x}_\theta) < \min_{\theta\in\Theta_i} \mathbb{E}u(w_0 + \alpha\tilde{x}_\theta)$$

and therefore $\alpha_1^* < \alpha_i^*$. Note that the assumption that $i \neq 1$ is without the loss of generality for if α_1^* is an inner solution, then $\alpha_i^* > \alpha_1^*$ for every $i = 2, \ldots, m$. Now as a consequence we get that $\alpha_i^* \geq \alpha_1^*$ for all i and $\alpha_i^* > \alpha_1^*$ for some i. In this case we get that $\mathbb{E}\boldsymbol{\alpha}^* > \alpha^*$. □

It is essential to observe that the first part of Proposition 2 is independent of the information partition. The *quality* of information therefore is irrelevant in the sense that in the limit the decision maker will always invest on the average at least as much to the ambiguous asset as she had invested under no additional information. On the other hand average investment is strictly higher in the limit only if the information is valuable to the investor, a property of the *given* information partition. The following example demonstrates this.

4.3 Example 2

Let there be two states of nature. In the *bad state* the return of the ambiguous asset is -0.1 and in the *good state* the return is 0.3. The investor perceives four possible probability distributions

- $f_1 = (0.95, 0.05)$
- $f_2 = (0.8, 0.2)$
- $f_3 = (0.6, 0.4)$
- $f_4 = (0.4, 0.6)$

and so the parameter space is again $\Theta = \{1, 2, 3, 4\}$. The subjective beliefs about these priors are uniform so $q_\theta = 1/4$ for $\theta \in \Theta$. The initial wealth level is $w_0 = 1$, the ϕ function admits constant ambiguity aversion and the utility function has the form of constant relative risk aversion

- $u(z) = \frac{z^{1-\rho}}{1-\rho}$

where the risk parameter $\rho = 3$. We want to observe how acquiring different kind of information affects the average investment in the ambiguous asset as ambiguity aversion increases. For this purpose consider the following two information partitions

- $\Theta^a = \{\{1,3\},\{2,4\}\}$
- $\Theta^b = \{\{1,2\},\{3,4\}\}$

Intuitively what is happening here is the following. Distributions f_1 and f_2 are really unfavorable as both yield negative expected returns. If the decision maker knew with certainty that the returns are distributed according to either f_1 or f_2, then her demand in both cases would be zero for the risky asset, since she is risk averse. The other two distributions f_3 and f_4 are the favorable ones with positive expected return. Now if information Θ^1 is considered then the reduction of ambiguity is not so prominent because both possible parameter restrictions – elements of Θ^1 – contain an unfavorable as well as a favorable distribution. The other information Θ however separates the unfavorable and the favorable distributions. The investor ex ante believes with probability $1/2$ that the expert is going to restrict the plausible distributions to those with negative expected return, and with probability $1/2$ that only the distributions with positive expected returns are relevant.

