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FAKULTÄT
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Many-valued logics for games

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Matthias Paul Lanzinger

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Abstract

Logics for games provide systems of formal reasoning for game theory. While games commonly have more than two possible payoff values, most logics for games are two-valued. This limits their capacity to directly represent different payoff values or corresponding preferences. By employing many-valued logics one can avoid the introduction of specific predicates for encoding this payoff information.

This thesis generalizes the propositional forcing power logics introduced by van Benthem beyond two truth-values. Instead of extending classical propositional logic by modal operators for forcing power in games, generalized versions of the modal operators are used to extend Łukasiewicz logics. These many-valued forcing power logics are shown to preserve the main property of the two-valued case: invariance under power bisimulation. Moreover one arrives at a new result expressing that for a restricted class of formulas the lower bound of truth values is preserved under power simulation. This power simulation is also shown to lead to a further notion of equivalence, different from power bisimulation. Furthermore, the minimax theorem of game theory is related to modal duality in finite two-player perfect information games. A notion of game reduction is introduced and used to transfer properties of two-player perfect information games to broader classes of games.

Kurzfassung

Spiellogiken ermöglichen formales Schließen im Kontext der Spieltheorie. Während allerdings in Spielen oft mehr als zwei mögliche Auszahlungswerte vorkommen, sind die meisten Spiellogiken nur zweiwertig. Daher ist es in diesen Logiken nur beschränkt möglich, Auszahlungswerte oder die zugehörigen Präferenzen direkt zu repräsentieren. Durch den Einsatz von mehrwertigen Logiken kann diese Auszahlungsinformation direkt, ohne spezifische Prädikate, codiert werden.

Diese Arbeit befasst sich mit der mehrwertigen Verallgemeinerung der Forcing-Power-Aussagenlogik von van Benthem. Anstatt klassische Aussagenlogik mit einem Modaloperator für die erzwingbaren Ausgänge eines Spiels zu erweitern, wird in dieser Arbeit Łukasiewicz-Logik mit einer generalisierten Form des Modaloperators erweitert. Es wird gezeigt, dass die zentrale Eigenschaft des zweiwertigen Falles, Invarianz unter Power-Bisimulation, auch im mehrwertigen Fall erhalten bleibt. Darüber hinaus wird ein neues Resultat präsentiert, das zeigt, dass für eine eingeschränkte Klasse an Formeln die untere Schranke des Wahrheitswerts unter Power-Simulation erhalten bleibt. Auf Basis dieser Power-Simulation wird eine weitere Form der Spieläquivalenz eingeführt, die sich von Power-Bisimulation unterscheidet. Des Weiteren wird das Min-Max-Theorem mit der Dualität des Modaloperators in Zwei-Personen Spielen mit perfekter Information in Verbindung gebracht. Ein Begriff der Spielreduktion wird eingeführt und verwendet um von Eigenschaften von Zwei-Personen-Spielen mit perfekter Information auf Eigenschaften von allgemeineren Klassen von Spielen zu schließen.

Contents

CHAPTER 1

Introduction

1.1 Motivation & Problem statement

Game theory has proven to be of use in a great variety of fields of study including psychology, economics, biology, political science and computer science. Of noted interest are the varied applications in the study of logic, some examples include the Ehrenfeucht-Fraïssé game which plays an important role in finite model theory [12], Giles's game as a proof system of Łukasiewicz logic [14] or Väänänen's approach to model theory [51].

In the literature there exist many different logics for games (see section 2.2). The multitude of different approaches is motivated by the diversity of game theoretic applications. Because of the complex structure of games, it is challenging to build logics for games that are powerful enough to cover a large number of different applications. The development of more general logics for games is an ongoing area of research (cf. [55]). Beyond formal reasoning, there is further potential in the study of logics for games. They provide an extra layer on which connections between games can be investigated. As an example, a classic point of interest in game theory is equivalence between games (cf. [49, 25]). Correspondingly, the equivalence of structures is an important aspect of a logic's model theory.

Current approaches (see chapter 2) are often limited to boolean evaluation. In contrast, many common games are inherently many-valued, e.g., games involving costs, resources, pay-offs or sharing. With a two-valued formalism it becomes problematic to model these types of goals: they would either require complex encoding or a simplification to win/lose cases which leads to a loss of information. Many-valued logics for games allow for a more direct encoding of such goals as truth degrees. This not only improves usability but also makes the interpretation of results more intuitive. Beyond these considerations, a generalization to many-valuedness also serves to examine the possible generalization of two-valued results.

1.2 Aim of the work

The central aim of this thesis is the development of a family of many-valued modal logics for games. In particular, the objective is to generalize a suitable existing logic for games to many-valuedness. The motivations for selecting the logic that is generalized are presented through a discussion of the existing research in logics for games. To make the thesis more self-contained, it also contains an introduction to the required foundations and definitions of game theory and many-valued logics.

The generalization to many-valuedness brings with it an important goal; namely to investigate which properties of the two-valued logic still hold in the more general setting. To this end the main results of the original logic are shown to generalize to the new many-valued systems. Exploring properties beyond the results of the original logic is a further goal. Possible topics of such further research are the investigation of different kinds of structural equivalence, the representability of common game theoretic characteristics (e.g, existence of a pure-strategy Nash equilibrium, zero-sum, dictatorship) and the construction of a proof system. At the same time, this thesis contains interpretations of its results in the context of game theory. Their application to games, as well as their more general theoretic meaning, is to be discussed. Particular focus falls on how the differences between certain families of games, e.g., perfect/imperfect information or 2-player/n-player, manifest themselves in the developed logics.

Building on our findings, an assessment is made concerning possible applications and perceived weaknesses. The thesis concludes with an evaluation of the developed system based on this assessment and a discussion of opportunities and possible directions for further research based on the work presented here.

1.3 Methodological approach

To reach the expected results the following methodological approach is applied:

1. In the first step, a review and analysis of existing literature is performed. The focus of this analysis is on existing logics for games, many-valued logics and logics for structurally similar systems. From this analysis, an existing two-valued logic is selected as the reference point for the many-valued logics of this work.

2. The known properties of the two-valued logic are investigated for the many-valued case. This means to either show that a property still holds or to demonstrate that it no longer holds in the new many-valued system. The latter case may lead to further analysis into the precise reasons why the property no longer holds.

3. Attempts are made to complement and extend the results of the previous step by new results. Of particular interest are properties that make use of the many-valuedness of the proposed logics.

1.4 Structure of the work

Beyond this introductory chapter, this thesis is structured into three parts. Chapter 2 provides preliminaries and a survey of the literature on logics for games. The chapter begins with an introduction to game theory. The section aims to provide some context for the logics discussed later as well as a formal basis for the concepts used in the following chapter. Section 2.2 gives an overview of the state of the art in logics for games. A foundation for working with Łukasiewicz logic in the context of this thesis is provided in section 2.3.

Chapter 3 is an exposition of the many-valued generalization of forcing power logics. Its four sections investigate different aspects of the proposed logics. Section 3.1 builds on the introductory chapter to formally define the language and semantics of the logics. This is followed by a discussion on how the logics relate to normal modal logics in section 3.2. The effect of information in games and how this relates to the duality of modal operators is the theme of section 3.3. The generalization of the two-valued power bisimulation result is presented in section 3.4. The section also contains discussion of other types of equivalences for structures.

The last chapter provides a summary of the work and analyses it with regard to the stated aims. Following that, recommendations for further research are presented.

CHAPTER 2

State of the art

2.1 Game theoretic preliminaries

The core of this chapter reviews concepts from the literature on the various existing instances of logics for games. To describe these logics in appropriate detail and to discuss their strengths and weaknesses will require the introduction of some game theoretic notions beforehand. The natural starting point is a clarification of what exactly is understood as a *game* and a formal introduction to some fundamental properties of games. This also serves as a foundation for the analysis of the logics for games introduced in the next section. Further aspects of game theory will be introduced throughout the thesis when needed. A comprehensive review of concrete games in the literature will not be part of this work. From the perspective of logics for games, any game is just a specific structure. A detailed discussion of many different games and their properties can be found in [17]. Some frequently studied games will be introduced throughout this work as motivating and illustrating examples.

This thesis considers only finite games. If not stated otherwise, any reference to an extensive game form implicitly refers to a *finite* extensive game form. The same applies to finite extensive games. Later sections discuss some of the problems that would arise if infinite games were allowed. The definitions in this section have been adapted from those used by Osborne and Rubenstein in [36].

Definition 2.1. A *finite sequence* is an ordered list of objects written as $(a_1, a_2, \ldots, a_K)$ or $(a_k)_{k=1,\ldots,K}$ for a sequence of length K. For the empty sequence ($K = 0$), we write $()$.

The *concatenation* operator for adding elements to a sequence is denoted by $\circ$:

$$(a_1, \ldots, a_K) \circ b = (a_1, \ldots, a_K, b)$$

Definition 2.2. A *(finite) extensive game form* $\mathbf{G}$ is a tuple

$$\left\langle N, M, H, t, \{\sim_n \subseteq H^2 \,|\, n \in N\}\right\rangle$$

where:

- N is a non-empty finite set of *players*.
- M is a non-empty finite set of *moves*.
- H is the set of *histories*, those finite sequences of moves that are possible. It satisfies the following properties:
  - $() \in H$.
  - If $(a_k)_{k=1,\ldots,K} \in H$ and $L < K$ then $(a_k)_{k=1,\ldots,L} \in H$.
- $t : H \to N$, the *turn fuction* is a partial function mapping every non-terminal history to the player whose turn it is to make the next move.
- For every player n, $\sim_n$ is an equivalence relation such that if for any two non-terminal histories h_1, h_2, where $t(h_1) = t(h_2) = n$, player n can not distinguish between them iff $h_1 \sim_n h_2$. In particular, for any move $m \in M$, if $h_1 \sim_n h_2$ then $h_1 \circ m \in H \Rightarrow h_2 \circ m \in H$.

Definition 2.3. An *(finite) extensive game* $\mathcal{G}$ is a tuple $\langle \mathbf{G}, \{\leq_n \,|\, n \in N\}\rangle$ where $\mathbf{G}$ is a finite extensive game form with terminal histories T, and $\leq_n \subseteq T^2$ is a linear order expressing player n's preference.

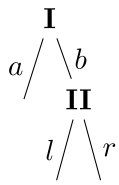
The game forms can be thought of as the rules defining the way the game is played. To analyze decision making in the game, some kind of evaluation of the outcomes has to be added to game forms. In the defined extensive games this is provided by the linear ordering of terminal histories. Transitivity and antisymmetry are natural properties of any notion of preference and motivate the restriction to linear orders.

Definition 2.4. For an extensive game form $\mathbf{G}$, let T be the set of terminal histories. A *payoff function* for a player n in $\mathbf{G}$ is a function $u_n : T \to \mathbb{R}$.

Often a payoff is more natural and easier to define than the raw preference relation. In those cases where only payoffs are given the preference is implicitly understood to be the ordering of the payoffs, i.e., for terminal histories t_1, t_2 and player n:

$$t_1 \leq_n t_2 \iff u_n(t_1) \leq_{\mathbb{R}} u_n(t_2)$$

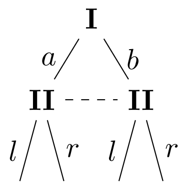
Extensive definitions of games correspond to defining games as rooted trees. Figure 2.1 gives examples of two such trees. Intuitively, vertices indicate turns, edges are the possible moves and a play starts at the root. The histories are all the paths from the root. In the context of game trees, it is sometimes simpler to think of states of the game (vertices of the tree) instead of histories, the two concepts are equivalent (every vertex is the end of a unique history) and will be used interchangeably. Note that the leaves of the game tree correspond to the terminal histories. The empty sequence () will also be referred to as the *root history*. The relations $\sim_n$ are indicated by dashed lines between states like in figure 2.1c. Game forms will be defined graphically by such a tree when complexity permits. The graphic definition is often easier to process, and the extensive game form can be uniquely determined from the tree.



(a)

$$
\begin{aligned}
&N = \{\mathbf{I}, \mathbf{II}\}, \quad M = \{a, b, l, r\} \\
&H = \{(), (a), (b), (b, l), (b, r)\} \\
&t(()) = \mathbf{I}, \quad t((b)) = \mathbf{II} \\
&\sim_{\mathbf{I}} = \sim_{\mathbf{II}} = \{(h, h) \mid h \in H\}
\end{aligned}
$$

(b) Formal game form of 2.1a



(c)

$$
\begin{aligned}
&N = \{\mathbf{I}, \mathbf{II}\}, \quad M = \{a, b, l, r\} \\
&H = \{(), (a), (b), (a, l), (a, r), (b, l), (b, r)\} \\
&t(()) = \mathbf{I}, \quad t((a)) = \mathbf{II}, \quad t((b)) = \mathbf{II} \\
&\sim_{\mathbf{I}} = \{(h, h) \mid h \in H\} \\
&\sim_{\mathbf{II}} = \{(h, h) \mid h \in H\} \cup \{((a), (b)), ((b), (a))\}
\end{aligned}
$$

(d) Formal game form of 2.1c

Figure 2.1: Simple game trees and their extensive form definitions.

Definition 2.5. The equivalence classes formed by the $\sim$ relations are called *information sets*. The set of a game form $\mathbf{G}$'s information sets for player n is denoted by $\iota_n(\mathbf{G})$.

If an information set has exactly one element, it is called *trivial*.

In figure 2.1c a non-trivial information set exists. It expresses that **II** does not know whether a or b was played in the first move. Thus **II** is not able to act differently depending on what was played first. This inability to distinguish histories will be made clearer by formalizing strategies. With regards to graphical notation this also illustrates that in information sets the labels of edges, determining the moves, matter. For simplicity, when edge labels are omitted, moves are assumed to be ordered left to right in the same way for all histories of the same information set.

Definition 2.6. A *strategy* σ_n of a player n in a game form $\mathbf{G}$ is a function $\iota_n(\mathbf{G}) \to M$ such that:

$$\forall X \in \iota_n(\mathbf{G}).\, \forall h \in X.\, h \circ \sigma_n(X) \in H$$

The set of all strategies of a player n is denoted by Σ_n.

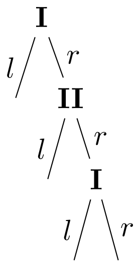
Definition 2.7. A *strategy profile* s of a game form $\mathbf{G}$ with player set $N = \{n_1, n_2, \ldots, n_k\}$ is a tuple

$$s = (\sigma_{n_1}, \sigma_{n_2}, \ldots, \sigma_{n_k})$$

where σ_n is a strategy of player n in $\mathbf{G}$.

Under the assumption that players will play the strategies in a given profile, the profile leads to at most one terminal history (none only if the play goes on infinitely) because the strategy functions are deterministic and only allow moves extending the history. Hence, a game's preference relation can be extended directly to strategy profiles by ordering them according to their corresponding terminal history.

Remark. Definition 2.6 requires a strategy to assign a move even to unreachable states. As an example consider the two strategies $\sigma_{\mathbf{I}}, \sigma'_{\mathbf{I}}$ for the game form of figure 2.2. By the



$$\begin{aligned} \sigma_{\mathbf{I}}(()) &= l, &\qquad \sigma'_{\mathbf{I}}((r,r)) &= r \\ \sigma'_{\mathbf{I}}(()) &= l, &\qquad \sigma'_{\mathbf{I}}((r,r)) &= l \end{aligned}$$

Figure 2.2

given definition, $\sigma_{\mathbf{I}}$ and $\sigma'_{\mathbf{I}}$ are different strategies. But they only differ at a history that will never be reached when following either of the strategies. This can lead to unintuitive situations, especially when counting strategies. There exist alternative definitions of strategies that avoid this issue. For example, when defining games directly as game trees, strategies can be defined as particular subtrees (cf. [33]).

The effect of information sets on strategies is illustrated by the game in figure 2.3. In the end points **w**/**l** represent win/lose scenarios for **II** respectively. Without the information set linking both of the possible turns of **II** the strategy to win would be to answer a with r and b with l. However, **II** does not know what was played because $(a) \sim_{\mathbf{II}} (b)$. The only possible strategies for **II** are thus to either always play l or always play r.

The existence of non-trivial information sets has a deep effect on many theoretical properties of games and strategies. It makes sense to distinguish between these two types of games.

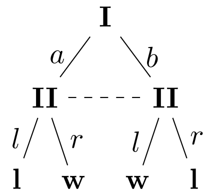


Figure 2.3

Definition 2.8. A *perfect information game form* is an extensive game form where

$$\forall n \in N.\ \sim_n = \{(h,h) \mid h \in H\}.$$

Extensive games that do not satisfy this condition are called *imperfect information game form*.

When considering games with payoffs, there are additional possibilities of classifying games according to the properties of the payoff functions. Consider the example of chess, if white wins, black loses and vice versa; if white draws so does black. This type of symmetry is intriguing and appears in a variety of situations. In broad terms, it represents the fact that one player can only gain something at the expense of another player's loss. This is an interesting property because moves that are good for a player will always be bad for another, creating a situation where players can not escape competitive behavior without sacrificing self-interest. This property is formalized in the form of constant-sum games below.

Definition 2.9. A extensive game $\mathcal{G}$ with payoff functions $\{u_n \,|\, n \in N\}$ is called *constant-sum* if for all terminal histories t there is a $c \in \mathbb{R}$ such that:

$$\sum_{n \in N} u_n(t) = c$$

When $c = 0$, $\mathcal{G}$ is also called *zero-sum*.

Besides extensive form there exists another common representation of games, so-called *strategic form* (also called *normal form*). A game in strategic form only consists of players, all their possible strategies and payoff functions for the strategy profiles. Strategic form is particularly useful in games where strategies are independent of the other players' moves such as the game of figure 2.3 discussed above. A definition of this game in strategic form is given in figure 2.4. Rows represent the strategies of **I**, columns those of **II**, and the outcome of a strategy profile is seen at the crossing of its components. This, often convenient, representation is also called a *payoff matrix* or a *matrix game*.

I \ II	l	r
a	**l**	**w**
b	**w**	**l**

Figure 2.4: The game of figure 2.3 in strategic form.

Rationality & Game solutions

The end product of many game theoretical analyses is an evaluation of how well a game can or will end for a player and an assessment of the relative merit of strategies. Analysis in this direction usually presupposes a fixed view of what it means for players to act rationally. Figure 2.5a is used as a guiding example, with the pair at the terminal position representing the payoff for **I**, **II** respectively. If **I** takes the move to the right, **II** could play left and leave **I** with no payoff at all. A cautious **I** would play the left move even though both players would prefer outcome $(2, 2)$ over $(1, 0)$. This pessimistic view where a player assumes the worst outcome from other players actions is the default type of rationality considered in games. It is the most basic and general concept of rationality, it considers all moves of an opponent possible without presuming any preference or rationality of the opponent. Even in figure 2.5b where the rational move of **II** would be to play right a rational **I** can not act under this assumption. This model of rationality will be the one used throughout this work.

(a) (b)

Figure 2.5: Rationality examples

There is no single notion of a solution of a game. The precise meaning varies depending on the application and the characteristics of the game itself. In general, a solution represents an expected outcome of a game. Such an outcome may be the strategies chosen by rational players, expected payoffs, points of equilibrium and others. One fundamental concept for finding solutions is that of *dominating strategies* (defined below). These are strategies that perform at least as good as any other strategy irrespective of opponent actions. If such a strategy exists it is expected to be played because it is, by definition, always better than any alternative.

On the other hand, the dual concept of *dominated strategies* can be used to limit the strategy space to analyze. Recognizing dominated strategies is the basis of a common algorithm for solving strategic games called Iterated elimination of dominated strategies

(IEDS). By removing strategies that a rational player would never play, a new, smaller game is created. This new game may again contain a dominated strategy and if so the process is repeated. An in-depth analysis of IEDS variants and results can be found in most game theory texts, e.g., [36].

Notation. For a strategy profile s, we write $s[n/\sigma]$ for the strategy profile where player n's strategy in s is replaced by the strategy σ.

Definition 2.10. Let $\mathcal{G}$ be an extensive game and σ^*, σ two strategies of a player n. σ^* *dominates* σ if for every strategy profile s:

$$s[n/\sigma^*] \geq_n s[n/\sigma]$$

A strategy of a player n is a *dominating strategy* if it dominates all other strategies of player n.

σ^* *is dominated by* σ if for every strategy profile s:

$$s[n/\sigma^*] <_n s[n/\sigma]$$

A strategy of a player n is a *dominated strategy* if it is dominated by at least one other strategy of player n.

