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Isoperimetric inequalities for the anisotropic and fractional perimeter

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Contents

Introduction

Throughout the history of mankind, reaching back to ancient Egypt and Greece, natural philosophers and mathematicians were interested in the study of the measurement of the geometric properties of volume and surface area, and their relationship manifested in isoperimetric inequalities. The modern and most commonly used definition of volume is provided by the Lebesgue measure, whereas there are various definitions of surface area for different classes of sets:
The $n-1$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$, for example, leads to a suitable measurement of the area of parametrized hypersurfaces in $\mathbb{R}^n$, but may change drastically if the set in question is altered by a set of Lebesgue measure 0. De Giorgi [10] introduced the notion of the *perimeter* in 1953, which is invariant under such changes; for a Lebesgue measurable set $E \subseteq \mathbb{R}^n$ it is defined as

$$P(E) := \sup \left\{ \int_E \operatorname{div} T(x) \, \mathrm{d}x : T \in C_c^1(\mathbb{R}^n), \ |T| \leq 1 \right\}$$

and if this quantity is finite, one can use the notion of the reduced boundary $\partial^* E$ to express it as

$$P(E) = \int_{\partial^* E} |\nu_E(x)| \, \mathrm{d}\mathcal{H}^{n-1}(x),$$

where $\nu_E(x)$ is the measure-theoretic outer unit normal at $x \in \partial^* E$. This notion turned out fruitful in the study of geometric variational problems, such as Plateau-type problems and already mentioned isoperimetric inequalities, which provide the answer to questions of the sort "Which sets minimize their surface area for a given volume?". The isoperimetric inequality for the perimeter takes the form

$$P(E) \geq n\omega_n^{\frac{1}{n}} |E|^{\frac{n-1}{n}},$$

where ω_n is the volume of the Euclidean unit ball, and equality holds precisely for those sets E that differ from a ball on a set of volume 0.
As the surface area is closely related to certain forms of energies in physics, isoperimetric inequalities can be utilized for the explanation why certain materials take on their distincitve shapes. In the case of crystals, Wulff conjectured in 1901 that their shapes minimize a corresponding surface energy functional, and Taylor [28] could identify these shapes as the unique minimizers of the functional in 1978. The surface energy functional can be modeled by replacing the Euclidean length by an arbitrary norm $\|\cdot\|_K$ with unit ball K on $\mathbb{R}^n$, leading to the notion of the *anisotropic perimeter* $P_K(E)$ of a set E,

$$P_K(E) = \int_{\partial^* E} \|\nu_E(x)\|_{K^*} \, \mathrm{d}\mathcal{H}^{n-1}(x),$$

where K^* is the polar body of K.

Another generalization of the perimeter which is useful in the study of phase transition problems with long-range interactions, is the *fractional perimeter*,

$$P_s(E) = \int_E \int_{E^c} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}x \, \mathrm{d}y,$$

where $0 < s < 1$. It was first thoroughly introduced by Caffarelli, Roquejoffre and Savin [7] in 2010 in the study of corresponding minimizers. Closely related to the theory of fractional perimeters is the notion of fractional Sobolev spaces, which arise naturally as image of trace operators on Sobolev spaces and thus have important applications in many fields dealing with partial differential equations. The theory of fractional Sobolev spaces is much older, reaching back to the 1950s, when they were almost simultaneously introduced by Aronszajn, Gagliardo and Slobodeckij (see [11] and references therein).
A combination of both concepts of anisotropic and fractional perimeter can be obtained by the definition

$$P_{s,K}(E) = \int_E \int_{E^c} \frac{1}{\|x-y\|_K^{n+s}} \, \mathrm{d}x \, \mathrm{d}y,$$

called *anisotropic fractional perimeter*. It was first studied by Ludwig [19] in 2014.

This thesis aims to properly define all the various concepts of surface area listed above, examine their basic properties and, ultimately, formulate and prove corresponding isoperimetric inequalities. It is structured as follows:

In Chapter 1 we first state basic results dealing with the perimeter and reduced boundary, as well as crucial facts about convex bodies. We devote one section to prove that the classical isoperimetric inequality and its analytical counterpart, the classical Sobolev inequality, are equivalent.

Chapter 2 consists of the examination of the anisotropic perimeter and the anisotropic isoperimetric inequality. The original proof by Taylor [28] heavily relies on the use of currents, so we use more straightforward techniques, established by Maggi [22], instead. The biggest drawback of this method, however, is that we cannot classify all cases for which equality holds.

Fractional perimeters are the subject of examination in Chapter 3: The main results we will prove therein deal with Gagliardo seminorms of functions, so we will establish a link between this seminorm and the fractional perimeters first.
Then we discuss, in which sense fractional perimeters can be understood as generalizations of the classical perimeter: To this end, we reenact the proofs by Bourgain, Brezis and Mironescu [5], and Maz'ya and Shaposhnikova [23] for their limiting results as $s \to 1^-$ and $s \to 0^+$: Up to constant factors, $(1-s)P_s(\cdot)$ converges to the perimeter as $s \to 1^+$ and $sP_s(\cdot)$ converges to the volume as $s \to 0^+$.
The last part of this chapter consists of the proof of the fractional isoperimetric inequalty. We follow along the lines of Frank and Seiringer [12] and prove the fractional Hardy and the fractional Sobolev inequality along the way. The equality cases are obtained via the method of symmetric decreasing rearrangement of functions.

In Chapter 4 we combine the theories established in Chapter 2 and 3 and introduce the notion of anisotropic fractional perimeters. Using the ideas of Ludwig [19] we first show convergence results as $s \to 1^-$ and $s \to 0^+$ by reducing the statements to the one-dimensional setting. The statement of a correspoding isoperimetric inequality concludes the discussion of anisotropic fractional perimeters.

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1 Preliminary results from geometric measure theory and convex geometry

This chapter aims to provide the basic notions and facts from geometric measure theory and convex geometry we will need throughout the rest of the thesis.
We start in section 1.1 by settling the notation of basic mathematical objects any reader familiar with analysis and measure theory should know.
In section 1.2 we will recall the definition and basic facts about the perimeter, a measure-theoretic way of measuring the surface area of a set that has its roots in the Gauss-Green theorem, which relates the integration of a vector field over a surface to the integration of its divergence over a volume.
The famous isoperimetric inequality for sets and Sobolev inequality for functions will be stated in section 1.3 and we will prove that both inequalities are equivalent, meaning each of both inequalities can be derived from the other one.
In section 1.4 we want to investigate how norms on finite dimensional real vector spaces can be identified by certain convex bodies and vice versa. The notions of support function and polar body of a convex body will be crucial to this task and their definitions as well as simple properties will be stated. The Minkowski theorem and the introduction of mixed volumes will conclude this section; mixed volumes are closely related to the anisotropic perimeter discussed in chapter 2.

1.1 Basic notation

We provide a short summary of all mathematical objects we will heavily use throughout the thesis and settle their notation in this section.

Analysis

Throughout the thesis we are interested in the study of functions and measures defined on subsets of the n-dimensional Euclidean space $\mathbb{R}^n, n \geq 1$. For a vectors $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$ (which we always assume to be column vectors) we denote the Euclidean scalar (or inner) product on $\mathbb{R}^n$ by

$$x \cdot y := \sum_{j=1}^{n} x_j y_j$$

and the induced Euclidean norm on $\mathbb{R}^n$ by

$$|x| := \sqrt{x \cdot x}.$$

The $n-1$-dimensional unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ is defined by

$$\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}.$$

If $x \in \mathbb{R}^n$ and $r > 0$, then by $B_r(x)$ or $B(x,r)$ we denote the open ball centered at x with radius r,

$$B(x,r) = B_r(x) := \{y \in \mathbb{R}^n : |y-x| < r\}.$$

If $x = 0$, we simply write B_r.
The indicator function $\mathbb{1}_E$ of a set $E \subseteq \mathbb{R}^n$ is defined by $\mathbb{1}_E(x) = 1$ for $x \in E$ and $\mathbb{1}_E(x) = 0$ for $x \in E^c$.
Let $\Omega \subseteq \mathbb{R}^n$ be an open set. For a differentiable function $f : \Omega \to \mathbb{R}$ the partial derivative w.r.t the j-th coordinate at a point $x \in \Omega$ is denoted by $\partial_j f(x)$ and we will identify its differential $df(x)$ with the matrix consisting of all partial derivatives:

$$df(x) = (\partial_1 f(x), \dots, \partial_n f(x)).$$

The gradient $\nabla f(x)$ of f at x is defined as the transpose of $df(x)$, i.e. $\nabla f(x) = df(x)^T$. Similarly, if $g : \Omega \to \mathbb{R}^m, m \geq 1$ is a differentiable function and we denote its component functions as $g^{(j)}$, i.e. $g = (g^{(1)}, \dots, g^{(m)})^T$, then the differential $dg(x)$ is an $m \times n$-matrix, $dg(x) = (\partial_k g^{(j)}(x)) \in \mathbb{R}^{m \times n}$. If $m = n$, the divergence of g at x, written $\operatorname{div} g(x)$, is defined as

$$\operatorname{div} g(x) := \sum_{j=1}^{n} \partial_j g^{(j)}(x).$$

For $k \in \mathbb{N} \cup \{0, \infty\}$ we denote by $C^k(\Omega)$ ($C^k(\Omega; \mathbb{R}^m)$) the class of all k-times continuously differentiable functions (with values in $\mathbb{R}^m$). In case of $k = 0$, i.e. for continuous functions, we oftentimes omit the exponent k, e.g. $C(\Omega) = C^0(\Omega)$.
The support of a function $T : \mathbb{R}^n \to \mathbb{R}^m$, denoted by $\operatorname{spt} T$, is defined as

$$\operatorname{spt} T := \overline{\{x \in \Omega : T(x) \neq 0\}},$$

where $\overline{A}$ denotes the closure of $A \subseteq \mathbb{R}^n$. The class of all k-times continuously differentiable functions (with values in $\mathbb{R}^m$) with compact support in Ω is denoted by $C_c^k(\Omega)$ ($C_c^k(\Omega; \mathbb{R}^m)$).
In some of our arguments we will need to smoothen functions before applying a suitable result (this is mainly the case if the result needs some kind of regularity): By a family of standard mollifier $\rho_\varepsilon, \varepsilon > 0$, we mean functions in $C_c^\infty(B_\varepsilon; [0, \infty))$ such that $\int_{\mathbb{R}^n} \rho_\varepsilon(x) \, \mathrm{d}x = 1$. Furthermore, we define the convolution of a function $u \in L^p(\mathbb{R}^n), 1 \leq p < \infty$, with a mollifier ρ_ε as

$$u * \rho_\varepsilon(x) := \int_{\mathbb{R}} u(x-y) \rho_\varepsilon(y) \, \mathrm{d}y,$$

and this convolution satisfies $u * \rho_\varepsilon \to u$ in $L^p(\mathbb{R}^n)$.

Measure theory

If $E \subseteq \mathbb{R}^n$ is a Lebesgue measurable subset, then we denote its Lebesgue measure by $|E|$ or $\mathcal{L}^n(E)$. The last notation for the Lebesgue measure is mostly used in integrals, where we do not want to explicitly write down the integration variables. For $k \in \mathbb{N} \cup \{0\}$ the k-dimensional Hausdorff measure of E is denoted by $\mathcal{H}^k(E)$. If there is no chance of confusion with the Lebesgue measure, as it is for instance the case if k is

such that $0 < \mathcal{H}^k(E) < \infty$, then we comfortably write $|E|$ for the k-dimensional Hausdorff measure of E. For example, by $|\mathbb{S}^{n-1}|$ we mean the $n-1$-dimensional Hausdorff measure of the unit sphere as opposed to its Lebesgue measure, which would be 0.
The restriction $\mu \restriction A$ of a measure μ to a measurable set $A \subseteq \mathbb{R}^n$ is the measure defined by $\mu \restriction A(E) := \mu(A \cap E), E \subseteq \mathbb{R}^n$ measurable.
Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a measurable function. Then the pushforward $f_\# \mu$ of a measure μ is defined by $f_\# \mu(E) := \mu(f^{-1}(E)), E \subseteq \mathbb{R}^n$ measurable.
A $\mathbb{R}^m$-valued Radon measure μ on $\mathbb{R}^n$ is a bounded linear functional on $C_c^0(\mathbb{R}^n; \mathbb{R}^m)$, i.e. $\mu : C_c^0(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R}$:

$$T \mapsto \mu(T) = \int_{\mathbb{R}^n} T \,\mathrm{d}\mu,$$

where boundedness means that

$$\sup \left\{ \int_{\mathbb{R}^n} T \,\mathrm{d}\mu : T \in C_c^0(\mathbb{R}^n; \mathbb{R}^m), \operatorname{spt} T \subseteq K, |T| \leq 1 \right\}$$

is finite for every compact set $K \subset \mathbb{R}^n$. The total variation $|\mu|$ of a $\mathbb{R}^m$-valued Radon measure μ on $\mathbb{R}^n$ is the outer measure defined by

$$|\mu|(A) := \sup \left\{ \int_{\mathbb{R}^n} T \,\mathrm{d}\mu : T \in C_c^0(\mathbb{R}^n; \mathbb{R}^m), \operatorname{spt} T \subseteq A, |T| \leq 1 \right\}, \ A \subseteq \mathbb{R}^n \text{ open},$$
$$|\mu|(E) := \inf \{ |\mu|(A) : E \subseteq A, A \text{ open} \} \ E \subseteq \mathbb{R}^n.$$

Riesz's representation theorem (c.f. [22, Theorem 4.7]) asserts the existence of a $|\mu|$-measurable function $g : \mathbb{R}^n \to \mathbb{R}^m$ with $|g| = 1$ $|\mu|$-a.e. and

$$\int_{\mathbb{R}^n} T \,\mathrm{d}\mu = \int_{\mathbb{R}^n} T \cdot g \,\mathrm{d}|\mu|$$

for every $T \in C_c^0(\mathbb{R}^n; \mathbb{R}^m)$. If $E \subseteq \mathbb{R}^n$ is a bounded Borel set we understand by $\mu(E) \in \mathbb{R}^m$ the expression $\int_E g \,\mathrm{d}|\mu|$.
For a $\mathbb{R}^m$-valued Radon measure μ on $\mathbb{R}^n$ and a bounded Borel function $f : \mathbb{R}^n \to \mathbb{R}^m$ we can construct a new $\mathbb{R}$-valued Radon measure $f \cdot \mu$ by setting

$$f \cdot \mu(\varphi) := \int_{\mathbb{R}^n} f \varphi \,\mathrm{d}\mu$$

for $\varphi \in C_c^0(\mathbb{R}^n)$. For its total variation we have the formula $|f \cdot \mu| = |f||\mu|$. An analogous construction can be made for positive measures μ, resulting in a $\mathbb{R}^m$-valued Radon measure.
The support $\operatorname{spt} \mu$ of a $\mathbb{R}^m$-valued Radon measure μ on $\mathbb{R}^n$ is defined as the intersection of all closed sets $C \subseteq \mathbb{R}^n$, such that $\mu(\mathbb{R}^n \backslash C) = 0$.
The Lebesgue-Besicovitch differentiation theorem (c.f. [22, Theorem 5.8]) ensures that

$$D_{|\mu|}\mu(x) := \lim_{r \to 0^+} \frac{\mu(B_r(x))}{|\mu|(B_r(x))}$$

exists for $|\mu|$-a.e. $x \in \operatorname{spt} \mu$. If we extend this function with 0 outside of $\operatorname{spt} \mu$, then it coincides with the function g from Riesz's representation theorem $|\mu|$-a.e.
Ultimately, let $\mu_j, j \in \mathbb{N}$, and μ be $\mathbb{R}^m$-valued Radon measures on $\mathbb{R}^n$. We say that μ_j weak-star converges to μ, denoted by $\mu_j \overset{*}{\rightharpoonup} \mu$, if

$$\lim_{j \to \infty} \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\mu_j = \int_{\mathbb{R}^n} \varphi \,\mathrm{d}\mu$$

for every $\varphi \in C_c^0(\mathbb{R}^n; \mathbb{R}^m)$.

1.2 The perimeter

All definitions and statements in this section are taken from [22], unless noted otherwise.

1.1 Definition. Let $E \subseteq \mathbb{R}^n$ be a Lebesgue measurable set; we say that E is a *set of locally finite perimeter* if for every compact $K \subseteq \mathbb{R}^n$

$$\sup\left\{\int_E \operatorname{div} T(x)\,\mathrm{d}x : T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ \operatorname{spt} T \subseteq K,\ |T| \leq 1\right\} < \infty.$$

If the above quantity can be bound by a constant independent of the choice of the compact set K, then we say E is a *set of finite perimeter* .

1.2 Theorem (Sets of locally finite perimeter and Radon measures). *Let $E \subseteq \mathbb{R}^n$ be a Lebesgue measurable set. Then E is a set of locally finite perimeter if and only if there exists a $\mathbb{R}^n$-valued Radon measure μ_E on $\mathbb{R}^n$ such that*

$$\int_E \operatorname{div} T(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} T\,\mathrm{d}\mu_E \quad \forall T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n).$$

μ_E is called Gauß-Green measure *of E.*
Moreover, E is of finite perimeter if and only if $|\mu_E|(\mathbb{R}^n) < \infty$.

Now we can define the *relative perimeter* $P(E;F)$ of E in $F \subseteq \mathbb{R}^n$ and the *perimeter* of E as

$$P(E;F) := |\mu_E|(F), \quad P(E) := |\mu_E|(\mathbb{R}^n).$$

In particular, for $\Omega \subseteq \mathbb{R}^n$ open, we have

$$P(E;\Omega) = \sup\left\{\int_E \operatorname{div} T(x)\,\mathrm{d}x : T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ \operatorname{spt} T \subseteq \Omega,\ |T| \leq 1\right\}.$$

1.3 Example. If $E \subseteq \mathbb{R}^n$ is an open set with C^1-boundary and outer unit normal ν_E, then the Gauß-Green theorem asserts that

$$\int_E \operatorname{div} T(x)\,\mathrm{d}x = \int_{\partial E} T \cdot \nu_E\,\mathrm{d}\mathcal{H}^{n-1}\ \forall T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),$$

so $\mu_E = \nu_E \mathcal{H}^{n-1} \restriction \partial E$ and $P(E;F) = \mathcal{H}^{n-1}(F \cap \partial E)$. This justifies the notion of the perimeter as suitable measure for the surface area. ▲

A generalization of the above condition from sets to functions is provided by the notion of variation of a function (see [4, Chapter 3]):

1.4 Definition. We say a function $u \in L^1(\Omega), \Omega \subseteq \mathbb{R}^n$ open, is of *bounded variation*, in symbols $u \in BV(\Omega)$, if there exists a finite $\mathbb{R}^n$-valued Radon measure Du (i.e $|Du|(\Omega) < \infty$) on Ω, such that

$$\int_\Omega u \operatorname{div} T\,\mathrm{d}x = -\int_\Omega T\,\mathrm{d}Du \quad \forall T \in C_c^1(\Omega;\mathbb{R}^n).$$

Furthermore, we define the *variation* $V(u;\Omega)$ of a function $u \in L^1_{loc}(\Omega)$ in Ω as the quantity

$$V(u;\Omega) := \sup\left\{\int_\Omega u \operatorname{div} T\,\mathrm{d}x : T \in C_c^1(\Omega;\mathbb{R}^n),\ |T| \leq 1\right\}.$$

For $\Omega = \mathbb{R}^n$ we simply write $V(u) = V(u;\Omega)$.

1.5 Remark.

1. If $u \in C^1(\Omega)$, then $V(u;\Omega) = \int_\Omega |\nabla u| \,\mathrm{d}x$.
2. A function $u \in L^1(\Omega)$ is of bounded variation if and only if $V(u;\Omega) < \infty$. In this case the variation $V(u;\Omega)$ coincides with $|Du|(\Omega)$.
3. Functions of bounded variation can be approximated by smooth functions: For each $u \in BV(\Omega) \cap L^p(\Omega)$ (with compact support), $1 \leq p < \infty$, there exists a sequence $(u_j)_{j=1}^\infty$ with $u_j \in C^\infty(\Omega)$ (with compact support) such that $u_j \to u$ in $L^p(\Omega)$ and
$$\int_\Omega |\nabla u_j| \,\mathrm{d}x \to V(u;\Omega), \quad j \to \infty$$

We are now in the situation to establish the link between the notions of perimeter and variation:
The perimeter of a Lebesgue measurable set $E \subseteq \mathbb{R}^n$ in Ω is the variation of its indicator function $\mathbb{1}_E$, i.e. $P(E;\Omega) = V(\mathbb{1}_E;\Omega)$.

Another useful tool is the following approximation result for the Gauss-Green measure:

1.6 Proposition. *Let $E \subseteq \mathbb{R}^n$ be a set of locally finite perimeter. Then*
$$\begin{aligned} -\nabla(\mathbb{1}_E * \rho_\varepsilon)\mathcal{L}^n &\overset{*}{\rightharpoonup} \mu_E, \text{ and} \\ |\nabla(\mathbb{1}_E * \rho_\varepsilon)|\mathcal{L}^n &\overset{*}{\rightharpoonup} |\mu_E|. \end{aligned}$$

The last part of this section is devoted to present some basic facts about the reduced boundary. We first give its definition:

1.7 Definition. Let $E \subseteq \mathbb{R}^n$ be a set of locally finite perimeter and μ_E its Gauß-Green measure. Then the *reduced boundary* $\partial^* E$ is defined as the set of all points $x \in \operatorname{spt} \mu_E$ such that the limit
$$\nu_E(x) := \lim_{r \to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))}$$
exists and is an element of $\mathbb{S}^{n-1}$, i.e. $|\nu_E(x)| = 1$.

The result we will most heavily rely on is the following connection between perimeter and Hausdorff measure:

1.8 Proposition. *Let $E \subseteq \mathbb{R}^n$ be a set of locally finite perimeter. Then for $F \subseteq \mathbb{R}^n$*
$$P(E;F) = |\mu_E|(F) = \mathcal{H}^{n-1}(F \cap \partial^* F). \tag{1.1}$$

1.3 Equivalence of the Sobolev and isoperimetric inequality

We first state the inequalities of interest in this chapter, starting with the Sobolev inequality, which is the basis for various embedding results (c.f. [1]):

1.9 Theorem (classical Sobolev inequality). *For every $f \in C_c^1(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} |\nabla f(x)| \,\mathrm{d}x \geq n\omega_n \left(\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}} \,\mathrm{d}x \right)^{\frac{n-1}{n}}, \tag{1.2}$$

where ω_n denotes the volume of the n-dimensional unit ball.

Though the inequality is strict for all $f \in C_c^1(\mathbb{R}^n)$, it is sharp as can be seen by an approximation of indicator functions in $BV(\mathbb{R}^n)$ by functions in $C_c^1(\mathbb{R}^n)$, see [27].

On the other hand, we consider the isoperimetric inequality, which connects the geometric properties of volume and perimeter of a set (c.f. [22, Theorem 14.1]):

1.10 Theorem (classical isoperimetric inequality). *Let $E \subseteq \mathbb{R}^n$ be a Lebesgue measurable set with $|E| < \infty$. Then,*

$$P(E) \geq n\omega_n^{\frac{1}{n}} |E|^{\frac{n-1}{n}}, \tag{1.3}$$

with equality iff E is equivalent to a ball.

Before we can established the equivalence of the inequalities we have stated before, we state the Minkowski inequality related to exponentiation of double integrals (cf. [16, (6.13.9)]):

1.11 Theorem (Minkowski inequality). *Let $f : \Omega_1 \times \Omega_2 \to [0, \infty)$ measurable, where $\Omega_1 \subseteq \mathbb{R}^m, \Omega_2 \subseteq \mathbb{R}^n$ are open sets. Then for $p > 1$ we have*

$$\left(\int_{\Omega_1} \left(\int_{\Omega_2} f(x,y) \,\mathrm{d}y \right)^p \mathrm{d}x \right)^{\frac{1}{p}} \leq \int_{\Omega_2} \left(\int_{\Omega_1} f(x,y)^p \,\mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}y.$$

The main result of this chapter states that each of the above inequalities can be used to prove the other one, i.e. both inequalities are equivalent. To this end, we follow the proof of [13].

1.12 Theorem. *The classical Sobolev inequality* (1.2) *is equivalent to the classical isoperimetric inequality* (1.3) *for compact sets with C^1-boundary.*

Proof . We first consider the implication Sobolev $\Rightarrow$ isoperimetric. Any compact set K with C^1-boundary has finite perimeter, so the indicator function $\mathbb{1}_K$ is of bounded variation. Therefore, by remark 1.5 we can approximate $\mathbb{1}_K$ by functions $u_j \in C_c^\infty(\mathbb{R}^n)$, such that $u_j \to \mathbb{1}_K$ in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} |\nabla u_j(x)| \,\mathrm{d}x \to P(K)$ as $j \to \infty$. Application of the Sobolev inequality yields

$$\begin{aligned} P(K) = \lim_{j\to\infty} \int_{\mathbb{R}^n} |\nabla u_j(x)| \,\mathrm{d}x \quad &\geq \quad n\omega_n^{\frac{1}{n}} \lim_{j\to\infty} \left(\int_{\mathbb{R}^n} |u_j(x)|^{\frac{n}{n-1}} \,\mathrm{d}x \right)^{\frac{n-1}{n}} = \\ &= \quad n\omega_n^{\frac{1}{n}} \left(\int_{\mathbb{R}^n} \mathbb{1}_K(x) \,\mathrm{d}x \right)^{\frac{n-1}{n}} = n\omega_n^{\frac{1}{n}} |K|^{\frac{n-1}{n}}. \end{aligned}$$

For the proof of isoperimetric $\Rightarrow$ Sobolev we make use of the coarea formula:

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla f(x)| \,\mathrm{d}x &= \int_{\mathbb{R}} P(\{f > t\}) \,\mathrm{d}t = \int_0^\infty P(\{f > t\}) + P(\{f < -t\}) \,\mathrm{d}t = \\ &= \int_0^\infty P(\{|f| > t\}) \,\mathrm{d}t.\end{aligned}$$

The last equality holds, because for almost every $t > 0$ the sets $\{f > t\}$ and $\{f < -t\}$ are of finite perimeter and lie at mutually positive distance. Now we can deduce the Sobolev inequality by successively applying the isoperimetric and Minkowski inequality:

$$\begin{aligned}\int_0^\infty P(\{|f| > t\}) \,\mathrm{d}t &\geq n\omega_n^{\frac{1}{n}} \int_0^\infty |\{|f| > t\}|^{\frac{n-1}{n}} \,\mathrm{d}t = \\ &= n\omega_n^{\frac{1}{n}} \int_0^\infty \left(\int_{\mathbb{R}^n} \mathbb{1}_{\{|f|>t\}}^{\frac{n}{n-1}}(x) \,\mathrm{d}x \right)^{\frac{n-1}{n}} \mathrm{d}t \geq \\ &\geq n\omega_n^{\frac{1}{n}} \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \mathbb{1}_{\{|f|>t\}}(x) \,\mathrm{d}t \right)^{\frac{n}{n-1}} \mathrm{d}x \right)^{\frac{n-1}{n}} = \\ &= n\omega_n^{\frac{1}{n}} \left(\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}} \,\mathrm{d}x \right)^{\frac{n-1}{n}}.\end{aligned}$$

■

1.4 Convex bodies and norms on $\mathbb{R}^n$

We first state some simple observations on the space $\mathbb{R}^n$ equipped with an arbitrary norm $\|\cdot\|$ (for a general reference see for instance [29]):

1.13 Facts. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$.

1. $\|\cdot\|$ can unambiguously be identified with its closed unit ball

   $$K := \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

   From now on, we will use the notation $\|\cdot\|_K$ to denote the norm with unit ball K.