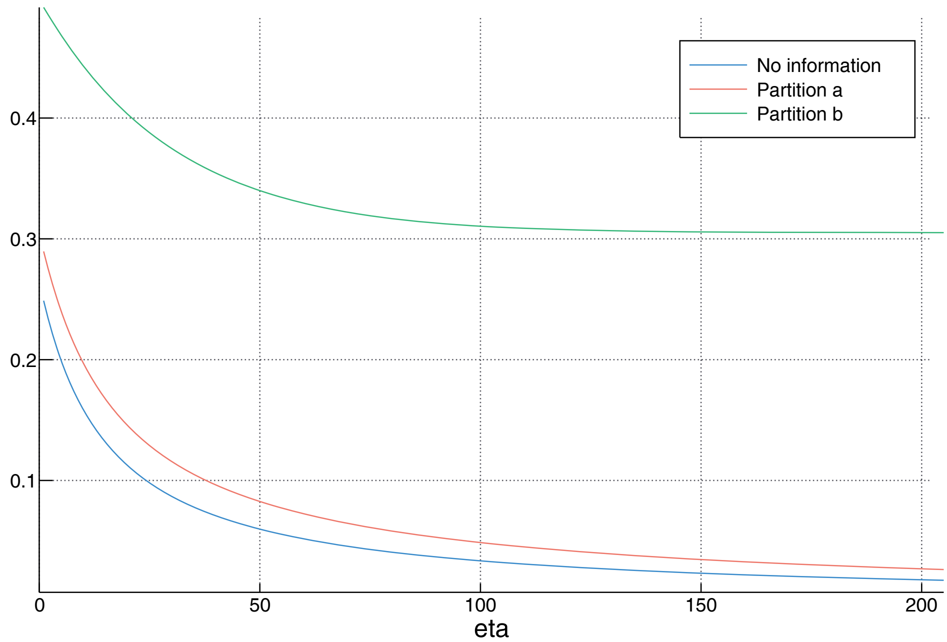
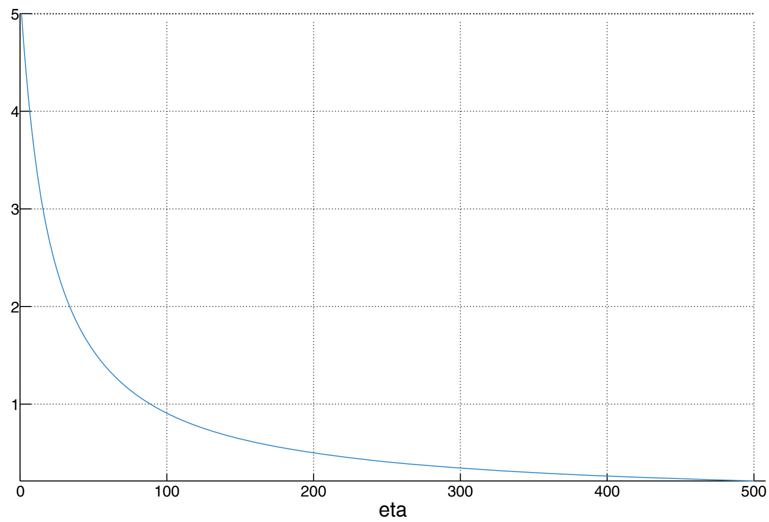
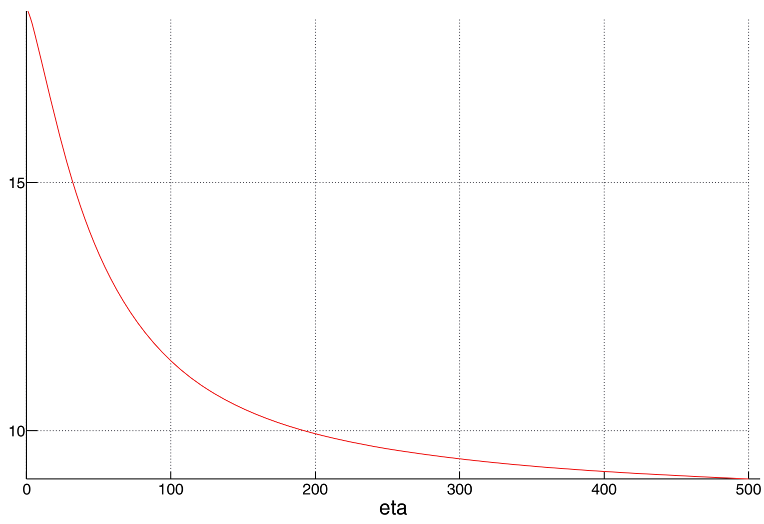