Dominance is a strong property for a strategy to have, and often no dominant strategy exists. A different way of judging the outcome of a game is that of a Nash equilibrium. It weakens the global nature of dominance by considering a sort of dominance local to fixed opponent strategies. Such local dominance of a strategy is too specific to be useful in most cases. Instead, one is interested in strategy profiles where every player has a locally dominant strategy.

Definition 2.11. A strategy profile s is called a *pure Nash equilibrium* of an extensive game $\mathcal{G}$ if:

$$\forall n \in N. \forall \sigma \in \Sigma_n.\ s \geq_n s[n/\sigma]$$

where Σ_n is the set of all strategies of player n.

A pure equilibrium can be thought of as a strategy profile where, in retrospect, no player regrets their choice of strategy. Figure 2.6a describes a much-studied game commonly called Chicken [30]. Assume that two players each drive a car towards each other. They have the option of staying straight on a collision course or evading the other. Evading is interpreted as cowardice, a combination of one player going straight with the other evading is construed as a negative result for the evader, positive for the other. If both evade, the outcome is considered neutral. If nobody evades, the result is a crash which is seen as the worst outcome. The pure Nash equilibria of Chicken are STRAIGHT/EVADE and EVADE/STRAIGHT. Take the profile where **I** played STRAIGHT and **II** EVADE as an example. If **II** had played STRAIGHT instead, the payoff would be -5 instead of -1. If **I** had played EVADE instead, the payoff would decrease from $+1$ to 0.

Thus, in hindsight nobody regrets their move. Importantly, every player considers only changes to their own strategy while assuming the other players will not change theirs. Note, that in this game, no player has a dominating strategy.

Figure 2.6b gives a game with no pure Nash equilibrium. In this game, two players each take a coin and select a side of the coin. They then reveal their choices at the same time. If both choose the same side, **I** gets both coins; else **II** gets them. For every strategy profile, the loser will regret his choice because a change in strategy would always make him the winner.

I \ II	STRAIGHT	EVADE
STRAIGHT	$-5, -5$	$+1, -1$
EVADE	$-1, +1$	$0, 0$

(a) Chicken.

I \ II	HEADS	TAILS
HEADS	$+1, -1$	$-1, +1$
TAILS	$-1, +1$	$+1, -1$

(b) Matching pennies.

Figure 2.6

An important extension to the defined pure strategies is that of *mixed strategies*. A *mixed strategy* is the set of pure strategies together with a probability distribution that determines the likelihood of playing a strategy. For example, in the matching pennies game of figure 2.6b there exist the mixed strategies of playing HEADS with probability p and TAILS with probability $1 - p$ for $0 \leq p \leq 1$. The payoff of a mixed strategy profile is the expected value of the pure strategy profile under consideration of the probability distribution. This way pure Nash equilibria can naturally be extended to mixed Nash equilibria. What makes mixed strategies important is the fact that every finite non-cooperative game has at least one mixed Nash equilibrium [34]. For the matching pennies game, the equilibrium is for both players to play each strategy with equal probability. The resulting mixed strategy profile has an expected payoff of 0 for both players. Formal details will be foregone at this point as the content of this thesis is concerned only with pure strategies.

The previous examples for pure Nash equilibria demonstrated that in imperfect information games, the existence of pure equilibria depends on the particular preference relation. In the case of perfect information, the situation is different. Again consider the intuition of not regretting the choice of strategy. Under perfect information, a strategy can individually respond to every move, and every player knows all continuations of every history. Therefore, hindsight is the same as foresight. Regretting a choice can only be the result of a mistake as every possible outcome could have been considered beforehand. Thus, a theoretical player, who is assumed not to make mistakes, would always have the possibility of playing regret-free. This intuition holds up in general and leads to one of the classic results of game theory.

Theorem 2.1 (Zermelo's Theorem)**.** *Every finite perfect information extensive game has a pure Nash equilibrium.*

The result in this form is actually due to Kuhn [27]. The original result by Zermelo [57] predates the field of game theory. Zermelo proved that in a game of chess either there exists a strategy for white to win or a strategy for black to win or both sides can force a draw. The proof does hint at the general applicability of the proof strategy for a class of games that roughly corresponds to two player games with perfect information, hence the theorem is usually named after him. In fact, different versions of this result and variations are all called Zermelo's theorem in the literature (cf. [46]).

It should be emphasized that the presented introduction only scratches the surface of the current state of game theory. Furthermore, the selection of topics here is not representative of their relative importance. Mixed strategies and equilibria play a central role in many applications of game theory. However, for the logics developed in this thesis, the probabilistic nature of mixed strategies is problematic. Therefore, this section is biased towards the discussions of pure strategies.

2.2 Logics for games

As mentioned in the introduction, games have become an accepted tool in various mathematical applications. Demand for the study of games as structures is a natural consequence that is addressed by a logical approach. How logic has been used to study games will be the content of this section. Games are very rich structures, their main components are all very different types of mathematical objects, information sets are equivalence classes, preferences are orders and histories are particular sequences. It is difficult to develop formalisms that handle all these different objects at the same time and the same level. Therefore, it seems sensible to develop different logics for different purposes. This has lead to a large variety of logics for games being proposed.

This section will attempt to provide an overview of two of the main threads of research in logics for games – namely, the study of game equivalence via invariance of formulas and the study of strategies. Accordingly, the contents of this section are split into two categories. First, under the umbrella of structure oriented, are logics that either operate on extensive game forms directly or are at least aimed at reasoning about the structures themselves. The other group is made up of the strategy oriented logics for games. Here strategies or strategy profiles are the principal objects of study. These categories should not be seen as strictly separate. There are some areas of overlap between the two groups. A more extensive survey of logics for games are can be found in a recent monograph [55] and a survey article in the Handbook of Modal Logic [56]. They both provide thorough extensions to this section.

2.2.1 Structure oriented logics

The study of equality of objects is a common theme in many mathematical fields including game theory. The initial study of game equivalence was focused on structural equivalence. Such a notion of equivalent game structures is proposed in [25], and a set of structural transformations that preserve this equivalence are presented by Thompson in [49]. In logic, the equivalence of structures is often based on invariant formula evaluations across structures. Many proposed logics for games with a structural orientation aim at providing further types of game equivalence.

In [53], van Benthem proposes multiple logics to such an end. The work is based on the similarity of extensive game forms to the process models that are used, e.g., in model checking (cf. [48, 10]). First, different levels of a modal action language, with actions corresponding to moves in the game, are presented for perfect information games. This goes as far as allowing for μ recursion and composition of actions. The ultimate goal in adapting these languages from process models was to also adapt the use of bisimulation as a notion of equivalence. The formulas of these action languages are shown to be invariant under bisimulation.

In the same paper, another approach is suggested that is based on the concept of forcing powers. These forcing powers represent the different sets of outcomes in which a player can force the game to end in (see also section 3.1). The associated modal language then expresses the fact that the truth of a statement can be forced, i.e., play can be steered in such a way that the statement is guaranteed to be true at the end of the game. Because a player can, in general, not force a single outcome but only a set of outcomes the models of this forcing power logic are not the usual modal frames with world to world accessibility. Instead, the more unusual world to set accessibility is used. Analog to the bisimulation of the action languages, power bisimulation (cf. section 3.4) is introduced for these structures with world to set accessibility. The forcing power language is shown to be invariant under power bisimulation.

The modal action language from above was limited to perfect information games. In [52] an epistemic action language is proposed that allows for similar reasoning under perfect and imperfect information. An epistemic modal operator is introduced that expresses truth in all the states of an information set, i.e., a player knows that a formula holds at a state s iff it holds at every state in the information set that contains s. Again an appropriate version of a bisimulation was developed (with an extra back-and-forth condition for information) that preserves formula invariance.

The logics discussed above consider only the capabilities and interests of individual players. This type of behavior sometimes leads to unintuitive results in game theory, e.g., games were cooperation would yield a better payoff for all players than pursuing their individual strategies. Pauly proposed a logic for reasoning about the capabilities of groups of actors that cooperate called coalition logic [40]. However, there are some noteworthy differences to the logics mentioned before. The structures of coalition logics are not games. Rather they consist of a set of states where each state is assigned its own strategic game form. The transitions between states are determined by the played strategy profiles. Therefore, the structures are not extensive game forms as before, but graphs. Notably, there is also no terminal state in coalition logics. Again an appropriate bisimulation has been shown to preserve formula valuation. It is also related to the social software programme [39], which proposes a study of social procedures using the same methods as the study of computer programs. Social software provides varied possibilities of application of logics for games. Coalition logic has recently been generalized to many-valuedness [26]. In particular, the question of whether a coalition is effective for a formula has been generalized beyond two truth values.

2.2.2 Strategy oriented logics

Strategies were introduced as functions on histories. This way of defining strategies is often convenient, but not well suited for the study of strategies themselves. Intuitively a strategy in a game is thought of in terms of replies to opponent moves, i.e., "if my opponent does a, I will play b, but if they do c, I will reply with d." This structure can only be observed indirectly when strategies are represented by functions. Thus, strategy oriented logics focus on other possibilities of representing strategies.

A popular approach is to express strategies via propositional dynamic logic (PDL) [22].

Single moves of a game are taken as atomic actions which can then be combined by operations such as choice, conditional execution, dual and composition. The resulting action terms can be seen as programs of sorts. In [54] the use of such PDL formulas as explicit strategies is discussed. The same work also extends the aforementioned modal action logics of [53] to allow for compound actions in the form of PDL formulas. Instead of using moves as atomic actions, one can also use the operations of PDL to combine games to form more complex games. Game Logic (GL) [38, 41] is based on this idea. Its modal operators are based on the same forcing powers that were mentioned before, but instead of having fixed games as structures PDL formulas are used to build up different games from atomic games. Applications of PDL in games are limited by the fact that role change is expressed by the dual operation. This is only enough to switch between two roles, i.e., two players. Because of this GL is a logic specifically for two-player games.

The boolean games proposed by Harrenstein, van der Hoek, Meyer and Witteven in [23] provide a way of expressing strategies in propositional logic. Boolean games are strategic games in which the players assign truth values to variables. Players are assigned formulas that represent their payoff function and a set of variables over which they have control. A strategy profile then represents a full variable assignment under which the payoff formulas can be evaluated. Using these logical representations of payoffs, formula schemata, parameterized by strategy profiles, can be used to express properties of profiles. This approach is limited by the fact that it only works with payoff functions expressible in propositional logic. As a result, this approach has been generalized to many-valuedness by the Łukasiewicz games of Marchioni and Wooldridge [29] and further to a more general class of many-valued logics by Běhounek, Cintula, Fermüller and Kroupa [3]. The latter also contains general ways to construct formulas that express pure and mixed Nash equilibria for a wide variety of cases. In contrast to most of this section, the logics of this paragraph are standard propositional languages without any extensions for games.

Another, different approach is based on temporal logics. Alternating-time temporal logic (ATL) was proposed by Alur et al. [2] as an extension of computation tree logic (CTL). The trees on which the logic operates are the full expansions of play for all strategy combinations. That is, universal quantification refers to all possible moves and all possible answers. The difference to CTL is that in ATL branch quantifiers are parameterized by sets of players. Thus, ATL is a system for reasoning directly on strategies. The work demonstrates how various problems of game theory correspond to model checking problems in ATL.

Strategy logic by Chatterjee, Henzinger and Piterman [5] goes one step further than ATL. Strategy logic is a first-order language where the principal objects are strategies while also retaining the capability for temporal reasoning by allowing for operations of linear temporal logic (LTL). Instead of quantifying over trees generated by strategies as in ATL, strategy logic directly allows for quantification over strategies through first-order quantification. ATL can, in fact, be fully expressed in strategy logic. Meta-level quantification over strategies frequently happens in definitions of game theoretic properties, e.g., dominating strategies. This provides a clear motivation for such an approach for reasoning about strategies. Strategy logic has been shown to be decidable, albeit with nonelementary complexity. A further, similar, strategy logic was proposed by Mogavero [32] based on perceived shortcomings of the strategy logic of Chatterjee, et al.

Viewing strategies akin to programs is an apparent theme of many of the mentioned approaches. The variety of the different existing models of computation leads to different possibilities of representing strategies by the programs or machines that play them. One example in this area is the treatment of strategies as automata as proposed by Ramanujam [43, 42]. A further link between games and computations is shown by Japaridze's computability logic [24]. There, game-semantics, on the basis of a Turing machine playing against an environment, are used to build a logical framework for computations.

2.2.3 Conclusion

This section summarized a variety of existing logics for games. It is apparent that there is a broad spectrum of approaches with differing goals and for most of them a many-valued generalization is still open. In general, the logics here are also not widely studied. For most of them, there exists little literature beyond the initial results, which makes a comparison between them difficult.

Forcing power logic was chosen as the focus of our own approach because it is conceptually based on the possible outcomes of games. This fits well with the initially stated primary motivation for a many-valued system, modeling outcome values naturally. Beyond that, invariance under power bisimulation becomes even more significant with the ability to directly relate payoffs. The presented modal action logic shares these characteristics for the most part. Ultimately, its limitations with regards to imperfect information motivated the use of forcing power logic in this thesis.

2.3 Łukasiewicz logic

If not stated otherwise, all the logics mentioned in the previous section are *two-valued logics*. That is, their formulas can be just either true or false. This can work well for statements such as "**I** wins the game" or "playing the game results in positive value for **I**" that are binary in nature. However, in general, games can have a fine-grained preference of outcomes. Take the two statement φ = "I win 2€" and ψ="**II** wins 2€" in a two player game where every player can either win 2€, win 1€ or win nothing. Say the game is played, **I** wins 1€, **II** wins nothing. Now, φ is obviously not fully true, but one could argue that it is more true than it would be if **I** had won nothing. The statement can be considered to be half true. By adding this additional truth degree, richer comparisons between φ and ψ are possible, e.g., φ being more true than ψ represents that **I** did better in the game than **II**. With only two degrees of truth, φ and ψ would both be equally false, and the additional information would be lost. Of course the fact that I win more money than my opponent can also be modeled in a two-valued system. The point of the example is to demonstrate that many-valued systems can be a suitable tool to make use of the full preference relation in a practical and flexible way.

This section will introduce the many-valued system used by the logics of this thesis and aims to provide a basic working knowledge of it. Some results from the literature that are of particular relevance to this thesis will be presented. Further details will be introduced in chapter 3 where needed. This section is based on [6, 18, 4].

The work in this thesis is based on propositional Łukasiewicz logic, first proposed by Jan Łukasiewicz [28]. A finite Łukasiewicz logic Ł_k ($k \in \mathbb{N}, k > 1$) has the set of truth values $\{\frac{i}{k-1} \,|\, 0 \leq i \leq k-1\}$. For infinite Łukasiewicz logic Ł_∞, the set of truth values is the closed interval $[0, 1]$. The name of a logic will also be used to denote its set of truth values. When a statement refers to a general Ł_k the intention is that this includes Ł_∞. The truth values 0 and 1 will also be referred to as absolute false and absolute true respectively.

Given a set of propositional variables $\mathcal{PV}$, the syntax of Łukasiewicz logics formulas is defined by:

$$\varphi ::= p|\top|\bot|\neg\varphi|\varphi \supset \varphi|\varphi \wedge \varphi|\varphi \vee \varphi|\varphi \,\&\, \varphi|\varphi \oplus \varphi$$

where $p \in \mathcal{PV}$. A *valuation function* v for a Ł_k is a function that assigns a truth value to every formulas. Every propositional variable is assigned a truth value, for compound formulas v is defined in table 2.1.

Note that the language can be defined from only $\supset$ and $\bot$ with the following identities [6].

$$\begin{aligned}
\neg\varphi &\equiv \varphi \supset \bot, & \top &\equiv \neg\bot \\
\varphi \oplus \psi &\equiv \neg\varphi \supset \psi, & \varphi \,\&\, \psi &\equiv \neg(\neg\varphi \oplus \neg\psi) \\
\varphi \vee \psi &\equiv (\varphi \,\&\, \neg\psi) \oplus \psi, & \varphi \vee \psi &\equiv \neg(\neg\varphi \vee \neg\psi)
\end{aligned}$$

Symbol	v	Name
$\bot$	0	Falsum
$\top$	1	Verum
$\neg$	$v(\neg\varphi) = 1 - v(\varphi)$	Negation
$\supset$	$v(\varphi \supset \psi = \min\{1, 1 - v(\varphi) + v(\psi)\}$	Implication
$\wedge$	$v(\varphi \wedge \psi) = \min\{v(\varphi), v(\psi)\}$	Weak Conjunction
$\vee$	$v(\varphi \vee \psi) = \max\{v(\varphi), v(\psi)\}$	Weak Disjunction
$\&$	$v(\varphi \,\&\, \psi) = \max\{0, v(\varphi) + v(\psi) - 1\}$	Strong Conjunction
$\oplus$	$v(\varphi \oplus \psi) = \min\{1, v(\varphi) + v(\psi)\}$	Strong Disjunction

Table 2.1: Operators of Łukasiewicz logic.

An axiomatization for $Ł_\infty$ is given by the following axiom schemata with modus ponens.

$$\varphi \supset (\psi \supset \varphi) \tag{2.1}$$

$$(\varphi \supset \psi) \supset ((\psi \supset \chi) \supset (\varphi \supset \chi)) \tag{2.2}$$

$$((\varphi \supset \psi) \supset \psi) \supset ((\psi \supset \varphi) \supset \varphi) \tag{2.3}$$

$$((\varphi \supset \bot) \supset (\psi \supset \bot)) \supset (\psi \supset \varphi) \tag{2.4}$$

It was shown to be complete by Rose and Rosser in [44]. Complete axiomatizations for finite Łukasiewicz logics exist, but are more complex, cf. [50].

Among other things, we will use formulas to represent goals in a game by combining subgoals with logical connectives. For example, a player has two distinct goals, but ultimately only wants to have as much success as possible in either one, the worse one is ignored. Combining the two goals via a weak disjunction would express this. The question then arises what functions can be expressed by formulas of Łukasiewicz logic. McNaughton showed that in $Ł_\infty$ for every function $f : [0,1]^n \to [0,1]$ of the form:

$$f(x_1, \ldots, x_n) = \min\{1, \max\{0, b + m_1 x_1 + \cdots + m_n x_n\}\} \qquad b, m_1, \ldots, m_n \in \mathbb{Z}$$

there exists a formula φ with variables $p_1, \ldots, p_n$ such that $v(\varphi) = f(x_1, \ldots, x_n)$ when $v(p_i) = x_i$ for $1 \leq i \leq n$ [31].

For some many-valued logics, there already exist strong connections to game theory. For Łukasiewicz logic, this connection comes in the form of Giles's game [1]. The game is played by two players and starts with each asserting any number of propositional formulas. The moves of the game then consist of players attacking and defending such asserted formulas. Through these moves, the formulas are ultimately reduced to their atomic parts. Asserting an atom is interpreted as betting on it to be true. To evaluate the outcome of these bets, each atom is also assigned a risk of losing the bet. At the end of play the player with the lower sum of risks wins as they are expected to win money from the betting. Ultimately, the important part is that the validity of a formula $Ł_\infty$ was shown to be equivalent to the existence of a winning strategy for one of the players for every risk assignment. These results have been extended to other many-valued logics, including finite-valued Łukasiewicz logics, in [13, 15]. Further work also directly relates strategies in Giles's game to derivations in hypersequent calculi [14]. We extend Łukasiewicz logic with a modal operator that expresses a player having the power to make a formulas true. Extending a game-based calculus, like Giles's game, could be a natural way of handling the modal operator in a proof system. One step further, this would also enable the proposed logics to handle formal proofs as structures.

Many-valued logics in general are studied under the umbrella of mathematical fuzzy logic [8, 7]. An important part of this field is the study of *(continuous) t-norm fuzzy logics*. These form a family of logics with the interval real [0,1] as their truth values and a binary operator that follows certain rules called the *t-norm*. $Ł_\infty$ is a prominent t-norm fuzzy logic (& is the t-norm). Other well-known examples include Gödel-Dummet logic and Product logic, which have the minimum and the real product as their respective t-norms. Notably, all continuous t-norm fuzzy logics allow one to define the semantics of weak disjunction and conjunction as maximum and minimum respectively. Keeping this in mind it is apparent that some of the results of chapter 3 would also hold with the semantics of a different t-norm fuzzy logic. T-norm fuzzy logics and other proposals deal with many-valued logics in a more general way (e.g., [9]) that would provide the natural next step to generalize the logics of this work beyond Łukasiewicz semantics.