2. The unit ball K of $\|\cdot\|_K$ is an origin-symmetric, compact and convex set with non-empty interior.

3. If conversely L is an origin-symmetric, compact and convex set with non-empty interior, then the function

   $$x \mapsto \inf \{\lambda > 0 : x \in \lambda L\},\ x \in \mathbb{R}^n,$$

   is the norm with unit ball L. It is also called *Minkowski functional* of L.

4. $\|\cdot\|_K$ is equivalent to the Euclidean norm $|\cdot|$, i.e. there exist constants $\alpha, \beta > 0$ such that
$$\alpha\,|x| \leq \|x\|_K \leq \beta\,|x| \quad \forall\, x \in \mathbb{R}^n.$$
Consequently, we can inscribe the closed ball $B_{\frac{1}{\beta}}$ into K and circumscribe $B_{\frac{1}{\alpha}}$ around K:
$$B_{\frac{1}{\beta}} \subseteq K \subseteq B_{\frac{1}{\alpha}}.$$

5. By the Riesz-Fischer theorem from functional analysis every element of the dual space $(\mathbb{R}^n)^*$ can be identified as an element of $\mathbb{R}^n$, and its norm can be computed via the scalar product
$$\|u\|_K^* := \max_{\|x\|_K \leq 1} x \cdot u = \max_{x \in K} x \cdot u, \quad u \in \mathbb{R}^n \simeq (\mathbb{R}^n)^*.$$

We have seen that certain convex sets arise naturally in the study of norms in $\mathbb{R}^n$, namely in the role of unit balls. The rest of this chapter consists of a short exposition to convex geometry. For further reading, see for example the books of Gruber [15] or Schneider [25].
We call a compact convex set a *convex body*. Let $\mathcal{C}^n$ denote the set of all convex bodies in $\mathbb{R}^n$. We call a convex body *proper*, if it has non-empty interior.
In the following we want to study the interplay of our norms with the Euclidean structure on $\mathbb{R}^n$ given by the Euclidean scalar product. The following concepts are important tools to this end:

1.14 Definition. Let $K \in \mathcal{C}^n$ be a convex body.

1. The function $h_K : \mathbb{R}^n \to \mathbb{R}$, defined by
$$h_K(u) := \max_{x \in K} x \cdot u,$$
is called *support function* of K.

2. The convex body
$$K^* := \{u \in \mathbb{R}^n : x \cdot u \leq 1 \text{ for all } x \in K\}$$
is called the *polar body* of K.

1.15 Facts.

1. For a unit vector $u \in \mathbb{S}^{n-1}$ the support function $h_K(u)$ is the distance from the unique support hyperplane with normal vector u to the origin. In particular, $x \cdot u = h_K(u)$ implies $x \in \partial K$ and u is a normal vector to K in x.

2. In the case that K is an origin-symmetric proper convex body, h_K is a norm on $\mathbb{R}^n$ and the polar body K^* is also an origin-symmetric proper convex body.

3. $u \in K^*$ if and only if $h_K(u) \leq 1$.

4. The dual norm $\|u\|_K^*$ of an element $u \in \mathbb{R}^n$ coincides with the support function $h_K(u)$ of the unit ball. Taking into account facts 2 and 3, we have the following identities ($x \in \mathbb{R}^n$):

$$\begin{aligned} \|u\|_{K^*} &= h_K(u) = \|u\|_K^*, \\ \|x\|_K &= h_{K^*}(x) = \|x\|_{K^*}^*. \end{aligned}$$

5. Writing $x \cdot u = \|x\|_K \dfrac{x}{\|x\|_K} \cdot u \leq \|x\|_K \max\limits_{\xi \in K} \xi \cdot u$ for $x \in \mathbb{R}^n \setminus \{0\}$, by definition of the support function the following analogue to the Cauchy-Schwarz inequality in Euclidean space holds:

$$x \cdot u \leq \|x\|_K \|u\|_{K^*}.$$

If $x \in \partial^* K$ and $u \in \mathbb{S}^{n-1}$, then equality holds if and only if u is the outer unit normal of K at x, i.e. $u = \nu_K(x)$ (see fact 1).

We now present Minkowski's famous result which describes the volume of an arbitrary linear combination of convex bodies as a polynomial in its scaling factors:

1.16 Theorem. *There exists a nonnegative, symmetric function $V : \mathcal{C}^n \to [0, \infty)$, called* mixed volume, *such that for any finite family of convex bodies $C_1, \dots, C_m$ and corresponding positive scalars $\lambda_1, \dots, \lambda_m \geq 0$*

$$|\lambda_1 C_1 + \cdots + \lambda_m C_m| = \sum_{i_1, \dots, i_n = 1}^{m} V(C_{i_1}, \dots, C_{i_n}) \lambda_{i_1} \dots \lambda_{i_n},$$

where the left hand side denotes the volume (Lebesgue measure) of $\lambda_1 C_1 + \cdots + \lambda_m C_m$ as usual.

Now let us fix two convex bodies $K, L \subseteq \mathbb{R}^n$ and a positive $\lambda > 0$ and compute the volume of the combination $K + \lambda L$ by means of Minkowski's theorem:

$$\begin{aligned} |K + \lambda L| = V(K, \dots, K) &+ nV(L, \underbrace{K, \dots, K}_{n-1 \text{ times}})\lambda + \binom{n}{2} V(L, L, \underbrace{K, \dots, K}_{n-2 \text{ times}})\lambda^2 + \cdots + \\ &+ V(L, \dots, L)\lambda^n = \\ &= \sum_{j=0}^{n} \binom{n}{j} V(\underbrace{K, \dots, K}_{n-j \text{ times}}, \underbrace{L, \dots, L}_{j \text{ times}})\lambda^j. \end{aligned}$$

The mixed volumes appearing on the right-hand side of the formula above are conveniently abbreviated by

$$V_j(K, L) := V(\underbrace{K, \dots, K}_{n-j \text{ times}}, \underbrace{L, \dots, L}_{j \text{ times}}), \ j = 0, \dots, n,$$

and called the *j-th mixed volume* of K and L. If in particular $L = B_1(0)$, then the above result is called *Steiner's formula* for parallel bodies and the j-th mixed volume is called *quermassintegral* $W_j(K)$.

An important consequence of Steiner's formula is the existence of the *relative Minkowski content* , which is the quantity defined by

$$\mathcal{M}_L(K) := \lim_{\lambda \searrow 0} \frac{|K + \lambda L| - |K|}{\lambda}$$

and can be expressed in terms of mixed volumes as

$$\mathcal{M}_L(K) = nV_1(K, L). \tag{1.4}$$

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2 The anisotropic perimeter

The definition of the perimeter depends on the norm we impose on our space $\mathbb{R}^n$; in our previous observations we used the Euclidean norm $|\cdot|$ which provides the standard (Euclidean) perimeter P.
The starting point of the following observations will be a generalization of the perimeter where we admit arbitrary norms in the definition. We will then give a link between this so-called anisotropic perimeter and the Euclidean perimeter via the equivalence of the corresponding norms. As it turns out, the analogue of formula (1.1) for the calculation of the perimeter also holds true in the anisotropic case. The anisotropic perimeter of a convex body is up to a factor equal to the more well known mixed volumes and Minkowski content of convex geometry.
The section 2.1 is devoted to the study of the anisotropic isoperimetric inequality. We will follow the approach of Maggi [22, chapter 20] for its proof.

We will now give the definition of the anisotropic perimeter of a set with respect to the unit ball K of the norm $\|\cdot\|_K$ (see [3, Definition 3.1]):

2.1 Definition. Let $\|\cdot\|_K$ be an arbitrary norm on $\mathbb{R}^n$ with corresponding unit ball K and $\Omega \subseteq \mathbb{R}^n$ an open set. Further let $E \subseteq \mathbb{R}^n$ be a Lebesgue measurable set. Then the *anisotropic perimeter* of E in Ω with respect to K is defined as

$$P_K(E;\Omega) := \sup\left\{\int_E \operatorname{div} T(x)\,\mathrm{d}x : T \in C_c^1(\Omega;\mathbb{R}^n),\ \|T\|_K \le 1\right\}.$$

2.2 Remark. By the inclusion relation of the unit ball K and certain Euclidean balls stated in 1.13, we can compare the anisotropic perimeter with the standard perimeter via

$$\frac{1}{\beta}P(E;\Omega) \le P_K(E;\Omega) \le \frac{1}{\alpha}P(E;\Omega),$$

and E has finite anisotropic perimeter with respect to K if and only if E is of finite perimeter.

We now state the anisotropic counterpart of Proposition 1.8, whose proof can be found in [3]:

2.3 Proposition. *Let $E \subset \mathbb{R}^n$ a set of finite perimeter. Then*

$$P_K(E;\Omega) = \int_{\Omega\cap\partial^* E} \|\nu_E(x)\|_{K^*}\,\mathrm{d}\mathcal{H}^{n-1}(x).$$

We now come back to the theory of mixed volumes introduced in chapter 1.4. At the beginning we first state, how the first mixed volume $V_1(K,L)$ can be obtained up to a factor by integrating the support function of h_L over the unit sphere with respect to the surface area measure S_{n-1} (see [25, (5.34)]):

$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} h_L(u) S_{n-1}(K, du).$$

The surface area measure of K is the $(n-1)$-dimensional Hausdorff measure of the reverse spherical image, so if $\nu_K : \partial^* K \to \mathbb{S}^{n-1}$ maps every regular x point of K onto its unit normal vector $\nu_K(x)$, then $S_{n-1} = (\nu_K)_\# \mathcal{H}^{n-1}$ and

$$\begin{aligned} V_1(K,L) &= \frac{1}{n}\int_{S^{n-1}} h_L(x)\,\mathrm{d}(\nu_K)_\#\mathcal{H}^{n-1}(x) = \\ &= \frac{1}{n}\int_{\partial^*\, K} h_L(\nu_K(x))\,\mathrm{d}\mathcal{H}^{n-1}(x) = \\ &= \frac{1}{n}\int_{\partial^* K} \|\nu_K(x)\|_{L^*}\,\mathrm{d}\mathcal{H}^{n-1}(x) = \frac{1}{n}P_L(K). \end{aligned}$$

By formula (1.4) we also have $\mathcal{M}_L(K) = P_L(K)$.

The observations on the relationship between anisotropic perimeter, first mixed volume and Minkowski content are summarized in the following proposition:

2.4 Proposition. *Let $K, L \subseteq \mathbb{R}^n$ be proper convex bodies, and additionally L be origin-symmetric. Then*

$$P_L(K) = \mathcal{M}_L(K) = nV_1(K,L).$$

2.1 The anisotropic isoperimetric inequality

For the following results we refer to [22, Chapter 20]:
In this section our main goal will be the outline of the proof for following theorem:

2.5 Theorem. *If $E \subseteq \mathbb{R}^n$ is a set of finite perimeter such that $|E| < \infty$, and $K \subseteq \mathbb{R}^n$ is the closed unit ball of the norm $\|\cdot\|_K$ on $\mathbb{R}^n$, then*

$$P_K(E) \geq n\,|K|^{\frac{1}{n}}\,|E|^{\frac{n-1}{n}}. \tag{2.1}$$

Equality holds precisely for sets E homothetic to K, up to a set of measure zero.

The first easy task we are concerned with is showing, that for the unit ball K equality actually holds in (2.1):

2.6 Proposition. *The unit ball K of the norm $\|\cdot\|_K$ satisfies*

$$P_K(K) = n\,|K|\,,$$

so for $E = K$, equality holds in (2.1).

Proof. We apply the divergence theorem to the identity map id_K on K, where $\operatorname{div}\mathrm{id}_K = n$. Furthermore, observe that for $x \in \partial^* K$ we have $x \cdot \nu_K(x) = \|x\|_K \|\nu_K(x)\|_{K^*}$ by Facts

1.15:

$$n\,|K| = \int_K \operatorname{div} \mathrm{id}_K(x)\,\mathrm{d}x = \int_{\partial^* K} x \cdot \nu_K(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) =$$
$$= \int_{\partial^* K} \underbrace{\|x\|_K}_{=1} \|\nu_K(x)\|_{K^*}\,\mathrm{d}\mathcal{H}^{n-1}(x) = P_K(K).$$

■

2.7 Lemma. *Let $\varphi \in C_c^1(\mathbb{R}^n)$ with $\varphi \geq 0$. Then there exists a diffeomorphism $g : \{\varphi > 0\} \to (0,1)^n$, such that for every x with $\varphi(x) > 0$ its Jacobian matrix $dg(x)$ is lower triangular with diagonal entries $\partial_k g^{(k)}(x) > 0$ and*

$$\det dg(x) = \frac{\varphi(x)}{\int_{\mathbb{R}^n} \varphi\,\mathrm{d}\mathcal{L}^n}.$$

Proof .
For a point $x \in \mathbb{R}^n$, a real value $t \in \mathbb{R}$, and $k = 1, \dots, n$ we define the halfspaces $H_k(t)$ and hyperplanes $I_k(t)$ parallel to the coordinate axes and containing x as

$$H_k(t) := \{y \in \mathbb{R}^n : y_k < x_k\}\,, \quad I_k(t) := \{y \in \mathbb{R}^n : y_k = x_k\}\,.$$

For $\varphi \in C_c^1(\mathbb{R}^n), \varphi \geq 0$, we define a map $g \in C^1(\mathbb{R}^n; [0,1]^n)$ via

$$g^{(1)}(x) := \frac{\int_{H_1(x_1)} \varphi\,\mathrm{d}\mathcal{L}^n}{\int_{\mathbb{R}^n} \varphi\,\mathrm{d}\mathcal{L}^n} \in [0,1],$$
$$g^{(k)}(x) := \frac{\int_{H_k(x_k) \cap I_1(x_1) \cap \ldots \cap I_{k-1}(x_{k-1})} \varphi\,\mathrm{d}\mathcal{H}^{n-k+1}}{\int_{I_1(x_1) \cap \ldots \cap I_{k-1}(x_{k-1})} \varphi\,\mathrm{d}\mathcal{H}^{n-k+1}} \in [0,1],$$

where $k = 2, \dots, n$ and $g^{(k)}$ is the k-th component of g. Note that the k-th component $g^{(k)}$ only depends on the first k variables $x_1, \dots, x_k$. We observe, that by Fubini's theorem each of the integrals in the nominator can be written as

$$\int_{H_k(x_k) \cap I_1(x_1) \cap \ldots \cap I_{k-1}(x_{k-1})} \varphi\,\mathrm{d}\mathcal{H}^{n-k+1} = \int_{-\infty}^{x_k} \int_{I_1(x_1) \cap \ldots \cap I_k(s)} \varphi\,\mathrm{d}\mathcal{H}^{n-k}\,\mathrm{d}s.$$

We immediately see, that g is a bijective function from $\{\varphi > 0\}$ to $(0,1)^n$. Furthermore, g is continuously differentiable and

$$\partial_k g^{(k)}(x) = \frac{\int_{I_1(x_1) \cap \ldots \cap I_k(x_k)} \varphi\,\mathrm{d}\mathcal{H}^{n-k}}{\int_{I_1(x_1) \cap \ldots \cap I_{k-1}(x_{k-1})} \varphi\,\mathrm{d}\mathcal{H}^{n-k+1}}, \qquad k = 1, \dots, n$$
$$\partial_j g^{(k)}(x) = 0, \qquad k = 1, \dots, n-1,\ k < j \leq n$$

for all $x \in \mathbb{R}^n$. We infer, that the matrix corresponding to the differential $dg(x)$ is lower triangular and

$$\partial_k g^{(k)}(x) > 0, \quad \forall\, x \in \{\varphi > 0\}\,,$$
$$\det dg(x) = \prod_{k=1}^n \partial_k g^{(k)}(x) = \frac{\varphi(x)}{\int_{\mathbb{R}^n} \varphi\,\mathrm{d}x}, \quad \forall\, x \in \{\varphi > 0\}\,.$$

The previous relations show that g is a diffeomorphism from $\{\varphi > 0\}$ onto $(0,1)^n$. ■

2.8 Lemma. *Let $u, v \in C_c^1(\mathbb{R}^n)$ with $u, v \geq 0$ and $\int_{\mathbb{R}^n} u^{\frac{n}{n-1}}\,\mathrm{d}x = \int_{\mathbb{R}^n} v^{\frac{n}{n-1}}\,\mathrm{d}x = 1$. Then*

$$n \int_{\mathbb{R}^n} v \,\mathrm{d}x \leq \max\{\|x\|_K : x \in \operatorname{spt} v\} \int_{\mathbb{R}^n} \|\nabla u\|_{K^*}\,\mathrm{d}x. \tag{2.2}$$

Proof. We apply Lemma 2.7 to the functions $\varphi = u^{\frac{n}{n-1}}$ and $\varphi = v^{\frac{n}{n-1}}$ to obtain maps g_u and g_v. The composition $f := g_v^{-1} \circ g_u : \{u > 0\} \to \{v > 0\}$ is a diffeomorphism and has an lower triangular Jacobian matrix with positive diagonal entries $\partial_j f^{(j)}$ on $\{u > 0\}$. A simple manipulation using the chain rule yields

$$df(x) = dg_v^{-1}(g_u(x))dg_u(x) = (dg_v(f(x)))^{-1}dg_u(x),$$

hence applying the determinant to both side we observe

$$\det df(x) = \frac{u^{\frac{n}{n-1}}(x)}{v^{\frac{n}{n-1}}(f(x))}.$$

We apply the transformation rule for integrals to establish a link between the functions u and v:

$$\int_{\mathbb{R}^n} v \,\mathrm{d}x = \int_{\{u>0\}} (v \circ f) \det df \,\mathrm{d}x = \int_{\{u>0\}} u(\det df)^{\frac{1}{n}}\,\mathrm{d}x. \tag{2.3}$$

The determinant on the right-hand side can be estimated with help of the arithmetic-geometric mean inequality:

$$(\det df)^{\frac{1}{n}} = \left(\prod_{k=1}^{n} \partial_k f^{(k)}\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{k=1}^{n} \partial_k f^{(k)} = \frac{\operatorname{div} f}{n}.$$

Finally, we apply this estimate to (2.3) and keep in mind, that f takes its values in $\operatorname{spt} v$, so all in all we recover the statement of our claim

$$\begin{aligned} n \int_{\mathbb{R}^n} v \,\mathrm{d}x &\leq \int_{\operatorname{spt}\, u} u \operatorname{div} f \,\mathrm{d}x = \int_{\mathbb{R}^n} u \operatorname{div} f \,\mathrm{d}x = \int_{\mathbb{R}^n} -(\nabla u)\cdot f \,\mathrm{d}x \\ &\leq \max\{\|x\|_K : x \in \operatorname{spt} v\} \int_{\mathbb{R}^n} \|\nabla u\|_{K^*}\,\mathrm{d}x. \end{aligned}$$

■

Before we turn to the proof of the anisotropic isoperimetric inequality (2.1), we provide the statement of Reshetnyak's continuity theorem without proof (for the proof see e.g. [22, Chapter 20.3]):

2.9 Theorem (Reshetnyak's continuity theorem). *Let $\Phi : \mathbb{S}^{n-1} \to [0,\infty)$ be continuous and $(\nu_j)_{j\in\mathbb{N}}$ and ν be $\mathbb{R}^n$-valued Radon measures on $\mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \Phi(D_{|\nu_j|}\nu_j(x))\,\mathrm{d}|\nu_j|(x) \overset{j\to\infty}{\to} \int_{\mathbb{R}^n} \Phi(D_{|\nu|}\nu(x))\,\mathrm{d}|\nu|(x),$$

whenever $\nu_j \overset{*}{\rightharpoonup} \nu$, $|\nu_j|(\mathbb{R}^n) \to |\nu|(\mathbb{R}^n)$ and $|\nu|(\mathbb{R}^n) < \infty$.

Proof of 2.5.
To obtain the anisotropic isoperimetric inequality for a bounded set $E \subseteq \mathbb{R}^n$ of finite perimeter, we apply inequality (2.2) to the functions

$$u_\varepsilon := \frac{\mathbb{1}_E \star \rho_\varepsilon}{\|\mathbb{1}_E \star \rho_\varepsilon\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}}, \quad v_\varepsilon := \frac{\mathbb{1}_K \star \rho_\varepsilon}{\|\mathbb{1}_K \star \rho_\varepsilon\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}}.$$

If we take a point $x \in \operatorname{spt} v_\varepsilon \subseteq K + \varepsilon B_1(0)$, then there exist $k \in K$ and $b \in B_1(0)$ such that $x = k + \varepsilon b$ and

$$\|k + \varepsilon b\|_K \leq \|k\|_K + \varepsilon \frac{1}{\beta} |b| \leq 1 + \frac{\varepsilon}{\beta},$$

which justifies

$$\lim_{\varepsilon \to 0^+} \max \{\|x\|_K : x \in \operatorname{spt} v_\varepsilon\} = 1.$$

The left-hand side of inequality (2.2) reads as

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} v_\varepsilon \,\mathrm{d}x = \frac{|K|}{|K|^{\frac{n-1}{n}}} = |K|^{\frac{1}{n}}$$

in the limit. For the right-hand side we have the convergence $\nu_\varepsilon := \nabla u_\varepsilon \mathcal{L}^n \overset{*}{\rightharpoonup} |E|^{-\frac{n-1}{n}} \mu_E$ and by Reshetnyak's continuity theorem 2.9 and the definition of the normal vector ν_E

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \|\nabla u_\varepsilon\|_{K^*} \,\mathrm{d}x = \frac{1}{|E|^{\frac{n-1}{n}}} \int_{\partial^* E} \|\nu_E\|_{K^*} \,\mathrm{d}\mathcal{H}^{n-1},$$

since the derivative $D_{|\nu_\varepsilon|}\nu_\varepsilon(x)$ equals to $\frac{\nabla u_\varepsilon(x)}{|\nabla u_\varepsilon(x)|}$ for $|\nu_\varepsilon|$-a.e. $x \in \mathbb{R}^n$.
The last task consists of reducing the case of unbounded sets to already proved statements: If $E \subseteq \mathbb{R}^n$ is a set of finite perimeter, then we can rewrite the reduced boundary of the intersection of E and any ball $B_R, R > 0$ as

$$\partial^*(E \cap B_R) = (E^{(1)} \cap \partial B_R) \cup (B_R \cap \partial^* E) \cup (\{\nu_E = \nu_{B_R}\}),$$

in the sense, that both sides are equal up to a set of $\mathcal{H}^{n-1}$-measure zero. Since $\mathcal{H}^{n-1} \restriction \partial^* E$ is a Radon measure, almost each set in the foliation $\{\nu_E = \nu_{B_R}\}, R > 0$, has $\mathcal{H}^{n-1}$-measure zero. Moreover, in the following integrals we have replaced $E^{(1)}$ by the original set E, because $\mathcal{H}^{n-1}((E \triangle E^{(1)}) \cap \partial B_R) = 0$ for almost every $R > 0$. Ultimately, our inequality for bounded sets yields for a.e. $R > 0$

$$\begin{aligned} n\,|K|^{\frac{1}{n}}\,|E \cap B_R|^{\frac{n-1}{n}} &\leq \int_{B_R \cap \partial^* E} \|\nu_E\|_{K^*} \,\mathrm{d}\mathcal{H}^{n-1} + \int_{E \cap \partial B_R} \|\nu_E\|_{K^*} \,\mathrm{d}\mathcal{H}^{n-1} \leq \\ &\leq P_K(E) + \beta \mathcal{H}^{n-1}(E \cap \partial B_R). \end{aligned}$$

Again, the summand $\mathcal{H}^{n-1}(E \cap \partial B_R) = 0$ for a.e. $R > 0$. By choosing a suitable sequence $R_i \to \infty$ of radii, we obtain the stated result. ■

2.10 Remark. The last statement in this chapter deals with a functional counterpart to the anisotropic isoperimetric inequality (see [8]). The natural space for our functions is the *homogeneous Sobolev space* , defined for $p \in [1, n)$ and $p^\star := \frac{np}{n-p}$ as

$$\dot{W}^{1,p}(\mathbb{R}^n) := \left\{ f \in L^{p^\star}(\mathbb{R}^n) : \nabla f \in L^p(\mathbb{R}^n) \right\}.$$

The extremal functions, for which equality holds, are given by

$$h_p(x) := \frac{1}{(\sigma_p + \|x\|_K^q)^{\frac{n-p}{p}}}, \quad p > 1,$$
$$h_1(x) := \frac{\mathbb{1}_K(x)}{|K|^{\frac{n-1}{n}}},$$

where $q := \frac{p}{p-1}$ is the dual exponent of p and $\sigma_p > 0$ is chosen so that $\|h_p\|_{L^{p^\star}} = 1$. By $\|\nabla f\|_{L^p}$ we mean - similarly to the Euclidean case - the expression

$$\|\nabla f\|_{L^p} := \left(\int_{\mathbb{R}^n} \|\nabla f\|_{K^*}^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

The *anisotropic Sobolev inequality* now reads as:
If $p \in (1, n)$ *and* $f \neq 0$ *lies in* $\dot{W}^{1,p}(\mathbb{R}^n)$, *then*

$$\|\nabla f\|_{L^p} \geq \|\nabla h_p\|_{L^p} \|f\|_{L^{p^\star}}.$$

3 The fractional perimeter

Sobolev spaces play a big role in the discussion of partial differential equations, where oftentimes some irregularity in the solution is desired to accurately describe a real world phenomenon (e.g. in traffic flow models the initial condition might have a point of discontinuity at a traffic light). The need for a suitable function space arises, in which the boundary conditions can be described, i.e. the range of a linear operator (called trace operator) mapping smooth functions to the restriction on the boundary, which is additionaly continuous on the whole Sobolev space. It turns out, that fractional Sobolev spaces are the ideal framework for the image of such traces.

