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Some variants of the Aubin-Lions Lemma

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Wien, Februar 2017
Philipp Holzinger

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Abstract

The purpose of this thesis is to write out and extend the paper [Mou16] by Ayman Moussa, which provides a modern approach to the classical Aubin-Lions Lemma. The author of [Mou16] states and proves in his work two generalizations of the Aubin-Lions Lemma, which is an indispensable tool in the studies of nonlinear parabolic differential equations. The two versions handle the problems, delivered by the estimates of degenerated evolution equations and incompressible Navier-Stokes equations, the latter being considered on a non-cylindrical domain. The interesting fact about his work is his totally different approach to these problems, which where already studied by many other authors, without using the Aubin-Lions Lemma itself. We prepare appropriate theory, use most of the ideas and strategies of [Mou16] and carry out the proofs in [Mou16] substantially equal.

Contents

Introduction

One of the most powerful tools in the studies of nonlinear parabolic differential equations is the Aubin-Lions Lemma. This result is named after the French mathematicians Jean-Pierre Aubin and Jacques-Louis Lions. The original statement and the proof can be found in the work of Aubin [Aub63]. It provides a compactness criterion in the theory of Lebesgue spaces of Banach space valued functions. More precisely suppose B, X, Y are Banach spaces of functions defined on a set $\Omega \subseteq \mathbb{R}^d$ and let $I \subset \mathbb{R}^d$ be a non empty and bounded interval. Consider a sequence of functions $(u_n)_{n\in\mathbb{N}}$ with $u_n : I \to B$ such that for $p \in [1, \infty)$ the p-th power of the norm $\|u_n(t)\|_B : I \to \mathbb{R}$ is integrable. If

i) X is compactly embedded in B, which is in turn continuously embedded in Y,

ii) $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^p(I; X)$,

iii) $(\partial_t u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^r(I; Y)$, for $r > 1$,

then the sequence $(u_n)_{n\in\mathbb{N}}$ has a converging subsequence in $\mathrm{L}^p(I; B)$.
Why is this theorem so important? The Aubin-Lions Lemma gives us the possibility to prove the existence of a solution in many evolution equations. The usual way to apply the theorem is to approximate a parabolic problem itself with a Galerkin method or a suitable regularization. These approximation methods provide a bunch of solvable problems and therefore we obtain a family of solutions. In general this family of solutions possesses a-priori uniform estimates securing the hypotheses of the Aubin-Lions Lemma and therefore, applying this theorem renders a converging subsequence, with the limit constituting a solution for the original problem. With this in mind we construct two similar results.

In chapter 1 we give a brief overview of Radon measures, Sobolev spaces, Lebesgue spaces for vector valued functions and we also define most of our notation there.

For the first version of the Aubin-Lions Lemma we modify the classical result by replacing condition ii) with another boundedness condition. In most of the cases the Banach space X is usally chosen as the Sobolev space $\mathrm{W}^{1,p}(\Omega)$. Therefore we have some information about the sequence of weak gradients $(\nabla_x u_n)_{n\in\mathbb{N}}$, where ∇_x denotes the gradient in the space variable. We want to weaken this condition. So let $\Phi : \mathbb{R} \to \mathbb{R}$ be a nonlinear function with appropriate conditions and exchange condition ii) above with

$\overline{\text{ii)}}$ $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^p(I; B)$ and $(\nabla_x \Phi(u_n))_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^p(I; B)$.

Inthsi case the problem arises that we cannot get easily a boundedness result for the gradient of u_n. If the derivative $\Phi'(u_n(t, x))$ is nowhere zero, we could write the gradient of u_n as follows:

$$\nabla_x u_n = \frac{1}{\Phi'(u_n)} \nabla_x \Phi(u_n).$$

We see, if u_n approaches a critical point of Φ, the expression degenerates and the usual Aubin Lions Lemma fails here.

We deal with this nonlinear version in chapter 2. The chapter starts with a characterisation for Sobolev functions and progresses with the commutativity of the weak limit and multiplication of two functions. If a sequence $(u_n)_{n\in\mathbb{N}}$ converges weakly towards a function u and a second sequence $(v_n)_{n\in\mathbb{N}}$ weakly-* converges towards a function v we obtain that

$$(u_n v_n) \xrightharpoonup{n\to\infty} uv$$

At the end of the chapter we show the proof of our first version applying this weak convergence result.

For the second version of the Aubin-Lions Lemma we change the setting from a fixed spatial domain into a moving spatial domain. We start with functions defined on the time/space domain $I \times \Omega$, therefore we could say that for each $t \in I$ the evaluation of $u(t)$ is a function from Ω to $\mathbb{R}$. This is just possible, because $I \times \Omega$ is a so called *cylindrical domain*, in the sense, that we have a fixed domain Ω for each moment $t \in I$.
In contrast consider a family of domains $(\Omega^t)_{t\in I}$, representing the motion of a spatial domain and the corresponding *non cylindrical* time/space domain is given by

$$\hat{\Omega} \coloneqq \bigcup_{t\in I} \{t\} \times \Omega^t.$$

To illustrate the difference of these two domains we present the figure, taken from [Mou16] on page 3, where on the left hand side we have an example for a cylindrical domain and on the right hand side we have a non cylindrical domain.

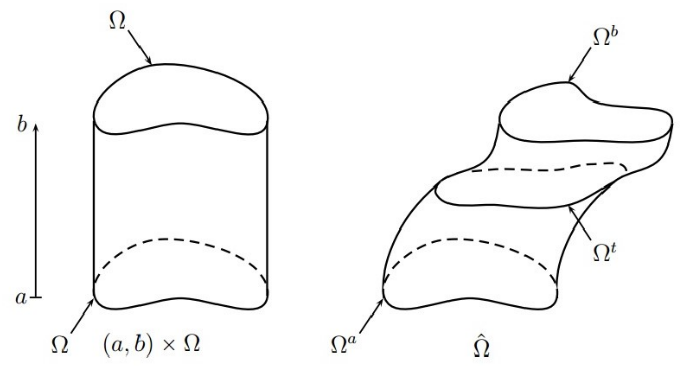

Figure 0.1: Illustration of a cylindrical (left) and a non cylindrical domain (right).

The arising problem with the non cylindrical time/space domain is that we cannot assume functions $u : (t, x) \mapsto u(t, x)$ defined on $\hat{\Omega}$ as functions of the time variable with values in some Banach space. Therefore the statement of the Aubin-Lions Lemma is already problematic.
In chapter 3 we give specific regularity conditions for the motion of the family $(\Omega^t)_{t\in I}$, for example we claim that each Ω^t is diffeomorph to a bounded domain $\Omega \subset \mathbb{R}^d$. With these assumptions we obtain a lot of facts for the family of sets, e.g. we find a common Poincaré constant independent of $t \in I$.
The first result we get is close to a Aubin-Lions Lemma for non cylindrical domains and the proof of this statement applies also for cylindrical domains, hence we could prove a similar

statement to the Aubin-Lions Lemma.
The main theorem of this chapter deals with a compactness result for divergence free vector fields, which arise in the theory of incompressible Navier-Stokes equations on a moving domain. For a better understanding we introduce the space of all square integrable divergence free vector fields $\mathrm{L}^2_{\mathrm{div}}(\hat{\Omega})^d$ and the important subspace of vector fields in $\mathrm{L}^2_{\mathrm{div}}(\hat{\Omega})^d$ satisfying a homogeneous Neumann condition on $\partial\hat{\Omega}$. At the end of the thesis we present the proof of the main theorem, where we comprehensively go through the ideas of [Mou16].

1 Prerequisites

In this chapter we want to give a summary about the most important theory we are going to use in this thesis. Every reader should know the basics of topological spaces, Banach spaces, metric spaces (for these we refer to [Kal14]) and also measure theory (for this theory we suggest [Bog07]). Mostly we work with functions defined on the euclidean space $\mathbb{R}^d$ with the topology induced by the euclidean norm. Let $\Omega \subseteq \mathbb{R}^d$, then we denote the space of all continuous functions $f : \Omega \to \mathbb{R}$ as $\mathrm{C}^0(\Omega)$ and the space of all functions $f : \Omega \to \mathbb{R}$, which are k-times continuously differentiable will be denoted as $\mathrm{C}^k(\Omega)$, where k takes values in $\mathbb{N} \cup \{\infty\}$. We define the *support* of a function $f : \mathbb{R}^d \to \mathbb{R}$ as the set

$$\operatorname{supp} f \coloneqq \overline{\{x \in \mathbb{R}^m : f(x) \neq 0\}}.$$

The subset of $\mathrm{C}^k(\mathbb{R}^d)$ of all functions f with compact support in $\mathbb{R}^d$, will be denoted as $\mathrm{C}^k_c(\mathbb{R}^d)$. For a subset $\Omega \subset \mathbb{R}^d$ we often write $\mathcal{D}(\Omega)$ for the space $\mathrm{C}^\infty_c(\Omega)$ and call it the *test functions* We want to give a criterion for the smoothness of the boundary of a bounded subset $\Omega \subset \mathbb{R}^d$ (not:$\partial\Omega$).

Definition 1.0.1. (C^k /Lipschitz boundary) Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set and $Q \coloneqq \left\{x \in \mathbb{R}^d : |x_i| < 1 \text{ for } i = 1, \ldots, d\right\}$. We say that Ω has a C^k boundary, if for every $x \in \partial\Omega$, there exist a neighborhood $U \subset \mathbb{R}^d$ of x and a map $H : Q \to U$ satisfying:

i) H is bijective,

ii) $H \in \mathrm{C}^k(\overline{Q})$ and $H^{-1} \in \mathrm{C}^k(\overline{U})$,

iii) $H(Q_+) = U \cap \Omega$ and $H(Q_0) = U \cap \partial\Omega$,

where $Q_+ \coloneqq \{x \in Q : x_n > 0\}$ and $Q_0 \coloneqq \{x \in Q : x_n = 0\}$. If H is just Lipschitz continuous, we call $\partial\Omega$ a Lipschitz boundary.

Let us start with some measure theory.

1.1 Measure theory

The *Borel σ-algebra* of a topological space X, which is the σ-algebra generated by all open sets of X, will be denoted as $\mathcal{B}(X)$. Let additionally Y be a metric space, and let $f : X \to Y$. We say that f is a *Borel function* , if $f^{-1}(O) \in \mathcal{B}(X)$ for every open set $O \subseteq Y$.
One of the most used measures on $\mathbb{R}^d$, is the d-dimensional *Lebesgue measure* and it will be dentoted as λ_d. Everytime we integrate with respect to the *Lebesgue measure*, we write dx instead of $d\lambda_d(x)$. The space of all functions for which the p-th power of the absolute value is Lebesgue integrable in $\Omega \subseteq \mathbb{R}^d$ will be denoted usally as $\mathrm{L}^p(\Omega)$ for all $p \in [1, \infty]$ and its norm will be denoted as $\|.\|_{\mathrm{L}^p(\Omega)}$. We denote the *conjugate exponent* of $p \in [1, \infty)$ as p', which fulfills $\frac{1}{p} + \frac{1}{p'} = 1$ and where $\mathrm{L}^{p'}(\Omega)$ is the topological dual of $\mathrm{L}^p(\Omega)$. The set of all locally integrable functions on Ω will be denoted as $\mathrm{L}^1_{loc}(\Omega)$. Since we will use them quite often, we recall Fubini and the Dominated Convergence Theorem.

Theorem 1.1.1. (Fubini) *Let μ and ν be $\sigma-$finite nonnegative measures on the spaces X and Y. Suppose that the function $f : X \times Y \to \mathbb{R}$ is integrable with respect to the product measure $\mu \otimes \nu$. Then the function $y \mapsto f(x, y)$ is integrable with respect to ν for $\mu-$a.e x, the function $x \mapsto f(x, y)$ is integrable with respect to μ with $\nu-$a.e y, the functions*

$$x \mapsto \int_Y f(x,y)\, d\nu(y), \qquad y \mapsto \int_X f(x,y)\, d\mu(x)$$

are integrable on the corresponding spaces, and one has

$$\int_{X\times Y} f(x,y)\, d\mu \otimes \nu = \int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x) = \int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(y).$$

Proof. See [Bog07],Theorem 3.4.4 on page 185. □

Theorem 1.1.2. (Dominated Convergence Theorem) *Let λ_d be the d-dimensional Lebesgue measure on $\mathbb{R}^d$ and let $\Omega \subseteq \mathbb{R}^d$ be an open set. Suppose that $(f_n)_{n\in\mathbb{N}} \in \mathrm{L}^p(\Omega)$ converges to a function f almost everywhere (not: a.e.) in Ω. If there exists a function $g \in \mathrm{L}^p(\Omega)$ such that*

$$|f_n(x)| \le g(x) \qquad \text{a.e in } \Omega \text{ for every } n \in \mathbb{N},$$

then the function f lies in $\mathrm{L}^p(\Omega)$ and

$$f_n \xrightarrow{n\to\infty} f \qquad \text{in } \mathrm{L}^p(\Omega).$$

Let us give a more general version of the Hölder inequality, which is the key element in the proof of the Lyapunov inequality.

Theorem 1.1.3. (Hölder inequality) *Assume that $\Omega \subseteq \mathbb{R}^d$ and $p, q, r \in [1, \infty]$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If $f \in \mathrm{L}^p(\Omega)$ and $g \in \mathrm{L}^q(\Omega)$, then $fg \in \mathrm{L}^r(\Omega)$ and*

$$\|fg\|_{\mathrm{L}^r(\Omega)} \le \|f\|_{\mathrm{L}^p(\Omega)} \|g\|_{\mathrm{L}^q(\Omega)}.$$

Theorem 1.1.4. (Lyapunov inequality) *If $f \in \mathrm{L}^p \cap \mathrm{L}^q$ with $1 \le p \le q \le \infty$, then $f \in \mathrm{L}^r$, for all r with $p \le r \le q$ and there exists $\beta \in [0, 1]$ such that*

$$\|f\|_r \le \|f\|_p^{\beta} \|f\|_q^{1-\beta}, \text{ where } \frac{1}{r} = \frac{\beta}{p} + \frac{1-\beta}{q}.$$

We give now a short summary about Radon measures, which can be identified as elements of the dual space of the set of all continuous functions with compact support. For the interested reader we recommend the books [AFP00], [Mag12] and [EG15].

Definition 1.1.5. Let $(X, \mathcal{E})$ be a measure space and let $d \in \mathbb{N}, d \ge 1$.

a) We say that $\mu : \mathcal{E} \to \mathbb{R}^d$ is a *measure*, if $\mu(\emptyset) = 0$ and for any sequence $(E_h)_{h\in\mathbb{N}}$ of pairwise disjoint elements of $\mathcal{E}$

$$\mu\left(\bigcup_{h=0}^{\infty} E_h\right) = \sum_{h=0}^{\infty} \mu(E_h).$$

If $d = 1$ we say, that μ is a *real measure*, if $d > 1$ we say that μ is a *vector measure* .

b) If μ is a measure, we define the *total variation* $|\mu|$ for every $E \in \mathcal{E}$ as follows:

$$|\mu|(E) := \sup\left\{\sum_{h=0}^{\infty} |\mu(E_h)| : E_h \in \mathcal{E} \text{ pairwise disjoint, } E = \bigcup_{h=0}^{\infty} E_h\right\}.$$

Definition 1.1.6. Let X be a locally compact separable metric space, $\mathcal{B}(X)$ its Borel σ-algebra, and consider the measure space $(X, \mathcal{B}(X))$.

(i) A positive measure on $(X, \mathcal{B}(X))$ is called a *Borel measure*. If a Borel measure is finite on all compact sets (or in other words if a Borel measure is *locally finite*), it is called a *positive Radon measure*.

(ii) A (real- or vector-valued) set function defined on the relatively compact Borel subsets of X, that is a measure on $(K, \mathcal{B}(K))$ for every compact set $K \subseteq X$, is called a *(real- or vector-valued) Radon measure* on X. If $\mu : \mathcal{B}(X) \to \mathbb{R}^d$ is a measure, according to Definition 1.1.5, then we say that μ is a *finite Radon measure.*

If O is an open set in $\mathbb{R}^d$, then we denote by $\mathcal{M}(O)$ (resp.($\mathcal{M}(\overline{O})$) the set of finite Radon measures on O (resp.$\overline{O}$).

The *Riesz Representation Theorem* is a very strong tool to handle the topological dual space of all continuous functions with compact support. One can find the theory for this theorem in [Mag12]. We only want to use the outcome, which is stated in the next remark.

Remark 1.1.7. Thanks to the *Riesz Representation Theorem*, we identify the topological dual space of $\mathrm{C}_c(O)$ with $\mathcal{M}(O)$. The evaluation of a Radon measure $\mu \in \mathcal{M}(O)$ at a function $f \in \mathrm{C}_c(O)$ takes the form:

$$\langle \mu, f\rangle_{\mathrm{C}_c(O)} = \int_O f\, d\mu.$$

Conversely, if $g : O \to \mathbb{R}$ is a bounded Borel function and if we define $\mu_g(A) = \int_A g(x)\, dx$ for all $A \subseteq O$, then μ_g is clearly a finite Radon measure and we simply write $g \in \mathcal{M}(O)$. We also denote with $\langle g, f\rangle$ the evaluation $\langle \mu_g, f\rangle$, in the sense

$$\langle g, f\rangle = \int_O f(x)g(x)\, dx.$$

The interpretation of $\mathcal{M}(O)$ as dual space of $\mathrm{C}_c(O)$ renders also an equivalent definition for the *total variation* of a measure $\mu \in \mathcal{M}(O)$, in particular we have for every open set $A \subseteq O$

$$|\mu|(A) := \sup\{\langle \mu, \varphi\rangle : \varphi \in \mathrm{C}_c(A),\ \|\varphi\|_\infty \le 1\}$$

and for $E \subset O$ arbitrary

$$|\mu|(E) := \inf \{|\mu|(A) : E \subseteq \text{ and } A \text{ is open}\}.$$

■

For Chapter 2, we will need the definition of the vague Topology on $\mathcal{M}(\mathbb{R}^d)$, with which we can define the weak convergence in $\mathcal{M}(\mathbb{R}^d)$.

Definition 1.1.8. Let $O \subseteq \mathbb{R}^d$ be an open set, then we introduce the *vague topology* on $\mathcal{M}(O)$, which is generated by the mappings $\mu \mapsto \langle \mu, f \rangle = \int_O f d\mu$ for all $f \in \mathrm{C}_c(O)$. In particular, μ_n *vaguely converges* to μ, if and only if

$$\langle \mu_n, f \rangle \to \langle \mu, f \rangle, \qquad \forall f \in \mathrm{C}_c(O). \tag{1.1}$$

We state the property for Borel measures, that every Borel set with bounded measure can be approximated by compact sets. We use this property especially for the Lebesgue measure, but we state the generell result.

Theorem 1.1.9. (Inner approximation by compact sets) *If μ is a Borel measure on $\mathbb{R}^d$, and E is a Borel set in $\mathbb{R}^d$, with $\mu(E) < \infty$, then for every $\varepsilon > 0$ there exists a compact set $K \subseteq E$ such that $\mu(E \setminus K) \leq \varepsilon$. In particular,*

$$\mu(E) = \sup \{\mu(K) : K \subseteq E, K \textit{ is compact}\}.$$

Proof. See [Mag12], proof of Theorem 2.8 on page 18. □

At the end, we define the space of all functions of bounded variation and give an important result. For the interested reader we suggest the book [AFP00].

Definition 1.1.10. Let Ω be an open set in $\mathbb{R}^d$ and $u \in \mathrm{L}^1(\Omega)$. We say u is a *function of bounded variation in* Ω, if the distributional derivative of u is representable by a finite Radon measure on Ω, i.e. if

$$\int_\Omega u \cdot \partial_{x_i} \phi \, dx = - \int_\Omega \phi \, dD_i u \qquad \forall \phi \in \mathrm{C}_c^\infty(\Omega), \; i = 1, \dots, n \tag{1.2}$$

for some $\mathbb{R}^d$-valued measure $Du = (D_1 u, \dots, D_n u)$ in Ω. The vector space of all functions of bounded variation in Ω is denoted by $BV(\Omega)$.

We notice that $BV(\Omega)$ endowed with the norm

$$\|u\|_{BV\Omega} = \|u\|_{\mathrm{L}^1(\Omega)} + |Du|(\Omega),$$

is a Banach space. Observe that Du is a functional on $\mathrm{C}_c(\Omega)$, in the sense that

$$\langle Du, \phi \rangle = \sum_{i=1}^d \langle D_i u, \phi \rangle \qquad \forall \phi \in \mathrm{C}_c(\Omega)$$

and the expression $|Du|(\Omega)$ can be, thanks to the Riesz Representation Theorem, interpreted as

$$|Du|(\Omega) = \sup_{\varphi \in \mathrm{C}_c^\infty(\Omega), \|\varphi\| \leq 1} \sum_{i=1}^d |\langle D_i u, \varphi_i \rangle|$$

The next theorem gives a compactness result for a bounded sequence in $BV(\Omega)$.

Theorem 1.1.11. (Compactness for BV functions) *Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded, with Lipschitz boundary $\partial\Omega$. Assume $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $BV(\Omega)$. Then there exists a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ and a function $f \in BV(\Omega)$ such that*

$$(f_{n_j})_{j\in\mathbb{N}} \to f \qquad \textit{in } \mathrm{L}^1(\Omega).$$

as $j \to \infty$.

Proof. See [EG15] proof of Theorem 5.5, page 203. □

1.2 Sobolev spaces

1.2.1 A short recap

In this section we recall the definition of Sobolev spaces and provide an overview of the relevant statements. Suppose $\Omega \subseteq \mathbb{R}^d$ open, $u, v \in \mathrm{L}^1_{loc}(\Omega)$ and $\alpha = (\alpha_1, \ldots, \alpha_2)$ is a multiindex of order $|\alpha| = \alpha_1 + \ldots + \alpha_d = k$. We say that v is the α^{th}- *weak partial derivative* of u, written $D^\alpha u = v$, if

$$\int_\Omega u(x) D^\alpha \phi(x)\, dx = (-1)^{|\alpha|} \int_\Omega v(x)\phi(x)\, dx, \qquad \forall \phi \in \mathrm{C}^\infty_c(\Omega).$$

Definition 1.2.1. Let $p \in [1, \infty]$, $m \in \mathbb{N}_0$ and $\Omega \subseteq \mathbb{R}^d$, then we define the *Sobolev spaces* as

$$\mathrm{W}^{m,p}(\Omega) := \{u \in \mathrm{L}^p(\Omega) : D^\alpha u \in \mathrm{L}^p(\Omega)\}$$

where α is a multi-index and $D^\alpha u$ is the appropriate weak partial derivative of u. With this we are able to define the Sobolev spaces.
The space $\mathrm{W}^{m,p}(\Omega)$ actually consists of equivalence classes of functions, which coincide on Ω except on a null set. We define the norm on $\mathrm{W}^{m,p}(\Omega)$ as:

$$\begin{aligned} \|u\|^p_{\mathrm{W}^{m,p}(\Omega)} &= \sum_{|\alpha|\le m} \|D^\alpha u\|^p_{\mathrm{L}^p(\Omega)}, &&\text{if } p < \infty, \\ \|u\|_{\mathrm{W}^{m,p}(\Omega)} &= \max_{|\alpha|\le m} \|D^\alpha u\|_{\mathrm{L}^p(\Omega)}, &&\text{if } p = \infty, \end{aligned}$$

It is common to write $\mathrm{H}^k(\Omega)$ instead of $\mathrm{W}^{k,2}(\Omega)$ for all $k \in \mathbb{N}_0 \cup \{\infty\}$.
We have clearly that each k-times continuously differentiable function f with compact support in $\Omega \subset \mathbb{R}^d$, lies in $\mathrm{W}^{k,p}(\Omega)$ for all $p \in [1, \infty]$. We call a function f a *test function* if $f \in \mathrm{C}^\infty_c(\Omega)$ and we will write $\mathcal{D}(\Omega)$ instead of $\mathrm{C}^\infty_c(\Omega)$. The closure of $\mathrm{C}^\infty_c(\Omega)$ in $\mathrm{W}^{k,p}(\Omega)$ will be denoted as

$$\mathrm{W}^{k,p}_0 := \overline{\mathrm{C}^\infty_c(\Omega)}^{\|\cdot\|_{\mathrm{W}^{k,p}(\Omega)}}$$

For an open set $\Omega \subseteq \mathbb{R}^d$ we establish the our first density result stated in [Bre10]. We need the following

Definition 1.2.2. Let $\Omega \subset \mathbb{R}^d$ be an open set. We say an open subset $\omega \subset \Omega$ is *strongly included* in Ω if $\overline{\omega} \subset \Omega$ and $\overline{\omega}$ is compact

Lemma 1.2.3. (Friedrichs Lemma) *Let Ω be an open subset of $\mathbb{R}^d$ and $u \in \mathrm{W}^{1,p}(\Omega)$ with $1 \leq p < \infty$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{C}_c^\infty(\mathbb{R}^d)$ such that*

$$u_n|_\Omega \xrightarrow[n\to\infty]{} u \qquad \text{in } \mathrm{L}^p(\Omega),$$

$$\nabla u_n|_\omega \xrightarrow[n\to\infty]{} \nabla u|_\omega \qquad \text{in } \mathrm{L}^p(\omega)^d \qquad \text{for all strongly included subsets } \omega \subset \Omega.$$

In the case $\Omega = \mathbb{R}^d$, there exists a subsequence $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{C}_c^\infty(\mathbb{R}^d)$ such that

$$u_n \xrightarrow[n\to\infty]{} u \qquad \text{in } \mathrm{L}^p(\Omega),$$

$$\nabla u_n \xrightarrow[n\to\infty]{} \nabla u \qquad \text{in } \mathrm{L}^p(\omega)^d.$$

Proof. See [Bre10], page 265, Theorem 9.2 □

Most of the time we will need continuous Sobolev embeddings for all $p \in [1, \infty]$ and for this purpose we want to recall the definitions of a continuous and a compact embedding.

Definition 1.2.4. Let X and Y be Banach spaces, $X \subseteq Y$. We say that X is *continuously embedded* in Y, denoted by $X \hookrightarrow Y$, provided

$$\|x\|_Y \leq C\|x\|_X \qquad \forall x \in X$$

Definition 1.2.5. Let (X, d) be a metric space. A subset M of X is *totally bounded*, if and only if for every $\epsilon > 0$, there exists a finite collection of open balls in M of radius ϵ, whose union contains M.

Definition 1.2.6. Let X and Y be Banach spaces, $X \subseteq Y$. We say that X is *compactly embedded* in Y, denoted by $X \subset\subset Y$, provided

(i) X embeds continuously in Y,

(ii) any bounded set in X is totally bounded in Y, i.e. every sequence in a bounded subset of X has a subsequence that is Cauchy in the norm $\|.\|_Y$.

At this point we define the Sobolev exponent and state one of the main embedding theorems for Sobolev spaces.

Definition 1.2.7. Let d be the dimension of the underlying space $\mathbb{R}^d$. If $1 \leq p < d$, the *Sobolev exponent* of p is

$$p^\star := \frac{dp}{d-p}.$$

We adopt the convention $p^\star = \infty$, when $p = d$.

Theorem 1.2.8. (Rellich-Kondrachov Compactness Theorem) *Assume Ω is a bounded open subset of $\mathbb{R}^d$, and $\partial\Omega$ is C^1. Suppose $1 \leq p < d$. Then*

$$\mathrm{W}^{1,p}(\Omega) \subset\subset \mathrm{L}^q(\Omega)$$

for each $1 \leq q < p^\star$.

Proof. See [Eva98] page 272. $\square$

The next two corollaries, taken from [Bre10], give an overview of all continuous embeddings, which are important for us.

Corollary 1.2.9. *Suppose that Ω is an open and bounded subset of $\mathbb{R}^d$ with $\partial\Omega \in C^1$. Let $p \in [1, \infty]$, then we have*

$$\begin{aligned} &\mathrm{W}^{1,p}(\Omega) \subset \mathrm{L}^{p^\star}(\Omega) && && \text{if } p < d, \\ &\mathrm{W}^{1,p}(\Omega) \subset \mathrm{L}^{q}(\Omega) && \forall q \in [p, \infty), && \text{if } p = d, \\ &\mathrm{W}^{1,p}(\Omega) \subset \mathrm{L}^{\infty}(\Omega) && && \text{if } p > d, \end{aligned}$$

and all these embeddings are continuous.

Proof. See Corollary 9.14 in [Bre10], on page 285. $\square$

Sometimes we need Sobolev embeddings for the whole space $\mathbb{R}^d$. The following results can be found in [Bre10].

Corollary 1.2.10. *Let $m \geq 1$ be an integer and let $p \in [1, \infty)$. We have*

$$\begin{aligned} &\mathrm{W}^{m,p}(\mathbb{R}^d) \subset \mathrm{L}^{q}(\mathbb{R}^d), && \text{where } \frac{1}{q} = \frac{1}{p} - \frac{m}{d}, && \text{if } \frac{1}{p} - \frac{m}{d} > 0, \\ &\mathrm{W}^{m,p}(\mathbb{R}^d) \subset \mathrm{L}^{q}(\mathbb{R}^d), && \forall q \in [p, \infty), && \text{if } \frac{1}{p} - \frac{m}{d} = 0, \\ &\mathrm{W}^{m,p}(\mathbb{R}^d) \subset \mathrm{L}^{\infty}(\mathbb{R}^d), && && \text{if } \frac{1}{p} - \frac{m}{d} < 0, \end{aligned}$$

and all these embeddings are continuous.

Proof. See Corollary 9.13 in [Bre10], on page 284. $\square$

At the end we give a generally version of the *Poincaré inequality* . For this version we need the definition of the *mean value* of a function and we give a meaning to the expression $\|\nabla u\|_{\mathrm{L}^p(\Omega)}$. Let $\Omega \subset \mathbb{R}^d$ and denote with $|.|$ the *p-norm* in $\mathbb{R}^d$, then

$$(u)_\Omega := \int_\Omega u \, dx, \qquad \|\nabla u\|^p_{\mathrm{L}^p(\Omega)} = \int_\Omega |\nabla u|^p \, dx.$$

Theorem 1.2.11. (Poincaré inequality) *Let Ω be a bounded, connected and open subset of $\mathbb{R}^d$, with Lipschitz boundary $\partial\Omega$. Assume $p \in [1, \infty]$. Then there exists a constant C_Ω, depending only on d, p and Ω, such that*

$$\|u - (u)_\Omega\|_{\mathrm{L}^p(\Omega)} \leq C_\Omega \|\nabla u\|_{\mathrm{L}^p(\Omega)},$$

for each function $u \in \mathrm{W}^{1,p}(\Omega)$.

