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DISSERTATION

Martingale Decomposition Theorems and the Structure of No Arbitrage

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Kurzfassung der Dissertation

Im ersten Kapitel beweisen wir einen Martingalzerlegungssatz für beliebige lokale Martingale. Wir vergleichen diese Zerlegung, die wir Radon–Nikodym–Zerlegung nennen, mit der wohlbekannten Kunita–Watanabe–Zerlegung. Anhand verschiedener Beispiele illustrieren wir, dass die Kunita–Watanabe–Zerlegung im Allgemeinen nicht existiert. Nichtsdestotrotz, die Radon–Nikodym–Zerlegung existiert, und wir geben diese für die speziellen Beispiele an.

In Kapitel 2 widmen wir uns der Struktur lokal quadratintegrierbarer Semimartingale. In mehreren Strukturtheoremen beleuchten wir die Verbindung zwischen der Struktur lokal quadratintegrierbarer Semimartingale einerseits und der Struktur strikt positiver σ–Martingaldichten andererseits. Während die Struktur lokal quadratintegrierbarer Semimartingale mit Hilfe sogenannter Strukturbedingungen beschrieben werden kann, lässt sich die Struktur der σ–Martingaldichten mit Hilfe verschiedener Martingalzerlegungsätze beschreiben. Wir vergleichen diese neuen Strukturbedingungen mit der wohlbekannten Strukturbedingung **(SC)** und überdies mit der schwachen Strukturbedingung **(SC′)**. Anhand zahlreicher Beispiele illustrieren wir, wie diese neuen Strukturbedingungen verwendet werden können, um strikt positive σ–Martingaldichten zu finden.

Im dritten Kapitel wenden wir uns der Modellierung mehrerer Phänomene zu, die im Zusammenhang mit der Anwesenheit großer Händler in Finanzmärkten stehen. Um diese Phänomene sauber trennen zu können, orientieren wir uns am Baukastenprinzip. Hierbei steht jeder einzelne Baustein für ein im Zusammenhang mit der Anwesenheit eines großen Händlers auftretendes Phänomen. Wir konzentrieren uns hier auf zwei Phänomene. Das erste betrifft die Art und Weise, wie ein großer Händler Einfluss auf den Preisprozess nehmen kann. Um ein Finanzmarktmodell mit großem Händler einführen zu können, das nicht im Widerspruch zu gängigen *no–arbitrage*–Annahmen für den kleinen Händler steht, widmen wir uns im zweiten Teil des dritten Kapitels diesem Phänomen. Schließlich untersuchen wir das Nutzenmaximierungsproblem des großen Händlers. Dabei stellt sich heraus, dass ein großer Händler, trotz erfüllter *no–arbitrage*–Bedingungen, einen Finanzmarkt destabilisieren kann.

Abstract

In Chapter 1, we provide a martingale decomposition theorem for arbitrary local martingales. Moreover, we compare this decomposition, named *Radon–Nikodym decomposition*, to the well known Kunita–Watanabe decomposition. Furthermore, we give examples in which the Kunita–Watanabe decomposition does not exist. Finally, we provide the Radon–Nikodym decomposition in these particular examples.

Chapter 2 is dedicated to *structure conditions* for locally square–integrable semimartingales. In several *structure theorems*, we highlight the connection between the structure of locally square–integrable semimartingales, encoded in different *structure conditions*, and different *martingale decomposition theorems* of strictly positive σ–martingale densities with respect to the local martingale part of the semimartingale under consideration. We compare these new structure conditions to the well known structure condition **(SC)** and the weak structure condition $\mathbf{(SC')}$. Through numerous examples we highlight how these new structure conditions can be used in order to find strictly positive σ–martingale densities.

In Chapter 3, we provide a modular model approach to large traders. The idea is to 'decompose' the different phenomena, related to the presence of a large trader in a financial market, into several modules. Here, we consider the 'price module' and the 'no arbitrage for the small trader' module. In the first one, we provide a flexible model that allows us to model the impact of the large trader on the price process. In the second module, we provide minimal assumptions that ensure that the turbulences, caused by the large trader's actions, do not lead to arbitrage opportunities for the small trader. With the help of the structure condition **(SC)**, we provide sufficient conditions that ensure that these results hold for a large class of large trader strategies. Finally, we consider the large trader utility maximization problem. We discover new phenomena that reveal that the presence of a large trader might destabilize the financial market. These phenomena appear even though the large trader strategy is not an arbitrage strategy in the sense of the classical no arbitrage condition.

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Contents

Introduction

"It is by logic that we prove, but by intuition that we discover.
To know how to criticize is good, but to know how to create is better."

H. Poincaré
in "Science and Method"

"[...] the Black–Scholes model, which has, since it was first proposed by Samuelson [44], established itself as the undoubted benchmark model and changed the whole industry, as was pointed out in the justification of the Nobel Foundation for awarding the 1997 Nobel prize in economics to F. Black and M. Scholes, two of the main contributors (along with R. Merton) of the theory behind the model [21]. The model was the first to offer a *convincing* principle to find unique option prices based on the argument of no arbitrage."[1] Here, we want to add that the principle is, in this particular model, *intuitive*, too.

On the one hand, this principle is convincing, as it can be proven by *logic*. On the other hand, this principle is *intuitive* due to the 'geometry' provided by the underlying Brownian motion. The Black–Scholes model does not only allow for a unique price for an option, but it also allows for a complete replication of the option. The latter is a consequence of a martingale representation theorem for the Brownian motion. It allows for replicating the payment stream of an option without making any hedging error at all. Moreover, as the price process in the Black–Scholes model is continuous, it has to satisfy the so–called structure condition **(SC)** to allow for the existence of an equivalent local martingale measure. The structure condition in turn relates the price process in an (almost) direct way to the 'convincing principle to find option prices', the equivalent local martingale measure. Again, this connection is provided by a *structure theorem* that relates the 'geometry of the price process', encoded in the structure condition **(SC)**, to the 'geometry of the equivalent local martingale measure'. The geometry of the latter is essentially provided by its Kunita–Watanabe decomposition with respect to the local martingale part of the price process; see [12]. Hence, the Black–Scholes model is a good mathematical instrument for the financial engineer in the sense of Poincaré: "The engineer must receive a complete mathematical training, but of what use is it to be to him, except

[1] Blum; [8].

to enable him to see the different aspects of things and to see them quickly? He has no time to split hairs. In the complex physical objects that present themselves to him, he must promptly recognize the point where he can apply the mathematical instruments we have put in his hands. How could he do this if we [*mathematicians*] left between the former and the latter that deep gulf dug by the logicians?"[2]

Thus, the Black–Scholes model is, from an *intuitive*, as well as from a *logical* point of view, a beautiful mathematical instrument for a financial engineer. But, there are also downsides. Two, amongst others, are the normally distributed logarithmic returns and the absence of jumps. As pointed out in [24], these facts lead to an underestimation of rare events.

To overcome these downsides, one has to consider far more general processes. Thanks to the *Fundamental Theorem of Asset Pricing* **(FTAP)** [17], the connection between a no arbitrage principle on the one hand, and the existence of a convincing pricing operator on the other, remains stable for the large class of semimartingale price processes.

Unfortunately, this is not true for the *geometric interpretation* mentioned in the context of the Black–Scholes model. Although the *geometric connection*, provided by the structure theorem of Choulli and Stricker [12], remains stable for a sufficiently large class of semimartingales, it *neither* provides an *intuitive* link to a *strictly positive* pricing operator, *nor* does it ensure the existence of the latter. In this regard, the **(FTAP)**, although undoubtedly correct due to a 'proof by logic', lacks a 'simple proof by *intuition*' as in the Black–Scholes model.[3] But, as Poincaré pointed out: "It is through it [*intuition*] that the mathematical world remains in touch with the real world, and even if pure mathematics could do without it [*intuition*], we should still have to have recourse to it to fill up the gulf that separates the symbol from reality. The practitioner will always need it [*intuition*], and for every pure geometrician[4] there must be a hundred practitioners."[5]

The purpose of the first part of this thesis is, in the words of Poincaré, to provide first steps towards a recourse to intuition to fill up the gulf that separates the symbol from reality.

In order to do so, we have to adapt the ideas of the structure theorem by Choulli and Stricker to not necessarily continuous semimartingales. On the one hand, we have to look at special semimartingales and their unique decompositions in a dif-

[2]Poincaré; [41]. Here, the logicians are those that mainly focus on a logical proof and neglect the intuition behind it.

[3]This is an allusion to the quotation of Poincaré at the beginning of this introduction! In no way this statement means to degrade the great achievement that the rigorous proof of the **(FTAP)** constitutes!

[4]In the context of this thesis, the term 'pure geometrician' should be thought of as 'a mathematician in the (academic) field of mathematical finance'.

[5]Poincaré; [41].

ferent light. In which sense are these decompositions unique? Is it possible to look at this decomposition from different points of view to end up with more than one decomposition of the same special semimartingale? The answer to this question will lead us to several new structure conditions. On the other hand, we need a decomposition theorem for arbitrary local martingales in order to provide a link between strictly positive σ–martingale densities and the semimartingale under consideration. These links are provided in several *structure theorems*. Each of them focuses on a different geometric aspect of the connection between semimartingales on the one hand, and σ–martingale densities on the other hand. Moreover, each of these structure theorems serves as a tool that helps to find strictly positive σ–martingale densities. Through numerous examples, we highlight the different mechanisms of these structure conditions and how powerful they might be, if they act in concert.

In the second part of this thesis, Chapter 3, we leave the pricing theory of frictionless small trader markets behind. Several studies, amongst them [32], point out that, in general, the competitive market paradigm is not justified. As a consequence, there has to be a large trader (or a group of small traders that act in concert) that has an impact on the evolution of the price process. Our goal is to incorporate different aspects of this impact into different modules. Assembling these modules leads to the large trader modular model. This modular model approach allows us, for instance, to incorporate different phenomena such as liquidity risks, related to market depth and market resiliency, into the model. The famous Almgren–Chriss model [2, 1], which seems to be the benchmark large trader model, is one particular example that is covered by our modular model approach.

Apart from modelling different types of liquidity risk, we address the question of whether or not the presence of a large trader might lead to arbitrage opportunities for the small traders. It is remarkable that, in order to extend certain no arbitrage assumptions for the small trader to a large set of large trader strategies, the structure conditions are, again, a powerful tool to achieve this goal.

1. A martingale decomposition theorem

1.1. Introduction

In [37], Kunita and Watanabe provide a decomposition theorem for square–integrable martingales. This decomposition is unique and characterized in the following way. Given two square–integrable martingales N, M, the theorem ensures the existence of a predictable process $\tilde{\lambda}$ and a martingale $\tilde{L}$ such that

$$N = \int \tilde{\lambda} \, dM + \tilde{L},$$

where $[\tilde{L}, M]$ is a martingale. Moreover, this representation property is a symmetric property, i.e. there also exists a decomposition of M with respect to N. This result can be extended to the case of locally square–integrable martingales by localization. The proof for the existence of this decomposition strongly relies on the Hilbert space structure of the space of square–integrable martingales. Therefore, it is not surprising that the class of locally square–integrable martingales is essentially the most general class that allows for such a decomposition for all its elements. This fact is proved by Ansel and Stricker [3] by means of counterexamples. In order to provide a similar decomposition theorem for arbitrary local martingales, one has two options. Either one changes the assumptions on the orthogonality of $\tilde{L}$ and M, or one uses a more general concept of stochastic integration.

We take the second option. More precisely, we use the compensated stochastic integral, introduced by Meyer in [40], to provide an integral decomposition for arbitrary local martingales N and M of the following form. There exists an optional integrand H such that the compensated stochastic integral $H \underset{c}{\bullet} M$ exists and

$$N = H \underset{c}{\bullet} M + L, \tag{1.1}$$

where $[L, M]$ is a σ–martingale. The integrand in this decomposition arises from the relation of N and M in a very natural way. It is the Radon–Nikodym derivative of the quadratic covariation of N and M with respect to the quadratic variation of M, i.e. $H.[M] = [N, M]$. For this reason we call the decomposition (1.1) the

Radon–Nikodym decomposition of N with respect to M. Moreover, we provide a more detailed characterization of (1.1). If we denote by

$$N^c = \int \tilde{\lambda}\, dM^c + \tilde{L}$$

the Kunita–Watanabe decomposition of the continuous local martingale part N^c of N w.r.t. M^c, then

$$(H \underset{c}{\bullet} M)^c = \int \tilde{\lambda}\, dM^c \quad \text{and} \quad L^c = \tilde{L}$$

holds up to indistinguishability.

The chapter is organized as follows. In Section 2, we give an overview of the compensated stochastic integral of Meyer. Besides, we fix notations and provide a short collection of definitions, including σ–martingales and compensable processes. Furthermore, we recall some well known results related to these processes. Section 3 is the main part of this first chapter. Apart from the results on the Radon–Nikodym decompositions for local martingales, it contains an inequality (needed for the proof of existence of the Radon–Nikodym decomposition) being of interest in its own right. In Section 4, we compare the Kunita–Watanabe decomposition and the Radon–Nikodym decomposition for locally square–integrable martingales. Finally, Section 5 contains the examples presented in [3]. They highlight that the Kunita–Watanabe decomposition does not exist in general. We complete these examples by computing the corresponding Radon–Nikodym decompositions.

1.2. Definition and preliminary results

Throughout this chapter we consider a complete stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions. Furthermore, all semimartingales are assumed to have càdlàg paths. For unexplained notation we refer to [27].

In this section, we briefly recall the definition of the stochastic integral w.r.t.[1] local martingales for predictable integrands. Moreover, we provide a more detailed overview of stochastic integration theory w.r.t. local martingales for optional integrands. This stochastic integral is referred to as compensated stochastic integral. In the third subsection, we introduce a quotient space which we need in order to define a certain Hilbert space in Section 1.4. Finally, we provide definitions and basic results on σ–martingales and compensable processes in the last subsection.

[1] abbr.: with respect to

1.2.1. Stochastic integral w.r.t. local martingales

We briefly recall the classical definition of stochastic integrals w.r.t. local martingales for predictable integrands. For details we refer to [27].

Definition 1.1 ([27, 9.1 Definition])**:** *Let M be a local martingale, and H be a predictable process. If there exists a local martingale L such that*

$$[L, N] = H.\,[M, N] \tag{1.2}$$

holds for every local martingale N, then we say that H is integrable *w.r.t. M in the domain of local martingales (or simply,* integrable*), and L (it is uniquely determined by the above equation) is called the* stochastic integral *of H w.r.t. M, and denoted by $H.M$. The collection of all predictable processes which are integrable w.r.t. M is denoted by $L_m(M)$.*

The elements of $L_m(M)$ can be characterized in the following way.

Theorem 1.2 ([27, 9.2 Theorem])**:** *Let M be a local martingale, and H be a predictable process. Then $H \in L_m(M)$ if and only if $\sqrt{H^2.[M]} \in \mathcal{A}^+_{loc}$.*

We end this subsection with the following lemma.

Lemma 1.3: *Let $H, K \in L_m(M)$. Further, let $(T_n)_{n\geq 1}$ be a non–decreasing sequence of stopping times that tends to ∞ a.s. and that localizes $\sqrt{H^2.[M]}$ and $\sqrt{K^2.[M]}$. If $E\left[\sqrt{(H-K)^2.[M^{T_n}]_\infty}\right] = 0$ for all $n \geq 1$, then $H.M$ and $K.M$ are indistinguishable.*

Proof: The Kunita–Watanabe inequality, see [27, 8.3 Theorem], and Definition 1.1 ensure that $[H.M - K.M, N] = 0$ for all $N \in \mathcal{M}_{loc}$. Due to [27, 7.36 Theorem], $H.M$ and $K.M$ are indistinguishable. □

1.2.2. Compensated stochastic integral w.r.t. local martingales

There are various approaches to extend the definition of stochastic integrals to a broader class of integrands. Here, we recall the definition of the compensated stochastic integral as given in [27, Chapter IX §2.]. Although the integral is defined for progressive integrands, we are only interested in optional integrands. For a more detailed analysis we refer to [20, Chapter VIII §2.] and the French literature [40, Chapitre V]. We start the overview with the following theorem. It basically says that the stochastic integral and the compensated stochastic integral are the same for continuous local martingales.

Theorem 1.4 ([27, 9.6 Lemma])**:** *Let M be a continuous martingale, and H be an optional process. Then there exists a $L \in \mathcal{M}_{loc}$ such that (1.2) holds for all $N \in \mathcal{M}_{loc}$ if and only if $H^2.[M] \in \mathcal{V}^+$. In this case there exists a predictable process $K \in L_m(M)$ such that $K.M = L$. We say that H is integrable w.r.t. M, and L is called the stochastic integral of H w.r.t. M, denoted by $H.M$.*

The main difference between the stochastic integral and the compensated stochastic integral becomes apparent when the integrator is a purely discontinuous local martingale.

Definition 1.5 ([27, 9.7 Definition])**:** *Let M be a purely discontinuous local martingale, and H be an optional process. If $H\Delta M$ has a predictable projection, and there exists a purely discontinuous local martingale L such that $\Delta L = H\Delta M - {}^p(H\Delta M)$, we call L the* compensated stochastic integral *of H w.r.t. M, and denote $L = H \underset{c}{\bullet} M$.*

Remark 1.6: *Note that if H is predictable, Definition 1.5 and Definition 1.1 coincide. Indeed,* [27, 7.13 Theorem] *ensures that ${}^p(H\Delta M) = H^p(\Delta M) = 0$.*

The next lemma characterizes the jumps of a compensated stochastic integral.

Lemma 1.7: *Let M be a purely discontinuous local martingale starting in zero, H be an optional process and T be a stopping time. Suppose that the compensated stochastic integral of H w.r.t. M exists. Then we have on $\{T < \infty\}$:*

$$\Delta \left(H \underset{c}{\bullet} M\right)_T = \begin{cases} H\Delta M_T - E\left[H\Delta M_T | \mathcal{F}_{T-}\right] \ a.s., & \text{if } T \text{ is predictable,} \\ H\Delta M_T \ a.s., & \text{if } T \text{ is totally inaccessible.} \end{cases}$$

Proof: If T is predictable, the statement results from the definition of the predictable projection; see [27, 5.2 Theorem]. Let T be totally inaccessible. First note that ${}^p(H\Delta M)$ is a predictable thin process. Indeed, the compensated stochastic integral $H \underset{c}{\bullet} M$ is a purely discontinuous local martingale such that $\Delta H \underset{c}{\bullet} M = H\Delta M - {}^p(H\Delta M)$. Therefore, we get

$$\begin{aligned} &\{{}^p(H\Delta M) \neq 0\} = \\ &\quad = \{{}^p(H\Delta M) \neq 0\} \cap \left(\{H\Delta M - {}^p(H\Delta M) \neq 0\} \cup \{H\Delta M = {}^p(H\Delta M)\}\right) \\ &\quad \subset \{\Delta H \underset{c}{\bullet} M \neq 0\} \cup \{\Delta M \neq 0\}. \end{aligned}$$

Due to [27, Theorem 3.32], the r.h.s.[2] of the equation above is a thin set. [27, Theorem 3.19] ensures that the set $\{{}^p(H\Delta M) \neq 0\}$, as a subset of a thin set, is itself a thin set. Since the predictable projection ${}^p(H\Delta M)$ is by definition a predictable process, it has to be a predictable thin process. [30, 2.23 Lemma] ensures that there

[2] abbr.: right hand side

exists a sequence $(T_n)_{n\geq 1}$ of predictable stopping times with disjoint graphs such that $\{{}^p(H\Delta M) \neq 0\} = \bigcup_{n\geq 1} [\![T_n]\!]$. Accordingly, it appears that

$$^p(H\Delta M)_T = {}^p(H\Delta M)_T \mathbb{1}_{\bigcup_{n\geq 1}[\![T_n]\!]}(T) = {}^p(H\Delta M)_T \mathbb{1}_{\bigcup_{n\geq 1}\{T_n = T\}}.$$

Since T is totally inaccessible and $(T_n)_{n\geq 1}$ are predictable stopping times, we have

$$P\left(\bigcup_{n\geq 1} \{T = T_n < \infty\}\right) = 0.$$

Combining the last two equations, we get

$$^p(H\Delta M)_T = 0 \qquad \text{a.s. on } \{T < \infty\}.$$

□

It is well known that a local martingale can be uniquely decomposed into a continuous local martingale and a purely discontinuous local martingale; see [27, 7.25 Theorem]. Thus, it is natural that the compensated stochastic integral w.r.t. arbitrary local martingales is essentially a composition of Theorem 1.4 and Definition 1.5.

Definition 1.8 ([27, 9.9 Definition])**:** *Let M be a local martingale, and H be an optional process. If $H^2.[M^c] \in \mathcal{V}^+$, ${}^p(H\Delta M)$ exists and*

$$\sqrt{\sum (H\Delta M - {}^p(H\Delta M))^2} \in \mathcal{A}^+_{loc},$$

define

$$H \underset{c}{\bullet} M = H_0 M_0 + H.M^c + H \underset{c}{\bullet} M^d.$$

$H \underset{c}{\bullet} M$ is called the compensated stochastic integral *of H w.r.t. M.*

Example 1.9 ([27, 9.8 Lemma])**:** Let M be a purely discontinuous local martingale. Put $H = \mathbb{1}_{\{\Delta M \neq 0\}}$. Then the compensated stochastic integral of H w.r.t. M exists and $H \underset{c}{\bullet} M = M$.

The subsequent theorem is the definition of the compensated stochastic integral as given by Meyer in [40]. Note that it is closely related to the Definition 1.1 and the characterisation (Theorem 1.2) of the stochastic integral for predictable integrands.

Theorem 1.10 ([27, 9.10 Theorem])**:** *Let M be a local martingale, and H be an optional process. If $\sqrt{H^2.[M]} \in \mathcal{A}^+_{loc}$, then $H \underset{c}{\bullet} M$ exists, and it is the unique local martingale L such that for every bounded martingale N, $[L, N] - H.[M, N] \in \mathcal{M}_{loc,0}$.*

Remark 1.11: *In the following, all compensated stochastic integrals $H \underset{c}{\bullet} M$ will satisfy $\sqrt{H^2.[M]} \in \mathcal{A}^+_{loc}$.*

The next lemma is an analogue to Lemma 1.3.

Lemma 1.12: *Let H, K be optional processes such that $\sqrt{H^2.[M]} \in \mathcal{A}^+_{loc}$ and $\sqrt{K^2.[M]} \in \mathcal{A}^+_{loc}$ holds. Let further $(T_n)_{n\geq 1}$ be a non–decreasing sequence of stopping times that tends to ∞ a.s. and that localizes $\sqrt{H^2.[M]}$ and $\sqrt{K^2.[M]}$. If we have $E\left[\sqrt{(H-K)^2.[M^{T_n}]_\infty}\right] = 0$ for all $n \geq 1$, then $H \underset{c}{\bullet} M$ and $K \underset{c}{\bullet} M$ are indistinguishable.*

Proof: The proof is similar to the proof of Lemma 1.3. Due to the Kunita–Watanabe inequality, we find that $(H-K).[M,N] = 0$ for all bounded martingales $N \in \mathcal{M}_{loc}$. The unique characterization of $H \underset{c}{\bullet} M$ and $K \underset{c}{\bullet} M$ in Theorem 1.10 results in the fact that $H \underset{c}{\bullet} M$ and $K \underset{c}{\bullet} M$ are indistinguishable. □

The following example is probably the archetype example for the compensated stochastic integral.

Example 1.13 ([40, Chapitre V, 21 Theoreme])**:** Let M be a local martingale starting in zero. Then $\sqrt{(\Delta M)^2.[M]} \in \mathcal{A}^+_{loc}$ holds if and only if M is locally square–integrable. Furthermore,

$$(\Delta M) \underset{c}{\bullet} M = [M] - \langle M \rangle,$$

where $\langle M \rangle$ denotes the predictable quadratic variation of M.

The generalisation of the stochastic integral to the compensated stochastic integral as presented so far has some drawbacks. For example, the compensated stochastic integral is not associative in general. To overcome this drawback, Yor [54] suggests a different definition of the stochastic integral for suitably chosen optional integrands. However, for predictable integrands we have the following associativity formula.

Lemma 1.14 ([20, Chapter VIII.2, 40 (c) Associativity formula])**:** *Let H be an optional process and K be a predictable process. If $\sqrt{H^2.[M]} \in \mathcal{A}^+_{loc}$ and K is locally bounded, the following three compensated integrals exist and are equal:*

$$K.(H \underset{c}{\bullet} M) = (KH) \underset{c}{\bullet} M = H \underset{c}{\bullet} (K.M).$$

Remark 1.15: *For our purposes the most important consequence of Lemma 1.14 is the following stopping rule. Let T be a stopping time and set $K := \mathbb{1}_{[\![0,T]\!]}$. Due to Lemma 1.14, we get*

$$(H \underset{c}{\bullet} M)^T = (\mathbb{1}_{[\![0,T]\!]} H) \underset{c}{\bullet} M = H \underset{c}{\bullet} M^T.$$

We end the overview with the following remark. It strengthens the characterisation of the compensated stochastic integral as given in Theorem 1.10 under certain integrability conditions.

Remark 1.16: *For $M \in \mathcal{M}^2_{loc}$ and $H^2.[M] \in \mathcal{A}^+_{loc}$ the characterisation of the compensated stochastic integral can be sharpened. Denote by $(T_n)_{n\geq 1}$ a non–decreasing sequence of stopping times tending to ∞ a.s. that localizes $M \in \mathcal{M}^2_{loc}$ and $H^2.[M] \in \mathcal{A}^+_{loc}$. Due to* [20, Chapter VIII.2, 33 Theorem] *and Lemma 1.14, we get*

$$E\left[[H \underset{c}{\bullet} M^{T_n}]_\infty\right] \leq E\left[\int\limits_0^\infty H_u^2 \, d[M^{T_n}]_u\right]$$

for all $n \geq 1$. Furthermore,

$$[H \underset{c}{\bullet} M, N] - H.[M,N]$$

is a local martingale for all $N \in \mathcal{M}^2_{loc}$.

1.2.3. Quotient space of optional integrands

Lemma 1.12 indicates that two compensated stochastic integrals are indistinguishable, if the integrands are essentially the same. In the following, we do not want to distinguish between optional integrands (in the sense of Theorem 1.10) generating the same compensated stochastic integral. Due to Lemma 1.12, we can accomplish this task by defining a proper quotient space. For a local martingale M we define

$$\mathcal{B}^{\mathcal{O}} := \mathcal{B}^{\mathcal{O}}(M) := \left\{ H \in \mathcal{O} : \sqrt{\int\limits_0^{\cdot} H_u^2 \, d[M]_u} \in \mathcal{A}^+ \right\}.$$

It is straightforward to check that $\mathcal{B}^{\mathcal{O}}$ is a $\mathbb{R}$–vector space and that the functional

$$\begin{aligned} \|.\| : \mathcal{B}^{\mathcal{O}} &\longrightarrow \mathbb{R}_{\geq 0} \\ H &\longmapsto E\left[\sqrt{\int\limits_0^\infty H_u^2 \, d[M]_u}\right] \end{aligned}$$

defines a seminorm on $\mathcal{B}^{\mathcal{O}}$. Moreover,

$$\mathcal{N} := \left\{ H \in \mathcal{B}^{\mathcal{O}} : \|H\| = 0 \right\}$$

is a linear subspace of $\mathcal{B}^{\mathcal{O}}$. For $H, K \in \mathcal{B}^{\mathcal{O}}$ we define the equivalence relation

$$H \sim K \quad :\Longleftrightarrow \quad H - K \in \mathcal{N}.$$

Definition 1.17: *Let M be a local martingale. We denote by*

$${}^oL := {}^oL(M) := \mathcal{B}^{\mathcal{O}} / \mathcal{N}$$

the quotient space of $\mathcal{B}^{\mathcal{O}}$ and $\mathcal{N}$. The localized space is denoted by ${}^oL_{loc} := {}^oL_{loc}(M)$.

Remark 1.18:

1. *With the usual abuse of notation, we call the elements of* ${}^{o}L_{loc}(M)$ *optional processes or optional integrands w.r.t.* M.

2. *Using the same procedure for predictable integrands, we can define the subspaces* ${}^{p}L(M) \subset {}^{o}L(M)$ *and* ${}^{p}L_{loc}(M) \subset {}^{o}L_{loc}(M)$ *of predictable integrands in a similar way.*

As a consequence, we get the following result.

Lemma 1.19: $\left({}^{o}L\left(M\right), \|.\|\right)$ *and* $\left({}^{p}L\left(M\right), \|.\|\right)$ *are normed* $\mathbb{R}$*–vector spaces.*

1.2.4. σ–martingales and compensable processes

The concepts of σ–martingales and compensable processes trace back to the work of Chou [14] and Émery [22]. Both authors consider a certain subclass of semimartingales. Their elements are called *semimartingales de la classe* (Σ). This class has also been studied by Kallsen [33]. We are interested in the subclass $(\Sigma_m) \subset (\Sigma)$. In the English literature the elements of (Σ_m) are most often called σ–martingales. We work with the following definition. For simplicity, we only consider processes starting in zero a.s..

Definition 1.20: *Let* X *be a semimartingale starting in zero.* X *is a* σ*–martingale, if there exists a strictly positive, bounded, and predictable process* K*, such that* $K.X$ *is a uniformly integrable martingale.*

Several equivalent characterisations of σ–martingales are provided in the next proposition.

Proposition 1.21: *Let* X *be a semimartingale starting in zero a.s.. The following statements are equivalent.*

1. X *is a* σ*–martingale.*

2. *There exists a predictable partition* $(A_n)_{n\geq 1} \subset \mathcal{P}$ *of* $\Omega \times \mathbb{R}_+$ *such that* $\mathbb{1}_{A_n}.X$ *is a local martingale for all* $n \geq 1$.

3. *There exists a predictable partition* $(D_n)_{n\geq 1} \subset \mathcal{P}$ *of* $\Omega \times \mathbb{R}_+$ *such that* $\mathbb{1}_{D_n}.X$ *is a uniformly integrable martingale for all* $n \geq 1$.

4. *There exists a non–decreasing sequence* $(\tilde{A}_n)_{n\geq 1} \subset \mathcal{P}$ *such that* $\bigcup_{n\geq 1} \tilde{A}_n = \Omega \times \mathbb{R}_+$ *and* $\mathbb{1}_{\tilde{A}_n}.X$ *is a local martingale for all* $n \geq 1$.

5. *There exists a non–decreasing sequence* $(\tilde{D}_n)_{n\geq 1} \subset \mathcal{P}$ *such that* $\bigcup_{n\geq 1} \tilde{D}_n = \Omega \times \mathbb{R}_+$ *and* $\mathbb{1}_{\tilde{D}_n}.X$ *is a uniformly integrable martingale for all* $n \geq 1$.

We call the sequence $(\tilde{D}_n)_{n\geq 1} \subset \mathcal{P}$, that satisfies 5., a localizing sequence *for X.*

Proof: The equivalence of the first three statements is exactly [22, Proposition 2]. To prove '*2.* $\Rightarrow$ *4.*' just set $\tilde{A}_n := \bigcup_{m\leq n} A_m$. To see that '*4.* $\Rightarrow$ *2.*' holds, we define $A_0 := \emptyset$ and $A_n := \tilde{A}_n \setminus \tilde{A}_{n-1}$ for $n \geq 1$. '*3.* $\Leftrightarrow$ *5.*' follows in the same way. □

The following concept has been introduced by Émery to characterize those semimartingales that can be decomposed into the sum of a σ–martingale and a predictable process of finite variation.