This chapter is structured as follows:

In section 3.1 we will define the fractional Sobolev spaces and the (semi)norm thereon. In search of basic examples, we will prove that $C_c^\infty(\Omega), \Omega \subseteq \mathbb{R}^n$ open, is a subspace of all fractional Sobolev spaces. Then we turn to the completion of $C_c^\infty(\Omega)$ with respect to the seminorm, called homogeneous fractional Sobolev space, and explicitly present a sequence of smooth compactly supported functions converging to a given function in this space. The definition of the fractional perimeter and the relationship to the previously defined Gagliardo seminorm conclude the section.

Section 3.2 illustrates the result of Bourgain, Brezis and Mironescu [5] dealing with the limit of the Gagliardo seminorm as $s \to 1^-$, as well as all the steps needed for its proof. We will then state a similar result established by Dávila [9], who calculated the limit for functions of bounded variation. Since sets of finite perimeter correspond to indicator functions with bounded variation, this gives use a tool to obtain a convergence result for fractional perimeters.

An analogue to the Bourgain, Brezis and Mironescu result for the limit $s \to 0^+$ is presented in section 3.3. It was proved by Maz'ya and Shaposhnikova in [23].

In section 3.4 we reenact the proof of the fractional Hardy inequality given by Frank and Seiringer [12], ultimately leading to the fractional Sobolev inequality and the fractional isoperimetric inequality. The classification of equality cases in all of these inequality heavily relies on the notion of symmetric decreasing rearrangement of a function. We will show that the Gagliardo seminorm does not increase under symmetric decreasing rearrangement.

$\boxed{\text{Throughout this chapter } \Omega \text{ denotes an open set in } \mathbb{R}^n.}$

3.1 Defintion and basic facts

The following introduction to fractional Sobolev spaces is based on [11].

3.1 Definition. For a fractional exponent $s \in (0, 1)$, and $p \in [1, \infty)$ we define the *fractional Sobolev space* $W^{s,p}(\Omega)$ by

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega \times \Omega) \right\}.$$

We endow $W^{s,p}(\Omega)$ with the *Gagliardo seminorm*

$$[u]_{W^{s,p}} := \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{p}},$$

or with the *Gagliardo norm*

$$\|u\|_{W^{s,p}} := \left(\int_\Omega |u|^p \,\mathrm{d}x + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{p}},$$

respectively.

3.2 Remark (Rewriting the integrand in the Gagliardo seminorm). By the transformation rule for integrals, the double integral in the definition of the Gagliardo seminorm can be rewritten as

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \int_{\Omega-y} \frac{|u(y+h)-u(y)|^p}{|h|^{n+sp}} \,\mathrm{d}h\,\mathrm{d}y$$

by means of the transformation $(h,y) \mapsto (y+h,y)$ from $\{(h,y) : h \in \Omega - y\}$ to $\Omega \times \Omega$. This simple transformation will be a useful tool in obtaining inequalities for Gagliardo seminorms.

We first are interested in some familiar examples of functions that belong to fractional Sobolev spaces and we will see that all smooth functions with compact support have finite Gagliardo norm. To this end, we now state a short lemma relating the difference of function values to an integral (see the proof of Proposition 9.3 in [6]):

3.3 Lemma. *Let* $u \in C^1(\mathbb{R}^n)$*, then for* $x, h \in \mathbb{R}^n$ *we have*

$$u(x+h) - u(x) = \int_0^1 \nabla u(x+th) \cdot h \,\mathrm{d}x. \tag{3.1}$$

Proof . We consider the function

$$v(t) := u(x+th), \quad t \in \mathbb{R}$$

with derivative $v'(t) = h \cdot \nabla u(x+th)$. Then, by the fundamental theorem of calculus

$$u(x+h) - u(x) = v(1) - v(0) = \int_0^1 v'(t) \,\mathrm{d}t = \int_0^1 h \cdot \nabla u(x+th) \,\mathrm{d}t.$$

■

The proof of the following statement, that smooth functions with compact support belong to all fractional Sobolev spaces, utilizes ideas from [11, Chapter 2]:

3.4 Example. Every smooth, compactly supported function $u \in C_c^\infty(\Omega)$ belongs to $W^{s,p}(\Omega)$, where $0 < s < 1$ and $p \in [1, \infty)$:
To show this statement, we split the seminorm into two summands as follows

$$\begin{aligned}[u]_{W^{s,p}}^p &= \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y = \\ &= \int_\Omega \int_{\Omega \cap \{|x-y|<1\}} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \int_{\Omega \cap \{|x-y|\geq 1\}} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We apply the transformation in Remark 3.2 and identity (3.1) to the first summand, and obtain

$$\begin{aligned}\int_\Omega \int_{\Omega \cap \{|x-y|<1\}} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y &\leq \int_\Omega \int_{B_1} \left(\frac{|u(x+h) - u(x)|}{|h|} \right)^p \frac{1}{|h|^{n-(1-s)p}} \,\mathrm{d}h\,\mathrm{d}x \leq \\ &\leq \int_\Omega \int_{B_1} \left(\int_0^1 \frac{|\nabla u(x+th)|}{|h|^{\frac{n}{p}-(1-s)}} \,\mathrm{d}t \right)^p \mathrm{d}h\,\mathrm{d}x \leq \\ &\leq \int_{\mathbb{R}^n} \int_{B_1} \int_0^1 \frac{|\nabla u(x+th)|^p}{|h|^{n-(1-s)p}} \,\mathrm{d}t\,\mathrm{d}h\,\mathrm{d}x = \\ &= \int_{B_1} \int_0^1 \frac{\|\nabla u\|_{L^p(\mathbb{R}^n)}^p}{|h|^{n-(1-s)p}} \,\mathrm{d}t\,\mathrm{d}h = \\ &= \frac{|\mathbb{S}^{n-1}|}{(1-s)p} \|\nabla u\|_{L^p(\mathbb{R}^n)}^p < \infty.\end{aligned}$$

Note, that we have tacitly extended u by 0 outside of Ω.
For the second summand we exploit the convexity of the function $t \mapsto |t|^p$ on $\mathbb{R}$, so we get

$$\begin{aligned}\int_\Omega \int_{\Omega \cap \{|x-y|\geq 1\}} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y &\leq 2^{p-1} \int_\Omega \int_{\Omega \cap \{|x-y|\geq 1\}} \frac{|u(x)|^p + |u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y \leq \\ &\leq 2^p \int_\Omega \int_{|h|\geq 1} \frac{|u(x)|^p}{|h|^{n+sp}} \,\mathrm{d}h\,\mathrm{d}x = \\ &= \frac{2^p |\mathbb{S}^{n-1}|}{sp} \|u\|_{L^p(\Omega)}^p < \infty.\end{aligned}$$

Putting both estimates together, we see that the Gagliardo seminorm and thus the Gagliardo norm of u is finite. ▲

3.5 Remark. The above proof can be repeated for all class of functions as long as they satisfy certain differentiability conditions and can be extended onto $\mathbb{R}^n$, such that the extension operator is continuous.
This means, for example, that if Ω has bounded Lipschitz boundary, then $W^{1,p}(\Omega) \subseteq W^{s,p}(\Omega)$. Furthermore, we have that the embedding is continous (see Proposition 2.2 in [11]; Example 9.1 therein shows, that such inclusions fail to hold in general, if the assumption that Ω has Lipschitz boundary is dropped).

The even stronger statement, that $C_c^\infty(\mathbb{R}^n)$ is dense in every $W^{s,p}(\mathbb{R}^n)$, holds. In the following remark we give a sequence of approximating functions in $C_c^\infty(\mathbb{R}^n)$:

3.6 Remark. We want to explicitly construct a sequence of smooth, compactly supported functions that converges to a given $u \in W^{s,p}(\mathbb{R}^n)$ (see [1, Theorem 7.38]): Take $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\psi(t) = 1$ if $t \leq 0$ and $\psi(t) = 0$ for $t \geq 1$, and then set

$$\psi_j(x) := \psi(|x| - j)$$

for $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$. The ψ_j defined that way can be thought of as bump functions, meaning that $\psi_j \in C_c^\infty(\mathbb{R}^n)$ with $\psi_j = 1$ on $B_j(0)$ and $\psi_j = 0$ on $B_{j+1}^c(0)$.
Furthermore, let $(\rho_j)_{j\in\mathbb{N}}$ be a family of smooth mollfiers. Then

$$u_j := (\psi_j u) * \rho_j$$

converges to u with respect to the Gagliardo seminorm as well as

$$\int_{\mathbb{R}^n} |u - u_j|^p \,\mathrm{d}x \overset{j\to\infty}{\to} 0$$

for every $p \geq 1$.

3.7 Definition. Let $E \subseteq \mathbb{R}^n$ be a Borel set, and $0 < s < 1$ a fractional exponent. Then the *fractional s-perimeter* $P_s(E)$ of E is defined as

$$P_s(E) := \int_E \int_{E^c} \frac{1}{|x-y|^{n+s}} \,\mathrm{d}x\,\mathrm{d}y.$$

3.8 Remark (Relation between seminorm and s-perimeter). The fractional s-perimeter of a set $E \subseteq \mathbb{R}^n$ coincides with half the seminorm of its indicator function with $p = 1$, i.e.

$$P_s(E) = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\mathbb{1}_E(x) - \mathbb{1}_E(y)|}{|x-y|^{n+s}} \,\mathrm{d}x\,\mathrm{d}y, \tag{3.2}$$

since the expression $|\mathbb{1}_E(x) - \mathbb{1}_E(y)|$ in the nominator of the integrand in $[\mathbb{1}_E]_{W^{s,1}}$ is equal to 1 precisely for $x \in E$ and $y \in E^c$ or vice versa.

3.2 The limiting case $s \to 1^-$

All results of the following section are due to Brezis, Bourgain and Mironescu [5], unless stated otherwise.

3.9 Definition. A sequence (ρ_j) of non-negative functions $\rho_j : \mathbb{R}^n \to [0, \infty)$ is called a sequence of *radial mollifiers* if for every $j \in \mathbb{N}$

- $\rho_j(x)$ only depends on the length $|x|$ of x, so by abuse of notation $\rho_j(x) = \rho_j(|x|)$,
- $\int_{\mathbb{R}^n} \rho_j(x) \,\mathrm{d}x = 1$,
- for every $\delta > 0$
$$\lim_{j\to\infty} \int_\delta^\infty \rho_j(r) r^{n-1} \,\mathrm{d}r = 0. \tag{3.3}$$

Analogously, we can define a family of radial mollifiers $(\rho_\varepsilon), \varepsilon > 0$, where we consider the limit $\varepsilon \to 0^+$.

Before we show some preparatory convergence results for integrals involving radial mollifiers, we want to present an important family of those mollifiers:

3.10 Example. For $0 < s < 1, p \in [1, \infty)$ and $\varepsilon > 0$ the functions

$$\rho_\varepsilon(x) := C_\varepsilon \frac{1}{|x|^{n-\varepsilon p}} \mathbb{1}_{B(1/\varepsilon)}(x), \tag{3.4}$$

where $B(1/\varepsilon) := \left\{|x| < \frac{1}{\varepsilon}\right\}$ and $C_\varepsilon = \frac{\varepsilon^{\varepsilon p+1} p}{|\mathbb{S}^{n-1}|}$ make up a family of radial mollifiers. To see this, we first compute the integrals over $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} \rho_\varepsilon(x) \, \mathrm{d}x = C_\varepsilon |\mathbb{S}^{n-1}| \int_0^{\frac{1}{\varepsilon}} r^{\varepsilon p - 1} \, \mathrm{d}r = C_\varepsilon \frac{|\mathbb{S}^{n-1}|}{\varepsilon^{\varepsilon p+1} p} = 1.$$

In a similar fashion, we check for $\delta > 0$ and $\frac{1}{\varepsilon} > \delta$

$$\int_\delta^{\frac{1}{\varepsilon}} \rho_\varepsilon(r) r^{n-1} \, \mathrm{d}r = \frac{\varepsilon^{\varepsilon p+1} p}{|\mathbb{S}^{n-1}|} \frac{1}{\varepsilon p} \left(\frac{1}{\varepsilon^{\varepsilon p}} - \delta^{\varepsilon p} \right) = \frac{1}{|\mathbb{S}^{n-1}|} - \frac{\varepsilon^{\varepsilon p} \delta^{\varepsilon p}}{|\mathbb{S}^{n-1}|} \overset{\varepsilon \to 0}{\to} 0.$$

▲

The following lemma is taken from [6, Proposition 9.3]:

3.11 Lemma. *Let $u \in W^{1,p}(\mathbb{R}^n), 1 \leq p < \infty$ and for each $h \in \mathbb{R}^n$ define the function $u_h(x) := u(x+h)$. Then the map $h \mapsto u_h$ is a uniformly continuous map from $\mathbb{R}^n$ to $L^p(\mathbb{R}^n)$. In particular, there exists a constant $C > 0$ such that for all $h \in \mathbb{R}^n$*

$$\|u_h - u\|_{L^p} \leq C \, |h| \, . \tag{3.5}$$

Proof. We first show the formula for any $u \in C_c^\infty(\mathbb{R}^n)$ and then proceed by a standard density argument: For $u \in C_c^\infty(\mathbb{R}^n)$ we can apply Lemma 3.3, so

$$\begin{aligned}
\int_{\mathbb{R}^n} |u(x+h) - u(x)|^p \, \mathrm{d}x \leq |h|^p \int_{\mathbb{R}^n} \int_0^1 |\nabla u(x+th)|^p \, \mathrm{d}t \, \mathrm{d}x = \\
= |h|^p \int_0^1 \int_{\mathbb{R}^n} |\nabla u(x+th)|^p \, \mathrm{d}x \, \mathrm{d}t = \\
= |h|^p \int_0^1 \int_{\mathbb{R}^n} |\nabla u(y)|^p \, \mathrm{d}y \, \mathrm{d}t = |h|^p \, \|\nabla u\|_{L^p}^p .
\end{aligned}$$

For the general case that $u \in W^{1,p}(\mathbb{R}^n)$ there exists a sequence (u_i) of functions in $C_c^\infty(\mathbb{R}^n)$ such that $u_i \to u$ in $L^p(\mathbb{R}^n)$ and $\nabla u_i \to \nabla u$ in $L^p(\mathbb{R}^n)$. The inequality (3.5) also holds in the limit, since

$$\left(\int_{\mathbb{R}^n} |u(x+h)-u(x)-(u_i(x+h)-u_i(x))|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \leq$$
$$\leq \left(\int_{\mathbb{R}^n} |u(x+h)-u_i(x+h)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^n} |u(x)-u_i(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \stackrel{i\to\infty}{\to} 0.$$

■

3.12 Lemma. *Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded set, $u \in W^{1,p}(\Omega)$, and $1 \leq p < \infty$. Furthermore, let $\rho \in L^1(\mathbb{R}^n)$, $\rho \geq 0$. Then*

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho(x-y) \,\mathrm{d}x \,\mathrm{d}y \leq C \|\nabla u\|_{L^p}^p \|\rho\|_{L^1},$$

where C only depends on p and Ω.

Proof . It is well known that each function $u \in W^{1,p}(\Omega)$ can be extended to a function $u \in W^{1,p}(\mathbb{R}^n)$ in such a way, that $\|u\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}$ with a constant $C > 0$ depending only on p and Ω. By Lemma 3.11 we get

$$\left(\int_{\mathbb{R}^n} |u(x+h)-u(x)| \,\mathrm{d}x\right)^{\frac{1}{p}} \leq |h| \, \|\nabla u\|_{L^p(\mathbb{R}^n)} \leq C \, |h| \, \|\nabla u\|_{L^p(\Omega)},$$

for all $h \in \mathbb{R}^n$. Now we apply the transformation of Remark 3.2 to obtain

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho(x-y) \,\mathrm{d}x \,\mathrm{d}y \leq \int_{\mathbb{R}^n} \frac{\rho(h)}{|h|^p} \int_{\mathbb{R}^n} |u(x+h)-u(x)|^p \,\mathrm{d}x \,\mathrm{d}h \leq$$
$$\leq C^p \|\nabla u\|_{L^p}^p \int_{\mathbb{R}^n} \rho(h) \,\mathrm{d}h = C^p \|\nabla u\|_{L^p}^p \|\rho\|_{L^1}.$$

■

3.13 Lemma. *Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded set, $u \in L^p(\Omega)$, and $1 < p < \infty$. Then*

$$\lim_{j\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho_j(x-y) \,\mathrm{d}x \,\mathrm{d}y = C_{n,p} \|\nabla u\|_{L^p}^p, \tag{3.6}$$

where $\|\nabla u\|_{L^p} = \infty$ if $u \notin W^{1,p}$, and the constant $C_{n,p}$ only depends on n and p.

Proof .
For an arbitrary function $u \in L^p$ we define

$$U_j(x,y) := \frac{|u(x)-u(y)|}{|x-y|} \rho_j^{\frac{1}{p}}(x-y),$$

where (ρ_j) is a sequence of radial mollifiers. We first deal with the case $u \in W^{1,p}$: if $v \in W^{1,p}$ is another function and V_j defined accordingly, then

$$|\|U_j\|_{L^p} - \|V_j\|_{L^p}| \leq \|U_j - V_j\|_{L^p} = \tag{3.7}$$
$$= \left(\int_\Omega \int_\Omega \frac{||u(x)-u(y)| - |v(x)-v(y)||^p}{|x-y|^p} \rho_j(x-y) \,\mathrm{d}x \,\mathrm{d}y\right)^{\frac{1}{p}} \leq$$
$$\leq \left(\int_\Omega \int_\Omega \frac{|(u(x)-v(x)) - (u(y)-v(y))|^p}{|x-y|^p} \rho_j(x-y) \,\mathrm{d}x \,\mathrm{d}y\right)^{\frac{1}{p}} \leq$$
$$\leq C \|\nabla(u-v)\|_{L^p} \underbrace{\|\rho_j\|_{L^1}}_{=1} \tag{3.8}$$

by Lemma 3.12. If we can show that

$$\lim_{j\to\infty} \|U_j\|_{L^p}^p = K\|\nabla u\|_{L^p}^p \tag{3.9}$$

holds for a dense subset of $W^{1,p}$, then the above inequality asserts the validity for any function in $W^{1,p}$. By density, for each $\varepsilon > 0$ there exists a function u_ε in the dense set such that $\|v - u_\varepsilon\|_{W^{1,p}} < \varepsilon$. Then we can estimate

$$\Big|\|V_j\|_{L^p} - K\|\nabla v\|_{L^p}\Big| \le \underbrace{\Big|\|V_j\|_{L^p} - \|U_{\varepsilon,j}\|_{L^p}\Big|}_{<C\varepsilon,\text{ see(3.8)}} + \underbrace{\Big|\|U_{\varepsilon,j}\|_{L^p} - K\|\nabla u_\varepsilon\|_{L^p}\Big|}_{<\varepsilon,\text{ by assumption}} +$$
$$+ \underbrace{\Big|K\|\nabla u_\varepsilon\|_{L^p} - K\|\nabla v\|_{L^p}\Big|}_{<K\varepsilon,\text{ by density}} < \tilde{C}\varepsilon$$

for a constant $\tilde{C} > 0$ and sufficiently large indices j.
We will now show equality (3.9) for the dense space $C^2(\overline{\Omega})$; here $C^2(\overline{\Omega})$ denotes the space of all twice continuously differentiable functions on Ω, which themselves and whose partial derivatives have a uniformly continuous extension onto $\overline{\Omega}$.
Taylor expansion of a function $u \in C^2(\overline{\Omega})$ yields

$$\frac{|u(x) - u(y)|}{|x - y|} = \left|\nabla u(x) \cdot \frac{x - y}{|x - y|}\right| + O(|x - y|)$$

for all $x, y \in \Omega$. Now we split the integral with respect to y in (3.6) into two parts: Let $x \in \Omega$ and $R := \operatorname{dist}(x, \partial\Omega)$:

$$\int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^p}\rho_j(x - y)\,\mathrm{d}y =$$
$$\int_{B(x,R)} \frac{|u(x) - u(y)|^p}{|x - y|^p}\rho_j(x - y)\,\mathrm{d}y + \int_{\Omega\setminus B(x,R)} \frac{|u(x) - u(y)|^p}{|x - y|^p}\rho_j(x - y)\,\mathrm{d}y. \tag{3.10}$$

The second integral converges to 0 as $j \to \infty$: we can assume that the integrand is defined on $\mathbb{R}^n\setminus B(x,R)$ by extension with 0 outside the domain of integration. Using spherical coordinates yields

$$\int_{\Omega\setminus B(x,R)} \frac{|u(x) - u(y)|^p}{|x - y|^p}\rho_j(x - y)\,\mathrm{d}y \le C\int_R^\infty \rho_j(r)r^{n-1}\,\mathrm{d}r,$$

where the constant is derived from the boundedness of $\frac{|u(x)-u(y)|^p}{|x-y|^p}$ on $\Omega\setminus B(x,R)$ and the integration with respect to the angular coordinates. Since ρ_j is a radial mollifier, by (3.3) the right hand side tends to 0.
For the first summand in (3.10) we have

$$\int_{B(x,R)} \frac{|u(x) - u(y)|^p}{|x - y|^p}\rho_j(x - y)\,\mathrm{d}y =$$
$$= \int_0^R \rho_j(r)\int_{|y-x|=r}\left(\left|\nabla u(x)\cdot\frac{x-y}{|x-y|}\right|^p + O(|x-y|^p)\right)\mathrm{d}\mathcal{H}^{n-1}(y)\,\mathrm{d}r =$$
$$= \int_0^R \rho_j(r)\int_{|\omega|=r}\left(\left|\nabla u(x)\cdot\frac{\omega}{|\omega|}\right|^p + O(|\omega|^p)\right)\mathrm{d}\mathcal{H}^{n-1}(\omega)\,\mathrm{d}r =$$
$$= \int_0^R \rho_j(r)\int_{\mathbb{S}^{n-1}}\left(|\nabla u(x)\cdot\omega|^p + O(r^p)\right)r^{n-1}\,\mathrm{d}\mathcal{H}^{n-1}(\omega)\,\mathrm{d}r. \tag{3.11}$$

In case $\nabla u(x) \neq 0$ then

$$\int_{\mathbb{S}^{n-1}} |\nabla u(x) \cdot \omega|^p \, \mathrm{d}\mathcal{H}^{n-1}(\omega) = |\mathbb{S}^{n-1}||\nabla u(x)|^p \underbrace{\frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \left| \frac{\nabla u(x)}{|\nabla u(x)|} \cdot \omega \right|^p \mathrm{d}\mathcal{H}^{n-1}(\omega)}_{=:C_{n,p}},$$

and we can use any unit vector in $\mathbb{R}^n$ instead of $\frac{\nabla u(x)}{|\nabla u(x)|}$ in the definition of the constant $C_{n,p}$, since the occurring integral is invariant under rotation. In particular, above identity also holds whenever $\nabla u(x) = 0$. All in all, we have

$$\begin{aligned} &\int_{B(x,R)} \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_j(x - y) \, \mathrm{d}y = \\ &= C_{n,p} |\nabla u(x)|^p \int_0^R |\mathbb{S}^{n-1}| r^{n-1} \rho_j(r) \, \mathrm{d}r + \int_0^R O(r^{n+p-1}) \rho_j(r) \, \mathrm{d}r. \end{aligned}$$

By normalization of the radial mollifier

$$\begin{aligned} 1 &= \int_{\mathbb{R}^n} \rho_j(x) \, \mathrm{d}x = |\mathbb{S}^{n-1}| \int_0^\infty \rho_j(r) r^{n-1} \, \mathrm{d}r = \\ &= |\mathbb{S}^{n-1}| \Big(\int_0^R \rho_j(r) r^{n-1} \, \mathrm{d}r + \underbrace{\int_R^\infty \rho_j(r) r^{n-1} \, \mathrm{d}r}_{\overset{j\to\infty}{\to} 0} \Big) \quad \forall \, j \in \mathbb{N}, \end{aligned} \tag{3.12}$$

and for sufficiently small $\varepsilon > 0$

$$\begin{aligned} \int_0^R r^{n+p-1} \rho_j(r) \, \mathrm{d}r &\leq \int_0^\varepsilon r^{n+p-1} \rho_j(r) \, \mathrm{d}r + \int_\varepsilon^R r^{n+p-1} \rho_j(r) \, \mathrm{d}r \leq \\ &\leq \varepsilon^p \int_0^\varepsilon r^{n-1} \rho_j(r) \, \mathrm{d}r + R^p \int_\varepsilon^\infty r^{n-1} \rho_j(r) \, \mathrm{d}r \leq \\ &\leq \varepsilon^p \underbrace{\int_0^\infty r^{n-1} \rho_j(r) \, \mathrm{d}r}_{=\frac{1}{|\mathbb{S}^{n-1}|}} + R^p \int_\varepsilon^\infty r^{n-1} \rho_j(r) \, \mathrm{d}r \overset{j\to\infty}{\to} \frac{\varepsilon^p}{|\mathbb{S}^{n-1}|}. \end{aligned} \tag{3.13}$$

In conclusion, taking the limit $j \to \infty$ in (3.10) yields

$$\lim_{j\to\infty} \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_j(x - y) \, \mathrm{d}y = C_{n,p} |\nabla u(x)|^p.$$

Since $u \in C^2(\overline{\Omega})$, u is a Lipschitz function on Ω with Lipschitz constant $L > 0$, and we have

$$\int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_j(x - y) \, \mathrm{d}y \leq L^p,$$

so the dominated convergence theorem asserts (3.6) for $u \in C^2(\overline{\Omega})$, and by density for $u \in W^{1,p}(\Omega)$.