Proof. See [Eva98] Theorem 1 on page 275. $\square$

Remark 1.2.12. It is easy to check that the norm of the Sobolev space $\mathrm{H}^1(\Omega)$, for arbitrary $\Omega \subset \mathbb{R}^d$, holds

$$\|u\|^2_{\mathrm{H}^1(\Omega)} = \|u\|^2_{\mathrm{L}^2(\Omega)} + \sum_{i=1}^{d} \|\partial_{x_i} u\|^2_{\mathrm{L}^2(\Omega)} = \|u\|^2_{\mathrm{L}^2(\Omega)} + \|\nabla u\|^2_{\mathrm{L}^2(\Omega)}.$$

Hence we obtain with the Poincaré inequality the following estimate

$$\|u - (u)_\Omega\|_{\mathrm{H}^1(\Omega)} \leq \sqrt{(1 + C_\Omega)} \|\nabla u\|_{\mathrm{L}^2(\Omega)}. \tag{1.3}$$

$\blacksquare$

1.2.2 Traces

We are going to define the *trace-operator* for Sobolev functions. It assigns “boundary values” to functions $u \in \mathrm{W}^{1,p}(U)$, assuming that ∂U is C^1. If $u \in C(\overline{U})$, the evaluation of u at a boundary point of U is well defined. Therefore the restriction of the trace-operator γ on $\mathrm{W}^{1,p}(U) \cap C(\overline{U})$ has to satisfy $\gamma u = u|_{\partial U}$. The next theorem guarantees the existence of such an operator. If it is not otherwise stated, assume $p \in [1, \infty)$.

Theorem 1.2.13. (Trace-Theorem) *Assume that $U \subset \mathbb{R}^d$ is bounded and ∂U is C^1. Then there exists a bounded linear operator*

$$\gamma : \mathrm{W}^{1,p}(U) \to \mathrm{L}^p(\partial U)$$

such that

(i) $\gamma u = u|_{\partial U}$ if $u \in \mathrm{W}^{1,p}(U) \cap C(\overline{U})$ and

(ii) $\|\gamma u\|_{\mathrm{L}^p(\partial U)} \leq C \|u\|_{\mathrm{W}^{1,p}(U)}$

for each $u \in \mathrm{W}^{1,p}(U)$, with the constant C depending only on p and U.

Proof. See [Eva98] page 258. □

Definition 1.2.14. Let U be an open set in $\mathbb{R}^d$ and ∂U is C^k and let $u \in \mathrm{H}^m(U)$ for $m \in \mathbb{N} \cup \{\infty\}$, then

(i) we call γu the *trace* of u on ∂U, and

(ii) we define the space $\mathrm{H}^{m-\frac{1}{2}}(\partial U) \coloneqq \gamma(\mathrm{H}^m(U))$ and endowe it with the norm

$$\|g\|_{\mathrm{H}^{m-\frac{1}{2}}(\partial U)} = \inf_{u \in \mathrm{H}^p(U), \gamma u = g} \|u\|_{\mathrm{H}^m(U)}.$$

We will not need the next result for this thesis, but it gives some intuition on how functions with trace equal to 0 can be understood.

Theorem 1.2.15. *Assume $U \subset \mathbb{R}^d$ is bounded and ∂U is C^1. Suppose furthermore that $u \in \mathrm{W}^{1,p}(U)$. Then*

$$u \in \mathrm{W}^{1,p}_0(U) \qquad \textit{if and only if} \qquad \gamma u = 0 \textit{ on } \partial U$$

Proof. See [Eva98] page 259. □

1.2.3 Regularization

We shall introduce some elementary results concerning the approximation of functions by smooth functions. Let $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ be a smooth, even and non-negative function with support in the unit ball $B_1(0)$ of $\mathbb{R}^d$ and $\|\varphi\|_{\mathrm{L}^1(\mathbb{R}^d)} = 1$. We define for each the sequence of non-negative even *mollifiers* $\varphi_\eta \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ as $\varphi_\eta(\mathbf{x}) \coloneqq \frac{1}{\eta^d}\varphi(\frac{x}{\eta})$, $\eta > 0$, such that the properties

(i) $\varphi_\eta \geq 0$,

(ii) $\varphi_\eta(\mathbf{x}) = \varphi_\eta(-\mathbf{x})\ \forall \mathbf{x} \in \mathbb{R}^d$,

(iii) $\operatorname{supp}(\varphi_\eta) \subset B_\eta(\mathbf{0})$,

(iv) $\int_{\mathbb{R}^d} \varphi_\eta = 1$.

hold. Let $f, g : \mathbb{R}^d \to \mathbb{R}$ be two Borel functions. We write the convolution of these two functions as

$$f \star g(x) := \int_{\mathbb{R}^d} f(x-y)g(y)\,dy.$$

We assume that the reader knows the basic properties of convolutions. Otherwise we suggest [Bre10] for more information. The mollifiers have for this thesis relevant results, like the regularization of a function $f \in \mathrm{L}^p(\Omega)$, where Ω is an open subset of $\mathbb{R}^d$. With this regularization we are able to approximate such a function arbitrarily close.

Definition 1.2.16. Let f be in $\mathrm{L}^p(\mathbb{R}^d)$ for $1 \le p < \infty$, then we define the *$\eta-$regularization* as the convolution $f_\eta := f \star \varphi_\eta$.

Lemma 1.2.17. *For each $\eta \in (0,1)$ we have that $\operatorname{supp} f_\eta \subset \operatorname{supp} f + \overline{B_\eta(0)}$ and $f_\eta \in \mathrm{C}^\infty(\mathbb{R}^d)$.*

Proof. See [Sho94] page 31. □

Lemma 1.2.18. *If $f \in C_c(\Omega)$, where $\Omega \subseteq \mathbb{R}^d$ is open, then $f_\eta \to f$ uniformly in Ω for $\eta \to 0$. If $f \in \mathrm{L}^p(\Omega)$, $1 \le p < \infty$, then $\|f_\eta\|_{\mathrm{L}^p(\Omega)} \le \|f\|_{\mathrm{L}^p(\Omega)}$ and $f_\eta \to f$ in $\mathrm{L}^p(\Omega)$ for $\eta \to 0$.*

Proof. See [Sho94] page 32. □

1.3 The Lebesgue Space $\mathrm{L}^p([0,T];B)$ of Banach space Valued Functions

Since solutions of parabolic equations are functions in space and time, it may occur that the regularity of the time is different to the regularity of the space. Therefore we need Sobolev spaces which can differentiate between the space- and time variable. We say a function $[0,T] \times B \to \mathbb{R}$ takes values in a Banach space B, if $u(t) \in B$ for all $t \in [0,T]$. Lets take a closer look on these functions.

Definition 1.3.1. Let B be a Banach space and $T > 0$

(i) The space $\mathrm{C}^k([0,T];B)$, $k \in \mathbb{N}_0 \cup \{\infty\}$ is the set of all k-times continuously differentiable functions $u : [0,T] \to B$. The norm is given by

$$\|u\|_{\mathrm{C}^k([0,T];B)} = \sum_{i=0}^{k} \max_{0 \le t \le T} \|\frac{\partial^i u}{\partial t^i}(t)\|_B$$

(ii) The space $\mathrm{L}^p((0,T);B)$ for $p \in [1,\infty]$ is the space of all (equivalence classes of) measurable functions $u : (0,T) \to B$, that satisfy

$$\|u\|^p_{\mathrm{L}^p((0,T);B)} = \int_0^T \|u(l)\|_B^p \, dl < \infty \qquad \text{for } p \in [0,\infty)$$
$$\|u\|_{\mathrm{L}^\infty((0,T);B)} = \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_B < \infty \qquad \text{for } p = \infty.$$

We give some properties of the Lebesgue space concerning their topological structure. For the proofs and further information see [Zei90a], page 407.

Proposition 1.3.2. (Properties of Lebesgue Spaces) *Let $m \in \mathbb{N}_0$ and $1 \leq p \leq \infty$. Let B and X be Banach spaces over $\mathbb{R}$. Then*

(i) $\mathrm{C}^m([0,T];B)$, *endowed with the norm $\|.\|_{\mathrm{C}^m([0,T];B)}$, is a Banach space over $\mathbb{R}$.*

(ii) $\mathrm{L}^p([0,T];B)$ *endowed with the norm $\|.\|_{\mathrm{L}^p([0,T];B)}$, is a Banach space over $\mathbb{R}$, in the case where one identifies functions that are equal almost everywhere on $[0,T]$. Moreover, the set of all step functions $u : [0,T] \to B$ is dense in $\mathrm{L}^p([0,T];B)$.*

(iii) $\mathrm{C}([0,T];B)$ *is dense in* $\mathrm{L}^p([0,T];B)$, *and the embedding*

$$\mathrm{C}([0,T];B) \hookrightarrow \mathrm{L}^p([0,T];B)$$

is continuous.

(iv) If H is a Hilbert space with scalar product $(.,.)_H$, then $\mathrm{L}^2([0,T];H)$ is also a Hilbert space with the scalar product

$$(u,v) = \int_0^T (u(t),v(t))_H \, dt.$$

(v) For $1 \leq p < \infty$ the space$\mathrm{L}^p([0,T];B)$ is separable, if B is separable .

(vi) If the embedding $X \hookrightarrow B$ is continuous, then the embedding

$$\mathrm{L}^r([0,T];X) \hookrightarrow \mathrm{L}^p([0,T];B) \qquad 1 \leq p \leq r \leq \infty$$

is also continuous.

(vii) Let $\Omega \subseteq \mathbb{R}^d$ and let $B = \mathrm{L}^p(\Omega)$ with $1 \leq p < \infty$. Then we can identify $\mathrm{L}^p([0,T];\mathrm{L}^p(\Omega))$ with $\mathrm{L}^p(I \times \Omega)$. This does not hold for $p = \infty$.

Let B be a Banach space with norm $\|.\|$. We first recall the definition of the dual space. It is defined by $B' := \{f : B \to \mathbb{R} : f \text{ is linear and continuous}\}$ and its norm is given by

$$\|f\|_{B'} = \sup_{\|u\| \leq 1} |f(u)|.$$

The following describes the dual space of $\mathrm{L}^p(0,T;B)$, in the terms of another Lebesgue space.

Proposition 1.3.3. *Let B be a reflexive and separable Banach space, $1 \leq p < \infty$ and p' be the conjugate exponent of p. Then we the following identification:*

$$(\mathrm{L}^p([0,T];B))' = \mathrm{L}^{p'}([0,T];B')$$

We point out that there exists also a Hölder inequality for Lebesgue spaces.

Proposition 1.3.4. (Hölder inequality) *Let B be a Banach space, then for all $u \in \mathrm{L}^p([0,T];B)$ and for all $v \in \mathrm{L}^{p'}([0,T];B')$ the adapted Hölder inequality holds:*

$$\|uv\|_{\mathrm{L}^1([0,T];B)} \leq \|u\|_{\mathrm{L}^p([0,T];B)} \|v\|_{\mathrm{L}^{p'}([0,T];B')}, \tag{1.4}$$

and all these integrals do exist.

Proof. For the proof see [Zei90a], proof of Proposition 23.6 on page 411. □

At the end we want to state an usefull proposition from functional analysis, without a proof.

Proposition 1.3.5. *Let B and X be Banach spaces such that B is dense and continuously embedded. Then the embedding $X' \hookrightarrow B'$ is continuous. Additionally if X is reflexive, then X' is dense in B'.*

2 First Version

2.1 The nonlinear version

Let us begin this chapter with the statement of the first version of the Aubin-Lions Lemma. This version deals with the lack of knowledge of the compactness in the space variable, where we only have information about some function on it. We denote by $\mathcal{M}(I, \mathrm{H}^{-m}(\Omega))$ the dual space of $\mathrm{C}_c(I, \mathrm{H}^m(\Omega))$ for any arbitrary subset Ω in $\mathbb{R}^d$.

Theorem 2.1.1. *Consider a non-empty closed and bounded interval $I \subset \mathbb{R}$, and a bounded open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. Furthermore consider a function $\Phi \in \mathrm{W}^{1,1}_{loc}(\mathbb{R}, \mathbb{R})$ such that $(\mathbb{1}_{|\Phi'|<\delta})_{\delta>0}$ converges to 0 in $\mathrm{L}^1(\mathbb{R})$, as $\delta \to 0$. If a sequence of functions $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^2(I \times \Omega)$ satisfies, that*

(i) $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(I \times \Omega)$,

(ii) the sequence of time derivative $(\partial_t(u_n))_{n\in\mathbb{N}}$ is bounded in $\mathcal{M}(I; \mathrm{H}^{-m}(\Omega))$, and

(iii) $(\nabla_{\mathbf{x}}\Phi(u_n))_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(I \times \Omega)$,

then $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathrm{L}^2(I \times \Omega)$.

To prove this theorem, we need to show that the product of a weakly converging sequence and a weakly-* converging sequence, which satisfy additional properties, is weakly convergent. This convergence result will be shown in the section below.

2.2 Weak convergence of a product

This section starts with the Lyapunov inequality for functions in vector valued Lebesgue spaces, then we show a characterisation for Sobolev functions, which is the key element of the Commutator Lemma. With all this preparation we are able to show the desired result of this section.

Lemma 2.2.1. (Lyapunov inequality for vector valued functions) *Let $I \subset \mathbb{R}$ be a bounded interval, $u \in \mathrm{L}^q(I; \mathrm{L}^{p_1}(\mathbb{R}^d)) \cap \mathrm{L}^q(I; \mathrm{L}^{p_2}(\mathbb{R}^d))$ with $q \in [1, \infty]$ and $1 \le p_1 \le p_2 \le \infty$. Then $u \in \mathrm{L}^q(I; \mathrm{L}^r(\mathbb{R}^d))$ for all $r \in [p_1, p_2]$ and for the number $\beta \in [0, 1]$ which satisfies $\frac{1}{r} = \frac{\beta}{p_1} + \frac{1-\beta}{p_2}$, we have*

$$\|u\|_{\mathrm{L}^q(I;\mathrm{L}^r(\mathbb{R}^d))} \le \|u\|^{\beta}_{\mathrm{L}^q(I;\mathrm{L}^{p_1}(\mathbb{R}^d))} \|u\|^{1-\beta}_{\mathrm{L}^q(I;\mathrm{L}^{p_2}(\mathbb{R}^d))}.$$

Proof. The only cases where $\beta = 1$ or $\beta = 0$, occur when $r = p_1$ or $r = p_2$ or $p_1 = p_2$. In all these cases we clearly have equality. Therefore let $r \in (p_1, p_2)$ with $p_1 \ne p_2$.
There exists $\beta \in (0, 1)$ such that $\frac{1}{r} = \frac{\beta}{p_1} + \frac{1-\beta}{p_2}$. Since $u(t) \in \mathrm{L}^{p_1}(\mathbb{R}^d) \cap \mathrm{L}^{p_2}(\mathbb{R}^d)$ for all $t \in I$, we are able to apply the Lyapunov inequality , Theorem 1.1.4, and obtain for all $t \in I$

$$\|u(t)\|_{\mathrm{L}^r(\mathbb{R}^d)} \le \|u(t)\|^{\beta}_{\mathrm{L}^{p_1}(\mathbb{R}^d)} \|u(t)\|^{1-\beta}_{\mathrm{L}^{p_2}(\mathbb{R}^d)}. \tag{2.1}$$

Using (2.1) we get

$$\|u\|^q_{\mathrm{L}^q(I;\mathrm{L}^r(\mathbb{R}^d))} = \int_I \|u(t)\|^q_{\mathrm{L}^r(\mathbb{R}^d)}\,dt \le \int_I \|u(t)\|^{q\beta}_{\mathrm{L}^{p_1}(\mathbb{R}^d)}\|u(t)\|^{q(1-\beta)}_{\mathrm{L}^{p_2}(\mathbb{R}^d)}\,dt. \tag{2.2}$$

Since

$$\int_I (\|u(t)\|^{q\beta}_{\mathrm{L}^{p_1}(\mathbb{R}^d)})^{\frac{1}{\beta}}\,dt < \infty \quad\text{and}\quad \int_I (\|u(t)\|^{q(1-\beta)}_{\mathrm{L}^{p_2}(\mathbb{R}^d)})^{\frac{1}{1-\beta}}\,dt < \infty,$$

we have $\|u(t)\|^{q\beta}_{\mathrm{L}^{p_1}(\mathbb{R}^d)} \in \mathrm{L}^{\frac{1}{\beta}}(I)$ and $\|u(t)\|^{q(1-\beta)}_{\mathrm{L}^{p_2}(\mathbb{R}^d)} \in \mathrm{L}^{\frac{1}{1-\beta}}(I)$. Since $(\frac{1}{\beta})^{-1} + (\frac{1}{1-\beta})^{-1} = 1$, we can apply the Hölder inequality on (2.2) and obtain

$$\begin{aligned}\int_I \|u(t)\|^{q\beta}_{\mathrm{L}^{p_1}(\mathbb{R}^d)}\|u(t)\|^{q(1-\beta)}_{\mathrm{L}^{p_2}(\mathbb{R}^d)}\,dt &\le \left(\int_I (\|u(t)\|^{q\beta}_{\mathrm{L}^{p_1}(\mathbb{R}^d)})^{\frac{1}{\beta}}\,dt\right)^{\beta}\left(\int_I (\|u(t)\|^{q(1-\beta)}_{\mathrm{L}^{p_2}(\mathbb{R}^d)})^{\frac{1}{(1-\beta)}}\,dt\right)^{1-\beta}\\ &= \|u\|^{q\beta}_{\mathrm{L}^q(I;\mathrm{L}^{p_1}(\mathbb{R}^d))}\|u\|^{q(1-\beta)}_{\mathrm{L}^q(I;\mathrm{L}^{p_2}(\mathbb{R}^d))}.\end{aligned}$$

This shows the desired estimate for all $q \in [1,\infty)$. Since all arguments also hold for $q = \infty$, the proof can be done the same way. □

For the characterisation of Sobolev functions we give the definition of the shift operator and show that it is an isometric linear operator.

Definition 2.2.2. We define the *shift operator* $\tau_h : \mathbb{R}^{\mathbb{R}^d} \to \mathbb{R}^{\mathbb{R}^d}$ for $h \in \mathbb{R}^d$. If f is some function defined on $\mathbb{R}^d$, then $\tau_h f(x) = f(x-h)$.

Lemma 2.2.3. *Let $\tau_y : \mathrm{L}^p(\mathbb{R})^d \to \mathrm{L}^p(\mathbb{R})^d$ for $p \ge 1$, then τ_y is an isometric linear operator.*

Proof. The linearity of τ_y is clear. Define the shift operator for $y \in \mathbb{R}^d$ as $\tilde{\tau}_y(x) = x - y$ for all $x \in \mathbb{R}^d$. We have that $\det D\tilde{\tau}_y = 1$ for all $y \in \mathbb{R}^d$ and with the Transformation rule (see Theorem 3.1.3) we obtain for all $u \in \mathrm{L}^p(\mathbb{R}^d)$ and $z = x - y$

$$\|\tau_y u\|^p_{\mathrm{L}^p(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |u(\tilde{\tau}_y(x))|^p\,dx = \int_{\mathbb{R}^d} |u(\tilde{\tau}_y(\tilde{\tau}_{-y}(z))|^p\,dz = \|u\|^p_{\mathrm{L}^p(\mathbb{R}^d)}.$$

□

Proposition 2.2.4. *For $h \in \mathbb{R}^d$ let $\tau_h : \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d)$ be the shift operator as defined in Definition 2.2.2, and let $u \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ with $1 \le p \le \infty$. Then we have*

$$\|\tau_h u - u\|_{\mathrm{L}^p(\mathbb{R}^d)} \le |\mathbf{h}|\|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^d)}. \tag{2.3}$$

Proof. We want to reproduce the proof of [Bre10] on page 267.

Assume first that $u \in \mathrm{C}^\infty_c(\mathbb{R}^d)$. Let $h \in \mathbb{R}^d$ and set

$$v(t) := u(x + th), \qquad t \in \mathbb{R}.$$

Then $v(t)' = h \cdot \nabla u(x + th)$ and thus

$$u(x+h) - u(x) = v(1) - v(0) = \int_0^1 v'(t)\,dt = \int_0^1 h \cdot \nabla u(x+th)\,dt.$$

With Jensen's inequality we obtain for $1 \leq p \leq \infty$ that

$$|\tau_h u - u|^p \leq |h|^p \int_0^1 |\nabla u(x + th)|^p \, dt.$$

and

$$\begin{aligned}\int_{\mathbb{R}^d} |\tau_h u(x) - u(x)|^p dx &\leq |h|^p \int_{\mathbb{R}^d} \int_0^1 |\nabla u(x + th)|^p \, dt \, dx \\ &\leq |h|^p \int_0^1 \int_{\mathbb{R}^d} |\nabla u(x + th)|^p \, dx \, dt \\ &\leq |h|^p \int_0^1 \int_{\mathbb{R}^d} |\nabla u(y)|^p \, dy \, dt\end{aligned}$$

and thus

$$\|\tau_h u - u\|^p_{\mathrm{L}^p(\mathbb{R}^d)} \leq |h|^p \|\nabla u\|^p_{\mathrm{L}^p(\mathbb{R}^d)}.$$

This proves (2.3) for all $u \in \mathrm{C}_c^\infty(\mathbb{R}^d)$. Assume now that $u \in \mathrm{W}^{1,p}(\mathbb{R}^d)$, with $1 \leq p < \infty$. Thanks to the Friedrichs Lemma (Lemma 1.2.3) we get a sequence $(u_n)_{n\in\mathbb{N}} \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ with $u_n \to u$ in $\mathrm{L}^p(\mathbb{R}^d)$ and $\nabla u_n \to \nabla u$ in $\mathrm{L}^p(\mathbb{R}^d)^d$. The last inequality holds for all u_n and we conclude the proof with passing to the limit. ☐

It is not clear that one can apply Proposition 2.2.4 above for all non-empty open and bounded subsets $O \subseteq \mathbb{R}^d$, because if $u \in \mathrm{L}^p(O)$, the function $\tau_h u$ would not be well defined. This Problem in mind we give a refinement of the previous Proposition, which we will use in another chapter.

Theorem 2.2.5. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\omega \subset \Omega$ be compact or strongly included (see Definition 1.2.2). Furthermore let $u \in \mathrm{W}^{1,p}(\Omega)$ with $1 \leq p \leq \infty$, then we have for all $h \in \mathbb{R}^d$ with $|h| \leq d(\overline{\omega}, \partial\Omega)$*

$$\|\tau_h u - u\|_{\mathrm{L}^p(\omega)} \leq \|\nabla u\|_{\mathrm{L}^p(\Omega)} |h|.$$

Proof. The proof is very similar to the prove of Proposition 2.2.4. For a detailed proof see [Bre10] Proposition 9.3 on page 267. ☐

To follow Moussa [Mou16] we will repeatedly use the sequence $(\varphi_\eta)_{\eta\in(0,1)}$ of nonnegative even mollifiers as defined in Subsection 1.2.3. If we convolute such a mollifier with a function $u : I \times \mathbb{R}^d \to \mathbb{R}$, where $I \subset \mathbb{R}$ is a non empty and bounded interval, then we understand this convolution as a convolution in the space variable. For example let $u \in \mathrm{L}^p(I \times \mathbb{R}^d)$ for some $p \in [1, \infty]$ and let φ_η, $\eta > 0$, be a sequence of mollifiers as defined in Subsection 1.2.3, then the convolution of both functions is defined as

$$u \star \varphi_\eta(t, x) := \int_{\mathbb{R}^d} u(t, x - y)\varphi_\eta(y) \, dy = \int_{\mathbb{R}^d} u(t, y)\varphi_\eta(x - y) \, dy.$$

Lemma 2.2.6. (Commutator Lemma) *Let $q \in [1, \infty)$, $p \in [1, d]$, $r \in [1, p^\star)$ and $I := [a, b] \subset \mathbb{R}$ be a non-empty closed and bounded interval. Consider a bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^q(I; \mathrm{W}^{1,p}(\mathbb{R}^d))$ and a bounded sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^{q'}(I; \mathrm{L}^{r'}(\mathbb{R}^d))$. Let $(\varphi_\eta)_{\eta>0}$ be defined as in Subsection 1.2.3, t hen the commutator (convolution in the space variable only)*

$$S_{n,\eta} := u_n(v_n \star \varphi_\eta) - (u_n v_n) \star \varphi_\eta$$

goes to 0 *in* $\mathrm{L}^1(I \times \mathbb{R}^d)$ *as* $\eta \to 0$, *uniformly in* n.

Proof. We will use the shift operator τ_h, as defined in Definition 2.2.2. We want to show, that for all $r < p^\star$

$$\tau_h u_n - u_n \xrightarrow[h \to 0]{} 0 \qquad \text{in } \mathrm{L}^q(I; \mathrm{L}^r(\mathbb{R}^d)) \text{ uniformly in } n. \tag{2.4}$$

Since $\mathrm{W}^{1,p}(\mathbb{R}^d)$ embeds continuously into $\mathrm{L}^r(\mathbb{R}^d)$ for all $r \in [1, p^\star]$ and thanks to Proposition 1.3.2 we know, that the sequence $(u_n)_{n\in\mathbb{N}}$ is also bounded in $\mathrm{L}^q(I; \mathrm{L}^r(\mathbb{R}^d))$ for all $r \in [1, p^\star]$. Therefore we get that for all $h \in \mathbb{R}^d$ the sequence $(\tau_h u_n - u_n)_{n\in\mathbb{N}}$ is also bounded in $\mathrm{L}^q(I; \mathrm{L}^r(\mathbb{R}^d))$ for all $r \in [1, p^\star]$. Especially we find $C_1, C_{p^\star} > 0$, such that

$$\begin{aligned} &\|\tau_h u_n - u_n\|_{\mathrm{L}^q(I;\mathrm{L}^1(\mathbb{R}^d))} \le C_1 \qquad \forall n \in \mathbb{N}, \\ &\|\tau_h u_n - u_n\|_{\mathrm{L}^q(I;\mathrm{L}^{p^\star}(\mathbb{R}^d))} \le C_{p^\star} \qquad \forall n \in \mathbb{N}. \end{aligned}$$

We want to apply Lemma 2.2.1. Since for all $n \in \mathbb{N}$ we have $\tau_h u_n - u_n \in \mathrm{L}^q(I; \mathrm{L}^1(\mathbb{R}^d)) \cap \mathrm{L}^q(I; \mathrm{L}^p(\mathbb{R}^d))$ and $\tau_h u_n - u_n \in \mathrm{L}^q(I; \mathrm{L}^p(\mathbb{R}^d)) \cap \mathrm{L}^q(I; \mathrm{L}^{p^\star}(\mathbb{R}^d))$, we find $\beta_1, \beta_{p^\star} \in [0, 1]$ such that the following two estimates hold for all $n \in \mathbb{N}$:

$$\|\tau_h u_n - u_n\|_{\mathrm{L}^q(I;\mathrm{L}^r(\mathbb{R}^d))} \le \|\tau_h u_n - u_n\|^{\beta_1}_{\mathrm{L}^q(I;\mathrm{L}^1(\mathbb{R}^d))} \|\tau_h u_n - u_n\|^{(1-\beta_1)}_{\mathrm{L}^q(I;\mathrm{L}^p(\mathbb{R}^d))} \qquad \forall r \in [1, p], \tag{2.5}$$

$$\|\tau_h u_n - u_n\|_{\mathrm{L}^q(I;\mathrm{L}^r(\mathbb{R}^d))} \le \|\tau_h u_n - u_n\|^{\beta_{p^\star}}_{\mathrm{L}^q(I;\mathrm{L}^p(\mathbb{R}^d))} \|\tau_h u_n - u_n\|^{(1-\beta_{p^\star})}_{\mathrm{L}^q(I;\mathrm{L}^{p^\star}(\mathbb{R}^d))} \qquad \forall r \in [p, p^\star]. \tag{2.6}$$

Setting $C \coloneqq \max(C_1^{\beta_1}, C_{p^\star}^{(1-\beta_{p^\star})})$ and combining (2.5) and (2.6) we have for all $n \in \mathbb{N}$:

$$\|\tau_h u_n - u_n\|_{\mathrm{L}^q(I;\mathrm{L}^r(\mathbb{R}^d))} \le C \max_{i \in \{(1-\beta_1), \beta_{p^\star}\}} (\|\tau_h u_n - u_n\|^i_{\mathrm{L}^q(I;\mathrm{L}^p(\mathbb{R}^d))}) \qquad \forall r \in [1, p^\star]. \tag{2.7}$$

So it is sufficient for (2.4) to prove that $\|\tau_h u_n - u_n\|_{\mathrm{L}^q(I;\mathrm{L}^p(\mathbb{R}^d))}$ goes to zero for $h \to 0$ uniformly in n. Let us denote with ∇_x the gradient in $\mathbb{R}^d$, then thanks to Proposition 2.2.4 we obtain

$$\begin{aligned} \|\tau_h u_n - u_n\|^q_{\mathrm{L}^q(I;\mathrm{L}^p(\mathbb{R}^d))} &= \int_I \|\tau_h u_n(t) - u_n(t)\|^q_{\mathrm{L}^p(\mathbb{R}^d)} \, dt \\ &\le \int_I |h|^q \|\nabla_x u_n(t)\|^q_{\mathrm{L}^p(\mathbb{R}^d)} \, dt = |h|^q \|\nabla_x u_n\|^q_{\mathrm{L}^q(I;\mathrm{L}^p(\mathbb{R}^d))}. \end{aligned}$$

The sequence $(\nabla_x u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^q(I; \mathrm{L}^p(\mathbb{R}^d))$ by some constant $\tilde{C} > 0$, so that for all $r \in [1, p^\star]$ we finally get with (2.7)

$$\begin{aligned} \|\tau_h u_n - u_n\|_{\mathrm{L}^q(I;\mathrm{L}^r(\mathbb{R}^d))} &\le C \max_{i \in \{(1-\beta_1), \beta_{p^\star}\}} (\|\tau_h u_n - u_n\|^i_{\mathrm{L}^q(I;\mathrm{L}^p(\mathbb{R}^d))}) \\ &\le C \max_{i \in \{(1-\beta_1), \beta_{p^\star}\}} (\|\nabla_x u_n\|^i_{\mathrm{L}^q(I;\mathrm{L}^p(\mathbb{R}^d))} |h|^i) \\ &\le C \max_{i \in \{(1-\beta_1), \beta_{p^\star}\}} (\tilde{C}^i |h|^i). \end{aligned}$$

Passing to the limit $h \to 0$, we obtain (2.4).