Figure 3: Average investment level under different information partitions

Figure 3 presents the numerical results for the average demand for the ambiguous asset under different information. As observable even if Θ^a does not bring well-structured information it still manages to raise the average investment for all degrees of ambiguity aversion. It is not surprising that the other information partition Θ^b yields a higher average investment level since it separates the favorable priors from the unfavorable ones. As we mentioned before when $\eta \to \infty$, the contingent investment plans are assessed according to the maxmin expected utility so in the limit the most unfavorable priors determine the welfare. In the case of Θ^a this means that both contingencies are going to be evaluated using the priors with negative expected returns, namely f_1 and f_2. But because of the borrowing constraint the demand cannot be negative so in both contingency of Θ^a the limit investment level is zero. The limit investment level of the no information case is also zero as the universal worst prior is f_1. This example shows that even if information raises the average demand for the ambiguous asset for all degrees of ambiguity aversion, in the limit they coincide.



(a) Partition a



(b) Partition b

Figure 4: Difference in values under different information

Just like in Example 1 we compute and plot the differences of values between the two information structures and the original value under no information. The value differences are shown on Figure 4a and 4b. Under both information partitions the difference of the values is decreasing in the degree of ambiguity aversion.

4.4 Example 3

This example heavily draws on the second example in Gollier (2011), where he shows that increasing ambiguity aversion can in fact raise the demand for the ambiguous asset. The following example can be considered as an extended version of his, in a sense that we need to modify it a little bit to fit it in our framework. The original example consisted of two return distributions so in order to be able to partition the parameter space into two disjoint sets with cardinality greater than 2 we need at least four plausible distributions. They are the following

- $\tilde{x}_1 \sim (-1, 2/10; , -0.25, 3/20; 0.75, 7/20; 1.25, 3/10)$
- $\tilde{x}_2 \sim (-1, 1/5; 0, 1/5; 1, 3/5)$
- $\tilde{x}_3 \sim (-1, 2/10; , -0.25 - \gamma, 3/20 - \delta; 0.75 + \gamma, 7/20 + \delta; 1.25, 3/10)$
- $\tilde{x}_4 \sim (-1 - \varepsilon, 1/5; 0, 1/5; 1 + \varepsilon, 3/5)$

The above notation should be read as $\tilde{x}_1 = -1$ with probability 2/10 and so on. We basically duplicate the excess return distributions $\tilde{x}_1$ and $\tilde{x}_2$ with a slight modification, embodied in the parameters γ, δ and ε. Since the original example is so carefully constructed it is really sensitive to bigger changes in probabilities and the return values, so we keep $\gamma = \delta = \varepsilon = 0.05$. The utility function is of the form

- $u(z) = \min\{z, 3 + 0.3(z - 3)\}$

This piecewise linear function has a kink at $z = 3$, it is concave however not strictly. The ϕ function is of the constant absolute ambiguity aversion form with parameter η. The prior beliefs are

- $q_1 = 2.5\%,\ q_2 = 47.5\%,\ q_3 = 2.5\%,\ q_4 = 47.5\%$

This way we preserved the ratio of probabilities q_1 and q_2 from the original example. The information the decision maker obtains is

- $\Theta^a = \{\{1, 2\}, \{3, 4\}\}$

This means that the first parameter restriction leads back to the original case of Gollier (2011), while the other leads to a slightly modified one. The numerical results are shown on Figure 5 for the simulation range $\eta \in [1, 300]$.