Definition 1.22: *Let V be a process of finite variation starting in zero. V is called* compensable, *if there exists a predictable process C of finite variation starting in zero such that process $V - C$ is a σ–martingale. C is called the* compensator *of V.*

The next proposition is similar to Proposition 1.21. It provides several criteria to identify the compensator of a compensable process.

Proposition 1.23 ([22, Proposition 3])**:** *Let V be a process of finite variation starting in zero and C be a predictable process of finite variation. The following statements are equivalent.*

1. *$V - C$ is a σ–martingale.*
2. *There exists a predictable partition $(A_n)_{n\geq 1} \subset \mathcal{P}$ of $\Omega \times \mathbb{R}_+$ such that $\mathbb{1}_{A_n}.V$ is of locally integrable variation and its compensator is given by $\mathbb{1}_{A_n}.C$ for all $n \geq 1$.*
3. *There exists a predictable partition $(D_n)_{n\geq 1} \subset \mathcal{P}$ of $\Omega \times \mathbb{R}_+$ such that $\mathbb{1}_{D_n}.V$ is of integrable variation and its compensator is given by $\mathbb{1}_{D_n}.C$ for all $n \geq 1$.*
4. *There exists a strictly positive, bounded, and predictable process K such that the Stieltjes integral w.r.t. V and C is well defined and $K.V - K.C$ is a uniformly integrable martingale.*

Furthermore, for all processes of finite variation there exists at most one predictable process of finite variation satisfying the conditions above.

Remark 1.24: *As in Proposition 1.21, the predictable partition in 2. and 3. can be replaced by a non–decreasing sequence of predictable sets tending to $\Omega \times \mathbb{R}_+$.*

We collect several results concerning the existence of a compensator and its shape in the following lemmas. The first lemma gives a sufficient condition for the existence of a compensator.

Lemma 1.25: *Let $V \in \mathcal{V}$. $V \in \mathcal{A}_{loc}$ if and only if there exists a predictable process C of finite variation process such that $V - C$ is a local martingale. If either of the conditions holds, V is compensable.*

Proof: [27, 7.20 Corollary]. □

If a process of finite variation is non–decreasing, Lemma 1.25 can be sharpened.

Lemma 1.26: *Let V be a non–decreasing process of finite variation. V is compensable if and only if $V \in \mathcal{A}^+_{loc}$.*

Proof: Due to Lemma 1.25, the 'if' part is clear. Let V be compensable and denote its compensator by C. Since C is predictable and of finite variation, [27, 5.19 Theorem] ensures that there exists a non–decreasing sequence $(T_m)_{m\geq 1}$ of stopping times such that $T_m \uparrow \infty$ a.s. and $C^{T_m} \in \mathcal{A}$ for all $m \geq 1$. Denote by $(D_n)_{n\geq 1}$ the localizing sequence of predictable sets such that $\mathbb{1}_{D_n}.V - \mathbb{1}_{D_n}.C$ is a uniformly integrable martingale for all $n \geq 1$. Due to the monotone convergence theorem, we get

$$E\left[V^{T_m}\right] = \lim_{n\to\infty} E\left[\mathbb{1}_{D_n}.V^{T_m}\right] = \lim_{n\to\infty} E\left[\mathbb{1}_{D_n}.C^{T_m}\right] = E\left[C^{T_m}\right] < \infty.$$

□

Thanks to Example 1.13, we know that for a locally square integrable martingale M the process $[M]-\langle M\rangle$ is a local martingale. In particular, $[M]$ is compensable and its compensator is given by $\langle M\rangle$. The next lemma characterizes the compensator of a Stieltjes integral $\lambda.[M]$ having predictable integrands.

Lemma 1.27: *Let λ be a predictable process and $M \in \mathcal{M}^2_{loc}$. Denote by $\langle M\rangle$ the predictable quadratic variation of M.*

1. *$V := \int |\lambda|\, d[M] \in \mathcal{A}^+_{loc}$ if and only if $C := \int |\lambda|\, d\langle M\rangle \in \mathcal{A}^+_{loc}$. If either of the conditions hold, $V - C$ is a local martingale.*

2. *If either of the above conditions hold, $\lambda \in L_m([M] - \langle M\rangle)$.*

Proof: We start with the first item. Let $V \in \mathcal{A}^+_{loc}$. Due to Lemma 1.26, we only have to prove that the compensator is given by C. This follows from [27, 5.23 Theorem 2)] and the uniqueness of the compensator. Let $C \in \mathcal{A}^+_{loc}$. Since $C = \int |\lambda|\, d\langle M\rangle = \int (\sqrt{|\lambda|})^2\, d\langle M\rangle$, we know that $\sqrt{|\lambda|}.M \in \mathcal{M}^2_{loc}$. Due to the Burkholder–Davis–Gundy inequality, see [27, 10.36 Theorem], we conclude that $\int |\lambda|\, d[M] \in \mathcal{A}^+_{loc}$. The second statement follows from [27, 9.5 Theorem]. □

1.3. Radon–Nikodym decomposition for local martingales

Now we are in the position to state and prove the main results of this chapter. We prove the following decomposition theorem for local martingales.

Theorem 1.34 : Let N and M be local martingales starting in zero. Let H be the Radon–Nikodym derivative of $d[N, M]$ w.r.t. $d[M]$. Define $L := N - H \underset{c}{\bullet} M \in \mathcal{M}_{loc}$. Then $[L, M]$ is a σ–martingale and the decomposition

$$N = H \underset{c}{\bullet} M + L$$

is called the *Radon–Nikodym decomposition of* N *w.r.t.* M.

Throughout this chapter we work with the following definition of a Kunita–Watanabe decomposition.

Definition 1.28: *Let N and M be local martingales starting in zero. N features a* Kunita–Watanabe decomposition *w.r.t. M, if there exists a predictable process $K \in L_m(M)$ such that*

$$N = K.M + \tilde{L}$$

and $[\tilde{L}, M]$ is a local martingale.

The Radon–Nikodym decomposition

In order to prove the existence of the Radon–Nikodym decomposition, we need to know whether the Radon–Nikodym derivative of $d[N, M]$ w.r.t. $d[M]$ is integrable w.r.t. M. We formulate this statement in the next theorem being of interest in its own right.

Theorem 1.29: *Let X and Y be two semimartingales starting in zero. Then there exists an optional process H such that $[Y, X]$ and $H.[X]$ are indistinguishable. Moreover,*

$$\sqrt{H^2.[X]} \leq \sqrt{[Y]} \tag{1.3}$$

holds up to indistinguishability.

The proof of Theorem 1.29 is essentially an application of the Kunita–Watanabe inequality and the following lemma.

Lemma 1.30 (compare [9, p. 4.7.102.])**:** *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space, $p \in (1, \infty)$, and $f \in L^1(\Omega, \mathcal{F}, \mu)$. Then $f \in L^p(\Omega, \mathcal{F}, \mu)$ if and only if there exists a constant $C > 0$ such that*

$$\sum_{k=0}^{M} \mu(A_k)^{1-p} \left| \int_{A_k} f \, d\mu \right|^p \leq C \tag{1.4}$$

holds for every finite partition of Ω into disjoint measurable sets A_k with positive measure. In addition, the smallest possible constant equals $\|f\|_p^p$.

Proof: By Hölder's inequality we get for $A \in \mathcal{F}$ and $\frac{p}{q} = p - 1$

$$\left| \int\limits_A f \, d\mu \right|^p \leq \|\mathbb{1}_A\|_q^p \|\mathbb{1}_A f\|_p^p = \mu(A)^{p-1} \|\mathbb{1}_A f\|_p^p. \tag{1.5}$$

Therefore, it suffices to prove that (1.4) implies $f \in L^p(\Omega, \mathcal{F}, \mu)$. By considering separately the sets $\{f \geq 0\}$ and $\{f < 0\}$, we may and do assume that $f \geq 0$ μ–a.s.. Define $f_N := \min\{f, N\}$. Since μ is finite, $f_N \in L^p(\Omega, \mathcal{F}, \mu)$ and due to (1.5),

$$\sum_{k=0}^{M} \mu(A_k)^{1-p} \left| \int\limits_{A_k} f \, d\mu \right|^p \leq \|f_N\|_p^p$$

holds for every finite disjoint partition $(A_k)_{k \leq M}$ of Ω. Define for $n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$ the sets $A_{k,n} := \{\frac{k}{n} N \leq f_N < \frac{k+1}{n} N\}$. By choosing a subsequence, if necessary, we can assume that $\mu(A_{k,n}) > 0$. Due to the definition of the sequence $(A_{k,n})_{k \leq n}$ and the mean value theorem, we get

$$\begin{aligned}
\|f_N\|_p^p &= \sum_{k=0}^{n} \int\limits_{A_{k,n}} f_N^p \, d\mu \\
&\leq N^p \sum_{k=0}^{n} \mu(A_{k,n}) \left(\frac{k}{n}\right)^p + N^p \sum_{k=0}^{n} \mu(A_{k,n}) \frac{\left(\frac{k+1}{n}\right)^p - \left(\frac{k}{n}\right)^p}{\frac{k+1}{n} - \frac{k}{n}} \frac{1}{n} \\
&\leq N^p \sum_{k=0}^{n} \mu(A_{k,n}) \left(\frac{k}{n}\right)^p + \frac{1}{n} N^p \mu(\Omega) \left(\frac{n+1}{n}\right)^{p-1}.
\end{aligned}$$

Now let $C_{min} > 0$ be the smallest constant such that (1.4) holds for f. As a consequence of monotonicity and the definition of $(A_{k,n})_{k \leq n}$, we get

$$\begin{aligned}
C_{min} &\geq \sum_{k=0}^{M} \mu(A_{k,n})^{1-p} \left| \int\limits_{A_{k,n}} f_N \, d\mu \right|^p \geq \sum_{k=0}^{M} \mu(A_{k,n})^{1-p} \left| \frac{k}{n} N \mu(A_{k,n}) \right|^p \\
&= N^p \sum_{k=0}^{n} \mu(A_{k,n}) \left(\frac{k}{n}\right)^p.
\end{aligned}$$

Combining the two estimations, we get for all $n \in \mathbb{N}$

$$\|f_N\|_p^p \leq C_{min} + \frac{1}{n} N^p \mu(\Omega) \left(\frac{n+1}{n}\right)^{p-1}.$$

Hence, $\|f_N\|_p^p \leq C_{min}$ for all $N \in \mathbb{N}$. Due to Fatou's lemma, we get $\|f\|_p^p \leq C_{min}$. Applying Hölder's inequality to (1.4), we get $\|f\|_p^p \geq C_{min}$ which proves the claim. $\square$

Proof of Theorem 1.29: The Kunita–Watanabe inequality and [27, 5.14 Theorem] ensure the existence of an optional process H such that $[Y,X] = H.[X]$ holds. Since $[Y,X] = H.[X]$, the total variation process of $[Y,X]$ is given by $|H|.[X]$. Thus, the Kunita–Watanabe inequality ensures that we can find a set $\tilde{N}$ of measure zero such that for $\omega \notin \tilde{N}$ and for all pairs of rational numbers $s \leq t$ we have

$$\left(\int\limits_{(s,t]} |H_u|(\omega)\, d[X]_u(\omega) \right)^2 \leq \int\limits_{(s,t]} d[X]_u(\omega) \int\limits_{(s,t]} d[Y]_u(\omega).$$

Since $d[X]_.(\omega)$ as well as $d[Y]_.(\omega)$ are finite measures on each compact subset of $\mathbb{R}_+$, the dominated convergence theorem ensures that the above inequality also holds for any Borel–measurable set $A \in \mathcal{B}(\mathbb{R}_+)$ being a subset of a compact set. In particular, we get for all $A \in \mathcal{B}(\boldsymbol{R}_+)$ and all $T \in [0,\infty)$

$$\left(\int\limits_{A\cap[0,T]} |H_u|(\omega)\, d[X]_u(\omega) \right)^2 \leq \int\limits_{A\cap[0,T]} d[X]_u(\omega) \int\limits_{A\cap[0,T]} d[Y]_u(\omega).$$

As for every finite partition of $[0,T]$ into disjoint measurable sets A_k we have

$$\sum_{k=0}^{M} \int\limits_{A_k} d[Y]_u(\omega) \leq [Y]_T(\omega),$$

we may apply Lemma 1.30 for $p = 2$. Due to the *additional property* at the end of Lemma 1.30, we can conclude that

$$\int\limits_0^T H_u^2(\omega).[X]_u(\omega) \leq [Y]_T(\omega)$$

holds for all $T \in [0,\infty)$. By taking the square–root on both sides we get the desired result (1.3). □

Using the Kunita–Watanabe inequality for the predictable covariation [27, Remark p. 210] the same line of arguments leads to the following corollary.

Corollary 1.31: *Let X and Y be two locally square–integrable semimartingales starting in zero. Then there exists a predictable process K such that $\langle Y, X\rangle$ and $K.\langle X\rangle$ are indistinguishable. Moreover,*

$$\sqrt{K^2.\langle X\rangle} \leq \sqrt{\langle Y\rangle}$$

holds up to indistinguishability.

Remark 1.32: *For locally square–integrable martingales M and N, this corollary leads directly to the Kunita–Watanabe decomposition. Indeed, the corollary ensures that there exists $K \in \mathcal{P}$ such that $\langle N, M\rangle = K.\langle M\rangle$ and $K.M \in \mathcal{M}^2_{loc}$. Hence, for $L := N - K.M \in \mathcal{M}^2_{loc}$ we get $\langle L, M\rangle = \langle N, M\rangle - K.\langle M\rangle = 0$.*

It is well known, see [27, 7.30 Theorem], that $\sqrt{[N]} \in \mathcal{A}^+_{loc}$ holds for all local martingales $N \in \mathcal{M}_{loc}$. Accordingly, the next corollary follows immediately from Theorem 1.29.

Corollary 1.33: *Let N and M be two local martingales starting in zero. Then the Radon–Nikodym derivative H of $d[N, M]$ w.r.t. $d[M]$ satisfies $\sqrt{H^2.[M]} \in \mathcal{A}^+_{loc}$.*

Now we have all tools at hand to prove the main result, the existence of the Radon–Nikodym decomposition for local martingales.

Theorem 1.34 (Radon–Nikodym decomposition)**:** *Let N and M be local martingales starting in zero. Let H be the* Radon–Nikodym derivative *of $d[N, M]$ w.r.t. $d[M]$. The compensated stochastic integral $H \underset{c}{\bullet} M \in \mathcal{M}_{loc}$ is well defined. Moreover, if $L := N - H \underset{c}{\bullet} M \in \mathcal{M}_{loc}$, then $[L, M]$ is a σ–martingale and the decomposition*

$$N = H \underset{c}{\bullet} M + L$$

is called the Radon–Nikodym decomposition of N w.r.t. M.

Proof: Due to Corollary 1.33 and Theorem 1.10, the local martingale $H \underset{c}{\bullet} M \in \mathcal{M}_{loc}$ is well defined. Accordingly, it suffices to prove that $[L, M]$ is a σ–martingale. As a result of Theorem 1.4 and Definition 1.5 we get

$$\begin{aligned}
[L, M] &= H.[M] - [H \underset{c}{\bullet} M, M] \\
&= H.[M^c, M^c] + H.[M^d, M^d] - [(H \underset{c}{\bullet} M)^c, M^c] - [(H \underset{c}{\bullet} M)^d, M^d] \\
&= H.[M^d, M^d] - [H \underset{c}{\bullet} M^d, M^d] \\
&= \sum {}^p(H\Delta M^d)\Delta M^d. \qquad (1.6)
\end{aligned}$$

Recalling the proof of Lemma 1.7, we know that ${}^p(H\Delta M^d)$ is a predictable thin process. Due to [30, 2.23 Lemma], there exists a sequence $(T_n)_{n\geq 1}$ of predictable stopping times with disjoint graphs such that $\left\{{}^p(H\Delta M^d) \neq 0\right\} = \bigcup_{n\geq 1}[\![T_n]\!]$. Define

$$\varphi := \mathbb{1}_{(\bigcup_{n\geq 1}[\![T_n]\!])^c} + \sum_{n\geq 1} \frac{1}{2^{n+1}} \mathbb{1}_{[\![T_n]\!]} \frac{1}{1 + |{}^p(H\Delta M^d)|}$$

and note that φ is a predictable, bounded, and strictly positive process. Further, we define

$$A := \sum_{n\geq 1} \frac{1}{2^{n+1}} \mathbb{1}_{[\![T_n,\infty]\!]} \frac{{}^p(H\Delta M^d)}{1 + |{}^p(H\Delta M^d)|}.$$

Since A is an adapted, predictable process of finite variation, [27, 7.7 Theorem] ensures that A is locally bounded. Therefore, $A \in \mathcal{A}_{loc}$ and its jumps are given by

$$\Delta A_s = \sum_{n\geq 1} \frac{1}{2^{n+1}} \mathbb{1}_{[\![T_n]\!]}(s) \frac{{}^p(H\Delta M^d)_s}{1+|{}^p(H\Delta M^d)_s|}.$$

Moreover,

$$\begin{aligned} \varphi.[L,M] &= \sum_{s\leq \cdot} \left(\sum_{n\geq 1} \frac{1}{2^{n+1}} \mathbb{1}_{[\![T_n]\!]}(s) \frac{{}^p(H\Delta M^d)_s}{1+|{}^p(H\Delta M^d)_s|} \right) \Delta M^d_s \\ &= \sum_{s\leq \cdot} \Delta A_s \Delta M^d_s = [A, M^d] \end{aligned}$$

holds and Yoeurp's lemma [27, 9.4 Example 1)] ensures that $[L,M]$ is a σ–martingale.

□

Remark 1.35: *Note that the existence of the Radon–Nikodym decomposition of two local martingales M, N, is a symmetric property. Indeed, the Radon–Nikodym decomposition of N w.r.t. M exists if and only if the Radon–Nikodym decomposition of M w.r.t. N exists.*

The next theorem gives a more detailed characterisation of the Radon–Nikodym decomposition of local martingales.

Theorem 1.36: *Let N and M be local martingales starting in zero and denote by $N = H \underset{c}{\bullet} M + L$ the Radon–Nikodym decomposition of N w.r.t. M. Moreover, let*

$$N^c = \lambda.M^c + \tilde{L}$$

be the Kunita–Watanabe decomposition of N^c w.r.t. M^c, where $\lambda \in L_m(M^c)$, $\tilde{L} \in \mathcal{M}^c_{loc}$, and $[M^c, \tilde{L}] = 0$. Then $(H \underset{c}{\bullet} M)^c = \lambda.M^c$, $L^c = \tilde{L}$, and the process $[L^d, M^d]$ is a σ–martingale.

Proof: Recall that a process of finite variation can be decomposed in a unique way into a continuous– and a purely discontinuous process of finite variation. Since

$$[N,M] = H.[M] = H.[M^c] + H.[M^d]$$

and

$$[N,M] = \langle N^c, M^c\rangle + [N^d, M^d],$$

we can conclude that

$$H.[M^c] = \langle N^c, M^c\rangle.$$

Due to Remark 1.32, we know that there exists a unique $\lambda \in L_m(M^c)$ such that

$$H.[M^c] = \langle N^c, M^c\rangle = \lambda.[M^c] \tag{1.7}$$

holds up to indistinguishability. Furthermore, the Kunita–Watanabe decomposition of N^c w.r.t. M^c is given by

$$N^c = \lambda.M^c + \tilde{L},$$

where $\tilde{L} \in \mathcal{M}^c_{loc}$ and $[M^c, \tilde{L}] = 0$. Hence, it remains to prove that $H.M^c = \lambda.M^c$. Because of the Kunita–Watanabe inequality, we know that

$$d[M^c, R] \ll d[M^c], \qquad \forall\, R \in \mathcal{M}_{loc}. \tag{1.8}$$

Combining (1.7) and (1.8), we can conclude that

$$H.[M^c, R] = \lambda.[M^c, R], \qquad \forall\, R \in \mathcal{M}_{loc}.$$

Due to the definition of $H.M^c$, the above equation ensures that

$$[H.M^c - \lambda.M^c, R] = 0, \qquad \forall\, R \in \mathcal{M}_{loc}.$$

This implies that

$$H.M^c = \lambda.M^c$$

holds; see [27, 7.36 Theorem]. In turn, this ensures that $L^c = \tilde{L}$ and $[L, M] = [L^d, M^d]$. Finally, Theorem 1.34 guarantees that $[L^d, M^d]$ is a σ–martingale. □

Under certain regularity assumptions we can sharpen Theorem 1.36.

Corollary 1.37: *Let N and M be local martingales starting in zero and denote by $N = H \underset{c}{\bullet} M + L$ the Radon–Nikodym decomposition of N w.r.t. M. If M is quasi–left–continuous, the quadratic covariation $[L, M]$ is zero on $[\![0, \infty[\![$.*

Proof: Following the lines of the proof of Theorem 1.34, we find that

$$[L, M] = \sum {}^p(H\Delta M^d)\Delta M^d$$

is a thin process. Furthermore, there exists a sequence $(T_n)_{n\geq 1}$ of predictable stopping times with disjoint graphs such that $\left\{{}^p(H\Delta M^d) \neq 0\right\} = \bigcup_{n\geq 1}[\![T_n]\!]$. This implies that $\{[L, M] \neq 0\} \subset \bigcup_{n\geq 1}[\![T_n]\!]$. Since M is quasi–left–continuous, [27, 4.23 Theorem] ensures that $[L, M]$ is quasi–left–continuous, too. Due to [27, Remark p. 122], there exists a sequence $(S_m)_{m\geq 1}$ of totally inaccessible stopping times such that $\{[L, M] \neq 0\} = \bigcup_{m\geq 1}[\![S_m]\!]$. Therefore,

$$\{[L, M] \neq 0\} \subset \left(\bigcup_{m\geq 1}[\![S_m]\!]\right) \cap \left(\bigcup_{n\geq 1}[\![T_n]\!]\right) = \bigcup_{m\geq 1}\bigcup_{n\geq 1}\{S_m = T_n\}.$$

This implies that

$$\left\{\mathbb{1}_{[\![0,\infty[\![}[L, M] \neq 0\right\} \subset \bigcup_{m\geq 1}\bigcup_{n\geq 1}\{S_m = T_n < \infty\}.$$

Since $(S_m)_{m\geq 1}$ is a sequence of totally inaccessible stopping times and $(T_n)_{n\geq 1}$ are predictable stopping times we have $P(S_m = T_n < \infty) = 0$ for all $m, n \geq 1$. □

The next corollary is an immediate consequence of Theorem 1.36.

Corollary 1.38: *For $N, M \in \mathcal{M}^c_{loc}$ the Kunita–Watanabe decomposition and the Radon–Nikodym decomposition of N w.r.t. M exist and are indistinguishable.*

1.4. Radon–Nikodym decomposition vs. Kunita–Watanabe decomposition: the locally square–integrable case

Throughout this section, we assume that N and M are locally square integrable martingales. The classical proof for the existence of the Kunita–Watanabe decomposition relies on the Hilbert space structure of the space of square–integrable martingales. We provide an alternative proof which relies on a different Hilbert space. In particular, this approach enables us to compare the Kunita–Watanabe decomposition and the Radon–Nikodym decomposition. The elements of this specific Hilbert space are defined in the next definition.

Definition 1.39: *For a locally square integrable martingale $M \in \mathcal{M}^2_{loc}$, we define*

$$ {}^oL^2 := {}^oL^2\left(M\right) := \left\{ H \in {}^oL(M) : \int_0^{\cdot} H_u^2 \, d[M]_u \in \mathcal{A} \right\} \tag{1.9} $$

and

$$ {}^pL^2 := {}^pL^2\left(M\right) := \left\{ H \in {}^oL^2\left(M\right) : \, H \in \mathcal{P} \right\}. \tag{1.10} $$

Furthermore, we denote by ${}^oL^2_{loc}\left(M\right)$ and ${}^pL^2_{loc}\left(M\right)$ the localized classes.

The following theorem ensures that ${}^oL^2\left(M\right)$ and ${}^pL^2\left(M\right)$, being equipped with a properly defined inner product, are indeed Hilbert spaces.

Theorem 1.40: *The mapping*

$$ \begin{aligned} \langle .,. \rangle : {}^oL^2\left(M\right) \times {}^oL^2\left(M\right) &\longrightarrow \mathbb{R} \\ (H, K) &\longmapsto E\left[\int_0^{\infty} H_u K_u \, d[M]_u \right] \end{aligned} $$

defines an inner product on ${}^oL^2\left(M\right)$. Moreover, the spaces $\left({}^oL^2\left(M\right), \langle .,. \rangle\right)$ and $\left({}^pL^2\left(M\right), \langle .,. \rangle\right)$ are Hilbert spaces.

Proof: It is straightforward to check that $\left({}^{o}L^{2}\left(M\right), \langle .,.\rangle\right)$ is indeed an inner product space. Additionally,

$$\begin{aligned} m : \mathcal{O} &\longrightarrow [0,\infty] \\ B &\longmapsto E\left[\mathbb{1}_B.[M]\right] \end{aligned}$$

defines a σ-finite measure on the optional σ-algebra $\mathcal{O}$. Due to the Riesz–Fischer theorem, we know that ${}^{o}L^{2}\left(M\right)$ is a Hilbert space. By restricting the σ–finite measure m to the sub–σ–algebra $\mathcal{P} \subset \mathcal{O}$ of predictable processes, we can conclude, using the same arguments as before, that $\left({}^{p}L^{2}\left(M\right), \langle .,.\rangle\right)$ is a Hilbert space, too. □

Remark 1.41:

1. *To prove that* $\left({}^{p}L^{2}\left(M\right), \langle .,.\rangle\right)$ *is a Hilbert space one can also use Lemma 1.27. Indeed, it enables us to define an isometry between the inner product space* $\left({}^{p}L^{2}\left(M\right), \langle .,.\rangle\right)$ *and the well known Hilbert space* $L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, \tilde{m})$*. Here,* $\mathcal{P}$ *denotes the predictable* σ*–algebra and the* σ*–finite measure on* $\mathcal{P}$ *is defined via*

$$\begin{aligned} \tilde{m} : \mathcal{P} &\longrightarrow [0,\infty] \\ B &\longmapsto E\left[\mathbb{1}_B.\langle M\rangle\right]. \end{aligned}$$

2. *Ito's isometry is an isometry between the Hilbert space* $\left({}^{p}L^{2}\left(M\right), \langle .,.\rangle\right)$ *and the subspace*

$$\mathbb{H} := \left\{N \in \mathcal{M}^2 \middle|\, \exists\, K \in {}^{p}L^{2}\left(M\right) : \;N = K.M\right\} \subset \mathcal{M}^2$$

of the Hilbert space $\mathcal{M}^2$ *of square integrable martingales. Consequently,* $\mathbb{H} \subset \mathcal{M}^2$ *is closed. Using the orthogonal projection of* $N \in \mathcal{M}^2$ *onto* $\mathbb{H}$*, this leads to the classical proof of the Kunita–Watanabe decomposition of* N *w.r.t.* M*.*

3. *Note that the technical modification of Section 1.2.3 is essential for the proof of the theorem. Otherwise, the mapping defined in the theorem above would not be a norm.*

Since ${}^{o}L^{2}\left(M\right)$ and ${}^{p}L^{2}\left(M\right)$ are Hilbert spaces and ${}^{p}L^{2}\left(M\right) \subset {}^{o}L^{2}\left(M\right)$, we know that for all $H \in {}^{o}L^{2}\left(M\right)$ there exists an orthogonal projection of H onto ${}^{p}L^{2}\left(M\right)$. The next lemma ensures the existence of a 'local orthogonal projection' for elements in ${}^{o}L^{2}_{loc}\left(M\right)$.

Lemma 1.42: *Let* $H \in {}^{o}L^{2}_{loc}\left(M\right)$*. There exists a unique process* $\lambda \in {}^{p}L^{2}_{loc}\left(M\right)$ *such that for all* $K \in {}^{p}L^{2}_{loc}\left(M\right)$ *there exists a sequence* $(T_n)_{n\geq 1}$ *of stopping times that tends to* ∞ *a.s. and localizes* $\int H^2\, d[M]$*,* $\int \lambda^2\, d[M]$*, and* $\int K^2\, d[M]$*. Furthermore,*

$$E\left[\int_0^\infty \left(H_u - \lambda_u\right) K_u \; d[M^{T_n}]_u\right] = 0, \qquad \text{for all } n \geq 1. \tag{1.11}$$

Proof: Let $(S_n)_{n\geq 1}$ be a non–decreasing sequence of stopping times that tends to ∞ a.s. and localizes H, i.e. $H\mathbb{1}_{[\![0,S_n]\!]} \in {}^{o}L^2(M)$. Denote by $\lambda^n \in {}^{p}L^2(M)$ the orthogonal projection of $H\mathbb{1}_{[\![0,S_n]\!]}$ onto ${}^{p}L^2(M)$. Since $(S_n)_{n\geq 1}$ is non–decreasing, the uniqueness of the orthogonal projection ensures that $\lambda^n = \lambda^m \mathbb{1}_{[\![0,S_n]\!]}$ for all natural numbers $n \leq m$. Set $S_0 = 0$ and define

$$\lambda := \mathbb{1}_{[\![0]\!]}\lambda_0^1 + \sum_{n\geq 1} \lambda^n \mathbb{1}_{]\!]S_{n-1},S_n]\!]}.$$

By definition, we have $\lambda \in {}^{p}L^2_{loc}(M)$. Let $(R_n)_{n\geq 1}$ be a non–decreasing sequence of stopping times that tends to ∞ a.s. and localizes K, i.e. $K\mathbb{1}_{[\![0,R_n]\!]} \in {}^{p}L^2(M)$. It is clear that the sequence $(T_n)_{n\geq 1}$ of stopping times, where $T_n := \min\{S_n, R_n\}$, satisfies all desired properties. The uniqueness follows from the uniqueness of the (local) orthogonal projections. □

Definition 1.43: *Let $H \in {}^{o}L^2_{loc}(M)$. The process $\lambda \in {}^{p}L^2_{loc}(M)$ given by Lemma 1.42 is called the* local orthogonal projection *of H onto ${}^{p}L^2_{loc}(M)$.*

The next theorem is the main result of this section. It ensures the existence of the Kunita–Watanabe decomposition for locally square integrable martingales and explains the connection to the Radon–Nikodym decomposition.

Theorem 1.44: *Let $N, M \in \mathcal{M}^2_{loc}$. The Radon–Nikodym derivative H of $d[N,M]$ w.r.t. $d[M]$ satisfies $H \in {}^{o}L^2_{loc}(M)$. Furthermore, the following statements hold:*

1. *The Kunita–Watanabe decomposition of N w.r.t. M is given by*

$$N = \lambda.M + \tilde{L},$$

where $\lambda \in {}^{p}L^2_{loc}(M)$ is the local orthogonal projection of H onto ${}^{p}L^2_{loc}(M)$.

