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Regularization of local volatility models

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Abstract

This thesis studies a regularization technique of Dupire's formula for Levy jump diffusion models. In particular, a procedure is introduced to reobtain the option prices with a local volatility model.
These results are applied to Kou's model, and for that purpose representations of the option price surface and some derivatives are stated.
The derived results are then implemented numerically, and the functionality of the introduced procedure is proved using the programming language Matlab. Furthermore issues that arise during this implementation are addressed, such as errors stemming from numerical integration.

This work is largely based on the paper *How to make Dupire's formula work with jumps* by Friz, Gerhold and Yor.

Keywords: Fourier transform, moment generating function, option pricing, Kou model, Dupire formula

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Chapter 1

Introduction

Financial mathematics and stochastic analysis are two relatively new fields of mathematics, but have lately become an important field of research. This is due to the rising interest in derivatives and stock prices which require financial modelling.
Louis Bachelier was the first to discuss stock price modelling and option pricing in 1900 in his thesis 'Theory of Speculation', [Bac00]. In his paper he introduced a model to describe the discounted stock price process $(\tilde{S}_t)_{t\in[0,T]}$ as a Brownian motion with volatility σ and initial value S_0 by

$$\tilde{S}_t = S_0 + S_0\sigma W_t,$$

where $(W_t)_{t\in[0,T]}$ is a standard Brownian motion. In 1965, Samuelsson proposed a financial model using geometric Brownian motion which became famous when Black and Scholes used this model in their seminal paper, [BS73]. In the last few decades, the interest in financial mathematics and stochastic modelling increased, and with this some weaknesses of stochastic models based on Brownian motions, and therefore especially of the Black-Scholes model were revealed. Especially the absence of heavy tails in the log-returns and the lack of volatility smiles and jumps were reasons to introduce new models that satisfy these requirements.
One generalization that can makes adaption to all weaknesses mentioned above possible, is to use the large class of Levy processes, and exponential Levy models. This class includes the Black-Scholes model as a special case, when the Levy process is a (non standard) Brownian motion. Levy models and especially exponential Levy models are commonly used due to the fact that these processes form semimartingales and that their distributions possess special properties. They can also include jumps, making stock price modelling more accurate but harder to handle.
Another approach is, to model the assets on behalf of a so called *implied*

volatility model. There, the underlying is defined via a stochastic differential equation (SDE), where the volatility is a function of time and the strike price. This function is computed such, that the input in a call price model returns the corresponding market price. This can overcome the problems regarding volatility smiles, but obviously does not include jumps.
A similar approach is, to use a *local volatility model* for the underlying. Within this model, the asset is again defined via an SDE, where the volatility function depends on time and the momentary asset price. This volatility function is defined via Dupire's formula, using a given (market) call price surface.

Let $(S_t)_{t\in[0,T]}$ be a stochastic process modelling an asset, and $(B_t)_{t\in[0,T]}$ a riskless security with

$$B_t := \mathrm{e}^{rt}, \quad 0 \leq t \leq T,$$

where r denotes the riskless interest rate. Then a European call option with underlying S, time to maturity $T > 0$ and strike $K > 0$ is defined as

$$C(k,T) = \mathbb{E}[B_T^{-1}(S_T - K)^+],$$

with $(.)^+ = \max(., 0)$ defining the positive part.
Throughout this thesis, w.l.o.g the initial value is set to 1,$S_0 = 1$. To get call prices for initial values $S_0 > 0$, one only has to multiply this value with the call price, with modified strike price $\widetilde{K} := \frac{K}{S_0}$.

$$\mathbb{E}[B_T^{-1}(S_0 S_T - K)^+] = S_0 \mathbb{E}[B_T^{-1}(S_T - \widetilde{K})^+]$$

In this thesis the mentioned asset S is modelled by Kou's model, [Kou02]. With this definition a call price surface is generated using the approach of [Lee04]. The mathematical requirements to do this are presented in the following chapter. Then, in Chapter 3, Levy processes and some properties are presented and proved. Furthermore some exponential Levy models are introduced, focusing on Kou's model, which is described in detail in Section 3.2.1. After the mathematical foundations are set, the theory behind the paper of Gerhold et al.,[FGY13], is presented. In Chapter 5, all these results and theory is combined and applied numerically to Kou's model, highlighting some numerical problems and performing some backtests. This thesis is concluded with a short resume.

Chapter 2

Preliminary definitions and results

2.1 Moment generating functions and Characteristic functions

Definition 2.1. For a given random variable X the moment generating function (*mgf*) is defined by

$$M_X(t) := \mathbb{E}[\mathrm{e}^{tX}],$$

wherever it exists.

Theorem 2.2. *Let X, Y be two random variables, and M_X, M_Y their corresponding moment generating functions. Then the following properties are satisfied:*

i) X and Y follow the same distribution, if and only if the two moment generating functions are equal on their domain.

$$X \overset{d}{=} Y \Leftrightarrow M_X = M_Y$$

ii) If furthermore the moment generating function of X exists in an open interval around zero, then all moments of X can be calculated as derivative of M_X.

$$\mathbb{E}[X^n] = \frac{\mathrm{d}^n M_X}{\mathrm{d}t^n}(0),\ n \in \mathbb{N}$$

Sketch of Proof.

i) "⇒": This direction is trivial as equality in distribution implies that $f(X) \stackrel{d}{=} f(Y)$, for all functions f: $\mathbb{R} \to \mathbb{R}$. By choosing $f_t : x \mapsto \mathrm{e}^{tx}$ and taking the expectation follows, that $\mathbb{E}[\mathrm{e}^{tX}] = \mathbb{E}[\mathrm{e}^{tY}]$, $\forall t$.
"⇐": This proof is done for characteristic functions in [JP03, Theorem 14.1]. Using Remark 2.7 yields the result for moment generating functions.

ii) If the moment generating function exists in an open interval around 0, then the derivative is well defined for $t = 0$. The statement follows by interchanging the differentiation and the expectation and evaluating at $t = 0$. □

Remark 2.3. Theorem 2.2 i) implies, that the moment generating function of a random variable uniquely determines the underlying probability distribution and vice versa.

Another important property of the mgf relates to the sum of independent random variables.

Theorem 2.4. *Let $(X_j)_{j\in I}$ be a set of independent random variables with well defined mgfs, and I an arbitrary index set. Then the mgf of the sum of these random variables $S := \sum_{j\in I} X_j$ is well defined and given by*

$$M_S(t) = \mathbb{E}[e^{\sum_{j\in I} tX_j}] = \prod_{j\in I} M_{X_j}(t).$$

Proof. The statement follows by using the power law of the exponential function and the independence of the random variables.

$$\mathbb{E}\left[\mathrm{e}^{\sum_{j\in I} tX_j}\right] = \mathbb{E}\Big[\prod_{j\in I} \mathrm{e}^{tX_j}\Big] = \prod_{j\in I} \mathbb{E}[\mathrm{e}^{tX_j}] = \prod_{j\in I} M_{X_j}(t)$$

□

As mentioned above, the moment generating function does not always exist (cf. Cauchy distribution). Therefore the characteristic function is often used instead.

Definition 2.5. For a given, $\mathbb{R}^n$ - valued random variable X the characteristic function is defined by

$$\varphi_X : \begin{cases} \mathbb{R}^n \to \mathbb{C} \\ \varphi_X(t) = \mathbb{E}[\mathrm{e}^{it\cdot X}], \end{cases}$$

where i is the imaginary unit.

Lemma 2.6. *The characteristic function exists for each random variable with an arbitrary power law, in fact it is even bounded by* 1*,*

$$\varphi_X(t) \leq 1.$$

Proof. To show the existence of the characteristic function it is sufficient to show the boundedness for all $t \in \mathbb{R}^n$.

$$|\mathbb{E}\left[\mathrm{e}^{it\cdot X}\right]| \leq \mathbb{E}\left[|\mathrm{e}^{it\cdot X}|\right] = 1$$

□

Remark 2.7. If the moment generating function of a random variable exists, then the domain of its characteristic function can be extended to the complex numbers and $\varphi_X(-iz) = M_X(z)$ for all $z \in \mathbb{C}$.

Due to this property characteristic functions and moment generating functions share the most important properties, so Theorem 2.4 can be adapted to characteristic functions.

Lemma 2.8. *The characteristic function of a random variable X uniquely determines the distribution of X.*

Lemma 2.9. *Let X be a real value random variable, and φ_X its characteristic function.*
If the k-th moment of X exists, then it can be calculated by:

$$\mathbb{E}\left[X^k\right] = (-i)^k \frac{\mathrm{d}^k \varphi_X}{\mathrm{d}t^k}(0).$$

Proof. This follows by applying Theorem 2.2 ii) and Remark 2.7 or by following the proof of Theorem 2.2 ii). □

In mathematical finance stock prices or other underlyings are often modelled as exponential stochastic variables. Usually these processes form martingales, if discounted. Therefore it is common to use a generalized and discounted characteristic function, to be able to use these properties.

Definition 2.10. Let X denote a $\mathbb{R}^n$ - valued random variable, and A_X the interior of the set $\{\nu \in \mathbb{R}^n : \ \mathbb{E}\left[\mathrm{e}^{\nu\cdot X}\right] < \infty\}$.
For every complex vector $\theta \in \Lambda_X := \{\xi \in \mathbb{C}^n : \ -\mathrm{Im}(\xi) \in A_X\}$ the generalized discounted characteristic function of X can be defined by:

$$\varphi_X : \Lambda_X \to \mathbb{C} \text{ with } \varphi_X(\theta) := \mathbb{E}\left[\mathrm{e}^{-\int_0^T r_t \mathrm{d}t} \mathrm{e}^{i\theta\cdot X}\right],$$

where $\mathrm{e}^{-\int_0^T r_t \mathrm{d}t}$ is the corresponding discounting factor.

Remark 2.11. The existence of this generalized version is trivial, as the corresponding domains are chosen exactly such, that all parts are finite. To be exact, the discounted factor does not play a role for existence, and the characteristic function itself can be split into two exponentials. One with a mere imaginary exponent, which is therefore bounded, and one with a real exponent, which is restricted such that it is finite.

In the following, only this generalized version will be used and it will be referred to as the characteristic function.
In case that a probability distribution admits a probability density function, its characteristic function can be interpreted as the Fourier transform of this very density. For this reason a brief introduction to the theory of Fourier transformation given.

2.2 Fourier transformation & the damped option price

The Fourier transformation is an important tool when pricing financial instruments, especially options.
The general idea behind Fourier pricing is, that if the characteristic function of an underlying is known in closed form, the characteristic function/Fourier transform of the option and finally the option price itself can be computed. This section will introduce a way to calculate the original function out of its Fourier transformed, known as the Fourier inversion theorem. Furthermore formulas for calculating the characteristic function of the (damped) option price and the option price itself will be presented in terms of the characteristic function of the underlying.

2.2.1 Fourier transformation and properties

At first the Fourier transformation for general $L^1(\mathbb{R})$ functions is introduced and some properties and results are stated.

Definition 2.12. For an integrable function $f \in L^1(\mathbb{R})$ the Fourier transform $\hat{f}: \mathbb{R} \to \mathbb{C}$ is defined as follows.

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(y) \mathrm{e}^{ixy} \mathrm{d}y$$

The Fourier transform $\hat{f}$ of any integrable function f is well defined, due to the fact that

$$|\hat{f}(x)| = |\int_{-\infty}^{\infty} f(y)\mathrm{e}^{ixy}\mathrm{d}y| \leq \int_{-\infty}^{\infty} |f(y)\mathrm{e}^{ixy}|\mathrm{d}y = \int_{-\infty}^{\infty} |f(y)|\mathrm{d}y < \infty \; \forall y \in \mathbb{R}\,.$$

Proposition 2.13. *Let $f, g \in L^1(\mathbb{R})$ be integrable functions, and $\hat{f}, \hat{g}$ their corresponding Fourier transforms. Then the following properties are satisfied.*

i) The Fourier transform is linear, i.e. $\forall a, b \in \mathbb{C}: \; \widehat{(af + bg)} = a\hat{f} + b\hat{g}$.

ii) $\hat{f}$ *is uniformly continuous and* $\|\hat{f}\|_\infty \leq \|f\|_{L^1}$

iii) $\widehat{f * g} = \hat{f} \cdot \hat{g}$

iv) For a real number $t \in \mathbb{R}$ *define* $h(x) := f(x+t)$*, then* $\hat{h}(x) = e^{-ixt}\hat{f}(x)$.

Proof. See [Rud87, Chapter 9]. Note that the definition of the characteristic function there, $\hat{f}^*$ varies from Definition 2.12. To be precise, it is defined as

$$\hat{f}^*(x) := \frac{1}{\sqrt{2\pi}}\hat{f}(-x).$$

□

If the Fourier transform satisfies certain conditions, it is possible to compute the original function out of it with the Fourier inversion theorem. This theorem is fundamental for the rest of the thesis and option pricing in general.

Theorem 2.14. *For f and $\hat{f}$ in $L^1(\mathbb{R})$, the following equation is valid.*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixy}\hat{f}(y)\mathrm{d}y$$

Proof. See [Rud87, Chapter 9]. □

Note, that if a random variable admits a probability density, its characteristic function can be interpreted as the Fourier transform of this density. With this theorem the characteristic function uniquely determines the probability density function, and therefore the probability distribution.

2.2.2 Fourier transform of the damped option price

In the previous section a condition for a Fourier transform to exist, was that the function is integrable. This causes problems when considering pricing functions such as the call price function $(x,k) := (x-k)^+$. Therefore the idea of the damped option price is introduced. This section is mainly taken from [Lee04].

The payoff of a european call option with strike K, underlying S and maturity $T > 0$ is given by $C_T = (S_T - K)^+$. In case of a positive strike and an almost surely positive underlying, it can be expressed via the log-underlying $X_T := \log(S_T)$ and the log-strike $k := \log(K)$, $C_T = \left(\mathrm{e}^{X_T} - \mathrm{e}^k\right)^+$.

Let now $G : \mathbb{R}^2 \to \mathbb{R}$ denote the pricing function defined by $G(x,k) = \left(\mathrm{e}^x - \mathrm{e}^k\right)^+$. Further let B_t be the time t value of a discount Bond maturing at T, $B_t = \mathbb{E}\left[\mathrm{e}^{-\int_t^T r_t dt}\right]$, where $(r_t)_{t\in[0,T]}$ is the - possibly stochastic - interest rate.
Now the time zero price of a european call option can be written by

$$C(k,t) = B_0\mathbb{E}[G(X_t,k)] \ .$$

As mentioned before, the call price is not integrable, and therefore the Fourier transform cannot be formed. To do this, the *damped option price function* $C_\alpha(k,t)$ is introduced:

$$C_\alpha(k,t) := \mathrm{e}^{\alpha k}C(k,t) \ .$$

Theorem 2.15. *If there exists a $p > 0$ with $\mathbb{E}[B_0 e^{(p+1)X}] < \infty$ then for $0 < \alpha \le p$, the damped option price function is integrable, $C_\alpha(.,t) \in L^1(\mathbb{R})$, $\forall t \ge 0$. Therefore the Fourier transform $\hat{c}_\alpha(k)$ exists, and is given by*

$$\hat{c}_\alpha(k,t) = \frac{\varphi_X^t(k-(\alpha+1)i)}{(\alpha+ik)(\alpha+1+ik)},$$

where φ_X^t is the (generalized, damped) characteristic function of X_t.

Proof.

$$\int_{\mathbb{R}} |C_\alpha(u,t)|\mathrm{d}u = \int_{\mathbb{R}} \mathrm{e}^{\alpha u}B_0\mathbb{E}[(\mathrm{e}^{X_t} - \mathrm{e}^u)^+]\mathrm{d}u = \mathbb{E}\left[B_0\int_{\mathbb{R}} \mathrm{e}^{\alpha u}(\mathrm{e}^{X_t} - \mathrm{e}^u)^+\mathrm{d}u\right] =$$
$$= \mathbb{E}\left[B_0\int_{-\infty}^{X_t} \mathrm{e}^{\alpha u}(\mathrm{e}^{X_t} - \mathrm{e}^u)^+\mathrm{d}u\right] \le \mathbb{E}\left[B_0\int_{-\infty}^{X_t} \mathrm{e}^{\alpha u}\mathrm{e}^{X_t}\mathrm{d}u\right] \le$$
$$\le \mathbb{E}\left[B_0\mathrm{e}^{(\alpha+1)X_t}\right] \le \mathbb{E}\left[B_0\mathrm{e}^{(p+1)X_t}\right] < \infty$$

The interchange of the expectation and the integral is provided by Fubini's theorem using the non-negativity of the integrand. Therefore the Fourier transform exists.

Let F_t denote the distribution of $X_t,\ X_t \sim F_t$. Then again, by using Fubini's theorem for non negative integrands, the Fourier transform can be calculated.

$$\begin{aligned}
\hat{c}_\alpha(k,t) &= \int_{\mathbb{R}} \mathrm{e}^{iuk}\mathrm{e}^{\alpha u} B_0 \mathbb{E}[(\mathrm{e}^{X_t} - \mathrm{e}^u)^+]\mathrm{d}u = \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{(\alpha+ik)u} B_0 \int_{\mathbb{R}} (\mathrm{e}^x - \mathrm{e}^u)^+ \mathrm{d}F_t(x)\mathrm{d}u = \\
&= \int_{\mathbb{R}} \int_{-\infty}^{x} \mathrm{e}^{(\alpha+ik)u} B_0 (\mathrm{e}^x - \mathrm{e}^u)\mathrm{d}u\ \mathrm{d}F_t(x) = \\
&= \int_{\mathbb{R}} B_0 \left(\frac{\mathrm{e}^{(\alpha+1+ik)x}}{\alpha+ik} - \frac{\mathrm{e}^{(\alpha+1+ik)x}}{\alpha+1+ik} \right) dF_t(x) \\
&= \frac{\mathbb{E}\left[B_0 \mathrm{e}^{(\alpha+1+ik)X_t}\right]}{(\alpha+ik)(\alpha+1+ik)} = \frac{\varphi_X^t(k-(\alpha+1)i)}{(\alpha+ik)(\alpha+1+ik)}
\end{aligned}$$

□

This theorem shows the need of the generalized characteristic function introduced in Section 2.1.
The main purpose however, is calculating the call price function rather than its Fourier transform. To be able to apply the Fourier inversion theorem it is necessary that $\hat{c}_\alpha(k,t)$ is again integrable.