CHAPTER 3

$\mathcal{FP}$ – A family of many-valued forcing power logics

This chapter contains the presentation of a family of many-valued logics for finite extensive game forms. $\mathcal{FP}$ is a many-valued extension of the forcing powers logic of van Benthem [53] (cf. 2.2.1). The motivation to use this logic as a basis is multifold: Forcing powers provide an elegant way to handle games with imperfect information in the same way as perfect information games. This is in contrast to many other structure-oriented logics for games in which imperfect information leads to complications. Furthermore, the initial bisimulation result for propositional forcing power logic is intriguing, a generalization to MV could provide a promising type of game equivalence. As a final point, aside from the initial results by van Benthem, there is little study of forcing power logics in the literature. A broader study, beyond the generalization of results to many-valuedness may be worthwhile.

Section 3.1 provides a formal foundation of the language and its core elements as well as some introductory examples. The logics in $\mathcal{FP}$ are non-normal modal logics. Details as to what parts of normal modal logics get lost, and their connection to games, are presented in section 3.2. Section 3.3 investigates the role of information in the proposed logics. It also introduces reduced game structures as a proof technique for connecting different classes of games. The final section 3.4 is on the topic of game equivalence. In particular, it also contains a generalization of the aforementioned bisimulation result to the logics of $\mathcal{FP}$.

3.1 The language of $\mathcal{FP}$

3.1.1 Forcing Powers

Definition 3.1. Let $\mathbf{G}$ be a finite extensive game form with terminal histories T and the set of all its strategy profiles S. A *terminal function* $\tau : H \times S \to T$ is a function that for a given strategy profile s and history h assigns the terminal history at which the play starting from h will end if all players play according to their strategies in s.

Lemma 3.1. *For every finite extensive game form* $\mathbf{G}$*, the terminal function* τ *is uniquely determined.*

Proof. The proof proceeds by backwards induction on the length of histories. For the base case, let t be a terminal history. In a terminal history no more moves can be made, all plays starting from t will also immediately end at t, so $\tau(t, s) = t$ for all strategy profiles s.

For the induction step, consider an arbitrary non-terminal history h. If every h' that extends h by exactly one turn has a unique $\tau(h', s)$ for all strategy profiles s, then, $\tau(h, s)$ is also unique for all s: Let h be the history s.t. $h \circ m = h^*$ where m is a single move. For every strategy profile s, $\tau(h, s)$ is unique:

The turn function assigns exactly one player n to h. Let σ_n be the strategy of player n in s. Recall that strategies are functions, so $h \circ \sigma_n(h)$ is the uniquely determined next history of the play. By the induction hypothesis τ is uniquely determined for this successor history. $\square$

Notation. The notation for *forcing relations* is carried over from the two-valued forcing power logic (cf. [51]). For a binary relation R, we write $R\,a, b$ to denote that $(a, b) \in R$.

Definition 3.2. Let $\mathbf{G}$ be an extensive game form with a player n, the player's set of strategies Σ_n, the set of strategy profiles S and terminal function τ. The *forcing relation* $\rho^n_{\mathbf{G}}$ for n in $\mathbf{G}$ is defined as:

$$\rho^n_{\mathbf{G}} h, X \iff \exists \sigma \in \Sigma_n . \forall s \in S.\, \tau(h, s[n/\sigma]) \in X$$

If $\rho^n_{\mathbf{G}} h, X$ holds, then we say that X *can be forced* from h by n. Such X will also be called *forceable sets* (for player n).

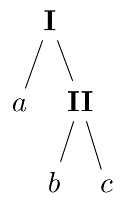
The *forcing power* $\Phi^n_{\mathbf{G}}(h)$ of a player n in $\mathbf{G}$ from h is defined as:

$$\Phi^n_{\mathbf{G}}(h) \equiv \{X \in \mathcal{P}(H) \mid \rho^n_{\mathbf{G}} h, X\}$$

Notation. When defining games graphically it is convenient to use symbolic names for the terminal histories instead of referring to them in sequence form. The names of the histories will be noted as labels at the leaves of a game tree like in figure 3.1. (Recall that terminal histories correspond to leaves in a game tree.)

Figure 3.1 provides an example for the above definitions. At the root, clearly player **I** can force set $\{a\}$ by choosing the left move. If **I** plays the right-side move it is guaranteed that the play will end at either b or c, giving a forceable set $\{b, c\}$, but **I** has no further control over the outcome so $\{b\}$ and $\{c\}$ are not forceable.

Player **II** is in a different situation. **I** can play left at the start without **II** having any control over the outcome. Therefore, every forceable set of **II** must contain a. If **I** plays right, **II** can chose between b or c, resulting in the two forceable sets $\{a, b\}, \{a, c\}$. The set $\{a, b, c\}$ is trivially forceable by both players as play will always end at some terminal history and the set contains all of them.



(a) The game form **G**

$$\Phi^{\mathbf{I}}_{\mathbf{G}}(()) = \{\{a\}, \{b, c\}, \{a, b, c\}\}$$
$$\Phi^{\mathbf{II}}_{\mathbf{G}}(()) = \{\{a, b\}, \{a, c\}, \{a, b, c\}\}$$

(b) Forcing powers of 3.1a

Figure 3.1: A game form and its forcing powers.

Lemma 3.2 ([53]). *Forcing relations are closed under supersets, i.e, for any* $\mathbf{G}, n, h, X, Y$ *if* $\rho^n_{\mathbf{G}} h, X$ *and* $X \subseteq Y$*, then* $\rho^n_{\mathbf{G}} h, Y$.

Lemma 3.3 ([53]). *If* $\rho^i_{\mathbf{G}} h, Y$ *and* $\rho^j_{\mathbf{G}} h, Z$*, then* Y *and* Z *overlap.*

Remark. Building a logic on the foundation of forcing powers makes the handling of games with histories of infinite length problematic. The given definition of the terminal function does not work because there can be histories that can never be extended to a terminal history. Beyond formal details it remains unclear how to semantically reflect the possibility of never reaching a payoff. Especially when language semantics are motivated by notions of rationality and preference like those of $\mathcal{FP}$. When does a player prefer a game going on forever over some payoff?

3.1.2 Syntax & Semantics

All logics of $\mathcal{FP}$ share the same syntax except for the number of modalities. There is a modal operator for every player. Therefore, the set of players has to be a parameter of the language. The underlying many-valued reasoning system of $\mathcal{FP}$ logics is that of Łukasiewicz logic. A second parameter of the logic specifies its number of truth values.

Often the parameters can either be inferred from context or are general. In those cases the parameters are ommited and the following convention is followed: Let $\mathcal{L}$ be a logic of the form $\mathcal{L}(N, k)$. Say $\mathcal{L}$ *is a logic for* $\mathbf{G}$, if $\mathbf{G}$ is a finite extensive game form with players N. Also, we say $\mathcal{L}$ *is based on* $Ł_k$, which means that it has the same set of truth values as $Ł_k$. In both cases only one parameter is fixed, e.g., an arbitrary $\mathcal{L}$ for **G** is general in the number of truth values and fixed in its players.

Definition 3.3. For every logic $\mathcal{L}(N, k) \in \mathcal{FP}$, with $k \in \mathbb{N} \cup \{\infty\}$, its formulas are defined in two layers, where $\mathcal{PV}$ is a countable set of propositional variables:

Its set of *base formulas* $BForm(\mathcal{L}(N, k))$ is defined inductively as the smallest set satisfying the following formation rules:

- $\bot \in BForm(\mathcal{L}(N, k))$
- If $p \in \mathcal{PV}$, then $p \in BForm(\mathcal{L}(N, k))$.
- If $\varphi, \psi \in BForm(\mathcal{L}(N, k))$, then $\varphi \supset \psi \in BForm(\mathcal{L}(N, k))$.

Variables and $\bot$ are also referred to as *atoms*.

Its set of *game formulas* $GForm(\mathcal{L}(N, k))$ is defined inductively as the smallest set satisfying the following formation rules:

- If $\varphi \in BForm(\mathcal{L}(N, k)) \cup GForm(\mathcal{L}(N, k))$, then $\{G, n\}\varphi \in GForm(\mathcal{L}(N, k))$ for all $n \in N$.
- If $\varphi, \psi \in GForm(\mathcal{L}(N, k))$, then $\varphi \supset \psi \in GForm(\mathcal{L}(N, k))$.

Formulas of the form $\{G, n\}\varphi$ are called *atomic game formulas*, where φ is called the *inner formula*.

Finally, $Form(\mathcal{L}(N, k))$ denotes the set of all formulas of $\mathcal{L}(N, k)$:

$$Form(\mathcal{L}(N, k)) = BForm(\mathcal{L}(N, k)) \cup GForm(\mathcal{L}(N, k))$$

In general, the name G will be used in the modal operator. When talking about specific games the name may correspond to the name of the game for clarity.

The other usual operators of Łukasiewicz logic are available as well. They can be defined from $\bot$ and $\supset$ as shown in section 2.3. For binary connectives, the same restrictions that exist for $\supset$, with respect to mixing base formulas with game formulas, apply. Additionally, $\varphi \equiv \psi$ will be used as a shortcut for $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$.

The languages of the logics of $\mathcal{FP}$ are propositional languages extended by unary operators $\{G, \cdot\}$. This operator will be called the *forcing modality*. Note that game formulas are not on the same level as base formulas. For $p, q \in \mathcal{PV}$, $p \supset \{G, \mathbf{I}\}(q)$ is not a formula of any $\mathcal{FP}$ logic because p is not a game formula. The motivation for excluding such formulas stems from the fact that we do not allow for variable assignments at non-terminal histories (cf. definition 3.4). This leads to base formulas only having a defined truth value at terminal histories while the truth of game formulas is defined at every history.

Definition 3.4 ($\mathcal{L}$-structures)**.** An $\mathcal{L}(N,k)$-*structure* $\mathcal{G}$ of a logic $\mathcal{L}(N,k) \in \mathcal{FP}$, is a tuple $\langle \mathbf{G}, v\rangle$ where $\mathbf{G}$ is a finite extensive game form with players N and terminal histories T. The *assignment* $v : \mathcal{PV} \times T \to V_k$ maps every propositional variable at any terminal history to a truth value in V_k, the set of truth values of $Ł_k$.

In general, the term $\mathcal{L}$-*structure* is used used for a logic $\mathcal{L}$ of $\mathcal{FP}$. The term *structure*, without reference to a logic, is also used in some contexts.

Definition 3.5. Let $\langle \mathbf{G}, v\rangle$ be an $\mathcal{L}(N,k)$-structure. Let H be the set of histories of $\mathbf{G}$ and T the set of terminal histories. The assignment v is extended as follows.

For all $t \in T$:

- $v(\varphi \supset \psi,\, t) = \min\{1, 1 - v(\varphi,\, t) + v(\psi,\, t)\}$

where $\varphi, \psi \in BForm(\mathcal{L}(N,k))$.

For all $h \in H$:

- $v(\bot,\, h) = 0$
- $v(\varphi \supset \psi,\, h) = \min\{1, 1 - v(\varphi,\, h) + v(\psi,\, h)\}$
- $v(\{G,n\}\varphi,\, h) = \max\limits_{T\in\Phi^n_{\mathbf{G}}(h)} \{\min\limits_{t\in T}\{v(\varphi,\, t)\}\}$

where $\varphi, \psi \in GForm(\mathcal{L}(N,k))$ and $n \in N$.

The extended version of v is called the *valuation function.*

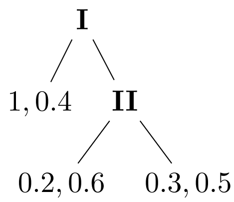
The semantics of the operators are those of Łukasiewicz logic. The semantics of forcing modalities are a generalization of the semantics of van Benthem's forcing logics. In the two-valued case a modality is true if there is a forceable set such that the inner formula evaluates to true in every state. On a meta-level this can be formulated as $\exists X \in \Phi^n_{\mathbf{G}}(h).\forall x \in X.v(\cdot,\, h) = \mathit{true}$. In first-order Łukasiewicz logic $\forall$ and $\exists$ are interpreted as supremum and infinum respectively. Because the structures of $\mathcal{FP}$ are finite this can be simplified to maximum and minimum to arrive at the final generalized semantics. This simplification is actually one of the reasons to limit the structures to finite game forms. Some problems that appear otherwise are discussed below, after some more context has been provided.

Notation. The following naming conventions are used throughout this work:

- p, q for propositional variables.
- φ, ψ for formulas.
- $\mathcal{L}$ for logics.
- $\mathcal{G}, \mathcal{M}, \mathcal{N}$ for $\mathcal{L}$-structures and $\mathbf{G}, \mathbf{M}, \mathbf{N}$ for their respective game forms.

- n, o for players in general.
- Roman numerals for specific players (e.g., in examples).

Intuitively, the semantics of the forcing modality correspond to the notion of being able to force a result at least as good as the valuation. A player can choose the forceable set in which play terminates, but not the specific history inside the set. Thus a player assumes to get the worst inside the forceable set (the minimum) and chooses the forceable set where that value is highest. Some example valuations are shown in figure 3.2. Defining the assignment by placing the values for propositional variables at the terminal states will be the default from here on.



(a) $\langle \mathbf{G}, v \rangle$

Values at end states are the assignments for p, q respectively (3.1)

$$v(\{G, \mathbf{I}\}p,\ ()) = 1 \qquad v(\{G, \mathbf{II}\}p,\ ()) = 0.3 \tag{3.2}$$
$$v(\{G, \mathbf{I}\}q,\ ()) = 0.5 \qquad v(\{G, \mathbf{II}\}q,\ ()) = 0.4 \tag{3.3}$$
$$v(\neg\{G, \mathbf{I}\}p,\ ()) = 0 \qquad v(\{G, \mathbf{I}\}\neg p,\ ()) = 0.7 \tag{3.4}$$
$$v(\{G, \mathbf{I}\}(p \wedge q),\ ()) = 0.4 \tag{3.5}$$
$$v(\{G, \mathbf{I}\}p \wedge \{G, \mathbf{I}\}q,\ ()) = 0.5 \tag{3.6}$$

(b) Formula evaluations for 3.2a

Figure 3.2: Valuation examples

For every extensive game (with payoffs in the interval $[0, 1]$) there is a natural, direct translation to an $\mathcal{L}$-structure: For every player n with payoff function u_n create a new variable p_n. There exists an assignment v such that $v(p_n, t) = u_n(t)$ at every terminal history t. Such an assignment, together with the underlying game form, is a direct representation of the game as a logical structure. The ability to reason about payoffs in these representations is the immediate application of the logics introduced in this chapter. Consider the representation of a three player game as described above. Formula 3.7 is an example of a statement that can be made about payoffs that would be complex to make in natural language. Note that it is also possible to consider how high a player can force the payoffs of other players to be, e.g., $\{G, \mathbf{I}\}p_{\mathbf{II}}$ or $\{G, \mathbf{I}\}\,(p_{\mathbf{II}} \wedge p_{\mathbf{III}})$.

$$((\{G, \mathbf{I}\}p_{\mathbf{I}} \wedge \{G, \mathbf{II}\}p_{\mathbf{II}}) \supset \{G, \mathbf{III}\}p_{\mathbf{III}}) \supset (\{G, \mathbf{I}\}p_{\mathbf{I}} \oplus \{G, \mathbf{III}\}p_{\mathbf{III}}) \tag{3.7}$$

Beyond such a direct representation of payoffs, it is possible to formulate the payoff function directly in the logic for some cases. Consider a game where two players each play a number from a finite subset of the interval $[0, 1]$. Let m_n refer to the number played by player n. The payoff functions for the two players $\mathbf{I}, \mathbf{II}$ are defined as:

$$u_{\mathbf{I}}((m_{\mathbf{I}}, m_{\mathbf{II}})) = \max\{m_{\mathbf{I}} - m_{\mathbf{II}}, 0\}, \qquad u_{\mathbf{II}}((m_{\mathbf{I}}, m_{\mathbf{II}})) = \min\{1 - m_{\mathbf{I}} + m_{\mathbf{II}}, 1\}$$

Instead of directly representing the payoff values in the assignment, we could encode their calculation via corresponding formulas and represent the parameters used for the calculation in the assignment. In the given example, take the assignment that, at every terminal history $(m_{\mathbf{I}}, m_{\mathbf{II}})$, assigns $m_{\mathbf{I}}$ to p and $m_{\mathbf{II}}$ to q. The actual payoffs are then represented by the following formulas:

$$\varphi_{\mathbf{I}} = p \,\&\, \neg q, \qquad \varphi_{\mathbf{II}} = p \supset q$$

This makes it possible to incorporate the internals of the payoff function when reasoning about payoffs.

Because structures are based on game forms instead of games, our structures do not have to be used to represent a game. Truth assignments can also be used to encode information other than payoffs. In the previous example p and q were used to encode the first and second move respectively. This is possible in general by identifying moves by truth values and assigning variables p_i the values of the ith move made. A further possibility is to identify individual terminal histories by assigning a variable a unique truth value at every terminal history. Using such different encodings allows us to reason about many other things beyond just payoffs. However, the semantics of the forcing modality might not be appropriate for some of those applications.

A further application is the study of the relations between games that have the same game form. By basing assignments on payoffs like in the aforementioned direct representation of a game as an $\mathcal{L}$-structure, one can also encode the payoffs of multiple games (that all share the same underlying game form) in a single assignment. Consider the last two formulas of figure 3.2 as an example. Say the two variables p and q encode the payoffs for **I** in two different games. Formula 3.5 can be interpreted as the forceable least payoff for a **I** who does not know which of the two games is being played. On the other hand, formula 3.6 describes the case where **I** has to play both games and receives the lower of the two payoffs. In this case the latter situation is apparently preferable for **I**. General statements for these types of connections will be presented in section 3.2.3.

At this point a formal link between $\mathcal{FP}$ semantics and the forceable payoff in a game is still open. The following lemma corrects this. The evaluation of an atomic game formula provides a lower bound that the respective player can always reach for the inner formula. From the semantics it is also clear that this bound corresponds to a real outcome, and that this is the highest such bound.

Lemma 3.4. *Let $\langle \mathbf{G}, v\rangle$ be a $\mathcal{L}$-structure with histories H. For every positive atomic game formula $\{G, n\}\varphi$, player n has a strategy σ such that for any strategy profile s:*

$$\forall h \in H.\, v(\varphi,\, \tau(h, s[n/\sigma])) \geq v(\{G, n\}\varphi,\, h)$$

Proof. By the semantics of $\{G, n\}\varphi$ for any h there is a set T in $\Phi^n_{\mathbf{G}}(h)$ such that

$$\min_{t \in T}\{v(\varphi,\, t)\} = v(\{G, n\}\varphi,\, h).$$

By definition of $\Phi^n_{\mathbf{G}}(h)$ there is a strategy σ such that for any strategy profile s, $\tau(h, s[n/\sigma]) \in T$. As $v(\{G, n\}\varphi,\, h)$ is the minimum of the valuations of terminals $\in T$, formula φ will always evaluate greater or equal at terminals reached by n playing σ than at h. □

The fact that forcing relations are closed under supersets is inconvenient. The number of forcing powers explodes with the size of the game. This is of particular concern seeing that the proposed evaluation of forcing modalities includes the evaluation of every member of the forcing powers. However, the following lemma shows that only the smallest sets need to be considered for evaluation, making formula evaluation less complex. This also makes a full enumeration of forcing powers superfluous in most cases and from here on forcing powers will usually be given only as their *Core*.

Notation. A forcing powers is defined relative to an extensive game form, a history and a player. The following definition and lemmata are general in all these parameters and they are therefore dropped from the notation. The general use of forcing power here is intended to pertain to all forcing powers for all game forms, histories and players.

Definition 3.6. Let Φ be a forcing power. The corresponding *minimal forcing power* $Core(\Phi)$ is the set that contains exactly those sets $X \in \Phi$ that have no proper subsets in Φ.

Lemma 3.5. *For every forcing power* Φ*:*

(1) $Core(\Phi) \subseteq \Phi$

(2) $Core(\Phi)$ *is uniquely determined.*

(3) $\forall X \in \Phi. \exists Y \in Core(\Phi).\, Y \subseteq X$

Proof. (1) and (2) are trivial.