Now we consider the case that $u \in L^p(\Omega)$ and the left hand side of (3.6) is finite. We will show that this automatically implies $u \in W^{1,p}(\Omega)$. To this end, we will first prove following lemma:

3.14 Lemma. *Let $u \in L^1(\mathbb{R}^n), \varphi \in C_c^\infty(\mathbb{R}^n)$, and $\rho \in L^1(\mathbb{R}^n)$, ρ radial with $\rho \geq 0$. Furthermore, let $e \in \mathbb{R}^n$ be a unit vector. Then*

$$\begin{aligned}\left|\int_{\mathbb{R}^n} u(x) \int_{(y-x)\cdot e \geq 0} \frac{\varphi(y)-\varphi(x)}{|y-x|}\rho(y-x)\,\mathrm{d}y\,\mathrm{d}x\right| &\leq \\ \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|}{|x-y|}|\varphi(y)|\rho(x-y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{3.14}$$

Proof . The integral on the left-hand side of the inequality exists and is finite, since φ satisfies the Lipschitz condition $\varphi(x) - \varphi(y) \leq L|x-y|$ for a constant $L > 0$.
We prove a special case of the inequality, namely for the situation where ρ is replaced by the function ρ_δ defined by

$$\rho_\delta(r) := \begin{cases} 0, & \text{for } r < \delta, \\ \rho(t), & \text{for } r \geq \delta. \end{cases}$$

Then inequality (3.14) follows by $\rho_\delta \leq \rho$ for all $\delta > 0$ and by dominated convergence. This process of "cutting off" the function ρ in a neighbourhood of 0 ensures the integrability of the functions

(i) $|u(x)||\varphi(y)|\frac{\rho_\delta(y-x)}{|y-x|}$ and
(ii) $|u(x)||\varphi(x)|\frac{\rho_\delta(y-x)}{|y-x|}$

on $\mathbb{R}^n \times \mathbb{R}^n$. To see this for (i), we compute

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)|\underbrace{|\varphi(y)|}_{\leq C}\frac{\rho_\delta(y-x)}{|y-x|}\,\mathrm{d}(x,y) &\leq C\int_{\mathbb{R}^n}|u(x)|\int_{\mathbb{R}^n}\frac{\rho_\delta(y-x)}{|y-x|}\,\mathrm{d}y\,\mathrm{d}x = \\ &= \tilde{C}\int_{\mathbb{R}^n}|u(x)|\,\mathrm{d}x\int_0^\infty \frac{\rho_\delta(r)}{r}r^{n-1}\,\mathrm{d}r \leq \\ &\leq \frac{\tilde{C}}{\delta}\int_{\mathbb{R}^n}|u(x)|\,\mathrm{d}x\int_\delta^\infty \rho_\delta(r)r^{n-1}\,\mathrm{d}r \leq \\ &\leq \frac{C}{\delta}\int_{\mathbb{R}^n}|u(x)|\,\mathrm{d}x\int_{\mathbb{R}^n}\rho(y)\,\mathrm{d}y < \infty.\end{aligned}$$

The same argumentation can be repeated verbatim for the second function. Now we can split the following integral into two parts:

$$\begin{aligned} I &:= \int_{\mathbb{R}^n} u(x)\int_{(y-x)\cdot e\geq 0}\frac{\varphi(y)-\varphi(x)}{|y-x|}\rho_\delta(y-x)\,\mathrm{d}y\,\mathrm{d}x = \\ &= \int_{(y-x)\cdot e\geq 0} u(x)\varphi(y)\frac{\rho_\delta(y-x)}{|y-x|}\,\mathrm{d}(x,y) - \underbrace{\int_{(y-x)\cdot e\geq 0} u(x)\varphi(x)\frac{\rho_\delta(y-x)}{|y-x|}\,\mathrm{d}(x,y)}_{:=I_2}.\end{aligned}$$

By interchanging variables and since ρ_δ is radial we can rewrite I_2 as

$$
\begin{aligned}
I_2 &= \int_{(x-y)\cdot e\geq 0} u(y)\varphi(y)\frac{\rho_\delta(x-y)}{|x-y|}\,\mathrm{d}(x,y) = \int_{\mathbb{R}^n} u(y)\varphi(y)\int_{z\cdot e\geq 0}\frac{\rho_\delta(z)}{|z|}\,\mathrm{d}z\,\mathrm{d}y = \\
&= \int_{\mathbb{R}^n} u(y)\varphi(y)\int_{(-z)\cdot e\geq 0}\frac{\rho_\delta(z)}{|z|}\,\mathrm{d}z\,\mathrm{d}y = \int_{(x-y)\cdot e\leq 0} u(y)\varphi(y)\frac{\rho_\delta(x-y)}{|x-y|}\,\mathrm{d}(x,y).
\end{aligned}
$$

Using this simple manipulation we can swap the the roles of the functions u and φ: instead of subtracting values of φ in I we can subtract values of u and conclude the proof:

$$
\begin{aligned}
I &= \int_{(y-x)\cdot e\geq 0}\varphi(y)\frac{u(x)-u(y)}{|y-x|}\rho_\delta(y-x)\,\mathrm{d}(x,y) \leq \\
&\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|}{|y-x|}|\varphi(y)|\rho_\delta(y-x)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

■

Proof of 3.13 continued: We now continue the proof of 3.13 in case the quantity

$$
A_p := \liminf_{j\to\infty}\left(\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^p}\rho_j(x-y)\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}}
$$

is finite.
Let $\varphi\in C_c^\infty(\Omega)$ (which we assume to be extended by 0 outside of Ω) and let $e\in\mathbb{R}^n$ be a unit vector. We repeat the calculations in 3.11 for $x\in\mathbb{R}^n$ to obtain

$$
\begin{aligned}
&\int_{(y-x)\cdot e\geq 0}\frac{\varphi(y)-\varphi(x)}{|y-x|}\rho_j(y-x)\,\mathrm{d}y = \\
&= \int_0^\infty\rho_j(r)\int_{\mathbb{S}^{n-1}\cap\{\omega\cdot e\geq 0\}}\left(\nabla\varphi(x)\cdot\omega+O(r^p)\right)r^{n-1}\,\mathrm{d}\mathcal{H}^{n-1}(\omega)\,\mathrm{d}r.
\end{aligned}
$$

We decompose ω into its components parallel and orthogonal to e by

$$
\omega = \underbrace{(\omega\cdot e)e}_{=:\omega_e} + \underbrace{[\omega-(\omega\cdot e)e]}_{=:\omega_\perp}
$$

and since the halfsphere $\mathbb{S}^{n-1}\cap\{\omega\cdot e\geq 0\}$ is symmetric with respect to the axis spanned by e

$$
\int_{\mathbb{S}^{n-1}\cap\{\omega\cdot e\geq 0\}}\nabla\varphi(x)\cdot\omega_\perp\,\mathrm{d}\mathcal{H}^{n-1}(\omega) = 0.
$$

What is left, we can rewrite as

$$
\int_{\mathbb{S}^{n-1}\cap\{\omega\cdot e\geq 0\}}\nabla\varphi(x)\cdot\omega\,\mathrm{d}\mathcal{H}^{n-1}(\omega) = \nabla\varphi(x)\cdot e\cdot\frac{1}{2|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}|\omega\cdot e|\,\mathrm{d}\mathcal{H}^{n-1}(\omega),
$$

so we can pass to the limit using 3.12 and 3.13,

$$
\lim_{j\to\infty}\int_{(y-x)\cdot e\geq 0}\frac{\varphi(y)-\varphi(x)}{|y-x|}\rho_j(y-x)\,\mathrm{d}y = K\nabla\varphi(x)\cdot e,
$$

with a constant $K > 0$ only depending on the dimension n.
If we extend f by 0 outside of Ω, we can apply Lemma 3.14:

$$\begin{aligned} &\left| \int_\Omega u(x) \int_{(y-x)\cdot e \geq 0} \frac{\varphi(y) - \varphi(x)}{|y-x|} \rho_j(y-x) \,\mathrm{d}y \,\mathrm{d}x \right| \leq \\ &\leq \int_{\mathbb{R}^n} \int_{\operatorname{spt} \varphi} \frac{|u(x) - u(y)|}{|x-y|} |\varphi(y)| \rho_j(x-y) \,\mathrm{d}x \,\mathrm{d}y = \\ &= \underbrace{\int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x-y|} |\varphi(y)| \rho_j(x-y) \,\mathrm{d}x \,\mathrm{d}y}_{=: J_{1,j}} + \underbrace{\int_{\mathbb{R}^n \setminus \Omega} \int_{\operatorname{spt} \varphi} |u(y)| |\varphi(y)| \frac{\rho_j(x-y)}{|x-y|} \,\mathrm{d}x \,\mathrm{d}y}_{=: J_{2,j}} . \end{aligned} \tag{3.15}$$

For fixed $y \in \Omega$ the measure defined by $A \mapsto \int_A \rho_j(x-y) \,\mathrm{d}x$, $A \subseteq \mathbb{R}^n$ Borel, is a probability measure. Thus, we can apply Jensen's inequality,

$$\left(\int_\Omega \frac{|u(x) - u(y)|}{|x-y|} \rho_j(x-y) \,\mathrm{d}x \right)^p \leq \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^p} \rho_j(x-y) \,\mathrm{d}x,$$

so application of the Hölder inequality for $J_{1,j}$ yields

$$J_{1,j} \leq \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^p} \rho_j(x-y) \,\mathrm{d}x \,\mathrm{d}y \right)^{\frac{1}{p}} \|\varphi\|_{L^{p'}} .$$

Let $d := \operatorname{dist}(\mathbb{R}^n \setminus \Omega, \operatorname{spt} \varphi) > 0$, then $J_{2,j}$ can be estimated as follows

$$\begin{aligned} J_{2,j} &= \int_{\operatorname{spt} \varphi} \int_{(\mathbb{R}^n \setminus \Omega) - y} |u(y)| |\varphi(y)| \frac{\rho_j(\xi)}{|\xi|} \,\mathrm{d}\xi \,\mathrm{d}y \leq \\ &\leq \frac{1}{d} \int_{\operatorname{spt} \varphi} |u(y)| |\varphi(y)| \,\mathrm{d}y \int_{|\xi| > d} \rho_j(\xi) \,\mathrm{d}\xi \overset{j \to \infty}{\to} 0. \end{aligned}$$

Now we pass to the limit in both sides of 3.15 to obtain

$$K \left| \int_\Omega u(x) (\nabla \varphi(x) \cdot e) \,\mathrm{d}x \right| \leq A_p \|\varphi\|_{L^{p'}} .$$

For the special cases that $e = e_i, i = 1, \ldots, n$, we see, that the linear functionals

$$L_i : \begin{cases} L^{p'} & \to \mathbb{R} \\ \varphi & \mapsto \int_\Omega u \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}\mathcal{L}^n \end{cases}$$

are continuous and by duality, there exist functions $v_i \in L^p$ such that

$$\langle L_i, \varphi \rangle = \int_\Omega v_i \varphi \,\mathrm{d}\mathcal{L}^n .$$

We conclude, that u is weakly differentiable with derivatives in L^p, i.e. $u \in W^{1,p}$. ■

3.15 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded set, $u \in L^p(\Omega)$, and $1 < p < \infty$. Then there exists a constant $C_{n,p}$ only depending on n and p such that*

$$\lim_{s \to 1} (1-s) \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x \,\mathrm{d}y = C_{n,p} \|\nabla u\|_{L^p}^p .$$

Proof . We apply Lemma 3.13 to the Gagliardo seminorm, where we can split the denominator in the integral in such a way that one factor is equal to a radial mollifier defined in Example 3.10 ($\varepsilon > 0$ small enough such that $1/\varepsilon > \operatorname{diam}(\Omega)$):

$$\varepsilon \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+(1-\varepsilon)p}} \,\mathrm{d}x\,\mathrm{d}y = \frac{|\mathbb{S}^{n-1}|}{\varepsilon^{\varepsilon p} p} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \underbrace{\frac{C_\varepsilon}{|x-y|^{n-\varepsilon p}}}_{=\rho_\varepsilon(x-y)} \,\mathrm{d}x\,\mathrm{d}y$$

$$\overset{\varepsilon \to 0}{\to} \frac{|\mathbb{S}^{n-1}|}{p} C_{n,p} \|\nabla u\|^p_{L^p(\Omega)}.$$

Substituting $s := 1-\varepsilon$ provides the desired result. ■

The next result is the analogue of Lemma 3.13 for functions of bounded variation; it was proved by Dávila in [9].

3.16 Lemma. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set with Lipschitz boundary, and let $u \in BV(\Omega)$. Then*

$$\lim_{j\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \rho_j(x-y) \,\mathrm{d}x\,\mathrm{d}y = C_{1,n} V(u;\Omega), \tag{3.16}$$

where $V(u;\Omega)$ denotes the variation of u in Ω (see Definition 1.4) and $C_{1,n} > 0$ is a constant only depending on n.

There is an immediate application of this lemma to the mollifiers discussed in Example 3.10; it can be shown by repeating the proof of Theorem 3.15 verbatim with $p = 1$.

3.17 Corollary. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set with Lipschitz boundary, and $u \in BV(\Omega)$. Then there exist a constant $C_{1,n} > 0$ only depending on n such that*

$$\lim_{s\to 1^-} (1-s) \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|^{n+s}} \,\mathrm{d}x\,\mathrm{d}y = C_{1,n} V(u;\Omega).$$

Now we can translate our results to the language of perimeters:

3.18 Corollary. *Let $E \subseteq \mathbb{R}^n$ be a bounded Borel set of finite perimeter. Then*

$$\lim_{s\to 1^-} (1-s) P_s(E) = \alpha_n P(E),$$

where the constant α_n only depends on the dimension n.

Proof . Since E has finite perimeter, the indicator function $\mathbb{1}_E$ belongs to $BV(\mathbb{R}^n)$ and $V(\mathbb{1}_E) = P(E)$.
On the other hand, the fractional perimeter $P_s(E)$ can be rewritten by (3.2), so the limit can be expressed as follows

$$\lim_{s\to 1^-} (1-s) P_s(E) = \frac{1}{2} \lim_{s\to 1^-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\mathbb{1}_E(x) - \mathbb{1}_E(y)|}{|x-y|^{n+s}} \,\mathrm{d}x\,\mathrm{d}y = \\ = \alpha_n V(\mathbb{1}_E) = \alpha_n P(E),$$

where we have applied Corollary 3.18. ■

3.3 The limiting case $s \to 0^+$

The next result is due to Maz'ya and Shaposhnikova [23]:

3.19 Theorem. *Let $p \geq 1$ and $u \in W^{s_0,p}(\mathbb{R}^n)$ for an $s_0 \in (0,1)$. Then*

$$\lim_{s \to 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}x \, \mathrm{d}y = \frac{2}{p} |\mathbb{S}^{n-1}| \|u\|_{L^p}^p. \tag{3.17}$$

Proof. We first calculate

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{|x-y|>2|x|} \frac{|u(x)|^p}{|x-y|^{n+sp}} \, \mathrm{d}y \, \mathrm{d}x = |\mathbb{S}^{n-1}| \int_{\mathbb{R}^n} |u(x)|^p \int_{2|x|}^{\infty} r^{-sp-1} \, \mathrm{d}r \, \mathrm{d}x = \\ = \frac{|\mathbb{S}^{n-1}|}{2^{sp} sp} \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}} \, \mathrm{d}x. \end{aligned} \tag{3.18}$$

The domain of integration expressed by the inequality $|x-y| > 2|x|$ can be suitably expanded, since it implies

$$|x-y| > 2|x-y+y| \geq 2|x-y| - 2|y|,$$

and therefore

$$|x-y| < 2|y| \tag{3.19}$$

on one hand, and

$$\frac{2}{3}|y| \leq \frac{2}{3}|x-y| + \frac{2}{3}|x| < |x-y|$$

on the other hand. Plugging in $u(y)$ into the first integral of (3.18) and using the simple inequalities established above yields

$$\begin{aligned} \left(\frac{|\mathbb{S}^{n-1}|}{2^{sp} sp} \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}} \, \mathrm{d}x \right)^{\frac{1}{p}} \leq \overbrace{\left(\int_{\mathbb{R}^n} \int_{|x-y|>2|x|} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}y \, \mathrm{d}x \right)^{\frac{1}{p}}}^{=:I_1} + \\ + \underbrace{\left(\int_{\mathbb{R}^n} \int_{\frac{2}{3}|y|<|x-y|<2|y|} \frac{|u(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{p}}}_{=:I_2}. \end{aligned}$$

We observe that the integrand of I_1 is symmetrical in x and y and the domains $\{|x-y| > 2|x|\}$ and $\{|x-y| > 2|y|\}$ are disjoint by inequality (3.19), so

$$\begin{aligned} 2I_1^p = \int_{\mathbb{R}^n} \int_{|x-y|>2|x|} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}y \, \mathrm{d}x + \int_{\mathbb{R}^n} \int_{|x-y|>2|y|} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}y \, \mathrm{d}x \leq \\ \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}y \, \mathrm{d}x. \end{aligned}$$

For the second integral I_2 transformation into spherical coordinates yields

$$I_2^p = |\mathbb{S}^{n-1}| \frac{3^{sp} - 1}{2^{sp} sp} \int_{\mathbb{R}^n} \frac{|u(y)|^p}{|y|^{sp}} \, \mathrm{d}y,$$

so we get

$$\left(\frac{|\mathbb{S}^{n-1}|}{2^{sp}sp}\right)^{\frac{1}{p}}(1-(3^{sp}-1)^{\frac{1}{p}})\left(\int_{\mathbb{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,\mathrm{d}x\right)^{\frac{1}{p}}\leq 2^{-\frac{1}{p}}[u]_{W^{s,p}}.$$

For $\delta\in(0,1)$ the factor $(1-(3^{sp}-1)^{\frac{1}{p}})$ on the left-hand side surely is positive if $s<\frac{\ln(\delta^p+1)}{p\ln 3}$, and by this choice of s it can be estimated by $(1-\delta)$ from below, such that

$$\frac{|\mathbb{S}^{n-1}|}{2^{sp-1}p}(1-\delta)^p\int_{\mathbb{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,\mathrm{d}x\leq s[u]^p_{W^{s,p}} \tag{3.20}$$

holds. Before we take the limit $s\to 0$ we want to take a closer look at the integral on the left-hand side:

$$\int_{\mathbb{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,\mathrm{d}x\geq\int_{|x|\leq 1}|u(x)|^p\,\mathrm{d}x+\underbrace{\int_{|x|>1}\frac{|u(x)|^p}{|x|^{sp}}\,\mathrm{d}x}_{\to\int_{|x|>1}|u(x)|^p\,\mathrm{d}x},$$

where the limit of the last integral can be comfortably obtained by dominated convergence. Going back to our original inequality (3.20), letting $s\to 0$ we arrive at

$$\frac{2}{p}|\mathbb{S}^{n-1}|\|u\|^p_{L^p}\leq\liminf_{s\to 0}s[u]^p_{W^{s,p}},$$

since δ can be arbitrarily small. From now on we assume $u\in L^p(\mathbb{R}^n)$, otherwise the statement holds with both sides equal to infinity. We proceed by parting the domain of integration:

$$\begin{aligned}[u]^p_{W^{s,p}}&=2\int_{\mathbb{R}^n}\int_{|y|\geq 2|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x+\int_{\mathbb{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x\leq\\&\leq 2\left[\left(\int_{\mathbb{R}^n}\int_{|y|\geq 2|x|}\frac{|u(x)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{p}}+\left(\int_{\mathbb{R}^n}\int_{|y|\geq 2|x|}\frac{|u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}}\right]^p+\\&+\int_{\mathbb{R}^n}\int_{|x|<|y|<2|x|}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Let us denote the integrals together with their corresponding exponents on the right-hand side by J_1, J_2 and J_3 after their order of appearance.
For an estimation of J_1 we observe, that for each $x\in\mathbb{R}^n\setminus\{0\}$ every element of the set $\{y\in\mathbb{R}^n:|y|\geq 2|x|\}$ can be written in the form $y=x+\rho\frac{x}{|x|}$, where $\rho\geq|x|$. This observation can be applied to the integral J_1 in the following way:

$$\begin{aligned}s^{\frac{1}{p}}J_1&\leq\left(s\int_{\mathbb{R}^n}|u(x)|^p\int_{\mathbb{S}^{n-1}}\int_{|x|}^{\infty}\frac{\rho^{n-1}}{|\rho v|^{n+sp}}\,\mathrm{d}\rho\,\mathrm{d}\mathcal{H}^{n-1}(v)\,\mathrm{d}x\right)^{\frac{1}{p}}=\\&=\left(s|\mathbb{S}^{n-1}|\int_{\mathbb{R}^n}|u(x)|^p\int_{|x|}^{\infty}\rho^{-sp-1}\,\mathrm{d}\rho\,\mathrm{d}x\right)^{\frac{1}{p}}=\\&=\frac{|\mathbb{S}^{n-1}|^{\frac{1}{p}}}{p^{\frac{1}{p}}}\left(\int_{\mathbb{R}^n}\frac{|u(x)|^p}{|x|^{sp}}\,\mathrm{d}x\right)^{\frac{1}{p}}.\end{aligned}$$

Since there exists a fractional exponent $s_0 \in (0,1)$ such that $u \in W^{s_0,p}(\mathbb{R}^n)$, the expression on the right-hand side of

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}} \,\mathrm{d}x \leq \int_{|x|\leq 1} \frac{|u(x)|^p}{|x|^{s_0 p}} \,\mathrm{d}x + \int_{|x|>1} |u(x)|^p \,\mathrm{d}x$$

is finite ($s < s_0$). Therefore, by dominated convergence

$$\limsup_{s\to 0^+} s^{\frac{1}{p}} J_1 \leq \frac{|\mathbb{S}^{n-1}|^{\frac{1}{p}}}{p^{\frac{1}{p}}} \|u\|_{L^p}.$$

For the inner integral in J_2 we can apply the reverse triangle inequality in the denominator which results in

$$\int_{|y|\geq 2|x|} \frac{1}{|x-y|^{n+sp}} \,\mathrm{d}x \leq \int_{|y|\geq 2|x|} \frac{1}{(|y|-|x|)^{n+sp}} \,\mathrm{d}x \overset{|x|\leq 2^{-1}|y|}{\leq} \frac{2^{n+sp}}{|y|^{n+sp}} \int_{|y|\geq 2|x|} \,\mathrm{d}x = \\ = \frac{2^{sp}}{|y|^{sp}} \omega_n.$$

We can estimate $s^{\frac{1}{p}} J_2$ now by

$$s^{\frac{1}{p}} J_2 \leq 2^s (s\omega_n)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} \frac{|u(y)|^p}{|y|^{sp}} \,\mathrm{d}y \right)^{\frac{1}{p}},$$

which tends to zero as $s \to 0$.
For the last integral J_3 we make use that there exists a number $\tau \in (0,1)$ such that $u \in W^{\tau,p}(\mathbb{R}^n)$. Furthermore, we now only consider $s < \tau$ and we let $N > 1$. Then

$$sJ_3 = s \int_{\mathbb{R}^n} \int_{\substack{|x|<|y|<2|x| \\ |x-y|\leq N}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+\tau p}} \frac{1}{|x-y|^{(s-\tau)p}} \,\mathrm{d}y \,\mathrm{d}x + \\ + s \int_{\mathbb{R}^n} \int_{\substack{|x|<|y|<2|x| \\ |x-y|> N}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}y \,\mathrm{d}x.$$

The first summand is less than or equal to

$$sN^{(\tau-s)p} \int_{\mathbb{R}^n} \int_{\substack{|x|<|y|<2|x| \\ |x-y|\leq N}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}y \,\mathrm{d}x,$$

which converges to 0 as $s \to 0$. Since both inequalites $|x| < |y| < 2|x|$ and $|x-y| > N$ imply

$$|x| > |x-y| - |y| > N - 2|x|,$$

the second summand can be estimated as follows:

$$s\int_{\mathbb{R}^n}\int_{\substack{|x|<|y|<2|x|\\|x-y|>N}}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x \le 2^{p-1}s\left(\int_{\mathbb{R}^n}\int_{\substack{|x|<|y|<2|x|\\|x-y|>N}}\frac{|u(x)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x+\right.$$

$$\left.+\int_{\mathbb{R}^n}\int_{\substack{|x|<|y|<2|x|\\|x-y|>N}}\frac{|u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x\right)\le$$

$$\le 2^{p-1}s\left(\int_{|x|>N/3}\int_{|x-y|>N}\frac{|u(x)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x+\int_{|y|>N/3}\int_{|x-y|>N}\frac{|u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)=$$

$$=2^p s\int_{|x|>N/3}\int_{|x-y|>N}\frac{|u(x)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x=\frac{2^p}{N^{sp}p}\int_{|x|>N/3}|u(x)|^p\,\mathrm{d}x.$$

N can be chosen arbitrarily large, so our second summand can be made arbitrarily small. In conclusion, $sJ_3\to 0$ as $s\to 0$, and

$$\limsup_{s\to 0} s[u]^p_{W^{s,p}}\le\frac{2}{p}|\mathbb{S}^{n-1}|\|u\|^p_{L^p},$$

which concludes the proof. ■

3.20 Corollary. *Let $E\subseteq\mathbb{R}^n$ be a bounded Borel set of finite fractional s'-perimeter for every $s'\in(0,1)$. Then*

$$\lim_{s\to 0^+} sP_s(E)=n\omega_n\,|E|\,.$$

Proof. By formula (3.2) we have

$$sP_s(E)=\frac{s}{2}[\mathbb{1}_E]_{W^{s,1}}\overset{s\to 0^+}{\to}|\mathbb{S}^{n-1}|\|\mathbb{1}_E\|_{L^1}=n\omega_n|E|,$$

where we apply the convergence result Theorem 3.19 we have established for the seminorm. ■

3.4 The fractional Sobolev inequality

In this section we explain the results of Frank and Seiringer [12].