Theorem 2.16. *If the conditions from Theorem 2.15 are satisfied, then* $\hat{c}_\alpha(.,t) \in L^1(\mathbb{R})$.

Proof. The following inequation follows by using properties of the absolute value of a complex number. The functions $\mathbb{1}_{U_\epsilon(0)}$ and $\mathbb{1}_{U_\epsilon^C(0)}$ describe the indicator function on an epsilon neighbourhood around zero and its complement, respectively.

$$\begin{aligned}
|\hat{c}_\alpha(k,t)| &\leq \frac{|\mathbb{E}[\mathrm{e}^{(\alpha+1+ik)X_t}]|}{|(\alpha+ik)(\alpha+1+ik)|} \leq \frac{\mathbb{E}[|\mathrm{e}^{(\alpha+1)X_t}|]}{|\alpha+ik| \cdot \underbrace{|\alpha+1+ik|}_{\geq|\alpha+ik|}} \leq \frac{\mathbb{E}[|\mathrm{e}^{(\alpha+1)X_t}|]}{|\alpha+ik|^2} \leq \\
&\leq \frac{\mathbb{E}[|\mathrm{e}^{(\alpha+1)X_t}|]}{\alpha^2+k^2} \leq \underbrace{\mathbb{E}[|\mathrm{e}^{(\alpha+1)X_t}|]}_{<\infty} \left(\underbrace{\frac{1}{\alpha^2}\mathbb{1}_{U_\epsilon(0)}}_{\in L^1(\mathbb{R})} + \underbrace{\frac{1}{k^2}\mathbb{1}_{U_\epsilon^C(0)}}_{\in L^1(\mathbb{R})} \right)
\end{aligned}$$

Therefore, as $\hat{c}_\alpha(.,t)$ is bounded by a $L^1(\mathbb{R})$ function, it is integrable itself. □

Now, the Fourier inversion theorem can be applied to compute the original call price function out of its Fourier transform.

Theorem 2.17. *If there exists a $p > 0$ with $\mathbb{E}[B_0 e^{(p+1)X_t}] < \infty$ then for $0 < \alpha \leq p$ the call price function is given by*

$$C(k,t) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \frac{\varphi_X^t(u-(\alpha+1)i)}{(\alpha+iu)(\alpha+1+iu)} \mathrm{d}u = \tag{2.1}$$

$$= \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} \mathrm{Re}\left(e^{-iuk} \frac{\varphi_X^t(u-(\alpha+1)i)}{(\alpha+iu)(\alpha+1+iu)} \right) \mathrm{d}u \tag{2.2}$$

Proof. The conditions of Theorem 2.16 are satisfied, therefore Theorem 2.14 can be applied. Multiplying with $\mathrm{e}^{-\alpha k}$ yields the first equation, the second equation then follows as the real part of the integrand is an even function, and its imaginary part is odd. □

Remark 2.18. In the following chapters it will be necessary to compute the derivatives of the call price with respect to the strike K. To make these calculations easier, a representation of the call price as function of the strike $K := \mathrm{e}^k$ is stated.

$$C(K,t) = \frac{K^{-\alpha}}{2\pi} \int_{-\infty}^{\infty} K^{-iu} \frac{\varphi_X^t(u-(\alpha+1)i)}{(\alpha+iu)(\alpha+1+iu)} \mathrm{d}u = \tag{2.3}$$

$$= \frac{K^{-\alpha}}{\pi} \int_0^{\infty} \mathrm{Re}\left(K^{-iu} \frac{\varphi_X^t(u-(\alpha+1)i)}{(\alpha+iu)(\alpha+1+iu)} \right) \mathrm{d}u \tag{2.4}$$

Chapter 3

Levy Processes and jump diffusion models

In mathematical finance one of the most important things is the underlying model one uses for stock prices. Nevertheless it is hard to create a model that is both, simple enough to get formulas for matters of calculation and complex enough to describe the market well. Due to that reason, models, such as the Black Scholes model or similar that assure easy calculation of various option prices etc., are not useful in practice as they are rather simple and have very strict assumptions.
One thing that is often criticized is that these models do not provide jumps - which do appear in real markets. Hence this thesis will focus on Levy jump diffusion models and especially on the Kou model defined in [Kou02]. For this purpose Levy processes and some results are presented before defining the jump diffusion models.

3.1 Levy Processes

This section will provide all necessary definitions and results for further calculations with Levy processes. The structure mainly follows [Pap08].
For all further calculations let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtrated probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, where the time horizon $T > 0$ can be infinite.[1]

Definition 3.1. A real valued stochastic process $X := (X_t)_{t\in[0,T]}$ is called a *Levy process* if it satisfies the following properties:

(L0) $X_0 = 0$ a.s.

[1] If $T = +\infty$, then each interval $[0, T]$ has to be replaced by $[0, T] \cap [0, \infty) = [0, T)$.

(L1) Independence of increments: For any $0 \leq t_1 < t_2 < \ldots < t_n \leq T$, $X_{t_i} - X_{t_{i-1}}$ and $X_{t_j} - X_{t_{j-1}}$, $i, j \in \{2, \ldots, n\}$, $i \neq j$ are independent.

(L2) Stationarity of increments: $X_t - X_s \stackrel{d}{=} X_{t-s}$, for any $0 \leq s < t \leq T$.

(L3) Stochastic continuity: $\lim_{s \to t} \mathbb{P}(|X_t - X_s| > \epsilon) = 0$, $\forall \epsilon > 0$ and $t \geq 0$.

Remark 3.2. Note that in some papers a Levy process is needed to be cadlag, however this condition is omitted as for every Levy process X there exists an almost surely (a.s.) cadlag version of it. For further results this version will be taken.
Further note that the condition (L3) does not imply that the paths are continuous, but rather says that there are no jumps at a fixed time t a.s. .

Example 3.3. To illustrate that the possibility of jumps is conform with Definition 3.1, consider $(N_t)_{t \geq 0}$ a homogeneous *Poisson process*. Then per definition $N_0 = 0$ holds, and the process has independent and stationary increments. So the only thing that needs to be shown, is that (L3) is satisfied. As the increments of a Poisson process are Poisson distributed, the following equation holds.

$$\mathbb{P}[N_t - N_s = k] = \mathrm{e}^{-\lambda(t-s)} \frac{(\lambda(t-s)^k)}{k!}, \quad \forall k \in \mathbb{N}_0,\ t > s \geq 0$$

With this the validity of (L3) follows.

$$\lim_{s \to t} \mathbb{P}[N_t - N_s > 0] = \lim_{s \to t} 1 - \mathrm{e}^{-\lambda(t-s)} = 0 \quad a.s.$$

One important property of Levy processes specifies the characteristic function(s) of the process.

Theorem 3.4. *Let $(X_t)_{t \geq 0}$ be a Levy process, and φ_{X_t}, $t \geq 0$ denote the characteristic function of X_t, $\varphi_{X_t}(\theta) = \mathbb{E}[e^{i\theta X_t}]$.*
Then φ_{X_t}, $t \geq 0$ can be expressed by means of φ_{X_1} as follows.

$$\varphi_{X_t}(\theta) = \varphi_{X_1}(\theta)^t = \mathbb{E}[e^{i\theta X_1}]^t$$

Proof. Let $n \in \mathbb{N}$ and $t \geq 0$ be fixed but arbitrary. Then X_t can be written as a telescopic sum of increments of the process.

$$X_t = X_{\frac{t}{n}} + \left(X_{\frac{2t}{n}} - X_{\frac{t}{n}}\right) + \ldots + \left(X_t - X_{\frac{t(n-1)}{n}}\right)$$

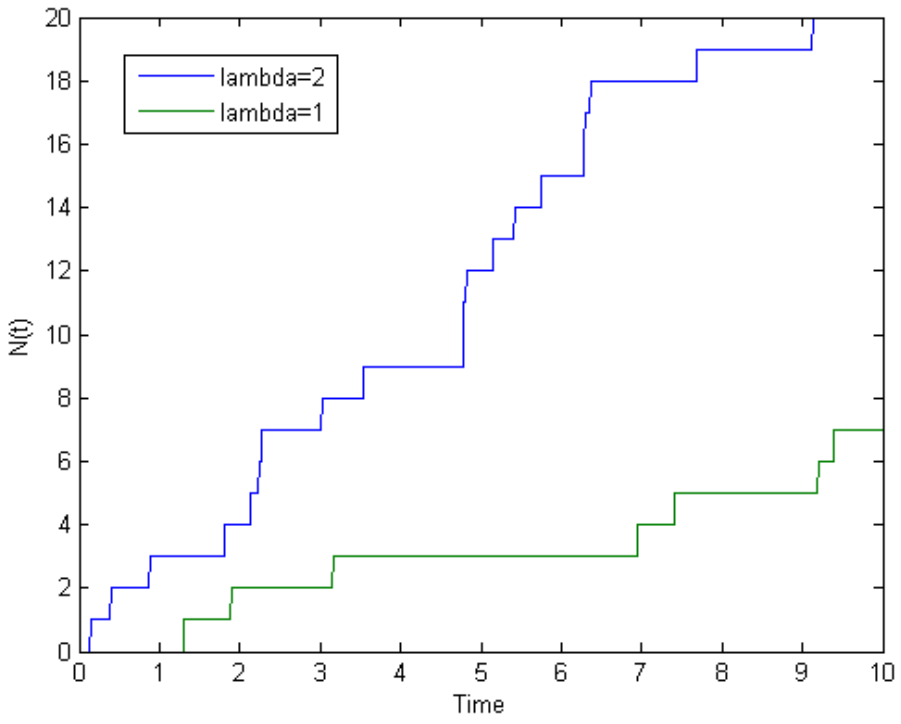

Figure 3.1: Two sample paths of homogeneous Poisson processes.

Due to the stationarity of increments (L2), each of the terms n in the brackets is equal to $X_{\frac{t}{n}}$ in distribution.
Using first the independence and the stationarity of increments of a Levy process yields the following.

$$\mathbb{E}\left[\mathrm{e}^{i\theta X_m}\right] = \mathbb{E}[\mathrm{e}^{i\theta X_1}] \cdot \mathbb{E}[\mathrm{e}^{i\theta(X_2 - X_1)}] \cdots \mathbb{E}[\mathrm{e}^{i\theta(X_m - X_{m-1})}] = \mathbb{E}\left[\mathrm{e}^{i\theta X_1}\right]^m$$

$$\mathbb{E}\left[\mathrm{e}^{i\theta X_1}\right]^{\frac{1}{n}} = \mathbb{E}\left[\mathrm{e}^{i\theta X_{\frac{1}{n}}}\right]^{\frac{1}{n}} \cdot \mathbb{E}\left[\mathrm{e}^{i\theta(X_{\frac{2}{n}} - X_{\frac{1}{n}})}\right]^{\frac{1}{n}} \cdots \mathbb{E}\left[\mathrm{e}^{i\theta(X_1 - X_{\frac{n-1}{n}})}\right]^{\frac{1}{n}} = \mathbb{E}\left[\mathrm{e}^{i\theta X_{\frac{1}{n}}}\right]$$

Combining these two results, proves the lemma for any positive rational number.
To get the result for any positive irrational number t, take a series of rational numbers $(q_n)_{n\in\mathbb{N}}$ decreasing to t. Due to dominated convergence and the a.s. right continuity of X_u in u, the statement of the lemma follows.

$$\mathbb{E}[\mathrm{e}^{i\theta X_t}] = \lim_{n\to\infty} \mathbb{E}[\mathrm{e}^{i\theta X_{q_n}}] = \lim_{n\to\infty} \mathbb{E}[\mathrm{e}^{i\theta X_1}]^{q_n} = \mathbb{E}[\mathrm{e}^{i\theta X_1}]^t$$

□

Definition 3.5. Let $X := (X_t)_{t\geq 0}$ be a Levy process. Then the *characteristic exponent* of this process is defined by

$$\psi(\theta) := -\log\left(\mathbb{E}\left[\mathrm{e}^{i\theta X_1}\right]\right). \tag{3.1}$$

With this definition, Theorem 3.4 can be written differently.

Lemma 3.6. *Let $(X_t)_{t\geq 0}$ be a Levy process, and ψ denote the characteristic exponent of X defined in* (3.1).
Then the characteristic function $\varphi_{X_t},\ t \geq 0$ can be written as

$$\varphi_{X_t}(\theta) = e^{-t\psi(\theta)}. \tag{3.2}$$

The lemma above shows that the characteristic function a Levy process is of a special type. Due to Theorem 2.2 this also restricts the class of possible distributions of Levy processes. Precisely, the family of distributions of a Levy process fulfills the following property.

Definition 3.7. A probability distribution F of a random variable is called *infinitely divisible*, if, for any $n \in \mathbb{N}$, there exists a probability distribution F_n such, that F is given as the n-th convolution of F_n

$$F_X = F^{*(n)} = F_n * \ldots * F_n\ .$$

A random variable X is called *infinitely divisible*, if for any $n \in \mathbb{N}$ there exits a set of iid random variables $(X_j^{(n)})_{j=1,\dots,n}$, $X_j^{(n)} \sim F_n$, $\forall j = 1, \dots, n$, such that

$$X \stackrel{d}{=} X_1^{(n)} + \ldots + X_n^{(n)} \ .$$

Lemma 3.8. *Let X be a random variable and F_X denote its distribution, $X \sim F_X$. Then X is infinitely divisible if and only if F_X is.*

Proof. Let $(X_k)_{k\in\mathbb{N}}$ be a family of iid random variables with $X_k \sim F$ for all k. Then the distribution of the sum $\sum_{k=1}^n X_k$ is given as the n-th convolution of F, for any $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} X_k \sim F^{*n}.$$

With this relation the statement follows. □

Before the relation of Levy processes and infinitely distributions can be shown, another result is required.

Lemma 3.9. *For two infinitely divisible distributions F and G, their convolution $F * G$ again is infinitely divisible.*

Proof. Suppose two random variables $X \sim F$ and $Y \sim G$, $n \in \mathbb{N}$ and $Z := X + Y$.

$$Z = \underbrace{X}_{\stackrel{d}{=}\sum_{j=1}^n X_j^n} + \underbrace{Y}_{\stackrel{d}{=}\sum_{j=1}^n Y_j^n} \stackrel{d}{=} \sum_{j=1}^n \underbrace{X_j^n + Y_j^n}_{:=Z_j^{(n)}}$$

As the set $(Z_j^{(n)})_{j=1,\dots,n}$ is iid, the infinitely divisibility follows. □

Now it can be shown that the class of infinitely divisible distributions covers all possible distributions of Levy processes. With Lemma 3.8, this implies that each Levy process itself is infinitely divisible.

Theorem 3.10. *Let $(X_t)_{t\in[0,T]}$ denote a Levy process, then for every $t \in [0, T]$ there exists an infinitely divisible distribution F_t with $X_t \sim F_t$. Also, for every infinitely divisible distribution F, there exists a Levy process $(X_t)_{t\in[0,T]}$ with $X_1 \sim F$.*

Proof. W.l.o.g. let $n \in \mathbb{N}$ and $t > 0$ be arbitrary but fixed, and define random variables $Y_1, \dots, Y_n$ by

$$Y_j := X_{\frac{jt}{n}} - X_{\frac{(j-1)t}{n}}, \quad j = 1, \dots, n.$$

Then with (L1) and (L2) follows, that $(Y_j)_{j=1,\dots,n}$ is a set of iid random variables with $Y_1 \stackrel{d}{=} \dots \stackrel{d}{=} Y_n \stackrel{d}{=} X_{\frac{t}{n}}$.
By using the incremental notation of X_t it can be written as $X_t \stackrel{d}{=} \sum_{j=1}^{n} Y_j$. As the choice of n was arbitrary, the distribution of X_t is infinitely divisible.

Otherwise, for any infinitely divisible probability distribution F define $X_1 \sim F$. With Theorem 3.4 the characteristic function of every X_t, $t \geq 0$ and therefore their distribution is known. Then the properties of a Levy process have to be shown.
The exact prove is done in [Sat99, Theorem 7.10(ii)]. □

Example 3.11. Important examples of infinitely divisible distributions, and their corresponding Levy processes are presented in the following.

- The degenerate distribution δ_x:
  Take a random variable $Z \sim \delta_z$, this means that $Z = z$ a.s., for $z \in \mathbb{R}$. Then for any $n \in \mathbb{N}$, Z can be decomposed to a sum of iid random variables $Z_1^n, \dots, Z_n^n \sim \delta_{z/n}$:

  $$Z \stackrel{d}{=} Z_1^n + \dots + Z_1^n.$$

  As shown in Theorem 3.10 a Levy process X exists with $X_1 \sim \delta_x$. The characteristic function of X_1 is given by $\varphi_1^X(\theta) = \mathbb{E}[\mathrm{e}^{i\theta X_1}] = \mathrm{e}^{i\theta x}$, and with Theorem 3.4 it can be calculated for X_t, $t \geq 0$: $\varphi_{X_t}(\theta) = \mathrm{e}^{i\theta xt}$. As the characteristic function defines the distribution, the whole process $(X_t)_{t\geq 0}$ is known with $X_t = xt$ a.s.
  It is easy to verify that (L1-3) are valid, therefore this process really is a Levy process.

- The Poisson distribution $Poi(\lambda)$:
  Let $(Z_i)_{i=1,\dots,n}$ be a set of iid Poisson distributed random variables, $Z_i \sim Poi(\lambda_i)$, $i = 1, \dots, n$. Then the sum of these random variables again is Poisson distributed.[2]

  $$Z := \sum_{i=1}^{n} Z_i \text{ and } \lambda = \sum_{i=1}^{n} \lambda_i, \text{ then } Z \sim Poi(\lambda)$$

  With this, any Poisson distributed random variable can be written as sum of n iid Poisson random variables, and the infinitely divisibility follows.