2. *Let $N = H \underset{c}{\bullet} M + L$ denote the Radon–Nikodym decomposition of N w.r.t. M. Then $[M, L]$ is a local martingale.*

3. *The compensator of $[N,M] = H.[M]$ is given by $\int \lambda \, d\langle M\rangle$.*

Proof: First note that $H \in {}^{o}L^2_{loc}(M)$ holds due to Theorem 1.29. To prove *1.*, we only have to prove that $[M, \tilde{L}]$ is a local martingale. Since $[M, \tilde{L}] = H.[M] - \lambda.[M]$, we know that $[M, \tilde{L}] \in \mathcal{A}^+_{loc}$. Denote the localizing sequence of $[M, \tilde{L}]$ by $(T_n)_{n\geq 1}$. Let $s \leq t$ and $A \in \mathcal{F}_s$. Due to Lemma 1.42, we have

$$E\left[\mathbb{1}_A\left([M^{T_n}, \tilde{L}]_t - [M^{T_n}, \tilde{L}]_s\right)\right] = E\left[\int_0^\infty (H_u - \lambda_u)\,\mathbb{1}_A \mathbb{1}_{(s,t]}(u)\, d[M^{T_n}]_u\right] = 0$$

for all $n \geq 1$. Therefore, $[M, \tilde{L}]$ is a local martingale. Next we prove *2.*. In consequence of Theorem 1.34, it suffices to prove that $[M, L] \in \mathcal{A}_{loc}$. Since $[M, L] = [N, M] - [H \underset{c}{\bullet} M, M]$, item *2.* follows from the Kunita–Watanabe inequality and Remark 1.16. To prove item *3.*, we use the Kunita–Watanabe decomposition of N w.r.t. M to get

$$H.[M] = [N, M] = \lambda.[M] + [M, \tilde{L}] = \lambda.\left([M] - \langle M \rangle\right) + \lambda.\langle M \rangle + [M, \tilde{L}].$$

Rearranging the equation results in

$$H.[M] - \lambda.\langle M \rangle = \lambda.\left([M] - \langle M \rangle\right) + [M, \tilde{L}].$$

Because of Lemma 1.27 and the first part of the proof, the r.h.s. of the above equation is a local martingale. □

Remark 1.45: *Notice the difference between the classical proof of the Kunita–Watanabe decomposition as described in Remark 1.41 and the proof presented above. In the proof of Theorem 1.44, we used the fact that the Radon–Nikodym derivative $H \in {}^{o}L^2_{loc}(M)$ of $d[N, M]$ w.r.t. $d[M]$ is locally situated in the Hilbert space ${}^{o}L^2(M)$ and that its local orthogonal projection onto ${}^{p}L^2_{loc}(M)$ satisfies (1.11).*

1.5. Examples in which the Kunita–Watanabe decomposition does not exist

As we have seen in Theorem 1.34, the Radon–Nikodym decomposition always exists. As pointed out in Remark 1.35, the existence of the Radon–Nikodym decomposition for two local martingales is a symmetric property. However, the Kunita–Watanabe decomposition does not satisfy these properties in general. In [3], Ansel and Stricker consider two different cases in which the Kunita–Watanabe decomposition does not exist in general. We recall these examples and provide the Radon–Nikodym decompositions. We also show that the existence of the Kunita–Watanabe decomposition is not a symmetric property in general. In all cases we consider local martingales $N, M \in \mathcal{M}_{loc}$ and we show that the Kunita–Watanabe decomposition of N w.r.t. M does not exist.

1.5.1. Case 1: N square integrable, M arbitrary

The example below shows that for a square–integrable martingale N and an arbitrary martingale M the Kunita–Watanabe decomposition does not exist in general. This example is a combination of [29, Exercices 1.1, p. 23] and [29, Exercices 4.10, p. 141]. We start with a lemma which is an extension and slight modification of [29, Exercices 1.1, p. 23]. It builds the core of the current example.

Lemma 1.46: *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Denote by $\mathcal{N} \subset \mathcal{F}$ the σ–algebra generated by the null sets of $\mathcal{F}$ w.r.t. P. Furthermore, let $\mathcal{N} \subsetneq \mathcal{F}$. Define the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ via*

$$\mathcal{F}_t = \begin{cases} \mathcal{N}, & \forall\, t < 1, \\ \mathcal{F}, & \forall\, t \geq 1. \end{cases}$$

Then the following statements hold:

1. *$\mathbb{F}$ satisfies the usual conditions.*
2. *If T is a stopping time and $t \in [0,1]$, then $P(T < t) \in \{0,1\}$.*
3. *All optional processes on $[0,1)$ are indistinguishable from a deterministic measurable function on $[0,1)$.*
4. *All predictable processes on $[0,1]$ are indistinguishable from a deterministic measurable function on $[0,1]$.*
5. *M is a uniformly integrable martingale if and only if there exists a random variable $Z \in L^1(\Omega, \mathcal{F}, P)$ such that $M = E[Z] + (Z - E[Z])\mathbb{1}_{[\![1,\infty[\![}$.*
6. *All local martingales are uniformly integrable martingales.*
7. *All semimartingales are of finite variation.*
8. *All σ–martingales are uniformly integrable martingales.*
9. *Let M be a local martingale starting in zero and let $K \in L_m(M)$. Then K_1 is a.s. constant and*
$$K.M = K_1 M_1 \mathbb{1}_{[\![1,\infty[\![}.$$

Proof: *Item 1.*: The first statement holds due to the definition of $\mathbb{F}$.

Item 2.: Since T is a stopping time, we get for $t \in [0,1]$

$$\{T < t\} = \bigcup_{\substack{n \geq m \\ t - \frac{1}{1+m} \geq 0}} \underbrace{\left\{T \leq t - \frac{1}{n+1}\right\}}_{\in \mathcal{F}_{t - \frac{1}{1+n}} = \mathcal{N}} \in \mathcal{N}.$$

By definition, $\mathcal{N}$ contains only sets of measure zero or one. Therefore, *2.* holds.

Item 3.: Due to [27, 3.17 Theorem], the optional σ–algebra on $[0,1)$ equals

$$\sigma\left([\![T, 1[\![;\ T \text{ is a stopping time}\right).$$

Due to *2.*, we can conclude that for any stopping time T and all $t \in [0,1)$

$$\{T = t\} = \{T \leq t\} \cap \{T < t\}^c \in \mathcal{N}.$$

Therefore,

$$[\![T,1[\![= N \times [t,1), \qquad \text{for } N \in \mathcal{N} \text{ and } t \in [0,1),$$

where $\emptyset \times [t,1) := N \times \emptyset := \emptyset$ for all $t \in [0,1)$ and $N \in \mathcal{N}$. Consequently, all optional processes on $[0,1)$ are indistinguishable from a deterministic measurable function.

Item 4.: This holds thanks to the characterisation of the predictable σ–algebra in [27, 3.21 Theorem].

Item 5.: The 'if' part is clear. Let M be a uniformly integrable martingale. Due to *3.*, M has to be constant on $[0,1)$. Thanks to the definition of the filtration and the martingale property, we know that $M_t = M_1$ a.s. for all $t \geq 1$. Since uniformly integrable martingales have a constant expectation and $Z := M_1 \in L^1(P)$, we get the desired result.

Item 6.: Let M be a local martingale and denote by $(T_n)_{n\geq 1}$ a non–decreasing sequence of stopping times tending to ∞ a.s. that localizes M. Thanks to item *2.*, there exists a $N \in \mathbb{N}$ such that $P(T_n < 1) = 0$ for all $n \geq N$. Therefore, $M_1 \in L^1(P)$. Due to the definition of the filtration and the martingale property of M^{T_n}, we get $M_{t\wedge T_n} = M_1$ a.s. for all $t \geq 1$ and all $n \geq N$. Since $T_n \uparrow \infty$ a.s., we get $M_t = M_1$ a.s. for all $t \geq 1$. Thanks to *5.*, M is a uniformly integrable martingale.

Item 7.: This follows from the definition of semimartingales, *5.*, and *6.*

Item 8.: Due to *3.*, we may assume w.l.o.g.[3] that $X_0 = 0$ a.s.. By definition there exists a strictly positive, bounded, and predictable process K such that $K.X$ is a uniformly integrable martingale. Due to item *5.*, $K.X$ is zero on $[0,1)$. Since K is strictly positive and deterministic on $[0,1)$, X has to be zero on $[0,1)$ up to indistinguishablility, too. Indeed, due to *7.*, we know that X is of finite variation. Since $K.X$ is indistinguishable from zero on $[0,1)$, we can conclude that there exists a null set $\tilde{N}$ such that for all $\omega \in \tilde{N}^c$ and all $A \in \mathcal{B}([0,1))$

$$\int_A K(\omega)\, dX(\omega) = 0. \tag{1.12}$$

We denote by $N, P \in \mathcal{B}([0,1))$ the Hahn–decomposition of the signed measure $dX(\omega)$ on $[0,1)$. Due to (1.12), we have

$$0 = \int_{N\cap A} K(\omega)\, dX(\omega) = \int_{N\cap A} K(\omega)\, d\left(-X(\omega)\right)$$
$$0 = \int_{P\cap A} K(\omega)\, dX(\omega)$$

for all $A \in \mathcal{B}([0,1))$. Since $K(\omega)$ is strictly positive, this ensures that $X(\omega)$ has to be zero on $[0,1)$.

[3] abbr.: without loss of generality

Furthermore, K_1 is a.s. constant due to *4.* and $K.X_1 = K_1X_1 - K_1X_{1-} + K.X_{1-}$. Since $K_1 > 0$ a.s., this implies in particular that $X_1 \in L^1(P)$. Because of the martingale property of $K.X$ and the definition of the filtration, we get $K.X_1 = K.X_t$ a.s. for all $t \geq 1$. The same argument as before ensures that $X_1 = X_t$ a.s. for all $t \geq 1$. Due to *5.*, the claim is proven.

Item 9.: Since M is a local martingale starting in zero, we know that

$$M = M_1 \mathbb{1}_{[\![1,\infty[\![}.$$

As a result of *4.*, K_1 is a.s. constant. Moreover, $K.M$ is a local martingale such that $K.M_0 = 0$ and $\Delta K.M_1 = K_1M_1$. Due to *5.* and *6.*, we have

$$K.M = K_1M_1\mathbb{1}_{[\![1,\infty[\![}.$$

□

The next example shows that the Kunita–Watanabe decomposition does not exist in general, if N is assumed to be square–integrable and M is arbitrary.

Example 1.47 (compare [29, Exercices 4.10, p. 141])**:** Consider the setting of Lemma 1.46. Let $U, V \in L^1(\Omega, \mathcal{F}, P)$ be centred random variables. Moreover, let $V \in L^\infty(P)$, $U \notin L^2(P)$, and $E[UV] \neq 0$. Define $N = V\mathbb{1}_{[\![1,\infty[\![}$ and $M = U\mathbb{1}_{[\![1,\infty[\![}$. Note that for $K \in L_m(M)$

$$[N - K.M, M] = VU\mathbb{1}_{[\![1,\infty[\![} - K_1U^2\mathbb{1}_{[\![1,\infty[\![},$$

where K_1 is a.s. constant. Due to Lemma 1.46 and the assumptions, $[N - K.M, M]$ cannot be a σ–martingale for arbitrary $K \in L_m(M)$. Hence, the Kunita–Watanabe decomposition of N w.r.t. M does not exist.

Though the Kunita–Watanabe decomposition does not exit, the Radon–Nikodym decomposition does exit.

Example 1.48: Consider the setting of Example 1.47 and note that

$$[N, M] = VU\mathbb{1}_{[\![1,\infty[\![} = \frac{V\mathbb{1}_{\{U\neq 0\}}}{U}\mathbb{1}_{[\![1]\!]}.[M], \quad a.s..$$

Denote the Radon–Nikodym derivative of $d[N, M]$ w.r.t. $d[M]$ by $H := \frac{V\mathbb{1}_{\{U\neq 0\}}}{U}\mathbb{1}_{[\![1]\!]}$. Due to Lemma 1.46, it suffices to compute $\Delta H \underset{c}{\bullet} M_1$ in order to characterize $H \underset{c}{\bullet} M$. By the definition of the compensated stochastic integral and the fact that $\Delta M_1 = M_1$, we can conclude that

$$\Delta H \underset{c}{\bullet} M_1 = H_1 \Delta M_1 - {}^p(H\Delta M)_1 = V\mathbb{1}_{\{U\neq 0\}} - {}^p(H\Delta M)_1.$$

Lemma 1.46 guarantees that

$$\left(V\mathbb{1}_{\{U\neq 0\}} - E\big[V\mathbb{1}_{\{U\neq 0\}}\big]\right)\mathbb{1}_{[\![1,\infty[\![}$$

is a uniformly integrable martingale. Hence, [27, 7.13 Theorem] ensures that

$${}^{p}\left(\Delta\left(V\mathbb{1}_{\{U\neq 0\}} - E\big[V\mathbb{1}_{\{U\neq 0\}}\big]\right)\mathbb{1}_{[\![1,\infty[\![}\right) = 0$$

and

$$\begin{aligned}{}^{p}(H\Delta M)_1 &= {}^{p}\left(\Delta\left(V\mathbb{1}_{\{U\neq 0\}} - E\big[V\mathbb{1}_{\{U\neq 0\}}\big]\right)\mathbb{1}_{[\![1,\infty[\![}\right)_1 + E\left[V\mathbb{1}_{\{U\neq 0\}}\right]\\ &= E\left[V\mathbb{1}_{\{U\neq 0\}}\right].\end{aligned}$$

Combining the results we end up with

$$\Delta H\underset{c}{\bullet}M_1 = V\mathbb{1}_{\{U\neq 0\}} - E\left[V\mathbb{1}_{\{U\neq 0\}}\right].$$

Due to Lemma 1.46, we get

$$H\underset{c}{\bullet}M = \left(V\mathbb{1}_{\{U\neq 0\}} - E\left[V\mathbb{1}_{\{U\neq 0\}}\right]\right)\mathbb{1}_{[\![1,\infty[\![}.$$

Consequently, $L := N - H\underset{c}{\bullet}M$ satisfies

$$\begin{aligned}[L, M] &= [N - H\underset{c}{\bullet}M, M] = V\mathbb{1}_{\{U=0\}}U\mathbb{1}_{[\![1,\infty[\![} - E\left[V\mathbb{1}_{\{U\neq 0\}}\right]U\mathbb{1}_{[\![1,\infty[\![}\\ &= -E\left[V\mathbb{1}_{\{U\neq 0\}}\right]U\mathbb{1}_{[\![1,\infty[\![}.\end{aligned}$$

Since U is centred, Lemma 1.46 ensures that $[L, M]$ is a martingale.

1.5.2. Case 2: N arbitrary, M not continuous

The Kunita–Watanabe decomposition also does not exist in general in Case 2, i.e. if N is arbitrary and M is not continuous. To set the example we need the next lemma. It makes use of the following notation. A Borel–measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ is an element of $L^p_{loc}(\mathbb{R}_+, dt)$ for $p \geq 1$, if $f\mathbb{1}_{[0,n]} \in L^p(\mathbb{R}_+, dt)$ for all $n \in \mathbb{N}$. In the next lemma, we collect some technical results which we need for the following example.

Lemma 1.49: *Let B be a standard Brownian motion, P be a Poisson process with intensity 1, and $f \in L^1(\mathbb{R}_+, dt) \setminus L^2_{loc}(\mathbb{R}_+, dt)$. Denote by $(T_n)_{n\geq 1}$ the sequence of jump times of P and define the local martingale M by $M = B + P - t$. Then $f \in L_m(P - t)$, $(\sum_{n\geq 1} f(T_n)\mathbb{1}_{[\![T_n]\!]}) \in {}^{o}L_{loc}(M)$, and $f \notin L_m(M)$.*

Proof: First note that

$$\begin{aligned}[(P-t)]_t &= \sum_{s\leq t}\Delta P_s^2 = \sum_{n\geq 1}\mathbb{1}_{[T_n,\infty)}(t),\\ f^2.\,[(P-t)]_t &= \sum_{n\geq 1} f^2(T_n)\mathbb{1}_{[T_n,\infty)}(t).\end{aligned}$$

Furthermore, we have

$$\left(\sum_{n\geq 1} f(T_n)\mathbb{1}_{[\![T_n]\!]}\right)^2 . [M]_t = f^2 . [(P-t)]_t .$$

Due to the sub–additivity of the square root, see Lemma A.20, we get

$$\left(f^2 . [(P-t)]_t\right)^{\frac{1}{2}} \leq |f| . [(P-t)]_t .$$

Lemma 1.27 ensures that

$$E\Big[|f| . [(P-t)]_t\Big] = \int_0^t |f_u| \, du.$$

Combining the last three equations, we get $\sum_{n\geq 1} f(T_n)\mathbb{1}_{[\![T_n]\!]} \in {}^oL_{loc}(M)$ and $f \in L_m(P-t)$. By assumption $f \notin L_m(B)$. Due to [27, 9.3 Theorem], we know that $L_m(M) = L_m(B) \cap L_m(P-t)$. Hence, $f \notin L_m(M)$. □

Here is another example which shows that the Kunita–Watanabe decomposition does not exist in general.

Example 1.50 ([3, D.K.W. cas 4:])**:** Consider the setting of Lemma 1.49. Define $N = f.(P-t)$ and let $K \in L_m(M)$. Since M is locally bounded, Fefferman's inequality, see [27, 10.17 Theorem], ensures that $[N - K.M, M] \in \mathcal{A}_{loc}$. Suppose that there exists $K \in L_m(M)$ such that $[N - K.M, M]$ is a local martingale. Lemma 1.27 and Fefferman's inequality ensure that the l.h.s.[4] of the equation

$$\begin{aligned}
[N - K.M, M] + K.([M] - \langle M\rangle) &= f.[(P-t)] - K.\langle M\rangle \\
&= f.[(P-t)] - 2K.\langle (P-t)\rangle
\end{aligned}$$

is a local martingale. As a result of the uniqueness of the compensator, we get $2K = f$ a.s. (w.r.t. the Lebesgue–measure). This is a contradiction to $f \notin L_m(M)$. Therefore, N does not admit a Kunita–Watanabe decomposition w.r.t. M.

The Radon–Nikodym decomposition that corresponds to Example 1.50 is provided in the next example.

Example 1.51: Consider the setting of Example 1.50 and note that

$$[N, M] = f.[(P-t)] = \left(\sum_{n\geq 1} f(T_n)\mathbb{1}_{[\![T_n]\!]}\right) .[M].$$

[4] abbr.: left hand side

Due to Lemma 1.49, we know that $H = \left(\sum_{n\geq 1} f(T_n)\mathbb{1}_{[\![T_n]\!]}\right) \in {}^oL_{loc}(M)$. Furthermore, Theorem 1.34 ensures that $N = H \underset{c}{\bullet} M + L$ and $[L, M]$ is a σ–martingale. Since M is locally bounded, we can apply Fefferman's inequality to conclude that $[L, M]$ even is a local martingale. Note that the more detailed characterization of Theorem 1.36 allows us to conclude that $L^c = 0$ and $(H \underset{c}{\bullet} M)^c = 0$ up to indistinguishability. Hence, by definition of the compensated integral for purely discontinuous local martingales we can conclude that $L^d = 0$ and

$$N = H \underset{c}{\bullet} M$$

holds up to indistinguishability.

Although Example 1.50 shows that the Kunita–Watanabe decomposition of N w.r.t. M does not exist, the next example shows that the Kunita–Watanabe decomposition of M w.r.t. N does exist.

Example 1.52: Consider the setting of Example 1.50. If f is strictly positive, [27, 9.3 Theorem] ensures that $f^{-1} \in L_m(N)$ and $f^{-1}.N = (P - t)$. Thus, M admits a Kunita–Watanabe decomposition w.r.t. N. Indeed,

$$M = B + (P - t) = f^{-1}.N + B$$

and $[N, B] \equiv 0$. Combining this result with Example 1.50 shows that the existence of the Kunita–Watanabe decomposition is not a symmetric property in general.

2. A structured approach to structure conditions

2.1. Introduction

The Fundamental Theorem of Asset Pricing **(FTAP)** is for sure one of the most important results in mathematical finance. It provides an economically meaningful no arbitrage condition, the *no free lunch with vanishing risk* condition **(NFLVR)**. Moreover, it ensures that **(NFLVR)** is necessary and sufficient for the existence of a special pricing operator, an equivalent σ–martingale measure. For references on the history of the Fundamental Theorem of Asset Pricing and a proof of **(FTAP)**, we refer to [17] and [18]. The building blocks of **(NFLVR)** are the following weaker conditions: the *no arbitrage* condition **(NA)** and the *no unbounded profit with bounded risk* condition **(NUPBR)**. During the last decade, several authors provided equivalent reformulations of the **(NUPBR)** condition. The most popular of these reformulations is probably the existence of the so–called *numéraire portfolio*; see [34]. Most recently, Schweizer and Takaoka [52] proved that the **(NUPBR)** condition is also equivalent to the existence of a *strictly positive σ–martingale density* for the underlying semimartingale. A beautiful overview of this topic can be found in [28]. The main point of these equivalent reformulations is the fact that they all ensure the existence of a reasonable pricing operator to price essentially the terminal wealth of all 1–admissible trading strategies. But how can we find a natural candidate for e.g. a strictly positive σ–martingale density for an arbitrary, locally square–integrable semimartingale $S = M + A$, where M denotes the local martingale of the canonical decomposition of S?

For continuous semimartingales the structure condition **(SC)** is a very good tool that leads *directly* to a natural candidate for a strictly positive σ–martingale density, the *minimal martingale measure*. In order to explain the importance of **(SC)** as a good tool for finding strictly positive σ–martingale densities in the continuous paths case, we recall its definition. S satisfies the structure condition **(SC)**, if

$$S = M + \int \tilde{\lambda}\, d\langle M\rangle,$$

where $\tilde{\lambda} \in \mathcal{P}$ can be chosen such that $\int \tilde{\lambda}\, dM$ is a locally square–integrable martingale. Due to this *specific structure* of the semimartingale S, one can show that all strictly positive σ–martingale densities $\mathcal{E}(-N)$ for S feature a *specific* Kunita–Watanabe *decomposition*. More precisely,

$$N = \int \tilde{\lambda}\, dM + L, \tag{2.1}$$

where L is a local martingale, orthogonal to M, and $\int \tilde{\lambda}\, dM$ is a *locally square–integrable* martingale. Throughout this chapter, we call this type of theorems, linking the structure of a semimartingale S to the specific structure of strictly positive σ–martingale densities $\mathcal{E}(-N)$ of S, *structure theorems*.

Let us get back to the structure theorem presented above. The crucial point is the additional assumption that S has continuous paths. This ensures that the minimal martingale measure

$$\mathcal{E}\left(-\int \tilde{\lambda}\, dM\right)$$

is a *strictly positive* σ–martingale density for S. Although this particular structure theorem also holds for arbitrary, locally square–integrable semimartingales, it loses its importance as an indicator for a natural candidate for a *strictly positive* σ–martingale density. Moreover, the degree of flexibility provided by the structure theorem (we can vary the orthogonal local martingale in (2.1)), in order to find a strictly positive σ–martingale density, is rather low. For a proof of this particular structure theorem, we refer to [15]. Similar results can be found in [50].

Another problem that arises if one translates the ideas of **(SC)** in a direct way to arbitrary, locally square–integrable semimartingales, is the fact that **(SC)** is neither necessary nor sufficient for the existence of a strictly positive σ–martingale density. Besides, the structure condition **(SC)** is not invariant under (proper) equivalent measure changes. All these drawbacks of **(SC)** are well known. So far, the only detailed discussion on pros and cons of **(SC)** can be found in [15]. Summarizing the above arguments leads to the conclusion that the structure condition **(SC)** and the structure theorem related to **(SC)** are very good tools for finding strictly positive σ–martingale densities for *continuous* semimartingales. However, for arbitrary, locally square–integrable semimartingales it loses its importance as a good tool for finding strictly positive σ–martingale densities.

Of course, it would be very nice to have a complete characterisation of the **(NUPBR)** condition (or equivalently the existence of a strictly positive σ–martingale density) in terms of a structure condition, not only for continuous but for all semimartingales. All objectives of this chapter go into this direction.

The first goal of this chapter is to explain the connection between 'martingale decomposition theorems' of a (strictly positive) σ–martingale density for a semimartingale S on the one hand, and 'structure conditions' of S on the other hand.

These connections are provided in several 'structure theorems'. We use the Radon–Nikodym decomposition of a strictly positive σ–martingale density $\mathcal{E}(-N)$ of locally square–integrable semimartingale in order to provide our 1^{st} *Structure Theorem*. Moreover, we provide a geometric interpretation of the 1^{st} Structure Theorem that leads in a very natural way to our first structure condition, the *minimal structure condition* **(MSC)**. It turns out that **(MSC)** has several desirable properties. For example, **(MSC)** is invariant under equivalent measure changes and equivalent to **(SC)** for continuous semimartingales. By means of *toy examples* and different lemmata, we highlight the connections and differences between **(MSC)**, **(SC)**, and the *weak structure condition* **(SC′)**. Using the insights of the 1^{st} Structure Theorem and the minimal structure condition, we introduce an additional *natural structure condition* **(NSC)**. Under the assumption that a semimartingale satisfies the natural structure condition, we provide our 2^{nd} *Structure Theorem*. Its particular importance lies in its flexibility. Indeed, it takes into account that the predictable, finite variation part A of a locally square–integrable semimartingale $S = M + A$ might have *several* possible decompositions. Furthermore, it links each of these decompositions to a natural candidate $\mathcal{E}(-N)$ for a strictly positive σ–martingale density. These candidates, more precisely N, all feature a *natural Kunita–Watanabe decomposition* w.r.t. M. Taking into account the different decompositions of A makes our 2^{nd} Structure Theorem a remarkably powerful tool for finding possible candidates for strictly positive σ–martingale densities.

In the second part of this chapter, we use the new insights achieved so far to provide further structure conditions that are sufficient for the existence of a strictly positive σ–martingale density. Using the refined version of the Radon–Nikodym decomposition, see Theorem 1.36, we derive our 3^{rd} *Structure Theorem*. Taking into consideration the insights provided by this theorem, we introduce our third structure condition, the *strong structure condition* **(SSC)**. In several examples we highlight the different amounts of 'sensitivity' of the different structure conditions **(SC)**, **(NSC)**, and **(SSC)** as tools for finding a strictly positive σ–martingale density. As an application of the insights achieved so far, we provide a full structural characterisation of the **(NUPBR)** condition for a class of toy examples. Finally, we introduce our last structure condition, the *floating structure condtion* **(FSC)**. The condition **(FSC)** provides, similar as **(SC)** in the continuous paths case, a natural candidate for a strictly positive σ–martingale density, the *floating martingale density*. Moreover, it turns out that for continuous semimartingales the minimal martingale measure and the floating martingale density coincide.

The chapter is structured as follows. Section 2 provides an overview of several classical no arbitrage conditions, definitions of σ–martingales and the classical structure conditions **(SC)** and **(SC′)**. Besides, we recall some invariance properties of these objects under equivalent measure changes. In Section 3, we use the Radon–

Nikodym decomposition to derive the minimal structure condition. Furthermore, we introduce the natural structure condition **(NSC)** for a semimartingale S and explain its connection to the natural Kunita–Watanabe decomposition of strictly positive σ–martingale densities for S. In Section 4, we focus on the practical use of structure conditions as indicators for the existence of a strictly positive σ–martingale density. This is done by introducing the strong structure condition **(SSC)** and the floating structure condition **(FSC)**. We end the chapter with Section 5. It provides a short conclusion and some remarks on possible generalizations of the results provided before.

2.2. Definition and preliminary results

In this section, we collect several definitions of no arbitrage conditions and structure conditions. Moreover, we summarize some well known results on measure changes and stochastic integration for reference purposes. The reader, who is familiar with this theory can skip this section. In order to allow the reader to easily compare the results of this chapter to well known results in the literature, we adapt the notation and setup of the survey article [28]. Throughout this chapter, we consider a complete stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}$ satisfies the usual conditions and the time horizon is finite, i.e. $T \in (0, \infty)$. Results from the literature, that are formulated for an infinite time horizon, are used by applying the respective result to the stopped process. We consider a financial market with $d+1$ assets. One of these assets serves as a numéraire and is denoted by S^0. All other quantities are expressed in units of this numéraire. W.l.o.g. we assume that $S^0 \equiv 1$. The d risky assets are modelled by a $\mathbb{R}^d$–valued semimartingale $S = (S_t)_{0\leq t\leq T}$, where S^i denotes the price of the i^{th} risky asset. We suppose that trading in the financial market is frictionless. Moreover, we only allow self–financing strategies, i.e. a trading strategy is given by a pair (x, ϑ), where $x \in \mathbb{R}$ is the initial capital and $\vartheta = (\vartheta_t)_{0\leq t\leq T} \in L(S)$. As usual, $L(S)$ denotes the set of $\mathbb{R}^d$–valued predictable and S–integrable processes. The vector–stochastic integral of $\vartheta \in L(S)$ w.r.t. S is denoted by $\vartheta \cdot S := \int \vartheta\, dS$. For details on the vector–stochastic integral, we refer to [12], [30], and [33]. The wealth–process associated with a self–financing trading strategy (x, ϑ) is denoted by

$$X_t^{x,\vartheta} := x + \vartheta \cdot S_t = x + \int_0^t \vartheta_u\, dS_u, \qquad 0 \leq t \leq T.$$

A strategy $\vartheta \in L(S)$ is called *a–admissible* for $a \geq 0$, if $\vartheta \cdot S \geq -a$. For $x \geq 0$ we define

$$\mathcal{X}^x := \left\{X^{x,\vartheta} \mid \vartheta \in L(S) \text{ and } X^{x,\vartheta} \geq 0\right\}$$

and set $\mathcal{X}_T^x := \{X_T^{x,\vartheta} \mid X^{x,\vartheta} \in \mathcal{X}^x\}$.

2.2.1. Strictly positive σ–martingale density and no arbitrage conditions

Let us recall the definitions of the most popular no arbitrage conditions.

Definition 2.1 ([52, Definition 2.3])**:** *The semimartingale S is said to satisfy the* no arbitrage *condition* **(NA)***, if for all* 1*–admissible strategy $\vartheta \in L(S)$ the following implication holds:*

$$\vartheta \cdot S_T \geq 0 \ a.s. \quad \Longrightarrow \quad \vartheta \cdot S_T = 0 \ a.s..$$

The next condition is equivalent to **(NA)** in the discrete time setting; see [31]. However, in a continuous time setting this does not hold in general.

Definition 2.2 ([28, Remark 3), p. 11])**:** *The semimartingale S is said to satisfy the condition* **(NA$_+$)***, if for any* 0*–admissible strategy $\vartheta \in L(S)$ the following implication holds:*

$$\vartheta \cdot S_T \geq 0 \ a.s. \quad \Longrightarrow \quad \vartheta \cdot S \equiv 0.$$

For more details on **(NA$_+$)** we refer to [51]. The next definition provides a mathematical characterisation of the **(NUPBR)** condition.

Definition 2.3 ([52, Definition 2.2])**:** *The semimartingale S is said to satisfy the* no unbounded profit with bounded risk **(NUPBR)** *condition, if the set $\mathcal{X}_T^1$ is bounded in L^0, i.e.*

$$\lim_{c\to\infty} \sup_{\vartheta \text{ is 1–admissible}} P\left(|\vartheta \cdot S_T| > c\right) = 0.$$

The terms 'strictly positive σ–martingale density' and 'equivalent σ–martingale measure' are characterized in the next definition.

Definition 2.4 (compare [28, Definition p. 5])**:** *A* strictly positive σ–martingale density *(or* strictly positive local martingale density*) for S is a local P–martingale $Z = (Z_t)_{0\leq t\leq T}$ with the following properties:*

1. *$Z_0 = 1$ a.s..*
2. *$Z > 0$ up to indistinguishability.*
3. *ZS^i is a P–σ–martingale (a local P–martingale) for each $i \in \{1, \ldots, d\}$.*

If a strictly positive σ–martingale density (or a strictly positive local martingale density) Z satisfies $E[Z_T] = 1$, the measure $dQ := Z_T\, dP$ is called an equivalent σ–martingale measure *(*equivalent local martingale measure*) for S.*

Remark 2.5: *Throughout this chapter, we frequently use the following fact. If Z is a strictly positive local martingale, where $Z_0 = 1$ a.s., then there exists an (up to indistinguishability) unique local martingale N such that $N_0 = 0$ a.s. and $Z = \mathcal{E}(-N)$ hold; see* [30, 8.3 Theorem].