(3) For every $X \in \Phi$ there are two cases: If X has no proper subset in Φ, then it is in $Core(\Phi)$ by definition. Otherwise, X must have at least one minimal subset in Φ because every set in Φ is finite. That subset is then in $Core(\Phi)$ by definition.

□

Lemma 3.6. *For any forcing power* Φ*, it holds that*

$$\max_{T\in\Phi}\{\min_{t\in T}\{v(\varphi,\, t)\}\} = \max_{T\in Core(\Phi)}\{\min_{t\in T}\{v(\varphi,\, t)\}\}$$

Proof. From $\Phi \supseteq Core(\Phi)$, it immediately follows that

$$\max_{T\in\Phi}\{\min_{t\in T}\{v(\varphi,\, t)\}\} \geq \max_{T\in Core(\Phi)}\{\min_{t\in T}\{v(\varphi,\, t)\}\}.$$

Assume the left side were strictly greater than the right. Then there is an $X \in \Phi \setminus Core(\Phi)$ such that

$$\min_{x \in X}\{v(\varphi,\, x)\} > \max_{T \in Core(\Phi)}\{\min_{t \in T}\{v(\varphi,\, t)\}\}.$$

By lemma 3.5 there is a nonempty $X^* \in Core(\Phi)$ s.t. $X^* \subseteq X$. From $X^* \subseteq X$ it follows that:

$$\min_{x \in X}\{v(\varphi,\, x)\} \leq \min_{x^* \in X^*}\{v(\varphi,\, x^*)\}$$

and because $X^* \in Core(\Phi)$ this is a contradiction to the assertion that the left side is greater. □

Remark. Assume that structures were not limited to finite sets of moves. Let $\langle \mathbf{G}, v\rangle$ be a $\mathcal{L}$-structure based on Ł$_\infty$. On the first turn a player **I** can play any rational $m \in [0, 1)$ and then **G** ends in a terminal history at which the variable p is assigned the truth value m. The minimal forcing power at the root for **I** is

$$Core(\Phi^{\mathbf{I}}_{\mathbf{G}}(())) = \{\{m\} \mid 0 \leq m < 1, m \in \mathbb{Q}\}.$$

Now consider the evaluation of the formula $\{G, \mathbf{I}\}p$, as there is a one element set for every move the formula evaluates to $\sup\{m \mid 0 \leq m < 1\} = 1$ but no terminal history where $v(p,\, \cdot) = 1$ exists. In other words, this would mean that **I** has the power to make p absolutely true, even though p is never absolutely true. The link between feasible play and the semantics of $\mathcal{FP}$ provided by lemma 3.4 does not hold for games that have an infinite number of moves.

3.1.3 Satisfiability, Validity & Entailment

In modal logics it makes sense to distinguish between satisfiability at a specific world, or in all worlds of a given frame. What is usually called world and frame in modal logics corresponds to histories and extensive game forms in $\mathcal{L}$-structures. Similar distinctions are also made for validity. This section formally defines the types of satisfiability and validity of $\mathcal{FP}$ logics. Furthermore, the step from two-valued to many-valued logics adds further adds an additional distinction. The usual question of satisfiability – *"Can this formula be true?"* – can be extended in two ways to the presented many-valued logics. The two questions then are: *"Can this formula evaluate absolutely true?"* and *"Can this formula evaluate at least partially true?"*.

Definition 3.7 (Satisfiability)**.** Given an extensive game form **G** with histories H, terminal histories T and a logic $\mathcal{L}$ for **G**. A formula φ is called *locally 1-satisfiable* in **G** if:

- $\varphi \in BForm(\mathcal{L})$ and there exists an assignment v and a $t \in T$ s.t. $\langle \mathbf{G}, v\rangle$ is a $\mathcal{L}$-structure and
$$v(\varphi,\, t) = 1.$$

- $\varphi \in GForm(\mathcal{L})$ and there exists an assignment v and a $h \in H$ s.t. $\langle \mathbf{G}, v \rangle$ is a $\mathcal{L}$-structure and
$$v(\varphi,\, h) = 1.$$

A formula $\varphi \in GForm(\mathcal{L})$ is called *globally 1-satisfiable* in $\mathbf{G}$ if there exists an assignment v s.t. $\langle \mathbf{G}, v \rangle$ is a $\mathcal{L}$-structure and

$$v(\varphi,\, h) = 1 \qquad \text{for every } h \in H.$$

The definitions of *locally/globally positive-satisfiable* formulas are analogous to those of 1-satisfiability. Instead of checking whether the valuation is exactly 1, it suffices to check whether the valuation is greater 0 for positive-satisfiability.

Definition 3.8 (Validity)**.** Given an extensive game form $\mathbf{G}$ with histories H, terminal histories T and a logic $\mathcal{L}$ for $\mathbf{G}$. A formula φ is called *locally valid* at h in $\mathbf{G}$ if:

- $\varphi \in BForm(\mathcal{L}), h \in T$ and for all assignments v s.t. $\langle \mathbf{G}, v \rangle$ is a $\mathcal{L}$-structure it holds that:
$$v(\varphi,\, h) = 1.$$
- $\varphi \in GForm(\mathcal{L}), h \in H$ and for all assignments v s.t. $\langle \mathbf{G}, v \rangle$ is a $\mathcal{L}$-structure it holds that:
$$v(\varphi,\, h) = 1.$$

Given an extensive game form $\mathbf{G}$ with histories H and a logic $\mathcal{L}$ for $\mathbf{G}$. A formula $\varphi \in GForm(\mathcal{L})$ is called *globally valid* in $\mathbf{G}$ if for all assignments v s.t. $\langle \mathbf{G}, v \rangle$ is a $\mathcal{L}$-structure it holds that:

$$v(\varphi,\, h) = 1 \qquad \text{for every } h \in H.$$

Remark. There is no need to consider global satisfiability or validity for base formulas. The problems are equivalent to the local cases because base formulas can only depend on the assignment at a single terminal history. (Recall, base formulas contain no modal operators)

While this thesis is mostly focused on 1-satisfiable and valid formulas, their positive counterparts are just as important. In particular, deciding whether a formula is valid is the dual problem to deciding whether it is positive-satisfiable. The distinction is also important in terms of complexity. In Łukasiewicz logic deciding 1-satisfiability is known to be NP-complete for the finite-valued and the infinite-valued case. Positive-satisfiability is also NP-complete in the finite-valued case. However, for infinite-valued Łukasiewicz logic it is Σ_2-complete [20, 21].

The $\models$ relation is overloaded in the usual way. Technically different relations are expressed by the same symbol and differentiated by context.

Definition 3.9 (Entailment)**.** For an $\mathcal{L}$-structure $\mathcal{G} = \langle \mathbf{G}, v \rangle$ with terminal histories T, $t \in T$ and $\varphi \in BForm(\mathcal{L})$ say:

- $\mathcal{G}, t \vDash_{\mathcal{L}} \varphi$ iff $v(\varphi, t) = 1$.
- $\mathcal{G} \vDash_{\mathcal{L}} \varphi$ iff $\mathcal{G}, t' \vDash_{\mathcal{L}} \varphi$ for all $t' \in T$.

For an $\mathcal{L}$-structure $\mathcal{G} = \langle \mathbf{G}, v \rangle$ with histories H, $h \in H$ and $\varphi \in GForm(\mathcal{L})$ say:

- $\mathcal{G}, h \vDash_{\mathcal{L}} \varphi$ iff $v(\varphi, h) = 1$.
- $\mathcal{G} \vDash_{\mathcal{L}} \varphi$ iff $\mathcal{G}, h' \vDash_{\mathcal{L}} \varphi$ for all $h' \in H$.

For an extensive game form $\mathbf{G}$ with histories H, $h \in H$, a logic $\mathcal{L}$ for $\mathbf{G}$ and $\varphi \in Form(\mathcal{L})$ say:

- $\mathbf{G}, h \vDash_{\mathcal{L}} \varphi$ iff φ is valid at h.
- $\mathbf{G} \vDash_{\mathcal{L}} \varphi$ iff φ is valid in $\mathbf{G}$.
- $\vDash_{\mathcal{L}} \varphi$ iff φ is valid for all finite extensive game forms of which $\mathcal{L}$ is a logic.

Notation. In all the above cases We write $\nvDash$ to state that the entailment does not hold.

3.2 Neighborhood semantics and basic reasoning

This section investigates how various common properties of modal logics apply to $\mathcal{FP}$ logics. This serves to illustrate similarities and differences to other modal logics. Many well-studied modal logics are used in the study of topics related to game theory, e.g., knowledge, belief or temporal relations. Relating them to $\mathcal{FP}$ on a logical level may further deepen the understanding of their role in $\mathcal{FP}$ logics. Furthermore, this section provides basic tools for formal reasoning in $\mathcal{FP}$ logics. A complete calculus is not known at this time. The work here can be seen as the first steps toward a Hilbert-type calculus.

3.2.1 Neighborhoods and alternative modal operators

The first thing to note is the fact that $\mathcal{FP}$ logics are based on forcing relations, which relate histories to sets of histories. This means that the semantics of these logics do not fit with the commonly used world to world accessibility of Kripke semantics. The generalization to world-to-set accessibility is referred to as neighborhood semantics. A general exposition of neighborhood semantics is not given at this point, a detailed introduction, including various example logics, can be found in [37].

One of the interesting extensions neighborhood semantics bring to modal logic is a natural addition of further modal operators. Recall, in the previous section it was discussed that in the two-valued case the truth of a game formula $\{G, n\}\varphi$ can be expressed as:

$$\exists X \in \Phi^n_{\mathbf{G}}(h).\forall x \in X.v(\varphi, x) = 1.$$

It's dual would have the quantifiers swapped. There is a further natural possibility for the semantics of a modal operator, namely:

$$\forall X \in Core(\Phi^n_{\mathbf{G}}(h)).\forall x \in X.v(\varphi, x) = 1.$$

Intuitively this may seem to express that even with full cooperation of all players the formula can not be made false. In the many-valued case ($\forall$ becomes the infinum) this corresponds to a cooperative effort to minimize truth. The sets of forcing powers have to be limited to the minimal forcing powers because closure under supersets becomes a problem for this modal operator. Because the set of all terminal histories is always forceable an evaluation on the full forcing power would be independent of n and h. It would always evaluate to the minimal truth-value for φ at any terminal history. Note that this modal operator also entails a dual operator where both quantifications are existential, respectively expressing the outcome of a cooperative effort by all to maximize the truth of a formula.

In general, these alternative modal operators express what is possible under full cooperation between all players. This does not fit well with the "cautious" sense of rationality that $\mathcal{FP}$ is built on. Allowing these cooperative modalities would thus allow for mixing different assumptions of rationality. Such a step would require many further considerations and a system of working with different systems of rationality and does not fit the scope of this work.

A related topic is the interpretation of the dual of the forcing modality. Its semantics is:

$$v(\neg\{G, n\}\neg\varphi,\, h) = \min_{T \in \Phi^n_{\mathbf{G}}(h)} \{\max_{t \in T}\{v(\varphi,\, t)\}\}.$$

This follows directly from the bounded truth domains which lead to the identity

$$1 - \max_{x \in X}\{x\} = \min_{x \in X}\{1 - x\}.$$

Minimizing over the strategic possibilities corresponds to considering the *worst* strategy (with respect to maximizing the truth of φ) to be played from h. The dual semantics can then be read as the lowest upper bound of truth n can force for φ. To relate this to the forcing modality interpretation of "n can force φ (from h)", the dual evaluation can then informally be seen as the truth-value of the statement "n can not force the negation of φ (from h)" which can also be seen the lowest forceable upper bound of the truth value of φ (in contrast to the primal representing the highest forceable lower bound). If $\neg\{G, n\}\neg\varphi$ is absolutely true, that means that n can only enforce an upper bound of 1 for the valuation of φ.

3.2.2 Basic reasoning in $\mathcal{FP}$

The logics of $\mathcal{FP}$ are based on Łukasiewicz logics and the semantics of connectives and negation are equivalent to those of the propositional Łukasiewicz logic with the same number of truth-values. It follows that the valid formulas of a Łukasiwicz logic $Ł_n$ are also valid basic formulas in the $\mathcal{FP}$ logics based on $Ł_n$. In the same way it also follows that modus ponens (MP) is a sound rule in $\mathcal{FP}$ logics. Because of the two tiers of formulas some care has to be taken with respect to substitution. Take the valid formula $p \supset (q \supset p)$, substituting p for a game formula, say $\{G, \mathbf{I}\}\psi$, does not yield a $\mathcal{FP}$-formula. The substitution is only sound if q is also substituted by a game formula at the same time. These considerations are made more precise in by the lemmata 3.7 and 3.8 below.

Lemma 3.7. *Let* $Ł_k$ *be the* k*-valued propositional* Łukasiewicz *logic,* $\mathcal{L} \in \mathcal{FP}$ *based on* $Ł_k$ *and* φ *a formula of* $Ł_k$ *(therefore, also* $\varphi \in BForm(\mathcal{L})$*).*

$$\textit{If} \quad \vDash_{Ł_k} \varphi \quad \textit{, then} \quad \vDash_{\mathcal{L}} \varphi.$$

Proof. Any terminal history can be seen as an individual interpretation of propositional $Ł_k$. Base formulas semantics are equivalent to those of $Ł_k$, a valid formula in $Ł_k$ will then also be true at every terminal history. □

Notation (Substitution). Given formulas φ, ψ and a propositional variable p that occurs in φ. Write $\varphi[\psi/p]$ for the formula that is obtained by replacing every occurrence of p in φ with ψ.

Lemma 3.8. *For every $\mathcal{L}$-structure $\mathcal{G}$ and $\varphi, \psi_1, \psi, \dots \in Form(\mathcal{L})$:*

$$\text{If } \mathcal{G} \vDash_{\mathcal{L}} \varphi \text{ and } \varphi[\psi_1/p_1][\psi_2/p_2]\cdots \in Form(\mathcal{L}), \text{ then } \mathcal{G} \vDash_{\mathcal{L}} \varphi[\psi_1/p_1][\psi_2/p_2]\cdots$$

where $p_1, p_2, \dots$ are propositional variables occurring in φ.

Proof. Follows from the truth-functionality of $\mathcal{FP}$ semantics, φ is valid no matter what p_i evaluates to. The substitute formula will evaluate to one of the same values that p_i can evaluate to and can not change the overall evaluation. □

Lemma 3.9. *For any $\mathcal{L} \in \mathcal{FP}$ and any pair of players n, m of $\mathcal{L}$:*

$$\vDash_{\mathcal{L}} \{G, n\}\{G, m\}\varphi \equiv \{G, n\}\varphi$$

Proof. Consider the evaluation of a formula $\{G, n\}\{G, m\}\varphi$ at a history h in a $\mathcal{L}$-structure $\mathcal{G} = \langle \mathbf{G}, v\rangle$.

$$v(\{G, n\}\{G, m\}\varphi,\, h) = \max_{T \in \Phi^n_{\mathbf{G}}(h)} \{\min_{t \in T}\{v(\{G, m\}\varphi,\, t)\}\}$$

Then, for any terminal history t clearly $Core(\Phi^{\mathbf{II}}_{\mathbf{G}}(t)) = \{\{t\}\}$ and by lemma 3.6 it follows that:

$$v(\{G, m\}\varphi,\, t) = v(\varphi,\, t).$$

Then, by substituting in the previous equality:

$$v(\{G, n\}\{G, m\}\varphi,\, h) = \max_{T \in \Phi^n_{\mathbf{G}}(h)} \{\min_{t \in T}\{v(\varphi,\, t)\}\} = v(\{G, n\}\varphi,\, h)$$ □

Lemma 3.9 will be implicitly applied when working with atomic game formulas. If something is proven for formulas $\{G, \cdot\}\varphi$ for base formulas φ it holds for all atomic game formulas. This is convenient because all $\mathcal{FP}$ logics have the same base formulas. In particular, section 3.3 will make use of this simplification.

Going back to the intended meaning of the forcing modality this is an expected result. The truth degree of $\{G, n\}\{G, m\}\varphi$ can be interpreted as the extent with which n can force $\{G, m\}\varphi$ to be true. But players have no control over each others strategic decisions, they can only try to maximize the truth value within their own strategic possibilities. This intuition suggests that the degree to which n can force $\{G, m\}\varphi$ true is no different than the degree to which n can force φ to be true. Lemma 3.9 confirms this intuition.

Lemma 3.10. *The monotonicity rule is sound in all logics of $\mathcal{FP}$, for all players n.*

$$\frac{\varphi \supset \psi}{\{G,n\}\varphi \supset \{G,n\}\psi}\ \text{Mon}$$

Proof. Let $\langle \mathbf{G}, v\rangle$ be an arbitrary $\mathcal{L}$-structure. Note that the validity of $\varphi \supset \psi$ implies $v(\varphi, t) \leq v(\psi, t)$ for every terminal history t. Let $\alpha = v(\{G,n\}\varphi, h), \beta = v(\{G,n\}\psi, h)$ for some history h. Note that by definition α is the minimum valuation of some (subset-) minimal set in $\Phi^n_{\mathbf{G}}(h)$. Now, at the same world where φ evaluates to α, by assumption, ψ evaluates to a greater truth-value. Thus, the minimum evaluation of that set is greater than α and in turn so is β. □

Monotonicity together with the validity of $\{G,\cdot\}\top$ also implies the validity of the necessitation rule (figure 3.3). The necessitation rule can be replaced by the following sequence: This is a common construction and can be found, e.g., in [16].

$$\cfrac{\cfrac{\cfrac{\varphi}{\top \supset \varphi}}{\{G,n\}\top \supset \{G,n\}\varphi}\ \text{Mon} \qquad \{G,n\}\top}{\{G,n\}\varphi}\ \text{MP}$$

$$\frac{\varphi}{\{G,n\}\varphi}$$

Figure 3.3: Necessitation

This establishes the soundness of some typical rules of reasoning for modal logics in $\mathcal{FP}$ logics. Aside from establishing basic tools for reasoning this section also serves to illustrate that $\mathcal{FP}$ logics behave like standard modal logics in many ways.

3.2.3 Sub- and Superdistributivity of Forcing Modalities.

This section discusses laws of generalized types of distribution of forcing modalities over the various connectives in $\mathcal{FP}$ logics. Usually distributivity laws state the equivalence of formulas, e.g., in propositional logic $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee q)$. Modal logics commonly have valid distributivity laws for modal operators, e.g., $\Box(p \wedge q) \equiv \Box p \wedge \Box q$ is valid in all modal logics with Kripke semantics. In this sense the forcing modalities of $\mathcal{FP}$ do not distribute over any of the defined connectives. (Except for the trivial $\{G,n\}(\varphi \wedge \varphi) \equiv \{G,n\}\varphi \wedge \{G,n\}\varphi$ and the same for weak disjunction.)

A weaker form of distributivity, replacing equivalence by implication, does yield some results worthy of discussion. This will be referred to as sub- or superdistributivity, depending on the direction of the implication. This terminology is borrowed from other fields where equivalence is replaced by $\leq$ and $\geq$ [11].

The inner formulas in game formulas inherently express a type of goal, i.e. $\{G, \mathbf{I}\}\varphi$ expresses how true $\mathbf{I}$ can make φ. Therefore, the truth of a game formula can be seen as a measure of success. If an implication $\{G, \mathbf{I}\}\varphi \supset \{G, \mathbf{I}\}\psi$ is absolutely true this then can then be interpreted as $\mathbf{I}$ being more successful with goal ψ than goal φ. Under this view, sub- and superdistributivity is an appealing compensation for distributivity, a sub- or superdistributivity law relates success of a compound goal to its respective partial goals.

Furthermore, a game formula is linked to an independent play of the game. The formula $\{G, \mathbf{I}\}\varphi \wedge \{G, \mathbf{I}\}\psi$ can be interpreted as the conjunction of results from two separate, independent plays of the game. In contrast $\{G, \mathbf{I}\}(\varphi \wedge \psi)$ is the result of playing only once. A sub-/superdistributivity law thus allows one to relate the result of a single play to that of multiple plays, albeit with different goals.