[2] This can be shown on behalf of characteristic functions.

To get the corresponding Levy process with $X_1 \sim Poi(\lambda)$, Theorem 3.4 is again useful.
The characteristic function of a Poisson distributed random variable with parameter λ is given by

$$\varphi_{Poi}(\theta) = \mathrm{e}^{\lambda(i\theta-1)} = \varphi_{X_1}(\theta).$$

With this the characteristic function of $X_t,\ t \geq 0$ is given by $\varphi_{X_t}(\theta) = \left(\mathrm{e}^{\lambda(i\theta-1)}\right)^t = \mathrm{e}^{\lambda t(i\theta-1)}$. Therefore X_t again is Poisson distributed with parameter λt, and the resulting Levy process $(X_t)_{t\geq 0}$ is a *Poisson process*.

- The Compound Poisson distribution $CPoi(\lambda, F)$:
For $N \sim Poi(\lambda)$ and a series of iid random variables $(Z_n)_{n\in\mathbb{N}},\ Z_n \sim F$ with an arbitrary probability distribution F. Then $X := \sum_{n=1}^{N} Z_n$ is Compound Poisson $CPoi(\lambda, F)$ distributed. As a result of the addition stability of the Poisson distribution the $CPoi(\lambda, F)$ distribution is infinitely divisible:

$$X \overset{d}{=} \sum_{i=1}^{N_1} Z_{i,1} + \ldots + \sum_{j=1}^{N_n} Z_{j,n},$$

where $N_1, \ldots, N_n \sim Poi(\frac{\lambda}{n})$ and $(Z_{i,j})_{i\in\mathbb{N}} \sim F,\ \forall j = 1, \ldots, n$.
The characteristic function of a Compound Poisson process with parameters λ and F is given by $\varphi_{CP}(\theta) = \mathrm{e}^{\lambda(\varphi_F(\theta)-1)}$. Again a process is defined by $X_1 \sim CP(\lambda, F)$ and all other $X_t,\ t \geq 0$ via the characteristic function $\varphi_{X_t}(\theta) = \varphi_1^X(\theta)^t = \mathrm{e}^{\lambda t(i\varphi_F(\theta)-1)}$. Therefore X_t is again $CP(\lambda t, F)$ distributed, and the corresponding Levy process is a *Compound Poisson process*.

- The normal distribution $\mathcal{N}(\mu, \sigma^2)$:
The normal distribution is, like the Poisson distribution, stable regarding addition. Therefore a random variable $Z \sim \mathcal{N}(\mu, \sigma^2)$ can be decomposed into a sum of iid normal distributed stochastic variables: $Z \overset{d}{=} Z_1 + \ldots + Z_n,\ n \in \mathbb{N}$, where $Z_1, \ldots, Z_n \sim \mathcal{N}(\frac{\mu}{n}, \frac{\sigma^2}{n})$. With this the infinite divisibility follows.
Again a Levy process can be defined with $X_1 \sim \mathcal{N}(\mu, \sigma^2)$, and using Theorem 3.4 yields $X_t \sim \mathcal{N}(\mu t, \sigma^2 t)$. Therefore, together with (L1-3), the Levy process $(X_t)_{t\geq 0}$ is a *Brownian motion* with drift μ and diffusion σ, i.e. $X_t = \mu t + \sigma W_t$, where W_t is a standard Brownian motion.

If $X_1 \sim \mathcal{N}(0,1)$ is standard normal distributed, then $X_t \sim \mathcal{N}(0,t)$ and together with (L1-3) this shows that the process $(X_t)_{t\geq 0}$ is a *standard Brownian motion*.

- It is obvious that each distribution that is stable regarding addition is also infinitely divisible. Furthermore due to Lemma 3.9, the convolution of any two infinitely divisible distributions again is infinitely divisible.[3] This also shows that the sum of Levy processes again define a Levy process.

Another essential property of infinitely divisible distributions - and therefore Levy processes - is the representation of the characteristic function. This theorem is known as *Levy-Khintchine-formula*.

Theorem 3.12. *Let $(X_t)_{t\geq 0}$ be a Levy process, and $\varphi_1^X(\theta)$ the characteristic function of X_1. Then the following representation is valid:*

$$\varphi_1^X(\theta) = \exp\left(ai\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{i\theta x} - 1 - i\theta x \mathbb{1}_{|x|<1})\nu(\mathrm{d}x) \right), \qquad (3.3)$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and ν is a sigma-infinite measure, satisfying $\int_{\mathbb{R}\setminus\{0\}} 1 \wedge x^2 \ \nu(\mathrm{d}x) < \infty$.

Proof. See [Sat99, Theorem 8.1(i)]. Note that the theorem there only shows that for each infinitely divisible distribution, the characteristic function is of the form (3.3). Together with Theorem 3.10 this yields the statement of this theorem. □

This, together with Theorem 3.4 shows, that the characteristic function of a Levy process - and therefore the process itself - is fully determined by the triplet (a, σ, ν).

Definition 3.13. For a Levy process $(X_t)_{t\geq 0}$ with characteristic function stated in (3.3), the triplet (a, σ, ν) is called *Levy-(Khintchine) triplet*.
A Levy triplet is said to fulfill the usual conditions, if $a \in \mathbb{R}$, $\sigma \geq 0$ and ν is a sigma-infinite measure, satisfying $\int_{\mathbb{R}\setminus\{0\}} 1 \wedge x^2 \ \nu(\mathrm{d}x) < \infty$.

To be exact, $a \in \mathbb{R}$ is called the drift term, σ^2 the Gaussian coefficient and ν the Levy measure.

[3] As this is valid for a convolution of two distributions it is, by induction, valid for a n-fold convolution, $n \in \mathbb{N}$.

Remark 3.14. Note, that the Levy-Khintchine representation, and therefore the Levy triplet is unique.

As mentioned before, the sum of Levy processes is again a Levy process. Hence it is reasonable to to decompose a Levy process into a sum of some standard Levy processes. This in fact can be done on behalf of three Levy processes, and is called the *Levy-Itò decomposition*.

Theorem 3.15. *Let (a, σ, ν) be a Levy triplet that satisfies the usual conditions, and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Then three independent Levy processes $X^{(1)}, X^{(2)}, X^{(3)}$ on that probability space exist, such that: $X^{(1)}$ is a Brownian motion with drift, $X^{(2)}$ is a compound Poisson process and $X^{(3)}$ is a square integrable pure jump martingale with a.s. countable number of jumps on each finite time interval with magnitude less than 1.*
With $X := X^{(1)} + X^{(2)} + X^{(3)}$, there exists a Levy process $X = (X_t)_{t\geq 0}$ on that probability space with Levy triplet (a, σ, ν) and characteristic exponent

$$\psi_X(\theta) = -\left(ai\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{i\theta x} - 1 - i\theta x \mathbb{1}_{|x|<1})\nu(\mathrm{d}x)\right) \tag{3.4}$$

for $\theta \in \mathbb{R}$.

Remark 3.16. Note that sometimes the Levy process is decomposed into four parts, splitting the Brownian motion with drift into a drift term and a Brownian motion with diffusion σ.

The exact proof of this theorem is rather lengthy, and will not be stated here. A rigorous motivation of if, following [Pap08] is shown below.

Motivation of Theorem 3.15:
In Theorem 3.12 the characteristic function of a Levy process was stated to have a certain form. Therefore the characteristic exponent is defined by Equation (3.4). Now ψ_X is split into three parts $-\psi_X = \psi_X^1 + \psi_X^2 + \psi_X^3$.
$\psi_X^1(\theta) := ai\theta - \frac{1}{2}\sigma^2\theta^2$ is the characteristic exponent of a Brownian motion with drift a and diffusion σ, ψ_X^2 is defined via $\psi_X^2 := \int_{|x|\geq 1}(\mathrm{e}^{i\theta x} - 1)\nu(\mathrm{d}x)$ and $\psi_X^3 := \int_{0<|x|<1}(\mathrm{e}^{i\theta x} - 1 - i\theta x)\nu(\mathrm{d}x)$.
Now define $\lambda := \int_{|x|\geq 1}\nu(\mathrm{d}x)$, if $\lambda = 0$ then ψ_X^2 belongs to a compound Poisson process with intensity $\lambda = 0$ and arbitrary probability distribution F. Otherwise further define $F(\mathrm{d}x) = \frac{\mathbb{1}_{|x|\geq 1}\nu(\mathrm{d}x)}{\lambda}$. This obviously defines a probability distribution as $\int_{\mathbb{R}} F(\mathrm{d}x) = \frac{1}{\lambda}\int_{|x|\geq 1}\nu(\mathrm{d}x) = 1$. Now $\psi_X^{(2)}$ can be rewritten

$$\psi_X^2 = \int_{|x|\geq 1}(\mathrm{e}^{i\theta x} - 1)\nu(\mathrm{d}x) = \int_{\mathbb{R}} \lambda(\mathrm{e}^{i\theta x} - 1)F(\mathrm{d}x) = \lambda(\varphi_F(\theta) - 1).$$

This shows that this part belongs to a compound Poisson process $X^{(2)}$ with $X_1^{(2)} \sim CPoi(\lambda, F)$ (cf. Example 3.11).
To show that the last part ψ_X^3 belongs to a square integrable pure jump martingale is the hardest part, and is shown for example in [Pap08] or [Tan11].

As mentioned before the characteristic function $\varphi_X(\theta)$ does exist for all $\theta \in \mathbb{R}$ and an arbitrary random variable X. Due to the Levy-Ito representation theorem, one can show that for Levy processes the characteristic function even is integrable on $\mathbb{R}$.

Lemma 3.17. *Let $X := (X_t)_{t\in[0,T]}$ be a Levy process with characteristic function φ_{X_t}. If further the Levy triplet (a, σ, ν) satisfies the usual conditions, and X is no pure jump process ($\sigma > 0$), then the characteristic function of $X_t, t > 0$ is integrable on $\mathbb{R}$, $\varphi_{X_t} \in L^1(\mathbb{R})$, and*

$$|\varphi_{X_t}(\theta)| \leq e^{-\frac{1}{2}t\sigma^2\theta^2} \ \forall \theta, t$$

holds.

Proof. As the conditions of Theorem 3.15 are satisfied it can be applied, and the stated independent processes $X^{(1)}, X^{(2)}, X^{(3)}$ exist with

$$X \stackrel{d}{=} X^{(1)} + X^{(2)} + X^{(3)}.$$

Using the independence of these processes, and Jensen's inequality yields the following.

$$\begin{aligned} |\varphi_{X_t}| &= \left|\mathbb{E}\big[\mathrm{e}^{i\theta X_t^{(1)}}\big]\right| \left|\mathbb{E}\big[\mathrm{e}^{i\theta X_t^{(2)}}\big]\right| \left|\mathbb{E}\big[\mathrm{e}^{i\theta X_t^{(3)}}\big]\right| \leq \\ &\leq \left|\mathbb{E}\big[\mathrm{e}^{i\theta X_t^{(1)}}\big]\right| \mathbb{E}\big[\underbrace{|\mathrm{e}^{i\theta X_t^{(2)}}|}_{=1}\big]\mathbb{E}\big[\underbrace{|\mathrm{e}^{i\theta X_t^{(3)}}|}_{=1}\big] = \left|\mathbb{E}\big[\mathrm{e}^{i\theta X_t^{(1)}}\big]\right| \end{aligned}$$

As $X_t^{(1)} \sim \mathcal{N}(at, \sigma^2 t)$, the characteristic function is given by

$$\mathbb{E}\big[\mathrm{e}^{i\theta X_t^{(1)}}\big] = \mathrm{e}^{iat - \frac{1}{2}\sigma^2\theta^2}.$$

Taking the absolute value yields the stated inequality

$$|\varphi_{X_t}(\theta)| \leq \mathrm{e}^{-\frac{1}{2}t\sigma^2\theta^2} \ \forall \theta, t.$$

Now let $f_{(\mu,\sigma^2)}$ denote the probability density function of a $\mathcal{N}(\mu, \sigma^2)$ random variable.

$$f_{(\mu,\sigma^2)}(x) := \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

Then, using the inequality above and the assumptions $t > 0, \sigma > 0$ yields the supposed integrability of the characteristic function.

$$\int_{\mathbb{R}} |\varphi_{X_t}(\theta)| \mathrm{d}\theta \leq \int_{\mathbb{R}} \mathrm{e}^{-\frac{1}{2}t\sigma^2\theta^2} \mathrm{d}\theta = \sqrt{2\pi t\sigma^2} \underbrace{\int_{\mathbb{R}} f_{(0,\frac{1}{t\sigma^2})}(\theta) \mathrm{d}\theta}_{=1} = \sqrt{2\pi t\sigma^2} < \infty$$

□

3.2 Exponential Levy models

In mathematical finance, the most important thing is the model one uses for calculations of stock prices, derivatives etc. Complex models will fit the data better, but require more assumptions regarding parameters that may not be fulfilled and make calculations harder. Therefore one has to choose a model that on the one hand is sufficiently complex to reflect the real world circumstances well, and on the other hand is simple enough to enable easy price calculations of at least plain vanilla derivatives.
The Black-Scholes model has for a long time been the preferred model when it comes to pricing options, and in practice often still is. This model has been discussed in detail for its advantages and disadvantages, see for example [Gat06]. The main feature of this model is, that the asset $(S_t)_{t\geq 0}$ is modelled by a geometric Brownian motion and therefore the option prices can be calculated easily.

$$S_t := S_0 \mathrm{e}^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t},$$

where $\mu \in \mathbb{R},\ \sigma > 0$ and $(W_t)_{t\geq 0}$ is a Brownian motion.

However, there has been a lot of research regarding modifications of the Black-Scholes model to improve two aspects, see [KW04, Section 1].
For one thing, the log-returns of an underlying often do not follow a normal distribution as supposed in the Black-Scholes model. In fact, the distribution function tends to be left-skewed and leptokurtic which are two facts that indicate non gaussian log-returns. For another thing in the Black-Scholes model the implied volatility function is known to form the so-called volatility smile. This means that the constant implied volatility in the Black-Scholes model should rather be a convex function of the strike and time.
These two aspects motivate the use of more complex models, with other return distributions. Therefore the more general type of exponential Levy models is introduced.
Examples for other popular models are stochastic volatility models, GARCH

models or the constant elasticity of variance (CEV) model. However, the absence of jumps discards these models as jumps are an important issue when quantifying big changes in stock prices in short time intervals. Another reason to use jump models is the existence of jumps in observed prices such as exchange rates. Hence, this thesis focusses on exponential Levy (jump) models, and for this reason in the following some definitions and results for Levy processes are stated.

Definition 3.18. Let $(X_t)_{t\geq 0}$ be a Levy process with Levy triplet (a, σ, ν) [4]. The asset S is then said to follow an exponential Levy model, if

$$S_t := \mathrm{e}^{X_t} \ , \ t \geq 0.$$

Within these models two different types are distinguished.

Pure jump models

These models are specified by a Levy process with triplet (a, σ, ν), where $\sigma > 0$ and $\nu(\mathbb{R}) < \infty$, meaning that the Levy process has a diffusion and a jump part, with finite number of jumps. Some important examples are stated below.

- Variance-Gamma model: The Variance-Gamma process, also known as the Laplace motion is a pure jump Levy process with finite moments. Further, it has independent and are Variance Gamma distributed moments.
- Normal inverse Gaussian model: The Normal inverse Gauss process is a Levy process with Normal inverse Gaussian distributed increments. For further details see [TV09].

Jump diffusion models

These models are specified by a Levy process with triplet (a, σ, ν), where $\sigma > 0$ and $\nu(\mathbb{R}) < \infty$, meaning that the Levy process has a diffusion and a jump part, with finite number of jumps. The two most important models are stated below.

[4] See Definition 3.1 and Theorem 3.12.

- Merton model: Within this model, the corresponding Levy process, respectively its Levy measure has a Gaussian density,

$$\nu(x) = \frac{\lambda}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

  where λ defines the jump intensity, μ is the average jump size and σ the volatility.

- Kou model: This model will be discussed in detail in Section 3.2.1.

The Black-Scholes model can also be expressed as an exponential Levy model, more precisely it is given if the exponent X_t is a Levy process without jumps, $\nu(\mathbb{R}) = 0$.

3.2.1 Kou's model

Kou's model [Kou02] is a Levy jump diffusion model, where the asset price S_t has the following dynamics.

$$\frac{\mathrm{d}S_t}{S_{t-}} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}\left(\sum_{j=1}^{N_t} V_j - 1\right),$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and $(W_t)_{t\geq 0}$ is a standard Brownian motion. Further, $(N_t)_{t\geq 0}$ is a Poisson process with rate λ, and $(V_j)_{j\in\mathbb{N}}$ is a sequence of iid random variables $V_j \stackrel{d}{=} V, \quad \forall j \in \mathbb{N}$ with $V > 0$ *a.s.*, such that its logarithm has a double-exponential distribution:

$$Y := \log(V) \stackrel{d}{=} \begin{cases} \xi_1 & \text{with probability } p \\ \xi_2 & \text{with probability } 1-p \end{cases}$$

for $\xi_i \sim \mathrm{Exp}(\eta_i)$ with $\mathbb{E}[\xi_i] = \frac{1}{\eta_i},\ i \in \{1,2\}$.
This means that the probability density function of Y is given by

$$f_Y(y) = p\eta_1 \mathrm{e}^{-\eta_1 y} \mathbb{1}_{y\geq 0} + (1-p)\eta_2 \mathrm{e}^{\eta_2 y} \mathbb{1}_{y<0}\ ,$$

where $p \in [0,1]$ and $\eta_1 > 1,\ \eta_2 > 0$.