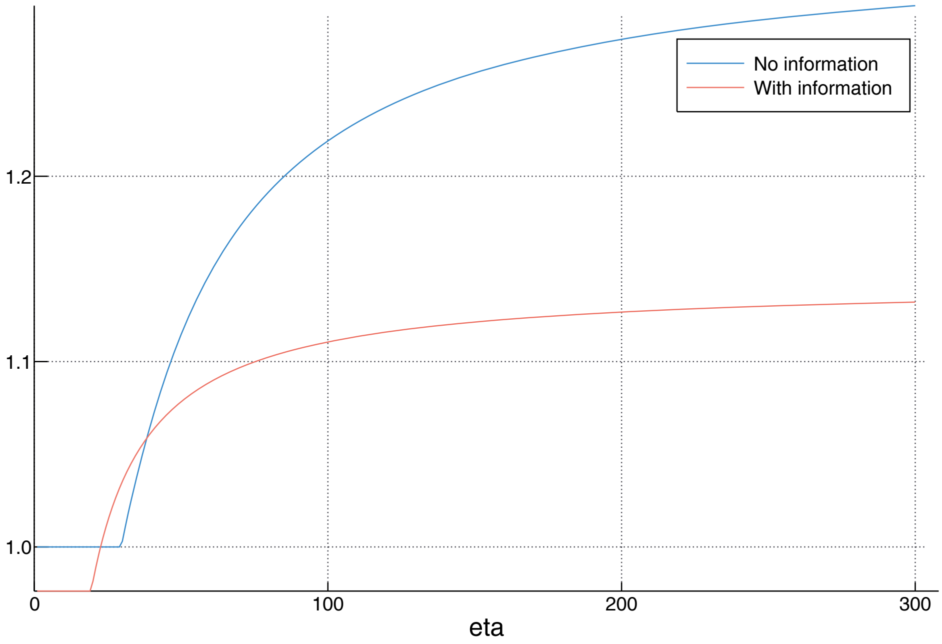


Figure 5: Average investment level under information

As in the original example it can be observed that after a certain level of η ambiguity aversion increases the investment. What is more important for us is that when ambiguity aversion is high enough the average ex ante demand for the ambiguous asset under information is always lower than the demand under information.

Figure 6 shows how the contingent investment plans change with ambiguity. Contingency 1 refers when the parameter restriction $\Theta_1 = \{1, 2\}$ is acquired and Contingency 2 refers when $\Theta_2 = \{3, 4\}$ is acquired. For smaller degrees of ambiguity aversion in all three cases the investment level is constant. In fact in Contingency 2 the investment level is constant for all values of η. In the other two cases after a threshold level of η the optimal investment level starts to increase. This threshold value of η is smaller when the set of parameters is smaller, that is when the parameter restriction Θ_1 is obtained.

4.5 Continuity of the demand

This result shows us that under certain conditions, if the ambiguity attitude tends to infinity then the average investment level exceeds the investment level of the original problem with no additional information. But there is a little bit more to say. For that we need to show that the optimal investment level is continuous in the parameter η. The way we will show this is by applying *Berge's Maximum Theorem* and check if its conditions are satisfied for our case. The version of the theorem we are using is from Sundaram (1996).