Lemma 3.11 (Superdistributivity over $\vee$)**.** *Let $\mathcal{L}(N, k)$ be a logic of $\mathcal{FP}$, for any $\varphi_1, \varphi_2 \in Form(\mathcal{L})$ and all $n \in N$:*

$$\vDash_{\mathcal{L}} (\{G, n\}\varphi_1 \vee \{G, n\}\varphi_2) \supset \{G, n\}(\varphi_1 \vee \varphi_2)$$

Proof. Let $\mathcal{G} = \langle \mathbf{G}, v \rangle$ be an arbitrary $\mathcal{L}$-structure. For any history h, assume without loss of generality that $\{G, n\}\varphi$ is at least as true as $\{G, n\}\psi$. Observe that for any terminal t, $v(\varphi, t) \leq v(\varphi \vee \psi, t)$. It follows that for every forceable set T:

$$\min_{t \in T}\{v(\varphi, t)\} \leq \min_{t \in T}\{v(\varphi \vee \psi, t)\}$$

Thus, the maximum over the forceable sets will also be less or equal for φ. □

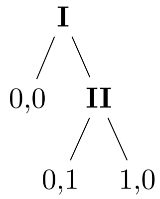
Corollary 3.12. *Let $\mathcal{L}(N, k)$ be a logic of $\mathcal{FP}$, for any $\Psi \subseteq Form(\mathcal{L})$ and $n \in N$:*

$$\vDash_{\mathcal{L}} \left(\bigvee_{\psi \in \Psi} \{G, n\}\psi \right) \supset \left(\{G, n\} \bigvee_{\psi \in \Psi} \psi \right)$$

Lemma 3.13. *The forcing modality does not distribute over $\vee$ in any logic of $\mathcal{FP}$ with at least two players.*

Proof. Figure 3.4 provides a counterexample for distribution over $\vee$. Note that the counterexample is applicable to every logic of $\mathcal{FP}$ because it only uses the truth values 0 and 1.

□



Labels at terminal histories denote truth values of p, q respectively.

$\mathcal{G}, () \not\models \{G, \mathbf{I}\}(p \lor q) \supset \{G, \mathbf{I}\}p \lor \{G, \mathbf{I}\}q$

Figure 3.4: Failure of distributing over $\lor$.

Forcing modalities share some intuition with the $\Diamond$ operator in Kripke semantics. In a way the forcing modality makes statements about the existence of strategies similar to the existence of reachable worlds expressed by $\Diamond$. For $\Diamond$ in minimal modal logic $\Diamond(\varphi \lor \psi) \supset \Diamond\varphi \lor \Diamond\psi$ is valid, in $\mathcal{FP}$ it is not. Van Benthem states that this difference "is precisely the point of forcing" [55]. This is best seen by examining the argument for why the formula is valid for $\Diamond$: Assuming there is a reachable world where $\varphi \lor \psi$ is true, then at least one of the two formulas is true, assume without loss of generality it is φ. Then for the right side the same world can be chosen for making $\Diamond\varphi$, and with it the consequent, true. The aspect of choosing the same world is where the argument breaks down for games. In a game another player might have the choice of the precise world/terminal history. The example in figure 3.4 illustrates this. At the root $\{G, \mathbf{I}\}(p \lor q)$ is clearly true, by playing right, **I** guarantees that either p or q will be true. However, the choice of the final world is up to **II**. $\{G, \mathbf{I}\}p$ is false because **II** could play right, $\{G, \mathbf{I}\}q$ is false because **II** could play left.

The conclusion from lemma 3.11 and its corollary is that players that maximize over subgoals do not profit from focusing on a single subgoal. Therefore, it is advantageous for a player to always consider all subgoals at the same time.

Lemma 3.14 (Subdistributivity of $\land$)**.** *Let $\mathcal{L}(N, k)$ be a logic of $\mathcal{FP}$, for any $\varphi_1, \varphi_2 \in Form(\mathcal{L})$ and all $n \in N$:*

$$\models_{\mathcal{L}} \{G, n\}(\varphi_1 \land \varphi_2) \supset (\{G, n\}\varphi_1 \land \{G, n\}\varphi_2)$$

Proof. Analogously to the proof of lemma 3.11. $\square$

Corollary 3.15. *Let $\mathcal{L}(N, k)$ be a logic of $\mathcal{FP}$, for any $\Psi \subseteq Form(\mathcal{L})$ and $n \in N$:*

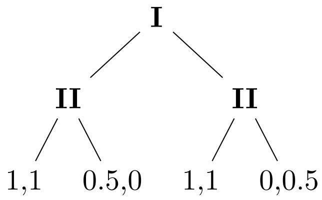
$$\models_{\mathcal{L}} \left(\{G, n\} \bigwedge_{\psi \in \Psi} \psi\right) \supset \left(\bigwedge_{\psi \in \Psi} \{G, n\}\psi\right)$$

Lemma 3.16. *The forcing modality does not distribute over $\land$ in any logic of $\mathcal{FP}$ with at least two players.*

Proof. The structure of figure 3.4 also provides a counterexample for distribution over $\land$. In particular, observe: $\mathcal{G}, () \not\models (\{G, \mathbf{II}\}\neg p \land \{G, \mathbf{II}\}\neg q) \supset \{G, \mathbf{II}\}(\neg p \land \neg q)$. $\square$

Weak conjunctions are a natural way to express outcomes that depend on each other. Say a manufacturer needs 10 nuts and bolts to use together and there are opponents who are competing for those resources. This situation can be seen as a game for those resources. The truth of getting 10 nuts and 10 bolts is given by p and q respectively. Say the logic has 11 truth values and the truth is proportional to the number of parts the manufacturer ends up with. The formula $p \wedge q$ is the actual goal of the manufacturer because a nut only has value if there is a bolt to fit it on and vice versa.

In the two-valued case strong and weak connectives coincide. For strong connectives in logics with more than two truth values, neither sub- nor superdistributivity holds. Figure 3.5 gives counterexamples to the directions that hold for the respective weak connectives. For the opposite directions, the counterexamples for the weak connectives from figure 3.4 also work for their strong counterparts.



Labels at terminal histories denote truth values of p, q respectively.

$\mathcal{G}, () \not\models \{G, \mathbf{I}\}p \oplus \{G, \mathbf{I}\}q \supset \{G, \mathbf{I}\}(p \oplus q)$
$\mathcal{G}, () \not\models \{G, \mathbf{II}\}(\neg p \,\&\, \neg q) \supset \{G, \mathbf{II}\}\neg p \,\&\{G, \mathbf{II}\}\neg q$

Figure 3.5: Counterexample for strong connectives.

Lemma 3.17. *Let $\mathcal{L} \in \mathcal{FP}$, n be a player in $\mathcal{L}$ and $\varphi \in Form(\mathcal{L})$, then:*

(1) $\{G, n\}(\varphi \oplus \varphi) \equiv \{G, n\}\varphi \oplus \{G, n\}\varphi$

(2) $\{G, n\}\varphi \,\&\{G, n\}\varphi \equiv \{G, n\}(\varphi \,\&\, \varphi)$

Proof.

(1) Recall, $v(\varphi \oplus \varphi,\, h) = \min\{1, v(\varphi,\, h) + v(\varphi,\, h)$. Clearly in every $\mathcal{L}$-structure for any terminal histories t_1, t_2:

$$v(\varphi,\, t_1) \geq v(\varphi,\, t_2) \Rightarrow v(\varphi \oplus \varphi,\, t_1) \geq v(\varphi \oplus \varphi,\, t_2)$$

Therefore, in every forceable set the same terminal history has the minimal valuation for both formulas. In the same way, the same terminal history will be the maximal minimum over the forcing powers. It follows that the forcing modality evaluates to the valuation of $\varphi \oplus \varphi$ at the same terminal history that would be taken for φ.

(2) Recall, $v(\varphi \,\&\, \varphi,\, h) = \max\{0, v(\varphi,\, h) + v(\varphi,\, h) - 1\}$. Again, in every $\mathcal{L}$-structure for any terminal histories t_1, t_2:

$$v(\varphi,\, t_1) \geq v(\varphi,\, t_2) \Rightarrow v(\varphi \,\&\, \varphi,\, t_1) \geq v(\varphi \,\&\, \varphi,\, t_2)$$

And the argument proceeds as above.

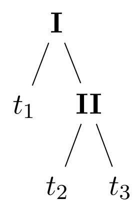
□

In the introduction of this section a similar statement to that of lemma 3.17 for weak connectives was dismissed as trivial because it follows from the idempotence of the operations. However, strong disjunction and conjunction are not idempotent. In a practical sense lemma 3.17 states that for subgoals that are linked by strong connectives multiple plays of a game can be collapsed to a single game with the same outcome. It should be clear from the proof that this would indeed apply to any operator that has the terminal monotonicity condition stated in the proof.

The valid formulas of this section are a good example of the type of reasoning gained through many-valued truth domains. These results have clear game theoretic interpretations when inner formulas represent payoffs. When considering two-valued encodings of the preference relation, the possibility of statements about the components of the payoff is lost, unless one codes payoffs in a first-order language.

3.2.4 Fixed extensive game forms

The structures for $\mathcal{FP}$ logics are defined to have a finite number of histories. This also means that, for a given extensive game form, the forcing powers and every forceable set is finite. It follows that, in a fixed extensive game form, game formulas can equivalently be stated as propositional formulas for a fixed history. The transformation is illustrated by the following example:



(a) The game form $\mathbf{G}$

$$Core(\Phi^{\mathbf{I}}_{\mathbf{G}}(())) = \{\{t_1\}, \{t_2, t_3\}\}$$

Figure 3.6

Consider a three-valued logic for $\mathbf{G}$ of figure 3.6a with a variable p and the question whether $\{G, \mathbf{I}\}p$ is 1-satisfiable at the root history: Is there an assignment v such that

$$\max\{v(p,\, t_1), \min\{v(p,\, t_2), v(p,\, t_3)\}\} = 1\,?$$

It is easy to see that this is equivalent to the question of 1-satisfiability of a formula

$$p_1 \vee (p_2 \wedge p_3)$$

in propositional Ł$_3$.

Definition 3.10. Let $\mathbf{G}$ be an extensive game form with terminal histories T and an atomic game formula of the form $\{G, n\}\varphi$. Let $\mathcal{PV}$ be the set of all propositional variables occurring in φ. For every $t \in T$, let φ_t denote the formula obtained by substituting every $p \in \mathcal{PV}$ by a fresh variable p_t. For a history h of $\mathbf{G}$, the formula

$$\bigvee_{T \in Core(\Phi^n_{\mathbf{G}}(h))} \bigwedge_{t \in T} \varphi_t$$

is called the *h-local propositional form* (h-lp form) of $\{G, n\}\varphi$.

For a game formula ψ in general the h-lp form is obtained by substituting every atomic game formula in ψ by its h-lp form.

Lemma 3.18. *Let $\mathcal{G} = \langle \mathbf{G}, v\rangle$ be a $\mathcal{L}$-structure for an $\mathcal{L}$ based on Ł$_k$, $\varphi \in GForm(\mathcal{L})$ and let $v_{Ł_k}$ be the assignment (and extending function of Ł$_k$. Let T be the set of terminal histories of $\mathbf{G}$. For every history h of $\mathbf{G}$:*

$$\textit{If } v(p,\, t) = v_{Ł_k}(p_t) \textit{ for all } p \in \mathcal{PV} \textit{ and } t \in T \quad \textit{then} \quad v(\varphi,\, h) = v_{Ł_k}(\varphi_h)$$

where φ_h is the h-lp form of φ.

Proof. Given $\mathcal{L}, \mathbf{G}, T$ as in the claim. Proof is by induction on the complexity of φ. For the base case let $\varphi = \{G, n\}\psi$ where $\psi \in BForm(\mathcal{L})$ and let ψ_t denote the formula obtained by substituting every $p \in \mathcal{PV}$ in ψ by a fresh variable p_t. For any t if $v(p,\, t) = v_{Ł_k}(p_t)$ for all variables p that occur in ψ, then $v(\psi,\, t) = v_{Ł_k}(\psi_t)$ because the evaluation of a base formula is equivalent to the evaluation of Łukasiewicz logic. It then folows that:

$$v(\varphi,\, h) = \max_{T \in Core(\Phi^n_{\mathbf{G}}(h))} \{\min_{t \in T}\{v(\psi,\, t)\}\} = v_{L_k}(\bigvee_{T \in Core(\Phi^n_{\mathbf{G}}(h))} \bigwedge_{t \in T} \varphi_t) = v_{L_k}(\varphi_t).$$

The assumption that $\psi \in BForm(\mathcal{L})$ can be made without loss of generality by lemma 3.9 so the first case covers all φ that are atomic game formulas.

For the step, assume the claim holds for $\varphi_1, \varphi_2 \in GForm(\mathcal{L})$, then it also holds for $\varphi_1 \supset \varphi_2$ and $\neg\varphi_1$. Both cases are a direct consequence of truth functionality. $\square$

The application of checking local satisfiability and validity using methods for Łukasiewicz logic is immediate. Using the transformations to local propositional forms can also be used to check global satisfiability in a given game form. This can be done simply by forming the disjunction of every h-lp form for every history h.

3.3 Information in $\mathcal{FP}$

The extensive game forms of $\mathcal{L}$-structures are only constrained by the fact that they need to be finite. As such the facts that hold for all structures are limited. Therefore, it is productive to also consider classes of games that are constrained to derive facts for the constrained class that go beyond what holds in general. This also leads to a better logical view of the effects the constraining properties have. In section 2.1 different such properties of games were discussed with one of the most impactful distinctions being that between perfect and imperfect information. The difference between the two cases can be seen in many aspects of games and it is especially clear in the step from the existence of mixed to pure Nash equilibria. This section will discuss the effects of information on $\mathcal{FP}$ formulas and forcing powers. This knowledge is then used in deriving duality of players for 2 player perfect information games. The proof technique of reducing games to less constrained ones is then introduced to generalize a weakened version of the duality result to all of $\mathcal{FP}$.

3.3.1 Forcing powers, information and dynamics

Forcing powers provide an effective abstraction of the information players have at their disposal. Different information sets in the same frame lead to different sets of possible strategies. The only place where strategies play a role in $\mathcal{FP}$ logics is in the definition of the forcing relation. So, formally there is no difference in treating perfect and imperfect information game forms in $\mathcal{FP}$, different information sets on the same frame simply lead to different forcing powers. On the language level, both cases are handled in the same way.

Figure 3.7 illustrates how a change in information is reflected in forcing powers. The strategic possibilities in a game similar to **N** were already discussed in section 2.1. There the only (pure) strategies of **II** are to always play x or always play y leading to the two minimal forceable sets $\{a, c\}$ or $\{b, d\}$ respectively. Due to the extra information in **M**, **II** gains two additional forceable sets. However, at history β the forcing powers are the same in both game forms. The example also hints at a way to systematically describe how forcing powers are combined from the powers at their successors. A formalization of this behavior gives a good foundation for proofs by structural induction, a very natural form of proof for extensive game forms (e.g., the proof of theorem 3.21).

Definition 3.11 (Union product)**.** For sets of sets R, S, the *union product* $R \star S$ is defined as

$$R \star S \equiv \{r \cup s \mid r \in R, s \in S\}.$$

A chain of union products $R_1 \star \ldots \star R_n$ is written as

$$\prod_{i=1}^{n} R_i$$

Note that $\star$ is associative and commutative.

(a) The game form **M**.

(b) The game form **N**.

$$Core(\Phi^{\mathbf{I}}_{\mathbf{M}}(\alpha)) = \{\{a,b\},\{c,d\}\}$$
$$Core(\Phi^{\mathbf{II}}_{\mathbf{M}}(\alpha)) = \{\{a,c\},\{a,d\},\{b,c\},\{b,d\}\}$$

$$Core(\Phi^{\mathbf{I}}_{\mathbf{M}}(\beta)) = \{\{a,b\}\}$$
$$Core(\Phi^{\mathbf{II}}_{\mathbf{M}}(\beta)) = \{\{a\},\{b\}\}$$

(c) Minimal forcing powers in **M**.

$$Core(\Phi^{\mathbf{I}}_{\mathbf{N}}(\alpha)) = \{\{a,b\},\{c,d\}\}$$
$$Core(\Phi^{\mathbf{II}}_{\mathbf{N}}(\alpha)) = \{\{a,c\},\{b,d\}\}$$

$$Core(\Phi^{\mathbf{I}}_{\mathbf{N}}(\beta)) = \{\{a,b\}\}$$
$$Core(\Phi^{\mathbf{II}}_{\mathbf{N}}(\beta)) = \{\{a\},\{b\}\}$$

(d) Minimal forcing powers in **N**.

Figure 3.7: Forcing powers and changes in information.

Lemma 3.19 (Power dynamics of perfect information)**.** *Let* $\mathbf{G} = \langle N, M, H, t, I\rangle$ *be a finite extensive game form with perfect information. For any non-terminal* $h \in H$ *and* $n \in N$ *with* $t(h) = n$*, the sets of forcing powers behave as follows:*

$$\Phi^n_{\mathbf{G}}(h) = \bigcup_{m \in M_h} \Phi^n_{\mathbf{G}}(h \circ m) \tag{3.8}$$

$$\Phi^o_{\mathbf{G}}(h) = \prod_{m \in M_h} \Phi^o_{\mathbf{G}}(h \circ m) \qquad \forall o \in N \setminus \{n\} \tag{3.9}$$

where M_h *is the set of all moves that are possible at* h*.*

Proof. For statement 3.8, $\supseteq$: By definition, for every $X \in \Phi^n_{\mathbf{G}}(h \circ m)$ where $m \in M_h$, there exists a strategy $\sigma_{m,X}$ that forces the game to end at an element of X from history $h \circ m$. Then, for any such X the strategy σ', defined as:

$$\sigma'(h) = m \quad \text{and} \quad \sigma'(u) = \sigma_{m,X}(u) \quad \forall u \in H \setminus \{h\}$$

is a strategy forcing X from h. This is easily verified: the next history in play (when following σ') will be $h \circ m$ and from there $\tau(h \circ m, s[n/\sigma']) \in X$ for any profile s by definition of $\sigma_{m,X}$. (τ is the terminal function of definition 3.1.)

Therefore, for player n, every set that is forceable from any history extending h by one move is also forceable from h.

For statement 3.8, $\subseteq$: Assume this false, then there must be an $X \in \Phi^n_{\mathbf{G}}(h)$ s.t. $X \notin \Phi^n_{\mathbf{G}}(m \circ h)$ for all $m \in M_h$. In other words, every possible move at h leads to a history from which X is not forceable by n. No strategy can exist to force X from h, a contradiction to $X \in \Phi^n_{\mathbf{G}}(h)$.

For statement 3.9, $\supseteq$: Every set

$$X \in \prod_{m \in M_h} \Phi^o_{\mathbf{G}}(h \circ m)$$

is of the form

$$\bigcup_{m \in M} X_m \quad \text{with} \quad X_m \in \Phi^o_{\mathbf{G}}(h \circ m) \text{ and } m \in M_h.$$

Let σ_{X_m} be a strategy that forces X_m from $h \circ m$ and let $(h \circ m)^*$ be the set of all histories extending $h \circ m$. For any such X define the strategy σ' such that:

$$\forall m \in M_h . \forall u \in (h \circ m)^* . \ \sigma'(u) = \sigma_{X_m}(u)$$

σ' forces X from h.

For statement 3.9, $\subseteq$: Assume this false, then there is an $X \in \Phi^o_{\mathbf{G}}(h)$, and a strategy σ to force it from h, that is not of the form

$$\bigcup_{m \in M} X_m \quad \text{with} \quad X_m \in \Phi^o_{\mathbf{G}}(h \circ m) \text{ and } m \in M_h.$$

That is, for some $m \in M_h$: $\nexists X_m \in \Phi^o_{\mathbf{G}}(h \circ m) : X_m \subseteq X$. Let s be a strategy profile where player n plays m at h and o follows σ. By assumption $\tau(h, s) \in X$, then also $\tau(h \circ m, s) \in X$ because thats the next history in play according to s. By definition then $X \in \Phi^o_{\mathbf{G}}(\gamma_m)$ but $X \subseteq X$, a contradiction to the initial assumption. $\square$

An important direct consequence of this lemma is that in perfect information, the forcing powers of a player n depend only on where it is n's turn. Changes in the turns of other player have no effect on n's forcing powers.

Corollary 3.20. *For any two finite perfect information extensive game forms **G**, **G'** with sets of histories H, H', turn functions t, t' and players $n \in N, n' \in N'$ respectively:*

If $H = H'$ and $t(h) = n \iff t'(h) = n'$ for all $h \in H$, then $\Phi^p_{\mathbf{G}}(h) = \Phi^{p'}_{\mathbf{G}'}(h)$ for all $h \in H$.