For reasons of simplicity, all sources of randomness, $(N_t)_{t\geq 0}$, $(W_t)_{t\geq 0}$ and $(Y_j)_{j\in\mathbb{N}}$, are set as independent and μ and σ are assumed constant.
The model can be generalized to multidimensional processes, and the assumptions regarding the independence and constancy can be loosened or dropped, respectively.

From the dynamics of the asset price S_t, the process can be computed by using Ito's formula on $X_t := \log(S_t)$:

$$\begin{aligned}
\mathrm{d}X_t &= \mathrm{d}(\log(S_t)) = \\
&= \frac{1}{S_t}\mathrm{d}S_t - \frac{1}{2S_t^2}\mathrm{d}\langle S_t, S_t\rangle + \mathrm{d}\left(\sum_{j=1}^{N_t} \log\left(S_t + S_t(V_j - 1)\right) - \log(S_t)\right) = \\
&= \frac{1}{S_t}\mathrm{d}S_t - \frac{1}{2S_t^2}\mathrm{d}\langle S_t, S_t\rangle + \mathrm{d}\left(\sum_{j=1}^{N_t} \underbrace{\log(V_j)}_{Y_j}\right).
\end{aligned}$$

Using integration, and as μ and σ are constant, the representation of X_t and S_t follow.

$$X_t = X_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t) + \sum_{j=1}^{N_t} Y_j \tag{3.5}$$

$$S_t = S_0 \mathrm{e}^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \prod_{j=1}^{N_t} V_j \tag{3.6}$$

Chapter 4

How to make Dupire's local volatility work with jump processes

Local volatility models are a commonly used tool for asset pricing. This is, as they can be calibrated to fit market data very well. Furthermore the local volatility can be computed directly out of option market prices with Dupire's formula. Anyhow, this only is feasible if the asset is continuous, and therefore has no jumps.
The drawbacks of such no jump models have been discussed in Section 3.2, and hence it is reasonable to adjust these jump processes in a way, such that Dupire's formula can be applied.
In the following section, at first local volatility is introduced and Dupire's formula is derived shortly. Then, the results of Gerhold's paper [FGY13] regarding Dupire's local volatility for Levy jump diffusions are summed up and applied to Kou modelled underlyings.

4.1 Local volatility & Dupire's formula

The idea to compute the dynamics of an asset on behalf of given call prices is a common task. Especially the calculation of the so called implied volatility $\sigma_{BS}(K,t)$ is a frequently used procedure [1]. Here, the idea is that the volatility of the underlying asset is a function of both, the option's strike price and time to maturity. The corresponding diffusion model is given by the following

[1] The implied volatility is calculated such, that the Black Scholes pricing formula equals the real prices for several points in time and strikes. For further information see [Gat06, MR97].

dynamics

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma_{BS}(K,t)\mathrm{d}W_t \ .$$

In practise it is interesting to fit a model to existing call (market) prices. Breeden and Litzenberger, [BL78] showed that european call prices define the underlying's density. Based on this, Dupire (1994) showed, that a unique diffusion process exists, matching this density.

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma(S_t,t)\mathrm{d}W_t \tag{4.1}$$

The function $\sigma(S_t,t)$ is then called *local volatility*.
For an arbitrage free call price surface, Dupire even showed that this model is uniquely determined by the so called Dupire formula. The following derivation is mainly taken from Gatheral, [Gat06].

Heuristic derivation of Dupire's formula

Consider the stock price model

$$\frac{\mathrm{d}S_t}{S_t} = r_t \mathrm{d}t + \sigma(S_t,t)\mathrm{d}W_t.$$

With $\varphi(S_t,t)$ defining the pseudo probability density, the expected payoff of a european call option with strike K and maturity T is given by

$$C(K,T) = \int_K^\infty \varphi(s,T)(s-K)\mathrm{d}s.$$

Then the call price is twice differentiable with respect to the strike, and once with respect to time, and the corresponding derivatives are given as follows.

$$\begin{aligned}
\frac{\partial C}{\partial K}(K,T) &= -\int_K^\infty \varphi(s,T)\mathrm{d}s \\
\frac{\partial^2 C}{\partial K^2}(K,T) &= \varphi(K,T) \\
\frac{\partial C}{\partial T}(K,T) &= \int_K^\infty \frac{\partial}{\partial T}\varphi(s,T)(s-K)\mathrm{d}s
\end{aligned}$$

Using the fact, that φ satisfies the Fokker-Planck equation,

$$\frac{\partial \varphi}{\partial T} = \frac{1}{2}\frac{\partial^2}{\partial S_t^2}(\sigma^2 S_T^2 \varphi) - S_T \frac{\partial}{\partial S_T}(r_T S_T \varphi),$$

and using partial integration twice yields the following equation for the derivative with respect to time.

$$\frac{\partial C}{\partial T}(K,T) = \frac{\sigma^2 K^2}{2}\varphi(K,T) + \int_K^\infty r_T s \varphi \mathrm{d}s = \frac{\sigma^2 K^2}{2}\frac{\partial^2 C}{\partial K^2}(K,T) - r_T K \frac{\partial C}{\partial K}(K,T)$$

Solving this equation for σ^2 gives Dupire's formula.

$$\sigma^2(K,T) = \frac{2(\partial_T C(K,T) + rK\partial_K C(K,T))}{K^2 \partial_{KK} C(K,T)}. \tag{4.2}$$

Note, that within this derivation no assumptions for interchanging differentiation with integration or applying the Fokker-Planck equation are made. For this reason the problems that occur when considering jumps are stressed.
For one thing it is questionable whether the call price function is twice continuously differentiable with respect to the strike when considering jump models for the underlying, and for another thing there is a blowup of the local volatility $\sigma(.,t) \to \infty$ for $t \to 0$ leaving the existence of a unique solution to (4.1) unsure.
Nevertheless, this formula is often used in practice even in presence of jumps after some regularizations are done. Therefore Gerhold et al. showed in [FGY13] that this approach is reasonable, and that Dupire's formula can be used even in presence of jumps using time shifting methods.

Theorem 4.1 ([FGY13], Theorem 1)**.** *Assume that $(S_t)_{t\geq 0}$ is a martingale (with jumps) and the corresponding smooth* [2] *call price surface*

$$C(K,T) = \mathbb{E}[(S_T - K)^+],\ \forall K, T \geq 0,$$

such that $\partial_T C > 0$ and $\partial_{KK} C > 0$. Define the ϵ-shifted local volatility by

$$\sigma_\epsilon^2(K,T) = \frac{2(\partial_T C(K,T+\epsilon) + rK\partial_K C(K,T+\epsilon))}{K^2 \partial_{KK} C(K,T+\epsilon)}. \tag{4.3}$$

Then $\frac{\mathrm{d}S_t^\epsilon}{S_t^\epsilon} = \sigma_\epsilon(S_t^\epsilon, t)\mathrm{d}W_t$, starting at randomized spot S_0^ϵ with distribution

$$\mathbb{P}[S_0^\epsilon \in \mathrm{d}K]/\mathrm{d}K = \partial_{KK} C(K,\epsilon),$$

admits a unique, non explosive strong SDE solution such that

$$\mathbb{E}[(S_T^\epsilon - K)^+] \to C(K,T),\ \textit{as}\ \epsilon \to 0,\ \forall K, T \geq 0. \tag{4.4}$$

[2] What is really needed here is that σ_ϵ as defined in 4.3 is locally Lipschitz in K and bounded, continuous on compacts in T.

This theorem is the basis of this thesis, and the following sections will focus on applying this to Kou's model and realizing it numerically. However, taking the limit for $\epsilon \to 0$ is hard to implement. Hence, another version of this theorem is provided by Gerhold et al.

Theorem 4.2 ([FGY13], Theorem 4)**.** *Suppose the setting of Theorem 4.1 with σ_{loc} defined by* (4.2) *and define the process $(\widetilde{S}^\epsilon_t)_{t\geq\epsilon}$ by*

$$\frac{\mathrm{d}\widetilde{S}^\epsilon_t}{\widetilde{S}^\epsilon_t} = \sigma_{loc}(\widetilde{S}^\epsilon_t, t)\mathrm{d}W_t,$$

starting at a random spot $\widetilde{S}^\epsilon_\epsilon$ with distribution

$$\mathbb{P}(\widetilde{S}^\epsilon_\epsilon \in \mathrm{d}K)/\mathrm{d}K = \partial_{KK}C(K,\epsilon).$$

Then, for $K \geq 0$ and $T \geq \epsilon$, S^ϵ rebuilds the given call price surface.

$$\mathbb{E}[(\widetilde{S^\epsilon}_T - K)^+] = C(K,T)$$

Proof. Let S^ϵ be the time shifted process defined in Theorem 4.1. Now define the process $Y = (Y_t)_{t\in[0,T]}$, as $Y_t := S^\epsilon_{t-\epsilon}$. Then the following equation holds

$$\mathbb{E}[(Y_T - K)^+] = C(K,T),$$

and using the SDE for S^ϵ yields a representation for Y.

$$Y_t = S^\epsilon_0 + \int_0^{t-\epsilon} S^\epsilon_u \sigma_\epsilon(S^\epsilon_u, u)\mathrm{d}W_u = Y^\epsilon_\epsilon + \int_\epsilon^t Y^\epsilon_s \sigma_\epsilon(Y^\epsilon_s, s)\mathrm{d}W_s$$

With the definitions of the initial values, it follows that

$$Y_\epsilon = S^\epsilon_0 \overset{d}{=} \widetilde{S}^\epsilon_\epsilon,$$

and with this

$$\widetilde{S}^\epsilon \overset{d}{=} Y.$$

Now the statement of the theorem follows.

$$\mathbb{E}[(\widetilde{S^\epsilon_T} - K)^+] = \mathbb{E}[(Y_T - K)^+] = C(K,T)$$

□

This version is preferred to Theorem 4.1, as the call price surface can be regained by Monte Carlo simulation for each time to maturity $T \geq \epsilon$.[3]

[3]The error between the price obtained by simulation and the actual call price (surface) can be explained by numerical errors, and a possibly to small simulation size.

Remark 4.3. Note, that although Theorem 4.2 only states the probability density of the 'initial value' S_ϵ^ϵ, the cumulative distribution function can be obtained by integration.

$$\mathbb{P}[S_\epsilon^\epsilon \leq x] = \int_0^x \partial_{KK} C(y, \epsilon) \mathrm{d}y = \partial_K C(x, \epsilon) - \partial_K C(0, \epsilon)$$

This will be used to simulate the initial value numerically.

The next sections will apply this theorem to Kou's model and establish an integral representation for the local volatility.

4.2 Local volatility in Kou's model

As described before, the computation of the local volatility happens in three steps:

1. Calculation of the characteristic function of the underlying.
2. Computing the Call price surface via inverse Fourier transformation.
3. Applying Dupire's formula on the Call price surface.

In the following the local volatility surface in Kou's model will be computed following the three steps stated before.

Assumption 1. *From now on the underlying S is considered a martingale. This can be done as a result from the no arbitrage condition.*

Characteristic function

As shown in Equation (3.6) the asset price S_t and its logarithm X_t have the following representation:

$$X_t = X_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t) + \sum_{j=1}^{N_t} Y_j$$

$$S_t = S_0 \mathrm{e}^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \prod_{j=1}^{N_t} V_j.$$

Out of this, one can easily compute the characteristic function of X_t, by applying the Levy-Khintchine formula, see Theorem 3.12. For reasons of integrity, the characteristic function is derived manually below.

$$\begin{aligned}\varphi_1^X(\theta) &= \mathbb{E}[\exp(i\theta X_1)] = \\ &= \mathbb{E}\Big[\exp\Big(i\theta \underbrace{(X_0 + \mu - \frac{1}{2}\sigma^2 + \sigma W_1)}_{:=X^{(1)}}\Big)\Big]\mathbb{E}\Big[\exp\Big(i\theta \underbrace{\sum_{j=1}^{N_1} Y_j}_{:=X^{(2)}}\Big)\Big],\end{aligned}$$

where the last equation follows due to the independence of $(W_t)_{t\geq 0}$, $(N_t)_t \geq 0$ and $(V_j)_{j\in\mathbb{N}}$. Then, by noticing that $X^{(1)} \sim \mathcal{N}(b, \sigma^2)$ with $b := X_0 + \mu - \frac{1}{2}\sigma^2$ the first part of the characteristic function is given by

$$\mathbb{E}[\exp(i\theta X^{(1)})] = \exp(i\theta b - \frac{1}{2}\theta^2\sigma^2).$$

For the second part it is important to notice that $X^{(2)}$ is compound Poisson distributed, $X^{(2)} \sim \mathcal{CP}(\lambda, F)$, where F is a double exponential distribution $DExp(p, \eta_1, \eta_2)$ with probability density f,

$$f(y) := p\eta_1 \mathrm{e}^{-\eta_1 y}\mathbb{1}_{y\geq 0} + (1-p)\eta_2 \mathrm{e}^{\eta_2 y}\mathbb{1}_{y<0}.$$

In Example 3.11 the characteristic function of a compound Poisson random variable $Y \sim CPoi(\lambda, F)$ was stated as

$$\varphi_Y(\theta) = \mathrm{e}^{\lambda(\varphi_F(\theta)-1)}.$$

Therefore only the characteristic function of the double exponential distribution F needs to be calculated.

$$\begin{aligned}\varphi_F(\theta) = \mathbb{E}[\mathrm{e}^{i\theta Y}] &= \int_{\mathbb{R}} \mathrm{e}^{i\theta y} f(y)\mathrm{d}y = \\ &= \int_0^\infty \mathrm{e}^{i\theta y} p\eta_1 \mathrm{e}^{-\eta_1 y}\mathrm{d}y + \int_{-\infty}^0 \mathrm{e}^{i\theta y}(1-p)\eta_2 \mathrm{e}^{\eta_2 y}\mathrm{d}y = \\ &= p\eta_1 \int_0^\infty \mathrm{e}^{(i\theta-\eta_1)y}\mathrm{d}y + (1-p)\eta_2 \int_{-\infty}^0 \mathrm{e}^{(i\theta+\eta_2)y}\mathrm{d}y = \frac{p\eta_1}{\eta_1 - i\theta} + \frac{(1-p)\eta_2}{\eta_2 + i\theta}\end{aligned}$$

With this, finally the characteristic function of the log asset X_t at time $t = 1$ is given by

$$\varphi_{X_1}(\theta) = \exp\left(i\theta b - \frac{1}{2}\theta^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - i\theta} + \frac{(1-p)\eta_2}{\eta_2 + i\theta} - 1\right)\right),$$

and the Levy exponent by

$$\psi_X(\theta) = i\theta b - \frac{1}{2}\theta^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - i\theta} + \frac{(1-p)\eta_2}{\eta_2 + i\theta} - 1\right) . \tag{4.5}$$

As $(X_t)_{t\geq 0}$ is a Levy process, the characteristic function of X_t follows by applying Theorem 3.4.

$$\varphi_{X_t}(\theta) = \exp\left(t\left(i\theta b - \frac{1}{2}\theta^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - i\theta} + \frac{(1-p)\eta_2}{\eta_2 + i\theta} - 1\right)\right)\right) \tag{4.6}$$

Due to Assumption 1, $(S_t)_{t\geq 0}$ can be considered a martingale. A sufficient condition for this is, that $\mathbb{E}[S_t] = \mathbb{E}[S_0 \mathrm{e}^{X_t}] = S_0\varphi_{X_t}(-i) = S_0$. Therefore it follows, that

$$b = -\left(\frac{\sigma^2}{2} + \lambda\left(\frac{p\eta_1}{\eta_1 - 1} + \frac{(1-p)\eta_2}{\eta_2 + 1} - 1\right)\right). \tag{4.7}$$

The mgf in the double exponential jump diffusion model follows analogously:

$$M_{X_t}(\theta) = \exp\left(t\left(\theta b + \frac{1}{2}\theta^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - \theta} + \frac{(1-p)\eta_2}{\eta_2 + \theta} - 1\right)\right)\right). \tag{4.8}$$

As $\eta_1 > 1$ and $\eta_2 > 0$, it follows that there exists a p with $1 < p < \eta_1$, such that $M_t^X(p) < \infty$. Therefore all assumptions for Theorem 2.15 are satisfied, and $\hat{c}_\alpha(k,t)$ can be computed.

The Call price surface

The calculation of the call price surface is done following Section 2.2.2. From Theorem 2.15 follows that the Fourier transform of the damped option price has the form

$$\hat{c}_\alpha(k,t) = \frac{\varphi_{X_t}(k - (\alpha+1)i)}{(\alpha + ik)(\alpha + 1 + ik)}. \tag{4.9}$$

Now, with Theorem 2.16, $\hat{c}_\alpha(k,t)$ is integrable and applying Theorem 2.17 provides

$$C(k,t) = \frac{\mathrm{e}^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-iuk}\hat{c}_\alpha(u,t)\mathrm{d}u = \frac{\mathrm{e}^{-\alpha k}}{\pi}\int_0^{\infty} \mathrm{Re}\left(\mathrm{e}^{-iuk}\hat{c}_\alpha(u,t)\right)\mathrm{d}u.$$

By plugging in the previously derived formulas for $\hat{c}_\alpha(u,t)$ and $\varphi_{X_t}(\theta)$ respectively yields an integral formula for the call price surface in Kou's model. This integral will be computed in Chapter 5 by means of numerical integration.

Applying Dupire's formula

In Dupire's formula (4.2), the first derivative with respect to T, $\partial_T C$ and the first and second derivative with respect to K, $\partial_K C$ and $\partial_{KK} C$ are used. Therefore it needs to be verified, that the conditions for interchanging integration with differentiation are satisfied. It is sufficient for the integrand to be integrable, differentiable with respect to the required variable and for the derivation to be bounded by an integrable function.
Hence, the call price surface is rewritten:

$$C(k,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k,t,u) \mathrm{d}u,$$

where

$$f(k,t,u) = \mathrm{e}^{-(\alpha+iu)k} \frac{\varphi_{X_t}(u-(\alpha+1)i)}{(\alpha+iu)(\alpha+1+iu)} \,. \tag{4.10}$$

So, as shown in Proposition 5.3, all required derivatives of f are bounded by an integrable function, and therefore the differentiation and integration can be interchanged. With this, the derivatives in Dupire's formula are given by

$$\begin{aligned}
\partial_K C(K,T) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \partial_K f(K,T,u) \mathrm{d}u \\
\partial_{KK} C(K,T) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \partial_{KK} f(K,T,u) \mathrm{d}u \\
\partial_T C(K,T) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \partial_T f(K,T,u) \mathrm{d}u
\end{aligned}$$

and Dupire's formula can be applied.[4]

[4]Note that, as $C(K,T)$ is computed numerically, these derivatives are computed the same way.