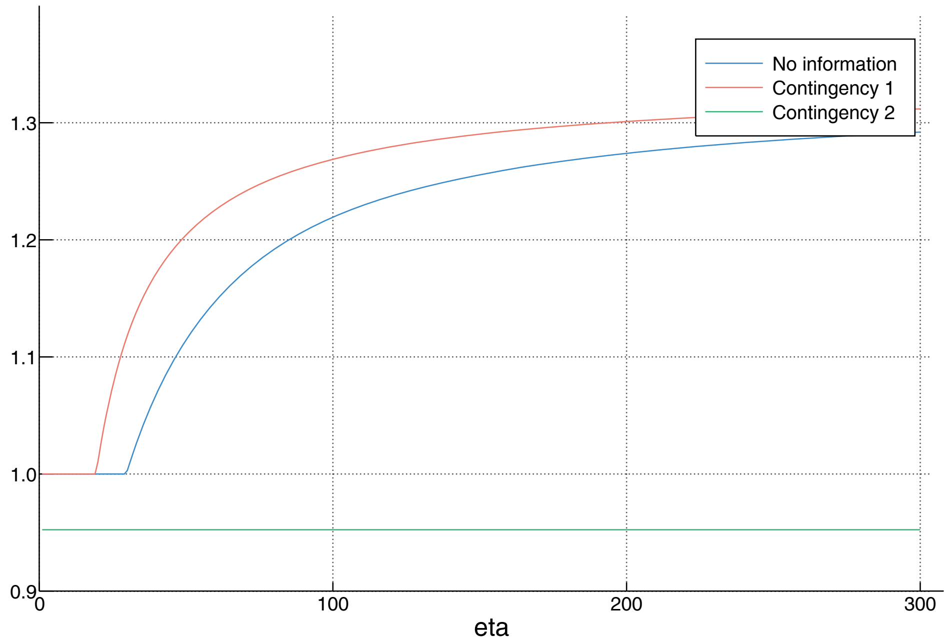


Figure 6: Contingent investment plans

The *Maximum Theorem* states that if we have a two metric spaces X and Y, a continuous function $f : X \times Y \to \mathbb{R}$ and a continuous, nonempty and compact valued correspondence $C : Y \rightrightarrows X$ and we define

$$f^*(y) = \max_{x \in C(y)} f(x, y)$$
$$C^*(y) = \operatorname*{argmax}_{x \in C(y)} f(x, y)$$

then f^* is continuous and C^* is upper hemicontinuous, nonempty and compact valued. Moreover if $f(\cdot, y)$ is strictly concave in the first argument for every value $y \in Y$ and C is additionally convex valued, meaning that for every $y \in Y$ its image is a convex set, then C^* is single valued and hence it is a continuous function.

Our maximization problem can be translated to the notation of the Maximum Theorem as follows. First define $f : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ such that

$$f(\alpha, \eta) = \sum_{\theta \in \Theta} q_\theta \left(-\frac{1}{\eta} \exp(-\eta \mathbb{E} u(w_0 + \alpha \tilde{x}_\theta)) \right)$$

This function is strictly concave in α for every value of η. To see this note that u is concavely increasing and the expectation operator does not affect concavity. Furthermore the transformation function $\phi(x) = -\frac{1}{\eta} \exp(-\eta x)$ is strictly concave and also increasing. Finally the composition of strictly concave and increasing function with a concave and increasing function is strictly concave. The constraint correspondence is $C : (0, \infty) \rightrightarrows \mathbb{R}$ such that

$$C(\eta) = [0, w_0] \quad \forall \eta > 0$$

This correspondence is obviously compact and convex valued. This means that all the conditions of the Maximum Theorem are satisfied. Therefore the demand for the ambiguous asset $C^*(\eta) = \text{argmax}_{\alpha \in C(\eta)} f(\alpha, \eta)$ is single valued and continuous. This result is summarized in the following lemma.

Lemma 3. *If ϕ admits constant absolute ambiguity aversion with parameter η, then the optimal investment level, given the degree of ambiguity aversion $\alpha^*(\eta)$ is a continuous function.*

This lemma is also valid for all contingent investment plans meaning that for an information partition set Θ_i the function $\alpha_i^*(\eta)$ is also continuous. A direct consequence of Proposition 2 is that for every investor with constant ambiguity aversion attitude there is a threshold value for the aversion parameter η such that for any higher value the ex ante average investment level exceeds the investment level of the no information case.

Proposition 3. *Let the assumptions of Proposition 2 be satisfied. If ϕ admits constant ambiguity aversion with parameter η and as $\eta \to \infty$ we have that $\mathbb{E}\boldsymbol{\alpha}^* > \alpha^*$ for a fixed information partition, then $\exists \eta_0 \in (0, \infty)$ such that*

$$\mathbb{E}\boldsymbol{\alpha}^*(\eta) > \alpha^*(\eta)$$

for every $\eta > \eta_0$.

The sketch of the proof is as follows. Proposition 3 of Klibanoff et al. (2005) basically establishes that if for a maxmin expected utility maximizing agent the act f yields a strictly higher utility than act g, then if the constant absolute ambiguity aversion parameter is high enough, the smooth model would also yield a strictly higher utility, given that the set of priors are the same. By assumption if $\mathbb{E}\boldsymbol{\alpha}^* > \alpha^*$ this implies that $\alpha_i^* \geq \alpha^*$ for every $i = 1, \ldots, m$ and $\alpha_j^* > \alpha^*$ for some j as it was established in Proposition 2. But this means that for η large enough we have $\alpha_j^*(\eta) > \alpha^*(\eta)$ and $\alpha_i^*(\eta) \geq \alpha^*(\eta)$ for all $i = 1, \ldots, m$.