We end this subsection with a lemma on strictly positive σ–martingale densities.

Lemma 2.6: *Let S be a $\mathbb{R}^d$–valued P–semimartingale and $Z = (Z_t)_{0 \leq t \leq T}$ a local P–martingale with $Z_0 = 1$ and $Z > 0$. Moreover, let φ be a $\mathbb{R}$–valued process that satisfies at least one of the following conditions:*

1. *φ is a strictly positive and locally bounded process with left–continuous paths.*
2. *φ is a predictable, bounded process that is bounded away from zero. I.e., there exists $\epsilon > 0$ such that $|\varphi| \geq \epsilon$.*

Then Z is a strictly positive σ–martingale density (or strictly positive local martingale density) for S if and only if $\varphi \cdot (ZS^i)$ is a P–σ–martingale (a local P–martingale) for each $i \in \{1, \dots, d\}$.

Proof: We only prove the statement for φ satisfying the first item. The proof for *2.* follows the same lines. Due to the properties of φ, we know φ^{-1} is a $\mathbb{R}$–valued, strictly positive, and locally bounded process with left–continuous paths, too. Moreover, the class of σ–martingales as well as the class of local martingales are stable w.r.t. stochastic integration of locally bounded integrands. Due to the associativity of the stochastic integral, we get

$$ZS^i = 1 \cdot \left(ZS^i\right) = \left(\varphi^{-1}\varphi\right) \cdot \left(ZS^i\right) = \varphi^{-1} \cdot \left(\varphi \cdot \left(ZS^i\right)\right) \qquad \text{for } i \in \{1, \dots, d\}.$$

□

2.2.2. Invariance principles and the classical structure conditions

Invariance principles

In mathematical finance equivalent measure changes are the most important tool. In the following, we collect some important results that are invariant under an equivalent change of measure.

Lemma 2.7: *Let S be a $\mathbb{R}^d$–valued semimartingale and let Q be a probability measure equivalent to P, i.e. $Q \sim P$. Then S satisfies* **(NUPBR)** *(*(**NA**)*,* (**NA$_+$**)*) w.r.t. P if and only if S satisfies* **(NUPBR)** *(*(**NA**)*,* (**NA$_+$**)*) w.r.t. Q.*

The existence of a strictly positive σ–martingale density is also invariant under equivalent measure changes.

Lemma 2.8: *Let S be a $\mathbb{R}^d$–valued semimartingale and $Q \sim P$. Then there exists a strictly positive σ–martingale density (or strictly positive local martingale density) for S under P if and only if there exists a strictly positive σ–martingale density (or strictly positive local martingale density) for S under Q.*

Proof: If there exists a strictly positive σ–martingale density (or strictly positive local martingale density) Z for S under P, we know that Z is a local P–martingale and ZS^i are P–σ–martingales (local P–martingales) for all $i \in \{1, \dots, d\}$. If we denote by $(D_t)_{t \leq T}$ the density process of P w.r.t. Q, [33, Proposition 5.1] ensures that DZ is a strictly positive local Q–martingale and DZS^i are Q–σ–martingales for all $i \in \{1, \dots, d\}$. □

All structure conditions being considered in this chapter rely on the existence of the predictable quadratic variation of the local martingale part of a special semimartingale. The predictable quadratic variation of the local martingale exists if and only if the local martingale is in fact locally square–integrable. Therefore, we need the following subclass of special semimartingales.

Definition 2.9: *Let S be a $\mathbb{R}^d$–valued semimartingale and $Q \sim P$. S is called a* locally square–integrable Q–semimartingale, *if it is a special semimartingale under Q with canonical decomposition $S = S_0 + M + A$ and the following properties hold:*

1. *M is a $\mathbb{R}^d$–valued, locally square–integrable Q–martingale with $M_0 = 0$.*

2. *A is a $\mathbb{R}^d$–valued, adapted, and predictable process of finite variation with $A_0 = 0$.*

We denote the set of locally square–integrable Q–semimartingales by $S \in \mathcal{S}^2_{loc}(Q)$.

The next lemma ensures that we can always find such a nice equivalent measure. Proofs of the statement can be found in [39] or [19].

Lemma 2.10: *Let S be a $\mathbb{R}^d$–valued semimartingale on $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Then there exists a probability measure $Q \sim P$, such that S is a locally square–integrable Q–semimartingale.*

The following lemma is the key ingredient to prove our structure theorems. Actually, it is an application of the product rule for semimartingales.

Key lemma 2.11: *Let S be a $\mathbb{R}^d$–valued P–semimartingale and $Q \sim P$. Further, let S be a special Q–semimartingale and denote its canonical decomposition (under Q) by $S = S_0 + M + A$. Moreover, let $Z = \mathcal{E}(-N)$ be a strictly positive local Q–martingale. Then Z is a strictly positive Q–σ–martingale (local Q–martingale) density for S if and only if $[N, M^i] - A^i$ is a Q–σ–martingale (local Q–martingale) for all $i \in \{1, \dots, d\}$.*

Proof: The definition of a strictly positive σ–martingale density is a 'component–wise' definition. Therefore, we can assume w.l.o.g. that $d = 1$. Since $Z = \mathcal{E}(-N) = 1 - \int \mathcal{E}(-N)_- \, dN$, the product rule leads to

$$\begin{aligned} ZS - S_0 &= \mathcal{E}(-N)_- \cdot \left(- \int S_- \, dN + M + A - [N, M] - [N, A] \right) \\ &= \mathcal{E}(-N)_- \cdot \left(- \int S_- \, dN + M - [N, A] \right) - \mathcal{E}(-N)_- \cdot ([N, M] - A) \, . \end{aligned}$$

Since $\mathcal{E}(-N)_-$ is left–continuous, locally bounded, and strictly positive, the process $\frac{1}{\mathcal{E}(-N)_-}$ is well defined and satisfies the same properties. Integrating $\frac{1}{\mathcal{E}(-N)_-}$ on both sides of the equation gives

$$\frac{1}{\mathcal{E}(-N)_-} \cdot (ZS - S_0) = \left(- \int S_- \, dN + M - [N, A] \right) - ([N, M] - A) \, .$$

Due to Yoeurp's lemma [27, 9.4 Examples. 1)], the first bracket term on the r.h.s. is a Q–local martingale. Thanks to Lemma 2.6, the l.h.s. is a Q–σ–martingale (a local Q–martingale) if and only if Z is a strictly positive Q–σ–martingale (local Q–martingale) density for S. Therefore, the claim is proven. □

Structure condition and weak structure condition

We already mentioned the classical structure conditions, **(SC)** and **(SC′)**, in the introduction. In this subsection, we provide rigorous definitions. Let S be a $\mathbb{R}^d$–valued, locally square–integrable P–semimartingale with canonical decomposition $S = S_0 + M + A$, where M is a $\mathbb{R}^d$–valued local martingale. Since M is locally square–integrable, the bracket process $\langle M, M \rangle$ exists. Note that $\langle M, M \rangle$ is a $\mathbb{R}^d \times \mathbb{R}^d$–valued, adapted, and predictable process. We denote its components by $\langle M^i, M^j \rangle$, where $i, j \in \{1, \ldots, d\}$. Moreover, we denote by $\langle M \rangle$ the $\mathbb{R}^d$–valued process, whose i^{th} component is given by $\langle M \rangle^i := \langle M^i, M^i \rangle$.

Definition 2.12 (compare [28, Definition p. 7])**:** *Let S be a $\mathbb{R}^d$–valued semimartingale. S is said to satisfy the* weak structure condition **(SC′)** *under P, if $S \in \mathcal{S}^2_{loc}(P)$ with canonical decomposition $S = S_0 + M + A$ and such that A is absolutely continuous w.r.t. $\langle M, M \rangle$. A is absolutely continuous w.r.t. $\langle M, M \rangle$, if there exists a $\mathbb{R}^d$–valued, predictable process $\hat{\lambda} = (\hat{\lambda}_t)_{0 \le t \le T}$ such that $A = \int \, d\langle M, M \rangle \hat{\lambda}$, i.e.*

$$A^i_t = \sum_{j=1}^{d} \int_0^t \hat{\lambda}^j_u \, d\langle M^i, M^j \rangle_u \qquad \textit{for } i \in \{1, \ldots, d\} \textit{ and } 0 \le t \le T.$$

$\hat{\lambda}$ is called the (instantaneous) market price of risk for S.

Let S be a $\mathbb{R}^d$–valued semimartingale that satisfies **(SC′)**. Denote its canonical decomposition by $S = S_0 + M + A$ and its instantaneous market price of risk by $\hat{\lambda} = (\hat{\lambda}_t)_{0\le t\le T}$. Due to the Kunita–Watanabe inequality, the Radon–Nikodym derivative $\bar{\lambda}^{i,j} = (\bar{\lambda}^{i,j}_t)_{0\le t\le T}$ of $d\langle M^i, M^j\rangle$ w.r.t. $d\langle M^i\rangle$ exists for all $i, j \in \{1, \dots, d\}$. Therefore, the $\mathbb{R}^d$–valued, predictable process $\tilde{\lambda} = (\tilde{\lambda}_t)_{0\le t\le T}$, where

$$\tilde{\lambda}^i_t := \sum_{j=1}^{d} \hat{\lambda}^j_t \bar{\lambda}^{i,j}_t \qquad \text{for } i \in \{1, \dots, d\} \text{ and } 0 \le t \le T, \tag{2.2}$$

is well–defined and satisfies

$$A^i_t = \int_0^t \tilde{\lambda}^i_u \, d\langle M^i, M^i\rangle_u = \int_0^t \tilde{\lambda}^i_u \, d\langle M^i\rangle_u \qquad \text{for } i \in \{1, \dots, d\} \text{ and } 0 \le t \le T.$$

Throughout the whole chapter, we work with this version of the weak structure condition. In order to refer to it, we state the result in the following lemma.

Lemma 2.13: *Let S be a $\mathbb{R}^d$–valued semimartingale that satisfies* **(SC′)** *and let $\tilde{\lambda} = (\tilde{\lambda}_t)_{0\le t\le T}$ be the $\mathbb{R}^d$–valued, predictable process defined in (2.2). Then S can be written as $S = S_0 + M + \int \tilde{\lambda} \, d\langle M\rangle$, where*

$$S^i_t = S^i_0 + M^i_t + \int_0^t \tilde{\lambda}^i_u \, d\langle M^i\rangle \qquad \textit{for } i \in \{1, \dots, d\} \textit{ and } 0 \le t \le T.$$

Remark 2.14: *One has to keep in mind that the predictable quadratic variation of a locally square–integrable martingale* is not invariant under equivalent measure changes *in general! To indicate that we mean the predictable quadratic variation under a measure $Q \sim P$, we write $\langle M\rangle^Q$.*

The next lemma states that any reasonable no arbitrage assumption implies the weak structure condition **(SC′)**.

Lemma 2.15 ([51, Theorem 2.2])**:** *Let S be a $\mathbb{R}^d$–valued P–semimartingale and $Q \sim P$ such that $S \in \mathcal{S}^2_{loc}(Q)$. If S satisfies either* **(NA)**, **(NA$_+$)**, *or* **(NUPBR)**, *then S satisfies the weak structure condition* **(SC′)** *under Q.*

The next definition characterizes the structure condition **(SC)**.

Definition 2.16 (compare [28, Definition p. 7])**:** *If S satisfies the weak structure condition* **(SC′)** *under P, we define*

$$\hat{K}_t := \int_0^t \hat{\lambda}^{tr}_u d\langle M, M\rangle \hat{\lambda}_u = \sum_{i,j=1}^{d} \int_0^t \hat{\lambda}^i_u \hat{\lambda}^j_u \, d\langle M^i, M^j\rangle_u, \qquad 0 \le t \le T,$$

and call $\hat{K} = (\hat{K}_t)_{0 \le t \le T}$ *the* mean–variance tradeoff process *of* S*. Because* $\langle M, M \rangle$ *is positive semidefinite, the process* $\hat{K}$ *is increasing and null at* 0*; but note that it may take the value* $+\infty$ *in general. We say that* S *satisfies the* structure condition **(SC)***, if* S *satisfies* **(SC′)** *and* $\hat{K}_T < \infty$ P*–a.s..*

The next lemma provides equivalent characterisations of **(SC)** that are also used in the literature; see e.g. [50].

Lemma 2.17: *Let* S *be a* $\mathbb{R}^d$*–valued semimartingale that satisfies the weak structure condition* **(SC′)** *under* P*. Denote by* $S = S_0 + M + A$ *its canonical decomposition and by* $\hat{\lambda}$ *the instantaneous market price of risk for* S*. The following statements are equivalent:*

1. *S satisfies* **(SC)***.*
2. *There exists an increasing sequence* $(T_n)_{n \ge 1}$ *of stopping times such that*
   a) $P(T_n = T) \to 1$ *for* $n \to \infty$*,*
   b) $\hat{K}_{T_n} = \sum_{i,j=1}^{d} \int_0^{T_n} \hat{\lambda}^i_u \hat{\lambda}^j_u \, d\langle M^i, M^j \rangle_u \le n$*.*
3. *There exists an increasing sequence* $(T_n)_{n \ge 1}$ *of stopping times such that*
   a) $P(T_n = T) \to 1$ *for* $n \to \infty$*,*
   b) $E\left[\hat{K}_{T_n}\right] < \infty$*.*
4. $\hat{\lambda} \in L^2_{loc}(M^T)$ *in the sense of* [29].

Proof: We first prove the equivalence of the first two statements. If S satisfies **(SC)**, the mean–variance tradeoff process $\hat{K}$ is a càdlàg, increasing, and predictable process. Due to [27, 7.7 Theorem], there exists an increasing sequence $(T_n)_{n \ge 1}$ of stopping times such that for all $n \ge 1$

$$\hat{K}_{T_n} = \sum_{i,j=1}^{d} \int_0^{T_n} \hat{\lambda}^i_u \hat{\lambda}^j_u \, d\langle M^i, M^j \rangle_u \le n$$

holds. Since

$$P\left(T_n \ne T\right) = P\left(K_T > n\right),$$

the equivalence of the first two statements follows. As *2.*⇒*3.*⇒*4.*, it remains to prove that *4.* implies the first item. Since $\hat{\lambda} \in {}^p L^2_{loc}(M^T)$, there exists an increasing sequence $(S_n)_{n \ge 1}$ of stopping times such that

1. $S_n \to \infty$ P–a.s. for $n \to \infty$,

2. $E\left[\hat{K}_{S_n\wedge T}\right] = E\left[\sum_{i,j=1}^{d}\int_0^{S_n\wedge T}\hat{\lambda}^i_u\hat{\lambda}^j_u \, d\langle M^i, M^j\rangle_u\right] < \infty.$

Moreover, the increasing sequence $(T_n)_{n\geq 1}$ of stopping times, where $T_n := S_n \wedge T$, satisfies $P(T_n = T) \to 1$ for $n \to \infty$. Since

$$P\left(\hat{K}_T < \infty\right) \geq P\left(\left\{\hat{K}_T < \infty\right\} \cap \{T_n = T\}\right), \qquad \forall\, n \geq 1,$$

the claim is proven. □

Remark 2.18:

1. *A sequence of stopping times satisfying the property 2.a) is called a γ–localizing sequence in* [13].

2. *The statement of the lemma is wrong, if we do not consider M as a* stopped, *locally square–integrable martingale on $\Omega \times \mathbb{R}_+$. Indeed, consider a standard Brownian motion $M := B$ on $\Omega\times[0,1]$ and let $\hat{\lambda} := (\mathbb{1}_{[0,1)}1/(1-u)+\mathbb{1}_{\{1\}})_{u\in[0,1]}$. Obviously, $\int_0^1 \hat{\lambda}_u^2 \, du = \infty$ holds. On the other hand, the sequence $(T_n)_{n\geq 1}$, where $T_n = 1-1/(1+n)$, satisfies $T_n \to 1$ a.s. and $|\hat{\lambda}_u \mathbb{1}_{[\![0,T_n]\!]}| \leq n+1$. Hence, $\int \hat{\lambda}\mathbb{1}_{[\![0,T_n]\!]} \, dB_u$ is a square–integrable martingale for all $n \geq 1$. Notice that $(T_n)_{n\geq 1}$ is not a γ–localizing sequence.*

Finally, we state the definition of the minimal martingale measure.

Definition 2.19: *Let S satisfy* **(SC)***. The process $\mathcal{E}\left(-\int \lambda \, dM\right)$ is called the* minimal martingale measure.

Remark 2.20: *Note that the minimal martingale measure is neither a uniformly integrable martingale, see* [46]*, nor non–negative (as we will see below) in general!*

2.3. Structure conditions and their connection to martingale decomposition theorems

In [52], Schweizer and Takaoka proved the following result on the **(NUPBR)** condition.

Theorem 2.21 ([52, Theorem 2.6])**:** *The $\mathbb{R}^d$–valued semimartingale S satisfies the condition* **(NUPBR)** *if and only if there exists a strictly positive σ–martingale density for S.*

This result ranks among a long sequence of equivalent reformulations of the condition **(NUPBR)**. All these results provide a reasonable pricing operator for wealth processes of 1–admissible trading strategies. The most prominent ones are the *growth optimal portfolio* and the *numéraire portfolio*; see [28] for references and details. For continuous semimartingales the **(NUPBR)** condition is closely tied to a structure condition of the underlying semimartingale. This condition is well known under the name *structure condition* **(SC)**. For continuous semimartingales the result above can be extended in the following way.

Lemma 2.22 ([15, Théorème 2.9])**:** *Let $S = S_0 + M + A$ be a continuous semimartingale, where M is the local martingale of the canonical decomposition. The following statements are equivalent:*

1. *There exists a strictly positive local martingale density for S.*
2. *S satisfies the* **(NUPBR)** *condition.*
3. *S satisfies the* **(SC)** *condition.*

There are several attempts to extend the ideas of the structure condition to arbitrary, locally square–integrable semimartingales; see [15] and [50]. Although some nice properties translate into this more general setting, see e.g. Corollary 2.43 below, the structure condition is neither necessary nor sufficient for the existence of a strictly positive σ–martingale density. Moreover, the structure condition **(SC)** lacks to be invariant under equivalent measure changes. This is highly unpleasant, since measure change techniques are the 'bread and butter' techniques in mathematical finance and appear in countless proofs and applications. For a nice and detailed analysis of **(SC)**, we refer to the work of Choulli and Stricker [15].

The goal of this section is to provide new structure conditions that overcome the disadvantages of the structure condition **(SC)**. We start with the *minimal structure condition* **(MSC)**. Apart from the fact that the minimal structure condition is necessary for the **(NUPBR)** condition, it is also invariant under equivalent measure changes. Furthermore, **(MSC)** is equivalent to **(SC)** for continuous semimartingales, a property that **(SC′)** does not possess. We derive the definition of **(MSC)** in a very natural and straightforward way from our first result, the 1^{st} *Structure Theorem*. This theorem provides a deep insight into the relation between strictly positive σ–martingale densities on the one hand, and the structure of locally square–integrable semimartingales and the potential decompositions of its predictable finite variation part on the other hand. A natural, additional assumption on the minimal structure condition leads to the *natural structure condition* **(NSC)**. Under **(NSC)**, we provide our 2^{nd} *Structure Theorem*. It gives a deep insight into the 'geometry' of strictly positive σ–martingale densities and semimartingales. Besides, a reformulation of the 2^{nd} Structure Theorem leads to the notation of a *natural Kunita–Watanabe decomposition*. It turns out that **(NSC)** is essentially equivalent to the fact that all strictly positive σ–martingale densities of a semimartingale feature a natural Kunita–Watanabe decomposition. This reformulation of the 2^{nd} Structure Theorem significantly generalizes the, up to now, most general result in this direction [15, Théorème 2.1], see Corollary 2.43.

2.3.1. From the Radon–Nikodym decomposition to the minimal structure condition (MSC)

The following conventions are important for the rest of this chapter. In order to refer to these conventions, we state them in the following remark.

Remark 2.23 (IMPORTANT CONVENTIONS)**:** *Since the definition of a strictly positive σ–martingale density for a $\mathbb{R}^d$–valued semimartingale is a component–wise definition, we assume w.l.o.g. that $d = 1$. If we compare our results with literature, this is always done under the standing assumption $d = 1$! Moreover, to simplify notation, we formulate results for general $Q \sim P$, only if the invariance of the result under equivalent measure changes is an important part of the statement. Finally, recall that we work with 'the version' of the weak structure condition given in Lemma 2.13.*

The Radon–Nikodym decomposition and the 1^{st} Structure Theorem

The 1^{st} Structure Theorem explains the connection between the Radon–Nikodym decomposition of a strictly positive σ–martingale density of a locally square integrable semimartingale $S = M + A$ and the predictable, finite variation process A. Furthermore, it provides a 'decomposition theorem' for A.

Theorem 2.24 (1^{st} Structure Theorem)**:** *Let S be a P–semimartingale and $Q \sim P$ be a probability measure such that $S \in \mathcal{S}^2_{loc}(Q)$. Denote the canonical decomposition of S by $S = S_0 + M + A$. Furthermore, denote by M^c and M^d the continuous and purely discontinuous local martingale part of M. Let $Z = \mathcal{E}(-N)$ be a strictly positive local Q–martingale and denote by*

$$N = \lambda.M^c + H \underset{c}{\bullet} M^d + L$$

the Radon–Nikodym decomposition of N w.r.t. M. Then the following statements are equivalent:

1. *Z is a strictly positive σ–martingale (local martingale) density for S under Q.*

2. a) *S satisfies the weak structure condition* $(\mathbf{SC}')$ *under Q, i.e. there exists a $\langle M\rangle^Q$–a.s. unique predictable process $\tilde{\lambda} \in L_m\left([M] - \langle M\rangle^Q\right)$ (under Q) such that $A = \int \tilde{\lambda}\, d\langle M\rangle^Q$.*

   b) *$\left(H - \tilde{\lambda}\right).[M]$ is a σ–martingale (local martingale) under Q.*

3. a) *There exists a unique process $\eta \in L_m\left([M^d] - \langle M^d\rangle^Q\right)$ (under Q) such that*
$$A = \int \lambda\, d\langle M^c\rangle^Q + \int \eta\, d\langle M^d\rangle^Q.$$
(If either M^c or M^d is identical zero, we choose $\lambda \equiv 0$ or $\eta \equiv 0$.)

   b) *If $M^c \equiv 0$ and $M^d = \eta.\left([M^d] - \langle M^d\rangle^Q\right)$, then $\eta.[M^d] \equiv 0$.*

   c) *$(H - \eta).[M^d]$ is a σ–martingale (local martingale) under Q.*

The key to the proof of Theorem 2.24 is the following lemma.

Lemma 2.25: *Let $Q \sim P$, $M \in \mathcal{M}^2_{loc}(Q)$ and H be an optional process, such that the path–wise Lebesgue–Stieltjes integral $H.[M]$ exists. Moreover, let $H.[M]$ be compensable and denote its compensator by C. There exists a $\langle M\rangle^Q$–a.s. unique predictable process $K \in L_m\left([M^d] - \langle M^d\rangle^Q\right)$ (under Q), such that $C = K.\langle M\rangle^Q$.*

Proof: If $H.[M] \in \mathcal{A}_{loc}$, the lemma's statement is exactly [27, Chapter V, §2. Remark p. 149]. As $H.[M]$ is compensable, Remark 1.24 ensures that there exists a non–decreasing sequence $(A_n)_{n\geq 1} \subset \mathcal{P}$ such that $\bigcup_{n\geq 1} A_n = \Omega \times [0,T]$ and $(\mathbb{1}_{A_n} H).[M] \in \mathcal{A}_{loc}$ for all $n \geq 1$. As a result of [27, Chapter V, §2. Remark p. 149], there exists a sequence $(K^n)_{n\geq 1} \subset \mathcal{P}$ such that $\mathbb{1}_{A_n}.C = \int K^n\ d\langle M\rangle^Q = \int \mathbb{1}_{A_n} K^n\ d\langle M\rangle^Q$. Due to the uniqueness of the compensator, we can conclude that $\mathbb{1}_{A_n} K^n = \mathbb{1}_{A_n} K^m$ $\langle M\rangle^Q$–a.s. for all $m \geq n$. Define
$$|K| := \lim_{m\to\infty} \mathbb{1}_{A_m} |K^m|$$
and denote by $TV(C)$ the total variation process of C. Due to the monotone convergence theorem, we get
$$\int |K|\ d\langle M\rangle^Q = \lim_{m\to\infty} \int \mathbb{1}_{A_m} |K^m|\ d\langle M\rangle^Q = \lim_{m\to\infty} \mathbb{1}_{A_m}.TV(C) = TV(C) \in \mathcal{A}_{loc}.$$
Therefore, $K := \lim_{m\to\infty} \mathbb{1}_{A_m} K^m$ is well–defined and $C = \int K\ d\langle M\rangle^Q$. Moreover, [27, 5.19 Theorem] ensures that $C \in \mathcal{A}_{loc}$. Finally, Lemma 1.27 allows us to conclude that $K \in L_m\left([M^d] - \langle M^d\rangle^Q\right)$ under Q. □

Proof of Theorem 2.24: We start proving the equivalence of *1.* and *2.*. Due to Key lemma 2.11, we know that Z is a strictly positive σ–martingale (local martingale) density for S if and only if $[N, M] - A = H.[M] - A$ is a σ–martingale (local martingale). Lemma 2.25 ensures that there exists a $\langle M\rangle^Q$–a.s. unique predictable process $\tilde{\lambda} \in L_m\left([M] - \langle M\rangle^Q\right)$ (under Q) such that $A = \tilde{\lambda}.\langle M\rangle^Q$. Hence,

$$\left(H - \tilde{\lambda}\right).[M] = H.[M] - \tilde{\lambda}.\langle M\rangle^Q - \tilde{\lambda}.\left([M] - \langle M\rangle^Q\right)$$

is a σ–martingale (a local martingale) under Q if and only if $H.[M] - A$ is a Q–σ–martingale (local martingale).

It remains to prove the equivalence of *1.* and *3.*. As before, Key lemma 2.11 ensures that Z is a strictly positive σ–martingale (local martingale) density for S if and only if $[N, M] - A = H.[M] - A$ is a σ–martingale (local martingale). Due to the Radon–Nikodym decomposition of N w.r.t. M, this is equivalent to

$$H.[M^d] + \lambda.\langle M^c\rangle^Q - A$$

being a σ–martingale (local martingale). As a result of Lemma 2.25, we know that there exists a $\langle M^d\rangle^Q$–a.s. unique predictable process $\eta \in L_m\left([M^d] - \langle M^d\rangle^Q\right)$ (under Q) such that

$$H.[M^d] - \eta.\langle M^d\rangle^Q$$

is a σ–martingale (local martingale). Hence, we can conclude that

$$A = \int \lambda \, d\langle M^c\rangle^Q + \int \eta \, d\langle M^d\rangle^Q.$$

Since

$$\left(H - \eta\right).[M^d] = H.[M^d] - \eta.\langle M^d\rangle^Q - \eta.\left([M^d] - \langle M^d\rangle^Q\right),$$

we have proven that *1.* and *3.a),3.c)* are equivalent. Therefore, it remains to prove the equivalence of *1.* and *3.b)*. Assume that $M^c \equiv 0$ and $M^d = \eta.\left([M^d] - \langle M^d\rangle^Q\right)$. Define

$$\operatorname{sign}(\eta)_t = \begin{cases} 1 & \text{if } \eta_t \geq 0, \\ -1 & \text{if } \eta_t < 0. \end{cases}$$

Since

$$\begin{aligned} \operatorname{sign}(\eta) \cdot S &= \operatorname{sign}(\eta) \cdot \left(M^c + M^d + \int \lambda \, d\langle M^c\rangle^Q + \int \eta \, d\langle M^d\rangle^Q\right) \\ &= \operatorname{sign}(\eta) \cdot \left(\eta.\left([M^d] - \langle M^d\rangle^Q\right) + \int \eta \, d\langle M^d\rangle\right) = \left(\operatorname{sign}(\eta)\eta\right).[M^d] \\ &= |\eta|.[M^d], \end{aligned}$$

Lemma 2.6 ensures that Z is a strictly positive σ–martingale (local martingale) density for S if and only if it is a strictly positive σ–martingale (local martingale) density for $|\eta|.[M^d]$. As $|\eta|.[M^d]$ is non–decreasing, this is possible if and only if $\eta.[M^d] \equiv 0$. $\square$

Remark 2.26: *Since*

$$H.[M^d] = [N^d, M^d] = [H \underset{c}{\bullet} M^d + L^d, M^d] = [H \underset{c}{\bullet} M^d, M^d] + [L^d, M^d],$$

Theorem 1.36 and the proof of the 1^{st} Structure Theorem ensure that $[H \underset{c}{\bullet} M^d, M^d] - \eta.\langle M^d\rangle^Q$ is a σ–martingale under Q.

Geometric interpretation of the 1^{st} Structure Theorem

The first step to provide a geometric interpretation of the connection between a strictly positive σ–martingale density and a semimartingale S, is a measure change $Q \sim P$ to ensure that $S \in \mathcal{S}^2_{loc}(Q)$. In the following presentation, we assume that $S \in \mathcal{S}^2_{loc}(P)$ and suppress the probability measure P to simplify notation. Moreover, we denote the canonical decomposition of S by $S = M + A$, where $A \in \mathcal{V} \cap \mathcal{P}$. Furthermore, let $\bar{\Omega} := \Omega \times [0, T]$ and denote by μ a σ–finite measure on $\big(\bar{\Omega}, \mathcal{P}\big)$. For $p \in \{1, 2\}$ we define

$$L^p_{loc}\left(\bar{\Omega}, \mathcal{P}, \mu\right) := \left\{ \int K \; d\mu : \; K \in \mathcal{P} \text{ and } \int |K|^p \; d\mu \in \mathcal{A}_{loc} \right\}.$$

Besides, we set

$$L^1_\sigma\left(\bar{\Omega}, \mathcal{O}, d[M]\right) := \left\{ \int K \; d[M] : \; K \in \mathcal{O} \text{ and } \int K \; d[M] \text{ is a } \sigma\text{–martingale} \right\}.$$

In the first step of the proof of Theorem 2.24, we translate the formulation '$Z = \mathcal{E}(-N)$ is a strictly positive σ–martingale density for S' into '$[N, M] - A$ is a σ–martingale'. Furthermore, by using the Radon–Nikodym derivative of $d[N, M]$ w.r.t. $d[M]$, this turns into the formulation

$$\text{'}H.[M] - A = \sigma\text{–martingale'}. \tag{2.3}$$

Since H is connected to N in a unique way, this is the 'equation' that contains all objects that are crucial to decide whether or not $\mathcal{E}(-N)$ is a (strictly positive) σ–martingale density for S. Equation (2.3) is the starting point of the geometric interpretation of Theorem 2.24 *2.* and Theorem 2.24 *3.*.

Geometric interpretation of Theorem 2.24 2.

In the first step, the weak structure condition $(\mathbf{SC'})$ allows us to identify $A \in \mathcal{V} \cap \mathcal{P}$ in a unique way with an element $\int \tilde{\lambda} \; d\langle M\rangle$ on 'the line' $L^1_{loc}(\bar{\Omega}, \mathcal{P}, d\langle M\rangle)$; see Figure 2.3.1. Due to (2.3) and Theorem 2.24 *2.b)*, this can be transformed, via adding and subtracting $\tilde{\lambda}.[M]$, into a '2–dimensional picture' (Figure 2.3.2), where the right angle in Figure 2.3.2 indicates that $(H - \tilde{\lambda}).[M]$ is a σ–martingale. Although this is a nice interpretation, Theorem 2.24 *3.* provides a much deeper understanding of the geometric link between strictly positive σ–martingale densities and semimartingales. In particular, these insights lead to a good minimal structure condition.