Since the mollifier φ_η has support in $B_\eta(0)$, we can rewrite the commutator as follows

$$\begin{aligned}
S_{n,\eta}(t,x) &= u_n(v_n \star \varphi_\eta)(t,x) - (u_n v_n) \star \varphi_\eta(t,x) \\
&= \int_{\mathbb{R}^d} u_n(t,x) v_n(t,x-y)\varphi_\eta(y)\,dy - \int_{\mathbb{R}^d} u_n(t,x-y)v_n(t,x-y)\varphi_\eta(y)\,dy \\
&= \int_{B_\eta(0)} (u_n(t,x) - u_n(t,x-y))v_n(t,x-y)\varphi_\eta(y)\,dy \\
&= \int_{B_\eta(0)} (u_n(t,x) - \tau_{-y}u_n(t,x))v_n(t,x-y)\varphi_\eta(y)\,dy.
\end{aligned}$$

Now for $\varepsilon > 0$ we find $\eta_0 > 0$ such that for all $|y| \le \eta_0$ and all $r \in [1, p^\star)$ we have $\|\tau_{-y}u_n - u_n\|_{\mathrm{L}^q(I;\mathrm{L}^r(\mathbb{R}^d))} \le \varepsilon$ for all $n \in \mathbb{N}$. Since the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^{q'}(I;\mathrm{L}^{r'}(\mathbb{R}^d))$, we find a constant $C_v > 0$ such that for all $n \in \mathbb{N}$ we have $\|v_n\|_{\mathrm{L}^{q'}(I;\mathrm{L}^{r'}(\mathbb{R}^d))} \le C_v$. Thanks to Fubini and the Hölder inequality (1.4) we obtain for all $\eta \le \eta_0$

$$\begin{aligned}
\|S_{n,\eta}\|_{\mathrm{L}^1(I\times\mathbb{R}^d)} = \|S_{n,\eta}\|_{\mathrm{L}^1(I;\mathrm{L}^1(\mathbb{R}^d))} &= \int_I \|u_n(t)(v_n(t)\star\varphi_\eta) - (u_n(t)v_n(t))\star\varphi_\eta\|_{\mathrm{L}^1(\mathbb{R}^d)}\,dt \\
&= \int_I\int_{\mathbb{R}^d} |u_n(t,x)(v_n(t)\star\varphi_\eta)(x) - ((u_n(t)v_n(t))\star\varphi_\eta)(x)|\,dx\,dt \\
&= \int_I\int_{\mathbb{R}^d} |\int_{B_\eta(0)} (u_n(t,x) - \tau_{-y}u_n(t,x))v_n(t,x-y)\varphi_\eta(y)|\,dy\,dx\,dt \\
&\le \int_{B_\eta(0)}\int_I\int_{\mathbb{R}^d} |(u_n(t,x) - \tau_{-y}u_n(t,x))v_n(t,x-y)\varphi_\eta(y)|\,dx\,dt\,dy \\
&\le \int_{|y|\le\eta} \|\tau_{-y}u_n - u_n\|_{\mathrm{L}^q(I;\mathrm{L}^r(\mathbb{R}^d))}\|v_n\|_{\mathrm{L}^{q'}(I;\mathrm{L}^{r'}(\mathbb{R}^d))}|\varphi_\eta(y)|\,dy \\
&\le \varepsilon\|v_n\|_{\mathrm{L}^{q'}(I;\mathrm{L}^{r'}(\mathbb{R}^d))}\int_{|y|\le\eta}|\varphi_\eta(y)|\,dy = \varepsilon\|v_n\|_{\mathrm{L}^{q'}(I;\mathrm{L}^{r'}(\mathbb{R}^d))} \\
&\le C_v\varepsilon,
\end{aligned}$$

which yields the desired uniform convergence. □

For the next proof we need a simple result from the measure theory.

Lemma 2.2.7. *Let $f_n, g_n, f, g : \Omega \to \mathbb{R}$ be measurable functions for all $n \in \mathbb{N}$ on a subset Ω in $\mathbb{R}^d$ satisfying*

- *f_n converges to f almost everywhere in Ω,*
- *the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^\infty(\Omega)$,*
- *g_n converges weakly towards g in $\mathrm{L}^1(\Omega)$,*

then the sequence of the products $(f_n g_n)_{n\in\mathbb{N}}$ converges weakly towards fg in $\mathrm{L}^1(\Omega)$.

Proof. Let $\phi \in \mathrm{L}^\infty(\Omega)$, then

$$\begin{aligned}
|\langle f_n g_n - fg, \phi\rangle| &= |\int_\Omega (f_n g_n - f_n g + f_n g - fg)\phi\,dx| \\
&\le |\int_\Omega f_n(g_n - g)\phi\,dx| + |\int_\Omega g(f_n - f)\phi\,dx| \\
&\le \|f_n\|_{\mathrm{L}^\infty(\Omega)}\int_\Omega |(g_n - g)\phi| + \int_\Omega |g(f_n - f)\phi|\,dx.
\end{aligned}$$

The first term converges to zero, thanks to the boundedness of $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^\infty(\Omega)$ and the weak convergence of $(g_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^1(\Omega)$. Since f_n converges to f almost everywhere, we have the following convergence

$$|u(x)\phi(x)(f_n(x)-f(x))| \xrightarrow{n\to\infty} 0 \qquad \text{a.e. in } \Omega.$$

Additionally we have clearly that the function $2|u|\|\phi\|_\infty\|f_n\|_{\mathrm{L}^\infty(\Omega)}$ lies in $\mathrm{L}^1(\Omega)$ and is an upper bound for the sequence $(|u\phi(f_n-f)|)_{n\in\mathbb{N}}$. Therefore we can apply the dominated convergence theorem and obtain that the sequence f_ng_n converges weakly towards fg. □

With all these tools, we are able to state and prove the main statement of this section.

Proposition 2.2.8. *Let $q\in[1,\infty)$, $p\in[1,d]$, $r\in[1,p^\star)$ and $I\subset\mathbb{R}$ a non-empty interval. Consider $(u_n)_{n\in\mathbb{N}}$ bounded in $\mathrm{L}^q(I;\mathrm{W}^{1,p}(\mathbb{R}^d))$ and $(v_n)_{n\in\mathbb{N}}$ bounded in $\mathrm{L}^{q'}(I;\mathrm{L}^{r'}(\mathbb{R}^d))$ and weakly or weakly-$*$ converging in these spaces to u and v respectively. If the sequence $(\partial_t v_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{M}(I;\mathrm{H}^{-m}(\mathbb{R}^d))$ for some $m\in\mathbb{N}$ then, up to a subsequence, we have the following vague convergence in $\mathcal{M}(I\times\mathbb{R}^d)$ (i.e. with $\mathrm{C}_c(I\times\mathbb{R}^d)$ test functions):*

$$(u_nv_n) \xrightharpoonup{n\to\infty} uv \tag{2.8}$$

Proof. For each $f\in\mathrm{C}_c(I\times\mathbb{R}^d)$ we find a compact set $K_f\subseteq I\times\mathbb{R}^d$ such that $\operatorname{supp} f\subseteq K_f$ and for K_f there exists an open Ball $B_N(0)$ with radius $N\in\mathbb{N}$, such that $K_f\subseteq I\times B_N(0)$. Let us define $O_N\coloneqq I\times B_N(0)$. If we find for each $N\in\mathbb{N}$ a converging subsequence, we can argue through a standard diagonal argument, that there exists a converging subsequence of $(u_nv_n)_{n\in\mathbb{N}}$, which converges vaguely in all $\mathcal{M}(O_N)$ and we obtain the desired vague convergence in $\mathcal{M}(I\times\mathbb{R}^d)$. Therefore it is sufficient to prove for all $N\in\mathbb{N}$ the existence of a subsequence of $(u_nv_n)_{n\in\mathbb{N}}$, which converges in the vague topology of $\mathcal{M}(O_N)$ towards uv. In particular we want to find for each $N\in\mathbb{N}$ a subsequence $(u_{n_i}v_{n_i})_{i\in\mathbb{N}}$ such that

$$\lim_{i\to\infty}\langle u_{n_i}v_{n_i},f\rangle=\lim_{i\to\infty}\int_{O_N}u_{n_i}(t,x)v_{n_i}(t,x)f(t,x)\,d(t,x)=0,\qquad \forall f\in\mathrm{C}_c(O_N). \tag{2.9}$$

To avoid a notational confusion, we will not write subindices. Now fix $N\in\mathbb{N}$ and define the sequence of mollifiers $(\varphi_\eta)_{\eta\in(0,1)}$ as in section 1.2.3, then we show (2.9) in the following 5 steps.

Step 1. In this step we want to prove that

$$u(v\star\varphi_\eta)\xrightarrow[\eta\to 0]{} uv \qquad \text{in } \mathrm{L}^1(O_N). \tag{2.10}$$

Since $\mathrm{W}^{1,p}(B_N(0))$ embeds continuously into $\mathrm{L}^r(B_N(0))$ we have that $u\in\mathrm{L}^1(I;\mathrm{L}^r(B_N(0)))$. Therefore we obtain with the Hölder inequality

$$\begin{aligned}\|u(v\star\varphi_\eta)-uv\|_{\mathrm{L}^1(O_N)} &= \int_I\|u(t)((v\star\varphi_\eta)(t)-v(t))\|_{\mathrm{L}^1(B_N(0))}\,dt\le\\ &\le\int_I\|u(t)\|_{\mathrm{L}^r(B_N(0))}\|(v\star\varphi_\eta)(t)-v(t)\|_{\mathrm{L}^{r'}(B_N(0))}\,dt\\ &\le\|u\|_{\mathrm{L}^q(I;\mathrm{L}^r(B_N(0)))}\|(v\star\varphi_\eta)-v\|_{\mathrm{L}^{q'}(I;\mathrm{L}^{r'}(B_N(0)))}.\end{aligned}$$

In the last step we applied Hölder on the functions $\|u(t)\|_{\mathrm{L}^r(B_N(0))}\in\mathrm{L}^q(I)$ and $\|(v\star\varphi_\eta)(t)-v(t)\|_{\mathrm{L}^{r'}(B_N(0))}\in\mathrm{L}^{q'}(I)$. Thanks to this estimate we can finish this step if

$$\|(v\star\varphi_\eta)-v\|_{\mathrm{L}^{q'}(I;\mathrm{L}^{r'}(B_N(0)))}\xrightarrow{\eta\to 0}0. \tag{2.11}$$

Interpret the expression $\|v \star \varphi_\eta(t) - v(t)\|_{\mathrm{L}^{r'}(B_N(0))}$ as function from I to $\mathbb{R}$, then, with applying Lemma 1.2.18, we obtain the pointwise convergence

$$\|v \star \varphi_\eta(t) - v(t)\|_{\mathrm{L}^{r'}(B_N(0))} \xrightarrow{\eta \to 0} 0 \qquad \forall t \in I.$$

Since $\|v\star\varphi_\eta(t)-v(t)\|_{\mathrm{L}^{r'}(B_N(0))}$ is bounded by the function $2\|v(t)\|_{\mathrm{L}^{r'}(B_N(0))}$, the conditions of the dominated convergence theorem are satisfied and we get (2.11).

Step 2. For this step we desire a weak converging subsequence of $(u_n(v_n \star \varphi_\eta))_{n\in\mathbb{N}}$, in particular we want for all $\eta \in (0,1)$

$$(u_n(v_n \star \varphi_\eta)) \xrightarrow{n\to\infty} u(v \star \varphi_\eta) \qquad \text{weakly in } \mathrm{L}^1(O_N). \tag{2.12}$$

For this we intend to apply Lemma 2.2.7 on the sequences $(v_n \star \varphi_\eta)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$.
Let $\eta \in (0,1)$ be arbitrary but fix. Thanks to Proposition 1.3.2 we have that
$\mathrm{L}^q(I; \mathrm{W}^{1,p}(B_N(0)))$ embeds continuously and dense into $\mathrm{L}^1(O_N)$ and therefore, thanks to Lemma 1.3.5, we have that

$$u_n \text{ converges weakly towards } u \text{ in } \mathrm{L}^1(O_N). \tag{2.13}$$

Now we take a closer look on the sequence $(v_n \star \varphi_\eta)_{n\in\mathbb{N}}$. First we want to show the existence of a subsequence such that

$$v_n \star \varphi_\eta \xrightarrow{n\to\infty} v \star \varphi_\eta \qquad \text{a.e. in } O_N. \tag{2.14}$$

Our strategy is to prove that $(v_n\star\varphi_\eta)_{n\in\mathbb{N}}$ is bounded in $BV(O_N)$ and deduce with Theorem 1.1.11 the existence of a converging subsequence in $\mathrm{L}^1(O_N)$, which in turn provides a converging subsequence a.e. in O_N.
Since $\mathrm{L}^{r'}(B_N)$ embeds continuously into $\mathrm{L}^1(B_N)$, there exists thanks to Proposition 1.3.2 an upper bound $C_v > 0$ of the sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^1(O_N)$. Since $\varphi_\eta \in \mathrm{C}_c^\infty(\mathbb{R}^d)$ we obtain for all $n \in \mathbb{N}$ the existence of the weak derivatives

$$\partial_{x_i}(v_n \star \varphi_\eta) = v_n \star \partial_{x_i}\varphi_\eta \qquad \forall i \in \{1, \ldots, d\}.$$

We have for all $i \in \{1, \ldots, d\}$ that the sequence $(\partial_{x_i}(v_n \star \varphi_\eta))_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^1(O_N)$, because of Young's inequality for convolutions we have

$$\begin{aligned}
\|\partial_{x_i}(v_n \star \varphi_\eta)\|_{\mathrm{L}^1(O_N)} &\le \int_I \|v_n(t) \star \partial_{x_i}\varphi_\eta\|_{\mathrm{L}^1(B_N(0))}\, dt \\
&\le \int_I \|v_n(t)\|_{\mathrm{L}^1(B_N(0))}\|\partial_{x_i}\varphi_\eta\|_{\mathrm{L}^1(B_N(0))}\, dt \\
&= \|\partial_{x_i}\varphi_\eta\|_{\mathrm{L}^1(B_N(0))}\|v_n\|_{\mathrm{L}^1(O_N)} \le \|\partial_{x_i}\varphi_\eta\|_{\mathrm{L}^1(B_N(0))} C_v.
\end{aligned}$$

Define the constant $C_{\eta_i} := \|\partial_{x_i}\varphi_\eta\|_{\mathrm{L}^1(B_N(0))}$. Since $(\partial_{x_i}(v_n\star\varphi_\eta))_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^1(O_N)$ and since $(\partial_{x_i}(v_n \star \varphi_\eta))_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}(O_N)$, the total variation of each element coincide with the L^1-norm. Therefore we have for each $i \in \{1, \ldots, d\}$ that the sequence of total variations $(|\partial_{x_i}(v_n \star \varphi_\eta)|(O_N))_{n\in\mathbb{N}}$ is bounded by a constant depending only on the mollifier φ_η .

Now let C_{∂_t} denote the upper bound of the total variations $(|\partial_t(v_n \star \varphi_\eta)|(O_N))_{n\in\mathbb{N}}$. For the sequence of the time derivative $(\partial_t(v_n \star \varphi_\eta))_{n\in\mathbb{N}}$ we get for any $\phi \in \mathrm{C}_c^\infty(O_N)$

$$\begin{aligned}|\langle \partial_t(v_n \star \varphi_\eta), \phi\rangle| &= |\langle (v_n \star \varphi_\eta), \partial_t\phi\rangle| = |\langle v_n, \partial_t\phi \star \varphi_\eta\rangle| = |\langle v_n, \partial_t(\phi \star \varphi_\eta)\rangle| \\ &= |\langle \partial_t v_n, \phi \star \varphi_\eta\rangle| \le \|\partial_t v_n\| \|\phi \star \varphi_\eta\|_{\mathrm{C}_c(I;\mathrm{H}^m(B_N(0)))} \\ &\le C_{\partial_t} C_{\eta_t} \|\phi\|_\infty,\end{aligned}$$

where C_{η_t} is a constant depending on the mollifier φ_η. Thanks to the dense and continuous embedding $\mathrm{C}_c^\infty(O_N)$ into $\mathrm{C}_c(O_N)$, we obtain for all $n \in \mathbb{N}$

$$\sup_{\phi\in \mathrm{C}_c(O_N), \|\phi\|\le 1} |\langle \partial_t(v_n \star \varphi_\eta), \phi\rangle| \le C_{\partial_t} C_{\eta_t}.$$

Therefore we get that the sequence of total variations $(|\partial_t(v_n \star \varphi_\eta)|(O_N))_{n\in\mathbb{N}}$ is bounded by a constant depending only on the mollifier φ_η. Eventually we obtain that

$$\sup_{n\in\mathbb{N}} \|v_n \star \varphi_\eta\|_{BV(O_N)} \le C_\eta$$

and deduce with Theorem 1.1.11 that (2.14) is true. Since the sequence $(v_n \star \varphi_\eta)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^\infty(O_N)$, what is a direct consequence the bounds of $(v_n)_{n\in\mathbb{N}}$ and $(\partial_t v_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^{q'}(I;\mathrm{L}^{r'}(B_N(0)))$ and $\mathcal{M}(I;\mathrm{H}^{-m}(B_N(0)))$ respectively and the fact that $\varphi_\eta \in \mathrm{C}_c(\mathbb{R}^d)$, the conditions of Lemma 2.2.7 are satisfied, hence the desired weak convergence (2.12) holds.

Step 3. From Lemma 2.2.6 we infer

$$\sup_{n\in\mathbb{N}} \|u_n(v_n \star \varphi_\eta) - (u_n v_n) \star \varphi_\eta\|_{\mathrm{L}^1(O_N)} \xrightarrow{\eta\to 0} 0$$

Step 4. The aim of this step is to prove that for all $f \in \mathrm{C}_c(O_N)$

$$\langle u_n v_n \star \varphi_\eta - u_n v_n, f\rangle_{\mathrm{C}_c(O_N)} \xrightarrow{\eta\to 0} 0 \tag{2.15}$$

uniformly in n. Since $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^q(I;\mathrm{W}^{1,p}(\mathbb{R}^d))$ and $\mathrm{L}^q(I;\mathrm{W}^{1,p}(\mathbb{R}^d)) \hookrightarrow \mathrm{L}^1(I;\mathrm{L}^r(\mathbb{R}^d))$, we get

$$\int_I \|u_n(t)v_n(t)\|_{\mathrm{L}^1(\mathbb{R}^d)}\, dt \le \int_I \|u_n(t)\|_{\mathrm{L}^r(\mathbb{R}^d)} \|v_n(t)\|_{\mathrm{L}^{r'}(\mathbb{R}^d)}\, dt.$$

With the boundedness of $(v_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^1(I;\mathrm{L}^r(\mathbb{R}^d))$, the inequality above and Proposition 1.3.2 we obtain that $(u_n v_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^1(I \times \mathbb{R}^d)$. Since φ_η is a mollifier, we also obtain that $(u_n v_n \star \varphi_\eta)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^1(B_N(0))$. Thanks to Remark 1.1.7 and the fact that φ_η is even, we get for arbitrary $f \in \mathrm{C}_c(O_N)$

$$\begin{aligned}\langle u_n v_n \star \varphi_\eta - u_n v_n, f\rangle_{\mathrm{C}_c(O_N)} &= \int_{O_N} (u_n(t,x)v_n(t,x) \star \varphi_\eta(x)) f(t,x)\, d(t,x) - \\ &- \int_{O_N} u_n(t,x)v_n(t,x) f(t,x)\, d(t,x) = \\ &= \langle u_n v_n, f \star \varphi_\eta - f\rangle_{\mathrm{C}_c(O_N)} \le \\ &\le \|u_n v_n\|_{\mathrm{L}^1(I\times\mathbb{R}^d)} \|f \star \varphi_\eta - f\|_{\mathrm{L}^\infty(I\times\mathbb{R}^d)}\end{aligned}$$

Since $f \star \varphi_\eta - f$ converges, due to Lemma 1.2.18, uniformly to zero w.r.t. $\|.\|_{\mathrm{C}_c(I\times\mathbb{R}^d)}$, it also converges uniformly to zero in $\mathrm{L}^\infty(I \times \mathbb{R}^d)$. This shows the uniform convergence (2.15).

Step 5. We are now able to show the desired convergence. For that reason let $f\mathrm{C}_c(O_N)$ be arbitrary, then

$$\begin{aligned}\langle uv - u_n v_n, f\rangle_{\mathrm{C}_c(O_N)} &= \langle uv - u(v \star \varphi_\eta), f\rangle_{\mathrm{C}_c(O_N)} && (2.16)\\ &+ \langle u(v \star \varphi_\eta) - u_n(v_n \star \varphi_\eta), f\rangle_{\mathrm{C}_c(O_N)} && (2.17)\\ &+ \langle u_n(v_n \star \varphi_\eta) - (u_n v_n) \star \varphi_\eta, f\rangle_{\mathrm{C}_c(O_N)} && (2.18)\\ &+ \langle (u_n v_n) \star \varphi_\eta - u_n v_n, f\rangle_{\mathrm{C}_c(O_N)} && (2.19)\end{aligned}$$

Let $\varepsilon > 0$, then we can choose $\eta > 0$, such that the summands (2.16),(2.18) and (2.19) are smaller than $\frac{\varepsilon}{4}$ for all $n \in \mathbb{N}$, thanks to the steps 1, 3 and 4. Due to step 2 we find $n_\eta \in \mathbb{N}$, such that the summand (2.17) is smaller than $\frac{\varepsilon}{4C_\eta}$.

□

To prove our main theorem we give a simple variation of Proposition 2.2.8 in the case of an open and bounded domain $\Omega \subseteq \mathbb{R}^d$. We give a definition of bump functions, which are needed for the proof.

Definition 2.2.9. Let $O \subseteq \mathbb{R}^d$ be an open set. Then for every compact subset $K \subset O$ there exists a smooth function $\rho : \mathbb{R}^d \to \mathbb{R}$, with $\rho(x) = 1$ for all $x \in K$, $0 < \rho(x) < 1$ for $x \in O \setminus K$ and $\rho(x) = 0$ for $x \in \mathbb{R}^d \setminus O$. We call such functions *bump functions*.

Proposition 2.2.10. *Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set with Lipschitz boundary, $q \in [1,\infty)$, $p \in [1,d]$, $r \in [1,p^\star]$ and $I \subset \mathbb{R}$ a closed and bounded interval. Consider two sequences $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ bounded in $\mathrm{L}^q(I;\mathrm{W}^p(\Omega))$ and $\mathrm{L}^{q'}(I;\mathrm{L}^{r'}(\Omega))$, respectively weakly and weak−* converging in this spaces to u and v. If $(\partial_t v_n)_{n\in\mathbb{N}}$ is bounded in $\mathcal{M}(I;\mathrm{H}^{-m}(\Omega))$ for some $m \in \mathbb{N}$, then up to a subsequence, we have the following weak−* convergence in $\mathcal{M}(I \times \Omega)$ (i.e. with $\mathrm{C}\ (I \times \Omega)$)*

Proof. Notice that $(u_n v_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^1(I \times \Omega)$. For the proof, read *step 4* in the proof of Proposition 2.2.8 and replace $\mathbb{R}^d$ with Ω.
Since λ_d is a Radon measure we can apply Theorem 1.1.9 to find a sequence of compact sets $(K_k)_{k\in\mathbb{N}}$, such that $\lambda_d(\Omega \setminus K_k) < \frac{1}{k}$ for all $k \in \mathbb{N}$. For each K_k we define the bump function $\rho_k : \mathbb{R}^d \to \mathbb{R}$. It is easy to prove the following convergences:

$$\begin{aligned}\rho_k uv &\xrightarrow{k\to\infty} uv && \text{in}\, \mathrm{L}^1(I \times \Omega) && (2.20)\\ \rho_k u_n &\xrightharpoonup{n\to\infty} \rho_k u && \text{weakly in}\, \mathrm{L}^q(I;\mathrm{W}^{1,p}(\Omega)) && (2.21)\\ \rho_k v_n &\xrightharpoonup{n\to\infty} \rho_k v && \text{weakly-* in}\, \mathrm{L}^q(I;\mathrm{W}^{1,p}(\Omega)) && (2.22)\end{aligned}$$

The two sequences $(\rho_k u_n)_{n\in\mathbb{N}}$ and $(\rho_k v_n)_{n\in\mathbb{N}}$ satisfy all the assumptions of Proposition 2.2.8, therefore we get that $(\rho_k^2 u_n v_n)_{n\in\mathbb{N}}$ converges weakly to $\rho_k^2 uv$ in $\mathcal{M}(I \times \mathbb{R}^d)$ or $\mathcal{M}(I \times \Omega)$ respectively. For every $f \in \mathrm{C}^0(I \times \overline{\Omega})$, all $k \in \mathbb{N}$ and all $n \in \mathbb{N}$ the following holds:

$$\begin{aligned}|\langle u_n v_n, f\rangle_{\mathrm{C}^0(I\times\overline{\Omega})} - \langle uv, f\rangle_{\mathrm{C}^0(I\times\overline{\Omega})}| &= |\langle u_n v_n, (1-\rho_k^2)f\rangle + \langle \rho_k^2 u_n v_n, f\rangle - \langle uv, f\rangle| \leq\\ &\leq |\langle u_n v_n, (1-\rho_k^2)f\rangle| + |\langle \rho_k^2 u_n v_n, f\rangle - \langle \rho_k uv, f\rangle| + |\langle \rho_k uv, f\rangle - \langle uv, f\rangle|. \qquad (2.23)\end{aligned}$$

Lets take a closer look on the first summand of (2.23). Since f is continuous and has compact support, it is bounded by a constant $C_f \in \mathbb{R}$. We have also $(1-\rho_k^2) = 0$ on K_k and therefore we have for all $n \in \mathbb{N}$

$$|\langle u_n v_n, (1-\rho_k^2) f\rangle| \leq C_f \int_{I\times(\overline{\Omega}\setminus K_k)} |u_n v_n| \, d(t,x) \leq C_f \cdot C\lambda_d(I \times (\overline{\Omega} \setminus K_k)).$$

This term converges to zero for k to infinity, uniformly in n. For every $\varepsilon > 0$ there exists a $k_0 \in \mathbb{N}$, such that for all $k \geq k_0$ and all $n \in \mathbb{N}$ we have $\langle u_n v_n, (1-\rho_k^2) f\rangle \leq \frac{\varepsilon}{2}$.
The other summands of (2.23) are treated as follows. Together with (2.20) we find for every $\varepsilon > 0$ a $k_1 \in \mathbb{N}$, such that $|\langle \rho_k^2 uv, f\rangle - \langle uv, f\rangle| \leq \frac{\varepsilon}{4}$ for all $k \geq k_1$. Define $k_{f,\varepsilon} := \max(k_0, k_1)$, then we are able to choose $n_{f,\varepsilon}$, such that for all $n \geq n_{f,\varepsilon}$ we obtain $|\langle \rho_{k_{f,\varepsilon}}^2 u_n v_n, f\rangle - \langle \rho_{k_{f,\varepsilon}}^2 uv, f\rangle| \leq \frac{\varepsilon}{4}$.
This shows that we find for every $f \in \mathrm{C}^0(I \times \overline{\Omega})$ and every $\varepsilon > 0$ a $n_{f,\varepsilon}$, such that for all $n > n_{f,\varepsilon}$ we get $|\langle u_n v_n, f\rangle_{\mathrm{C}^0(I\times\overline{\Omega})} - \langle uv, f\rangle_{\mathrm{C}^0(I\times\overline{\Omega})}| < \varepsilon$, which shows the weak convergence of the sequence $(u_n v_n)_{n\in\mathbb{N}}$. □

2.3 Proof of Theorem 2.1.1

At this point we want to remind the reader of *Rademacher's Theorem*, which says that every Lipschitz function $f : U \to \mathbb{R}^m$, where U is a subset of $\mathbb{R}^n$ and n, m are natural numbers, is a.e. differentiable and its derivative is bounded by its Lipschitz constant. Further we recall a variation of the chain rule, where f is a Lipschitz continuous function and u has a weak derivative. For more detailed information we suggest [Zie89].

Theorem 2.3.1. (Chain Rule for Sobolev functions) *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and $u \in \mathrm{W}^{1,p}(\Omega)$, where Ω is some open set in $\mathbb{R}^d$ and $p \in [1, \infty]$. If $f \circ u \in \mathrm{L}^p(\Omega)$, then $f \circ u \in \mathrm{W}^{1,p}(\Omega)$ and for almost all $x \in \Omega$ it holds*

$$D(f \circ u)(x) = f'(u(x))Du(x).$$

Proof. See [Zie89], proof of Theorem 2.1.11 on page 48. □

Definition 2.3.2. Let X be a normed space $(X, \|.\|)$. A function $f : X \to \mathbb{R} \cup \{-\infty, \infty\}$ is called *lower (upper) semi-continuous* at $x_0 \in X$ if for everey sequence $(x_n)_{n\in\mathbb{N}}$ converging to x_0, i.e. $x_n \to x$ holds

$$f(x_0) \leq \liminf_{n\to\infty}(f(x_n)) \qquad (\limsup_{n\to\infty} f(x_n) \leq f(x_n)).$$

We call f *weakly lower (upper) semicontinous* if we replace in the definition above convergence in the norm by weak convergence, i.e. $x_n \rightharpoonup x_0$.