We already dealt with two Sobolev-type inequalities (and the corresponding isoperimetric inequalities as their geometric counterparts) in Theorem 1.9 for the Euclidean setting and in Remark 2.10 for general norms on $\mathbb{R}^n$. Therein we measured the functions by the usual Lebesgue norm on one side and compared this quantity with an integral involving their derivatives. For the fractional Sobolev inequality we replace this Lebesgue norm by a more general type of norms, namely Lorentz norms:

3.21 Definition. For $1\le q<\infty, 1\le r\le\infty$ we define the *Lorentz space* $L_{q,r}(\mathbb{R}^n)$ as the set of those measurable functions u on $\mathbb{R}^n$ for which the quantity

$$\|u\|_{q,r} := \left(q \int_0^\infty \mu_u(t)^{\frac{r}{q}} t^{r-1} \, \mathrm{d}t \right)^{\frac{1}{r}} \quad \text{for } 1 \le r < \infty,$$
$$\|u\|_{q,\infty} := \sup_{t>0} \mu_u(t)^{\frac{1}{q}} t, \quad \text{for } r = \infty,$$

is finite. Here $\mu_u(t) := |\{x \in \mathbb{R}^n : |u(x)| > t\}|$ is the *distribution function* of u.

For the discussion of equality cases in the fractional Sobolev inequality to come, we need the notion of the symmetric decreasing rearrangement of a function. We motivate this definition with the following simple lemma:

3.22 Lemma. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a function. Then the value of u at $x \in \mathbb{R}^n$ can be obtained by the* layer-cake representation

$$u(x) = \int_0^\infty \mathbb{1}_{\{u>t\}}(x) \, \mathrm{d}t. \tag{3.21}$$

Proof. The indicator function in the integral equals 1 precisely for $0 < t < u(x)$. ■

3.23 Definition.

1. Let $A \subseteq \mathbb{R}^n$ be a Lebesgue measurable set with $|A| < \infty$. Then the set
$$A^\# := \{x \in \mathbb{R}^n : \omega_n |x|^n < |A|\}$$
is called the *symmetric rearrangement* of A.

2. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a measurable function such that $\{|u| > t\}$ has finite Lebesgue measure for every $t > 0$. Then the function $u^\# : \mathbb{R}^n \to [0, \infty)$ defined as
$$u^\#(x) := \int_0^\infty \mathbb{1}_{\{|u|>t\}^\#}(x) \, \mathrm{d}t$$
is called the *symmetric decreasing rearrangement* of u.

In the spirit of [18, chapter 3.3] we now state some simple facts concerning symmetric decreasing rearrangements:

3.24 Remark.

1. $A^\#$ is the open ball centred at the origin with the same volume as A. Especially $|A^\#| = |A|$.

2. Since the sets $\{|u| > t\}^\#$ are balls, $u^\#(x)$ clearly only depends on $|x|$. Furthermore, if $|x| \le |y|$, then $\mathbb{1}_{\{|u|>t\}^\#}(y) = 1$ implies $\mathbb{1}_{\{|u|>t\}^\#}(x) = 1$, and we have $u(x) \ge u(y)$. This justifies our nomenclature of $u^\#$ as symmetric decreasing rearrangement.

3. If $t_1 < t_2$, then $\{|u| > t_2\}^\# \subseteq \{|u| > t_1\}^\#$, and therefore for $x \in \mathbb{R}^n$
$$\mathbb{1}_{\{|u|>t\}^\#}(x) = \begin{cases} 1, & t < u^\#(x) \\ 0, & t > u^\#(x). \end{cases}$$

4. It holds that
$$\{|u| > s\}^{\#} = \left\{u^{\#} > s\right\}, \ s > 0,$$
since for all $x \in \mathbb{R}^n$ the fact that $\int_0^\infty \mathbb{1}_{\{|u|>t\}^{\#}}(x)\,\mathrm{d}t > s$ is equivalent to $\mathbb{1}_{\{|u|>s\}^{\#}}(x) = 1$. In particular,
$$\left|\left\{u^{\#} > s\right\}\right| = |\,\{|u| > s\}\,|, \ s > 0,$$
and the distribution functions μ_u and $\mu_{u^{\#}}$ coincide. Ultimately, for Lorentz norms with $1 \le q < \infty, 1 \le r \le \infty$
$$\|u^{\#}\|_{q,r} = \|u\|_{q,r}.$$

5. For any Lebesgue measurable set A with $|A| < \infty$ we have
$$(\mathbb{1}_A)^{\#} = \mathbb{1}_{A^{\#}}.$$
By the previous remark we get for level sets
$$(\mathbb{1}_{\{|u|>s\}})^{\#} = \mathbb{1}_{\left\{u^{\#}>s\right\}}.$$

6. We can use the super-level sets $\{|u| \ge t\}$, where equality is also allowed, in our definition of $u^{\#}$. Clearly,
$$u^{\#}(x) \le \int_0^\infty \mathbb{1}_{\{|u|\ge t\}^{\#}}(x)\,\mathrm{d}t.$$
If we denote the points of discontinuity of $\mathbb{1}_{\{|u|>t\}^{\#}}(x)$ and $\mathbb{1}_{\{|u|\ge t\}^{\#}}(x)$ (as functions in t) as t_1 and t_2 respectively, then strict inequality would imply $t_1 < t_2$. Then for each $t \in (t_1, t_2)$, the set $\{|u| = t\}$ would have measure greater than 0, which is not possible.

7. If $\Phi : [0,\infty) \to [0,\infty)$ is non-decreasing, then
$$(\Phi \circ |u|)^{\#} = \Phi \circ u^{\#}. \tag{3.22}$$
Indeed, the sets $\{|u| \ge t\}$ and $\{\Phi \circ |u| \ge \Phi(t)\}$ coincide, so the discontinuity of $\mathbb{1}_{\{|u|\ge t\}^{\#}}(x)$ at $t = u^{\#}(x)$ carries over to a discontinuity of $\mathbb{1}_{\{\Phi\circ|u|\ge t\}^{\#}}(x)$ at $t = \Phi(u^{\#}(x))$, which shows the equation.

We now state the fractional Sobolev inequality together with a bundle of other important inequalities and tools needed for its proof, which we then will carry out immediately; the proofs of those tools will be postponed to later sections for the most part, so the rest of this section can be thought of a rough outline of all steps which we will refine later on (in subsections 3.4.1 - 3.4.2).

3.25 Theorem (fractional Sobolev inequality). *Let $n \in \mathbb{N}$, $0 < s < 1$ the fractional exponent, $1 \le p < \frac{n}{s}$, and $p^\star := \frac{np}{n-sp}$. Then $W^{s,p}(\mathbb{R}^n)$ is continuously embedded into $L_{p^\star,p}(\mathbb{R}^n)$ by means of the inequality*
$$\|u\|_{p^\star,p} \le \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{\frac{s}{n}} C_{n,s,p}^{-\frac{1}{p}} \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}},$$

where $u \in W^{s,p}(\mathbb{R}^n)$, and $|\mathbb{S}^{n-1}|$ denotes the surface area of the n-dimensional unit sphere. The constant $C_{n,s,p}$ is given by

$$C_{n,s,p} := 2\int_0^1 r^{sp-1}|1 - r^{(n-sp)/p}|^p \Phi_{n,s,p}(r)\,\mathrm{d}r, \tag{3.23}$$

where

$$\Phi_{n,s,p}(r) := |\mathbb{S}^{n-2}| \int_{-1}^1 \frac{(1-t^2)^{\frac{n-3}{2}}}{(1-2rt+r^2)^{\frac{n+sp}{2}}}\,\mathrm{d}t, n \geq 2, \tag{3.24}$$

$$\Phi_{1,s,p}(r) := \left(\frac{1}{(1-r)^{1+sp}} + \frac{1}{(1+r)^{1+sp}}\right), n = 1. \tag{3.25}$$

This inequality is sharp. In the particular case $p = 1$ equality holds, if and only if u is proportional to a non-negative function v, such that the super-level sets $\{v > t\}$ are balls for almost every $t \geq 0$. For $p > 1$ the inequality is strict for any u not identically zero.

3.26 Theorem (fractional Hardy inequality). *Let $n \in \mathbb{N}$, $0 < s < 1$ the fractional Sobolev exponent, and $1 \leq p < \frac{n}{s}$. Then for all $u \in W^{s,p}(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \geq C_{n,s,p}\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{sp}}\,\mathrm{d}x \tag{3.26}$$

with the same constant $C_{n,p,s}$ as given in Theorem 3.25. This constant is optimal. For $p = 1$ equality holds if and only if u is proportional to a symmetric decreasing function. For $p > 1$, the inequality is strict for any function u in $W^{s,p}(\mathbb{R}^n)$ not identically 0 almost everywhere.

3.27 Lemma. *Let $0 < s \leq 1$ and $1 \leq p < \frac{n}{s}$. Then for any non-negative symmetric decreasing function u on $\mathbb{R}^n$*

$$\|u\|_{p^\star,p} = \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{\frac{s}{n}} \left(\int_{\mathbb{R}^n} \frac{u^p}{|x|^{sp}}\,\mathrm{d}x\right)^{\frac{1}{p}}. \tag{3.27}$$

Proof . We first calculate the Lorentz norm on the the left-hand side by substituting $\tau := t^p$ and noting that $\mu_u(t) = \mu_{u^p}(\tau)$,

$$\|u\|_{p^\star,p} = \left(p^\star \int_0^\infty \mu_u(t)^{\frac{p}{p^\star}} t^{p-1}\,\mathrm{d}t\right)^{\frac{1}{p}} = \left(\frac{p^\star}{p}\int_0^\infty \mu_{u^p}(t)^{\frac{p}{p^\star}}\,\mathrm{d}t\right)^{\frac{1}{p}}. \tag{3.28}$$

u^p is again a symmetric decreasing function, so $\mu_{u^p}(t)$ is the volume of a ball $B(t)$ with radius $\left(\frac{n\mu_{u^p}(t)}{|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n}}$ and after using the layer-cake formula for u^p on the integral on right-hand side of (3.27) we comfortably can transform into spherical coordinates:

$$\left(\int_{\mathbb{R}^n} \frac{u^p(x)}{|x|^{sp}}\,\mathrm{d}x\right)^{\frac{1}{p}} = \left(\int_0^\infty \int_{\mathbb{R}^n} \frac{\mathbb{1}_{B(t)}(x)}{|x|^{sp}}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}} = \frac{n^{\frac{n-sp}{np}}|\mathbb{S}^{n-1}|^{\frac{s}{n}}}{(n-sp)^{\frac{1}{p}}}\left(\int_0^\infty \mu_{u^p}(t)^{\frac{p(n-sp)}{np}}\,\mathrm{d}t\right)^{\frac{1}{p}}.$$

Now we are nearly done with the proof: multiplying with $\left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{\frac{s}{n}}$ on both sides and rewriting all expressions containing the dimension n in terms of p and $p^\star$ will lead back to the right-hand side of (3.28). ■

3.28 Theorem. *Let $n \in \mathbb{N}$, $0 < s < 1$, $1 \leq p < \frac{n}{s}$, and $u \in W^{s,p}(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y \geq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u^{\#}(x) - u^{\#}(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y. \tag{3.29}$$

If $p = 1$, then equality holds if and only if u is proportional to a non-negative function v such that the super-level sets $\{v > \tau\}$ are balls for a. e. $\tau > 0$. If $p > 1$, then equality holds if and only if u is proportional to a translate of a symmetric decreasing function.

Proof of 3.25. First, recall that the Lorentz norm does not change under symmetric decreasing rearrangement, see Remark 3.24, so if we can show the inequality for symmetric decreasing u first, we can immediately obtain the result for general u from 3.28. Thus, let u be symmetric decreasing. By Lemma 3.27 and the fractional Hardy inequality 3.26 we infer

$$\begin{aligned} \|u\|_{p^\star,p} &= \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{s/n} \left(\int_{\mathbb{R}^n} \frac{u^p}{|x|^{sp}} \,\mathrm{d}x^{1/p}\right) \leq \\ &\leq \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{s/n} C_{n,s,p}^{-1/p} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}. \end{aligned}$$

Note, that for $p = 1$ all sides in the chain of inequalities above are equal, so for general u we have equality in the fractional Sobolev inequality if and only if there is equality in the rearrangement inequality 3.28, i. e. if u is proportional to a non-negative function v such that the super-level sets $\{v > \tau\}$ are balls for a. e. $\tau > 0$. In the same fashion we see, that in case of $p > 1$ the inequality is strict for any function not identically 0 almost everywhere. ∎

3.29 Corollary (fractional isoperimetric inequality). *Let $E \subseteq \mathbb{R}^n$ be a set of finite perimeter with $|E| < \infty$. Then*

$$|E|^{\frac{n-s}{n}} \leq \frac{2(n-s)}{nC_{n,s,1}} \left(\frac{n}{|\mathbb{S}^{n-1}|}\right)^{\frac{s}{n}} P_s(E), \tag{3.30}$$

with equality if and only if E is equivalent to a ball.

Proof. This is a simple consequence of the fractional Sobolev inequality when u is equal to the indicator function of E and $p = 1$: we calculate the Lorentz norm $\|\mathbb{1}_E\|_{\frac{n}{n-s},1}$ of the indicator function $\mathbb{1}_E$:

$$\begin{aligned} \|\mathbb{1}_E\|_{\frac{n}{n-s},1} &= \frac{n}{n-s} \int_0^\infty |\{\mathbb{1}_E > t\}|^{\frac{n-s}{n}} \,\mathrm{d}t = \\ &= \frac{n}{n-s} |E|^{\frac{n-s}{n}}, \end{aligned}$$

whereas the integral on the right hand side of (3.30) is equal to the fractional perimeter as shown in Remark 3.8.

The use of the indicator function also settles the equality case. ∎

3.4.1 Proof of the fractional Hardy inequality

Our main goal for the proof of the fractional Hardy inequality will be establishing a more general framework, which obviously has the advantage of providing more general results that might also be succesfully applied to other problems, but also gives us a nice level of abstraction where our goal can be approached more systematically. The big drawback of this method, however, is that we lose information regarding the constant $C_{n,s,p}$, so we will show the inequality this way, but need an extra result concerning the optimality of $C_{n,s,p}$.
We now state assumptions that remain valid for the rest of this section:

3.30 Assumptions.

- Let $n \in \mathbb{N}$ be the dimension of the ambient space as always, $p \geq 1$, and $\Omega \subseteq \mathbb{R}^n$ be open.
- Let k be a non-negative, measurable function defined almost everywhere on $\Omega \times \Omega$ such that $k(x,y) = k(y,x)$ for almost all pairs $(x,y) \in \Omega \times \Omega$.
- Let $k_\varepsilon, \varepsilon > 0$, be a family of measurable functions defined a.e. on $\Omega \times \Omega$ such that $k_\varepsilon(x,y) = k_\varepsilon(y,x)$, as well as $0 \leq k_{\varepsilon_1}(x,y) \leq k_{\varepsilon_2}(x,y) \leq k(x,y)$ for $\varepsilon_1 \leq \varepsilon_2$ and
$$\lim_{\varepsilon \to 0} k_\varepsilon(x,y) = k(x,y) \tag{3.31}$$
for a.e. $(x,y) \in \Omega \times \Omega$.
- Let ω be a positive, measurable function on Ω.
- Define
$$V_\varepsilon(x) := 2\omega(x)^{-p+1} \int_\Omega (\omega(x) - \omega(y))|\omega(x) - \omega(y)|^{p-2} k_\varepsilon(x,y)\,\mathrm{d}y. \tag{3.32}$$
Then we impose,
  - the integral occuring in $V_\varepsilon(x)$ is absolutely convergent for a.e. $x \in \Omega$,
  - V_ε belongs to $L^1_{loc}(\Omega)$, and
  - $V := \lim_{\varepsilon \to 0} V_\varepsilon$ exists weakly in $L^1_{loc}(\Omega)$, i.e. $\int_\Omega V_\varepsilon g\,\mathrm{d}x \to \int_\Omega V g\,\mathrm{d}x$ for any bounded g with compact support in Ω.
- Finally, define
$$E[u] := \int_\Omega \int_\Omega |u(x) - u(y)|^p k(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.33}$$

3.31 Proposition. *Under the assumptions 3.30, for any measurable function u on Ω with compact support $\operatorname{spt} u \subseteq \Omega$ and both $E[u]$ and $\int_\Omega V_+|u|^p\,\mathrm{d}x$ finite, one has*
$$E[u] \geq \int_\Omega V(x)|u(x)|^p\,\mathrm{d}x. \tag{3.34}$$

In order to prove the preceding Proposition 3.31, we first need an estimate for the distance of a complex number to a number lying in the unit interval, all raised to the p-th power:

3.32 Lemma. *Let $p \geq 1$. Then for all $a \in \mathbb{C}$ and $t \in [0,1]$ one has*

$$|a-t|^p \geq (1-t)^{p-1}(|a|^p - t). \tag{3.35}$$

In particular, if $p > 1$, then equality holds precisely for $a = 1$ or $t = 0$.

Proof . In the case $p = 1$ the inequality simply boils down to the reverse triangle inequality for complex numbers.
In case that $p > 1$ one can easily show the inequality for certain configurations of a and t: If $t = 0$, then both sides are clearly equal. If $t = 1$ or $|a|^p \leq t$, then the right-hand side is less or equal to 0 and the inequality holds trivially. Furthermore, since t lies in the interval $[0,1]$, for fixed $|a|$ the left-hand side is minimal for a real and non-negative. Hence we only need to consider the case $0 < t < 1$ and $a > \sqrt[p]{t}$.
Let f be the function on $(\sqrt[p]{t}, \infty)$ defined by

$$f(a) := \frac{(a-t)^p}{a^p - t}.$$

We want to find the infimum of f on its domain, so first we look for values of a where the derivative vanishes:

$$f'(a) = \frac{pt(a-t)^{p-1}(a^{p-1}-1)}{(a^p-t)^2}$$

can only be 0 for $a = 1$, the factor $(a-t)^{p-1}$ is greater than 0 by $a > \sqrt[p]{t} \geq t$. Furthermore, $f'(a) < 0$ if $a \in (\sqrt[p]{t}, 1)$ and $f'(a) > 0$ for $a \in (1, \infty)$, i.e. f has a global minimum at $a = 1$ with $f(1) = (1-t)^{p-1}$. This establishes the desired inequality together with the equality cases. ∎

Proof of 3.31. If $\int_\Omega V_- |u|^p \, \mathrm{d}x = \infty$, then the inequality trivially holds with $-\infty$ on the right-hand side, so from now on we assume $\int_\Omega V_- |u|^p \, \mathrm{d}x < \infty$. Furthermore, we can approximate u by the sequence $u_M := u \min(1, M|u^{-1}|)$ such that both $|u_M(x)|^p$ and the factor $|u_M(x) - u_M(y)|^p$ in $E[u_M]$ are monotonically increasing. Hence, if we can show our inequality for bounded functions, it follows for general functions via the monotone convergence theorem.
We set $v := \frac{u}{\omega}$, multiply (3.32) by $\omega(x)^p |v(x)|^p = |u(x)|^p$ on both sides and integrate with respect to x:

$$\int_\Omega V_\varepsilon(x)|u(x)|^p \, \mathrm{d}x = 2 \int_\Omega \int_\Omega \omega(x)|v(x)|^p (\omega(x) - \omega(y))|\omega(x) - \omega(y)|^{p-2} k_\varepsilon(x,y) \, \mathrm{d}y \, \mathrm{d}x. \tag{3.36}$$

Note that the resulting double integral is convergent, since V_ε belongs to $L^1_{loc}(\Omega)$ and u has compact support. Interchangig the variables reverses the sign of the integrals, so the factor 2 can be obtained by summation of two integrals - one with the order of variables as in (3.36), and one with reversed order:

$$\int_\Omega V_\varepsilon(x)|u(x)|^p \, \mathrm{d}x =$$
$$\int_\Omega \int_\Omega \big(\omega(x)|v(x)|^p - \omega(y)|v(y)|^p\big)(\omega(x) - \omega(y))|\omega(x) - \omega(y)|^{p-2} k_\varepsilon(x,y) \, \mathrm{d}x \, \mathrm{d}y.$$

Defining

$$\Phi_u(x,y) := |\omega(x)v(x) - \omega(y)v(y)|^p - \\ - \big(\omega(x)|v(x)|^p - \omega(y)|v(y)|^p\big)(\omega(x) - \omega(y))|\omega(x) - \omega(y)|^{p-2}$$

we can rewrite the above identity as

$$\int_\Omega\int_\Omega \Phi_u(x,y)k_\varepsilon(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega V_\varepsilon|u|^p\,\mathrm{d}x = \int_\Omega\int_\Omega |u(x)-u(y)|^p k_\varepsilon(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.37}$$

We claim that the first integral is non-negative, by showing the even stronger result that $\Phi_u \geq 0$ pointwise. Since $\Phi_u(x,y) = \Phi_u(y,x)$, we can without loss of generality assume that $\omega(x) \geq \omega(y)$. By suitably factorizing and applying lemma 3.32 with $a = \frac{v(x)}{v(y)}$ and $t = \frac{\omega(y)}{\omega(x)}$ we have

$$\omega(x)^p|v(y)|^p\left|\frac{v(x)}{v(y)} - \frac{\omega(y)}{\omega(x)}\right|^p \geq \omega(x)^p|v(y)|^p\left(1-\frac{\omega(y)}{\omega(x)}\right)^{p-1}\frac{\omega(x)|v(x)|^p - \omega(y)|v(y)|^p}{\omega(x)|v(y)|^p} = \\ = \big(\omega(x)|v(x)|^p - \omega(y)|v(y)|^p\big)(\omega(x)-\omega(y))^{p-1},$$

which affirms our claim.
In the final step, we pass to the limit $\varepsilon \to 0$ in (3.37) and obtain

$$\int_\Omega\int_\Omega \Phi_u(x,y)k(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_\Omega V|u|^p\,\mathrm{d}x = \int_\Omega\int_\Omega |u(x)-u(y)|^p k(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.38}$$

The integral $\int_\Omega V_\varepsilon|u|^p\,\mathrm{d}x$ converges to $\int_\Omega V|u|^p\,\mathrm{d}x$, because of our assumption that V_ε converges weakly to V, and $|u|^p$ is bounded with compact support. The remaining integrals converge by monotone convergence, since k_ε is a family of monotonically increasing functions. For the left-hand side, we can apply the dominated convergence theorem with majorant $k(x,y) \geq k_\varepsilon(x,y)$ to see, that it converges to $E[u]$. ∎

3.33 Remark. We want to emphasise the fact, that each of the following integrals that results by splitting Φ_u into separate summands, namely

$$\int_\Omega\int_\Omega |u(x)-u(y)|^p k(x,y)\,\mathrm{d}x\,\mathrm{d}y, \text{ and}$$
$$\int_\Omega\int_\Omega \left(\frac{|u(x)|^p}{\omega(x)^{p-1}} - \frac{|u(y)|^p}{\omega(y)^{p-1}}\right)(\omega(x)-\omega(y))|\omega(x)-\omega(y)|^{p-2}k(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

is finite. We will need this result later to ascertain the convergence of suitable approximations.

Now we want to apply the preceding proposition to our case, namely to the case of Gagliardo seminorms. If we set $\Omega := \mathbb{R}^n \setminus \{0\}$, and $k(x,y) := |x-y|^{-n-sp}$, the functional $E[u]$ results in

$$E[u] = \int_{\mathbb{R}^n\setminus\{0\}}\int_{\mathbb{R}^n\setminus\{0\}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y$$

whose domain of integration differs from the left-hand side of the Hardy inequality (3.26) only by a set of measure zero. To recover the right-hand side, we first define for $p < n/s$

$$\alpha := \frac{n-sp}{p}\ (>0),$$

and then choose $\omega(x) := |x|^{-\alpha}$. This will result in $V(x) = C_{n,s,p}|x|^{-sp}$ as desired. First, however, we need to check all assumptions of 3.30; we start with the integrability properties of $V_\varepsilon(x)$:

3.34 Lemma. *For $n \in \mathbb{N}, 0 < s < 1$, and $1 \le p < n/s$, as well as $\varepsilon > 0$, the integral in*

$$V_\varepsilon(x) := \frac{2}{|x|^{\alpha(1-p)}} \int_{||x|-|y||>\varepsilon} \left(\frac{1}{|x|^\alpha} - \frac{1}{|y|^\alpha}\right) \left|\frac{1}{|x|^\alpha} - \frac{1}{|y|^\alpha}\right|^{p-2} \frac{1}{|x-y|^{n+sp}} \,\mathrm{d}y \tag{3.39}$$

is absolutely convergent for a.e. $x \in \mathbb{R}^n \setminus \{0\}$. Furthermore, $V_\varepsilon \in L^1_{loc}(\mathbb{R}^n \setminus \{0\})$.