Chapter 5

Numerical realisation

The previous chapters provide all necessary results and formulas to be able to compute the local volatility theoretically. Anyways, to be able to do this numerically there are still some issues that need to be taken care of, such as numerical integration or Fourier inversion. This chapter will therefore focus on the numerical problems that occur while computation, and describe the most important steps of the algorithm. The aim of the following algorithm is, to show that Theorem 4.2 is applicable for jump diffusion processes. Therefore Kou's model is selected as model for the underlying. Once the model is chosen and all parameters are set, all required results can be computed following Section 4.2, and with this the damped Fourier transform of the option price, and the call price surface itself are calculated.

5.1 Optimal damping factor

In Section 2.2.2 the damped option price was introduced, to be able to compute either its Fourier transform and the call price surface itself. As the damping factor α does not have any impact on the resulting option price in theory, the choice of α has not been an issue in this paper. The only restrictions to it, are the ones resulting from Theorems 2.15 and 2.17.
Nevertheless, this factor α does have a big impact on the oscillation and absolute value of the integrand, and therefore is important for numerical integration. This issue has been discussed in detail in [LK07]. The results of this paper are now shortly presented, and then applied to Kou's model to get the optimal damping factor for Fourier inversion.

The first restriction results from the strip of regularity of the characteristic function. The generalized version of the characteristic function is well defined on $\Lambda_X := \{\xi \in \mathbb{C} : \ -\text{Im}(\xi) \in A_X\}$, cf. Definition 2.10, with

$A_X := (\zeta_-, \zeta_+) \subseteq \mathbb{R}$. As the Fourier transform of the call price is given by Equation (4.9), the permitted interval for alpha follows easily.

$$\alpha \in (\zeta_- - 1, \zeta_+ - 1)$$

Computing the strip of regularity for Kou's model

Lemma 5.1. *Let X be a random variable and M_X its mgf, and let (ζ_-, ζ_+) be the maximal interval where the mgf exists.*
Then these bounds are given as follows.

$$\zeta_- := \sup\{\theta < 0 | \log(M_X(\theta)) = \infty\}$$
$$\zeta_+ := \inf\{\theta > 0 | \log(M_X(\theta)) = \infty\}$$

Proof. As the logarithm is a strictly increasing function, one can see that $M_X(\theta) < \infty$, for all θ with $\log(M_X(\theta)) \neq \infty$. Let Z be the set of all values, where the mgf does not exist,

$$Z := \{\theta \in \mathbb{R} | \log(M_X(\theta)) = \infty\}.$$

As $M_X(0) = 1 < \infty$ for any random variable X, it follows that $0 \in (\zeta_-, \zeta_+)$, and therefore $\zeta_- \leq 0$ and $\zeta_+ \geq 0$.
Now define ζ_- and ζ_+ as stated above. Then $(\zeta_-, \zeta_+) \cap Z = \emptyset$, and therefore $M_X(\theta) < \infty$ for all $\theta \in (\zeta_-, \zeta_+)$.
Let $I := (a, b) \supsetneq (\zeta_-, \zeta_+)$ be another interval, and let w.l.o.g. $a < \zeta_-$, and $(z_n)_{n\in\mathbb{N}} \in Z^{\mathbb{N}}$ with $z_n \nearrow \zeta_-$ a series that converges from below to ζ_-. Then there exists an index $N \in \mathbb{N}$ such that $z_N \in I$, and therefore the mgf does not exist on the whole interval.
The same steps can be done if $b > \zeta_+$ with a series in Z decreasing to ζ_+, and with this (ζ_-, ζ_+) must be the maximal interval. □

Now, this lemma can be applied to the moment generating function stated in (4.8).

$$\log(M_t^X(\theta)) = t\left(\theta b + \frac{1}{2}\theta^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - \theta} + \frac{(1-p)\eta_2}{\eta_2 + \theta} - 1\right)\right)$$

This function is continuous, and therefore finite, on $\mathbb{R} \setminus \{-\eta_2, \eta_1\}$. So the strip of regularity is given by

$$(\zeta_-, \zeta_+) := (-\eta_2, \eta_1) \supset [0, 1],$$

and with this

$$\alpha \in (\alpha_-, \alpha_+) := (-\eta_2 - 1, \eta_1 - 1). \tag{5.1}$$

To get the optimal choice of α, the integrand only needs to be minimized over this interval.

Optimal choice of α suggested by Lord & Kahl

As mentioned before the choice of alpha is not imported when calculating the option prices in theory. However, due to numerical errors it does have an impact when integrating numerically. These numerical errors are caused by both, highly oscillating integrands and big absolute values of the integrand. Lord and Kahl [LK07] optimized the choice of the damping parameter α regarding big absolute values. These results have been been taken up by Peterseil [Pet14] and were presented in a mathematically more accurate way. The following proposition presents the result from these discussions.

Proposition 5.2. *[Pet14, Proposition 5.3.4]*
Let X be a random variable with φ_X its corresponding moment generating function, and let further ζ_- and ζ_+ be defined as in Lemma 5.1. Then $\alpha^* \in (\alpha_-, \alpha_+) := (\zeta_- - 1, \zeta_+ - 1)$ minimizes

$$\sup_{u\in(0,\infty)} \left| e^{-(\alpha+iu)k} \frac{\varphi_X(u-(\alpha+1)i)}{(\alpha+iu)(\alpha+1+iu)} \right| \tag{5.2}$$

if and only if it is a solution to

$$\min_{\alpha\in[\alpha_-,\alpha_+]\setminus\{-1,0\}} \log M_X(\alpha+1) - \frac{1}{2}\log \alpha^2(\alpha+1)^2 - \alpha k. \tag{5.3}$$

In case one of the bounds $\alpha_-,\ \alpha_+$ is infinite, additionally assume

$$\begin{aligned} &\mathbb{P}(X < k) > 0 \quad \text{if } \alpha_- = -\infty \text{ and} \\ &\mathbb{P}(X > k) > 0 \quad \text{if } \alpha_+ = +\infty. \end{aligned}$$

Then the minimization problem in (5.3) *has a unique solution in each of the intervals*

$$\begin{aligned} &\circ\ [\alpha_-, -1) \cap (-\infty, -1), \\ &\circ\ (-1, 0) \cap (-1, \alpha_+] \text{ and} \\ &\circ\ (0, \alpha_+] \cap (0, +\infty). \end{aligned}$$

Proof. See [Pet14, Proposition 5.3.4]. Note that the notation there is slightly different, as α is replaced with $\alpha + 1$ and therefore the intervals are shifted accordingly. Furthermore note that with Remark 2.7, $\varphi_X(u-(\alpha+1)i) = M_X(\alpha+1-iu)$, wherever it exists. □

With this theorem calculating the optimal choice of alpha can be done by numerical optimization of expression (5.3). However, this choice of α is only optimal for the integrand stated in (5.2), and therefore when calculating the call price surface $C(K,T)$. As Dupire's formula (4.2) requires numerical calculation of some derivatives first an integral representation is derived and similar results for these functions are presented.

Proposition 5.3. *Let the log-underlying $X := (X_t)_{t\in[0,T]}$ be a non pure-jump Levy process with characteristic function φ_{X_t}, and let its Levy triplet (a,σ,ν) satisfy the usual conditions with $\sigma > 0$. Further let α_-, α_+ be defined as in Proposition 5.2 and K be the strike of a european call option $C(K,t)$ stated in (2.3).*
Then $C(.,.) \in \mathcal{C}^{2,1}$ [1], and the corresponding derivatives are given by

$$\frac{\partial C}{\partial K}(K,t) = -\frac{1}{\pi}\int_0^\infty \operatorname{Re}\left(K^{-(\alpha+1+iu)}\frac{\varphi_{X_t}(u-(\alpha+1)i)}{(\alpha+1+iu)}\right)\mathrm{d}u$$

$$\frac{\partial^2 C}{\partial K^2}(K,t) = \frac{1}{\pi}\int_0^\infty \operatorname{Re}\left(K^{-(\alpha+2+iu)}\varphi_{X_t}(u-(\alpha+1)i)\right)\mathrm{d}u$$

$$\frac{\partial C}{\partial t}(K,t) = \frac{1}{\pi}\int_0^\infty \operatorname{Re}\left(-\psi(u-(\alpha+1)i)K^{-(\alpha+iu)}\frac{\varphi_{X_t}(u-(\alpha+1)i)}{(\alpha+iu)(\alpha+1+iu)}\right)\mathrm{d}u,$$

where $\psi(.)$ is the characteristic exponent stated in (3.2), and $0 < \alpha \leq \alpha_+$.

Proof. With the assumptions made, all conditions for Theorem 2.17 are satisfied, and the call price is given by (2.2). The key point in this proof is to show that the integral and the differential can be interchanged. To do this, the integrands have to be differentiable and the corresponding derivatives have to be absolutely bounded by an integrable function.
For this purpose, let $f_\alpha : \mathbb{R}^+ \times [0,T] \times \mathbb{R} \to \mathbb{C}$ denote the integrand as a function of the strike $K := \mathrm{e}^k$ for each valid α,

$$f_\alpha(K,t,u) := K^{-(\alpha+1+iu)}\frac{\varphi_{X_t}(u-(\alpha+1)i)}{(\alpha+iu)(\alpha+1+iu)}.$$

To see the dependence on t better, f_α is rewritten using (3.2), and the characteristic exponent $\psi : \mathbb{C} \to \mathbb{C}$.

$$f_\alpha(K,t,u) := K^{-(\alpha+1+iu)}\frac{\mathrm{e}^{t\psi(u-(\alpha+1)i)}}{(\alpha+iu)(\alpha+1+iu)}\ .$$

[1] Note that this means that $C(.,.)$ is twice differentiable with respect to K. Anyhow, due to the chain rule of derivations $C(\log(K),t)$ is also twice differentiable with respect to k.

Then f_α is clearly twice differentiable with respect to K, and once with respect to t, $f_\alpha(.,.,u) \in C^{2,1}$, with the following derivatives.

$$\begin{aligned}
\frac{\partial f_\alpha}{\partial K}(K,t,u) &= -K^{-(\alpha+1+iu)}\frac{\varphi_{X_t}(u-(\alpha+1)i)}{(\alpha+1+iu)} \\
\frac{\partial^2 f_\alpha}{\partial K^2}(K,t,u) &= K^{-(\alpha+2+iu)}\varphi_{X_t}(u-(\alpha+1)i) \\
\frac{\partial f_\alpha}{\partial t}(K,t,u) &= -\psi(u-(\alpha+1)i)K^{-(\alpha+iu)}\frac{\varphi_{X_t}(u-(\alpha+1)i)}{(\alpha+iu)(\alpha+1+iu)}
\end{aligned}$$

These three functions exactly describe the corresponding derivations of the integrand of $C(K,t)$. Therefore the only thing that needs to be shown is, that these derivatives are bounded by integrable functions, then derivation and integration can be interchanged. Using that $|z| \geq |Re(z)|$ and $|a^z| = |a^{Re(z)}|$ is valid for all $z \in \mathbb{C}$ yields the following inequalities.

$$\left|\frac{\partial f_\alpha}{\partial K}(K,t,u)\right| \leq K^{-(\alpha+1)}\frac{|\varphi_{X_t}(u-(\alpha+1)i)|}{|\alpha+1|} \tag{5.4}$$

$$\left|\frac{\partial^2 f_\alpha}{\partial K^2}(K,t,u)\right| \leq K^{-(\alpha+2)}|\varphi_{X_t}(u-(\alpha+1)i)| \tag{5.5}$$

To get an estimate for the derivative with respect to t, first a bound for $\psi(z) := -\log \mathbb{E}[e^{izX_1}], z \in \mathbb{C}$ needs to be found. Note therefore, that here log is the complex logarithm defined as $\log : \mathbb{C} \setminus \{0\} \to [0,\infty) \times i\ (-\pi,\pi]$ [2]. With this definition, the following inequality holds for all $z \in \mathbb{C}$,

$$|\log z| \leq |\log |z|| + \pi.$$

Now this can be applied to the characteristic exponent.

$$\begin{aligned}
|\psi(z)| &= |\log|\mathbb{E}[e^{izX_1}]|| + \pi \leq |\log \mathbb{E}[|e^{izX_1}|]| + \pi = \\
&= \mathbb{E}[e^{-\mathrm{Im}(z)X_1}] + \pi := c^* < \infty
\end{aligned}$$

With these results an upper bound can be established.

$$\begin{aligned}
\left|\frac{\partial f_\alpha}{\partial t}(K,t,u)\right| &\leq \underbrace{|\psi(u-(\alpha+1)i)|}_{\leq c^*} K^{-\alpha}\frac{|\varphi_{X_t}(u-(\alpha+1)i)|}{|\alpha^2+\alpha|} \\
&\leq c^* K^{-\alpha}\frac{|\varphi_{X_t}(u-(\alpha+1)i)|}{|\alpha^2+\alpha|}
\end{aligned} \tag{5.6}$$

[2] This is often referred to as the principal value of the complex logarithm.

As X was assumed to be a non pure jump process with $\sigma > 0$, and the Levy triplet fulfills the usual conditions, Lemma 3.17 can be applied. Therefore φ_{X_t} is integrable, and with this all stated derivatives are integrable with respect to u.
This was the main condition for interchanging the integral and the differential.

$$\frac{\partial C}{\partial K}(K,t) = \frac{\partial}{\partial K}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} K^{-(\alpha+iu)}\frac{\varphi_{X_t}(u-(\alpha+1)i)}{(\alpha+iu)(\alpha+1+iu)}\mathrm{d}u\right) =$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial f_\alpha}{\partial K}(K,t,u)\mathrm{d}u = -\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left(K^{-(\alpha+1+iu)}\frac{\varphi_{X_t}(u-(\alpha+1)i)}{(\alpha+1+iu)}\right)\mathrm{d}u$$

The last equation follows, as the imaginary part of the integrand is an odd function, and therefore the integral equals zero. The same steps can be done for the other derivatives as well, which yields the stated representations. □

Now, one can see that these representations again are integral notations that need to be computed numerically. For this reason it is once more required to find an optimal damping factor for each derivative that minimizes computation errors. As mentioned for the call price itself, these functions also do not depend on the choice of α in theory, but might do when calculations are done numerically.
However, in the following only the derivatives with respect to the strike are considered. The derivative with respect to the time to maturity t is not optimized individually, but rather computed with the optimal alpha from Proposition 5.2. This is done, as getting a similar result for $\frac{\partial C}{\partial T}$ to is not easy when optimizing this derivative, and doing so would cause a remarkably higher computation time.
In the following, the approach from [Pet14] is modified to get similar results for these derivations.

Lemma 5.4. *Let $X := (X_t)_{t\in[0,T]}$ fulfill the assumptions of Proposition 5.3, with characteristic function φ_{X_t}. Further, let $\alpha \in \mathbb{R}$ be such, that $\varphi_{X_t}(-(\alpha+1)i) < \infty$. If $\alpha \neq -1$ then*

$$\sup_{u\in(0,\infty)}\left|K^{-(\alpha+1+iu)}\frac{\varphi_{X_t}(u-(\alpha+1)i)}{(\alpha+1+iu)}\right| = K^{-(\alpha+1)}\frac{\varphi_{X_t}(-(\alpha+1)i)}{|\alpha+1|},$$

and if $\alpha = -1$ then

$$\sup_{u\in(0,\infty)}\left|K^{-(\alpha+1+iu)}\frac{\varphi_{X_t}(u-(\alpha+1)i)}{(\alpha+1+iu)}\right| = +\infty.$$

Proof. In case $\alpha \neq -1$, one gets

$$\left| K^{-(\alpha+1+iu)} \frac{\varphi_{X_t}(u-(\alpha+1)i)}{(\alpha+1+iu)} \right| \leq K^{-(\alpha+1)} \frac{\varphi_{X_t}(-(\alpha+1)i)}{|\alpha+1|}.$$

As φ_{X_t} is continuous wherever it exists, the whole expression is and taking the right limit yields

$$\lim_{u \searrow 0} \left| K^{-(\alpha+1+iu)} \frac{\varphi_{X_t}(u-(\alpha+1)i)}{(\alpha+1+iu)} \right| = K^{-(\alpha+1)} \frac{\varphi_{X_t}(-(\alpha+1)i)}{|\alpha+1|},$$

and therefore the stated equation holds.
If $\alpha = -1$, then $|\alpha+1+iu| = |u|$, and with this the second equation follows.

$$\lim_{u \searrow 0} \left| K^{-(\alpha+1+iu)} \frac{\varphi_{X_t}(u-(\alpha+1)i)}{(\alpha+1+iu)} \right| = K^{-(\alpha+1)} \lim_{u \searrow 0} \frac{\varphi_{X_t}(-(\alpha+1)i)}{|u|} = +\infty$$

□

Lemma 5.5. *For $\mu \in \mathbb{R}$ define $f_\mu : \mathbb{R} \setminus \{-1\} \to \mathbb{R}^+$ as*

$$f_\mu(x) := \frac{e^{\mu x}}{|x+1|}. \tag{5.7}$$

Then

$$f''(x) > 0, \ \forall x \in \mathbb{R} \setminus \{-1\}.$$

Proof. At first, it is helpful to write this function in a different way.

$$f_\mu(x) := \begin{cases} \dfrac{\mathrm{e}^{\mu x}}{-(x+1)} & x < -1 \\ \dfrac{\mathrm{e}^{\mu x}}{x+1} & x > -1 \end{cases}$$

Now the second derivatives can be computed for each of these cases by using the quotient rule of derivation.

$$f''_\mu(x) = \begin{cases} \dfrac{\mathrm{e}^{\mu x}(\mu(x+1)-1)^2 + \mathrm{e}^{\mu x}}{-(x+1)^3} & x < -1 \\ \dfrac{\mathrm{e}^{\mu x}(\mu(x+1)-1)^2 + \mathrm{e}^{\mu x}}{(x+1)^3} & x > -1 \end{cases}$$

Combining these two cases gives a closed form for f'',

$$f''_\mu(x) = \frac{\mathrm{e}^{\mu x}(\mu(x+1)-1)^2 + \mathrm{e}^{\mu x}}{|x+1|^3}, \ x \in \mathbb{R} \setminus \{-1\}.$$

With this representation, obviously, $f''_\mu(x) > 0$ is satisfied for $x \neq -1$. □

The following lemma is an equivalent to Proposition 5.2 for the integrand of the first derivative of the call price with respect to the strike.