To prove Theorem 2.1.1 we define the critical points of Φ. A point $p_0 \in \mathbb{R}$ is called a *critical point of* Φ , if $\Phi'(p_0) = 0$ and a *regular point of* Φ otherwise. We call a function $f : U \to \mathbb{R}$ *regular* where $U \subseteq \mathbb{R}^d$, if $\Phi'(f(x)) \neq 0$ for all $x \in \mathbb{R}^d$.
The main problem of Theorem 2.1.1 is that we know almost nothing about the space derivatives of $(u_n)_{n\in\mathbb{N}}$, just that for some function $\Phi \in \mathrm{W}^{1,1}(\mathbb{R}, \mathbb{R})$ the sequence of gradients $(\nabla_{\mathbf{x}}\Phi(u_n))_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(I \times \Omega)$. If every $u_n(t, x)$ would be regular for all $t \in I$, we could write the space derivative of each u_n as follows:

$$\nabla_{\mathbf{x}} u_n = \frac{1}{\Phi'(u_n)} \nabla_{\mathbf{x}}\Phi(u_n).$$

In general the functions u_n are not regular, so that we cannot use this equality. Our aim for the proof is to find a function $\zeta(u_n) : \mathbb{R} \to \mathbb{R}$, such that ζ erases all critical values of u_n for all $n \in \mathbb{N}$ and that the weak limit of $u_n\zeta(u_n)$ is not far from the quadratic weak limit of u_n (so ζ has to approach the identity map in a suitable way).
Proof (of Theorem 2.1.1). We follow [Mou16]. Let us first notice that, without loss of generality, we can assume that Φ is an increasing function. Indeed, if one defines

$$\tilde{\Phi}(z) := \int_0^z |\Phi'(r)|\, dr,$$

then $\tilde{\Phi}$ is obviously increasing, satisfies $\tilde{\Phi}' = |\Phi'|$ whence $|\nabla_{\mathbf{x}}\tilde{\Phi}(u_n)| = |\nabla_{\mathbf{x}}\Phi(u_n)|$ pointwise. In particular, the assumptions $\Phi \in \mathrm{W}^{1,1}(\mathbb{R}, \mathbb{R})$, $(\mathbb{1}_{|\Phi'|<\delta})_{\delta>0} \to 0$ as $\delta \to 0$ and $(\nabla_{\mathbf{x}}(u_n))_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(I \times \Omega)$ are all satisfied by replacing Φ by $\tilde{\Phi}$.
Recalling the above discussion, we would like to find a function ζ_ε, which somehow approaches the identity function $\mathrm{Id} : \mathbb{R} \to \mathbb{R}$, but also erases the critical values of u_n. The following function does the job

$$\zeta_\varepsilon(z) := \int_0^z \min\{1, \frac{\Phi'(r)}{\varepsilon}\}\, dr.$$

Indeed, we first have

$$\zeta_\varepsilon(z) - z \leq \int_0^z \frac{\Phi'(r)}{\varepsilon} \mathbb{1}_{\Phi'(r)<\varepsilon}\, dr,$$

whence, by assumption,

$$\|\zeta_\varepsilon - \mathrm{Id}\,\|_\infty \leq \|\mathbb{1}_{\Phi'<\varepsilon}\|_{\mathrm{L}^1(\mathbb{R})} \xrightarrow{\varepsilon\to 0} 0 \tag{2.24}$$

We obtain, that the function ζ_ε approaches the identity function in a satisfying way. It is left to show that it also erases the critical values of u_n. Since Φ is increasing, one can introduce Φ^{-1}, for example $\Phi^{-1}(p) := \inf\{z \in \mathbb{R} : \Phi(z) \geq p\}$. Consider the function $\Psi_\varepsilon := \zeta_\varepsilon \circ \Phi^{-1}$. We are going to show that Ψ_ε is Lipschitz continuous. For $p_1, p_2 \in \Phi(\mathbb{R})$, there exists $z_1, z_2 \in \mathbb{R}$ such that $\Phi(z_i) = p_i$ and if $p_1 > p_2$, then $z_1 > z_2$. Without loss of generality we can assume that $p_1 > p_2$.

$$\begin{aligned}|\Psi_\varepsilon(p_1) - \Psi_\varepsilon(p_2)| = |\Psi_\varepsilon(\Phi(z_1)) - \Psi_\varepsilon(\Phi(z_2))| = |\zeta_\varepsilon(z_1) - \zeta_\varepsilon(z_2)| \leq \\ \leq |\int_{z_2}^{z_1} \frac{1}{\varepsilon}\Phi'(r)\, dr| = \frac{1}{\varepsilon}|\Phi(z_1) - \Phi(z_2)| = \frac{1}{\varepsilon}|p_1 - p_2|.\end{aligned}$$

At this point we can apply the chain rule (see Theorem 2.3.1) to the functions Ψ_ε and Φ and gain the following equality for all $n \in \mathbb{N}$

$$\nabla_{\mathbf{x}}\zeta_\varepsilon = \Psi'_\varepsilon(\Phi(u_n))\nabla_{\mathbf{x}}\Phi(u_n).$$

We know that $\|\Psi'\|_\infty < \frac{1}{\varepsilon}$ a.e., and that $(\nabla_{\mathbf{x}}\Phi(u_n))_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(I \times \Omega)$, from which we infer that $(\nabla_{\mathbf{x}}\zeta_\varepsilon(u_n))_{n\in\mathbb{N}}$ is also bounded in $\mathrm{L}^2(I \times \Omega)$. Since $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(I \times \Omega)$ by a constant $C \in \mathbb{R}$ we obtain that $(\zeta_\varepsilon(u_n))_{n\in\mathbb{N}}$ is also bounded in $\mathrm{L}^2(I \times \Omega)$, because with (2.24) we have

$$\|\zeta_\varepsilon(u_n)\|_{\mathrm{L}^2} = \|\zeta_\varepsilon(u_n) - u_n + u_n\|_{\mathrm{L}^2} \leq \|\zeta_\varepsilon(u_n) - u_n\|_{\mathrm{L}^2} + \|u_n\|_{\mathrm{L}^2} \leq \|\mathbb{1}_{\Phi'<\varepsilon}\|_{\mathrm{L}^1}\lambda(I \times \Omega)^{\frac{1}{2}} + C.$$

With the reflexivity of L^2, we obtain weak convergent subsequences of $(u_n)_{n\in\mathbb{N}}$ and $(\zeta_\varepsilon(u_n))_{n\in\mathbb{N}}$ respectively. Let us denote the corresponding limits by u and u_ε. Using this weak convergence and the fact that $(\zeta_\varepsilon(u_n))_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(I;\mathrm{H}^1(\Omega))$ and $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(I;\mathrm{L}^2(\Omega))$ with $(\partial_t(u_n))_{n\in\mathbb{N}}$ bounded in $\mathcal{M}(I;\mathrm{H}^{-m}(\Omega))$, one can apply Proposition 2.2.10 and obtains with $\mathbb{1}\in\mathrm{C}^0(I\times\overline{\Omega})$ as a test function

$$\int_{I\times\Omega}\zeta_\varepsilon(u_n)u_n \xrightarrow{n\to\infty} \int_{I\times\Omega} u_\varepsilon u.$$

Again with (2.24) and the lower semicontinuity of the norm we get

$$\|u-u_\varepsilon\|_{\mathrm{L}^2}\le\liminf_{n\to\infty}\|u_n-\zeta_\varepsilon(u_n)\|_{\mathrm{L}^2}\le\|\mathbb{1}_{\Phi'<\varepsilon}\|_{\mathrm{L}^1}\lambda(I\times\Omega)^{\frac{1}{2}}$$

Let us again denote with $C\in\mathbb{R}$ the upper barrier of $(u_n)_{n\in\mathbb{N}}$ regarding to $\|.\|_{\mathrm{L}^2}$, then we get with help of the Cauchy-Schwarz inequality

$$\begin{aligned}\limsup_{n\to\infty}\|u_n\|_{\mathrm{L}^2}&\le\limsup_{n\to\infty}\int_{I\times\Omega}\zeta_\varepsilon(u_n)u_n+\limsup_{n\to\infty}\int_{I\times\Omega}(u_n-\zeta_\varepsilon(u_n))u_n\le\\&\le\int_{I\times\Omega}u_\varepsilon u+C\|\mathbb{1}_{\Phi'\varepsilon}\|_{\mathrm{L}^1}=\int_{I\times\Omega}(u^2-(u_\varepsilon-u)u)+C\lambda(I\times\Omega)^{\frac{1}{2}}\|\mathbb{1}_{\Phi'<\varepsilon}\|_{\mathrm{L}^1}\le\\&\le\|u\|_{\mathrm{L}^2}^2+\langle u_\varepsilon-u,u\rangle_{\mathrm{L}^2}+\tilde{C}\|\mathbb{1}_{\Phi'\varepsilon}\|_{\mathrm{L}^1}\le\|u\|_{\mathrm{L}^2}^2+2\tilde{C}\|\mathbb{1}_{\Phi'<\varepsilon}\|_{\mathrm{L}^1}\le\\&\le\liminf_{n\to\infty}\|u_n\|_{\mathrm{L}^2}+2\tilde{C}\|\mathbb{1}_{\Phi'<\varepsilon}\|_{\mathrm{L}^1}.\end{aligned}$$

For $\varepsilon\to0$ we get with the condition $\|\mathbb{1}_{\Phi'<\varepsilon}\|_{\mathrm{L}^1}\to0$ as ε to zero

$$\limsup_{n\to\infty}\|u_n\|_{\mathrm{L}^2}\le\|u\|_{\mathrm{L}^2}\le\liminf_{n\to\infty}\|u_n\|_{\mathrm{L}^2},$$

which shows, that there exists a converging subsequence of $(u_n)_{n\in\mathbb{N}}$. $\square$

3 Second Version

The purpose of this chapter is to adapt the theory we know to a different domain than the typical cylindrical domain we are used to and state two statements. The first statement is concerned with a compactness criteria in the sense of the Aubin-Lions Lemma. The motivation for the second statement is to get a compactness criteria for divergence free vector fields. The latter appear mainly in the theory of incompressible Navier Stokes equations in a time-depended domain, which is concerned about finding a solution for the nonlinear problem

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_x \mathbf{u} - \Delta_x \mathbf{u} + \nabla_x p &= 0, \\ \operatorname{div}_x \mathbf{u} &= 0, \end{aligned}$$

where p is some expression for the pressure. To deal with the pressure term, from which is usually very less known, one has to test in the weak formulation against divergence free test functions. The problem is that the set of these test functions is in general not dense in the Lebesgue spaces. Therefore we need the definition of the space $\mathrm{L}^2_{\mathrm{div}}(\Omega)$. We do not go deeper into this theory, because this would to go beyond the scope of this work. For the interested reader we suggest [BGM17].
Before we can state these two theorems we start with some preliminaries and then go over to the definition of our setting.

3.1 Preliminaries

In this section we want to recall some theorems from the classic analysis we are going to use in this chapter. For this section let F be a function from O to U, with $O, U \subseteq \mathbb{R}^d$ open. We call F a *diffeomorphism* if F is bijective such that F and F^{-1} are continuous differentiable. The *Jacobian* of a function $f \in \mathrm{C}^1(\mathbb{R}^m, \mathbb{R}^n)$ will be denoted as Df which lives in $\mathrm{C}^0(\mathbb{R}^m; \mathfrak{L}(\mathbb{R}^m; \mathbb{R}^n))$, where $\mathfrak{L}(\mathbb{R}^m; \mathbb{R}^n)$ is the space of all linear and bounded functions from $\mathbb{R}^m$ to $\mathbb{R}^n$. We start with the integral formula of the mean value.

Theorem 3.1.1. (Integral formula of the mean value) *Let $U \subset \mathbb{R}^d$ be open, $f : U \to \mathbb{R}^d$ a differentiable function and $\gamma : [\alpha, \beta] \to U$ a differentiable curve with $\gamma(\alpha) = x_1$ and $\gamma(\beta) = x_2$. Then the following holds:*

$$f(x_2) - f(x_1) = \int_\alpha^\beta Df(\gamma(t)) \circ \gamma'(t) \, dt.$$

The next two theorems will help us to handle the upcoming diffeomorphism.

Theorem 3.1.2. (Inverse Function Theorem) *Let $f : C \to \mathbb{R}^d$ be continuously differentiable on the open set $C \subseteq \mathbb{R}^d$. Furthermore let $c \in C$, such that*

$$\det Df(c) \neq 0$$

and let $E \subseteq \mathbb{R}^d$ be open with $f(c) \in E$. Then there exists open sets $O \subseteq C$ and $U \subseteq E$ with the following properties:

- *$c \in O$ and $f(c) \in U$.*
- *The restriction $f|_O$ is bijective from O to U.*
- *The inverse mapping $g : U \to O$ of $f|_O : O \to U$ is continuously differentiable with $Dg(f(t)) = Df(t)^{-1}$ for all $t \in O$.*

Theorem 3.1.3. (Transformation-Rule) *Let $F : O \to U$ be a diffeomorphism between the open sets $O, U \subseteq \mathbb{R}^d$. If $B \subseteq O$, B Borel, then we have*

$$\lambda_d(F(B)) = \int_B |\det DF(x)| \, d\lambda_d(x).$$

A function $f : U \to \mathbb{R}$ is integrable, if and only if the function $f \circ F \cdot |\det DF|$ is also integrable. In that case we have

$$\int_U f(x)\, dx = \int_O f(F(y)) \cdot |\det DF(y)| \, dy.$$

At the end we give the definition of ε- interior sets, because we need these sets through the whole chapter

Definition 3.1.4. If A is a connected open set in $\mathbb{R}^d$ and $\varepsilon > 0$, we define the $\varepsilon-$*interior* of A as $A_\varepsilon := \{x \in A : d(x, A^c) > \varepsilon)\}$, while $A_{-\varepsilon}$ denotes the $\varepsilon-$*exterior* of A, that is $A_{-\varepsilon} := A + B_\varepsilon(0)$.

3.2 The setting

At the beginning we recall the definition of a domain. We call an open and connected subset $\Omega \subseteq \mathbb{R}^d$ *domain* . Let $I := [a, b]$ be a closed interval with $a, b \in \mathbb{R}$. For this chapter we consider the open subset of $\mathbb{R} \times \mathbb{R}^d$

$$\hat{\Omega} := \bigcup_{a<t<b} \{t\} \times \Omega^t,$$

which is restricted to some conditions:

[C1] For all $t \in [a, b]$, Ω^t is a bounded domain with Lipschitz boundary.

[C2] There exists a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary such that for all $t \in [a, b]$ there exists a C^1-diffeomorphism $\vartheta_t : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\vartheta_t(\Omega) = \Omega^t.$$

[C3] The function $\Theta : (t, \mathbf{x}) \to \vartheta_t(\mathbf{x})$ lies in $\mathrm{C}^0([a, b]; \mathrm{C}^1(\mathbb{R}^d, \mathbb{R}^d))$

Facts 3.2.1.

1. With the continuity of $\det : \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^d) \to \mathbb{R}$ and the continuity of $D\vartheta_t$, we get $|\det D\vartheta_t| \in \mathrm{C}^0(\mathbb{R}^d, \mathbb{R})$ such that the mapping $\tilde{\Theta} : (t, \mathbf{y}) \to |\det D\vartheta_t(\mathbf{y})|$ lies in $\mathrm{C}^0(I \times \mathbb{R}^d, \mathbb{R})$. Since Ω is bounded we know that $I \times \overline{\Omega}$ is compact in $\mathbb{R}^{d+1}$ and we find a constant $\beta > 0$ such that $\sup_{(t,\mathbf{y}) \in I \times \overline{\Omega}} |\det D\vartheta_t(\mathbf{y})| \leq \beta$.
   If we do the same for $\overline{\Theta} : (t, \mathbf{y}) \to |\det D\vartheta_t^{-1}(\mathbf{y})|$ we find another constant $\tilde{\alpha} > 0$, such that $\sup_{(t,\mathbf{y}) \in I \times \overline{\Omega}} |\det D\vartheta_t^{-1}(\mathbf{y})| \leq \tilde{\alpha}$. Thanks to the *Inverse Function Theorem* (Theorem 3.1.2.) we have for $\alpha := \frac{1}{\tilde{\alpha}}$ and for all $(t, \mathbf{y}) \in I \times \overline{\Omega}$
$$0 < \alpha \leq |\det D\vartheta_t(\mathbf{y})| \leq \beta. \tag{3.1}$$

2. The previous estimate will allow us to use the change of variable $\mathbf{x} = \vartheta_t(\mathbf{y})$ to transport estimates from Ω to Ω^t. If we set $S_{p,\Omega}$ as the Sobolev constant of the embedding $\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}^{p^\star}(\Omega)$, we get for instance for $p < d$ and for all $v \in \mathrm{W}^{1,p}(\Omega^t)$ with the *Transformation-Rule* (Theorem 3.1.3.):
$$\begin{aligned}\|v\|^{p^\star}_{\mathrm{L}^{p^\star}(\Omega^t)} &= \int_{\Omega^t} v^{p^\star}(\mathbf{x})\, d\mathbf{x} = \int_{\Omega} v(\vartheta_t(\mathbf{y}))^{p^\star} |\det D\vartheta_t(\mathbf{y})|\, d\mathbf{y} \leq \beta \|v(\vartheta_t(\mathbf{y}))\|^{p^\star}_{\mathrm{L}^{p^\star}(\Omega)} \\ &\leq \beta S^{p^\star}_{p,\Omega} \|v(\vartheta_t(\mathbf{y}))\|^{p^\star}_{\mathrm{W}^{1,p}(\Omega)} = \beta S^{p^\star}_{p,\Omega} \|v \cdot |\det D\vartheta_t^{-1}|\|^{p^\star}_{\mathrm{W}^{1,p}(\Omega^t)} \\ &\leq \beta S^{p^\star}_{p,\Omega} \alpha^{-\frac{p^\star}{p}} \|v\|^{p^\star}_{\mathrm{W}^{1,p}(\Omega^t)}.\end{aligned}$$
   If we set $K_p := S_{p,\Omega} \beta^{\frac{1}{p^\star}} \alpha^{\frac{-1}{p}}$, we get the uniform estimate for all $p < d$:
$$\forall t \in [a, b], \qquad \forall v \in \mathrm{W}^{1,p}(\Omega^t), \qquad \|v\|_{\mathrm{L}^{p^\star}(\Omega^t)} \leq K_p \|v\|_{\mathrm{W}^{1,p}(\Omega^t)} \tag{3.2}$$
   Thanks to Corollary 1.2.9 we can do the same estimate as above for $p \geq d$, if we replace $p^\star$ by $p + 1$, and obtain
$$\forall t \in [a, b], \qquad \forall v \in \mathrm{W}^{1,p}(\Omega^t) \qquad \|v\|_{\mathrm{L}^{p+1}(\Omega^t)} \leq \tilde{K}_p \|v\|_{\mathrm{W}^{1,p}(\Omega^t)}, \tag{3.3}$$
   where $\tilde{K}_p := \tilde{S}_{p,\Omega} \beta^{\frac{1}{p+1}} \alpha^{\frac{-1}{p}}$ and $\tilde{S}_{p,\Omega}$ is the constant of the continuous embedding of $\mathrm{W}^{1,p}(\Omega) \hookrightarrow \mathrm{L}^{p+1}(\Omega)$.

3. We are going to work with functions $u \in \mathrm{L}^p(\hat{\Omega})$. Let us take a closer look on the integral
$$\int_{\hat{\Omega}} |u|^p \, d(t, x).$$
   Since $u \in \mathrm{L}^p(\hat{\Omega})$, we have $|u|^p \mathbb{1}_{\hat{\Omega}} \in \mathrm{L}^1(\mathbb{R} \times \mathbb{R}^d)$. We set the following:
$$X = \mathbb{R}, \qquad Y = \mathbb{R}^d, \qquad \mu(t) = \lambda(t), \qquad \nu(x) = \lambda_d(x) \qquad f(t, x) = |u(t, x)|^p \mathbb{1}_{\hat{\Omega}}(t, x).$$
   We can write the indicator-function above in the following way
$$\mathbb{1}_{\hat{\Omega}}(t, x) = \sum_{s \in I} \mathbb{1}_s(t) \cdot \mathbb{1}_{\Omega^s}(x) = \begin{cases} \mathbb{1}_{\Omega^t}(x) & s = t \\ 0 & else \end{cases} = \mathbb{1}_{\Omega^t}(x).$$

Applying Fubini's theorem (Theorem 1.1.1) we obtain, that the functions $x \to f(t,x)$ and $t \to f(t,x)$ are integrable with respect to $\lambda_d(x)$ and $\lambda(t)$ respectively. Since the product measure of $\lambda \otimes \lambda_d$ is the $d+1$ dimensional Lebesgue measure $\lambda_{d+1}(t,x)$, we get:

$$\int_{\hat{\Omega}} |u(t,x)|^p \, d(t,x) = \int_{\mathbb{R}\times\mathbb{R}^d} |u(t,x)|^p \mathbb{1}_{\hat{\Omega}}(t,x) \, d(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(t,x)|^p \mathbb{1}_{\Omega^t}(x) \, d(t,x)$$
$$= \int_I \int_{\Omega^t} |u(t,x)|^p \, d(t,x) = \int_I \|u(t)\|^p_{\mathrm{L}^p(\Omega^t)} \, dt.$$

■

At this point we want to introduce a useful notation for $\hat{\Omega}$ (in contradiction with the one used for ε-interior sets defined in Defintion 3.1.4):

$$\begin{aligned}\hat{\Omega}_\varepsilon &\coloneqq \bigcup_{a<t<b} \{t\} \times \vartheta_t(\Omega_\varepsilon),\\ \partial\hat{\Omega} &\coloneqq \bigcup_{a<t<b} \{t\} \times \partial\Omega^t,\\ \partial\hat{\Omega}_\varepsilon &\coloneqq \bigcup_{a<t<b} \{t\} \times \vartheta_t(\partial\Omega_\varepsilon).\end{aligned}$$

We want to point out that $\hat{\Omega}_\varepsilon$ is not the ε-interior of $\hat{\Omega}$ and that $\partial\hat{\Omega}$ and $\partial\hat{\Omega}_\varepsilon$ are not the boundaries of $\hat{\Omega}$ and $\hat{\Omega}_\varepsilon$ in the classical sense respectively.

3.3 First statement

The motivation of the next theorem is to get a to the Aubin-Lions Lemma similar result for moving domains.

Theorem 3.3.1. *Let $\hat{\Omega}$ satisfy the assumptions* **[C1]-[C3]**. *Let $p \in [1,\infty)$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions, that is bounded in $\mathrm{L}^p(\hat{\Omega})$ and such that the sequence of gradients $(\nabla_{\mathbf{x}} u_n)_{n\in\mathbb{N}}$ is also bounded in $\mathrm{L}^p(\hat{\Omega})$. We assume the existence of a constant $C > 0$ and an integer $N \in \mathbb{N}$ such that, for any test function ψ,*

$$|\langle \partial_t u_n, \psi \rangle| \leq C \sum_{|\alpha|\leq N} \|\partial^\alpha_{\mathbf{x}} \psi\|_{\mathrm{L}^2(\hat{\Omega})}. \tag{3.4}$$

Then the sequence $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathrm{L}^p(\hat{\Omega})$.

Remark 3.3.2. There are two important facts to note here. First of all the partial derivative $\partial_t u_n$ exists only in the distributional sense, that means

$$\langle \partial_t u_n, \psi \rangle = -\langle u_n, \partial_t \psi \rangle \qquad \forall \psi \in \mathcal{D}.$$

Secondly we cannot find a fixed Banach space Y, such that we can write $\partial_t u_n \in \mathrm{L}^p(I;Y)$, because we don't have a cylindrical domain. ■

If one defines $\Omega^t = \tilde{\Omega}$, where $\tilde{\Omega}$ is a Lipschitz domain, we get nearly the Aubin-Lions Lemma, because we can identify the spaces $\mathrm{L}^p(I \times \tilde{\Omega})$ with $\mathrm{L}^p(I;\mathrm{L}^p(\tilde{\Omega}))$. Then the conditions turn into

- $\mathrm{W}^{1,p}(\tilde{\Omega})$ embeds compactly in $\mathrm{L}^p(\tilde{\Omega})$,

- $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^p(I;\mathrm{W}^{1,p}(\tilde{\Omega}))$,
- condition (3.4) holds for the sequence $(\partial_t u_n)_{n\in\mathbb{N}}$,

then we find a converging subsequence of $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^p(I;\mathrm{L}^r(\tilde{\Omega}))$ for an appropriate $r \geq 1$. This looks very familiar to the original Aubin-Lions Lemma. The proof for our statement can also be done for cylindrical domains.

3.3.1 A common Poincaré constant.

Before we start working for the proof of Theorem 3.3.1, we take a closer look on the interior sets Ω_ε, with the aim to find a common Poincaré constant for these sets. First let us recall the definition of the Lipschitz constant of $\partial\Omega$.

Definition 3.3.3. If Ω is open and bounded, then $\partial\Omega$ has to be bounded and closed, so we get that $\partial\Omega$ is compact. If Ω has Lipschitz boundary, we find for each open cover $\bigcup_{x\in\partial\Omega} U_x$ of $\partial\Omega$ a finite cover $U_{x_1}, \cup \ldots \cup U_{x_m}$ of $\partial\Omega$. Let H_{x_i} be the approtiate Lipschitz functions for this cover and L_{x_i} the Lipschitz constants respectively. Then we have with $\mathrm{L}{:=} \max_{i\in\{1\ldots m\}}\mathrm{L}_{x_i}$ a constant, such that for all $x, y \in Q$ and $i \in \{1, \ldots, m\}$: $|H_{x_i}(x) - H_{x_i}(y)| \leq \mathrm{L}|x-y|$. We define the *Lipschitz constant of* Ω as the infimum of $\max_{i\in\{1\ldots m\}}\mathrm{L}_{x_i}$ over all such coverings.

It is possible to find $\gamma > 0$ such that for all $\varepsilon \in [0, \gamma)$ the interior sets Ω_ε share a common Poincaré constant. To show this, we state a geometry result without a proof before.

Proposition 3.3.4. *There exists $\gamma > 0$ and $C_\gamma > 0$, such that for any $\varepsilon \in [0, \gamma]$ the open set Ω_ε is Lipschitz, with a constant not exceeding C_γ.*

With this proposition we are now able to state the desired result for the interior sets.

Proposition 3.3.5. *Consider γ the positive number defined in Proposition 3.3.4. Then for $\varepsilon \in [0, \gamma]$, the open sets Ω_ε share a common Poincaré-Wirtinger constant, that is: there exists a positive constant $C_{\Omega,\gamma}$ depending only on Ω and γ, such that for all $v \in \mathrm{H}^1(\Omega_\varepsilon)$ with $\int_{\Omega_\varepsilon} v = 0$, we have*

$$\|v\|_{\mathrm{L}^2(\Omega_\varepsilon)} \leq C_{\Omega,\gamma}\|\nabla v\|_{\mathrm{L}^2(\Omega_\varepsilon)}.$$

For the proof we need an extension theorem and a result from geometric analysis, which can be found in [Ste70] or [Mag12] respectively.

Theorem 3.3.6. *Let Ω be a bounded Lipschitz domain, then there exists a linear operator $\mathrm{P}_\Omega : \mathrm{W}^{k,p}(\Omega) \to \mathrm{W}^{k,p}(\mathbb{R}^d)$ for all $k \in \mathbb{N}$ and $p \in [1, \infty]$, with the following properties for all $u \in \mathrm{W}^{k,p}(\Omega)$:*

i) $\mathrm{P}_\Omega(u)|_\Omega = u$,

ii) $\|\mathrm{P}_\Omega u\|_{\mathrm{W}^{k,p}(\mathbb{R}^d)} \leq C_{L_\Omega,d,k}\|u\|_{\mathrm{W}^{k,p}(\Omega)}$,

where the constant $C_{L_\Omega,d,k}$ depends only on the dimension d, the differentiability k and L_Ω, which is the Lipschitz constant of Ω.

Proof. See [Ste70] Theorem 5 and Theorem 5' on page 181. □

The next Lemma deals with vanishing gradients. If a function $u \in \mathrm{L}^1_{loc}$ has a vanishing gradient (in distributional sense) in an open and bounded set, then the function is constant on this set almost everywhere.

Lemma 3.3.7. (Vanishing weak gradient) *If $u \in \mathrm{L}^2_{loc}(\mathbb{R}^d)$, $A \subset \mathbb{R}^d$ is open and connected, and*

$$\int_{\mathbb{R}^d} u \nabla \psi \, d\lambda = 0 \qquad \forall \psi \in \mathcal{D}(A) \text{ and } \forall i \in \{1, \dots, d\}, \tag{3.5}$$

then there exists $c \in \mathbb{R}$ such that $u = c$ a.e in A.

Proof. See [Mag12] Lemma 7.5 on page 72. □

Remark 3.3.8. The condition (3.5) in Lemma 3.3.7 is to understand component wise, in the sense that for all $\psi \in \mathcal{D}(A)$

$$\int_{\mathbb{R}^d} u \nabla \psi \, d\lambda = 0 \qquad \Leftrightarrow \qquad \int_{\mathbb{R}^d} u \frac{\partial \psi}{\partial x_i} \, d\lambda = 0 \qquad \forall i \in \{1, \dots, d\}.$$

We say that u has a vanishing gradient in distributional sense. ■

Proof of Proposition 3.3.5. Let γ be the constant defined in Proposition 3.3.4, then the interior sets Ω_ε share a common Lipschitz constant M_γ for all $\varepsilon \in [0, \gamma)$. Hence the extensions $\mathrm{P}_{\Omega_\varepsilon} : \mathrm{H}^1(\Omega_\varepsilon) \to \mathrm{H}^1(\mathbb{R}^d)$ are bounded by some constant depending only on γ as follows

$$\| \mathrm{P}_\Omega u \|_{\mathrm{H}^1(\mathbb{R}^d)} \le C_{M_\gamma} \| u \|_{\mathrm{H}^1(\Omega)}. \qquad \forall u \in \mathrm{H}^1(\Omega).$$

We argue by contradiction. If we assume the opposite statement, there would be a sequence $(\varepsilon_n)_{n \in \mathbb{N}} \in [0, \gamma)$ and a sequence $(u_n)_{n \in \mathbb{N}} \in \mathrm{H}^1(\Omega_{\varepsilon_n})$, such that

$$\| u_n \|_{\mathrm{L}^2(\Omega_{\varepsilon_n})} = 1, \qquad \| \nabla u_n \|_{\mathrm{L}^2(\Omega_{\varepsilon_n})} \le 1/n, \qquad \int_{\Omega_{\varepsilon_n}} u_n(x) \, dx = 0.$$

We can assume, without loss of generality, that the sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ converges monotonically to some $\varepsilon \in [0, \gamma]$. Let us define the sequence $(v_n)_{n \in \mathbb{N}} \in \mathrm{H}^1(\mathbb{R}^d)$, where $v_n := \mathrm{P}_{\Omega_{\varepsilon_n}} u_n$. Our next step is to show that $(v_n)_{n \in \mathbb{N}}$ is relatively bounded in $\mathrm{L}^2_{loc}(\mathbb{R}^d)$. Thanks to Theorem 3.6.7 it is sufficient to prove that

$$\| \tau_h v_n - v_n \|_{\mathrm{L}^2(\mathbb{R}^d)} \xrightarrow{h \to 0} 0 \text{ uniformly in } n. \tag{3.6}$$

The second property of $\mathrm{P}_{\Omega_{\varepsilon_n}}$ guarantees, that the sequence $(v_n)_{n \in \mathbb{N}}$ is bounded in $\mathrm{H}^1(\mathbb{R}^d)$ by some constant B. Because of Proposition 2.2.4 we get

$$\| \tau_h v_n - v_n \|_{\mathrm{L}^2(\mathbb{R}^d)} \le \| \nabla v_n \|_{\mathrm{L}^2(\mathbb{R}^d)} |h| \le |h| B \xrightarrow{h \to 0} 0 \qquad \text{uniformly in n.}$$

With (3.6) we find a converging subsequence of $(v_n)_{n \in \mathbb{N}}$, which we denote again with $(v_n)_{n \in \mathbb{N}}$, with limit v. Since v_n equals u_n on Ω_{ε_n}, we have that the sequence $(\| v_n \|_{\mathrm{L}^2(\Omega_{\varepsilon_n})})_{n \in \mathbb{N}}$ converges to 0. The sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ is monoton, so we have that either Ω_{ε_n} is included in $\Omega_{\varepsilon_{n+1}}$ for all $n \in \mathbb{N}$, either $\Omega_{\varepsilon_{n+1}}$ is included in Ω_{ε_n} for all $n \in \mathbb{N}$. We want to show that in both cases we obtain $\nabla v = 0$ in the sense of Lemma 3.3.7.