Proof . If $\varepsilon \ge |x|$, then the domain of integration is the complement of a ball centered at the origin with radius $|x| + \varepsilon$, whereas, if $\varepsilon < |x|$, the domain consists of a further component. We first consider the integral over all y with $|y| > |x| + \varepsilon$:

$$\int_{|y|>|x|+\varepsilon} \left(\frac{1}{|x|^\alpha} - \frac{1}{|y|^\alpha}\right)^{p-1} \frac{1}{|x-y|^{n+sp}} \,\mathrm{d}y \le \frac{1}{|x|^{\alpha(p-1)}} \int_{|y|>|x|+\varepsilon} \frac{1}{|x-y|^{n+sp}} \,\mathrm{d}y.$$

Substituting y with $y - x$ and observing that $|y - x| > |x| + \varepsilon$ implies $|y| > \varepsilon$, we can further estimate and evaluate the integral:

$$\frac{1}{|x|^{\alpha(p-1)}} \int_{|y|>\varepsilon} |y|^{-n-sp} \,\mathrm{d}y = \frac{|\mathbb{S}^{n-1}|}{|x|^{\alpha(p-1)} sp \cdot \varepsilon^{sp}} < \infty.$$

Using our estimates, we can cancel the factor $\frac{1}{|x|^{\alpha(1-p)}}$ in front of the integral in $V_\varepsilon(x)$, so the expression does not depend on x any longer and therefore is integrable w.r.t x over every compact set.
For the case that $\varepsilon < |x|$ the integral over the inner ball can be estimated for absolute convergence as follows:

$$\int_{|y|<|x|-\varepsilon} \left(\frac{1}{|y|^\alpha} - \frac{1}{|x|^\alpha}\right)^{p-1} \frac{1}{|x-y|^{n+sp}} \,\mathrm{d}y \le \frac{1}{\varepsilon^{n+sp}} \int_{|y|<|x|-\varepsilon} \left(\frac{1}{|y|^\alpha}\right)^{p-1} \mathrm{d}y. \tag{3.40}$$

Transforming into spherical coordinates yields

$$\frac{|\mathbb{S}^{n-1}|}{\varepsilon^{n+sp}} \int_0^{|x|-\varepsilon} r^{\frac{n}{p}+s(p-1)-1} \,\mathrm{d}r$$

for the right-hand side. Since $\frac{n}{p} + s(p-1) - 1 > -1$, the integral converges, even after integrating w.r.t x over a compact set $K \subseteq \mathbb{R}^n \setminus \{0\}$, in which case we only need to extend the upper limit to a constant and estimate the factor $\frac{1}{|x|^{\alpha(1-p)}}$ in front of $V_\varepsilon(x)$ by another constant, both independent of x. ■
Next, we will show that our V_ε from (3.39) weakly converges to $V(x) = C_{n,s,p}|x|^{-sp}$, and prove the even stronger result of uniform convergence on every compact set.

3.35 Remark. If a family of functions $u_\varepsilon, \varepsilon > 0$, on Ω converges uniformly to u on every compact subset of Ω, then it also converges weakly in $L^1_{loc}(\Omega)$: To see this, we calculate for every bounded g with compact support

$$\left|\int_\Omega u_\varepsilon g \,\mathrm{d}x - \int_\Omega ug \,\mathrm{d}x\right| \le \int_{\mathrm{spt}\ g} |u_\varepsilon - u||g| \,\mathrm{d}x \le |\mathrm{spt}\ g| \sup_{\mathrm{spt}\ g} |u_\varepsilon - u| \sup_{\mathrm{spt}\ g} |g|,$$

which tends to 0 as $\varepsilon \to 0$.

3.36 Lemma. *Let V_ε be defined as in* (3.39). *Then*

$$\lim_{\varepsilon\to 0} V_\varepsilon(x) = C_{n,s,p}|x|^{-sp} \tag{3.41}$$

uniformly for every compact subset of $\mathbb{R}^n\setminus\{0\}$. Here $C_{n,s,p}$ denotes the constant defined in (3.23).

Proof .
Step 1: Pointwise convergence of $V_\varepsilon(x)$

We start by transforming into spherical coordinates[1] , i.e. $x = ru, y = \rho v$ with $r, \rho > 0$ and $u, v \in \mathbb{S}^{n-1}$:

$$V_\varepsilon(x) = \frac{2}{r^{\alpha(1-p)}} \int_{|\rho-r|>\varepsilon} \rho^{n-1}\mathrm{sgn}(\rho^\alpha - r^\alpha) \left|\frac{1}{r^\alpha} - \frac{1}{\rho^\alpha}\right|^{p-1} \int_{\mathbb{S}^{n-1}} \frac{1}{|ru-\rho v|^{n+sp}} \,\mathrm{d}\mathcal{H}^{n-1}(v)\,\mathrm{d}\rho. \tag{3.42}$$

In case $n \geq 2$ we make use of the following formula for the surface integral over the unit sphere $\mathbb{S}^{n-1}$, where $e \in \mathbb{S}^{n-1}$ is an arbitrary unit vector (see [24], formula (§1.41)[2]):

$$\int_{\mathbb{S}^{n-1}} f(v)\,\mathrm{d}\mathcal{H}^{n-1}(v) = \int_{-1}^{1}\int_{\mathbb{S}^{n-1}\cap e^\perp} f(te + \sqrt{1-t^2}w)(1-t^2)^{\frac{n-3}{2}}\,\mathrm{d}\mathcal{H}^{n-2}(w)\,\mathrm{d}t. \tag{3.43}$$

This yields with $e := u$

$$I(\rho) := \int_{-1}^{1}\int_{\mathbb{S}^{n-1}\cap u^\perp} \frac{(1-t^2)^{\frac{n-3}{2}}}{|ru - \rho t u - \rho\sqrt{1-t^2}w|^{n+sp}}\,\mathrm{d}\mathcal{H}^{n-2}(w)\,\mathrm{d}t$$

for the surface integral in (3.42). Now we rewrite the denominator as

$$\begin{aligned} r^{n+sp}\left|\left(1-\frac{\rho}{r}t\right)u - \frac{\rho}{r}\sqrt{1-t^2}w\right|^{2\frac{n+sp}{2}} &= r^{n+sp}\left[1-2\frac{\rho}{r}t + \left(\frac{\rho}{r}\right)^2\right]^{\frac{n+sp}{2}}, &&\text{if } \rho < r,\\ \rho^{n+sp}\left|\left(\frac{r}{\rho}-t\right)u - \sqrt{1-t^2}w\right|^{2\frac{n+sp}{2}} &= \rho^{n+sp}\left[1-2\frac{r}{\rho}t + \left(\frac{r}{\rho}\right)^2\right]^{\frac{n+sp}{2}}, &&\text{if } \rho > r, \end{aligned}$$

so this integral can be expressed in terms of the function $\Phi_{n,s,p} =: \Phi$ in (3.24):

$$I(\rho) = \begin{cases} r^{-n-sp}\Phi\left(\frac{\rho}{r}\right), & \text{if } \rho < r\\ \rho^{-n-sp}\Phi\left(\frac{r}{\rho}\right), & \text{if } \rho > r. \end{cases} \tag{3.44}$$

[1] formula (3.42) also holds true in case $n = 1$, where the surface integral is simply the sum of two real values; we remark this fact explicitly, as in the following we will discuss the cases $n = 1$ and $n \geq 2$ individually.

[2] strictly speaking, this identity is only proved for dimensions $n \geq 3$ therein; for $n = 2$ it can easily be seen by parametrizing both left and right semicircle the same way as in (3.43) and applying the area formula for surface integrals, with corresponding Jacobian $(1-t^2)^{-1/2}$

As for the remaining integral w.r.t. ρ in (3.42) we plug in the factor $|\rho - r|^{1+sp}$ in both nominator, where we write it as $r^{1+sp}(1-\rho/r)^{1+sp}$ or $\rho^{1+sp}(1-r/\rho)^{1+sp}$ respectively, and denominator, which yields

$$\frac{r^{\alpha(1-p)}}{2} V_\varepsilon(x) = r^{-n+1} \int_{|\rho - r| > \varepsilon} \frac{\operatorname{sgn}(\rho^\alpha - r^\alpha)}{|\rho - r|^{2-(1-s)p}} \varphi(\rho, r) \,\mathrm{d}\rho, \tag{3.45}$$

where

$$\varphi(\rho, r) = \left| \frac{\rho^{-\alpha} - r^{-\alpha}}{\rho - r} \right|^{p-1} \cdot \begin{cases} \rho^{n-1} \left(1 - \frac{\rho}{r}\right)^{1+sp} \Phi(\frac{\rho}{r}), & \text{if } \rho < r, \\ r^{n-1} \left(1 - \frac{r}{\rho}\right)^{1+sp} \Phi(\frac{r}{\rho}), & \text{if } \rho > r. \end{cases} \tag{3.46}$$

Note, that if we substitute $z := \frac{\rho}{r}$ and use the identity $\varphi(\rho, r) = r^{\alpha - p(1-s)} \varphi(\frac{\rho}{r}, 1)$ (which holds true in both cases $\rho < r$ and $\rho > r$), we can write

$$\begin{aligned} V_\varepsilon(x) &= r^{-sp} \cdot 2 \int_{|z-1| > \frac{\varepsilon}{r}} \frac{\operatorname{sgn}(z^\alpha - 1)}{|z-1|^{2-p(1-s)}} \varphi(z, 1) \,\mathrm{d}z = \\ &= 2r^{-sp} \left[-\int_0^{1-\frac{\varepsilon}{r}} \frac{\varphi(z,1)}{(1-z)^{2-(1-s)p}} \,\mathrm{d}z + \int_{1+\frac{\varepsilon}{r}}^\infty \frac{\varphi(z,1)}{(z-1)^{2-(1-s)p}} \,\mathrm{d}z \right]. \end{aligned} \tag{3.47}$$

The second integral can be rewritten by substituting z with z^{-1}, and we obtain

$$\begin{aligned} V_\varepsilon(x) &= 2r^{-sp} \left[-\int_0^{1-\frac{\varepsilon}{r}} \frac{\varphi(z,1)}{(1-z)^{2-(1-s)p}} \,\mathrm{d}z + \int_0^{1-\frac{\varepsilon}{r+\varepsilon}} \frac{z^{-(1-s)p} \varphi(z^{-1},1)}{(1-z)^{2-(1-s)p}} \,\mathrm{d}z \right] = \\ &= 2r^{-sp} \left[\int_0^{1-\frac{\varepsilon}{r}} \frac{z^{-(1-s)p} \varphi(z^{-1},1) - \varphi(z,1)}{(1-z)^{2-(1-s)p}} \,\mathrm{d}z + \int_{1-\frac{\varepsilon}{r}}^{1-\frac{\varepsilon}{r+\varepsilon}} \frac{z^{-(1-s)p} \varphi(z^{-1},1)}{(1-z)^{2-(1-s)p}} \,\mathrm{d}z \right]. \end{aligned} \tag{3.48}$$

In step 2 we will show, that $\varphi(z,1)$ is bounded in a neighbourhood of $z = 1$. This implies that the second integral converges to 0 as $\varepsilon \to 0^+$ since the length of the interval, over which we integrate, is of the order $O(\varepsilon^2)$. For the first integral, we can explicitly write down the values of φ appearing in the nominator,

$$\begin{aligned} z^{-p(1-s)} \varphi(z^{-1}, 1) &= z^{sp-1} (1 - z^\alpha)^{p-1} (1-z)^{2-p(1-s)} \Phi(z), \\ \varphi(z,1) &= z^{sp-1+\alpha} (1 - z^\alpha)^{p-1} (1-z)^{2-p(1-s)} \Phi(z), \end{aligned}$$

so it is equal to

$$\int_0^{1-\frac{\varepsilon}{r}} z^{sp-1} (1 - z^\alpha)^p \Phi(z) \,\mathrm{d}z = \int_0^{1-\frac{\varepsilon}{r}} \frac{z^{sp-1}}{(1-z)^{1-(1-s)p}} \left(\frac{1-z^\alpha}{1-z} \right)^p (1-z)^{1+sp} \Phi(z) \,\mathrm{d}z.$$

The factor $\left(\frac{1-z^\alpha}{1-z}\right)^p$ converges to α^p as $z \to 1$ and is therefore bounded in a neighbourhood of $z = 1$. In addition, we will see in step 2 that $(1-z)^{1+sp}\Phi(z)$ is bounded at $z = 1$ too. This ultimately yields that

$$C \int_\delta^1 \frac{z^{sp-1}}{(1-z)^{1-(1-s)p}} \,\mathrm{d}z$$

with a suitable constant $C > 0$ and $\delta > 0$ is an integrable majorant for the first integral in (3.48) and it thus converges as $\varepsilon \to 0^+$. This shows the pointwise convergence of V_ε in case $n \geq 2$ with the right constant $C_{n,s,p}$ defined in (3.23).
The proof for $n = 1$ is rather similar: Explicitly writing down the domain of integration in $V_\varepsilon(x)$ as union of intervals and suitably reversing the signs we obtain

$$V_\varepsilon(x) = \frac{2}{|x|^{\alpha(1-p)}} \left[-\int_0^{|x|-\varepsilon} (y^{-\alpha} - |x|^{-\alpha})^{p-1} \left(\frac{1}{|x+y|^{1+sp}} + \frac{1}{|x-y|^{1+sp}} \right) \mathrm{d}y + \right.$$
$$\left. + \int_{|x|+\varepsilon}^{\infty} (|x|^{-\alpha} - y^{-\alpha})^{p-1} \left(\frac{1}{|x+y|^{1+sp}} + \frac{1}{|x-y|^{1+sp}} \right) \mathrm{d}y \right].$$

We factor $|x|$ out and substitute $z := \frac{y}{|x|}$ in both integrals, and in the last integral we further substitute z with z^{-1}, which yields

$$V_\varepsilon(x) = \frac{2}{|x|^{\alpha(1-p)}} \left[-\int_0^{1-\frac{\varepsilon}{|x|}} |x|^{\alpha-1} (z^{-\alpha} - 1)^{p-1} \left(\frac{1}{|1+z|^{1+sp}} + \frac{1}{|1-z|^{1+sp}} \right) \mathrm{d}z + \right.$$
$$\left. + \int_0^{\frac{|x|}{|x|+\varepsilon}} |x|^{\alpha-1} z^{-\alpha} (z^{-\alpha} - 1)^{p-1} \left(\frac{1}{|1+z|^{1+sp}} + \frac{1}{|1-z|^{1+sp}} \right) \mathrm{d}z \right].$$

An integrable majorant can be established the same way as in the case $n \geq 2$ and passing to the limit $\varepsilon \to 0^+$ we get

$$\lim_{\varepsilon \to 0^+} V_\varepsilon(x) = |x|^{-sp} \cdot 2 \int_0^1 z^{sp-1} (1 - z^\alpha)^p \left(\frac{1}{|1+z|^{1+sp}} + \frac{1}{|1-z|^{1+sp}} \right) \mathrm{d}z,$$

recovering the constant $C_{1,s,p}$ from (3.23).

Step 2: Boundedness of $\varphi(z, 1)$

We first take a closer look at the function Φ: After the trigonometric substitution $t = -\cos\tau$ in (3.24) we have

$$\Phi(z) = |\mathbb{S}^{n-2}| \int_0^\pi \frac{\sin^{n-2}\tau}{(1 + 2z\cos\tau + z^2)^{\frac{n+sp}{2}}} \mathrm{d}\tau,$$

which can be rewritten by formula (3.665) in [14] as a product of the Beta function B and the hypergeometric function F as follows:

$$\Phi(z) = |\mathbb{S}^{n-2}| B\left(\frac{n-1}{2}, \frac{1}{2}\right) F\left(\frac{n+sp}{2}, \frac{2+sp}{2}, \frac{n}{2}; z^2\right). \tag{3.49}$$

By [21], formula (2) in chapter 6.8, and formula (5) in chapter 6.2.1, if $a+b-c > 1$, then

$$\lim_{z \to 1^-} (1-z)^{a+b-c} F(a, b, c; z) = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \tag{3.50}$$

where Γ denotes the Gamma function. In our constellation (3.49), we have $a + b - c = 1 + sp > 1$, so by setting $C := |\mathbb{S}^{n-1}| B(\frac{n-1}{2}, \frac{1}{2})$ and using (3.50)

$$\lim_{z \to 1^-} (1-z)^{1+sp} \Phi(z) = \lim_{z \to 1^-} \frac{C}{(1+z)^{1+sp}} F\left(\frac{n+sp}{2}, \frac{2+sp}{2}, \frac{n}{2}; z^2\right) =$$
$$= \frac{C}{2^{1+sp}} \frac{\Gamma(n/2)\Gamma(1+sp)}{\Gamma((n+sp)/2)\Gamma((2+sp)/2)}.$$

Thus, $(1-z)^{1+sp}\Phi(z)$ is bounded in a neighbourhood of $z=1$.
The remaining factor in our definition of $\varphi(z,1)$ resembles a differential quotient,

$$\left|\frac{z^{-\alpha}-1}{z-1}\right|^{p-1} \overset{z\to 1^-}{\to} \alpha^{p-1}$$

and therefore is bounded.

Step 3: Uniform convergence on compact sets

We want to prove that for every compact $K \subset \mathbb{R}^n \setminus \{0\}$ the family V_ε of functions is Cauchy, meaning that

$$\sup_{x\in K} |V_{\varepsilon_1}(x) - V_{\varepsilon_2}(x)|$$

is arbitarily small for $\varepsilon_1, \varepsilon_2$ small enough (we assume w.l.o.g. $\varepsilon_1 < \varepsilon_2$). To do so, we utilize the following scaling result: If $x \in \mathbb{R}^n \setminus \{0\}$ and $\lambda > 0$, then by substituting $\tilde{y} = \lambda y$

$$\begin{aligned}&\int_{\varepsilon_1 < ||\lambda x| - |\tilde{y}|| < \varepsilon_2} |\omega(\lambda x) - \omega(\tilde{y})|^{p-1} k(\lambda x, \tilde{y})\, \mathrm{d}\tilde{y} \\ &= \lambda^{-\alpha(p-1)-sp} \int_{\frac{\varepsilon_1}{\lambda} < ||x|-|y|| < \frac{\varepsilon_2}{\lambda}} |\omega(x) - \omega(y)|^{p-1} k(x,y)\, \mathrm{d}y.\end{aligned}$$

Since K is contained in a certain annulus, $K \subseteq \{x \in \mathbb{R}^n : r \le |x| \le R\}, 0 < r < R$, it suffices to consider

$$\sup_{\substack{r\le\lambda\le R \\ \xi\in\mathbb{S}^{n-1}}} \lambda^{-\alpha(p-1)-sp} \int_{\frac{\varepsilon_1}{R} < |1-|y|| < \frac{\varepsilon_2}{r}} |1-\omega(y)|^{p-1} \frac{1}{|\xi - y|^{n+sp}}\, \mathrm{d}y.$$

The factor $\lambda^{-\alpha(p-1)-sp}$ attains its maximum value at one of the boundary points r or R, while for the remaining factor we observe that the map

$$\xi \in \mathbb{S}^{n-1} \mapsto \int_{\frac{\varepsilon_1}{R} < |1-|y|| < \frac{\varepsilon_2}{r}} |1-\omega(y)|^{p-1} \frac{1}{|\xi - y|^{n+sp}}\, \mathrm{d}y$$

is continuous as can easily be seen from the majorant established in (3.40). Therefore there exists a $\xi_0 \in \mathbb{S}^{n-1}$ such that it attains its maximal value there. Since we have established pointwise convergence for all $x \in \mathbb{R}^n \setminus \{0\}$, the supremum can be made arbitrarily small if ε_1 and ε_2 are chosen small enough. ∎

Proof of the Hardy inequality (3.26) *and equality cases for* $p > 1$.
By our choice of the open set Ω and functions k, ω and V, as well as setting

$$k_\varepsilon(x,y) := \begin{cases} k(x,y), & \text{if } ||x|-|y|| > \varepsilon, \\ 0, & \text{if } ||x|-|y|| \le \varepsilon \end{cases}$$

we could assure the validity of all the assumptions in 3.30 in Lemma 3.34 and 3.36. Furthermore, for functions $u \in C_c^\infty(\mathbb{R}^n)$ the quantity $E[u]$ is finite (see example 3.4) and by the boundedness of u and since $\operatorname{spt} u \subseteq B_R$ for a $R > 0$, we also have

$$\int_{\mathbb{R}^n\setminus\{0\}} |V||u|^p\, \mathrm{d}x \le C \int_0^R \rho^{n-sp-1}\, \mathrm{d}\rho < \infty$$

For the sequence (u_j) of approximations of u defined in Remark 3.6 we have

$$\int_{|x|\geq 1} \frac{|u(x)-u_j(x)|^p}{|x|^{sp}}\,\mathrm{d}x \leq \int_{|x|\geq 1} |u(x)-u_j(x)|^p\,\mathrm{d}x \overset{j\to\infty}{\to} 0.$$

On the other hand, since $p < n/s$, there exists a $\delta > 1$ such that $n - \delta sp > 0$; let $\delta' > 1$ the conjugate exponent, i.e. $\frac{1}{\delta} + \frac{1}{\delta'} = 1$. Then, by the Hölder inequality

$$\int_{|x|<1} \frac{|u(x)-u_j(x)|^p}{|x|^{sp}}\,\mathrm{d}x \leq \left(\int_{|x|<1} |x|^{-\delta sp}\,\mathrm{d}x\right)^{\frac{1}{\delta}} \left(\int_{|x|<1} |u(x)-u_j(x)|^{\delta' p}\,\mathrm{d}x\right)^{\frac{1}{\delta'}},$$

where the right-hand side converges to 0 as $j \to \infty$.
In a similar fashion, namely by approximation, we will tackle the proof of the equality cases for $p > 1$. Equality (3.38) in the proof of Proposition 3.31 was shown for bounded u with compact support, such that $\int V|u|^p\,\mathrm{d}x$ and $\int\int |u(x)-u(y)|k(x,y)\,\mathrm{d}x\,\mathrm{d}y$ are finite. By approximation, this equality holds true for all functions $u \in W^{s,p}(\mathbb{R}^n)$ (see also Remark 3.33).
If there is equality in (3.31) for some $u \in W^{s,p}(\mathbb{R}^n)$, then this identity remains valid if u is replaced by $|u|$, since

$$[u]^p_{W^{s,p}} \geq [|u|]^p_{W^{s,p}} \geq \int_{\mathbb{R}^n} V|u|^p\,\mathrm{d}x = [u]^p_{W^{s,p}}.$$

From identity (3.38)

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \underbrace{\Phi_{|u|}(x,y)}_{\geq 0}\underbrace{k(x,y)}_{>0}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^n} V|u|^p\,\mathrm{d}x = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} ||u|(x)-|u|(y)|^p k(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

we see that $\Phi_{|u|} = 0$, which means that we have equality in the situation of Lemma 3.32 and we can rule out the case that $t = \frac{\omega(y)}{\omega(x)} = 0$, since ω is strictly positive. The only remaining possibility is, that $a = \left|\frac{v(x)}{v(y)}\right| = 1$. This means that $|v(x)| = \frac{|u(x)|}{\omega(x)}$ is constant, further implying that $|u(x)| = c\omega(x) = c|x|^{-\frac{n-sp}{p}}$ for a $c \in \mathbb{R}$. Since we demand $u \in W^{s,p}(\mathbb{R}^n)$, the constant c must be equal to 0, so the Hardy inequality is strict for every $0 \not\equiv u \in W^{s,p}(\mathbb{R}^n)$. ■
Since for $p > 1$ equality holds only for $u \equiv 0$, we need to prove the optimality of the constant $C_{n,s,p}$ separately:

Proof of the optimality of the constant $C_{n,s,p}$ for $p > 1$. In the following we give a sequence $(u_j)_{j\in\mathbb{N}}$ of $W^{s,p}(\mathbb{R}^n)$-functions such that

$$\frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u_j(x)-u_j(y)|^p k(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\int_{\mathbb{R}^n} |u_j(x)|^p|x|^{-sp}\,\mathrm{d}x} \to C_{n,s,p}, \text{ as } j \to \infty.$$

For $j \in \mathbb{N}$ we define the three regions

$$\begin{aligned} B &:= \{x \in \mathbb{R}^n : |x| < 1\}, \\ M_j &= \{x \in \mathbb{R}^n : 1 \leq |x| \leq j\}, \\ O_j &= \{x \in \mathbb{R}^n : |x| \geq j\}, \end{aligned}$$

and set

$$u_j(x) := \begin{cases} 1 - j^{-\alpha} & \text{if } x \in B, \\ |x|^{-\alpha} - j^{-\alpha} & \text{if } x \in M_j, \\ 0 & \text{if } x \in O_j. \end{cases}$$

Next, we take formula (3.41), multiply it with $u_j(x)$ and integrate w.r.t x on both sides. This yields

$$\begin{aligned} &2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u_j(x)(\omega(x)-\omega(y))|\omega(x)-\omega(y)|^{p-2}k(x,y)\,\mathrm{d}x\,\mathrm{d}y = \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u_j(x)-u_j(y))(\omega(x)-\omega(y))|\omega(x)-\omega(y)|^{p-2}k(x,y)\,\mathrm{d}x\,\mathrm{d}y = \\ &= C_{n,s,p}\int_{\mathbb{R}^n} \frac{u_j(x)\omega(x)^{p-1}}{|x|^{sp}}\,\mathrm{d}x, \end{aligned}$$

where we interchanged the variables x and y for one integral in the first equality. In case that $x, y \in M_j$, we have $u_j(x) - u_j(y) = \omega(x) - \omega(y)$, so the second expression in this chain of equalities can be rewritten as

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u_j(x)-u_j(y)|^p k(x,y)\,\mathrm{d}x\,\mathrm{d}x + 2R_0,$$

where

$$\begin{aligned} R_0 := &\int_{M_j}\int_B (1-\omega(y))((\omega(x)-\omega(y))^{p-1} - (1-\omega(y))^{p-1})k(x,y)\,\mathrm{d}x\,\mathrm{d}y + \\ &+ \int_{O_j}\int_{M_j} (\omega(x)-j^{-\alpha})((\omega(x)-\omega(y))^{p-1} - (\omega(x)-j^{-\alpha})^{p-1})k(x,y)\,\mathrm{d}x\,\mathrm{d}y + \\ &+ \int_{O_j}\int_B (1-j^{-\alpha})((\omega(x)-\omega(y))^{p-1} - (1-j^{-\alpha})^{p-1})k(x,y)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Since ω is monotonically decreasing in $|x|$, all integrands are pointwise non-negative, so that $R_0 \geq 0$.
The integral in the last expression in the chain of equalities can be expressed via

$$\int_{\mathbb{R}^n} \frac{u_j^p(x)}{|x|^{sp}}\,\mathrm{d}x + R_1 + R_2$$

with

$$\begin{aligned} R_1 &:= \int_B (1-j^{-\alpha})\left(\omega(x)^{p-1} - (1-j^{-\alpha})^{p-1}\right)|x|^{-sp}\,\mathrm{d}x, \\ R_2 &:= \int_{M_j} (\omega(x)-j^{-\alpha})\left(\omega(x)^{p-1} - (\omega(x)-j^{-\alpha})^{p-1}\right)|x|^{-sp}\,\mathrm{d}x. \end{aligned}$$