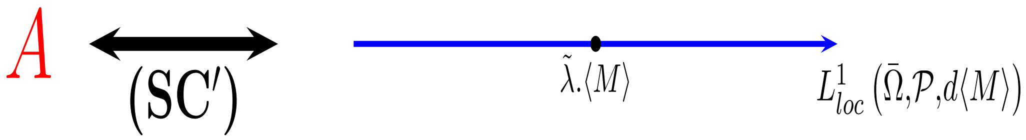

Figure 2.3.1.:

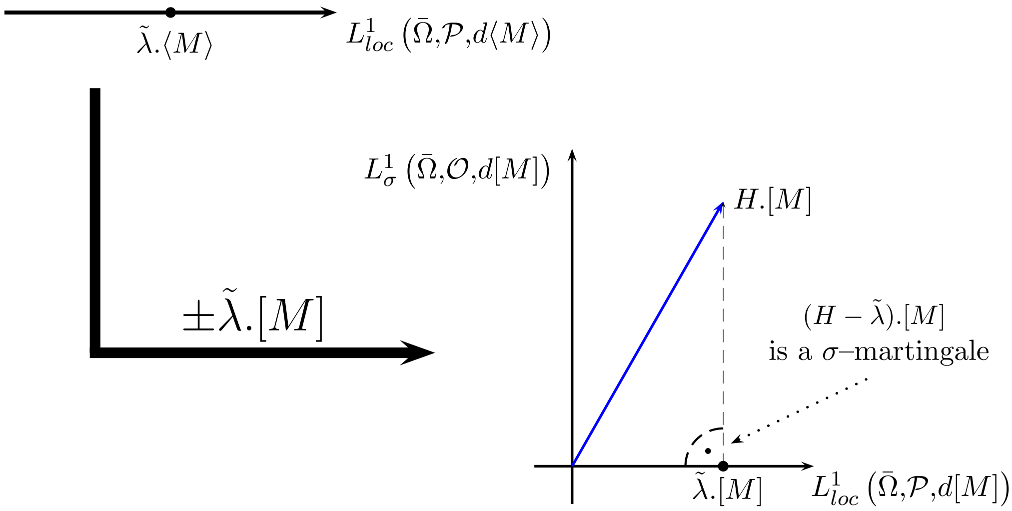

Figure 2.3.2.:

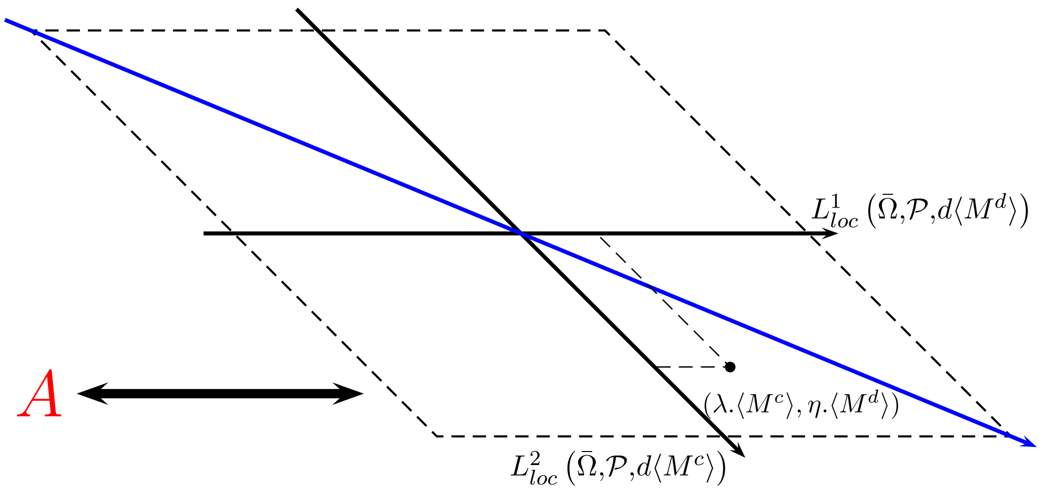

Figure 2.3.3.:

Geometric interpretation of Theorem 2.24 3.

The key advantage of Theorem 2.24 *3.* is to consider $H.[M^c]$ and $H.[M^d]$ separately. While Figure 2.3.1 identifies the predictable finite variation part A of S with an element on the 'predictable line' $L^1_{loc}(\bar{\Omega}, \mathcal{P}, d\langle M\rangle)$, Theorem 2.24 *3.* allows us to identify A with an element on the 'predictable plane'

$$L^2_{loc}(\bar{\Omega}, \mathcal{P}, d\langle M^c\rangle) + L^1_{loc}(\bar{\Omega}, \mathcal{P}, d\langle M^d\rangle) \subset \mathcal{A}_{loc}.$$

Here, $A \in L^2_{loc}(\bar{\Omega}, \mathcal{P}, d\langle M^c\rangle) + L^1_{loc}(\bar{\Omega}, \mathcal{P}, d\langle M^d\rangle)$ means that there exist $\lambda.\langle M^c\rangle \in L^2_{loc}(\bar{\Omega}, \mathcal{P}, d\langle M^c\rangle)$ and $\eta.\langle M^d\rangle \in L^1_{loc}(\bar{\Omega}, \mathcal{P}, d\langle M^d\rangle)$ such that

$$A = \lambda.\langle M^c\rangle + \eta.\langle M^d\rangle.$$

With an abuse of notation, we also write

$$(\lambda.\langle M^c\rangle, \eta.\langle M^d\rangle) \in L^2_{loc}(\bar{\Omega}, \mathcal{P}, d\langle M^c\rangle) + L^1_{loc}(\bar{\Omega}, \mathcal{P}, d\langle M^d\rangle).$$

With this notation we can translate Theorem 2.24 into Figure 2.3.3. While the weak structure condition **(SC′)** suggests to consider A as an element

$$(\tilde{\lambda}.\langle M^c\rangle, \tilde{\lambda}.\langle M^d\rangle) \in L^1_{loc}(\bar{\Omega}, \mathcal{P}, d\langle M^c\rangle) + L^1_{loc}(\bar{\Omega}, \mathcal{P}, d\langle M^d\rangle)$$

of the 'diagonal', Theorem 2.24 *3.* indicates that it is much more reasonable to consider A as an element

$$(\lambda.\langle M^c\rangle, \eta.\langle M^d\rangle) \in L^2_{loc}(\bar{\Omega}, \mathcal{P}, d\langle M^c\rangle) + L^1_{loc}(\bar{\Omega}, \mathcal{P}, d\langle M^d\rangle)$$

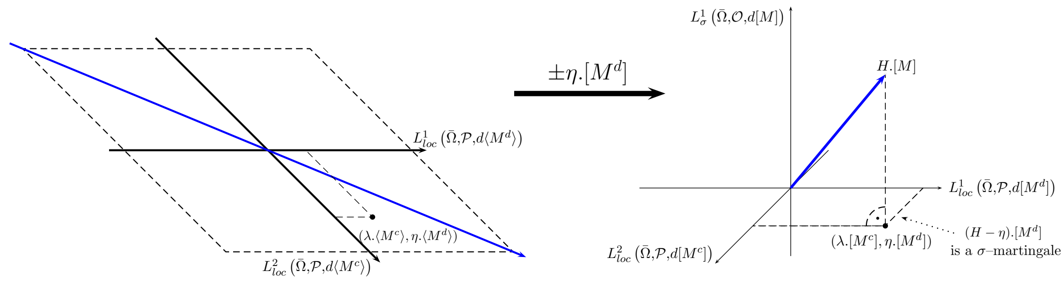

Figure 2.3.4.:

of the predictable plane. As in Figure 2.3.2, the transformation $\pm\eta.[M^d]$ relates a pair (λ,η) to H by 'adding a further dimension'. Indeed, let $(T_n)_{n\geq 1}$ be a sequence of stopping times that exhausts the jumps of M. Then we get

$$\begin{aligned} H.[M] &= (H-\eta)\mathbb{1}_{\bigcup_{n\geq 1}[\![T_n]\!]}.[M] + H\mathbb{1}_{(\bigcup_{n\geq 1}[\![T_n]\!])^c}.[M] + \eta\mathbb{1}_{\bigcup_{n\geq 1}[\![T_n]\!]}.[M] \\ &= (H-\eta).[M^d] + \lambda.[M^c] + \eta.[M^d]. \end{aligned}$$

This embedding of the predictable plane into a '3–dimensional space' is illustrated in Figure 2.3.4. Again, the right angle sign in Figure 2.3.4 indicates that $(H-\eta).[M^d]$ is a σ–martingale.

The minimal structure condition (MSC) and a toy example

To define a minimal structure condition for a semimartingale $S \in \mathcal{S}^2_{loc}$, where $S = M + A$ denotes the canonical decomposition, the question is basically the following: 'What finite variation process $A \in \mathcal{V} \cap \mathcal{P}$ allows for the existence of a strictly positive σ–martingale density?'. Using the characterisation of Theorem 2.24 and its geometric interpretation, this question translates into the question: 'What subset $B \subset L^2_{loc}(\bar{\Omega},\mathcal{P},d\langle M^c\rangle) + L^1_{loc}(\bar{\Omega},\mathcal{P},d\langle M^d\rangle)$ allows for the existence of a strictly positive σ–martingale density?'; see Figure 2.3.5. Note that if there exist at least two representations of A, i.e.

$$\begin{aligned} A &= \int {}^1\lambda \, d\langle M^c\rangle + \int {}^1\eta \, d\langle M^d\rangle \\ &= \int {}^2\lambda \, d\langle M^c\rangle + \int {}^2\eta \, d\langle M^d\rangle, \end{aligned}$$

we get for $\alpha \in [0,1]$

$$A = \int \alpha\left({}^1\lambda\right) + (1-\alpha)\left({}^2\lambda\right) \, d\langle M^c\rangle + \int \alpha\left({}^1\eta\right) + (1-\alpha)\left({}^2\eta\right) \, d\langle M^d\rangle.$$

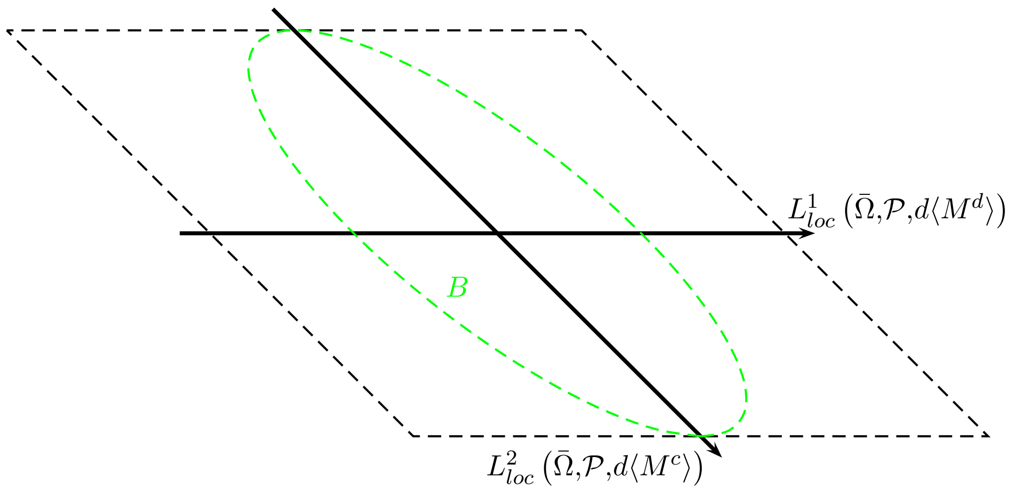

Figure 2.3.5.:

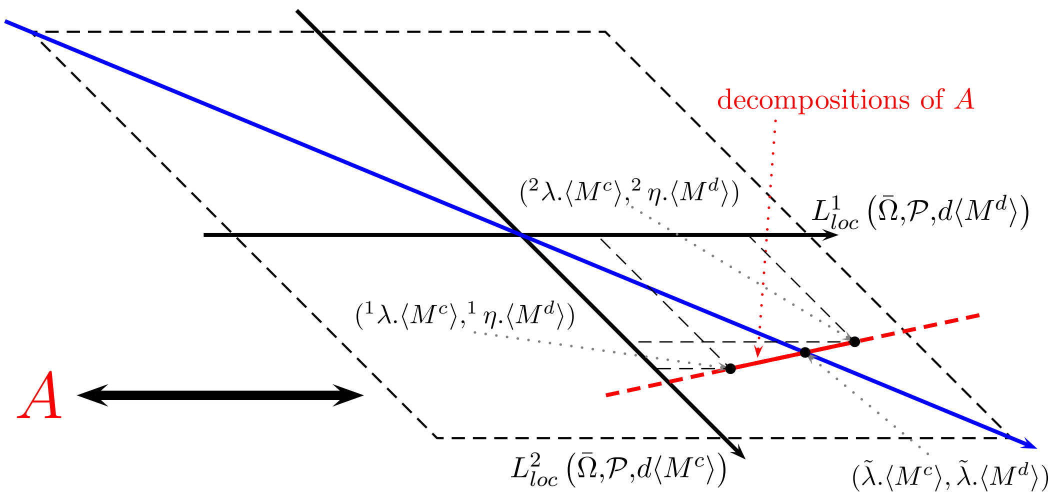

Figure 2.3.6.:

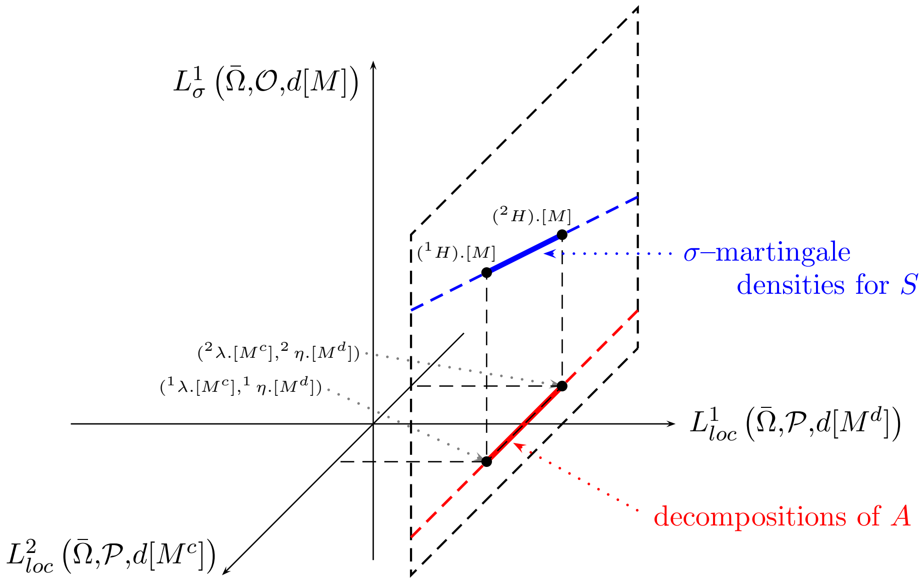

Figure 2.3.7.:

This in turn means that A can be interpreted as a 'line segment' (which might be a whole line or a point); see Figure 2.3.6. Notice that this is the key difference between **(SC′)** and the characterisation of Theorem 2.24 *3.a)*. While **(SC′)** considers *only one decomposition* of A, $A = \int \tilde{\lambda}\, d\langle M\rangle$, Theorem 2.24 *3.a)* indicates that we should look for *all potential decompositions* of A. Yet, how are these decompositions connected to different strictly positive σ–martingale densities for S? If $\mathcal{E}(-(^1N))$ and $\mathcal{E}(-(^2N))$ are strictly positive σ–martingale densities for the semimartingale S, Key lemma 2.11 ensures that $\mathcal{E}(-(^\alpha N))$, where

$$^\alpha N = \alpha(^1N) + (1-\alpha)(^2N), \qquad \alpha \in [0,1],$$

is also a strictly positive σ–martingale density for S. Hence, if $^\alpha H$ denotes the Radon–Nikodym derivative of $d[^\alpha N, M]$ w.r.t. $d[M]$ and $(^\alpha\lambda, {}^\alpha\eta)$ the unique pair corresponding to $^\alpha N$ (see Theorem 2.24), we see that the line segment

$$(^\alpha\lambda.[M^c], {}^\alpha\eta.[M^d]) = \alpha\left(^1\lambda.[M^c], {}^1\eta.[M^d]\right) + (1-\alpha)\left(^2\lambda.[M^c], {}^2\eta.[M^d]\right)$$

is the 'orthogonal projection' of the line segment

$$^\alpha H.[M] = \alpha\left(^1H\right).[M] + (1-\alpha)\left(^2H\right).[M]$$

onto the plane $L^2_{loc}(\bar{\Omega}, \mathcal{P}, d[M^c]) + L^1_{loc}(\bar{\Omega}, \mathcal{P}, d[M^d])$; see Figure 2.3.7. These ideas lead to the following natural definition of the minimal structure condition.

Definition 2.27 (Minimal structure condition)**:** *Let S be a P–semimartingale and $Q \sim P$ a probability measure. We say that S satisfies the* minimal structure condition **(MSC)** under Q, *if the following properties hold:*

1. *S is a locally square–integrable Q–semimartingale with canonical decomposition $S = S_0 + M + A$, where M^c and M^d are the continuous– and purely discontinuous local martingale parts of M.*

2. *There exist processes $\lambda \in L_m(M^c)$ and $\eta \in L_m([M^d] - \langle M^d \rangle^Q)$ under Q such that $A = \int \lambda \, d\langle M^c \rangle + \int \eta \, d\langle M^d \rangle$.*

3. *If $M^c \equiv 0$ and $M^d = \eta. \left([M^d] - \langle M^d \rangle^Q\right)$, then $\eta.[M] \equiv 0$.*

A pair (λ, η) that satisfies all conditions is called a version of **(MSC)** under Q.

Remark 2.28:

1. *The definition of* **(MSC)** *takes into account that $A \in \mathcal{V} \cap \mathcal{P}$ might have several decompositions into the sum of integrals w.r.t. $d\langle M^c \rangle^Q$ and $d\langle M^d \rangle^Q$. This is due to the fact that we do not insist on the uniqueness of the pair (λ, η) in 2.. This fact is crucial to understand the connection between semimartingales and their strictly positive σ–martingale densities.*

2. *The structure condition* **(SC)** *forces ($\langle M \rangle^Q$–a.s.) the uniqueness of the pair, as* **(SC)** *insists on a pair $(\tilde{\lambda}, \tilde{\lambda})$ on the diagonal.*

3. *If $S \in \mathcal{S}^2_{loc}$ does not satisfy the third property of the definition, the paths of S are of locally finite variation. Hence, all semimartingales whose paths are of infinite variation always satisfy the third property of the definition. For example, locally square–integrable Lévy–processes of Type C satisfy this property; see* [45, Theorem 21.9 (ii)].

In order to highlight connections and differences between **(MSC)**, **(SC′)**, **(SC)**, and a few other structure conditions, we introduce the following toy example.

Toy example 2.29 (Setting)**:** Let

$$
\begin{aligned}
S &= M + \int \tilde{\lambda} \, \langle M \rangle \\
&= c_1 B + c_2 N + \int \tilde{\lambda} \, \langle c_1 B + c_2 N \rangle
\end{aligned}
$$

be a semimartingale on the time interval $[0, 1]$, where B is a standard Brownian motion and $N = P - t$ a compensated Poisson process with intensity one. Furthermore, let $\tilde{\lambda} \in \mathcal{P}$ be a predictable process such that $E\left[\int_0^1 |\tilde{\lambda}_u| \, d\langle M \rangle_u\right] < \infty$. $\mathbb{F}$ is assumed to be the smallest filtration that satisfies the usual conditions and contains the natural filtration generated by S. Finally, $c_1, c_2 \in \{0,1\}$ are constants that enable us to switch on and off the continuous and the purely discontinuous local martingale part of M.

Remark 2.30: *In the following 'toy examples' the assumptions above are assumed to hold. To highlight specific points, we modify $c_1, c_2 \in \{0,1\}$ and $\tilde{\lambda} \in \mathcal{P}$.*

The first lemma explains the connection between **(MSC)** and **(SC′)**.

Lemma 2.31: *Let S be a locally square–integrable semimartingale. The following implication holds:*

$$\textbf{(MSC)} \qquad \Longrightarrow \qquad \textbf{(SC}'\textbf{)}.$$

If S is continuous, then **(SC)** *and* **(MSC)** *are equivalent.*

Proof: Let $S = S_0 + M + A$ be the canonical decomposition of S, where A denotes the predictable process of finite variation. Let (λ, η) be a version of **(MSC)**. Then $A = \int \lambda \, \langle M^c \rangle + \int \eta \, \langle M^d \rangle$. Hence, $dA \ll d\langle M \rangle$ holds and [27, 5.14 Theorem] ensures that there exists a predictable process $\tilde{\lambda} \in \mathcal{P}$ such that $A = \tilde{\lambda}.\langle M \rangle$. The second statement is obvious. □

The first toy example shows that the reverse implication in Lemma 2.31 does not hold in general.

Toy example 2.32: The following examples show that **(SC′)** is indeed weaker than **(MSC)**. Moreover, these examples highlight at what point **(SC′)** is too weak to ensure the existence of a strictly positive σ–martingale density.

1. (Continuous paths): Let $c_1 = 1$, $c_2 = 0$, and $\tilde{\lambda} \in L^1([0,1], du) \setminus L^2([0,1], du)$. Due to Definition 2.27, we know that for continuous semimartingales it is necessary that $\tilde{\lambda} \in L^2([0,1], du)$. Hence, **(SC′)** holds while **(MSC)** does not.

2. (Discontinuous paths): Let $c_1 = 0$, $c_2 = 1$, and $\tilde{\lambda} \equiv 1$. Then $S = N + t = P$ is an increasing process which, of course, cannot admit a strictly positive σ–martingale density. Moreover, it does not satisfy **(MSC)**.

Theorem 2.24 ensures that **(MSC)** is invariant under proper equivalent changes of measure. As this property is so important, we formulate it in the following theorem.

Theorem 2.33 (Invariance of **(MSC)** under equivalent measure change)**:** *Let S be a P–semimartingale and $Q \sim P$ be a probability measure. Further, let S be a locally square–integrable Q–semimartingale. If there exists a strictly positive σ–martingale density for S under P, S satisfies the minimal structure condition* **(MSC)** *under Q.*

Proof: Due to Lemma 2.8, the existence of a strictly positive σ–martingale density is invariant under equivalent measure changes. Hence, the statement follows immediately from Theorem 2.24. □

Remark 2.34: *If S is locally bounded, Theorem 2.33 holds for all $Q \sim P$. This property does not hold for* **(SC)** *in general; see Toy example 2.45.*

2.3.2. The natural Kunita–Watanabe decomposition and the natural structure condition (NSC)

Natural Kunita–Watanabe decomposition and the 2^{nd} Structure Theorem

Let S be a locally square–integrable semimartingale. If (λ, η) is a version of **(MSC)**, we know that $\int \lambda \, dM^c$ exists. Unfortunately, we do not know whether or not $\int \eta \, dM^d$ exists and whether or not it is a local martingale. The next definition focuses on those versions (λ, η) of **(MSC)** that satisfy this additional property.

Definition 2.35 (Natural structure condition)**:** *Let S be a locally square–integrable P–semimartingale with canonical decomposition $S = S_0 + M + A$ that satisfies the minimal structure condition* **(MSC)***. We say that S satisfies the* natural structure condition **(NSC)***, if there exists a version (λ, η) of* **(MSC)** *such that $\eta \in L_m(M^d)$. A pair (λ, η) satisfying this condition is called* a version of **(NSC)***.*

Remark 2.36:

1. *It is clear that for semimartingales having continuous paths,* **(NSC)** *and* **(SC)** *are equivalent.*

2. *If the toy example satisfies* **(MSC)***, it also satisfies* **(NSC)***.*

Definition 2.37: *Let N and M be local martingales starting in zero. Further, let M be locally square–integrable and denote by M^c and M^d its continuous– and purely discontinuous local martingale part. We say that N features a* natural Kunita–Watanabe decomposition *w.r.t. M, if there exists $\lambda \in L_m(M^c)$ and $\eta \in L_m(M^d) \cap L_m([M^d] - \langle M^d \rangle)$ such that*

$$N = \lambda . M^c + \eta . M^d + L,$$

where L is a local martingale and $[L, M]$ is a σ–martingale.

The first lemma highlights the connection between **(SC)**, **(MSC)**, and **(NSC)**.

Lemma 2.38: *Let $S \in \mathcal{S}^2_{loc}$ and denote by $S = S_0 + M + A$ its canonical decomposition. Moreover, let S satisfy* **(SC)***, i.e. $A = \int \tilde{\lambda} \, d\langle M \rangle$ with $\tilde{\lambda} \in {}^{p}L^2_{loc}(M)$. The following statements hold:*

1. *The pair* $(\tilde{\lambda}, \tilde{\lambda})$ *satisfies the first two properties of the definition of* **(MSC)**.

2. *If* $(\tilde{\lambda}, \tilde{\lambda})$ *is a version of* **(MSC)**, *then* $(\tilde{\lambda}, \tilde{\lambda})$ *is a version of* **(NSC)**.

Proof: Note that if we have proven the first statement, *2.* follows automatically. Hence, it remains to prove *1.*. The first property is clear. Since $\tilde{\lambda} \in {}^pL^2_{loc}(M)$, it follows that $\tilde{\lambda} \in L_m(M^d) \cap L_m(M^c)$. Moreover, the Kunita–Watanabe inequality ensures that

$$|\tilde{\lambda}|.[M^d] \leq \frac{1}{2}\left(\tilde{\lambda}^2.[M^d] + [M^d]\right).$$

Hence, $\int |\tilde{\lambda}|\, d[M^d] \in \mathcal{A}^+_{loc}$ holds. Finally, $\tilde{\lambda} \in L_m([M^d] - \langle M^d \rangle)$ is true thanks to Lemma 1.27. □

The next lemma gives a simple, sufficient condition for **(NSC)**.

Lemma 2.39: *Let* S *be a semimartingale that satisfies* **(MSC)**. *Furthermore, denote its canonical decomposition by* $S = S_0 + M + A$ *and let* (λ, η) *be a version of* **(MSC)**. *If there exists a constant* $\epsilon > 0$ *such that* $|\Delta M^d| \geq \epsilon$, *then* (λ, η) *is version of* **(NSC)**.

Proof: It remains to prove that $\eta \in L_m(M^d)$. Set $T_0 := 0$ and define $(T_n)_{n \geq 1}$ via

$$T_n := \inf\{t > T_{n-1} : \ \Delta M_t \neq 0\}, \qquad \text{for } n \geq 1.$$

Due to the sub–additivity of the square root, see Lemma A.20, we get

$$\sqrt{\eta^2.[M^d]^{T_n \wedge R}} \leq \sum_{i=1}^{n} \left|\eta_{T_i \wedge R} \Delta M^d_{T_i \wedge R}\right|$$

for all stopping times R. Hence, we get

$$\sqrt{\eta^2.[M^d]^{R}} \leq \sum_{i=1}^{\infty} \left|\eta_{T_i \wedge R} \Delta M^d_{T_i \wedge R}\right|.$$

By assumption, there exists $\epsilon > 0$ such that $|\Delta M^d| \geq \epsilon$. Moreover, $\eta.[M^d] \in \mathcal{A}_{loc}$ holds. Denote its localizing sequence by $(R_n)_{n \geq 1}$ and note that for all $i, n \geq 1$

$$\epsilon|\eta_{T_i \wedge R_n} \Delta M^d_{T_i \wedge R_n}| \leq |\eta_{T_i \wedge R_n} \Delta [M^d]_{T_i \wedge R_n}|$$

holds. Combining the last two equations, we conclude

$$\sqrt{\eta^2.[M^d]^{R_n}} \leq \sum_{i=1}^{\infty} \left|\eta_{T_i \wedge R_n} \Delta M^d_{T_i \wedge R_n}\right| \leq \frac{1}{\epsilon}|\eta|.[M^d]^{R_n} \in \mathcal{A}, \qquad \forall\, n \geq 1.$$

□

The next result is the main result of this subsection. Under the condition **(NSC)**, the theorem provides a deep connection between the structure of the drift of a semimartingale and the structure of its strictly positive σ–martingale densities.

Theorem 2.40 (2^{nd} Structure Theorem)**:** *Let S satisfy* **(NSC)** *and denote its canonical decomposition by $S = S_0 + M + A$. Moreover, let (λ, η) be a version of* **(NSC)** *and let $Z = \mathcal{E}(-N)$ be a strictly positive local martingale. Then Z is a strictly positive σ–martingale (local martingale) density for S if and only if N can be decomposed as*

$$N = \lambda.M^c + \eta.M^d + L,$$

where L is a local martingale and $[L, M]$ is a σ–martingale (local martingale).

Proof: Let (λ, η) be a version of **(NSC)** and define

$$L := N - \lambda.M^c - \eta.M^d \in \mathcal{M}_{loc}.$$

Due to Key lemma 2.11, we know that $\mathcal{E}(-N)$ is a strictly positive σ–martingale (local martingale) density for S if and only if

$$\begin{aligned}
[N, M] - A &= [N, M] - \lambda.\langle M^c\rangle - \eta.\langle M^d\rangle \\
&= [L + \lambda.M^c + \eta.M^d, M] - \lambda.\langle M^c\rangle - \eta.\langle M^d\rangle \\
&= [L, M] + \eta.\left([M^d] - \langle M^d\rangle\right)
\end{aligned}$$

is a σ–martingale (local martingale). Due to **(NSC)**, we can conclude that the process $\eta.\left([M^d] - \langle M^d\rangle\right)$ is a local martingale. Therefore, $\mathcal{E}(-N)$ is a strictly positive σ–martingale (local martingale) density for S if and only if $[L, M]$ is a σ–martingale (local martingale). Hence, the claim is proven. □

Remark 2.41:

1. *The theorem provides a link between martingale decomposition theorem(s) of the local martingale N and decomposition theorem(s) of the predictable finite variation process A.*

2. *This characterisation is of great benefit for practical applications! One way to apply it, in order to receive* strictly positive *σ–martingale densities, is presented in Section 2.4.2.*

The theorem can be reformulated as follows.

Theorem 2.42 (2^{nd} Structure Theorem; 2nd version)**:** *Let S be a semimartingale and $Z = \mathcal{E}(-N)$ be a strictly positive σ–martingale density for S. Then S satisfies* **(NSC)** *if and only if*

1. *$S \in \mathcal{S}^2_{loc}$; we denote the local martingale of its canonical decomposition by M.*

2. *N features a natural Kunita–Watanabe decomposition w.r.t. M.*

The second version of the 2nd Structure Theorem is a remarkable generalisation of [15, Théorème 2.1] by Ansel and Stricker. We state their theorem in the following corollary.

Corollary 2.43 ([15, Théorème 2.1])**:** *Let S be a semimartingale and $Z = \mathcal{E}(-N)$ be a strictly positive local martingale density for S. Then S satisfies* **(SC)** *iff*[1]

1. *$S \in \mathcal{S}^2_{loc}$; we denote the local martingale of its canonical decomposition by M.*

2. *N features a Kunita–Watanabe decomposition w.r.t. M. More precisely,*

$$N = \tilde{\lambda}.M + L,$$

where $\tilde{\lambda}.M$ is a locally square–integrable martingale and L as well as $[L, M]$ are local martingales.