Lemma 5.6. *Let X be a random variable and φ_X its corresponding moment generating function, and let further α_- and α_+ be defined as in Proposition 5.2. Then $\alpha^* \in (\alpha_-, \alpha_+)$ minimizes*

$$\sup_{u\in(0,\infty)} \left| e^{-(\alpha+1+iu)k} \frac{\varphi_X(u-(\alpha+1)i)}{\alpha+1+iu} \right| \tag{5.8}$$

if and only if it is a solution to

$$\min_{\alpha\in[\alpha_-,\alpha_+]\setminus\{-1\}} \log M_X(\alpha+1) - \frac{1}{2}\log(\alpha+1)^2 - (\alpha+1)k. \tag{5.9}$$

Additionally assume that both bounds α_-, α_+ are finite.
Then the minimization problem in (5.9) *has a unique solution in each of the intervals*

$$\begin{aligned} &\circ\ [\alpha_-, -1) \text{ and} \\ &\circ\ (-1, \alpha_+]. \end{aligned} \tag{5.10}$$

Again, the following proof is based on the approach in [Pet14, Proposition 5.3.4] and has been adapted to this purpose.

Proof. At first, note that every strictly convex function has at most exactly one minimum. Therefore, showing that the function of the optimization problem is convex yields a part of the statement.
To do this, note that the function f_μ defined in (5.7) is convex regardless on the choice of μ. Therefore it is convex when setting $\mu^* := X(\omega) - k$ for each $\omega \in \Omega$.

$$f_{\mu^*}(\alpha) := \frac{\mathrm{e}^{(X(\omega)-k)\alpha}}{|\alpha+1|},\ \alpha \in \mathbb{R}\setminus\{-1\}$$

This is equivalent to the fact that for all $\lambda \in (0,1)$ and $\alpha \neq \beta \in \mathbb{R}\setminus\{-1\}$, f_{μ^*} satisfies the following inequation.

$$f_{\mu^*}(\alpha\lambda + \beta(1-\lambda)) < \lambda f_{\mu^*}(\alpha) + (1-\lambda) f_{\mu^*}(\beta)$$

Now define the function $J : \mathbb{R}\setminus\{-1\} \to \mathbb{R}^+$ as

$$J(\alpha) := \mathbb{E}[f_{\mu^*}(\alpha+1)] = \mathrm{e}^{-k(\alpha+1)} \frac{M_X(\alpha+1)}{|\alpha+1|},$$

then taking the expectation on both sides of the inequation and multiplying with the positive factor e^{-k} yields the convexity of J.

$$J(\alpha\lambda + \beta(1-\lambda)) < \lambda J(\alpha) + (1-\lambda) J(\beta)$$

This, together with Lemma 5.4 shows that there exist at most one optimal alpha in each interval stated in (5.10) for the expression in (5.8). As the logarithm is a strictly monotonic increasing function, this minimization problem is equivalent to getting the minima (if existing) of

$$\log(J(\alpha)) = \log M_X(\alpha+1) - \frac{1}{2}\log(\alpha+1)^2 - (\alpha+1)k.$$

Now the only thing that needs to be shown, is that there really exists a unique minimum in each of the intervals. This statement is trivial, if either α_- or α_+ equals -1, so this case is excluded. With this the following is always valid

$$\lim_{\alpha \searrow -1} J(\alpha) = \lim_{\alpha \nearrow -1} J(\alpha) = +\infty. \tag{5.11}$$

As the function J is continuous on each, $[\alpha_-, -1]$ and $[-1, \alpha_+]$ there also exists a minimum in both of these intervals, and with (5.11) it follows that there exist one in the half open intervals. □

A similar result for the second derivative can be obtained analogously.

Lemma 5.7. *Let X be a random variable with φ_X its corresponding moment generating function, and let further α_- and α_+ be defined as in Proposition 5.2. Then $\alpha^* \in (\alpha_-, \alpha_+)$ minimizes*

$$\sup_{u\in(0,\infty)} \left| e^{-(\alpha+2+iu)k} \varphi_X(u-(\alpha+1)i) \right| \tag{5.12}$$

if and only if it is a solution to

$$\min_{\alpha\in[\alpha_-,\alpha_+]} \log M_X(\alpha+1) - (\alpha+2)k. \tag{5.13}$$

Additionally assume that both bounds α_-, α_+ are finite.
Then the minimization problem in (5.9) has a unique solution on $[\alpha_-, \alpha_+]$.

Proof. At first it needs to be shown, that the following equation is valid.

$$\sup_{u\in(0,\infty)} \left| K^{-(\alpha+2+iu)} \varphi_X(u-(\alpha+1)i) \right| = K^{-(\alpha+2)} \varphi_X(-(\alpha+1)i)$$

Clearly

$$\left| K^{-(\alpha+2+iu)} \varphi_X(u-(\alpha+1)i) \right| \leq K^{-(\alpha+2)} \varphi_X(-(\alpha+1)i)$$

is valid by using properties of complex numbers. On the other hand, taking the limit for $u \searrow 0$ yields

$$\lim_{u\searrow 0} \left| K^{-(\alpha+2+iu)} \varphi_X(u-(\alpha+1)i) \right| = K^{-(\alpha+2)} \varphi_X(-(\alpha+1)i),$$

which proofs the required equation.
Now, according to the proof of Lemma 5.6 define a function $f : (\alpha_-, \alpha_+) \to \mathbb{R}^+$ as

$$f(\alpha) := \mathrm{e}^{-k(\alpha+2)+X(\omega)(\alpha+1)}.$$

Then this function clearly is strictly convex for each $\omega \in \Omega$. Then, taking the expectation of this functions retains this property. As

$$\mathbb{E}[f(\alpha)] = \mathrm{e}^{-k(\alpha+2)}\varphi_X(-(\alpha+1)i) = \mathrm{e}^{-k(\alpha+2)}M_X(\alpha+1)$$

and this function converges to infinity at both interval bounds a unique minimum exists on (α_-, α_+). Due to the fact that the logarithm is monotonically increasing, the minimum of this function is equal to the minimum of its logarithm, which was to be shown. □

Definition 5.8. Consider the setting of Proposition 5.2, Lemma 5.6 or Lemma 5.7, with $\alpha_j^*, j \in I \subseteq \{1,2,3\}$ the unique solutions on the corresponding intervals of the minimization problem. Then each $\alpha \in \{\alpha_j, j \in I\}$ with

$$F(\alpha) = \min_{j \in I} F(\alpha_j^*)$$

is referred to as *optimal alpha* for this minimization problem.

Now this lemma can be applied in Kou's model to get the optimal damping factor α for computing the local volatility.

5.1.1 Application in Kou's model

For the following section suppose Kou's model, defined in (3.5), as the fixed model defining the underlying. Then define the log underlying, $X := (X_t)_{t\in[0,T]}$ with characteristic function φ_{X_t},

$$\varphi_{X_t}(\theta) = \exp\left(t\left(i\theta b - \frac{1}{2}\theta^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - i\theta} + \frac{(1-p)\eta_2}{\eta_2 + i\theta} - 1\right)\right)\right),$$

where b is given by (4.7) with $\sigma > 0$. With this the characteristic exponent has the following form

$$\psi(\theta) = i\theta b - \frac{1}{2}\theta^2\sigma^2 + \lambda\left(\frac{p\eta_1}{\eta_1 - i\theta} + \frac{(1-p)\eta_2}{\eta_2 + i\theta} - 1\right).$$

Then all assumptions in Proposition 5.3 are satisfied, and the call price $C(K,t) \in C^{2,1}$ has the following derivatives.

$$\frac{\partial C}{\partial K}(K,t) = -\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(K^{-(\alpha+1+iu)}\frac{\varphi_{X_t}(u-(\alpha+1)i)}{(\alpha+1+iu)}\right)\mathrm{d}u$$
$$\frac{\partial^2 C}{\partial K^2}(K,t) = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(K^{-(\alpha+2+iu)}\varphi_{X_t}(u-(\alpha+1)i)\right)\mathrm{d}u$$
$$\frac{\partial C}{\partial t}(K,t) = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(-\psi(u-(\alpha+1)i)K^{-(\alpha+iu)}\frac{\varphi_{X_t}(u-(\alpha+1)i)}{(\alpha+iu)(\alpha+1+iu)}\right)\mathrm{d}u.$$

Now further set the parameters in this model as follows.

σ	λ	p	η_1	η_2
0,21	1,4	0,04	3,7	1,8

Table 5.1: Model parameters

For this distinct set of parameters, the call price surface and its derivatives are plotted as function of α in Figure 5.1.
With this figure, one can observe three main things. The three poles at the interval bounds, the convexity in between and rather small differences in value besides close to the poles. This means, that the choice of alpha has only a small impact on the absolute value of the integrand. By minimizing this function due to Lemma 5.6, one gets the optimal choice of alpha, $\alpha^* = 0.997$. Anyways, both the plot and the optimal damping factor depend on the (log-) strike and the time to maturity t.
The most important property seen in this figure is the discontinuity around $k = 0$. This behaviour can be explained by two things, either the pole for $\alpha = 0$ which makes such value an improper choice, and the asymptotic behaviour of $\frac{\partial C}{\partial K}(K,t)$ for $K \to 0$.
For this reason consider

$$\frac{\partial C}{\partial K}(K,t) = -\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(K^{-(\alpha+1+iu)}\frac{\varphi_{X_t}(u-(\alpha+1)i)}{(\alpha+1+iu)}\right)\mathrm{d}u$$

as given in Proposition 5.3. Then assuming $t \in \mathbb{R}^+ \setminus \{0\}$ fixed and taking the limit $K \to 0$ gives the following asymptotic behaviour

$$\lim_{K\to 0}\frac{\partial C}{\partial K}(K,t) \approx K^{-(\alpha+1)}.$$

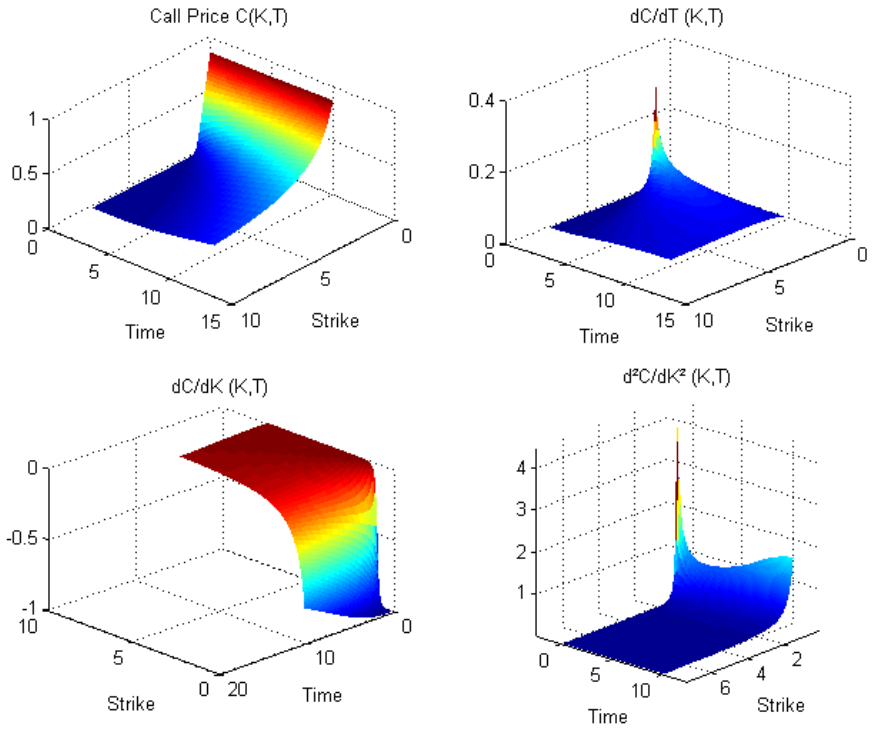

Figure 5.1: Call price surface and its derivatives in Kou's model.

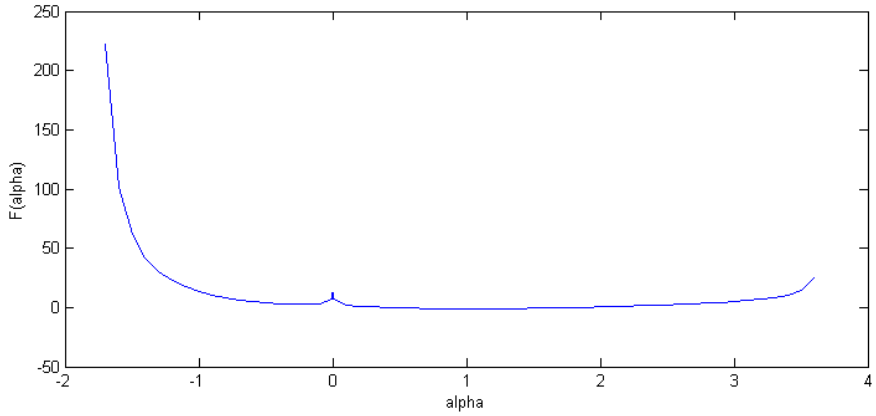

Figure 5.2: Optimizing function for $k = 1$, $t = 10$.

As mentioned before, the damping factor is only optimized for the integrand of $\frac{\partial C}{\partial K}$. Hence, to compute Dupire's local volatility with the highest numerical stability, one would tend to calculate an optimal damping factor for each

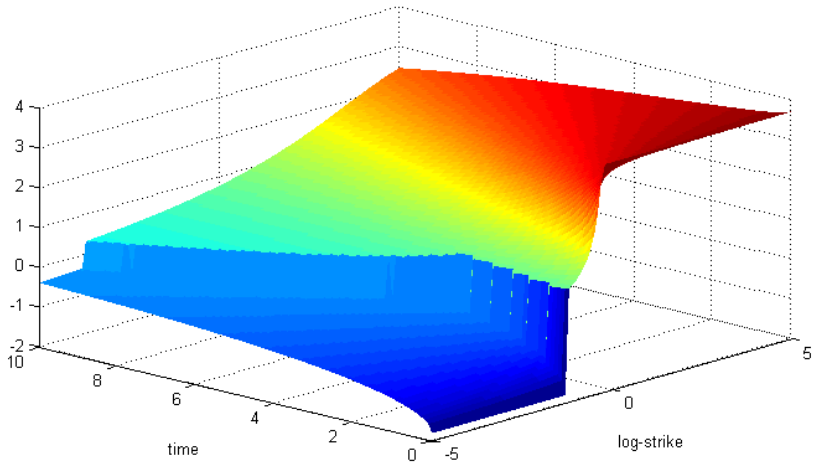

Figure 5.3: Optimal alpha: $\alpha^*(k,t)$

derivative in (4.2). Nevertheless, both $\frac{\partial^2 C}{\partial K^2}$ and $\frac{\partial C}{\partial T}$ are comparably flat as the function in Figure 5.3 so the choice of alpha is negligible as long as in a certain range. Furthermore getting a similar result to Lemma 5.6 is not that easy when optimizing $\frac{\partial C}{\partial T}$, and doing so would cause a remarkably higher computation time. For these two reasons Dupire's formula is calculated with one distinct alpha.
Once a distinct damping factor is chosen, one obtains the local volatility surface. Figure 5.4 shows the volatility surface in Kou's model with the parameters defined as in Table 5.1. It can be seen, that for $T \to 0$ the volatility surface has a blowup. This happens due to the fact that the underlying is a Levy process with a jump part. The jump part makes big differences possible, even in infinitesimal time intervals which results in high volatility as the local volatility model has to compensate this without jumps.

5.1.2 Monte Carlo simulation

Now, as the local volatility surface is computed numerically, the next step is to show that Theorem 4.2 is valid by numerical simulation. Hence, reconsider the main statement of this theorem.

For a given call price surface, and the resulting local volatility surface $\sigma^2(K,T)$ a random variable can be defined, by an SDE and a time-zero probability distribution, such that the original call price surface can be regained.