3.3.2 Perfect information with two players

Definition 3.12. $\mathcal{PI}2$ is the class of $\mathcal{FP}$ logics where the structures are limited to finite perfect information game forms with exactly two players.

Because renaming players has no effect, the names **I**, **II** will always be used to refer to the two players of the logics in $\mathcal{PI}2$, the generalization to arbitrary player names is implied.

In section 2.1 the notion of Nash equilibria was discussed and, in particular, the fact that finite perfect information games always have a pure Nash equilibrium. However, even then there can be multiple equilibria. Figure 3.8 shows a constant-sum game with three players that has three equilibria, and more importantly two different equilibrium values. The values at the leaves are payoffs for players **I**, **II**, **III** respectively. If **I** plays left then any variation in the strategies of the other players has no effect. This leads to two equilibria, one for each strategy of **II**. The strategy profile for which play ends at the terminal history after **II** plays left is also an equilibrium, having the maximum values for both players with any power in the game.

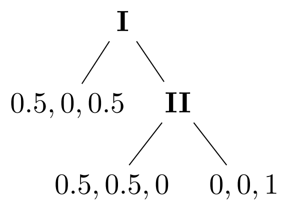


Figure 3.8: A perfect information constant-sum game with multiple pure equilibria.

As mentioned before, constant-sum games are interesting because the balance of payoffs represents a sort of forced competition. In the example, player **III** is used to balance out the sum of payoffs, which effectively leads to a situation where **I** and **II** are not competing for payoff. However, in perfect information games that are constant-sum and limited to two players this is not possible anymore. In those cases there is actually always a unique equilibrium value, i.e., there may be multiple equilibria but they all have identical payoff profiles. Many popular real-world board games fit these criteria, e.g., chess, go or checkers. That means that in these games the outcome would actually always be the same if played by perfect players. The fun and competition present in these games thus only comes from the fact that their complexity is too high to find the proper strategies. In the case of checkers this is actually no longer true, it has been shown that the equilibrium value in the game of checkers is a draw [45].

While the restriction to constant-sum is quite strict for games in general, it does always manifest itself in $\mathcal{FP}$ formulas when there are only two players. From the semantics of negation it follows that for any history h:

$$v(\varphi, h) + v(\neg\varphi, h) = 1$$

Negation thus naturally encodes the constant-sum property in two player games. Recall the notion of every inner formula inducing its own games. With exactly two players every base formula φ also induces a constant-sum game, the game where the opposing players preference comes from $\neg\varphi$. This leads to the following result for $\mathcal{PI}2$.

Theorem 3.21 (Player duality)**.** *For every logic $\mathcal{L} \in \mathcal{PI}2$:*

$$\vDash_{\mathcal{L}} \{G, \boldsymbol{I}\}\varphi \equiv \neg\{G, \boldsymbol{II}\}\neg\varphi$$

Proof. First some abbreviations are defined (Recall, in section 3.2 the dual of $\{G, \cdot\}$ was already discussed to be min max over the forceable sets):

$$f(h) = v(\{G, \mathbf{I}\}\varphi,\, h) = \max_{T \in \Phi^{\mathbf{I}}_{\mathbf{G}}(h)} \{\min_{t \in T}\{v(\varphi,\, t)\}\}$$
$$g(h) = v(\neg\{G, \mathbf{II}\}\neg\varphi,\, h) = \min_{T \in \Phi^{\mathbf{II}}_{\mathbf{G}}(h)} \{\max_{t \in T}\{v(\varphi,\, t)\}\}$$

The claim then is equivalent to $\forall h \in H : f(h) = g(h)$. It will be proven by backward induction on histories. The base case is for terminal histories t: Here, $\Phi^{\mathbf{I}}_{\mathbf{G}}(t) = \Phi^{\mathbf{II}}_{\mathbf{G}}(t)$ and then trivially $f(t) = g(t)$.

For the induction step, it is shown that if the claim holds for all $h \circ m \in H$, then it also holds for h:

Let M_h be the set of moves possible at history h. The proof is split in two cases, depending on who's turn it is at h:

$t(h) = \mathbf{I}$**:** By lemma 3.19, $\Phi^{\mathbf{I}}_{\mathbf{G}}(h) = \bigcup_{m \in M_h} \Phi^{\mathbf{I}}_{\mathbf{G}}(h \circ m)$, so $f(h) = f(h \circ k)$ for some $k \in M_h$. And because α is the largest minimum in the union:

$$\forall m \in M_h : f(h \circ k) \geq f(h \circ m).$$

Let $B \in \Phi^{\mathbf{II}}_{\mathbf{G}}(h)$ s.t. $g(h) = \max_{x \in B}\{v(\varphi,\, x)\}$. By lemma 3.19, $B = \bigcup_{m \in M_h} B_m$ with $B_m \in \Phi^{\mathbf{II}}_{\mathbf{G}}(h \circ m)$. Observe that $g(h) \geq \max_{x \in B_m}\{v(\varphi,\, x)\} \geq g(h \circ m)$ for all $m \in M_h$ because $B_m \subseteq B$ and $g(h \circ m)$ is the smallest maximum over a set containing B_m.

Also, there must be at least one B_l s.t. $g(h) = \max_{x \in B_l}\{v(\varphi,\, x)\}$. It follows that $g(h) = g(h \circ l)$: if for every such B_l there were a set X in $\Phi^{\mathbf{II}}_{\mathbf{G}}(h \circ l)$ where every element yielded a lower valuation than $g(h)$, then B is not the set yielding the smallest maximum. B_l could be replaced by X in the union, removing the highest element from B.

Now, combined with the induction hypothesis ($\forall m \in M_h : f(h \circ m) = g(h \circ m)$):

$$f(h) = f(h \circ k) = g(h \circ k) \leq g(h) \qquad g(h) = g(h \circ l) = f(h \circ l) \leq f(h)$$

Thus, $f(h) = g(h)$.

$t(h) = \mathbf{II}$**:** This time $\Phi^{\mathbf{II}}_{\mathbf{G}}(h) = \bigcup_{m \in M_h} \Phi^{\mathbf{II}}_{\mathbf{G}}(h \circ m)$. Therefore, directly $g(h) = g(h \circ l)$ for a $l \in M_h$. And $g(h)$ is the smallest maximum so $\forall m \in M_h : g(h) \leq g(h \circ m)$.

Analogue to the other case $\forall m \in M_h : f(h) \leq f(h \circ m)$ because minima can only become lower in supersets. The argument for the fact that there is a $h \circ k$ s.t. $f(h) = f(h \circ k)$ is again analogue to the argument for $g(h) = g(h \circ l)$ in the other case. Combined this gives:

$$f(h) = f(h \circ k) = g(h \circ k) \geq g(h) \qquad g(h) = g(h \circ l) = f(h \circ l) \geq f(h)$$

$\square$

This is a form of the minimax theorem of game theory [35], one of the field's foundational results. In fact there exist a variety of minimax theorems with different constraints on the principal sets and functions [47]. The theorems are usually stated in the form:

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y).$$

As an example application of the result in the context of $\mathcal{FP}$ logics it is used to prove Zermelo's classic result for chess: either white has a winning strategy, black has a winning strategy or both sides can guarantee a draw [57]: Consider a perfect information extensive game form $\mathbf{C}$, its moves and histories being those of chess and players **w,b**. To make sure the game form is finite, a game is considered to end in a draw once a position is repeated. In modern chess rules a player can actually decide whether to take the draw or play on after a threefold repetition of a position. From a theoretical perspective this choice is irrelevant. If a player were able to force a win from this position, then they would have also been able to do so from the previous history where the position appeared.

A propositional variable p will represent the statement "*white has won*", the induced assignment with base $Ł_3$ looks as follows:

$$v(p,\, t) = \begin{cases} 1 & \text{if } \mathbf{w} \text{ has won at } t. \\ 0.5 & \text{if there is a draw at } t. \\ 0 & \text{if } \mathbf{w} \text{ has lost at } t. \end{cases}$$

where t is a terminal history. Note that $\neg p$ represents the statement "*black has won*".

Together they give a structure $\mathcal{C} = \langle \mathbf{C}, v \rangle$ in which the result will be shown. Let $\mathbf{i}\varphi$ stand for $(\neg\varphi \supset \varphi) \wedge (\varphi \supset \neg\varphi)$, note that $\mathrm{V}(\mathbf{i}\varphi, h) = 1 \iff \mathrm{V}(\varphi, h) = 0.5$. The result can then be stated as:

$$\mathcal{C} \models \{C, \mathbf{w}\}p \vee \{C, \mathbf{b}\}\neg p \vee \mathbf{i}\{C, \mathbf{w}\}p \tag{3.10}$$

$$\mathcal{C}, h \not\models (\{C, \mathbf{w}\}p \wedge \{C, \mathbf{b}\}\neg p) \vee (\{C, \mathbf{w}\}p \wedge \mathbf{i}\{C, \mathbf{w}\}p) \vee (\{C, \mathbf{b}\}\neg p \wedge \mathbf{i}\{C, \mathbf{w}\}p) \tag{3.11}$$

for every history h.

$$\mathcal{C} \models \mathbf{i}\{C, \mathbf{w}\}p \equiv \mathbf{i}\{C, \mathbf{b}\}\neg p \tag{3.12}$$

The first statement says that at least one of white can force a win, black can force a win or white can draw is the case. By the second statement, no two of those possibilities can be true at the same time, i.e., exactly one of them is true. The third statement says that white can force a draw iff black can force a draw.

Statement 3.10 can be derived from the so-called principle of excluded fourth $\varphi \vee \neg\varphi \vee \mathbf{i}\varphi$, a tautology in $Ł_3$: By writing $\{C, \mathbf{w}\}p$ for φ it follows that:

$$\mathcal{C} \models \{C, \mathbf{w}\}p \vee \neg\{C, \mathbf{w}\}p \vee \mathbf{i}\{C, \mathbf{w}\}p$$

From theorem 3.21 $\mathcal{C} \vDash \neg\{C, \mathbf{w}\}p \equiv \{C, \mathbf{b}\}\neg p$ and by substituting equivalent formulas statement 3.10 is shown.

For statement 3.12, the same substitution gives the equivalent statement

$$\mathcal{C} \vDash \mathbf{i}\{C, \mathbf{w}\}p \equiv \mathbf{i}\neg\{C, \mathbf{w}\}p$$

From the structure of $\mathbf{i}$, by eliminating double negations and under commutativity of $\wedge$, it is clear that $\mathbf{i}\varphi \equiv \mathbf{i}\neg\varphi$.

For statement 3.11 consider the three disjuncts separately:

- $\{C, \mathbf{w}\}p \wedge \{C, \mathbf{b}\}\neg p$: Substituting $\{C, \mathbf{b}\}\neg p$ by the equivalent $\neg\{C, \mathbf{w}\}p$ makes it clear that this term can not be absolutely true.
- $\{C, \mathbf{w}\}p \wedge \mathbf{i}\{C, \mathbf{w}\}p$: The second conjunct is absolutely true iff the first conjunct evaluates to 0.5, their minimum can clearly not be 1.
- $\{C, \mathbf{b}\}\neg p \wedge \mathbf{i}\{C, \mathbf{w}\}p$: By again substituting $\{C, \mathbf{b}\}\neg p$ with $\neg\{C, \mathbf{w}\}p$ we get an analogous situation to the second disjunct.

It should be noted that the argument does not make direct use of the game having perfect information. The argument requires only player duality which can also hold under imperfect information for specific payoffs. Another significant aspect of the proof is that it leverages the three-valuedness of the logic. The law of excluded fourth becomes an enumeration of all possible outcomes, stating that one of the outcomes has to happen.

3.3.3 Imperfect information and n-players

Theorem 3.21 does not hold for more than two players. Of course it is inherently unclear which player would be dual to which other. Player duality also does not hold under imperfect information, examples for both these cases are given by the games in figures 3.9 and 3.10. In figure 3.10, let p evaluate to the payoffs at the respective terminal histories, at the root $\{G, \mathbf{I}\}p$ evaluates to 0.5, $\neg\{G, \mathbf{II}\}\neg p$ to 1. For figure 3.9, again let p correspond the the payoffs given. This gives the evaluations in the tables below.

Even though theorem 3.21 does not hold in these more general classes of games, it can be used for proving more general results by reducing games to specific two player perfect information games. In the following this mechanism is used to first prove a more general weaker version of theorem 3.21 for perfect information $\mathcal{L}$-structures with arbitrary numbers of players. By a reduction of imperfect information games to perfect information the result can then even be extended to all $\mathcal{L}$-structures.

Definition 3.13. For a finite extensive game form $\mathbf{G} = \langle N, M, H, t, I\rangle$ with perfect information, a partition of N into two non-empty sets S, T induces a *2-reduced game form*:

$$\mathbf{G}' = \langle\{S, T\}, M, H, t', I'\rangle \qquad \text{with } t'(h) = \begin{cases} S & \text{if } t(h) \in S \\ T & \text{if } t(h) \in T \end{cases}$$

φ	$v(\varphi, ())$
$\{G, \mathbf{I}\}p$	1
$\{G, \mathbf{II}\}p$	0.5
$\{G, \mathbf{III}\}p$	0.5

φ	$v(\varphi, ())$
$\neg\{G, \mathbf{I}\}\neg p$	0.75
$\neg\{G, \mathbf{II}\}\neg p$	1
$\neg\{G, \mathbf{III}\}\neg p$	1

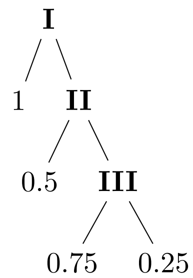


Figure 3.9: Three player game with no player duality.

$\mathbf{I}$ \ $\mathbf{II}$	A	B
A	1	0.5
B	0	1

Figure 3.10: Imperfect information game in strategic form with no player duality.

where $I' = \{\sim_S, \sim_T\}$, $\sim_S = \sim_T = \{(h, h) \mid h \in H\}$.

For a perfect information $\mathcal{L}$-structure $\langle \mathbf{G}, v\rangle$ call $\langle \mathbf{G}', v\rangle$ the *2-reduced $\mathcal{L}$-structure* of $\langle \mathbf{G}, v\rangle$ if $\mathbf{G}'$ is a 2-reduced game form of $\mathbf{G}$.

The constraint to perfect information avoids the possibly complex transformation of the information set relations. These reduced game forms are reminiscent of the coalition logics mentioned in section 2.2. In a way, one could consider the 2-reduced game to be the game played by coalitions S and T. Strictly speaking it is still a separate game than the one it is reduced from. However, these reductions are intended only as a technical tool to extend results from two player structures to structures with arbitrary numbers of players. The coalitional intuition or meaning behind them is not studied in this thesis. Also, note that the partition implicitly limits the applicability to game forms with at least two players.

Notation. For any structure $\langle \mathbf{G}, v\rangle$ a 2-reduced structure will have the same assignment v. However, because the forcing powers in the two structures can be different so can the valuation function that extends this assignment. To distinguish between the different valuation functions the assignments of reduced structures are named differently even though they are the same as the original assignment. This applies throughout this section.

Lemma 3.22. *For a perfect information $\mathcal{L}$-structure $\langle \mathbf{G}, v\rangle$ and a corresponding 2-reduced $\mathcal{L}$-structure $\mathcal{G} = \langle \mathbf{G}', v'\rangle$ with players S, T. Let H be the histories of $\mathbf{G}$ (and $\mathbf{G}'$) and φ be a base formulas, then the following hold:*

1. *$\mathbf{G}'$ is a perfect information game form.*
2. $\vDash_{\mathbf{G}'} \{G', T\}\varphi \equiv \neg\{G', S\}\neg\varphi$
3. $$\bigcup_{n\in Q} \Phi^n_{\mathbf{G}}(h) \subseteq \Phi^Q_{\mathbf{G}}(h) \qquad Q \in \{S, T\}, h \in H$$
4. $v(\{G, n\}\varphi,\, h) \leq v'(\{G', Q\}\varphi, h) \qquad n \in Q, Q \in \{S, T\}, h \in H$
5. *if* $Q = \{n\}$, *then* $v(\{G, n\}\varphi,\, h) = v'(\{G', Q\}\varphi, h) \qquad Q \in \{S, T\}, h \in H$
6. $$v(\bigvee_{n\in Q} \{G, n\}\varphi,\, h) \leq v'(\{G', Q\}\varphi, h) \qquad Q \in \{S, T\}, h \in H$$

Proof.

1. By definition there are only information sets of size 1 for both players.
2. By 1 theorem 3.21 applies to $\mathbf{G}'$.
3. Let σ be a strategy for player $n \in Q$ forcing a set X. Recall that this means that the play will end in a terminal history in X, no matter what is played when it is not n's turn. By definitions if it is n's turn in $\mathbf{G}$, it is also Q's turn in $\mathbf{G}'$. Therefore, there is a strategy σ' of Q where at any history h where $t(h) = n$: $\sigma'(h) = \sigma(h)$. Then σ' must also lead to terminal histories in X, no matter what Q plays at other histories. Therefore, every set forceable by a $n \in Q$ is also forceable by Q.
4. By statement 3, any set that n can force, Q can also force. Valuations are the same at terminal histories for both structures. Then, clearly Q can always force at least as good a result as n because Q can, in particular, force the best set of n with the same valuation.
5. If $Q = \{n\}$, then Q and n have the same forcing powers: By definition both players have the same turns, the set of histories is the same in $\mathbf{G}$ and $\mathbf{G}'$. By corrolary 3.20 the forcing powers are then the same.

   From equal sets of forcing powers and equal assignments it then follows that every formula must evaluate the same for Q and n at every history.

6. From 4 it follows that

$$\max_{n\in Q}\{v(\{G,n\}\varphi,\, h)\} \leq v'(\{G',Q\}\varphi, h) \qquad Q \in \{S,T\}, h \in H.$$

   By the semantics of $\vee$ this equals the statement to prove.

□

Lemma 3.23. *Let $\mathcal{G}$ be a perfect information $\mathcal{L}$-structure with set of players N. For every base formulas $\varphi \in Form(\mathcal{L})$ and $n \in N$:*

$$\mathcal{G} \vDash_{\mathcal{L}} \left(\bigvee_{o\in N\setminus\{n\}} \{G,o\}\varphi \right) \supset \neg\{G,n\}\neg\varphi$$

Proof. The case for $|N| = 1$ is trivial.

Let $\langle \mathbf{G}, v\rangle$ be an arbitrary perfect information $\mathcal{L}$-structure with set of players N and $|N| > 1$. Let $S = \{n\}, T = N \setminus \{n\}$ and let $\langle \mathbf{G}', v'\rangle$ be a 2-reduced $\mathcal{L}$-structure induced by the partitioning S, T.

By combining parts 5, 6 and 2 of lemma 3.22 the claim can be shown the following way:

1. For any base formula φ from $\vDash_{\mathbf{G}'} \{G',T\}\varphi \equiv \neg\{G',S\}\neg\varphi$ it holds for an arbitrary h that $v'(\{G',T\}\varphi, h) \leq v'(\neg\{G',S\}\neg\varphi, h)$.

2. Substitute part 5: $v'(\{G',T\}\varphi, h) \leq v(\neg\{G,n\}\neg\varphi,\, h)$.

3. Then from part 6 and the transitivity of $\leq$:

$$v(\bigvee_{t\in T}\{G,t\}\varphi,\, h) \leq v(\neg\{G,n\}\neg\varphi,\, h).$$

4. Now that both sides are back in G the inequality can again be stated as an implication:

$$\mathbf{G}, h \vDash \bigvee_{o\in N\setminus\{n\}} \{G,o\}\varphi \supset \neg\{G,n\}\neg\varphi$$

Finally, h, φ and n can be chosen freely, thus the claim holds. □

The proof and, in particular, its use of 2-reduced game forms reflects presumed rationality on which the semantics of $\mathcal{FP}$ is based on. A player acts as if the others all play in a way that is most detrimental for their payoff. The proof further illustrates this "me against the world" mentality present in this system of rationality.

Definition 3.14. For an extensive game form $\mathbf{G} = \langle N, M, H, t, I\rangle$, call $\mathbf{G}' = \langle N, M, H, t, I'\rangle$ its *pi-reduced game form.* where $I' = \{\sim_n \mid n \in N\}$ and $\sim_n = \{(h, h) \mid h \in H\}$ for all $n \in N$.

For a $\mathcal{L}$-structure $\langle \mathbf{G}, v\rangle$, call $\langle \mathbf{G}', v\rangle$ its *pi-reduced $\mathcal{L}$-structure* if$\mathbf{G}'$ is a pi-reduced game form of $\mathbf{G}$.