Proof: Note that the assumption **(SC)** as well as Corollary 2.43 *1.* imply that $S \in \mathcal{S}^2_{loc}$. We denote its canonical decomposition by $S = M + A$, where M is a local martingale. Moreover, note that the Kunita–Watanabe inequality ensures that $K \in {}^pL^2_{loc}(M)$ implies $K \in L_m(M^d) \cap L_m([M^d] - \langle M^d \rangle)$. Due to Key lemma 2.11, we know that $Z = \mathcal{E}(-N)$ is a strictly positive local martingale density for S if and only if $[N, M] - A$ is a local martingale. Now the rest of the proof follows the same lines as the proof of Theorem 2.40. □

Remark 2.44: *Corollary 2.43 also provides a connection between* one *decomposition theorem (Kunita–Watanabe decomposition) for the local martingale N and the drift A. Since the Kunita–Watanabe decomposition as well as the structure condition* **(SC)** *imply* uniqueness *of the particular decompositions, the practical use of Corollary 2.43 is rather low compared to Theorem 2.40.*

In the next toy example, we highlight some aspects concerning the relation of **(NSC)** and **(SC)**.

Toy example 2.45 (compare [15, Exemple 2.6.])**:** This toy example shows the following:

1. S satisfies **(NSC)**, but **(SC)** does not hold.

2. There exists an equivalent local martingale measure for S.

[1] abbr.: if and only if

3. **(SC)** is not necessary for the existence of equivalent local martingale measures.

4. **(SC)** is not invariant under an equivalent change of measure.

Let $c_1 = 0$, $c_2 = 1$, and $\tilde{\lambda} \in L^1([0,1], du) \setminus L^2([0,1], du)$ with $\tilde{\lambda} < 1$ du–a.s.. Since $N = [N] - \langle N \rangle$, the first statement holds. Note that if we have proven *2.*, statements *3.* and *4.* follow immediately. Indeed, *3.* is obvious. Moreover, since S is locally bounded, it satisfies **(SC)** under any equivalent local martingale measure. Therefore, it remains to prove *2.*. Since $\mathcal{E}\left(-\int \tilde{\lambda}\, dN\right)$ is strictly positive, Theorem 2.40 ensures that it is a strictly positive local martingale density. Hence, it remains to prove that $\mathcal{E}\left(-\int \tilde{\lambda}\, dN\right)$ is uniformly integrable. Denote by $(T_n)_{n\geq 1}$ a sequence of stopping times such that $T_n \uparrow 1$ a.s. and that $\mathcal{E}\left(-\int \tilde{\lambda}\, dN\right)^{T_n}$ is uniformly integrable for all $n \geq 1$. Due to the sub–additivity of the square root, we get for all $n \geq 1$

$$\sqrt{\left[\mathcal{E}\left(-\int \tilde{\lambda}\, dN\right)^{T_n}\right]_1} = \sqrt{\int_0^1 \left(\mathcal{E}\left(-\int \tilde{\lambda}\, dN\right)^{T_n}_{u-} |\tilde{\lambda}_u|\right)^2 d[N]_u}$$
$$\leq \sum_{u\leq 1} \mathcal{E}\left(-\int \tilde{\lambda}\, dN\right)^{T_n}_{u-} |\tilde{\lambda}_u| \Delta[N]_u = \int_0^1 \mathcal{E}\left(-\int \tilde{\lambda}\, dN\right)^{T_n}_{u-} |\tilde{\lambda}_u|\, d[N]_u.$$

As S satisfies **(NSC)**, we know that $\int |\tilde{\lambda}|\, d[N] \in \mathcal{A}_{loc}$. Since $\mathcal{E}\left(-\int \tilde{\lambda}\, dN\right)^{T_n}_{-}$ is predictable and locally bounded, $\int \mathcal{E}\left(-\int \tilde{\lambda}\, dN\right)^{T_n}_{-} |\tilde{\lambda}|\, d[N] \in \mathcal{A}_{loc}$, too. Moreover, [27, 5.26 Theorem] ensures that

$$E\left[\int_0^1 \mathcal{E}\left(-\int \tilde{\lambda}\, dN\right)^{T_n}_{u-} |\tilde{\lambda}_u|\, d[N]_u\right] = E\left[\int_0^1 \mathcal{E}\left(-\int \tilde{\lambda}\, dN\right)^{T_n}_{u-} |\tilde{\lambda}_u|\, du\right].$$

Due to [27, 5.3 Remark], the predictable projection of $\mathcal{E}\left(-\int \tilde{\lambda}\, dN\right)^{T_n}$ is given by $\mathcal{E}\left(-\int \tilde{\lambda}\, dN\right)^{T_n}_{-}$. Since $\mathcal{F}_0$ contains only sets of measure zero or one, [27, 5.26 Theorem] and [27, 5.32 Theorem] ensure that

$$\begin{aligned} E\left[\int_0^1 \mathcal{E}\left(-\int \tilde{\lambda}\, dN\right)^{T_n}_{u-} |\tilde{\lambda}_u|\, du\right] &= E\left[\int_0^1 \mathcal{E}\left(-\int \tilde{\lambda}\, dN\right)^{T_n}_{u} |\tilde{\lambda}_u|\, du\right] \\ &= E\left[\mathcal{E}\left(-\int \tilde{\lambda}\, dN\right)^{T_n}_{1} \int_0^1 |\tilde{\lambda}_u|\, du\right] \\ &= \int_0^1 |\tilde{\lambda}_u|\, du < \infty \end{aligned}$$

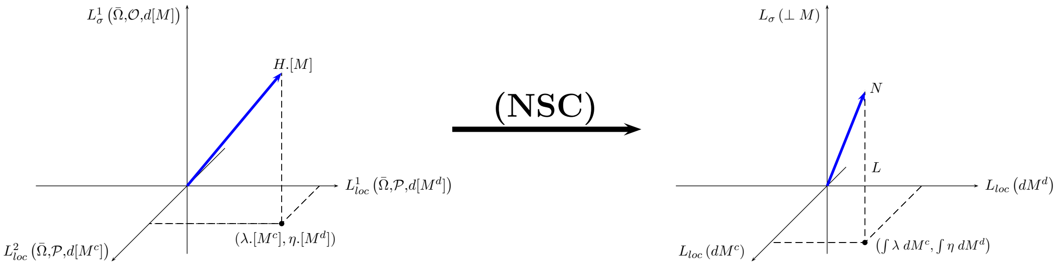

Figure 2.3.8.:

for all $n \geq 1$. Moreover, the monotone convergence theorem allows us to conclude that $E\left[\sqrt{\left[\mathcal{E}\left(-\int\tilde\lambda\, dN\right)\right]_1}\right] \leq \int_0^1 |\tilde\lambda_u|\; du < \infty$. Finally, the Burkholder–Davis–Gundy inequality ensures that $\mathcal{E}\left(-\int\tilde\lambda\, dN\right)$ is uniformly integrable.

Geometric interpretation of the 2^{nd} Structure Theorem

We end this section with a geometric interpretation of the 2^{nd} Structure Theorem and its relation to the 1^{st} Structure Theorem. The condition **(NSC)** and Theorem 2.40 enable us to translate the geometry of Figure 2.3.4 into a geometry in the space of local martingales. As in the geometric interpretation of the 1^{st} Structure Theorem, we consider $S \in \mathcal{S}^2_{loc}(P)$ and suppress the measure in the following notation. We denote the canonical decomposition of S by $S = M + A$, where $A \in \mathcal{V}\cap\mathcal{P}$. Moreover, we define the following spaces of local martingales:

$$
\begin{aligned}
L_{loc}(dM^c) &:= \left\{\int \lambda\, dM^c : \lambda \in L_m(M^c)\right\}, \\
L_{loc}\left(dM^d\right) &:= \left\{\int \eta\, dM^d : \eta \in L_m\left(M^d\right) \cap L_m\left([M^d] - \langle M^d\rangle\right)\right\}, \\
L_\sigma(\perp M) &:= \{L \in \mathcal{M}_{loc} : [L, M] \text{ is a } \sigma\text{–martingale}\}.
\end{aligned}
$$

If $\mathcal{E}(-N)$ is a strictly positive σ–martingale density for S, condition **(NSC)** 'translates' the geometry of Figure 2.3.4 into a geometry of the 3–dimensional space $L_{loc}(dM^c) + L_{loc}\left(dM^d\right) + L_\sigma(\perp M)$; see Figure 2.3.8. Under **(NSC)**, the interpretation of Figure 2.3.7 translates in a similar way; see Figure 2.3.9. Note that Corollary 2.43 and the structure condition **(SC)** allow for a similar transformation of Figure 2.3.2 in a 2–dimensional way.

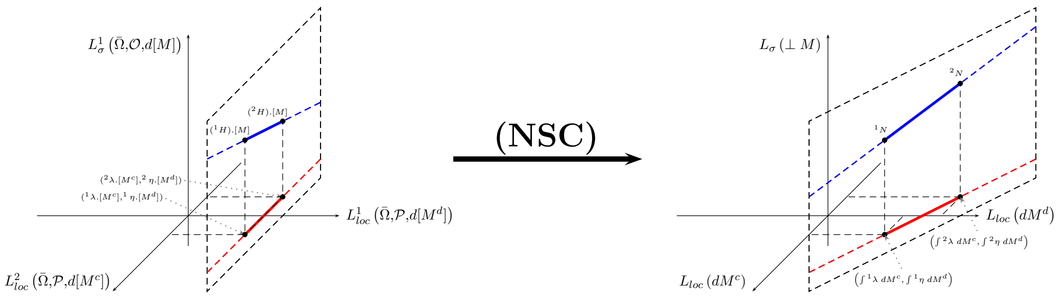

Figure 2.3.9.:

2.4. Necessary and sufficient structure conditions for (NUPBR)

In this section, we use the refined characterisation of the Radon–Nikodym theorem, Theorem 1.36, and Yor's formula to provide our 3^{rd} Structure Theorem. It leads directly to the definition of the *strong structure condition* **(SSC)**. Furthermore, we prove that **(SSC)** is necessary and sufficient for the existence of a strictly positive σ–martingale density. As an application, we provide a full structural characterisation of the **(NUPBR)** condition for the class of toy examples. It is worth mentioning that all examples provided by Choulli and Stricker in [15], highlighting the drawbacks of **(SC)**, fall into this class of toy examples. In the second part, we introduce the *floating structure condition* **(FSC)** and explain its connection to the other structure conditions. Moreover, we define the *floating martingale density* and explain its connection to the *minimal martingale measure*. We end this section with a simple, but nevertheless important observation: For continuous semimartingales all structure conditions introduced in this chapter are equivalent to the structure condition **(SC)**.

2.4.1. 3^{rd} Structure Theorem and the strong structure condition (SSC)

So far we introduced the structure conditions **(MSC)** and **(NSC)**. Both decompositions allow us to decompose a semimartingale $S = S_0 + M + A$ that satisfies either of these conditions into the sum

$$S = S_0 + S^c + S^d,$$

where

$$S^c := M^c + \int \lambda \, d[M^c],$$
$$S^d := M^d + \int \eta \, d\langle M^d \rangle,$$

and (λ, η) is a version of **(MSC)**. Due to Theorem 2.40, we know that there exists a strictly positive local martingale density for S^c for all versions of **(MSC)**. But what about the existence of a strictly positive σ–martingale density for S^d? Our 3^{rd} Structure Theorem relates the existence of a strictly positive σ–martingale density for S to the existence of a strictly positive σ–martingale density for S^d.

Theorem 2.46 (3^{rd} Structure Theorem)**:** *Let $Q \sim P$ be a probability measure and let S be a special Q–semimaritngale with canonical decomposition $S = S_0 + M + A$, where M is a local Q–martingale. Moreover, let N be a local Q–martingale and denote by*

$$N = \lambda . M^c + H \underset{c}{\bullet} M^d + L$$

the Radon–Nikodym decomposition of N w.r.t. M. Moreover, define

$$S^c := M^c + \int \lambda \, d\,[M^c] \qquad \text{and} \qquad S^d := M^d + A - \int \lambda \, d\,[M^c] \, .$$

Then the following statements are equivalent:

1. *$Z = \mathcal{E}(-N)$ is a strictly positive Q–σ–martingale density for S.*
2. *The process*

$$\mathcal{E}\left(-H \underset{c}{\bullet} M^d - L^d\right)$$

is a strictly positive Q–σ–martingale density for S^d.

Proof: First note that Yor's formula [42, Chap. II §8, Theorem 38] ensures that Z is strictly positive if and only if

$$\mathcal{E}\left(-H \underset{c}{\bullet} M^d - L^d\right)$$

is strictly positive. Due to Key lemma 2.11, we know that Z is a strictly positive Q–σ–martingale density for S if and only if

$$\begin{aligned}[N, M] - A &= [\lambda . M^c + H \underset{c}{\bullet} M^d + L, M] - A \\ &= [L^c, M^c] + [H \underset{c}{\bullet} M^d + L^d, M^d] - \left(A - \int \lambda \, d\,[M^c]\right)\end{aligned}$$

is a Q–σ–martingale. Because of Theorem 1.36, we can conclude that Z is a strictly positive Q–σ–martingale density for S if and only if

$$[H_{\overset{\bullet}{c}}M^d + L^d, M^d] - \left(A - \int \lambda \, d\,[M^c]\right)$$

is a Q–σ–martingale. Applying Key lemma 2.11 once more, we conclude that the last statement is equivalent to

$$\mathcal{E}\left(-H_{\overset{\bullet}{c}}M^d - L^d\right)$$

being a strictly positive Q–σ–martingale density for S^d. □

As we are interested in structure conditions that are necessary and sufficient for **(NUPBR)**, the theorem directly leads to the following definition.

Definition 2.47: *Let $Q \sim P$ and $S \in \mathcal{S}^2_{loc}$. Denote its canonical decomposition by $S = S_0 + M + A$, where M is a local Q–martingale. We say that S satisfies the* strong structure condition **(SSC)** under Q, *if there exists $\lambda \in L_m(M^c)$ under Q such that*

$$S^d := M^d + A - \int \lambda \, d\,[M^c]$$

satisfies **(NUPBR)**.

An immediate consequence of Theorem 2.21 and our 3^{rd} Structure Theorem, Theorem 2.46, is the following result.

Theorem 2.48: *Let $Q \sim P$ be a probability measure and let $S \in \mathcal{S}^2_{loc}(Q)$ with canonical decomposition $S = S_0 + M + A$, where M is a local Q–martingale. Then the following statements are equivalent:*

1. *There exists a strictly positive Q–σ–martingale density for S.*
2. *S satisfies the* **(NUPBR)** *condition.*
3. *S satisfies the strong structure condition* **(SSC)** *under Q.*

Proof: Due to Theorem 2.21 and Theorem 2.46, we know that '*2.⇒1.⇒3.*' holds. Hence, it remains to prove that '*3.⇒2.*' holds. Since S satisfies **(SSC)**, there exists $\lambda \in L_m(M^c)$ under Q such that

$$S^d := M^d + A - \int \lambda \, d\,[M^c]$$

satisfies **(NUPBR)**. Due to Theorem 2.21, there exists a strictly positive Q–σ–martingale density $\mathcal{E}(-N)$ for S^d. Denote by

$$N = H_{\overset{\bullet}{c}}M^d + L$$

the Radon–Nikodym decomposition of N w.r.t. M^d. Applying Theorem 2.46 (with $S = S^d$ and $S^c = 0$) ensures that

$$\mathcal{E}\left(-H \underset{c}{\bullet} M^d - L^d\right)$$

is a strictly positive Q–σ–martingale density for S^d. Applying Theorem 2.46 once more with $S = S^c + S^d$, where $S^c := M^c + \int \lambda \, d[M^c]$, we can conclude that

$$\mathcal{E}\left(-\lambda.M^c - H \underset{c}{\bullet} M^d - L^d\right)$$

is a strictly positive Q–σ–martingale density for S. Hence, S satisfies the **(NUPBR)** condition. □

The following lemma gives an equivalent reformulation of **(SSC)**.

Lemma 2.49: *Let $Q \sim P$ and $S \in \mathcal{S}^2_{loc}$. Denote its canonical decomposition by $S = S_0 + M + A$, where M is a local Q–martingale. The following statements are equivalent:*

1. *S satisfies the strong structure condition* **(SSC)** *under Q.*
2. *S satisfies* **(MSC)**. *Moreover, there exists a version (λ, η) of* **(MSC)** *such that*
$$\int \eta \, d\langle M^d\rangle^Q = A - \int \lambda \, d[M^c]$$
and $S^d := M^d + \int \eta \, d\langle M^d\rangle^Q$ satisfies the **(NUPBR)** *condition.*

Proof: This follows directly from Theorem 2.48 and Theorem 2.24. □

Due to the lemma, we can also talk about 'version of **(SSC)**'.

Definition 2.50: *Let $Q \sim P$ and let $S \in \mathcal{S}^2_{loc}$ satisfy* **(MSC)** *under Q. Moreover, let (λ, η) be a version of* **(MSC)** *and define*

$$S^c := M^c + \int \lambda \, d[M^c],$$
$$S^d := M^d + \int \eta \, d\langle M^d\rangle.$$

We call $S = S_0 + S^c + S^d$ the semimartingale decomposition of S w.r.t. (λ, η). Moreover, S^c and S^d are called the continuous and purely discontinuous semimartingale part of S w.r.t. (λ, η). *Finally, we call (λ, η)* a version of **(SSC)**, *if S^d satisfies* **(NUPBR)**.

The purpose of the next toy example is to highlight the different qualities of the structure conditions **(SC)**, **(NSC)**, and **(SSC)** as indicators for a strictly positive σ–martingale density.

Toy example 2.51: Let $c_1 = c_2 = 1$, $\tilde{\lambda} = 1$, and define for $\alpha \in [0,1]$ the following versions of **(NSC)**:

$$(\lambda^\alpha, \eta^\alpha) = (2\alpha, 2(1-\alpha)).$$

Now the structure conditions have the following different 'points of view' on the price process

$$S = B + N + 2t.$$

While **(SC)** considers S as

$$S = (B + N) + \int \tilde{\lambda}\, d\langle B + N\rangle,$$

(NSC) is more flexible. It considers 'different versions' of the drift of S, i.e. for $\alpha \in [0,1]$

$$S^\alpha = B + N + \int \lambda^\alpha\, d\langle B\rangle + \int \eta^\alpha\, d\langle N\rangle.$$

Finally, **(SSC)** considers all versions of the continuous and purely discontinuous part of S w.r.t. $(\lambda^\alpha, \eta^\alpha)$, i.e.

$$S^\alpha = S^{c,\alpha} + S^{d,\alpha},$$

where

$$S^{c,\alpha} := M^c + \int \lambda^\alpha\, d\langle M^c\rangle,$$
$$S^{d,\alpha} := M^d + \int \eta^\alpha\, d\langle M^d\rangle.$$

As a consequence of Theorem 2.48, it suffices to check whether or not

$$S^{d,\alpha} = M^d + \int \eta^\alpha\, d\langle M^d\rangle$$

satisfies the **(NUPBR)** condition. Due to Theorem 2.40, it is immediately clear that $S^{d,\alpha}$ satisfies the **(NUPBR)** condition if and only if $\alpha > 1/2$. Simply put, **(SSC)** selects the good versions $(\lambda^\alpha, \eta^\alpha)$ that directly lead to the strictly positive local martingale densities

$$\mathcal{E}\left(-\int \lambda^\alpha\, dB\right)\mathcal{E}\left(-\int \eta^\alpha\, dN\right).$$

Under **(NSC)**, we do not have Theorem 2.48 at our disposal. Thus, we have to take care of the 'whole process' S^α. Thanks to the simple structure of the example, Theorem 2.40 is still a powerful tool on its own. For each version $(\lambda^\alpha, \eta^\alpha)$ it suggests

$$\mathcal{E}\left(-\int \lambda^\alpha\, dB - \int \eta^\alpha\, dN\right)$$

as a natural candidate for a (strictly positive) local martingale density for S. After taking a closer look at the example, Theorem 2.40 almost pinpoints to the version $(2,0)$ of **(NSC)**. This leads to the strictly positive local martingale density

$$\mathcal{E}\left(-2B\right). \tag{2.4}$$

On the other hand, the structure condition **(SC)** suggests to choose

$$\mathcal{E}\left(-\int \tilde{\lambda}\, d(B+N)\right) = \mathcal{E}\left(-(B+N)\right)$$

as a candidate for a local martingale density. Unfortunately, $\mathcal{E}\left(-\int \tilde{\lambda}\, d(B+N)\right)$ is not strictly positive. Corollary 2.43 suggests to look for a local martingale L such that

$$\mathcal{E}\left(-\int \tilde{\lambda}\, d(B+N) - L\right)$$

is strictly positive and such that $[L, B+N]$ is a local martingale. In general, this characterisation does not seem to be a helpful tool to find strictly positive local martingale densities.

With the help of (2.4), we can guess a candidate that satisfies the requirements. If we choose $L = B - N$, a straightforward computation leads to

$$\mathcal{E}\left(-\int \tilde{\lambda}\, d(B+N) - L\right) = \mathcal{E}\left(-2B\right) \quad \text{and} \quad [L, B+N] = -N \in \mathcal{M}_{loc}.$$

Structural characterisation of (NUPBR) for the toy example

To highlight the power of the new insights on the relation between structure conditions and martingale decompositions of strictly positive σ–martingale densities, we provide a complete structural characterisation of the **(NUPBR)** condition for the toy example.

We start with the continuous paths case which follows directly from Lemma 2.31.

Theorem 2.52 (Toy example: $c_1 = 1$, $c_2 = 0$)**:** *Consider the toy example and let $c_1 = 1$ and $c_2 = 0$. Then it satisfies* **(NUPBR)** *if and only if $(\tilde{\lambda}, 0)$ is a version of* **(SSC)**. *Moreover,*

$$\mathcal{E}\left(-\int \tilde{\lambda}\, dB\right)$$

is the canonical choice of a strictly positive local martingale density.

The case $c_1 = 0$, $c_2 = 1$ is also rather obvious.

Theorem 2.53 (Toy example: $c_1 = 0$, $c_2 = 1$)**:** *Consider the toy example and let $c_1 = 0$ and $c_2 = 2$. If the toy example satisfies* **(MSC)**, *then the following statements are equivalent:*

1. *S satisfies* **(NUPBR)**.
2. *S satisfies* **(NA$_+$)**.
3. $\int \mathbb{1}_{\{\tilde{\lambda}\geq 1\}}\tilde{\lambda}\, d\langle N\rangle \equiv 0$.
4. *There exists a version $(0,\tilde{\eta})$ of* **(SSC)** *such that $\int \mathbb{1}_{\{\tilde{\eta}\geq 1\}}\tilde{\eta}\, d\langle N\rangle \equiv 0$.*

Moreover, the version $(0,\eta) = (0, \mathbb{1}_{\{\tilde{\lambda}<1\}}\tilde{\lambda}) = (0,\tilde{\lambda})$ of **(SSC)** *is the canonical choice that leads to the strictly positive local martingale density*

$$\mathcal{E}\left(-\int \eta\, dN\right).$$

Proof: Note, that if $\int \mathbb{1}_{\{\tilde{\lambda}\geq 1\}}\tilde{\lambda}\, d\langle N\rangle \not\equiv 0$ the sequence $(H(n))_{n\geq 1} \subset \mathcal{P}$, where

$$H(n) := n\tilde{\lambda}\mathbb{1}_{\{n\geq\tilde{\lambda}\geq 1\}}, \qquad n \geq 1,$$

is a sequence of 1–admissible trading strategies. Moreover,

$$H(n)\cdot S = n\left(\tilde{\lambda}\mathbb{1}_{\{n\geq\tilde{\lambda}\geq 1\}}\cdot P + \int \mathbb{1}_{\{n\geq\tilde{\lambda}\geq 1\}}\tilde{\lambda}\left(\tilde{\lambda}-1\right)\, d\langle N\rangle\right) \geq 0$$

for all $n \geq 1$. Furthermore, $\int \mathbb{1}_{\{\tilde{\lambda}\geq 1\}}\tilde{\lambda}\, d\langle N\rangle \not\equiv 0$ ensures that there exists $N \in \mathbb{N}$ such that $P(H(n)\cdot S_T > 0) > 0$ for all $n \geq N$. Therefore,

$$\textbf{(NUPBR)} \quad\Longrightarrow\quad \textbf{(NA}_+\textbf{)} \quad\Longrightarrow\quad \int \mathbb{1}_{\{\tilde{\lambda}\geq 1\}}\tilde{\lambda}\, d\langle N\rangle \equiv 0$$

holds. Due to Theorem 2.40, the strictly positive local martingale $\mathcal{E}\left(-\int \eta\, dN\right)$, where $\eta := \mathbb{1}_{\{\tilde{\lambda}<1\}}\tilde{\lambda}$, is a strictly positive local martingale density for S. Hence, the statement is proven. □

Intuitively, we achieve the structural characterisation of the **(NUPBR)** condition of the compound toy example, i.e. $c_1 = c_2 = 1$, by piecing together the structural characterisations found in the theorems above. Indeed, in the following theorem the **(NUPBR)** condition ensures that this educated guess is in fact true.

Theorem 2.54 (Toy example: $c_1 = c_2 = 1$)**:** *Let the toy example satisfy* **(MSC)** *and let $c_1 = c_2 = 1$. Then the following statements are equivalent:*

1. *S satisfies* **(NUPBR)**.

2. $\int \mathbb{1}_{\{\tilde{\lambda}\geq 1\}}\tilde{\lambda}^2 \, d\langle M\rangle \in \mathcal{A}_{loc}$.

3. $\int \mathbb{1}_{\{\tilde{\lambda}\geq \frac{1}{2}\}}\tilde{\lambda}^2 \, d\langle M\rangle \in \mathcal{A}_{loc}$.

4. There exists a version $(\tilde{\lambda}, \tilde{\eta})$ *of* **(SSC)** *such that* $\int \mathbb{1}_{\{\tilde{\eta}\geq 1\}}\tilde{\eta} \, d\langle N\rangle \equiv 0$.

Moreover, $(\lambda, \eta) = (2\mathbb{1}_{\{\tilde{\lambda}\geq \frac{1}{2}\}}\tilde{\lambda}, 2\mathbb{1}_{\{\tilde{\lambda}< \frac{1}{2}\}}\tilde{\lambda})$ *is a version of* **(SSC)** *that satisfies 4. and*

$$\mathcal{E}\left(-\int \lambda \, dB - \int \eta \, dN\right) = \mathcal{E}\left(-\int \lambda \, dB\right)\mathcal{E}\left(-\int \eta \, dN\right) \tag{2.5}$$

is the canonical choice of a strictly positive local martingale density.

Proof: ***1.*⇒*2.*:** We prove that **(NUPBR)** is violated, if $\int \mathbb{1}_{\{\tilde{\lambda}\geq 1\}}\tilde{\lambda}^2 \, d\langle M\rangle \notin \mathcal{A}_{loc}$ holds. The idea of this proof in borrowed from [15, Théorème 2.9 ii)]. Suppose that $\int \mathbb{1}_{\{\tilde{\lambda}\geq 1\}}\tilde{\lambda}^2 \, d\langle M\rangle \notin \mathcal{A}_{loc}$. Then there exists $\epsilon > 0$ such that

$$P\left(\int_0^1 \mathbb{1}_{\{\tilde{\lambda}\geq 1\}}\tilde{\lambda}_u^2 \, d\langle M\rangle_u = \infty\right) > \epsilon.$$

Define the sequence of bounded predictable processes $(\tilde{\lambda}(n))_{n\geq 1} \subset \mathcal{P}$ by $\tilde{\lambda}(n) := \tilde{\lambda}\mathbb{1}_{\{1\leq\tilde{\lambda}\leq n\}}$ for $n \geq 1$. As for all constants $c \geq 0$

$$\left\{\int_0^1 \tilde{\lambda}_u^2(n) \, d\langle M\rangle_u \geq c\right\} \nearrow \left\{\int_0^1 \tilde{\lambda}_u^2 \mathbb{1}_{\{\tilde{\lambda}\geq 1\}} \, d\langle M\rangle_u \geq c\right\}$$

holds, we can find a sequence $(c_n)_{n\geq 1} \subset \mathbb{N}$ such that $c_n \uparrow \infty$ and

$$P\left(\int_0^1 \tilde{\lambda}_u(n)\tilde{\lambda}_u \mathbb{1}_{\{\tilde{\lambda}\geq 1\}} \, d\langle M\rangle_u \geq c_n\right) > \epsilon, \qquad \forall\, n \geq N. \tag{2.6}$$

Now we define $(T_n)_{n\geq 1}$ and $(\alpha(n))_{n\geq 1}$ via

$$T_n := \inf\left\{t > 0 : \int_0^t \tilde{\lambda}_u(n)\tilde{\lambda}_u \mathbb{1}_{\{\tilde{\lambda}\geq 1\}} \, d\langle M\rangle_u > c_n\right\} \wedge 1,$$
$$\alpha(n) := c_n^{-\frac{3}{4}}\tilde{\lambda}(n)\mathbb{1}_{[\![0,T_n]\!]}.$$

Due to the definition of $(T_n)_{n\geq 1}$ and the continuity of $\langle M\rangle$, we get

$$\begin{aligned}\int_0^1 \alpha_u^2(n) \, d\langle M\rangle_u &= c_n^{-\frac{3}{2}} \int_0^1 \tilde{\lambda}_u^2(n)\mathbb{1}_{[\![0,T_n]\!]} \, d\langle M\rangle_u \\ &= c_n^{-\frac{3}{2}} \int_0^1 \tilde{\lambda}_u(n)\tilde{\lambda}_u \mathbb{1}_{\{\tilde{\lambda}\geq 1\}}\mathbb{1}_{[\![0,T_n]\!]} \, d\langle M\rangle_u \\ &\leq c_n^{-\frac{1}{2}}.\end{aligned} \tag{2.7}$$

Next we define the sequence of stopping times $(S_n)_{n\geq 1}$ and the bounded sequence $(H(n))_{n\geq 1} \subset \mathcal{P}$ via

$$\begin{aligned} S_n &:= \inf \left\{ t > 0 : \int_0^t \alpha_u(n)\, dB_u \leq -1 \right\} \wedge 1, \\ H(n) &:= \alpha(n) \mathbb{1}_{[\![0,S_n]\!]}. \end{aligned}$$