Case 1: Assume that $\Omega_{\varepsilon_{n+1}} \subseteq \Omega_{\varepsilon_n}$ for all $n \in \mathbb{N}$, then Ω_ε is included in all Ω_{ε_n}. Therefore we obtain for all $\psi \in \mathcal{D}(\Omega_\varepsilon)$

$$\begin{aligned} \left| \int_{\mathbb{R}^d} v \frac{\partial \psi}{\partial x_i} \, d\lambda \right| &= \left| \int_{\Omega_\varepsilon} v \frac{\partial \psi}{\partial x_i} \, d\lambda \right| = \left| (v, \frac{\partial \psi}{\partial x_i})_{\mathrm{L}^2(\Omega_\varepsilon)} \right| = \left| \lim_{n \to \infty} (v_n, \frac{\partial \psi}{\partial x_i})_{\mathrm{L}^2(\Omega_\varepsilon)} \right| \\ &= \lim_{n \to \infty} \left| (\frac{\partial v_n}{\partial x_i}, \psi)_{\mathrm{L}^2(\Omega_\varepsilon)} \right| \le \lim_{n \to \infty} \| \frac{\partial v_n}{\partial x_i} \|_{\mathrm{L}^2(\Omega_\varepsilon)} \| \psi \|_{\mathrm{L}^2(\Omega_\varepsilon)} \\ &\le \lim_{n \to \infty} \| \nabla v_n \|_{\mathrm{L}^2(\Omega_{\varepsilon_n})} \| \psi \|_{\mathrm{L}^2(\Omega_\varepsilon)} \le \lim_{n \to \infty} \frac{1}{n} \| \psi \|_{\mathrm{L}^2(\Omega_\varepsilon)} = 0. \end{aligned}$$

Thanks to Lemma 3.3.7 we get that there exists $c \in \mathbb{R}$ such that $v = c$ a.e. in Ω_ε.

Case 2: Assume now that $\Omega_{\varepsilon_n} \subseteq \Omega_{\varepsilon_{n+1}}$ for all $n \in \mathbb{N}$, then all sets Ω_{ε_n} are contained in Ω_ε. Now let $\psi \in \mathcal{D}(\Omega_\varepsilon)$ be arbitrary and define $K := \operatorname{supp}\psi$. K is a compact subset of Ω_ε, hence we find $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ we have that $K \subseteq \Omega_{\varepsilon_n}$. Going through the same calculation we did in Case 1, replacing Ω_ε with K, we obtain that there exists $c \in \mathbb{R}$ such that $v = c$ a.e in Ω_ε.

In both cases we got that v has to be equal to a constant $c \in \mathbb{R}$ on Ω_ε almost everywhere. On the other hand we can show that

$$v\mathbb{1}_{\Omega_\varepsilon} - v_n\mathbb{1}_{\Omega_{\varepsilon_n}} = (v - v_n)\mathbb{1}_{\Omega_{\varepsilon_n}} + v(\mathbb{1}_{\Omega_\varepsilon} - \mathbb{1}_{\Omega_{\varepsilon_n}}) \xrightarrow{n\to\infty} 0 \qquad \text{in } \mathrm{L}^2_{loc}(\mathbb{R}^d).$$

The first term on the right hand side converges to zero because v_n converges to v in $\mathrm{L}^2_{loc}(\mathbb{R}^d)$. For the second term observe that for case 1 we have $\mathbb{1}_{\Omega_\varepsilon} - \mathbb{1}_{\Omega_{\varepsilon_n}} = -\mathbb{1}_{\Omega_{\varepsilon_n}\setminus\Omega_\varepsilon}$ for all $n \in \mathbb{N}$ and for case 2 we have $\mathbb{1}_{\Omega_\varepsilon} - \mathbb{1}_{\Omega_{\varepsilon_n}} = \mathbb{1}_{\Omega_\varepsilon\setminus\Omega_{\varepsilon_n}}$ for all $n \in \mathbb{N}$. In both cases we have for all $x \in \mathbb{R}^d$

$$\mathbb{1}_{\Omega_{\varepsilon_n}\setminus\Omega_\varepsilon}(x) \xrightarrow{n\to\infty} 0, \qquad \mathbb{1}_{\Omega_\varepsilon\setminus\Omega_{\varepsilon_n}}(x) \xrightarrow{n\to\infty} 0.$$

We also have that both function sequences are bounded by $\mathbb{1}_\Omega$, hence we can apply the dominated convergence theorem. Therefore we obtain for all compact subsets $K \subset \mathbb{R}^d$, that $v(\mathbb{1}_{\Omega_\varepsilon} - \mathbb{1}_{\Omega_{\varepsilon_n}})$ converges to 0 in $\mathrm{L}^2(K)$ and eventually that $v_n\mathbb{1}_{\Omega_{\varepsilon_n}}$ converges to $v\mathbb{1}_{\Omega_\varepsilon}$ in $\mathrm{L}^2(K)$. Let K be compact with $\Omega_{\varepsilon_n} \subseteq K$ for all $n \in \mathbb{N}$ (for example $K := \overline{\Omega}$). Since $v_n\mathbb{1}_{\Omega_{\varepsilon_n}}$ converges to $v\mathbb{1}_{\Omega_\varepsilon}$ in $\mathrm{L}^2(K)$, the $\mathrm{L}^2(\Omega_\varepsilon)$ norm of v equals 1 and the mean-value of v vanishes on Ω_ε, but this is impossible if v is constant almost everywhere on Ω_ε.

□

The next lemma shows that the family $(\vartheta_t)_{t\in I}$ satisfies a Lipschitz condition independent of $t \in I$. We need this property for our last statement in this subsection as well as for the next subsection.

Lemma 3.3.9. *The family $(\vartheta_t)_{t\in[a,b]}$ is uniformly bilipschitz on a neighbourhood of Ω. More precisely, we find a constant $L \geq 1$ and $U \subseteq \mathbb{R}^d$, such that $\overline{\Omega} \subseteq U$ and for all $x, y \in U$ we get:*

$$\frac{1}{L}|x - y| \leq |\vartheta_t(x) - \vartheta_t(y)| \leq L|x - y| \qquad \forall t \in [a, b]. \tag{3.7}$$

Proof. Since Ω is bounded, we find a $r \geq 0$, such that the ball $B_r(0)$ contains $\overline{\Omega}$. We know that $\overline{B}_r(0)$ is convex, such that for all $x, y \in \overline{B}_r(0)$ exists a curve $\gamma_{x,y}(t) = x + (y - x)t$ thats image of the interval $[0, 1]$ is contained in the closed Ball.
Let $x, y \in \overline{B_r(0)}$ and $t \in [a, b]$ be arbitrary, then we get with Theorem 3.1.1:

$$\begin{aligned} |\vartheta_t(x) - \vartheta_t(y)| &\leq |\int_0^1 D\vartheta_t(\gamma_{x,y}(t))(x - y)\, dt| \\ &\leq \int_0^1 \|D\vartheta_t(\gamma_{x,y}(t))\|_{\mathfrak{L}(\mathbb{R}^d,\mathbb{R}^d)}|x - y|\, dt \\ &\leq \sup_{x\in\overline{B_r(0)}} \|D\vartheta_t(x)\|_{\mathfrak{L}(\mathbb{R}^d,\mathbb{R}^d)}|x - y|. \end{aligned}$$

We have $\Theta \in \mathrm{C}^0([a, b], \mathrm{C}^1(\mathbb{R}^d, \mathbb{R}^d))$, so that $D_x\Theta \in \mathrm{C}^0([a, b] \times \mathbb{R}^d, \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^d))$ with $D_x\Theta(t, x) = D\vartheta_t(x)$. Since $[a, b] \times \overline{B_r(0)}$ is compact in $[a, b] \times \mathbb{R}^d$, we get

$$\sup_{x\in\overline{B_r(0)}} \|\vartheta_t(x)\|_{\mathfrak{L}(\mathbb{R}^d,\mathbb{R}^d)} \leq \sup_{(t,x)\in[a,b]\times\overline{B_r(0)}} \|\vartheta_t(x)\|_{\mathfrak{L}(\mathbb{R}^d,\mathbb{R}^d)} = \|D_x\Theta\|_{\mathrm{C}^0([a,b]\times\overline{B_r(0)},\mathfrak{L}(\mathbb{R}^d,\mathbb{R}^d))} < \infty.$$

We define $\overline{L} := \|D_x\Theta\|$ and found a uniform Lipschitz constant for the family $(\vartheta_t)_{t\in[a,b]}$. If we do the same for the family $(\vartheta_t^{-1})_{t\in[a,b]}$, we obtain another uniform Lipschitz constant $\tilde{L}$. Define $L := \max(\overline{L}, \tilde{L})$, then we have:

$$\frac{1}{L}|x-y| = \frac{1}{L}|\vartheta_t^{-1}(\vartheta_t(x)) - \vartheta_t^{-1}(\vartheta_t(y))| \leq |\vartheta_t(x) - \vartheta_t(y)| \leq L|x-y|.$$

With the upper inequality, we see that $L^2 \geq 1$, so that L has to be greater than one. □

The next Lemma is a generalization of the chainrule for Sobolev functions (Theorem 2.3.1).

Lemma 3.3.10. *Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be bilipschitz and $u \in \mathrm{W}^{1,p}(\Omega)$, $p \geq 1$, then $v = u \circ T \in \mathrm{W}^{1,p}(T^{-1}(\Omega))$ and*

$$\nabla v(x) = \nabla u(T(x))DT(x),$$

for almost every $x \in \Omega$, where DT is the derivative of T.

Proof. See Theorem 2.2.2 in [Zie89] on page 52. □

For the sake of completeness, we give a version of Proposition 3.3.5 where we replace Ω with Ω^t and we will use this Proposition in a later section.

Proposition 3.3.11. *Let $\hat{\Omega}$ satisfy* **[C1]-[C3]**. *Consider γ the positive number defined in Proposition 3.3.4 . Then for $\varepsilon \in [0,\gamma]$, and $t \in [a,b]$, the open sets $\vartheta_t(\Omega_\varepsilon)$ share a common Poincaré-Wirtinger constant $C^{\Theta}_{\Omega,\gamma}$ (in the sense of Proposition 3.3.5).*

Proof. Fix $t \in [a,b]$, $\varepsilon \in [0,\gamma]$ and $u \in \mathrm{H}^1(\vartheta^t(\Omega_\varepsilon))$. Since ϑ_t is bilipschitz, we get with Lemma 3.3.10 that $v := u \circ \vartheta_t \in \mathrm{H}^1(\Omega_\varepsilon)$. Thanks to Proposition 3.3.5 we have

$$\int_{\Omega_\varepsilon} |v(y)|^2\,dy \leq C^2_{\Omega,\gamma}|\nabla v(y)|^2\,dy.$$

Considering the change of variable $y = \vartheta_t^{-1}(x)$ in the previous inequality, together with estimate (3.1) we get

$$\begin{aligned}
\frac{1}{\beta}\int_{\vartheta_t(\Omega_\varepsilon)} |u(x)|^2\,dx &\leq \int_{\vartheta_t(\Omega_\varepsilon)} |u(x)|^2|\det D\vartheta_t^{-1}(x)|\,dx = \int_{\Omega_\varepsilon} |v(y)|^2\,dy \\
&\leq C^2_{\Omega,\gamma}\int_{\Omega_\varepsilon} |\nabla v(y)|^2\,dy = C^2_{\Omega,\gamma}\int_{\vartheta_t(\Omega_\varepsilon)} |((\nabla v)\circ\vartheta_t^{-1})(x)|^2|\det D\vartheta_t^{-1}(x)|\,dx \\
&\leq \frac{1}{\alpha}C^2_{\Omega,\gamma}\int_{\vartheta(\Omega_\varepsilon)} |((\nabla v)\circ\vartheta_t^{-1})(x)|^2\,dx.
\end{aligned}$$

Because of the assumptions **[C1]-[C3]** we have $D\vartheta_t \in \mathrm{C}^0(\mathbb{R}^d, \mathfrak{L}(\mathbb{R}^d,\mathbb{R}^d))$ for all $t \in I$. Since $I \times \overline{\Omega}$ is a compact set in $\mathbb{R}^{d+1}$, there exists a constant $B \in \mathbb{R}$, where D_x denotes the space-derivative, such that

$$\begin{aligned}
\sup_{t\in I}\|D\vartheta_t\|_{\mathrm{C}^0(\mathbb{R}^d,\mathfrak{L}(\mathbb{R}^d,\mathbb{R}^d))} &= \|D_x\Theta\|_{\mathrm{C}^0(I\times\Omega,\mathrm{C}^0(\mathbb{R}^d,\mathfrak{L}(\mathbb{R}^d,\mathbb{R}^d)))} \\
&\leq \|D_x\Theta\|_{\mathrm{C}^0(I\times\overline{\Omega},\mathrm{C}^0(\mathbb{R}^d,\mathfrak{L}(\mathbb{R}^d,\mathbb{R}^d)))} \leq B.
\end{aligned}$$

Thanks to the definition of the operator norm and the chain-rule (Lemma 3.3.10), we have that $|\nabla v(y)| \leq \|D\vartheta_t\|_{\mathrm{C}^0(\mathbb{R}^d,\mathfrak{L}(\mathbb{R}^d,\mathbb{R}^d))}|\nabla u(\vartheta_t(y))|$, which leads us to

$$\begin{aligned}\frac{1}{\alpha}C_{\Omega,\gamma}^2\int_{\vartheta(\Omega_\varepsilon)}|(\nabla v)(\vartheta_t^{-1}(x))|^2\,dx &\leq \frac{1}{\alpha}C_{\Omega,\gamma}^2\|D\vartheta_t\|^2_{\mathrm{C}^0(\mathbb{R}^d,\mathfrak{L}(\mathbb{R}^d,\mathbb{R}^d))}\int_{\vartheta(\Omega_\varepsilon)}|\nabla u(\vartheta_t(\vartheta_t^{-1}(x)))|^2\,dx\\ &\leq \frac{1}{\alpha}C_{\Omega,\gamma}^2\|D_x\Theta\|^2_{\mathrm{C}^0}\|\nabla u\|^2_{\mathrm{L}^2(\vartheta_t(\Omega_\varepsilon))}.\end{aligned}$$

Bringing all together, setting $C_{\Omega,\gamma}^{\Theta} \coloneqq \sqrt{\beta/\alpha}C_{\Omega,\gamma}\|D_x\Theta\|_{\mathrm{C}^0}$, provides

$$\|u\|_{\mathrm{L}^2(\vartheta_t(\Omega_\varepsilon))} \leq C_{\Omega,\gamma}^{\Theta}\|\nabla u\|_{\mathrm{L}^2(\vartheta_t(\Omega_\varepsilon))} \qquad \forall t\in[a,b],\ \forall\varepsilon\in[0,\gamma].$$

□

3.3.2 Tools for the proof of the first statement.

In this subsection we look forward to the proof of Theorem 3.3.1 and state important results for the family $(\vartheta_t)_{t\in I}$ and for sequences of functions, which satisfy the conditions of Theorem 3.3.1.

Proposition 3.3.12. *Let $\hat{\Omega}$ satisfy* **[C1]-[C3]**, *then we have*

i) For all $t\in[a,b]$ and all $\varepsilon>0$, $\vartheta_t(\partial\Omega_\varepsilon)$ is the boundary in $\mathbb{R}^d$ of $\vartheta_t(\Omega_\varepsilon)$.

ii) There exists $\kappa\in(0,1]$ such that, for all $t\in[a,b]$ and for all $\varepsilon>0$, $\Omega^t_{\varepsilon/\kappa}\subset\vartheta_t(\Omega_\varepsilon)\subset\Omega^t_{\varepsilon\kappa}$.

iii) $\lambda_d(\Omega^t\setminus\vartheta_t(\Omega_\varepsilon))\to 0$ for $\varepsilon\to 0$ uniformly in t.

Proof.

i) Since ϑ_t is a diffeomorphism, ϑ_t is in particular a homeomorphism (a bijective function f, where f and f^{-1} are continuous). Let $O\subseteq\mathbb{R}^d$ be an open set. Since ϑ_t is continuous we have

$$\vartheta_t(\overline{O})\subseteq\overline{\vartheta_t(O)}.$$

The continuity of ϑ_t^{-1} provides that $\vartheta_t(\overline{O})$ is closed and contains $\vartheta_t(O)$. Since $\overline{\vartheta_t(O)}$ is the smallest closed set containing $\vartheta_t(O)$, we have

$$\overline{\vartheta_t(O)}=\vartheta(\overline{O}).$$

With this, we get for every open set $O\subseteq\mathbb{R}^d$, using that $\vartheta_t(O)$ is open:

$$\vartheta_t(\partial O)=\vartheta_t(\overline{O}\setminus O)=\vartheta_t(\overline{O})\cap\vartheta_t(O^c)=\overline{\vartheta_t(O)}\cap\vartheta_t(O)^c=\overline{\vartheta_t(O)}\setminus\vartheta_t(O)=\partial\vartheta_t(O).$$

ii) We want to use Lemma 3.3.9. Define $\kappa\coloneqq\frac{1}{L}$, where L is the constant used in formula (3.7). Let us show $\Omega^t_{\varepsilon/\kappa}\subseteq\vartheta_t(\Omega_\varepsilon)$ first. Let $w\in\Omega^t_{\varepsilon/\kappa}$, then there exists a $x\in\Omega$, such that $\vartheta_t(x)=w$. It is necessary to show that $x\in\Omega_\varepsilon$. According to i) we have:

$$\begin{aligned}L\cdot\varepsilon\leq d(w,\partial\Omega^t)&=d(\vartheta_t(x),\partial\vartheta_t(\Omega))=d(\vartheta_t(x),\vartheta_t(\partial\Omega))=\\ &=\inf_{y\in\partial\Omega}|\vartheta_t(x)-\vartheta_t(y)|\leq L\cdot\inf_{y\in\partial\Omega}|x-y|=L\cdot d(x,\partial\Omega).\end{aligned}$$

We get that x has to be in Ω_ε.
Second we want to show that $\vartheta_t(\Omega_\varepsilon) \subseteq \Omega^t_{\varepsilon\kappa}$. Let $z \in \vartheta_t(\Omega_\varepsilon)$, then we have $x \in \Omega_\varepsilon$ such that $\vartheta_t(x) = z$. We have to show that $z \in \Omega^t_{\varepsilon\kappa}$. Again with i) we obtain:

$$d(z, \partial\vartheta_t(\Omega)) = d(z, \vartheta_t(\partial\Omega)) = \inf_{y\in\partial\Omega} |\vartheta_t(x) - \vartheta_t(y)| \geq \inf_{y\in\partial\Omega} \frac{1}{L}|x-y| \geq \varepsilon\kappa.$$

Which shows that $z \in \Omega^t_{\varepsilon\kappa}$.

iii) First, we deal with the uniformity in t:
Since ϑ_t is bijective, we get $\Omega^t \setminus \vartheta_t(\Omega_\varepsilon) = \vartheta_t(\Omega) \setminus \vartheta_t(\Omega_\varepsilon) = \vartheta_t(\Omega \setminus \Omega_\varepsilon)$. Thanks to the Transformation Rule (Theorem 3.1.3) and formula (3.1), we get:

$$\lambda_d(\Omega^t \setminus \vartheta_t(\Omega_\varepsilon)) = \int_{\Omega^t\setminus\vartheta_t(\Omega_\varepsilon)} d\lambda_d(x) = \int_{\Omega\setminus\Omega_\varepsilon} |\det D\vartheta_t(x)|\, d\lambda_d(x) \leq \beta \cdot \lambda_d(\Omega \setminus \Omega_\varepsilon).$$

Whence it is sufficient to prove that the last term tends to zero for ε to zero. One has

$$\lambda_d(\Omega \setminus \Omega_\varepsilon) = \int_{\Omega\setminus\Omega_\varepsilon} dx = \int_{\mathbb{R}^d} \mathbb{1}_{\Omega\setminus\Omega_\varepsilon}(x)\, dx.$$

We have clearly, that $\mathbb{1}_{\Omega\setminus\Omega_\varepsilon}$ converges pointwise to 0 for ε to 0 and that $\mathbb{1}_{\Omega\setminus\Omega_\varepsilon} \leq \mathbb{1}_\Omega$ for all $\varepsilon \geq 0$, hence we can apply the dominated convergence theorem and obtain that

$$\lambda_d(\Omega \setminus \Omega_\varepsilon) = \int_{\mathbb{R}^d} \mathbb{1}_{\Omega\setminus\Omega_\varepsilon}(x)\, dx \xrightarrow{\varepsilon\to 0} 0.$$

Therefore we got the desired uniform convergence.

□

Next we give a result similar to the regularization in section 1.2.3. The difference is that we convolute in the space variable only.

Proposition 3.3.13. *Let $\hat\Omega$ satisfy* **[C1]-[C3]**, *$(u_k)_{k\in\mathbb{N}}$ and $(\nabla_{\mathbf{x}} u_k)_{k\in\mathbb{N}}$ be bounded sequences in $\mathrm{L}^p(\hat\Omega)$ for $p \in [1,\infty)$, where $\nabla_{\mathbf{x}}$ denotes the gradient in the last d coordinates. Let $(\varphi_\eta)_{\eta\in(0,1)}$ be the family of mollifiers in $\mathrm{C}^\infty_c(\mathbb{R}^d)$ defined in Subsection 1.2.3, then for every $\varepsilon > 0$ we have*

$$\lim_{\eta\to 0} \sup_{k\in\mathbb{N}} \|u_k - u_k \star \varphi_\eta\|_{\mathrm{L}^p(\hat\Omega_\varepsilon)} = 0,$$

where the convolution $\star$ has to be understood in the last d coordinates only.

Proof. First of all we define the measure $\mu_\eta(A) := \int_A \varphi_\eta(y)\, dy$ for all $A \subset \mathbb{R}^d$ and all $\eta \in (0,1)$. Observe that for all $\eta \in (0,1)$, μ_η is absolutely continuous with respect to the Lebesque measure with density φ_η and that μ_η is a probability measure on $\mathbb{R}^d$, because

$$\mu_\eta(\mathbb{R}^d) = \int_{\mathbb{R}^d} \varphi_\eta(y)\, dy = \int_{B_\eta(0)} \varphi_\eta(y)\, dy = 1.$$

Let $\varepsilon > 0$, then we have to show that for all $k \in \mathbb{N}$ the functions $(u_k \star \varphi_\eta)(t,x) = (u_k(t) \star \varphi_\eta)$ are well defined for all $(t,x) \in \hat\Omega_\varepsilon$ and all $\eta < C(\varepsilon)$, where $C(\varepsilon)$ is some constant depending on ε. We denote the restriction of u_k on $\hat\Omega_\varepsilon$ as $u_k|_{\hat\Omega_\varepsilon}$, hence it is easy to see that $u_k|_{\hat\Omega_\varepsilon}(t) = u_k(t)|_{\vartheta_t(\Omega_\varepsilon)}$. Thanks to Proposition 3.3.12, there exists a constant $\kappa \in (0,1]$, such that for all $t \in I$ we have $\vartheta_t(\Omega_\varepsilon) \subset \Omega^t_{\kappa\varepsilon} \subseteq \Omega^t_\varepsilon$. Thus we obtain with Lemma 1.2.17 for all $t \in I$ and all $\eta < \frac{\varepsilon}{2}$ that

$\operatorname{supp}(u(t)|_{\vartheta_t(\Omega_\varepsilon)} \star \varphi_\eta) \subseteq \Omega^t_\varepsilon + B_\eta(0) \subseteq \Omega^t$, hence the functions $u_k \star \varphi_\eta$ are well defined for all $k \in \mathbb{N}$.
We know, that $|.|^p$ is a convex function, therefore we can apply the Jensen inequality with $|.|^p$ and the probability measure μ_η. Thus we get for all $(t,x) \in \hat{\Omega}_\varepsilon$:

$$\begin{aligned}
|u_k(t,x) - u_k \star \varphi_\eta(t,x)|^p &= |\int_{\mathbb{R}^d} u_k(t,x)\varphi_\eta(y)\,dy - \int_{\mathbb{R}^d} u_k(t,x-y)\varphi_\eta(y)\,dy|^p \\
&= |\int_{\mathbb{R}^d} u_k(t,x) - u_k(t,x-y)\,d\mu_\eta(y)|^p \\
&\leq \int_{\mathbb{R}^d} |u_k(t,x) - \tau_{-y}u_k(t,x)|^p\,d\mu_\eta(y),
\end{aligned}$$

where τ_y is the shift operator defined in Definition 2.2.2. Thanks to Lemma 2.2.4, we know that $\|u_k(t) - \tau_{-y}u_k(t)\|_{\mathrm{L}^p(\vartheta_t(\Omega_\varepsilon))} \leq \|\nabla_x u_k(t)\|_{\mathrm{L}^p(\vartheta_t(\Omega_\varepsilon))}|y|$ for all $t \in I$. Integrating over $\hat{\Omega}_\varepsilon$ and using Fubini provides:

$$\begin{aligned}
\|u_k(t,x) - u_k \star \varphi_\eta(t,x)\|^p_{\mathrm{L}^p(\hat{\Omega}_\varepsilon)} &= \int_{\hat{\Omega}_\varepsilon} |u_k(t,x) - u_k \star \varphi_\eta(t,x)|^p\,d(t,x) \\
&\leq \int_{\hat{\Omega}_\varepsilon}\int_{\mathbb{R}^d} |u_k(t,x) - u_k(t,x-y)|^p\,d\mu_\eta(y)\,d(t,x) \\
&= \int_{\mathbb{R}^d}\int_{\hat{\Omega}_\varepsilon} |u_k(t,x) - u_k(t,x-y)|^p\varphi_\eta(y)\,d(t,x)\,dy \\
&= \int_{\mathbb{R}^d}\int_I\int_{\vartheta_t(\Omega_\varepsilon)} |u_k(t,x) - \tau_{-y}u_k(t,x)|^p\varphi_\eta(y)\,dx\,dt\,dy \\
&= \int_{\mathbb{R}^d}\int_I \|u_k(t) - \tau_{-y}u_k(t)\|^p_{\mathrm{L}^p(\vartheta_t(\Omega_\varepsilon))}\varphi_\eta(y)\,dt\,dy \\
&\leq \int_{\mathbb{R}^d}\int_I \|\nabla_x u_k(t)\|^p_{\mathrm{L}^p(\vartheta_t(\Omega_\varepsilon))}|y|^p\varphi_\eta(y)\,dt\,dy.
\end{aligned}$$

Our assumption guarantees the existence of a constant $C \in \mathbb{R}$, such that for all $k \in \mathbb{N}$ we have $\|\nabla_x u_k\|_{\mathrm{L}^p(\hat{\Omega})} \leq C$. For all $t \in I$, the set $\vartheta_t(\Omega_\varepsilon)$ is a subset of Ω^t, hence $\hat{\Omega}_\varepsilon \subset \hat{\Omega}$ and therefore we get $\|\nabla_x u_k\|_{\mathrm{L}^p(\hat{\Omega}_\varepsilon)} \leq C$. If we use Fubini in the same way as in Fact 3.2.1, 3., we obtain:

$$\begin{aligned}
\|u_k(t,x) - u_k \star \varphi_\eta(t,x)\|^p_{\mathrm{L}^p(\hat{\Omega}_\varepsilon)} &\leq \int_{\mathbb{R}^d}\int_I \|\nabla_x u_k(t)\|^p_{\mathrm{L}^p(\vartheta_t(\Omega_\varepsilon))}|y|^p\varphi_\eta(y)\,dt\,dy \\
&= \int_{\mathbb{R}^d} \|\nabla_x u_k\|^p_{\mathrm{L}^p(\hat{\Omega}_\varepsilon)}|y|^p\varphi_\eta(y)\,dy \\
&\leq C\int_{B_\eta(0)} |y|^p\varphi_\eta\,dy \leq C\eta^p
\end{aligned}$$

Letting η to 0, leads to the desired uniform convergence. □

The last statement of this section gives us, with sufficient conditions, a global convergence result for a only local converging sequence.

Proposition 3.3.14. *Let $\hat{\Omega}$ satisfy* **[C1]-[C3]** *and fix $p \in [1,\infty)$. Assume that $(u_n)_{n\in\mathbb{N}}$, $(\nabla_x u_n)_{n\in\mathbb{N}}$ are bounded in $\mathrm{L}^p(\hat{\Omega})$. If for all $\varepsilon > 0$, one has that $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathrm{L}^p(\hat{\Omega}_\varepsilon)$(local compactness), then $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathrm{L}^p(\hat{\Omega})$(global compactness).*

Remark 3.3.15. The sequence $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathrm{L}^p(\hat{\Omega}_\varepsilon)$ for all $\varepsilon > 0$, if and only if $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})$ for all $m \in \mathbb{N}$.