This means, that by Monte Carlo simulation with a sufficiently high amount

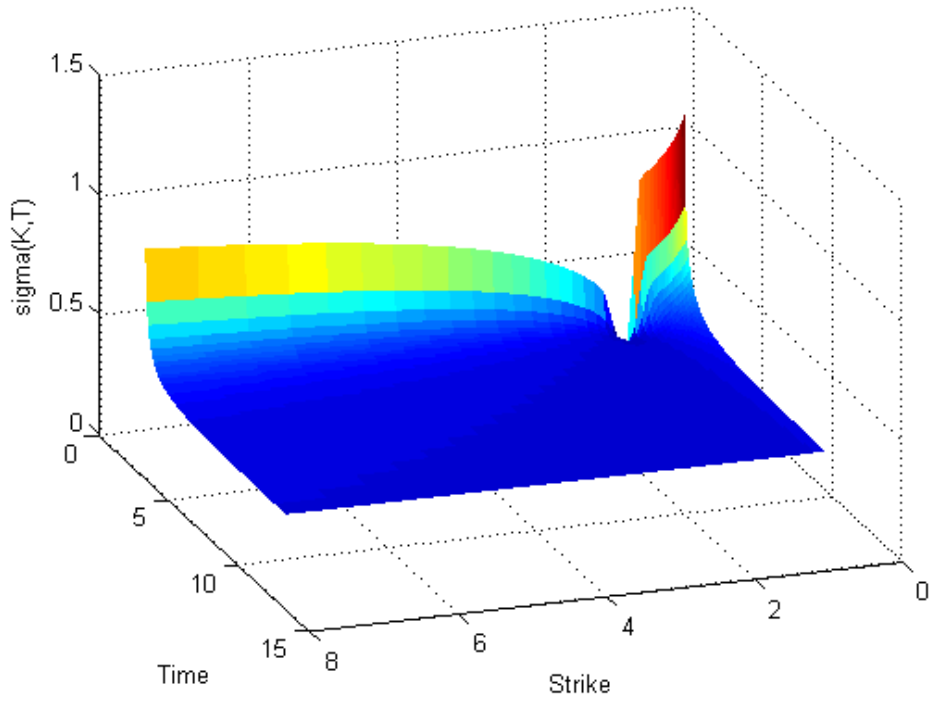

Figure 5.4: Local volatility surface in Kou's model.

of samples one should get exactly the same call prices, besides from numerical errors.
This simulation requires two steps of sampling, first the start value out its given distribution, and then the random variable out of the SDE.

Breeden and Litzenberger [BL78] showed, that the second derivative of a call price surface with respect to the strike equals the risk neutral probability density of the underlying multiplied by a factor. Once the probability density is known, samples of this distribution can be drawn using the approach stated below.

Inversion method

Sampling from a given probability distribution is done via the so called inversion method, described by the following theorem.

Theorem 5.9. *Let F be a probability distribution and define its generalized inverse by*

$$F^{-1}(u) := \inf\{x \in \mathbb{R} | F(x) \geq u\}.$$

If U is uniformly distributed on $[0,1]$, then $F^{-1}(U)$ has distribution function F.

Proof. To show this statement first note that

$$F^{-1}(U) \sim F \Leftrightarrow \mathbb{P}(F^{-1}(U) \leq x) = F(x).$$

Therefore it is sufficient to show that this equation holds.

$$\mathbb{P}(F^{-1}(U) \leq x) = \mathbb{P}(\inf\{y|F(y) \geq U\} \leq x) = \mathbb{P}(U \leq F(x)) = F(x)$$

□

With this result, one can easily sample from any kind of probability distribution once the cumulative distribution function (cdf) is known. For this purpose Remark 4.3 can be used to obtain the cdf and sample the initial value. The Matlab implementation of this is stated below.

```
function [X]=pdfrnd(x,cdf,size)
%x... values where pdf is calculated
%cdf...cdf(x)
%size .. sample size

cdf=cdf/max(cdf); %ensure cdf(end)=1

if cdf(1)>0
   cdf=[0,cdf];     %ensure cdf(1)=0
   x=[2*x(1)-x(2) x];
end

[cdf,idx,]=unique(cdf);
x=x(idx);

u=rand(size,1);
logical = arrayfun(@(s) any(s==cdf),u);

X=zeros(size,1);
X(not(logical))= interp1(cdf,x,u(not(logical)),'linear',0);

if any(logical)
    rd=zeros(sum(logical),1);
    i=1;
    for j = u(logical)
```

```
        idx= (j==cdf);
        rd(i)=y(idx(1))+(cd(idx(end))-cdf(idx(1)))*rand(1);
        i=i+1;
    end
    X(logical)=rd;
end
end
```

The Runge-Kutta scheme

Approximation of stochastic differential equations can be challenging numerically, especially when strong convergence is required. One famous way of doing this, is by using Taylor expansions to get time discrete approximations of the required SDEs.
The most popular examples are the Euler (- Maruyama) scheme and the Milstein scheme, where for the first one the order of strong convergence is 0.5, and for the latter 1 - see e.g. [KP92]. Hence the Milstein scheme should be preferred. However, this method requires computation of derivatives of the coefficient functions and is therefore discarded.

Consider the following stochastic differential equation

$$\mathrm{d}X_t = a(t, X_t)\mathrm{d}t + b(t, X_t)\mathrm{d}W_t \tag{5.14}$$

with initial condition $X_0 = x_0$, $x_0 \in \mathbb{R}^n$,

$$a : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n,$$
$$b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$$

and $W := (W_t)_{t\in[0,T]}$ a d dimensional Brownian motion.

Euler scheme:

For the SDE given in (5.14) the Euler-Maruyama method computes a discrete approximation in $N \in \mathbb{N}$ steps. The numerical solution $Y := (Y_n)_{n=0,\dots,N}$ is given as follows.

$$Y_0 := x_0 \tag{5.15}$$
$$Y_{n+1} := Y_n + a(\tau_n, Y_n)\Delta t + b(\tau_n, Y_n)\Delta W_n, \qquad n = 0, \dots, N-1 \tag{5.16}$$

where

$$\begin{aligned} \Delta t &:= \frac{T}{N}, \\ \tau_n &:= n\Delta t, && n = 0, \dots, N \text{ and} \\ \Delta W_n &:= W_{\tau_{n+1}} - W_{\tau_n}, && n = 0, \dots, N-1. \end{aligned}$$

The realization of this algorithm in Matlab is stated below.

```
function [X]=euler(mu,sigma,T,N,X0)
%calculate path of X: dX= mu(Xt,t) Xt dt + sigma(Xt,t) Xt dW
%where sims is the number of sample paths

sims=length(X0);
X=zeros(N+1,sims);
X(1,:)=X0;
b= zeros(1,sims);
a= b;

dt=T/N;
t=(0:N)'.*dt;

dW=sqrt(dt)*randn(N,sims);

for j=1:N
    a = arrayfun(@(x) mu(x,t(j)),X(j,:));
    b = arrayfun(@(x) sigma(x,t(j)),X(j,:));
    X(j+1,:)= X(j,:) + (a -1/2 * b.^2).*dt + b.*dW(j,:);
end

end
```

For non stochastic differential equations, the deterministic Runge-Kutta schemes avoid the use of derivatives. In the following a method will be introduced, which works similar to the these deterministic techniques, and will from now on be referred to as Runge-Kutta scheme. One method proposed by Platen [KP92] will be stated below. This method is based on the Milstein scheme, where the derivatives of the diffusion function are approximated through Taylor expansion.

Algorithm:

Then a discrete approximation of (5.14) $Y := (Y_n)_{n=0,1,\dots,N}$ in $N \in \mathbb{N}$ steps can be computed numerically, as follows.

$$Y_0 := x_0 \tag{5.17}$$

$$Y_{n+1} := Y_n + a(\tau_n, Y_n)\Delta t + b(\tau_n, Y_n)\Delta W_n + \tag{5.18}$$

$$\frac{1}{\sqrt{\Delta t}}(b(\tau_n, \overline{Y}_n) - b(\tau_n, Y_n))(\Delta W_n^2 - \Delta t),\ n = 0, \dots, N-1 \tag{5.19}$$

where

$$\begin{aligned}
\Delta t &:= \frac{T}{N}, \\
\tau_n &:= n\Delta t, && n = 0, \dots, N \\
\Delta W_n &:= W_{\tau_{n+1}} - W_{\tau_n}, && n = 0, \dots, N-1 \text{ and} \\
\overline{Y}_n &:= Y_n + a(\tau_n, Y_n)\Delta t + b(\tau_n, Y_n)\sqrt{\Delta t}, && n = 0, \dots, N-1.
\end{aligned}$$

Remark 5.10. Note, that like the Milstein scheme, the above presented method is an explicit scheme of strong order 1, and is therefore preferred to the Euler scheme.

This algorithm is again realized in Matlab, similar to the Euler method.

With either of these methods in combination with simulating the initial value, one can compute sample paths of the option's underlying.
When these algorithms are applied to exponential Levy models and their corresponding SDEs, it can be required to apply them on the log-underlying and taking the exponential afterwards to guarantee positivity. This can be done using Ito's formula for Levy processes.

Lemma 5.11. *Consider the following local volatility model for an underlying* S*.*

$$\begin{aligned}
\mathrm{d}S_t &= \sigma(t, S_t) S_t \ \mathrm{d}W_t, \ \textit{and} \\
S_0 &= s_0 \in \mathbb{R}^+
\end{aligned}$$

Then the model for the log-underlying $X_t := \log(S_t)$ *is given by*

$$\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}\sigma^2(t, e^{X_t})\mathrm{d}t + \sigma(t, e^{X_t}) \ \mathrm{d}W_t, \ \textit{and} \\
X_0 &= \log(s_0).
\end{aligned}$$

Proof. The equation for the initial value follows per definition.
The main statement of this lemma is an application of Ito's formula for Levy processes with $f : x \to \log(x)$. [Protter!] Then one obtains the following SDE.

$$\mathrm{d}X_t = \mathrm{d}(\log(S_t)) = \frac{1}{S_t}\mathrm{d}S_t + \frac{1}{2S_t^2}\mathrm{d}\langle S_t, S_t\rangle = -\frac{1}{2}\sigma^2(t, S_t)\mathrm{d}t + \sigma(t, S_t) \ \mathrm{d}W_t$$

Then using $S_t = \mathrm{e}^{X_t}$ yields the statement. □

The difference in these two algorithms mainly depends on the change of the Ito part $b(.,.)$ with respect to the underlying. Note further that in case of a deterministic function b, the expected difference is zero. This can be seen as the quadratic variation of the Brownian motion equals t.

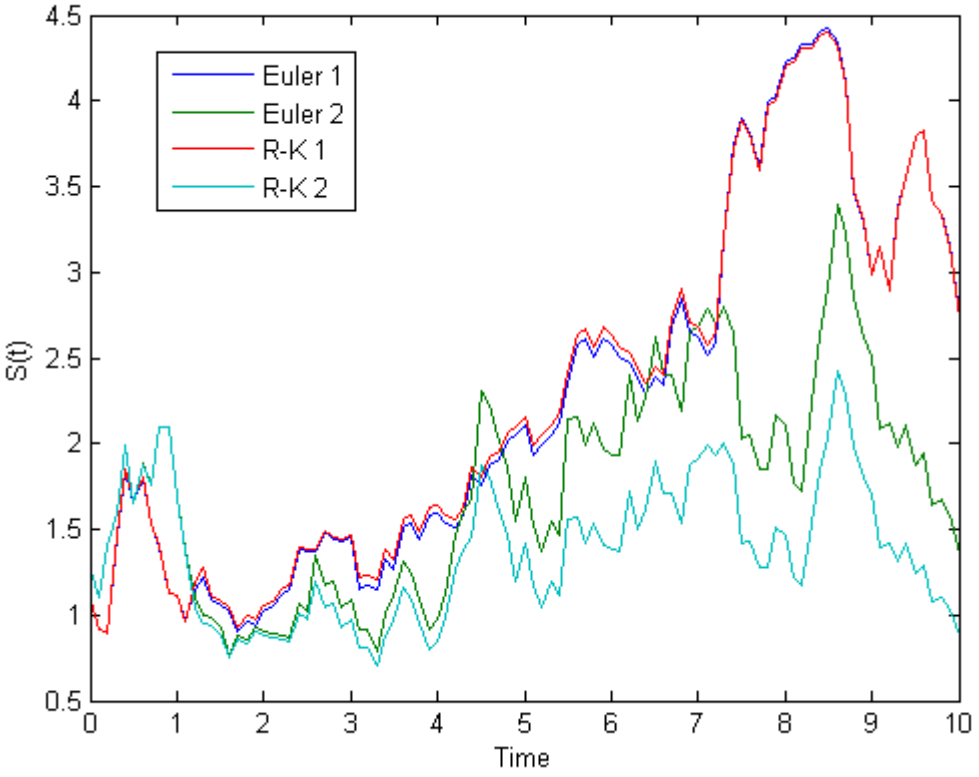

Figure 5.5: Comparison of Euler and Runge-Kutta algorithm in the local volatility model.

With this result, finally the underlying can be simulated such, that theorem 4.2 is satisfied and the original call prices can be obtained if the number of simulated paths is sufficiently high.
As mentioned before problems occur for T near zero, as the local volatility has a blow-up. This is avoided by time shifting the whole process. With this, the simulation starts at an arbitrary point in time $\epsilon > 0$. The approach is split into four steps stated below.

- At first a small time shift parameter $\epsilon > 0$ is chosen.

- Then, using the result from Breeden and Litzenberger [BL78] one gets the law of the underlying at time ϵ.
- Set S_ϵ as initial value and simulate the local volatility model from this time on.
- Then Theorem 4.2 ensures that for a sufficient number of simulations the original call price surface is obtained.

This approach is realized in Matlab including both, the Euler-Mayurama and Runge-Kutta simulation method. Note that the initial value is simulated for each realization, resulting in different starting points. Some sample paths from this simulation are plotted below using the Runge-Kutta scheme with $\epsilon = 0.1$.

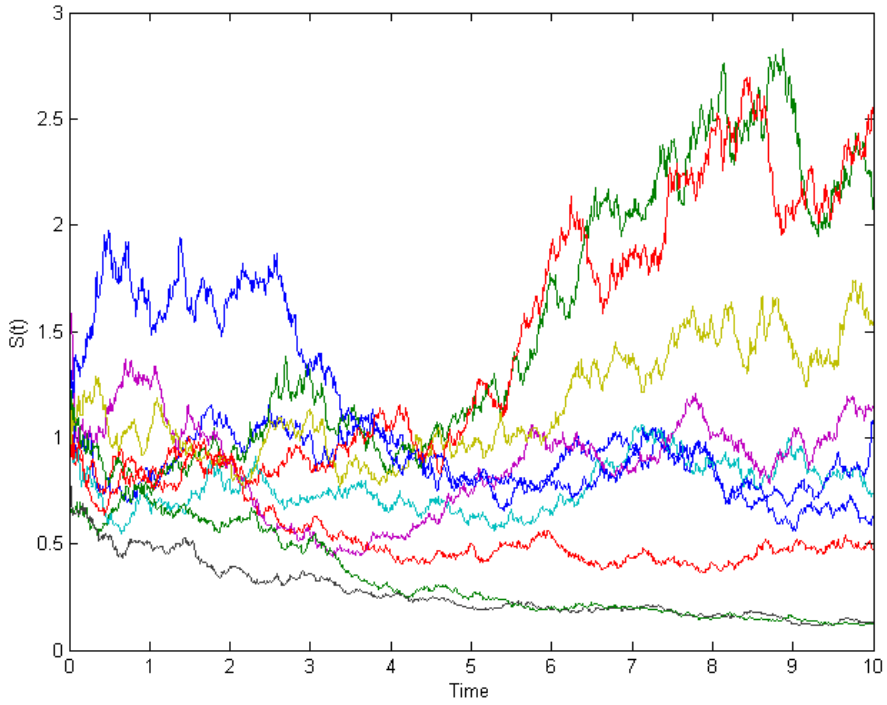

Figure 5.6: Sample paths generated from a local volatility model in Kou's model.

5.2 Backtesting

In order to guarantee the correct computation of the local volatility for a Kou-modelled call price surface, it is necessary to perform some test. One way of doing this is by showing that Theorem 4.2 holds. Anyways, this does neither prove that computation is done correctly, nor does it give an estimate for the numerical error. Getting these results is hard in general for Kou's model as calculating values of the local volatility is only possible by numerical methods due to the integral representation.
Nevertheless, it is possible to test the algorithm for a special setting of parameters - namely by setting the Compound Poisson intensity $\lambda = 0$. Doing this, the modelled process stated in (3.6) reduces to a geometric Brownian motion. Hence the Black-Scholes model can be applied, and the Call price surface and all derivatives, obtained by Fourier inversion, can be compared to the ones given by Black-Scholes formulas.

Proposition 5.12 (Black-Scholes)**.** *Let $(S_t)_{t\in[0,T]}$ denote the underlying of a european call option in Black-Scholes model, and $r > 0$ be the short rate. Then the arbitrage free price of this call option at time $t < T$ with strike $K > 0$ and maturity T in the Black Scholes model, is given by*

$$\begin{aligned} C_t &= c(K, S_t, T-t, \sigma_{BS}, r) = \\ &= S_t N(d_1(S_t, T-t)) - K e^{-r(T-t)} N(d_2(S_t, T-t)), \end{aligned} \tag{5.20}$$

where $N : \mathbb{R} \to [0,1]$ is the standard Gaussian cumulative distribution function, and $d_{1,2}$ are given by

$$d_{1,2}(s,\tau) = \frac{\log(\frac{s}{K}) + (r \pm \frac{1}{2}\sigma_{BS}^2\tau)}{\sigma_{BS}\sqrt{\tau}}.$$

Proof. See for example [BS73] or [MR97, Theorem 3.1.1]. □

Proposition 5.13. *Let $c(K, s, T, \sigma_{BS}, r)$ denote the price of a call option in Black-Scholes model stated in (5.20). Then it is twice differentiable with respect to the strike K, and once with respect to time T, and the corresponding derivatives are given as follows.*

$$\begin{aligned} \frac{\partial c}{\partial K}(K, s, T, \sigma_{BS}, r) &= -e^{-r(T-t)} N(d_2) \\ \frac{\partial^2 c}{\partial K^2}(K, s, T, \sigma_{BS}, r) &= n(d_2) \frac{e^{-r(T-t)}}{\sigma_{BS}\sqrt{T-t}K} \\ \frac{\partial c}{\partial T}(K, s, T, \sigma_{BS}, r) &= \frac{s\sigma_{BS}}{2\sqrt{T-t}} n(d_1) - Kr e^{-r(T-t)} N(d_2) \end{aligned}$$

Proof. These formulas follow by differentiation of (5.20), or see [MR97, Section 3.1.11] □

As mentioned before, setting $\lambda = 0$ in Kou's model results in Black-Scholes model. Hence, the call prices and all derivatives obtained by Fourier inversion should equal the ones given by the formulas stated in Propositions 5.12 and 5.13.
Comparison of the Call price surface and its derivatives are stated below for Kou's model with the setting described in Table 5.2.