Since

$$\begin{aligned} H(n) \cdot S &= \alpha(n) \cdot B^{S_n} + \alpha(n) \cdot N^{S_n} + \int \alpha(n) \tilde{\lambda}\, d\langle B + N \rangle^{S_n} \\ &= \alpha(n) \cdot B^{S_n} + \alpha(n) \cdot P^{S_n} + \\ &\quad + \int \alpha(n) \tilde{\lambda} \mathbb{1}_{\{\tilde{\lambda} \geq 1\}}\, d\langle B \rangle^{S_n} + \int \alpha(n) \left(\tilde{\lambda} - 1 \right) \mathbb{1}_{\{\tilde{\lambda} \geq 1\}}\, d\langle N \rangle^{S_n} \\ &\geq -1 \end{aligned} \tag{2.8}$$

for all $n \geq 1$, $(H(n))_{n\geq 1} \subset \mathcal{P}$ is a sequence of 1–admissible trading strategies. Moreover, (2.7) ensures that $\sup_{u\leq 1} |\alpha(n) \cdot B_u|$ tends to zero in $L^2(P)$. This in turn guarantees that

$$P\left(S_n = 1\right) \xrightarrow[n\to\infty]{} 1. \tag{2.9}$$

Furthermore, the definition of $(T_n)_{n\geq 1}$ and the continuity of $\langle M \rangle$ ensure that

$$\left\{ \omega \in \Omega : \int_0^1 \tilde{\lambda}_u(n) \tilde{\lambda}_u\, d\langle M \rangle_u \geq c_n \right\} = \left\{ \omega \in \Omega : \int_0^{T_n} \tilde{\lambda}_u(n) \tilde{\lambda}_u\, d\langle M \rangle_u \geq c_n \right\}$$

holds for all $n \in \mathbb{N}$. Hence, (2.9) and (2.6) imply that there exists $\tilde{N} \in \mathbb{N}$ such that for all $n \geq \tilde{N}$

$$\begin{aligned} &P\left(\int_0^1 H_u(n) \tilde{\lambda}_u \mathbb{1}_{\{\tilde{\lambda} \geq 1\}}\, d\langle M \rangle_u \geq c_n^{\frac{1}{4}} \right) \\ &\quad \geq P\left(\left\{ \int_0^1 \tilde{\lambda}_u(n) \tilde{\lambda}_u \mathbb{1}_{\{\tilde{\lambda} \geq 1\}}\, d\langle M \rangle_u \geq c_n \right\} \cap \{S_n = 1\} \right) \geq \epsilon \end{aligned} \tag{2.10}$$

holds. Decomposing $H(n) \cdot S$ in a similar way as in (2.8) we find that

$$1 + H(n) \cdot S_1 \geq -c_n^{-\frac{3}{4}} \int_0^1 \left| \tilde{\lambda}_u \right|\, d\langle N \rangle_u + \int_0^1 H_u(n) \tilde{\lambda}_u \mathbb{1}_{\{\tilde{\lambda} \geq 1\}}\, d\langle M \rangle_u.$$

Since by assumption $E\left[\int_0^1 \left| \tilde{\lambda}_u \right|\, d\langle N \rangle_u \right] < \infty$, (2.10) ensures that $(H(n))_{n\geq 1} \subset \mathcal{P}$ is a sequence of 1–admissible trading strategies that provides an unbounded profit with

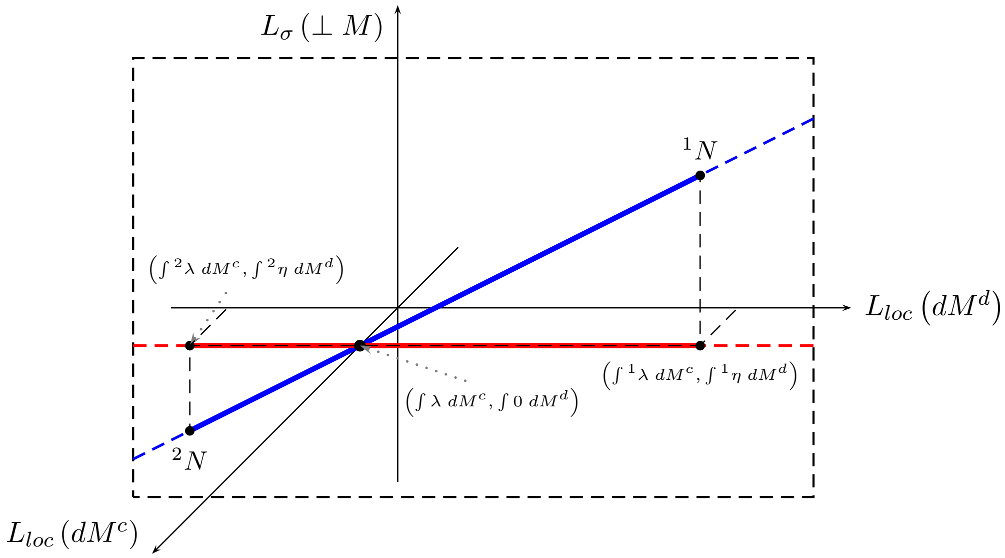

Figure 2.4.1.:

bounded risk. The rest of the proof is straightforward. It is clear that the implication ***2.***$\Rightarrow$***3.*** holds. Moreover, since $(\lambda, \eta) = (2\mathbb{1}_{\{\tilde{\lambda}\geq\frac{1}{2}\}}\tilde{\lambda}, 2\mathbb{1}_{\{\tilde{\lambda}<\frac{1}{2}\}}\tilde{\lambda})$ is a version of **(MSC)** that satisfies *4.*, Yor's formula [42, Chap. II §8, Theorem 38] and Theorem 2.40 ensure that

$$\mathcal{E}\left(-\int \lambda\, dB - \int \eta\, dN\right) = \mathcal{E}\left(-\int \lambda\, dB\right)\mathcal{E}\left(-\int \eta\, dN\right)$$

is a strictly positive σ–martingale density for S. This in turn implies the **(NUPBR)** condition. $\square$

2.4.2. Floating structure condition (FSC) and floating martingale density

Taking a closer look at Theorem 2.54 it becomes apparent that a proper decomposition of the drift $A = \int \tilde{\lambda}\, d\langle M\rangle$ of the locally square–integrable semimartingale $S = M + A$ can automatically lead to a *strictly positive* σ–martingale density for S. In Theorem 2.54 this goal is achieved by floating the 'critical mass' $\tilde{\lambda}\mathbb{1}_{\{\tilde{\lambda}\geq 1\}}$ of the drift towards the bracket process of the continuous martingale part. In this section, we consider the special case in which it is possible to float not only the critical mass of the drift but *all* its mass towards the bracket process of the continuous martingale part. The idea of the floating structure condition is highlighted in Figure 2.4.1. It is a special case of Figure 2.3.9 and ensures that $\mathcal{E}(-\int \lambda\, dM^c)$ is a local martingale density for $S = M + A$. These considerations lead to the following definition.

Definition 2.55: *Let S be a locally square–integrable semimartingale with canonical decomposition $S = S_0 + M + A$ that satisfies the minimal structure condition* **(MSC)**. *We say that S satisfies the* floating structure condition **(FSC)**, *if there exists $\lambda \in L_m(M^c)$ such that $(\lambda, 0)$ is a version of* **(MSC)**. *$\mathcal{E}(-\int \lambda\, dM^c)$ is called the* floating martingale density.

The next theorem ensures that the structure condition **(FSC)** is sufficient for the existence of a strictly positive σ–martingale for S.

Theorem 2.56: *If S satisfies* **(FSC)**, *there exists a strictly positive local martingale density for S.*

Proof: Suppose that **(FSC)** holds, and let $(\lambda, 0)$ be a version of **(MSC)**. Then $\mathcal{E}(-\int \lambda\, dM^c)$ is a strictly positive and locally bounded local martingale. Due to Theorem 2.40 the local martingale $\mathcal{E}(-\int \lambda\, dM^c)$ is a strictly positive local martingale density for S if and only if $[\int \lambda\, dM^c, M^c] - \lambda.\langle M^c \rangle$ is a local martingale. Hence, **(FSC)** is sufficient for the existence of a local martingale density for S. □

The following example highlights the 'floating property' of **(FSC)** in a simple example. Moreover, it points out the advantages of **(FSC)** and the *floating martingale density* compared to **(SC)** and the *minimal martingale measure*.

Toy example 2.57: As in Toy example 2.51, we choose $c_1 = c_2 = 1$ and $\tilde{\lambda} \equiv 1$. Clearly, **(MSC)** and **(SC)** are satisfied. Moreover, the decomposition of S according to the structure condition is given by

$$S = B + N + \langle B + N \rangle.$$

On the other hand, for all $\alpha \in [0, 1]$ the pair $(2\alpha, 2(1-\alpha))$ is a version of **(MSC)** and the corresponding decomposition is given by

$$S = B + N + 2\alpha \langle B \rangle + 2(1-\alpha)\langle N \rangle.$$

By floating α towards 1, we can float the drift towards the continuous martingale bracket. Since the stochastic exponential of a continuous local martingale is always strictly positive, $\mathcal{E}(-2B)$ is a canonical choice for a *strictly positive* local martingale density. On the other hand it is clear that $\mathcal{E}(-2\alpha B - 2(1-\alpha)N)$ hits zero for all $\alpha \leq \frac{1}{2}$. For $\alpha = \frac{1}{2}$ we get the minimal martingale measure, i.e. $\mathcal{E}(-\int 1\, dM) = \mathcal{E}(-2\alpha B - 2(1-\alpha)N)$. The structure condition **(SC)** indicates this 'martingale density' as a the natural candidate for a strictly positive σ–martingale density. Unfortunately, it is not strictly positive in general. Therefore, it is not a natural candidate for a pricing measure.

This idea also holds in a much more general setting.

Theorem 2.58: *Let S be a locally square–integrable semimartingale with canonical decomposition $S = S_0 + M + A$ that satisfies the minimal structure condition* **(MSC)**. *Moreover, let $d\langle M^d\rangle \ll d\langle M^c\rangle$, where $\langle M^d\rangle = \int \bar{\eta}\, d\langle M^c\rangle$. If there exists a version (λ, η) of* **(MSC)** *such that $\eta\bar{\eta} \in L_m(M^c)$, then S satisfies the floating structure condition* **(FSC)**.

Proof: Due to the assumptions of the lemma $(\lambda + \eta\bar{\eta}, 0)$ is a version of **(MSC)**. □

Example 2.59: Let $S = S_0 + M + A$ denote the canonical decomposition of a locally square–integrable semimartingale, where S is a Type C Lévy process with non vanishing continuous local martingale part. If S satisfies **(MSC)**, then $d\langle M^d\rangle \ll d\langle M^c\rangle$.

We end this section with the following lemma. It ensures that for continuous semimartingales all structure conditions introduced in this chapter boil down to the structure condition **(SC)**.

Lemma 2.60: *Let S be a continuous semimartingale. The following relations hold:*

$$\textbf{(SC)} \iff \textbf{(MSC)} \iff \textbf{(FSC)} \iff \textbf{(SSC)} \iff \textbf{(NSC)}$$

2.5. Conclusions and final remarks

During the lecture of this chapter it became apparent that we should consider 'structure conditions' as a *dynamic* concept. Combining this point of view with the insights on the connection between structure conditions on the one hand, and decompositions of strictly positive σ–martingale densities on the other hand, transform these new ideas into a powerful tool for finding strictly positive σ–martingale densities. Yet, the new structure conditions still allow for further developments. Indeed, similar to Definition 2.16 of **(SC)**, the natural structure condition **(NSC)** can be defined in a d–dimensional way. This generalization would allow for a 'simultaneous search' for a strictly positive σ–martingale densities for all d risky assets. Furthermore, these structure conditions can serve as a useful tool in other areas of mathematical finance. For example, we will provide an application of **(SC)** in the context of a large trader model in the next chapter.

3. A modular model approach to large traders

3.1. Introduction

Modelling a financial market with a large trader leads to a number of interesting and challenging problems. Similar to Jarrow [32], we see a large trader as a trader 'whose trades change prices'. The existence of a large trader goes hand in hand with dropping the 'competitive market paradigm'. This paradigm claims that any trader can buy or sell unlimited quantities of the stock under consideration, without influencing the stock's price.

In order to model a financial market with a large trader, one has to specify the *goal* of the large trader. In other words, one has to answer the following question:

'What is the large trader's motivation to trade?'

Schied and co–authors [48, 49, 26, 47] focus on finding optimal liquidation strategies for the large trader, while Bank and Baum [4] tie the motivation of the large trader to trade to a utility maximization problem. Regardless of the particular goal of the large trader, there is another important question associated with a large trader model:

'What affects the large trader's decision to achieve her goal?'

Clearly, any restriction of the large trader's actions due to *no arbitrage considerations for the large trader* influence the opportunities to achieve her goal. Apart from no arbitrage considerations for the large trader, *available information* and *liquidity risk* are further factors that have an impact on the 'quality' of the large trader's decision. Let us briefly comment on these points. Concerning the first point, it seems to be reasonable that the large trader *knows* at least *her current position* and *the current price* of the asset under consideration. As the large trader knows her strategy, she has a natural edge on information compared to the small trader. The latter only observes the current price of the asset. Given the large trader has additional (insider) information, it seems to be natural that this additional information improves the 'quality of the decision' she makes in order to achieve her goal.

Liquidity risk is a term used in the market micro–structure literature; see [38]. As pointed out in [11], liquidity risk is, roughly speaking, the additional risk due to the timing and size of a trade. Kyle [38] distinguishes three different types of liquidity risks. While 'market tightness' refers to the costs of turning over a position in a short period of time, 'market depth' refers to the ability of a market to absorb quantities without having a large effect on prices. The last one, 'market resiliency', refers to the speed with which a certain price impact of the large trader vanishes.

Furthermore, any kind of *trading restriction* affects the large traders opportunities to achieve her goal. First of all, such trading restrictions arise from liquidity risk, i.e. certain strategies might be too cost–intensive. Others might be discarded as they do not allow for a certain risk control for the large trader's wealth process. Finally, if the large trader is any large fund, it is likely that the trading strategies are, in general, not self–financing.

Apart from the factors that influence a large trader's decision, a reasonable question concerning the presence of a large trader is the following one:

'How does a large trader's strategy influence prices?'

This question is connected to the micro structure of the market. As pointed out above, this question is related to the 'market depth' and the 'market resiliency'. The probably most popular large trader market model that takes both aspects into account is the Almgren–Chriss model; see [2, 1].

Finally, the most important question is the following:

'How are these phenomena linked to each other?'

If we propose a certain large trader model, we put several assumptions on the model that lead to, or explain, the mechanisms of the various quantities above. Finding empirical evidence for certain connections and mechanisms is a completely different question.

The purpose of this chapter is to introduce a modular large trader model. The idea is to provide a module for each phenomenon connected to the presence of a large trader. In each module, we provide a definition of the particular phenomenon *for simple strategies*. We then extend these definitions using a proper limit procedure. Here, we focus on two modules. The first one is, of course, the 'price module'. Our definition of the price process, affected by a large trader strategy, is strongly connected to the *non–linear stochastic integration theory* of Carmona and Nualart; see [10]. Apart form the general definition and its extension, we provide several explicit examples, among those is the popular Almgren–Chriss model.

The second module is connected to no arbitrage considerations for the *small trader*. Again, we propose a minimal no arbitrage assumption that depends only on simple large trader strategies. It should be mentioned that our no arbitrage assumption is closely connected to the no arbitrage assumption in the discrete time setting of Jarrow; see [32]. Combining these two modules, the 'price module' and the 'no arbitrage' module, leads to the class of *reasonable large trader market models*. These large trader models feature, in our opinion, the two most important properties. Firstly, the price process affected by the large trader is a semimartingale. Secondly, the price process affected by the large trader is a *reasonable* arbitrage free model for a financial market with *small trader*.

As argued above, the driving force in a large trader model is the large traders motivation to trade. Usually, these goals are formulated using the large trader's real wealth process. Attaching the real wealth process to the class of reasonable large trader market models leads to the class of *minimal large trader market models*. Apart from the two features above, these large trader market models enable us to define a real wealth process for the large trader. Here, 'minimal' refers to the fact that these models allow us to model large trader phenomena without annihilating the usual no arbitrage assumptions made in the small trader literature. We compare this minimal large trader market model to the large trader model proposed by Bank and Baum; see [4]. Finally, we introduce an admissibility concept for the large trader and investigate the large trader's utility maximization problem in a basic minimal large trader market model. Despite its simple structure, it highlights new phenomena that are a result of the presence of a large trader.

The chapter is organized as follows. In Section 2, we provide the definitions of the price process, affected by a simple large trader strategy, and the minimal no arbitrage assumption for the small trader. We explain how to extend these definitions to general large trader strategies and provide several examples. In Section 3, we use the sophisticated non–linear integration theory of Carmona and Nualart [10], to provide a rich class of reasonable large trader market models. Section 4 provides the definition of the real wealth process of a large trader. Moreover, we discuss the connections and differences of the minimal large trader market model and the large trader model proposed in [4]. Section 5 is dedicated to the analysis of the large trader utility maximization problem. Moreover, it highlights new phenomena that arise in a financial market due to the presence of a large trader.

Finally, it should be mentioned that a short overview of strong non–linear integrators and the non–linear stochastic integration theory can be found in the Appendix.

3.2. Reasonable large trader market models

We start by introducing the two core modules of the large trader modular model. These are the 'price module' and the 'NA module'; of course, 'NA' is an acronym for 'no arbitrage'. As in all modules, the idea is to provide a minimal assumption that allows us to model the impact of the large trader on the module *for all simple strategies*. With this foundation, a proper definition of the module for simple strategies, we extend the definition to more general large trader strategies using a limit procedure. Summarizing the ideas of these to modules, we end this section with the definition of the class of *reasonable large trader market models*. In our opinion, this is the class of models that satisfies the two minimal requirements of a large trader model. Firstly, it allows to model the price process affected by the large trader as a semimartingale. Secondly, modelling the impact of a large trader can be done in a way that respects a (minimal) economic no arbitrage assumption for the *small trader*. Here, we choose the **(NFLVR)** condition as our minimal no arbitrage assumption for the small trader.

It might be helpful, if the reader is familiar with the basic definitions of strong non–linear integrators, strong non–linear integrals, as well as basic results on convergence in the semimartingale topology. A detailed introduction to these concepts is provided in the book by Carmona and Nualart; see [10]. An overview of these concepts can be found in the Appendix. As we work on a finite time interval, we use the following convention: Results from the literature formulated for an infinite time horizon are used by applying the corresponding result to the stopped process.

3.2.1. The price module: Price process affected by a large trader strategy

Our financial market consists of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ that satisfies the usual conditions. Besides, $\mathcal{F}_0$ is trivial apart from zero sets and T denotes some finite time horizon. At the core of our model, we have a family $(S(\vartheta, .))$, $\vartheta \in \mathbb{R}^d$, of $\mathbb{R}$–valued semimartingales, adapted to $\mathbb{F}$ and the following space of simple strategies.

Definition 3.1 ([10])**:** *Let θ be a predictable process with representation*

$$\theta(t) = \theta_{-1}\mathbb{1}_{\{0\}} + \sum_{i=0}^{n} \theta_i \mathbb{1}_{(\tau_i, \tau_{i+1}]}(t), \tag{3.1}$$

where $0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_{n+1} = T$ is a finite sequence of $(\mathcal{F}_t)$–stopping times, $\theta_{-1} \in \mathbb{R}^d$, and θ_i is for each $i \in \{0, \ldots, n\}$ a bounded, $\mathcal{F}_{\tau_i}$–measurable, $\mathbb{R}^d$–valued random variable. We call θ a simple strategy*, if the random variables θ_i as well as the stopping times τ_i take only finitely many values. The space of all $\mathbb{R}^d$–valued simple strategies is denoted by $\mathbf{S}(\mathbb{R}^d)$.*

Price process affected by simple large trader strategies

As pointed out in the introduction, there are different types of liquidity risk. These liquidity risks are caused by the presence of the large trader. In order to model the different types of liquidity risk, depending on the large trader strategy $\theta \in \mathbf{S}(\mathbb{R})$, we introduce a function

$$\begin{aligned} F : \mathbf{S}(\mathbb{R}) &\longrightarrow \mathbf{S}\left(\mathbb{R}^d\right), \\ \theta &\longmapsto F(\theta) =: \tilde{\theta}. \end{aligned}$$

While we interpret the process $\theta \in \mathbf{S}(\mathbb{R})$ as *the large trader strategy*, $\tilde{\theta} := F(\theta) \in \mathbf{S}\left(\mathbb{R}^d\right)$ is considered to be a breakdown of θ into those components that cause the impact on the price process. Furthermore, let $(\tau_i)_{i \leq n+1}$ denote the sequence of stopping times appearing in the decomposition of $\tilde{\theta}$. The stopping times $(\tau_i)_{i \leq n+1}$ are interpreted as those points in the future when the large trader's actions cause a change of the price dynamics. With these ideas in mind, we model the discounted price process, caused by those changes, by the elementary non–linear stochastic integral

$$P_t^\theta := \int_0^t S\left(F(\theta_s), ds\right) := S(\tilde{\theta}_{-1}, 0) + \sum_{i=0}^{n} \left\{ S\left(\tilde{\theta}_{\tau_i \wedge t+}, \tau_{i+1} \wedge t\right) - S\left(\tilde{\theta}_{\tau_i \wedge t+}, \tau_i \wedge t\right) \right\}. \tag{3.2}$$

There are two aspects that we have to take into account. First of all, the price processes, affected by a large trader strategy, should serve as a reasonable price process of a frictionless small trader model. As the class of semimartingales is the classical choice for modelling price processes in frictionless small trader markets, we work under the following standing assumption.

Assumption (P-I): For all $\theta \in \mathbf{S}(\mathbb{R})$, the process P^θ defined in (3.2) is a semimartingale. □

The second aspect is related to the representation of a simple strategy. In applications, we would like to define the function $F : \mathbf{S}(\mathbb{R}) \longrightarrow \mathbf{S}\left(\mathbb{R}^d\right)$ by using this representation. As the representation is not unique, we have to select a particular one.

Lemma 3.2: *Let $\theta \in \mathbf{S}(\mathbb{R})$. Then $\theta_{t+} \in \mathcal{F}_t$ for all $t \in [0, T]$. Moreover, define for $t \geq 0$*

$$\Delta^+ \theta_t := \lim_{h \downarrow 0} \left(\theta_{t+h} - \theta_t\right).$$

Denote by $t_1 < \ldots < t_K$ the jump times of $\theta \in \mathbf{S}(\mathbb{R})$, i.e. those $t \in [0, T]$ such that $\mathbb{P}(\Delta^+\theta_t \neq 0) > 0$. Besides, set

$$\Pi := \begin{cases} \{0 = t_0 \leq t_1 < \ldots < t_K < t_{K+1} = T\}, & \text{if } \theta \text{ jumps with positive probability,} \\ \{0, T\}, & \text{else.} \end{cases}$$

Then

$$\theta_{-1}\mathbb{1}_{\{0\}} + \sum_{\substack{i=0 \\ (t_i)\subset\Pi}}^{K} \theta_{t_i+}\mathbb{1}_{(t_i,t_{i+1}]} \tag{3.3}$$

is a representation of $\theta \in \mathbf{S}(\mathbb{R})$ *that is unique in the following sense:* Π *is the smallest deterministic partition of* $[0,T]$ *that contains* $0, T$*, and all jumps of* $\theta \in \mathbf{S}(\mathbb{R})$.

Proof: Recall that, by assumption, the filtration $\mathbb{F}$ satisfies the usual conditions. Hence, $\mathbb{F}$ is right–continuous. Since $\theta \in \mathbf{S}(\mathbb{R})$ is bounded, the right–continuity of $\mathbb{F}$ ensures that $\theta_{t+} \in \mathcal{F}_t$ for all $t \in [0,T]$. As $\theta \in \mathbf{S}(\mathbb{R})$, it jumps at most finitely many times. If we define Π as above, we know that it contains all jump times of $\theta \in \mathbf{S}(\mathbb{R})$. Hence, (3.3) is indeed a representation of $\theta \in \mathbf{S}(\mathbb{R})$. As $\mathbb{P}(\Delta^+\theta_{t_i} \neq 0) > 0$ for all $i \in \{1, \ldots, K\}$, the uniqueness is clear. $\square$

Definition 3.3: *For* $\theta \in \mathbf{S}(\mathbb{R})$ *we call* (3.3) *the* minimal representation of θ.

Now, let us consider some concrete examples.

Example 3.4 (Classical small trader setting)**:** If $S(\vartheta, .) = S(0, .)$ for all $\vartheta \in \mathbb{R}^d$, the definition coincides with the classical one for small traders. Indeed, the price process at time t is P_t^0. In this case, the trading activities of the large trader have no impact on the evolution of the price process.

Example 3.5 (Stochastic Differential Equations)**:** Let $d = 1$, $F = id$, and assume that the primal price processes $S(\vartheta, .)$ are given as strong solutions of the SDEs

$$dS(\vartheta, t) = b^{\vartheta}(S(\vartheta, t))\ dt + \sigma(S(\vartheta, t))\ dW_t.$$

Here, W is a Brownian motion, the function

$$\begin{aligned} b : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (\vartheta, x) &\longmapsto b^{\vartheta}(x) \end{aligned}$$

is assumed to be continuous and non–decreasing in the first argument and Lipschitz continuous in the second argument. Furthermore, we assume that σ is a function that is bounded from below by some $\varepsilon > 0$ and satisfies $|\sigma(x) - \sigma(y)|^2 \leq \rho(|x - y|)$ for some $\rho > 0$. Besides, let $S(\vartheta, 0) \leq S(\vartheta', 0)$ whenever $\vartheta \leq \vartheta'$. Hence, in this example, the trading decisions of the large trader influence the drift term instantaneously. Note that the comparison theorem [43, (3.7) Theorem] for SDEs ensures that the family $(S(\vartheta, .))_{\vartheta\in\mathbb{R}}$ satisfies the following condition:

Condition (O): $\vartheta \leq \vartheta'$ implies $S(\vartheta, .) \leq S(\vartheta', .)$.

This condition has been introduced by Bank and Baum; see [4]. Due to Theorem A.18, $(S(\vartheta,.))_{\vartheta\in\mathbb{R}}$ is a strong non–linear integrator. This in turn ensures that Assumption **(P-I)** holds. Finally, we want to emphasize the following point. Condition **(O)** has been introduced in [4] to exclude arbitrage opportunities for the large trader in *their* model. We do not need Condition **(O)** for any particular reason. It is just an additional feature of this primal family of price processes.

Example 3.6 (Reaction–Diffusion Setting)**:** Let $d = 1$ and $F = id$. Moreover, let $\psi(t,x,\vartheta)$ be a $\mathcal{C}^{1,2,1}$–function. Define, similarly as e.g. in [25], $S = (S(\vartheta,\cdot))_{\vartheta\in\mathbb{R}}$ via $S(\vartheta,t) = \psi(t,W_t,\vartheta)$, where the Brownian motion W models some fundamental state variable. Due to Ito's formula, the dynamics of the primitive price processes have the form

$$d\psi_t = \left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\psi_t\, dt + \frac{\partial}{\partial x}\psi_t\, dW_t.$$

We assume that $\partial\psi/\partial x$ is strictly positive. Note that $\partial\psi/\partial\vartheta \geq 0$ implies **(O)**. An explicit example for a reaction function is given by $\psi(t,W_t,\vartheta) = \exp(\sigma W_t + \kappa\vartheta t)$, where $\sigma,\kappa > 0$. It satisfies in particular condition **(O)**. Moreover, **(P-I)** holds due to Theorem A.18. Finally, the dynamics of $\psi(t,W_t,\vartheta) = \exp(\sigma W_t + \kappa\vartheta t)$ are given by

$$d\psi_t = \psi_t\left(\left(\kappa\vartheta + \frac{1}{2}\sigma^2\right)\, dt + \sigma\, dW_t\right).$$

Example 3.7 (Almgren–Chriss type model)**:** Let $d = 1$ and

$$\tilde{P} = \tilde{P}_0 + \sigma W,$$

where W is a Brownian motion, $\sigma > 0$ and $\tilde{P}_0 \in \mathbb{R}$. Besides, let $h, g : \mathbb{R} \to \mathbb{R}$ be non–decreasing, continuous functions with $g(0) = h(0) = 0$. We define $\int S\left(\tilde{\theta}, ds\right)$ for $\tilde{\theta} \in \mathbf{S}(\mathbb{R})$ as

$$\int_0^t S\left(\tilde{\theta}_s, ds\right) = \tilde{P}_t + \int_0^t \sum_{i=0}^n g(\tilde{\theta}_i)\mathbb{1}_{(\tau_i,\tau_{i+1}]}(u)\ du + \sum_{i=0}^n h(\tilde{\theta}_i)\mathbb{1}_{[\tau_i,\tau_{i+1})}(t).$$

Hence, **(P-I)** holds. To achieve an Almgren–Chriss type model for simple strategies let $\theta \in \mathbf{S}(\mathbb{R})$. $F : \mathbf{S}(\mathbb{R}) \to \mathbf{S}(\mathbb{R})$ is defined as the composition of two functions. The first one maps $\theta \in \mathbf{S}(\mathbb{R})$ to its minimal representation

$$\theta(t) = \theta_{-1}\mathbb{1}_{\{0\}} + \sum_{i=0}^n \theta_{t_i+}\mathbb{1}_{(t_i,t_{i+1}]}(t).$$

The second one maps the minimal representation to

$$\sum_{i=0}^n \frac{\theta_{t_i+} - \theta_{t_{i-1}+}}{t_i - t_{i-1}}\mathbb{1}_{(t_i,t_{i+1}]}(t),$$

where we use the convention $0 \cdot (\pm\infty) = 0$. Hence, we have

$$F(\theta) = \sum_{i=0}^{n} \frac{\theta_{t_i+} - \theta_{t_{i-1}+}}{t_i - t_{i-1}} \mathbb{1}_{(t_i,t_{i+1}]}(t). \tag{3.4}$$

As a result, we get

$$\begin{aligned} P_t^\theta &= \int_0^t S(F(\theta), ds) \\ &= \tilde{P}_t + \int_0^t \sum_{i=0}^{n} g\left(\frac{\theta_{t_i+} - \theta_{t_{i-1}+}}{t_i - t_{i-1}}\right) \mathbb{1}_{(t_i,t_{i+1}]}(u)\ du + \sum_{i=0}^{n} h\left(\frac{\theta_{t_i+} - \theta_{t_{i-1}+}}{t_i - t_{i-1}}\right) \mathbb{1}_{[t_i,t_{i+1})}(t). \end{aligned}$$

Price process affected by general large trader strategies

In all modules, the main idea how to extend the results for simple strategies to a general strategy is similar. We extend the definition to arbitrary strategies using a certain limit procedure. Of course, different limit procedures might lead to different classes of possible price processes affected by the large trader. Using Example 3.7, we explain how different choices can indeed influence the class of possible price processes affected by the large trader.

Intuitively, Example 3.7 leads to the Almgren–Chriss model [2, 1]. Indeed, let $\theta \in \mathbb{L}(\mathbb{R})$, where $\mathbb{L}(\mathbb{R}^d)$ denotes the space of all $\mathbb{R}^d$–valued, adapted processes having càglàd paths. Moreover, we assume that the paths of θ are continuously differentiable. Besides, define the sequence $(\theta^n)_{n\geq 1} \subset \mathbf{S}(\mathbb{R})$ via

$$\theta^n(t) = \theta_0 \mathbb{1}_{\{0\}} + \sum_{i=0}^{2^n-1} \theta_{\frac{i}{2^n}T} \mathbb{1}_{\left(\frac{i}{2^n}T, \frac{i+1}{2^n}T\right]}(t).$$

Recall the definition of F in (3.4) and note that

$$F(\theta^n) \underset{ucp}{\longrightarrow} \dot{\theta}.$$

Since $\dot{\theta} \in \mathbb{L}(\mathbb{R})$ has continuous paths, the dominated convergence theorem ensures that

$$P^{\theta^n} \underset{ucp}{\longrightarrow} \tilde{P} + \int g(\dot{\theta})\ du + h(\dot{\theta}).$$

Although $h(\dot{\theta})$ has continuous paths, it is not clear whether or not $h(\dot{\theta})$ is a semimartingale. Indeed, let $g \equiv 0$, $h \equiv id$, and θ be a deterministic and continuously differentiable function such that $\dot{\theta}$ is not of finite variation. Then the limit of P^{θ^n} in *ucp* is the sum of a martingale and a deterministic function that is not of finite variation. Hence, the limit cannot be a semimartingale. Nevertheless, for the set

$$\{\theta \in \mathbb{L}(\mathbb{R}) |\ \theta \text{ has twice continuously differentiable paths}\}$$

of possible large trader strategies, this limit procedure leads to reasonable *semimartingale* price processes affected by the large trader.