Let $\varepsilon \in (0,1)$, then there exists $m \in \mathbb{N}$, such that $\varepsilon \in [\frac{1}{m+1}, \frac{1}{m})$ and hence $\hat{\Omega}_\varepsilon \subseteq \hat{\Omega}_{\frac{1}{m+1}}$. Our assumption guarantees the existence of a subsequence $(u_{n_k})_{k\in\mathbb{N}}$, which converges in $\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m+1}})$ and hence $(u_{n_k})_{k\in\mathbb{N}}$ converges in $\mathrm{L}^p(\hat{\Omega}_\varepsilon)$ for all $\varepsilon \in [\frac{1}{m+1}, \frac{1}{m})$.
For $\varepsilon \geq 1$, we choose $m = 1$ and obtain that $\hat{\Omega}_\varepsilon \subseteq \hat{\Omega}_1$. Using the same argumentation as above, we find for every $\varepsilon > 0$ a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ that is converging in $\mathrm{L}^p(\hat{\Omega}_\varepsilon)$.
The other direction is clear. ■

Proof of Proposition 3.3.14. Thanks to Remark 3.3.15 we find in each $\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})$ a converging subsequence of $(u_n)_{n\in\mathbb{N}}$. By diagonal extraction we find a subsequence of $(u_n)_{n\in\mathbb{N}}$ that converges in all $\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})$ and therefore this sequence converges in $\mathrm{L}^p(\hat{\Omega}_\varepsilon)$ for all $\varepsilon > 0$. Let us denote this subsequence again by $(u_n)_{n\in\mathbb{N}}$ and its limit with u.
We begin the proof with showing that $u \in \mathrm{L}^p(\hat{\Omega})$. Since $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^p(\hat{\Omega})$ we find a constant $C \in \mathbb{R}$ such that $\sup_{n\in\mathbb{N}} \|u_n\|_{\mathrm{L}^p(\hat{\Omega}_\varepsilon)} \leq C$ for all $\varepsilon \geq 0$. Therefore we have $\|u\|_{\mathrm{L}^p(\hat{\Omega}_\varepsilon)} \leq C$ for all $\varepsilon > 0$. Passing to the limit $\varepsilon \to 0$, we obtain that $\|u\|_{\mathrm{L}^p(\hat{\Omega})} \leq C$.
To prove the convergence in $\mathrm{L}^p(\hat{\Omega})$ we split the proof in two cases, namely $p < d$ and $p \geq d$. Assume first $p < d$. Since the sequences $(u_n)_{n\in\mathbb{N}}$ and $(\nabla_x u_n)_{n\in\mathbb{N}}$ are bounded in $\mathrm{L}^p(\hat{\Omega})$, we hence deduce from estimate (3.2) the following:

$$\begin{aligned}\sup_{n\in\mathbb{N}} \int_I \|u_n(t)\|^p_{\mathrm{L}^{p^\star}(\Omega^t)}\, dt &\leq K_p^p \sup_{n\in\mathbb{N}} \int_I \|u_n(t)\|^p_{\mathrm{W}^{1,p}(\Omega^t)}\, dt \\ &\leq K_p^p (\sup_{n\in\mathbb{N}} \int_I \|u_n(t)\|^p_{\mathrm{L}^p(\Omega^t)} + \sup_{n\in\mathbb{N}} \int_I \|\nabla_x u_n(t)\|^p_{\mathrm{L}^p(\Omega^t)}\, dt) \\ &\leq K_p^p (\sup_{n\in\mathbb{N}} \|u_n\|_{\mathrm{L}^p(\hat{\Omega})} + \sup_{n\in\mathbb{N}} \|\nabla_x u_n\|_{\mathrm{L}^p(\hat{\Omega})}) < \infty.\end{aligned}$$

Now for arbitrary $\varepsilon > 0$, by Hölder inequality (Theorem 1.1.3) with $\frac{1}{p} = \frac{1}{p^\star} + \frac{p^\star - p}{pp^\star}$ we get

$$\begin{aligned}\|u_n\|^p_{\mathrm{L}^p(\hat{\Omega}\setminus\hat{\Omega}_\varepsilon)} &= \int_I \int_{\Omega^t} |u_n(t,x)|^p \mathbb{1}_{\Omega^t \setminus \vartheta_t(\Omega_\varepsilon)}(x)\, dx\, dt \leq \int_I \|u_n(t)\|^p_{\mathrm{L}^{p^\star}(\Omega^t)} \|\mathbb{1}_{\Omega^t\setminus\vartheta_t(\Omega_\varepsilon)}\|^p_{\mathrm{L}^{\frac{pp^\star}{p^\star - p}}(\Omega^t)}\, dt \\ &= \int_I \|u_n(t)\|^p_{\mathrm{L}^{p^\star}(\Omega^t)} \lambda_d(\Omega^t \setminus \vartheta_t(\Omega_\varepsilon))^{\frac{p^\star - p}{p^\star}}\, dt.\end{aligned}$$

Since $\hat{\Omega}$ is bounded, we obtain with the two upper estimates and Proposition 3.3.12:

$$\lim_{\varepsilon\to 0} \sup_{n\in\mathbb{N}} \|u_n\|_{\mathrm{L}^p(\hat{\Omega}\setminus\hat{\Omega}_\varepsilon)} \leq \lim_{\varepsilon\to 0} \sup_{t\in I} \lambda_d(\Omega^t \setminus \vartheta_t(\Omega_\varepsilon))^{\frac{p^\star - p}{p^\star}} \sup_{n\in\mathbb{N}} \int_I \|u_n(t)\|_{\mathrm{L}^{p^\star}(\Omega^t)}\, dt = 0. \tag{3.8}$$

We proved that $u \in \mathrm{L}^p(\hat{\Omega})$, and therefore we get again with Proposition 3.3.12:

$$\lim_{\varepsilon\to 0} \|u\|^p_{\mathrm{L}^p(\hat{\Omega}\setminus\hat{\Omega}_\varepsilon)} \leq \lim_{\varepsilon\to 0} \int_I \|u(t)\mathbb{1}_{\Omega^t\setminus\vartheta_t(\Omega_\varepsilon)}\|^p_{\mathrm{L}^p(\Omega^t)}\, dt \leq \lim_{\varepsilon\to 0} \sup_{t\in I} \lambda_d(\Omega^t \setminus \vartheta_t(\Omega_\varepsilon)) \|u\|^p_{\mathrm{L}^p(\hat{\Omega})} = 0. \tag{3.9}$$

Finally we have for arbitrary $\varepsilon > 0$, due to convergence of $(u_n)_{n\in\mathbb{N}}$ in all $\mathrm{L}^p(\hat{\Omega}_\varepsilon)$

$$\begin{aligned}\lim_{n\to\infty} \|u_n - u\|_{\mathrm{L}^p(\hat{\Omega})} &= \lim_{n\to\infty} \|u_n - u\|_{\mathrm{L}^p(\hat{\Omega}\setminus\hat{\Omega}_\varepsilon)} + \lim_{n\to\infty} \|u_n - u\|_{\mathrm{L}^p(\hat{\Omega}_\varepsilon)} \\ &\leq \sup_{n\in\mathbb{N}}(\|u_n\|_{\mathrm{L}^p(\hat{\Omega}\setminus\hat{\Omega}_\varepsilon)}) + \|u\|_{\mathrm{L}^p(\hat{\Omega}\setminus\hat{\Omega}_\varepsilon)}.\end{aligned}$$

Since ε was arbitrary we can pass to the limit $\varepsilon \to 0$ and obtain thanks to (3.8) and (3.9) the desired convergence.
Thanks to (3.3), the case $p \geq d$ is completely similar, replacing $p^\star$ by $p + 1$ and K_p by $\tilde{K}_p$ in the proof above. □

3.3.3 Proof of Theorem 3.3.1

The idea of the proof is to prove the statement for all $\hat{\Omega}_\varepsilon$ instead of $\hat{\Omega}$. Due to Proposition 3.3.14 this is sufficient to prove Theorem 3.3.1. The sequence of nonnegative even mollifiers $(\varphi_\eta)_{\eta\in(0,1)}$, defined in Subsection 1.2.3 will be again a great assistance for the proof. Now we are able to [Mou16].

Proof of Theorem 3.3.1. We only need to prove that for all integer $m \in \mathbb{N}$, $(u_n)_{n\in\mathbb{N}}$ is relatively bounded in $\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})$. The conclusion follows then by Remark 3.3.15 and with Proposition 3.3.14. Let $m \in \mathbb{N}$ be arbitrary. Since u_n is only defined on $\hat{\Omega}$, $u_n \star \varphi_\eta$ (convolution only in space variable) is well-defined only in a subset of $\hat{\Omega}$. Proposition 3.3.12 *ii)* provides a constant $\kappa \in (0,1)$ such that $\vartheta_t(\Omega_{\frac{1}{m}}) \subset \Omega^t_{\kappa\frac{1}{m}}$ for all $t \in [a,b]$, hence

$$\hat{\Omega}_{\frac{1}{m}} \subseteq \bigcup_{t\in I} \{t\} \times \Omega^t_{\kappa\frac{1}{m}}.$$

Furthermore we have for all $t \in [a,b]$ and $x \in \Omega^t_{\kappa\frac{1}{m}} + B_{\kappa\frac{1}{2m}}(0)$, that $d(x, \partial\Omega^t) \leq \kappa\frac{1}{2m}$. Since $\operatorname{supp}\varphi_\eta \subseteq B_{\kappa\frac{1}{2m}}$ for all $\eta \leq \frac{1}{2m}$, we have, thanks to Lemma 1.2.17,

$$\operatorname{supp}(u_n(t) \star \varphi_\eta) \subseteq \operatorname{supp} u_n + B_{\kappa\frac{1}{m}}(0).$$

Hence $u_n \star \varphi_\eta$ is well defined in $\hat{\Omega}_{\frac{1}{m}}$ for all $n \in \mathbb{N}$, if $\eta \leq \kappa\frac{1}{2m}$. In that case, for any $\psi \in \mathcal{D}(\hat{\Omega}_{\frac{1}{m}})$, one gets $\psi \star \varphi_\eta \in \mathcal{D}(\hat{\Omega})$.
Now fix $\eta \leq \frac{\kappa}{2m}$. Since $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^p(\hat{\Omega})$, the sequences $(u_n \star \varphi_\eta)_{n\in\mathbb{N}}$ and $(\nabla_x u_n \star \varphi_\eta)_{n\in\mathbb{N}}$ are both bounded in $\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})$. For the time derivative we just write for all $\psi \in \mathcal{D}(\hat{\Omega}_{\frac{1}{m}})$, using the fact that φ_η is even,

$$\langle \partial_t(u_n \star \varphi_\eta), \psi\rangle = \langle \partial_t u_n, \psi \star \varphi_\eta\rangle.$$

Now (since η is small enough), we have $\psi \star \varphi_\eta \in \mathcal{D}(\hat{\Omega})$ and it is therefore an admissible test-function for the estimate (3.4). Eventually, for any $\psi \in \mathcal{D}(\hat{\Omega}_{\frac{1}{m}})$, we have

$$\begin{aligned}
|\langle \partial_t(u_n \star \varphi_\eta), \psi\rangle| &= |\langle \partial_t u_n, \psi \star \varphi_\eta\rangle| \leq C \sum_{|\alpha|\leq N} \|\partial_x^\alpha(\psi \star \varphi_\eta)\|_{\mathrm{L}^2(\hat{\Omega})} = C \sum_{|\alpha|\leq N} \|\psi \star \partial_x^\alpha \varphi_\eta\|_{\mathrm{L}^2(\hat{\Omega})} = \\
&= C \sum_{|\alpha|\leq N} \left(\int_I \int_{\Omega^t} \left(\int_{\mathbb{R}^d} \psi(t,y) \partial_x^\alpha \varphi_\eta(t, x-y)\, dy \right)^2 dx\, dt \right)^{\frac{1}{2}} \\
&\leq C \sum_{|\alpha|\leq N} \|\partial_x^\alpha \varphi_\eta\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \left(\int_I \int_{\Omega_t} \|\psi(t)\|^2_{\mathrm{L}^1(\Omega^t_{\frac{1}{m}})}\, dx\, dt \right)^{\frac{1}{2}} \\
&\leq C(b-a)^{\frac{1}{2}} (\sup_{t\in I} \lambda_d(\Omega^t))^{\frac{1}{2}} \sum_{|\alpha|\leq N} \|\partial_x^\alpha \varphi_\eta\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \left(\int_I \|\psi(t)\|^2_{\mathrm{L}^1(\Omega^t_{\frac{1}{m}})} \frac{1}{b-a}\, dt \right)^{\frac{1}{2}} \\
&\leq C(b-a)^{\frac{1}{2}} (\sup_{t\in I} \lambda_d(\Omega^t))^{\frac{1}{2}} \sum_{|\alpha|\leq N} \|\partial_x^\alpha \varphi_\eta\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \int_I \|\psi(t)\|_{\mathrm{L}^1(\Omega^t_{\frac{1}{m}})} \frac{1}{b-a}\, dt \\
&\leq \|\psi\|_{\mathrm{L}^1(\hat{\Omega}_{\frac{1}{m}})} \tilde{C} (\sup_{t\in I} \lambda_d(\Omega^t))^{\frac{1}{2}} \sum_{|\alpha|\leq N} \|\partial_x^\alpha \varphi_\eta\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = \|\psi\|_{\mathrm{L}^1(\hat{\Omega}_{\frac{1}{m}})} C_{\varphi_\eta}.
\end{aligned}$$

The constant C_{φ_η} is bounded, because $\sup_{t\in I}\lambda_d(\Omega^t)$ is bounded, since $\hat{\Omega}$ is a bounded set, and φ_η lies in $\mathrm{C}_c^\infty(\mathbb{R}^d)$. This inequality shows that $\langle \partial_t(u_n \star \varphi_\eta), .\rangle$ is linear and bounded functional on a dense subset of $\mathrm{L}^1(\hat{\Omega}_{\frac{1}{m}})$, hence by duality $\partial_t(u_n\star\varphi_\eta) \in \mathrm{L}^\infty(\hat{\Omega}_{\frac{1}{m}})$. Observe that we obtained an estimate for all $n \in \mathbb{N}$, hence the sequence $\partial_t(u_n \star \varphi_\eta)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^\infty(\hat{\Omega}_{\frac{1}{m}})$ by C_{φ_η}. Due to this boundedness, we get that the sequence $(u_n \star \varphi_\eta)_{n\in\mathbb{N}}$ is also bounded in $\mathrm{W}^{1,p}(\hat{\Omega}_{\frac{1}{m}})$, therefore, thanks to *Rellich-Kondachrov* (Theorem 1.2.8), there exists a converging subsequence in $\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})$.
For every $l \geq 2m/\kappa$ we have that the sequence $(u_n \star \varphi_{\frac{1}{l}})_{n\in\mathbb{N}}$ has a converging subsequence in $\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})$ and by diagonal extraction we find a subsequence such that for any $k \in \mathbb{N}$ the sequence $(u_n \star \varphi_{\frac{1}{l_k}})_{n\in\mathbb{N}}$ has a convergent subsequence (without reindexing) in $\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})$. With all this preparation we are now able to prove that $(u_n)_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})$.
Let $\delta > 0$.
Then, thanks to Proposition 3.3.14, there exists $l_k \geq 2m/\kappa$ such that $\|u_n - u_n \star \varphi_{\frac{1}{l_k}}\|_{\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})} \leq \frac{\delta}{3}$ for all $n \in \mathbb{N}$. We are also able to find $n_\delta \in \mathbb{N}$ such that for all $i, j \geq n_\delta$ we have
$\|u_i \star \varphi_{\frac{1}{l_k}} - u_j \star \varphi_{\frac{1}{l_k}}\|_{\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})} \leq \frac{\delta}{3}$, hence for all $i, j \geq n_\delta$

$$\begin{aligned}\|u_i - u_j\|_{\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})} \leq \|u_i - u_i \star \varphi_{\frac{1}{l_k}}\|_{\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})} + \|u_i \star \varphi_{\frac{1}{l_k}} - u_j \star \varphi_{\frac{1}{l_k}}\|_{\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})} +\\ + \|u_j - u_j \star \varphi_{\frac{1}{l_k}}\|_{\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})} \leq \delta\end{aligned}$$

We conclude that $(u_n)_{n\in\mathbb{N}}$ has a subseqeunce that is Cauchy, since $\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})$ is a Banach space, that subsequence converges in $\mathrm{L}^p(\hat{\Omega}_{\frac{1}{m}})$. Since m was arbitrary, we have (thanks to Remark 3.3.15) for each $\varepsilon > 0$ that the sequence $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathrm{L}^p(\hat{\Omega}_\varepsilon)$ and with Proposition 3.3.14 we obtain the desired relativ compactness of $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^p(\hat{\Omega})$. ☐

3.4 A new Notation

From this point on we need to separate functions, which co-domains are $\mathbb{R}$ or $\mathbb{R}^d$. Functions with co-domain $\mathbb{R}^d$ are called *vector fields* and will be written in boldface, functions with co-domain $\mathbb{R}$ are written as usual. Let $\mathbf{u} : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field and let O be a subset of $\mathbb{R}^d$, then the $p-$th power of $\mathbf{u}$ is called integrable, if

$$\int_O |\mathbf{u}(x)|^p \, dx < \infty,$$

where $|.|$ denotes the $p-$norm in $\mathbb{R}^d$. For $p \in [1, \infty)$ the set of all vector fields, whose $p-$th power of its $p-$norm is integrable, is called $\mathrm{L}^p(O)^d$. We notice here, that with our definition we get

$$\mathbf{u} \in \mathrm{L}^p(O)^d \qquad \Leftrightarrow \qquad \mathbf{u}_i \in \mathrm{L}^p(O) \qquad \forall i \in \{1, \dots, d\}.$$

Since we write vector fields bold, we just write $\mathrm{L}^p(O)$ instead of $\mathrm{L}^p(O)^d$ and we will do so for all other spaces defined in this work. For $p = 2$ the space $\mathrm{L}^2(O)^d$ is a Hilbert space. The scalar product of $\mathrm{L}^2(O)^d$ is then

$$(\mathbf{u}, \mathbf{v})_{\mathrm{L}^2(O)} = \int_\Omega \mathbf{u}(x) \cdot \mathbf{v}(x) \, dx,$$

where "$\cdot$" denotes the Euclidean scalar product.

From now on we study vector fields in the space variable, in the sense $\mathbf{u} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. With $\mathbf{u} \in \mathrm{L}^p(\hat{\Omega})$ we mean that $\mathbf{u}(t) \in \mathrm{L}^p(\Omega^t)^d$ for all $t \in I$. If the gradient of $\mathbf{u}$ (only in space!) is in $\mathrm{L}^p(\hat{\Omega})$, we mean $\nabla_x \mathbf{u} \in \mathrm{L}^p(\hat{\Omega})^{d\times d}$ and identify $\mathrm{L}^p(\hat{\Omega})^{d\times d} \hat{=} \mathrm{L}^p(\hat{\Omega})^{d^2}$. Since we can work component-wise, we are able to adapt the theory of L^p −spaces simply to vector fields, which components lie in L^p.

3.5 The Space $\mathrm{L}^2_{\mathrm{div}}(O)$

Before we give the last version of the *Aubin-Lions-Lemma*, we have to define some special *Lebesgue-Spaces.* This spaces contain all vector fields, whose norm is *square-integrable* and whose *divergence* is also *square-integrable.* Let $O \subset \mathbb{R}^d$ be an open and bounded set with *Lipschitz-boundary* and let $\mathbf{u} \in \mathrm{L}^2(O)$ be a sufficiently smooth function, then we define the *divergence of* $\mathbf{u} = (u_1, \dots, u_d)$ as

$$\operatorname{div} \mathbf{u}(x) = \sum_{i=1}^{d} \frac{\partial u_i}{\partial x_i}(x).$$

This subsection relies on section 2.2 of the book [GR86]. We will give a short summary of this theory, focusing on the for us interesting parts. First of all we define the space of all divergence free test functions:

$$\mathcal{D}_{div}(O) \coloneqq \{\boldsymbol{\phi} \in \mathcal{D}(O) : \operatorname{div} \boldsymbol{\phi} = 0\},$$

where O is an arbitrary subset of $\mathbb{R}^d$.
In section 1.2.2 we introduced the trace operator $\gamma : \mathrm{H}^1(O) \to \mathrm{H}^{1/2}(\partial O)$, for O bounded and with a *Lipschitz-Boudary.* The topological dual space of $\mathrm{H}^{1/2}(\partial O)$ is denoted by $\mathrm{H}^{-1/2}(\partial O)$. We recall that in that case, there exists a normal trace operator

$$\begin{aligned} \gamma_n : \mathrm{C}^0(\overline{O}) &\to \mathrm{C}^0(\partial O) \\ \mathbf{v} &\mapsto \mathbf{v} \cdot \mathbf{n}|_{\partial O}, \end{aligned}$$

where $\mathbf{n}$ is the outward unit normal defined on the boundary ∂O.
Before we define $\mathrm{L}^2_{\mathrm{div}}(O)$, we introduce similar spaces, because the properties of $\mathrm{L}^2_{\mathrm{div}}(O)$ will follow nearly directly from the properties of these spaces.

Definition 3.5.1. Let O be an open and bounded subset of $\mathbb{R}^d$ with Lipschitz boundary, then we introduce the following spaces:

$$H(\mathrm{div}, O) \coloneqq \left\{\mathbf{u} \in \mathrm{L}^2(O) : \operatorname{div} \mathbf{u} \in \mathrm{L}^2(O)\right\}$$

we endow the space with the following norm:

$$\|\mathbf{v}\|_{H(\mathrm{div},O)} \coloneqq \left(\|\mathbf{v}\|^2_{\mathrm{L}^2(O)} + \|\operatorname{div} \mathbf{v}\|^2_{\mathrm{L}^2(O)}\right)^{\frac{1}{2}} \qquad \forall v \in H(\mathrm{div}, O).$$

and

$$\begin{aligned} H_0(\mathrm{div}, O) &\coloneqq \overline{\mathcal{D}(O)}^{\|\cdot\|_{H(\mathrm{div},O)}}. \\ \mathrm{L}^2_{\mathrm{div},0}(O) &\coloneqq \{\mathbf{u} \in H_0(\mathrm{div}, O) : \operatorname{div} \mathbf{u} = 0\}. \end{aligned}$$

There are some important and interesting coherences of these spaces, which we sum up in one theorem. For further information we suggest [GR86].

Theorem 3.5.2. *Let O be an open and bounded set with Lipschitz boudary ∂O. Then we have:*

i) *The space $H(\mathrm{div}, O)$ is a linear subspace of $\mathrm{L}^2(O)$. If we endow $H(\mathrm{div}, O)$ with the norm $\|.\|_{H(\mathrm{div},O)}$, it is a Hilbert-space.*

ii) *We have the following densities:*

$$\mathcal{D}_{\mathrm{div}}(O) \subseteq \mathrm{L}^2_{\mathrm{div},0}(O) = \overline{\mathcal{D}_{\mathrm{div}}(O)}^{\|.\|_{H(\mathrm{div},O)}}, \tag{3.10}$$

$$\mathcal{D}(\overline{O}) \subseteq H(\mathrm{div}, O) = \overline{\mathcal{D}(\overline{O})}^{\|.\|_{H(\mathrm{div},O)}}, \tag{3.11}$$

$$H_0(\mathrm{div}, O) = \{\mathbf{u} \in H(\mathrm{div}, O) : \mathbf{u} \cdot \mathbf{n}|_{\partial O} = 0\} \tag{3.12}$$

iii) *The space $H(\mathrm{div}, O)$ embeds continuously into $\mathrm{L}^2(O)$, with the constant $C = 1$.*

iv) *The space $\mathrm{L}^2_{\mathrm{div},0}(O)$ is a closed subspace of $\mathrm{L}^2(O)$, hence we have the decomposition*

$$\mathrm{L}^2(O) = \mathrm{L}^2_{\mathrm{div},0}(O) \oplus \mathrm{L}^2_{\mathrm{div},0}(O)^\perp,$$

where $\mathrm{L}^2_{\mathrm{div},0}(O)^\perp$ denotes the orthogonal complement of $\mathrm{L}^2_{\mathrm{div},0}(O)$ with respect to $(.,.)_{\mathrm{L}^2(O)}$.

v) *The mapping γ_n can be extended by continuity to a linear and continuous mapping, still denoted by γ_n from $H(\mathrm{div}, O)$ into $\mathrm{H}^{-1/2}(\partial O)$. This extension has the following properties:*

$$\gamma_n(H(\mathrm{div}, O)) = \mathrm{H}^{-1/2}(\partial O), \qquad \|\gamma_n\|_{\mathfrak{L}(H(\mathrm{div},O),\mathrm{H}^{-1/2}(\partial O))} = 1, \qquad \ker \gamma_n = H_0(\mathrm{div}, O).$$

Proof.

i) This is clear.

ii) The proof of (3.10) and (3.11) can be found in [GR86], Theorem 2.8 on page 30 and Theorem 2.4 on page 27 respectively. The proof of the equality (3.12) can be found in [GR86] Theorem 2.6 on page 29.

iii) For all $\mathbf{v} \in H(\mathrm{div}, O)$ we have

$$\|\mathbf{v}\|^2_{\mathrm{L}^2(O)} \leq \|\mathbf{v}\|^2_{\mathrm{L}^2(O)} + \|\,\mathrm{div}\,\mathbf{v}\|^2_{\mathrm{L}^2(O)} = \|\mathbf{v}\|^2_{H(\mathrm{div},O)}.$$

iv) Since for all $\mathbf{v} \in \mathrm{L}^2_{\mathrm{div},0}(O)$, $\mathrm{div}\,\mathbf{v} = 0$ holds, we have $\|\mathbf{v}\|_{\mathrm{L}^2(O)} = \|\mathbf{v}\|_{H(\mathrm{div},O)}$.

v) We find the prove of all results in [GR86], Theorem 2.5, Corollary 2.8 and Theorem 2.6.

□

With all this preparation, we are able to introduce the space of all vector fields in $\mathrm{L}^2(O)$, with a vanishing divergence:

$$\mathrm{L}^2_{\mathrm{div}}(O) := \left\{\mathbf{u} \in \mathrm{L}^2(O) : \mathrm{div}\,\mathbf{u} = 0\right\}. \tag{3.13}$$

Observe, that the space $\mathrm{L}^2_{\mathrm{div}}(O)$ does not coincide with $H_0(\mathrm{div}, O)$, because for $\mathbf{u} \in \mathrm{L}^2(O)$ does $\mathrm{div}\,\mathbf{u} = 0$ not imply that $\mathbf{u} \cdot \mathbf{n} = 0$ and otherwise either. We want to list the most important properties of this space.

Corollary 3.5.3. *Let O be an open and bounded set with a Lipschitz-Boudary ∂O. Then we have:*

i) $L^2_{\mathrm{div}}(O)$ *is the closure of* $\mathcal{D}_{\mathrm{div}}(\overline{O})$*, regarding to the* $\|.\|_{L^2(O)}$*-norm.*

ii) $L^2_{\mathrm{div}}(O)$ *is a closed subspace of* $H(\mathrm{div}, O)$ *and it is also a closed subspace in* $L^2(O)$*.*

iii) $L^2_{\mathrm{div},0}(O) = L^2_{\mathrm{div}}(O) \cap H_0(\mathrm{div}, O)$*.*

iv) The restriction of the mapping $\gamma_n : H(\mathrm{div}, O) \to H^{-1/2}(O)$ *on* $L^2_{\mathrm{div}}(O)$ *is still surjective and* $\ker \gamma_n|_{L^2_{\mathrm{div}}(O)} = L^2_{\mathrm{div},0}(O)$*.*

Proof.

i) This follows from $\overline{\mathcal{D}(\overline{O})}^{\|.\|_{H(\mathrm{div},O)}} = H(\mathrm{div}, O)$.

ii) Since the norms $\|.\|_{L^2(O)}$ and $\|.\|_{H(\mathrm{div},O)}$ coincide on $L^2_{\mathrm{div}}(O)$ ii), is clear.

iii) $L^2_{\mathrm{div},0}(O)$ is a subset of $H_0(\mathrm{div}, O)$ and $L^2_{\mathrm{div}}(O)$. Every vector field $\mathbf{u}$ in the Intersection has to fullfill that $\mathbf{u} \in L^2(O)$, $\mathrm{div}\, \mathbf{u} = 0$ and $\mathbf{u} \cdot \mathbf{n} = 0$, hence $\mathbf{u} \in L^2_{\mathrm{div},0}(O)$.

iv) For this point, we adapt the proof of Corollary 2.8 in [GR86]. To show that $\gamma_n(L^2_{\mathrm{div}}(O)) = H^{-1/2}(O)$ we want to find for every $\mu \in H^{-1/2}(O)$ a vector field $\mathbf{u} \in L^2_{\mathrm{div}}(O)$ such that:

$$\mathbf{u} \cdot \mathbf{n} = \mu \text{ on } \partial O$$

We want to solve the Neumann Laplacian problem: Find $\phi \in H^1(O)$ such that

$$\Delta\phi = 0 \text{ in } O,$$
$$\partial_{\mathbf{n}}\phi = \mu \text{ on } \partial O.$$

This problem has, up to a constant, a unique solution in $H^1(O)$. If we set $\nabla\phi = \mathbf{u}$, then $\mathbf{u} \in L^2_{\mathrm{div}}(O)$ and $\mathbf{u} \cdot \mathbf{n} = \mu$.
For the $\ker \gamma_n|_{L^2_{\mathrm{div}}(O)}$ we have the obvious equality

$$\ker \gamma_n|_{L^2_{\mathrm{div}}(O)} = \ker \gamma_n \cap L^2_{\mathrm{div}}(O) = H_0(\mathrm{div}, O) \cap L^2_{\mathrm{div}}(O) = L^2_{\mathrm{div},0}(O)$$

□

Remark 3.5.4. We want to adapt the previous notations for solenoidal vector fields depending on both time and space. Let O be an arbitrary open set of $\mathbb{R} \times \mathbb{R}^d$ (denote the first component with t and the last components with x). If there is no ambiguity on the time variable, we denote for the rest of this paper the space $\mathcal{D}_{\mathrm{div}}(\mathbb{R} \times \mathbb{R}^d)$ as the set of all test functions $\boldsymbol{\varphi} \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$, such that for all $t \in \mathbb{R}$, the functions $\boldsymbol{\varphi}(t) : x \mapsto \boldsymbol{\varphi}(t, x)$ are in $\mathcal{D}_{\mathrm{div}}(\mathbb{R}^d)$. $\mathcal{D}_{\mathrm{div}}(O)$ denotes the subspace of all test functions of $\mathcal{D}_{\mathrm{div}}(\mathbb{R} \times \mathbb{R}^d)$ having a compact support in O, while $L^2_{\mathrm{div}}(O)$ and $L^2_{\mathrm{div},0}(O)$ are respectively the closures of $\mathcal{D}_{\mathrm{div}}(\overline{O})$ and $\mathcal{D}_{\mathrm{div}}(O)$ in $L^2(O)$. With this notation we recover the definition we had without the time variable. ■

3.6 Second Statement

Differently to the other two sections, where we present the theorem at the beginning of the section, we have to do some work before we can state the last variant of the *Aubin-Lions-Lemma.* We start with a classical result from functional analysis.

Let H be an arbitrary Hilbert-space over $\mathbb{R}$ with scalar product $(.,.)_H$ and denote with $\|.\|_H$ the induced norm. The topological dual space of H will be denoted as H', which is endowed with the operatornorm $\|f\|_{H'} = \sup_{x\in H,\|x\|_H\le 1} |f(x)|$. We know ,that for every $x \in H$ the mapping $f_x : y \mapsto (x, y)_H$ is a bounded linear functional on H. In fact we have a much stronger result:

Lemma 3.6.1. *Let H be a Hilbert-space, then the mapping*

$$\begin{aligned} \Psi : H &\to H' \\ x &\mapsto f_x \end{aligned}$$

is an isometric ($\|f_x\|_{H'} = \|x\|_H$) linear bijection.

Since every Hilbert-space is *Hausdorff* (two different points in H can be separated by two disjoint open sets), it is enough to define a continuous function on a dense subset of H. For the rest of this section, if it is not otherwise stated, let O be an open and bounded subset of $\mathbb{R}^d$ with Lipschitz boundary ∂O. With Lemma 3.6.1 we obtain the *duality formula* for $\mathrm{L}^2(O)$ and its dense subset $\mathcal{D}(O)$

$$\|\mathbf{u}\|_{\mathrm{L}^2(O)} = \sup_{\boldsymbol{\varphi}\in\mathcal{D}(O),\|\boldsymbol{\varphi}\|_{\mathrm{L}^2(O)}\le 1} (\mathbf{u}, \boldsymbol{\varphi})_{\mathrm{L}^2(O)}. \tag{3.14}$$

We cannot expect a duality formula as (3.14), when testing only against divergence free test functions. As a matter of fact, there is a dual estimate of the same flavour for $\mathrm{L}^2_{\mathrm{div}}(O)$, but one has to take into account the normal trace.