We claim that the sum $R_1 + R_2$ is bounded in j, so that by the fact that

$$\int_{\mathbb{R}^n} u_j^p(x)|x|^{-sp}\,\mathrm{d}x \to \infty \text{ as } j \to \infty,$$

we are able to establish the convergence we described at the beginning of the proof:

$$\frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|u_j(x)-u_j(y)|^p k(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\int_{\mathbb{R}^n}|u_j(x)|^p|x|^{-sp}\,\mathrm{d}x} =$$
$$= C_{n,s,p}\left(1+\frac{R_1+R_2}{\int_{\mathbb{R}^n}|u_j(x)|^p|x|^{-sp}\,\mathrm{d}x}\right)-\frac{2R_0}{\int_{\mathbb{R}^n}|u_j(x)|^p|x|^{-sp}\,\mathrm{d}x}\to C_{n,s,p},$$

as $j\to\infty$.
Finally, we tend to the proof of the claim, that R_1+R_2 is bounded: As for R_1, we can simply omit the first factor in brackets as it is lower or equal to 1, as well as the negative summand in the second pair of brackets; this results in

$$R_1\le\int_B\omega(x)^{p-1}|x|^{-sp}\,\mathrm{d}x=\int_B|x|^{\alpha-n}\,\mathrm{d}x<\infty.$$

For the summand R_2 we can factorize in the second pair of brackets and obtain

$$\omega(x)^{p-1}(1-[1-(j/|x|)^{-\alpha}]^{p-1})$$

where $t:=(j/|x|)^{-\alpha}\le 1$. By virtue of the mean value theorem, we have

$$1-(1-t)^{p-1}\begin{cases}\le 1-(1-t)=t, & 1\le p\le 2,\\ =(p-1)(1-\tau)^{p-2}t\le(p-1)t, & p>2,\end{cases}$$

for a point τ between 0 and t; both cases can be unified to $1-(1-t)^{p-1}\le Ct$, whenever $0\le t\le 1$, with C only depending on p. Ultimately,

$$\begin{aligned}R_2&\le C\int_{M_j}\omega(x)^{p-1}(j/|x|)^{-\alpha}|x|^{-sp}\,\mathrm{d}x\le\\ &\le C\int_{M_j}j^{-\alpha}|x|^{\alpha-n}\,\mathrm{d}x=C\int_B|\xi|^{\alpha-n}\,\mathrm{d}\xi<\infty,\end{aligned}$$

where in the last equality we used the substitution $x=j\xi$. Putting both estimates for R_1 and R_2 together results in our boundedness claim. ■

Proof of equality cases for $p=1$. We only need to consider non-negative and symmetric decreasing functions u, since

$$[u]_{W^{s,1}}\overset{(1)}{\ge}[|u|]_{W^{s,1}}\overset{(2)}{\ge}[|u|^{\#}]_{W^{s,1}}\overset{(3)}{\ge}C_{n,s,1}\int_{\mathbb{R}^n}\frac{|u|^{\#}(x)}{|x|^s}\,\mathrm{d}x\overset{(4)}{\ge}C_{n,s,1}\int_{\mathbb{R}^n}\frac{|u(x)|}{|x|^s}\,\mathrm{d}x.$$

Inequality (1) stems from the reverse triangle inequality with equality iff u is proportional to a non-negative function. Theorem 3.28 provides (2) with equality iff $\{|u|>\tau\}$ is a ball for a.e. $\tau>0$. Inequality (3) is the fractional Hardy inequality of Theorem 3.26 and (4) follows from rearrangement inequality [18, Theorem 3.4] with equality iff u is symmetric decreasing. Thus, if for any function equality holds in the fractional Hardy inequality (3.26), then it must be proportional to a symmetric decreasing function.
Conversely, all symmetric decreasing functions satisfy (3.26) with equality: The symmetric decreasing function u allows a layer-cake representation $u(x)=\int_0^\infty \mathbb{1}_t(x)\,\mathrm{d}t$,

where $\mathbb{1}_t$ is the indicator function of a ball centered at the origin with radius $R(t)$. Hence, by Fubini

$$\int_{\mathbb{R}^n} \frac{u(x)}{|x|^s}\,\mathrm{d}x = |\mathbb{S}^{n-1}| \int_0^\infty \int_0^{R(t)} r^{n-s-1}\,\mathrm{d}r\,\mathrm{d}t = \frac{|\mathbb{S}^{n-1}|}{n-s} \int_0^\infty R(t)^{n-s}\,\mathrm{d}t,$$

whereas for the seminorm

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,\mathrm{d}x\,\mathrm{d}y &= 2\iint_{\{|x|<|y|\}} \frac{\int_0^\infty \mathbb{1}_t(x) - \mathbb{1}_t(y)\,\mathrm{d}t}{|x-y|^{n+s}}\mathrm{d}x\,\mathrm{d}y = \\ &= 2\iiint_{\{|x|<R(t)<|y|\}} |x-y|^{-n-s}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = \\ &= 2\iint_{\{|\xi|<1<|\eta|\}} |\xi-\eta|^{-n-s}\mathrm{d}\xi\,\mathrm{d}\eta \int_0^\infty R(t)^{n-s}\,\mathrm{d}t,\end{aligned}$$

where for the last equality we substituted $(x,y) = (R(t)^{-1}\xi, R(t)^{-1}\eta)$ for fixed $t>0$. It remains to show that

$$\int_{|\xi|<1}\int_{|\eta|>1} |\xi-\eta|^{-n-s}\,\mathrm{d}\eta\,\mathrm{d}\xi = \frac{|\mathbb{S}^{n-1}|}{2(n-s)} C_{n,s,1} = \frac{\mathbb{S}^{n-1}}{n-s}\int_0^1 t^{s-1}(1-t^{n-s})\Phi_{n,s,1}(t)\,\mathrm{d}t. \tag{3.51}$$

For the case $n \geq 2$, we abbreviate $\Phi := \Phi_{n,s,1}$, use spherical coordinates for ξ and η, writing $\xi = ru$ and $\eta = \rho v$, and evaluate the integral w.r.t v using formula (3.44):

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}}\int_0^1\int_{\mathbb{S}^{n-1}}\int_1^\infty \frac{r^{n-1}\rho^{n-1}}{|ru-\rho v|^{n+s}}\,\mathrm{d}\rho\,\mathrm{d}\mathcal{H}^{n-1}(v)\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^{n-1}(u) = \\ &= |\mathbb{S}^{n-1}|\int_0^1\int_1^\infty r^{n-1}\rho^{-s-1}\Phi(r/\rho)\,\mathrm{d}\rho\,\mathrm{d}r \overset{t:=\frac{r}{\rho}}{=} \\ &= |\mathbb{S}^{n-1}|\int_0^1\int_0^r r^{n-s-1}t^{s-1}\Phi(t)\,\mathrm{d}t\,\mathrm{d}r = \\ &= |\mathbb{S}^{n-1}|\int_0^1 t^{s-1}\Phi(t)\int_t^1 r^{n-s-1}\,\mathrm{d}r\,\mathrm{d}t = \\ &= \frac{|\mathbb{S}^{n-1}|}{n-s}\int_0^1 t^{s-1}(1-t^{n-s})\Phi(t)\,\mathrm{d}t.\end{aligned}$$

The calculation can be repeated verbatim for $n=1$ with the only difference that one already has two integrals in one dimension and thus does not need to transform into spherical coordinates. ∎

3.37 Remark. In the preceding proof of the equality cases in the fractional Hardy inequality for $p=1$ we have additionally derived in (3.51) an alternative expression for the fractional s-perimeter of the Euclidean unit ball. Indeed, the left hand side of (3.51) coincides with the s-perimeter of $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$.

3.4.2 Proof of the rearrangement inequality for $[\cdot]_{W^{s,p}}$

This part is devoted to the proof of Theorem 3.28, where we state, that the Gagliardo seminorm $[\cdot]_{W^{s,n}}$ does not increase under decreasing symmetric rearrangement. Of great help will be the following result on the change of certain integrals under rearrangement; the inequality is due to Riesz, the more recent classification of equality cases is due to Lieb [17, Lemma 3].

3.38 Proposition. *Let $f, g, h : \mathbb{R}^n \to \mathbb{R}$ be measurable functions. Then,*

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(x-y)h(y)\,\mathrm{d}x\,\mathrm{d}y\right| \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^{\#}(x)g^{\#}(x-y)h^{\#}(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.52}$$

Furthermore, if g is positive and symmetric decreasing, and the right hand-side is finite, then equality holds if and only if there exists a point $a \in \mathbb{R}^n$ such that

$$f(x) = f^{\#}(x-a), \text{ and } h(x) = h^{\#}(x-a)$$

for a.e. $x \in \mathbb{R}^n$.

The next statement deals with functionals that are similar to the Gagliardo seminorm. Note that we impose that $k \in L^1$ in contrast to our situation, where $k(\xi) = |\xi|^{-n-sp}$ does not enjoy this integrability.

3.39 Lemma. *Let $J : \mathbb{R} \to [0, \infty)$ be a convex function with $J(0) = 0$ and let $k \in L^1(\mathbb{R}^n)$ be symmetric decreasing. For a measurable function u on $\mathbb{R}^n$ we define*

$$E[u] := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J(u(x)-u(y))k(x-y)\,\mathrm{d}x\,\mathrm{d}y.$$

Then for all non-negative measurable functions u with $E[u]$ and $|\{u > \tau\}|$ finite for all $\tau > 0$ we have

$$E[u] \ge E[u^{\#}]. \tag{3.53}$$

If in addition, J is strictly convex and k is strictly decreasing, then equality holds if and only if there exists a point $x_0 \in \mathbb{R}^n$ and a symmetric decreasing function v such that $u(x) = v(x-x_0)$, i.e. u is a translate of a symmetric decreasing function.
If $J(t) = |t|$, then equality holds if and only if the super-level sets $\{u > \tau\}$ are balls for a.e. $\tau > 0$.

Proof .
Step 1: Reduction
First we decompose the convex function J into two parts "left" and "right" of the y-axis by $J = J_+ + J_-$, where $J_+(t) := J(t)$ for $t \ge 0$ and $J_+(t) := 0$ for $t < 0$. This decomposition instantly yields a corresponding representation of the functional E, which we will write as $E = E_+ + E_-$. Now observe, that we only need to prove inequality (3.53) for the functional E_+, since by $J_-(t) = J_+(-t)$ it immediately follows by interchanging the variables x and y and replacing J with $\tilde{J}(t) := J(-t)$, which again is convex and satisfies $\tilde{J}(0) = 0$:

$$\begin{aligned} E_-[v] &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_-(v(y)-v(x))k(y-x)\,\mathrm{d}y\,\mathrm{d}x = \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{J}_+(v(x)-v(y))k(x-y)\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

for any measurable function v.

Step 2: Proof for bounded functions u
Since J_+ is convex, it is differentiable almost everywhere on $\mathbb{R}$ with derivative J_+'. By the substitution $\xi = u(x) - \tau$ in the following calculation we obtain a formula for $J_+(u(x) - u(y))$:

$$J_+(u(x) - u(y)) = \int_{-\infty}^{u(x)-u(y)} J_+'(\xi)\,\mathrm{d}\xi = \int_{u(y)}^{\infty} J_+'(u(x) - \tau)\,\mathrm{d}\tau.$$

This expression can be used in conjunction with Fubini's theorem to write $E_+[u]$ as

$$E_+[u] = \int_0^\infty e_\tau^+[u]\,\mathrm{d}\tau,$$

where

$$e_\tau^+[u] := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_+'(u(x) - \tau)k(x - y)\mathbb{1}_{\{u\le\tau\}}(y)\,\mathrm{d}x\,\mathrm{d}y.$$

In the next step we would like to bring the integrals in $e_\tau^+[u]$ into a suitable form, such that Riesz's rearrangement inequality is easily applicable. To this end, we use $\mathbb{1}_{\{u\le\tau\}} = 1 - \mathbb{1}_{\{u>\tau\}}$ and split the integrals accordingly, which is made possible by the boundedness of u and the finiteness of $|\{u > \tau\}|$, since

$$\int_{\mathbb{R}^n} J_+'(u(x) - \tau)\,\mathrm{d}x = \int_{\{u>\tau\}} J_+'(u(x) - \tau)\,\mathrm{d}x \le |\{u > \tau\}|J_+'(C) < \infty,$$

where $C > 0$ is a constant. Therefore,

$$e_\tau^+[u] = \|k\|_{L^1}\int_{\mathbb{R}^n} J_+'(u(x) - \tau)\,\mathrm{d}x - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_+'(u(x) - \tau)k(x - y)\mathbb{1}_{\{u>\tau\}}(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Since the derivative J_+' is a non-decreasing function on its domain, by remark 3.24, 6., we have $(J_+'(u - \tau))^\# = J_+'(u^\# - \tau)$, so the first summand does not change under rearrangement. By Riesz's rearrangement inequality, the second integral does not decrease under rearrangement, which implies that $e_\tau^+[u] \ge e_\tau^+[u^\#]$ and ultimately $E_+[u] \ge E_+[u^\#]$.
Next we turn our attention to the cases of equality $E_+[u] = E_+[u^\#]$ under the additional assumption that k is strictly decreasing. By

$$\int_0^\infty \underbrace{(e_\tau^+[u] - e_\tau^+[u^\#])}_{\ge 0}\,\mathrm{d}x = 0$$

we have $e_\tau^\#[u] = e_\tau^\#[u^\#]$ for a.e. $\tau > 0$, resulting eventually in

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_+'(u(x){-}\tau)k(x{-}y)\mathbb{1}_{\{u>\tau\}}(y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (J_+')^\#(u(x){-}\tau)k(x{-}y)\mathbb{1}_{\{u>\tau\}}^\#(y)\,\mathrm{d}x\,\mathrm{d}y.$$

By Proposition 3.38 this is only possible if for a.e. $\tau > 0$ there exists a point $a_\tau \in \mathbb{R}^n$ such that both $\mathbb{1}_{\{u<\tau\}}(x) = \mathbb{1}_{\{u^\#<\tau\}}(x - a_\tau)$ and

$$J_\pm'(u(x) - \tau) = J_\pm'(u^\#(x - a_\tau) - \tau) \tag{3.54}$$

for a.e. $x \in \mathbb{R}^n$.

If we set $J(t) = t$, then $J_+'(t) = 1$ precisely for positive values of t, so by (3.54) the super-level sets $\{u > \tau\}$ are balls for a.e. $\tau > 0$.
If J_+ is strictly convex on $[0, \infty)$, then the derivative J_+' is strictly increasing thereon, which implies

$$(u(x) - \tau)_+ = (u^{\#}(x - a_\tau) - \tau)_+$$

for a.e. $x \in \mathbb{R}^n$ and $\tau > 0$. It follows, that a_τ must be constant in τ, so u is a translate of a symmetric decreasing function.

Step 3: Proof for general u
The inequality is easily shown by truncating the function u by setting $u_M := \min(u, M)$ for $M \in \mathbb{N}$. First, we observe that $(u_M)^{\#} = (u^{\#})_M =: u_M^{\#}$, because

$$(u_M)^{\#}(x) = \int_0^\infty \mathbb{1}_{\{u_M > t\}^{\#}}(x)\,\mathrm{d}t = \int_0^M \mathbb{1}_{\{u > t\}^{\#}}(x)\,\mathrm{d}t = \min(u^{\#}(x), M).$$

Since J is a non-negative convex function having a minimal point at 0, the expression $J(w_M(x) - w_M(y))$ is monotonically increasing in M for any non-negative measurable function w on $\mathbb{R}^n$ and all $x, y \in \mathbb{R}^n$. This implies that we can apply the monotone convergence theorem to both sides of the inequality $E[u_M] \geq E[u_M^{\#}]$, which we have established in step 2, to obtain inequality (3.53) also for possibly unbounded u.
The characterization of the equality cases remains to be proved: To this end, suppose that k is strictly decreasing and that $E_+[u] = E_+[u_M]$ for some non-negative u with $E[u]$ and $|\{u > \tau\}|$ finite for every $\tau > 0$. We want to reduce the situation to the case, where the functions involved are bounded, so we can apply our results of step 2: For any $M \in \mathbb{N}$ we decompose u into

$$u = u_M + v_M, \text{ where } u_M := \min(u, M),$$

so the functional E_+ can be expressed as

$$E_+[u] = E_+[u_M] + E_+[v_M] + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F_M(v_M(x), u_M(y)) k(x - y)\,\mathrm{d}x\,\mathrm{d}y \tag{3.55}$$

with

$$F_M(v, u) := J_+(v + M - u) - J_+(v) - J_+(M - u).$$

The interested reader can find the somewhat lengthy calculation (which mainly consists of simple algebraic manipulations) leading to this expression in the appendix.
For $0 \leq u \leq M$ and $v \geq 0$ we have $F_M(v, u) \geq 0$, since J_+ is convex with $J_+(0) = 0$. To see this, we suppose that $0 < M - u \leq v$, so

$$\frac{J_+(M - u) - J_+(0)}{M - u} \leq \frac{J_+(v + M - u) - J_+(v)}{M - u}.$$

An analoguous inequality holds for exchanged roles of $M - u$ and v; should at least one of the values $M - u$ and v be equal to 0, then also $F_M(v, u) = 0$.
We claim that the double integral in (3.55) does not increase when both u_M and v_M are replaced by $u_M^{\#}$ and $v_M^{\#}$. Then,

$$E_+[u] \geq E[u_M^{\#}] + E[v_M^{\#}] + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F_M(v_M^{\#}(x), u_M^{\#}(y)) k(x - y)\,\mathrm{d}x\,\mathrm{d}y = E_+[u^{\#}],$$

so $E_+[u_M] = E_+[u_M^\#]$ must hold true for every $M \in \mathbb{N}$. We can thereby transfer the characterization of equality cases for bounded functions in step 2 to unbounded functions.
We now show our claim that the double integral in (3.55) does not increase under rearrangement: Since J'_+ is by convexity monotonically increasing and continuous on the right, it can be viewed as distribution function of a non-negative measure μ, i.e. $J'_+(t) = \int_0^t \mathrm{d}\mu(\tau)$. This leads to an alterative expression of J_+ by

$$J_+(t) = \int_0^t \int_0^s \mathrm{d}\mu(\tau)\,\mathrm{d}s = \int_0^t \int_\tau^t \mathrm{d}s\,\mathrm{d}\mu(\tau) = \int_0^\infty (t-\tau)_+\,\mathrm{d}\mu(\tau).$$

$F_M(v,u)$ too can be rewritten as

$$F_M(v,u) = \int_0^\infty f_{M,\tau}(v,u)\,\mathrm{d}\mu(\tau),$$

where

$$f_{M,\tau}(v,u) := (v+M-u-\tau)_+ - (v-\tau)_+ - (M-u-\tau)_+.$$

For $0 \le u \le M$ and $v \le 0$ this is a non-negative function, which can be easily seen, if at least one of the summands $(v-\tau)_+$ and $(M-u-\tau)_+$ is equal to 0; else we can add all summands resulting in $\tau > 0$. Hence, by Fubini it suffices to show that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_{M,\tau}(v_M(x), u_M(y))k(x-y)\,\mathrm{d}x\,\mathrm{d}y$$

does not increase under rearrangement for all $\tau > 0$. We further split the function $f_{M,\tau} = f^{(1)}_{M,\tau} + f^{(2)}_{M,\tau}$, where

$$\begin{aligned} f^{(1)}_{M,\tau}(v) &:= v - (v-\tau)_+, \\ f^{(2)}_{M,\tau}(v,u) &:= v - (v+M-u-\tau)_+ + (M-u-\tau)_+ = \min(v, (u-M+\tau)_+). \end{aligned}$$

The first summand $f^{(1)}_{M,\tau}$ is monotonically increasing, bounded by τ and since by $|\{u > M\}| < \infty$ the support of v_M has finite measure. This implies that the integral

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^{(1)}_{M,\tau}(v_M(x))k(x-y)\,\mathrm{d}x\,\mathrm{d}y = \|k\|_{L^1}\int_{\mathbb{R}^n} f^{(1)}_{M,\tau}(v_M(x))\,\mathrm{d}x$$

is finite and does not change by replacing v_M with $v_M^\#$ (see Remark 3.24). The second occuring integral can be rewritten by the layer-cake formula (3.21) and Fubini as

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^{(2)}_{M,\tau}(v_M(x), u_M(y))k(x-y)\,\mathrm{d}x\,\mathrm{d}y = \\ &= \int_0^\infty \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbb{1}_{\{v_M > t\}}(x)k(x-y)\mathbb{1}_{\{(u_M - M+\tau)_+ > t\}}(y)\,\mathrm{d}x\,\mathrm{d}y\right)\mathrm{d}t. \end{aligned}$$

By Riesz' rearrangement inequality (3.52) it does not decrease under rearrangement, which shows our claim. ■

Proof of Theorem 3.28 (rearrangement inequality).
By the reverse triangle inequality we have $[u]^p_{W^{s,p}} \geq [|u|]^p_{W^{s,p}}$ with equality if and only if u is proportional to a non-negative function. Furthermore, by definition of the symmetric decreasing rearrangement, $[|u|^{\#}]^p_{W^{s,p}} = [u^{\#}]^p_{W^{s,p}}$, i.e. the right-hand side of inequality (3.29), which we want to prove, does not change by replacing u with $u^{\#}$. Thus, from now on we only have to prove it for non-negative functions. We rewrite the seminorm as follows (see [2], §9):

$$[u]^p_{W^{s,p}} = \frac{1}{\Gamma(\frac{n+sp}{2})} \int_0^\infty I_\alpha[u] \alpha^{\frac{n+sp}{2}-1} \, \mathrm{d}\alpha,$$

where

$$I_\alpha[u] := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^p e^{-\alpha|x-y|^2} \, \mathrm{d}x \, \mathrm{d}y.$$

Now we are in a situation to apply Lemma 3.39 with $J(t) = |t|^p$ and $k(\xi) = e^{-\alpha|\xi|^2}$, so $I_\alpha[u] \geq I_\alpha[u^{\#}]$ for all $\alpha > 0$ with equality for the same cases as in the aforementioned lemma. This concludes the proof. ■

4 The anisotropic fractional perimeter

We now combine the ideas developed in chapters 2 and 3; a general reference for the results about to follow can be found in [19] and [20].

In the following we will explore the following themes:
The introductory part for this chapter presents the notion of the anisotropic fractional perimeter as well as two types of convex bodies which will play a prominent role in the study of such perimeters, namely moment and centroid bodies.
Then in section 4.1 we develop some tools to reduce higher-dimensional integration to integration over lines. The Blaschke-Petkantschin formula provides a straightforward way of calculating integrals, whereas our results on slicing by lines relate the perimeter of a whole set to the perimeters of its slices. The conclusion of this section comprises the one-dimensional variants of the limiting results we have discussed in sections 3.2 and 3.3, and we explicitly derive estimates for the fractional perimeter in this setting.
Both sections 4.2 and 4.3 illustrate how the one-dimensional results we have previously shown can be applied to calculate the limit of anisotropic fractional s-perimeters as $s \to 1^-$ and $s \to 0^+$.
Ultimately, we will generalize the isoperimetric inequality in the last section 4.4.

From now on through the rest of the chapter, $K \subset \mathbb{R}^n$ denotes an origin-symmetric proper convex body.

We start with the definition of the anisotropic fractional s-perimeter:

4.1 Definition. Let $E \subseteq \mathbb{R}^n$ be a Lebesgue measurable set, $K \subseteq \mathbb{R}^n$ an origin-symmetric convex body, and $0 < s < 1$. Then the *anisotropic fractional s-perimeter* of E with respect to K is defined as

$$P_{s,K}(E) := \int_E \int_{E^c} \frac{1}{\|x-y\|_K^{n+s}} \,\mathrm{d}x\,\mathrm{d}y.$$

In the subsequent sections we will prove convergence results analogous to the results of Brezis, Bourgain and Mironescu (which we have discussed in 3.2), and the results of Maz'ya and Shaposhnikova (see section 3.3). Surprisingly, in the limit $s \to 1^-$ the anisotropic s-perimeter w.r.t. K does not converge to the anisotropic perimeter w.r.t. the same convex body. We will now present the type of resulting bodies and discuss their geometric meaning:

4.2 Definition. For a convex body $K \subset \mathbb{R}^n$ we define for $v \in \mathbb{R}^n$ the norm

$$\|v\|_{\mathrm{Z}^*K} := \frac{n+1}{2} \int_K |v \cdot x| \,\mathrm{d}x.$$

Then the convex body

$$\mathrm{Z}K := \{u \in \mathbb{R}^n : u \cdot v \leq 1 \text{ for all } \|v\|_{\mathrm{Z}^*K} \leq 1\},$$

i.e. the polar body of Z^*K, is called *moment body* of K.
The body

$$\Gamma K := \frac{2}{(n+1)|K|}\mathrm{Z}K$$

is called *centroid body* of K.

4.3 Remark. The centroid body of a convex body K can be interpreted geometrically as follows: For each unit vector $\xi \in \mathbb{S}^{n-1}$ we intersect K with the halfspace orthogonal to u. If we take the centroid of each of the intersections, then these centroids trace out twice the boundary of the centroid body.
A suitable multiple of the moment body results from this construction by taking the moment vectors of the intersections.
For a further discussion of these bodies see [25, Section 10.8].

4.1 From $\mathbb{R}^n$ to $\mathbb{R}^1$ and back: The Blaschke-Petkantschin formula and slicing by lines

For the proof of results concerning the limits of anisotropic fractional perimeters we will in the following take a detour to the one-dimensional case, where the results can easily be shown. This section has two main purposes: Firstly, to provide all the tools needed to make the descent from $\mathbb{R}^n$ to $\mathbb{R}^1$ such as slicing, and secondly, to prove the results of our interest in the one-dimensional base case.