Lemma 3.24. *Given a $\mathcal{L}$-structure $\langle \mathbf{G}, v\rangle$, its corresponding pi-reduced $\mathcal{L}$-structure $\langle \mathbf{G}', v'\rangle$ and a base formula φ. Let H be the histories and N be the players of $\mathbf{G}$ (and $\mathbf{G}'$). The following hold for every $h \in H, n \in N$:*

1. $\Phi^n_{\mathbf{G}}(h) \subseteq \Phi^n_{\mathbf{G}'}(h)$
2. $v(\{G, n\}\varphi,\, h) \leq v'(\{G', n\}\varphi, h)$
3. $v(\neg\{G, n\}\varphi,\, h) \geq v'(\neg\{G', n\}\varphi, h)$

Proof.

1. Recall that the strategies of player n (on $\mathbf{G}$) were defined as functions $\iota_n(\mathbf{G}) \to M$, cf. section 2.1. For every strategy s of n in $\mathbf{G}$ let s' be a strategy for n on $\mathbf{G}$' s.t.:
$$s(i) = m \iff s'(\{h\}) = m \qquad h \in i, i \in \iota_n(\mathbf{G})$$
Clearly, the strategy s' results in the same play as s at any history where it is n's turn. Therefore, k can force every set in $\mathbf{G}'$ that was forceable in $\mathbf{G}$.

2. The claim follows from the more general fact that for functions f:
$$\text{If } X \subseteq Y, \text{ then } \max_{x\in X}\{f(x)\} \leq \max_{y\in Y}\{f(y)\}.$$
Otherwise, there would be an $x \in X$ s.t. $f(x) > f(y)$ for all $y \in Y$, and thus, in particular, $f(x) > f(x)$. Following statement 1 the conclusion applies to evaluation of positive atomic game formulas over $\Phi^n_{\mathbf{G}}(h)$ and $\Phi^n_{\mathbf{G}'}(h)$.

3. Follows from the previous statement and the fact that if $a \leq b$, then $1 - a \geq 1 - b$ for $a, b \in [0, 1]$.

$\square$

Theorem 3.25. *For every $\mathcal{L}(N, k) \in \mathcal{FP}$, base formula φ and $n \in N$:*

$$\models_{\mathcal{L}(N,k)} \left(\bigvee_{o\in N\setminus\{n\}} \{G, n\}\varphi \right) \supset \neg\{G, n\}\neg\varphi$$

Proof. In the case of $|N| = 1$ empty disjunction is, as usual, interpreted as $\perp$. The claim then holds trivially.

For $|N| > 1$, let $\mathcal{G} = \langle \mathbf{G}, v \rangle$ be an arbitrary $\mathcal{L}$-structure and $\mathcal{G}' = \langle \mathbf{G}', v' \rangle$ its pi-reduced $\mathcal{L}$-structure. Let $N^* = N \setminus \{n\}$. Then by lemma 3.23:

$$\mathcal{G}' \vDash_{\mathcal{L}(N,k)} \bigvee_{o \in N^*} \{G', o\}\varphi \supset \neg\{G', n\}\neg\varphi$$

or as an inequality, for any history h:

$$\max_{o \in N^*} \{v'(\{G', o\}\varphi, h)\} \leq v'(\neg\{G', n\}\neg\varphi, h)$$

By lemma 3.24:

$$\begin{aligned} &v(\neg\{G, n\}\neg\varphi,\, h) \geq v'(\neg\{G', n\}\neg\varphi, h) \\ &v(\{G, o\}\varphi,\, h) \leq v'(\{G', o\}\varphi, h) \qquad o \in N^* \end{aligned}$$

Which leads to the following holding for any history h:

$$\begin{aligned} \max_{o \in N^*} \{v(\{G, o\}\varphi,\, h)\} &\leq \max_{o \in N^*} \{v'(\{G', o\}\varphi, h)\} \\ &\leq v'(\neg\{G', n\}\neg\varphi, h) \leq v(\neg\{G, n\}\neg\varphi,\, h) \end{aligned}$$

Thus, the formula holds for arbitrary $\mathcal{G}$.

□

3.4 Equivalence of $\mathcal{L}$-structures

The investigation of game equivalence was one of the main motivations stated at the beginning of this thesis. In section 2.2 game equivalence was discussed as an important aspect of existing research in logics for games. In particular, for two-valued forcing power logic a bisimulation exists that preserves formula evaluation. This section provides a generalization of this result to all $\mathcal{FP}$ logics as well as a new, weaker form of equivalence.

A common type of game equivalence in general game theory is the equivalence to *reduced strategic forms* [36]. To explain reduced strategic forms first requires a definition of equivalence for strategies:

Definition 3.15. For a preference relation $\leq_n$, let $=_n$ denote the corresponding equivalence relation. Two strategies σ_1, σ_2 of a player n are equal if for every strategy profile s: $s[n/\sigma_1] =_n s[n/\sigma_2]$. (The relation is extended to profiles as described in section 2.1.)

The reduced strategic form of a game is then obtained by translating to strategic form, but taking only one strategy from each equivalence class. An example of the reduced strategic form of a strategic zero-sum game is given by figure 3.11. The strategies of playing r and l are equal for **II**, only one is taken for the reduced form.

I \ II	l	c	r
a	**1**	**0.5**	**1**
b	**1**	**0**	**1**

(a) The full strategic game.

I \ II	l	c
a	**1**	**0.5**
b	**1**	**0**

(b) Corresponding reduced strategic form.

Figure 3.11: Reduced strategic form example.

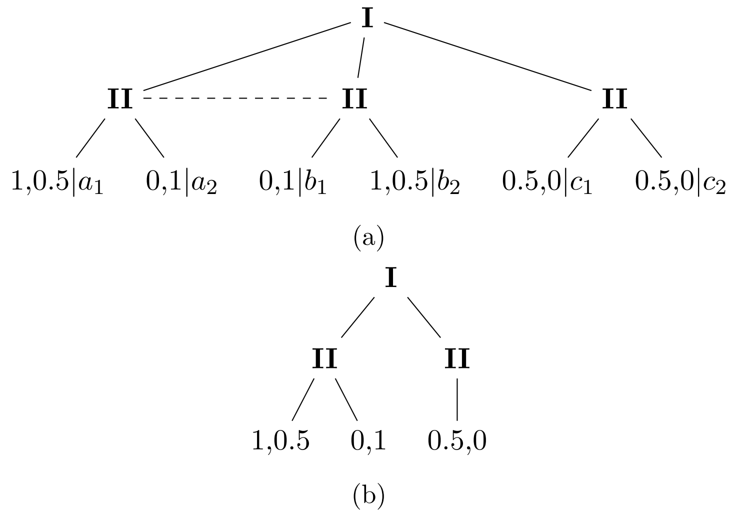
In a strictly formal sense there can be multiple reduced strategic forms for the same game because equivalent strategies could play different moves. Furthermore, the particular names of moves and players is of no importance at all. This sometimes leads to talking about *isomorphic reduced strategic forms* instead. Often these technicalities are ignored in favor of practicality.

The types of equivalences studied here are all based on $\mathcal{FP}$ logics. In particular, the common theme of considering logical structures equivalent when there is some level of evaluation invariance of formulas between them, is followed. Subsection 3.4.1 considers how the notion of reduced strategic forms is best expressed in the framework of $\mathcal{FP}$. In 3.4.2, van Benthem's bisimulation result for two-valued forcing power logics [53] is generalized to $\mathcal{FP}$ logics. In the final section 3.4.3, the constraints of power bisimulation are weakened to power simulation leading to a further invariance result for a fragment of $\mathcal{FP}$.

3.4.1 Valuation power equivalence

The notion of reduced strategic forms is the elimination of redundant strategies. This subsection adapts the notion for use with $\mathcal{L}$-structures. In particular we reduce redundant forcing powers and extend the equivalence of terminals to atomic agreement.

An immediate observation from the semantics of forcing modalities is the fact that if multiple terminal histories in the same forcing power share the same valuation of atoms, then some of them are redundant. The structures of figure 3.12a provide an example. The letters at the leaves are the names of the respective terminal histories. At the root **I** has the forceable set $\{c_1, c_2\}$. It is apparent that if p, q is the only propositional variables, all formulas evaluate equally at those two histories. One of them can be removed without it having an effect on any evaluation at the root. Similarly **I** also has the forceable sets $\{a_1, a_2\}$ and $\{b_1, b_2\}$ at the root. When considering the valuations there is no difference in the two sets and one can be removed. Note that the information set for **II** does not matter in this case as it does not affect the forcing power of **I**. By this argument it must follow that at the root of the structure in 3.12b, all formulas that depend only on the forcing powers of **I** evaluate equally as in structure 3.12a. In the following, the general case of this is captured under the concept of valuation power equivalence.



Values at terminal represent the assignment of p, q respectively.

Figure 3.12: Removal of redundant histories example.

Definition 3.16. Let $\mathcal{M} = \langle \mathbf{M}, v_M \rangle, \mathcal{N} = \langle \mathbf{N}, v_N \rangle$ be $\mathcal{L}$-structures.

Let U be a forceable set in **M** and W a forceable set in **N**. The sets U, W are called *valuation equivalent* if for every $p \in \mathcal{PV}$:

$$\forall u \in U \; \exists w \in W : v_M(p, u) = v_N(p, w)$$
$$\forall w \in W \; \exists u \in U : v_M(p, u) = v_N(p, w)$$

Two forcing powers Φ_M, Φ_N of $\mathbf{M}, \mathbf{N}$ respectively, are called *valuation power equivalent* if every $U \in \Phi_M$ has a valuation equivalent forceable set $W \in \Phi_N$ and vice versa.

Let $\mathcal{M} = \langle \mathbf{M}, v_M \rangle, \mathcal{N} = \langle \mathbf{N}, v_N \rangle$ be $\mathcal{L}$-structures. $\mathcal{M}$ and $\mathcal{N}$ are *valuation power equivalent* if for every player n:

$$\Phi^n_{\mathbf{M}}(()) \text{ is valuation power equivalent to } \Phi^n_{\mathbf{N}}(())$$

Lemma 3.26. *Given $\mathcal{M} = \langle \mathbf{M}, v_M \rangle, \mathcal{N} = \langle \mathbf{N}, v_N \rangle$ be $\mathcal{L}$-structures. Let U be and W be valuation equivalent forceable sets of $\mathbf{M}, \mathbf{N}$ respectively. For every $\varphi \in BForm(\mathcal{L})$:*

$$\min_{u \in U}\{v_M(\varphi, u)\} = \min_{w \in W}\{v_N(\varphi, w)\}$$

Proof. Let u^* be the terminal history in U that has the lowest valuation of φ. By definition of valuation equivalence there is a $w' \in W$ where all propositional variables are assigned the same as at u^*. It follows that

$$\min_{u \in U}\{v_M(\varphi, u)\} = v_M(\varphi, u^*) = v_N(\varphi, w') \geq \min_{w \in W}\{v_N(\varphi, w)\}.$$

By the analogous argument in the opposite direction it follows that

$$\min_{u \in U}\{v_M(\varphi, u)\} \leq \min_{w \in W}\{v_N(\varphi, w)\}.$$

□

Lemma 3.27. *Let $\mathcal{M} = \langle \mathbf{M}, v_M \rangle, \mathcal{N} = \langle \mathbf{N}, v_N \rangle$ be $\mathcal{L}$-structures. Let g, h be histories and i, j be players of $\mathbf{M}, \mathbf{N}$ respectively. For $\varphi \in BForm(\mathcal{L})$, if $\Phi^i_{\mathbf{M}}(g), \Phi^j_{\mathbf{N}}(h)$ are valuation power equivalent then:*

$$v_M(\{M, i\}\varphi, g) = v_N(\{N, j\}\varphi, h).$$

Proof. The claim can be formulated as

$$\max_{U \in \Phi^i_{\mathbf{M}}(g)}\{\min_{u \in U}\{v_M(\varphi, u)\}\} = \max_{W \in \Phi^j_{\mathbf{N}}(h)}\{\min_{w \in W}\{v_N(\varphi, w)\}\}.$$

Let $U^* \in \Phi^i_{\mathbf{M}}(g)$ be the forceable set for which the inner term of the left side is maximal. Because the forcing powers are valuation power equivalent there is a set $W' \in \Phi^j_{\mathbf{N}}(h)$ that is valuation equivalent to U^*. By lemma 3.26

$$\min_{u \in U^*}\{v_M(\varphi, u)\} = \min_{w \in W'}\{v_N(\varphi, w)\}.$$

It follows that

$$\max_{U \in \Phi^i_{\mathbf{M}}(g)}\{\min_{u \in U}\{v_M(\varphi, u)\}\} = \min_{u \in U^*}\{v_M(\varphi, u)\} = \min_{w \in W'}\{v_N(\varphi, w)\} \leq \max_{W \in \Phi^j_{\mathbf{N}}(h)}\{\min_{w \in W}\{v_N(\varphi, w)\}\}.$$

Again the opposite direction is analogous. □

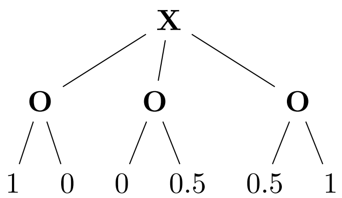
A possible application for valuation power equivalence of games can be found in complex games with only few possible outcomes. As an example the popular game *tic-tac-toe* is considered. The game has two players **X** and **O** who alternate in placing their symbol on a 3x3 grid. The objective is to place 3 of ones symbol in a straight or diagonal line. Let $\mathcal{T}$ be the $\mathcal{L}(\{\mathbf{X}, \mathbf{O}\}, 3)$-structure representing tic-tac-toe where p evaluates to $1, 0, 0.5$ for **X** wins, **O** wins and a draw respectively. The game has too many histories to handle manually. But, considering the few possible outcomes much of this complexity might be superfluous to formulas evaluated at the root history. For this example, the forcing powers were computed and reduced to their unique valuations of p by the program given in appendix A.

Notation. For a forcing power $\Phi^n_{\mathbf{G}}(h)$ we write $p(\Phi^n_{\mathbf{G}}(h))$ to denote the set that is obtained by mapping every terminal history in $\Phi^n_{\mathbf{G}}(h)$ to its assignment for the variable p.

With the terminal histories mapped to their valuations of p the set of forcing powers for the root history of $\mathcal{T}$ is:

$$p(\Phi^O_{\mathbf{T}}(())) = p(\Phi^X_{\mathbf{T}}(())) = \{\{1, 0\}, \{0, 0.5\}, \{1, 0\}, \{0, 1, 0.5\}\}.$$

This then allows for the construction of a simple game that is valuation power equivalent to $\mathcal{T}$. The game $\mathcal{G}$ given in figure 3.13 is such a game. Every game formula will evaluate equally at the roots of $\mathcal{T}$ and $\mathcal{G}$.



(a) $\mathcal{G}$

$$p(\Phi^X_{\mathbf{G}}(())) = \{\{1, 0\}, \{0, 0.5\}, \{1, 0\}, \{0, 1, 0.5\}\}$$
$$p(\Phi^O_{\mathbf{G}}(())) = \{\{1, 0\}, \{0, 0.5\}, \{1, 0\}, \{0, 1, 0.5\}\}$$

(b) Forcing powers mapped to valuations.

Figure 3.13: A game valuation power equivalent game to tic-tac-toe

3.4.2 Power Bisimulation

For the two-valued forcing power logic that provides the basis of $\mathcal{FP}$, there exists another way of equating structures. A common method of showing formulas to be invariant over two structures in modal logics with world to world accessibility relations is bisimulation. This concept was adapted to forcing power logics in the form of power bisimulation. It was shown that two-valued forcing power logic is invariant under power bisimulation [53]. Here, this power bisimulation result is shown to also hold in the many-valued case.

Definition 3.17 (Power Bisimulation)**.** Let $\mathcal{M} = \langle \mathbf{M}, v_M \rangle, \mathcal{N} = \langle \mathbf{N}, v_N \rangle$ be $\mathcal{L}$-structures of the same logic $\mathcal{L}$. Let H_M, H_N be the histories of $\mathbf{M}, \mathbf{N}$ respectively. A relation $E \subseteq H_M \times H_N$ is called a *power bisimulation* if the following conditions hold for all players i of $\mathcal{L}$:

1. If x is a terminal history and xEy, then y must also be a terminal history and $v_M(p,\, x) = v_N(p,\, y)$ for all propositional variables p.
2. If xEy and $\rho^i_{\mathbf{M}}x, U$ then there exists a set W with $\rho^i_{\mathbf{N}}y, W$ and
$$\forall w \in W\ \exists u \in U\ uEw.$$
3. If xEy and $\rho^i_{\mathbf{N}}y, W$ then there exists a set U with $\rho^i_{\mathbf{M}}x, U$ and
$$\forall u \in U\ \exists w \in W\ uEw.$$
4. $()E()$, i.e., the starting histories of both game forms are in relation to each other.

Conditions 2 and 3 together are called the back-and-forth condition.

If there exists a power bisimulation for structures $\mathcal{M}, \mathcal{N}$, we write $\mathcal{M} \equiv \mathcal{N}$.

Note. It is sufficient to consider only the minimal forcing powers (cf. section 3.1.2) for power bisimulation. Assume the conditions hold for all the minimal powers of some x, y with xEy. Then for any non-minimal U with $\rho^i_{\mathbf{M}}x, U$ there is a minimal $U' \subset U$ with $\rho^i_{\mathbf{M}}x, U'$ for which there exists a W', $\rho^i_{\mathbf{N}}y, W$ and $\forall w \in W\ \exists u \in U'\ uEw$ by assumption. Because U extends U' and u is quantified existentially it is also true that $\forall w \in W\ \exists u \in U\ uEw$.

Lemma 3.28. *Bisimilarity of $\mathcal{FP}$ structures is an equivalence relation.*

Proof. Reflexivity and symmetry are obvious. For transitivity, let $\mathcal{M} = \langle \mathbf{M}, v_M \rangle, \mathcal{N} = \langle \mathbf{N}, v_N \rangle, \mathcal{O} = \langle \mathbf{O}, v_O \rangle$ be $\mathcal{L}$-structures with bisimulations E between $\mathcal{M}$ and $\mathcal{N}$ and F between $\mathcal{N}$ and $\mathcal{O}$. A bisimulation D for $\mathcal{M}$ and $\mathcal{O}$ is given by:

$$D = \{(m, o) \,|\, (m, n) \in E \text{ and } (n, o) \in F\}$$

The back-and-forth condition requires verification: Assume xDz and $\rho^i_{\mathbf{M}}x, U$ for some player i. Then there is, by definition of D, a y with xEy. Because E is a bisimulation, there is a set V with $\rho^i_{\mathbf{N}}y, V$ and $\forall v \in V\ \exists u \in U\ uEv$. In the same way also yFz and there is a set W with $\rho^i_{\mathbf{O}}y, W$ and $\forall w \in W\ \exists v \in V\ vFw$. Combining all of this gives $\forall w \in W\ \exists u \in U\ uDv$ for the initial set U and half of the back-and-forth condition is verified. The proof for the other half proceeds in much the same way. $\square$

An example of such a power bisimulation is given in figure 3.14. The example demonstrates that power bisimulation is a broader relation than valuation power equivalence. The payoff 0.5 does not even occur in $\mathcal{N}$. Therefore, the two games can not be valuation power equivalent. Note that it is not important that the value that was removed was the middle value. The example could just as well have payoff 0.5 at c and then instead of δEc it would be γEc in the power bisimulation. How the forceable sets are related for conditions 2 and 3 of definition 3.17 is shown in detail in 3.14c.

(a) $\mathcal{M}$ (b) $\mathcal{N}$

$$Core(\Phi^{\mathbf{I}}_{\mathbf{M}}(())) = \{ \quad \{\alpha\}, \quad \{\beta,\gamma,\delta\} \quad \}$$

$$Core(\Phi^{\mathbf{I}}_{\mathbf{N}}(())) = \{ \quad \{a,b\}, \quad \{a,c\} \quad \}$$

$$Core(\Phi^{\mathbf{II}}_{\mathbf{M}}(())) = \{ \quad \{\alpha,\beta\}, \quad \{\alpha,\gamma\}, \quad \{\alpha,\delta\} \quad \}$$

$$Core(\Phi^{\mathbf{II}}_{\mathbf{N}}(())) = \{ \quad \{a\}, \quad \{b,c\} \quad \}$$

(c) Corresponding forceable sets.

$$E = \{((),()), (\alpha,a), (\alpha,b), (\beta,a), (\delta,c)\}$$

(d) Bisimulation relation.