Lemma 3.6.2. *Denote by C_O the Poincaré-Wirtinger constant of O. For all $\mathbf{u} \in \mathrm{L}^2_{\mathrm{div}}(O)$ one has*

$$\|\mathbf{u}\|_{\mathrm{L}^2(O)} \le \sup_{\boldsymbol{\varphi}\in\mathcal{D}_{\mathrm{div}}(O),\|\boldsymbol{\varphi}\|_{\mathrm{L}^2(O)}\le 1} (\mathbf{u}, \boldsymbol{\varphi})_{\mathrm{L}^2(O)} + (1 + C_O)\|\gamma_n \mathbf{u}\|_{\mathrm{H}^{-1/2}(\partial O)}. \tag{3.15}$$

Proof. It is sufficient to prove, that (3.15) holds for all $\mathbf{w} \in \mathcal{D}_{\mathrm{div}}(\overline{O})$, because $\mathcal{D}_{\mathrm{div}}(\overline{O})$ is dense in $\mathrm{L}^2_{\mathrm{div}}(O)$, hence we find for each $\mathbf{u} \in \mathrm{L}^2_{\mathrm{div}}(O)$ a sequence $(\mathbf{w}_n)_{n\in\mathbb{N}}$ in $\mathcal{D}_{\mathrm{div}}(\overline{O})$ such that $\mathbf{w}_n \to \mathbf{u}$ in $\mathrm{L}^2_{\mathrm{div}}(O)$. Since $\mathrm{L}^2_{\mathrm{div}}(O)$ is a closed subspace of $\mathrm{L}^2(O)$, we have especially that $(\mathbf{w}_n)_{n\in\mathbb{N}}$ converges to $\mathbf{u}$ in $\mathrm{L}^2(O)$.
Consider the orthogonal projection $\mathbb{P} : \mathrm{L}^2_{\mathrm{div}}(O) \to \mathrm{L}^2_{\mathrm{div},0}(O)$ and let $\mathbf{w} \in \mathcal{D}_{\mathrm{div}}(\overline{O})$, then we have

$$\|\mathbf{w}\|_{\mathrm{L}^2(O)} \le \|\mathbb{P}\mathbf{w}\|_{\mathrm{L}^2(O)} + \|\mathbf{w} - \mathbb{P}\mathbf{w}\|_{\mathrm{L}^2(O)}.$$

Our aim is to estimate both terms on the r.h.s. Let us start with $\|\mathbb{P}\mathbf{w}\|_{\mathrm{L}^2(O)}$. Since $\mathcal{D}_{\mathrm{div}}(O)$ is dense in $\mathrm{L}^2_{\mathrm{div},0}(O)$, $\mathrm{L}^2(O)$ is a Hilbert space and $\mathbb{P}\boldsymbol{\varphi} = \boldsymbol{\varphi}$ for all $\boldsymbol{\varphi} \in \mathcal{D}_{\mathrm{div}}(O)$, we obtain that

$$\begin{aligned} \|\mathbb{P}\mathbf{w}\|_{\mathrm{L}^2(O)} = \|\mathbb{P}\mathbf{w}\|_{\mathrm{L}^2_{\mathrm{div}}(O)} &= \sup_{\boldsymbol{\varphi}\in\mathcal{D}_{\mathrm{div}}(O),\|\boldsymbol{\varphi}\|_{\mathrm{L}^2}\le 1} \int_O \mathbb{P}\mathbf{w}(x)\cdot\boldsymbol{\varphi}(x)\,dx \\ &= \sup_{\boldsymbol{\varphi}\in\mathcal{D}_{\mathrm{div}}(O),\|\boldsymbol{\varphi}\|_{\mathrm{L}^2}\le 1} \int_O \mathbf{w}(x)\cdot\mathbb{P}\boldsymbol{\varphi}(x)\,dx \\ &= \sup_{\boldsymbol{\varphi}\in\mathcal{D}_{\mathrm{div}}(O),\|\boldsymbol{\varphi}\|_{\mathrm{L}^2}\le 1} (\mathbf{w}, \boldsymbol{\varphi})_{\mathrm{L}^2(O)}, \end{aligned}$$

It remains to estimate the second term $\|\mathbf{w} - \mathbb{P}\mathbf{w}\|_{\mathrm{L}^2(O)}$. If $\mathbf{w} \in \mathrm{L}^2_{\mathrm{div},0}(O)$, the estimate (3.15) holds, because $\gamma_n \mathbf{w} = 0$. Let $\mathbf{w} - \mathbb{P}\mathbf{w} \neq 0$, then one can solve the Neumann-Laplacian problem

$$\begin{aligned} \Delta\xi &= 0, \text{ on } O \\ \partial_{\mathbf{n}}\xi &= \gamma_n \mathbf{w} \text{ on } \partial O, \end{aligned}$$

and the initial condition

$$\int_O \xi = 0,$$

where $\Delta\xi = \operatorname{div}\nabla\xi$ and $\partial_{\mathbf{n}}\xi = \nabla\xi \cdot \mathbf{n}$. This boundary problem is well posed providing, a unique solution $v \in \mathrm{H}^1(O)$, with mean value $(v)_O = 0$. The variational formulation gives directly, $\|\nabla v\|^2_{\mathrm{L}^2(O)} \leq \|\gamma_n \mathbf{w}\|_{\mathrm{H}^{-1/2}(\partial O)} \|\gamma v\|_{\mathrm{H}^{1/2}(\partial O)}$, whence we obtain with the trace-theorem (Theorem 1.2.13) and estimate (1.3):

$$\|\nabla v\|^2_{\mathrm{L}^2(O)} \leq \|\gamma_n \mathbf{w}\|_{\mathrm{H}^{-1/2}(\partial O)} \|v\|_{\mathrm{H}^1(O)} \leq \sqrt{1 + C_O}\|\gamma_n \mathbf{w}\|_{\mathrm{H}^{-1/2}(\partial O)} \|\nabla v\|_{\mathrm{L}^2(O)}.$$

so that eventually $\|\nabla v\|_{\mathrm{L}^2(O)} \leq \sqrt{1 + C_O}\|\gamma_n \mathbf{w}\|_{\mathrm{H}^{-1/2}(O)}$. Obviously $\nabla v \in \mathrm{L}^2_{\mathrm{div},0}(O)^{\perp}$, because we have that $\mathbf{w} - \mathbb{P}\mathbf{w} \neq 0$ and thereby we obtain $0 \neq \gamma_n \mathbf{w} = \mathbf{n} \cdot \nabla v$. Since $\operatorname{div}(\mathbf{w} - \nabla v) = 0$ and $\mathbf{n} \cdot (\mathbf{w} - \nabla v) = \gamma_n \mathbf{w} - \partial_{\mathbf{n}} v = 0$, we get that $\mathbf{w} - \nabla v \in \mathrm{L}^2_{\mathrm{div},0}(O)$. With this results and

$$(\mathrm{Id}_{\mathrm{L}^2_{\mathrm{div}}(O)} - \mathbb{P})(\mathbf{w} - \nabla v) = 0 \qquad \Leftrightarrow \qquad (\mathrm{Id}_{\mathrm{L}^2_{\mathrm{div}}(O)} - \mathbb{P})\mathbf{w} = (\mathrm{Id}_{\mathrm{L}^2_{\mathrm{div}}(O)} - \mathbb{P})\nabla v,$$

we conclude that $(\mathrm{Id}_{\mathrm{L}^2_{\mathrm{div}}(O)} - \mathbb{P})\mathbf{w} = \nabla v$. At the end we obtain

$$\|\mathbf{w} - \mathbb{P}\mathbf{w}\|_{\mathrm{L}^2(O)} = \|\nabla v\|_{\mathrm{L}^2(O)} \leq \sqrt{1 + C_O}\|\gamma_n \mathbf{w}\|_{\mathrm{H}^{-1/2}(O)}.$$

The desired estimate (3.15) follows now from the density of $\mathcal{D}_{\mathrm{div}}(\overline{O})$ in $\mathrm{L}^2_{\mathrm{div}}(O)$. □

Remark 3.6.3. We see that with the continuity of γ_n (γ_n is linear and bounded), the r.h.s of (3.15) in Lemma 3.6.2 defines a norm, which is equivalent to the $\|.\|_{\mathrm{L}^2(O)}$ norm on $\mathrm{L}^2_{\mathrm{div}}(O)$. Therefore (3.15) is a generalization of (3.14). ■

Now let us get back to our setting defined in section 3.2. We want to adapt the spaces and results of this subsection to our setting. From now on, we understand under the divergence of a vector field, the divergence in the last d coordinates. All the results can of course be adapted to the case of divergence free (in the space variable x) vector fields on $\hat{\Omega}$. Recalling Remark 3.5.4 we define the following spaces:

$$\mathcal{D}_{\mathrm{div}}(\hat{\Omega}) := \left\{ \boldsymbol{\varphi} \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d) : \boldsymbol{\varphi}(t) \in \mathcal{D}_{\mathrm{div}}(\mathbb{R}^d)\ \forall t \in \mathbb{R},\ \operatorname{supp}\boldsymbol{\varphi} \subseteq \hat{\Omega} \right\}, \tag{3.16}$$

$$\mathrm{L}^2_{\mathrm{div}}(\hat{\Omega}) := \overline{\mathcal{D}_{\mathrm{div}}(\overline{\hat{\Omega}})} = \left\{ \mathbf{u} \in \mathrm{L}^2(\hat{\Omega}) : \operatorname{div}\mathbf{u}(t) = 0,\ \forall t \in (a,b) \right\} \tag{3.17}$$

For $t \in [a,b]$, Ω^t is a bounded set in $\mathbb{R}^d$ with Lipschitz boundary, hence there exists the outward unit normal n_t of Ω^t and the normal trace operator $\gamma_{n_t} : \mathrm{L}^2_{\mathrm{div}}(\Omega^t) \to \mathrm{H}^{-1/2}(\Omega^t)$. Thanks to Theorem 3.5.2 we know that the operatornorm of γ_{n_t} is not greater than 1, so that one has, for all $\boldsymbol{\varphi} \in \mathcal{D}_{\mathrm{div}}(\mathbb{R} \times \mathbb{R}^d)$,

$$\int_I \|\gamma_{n_t}\boldsymbol{\varphi}(t)\|_{\mathrm{H}^{-1/2}(\Omega^t)}\, dt \leq \int_I \|\boldsymbol{\varphi}(t)\|_{\mathrm{L}^2(\Omega^t)}\, dt = \|\boldsymbol{\varphi}\|_{\mathrm{L}^2(\hat{\Omega})}.$$

With this information we are able to define a normal trace operator on $\hat{\Omega}$. We defined $\partial\hat{\Omega}$ different to the boundary of $\hat{\Omega}$ in $\mathbb{R}^{d+1}$, therefore we have to define a slightly different trace operator. Let us define $\hat{n}$ as the outward normal of $\partial\hat{\Omega}$ such that $\hat{n}|_{\Omega^t} = n_t$ for all $t \in I$. This means, that $\hat{n}$ is orthogonal to $\partial\Omega^t$ in $\mathbb{R}^d$ and not to the boundary of $\hat{\Omega}$ in $\mathbb{R}^{d+1}$.
This allows us to define the normal trace operator on $\mathrm{L}^2_{\mathrm{div}}(\hat{\Omega})$, which lies in the space denoted $\mathrm{H}_x^{-1/2}(\partial\hat{\Omega})$, defined as the completion of $\mathrm{C}^\infty(\partial\hat{\Omega})$ for the norm

$$\|\psi\|_{\mathrm{H}_x^{-1/2}(\partial\hat{\Omega})} := \left(\int_I \|\psi(t)\|^2_{\mathrm{H}^{-1/2}(\partial\Omega^t)}\,dt\right)^{1/2}.$$

This normal trace operator will be denoted by $\gamma_{\hat{n}} : \mathrm{L}^2_{\mathrm{div}}(\hat{\Omega}) \to \mathrm{H}_x^{-1/2}(\partial\hat{\Omega})$. Let $\mathbf{u} \in \mathrm{L}^2_{\mathrm{div}}(\hat{\Omega})$ and $\phi \in \mathrm{H}^{1/2}(\partial\hat{\Omega})$ then the expression $\gamma_{\hat{n}}\mathbf{u}$ has to be understood as

$$\langle \gamma_{\hat{n}}\mathbf{u}, \phi\rangle_{\mathrm{H}^{1/2}(\partial\hat{\Omega})} = \int_{\partial\hat{\Omega}} \gamma_{\hat{n}}\mathbf{u}(t,x)\phi(t,x)\,d(t,x) = \int_I\int_{\partial\Omega^t} (n_t \cdot \mathbf{u}(t,x))\phi(t,x)\,dx\,dt.$$

With this new normal trace operator we are able to define the space $\mathrm{L}^2_{\mathrm{div},0}(\hat{\Omega})$, which is the kern of the map $\gamma_{\hat{n}}$.

$$\mathrm{L}^2_{\mathrm{div},0}(\hat{\Omega}) := \overline{\mathcal{D}_{\mathrm{div}}(\hat{\Omega})} = \left\{\mathbf{u} \in \mathrm{L}^2_{\mathrm{div}}(\hat{\Omega}) : \gamma_{\hat{n}}\mathbf{u} = 0\right\}$$

All this preparation allows us to formulate our last version of the Aubin-Lions Lemma.

Theorem 3.6.4. *Let $\hat{\Omega}$ fulfill the assumptions* **[C1]-[C3]**. *Consider a sequence $(\mathbf{u}_n)_{n\in\mathbb{N}}$ that lives in $\mathrm{L}^2_{\mathrm{div}}(\hat{\Omega})$. Assume that $(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$ and that the two sequences $(\mathbf{u}_n)_{n\in\mathbb{N}}$ and $(\nabla_{\mathbf{x}}\mathbf{u}_n)_{n\in\mathbb{N}}$ are bounded in $\mathrm{L}^2(\hat{\Omega})$. Assume furthermore that the non-cylindrical analogue of the normal trace operator fulfills*

$$\gamma_{\hat{n}}\mathbf{u}_n = 0, \tag{3.18}$$

and that there exists a constant $C > 0$ and an integer $N > 0$ such that for all divergence- free test functions $\boldsymbol{\psi}$

$$|\langle \partial_t \mathbf{u}_n, \boldsymbol{\psi}\rangle| \le C \sum_{|\alpha|\le N} \|\partial^\alpha_{\mathbf{x}}\boldsymbol{\psi}\|_{\mathrm{L}^2(\hat{\Omega})}. \tag{3.19}$$

Then the sequence $(\mathbf{u}_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathrm{L}^p(\hat{\Omega})$.

Observe that the condition 3.18 is a boundary condition, which is very important to obtain the following result

Proposition 3.6.5. *Let $(\mathbf{u}_n)_{n\in\mathbb{N}}$ fulfill the assumptions of Theorem 3.6.4, then we have for all $n \in \mathbb{N}$, that* $\operatorname{div} \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n \in \mathrm{L}^2(\mathbb{R};\mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d))$.

Proof. If a function $\mathbf{v} \in \mathrm{L}^2(\mathbb{R}\times\mathbb{R}^d)$ has also $\operatorname{div}\mathbf{v} \in \mathrm{L}^2(\mathbb{R}\times\mathbb{R}^d)$ (in the sense that $\operatorname{div}\mathbf{v}(t) \in \mathrm{L}^2(\mathbb{R}^d)$ for all $t \in \mathbb{R}$), then we have for all $\varphi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^d)$ the following:

$$\begin{aligned}\langle \mathbf{v}, \nabla_x\varphi\rangle_{\mathrm{L}^2} &= \int_{\mathbb{R}\times\mathbb{R}^d} \mathbf{v}\cdot\nabla_x\varphi = \int_{\operatorname{supp}\varphi} \mathbf{v}\cdot\nabla_x\varphi\\ &= \int_{\partial\operatorname{supp}\varphi} \varphi\mathbf{v}\cdot\mathbf{n}_{\partial\operatorname{supp}\varphi} - \int_{\operatorname{supp}\varphi} \operatorname{div}\mathbf{v}\varphi\\ &= -\langle \operatorname{div}\mathbf{v}, \varphi\rangle_{\mathrm{L}^2}\end{aligned}$$

Thanks to this, we can say a function $\mathbf{f} \in \mathrm{L}^2(\mathbb{R} \times \mathbb{R}^d)$ has a weak divergence, if there exists a function $g \in \mathrm{L}^2(\mathbb{R} \times \mathbb{R}^d)$ such that

$$\langle \mathbf{f}, \nabla_x \varphi \rangle_{\mathrm{L}^2} = -\langle g, \varphi \rangle_{\mathrm{L}^2}, \qquad \forall \varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d).$$

Now let $n \in \mathbb{N}$ be arbitrary. We have clearly that $\mathbb{1}_{\hat{\Omega}} \mathbf{u}_n \in \mathrm{L}^2(\mathbb{R} \times \mathbb{R}^d)$. Since $\operatorname{div} \mathbf{u}_n = 0$ on $\hat{\Omega}$ and $\gamma_{\hat{n}} \mathbf{u}_n = 0$, we obtain, with Green's formula, for all $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$

$$\begin{aligned}\langle \mathbb{1}_{\hat{\Omega}} \mathbf{u}_n, \nabla_x \varphi \rangle_{\mathrm{L}^2} &= \int_{\mathbb{R} \times \mathbb{R}^d} \mathbb{1}_{\hat{\Omega}} \mathbf{u}_n \cdot \nabla_x \varphi = \int_{\hat{\Omega}} \mathbf{u}_n \cdot \nabla_x \varphi \\ &= \int_{\partial \hat{\Omega}} \gamma_{\hat{n}}(\mathbf{u}_n) \varphi - \int_{\hat{\Omega}} \varphi \operatorname{div} \mathbf{u}_n = 0 = -\langle 0, \varphi \rangle_{\mathrm{L}^2}.\end{aligned}$$

Therefore we have $\operatorname{div} \mathbf{u}_n = 0$ and hence $\mathbf{u}_n \in \mathrm{L}^2(\mathbb{R}; \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d))$ for all $n \in \mathbb{N}$.

□

At the end of this section we give a version of Lemma 3.6.2 for the interior sets $\hat{\Omega}_\delta$ for $\delta \in [0, \gamma]$, where gamma is the constant of Proposition 3.3.4. Since Ω_δ are Lipschitz domains for $\delta \in [0, \gamma]$ (thanks to Proposition 3.3.4), we get that the diffeomorphic domains $\vartheta_t(\Omega_\delta)$ have also Lipschitz boundary. Hence we can apply the previous considerations for $\hat{\Omega}_\delta$ and define, in the same way as before, the normal trace operator $\gamma_{\hat{n},\delta} : \mathrm{L}^2_{\mathrm{div}}(\hat{\Omega}_\delta) \to \mathrm{H}_x^{-1/2}(\partial \hat{\Omega}_\delta)$.

Lemma 3.6.6. *Let $\hat{\Omega}$ fulfill the assumptions* **[C1]-[C3]**. *Recall the definition of γ in Proposition 3.3.4. For all $\delta \in [0, \gamma]$ and all $\mathbf{u} \in \mathrm{L}^2_{\mathrm{div}}(\hat{\Omega}_\delta)$*

$$\|\mathbf{u}\|_{\mathrm{L}^2(\hat{\Omega}_\delta)} \leq C_I \sup_{\boldsymbol{\psi} \in \mathcal{D}_{\mathrm{div}}(\hat{\Omega}_\delta), \|\boldsymbol{\psi}\|_{\mathrm{L}^2(\hat{\Omega}_\delta)} \leq 1} (\mathbf{u}, \boldsymbol{\psi})_{\mathrm{L}^2(\hat{\Omega})} + \ (C^{\Theta}_{\Omega,\gamma} + 1) \|\gamma_{\hat{n},\delta} \mathbf{u}\|_{\mathrm{H}_x^{-1/2}(\partial \hat{\Omega}_\delta)},$$

where C_I is a constant depending only on the length of the interval $[a, b]$ and $C^{\Theta}_{\Omega,\gamma}$ is the constant of Proposition 3.3.11.

Proof. Since the sets $\vartheta_t(\Omega_\delta)$ are all Lipschitz domains with a common Lipschitz constant $C^{\Theta}_{\Omega,\gamma}$ (in the sense of Proposition 3.3.11), we can apply Lemma 3.6.2 and obtain for all $\mathbf{u} \in \mathrm{L}^2_{\mathrm{div}}(\hat{\Omega}_\delta)$

$$\begin{aligned}\|\mathbf{u}(t)\|_{\mathrm{L}^2(\vartheta_t(\Omega_\delta))} \leq &\sup_{\boldsymbol{\psi} \in \mathcal{D}_{\mathrm{div}}(\hat{\Omega}_\delta), \|\boldsymbol{\psi}\|_{\mathrm{L}^2(\hat{\Omega})} \leq 1} (\mathbf{u}(t), \boldsymbol{\psi}(t))_{\mathrm{L}^2(\vartheta_t(\Omega_\delta))} \\ &+ \ (C^{\Theta}_{\Omega,\gamma} + 1) \|\gamma_{n_t} \mathbf{u}(t)\|_{\mathrm{H}^{-1/2}(\partial \vartheta_t(\Omega_\delta))}.\end{aligned}$$

All terms in the upper estimate are greater or equal zero, hence we can integrate in time on both sides and the estimate will still hold. With Jensen's inequality we get

$$\begin{aligned}\|\mathbf{u}\|_{\mathrm{L}^2(\hat{\Omega})} &= \sqrt{b-a} \left(\frac{1}{b-a} \int_I \|\mathbf{u}(t)\|^2_{\mathrm{L}^2(\vartheta_t(\Omega_\delta))} \, dt \right)^{(1/2)} \\ &\leq \frac{\sqrt{b-a}}{b-a} \int_I \|\mathbf{u}(t)\|_{\mathrm{L}^2(\vartheta_t(\Omega_\delta))} \, dt \\ &\leq \frac{\sqrt{b-a}}{b-a} \int_I \sup_{\boldsymbol{\psi} \in \mathcal{D}_{\mathrm{div}}(\hat{\Omega}_\delta), \|\boldsymbol{\psi}\|_{\mathrm{L}^2(\hat{\Omega})} \leq 1} (\mathbf{u}(t), \boldsymbol{\psi}(t))_{\mathrm{L}^2(\vartheta_t(\Omega_\delta))} \, dt \\ &+ (C^{\Theta}_{\Omega,\gamma} + 1) \sqrt{b-a} \left(\left(\frac{1}{b-a} \int_I \|\gamma_{n_t} \mathbf{u}(t)\|_{\mathrm{H}^{-1/2}(\partial \vartheta_t(\Omega_\delta))} \, dt \right)^2 \right)^{(1/2)} \\ &\leq \frac{\sqrt{b-a}}{b-a} \sup_{\boldsymbol{\psi} \in \mathcal{D}_{\mathrm{div}}(\hat{\Omega}_\delta), \|\boldsymbol{\psi}\|_{\mathrm{L}^2(\hat{\Omega})} \leq 1} (\mathbf{u}, \boldsymbol{\psi})_{\mathrm{L}^2(\hat{\Omega}_\delta)} + \ (C^{\Theta}_{\Omega,\gamma} + 1) \|\gamma_{\hat{n},\delta} \mathbf{u}\|_{\mathrm{H}_x^{-1/2}(\partial \hat{\Omega}_\delta)}\end{aligned}$$

□

3.6.1 Proof of Theorem 3.6.4

The main idea of the proof of Theorem 3.6.4 is to show that the sequence $(\mathbf{u}_n)_{n\in\mathbb{N}}$ fullfills some equi-continuity with respect to the $\mathrm{L}^2(\hat{\Omega})$ norm. This statement will be clarified by the next theorem, which can be found in [Bre10] on page 111.

Theorem 3.6.7. (Kolmogorov-M. Riez-Fréchet). *Let $\mathcal{F}$ be a bounded set in $\mathrm{L}^p(\mathbb{R}^d)$ with $p \in [1, \infty)$. Assume that*

$$\lim_{|h|\to 0} \|\tau_h f - f\|_{\mathrm{L}^p(\mathbb{R}^d)} = 0 \qquad \textit{uniformly in } f \in \mathcal{F}, \tag{3.20}$$

i.e., $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\|\tau_h f - f\|_{\mathrm{L}^p(\mathbb{R}^d)} < \varepsilon$ $\forall f \in \mathcal{F}$, $\forall h \in \mathbb{R}^d$ with $|h| < \delta$. Then the closure of $\mathcal{F}|_\Omega$ is compact for any measurable set $\Omega \subset \mathbb{R}^d$ with finite measure.

Proof. See [Bre10] on page 111. □

Remark 3.6.8. Let Ω be a bounded set in $\mathbb{R}^d$, then the function $\tau_h f$, for $f \in \mathrm{L}^p(\Omega)$, is maybe not well defined, hence we need a slight variation of Theorem 3.6.7. Since $f \in \mathrm{L}^p(\Omega)$, the function $\mathbb{1}_\Omega f$ lies in $\mathrm{L}^p(\mathbb{R}^d)$. When trying to establish that a family $\mathcal{F} \in \mathrm{L}^p(\Omega)$ has compact closure in $\mathrm{L}^p(\Omega)$ it is convenient to extend all functions $f \in \mathcal{F}$ to $\mathbb{1}_\Omega f$ and apply Theorem 3.6.7 to the set $\mathbb{1}_\Omega \mathcal{F} \subset \mathrm{L}^p(\mathbb{R}^d)$. ■

Proof of Theorem 3.6.4. The main idea of this proof is to apply Theorem 3.6.7. We want to show that the sequence $(\mathbf{u}_n)_{n\in\mathbb{N}}$ fullfills the equi-continuity condition (3.20). In order to use the shift operator correctly, we have to work, according to Remark 3.6.8, with the sequence of extensions $(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n)_{n\in\mathbb{N}}$. Our aim is to show that

$$\lim_{(s,y)\to 0} \|\tau_{(s,y)} \mathbb{1}_{\hat{\Omega}} \mathbf{u}_n - \mathbb{1}_{\hat{\Omega}} \mathbf{u}_n\|_{\mathrm{L}^2(\mathbb{R}\times\mathbb{R}^d)} = 0 \qquad \text{uniformly in } n. \tag{3.21}$$

To show (3.21), it is sufficient to prove the following 3 statements.

1.) *There exists a constant $r > 2$ such that the sequence $(\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^r(\hat{\Omega})$.*

2.) *We have for all compact subsets $K \subset \hat{\Omega}$*

$$\lim_{(0,y)\to 0} \|\tau_{(0,y)}(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n) - \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|_{\mathrm{L}^2(K)} = 0, \textit{ uniformly in } n. \tag{3.22}$$

3.) *We have for all compact subsets $K \subset \hat{\Omega}$*

$$\lim_{(s,0)\to 0} \|\tau_{(s,0)}(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n) - \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|_{\mathrm{L}^2(K)} = 0, \textit{ uniformly in } n. \tag{3.23}$$

Point 2.) and 3.) ist just another formulation to say, that the sequences converge in $\mathrm{L}^2_{loc}(\hat{\Omega})$. After we have proven this three statements we can conclude the prove (if the reader wants to jump directly to the conclusion, he can read the proves of these statements later).

1.

For $d > 2$ we set $q = 2^\star$ and for $d \leq 2$ we set $q = 3$. In both cases we have that $\mathrm{H}^1(\Omega^t)$ embeds continuously into $\mathrm{L}^q(\Omega^t)$ for all $t \in I$ and according to point 2 in Facts 3.2.1, there exists in both cases a constant $C \in \mathbb{R}$ such that for all $\mathbf{v} \in \mathrm{H}^1(\Omega^t)$

$$\|\mathbf{v}\|_{\mathrm{L}^q(\Omega^t)} \leq C\|\mathbf{v}\|_{\mathrm{H}^1(\Omega^t)} \qquad \forall t \in I.$$

This means we can say with no loss of generality, that there exists some $q > 2$ and $C \in \mathbb{R}$, such that for all $n \in \mathbb{N}$

$$\|\mathbf{u}_n\|_{\mathrm{L}^q(\Omega^t)} \leq C\|\mathbf{u}_n\|_{\mathrm{H}^1(\Omega^t)} \qquad \forall t \in I.$$

Since $(\mathbf{u}_n)_{n\in\mathbb{N}}$ and $(\nabla_x \mathbf{u}_n)_{n\in\mathbb{N}}$ are both bounded in $\mathrm{L}^2(\hat{\Omega})$ we obtain for all $n \in \mathbb{N}$

$$\begin{aligned}
\int_I \|\mathbf{u}_n(t)\|^2_{\mathrm{L}^q(\Omega^t)}\,dt &\leq \int_I C^2\|\mathbf{u}_n(t)\|^2_{\mathrm{H}^1(\Omega^t)}\,dt \\
&= C^2\int_I \|\mathbf{u}_n(t)\|^2_{\mathrm{L}^2(\Omega^t)}\,dt + C^2\int_I \|\nabla_x \mathbf{u}_n(t)\|^2_{\mathrm{L}^2(\Omega^t)}\,dt \\
&= C^2(\|\mathbf{u}_n\|^2_{\mathrm{L}^2(\hat{\Omega})} + \|\nabla_x \mathbf{u}_n\|^2_{\mathrm{L}^2(\hat{\Omega})}) < \infty.
\end{aligned}$$

Since this estimate holds for all n in $\mathbb{N}$ we have for some $q > 2$

$$\sup_{n\in\mathbb{N}} \int_I \|\mathbf{u}_n(t)\|^2_{\mathrm{L}^q(\Omega^t)}\,dt < \infty. \tag{3.24}$$

Now for all $r \in (2, q)$ we find $\beta \in (0, 1)$ such that $1/r = \beta/q + (1-\beta)/2$. It is possible to find $r \in (2, q)$, so that $r\beta \in [1, 2]$. If we interpret $\|\mathbf{u}_n(t)\|_{\mathrm{L}^q(\Omega^t)}$ as a function from I to $\mathbb{R}$, it lays, thanks to (3.24), in $\mathrm{L}^2(I)$ for all $n \in \mathbb{N}$. Since I is a bounded interval, we have the continuous embedding of $\mathrm{L}^2(I)$ in $\mathrm{L}^{r\beta}(I)$ and therefore we obtain the existence of a constant $\tilde{C}$ such that

$$\int_I \|\mathbf{u}_n\|^{r\beta}_{\mathrm{L}^q(\Omega^t)}\,dt \leq \tilde{C}\int_I \|\mathbf{u}_n\|^2_{\mathrm{L}^q(\Omega^t)}\,dt \qquad \forall n \in \mathbb{N}$$

With help of the Lyapunov inequality (Theorem 1.1.4) and the condition that the sequence $(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^\infty(\mathbb{R}; \mathrm{L}^2(\mathbb{R}^d))$, we get for all n in $\mathbb{N}$

$$\begin{aligned}
\|\mathbf{u}_n\|_{\mathrm{L}^r(\hat{\Omega})} &= \int_I \|\mathbf{u}_n(t)\|^r_{\mathrm{L}^r(\Omega^t)}\,dt \\
&\leq \int_I \|\mathbf{u}_n(t)\|^{r\beta}_{\mathrm{L}^q(\Omega^t)}\|\mathbf{u}_n\|^{r(1-\beta)}_{\mathrm{L}^2(\Omega^t)}\,dt \\
&\leq \|\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|^{r(1-\beta)}_{\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))}\int_I \|\mathbf{u}_n(t)\|^{r\beta}_{\mathrm{L}^q(\Omega^t)}\,dt \\
&\leq \|\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|^{r(1-\beta)}_{\mathrm{L}^\infty(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))}\tilde{C}\sup_{n\in\mathbb{N}}\int_I \|\mathbf{u}_n(t)\|^2_{\mathrm{L}^q(\Omega^t)}\,dt < \infty,
\end{aligned}$$

hence $(\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^r(\hat{\Omega})$ for some $r > 2$.