We want to start by gathering some basic facts about lines in $\mathbb{R}^n$ and their collection in the affine Grassmannian $\mathrm{Aff}(n,1)$ (see [26], 13.2) :
Each line $L \subset \mathbb{R}^n$ can be described as the set of all points $x + \lambda u$, $\lambda \in \mathbb{R}$, where $u \in \mathbb{S}^{n-1}$ is the direction unit vector and $x \in u^\perp$. Conveniently, we write $L = x + L_u$, where $L_u := \{\lambda u : \lambda \in \mathbb{R}\}$. We further remark, that the map

$$\Phi_L : \begin{cases} \mathbb{R} \to L \\ \lambda \mapsto x + \lambda u \end{cases} \tag{4.1}$$

is an immersion (meaning that $d\Phi_L(\lambda) \neq 0$ for all $\lambda \in \mathbb{R}$), that is also a homeomorphism, where L is equipped with the subspace topology.
The set of all lines, the *affine Grassmannian* $\mathrm{Aff}(n,1)$, can be endowed with a topology as follows: Let $L_0 \subset \mathbb{R}^n$ be a fixed linear one-dimensional subspace, and denote $SO(n)$ the special orthogonal group in $\mathbb{R}^n$ ($SO(n)$ carries the topology induced by $\mathbb{R}^{n^2}$). Then we call a set of lines open, if its preimage under the map

$$\begin{aligned} L_0^\perp \times SO(n) &\to \mathrm{Aff}(n,1) \\ (x, \vartheta) &\mapsto \vartheta(L_0 + x) \end{aligned}$$

is open in the product topology of $L_0^\perp \times SO(n)$. The resulting topology on $\mathrm{Aff}(n,1)$ is locally compact.

Furthermore we can equip $\mathrm{Aff}(n,1)$ and its Borel sets with a rigid motion invariant measure λ_1^n, that satisfies

$$\int_{\mathrm{Aff}(n,1)} h(L)\,\mathrm{d}\lambda_1^n(L) = \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{u^\perp} h(x+L_u)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}\mathcal{H}^{n-1}(u) \tag{4.2}$$

for all measurable functions $h : \mathrm{Aff}(n,1) \to [0,\infty)$.

We now have established the suitable framework to provide a link between one and higher dimensions, namely the *(affine) Blaschke-Petkantschin formula* :

4.4 Theorem. *Let $g : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ be a measurable function. Then,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x,y)\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}\mathcal{H}^{n-1}(y) = \int_{\mathrm{Aff}(n,1)}\int_L\int_L g(x,y)|x-y|^{n-1}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\lambda_1^n(L). \tag{4.3}$$

A more general statement for affine Grassmannians consisting of higher-dimensional subspaces as well as its proof can be found in [26], Theorem 7.2.7.

Our next result states that one-dimensional slices of sets of finite perimeter are again of finite perimeter almost everywhere:

4.5 Proposition. *Let $E \subseteq \mathbb{R}^n$ be a set of finite perimeter and $u \in \mathbb{R}^n$ a unit vector. Then, for almost every $x \in u^\perp$ the set $A_x := \Phi_L^{-1}(E \cap (x + L_u)) \subseteq \mathbb{R}$ with Φ_L defined in* (4.1) *is of finite perimeter. Furthermore,*

$$\int_{u^\perp} P(A_x)\,\mathrm{d}\mathcal{H}^{n-1}(x) \le P(E). \tag{4.4}$$

Proof . In [22, Proposition 14.5] the statement is shown for lines of the form $L = y + L_{e_n}$, where $e_n = (0,\dots,0,1) \in \mathbb{R}^n$, and $y \in e_n^\perp$.
If $\vartheta \in SO(n)$ is a rotation and $\xi = x + \lambda u$, $\lambda \in \mathbb{R}$, is a point of a line with $u \in \mathbb{S}^{n-1}$ and $x \in u^\perp$, then $\vartheta\xi = \vartheta x + \lambda\vartheta u$. The parameter λ does not change under rotation, hence $\Phi_L^{-1}(E \cap (x + L_u)) = \Phi_L^{-1}(\vartheta(E \cap (x + L_u)))$. We can choose a rotation $\vartheta \in SO(n)$, such that $\vartheta u = e_n$ and $\vartheta u^\perp = \mathbb{R}^{n-1} \times \{0\}$, so we can apply the result cited at the beginning of the proof to ϑL. ■
The following proposition is due to Wieacker [30, Theorem 1]:

4.6 Proposition. *Let $E \subseteq \mathbb{R}^n$ be a set of finite perimeter, and $u \in \mathbb{S}^{n-1}$. Then,*

$$\int_{\partial^* E} |u \cdot \nu_E(x)|\,\mathrm{d}\mathcal{H}^{n-1}(x) = \int_{E|u^\perp} \mathcal{H}^0(\partial^* E \cap (y + L_u))\,\mathrm{d}\mathcal{H}^{n-1}(y).$$

Now we state the results concerning the limits of the fractional s-perimeter as $s \to 1^-$ and $s \to 0^+$ in the one-dimensional case.

4.7 Lemma. *Let $A \subseteq \mathbb{R}$ be a bounded set of finite perimeter. Then,*

$$\lim_{s\to 1^-}(1-s)P_s(A) = \mathcal{H}^0(\partial^* A), \tag{4.5}$$

and

$$(1-s)P_s(A) \leq 8\mathcal{H}^0(\partial^* A)\max(1,\operatorname{diam}(A)) \tag{4.6}$$

for all $s \in [1/2, 1)$.

Proof . We have proved the convergence result (4.5) in Theorem 3.18, since $P(A) = \mathcal{H}^0(\partial^* A)$.
It remains to show the estimate (4.6). Since A has finite perimeter, it is - up to a set of measure zero - the disjoint union of finitely many intervals, i.e.

$$A = \bigcup_{j=1}^{M} I_j, \tag{4.7}$$

where $I_j = (a_j, b_j), j = 1, \ldots, M$ (see [22], Proposition 12.13). Furthermore, the reduced boundary of a set does not change under modifications by sets of measure zero, so from now on, we assume A to be of the form (4.7). We can write A^c as $A^c = \bigcup_{j=0}^{M} J_j$, where $J_j := [b_j, a_{j+1}]$ for $j = 1, \ldots, M-1$, $J_0 := (-\infty, a_1]$, $J_M := [b_M, \infty)$. The s-perimeter hence takes the following form:

$$P_s(A) = \sum_{j=0}^{M} \int_{J_j} \int_A \frac{1}{|x-y|^{s+1}} \,\mathrm{d}x\,\mathrm{d}y. \tag{4.8}$$

We estimate the summand for $j = 0$:

$$\begin{aligned}
\int_{J_0} \int_A \frac{1}{|x-y|^{s+1}} \,\mathrm{d}x\,\mathrm{d}y &\leq \int_{-\infty}^{a_1} \int_{a_1}^{b_M} \frac{1}{(x-y)^{s+1}} \,\mathrm{d}x\,\mathrm{d}y \overset{\text{Fub.}}{=} \int_{a_1}^{b_M} \int_{-\infty}^{a_1} \frac{1}{(x-y)^{s+1}} \,\mathrm{d}y\,\mathrm{d}x = \\
&= \int_{a_1}^{b_M} \frac{(x-a_1)^{-s}}{s} \,\mathrm{d}x = \frac{(b_M - a_1)^{1-s}}{s(1-s)} \leq \\
&\overset{s \geq 1/2}{\leq} \frac{2}{1-s} \max(1, \operatorname{diam}(A)).
\end{aligned} \tag{4.9}$$

The same estimate can be made with

$$\begin{aligned}
\int_{J_M} \int_A \frac{1}{|x-y|^{s+1}} \,\mathrm{d}x\,\mathrm{d}y &\leq \int_{b_M}^{\infty} \int_{a_1}^{b_M} \frac{1}{(y-x)^{s+1}} \,\mathrm{d}x\,\mathrm{d}y = \int_{-\infty}^{-b_M} \int_{-b_M}^{-a_M} \frac{1}{(x-y)^{s+1}} \,\mathrm{d}x\,\mathrm{d}y \leq \\
&\leq \frac{2}{1-s} \max(1, \operatorname{diam}(A)).
\end{aligned}$$

For $j = 1, \ldots, M-1$ we compute

$$\int_{J_j} \int_A \frac{1}{|x-y|^{s+1}} \,\mathrm{d}x\,\mathrm{d}y \leq \int_{a_1}^{b_j} \int_{b_j}^{a_{j+1}} \frac{1}{(y-x)^{s+1}} \,\mathrm{d}y\,\mathrm{d}x + \int_{a_{j+1}}^{b_M} \int_{b_j}^{a_{j+1}} \frac{1}{(x-y)^{s+1}} \,\mathrm{d}y\,\mathrm{d}x.$$

The first integral on the right-hand side equals to

$$\begin{aligned}
\int_{a_1}^{b_j} \int_{b_j}^{a_{j+1}} \frac{1}{(y-x)^{s+1}} \,\mathrm{d}y\,\mathrm{d}x &= \frac{1}{s(1-s)} \left[(a_{j+1} - b_j)^{1-s} + (b_j - a_1)^{1-s} - (a_{j+1} - a_1)^{1-s} \right] \leq \\
&\leq \frac{4}{1-s} \max(1, \operatorname{diam}(A)).
\end{aligned}$$

Similarly for the second integral,

$$\int_{a_{j+1}}^{b_M} \int_{b_j}^{a_{j+1}} \frac{1}{(x-y)^{s+1}} \,\mathrm{d}y \,\mathrm{d}x \le \frac{4}{1-s} \max(1, \operatorname{diam}(A)).$$

Ultimately, we add up all integrals in (4.8) and by $\mathcal{H}^0(\partial^* A) = 2M$ we get

$$(1-s)P_s(A) \le 8(M+1)\max(1, \operatorname{diam}(A)) \le 8\mathcal{H}^0(\partial^* A) \max(1, \operatorname{diam}(A)).$$

■

4.8 Lemma. *Let $A \subseteq \mathbb{R}^n$ be a bounded set of finite perimeter. Then,*

$$\lim_{s \to 0^+} sP_s(A) = 2|A|, \tag{4.10}$$

and

$$P_s(A) \le \frac{4}{s} \max(1, \operatorname{diam}(A)) + \operatorname{diam}(A)^2 + P_{s'}(A) \tag{4.11}$$

for $0 < s < s' < 1/2$.

Proof . The convergence result (4.10) is shown in Corollary 3.20. To prove the estimate (4.11), we set $a := \inf A$, $b := \sup A$ and $C := A^c \cap (a, b)$, so that

$$P_s(A) \le \int_{-\infty}^{a} \int_{a}^{b} \frac{1}{(x-y)^{s+1}} \,\mathrm{d}x \,\mathrm{d}y + \int_{C} \int_{a}^{b} \frac{1}{|x-y|^{s+1}} \,\mathrm{d}x \,\mathrm{d}y + \int_{b}^{\infty} \int_{a}^{b} \frac{1}{(y-x)^{s+1}} \,\mathrm{d}x \,\mathrm{d}y.$$

For the first and third summand on the right-hand side we can apply the same calculations as in (4.9) to see that both are less or equal to $\frac{2}{s} \max(1, \operatorname{diam}(A))$.
We further split the second summand into

$$\iint\limits_{\{|x-y| \ge 1\} \cap (A \times C)} \frac{1}{|x-y|^{1+s}} \,\mathrm{d}x \,\mathrm{d}y \le \operatorname{diam}(A)^2,$$

since $\operatorname{diam}(C) \le \operatorname{diam}(A)$, and

$$\iint\limits_{\{|x-y| < 1\} \cap (A \times C)} \frac{1}{|x-y|^{1+s}} \,\mathrm{d}x \,\mathrm{d}y \le \iint\limits_{\{|x-y| < 1\} \cap (A \times C)} \frac{1}{|x-y|^{1+s'}} \,\mathrm{d}x \,\mathrm{d}y \le P_{s'}(A).$$

Putting all estimates together concludes the proof. ■

4.2 The limiting case $s \to 1^-$

4.9 Theorem. *Let $E \subseteq \mathbb{R}^n$ be a bounded Borel set of finite perimeter. Then,*

$$\lim_{s \to 1^-} (1-s) P_{s,K}(E) = P_{\mathrm{Z}K}(E).$$

Proof . First we observe, that if $x, y \in L$, where $L \in \mathrm{Aff}(n,1)$ is a line, then $\|\frac{x-y}{|x-y|}\|_K$ does not depend on the choice of the points x and y, and calling this quantity $\|u(L)\|_K$ we have

$$\|x-y\|_K^{-n-s} = \|u(L)\|_K^{-n-s}|x-y|^{-n-s}.$$

We use the Blaschke-Petkantschin formula (4.3) and set $A_L := \Phi_L^{-1}(E \cap L)$ for $L \in \mathrm{Aff}(n,1)$ to obtain

$$\begin{aligned}
&\int_E \int_{E^c} \frac{1}{\|x-y\|_K^{n+s}} \,\mathrm{d}x\,\mathrm{d}y = \\
&= \int_{\mathrm{Aff}(n,1)} \|u(L)\|_K^{-n-s} \int_{E\cap L}\int_{E^c \cap L} |x-y|^{-s-1} \,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}\mathcal{H}^1(y)\,\mathrm{d}\lambda_1^n(L) = \\
&= \int_{\mathrm{Aff}(n,1)} \|u(L)\|_K^{-n-s} \int_{A_L}\int_{A_L^c} |\lambda-\mu|^{-s-1}\,\mathrm{d}\lambda\,\mathrm{d}\mu\,\mathrm{d}\lambda_1^n(L).
\end{aligned}$$

By Proposition 4.5, for a fixed direction $u \in \mathbb{S}^{n-1}$, the sets A_L and A_L^c with $L = x + L_u$ have finite perimeter for a.e. $x \in u^\perp$. Before we pass to the limit, we first want to find a suitable majorant with help of inequality (4.6); note, that by the equivalence of norms the factor $\|u(L)\|_K^{-n-s}$ can be estimated by an upper bound, which is independent of s (see also 1.13):

$$\|u(L)\|_K^{-n-s} \le \alpha^{-n-s} \underbrace{|u(L)|^{-n-s}}_{=1} \le \max(1, \alpha^{-n-1}).$$

Since E is bounded, $\mathrm{diam}(A_L)$ is uniformly bounded in $L \in \mathrm{Aff}(n,1)$ (e.g. by $\mathrm{diam}(E)$). Putting all estimates together and writing $A_{u,x} := A_{x+L_u}$, we have

$$\begin{aligned}
(1-s)P_{s,K}(E) &\le 8 \int_{\mathrm{Aff}(n,1)} \|u(L)\|_K^{-n-s}\mathcal{H}^0(\partial^* A_L)\max(1,\mathrm{diam}(A_L))\,\mathrm{d}\lambda_1^n(L) \le \\
&\le C \int_{\mathbb{S}^{n-1}} \int_{u^\perp} P(A_{u,x})\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}\mathcal{H}^{n-1}(u) \le \\
&\overset{(4.4)}{\le} C|\mathbb{S}^{n-1}|P(E),
\end{aligned} \tag{4.12}$$

which is finite. We now pass to the limit,

$$\lim_{s\to 1^-}(1-s)P_{s,K}(E) = \int_{\mathrm{Aff}(n,1)} \|u(L)\|_K^{-n-1}\mathcal{H}^0(\partial^* A_L)\,\mathrm{d}\lambda_1^n(L). \tag{4.13}$$

Since Φ_L is an isometry and hence preserves the topological boundary as well as the Hausdorff measure, we have $\mathcal{H}^0(\partial^* A_L) = \mathcal{H}^0(\partial^* E \cap L)$, as we can modify A_L to be a finite union of intervals leaving the $\mathcal{H}^0$-measure of the reduced boundary unchanged; the image of this union then differs from $\partial^* E \cap L$ only by a set of $\mathcal{H}^0$-measure 0, while preserving the number of boundary points of the union.

We rewrite the right-hand side of (4.13) by applying formula (4.2) for the measure λ_1^n, use Proposition 4.6, and finally transform from spherical into cartesian coordinates to establish the link to the moment body:

$$\begin{aligned}
\lim_{s\to1^-}(1-s)P_{s,K}(E) &= \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{E|u^\perp}\|u\|_K^{-n-1}\mathcal{H}^0(\partial^* E\cap(x+L_u))\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}\mathcal{H}^{n-1}(u) = \\
&= \frac{n+1}{2}\int_{\partial^* E}\int_{\mathbb{S}^{n-1}}|\nu_E(x)\cdot u|\frac{\|u\|_K^{-n-1}}{n+1}\,\mathrm{d}\mathcal{H}^{n-1}(u)\,\mathrm{d}\mathcal{H}^{n-1}(x) = \\
&= \frac{n+1}{2}\int_{\partial^* E}\int_{\mathbb{S}^{n-1}}|\nu_E(x)\cdot u|\int_0^{\|u\|_K^{-1}} r^n\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^{n-1}(u)\,\mathrm{d}\mathcal{H}^{n-1}(x) = \\
&= \frac{n+1}{2}\int_{\partial^* E}\int_K|\nu_E(x)\cdot y|\,\mathrm{d}y\,\mathrm{d}\mathcal{H}^{n-1}(x) = \\
&= \int_{\partial^* E}\|\nu_E(x)\|_{\mathrm{Z}^*K}\,\mathrm{d}\mathcal{H}^{n-1}(x) = P_{\mathrm{Z}K}(E).
\end{aligned}$$

■

4.3 The limiting case $s \to 0^+$

4.10 Theorem. *Let $E\subseteq\mathbb{R}^n$ be a bounded set of finite perimeter. Then,*

$$\lim_{s\to0^+} sP_{s,K}(E) = n|K||E|.$$

Proof . The Blascke-Petkantschin formula (4.3) together with a parametrization of the lines yield

$$P_{s,K}(E) = \int_{\mathrm{Aff}(n,1)}\|u(L)\|_K^{-n-s}\int_{A_L}\int_{A_L^c}|\lambda-\mu|^{-s-1}\,\mathrm{d}\lambda\,\mathrm{d}\mu\,\mathrm{d}\lambda_1^n(L)$$

for the perimeter, where $\|u(L)\|_K = \|\frac{x-y}{|x-y|}\|_K$, and $A_L\subseteq\mathbb{R}$ is the set of all parameters for $E\cap L, L\in\mathrm{Aff}(n,1)$. We estimate the right-hand side by inequality (4.11) and obtain

$$sP_{s,K}(E)\le 4\int_{\mathrm{Aff}(n,1)}\|u(L)\|_K^{-n-s}[\max(1,\mathrm{diam}(A_L))+\mathrm{diam}(A_L)^2+P_{s'}(A_L)]\,\mathrm{d}\lambda_1^n(L).$$

By the same calculations that lead to (4.12) we can see that the expression on the right is finite. This justifies exchanging passing to the limit and integration:

$$\begin{aligned}
\lim_{s\to0^+}P_{s,K}(E) &= 2\int_{\mathrm{Aff}(n,1)}\|u(L)\|_K^{-n}\mathcal{H}^1(A_L)\,\mathrm{d}\lambda_1^n(L) = \\
&= \int_{\mathbb{S}^{n-1}}\|u\|_K^{-n}\int_{E|u^\perp}\mathcal{H}^1(E\cap(x+L_u))\,\mathrm{d}\mathcal{H}^{n-1}(x)\,\mathrm{d}\mathcal{H}^{n-1}(u) = \\
&= n|E|\int_{\mathbb{S}^{n-1}}\frac{\|u\|_K^{-n}}{n}\,\mathrm{d}x = n|E|\int_{\mathbb{S}^{n-1}}\int_0^{\|u\|_K^{-1}} r^{n-1}\,\mathrm{d}r\,\mathrm{d}\mathcal{H}^{n-1}(u) = \\
&= n|K||E|.
\end{aligned}$$

■

4.4 An anisotropic fractional isoperimetric inequality

4.11 Theorem (anisotropic fractional isoperimetric inequality). *Let $E \subseteq \mathbb{R}^n$ be a set of finite perimeter with $|E| < \infty$, and $0 < s < 1$. Then*

$$P_{s,K}(E) \geq \frac{nC_{n,s,1}}{2\beta_K^{n+s}(n-s)} \left(\frac{|\mathbb{S}^{n-1}|}{n} \right)^{\frac{s}{n}} |E|^{\frac{n-s}{n}}, \tag{4.14}$$

where $C_{n,s,1}$ is the constant defined in (3.23)*, and the constant $\beta_K > 0$ is derived by the equivalence of norms*

$$\|x\|_K \leq \beta_K |x|, \; \forall \, x \in \mathbb{R}^n.$$

Proof . This is a simple application of the fractional isoperimetric inequality (3.30), since

$$P_{s,K}(E) = \int_E \int_{E^c} \frac{1}{\|x-y\|_K^{n+s}} \,\mathrm{d}x \,\mathrm{d}y \geq \int_E \int_{E^c} \frac{1}{\beta_K^{n+s}|x-y|^{n+s}} \,\mathrm{d}x \,\mathrm{d}y = \frac{1}{\beta_K^{n+s}} P_s(E).$$

∎

More generally, the sharp form of this inequality, namely

$$P_{s,K}(E) \geq \gamma_{s,K} |E|^{\frac{n-s}{n}}$$

holds, where

$$\gamma_{s,K} := \inf \left\{ P_{s,K}(E)|E|^{-\frac{n-s}{n}} : E \subseteq \mathbb{R}^n \text{ of finite perimeter}, \; 0 < |E| < \infty \right\}.$$

The author does not know, if $\gamma_{s,K}$ corresponds to the constant in (4.14), i.e. the inequality is sharp, and if all equality cases of the sharp version have been classified. However, Ludwig established following result in [19], which deals with convergence of minimizers as $s \to 1^-$:

4.12 Theorem. *Let $0 < s_j < 1, j \in \mathbb{N}$ be a sequence such that $\lim_{j\to\infty} s_j = 1$, and $E_{s_j} \subset \mathbb{R}^n$ be bounded Borel sets such that*

$$P_{s_j,K}(E_{s_j}) = \gamma_{s_j,K} |E_{s_j}|^{\frac{n-s_j}{n}}.$$

Furthermore, let $E_1 \subset \mathbb{R}^n$ be a bounded Borel set such that $E_{s_j} \to E_1$ as $j \to \infty$. Then there exists $c \geq 0$ such that $E_1 = cZK$ up to a set of measure zero.

Appendix

0.13 Lemma. *Let $J_+ : \mathbb{R} \to [0, \infty)$ be a convex function with $J_+(t) = 0$ for $t \leq 0$ and let $k \in L^1(\mathbb{R}^n)$ be symmetric decreasing. For a measurable function u on $\mathbb{R}^n$ we define*

$$E_+[u] := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J_+(u(x) - u(y))k(x - y)\,\mathrm{d}x\,\mathrm{d}y.$$

Furthermore for a measurable and non-negative function $u : \mathbb{R}^n \to [0, \infty)$ and $M \in \mathbb{N}$ we decompose

$$u = u_M + v_M, \text{ where } u_M := \min(u, M).$$

Then,

$$E_+[u] = E_+[u_M] + E_+[v_M] + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F_M(v_M(x), u_M(y))k(x - y)\,\mathrm{d}x\,\mathrm{d}y$$

with

$$F_M(v, u) := J_+(v + M - u) - J_+(v) - J_+(M - u).$$

Proof . We rewrite each of the summands on the right-hand side of the identity we want to prove, such that the domains of integration become more precise. We will use $u_M(x) = u(x)$ and $v_M(x) = 0$ if $u(x) \leq M$, and $u_M(x) = M$ and $v_M(x) > 0$ if $u(x) > M$ in the process:

- $$\begin{aligned} E_+[u_M] = &\int_{\{u(x)\leq M\}} \int_{\{u(y)\leq M\}} J_+(u(x) - u(y))k(x - y)\,\mathrm{d}y\,\mathrm{d}x + \\ &+ \int_{\{u(x)>M\}} \int_{\{u(y)\leq M\}} J_+(M - u(y))k(x - y)\,\mathrm{d}y\,\mathrm{d}x, \end{aligned}$$

- $$\begin{aligned} E_+[v_M] = &\int_{\{u(x)>M\}} \int_{\{u(y)\leq M\}} J_+(v_M(x))k(x - y)\,\mathrm{d}y\,\mathrm{d}x + \\ &+ \int_{\{u(x)>M\}} \int_{\{u(y)>M\}} J_+(u(x) - u(y))k(x - y)\,\mathrm{d}y\,\mathrm{d}x, \end{aligned}$$

- $$\begin{aligned} &\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J_+(v_M(x) + M - u_M(y))k(x - y)\,\mathrm{d}y\,\mathrm{d}x = \\ &= \int_{\{u(x)\leq M\}} \int_{\{u(y)\leq M\}} J_+(M - u(y))k(x - y)\,\mathrm{d}y\,\mathrm{d}x + \\ &+ \int_{\{u(x)>M\}} \int_{\{u(y)\leq M\}} J_+(u(x) - u(y))k(x - y)\,\mathrm{d}y\,\mathrm{d}x + \\ &+ \int_{\{u(x)>M\}} \int_{\{u(y)>M\}} J_+(v_M(x)))k(x - y)\,\mathrm{d}y\,\mathrm{d}x, \end{aligned}$$

- $\displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_+(v_M(x))k(x-y)\,\mathrm{d}y\,\mathrm{d}x =$
$$= \int_{\{u(x)>M\}}\int_{\mathbb{R}^n} J_+(v_M(x))k(x-y)\,\mathrm{d}y\,\mathrm{d}x,$$
- $\displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_+(M-u_M(y))k(x-y)\,\mathrm{d}y\,\mathrm{d}x =$
$$= \int_{\mathbb{R}^n}\int_{\{u(y)\le M\}} J_+(M-u(y)))k(x-y)\,\mathrm{d}y\,\mathrm{d}x.$$

All integrals where $u(x)-u(y)$ is *not* the argument of J_+ cancel out, whereas the sum of all integrals involving $u(x)-u(y)$ equals

$$E_+[u] - \underbrace{\int_{\{u(x)\le M\}}\int_{\{u(y)>M\}} J_+(u(x)-u(y))k(x-y)\,\mathrm{d}y\,\mathrm{d}x}_{=0}\,.$$

■

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Index