Figure 3.14: Bisimulation example.

Theorem 3.29. *Let $\mathcal{M} = \langle \mathbf{M}, v_M \rangle$, $\mathcal{N} = \langle \mathbf{N}, v_N \rangle$ be $\mathcal{L}$-structures with a power bisimulation E. Then for all all $\varphi \in Form(\mathcal{L})$:*

$$\textit{If } sEt, \textit{ then } \quad v_M(\varphi, s) = v_N(\varphi, t)$$

Proof. First consider only terminal histories s and t with sEt. At those histories atomic valuations are equal and because the semantics are truth functional all base formulas evaluate to the same value at both histories. Game formulas at terminal histories evaluate equally as the inner formulas, thus they also evaluate equally at terminal histories.

For the general case, consider two histories s, t such that sEt, an arbitrary player i and formula φ. By the semantics of $\{M, i\}$ there exists a set U with $\rho^i_{\mathbf{M}} s, U$ where $\min_{u \in U}\{v_M(\varphi, u)\} = v_M(\{M, i\}\varphi, s)$. Because E is a power bisimulation there exists a set W s.t. $\rho^i_{\mathbf{N}} t, W$ and $\forall w \in W\ \exists u \in U\ uEw$. From the previous argument it follows that $\forall w \in W\ v_N(\varphi, w) \geq v_M(\{M, i\}\varphi, s)$ because the evaluation value in every w is equal to the value at some $u \in U$ and $v_M(\{M, i\}\varphi,\, s)$ is the minimal evaluation in U. From this forceable set W it follows that $\min_{w \in W}\{v_N(\varphi,\, w)\} \geq v_M(\{M, i\}\varphi,\, s)$ and in turn $v_N(\{N, i\}\varphi,\, t) \geq v_M(\{M, i\}\varphi,\, s)$.

Because the back and forth clause also holds for the other direction the argument is the same to show $v_M(\{M, i\}\varphi,\, s) \geq v_N(\{N, i\}\varphi,\, t)$ and therefore, $v_M(\{M, i\}\varphi,\, s) = v_N(\{N, i\}\varphi,\, t)$. This shows that the claim holds also for atomic game formulas. Because of truth functionality the claim also holds for general game formulas. □

It is common in modal logics to regard properties of frames as being characterized by a formula or a set of formulas that are valid exactly in those frames were the property holds. In the logics of this thesis, no properties that refer only to extensive game forms, i.e., are independent of preference, can be characterized in such a way. An example of such a property is the existence of a a non-trivial information set. By applying theorem 3.29, it is easy to demonstrate this fact: For any two $\mathcal{L}$-structures there are assignments such that there is a bisimulation between them. In particular, this holds for any two assignments that assign every variable at every terminal history the same value. Consider two structures with such assignments, let H_1, H_2 be the non-terminal and T_1, T_2 be the terminal histories respectively. Because of these assignments it is easy to see that $H_1 \times H_2 \cup T_1 \times T_2$ is a bisimulation. Therefore, at any two histories $h_1 \in H_1, h_2 \in H_2$ every formula will evaluate equally.

Figure 3.14 already demonstrated that power bisimulation is different from valuation power equivalence. However, the valuation power equivalence of two $\mathcal{L}$-structures does imply the existence of a bismulation between them.

Lemma 3.30. *Let $\mathcal{M} = \langle \mathbf{M}, v_M \rangle, \mathcal{N} = \langle \mathbf{N}, v_N \rangle$ be $\mathcal{L}$-structures. If $\mathcal{M}$ is valuation power equivalent to $\mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.*

Let H_M, H_N be the histories of $\mathbf{M}, \mathbf{N}$ respectively. A power bisimulation for $\mathcal{M}, \mathcal{N}$ is given by:

$$E = \{((), ())\} \cup \{(g, h) \in H_M \times H_N \mid v_M(p,\, g) = v_N(p,\, h) \text{ for all propositional variables } p\}$$

Proof. Observe that conditions 2 and 3 of definition 3.17 hold if U and W are valuation equivalent forceable sets. Because the root forcing powers are valuation power equivalent every forceable set has a valuation equivalent in the other structure. Therefore, conditions 2 and 3 hold for E. Conditions 1 and 4 are trivial. □

3.4.3 Power Simulation

Motivated by the connection of simulation and bisimulation in modal logics the concept of power simulation is introduced. Analogous to power bisimulation the initial goal of the one-sided power simulation is to show that the $\leq$ relation on atoms can be extended to formulas. However, this can not be done for the full set of formulas, e.g., if $v_M(p,\, h) \leq v_N(p,\, h)$ then clearly $v_M(\neg p,\, h) \geq v_N(\neg p,\, h)$. The same problem also exists for implications. Therefore, a restricted language has to be formed to work around this problem. This brings a slight technical problem with it. In definition 3.3 of the syntax of $\mathcal{FP}$ logics, only $\supset$ and $\perp$ (and the modal operators) were part of the actual language while the other operators were defined as abbreviations. Explicitly making those operators part of the language is of course also possible and is the implied basis for the following definition.

Definition 3.18 (Power Simulation)**.** Let $\mathcal{M} = \langle \mathbf{M}, v_M \rangle, \mathcal{N} = \langle \mathbf{N}, v_N \rangle$ be $\mathcal{L}$-structures. Let H_M, H_N be the histories of $\mathbf{M}, \mathbf{N}$ respectively. A relation $E \subseteq H_M \times H_N$ is called a *power simulation* of $\mathcal{M}$ by $\mathcal{N}$ if the following conditions hold for all players i of $\mathcal{L}$:

1. If x is a terminal history and xEy, then y must also be a terminal history and $v_M(p,\, x) \leq v_N(p,\, y)$ for all propositional variables p.
2. If xEy and $\rho^i_{\mathbf{M}} x, U$, then there exists a set W with $\rho^i_{\mathbf{N}} y, W$ and
$$\forall w \in W\ \exists u \in U\ uEw.$$
3. $()E()$, i.e., the starting histories of both game forms are in relation to each other.

If there exists a power simulation of $\mathcal{M}$ by $\mathcal{N}$, we write $\mathcal{M} \leq \mathcal{N}$.

Definition 3.19 (Positive formulas)**.** For a $\mathcal{L} \in \mathcal{FP}$, let $Form^+(\mathcal{L})$ be the set of formulas that contain only the logical constants $\perp, \top, \wedge, \vee, \oplus, \&$ and forcing modalities of $\mathcal{L}$.

Theorem 3.31. *Let $\mathcal{M} = \langle \mathbf{M}, v_M \rangle, \mathcal{N} = \langle \mathbf{N}, v_N \rangle$ be $\mathcal{L}$-structures with a power simulation E of $\mathcal{M}$ by $\mathcal{N}$. Then for all $\varphi \in Form^+(\mathcal{L})$:*

$$\textit{If } sEt, \textit{ then } \quad v_M(\varphi,\, s) \leq v_N(\varphi,\, t)$$

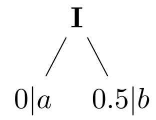
Proof. Observe that for atomic game formulas the proof is the same as one side of the proof of theorem 3.29. What has to be shown is that all the connectives preserve the $\leq$ relation, i.e.:

If $v_M(\varphi_1,\, s) \leq v_N(\varphi_1,\, t)$ and $v_M(\varphi_2,\, s) \leq v_N(\varphi_2,\, t)$, then $v_M(\varphi_1 \bullet \varphi_2,\, s) \leq v_N(\varphi_1 \bullet \varphi_2,\, t)$

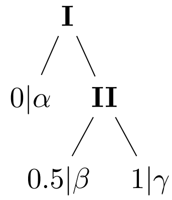
with $\bullet \in \{\vee, \wedge, \oplus, \&\}$. For $\wedge, \vee$ this is obvious. Considering that valuations are in the interval $[0, 1]$ the property also holds for $\bullet = +$. The property follows for $\oplus$ and $\&$.

By definition the $\leq$ relation holds for atoms and was shown to be preserved by every operator. $\square$

A maybe surprising fact is that $\mathcal{M} \leq \mathcal{N}$ and $\mathcal{N} \leq \mathcal{M}$ does *not* imply $\mathcal{M} \equiv \mathcal{N}$. The two games in figure 3.15 are an example where the implication fails. The two simulations are given in the figure. There can be no power bisimulation because $\mathbf{I}$ can force $\{b\}$ in $\mathcal{M}$ and there is no appropriate forceable set for $\mathbf{I}$ in $\mathcal{N}$. For the simulation, $\{\beta, \gamma\}$ works because both evaluate at least as high as b. Figure 3.16c gives a detailed view of which forceable sets are chosen as the W of condition 2 in definition 3.18. This distinction motivates the definition of a further distinct type of equivalent games, those that have a simulation in both directions, a *mutual simulation*. In that case the previous bounding result for positive formulas can easily be extended to positive formula invariance.



(a) $\mathcal{M}$



(b) $\mathcal{N}$

$$Core(\Phi^{\mathbf{I}}_{\mathbf{M}}(())) = \{ \quad \{a\}, \quad \{b\} \quad \}$$

$$Core(\Phi^{\mathbf{I}}_{\mathbf{N}}(())) = \{ \quad \{\alpha\}, \quad \{\beta, \gamma\} \quad \}$$

$$Core(\Phi^{\mathbf{II}}_{\mathbf{M}}(())) = \{ \quad \{a, b\} \quad \}$$

$$Core(\Phi^{\mathbf{II}}_{\mathbf{N}}(())) = \{ \quad \{\alpha, \beta\}, \quad \{\alpha, \gamma\} \quad \}$$

(c) Corresponding forceable sets for $\geq$.

$$E_{\leq} = \{(a, \alpha), (b, \beta), (b, \gamma)\} \tag{3.13}$$

$$E_{\geq} = \{(\alpha, a), (\alpha, b), (\beta, b)\} \tag{3.14}$$

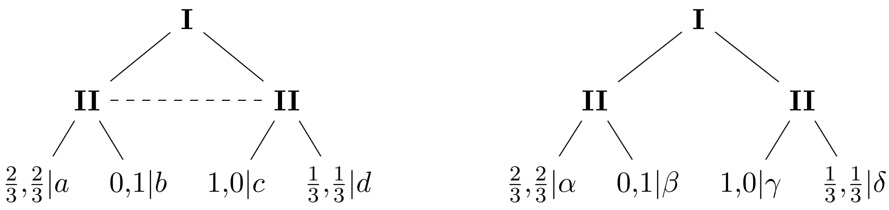
Figure 3.15: Mutual simulation. but no bismulation.

Corollary 3.32. *Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures with a power simulation E of $\mathcal{M}$ by $\mathcal{N}$ and power simulation D of $\mathcal{N}$ by $\mathcal{M}$. Then for all $\varphi \in Form^+(\mathcal{L})$:*

$$\textit{If } sEt \textit{ and } tDs, \textit{ then} \quad v_M(\varphi, s) = v_N(\varphi, t)$$

A similar connection to that of valuation power equivalence and power bisimulation also exists between power bisimulation and mutual simulation. Bisimulation does imply mutual simulation. It is easy to see from the definitions that a power bisimulation is in fact also a power simulation for both directions.

For a more general application of power simulation, recall the pi-reduced game forms of section 3.3. For every $\mathcal{L}$-structure $\mathcal{G}$ and a corresponding pi-reduced $\mathcal{L}$-structure $\mathcal{G}'$: $\mathcal{G} \leq \mathcal{G}'$. This follows directly from the fact that the forcing powers of the pi-reduced game form are a superset of the original game form at the same history (which was shown for lemma 3.24). This is illustrated in figure 3.16 for an extensive form version of the well-known *prisoner's dilemma* game.



(a) $\mathcal{M}$ (Prisoner's dilemma) (b) $\mathcal{N}$

$$Core(\Phi^{\mathbf{II}}_{\mathbf{M}}(())) = \{ \quad \{a, c\}, \quad \{b, d\} \quad \}$$

$$Core(\Phi^{\mathbf{II}}_{\mathbf{N}}(())) = \{ \quad \{\alpha, \gamma\}, \ \{\beta, \delta\}, \ \{\alpha, \delta\}, \ \{\beta, \gamma\}, \ \}$$

(c) Corresponding forceable sets. ($\Phi^{\mathbf{I}}_{\mathbf{M}}(()) = \Phi^{\mathbf{I}}_{\mathbf{N}}(())$)

$$E = \{((), ()), (a, \alpha), (b, \beta), (c, \gamma), (d, \delta)\}$$

Figure 3.16: Power Simulation for Prisoner's dilemma.

CHAPTER 4

Conclusion & Further work

This chapter provides a short review of this thesis. The summary (4.1) is used to establish answers to the research questions asked in the introduction. Furthermore, we provide an analysis of the observed weak and strong points of the approach. A second section (4.2) collects topics for further research.

4.1 Conclusion

This thesis develops a family of many-valued logics for games based on forcing powers. The introduced logics are non-normal modal logics who's accessibility relations correspond to forceable sets of outcomes in extensive game forms. The stated goal was the generalization of results from the two-valued version and the analysis of merit provided by many-valuedness.

Regarding the first goal, the central result of the two-valued case is invariance under power bisimulation. In section 3.4 it was shown that the same also holds for all the many-valued logics of this thesis. Beyond previous work, power simulation was introduced. It has been demonstrated that for positive formulas a lower-bound for truth values is preserved under power simulation. It was also shown that bidirectional power simulation (mutual simulation) is different from power bisimulation and provides even broader possibilities of relating games.

The step to many-valuedness was motivated by the ability to represent the payoffs of a game in a direct and natural way. From a formal perspective, there is little difference to the two-valued case. Aside from bisimulation, many other features have been shown to translate directly to the many-valued case (see sections 3.1, 3.2). However, the value of the direct encoding is apparent when relating logical results to game theory. Sections 3.2 and 3.3, as well as power (bi)simulation, provide clear examples of statements that are of particular interest when formulas are interpreted as payoffs. Under these considerations, the limitation to McNaughton functions is a weakness of the proposed system. The benefit in applications would be even stronger if a wider class of payoff functions could be encoded as formulas. Nonetheless, the many-valued forcing power logics of this thesis were shown to preserve many important properties of their two-valued counterpart, while at the very least providing more possibilities of modeling games as logical structures.

With regard to more general observations, forcing powers provide a convenient abstraction of the information available to players in a given game form. Perfect and imperfect information are handled in the same manner without requiring any additional logical mechanism for dealing with the difference. This ability to jointly deal with a very broad class of game forms is desirable and not always available in other logics for games. On the other hand, forcing powers have a complex structure, and it is hard to handle them generally. This structural complexity also explains the absence of a corresponding proof system in this work.

Forceable sets contain only terminal histories. They relate a history only to final states of a game and do not provide any information on the intermediate states. This makes it harder to work with aspects of games beyond payoffs. Currently there is no straightforward way to consider properties of strategies or profiles. However, further developments in this direction are possible (see section 4.2).

In conclusion, the generalization of forcing power logic to many-valuedness was shown to be worthwhile. Important facts about the two-valued case are preserved and have added meaning. Open questions and opportunities for further improvement are discussed in the following section.

4.2 Further work

The following list indicates some possibilities for further research, building on the work in this thesis.

Generalize to further classes of many-valued logics. A natural next step is to generalize the results of this thesis to a broader class of many-valued logics. Many of the statements in this work could be directly adapted for general continuous t-norm fuzzy logics. The framework for mathematical fuzzy logic by Cintula and Noguera [9] could provide a good basis for using an even broader class of many-valued logics.

Strategy profiles and extending histories. The truth of a game formula at a history h is naturally connected to its truth at histories that extend h. For histories that extend h by one move the general rule is clear. For forcing modalities of players who's turn it is at h the valuation can only become lower because the valuation at h was based on the best possible move. The reverse holds for forcing modalities of the other players.

A more detailed study of how the truth of formulas at a history h relates to its truth at a history that extends h may be worthwhile. For a motivating example, note that one can identify strategy profiles with the set of histories that they produce. Laws of how valuations in extending histories relate could then allow for a better study of strategy profiles in $\mathcal{FP}$. One possibility would be to logically characterize profiles by invariants over their histories.

Computational complexity. In this thesis complexity analysis is not considered. However, it is always valuable to know the computational complexity of the main decision problems of a logic.

Infinite game forms. A generalization of $\mathcal{FP}$ to allow infinite game forms might be worthwhile. Problems with the different ways a game can be infinite have been highlighted throughout the thesis. Some of the outlined issues may be solvable or irrelevant in particular contexts.

Giles's game. The motivation for extending Giles's game with rules for forcing modalities has already been explained in section 2.3. At this time no successful extensions for forcing modalities are known to the author. Preliminary efforts have run into difficulties in the bookkeeping required for modal formulas. Say one player asserts $\{G, \cdot\}p$ and the other $\{G, \cdot\}(q \supset p)$. Their evaluation can stem from different terminal histories, i.e., different instances of p. But at the same time they might not be independent of each other, as discussed in subsection 3.2.3.

Labelled tableaux. An alternative basis for a proof system could come from the work of Governatori and Luppi [19]. The authors present extensions of a labelled tableaux system for normal modal logics to classes of non-normal modal logics. However, the proposed proof systems are for two-valued logics and would need to be generalized even further. Furthermore, it is not easy to see whether forcing power logics would require further adaptions to the tableaux to consider the specific structure of game forms.

Fine-grained equivalence. The equivalences of section 3.4 can be adapted to allow for more fine-grained relations. One possibility would be to consider equivalence for single players instead of requiring conditions, e.g., the back-and-forth conditions of power bisimulation, to apply for every player. The presented invariance results can easily be adapted to only hold for formulas that are limited to a specific player's forcing modality. Such individual equivalences would present some interesting new questions. For example: How similar are the games in the equivalence class of a single player? If there is an individual power bisimulation for each player, does that imply the existence of a power bisimulation?

APPENDIX A

A valuation powers calculation program for *tic-tac-toe*

The following Python 2.7 code calculates the valuation powers of a structure that represents the game tic-tac-toe:

```
import itertools

pos_threes = [(0,1,2), (3,4,5), (6,7,8), (0,4,8), (2,4,6),
              (0,3,6), (1,4,7), (2,5,8)]

def turn_swap(cur_turn):
    return (cur_turn % 2) + 1

def make_unique(original_list):
    unique_list = []
    [unique_list.append(obj) for obj in original_list if obj not in unique_list]
    return unique_list

class Node:
    def __init__(self, free, board, turn):
        self.free = free
        self.board = board
        self.turn = turn
        self.won = self.check_won()
        self.children = list()
```

```python
    def check_won(self):
        for thr in pos_threes:
            m = self.board[thr[0]] * self.board[thr[1]] * self.board[thr[2]]
            if m == 1: return 1; # all three fields played by 1
            if m == 8: return 2; # all three fields played by 2
        return 0

    def terminal(self):
        return self.free == 0 or self.won != 0;

    def spawn_children(self):
        if self.terminal():
            return
        for i in range(0, len(self.board)):
            if self.board[i] == 0:
                ch = self.spawn(i)
                self.children.append(ch)
        self.expand_children()

    def expand_children(self):
        for ch in self.children:
            ch.spawn_children()

    def spawn(self, pos):
        nb = list(self.board)
        nb[pos] = self.turn
        return Node(self.free-1, nb, turn_swap(self.turn))

    def count_terminal(self):
        if self.terminal():
            return 1
        return sum(map(lambda ch: ch.count_terminal(), self.children))

    def gather_powers(self, player):
        if self.terminal():
            return [[self.won]]
        kids = map(lambda ch: ch.gather_powers(player), self.children)
        if player == self.turn:
            #union
            return make_unique(list(itertools.chain.from_iterable(kids)))
        else:
            # union product
            tuples = list(itertools.product(*kids))
            products = map(lambda t: list(itertools.chain.from_iterable(t)),
                           tuples)
            uniqf = make_unique(map(lambda p: sorted(make_unique(p)),
                                    products))
            return uniqf
```

```
if __name__ == "__main__":
    root = Node(9, [0]*9, 1)
    root.spawn_children()
    print "Count: ", root.count_terminal()
    print "Gathering"
    print "Powers of P1, ", root.gather_powers(1)
    print "Powers of P2, ", root.gather_powers(2)
```

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Index