5 Conclusion

In this paper we examined how the presence of information can affect the investment decisions in the portfolio choice model of Gollier (2011). Our first result shows that for ambiguity averse investors information is always valuable in the sense that ex ante welfare is never decreased when additional information is acquired. Under certain conditions on the transforming function ϕ and the values $V_i(\alpha_i^*)$ it is even strictly increased. This fact is independent of the quality of the given information, it only relies on the concavity of ϕ. Intuitively this result basically says that more information is always valuable.

The relation between ex ante average investment levels under different information settings is not so clear unfortunately. In various examples we show how different information and risk attitudes can change investment plans when ambiguity aversion is increased. It should be noted, however that we restricted our examples to the case of constant absolute ambiguity aversion because of its functional form is characterized by one parameter only. Furthermore it was established by Klibanoff et al. (2005) that when the parameter of ambiguity aversion tends to infinity, then we reach the maxmin expected utility model of Gilboa and Schmeidler (1989). Using the results of standard risk theory we put conditions on when average investment is increased in the limit when additional information is obtained. In future research one could examine our extended framework under more general assumptions on the transforming function ϕ. One possible approach could focus on finding conditions under which the relation between the average investment under information and the original investment level can be unambiguously determined.

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6 Appendix

In the examples we used the 0.5.1 version of the Julia programming language. The numerical optimization problems were solved using the Optim package. First we set up the model structure with all the methods necessary.

```
using Optim
using Plots; gr()

type Model
    u    # Utility function
    phi  # Transformation function
    Fs   # Priors (plausible distributions)
    qs   # Subjective beliefs about the priors
    S    # Returns
    w0   # Initial wealth

    function Model(u,phi,Fs,qs,S,w0)
        @assert size(Fs) == size(qs) == size(S) error
        if  isapprox(sum(qs),1.0) == false
            error("The qs don't add up to 1")
        end
        for i in 1:length(S)
            @assert length(S[i]) == length(Fs[i])
            if isapprox(sum(Fs[i]),1.0) == false
                error("Distribution ",i," doesn't add up to 1")
            end
        end
        new(u,phi,Fs,qs,S,w0)
    end
end

# Computes the equity premium
function equity_p(M::Model)
    ep = 0
    for i in 1:length(M.S)
        ep += M.qs[i]*sum(p*s for(p,s) in zip(M.Fs[i],M.S[i]))
    end
    return ep
end

# Expected utility under prior n, given investment level alpha
function U(M::Model, alpha, theta)
    dot(M.Fs[theta],M.u.(M.w0 + alpha*M.S[theta]))
end

# Value function under no additional information
function V0(M::Model,eta=10)
    V(alpha) = sum(q * M.phi(U(M,alpha,theta),eta) for (q,theta) in zip(M.qs,1:
        length(M.qs)))
end

# Value function of the info partition that contains indexes of one info set
function V_info(M::Model,partition,eta=10)
    qs = M.qs[partition]
    q_hats = qs/sum(qs)
    V(alpha) = sum(q * M.phi(U(M,alpha,theta),eta) for (q,theta) in zip(q_hats,
```

```
        partition))
end

# Computes the optimal investment of the value function V
function optimal_alpha(M::Model,V,eta=10)
    f = V(M,eta)
    o = optimize(x -> -f(x),0,M.w0)
end

# Determines the optimal investment level under prior theta
function maxmin_alpha(M::Model,theta)
    optimize(x->-U(M,x,theta),0,M.w0).minimizer
end
```

We defined the following utility and transforming ϕ functions

```
function phi(U,eta)
    -1/eta*exp(-U*eta)
end

function phi_inverse(U,eta)
    -1/eta*log(-eta*U)
end

function crra(x,rho=3)
    (x^(1-rho))/(1-rho)
end

function cara(x,rho=3)
    -exp(-rho*x)/rho
end
```