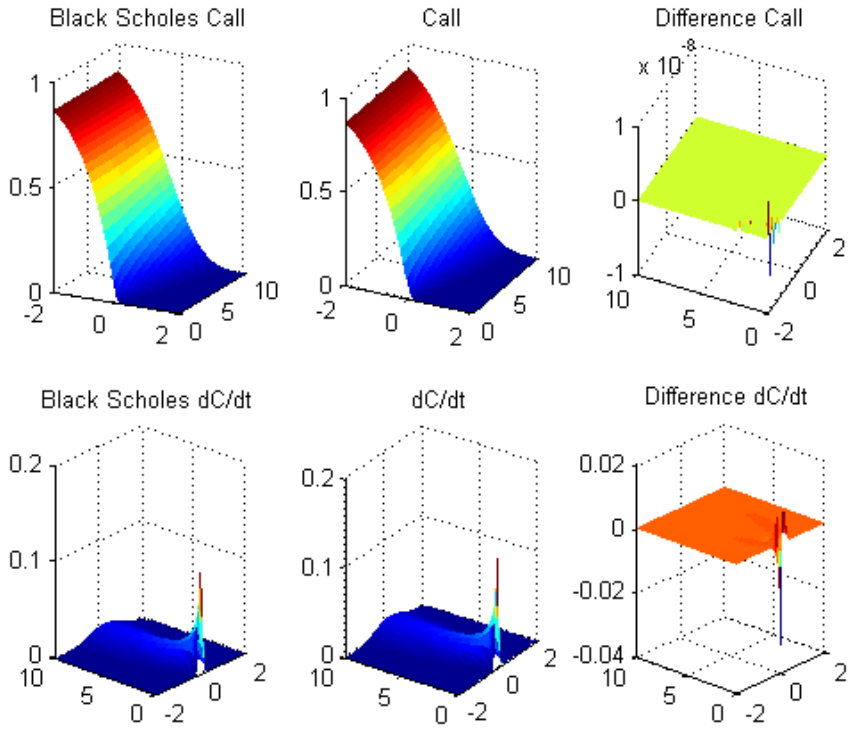

Figure 5.7: Comparison of Black Scholes model and Fourier option pricing.

Figure 5.7 shows that the differences between the Black-Scholes formula and the call price surface obtained by Fourier inversion are acceptable. Even the differences in the derivatives with respect to time can be explained by numerical errors.
This is an indication, that the algorithm works as expected for $\lambda = 0$.

Another way to test the quality and accuracy of the numerical method, is to compare the local volatility surfaces itself.

The local volatility surface itself, $\sigma(K,T)$ is given by Dupire's formula stated in (4.2). Then, due to Theorem 4.2, the stochastic process $(S_t)_{t\in[0,T]}$ defined by the SDE

$$\mathrm{d}S_t = S_t\sigma(S_t,t)\mathrm{d}W_t$$

rebuilds this original call price surface.
Applying this to the Black-Scholes model, where the underlying S^{BS} is given by $\mathrm{d}S_t^{BS} = S_t^{BS}\sigma_{BS}\mathrm{d}W_t$, should return the same volatility σ_{BS} as local volatility. This means that the local volatility obtained by the algorithm should be a flat plane equal the initial volatility,

$$\sigma(K,T) = \sigma_{BS}.$$

This gives evidence that the volatility surface is computed correctly, not considering numerical errors. Then Monte Carlo simulation with a sufficiently high amount of paths should yield the original call price surface for any parameter set, especially including the more interesting case with $\lambda \neq 0$.

Backtesting for $\lambda \neq 0$

As mentioned above, the previous tests give evidence, that the regularization method in [FGY13] can be applied numerically. To show this, all numerical methods and results mentioned before need to be combined. For this purpose, let the set of parameters be as stated in Table 5.1, and thus $\lambda \neq 0$.
At first the call price surface and all required derivatives are computed, using the choice of the damping factor as suggested in Section 5.1. Then a time shifting parameter needs to be chosen, $\epsilon := 0.1$ and the start value of the time shifted process, S_0^ϵ can be simulated $N \in \mathbb{N}$ times, where N is the number of simulations, with

$$\mathbb{P}[S_0^\epsilon \in \mathrm{d}K]/\mathrm{d}K = \partial_{KK}C(K,\epsilon).$$

Having done this, the time horizon $T^* > 0$, and a step size Δt for the Runge-Kutta scheme have to be set. This results in N paths of $(S_t^\epsilon)_{t\in[0,T^*]}$, and with this call prices for each strike $K > 0$ and time to maturity $0 < T \leq T^*$ can be estimated as arithmetic mean of the call prices of the individual simulations,

$$C(K,T) \approx \sum_{j=1}^{N} \frac{(S_T^{\epsilon,j} - K)^+}{N},$$

with $S^{\epsilon,j}$ the $j-th$ simulation of S^ϵ.

The following figures are created in Kou's model with the parameter set stated in Table 5.1, with step size $\Delta k = 0.1$.

Figure 5.8 below gives a comparison of the simulated call prices and the initial call prices as a function of the log strike for $T = 1$ with step size. The approximation error observed stems from two things, both the low number of simulations and errors due to numerical integration. The left plot compares the simulated call prices to the time shifted original call prices whereas the right plot gives a comparison of the non shifted ones. In particular, this means that the left side illustrates Theorem 4.2 and the right side Theorem 4.1 for one distinct ϵ.

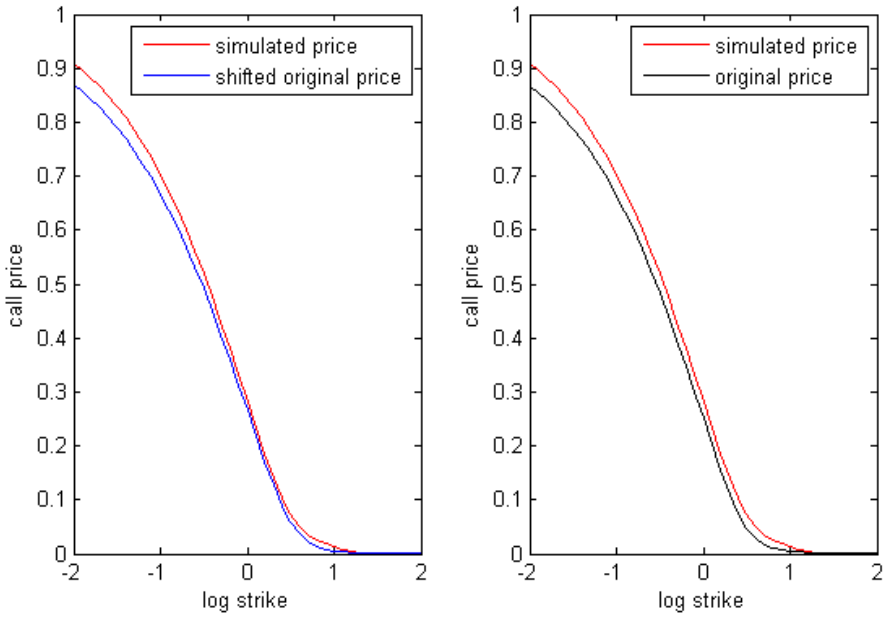

Figure 5.8: Comparison of call prices for $T = 1$ and $\epsilon = 0.1$ with 5000 simulations.

The relative error of the simulated call prices and the time shifted ones is plotted in the Figure 5.9. The increase for large log strike can be explained due to the fact, that the simulated call price converges to zero more slowly than the original price.
Figure 5.10 shows the initial call price surface compared to simulated ones with different simulation sizes. One can see, that with rising number, the error decreases, and the call price surface obtained by Monte Carlo simulation converges to the original one.

Theorem 4.1 requires to take the limit $\epsilon \to 0$ to obtain the exact call prices. To show numerically that the call prices converge to the original prices for

$\epsilon \to 0$, Figure 5.11 compares this application for three different time shifting parameters ϵ.

Again, the relative errors between each approximation in Figure 5.11 and the original non shifted call prices are illustrated in Figure 5.12.
This shows that Theorem 4.2 is applicable numerically, and gives a procedure to use Dupire's formula even in a jump diffusion setting.

σ	λ	p	η_1	η_2
0,21	0	0,04	3,7	1,8

Table 5.2: Model parameters

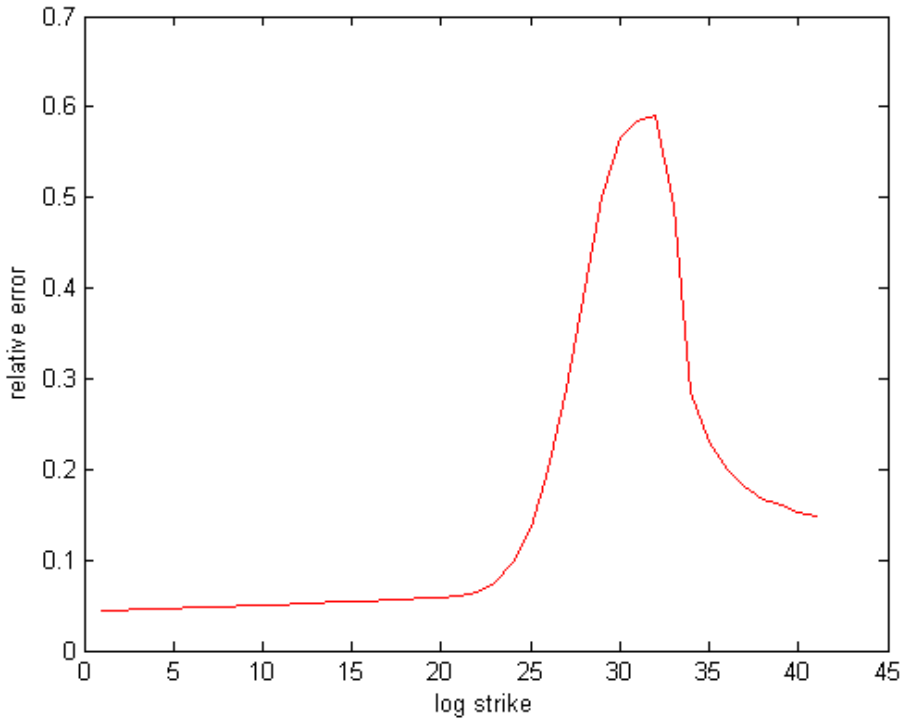

Figure 5.9: Relative error to the time shifted call prices with 5000 simulations.

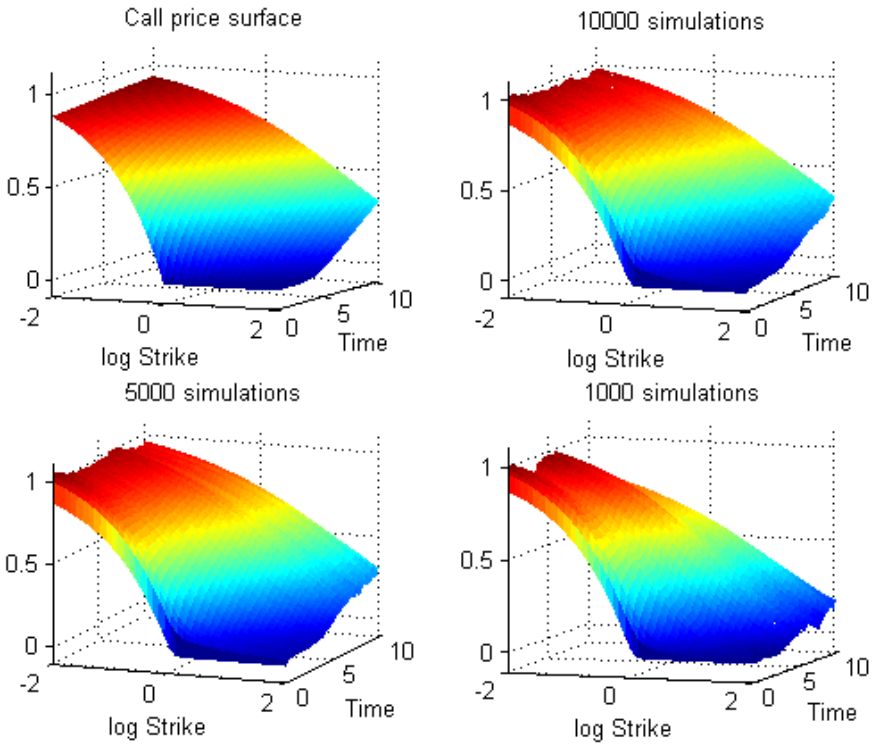

Figure 5.10: Simulated Call price surfaces in Kou's model.

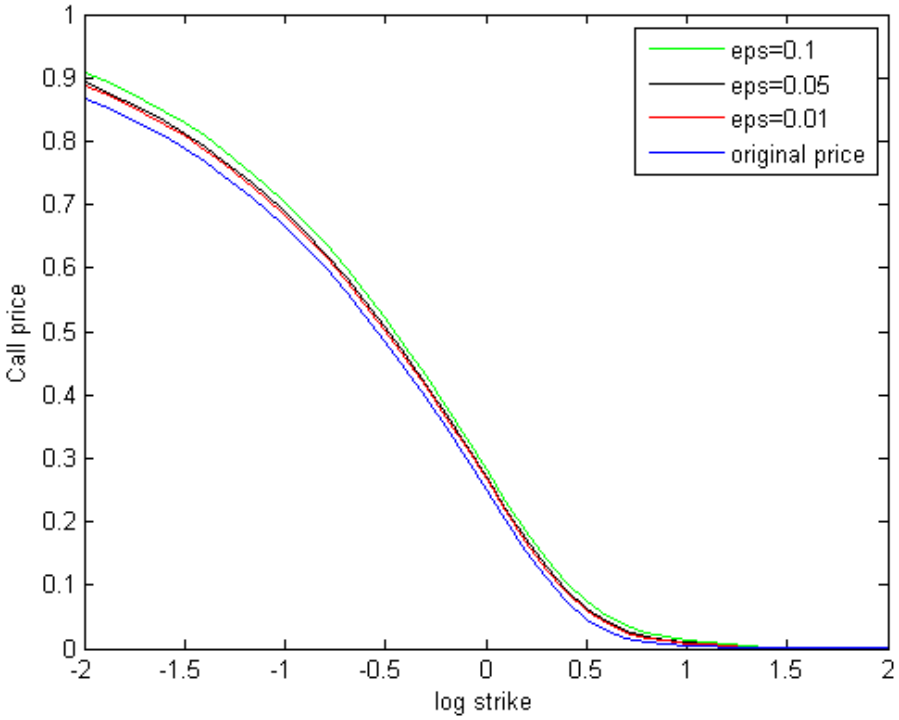

Figure 5.11: Comparison of call prices for $T = 1$ and different ϵ with 5000 simulations.

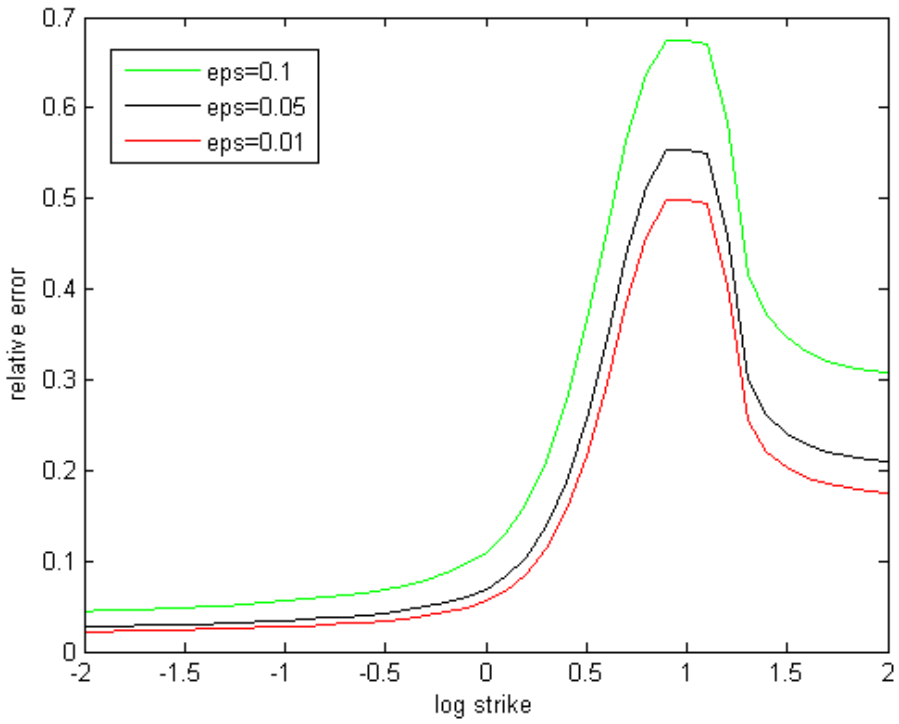

Figure 5.12: Relative errors for different ϵ with 5000 simulations.

Conclusio

Based on the results of Gerhold et al. in [FGY13], a numerical implementation of their procedure is presented. In particular, a regularization of Dupire's local volatility in the presence of jumps is applied numerically to Kou's model. To be able to do so, a rigorous derivation of a call price surface in this very model is done and representations for some derivatives of this surface are stated. This yields Dupire's local volatility surface stemming from Kou's model, which defines the according local volatiliy diffusion process. Due to [FGY13], the process stemming from this SDE with a stochastic initial value regains the original call price surface.
For the numerical implementation of this procedure, a sampling technique from SDEs is required. Hence, Euler's method and the Runge-Kutta scheme are introduced and compared. Further, to get sample initial values the inversion method is presented. These two results are then combined to get sample paths of the reproduced stochastic process, and Monte Carlo simulation is used to show that the original call prices can be obtained.
Additionally, a numerical problem is addressed that occurs in numerical integration and an improvement of this issue is stated and the numerical implementation is backtested to prove its correctness.

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