2.

Let K be an arbitrary compact subset of $\hat{\Omega}$. Then there exists $\delta_K > 0$ such that $d(K, \partial\hat{\Omega}) > \delta_K$. Let $(y_m)_{m\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d$, which converges to 0 for m to infinity. Then there exists m_0 such that for all $m > m_0$ we have that $|(0, y_m)| < \delta_K$. For all these m we have that $(K - y_m) \subset \hat{\Omega}$. Therefore we have for all $m > m_0$

$$\|\tau_{(0,y_m)}\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n - \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|_{\mathrm{L}^2(K)} = \|\tau_{(0,y_m)}\mathbf{u}_n - \mathbf{u}_n\|_{\mathrm{L}^2(K)}.$$

We describe the slices of K as $K^t := K \cap (\{t\} \times \Omega^t)$ and therefore we have clearly $\bigcup_{t\in I} K^t = K$. The sequence $(\nabla_x \mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^2(\hat{\Omega})$ by some constant $C_\nabla \in \mathbb{R}$. Thanks to Theorem 2.2.5 we obtain for all $m > m_0$

$$\begin{aligned}\|\tau_{(0,y_m)}\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n - \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|^2_{\mathrm{L}^2(K)} = \|\tau_{(0,y_m)}\mathbf{u}_n - \mathbf{u}_n\|_{\mathrm{L}^2(K)} &= \int_I \|\tau_{(0,y_m)}\mathbf{u}_n(t) - \mathbf{u}_n(t)\|^2_{\mathrm{L}^2(K^t)}\,dt \\ &\leq \int_I \|\nabla_x \mathbf{u}_n(t)\|^2_{\mathrm{L}^2(\Omega^t)} |(0,y_m)|^2\,dt = \|\nabla_x \mathbf{u}_n\|^2_{\mathrm{L}^2(\hat{\Omega})}|(0,y_m)|^2 \\ &\leq |(0,y_m)|^2 C_\nabla.\end{aligned}$$

Since K was arbitrary we get with the upper estimate clearly (3.22).

3.

Before we show (3.23) we want to clarify and simplify the upcoming steps, because this is the hardest part of the proof. Observe that the $\delta-$interior of Ω^t is not the same as $\vartheta_t(\Omega_\delta)$, but thanks to Proposition 3.3.12 point *ii)* we can frame $\vartheta_t(\Omega_\delta)$ between the two sets $\Omega^t_{\delta/\kappa}$ and $\Omega^t_{\delta\kappa}$ for some $\kappa \in (0, 1]$. The proof would be a notational mess if we take care about this fact. Therefore we simplify the proof by setting $\kappa = 1$, hence we have $\vartheta_t(\Omega^t) = \Omega^t_\delta$. To obtain the general case, we have to adapt the steps below line by line, but the core argument stays intact.

Notice that the assumption $\gamma_{\hat{n}}\mathbf{u}_n = 0$ gives us through Proposition 3.6.5 that $\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n \in \mathrm{L}^2(\mathbb{R}; \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d))$. From now on we will write $\mathbf{u}_n$, instead of $\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n$. Our boundary condition $\gamma_{\hat{n}}\mathbf{u}_n = 0$ does not have to apply for $\gamma_{\hat{n},\delta}\mathbf{u}_n$. Therefore we want to define, analogously to the proof of Lemma 3.6.2, the orthogonal projection

$$\mathbb{P}_\delta : \mathrm{L}^2_{\mathrm{div}}(\hat{\Omega}_\delta) \to \mathrm{L}^2_{\mathrm{div},0}(\hat{\Omega}_\delta).$$

Recall the definition of the mollifier $\varphi_\delta(x) := \delta^{-d}\varphi(\delta^{-1}x)$ with $\phi \in \mathcal{D}(\mathbb{R}^d)$ a nonnegative even function with a support in the unit ball (and integral 1). If we convolute the vector field $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}^d$ with the mollifier φ_δ we get the vector field

$$\mathbf{f} \star \varphi_\delta := (\mathbf{f}_1 \star \varphi_\delta, \ldots, \mathbf{f}_d \star \varphi_\delta).$$

Let us take a closer look on the convolution $\mathbf{u}_n \star \varphi_\delta$. Since $\mathbf{u}_n$ vanishes on the outside of $\hat{\Omega}$, the convolution $\mathbf{u}_n \star \varphi_\delta$ vanishes outside

$$\hat{\Omega}_{-\delta} := \bigcup_{t\in(a,b)} \{t\} \times \Omega^t_{-\delta},$$

where we recall Definition 3.1.4 $\Omega^t_{-\delta} = \Omega^t + B_\delta(0)$. This makes sense, because the support of φ_δ is the ball $B_\delta(0)$ and we have $\operatorname{supp} \mathbf{u}_n(t) \star \varphi_\delta = \Omega^t + B_\delta(0)$ for each $t \in (a, b)$. It is clear that $\mathbf{u}_n(t) \star \varphi_\delta \in \mathrm{L}^2(\mathbb{R}; \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d))$, because $\mathbf{u}_n \in \mathrm{L}^2(\mathbb{R}; \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d))$. We show (3.23) in the following four steps.

Step 1: For this step let $n \in \mathbb{N}$ be arbitrary. The convolution $\mathbf{u}_n(t) \star \varphi_\delta$ lies in $\mathrm{L}^2(\hat{\Omega}_{-\delta})$, hence we get for any $\varepsilon > 0$

$$\|\mathbf{u}_n(t) \star \varphi_\delta\|_{\mathrm{L}^2(\hat{\Omega}_\varepsilon)} \le \|\mathbf{u}_n(t) \star \varphi_\delta\|_{\mathrm{L}^2(\hat{\Omega}_{-\delta})} < \infty,$$

therefore $\mathbf{u}_n(t) \star \varphi_\delta \in \mathrm{L}^2(\hat{\Omega}_\varepsilon)$ for all $\varepsilon > 0$, especially for $\varepsilon = 2\delta$, where $\delta < \gamma/2$. Since $\mathbf{u}_n(t) \star \varphi_\delta$ has divergence zero, it lies also in $\mathrm{L}^2_{\mathrm{div}}(\hat{\Omega}_{2\delta})$ and therefore we can apply Lemma 3.6.6. Observe that $\mathcal{D}_{\mathrm{div}}(\hat{\Omega}_{2\delta})$ lies dense in $\mathrm{L}^2_{\mathrm{div},0}(\hat{\Omega}_{2\delta}) = \mathbb{P}_{2\delta}(\mathrm{L}^2_{\mathrm{div}}(\hat{\Omega}_{2\delta}))$ and thus we get the following estimate

$$\|\mathbf{u}_n(t) \star \varphi_\delta - \mathbb{P}_{2\delta}(\mathbf{u}_n(t) \star \varphi_\delta)\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})} \le 0 + (C^\Theta_{\Omega,\gamma})\|\gamma_{\hat{n},2\delta}(\mathbf{u}_n(t) \star \varphi_\delta)\|_{\mathrm{H}^{-1/2}_x(\partial\hat{\Omega}_{2\delta})}.$$

If $n_{t,2\delta}$ is the unit outward normal of $\Omega^t_{2\delta}$, we have that $-n_{t,2\delta}$ is the unit outward normal of $\mathbb{R}^d \setminus \Omega^t_{2\delta}$, hence the normal trace of $\mathbf{u}_n(t) \star \varphi_\delta \in \mathrm{L}^2(\Omega^t_{2\delta})$ is the opposite of the the normal trace as when we consider $\mathbf{u}_n(t) \star \varphi_\delta$ as an element of $\mathrm{L}^2(\mathbb{R}^d \setminus \Omega^t_{2\delta})$, but the norms of the traces coincide. Since $\|\gamma_{-n_t,2\delta}\| \le 1$ we get

$$\begin{aligned}
\|\gamma_{\hat{n},2\delta}(\mathbf{u}_n \star \varphi_\delta)\|^2_{\mathrm{H}^{-1/2}_x(\partial\hat{\Omega}_{2\delta})} &= \int_I \|\gamma_{n_t,2\delta}(\mathbf{u}_n(t) \star \varphi_\delta)\|^2_{\mathrm{H}^{-1/2}(\partial\Omega^t_{2\delta})}\, dt \\
&= \int_I \|\gamma_{-n_t,2\delta}(\mathbf{u}_n(t) \star \varphi_\delta)\|^2_{\mathrm{H}^{-1/2}(\partial(\mathbb{R}^d\setminus\Omega^t_{2\delta}))}\, dt \\
&\le \int_I \|\mathbf{u}_n(t) \star \varphi_\delta\|^2_{\mathrm{L}^2(\mathbb{R}^d\setminus\Omega^t_{2\delta})}\, dt = \|\mathbf{u}_n \star \varphi_\delta\|^2_{\mathrm{L}^2(\mathbb{R}^d\setminus\hat{\Omega}_{2\delta})}.
\end{aligned}$$

Combining these two estimates, using that $\operatorname{supp} \mathbf{u}_n \star \varphi_\delta \subseteq \hat{\Omega}_{-\delta}$, Lemma 1.2.18 and the Hölder inequality we obtain

$$\begin{aligned}
\|\mathbf{u}_n(t) \star \varphi_\delta - \mathbb{P}_{2\delta}(\mathbf{u}_n(t) \star \varphi_\delta)\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})} &\le (C^\Theta_{\Omega,\gamma} + 1)\|\mathbf{u}_n \star \varphi_\delta\|_{\mathrm{L}^2(\hat{\Omega}_{-\delta}\setminus\hat{\Omega}_{2\delta})} \\
&\le (C^\Theta_{\Omega,\gamma} + 1)\|\mathbf{u}_n\|_{\mathrm{L}^2(\hat{\Omega}_{-\delta}\setminus\hat{\Omega}_{2\delta})} \\
&\le (C^\Theta_{\Omega,\gamma} + 1)\lambda_{d+1}(\hat{\Omega}_{-\delta} \setminus \hat{\Omega}_{2\delta})^{(\frac{1}{2}-\frac{1}{r})}\|\mathbf{u}_n\|_{\mathrm{L}^r(\hat{\Omega})}.
\end{aligned}$$

It was shown previously that the sequence $(\mathbf{u}_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^r(\hat{\Omega})$ and since $\lambda_{d+1}(\hat{\Omega}_{-\delta} \setminus \hat{\Omega}_{2\delta}) \to 0$ for $\delta \to 0$, we conclude

$$\sup_{n\in\mathbb{N}} \|\mathbf{u}_n \star \varphi_\delta - \mathbb{P}_{2\delta}(\mathbf{u}_n \star \varphi_\delta)\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})} \xrightarrow{\delta\to 0} 0 \tag{3.25}$$

Step 2: In this step we want to prove that for all $\delta > 0$, there exists $\xi > 0$ such that

$$\forall \psi \in \mathcal{D}(\hat{\Omega}_{2\delta}) \text{ and } \forall \sigma \in (0, \xi] \text{ we have } \tau_{(-\sigma,0)}\psi \in \mathcal{D}(\hat{\Omega}_\delta).$$

That the function $\tau_{(-\sigma,0)}\psi$ is smooth as ψ is clear, so that we have only to show that $\operatorname{supp} \tau_{(-\sigma,0)}\psi \subset \hat{\Omega}_\delta$.

Fix $\delta > 0$. We know that $\Theta \in \mathrm{C}^0([a,b], \mathrm{C}^1(\mathbb{R}^d))$, hence there exists $\xi > 0$ such that $\|\Theta(t+\sigma) - \Theta(t)\|_{\mathrm{C}^1(\mathbb{R}^d)} < \delta$ for all $\sigma \in (0, \xi]$ and all $t \in [a,b]$. Therefore we have for all $x \in \Omega_{2\delta}$ that

$$d(\vartheta_{t+\sigma}(x), \vartheta_t(x)) = |\vartheta_{t+\sigma}(x) - \vartheta_t(x)| = |\Theta(t+\sigma, x) - \Theta(t, x)| < \delta, \qquad \forall \sigma \in (0, \xi].$$

This shows that $\vartheta_{t+\sigma}(x) \in \Omega^t_{2\delta} + \delta \subseteq \Omega^t_\delta = \vartheta_t(\Omega_\delta)$ and hence $\vartheta_{t+\sigma}(\Omega_{2\delta}) \subseteq \vartheta_t(\Omega_\delta)$ for all $t \in [a,b]$ and for all $\sigma \in (0,\xi]$.
Now consider $\psi \in \mathcal{D}(\hat{\Omega}_{2\delta})$. If (t,x) is not an element of $\hat{\Omega}_\delta$, then x lies not in $\vartheta_t(\Omega_\delta)$ and therefore $x \notin \vartheta_{t+\sigma}(\Omega_{2\delta})$, which is equivalent to $(t+\sigma,x) \notin \hat{\Omega}_{2\delta}$. We conclude that $\psi(t+\sigma,x)=0$, which means also $\tau_{(-\sigma,0)}\psi(t,x)=0$ and with that we have shown supp $\tau_{(-\sigma,0)}\psi \subset \hat{\Omega}_\delta$.

Step 3: The estimate (3.19) leads us for a fixed $\delta>0$ to the following inequality

$$\forall \boldsymbol{\psi} \in \mathcal{D}_{\mathrm{div}}(\hat{\Omega}_\delta), \qquad \langle \mathbf{u}_n \star \varphi_\delta, \partial_t \boldsymbol{\psi}\rangle_{\mathrm{L}^2(\hat{\Omega})} \le C_\delta \|\boldsymbol{\psi}\|_{\mathrm{L}^2(\hat{\Omega})}, \tag{3.26}$$

where C_δ is only depending only on the mollifier φ_δ. To show (3.26) we use our condition on the distributional time derivative (3.19) and follow the steps we did already in the proof of Theorem 3.3.1.

$$\begin{aligned} |\langle \mathbf{u}_n \star \varphi_\delta, \partial_t \boldsymbol{\psi}\rangle_{\mathrm{L}^2(\hat{\Omega})}| = |\langle \partial_t \mathbf{u}_n, \boldsymbol{\psi} \star \varphi_\delta\rangle_{\mathrm{L}^2(\hat{\Omega})}| &\le \sum_{\alpha \le N} \|\boldsymbol{\psi} \star \partial_x^\alpha \varphi_\delta\|_{\mathrm{L}^2(\hat{\Omega})} \\ &\le \|\boldsymbol{\psi}\|_{\mathrm{L}^2(\hat{\Omega})} \sum_{\alpha \le N} \|\partial_x^\alpha \varphi_\delta\|_{\mathrm{L}^\infty(\hat{\Omega})} = \|\boldsymbol{\psi}\|_{\mathrm{L}^2(\hat{\Omega})} C_\delta. \end{aligned}$$

For any pair $(\mathbf{v}, \boldsymbol{\Phi}) \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^d)^2$ a simple recalculation provides

$$\langle \tau_{(s,0)}\mathbf{v} - \mathbf{v}, \boldsymbol{\Phi}\rangle_{\mathrm{L}^2(\mathbb{R}\times\mathbb{R}^d)} = \langle (\tau_{(s,0)} - \mathrm{Id})\mathbf{v}, \boldsymbol{\Phi}\rangle_{\mathrm{L}^2(\mathbb{R}\times\mathbb{R}^d)} = s\int_0^1 \langle \mathbf{v}, \tau_{(-sz,0)}\partial_t \boldsymbol{\Phi}\rangle_{\mathrm{L}^2(\mathbb{R}\times\mathbb{R}^d)}\, dz.$$

If $\boldsymbol{\Phi} \in \mathcal{D}_{\mathrm{div}}(\hat{\Omega}_{2\delta})$, we have also that $\partial_t\boldsymbol{\Phi} \in \mathcal{D}_{\mathrm{div}}(\hat{\Omega}_{2\delta})$. According of step 2 we know that for s small enough and any $z \in [0,1]$ we obtain $\tau_{(-sz,0)}\partial_t\boldsymbol{\Phi} \in \mathcal{D}_{\mathrm{div}}(\hat{\Omega}_\delta)$ and hence the estimate (3.26) is usable for $\tau_{(-sz,0)}\partial_t\boldsymbol{\Phi}$. Since $\mathcal{D}(\mathbb{R}\times\mathbb{R}^d)$ is dense in $\mathrm{L}^2(\mathbb{R};\mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d))$ we can apply the upper formula to $\mathbf{v} = \mathbb{P}_{2\delta}(\mathbf{u}_n \star \varphi_\delta)$. Using Lemma 3.6.6 we get

$$\begin{aligned} \|(\tau_{(s,0)} - \mathrm{Id})\mathbb{P}_{2\delta}(\mathbf{u}_n\star\varphi_\delta)\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})} &\le C_I \sup_{\boldsymbol{\Phi}\in\mathcal{D}_{\mathrm{div}}(\hat{\Omega}_{2\delta}), \|\boldsymbol{\Phi}\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})}\le 1} \langle (\tau_{(s,0)} - \mathrm{Id})\mathbb{P}_{2\delta}(\mathbf{u}_n\star\varphi_\delta), \boldsymbol{\Phi}\rangle_{\mathrm{L}^2(\hat{\Omega}_{2\delta})} \\ &\quad + \|\gamma_{\hat{n}}(\tau_{(s,0)} - \mathrm{Id})\mathbb{P}_{2\delta}(\mathbf{u}_n\star\varphi_\delta)\|_{\mathrm{H}_x^{-1/2}(\partial\hat{\Omega}_{2\delta})} \\ &\le C_I \sup_{\boldsymbol{\Phi}\in\mathcal{D}_{\mathrm{div}}(\hat{\Omega}_{2\delta}), \|\boldsymbol{\Phi}\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})}\le 1} s\int_0^1 \langle \mathbf{v}, \tau_{(-sz,0)}\partial_t\boldsymbol{\Phi}\rangle_{\mathrm{L}^2}\, dz \\ &\quad + \|\gamma_{\hat{n}}(\tau_{(s,0)} - \mathrm{Id})\mathbb{P}_{2\delta}(\mathbf{u}_n\star\varphi_\delta)\|_{\mathrm{H}_x^{-1/2}(\partial\hat{\Omega}_{2\delta})} \\ &\le C_I \sup_{\boldsymbol{\Phi}\in\mathcal{D}_{\mathrm{div}}(\hat{\Omega}_{2\delta}), \|\boldsymbol{\Phi}\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})}\le 1} sC_\delta \|\tau_{(-sz,0)}\boldsymbol{\Phi}\|_{\mathrm{L}^2(\hat{\Omega}_\delta)} \\ &\le sC_I C_\delta. \end{aligned}$$

Since $n \in \mathbb{N}$ was arbitrary we get uniform convergence in n in the sense

$$\sup_{n\in\mathbb{N}} \|(\tau_{(s,0)} - \mathrm{Id})\mathbb{P}_{2\delta}(\mathbf{u}_n\star\varphi_\delta)\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})} \xrightarrow{s\to 0} 0. \tag{3.27}$$

Step 4: Putting all 3 steps together we are able to prove (3.23). So let $K \subset \hat{\Omega}$ be an arbitrary compact subset of $\hat{\Omega}$ and let $\delta \in (0, \gamma/2)$, then

$$\begin{aligned}
\tau_{(s,0)}\mathbf{u}_n - \mathbf{u}_n &= (\tau_{(s,0)} - \mathrm{Id})(\mathbf{u}_n - \mathbf{u}_n \star \varphi_\delta) &(3.28)\\
&+ (\tau_{(s,0)} - \mathrm{Id})(\mathbf{u}_n \star \varphi_\delta - \mathbb{P}_{2\delta}(\mathbf{u}_n \star \varphi_\delta)) &(3.29)\\
&+ (\tau_{(s,0)} - \mathrm{Id})\mathbb{P}_{2\delta}(\mathbf{u}_n \star \varphi_\delta). &(3.30)
\end{aligned}$$

We take δ small enough, such that $K \subset \hat{\Omega}_{2\delta}$ and we will proceed line by line in the $\mathrm{L}^2(\hat{\Omega}_\delta)$ norm.
Let $\varepsilon > 0$. For the first line (3.28) we can adapt Proposition (3.3.13) for $\mathbf{u}_n$ and $\hat{\Omega}$.Following the proof of Proposition (3.3.13) we obtain

$$\|\mathbf{u}_n - \mathbf{u}_n \star \varphi_\delta\|_{\mathrm{L}^2(K)} \leq \|\mathbf{u}_n - \mathbf{u}_n \star \varphi_\delta\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})} \leq \delta \|\nabla_x \mathbf{u}_n\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})} \leq \delta \|\nabla_x \mathbf{u}_n\|_{\mathrm{L}^2(\hat{\Omega})}.$$

Therefore we find $\delta_1 > 0$ such (3.28) becomes smaller than $\varepsilon/3$, independently from n (and s). Due step 1 we also find $\delta_2 > 0$ such that the second line 3.29 is smaller than $\varepsilon/3$ w.r.t. $\|.\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})}$ and for all $n \in \mathbb{N}$. Now let us set $\delta = \min(\delta_1, \delta_2)$. With δ_0 fixed, we are able to handle the third and last line, which convergence is independent of n and δ. Let s_0 such that for all $|s| < |s_0|$ the norm $\|(\tau_{(s,0)} - \mathrm{Id})\mathbb{P}_{2\delta}(\mathbf{u}_n \star \varphi_\delta)\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})}$ is smaller than $\varepsilon/3$, then we get for all $(s, 0) \in B_{s_0}(0)$

$$\begin{aligned}
\|\tau_{(s,0)}\mathbf{u}_n - \mathbf{u}_n\|_{\mathrm{L}^2(K)} &\leq \|\tau_{(s,0)}\mathbf{u}_n - \mathbf{u}_n\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})}\\
&\leq \|(\tau_{(s,0)} - \mathrm{Id})(\mathbf{u}_n - \mathbf{u}_n \star \varphi_\delta)\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})}\\
&+ \|(\tau_{(s,0)} - \mathrm{Id})(\mathbf{u}_n \star \varphi_\delta - \mathbb{P}_{2\delta}(\mathbf{u}_n \star \varphi_\delta))\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})}\\
&+ \|(\tau_{(s,0)} - \mathrm{Id})\mathbb{P}_{2\delta}(\mathbf{u}_n \star \varphi_\delta)\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})}\\
&\leq \|(\mathbf{u}_n - \mathbf{u}_n \star \varphi_\delta)\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})}\\
&+ \|(\mathbf{u}_n \star \varphi_\delta - \mathbb{P}_{2\delta}(\mathbf{u}_n \star \varphi_\delta))\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})}\\
&+ \|(\tau_{(s,0)} - \mathrm{Id})\mathbb{P}_{2\delta}(\mathbf{u}_n \star \varphi_\delta)\|_{\mathrm{L}^2(\hat{\Omega}_{2\delta})}\\
&< \varepsilon.
\end{aligned}$$

Since K was arbitrary we have shown, that $\tau_{(s,0)}\mathbf{u}_n - \mathbf{u}_n \xrightarrow{\mathrm{L}^2_{loc}(\hat{\Omega})} 0$ uniformly in n.

Conclusion

In this part we conclude our proof with showing

$$\forall \varepsilon > 0, \exists \delta_\varepsilon : \textit{such that } \|\tau_{(s,y)}\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n - \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|_{\mathrm{L}^2(\mathbb{R}\times\mathbb{R}^d)} < \varepsilon \qquad \forall n \in \mathbb{N}, \forall (s, y) \textit{ with } |(s, y)| < \delta_\varepsilon.$$

Recall $r > 2$ from point 1.), then we find a constant $R \in \mathbb{R}$ such that

$$\|\mathbf{u}_n\|_{\mathrm{L}^r(\hat{\Omega})} < R \qquad \forall n \in \mathbb{N}. \tag{3.31}$$

Let $\varepsilon > 0$. Thanks to Theorem 1.1.9, there exists a compact subset $K_\varepsilon \subset \hat{\Omega}$, such that

$$\lambda_{d+1}(\hat{\Omega} \setminus K_\varepsilon) < (\varepsilon^2/4R^r)^{(r-2)/2r}. \tag{3.32}$$

Due to point 2.) and 3.) we find $\delta_{\varepsilon/4} > 0$ and $\tilde{\delta}_{\varepsilon/4} > 0$ such that

$$\|\tau_{(0,y)}(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n) - \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|_{\mathrm{L}^2(K_\varepsilon)} < \frac{\varepsilon}{4} \qquad \forall|(0,y)| < \delta_{\varepsilon/4}, \tag{3.33}$$

$$\|\tau_{(s,0)}(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n) - \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|_{\mathrm{L}^2(K_\varepsilon)} < \frac{\varepsilon}{4} \qquad \forall|(s,0)| < \tilde{\delta}_{\varepsilon/4}, \tag{3.34}$$

Define $\delta_\varepsilon := \min(\delta_{\varepsilon/4}, \tilde{\delta}_{\varepsilon/4})$. If we use the linearity and the isometry of the shift operator, then we get, thanks to (3.33) and (3.34), for all $n \in \mathbb{N}$ and for all $(s,y) \in \mathbb{R} \times \mathbb{R}^d$ with $|(s,y)| < \delta_\varepsilon$

$$\begin{aligned}
&\|\tau_{(s,y)}(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n) - \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|_{\mathrm{L}^2(K_\varepsilon)} = \\
&\|\tau_{(0,y)}(\tau_{(s,0)}(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n)) - \tau_{(0,y)}(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n) + \tau_{(0,y)}(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n) - \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|_{\mathrm{L}^2(K_\varepsilon)} \leq \\
&\|\tau_{(0,y)}(\tau_{(s,0)}(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n) - \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n)\|_{\mathrm{L}^2(K_\varepsilon)} + \|\tau_{(0,y)}(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n) - \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|_{\mathrm{L}^2(K_\varepsilon)} = \\
&\|\tau_{(s,0)}(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n) - \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|_{\mathrm{L}^2(K_\varepsilon)} + \|\tau_{(0,y)}(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n) - \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|_{\mathrm{L}^2} < \\
&\frac{\varepsilon}{2}.
\end{aligned}$$

With this, (3.32) and the Hölder inequality (with $1/2 = 1/r + (r-2)/2r$) we obtain for all $n \in \mathbb{N}$ and for all $(s,y) \in \mathbb{R} \times \mathbb{R}^d$ with $|(s,y)| < \delta_\varepsilon$

$$\begin{aligned}
&\|\tau_{(s,y)}(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n) - \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|^2_{\mathrm{L}^2(\mathbb{R}\times\mathbb{R}^d)} = \\
&\|\tau_{(s,y)}(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n) - \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|^2_{\mathrm{L}^2(K_\varepsilon)} + \|\tau_{(s,y)}(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n) - \mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|^2_{\mathrm{L}^2(\mathbb{R}\times\mathbb{R}^d\setminus K_\varepsilon)} < \\
&\frac{\varepsilon^2}{4} + \|\tau_{(s,y)}\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|^2_{\mathrm{L}^2(\mathbb{R}\times\mathbb{R}^d\setminus K_\varepsilon)} + \|\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|^2_{\mathrm{L}^2(\mathbb{R}\times\mathbb{R}^d\setminus K_\varepsilon)} \leq \\
&\frac{\varepsilon^2}{4} + 2\|\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n\|^2_{\mathrm{L}^2(\mathbb{R}\times\mathbb{R}^d\setminus K_\varepsilon)} \leq \\
&\frac{\varepsilon^2}{4} + 2\|\mathbb{1}_{\hat{\Omega}\setminus K_\varepsilon}\mathbf{u}_n\|^2_{\mathrm{L}^2(\hat{\Omega})} \leq \\
&\frac{\varepsilon^2}{4} + 2\|\mathbf{u}_n\|^r_{\mathrm{L}^r(\hat{\Omega})}\lambda_{d+1}(\hat{\Omega} \setminus K_\varepsilon)^{\frac{2r}{r-2}} \leq \\
&\frac{\varepsilon^2}{4} + 2R^r\frac{\varepsilon^2}{4R^r} < \varepsilon^2
\end{aligned}$$

The choice of $K_\varepsilon \subset \hat{\Omega}$ depends only on ε, hence the choice of δ_ε is independend of n, therefore we have uniform convergence in n, which means that the condition (3.21) is satisfied.
At the end we obtain with Riesz-Fréchet-Kolmogorv's Theorem (Theorem 3.6.7) that the sequence $(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n)_{n\in\mathbb{N}}$ is relative compact in $\mathrm{L}^2(O)$ for all measurable sets $O \subset \mathbb{R} \times \mathbb{R}^d$ with finite measure. Since $(\mathbb{1}_{\hat{\Omega}}\mathbf{u}_n)_{n\in\mathbb{N}}$ coincide with $(\mathbf{u}_n)_{n\in\mathbb{N}}$ on $\hat{\Omega}$, we have at the end of the day that the sequence $(\mathbf{u}_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathrm{L}^2(\hat{\Omega})$. □

Remark 3.6.9. Theorem 3.6.4 has a homogeneous boundary condition (3.18). We want to mention that one can replace (3.18) with the weaker assumption:

$$(\gamma_{\hat{n}}\mathbf{u}_n)_{n\in\mathbb{N}} \;\text{ has a converging subsequence in }\; \mathrm{H}_x^{-1/2}(\partial\hat{\Omega}). \tag{3.35}$$

This condition is designed for non homogeneous boundary conditions. A. Moussa gives at the end of [Mou16] a short proof if one replaces the condition (3.18) in Theorem 3.6.4 with (3.35).
■

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