The code of the examples are the following

Example 1

```
w0 = 1.25

S  = [[-0.1,-0.05,0.1,0.2] for i in 1:4]

F1 = [0.35,0.2,0.3,0.15]
F2 = [0.35,0.25,0.25,0.15]
F3 = [0.3,0.2,0.25,0.25]
F4 = [0.15,0.15,0.3,0.4]

Fs = [F1,F2,F3,F4]

qs = [0.1,0.4,0.4,0.1]

M1 = Model(x->cara(x,3),phi,Fs,qs,S,w0)

etas = 1:500
alphas0 = Vector(length(etas))
alphas1 = Vector(length(etas))
alphas2 = Vector(length(etas))
alphas3 = Vector(length(etas))
alphas4 = Vector(length(etas))
```

```
for eta in etas
    alphas0[eta] = optimal_alpha(M1,V0,eta).minimizer
    alphas1[eta] = optimal_alpha(M1,(M,eta)->V_info(M,[1,2],eta),eta).minimizer
    alphas2[eta] = optimal_alpha(M1,(M,eta)->V_info(M,[3,4],eta),eta).minimizer
    alphas3[eta] = optimal_alpha(M1,(M,eta)->V_info(M,[1,4],eta),eta).minimizer
    alphas4[eta] = optimal_alpha(M1,(M,eta)->V_info(M,[2,3],eta),eta).minimizer
end

plot(etas,alphas0,xlims=(0,length(etas)+8),label="No information",xlab="eta")
plot!(etas,(qs[1]+qs[2])*alphas1+(qs[3]+qs[4])*alphas2, label="Partition a")
plot!(etas,(qs[1]+qs[4])*alphas3+(qs[2]+qs[3])*alphas4, label="Partition b")

V_diff1 = Vector(length(etas))
V_diff2 = Vector(length(etas))

for i in 1:length(etas)
    V_diff1[i] = (qs[1]+qs[2])*phi_inverse(V_info(M1,[1,2],etas[i])(alphas1[i]),
        etas[i]) +
                  (qs[3]+qs[4])*phi_inverse(V_info(M1,[3,4],etas[i])(alphas2[i]),
                      etas[i]) -
                  phi_inverse(V0(M1,etas[i])(alphas0[i]),etas[i])

    V_diff2[i] = (qs[1]+qs[4])*phi_inverse(V_info(M1,[1,4],etas[i])(alphas3[i]),
        etas[i]) +
                  (qs[2]+qs[3])*phi_inverse(V_info(M1,[2,3],etas[i])(alphas4[i]),
                      etas[i]) -
                  phi_inverse(V0(M1,etas[i])(alphas0[i]),etas[i])
end

plot(etas,10000*V_diff1,
     xlab="eta",

     legend=:none,
     xlims=(0,length(etas)+8))

plot(etas,10000*V_diff2,
     xlab="eta",
     legend=:none,
     xlims=(0,length(etas)+8),
     color="red")
```

Example 2

```
w0 = 1

S = [[-0.1,0.3] for i in 1:4]

F1 = [0.95,0.05]
F2 = [0.8,0.2]
F3 = [0.6,0.4]
F4 = [0.4,0.6]

Fs = [F1,F2,F3,F4]

qs = [0.25,0.25,0.25,0.25]
```

```
M3 = Model(x->crra(x,3),phi,Fs,qs,S,w0)

etas = 1:500
alphas0 = Vector(length(etas))
alphas1 = Vector(length(etas))
alphas2 = Vector(length(etas))
alphas3 = Vector(length(etas))
alphas4 = Vector(length(etas))

for eta in etas
    alphas0[eta] = optimal_alpha(M3,V0,eta).minimizer
    alphas1[eta] = optimal_alpha(M3,(M,eta)->V_info(M,[1,3],eta),eta).minimizer
    alphas2[eta] = optimal_alpha(M3,(M,eta)->V_info(M,[2,4],eta),eta).minimizer
    alphas3[eta] = optimal_alpha(M3,(M,eta)->V_info(M,[1,2],eta),eta).minimizer
    alphas4[eta] = optimal_alpha(M3,(M,eta)->V_info(M,[3,4],eta),eta).minimizer
end

plot(etas,alphas0,label="No information   ",xlims=(0,205),xlab="eta")
plot!(etas,(qs[1]+qs[3])*alphas1+(qs[2]+qs[4])*alphas2,label="Partition a")
plot!(etas,(qs[1]+qs[2])*alphas3+(qs[3]+qs[4])*alphas4,label="Partition b")

V_diff1 = Vector(length(etas))
V_diff2 = Vector(length(etas))

for i in 1:length(etas)
    V_diff1[i] = (qs[1]+qs[3])*phi_inverse(V_info(M3,[1,3],etas[i])(alphas1[i]),
        etas[i]) +
                 (qs[2]+qs[4])*phi_inverse(V_info(M3,[2,4],etas[i])(alphas2[i]),
                     etas[i]) -
                 phi_inverse(V0(M3,etas[i])(alphas0[i]),etas[i])

    V_diff2[i] = (qs[1]+qs[2])*phi_inverse(V_info(M3,[1,2],etas[i])(alphas3[i]),
        etas[i]) +
                 (qs[3]+qs[4])*phi_inverse(V_info(M3,[3,4],etas[i])(alphas4[i]),
                     etas[i]) -
                 phi_inverse(V0(M3,etas[i])(alphas0[i]),etas[i])
end

plot(etas,1000*V_diff1,legend=:none,xlims=(0,length(etas)+8),xlab="eta")

plot(etas,1000*V_diff2,legend=:none,xlims=(0,length(etas)+8),color="red",xlab="
    eta")
```

Example 3

```
w0 = 2

S1 = [-1,-0.25,0.75,1.25]
S2 = [-1,0.,1]
S3 = [-1,-0.3,0.8,1.25]
S4 = [-1.05,0.,1.05]

S = [S1,S2,S3,S4]

F1 = [2/10,3/20,7/20,3/10]
```

```
F2 = [1/5,1/5,3/5]
F3 = [2/10,3/20-0.05,7/20+0.05,3/10]
F4 = [1/5,1/5,3/5]

Fs = [F1,F2,F3,F4]

qs = [0.025,0.475,0.025,0.475]

M4 = Model(z->min(z,3+0.3*(z-3)),phi,Fs,qs,S,w0)

etas = 1:300
alphas0 = Vector(length(etas))
alphas1 = Vector(length(etas))
alphas2 = Vector(length(etas))
alphas3 = Vector(length(etas))
alphas4 = Vector(length(etas))

for eta in etas
    alphas0[eta] = optimal_alpha(M4,V0,eta).minimizer
    alphas1[eta] = optimal_alpha(M4,(M,eta)->V_info(M,[1,2],eta),eta).minimizer
    alphas2[eta] = optimal_alpha(M4,(M,eta)->V_info(M,[3,4],eta),eta).minimizer
    alphas3[eta] = optimal_alpha(M4,(M,eta)->V_info(M,[1,4],eta),eta).minimizer
    alphas4[eta] = optimal_alpha(M4,(M,eta)->V_info(M,[2,3],eta),eta).minimizer
end

plot(etas,alphas0,label="No information ",xlims=(0,length(etas)+8),xlab="eta")
plot!(etas,(qs[1]+qs[2])*alphas1+(qs[3]+qs[4])*alphas2,label="With information")

plot(etas,alphas0,label="No information",xlab="eta", ylims = (0.9,1.4),xlims=(0,
    length(etas)+8))
plot!(etas,alphas1,label ="Contingency 1")
plot!(etas,alphas2,label ="Contingency 2")
```