Due to these considerations, we suggest the following definition of a price process affected by a general large trader strategy.

Definition 3.8: *Let $(S(\vartheta,.))_{\vartheta\in\mathbb{R}^d}$ satisfy* **(P-I)**. *Moreover, let $F : \mathbf{S}(\mathbb{R}) \longrightarrow \mathbf{S}\left(\mathbb{R}^d\right)$ be a function and $\theta \in \mathbb{L}(\mathbb{R})$. If there exists a sequence $(\theta^n)_{n\geq 1} \subset \mathbf{S}(\mathbb{R})$ such that*

1. *$\theta^n \to \theta$ in ucp,*
2. *$(P^{\theta^n})_{n\geq 1}$ is a Cauchy–sequence in in the semimartingale topology,*

then the discounted price process P^θ affected by the large trader *strategy $\theta \in \mathbb{L}(\mathbb{R})$ is the limit of $(P^{\theta^n})_{n\geq 1}$ in the semimartingale topology, i.e.*

$$P^{\theta^n} \underset{\mathcal{SM}}{\longrightarrow} P^\theta.$$

Remark 3.9: *It should be mentioned that we do not know a priori whether or not the definition is, in general, independent of the approximating sequence.*

On the one hand, assuming that $(S(\vartheta,.))_{\vartheta\in\mathbb{R}^d}$ is a strong non–linear integrator might be too restrictive in some cases. On the other hand, it ensures the existence of price processes affected by the large trader for *arbitrary* strategies.

Theorem 3.10: *Let $(S(\vartheta,.))_{\vartheta\in\mathbb{R}^d}$ be a strong non–linear integrator. Then the family $(S(\vartheta,.))_{\vartheta\in\mathbb{R}^d}$ satisfies* **(P-I)**. *Moreover, if the function $F : \mathbb{L}(\mathbb{R}) \longrightarrow \mathbb{L}(\mathbb{R}^d)$ is continuous (w.r.t. the ucp–topology), then for all $\theta \in \mathbb{L}(\mathbb{R})$ there exists a sequence $(\theta^n)_{n\geq 1} \subset \mathbf{S}(\mathbb{R})$ such that the following properties hold:*

1. *$\theta^n \to \theta$ in ucp.*
2. *$(P^{\theta^n})_{n\geq 1}$ is a Cauchy–sequence in the semimartingale topology.*

Hence, the discounted price process P^θ affected by the large trader strategy $\theta \in \mathbb{L}(\mathbb{R})$ exists. Finally, the definition of P^θ is independent of the approximating sequence.

Proof: The statement is an immediate consequence of the definition of a strong non–linear integrator and Theorem A.16. □

3.2.2. The NA module: Incorporating no arbitrage considerations for the small trader into the price module

In this section, our financial market consists of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ that satisfies the usual conditions. Moreover, $\mathcal{F}_0$ is trivial apart from zero sets and $T \in (0, \infty)$. Furthermore, the family $(S(\vartheta,.))_{\vartheta\in\mathbb{R}^d}$ of

$\mathbb{R}$–valued semimartingales is adapted to $\mathbb{F}$ and satisfies **(P-I)**. Finally, $F : \mathbf{S}(\mathbb{R}) \longrightarrow \mathbf{S}(\mathbb{R}^d)$ is some fixed function.

Our goal is to incorporate no arbitrage considerations for the small trader into the 'price module'. In the introduction of Chapter 2, we already pointed out that there are various no arbitrage definitions that are linked to a pricing operator. Here, we consider the **(NFLVR)** condition. Thanks to the **(FTAP)**, this condition is equivalent to the existence of an equivalent σ–martingale measure for the price process P^θ. An important question at this point is: 'What filtration should we choose?'. There are three natural candidates.

We denote by

$$\mathbb{F}^{ST} = \left(\mathcal{F}_t^{ST}\right)_{t\in[0,T]}$$

the smallest filtration that satisfies the usual conditions and contains the filtration generated by P^θ. Here, the superscript 'ST' means 'small trader'. It emphasizes that $\mathbb{F}^{ST}$ is the natural choice for the small trader's information structure. The situation for the large trader, applying the strategy θ, is different. Her information structure is given by the smallest filtration

$$\mathbb{F}^{LT} = \left(\mathcal{F}_t^{LT}\right)_{t\in[0,T]}$$

that satisfies the usual conditions and, moreover, satisfies $\mathcal{F}_t^{ST} \vee \sigma(\theta_s : \ s \leq t) \subset \mathcal{F}_t^{LT}$ for all $t \in [0, T]$. In this case, the superscript 'LT' means 'large trader'. Finally, the most convenient choice for a filtration would be $\mathbb{F}$ itself.

In [32], Jarrow assumed that, "given the large trader's information, there are no arbitrage opportunities for the price taker". In our setting, this would be equivalent to assuming that there exists $\mathbb{Q}^\theta \sim \mathbb{P}$ such that P^θ is a $\mathbb{Q}^\theta$–σ–martingale w.r.t. $\mathbb{F}^{LT}$. Here, we work with the following weaker no arbitrage assumption *for the small trader*:

Assumption (NA-I): For all $\theta \in \mathbf{S}(\mathbb{R})$ there exists $\mathbb{Q}^\theta \sim \mathbb{P}$ such that P^θ is a $\mathbb{Q}^\theta$–σ–martingale w.r.t. $\mathbb{F}^{ST}$. □

We consider Assumption **(NA-I)** as our minimal standing assumption in this module. The next definition is a reformulation of this idea for general large trader strategies.

Definition 3.11: *Let $\theta \in \mathbb{L}(\mathbb{R})$, let $(S(\vartheta, .))_{\vartheta\in\mathbb{R}^d}$ be a family of $\mathbb{R}$–valued semimartingales adapted to $\mathbb{F}$, and let $F : \mathbf{S}(\mathbb{R}) \longrightarrow \mathbf{S}(\mathbb{R}^d)$ be a function. We say that the triple (S, F, θ) is a* reasonable large trader market model*, if the following conditions hold:*

1. *(S, F) satisfies Assumption* **(P-I)** *and Assumption* **(NA-I)**.
2. *The discounted price process P^θ exists in the sense of Definition 3.8.*

3. There exists $\mathbb{Q}^\theta \sim \mathbb{P}$ such that P^θ is a $\mathbb{Q}^\theta$–σ–martingale w.r.t. $\mathbb{F}^{ST}$.

We denote by (S, F, Ψ) the class of all reasonable large trader market models for the pair (S, F).

Remark 3.12: *Note that by definition $(S, F, \mathbf{S}(\mathbb{R}))$ is a subclass of (S, F, Ψ). The difficult question is: 'How big is (S, F, Ψ)?'.*

Without further restrictions, it seems to be almost impossible to give an answer to this question. Nevertheless, in concrete examples it is at least possible to show that (S, F, Ψ) is significantly bigger than $(S, F, \mathbf{S}(\mathbb{R}))$.

Example 3.13 (Classical small trader setting)**:** Under the assumptions of Example 3.4, we have $S(\vartheta, .) = S(0, .)$ for all $\vartheta \in \mathbb{R}^d$. Hence, for all functions $F : \mathbf{S}(\mathbb{R}) \longrightarrow \mathbf{S}(\mathbb{R}^d)$, Assumption **(NA-I)** holds if and only if P^0 features an equivalent σ–martingale measure w.r.t. $\mathbb{F}^{ST}$. In this case, **(NA-I)** is equivalent to $(S, F, \Psi) = (S, F, \mathbb{L}(\mathbb{R}))$.

Example 3.14 (Almgren–Chriss model)**:** Let $d = 3$ and let g be a non–decreasing continuous function such that $g(0) = 0$. Besides, let $h \in C^1(\mathbb{R}, \mathbb{R})$ be another non–decreasing function such that $h(0) = 0$. For $\vartheta \in \mathbb{R}^3$, we define the primal family $S = (S(\vartheta, .))_{\vartheta \in \mathbb{R}^3}$ by

$$S(\vartheta, .) = \tilde{P} + g(\vartheta_2)t + \vartheta_3 h'(\vartheta_2)t,$$

where $\tilde{P} = \tilde{P}_0 + \sigma W$ for a Brownian motion W and $\sigma > 0$. For $j \leq 3$ and $\tilde{\theta} \in \mathbf{S}(\mathbb{R}^3)$ with representation (3.1), we denote by $(\tilde{\theta}_i)_j$ the projection of $\tilde{\theta}_i$ onto its j^{th} coordinate. A straightforward computation reveals that

$$\begin{aligned} &\int_0^t S\left(\tilde{\theta}_s, ds\right) \\ &\quad = \tilde{P}_t + \int_0^t \sum_{i=0}^n g((\tilde{\theta}_i)_2) \mathbb{1}_{(\tau_i, \tau_{i+1}]}(u)\ du + \int_0^t \sum_{i=0}^n (\tilde{\theta}_i)_3 h'((\tilde{\theta}_i)_2) \mathbb{1}_{(\tau_i, \tau_{i+1}]}(u)\ du. \end{aligned}$$

Thus, Assumption **(P-I)** holds. Furthermore, Girsanov's theorem and Novikov's criterion ensure that **(NA-I)** is valid for any measurable function $F : \mathbf{S}(\mathbb{R}) \to \mathbf{S}(\mathbb{R}^3)$.

Let us choose a particular function. For $\theta \in \mathbf{S}(\mathbb{R})$, $F(\theta) = (F_1(\theta), F_2(\theta), F_3(\theta))$ is defined as follows. F_1 maps $\theta \in \mathbf{S}(\mathbb{R})$ to its minimal representation

$$\theta(t) = \theta_{-1} \mathbb{1}_{\{0\}} + \sum_{i=0}^n \theta_{t_{i+}} \mathbb{1}_{(t_i, t_{i+1}]}(t).$$

F_2 is defined as in the Almgren–Chriss model above; see (3.4). Finally, F_3 maps the minimal representation of $\theta \in \mathbf{S}(\mathbb{R})$ to

$$\sum_{i=0}^n \tilde{\theta}_i \mathbb{1}_{(t_i, t_{i+1}]}(t),$$

where for $i \leq n$

$$\tilde{\theta}_i = \begin{cases} \frac{\theta_{t_i+} - 2\theta_{t_{i-2}+} + \theta_{t_{i-4}+}}{\left(\frac{t_i - t_{i-4}}{2}\right)^2} & \text{if } i-4 \geq 0 \text{ and } i \leq n, \\ 0 & \text{else.} \end{cases} \tag{3.5}$$

At first glance, this definition might seem a little weird. But, for

$$\theta \in \{\theta \in \mathbb{L}(\mathbb{R}) | \text{ The paths of } \theta \text{ are twice continuously differentiable.}\},$$

we can choose an approximating sequence using the dyadic rationals. Indeed, let

$$\Pi^n := \left\{\frac{iT}{2^n} : 0 \leq i \leq 2^n\right\}$$

and define

$$\theta_u^n := \theta_0 \mathbb{1}_{\{0\}} + \sum_{i=0}^{2^n-1} \theta_{\frac{iT}{2^n}} \mathbb{1}_{\left(\frac{iT}{2^n}, \frac{(i+1)T}{2^n}\right]}(u)$$

For these strategies, (3.5) is given by

$$\tilde{\theta}_i^n = \begin{cases} \frac{\theta(2\frac{T}{2^n} + \frac{(i-2)T}{2^n}) - 2\theta(\frac{(i-2)T}{2^n}) + \theta(-2\frac{T}{2^n} + \frac{(i-2)T}{2^n})}{4\left(\frac{T}{2^n}\right)^2} & \text{if } i-4 \geq 0 \text{ and } i \leq n, \\ 0 & \text{else.} \end{cases} \tag{3.6}$$

Due to Taylor's theorem, for all $n \in \mathbb{N}$ and all $i \leq n$ there exist $\hat{\zeta}^{n,i}, \check{\zeta}^{n,i} \in [0,T]$ such that (3.6) is given by

$$\tilde{\theta}_i^n = \begin{cases} \frac{1}{2}\left(\ddot{\theta}(\hat{\zeta}^{n,i}) + \ddot{\theta}(\check{\zeta}^{n,i})\right) & \text{if } i-4 \geq 0 \text{ and } i \leq n, \\ 0 & \text{else.} \end{cases}$$

where $\ddot{\theta}$ denotes the second derivative of θ,

$$\frac{(i-2)T}{2^n} \leq \hat{\zeta}^{n,i} \leq \frac{iT}{2^n} \qquad \text{and} \qquad \frac{(i-4)T}{2^n} \leq \check{\zeta}^{n,i} \leq \frac{(i-2)T}{2^n}.$$

Hence, the dominated convergence theorem allows us to conclude that

$$\begin{aligned} P_t^\theta &= \tilde{P}_t + \int_0^t g(\dot{\theta}_u)\, du + \int_0^t \ddot{\theta}_u h'(\dot{\theta}_u)\, du \\ &= \tilde{P}_t + \int_0^t g(\dot{\theta}_u)\, du + h(\dot{\theta}_t) - h(\dot{\theta}_0). \end{aligned}$$

Moreover, Proposition A.6 ensures that

$$P^{\theta^n} \xrightarrow[\mathcal{SM}]{} P^\theta.$$

Hence, P^θ is a discounted price process in the sense of Definition 3.8 for all

$$\theta \in \{\theta \in \mathbb{L}(\mathbb{R}) | \text{ The paths of } \theta \text{ are twice continuously differentiable.}\} .$$

Moreover, if θ has bounded first and second derivatives, Girsanov's theorem and Novikov's criterion ensure that the triple (S, F, θ) is a *reasonable large trader market model*.

In order to provide a broader class of reasonable large trader market models, it is sensible to consider a family $(S(\vartheta, .))_{\vartheta \in \mathbb{R}^d}$ that satisfies certain regularity assumptions. At this point, the strong non–linear integrators seem to be tailor–made for our model and its minimal assumptions **(P-I)** and **(NA-I)**.

3.3. Strong non–linear integrators and the (LTMM)

In Section 3.2, we introduced Assumption **(P-I)**. It ensures, that all price processes, affected by a simple large trader strategy, are in fact semimartingales. As we pointed out above, this assumption is satisfied as soon as the family $S = (S(\vartheta, .))_{\vartheta \in \mathbb{R}^d}$ is a strong non–linear integrator. Besides, Theorem 3.10 guarantees that the price process P^θ exists for all $\theta \in \mathbb{L}(\mathbb{R})$ as soon as the function $F : \mathbb{L}(\mathbb{R}) \longrightarrow \mathbb{L}(\mathbb{R}^d)$ is continuous. Hence, for non–linear strong integrators $S = (S(\vartheta, .))_{\vartheta \in \mathbb{R}^d}$, it remains to provide conditions that ensure the existence of an equivalent σ–martingale measure for P^θ. This in turn enables us to characterize the class (S, F, Ψ) of reasonable large trader market models.

In the first part of this section, we introduce a no arbitrage assumption for the small trader that is *weaker* than **(NA-I)**. To be more precise, Assumption **(NA-II)** claims that for all *constant* large trader strategies $\vartheta \in \mathbb{R}$ there exists an equivalent σ–martingale measure for P^ϑ w.r.t. the underlying filtration $\mathbb{F}$. The fact that $S = (S(\vartheta, .))_{\vartheta \in \mathbb{R}^d}$ is a strong non–linear integrator enables us to prove that **(NA-II)** is sufficient for **(NA-I)**. In the second part, we use the definition of the strong non–linear integrator to provide sufficient conditions that guarantee the existence of an equivalent σ–martingale measure for P^θ, where $\theta \in \mathbb{L}(\mathbb{R})$ is a general large trader strategy.

As before, the financial market consists of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ that satisfies the usual conditions. Furthermore, $\mathcal{F}_0$ is trivial apart from zero sets and $T \in (0, \infty)$. Moreover, $F : \mathbb{S}(\mathbb{R}) \longrightarrow \mathbb{S}(\mathbb{R}^d)$ is some fixed continuous function and $S = (S(\vartheta, .))_{\vartheta \in \mathbb{R}^d}$ is a family of $\mathbb{R}$–valued semimartingales, adapted to $\mathbb{F}$, that satisfy the following assumption:

Assumption (P-II): $(S(\vartheta, .))_{\vartheta \in \mathbb{R}^d}$ is a strong non–linear integrator that satisfies the following conditions:

1. $S(\vartheta, .)$ is a continuous semimartingale for all $\vartheta \in \mathbb{R}^d$;
2. Outside of a $\mathbb{P}$–null set, S is continuous in the space parameter ϑ.

$\square$

Recall that for $\theta \in \mathbb{L}(\mathbb{R})$ the price process P^θ is defined as

$$P^\theta = \int S(F(\theta), ds),$$

where $F : \mathbb{L}(\mathbb{R}) \longrightarrow \mathbb{L}(\mathbb{R}^d)$ is a continuous function. Since we are not interested in a particular function F, we find it more convenient to consider $\int S(\theta, ds)$ as the price process affected by the large trader. Note that this entails that we have to use 'strategies' in $\mathbb{L}(\mathbb{R}^d)$! Throughout this section, we extensively use the notion '*extended simple strategy*' and '*convenient approximating sequence*'. Essentially, both notions refer to a slightly more general notion of simple strategies that have certain nice properties. For the definition of the space of extended simple strategies, $\mathbf{S}^e(\mathbb{R}^d)$, and the notion of 'convenient approximating sequence', we refer to Definition A.1 and Definition A.4. Finally, it should be mentioned that, if not otherwise stated, all (in–) equalities between random variables are understood as $\mathbb{P}$–a.s. (in–) equalities.

3.3.1. No arbitrage with simple strategies

Our main assumption is that there are no arbitrage opportunities for the small trader, if the large investor only employs *constant* strategies. A mathematical formulation is the following assumption:

Assumption (NA-II): For all $\vartheta \in \mathbb{R}^d$ there exists an equivalent local martingale measure $\mathbb{Q}^\vartheta$ for $S(\vartheta, .)$ w.r.t. $\mathbb{F}$. $\square$

The following example shows that in the 'Stochastic Differential Equation'–setting and in the 'Reaction–Diffusion Setting' Assumption **(NA-II)** is satisfied.

Example 3.15: Recall the assumptions of Example 3.5 and Example 3.6.

(i) Stochastic Differential Equations. The primal price processes $S(\vartheta, .)$ are given as strong solutions of the SDEs

$$dS(\vartheta, t) = b^\vartheta(S(\vartheta, t))\ dt + \sigma(S(\vartheta, t))\ dW_t,$$

where W is a Brownian motion. Furthermore, the function

$$\begin{aligned} b : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (\vartheta, x) &\longmapsto b^\vartheta(x) \end{aligned}$$

is continuous and non–decreasing in the first argument and Lipschitz continuous in the second argument. Moreover, σ is bounded from below by some $\varepsilon > 0$ and satisfies $|\sigma(x) - \sigma(y)|^2 \leq \rho(|x-y|)$ for some $\rho > 0$. Due to Theorem A.18 and Lemma A.10, we can conclude that $S(\vartheta,.)$ satisfies **(P-II)**. Besides, Girsanov's theorem and Novikov's criterion guarantee that assumption **(NA-II)** holds. Note, however, that there is in general *no universal* martingale measure *for all* $S(\vartheta,.)$. Hence, the analysis of Bank and Baum, see [4], does not apply to this situation.

(ii) Reaction–Diffusion Setting. Recall that the price process $S = (S(\vartheta,\cdot))_{\vartheta\in\mathbb{R}}$ is given by $S(\vartheta,t) = \psi(t,W_t,\vartheta)$, where the Brownian motion W models some fundamental state variable. Moreover, the dynamics of the primitive price processes have the form

$$d\psi_t = \left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\psi_t\, dt + \frac{\partial}{\partial x}\psi_t\, dW_t,$$

where $\partial\psi/\partial x$ is strictly positive. For the explicit reaction function $\psi(t,W_t,\vartheta) = \exp(\sigma W_t + \kappa\vartheta t)$, where $\sigma,\kappa > 0$, the dynamics are given by

$$d\psi_t = \psi_t\left(\left(\kappa\vartheta + \frac{1}{2}\sigma^2\right)\, dt + \sigma\, dW_t\right).$$

Here, condition **(O)** is valid. Moreover, Assumption **(P-II)** holds due to Theorem A.18 and Lemma A.10. Again, Girsanov's theorem and Novikov's criterion ensure that **(NA-II)** holds in this particular example. But, in general there does not exist a universal local martingale measure for all $S(\vartheta,.)$. Thus, the analysis of Bank and Baum does not apply to the Reaction–Diffusion Setting.

The next proposition is the main result of this subsection. It guarantees the existence of an equivalent local martingale measure, if the large trader's strategy is simple.

Proposition 3.16: *Let $\theta \in \mathbf{S}^e(\mathbb{R}^d)$ be an extended simple strategy of the large trader with representation* (A.1)*. Moreover, let $S = (S(\vartheta,.))_{\vartheta\in\mathbb{R}^d}$ satisfy* **(P-II)***. Under* **(NA-II)***, there exists an equivalent local martingale measure $\mathbb{Q}^\theta$ for the price process $\int S(\theta,ds)$. Moreover,* **(NA-II)** *is sufficient for* **(NA-I)***.*

Proof: For $\vartheta \in \mathbb{R}^d$, let Z^ϑ denote the density process of a local martingale measure $\mathbb{Q}^\vartheta$ for $S(\vartheta,.)$ which exists because of **(NA-II)**. As θ is a simple strategy, all the θ_i in the representation (A.1) assume only finitely many values denoted by $\{\vartheta_{i_1},...,\vartheta_{i_{m_i}}\}$. We may assume by localization that the $S(\vartheta,.)$ are $\mathbb{Q}^\vartheta$–martingales for each ϑ from this finite set. Define probability measures $\mathbb{Q}^{\theta_i}$, $i \in \{0,...,n\}$, by

$$Z_t^{\theta_i} := \left.\frac{d\mathbb{Q}^{\theta_i}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := c_i\sum_{j=1}^{m_i} Z_t^{\vartheta_{ij}}\mathbb{1}_{\{\theta_i=\vartheta_{ij}\}},$$

where the c_i are normalising constants. Then $\int S(\theta_i, ds)$ is $\mathbb{Q}^{\theta_i}$–martingale for all $i \in \{0, ..., n\}$. Indeed, for each $i \in \{0, ..., n\}$ and $t \in [0, T]$ we set

$$\begin{aligned} \Delta_t S^i &:= S(\theta_i, \tau_{i+1} \wedge t) - S(\theta_i, \tau_i \wedge t), \\ \Delta_t S^{i,j} &:= S(\vartheta_{ij}, \tau_{i+1} \wedge t) - S(\vartheta_{ij}, \tau_i \wedge t). \end{aligned}$$

Bayes' formula enables us to compute the conditional expectations under a measure change; for $i \in \{0, ..., n\}$ and $s < t$ we get

$$\begin{aligned} E_{\mathbb{Q}^{\theta_i}}\left[\Delta_t S^i \middle| \mathcal{F}_s\right] &= \frac{1}{Z_s^{\theta_i}} E_P\left[Z_T^{\theta_i} \Delta_t S^i \middle| \mathcal{F}_s\right] \\ &= \frac{1}{Z_s^{\theta_i}} c_i \sum_{j=1}^{m_i} E_P\left[Z_T^{\vartheta_{ij}} \Delta_t S^i \mathbb{1}_{\{\theta_i = \vartheta_{ij}\}} \middle| \mathcal{F}_s\right] \\ &= c_i \sum_{j=1}^{m_i} \frac{Z_s^{\vartheta_{ij}}}{Z_s^{\theta_i}} E_{\mathbb{Q}^{\vartheta_{ij}}}\left[\Delta_t S^{ij} \mathbb{1}_{\{\theta_i = \vartheta_{ij}\}} \middle| \mathcal{F}_s\right]. \end{aligned} \tag{3.7}$$

Moreover, for fixed $j \in \{1, ..., m_i\}$ we compute

$$\begin{aligned} E_{\mathbb{Q}^{\vartheta_{ij}}}\left[\Delta_t S^{ij} \mathbb{1}_{\{\theta_i = \vartheta_{ij}\}} \middle| \mathcal{F}_s\right] = {} & E_{\mathbb{Q}^{\vartheta_{ij}}}\left[\Delta_t S^{ij} \mathbb{1}_{\{\theta_i = \vartheta_{ij}\}} \mathbb{1}_{\{s \geq \tau_i\}} \middle| \mathcal{F}_s\right] + \\ & + E_{\mathbb{Q}^{\vartheta_{ij}}}\left[\Delta_t S^{ij} \mathbb{1}_{\{\theta_i = \vartheta_{ij}\}} \mathbb{1}_{\{s < \tau_i\}} \middle| \mathcal{F}_s\right]. \end{aligned} \tag{3.8}$$

First, let us compute the first term on the r.h.s.. Due to Assumption **(NA-II)**, we know that $\Delta_t S^{ij}$ are $\mathbb{Q}^{\vartheta_{ij}}$–martingales for all $j \in \{1, ..., m_i\}$. As θ_i is $\mathcal{F}_{\tau_i}$–measurable, we can conclude that

$$\begin{aligned} E_{\mathbb{Q}^{\vartheta_{ij}}}\left[\Delta_t S^{ij} \mathbb{1}_{\{\theta_i = \vartheta_{ij}\}} \mathbb{1}_{\{s \geq \tau_i\}} \middle| \mathcal{F}_s\right] &= E_{\mathbb{Q}^{\vartheta_{ij}}}\left[\Delta_t S^{ij} \middle| \mathcal{F}_s\right] \mathbb{1}_{\{\theta_i = \vartheta_{ij}\}} \mathbb{1}_{\{s \geq \tau_i\}} \\ &= \Delta_s S^{ij} \mathbb{1}_{\{\theta_i = \vartheta_{ij}\}} \mathbb{1}_{\{s \geq \tau_i\}}. \end{aligned}$$

In the second step, we compute the second term on the r.h.s. of (3.8). It is in fact equal to zero. Indeed, by conditioning on $\mathcal{F}_{\tau_i}$ and using the tower property of conditional expectation, we get

$$\begin{aligned} E_{\mathbb{Q}^{\vartheta_{ij}}}\left[\Delta_t S^{ij} \mathbb{1}_{\{\theta_i = \vartheta_{ij}\}} \mathbb{1}_{\{s < \tau_i\}} \middle| \mathcal{F}_s\right] &= E_{\mathbb{Q}^{\vartheta_{ij}}}\left[E_{\mathbb{Q}^{\vartheta_{ij}}}\left[\Delta_t S^{ij} \mathbb{1}_{\{\theta_i = \vartheta_{ij}\}} \mathbb{1}_{\{s < \tau_i\}} \middle| \mathcal{F}_{\tau_i}\right] \middle| \mathcal{F}_s\right] \\ &= E_{\mathbb{Q}^{\vartheta_{ij}}}\left[\mathbb{1}_{\{\theta_i = \vartheta_{ij}\}} \mathbb{1}_{\{s < \tau_i\}} E_{\mathbb{Q}^{\vartheta_{ij}}}\left[\Delta_t S^{ij} \middle| \mathcal{F}_{\tau_i}\right] \middle| \mathcal{F}_s\right]. \end{aligned}$$

Due to the martingale property of $S(\vartheta_{ij,}.)$, we can conclude that

$$\begin{aligned} E_{\mathbb{Q}^{\vartheta_{ij}}}\left[\mathbb{1}_{\{\theta_i = \vartheta_{ij}\}} \mathbb{1}_{\{s < \tau_i\}} E_{\mathbb{Q}^{\vartheta_{ij}}}\left[\Delta_t S^{ij} \middle| \mathcal{F}_{\tau_i}\right] \middle| \mathcal{F}_s\right] &= E_{\mathbb{Q}^{\vartheta_{ij}}}\left[\mathbb{1}_{\{\theta_i = \vartheta_{ij}\} \cap \{s < \tau_i\}} \Delta_{t \wedge \tau_i} S^{ij} \middle| \mathcal{F}_s\right] \\ &= 0. \end{aligned}$$

Since $\Delta_s S^{ij} \mathbb{1}_{\{\theta_i = \vartheta_{ij}\}} \mathbb{1}_{\{s < \tau_i\}}$ is zero, the computations above ensure that (3.8) is equivalent to

$$E_{\mathbb{Q}^{\vartheta_{ij}}}\left[\Delta_t S^{ij} \mathbb{1}_{\{\theta_i = \vartheta_{ij}\}} \middle| \mathcal{F}_s\right] = \Delta_s S^{ij} \mathbb{1}_{\{\theta_i = \vartheta_{ij}\}}.$$

Summing up over j, we deduce from (3.7) that

$$E_{\mathbb{Q}^{\theta_i}}\left[\Delta_t S^i \middle| \mathcal{F}_s\right] = \Delta_s S^i.$$

Hence, $\int S(\theta_i, ds)$ is a local martingale under $\mathbb{Q}^{\theta_i}$. Finally, we construct the density process Z^θ of $\mathbb{Q}^\theta$ on the whole time interval $[0,T]$ by concatenation,

$$Z_t^\theta := \prod_{i=0}^n \frac{Z_{t\wedge\tau_{i+1}}^{\theta_i}}{Z_{t\wedge\tau_i}^{\theta_i}}.$$

□

3.3.2. No arbitrage with dynamic strategies

We are looking for sufficient conditions that enable us to extend the results for extended simple strategies to general $\theta \in \mathbb{L}\left(\mathbb{R}^d\right)$. This in turn means that the class of reasonable large trader market models is given by $(S, F, \mathbb{L}(\mathbb{R}))$. Our main tool is the following version of the Fundamental Theorem of Asset Pricing.

Theorem 3.17: *Let $S = S_0 + M + \int \lambda \, d[M]$ be a continuous semimartingale.*

1. *S satisfies* **(SC)** *if and only if $\mathcal{E}\left(-\int \lambda \, dM\right)$ is a strictly positive local martingale density for S.*

2. *There exists an equivalent local martingale measure for S if and only if S satisfies* **(SC)** *and the classical* **(NA)***–condition.*

Proof: Lemma 2.22 ensures that the first statement holds. The second statement follows from the Fundamental Theorem of Asset Pricing [16]. □

Remark 3.18: *Let $\theta \in \mathbb{L}\left(\mathbb{R}^d\right)$. Due to the theorem it is clear that the structure condition* **(SC)** *is necessary for the price process $\int S(\theta, ds)$ in order to admit an equivalent local martingale measure. Hence, we start looking for sufficient conditions that ensure that $\int S(\theta, ds)$ satisfies* **(SC)***.*

The next lemma highlights that it is natural to use the definition of $\int S(\theta, ds)$ to find conditions that ensure the existence of an equivalent local martingale measure.

Lemma 3.19: *Let $S = (S(\vartheta, .))_{\vartheta\in\mathbb{R}^d}$ satisfy* **(P-II)** *and let Assumption* **(NA-II)** *hold. Further, let $\theta \in \mathbb{L}\left(\mathbb{R}^d\right)$, $(\theta^n)_{n\geq 1} \subset \mathbf{S}^e\left(\mathbb{R}^d\right)$, where*

$$\theta^n(t) = \theta_{-1}^n + \sum_{i=0}^{m_n} \theta_i^n \mathbb{1}_{\left(\tau_i^n, \tau_{i+1}^n\right]}(t).$$

Then the following statements hold:

1. *The canonical decompositions of* $\left(\int S(\theta^n, ds)\right)_{n\geq 1}$ *can be written as*

$$\int S(\theta^n, ds) = S_0^n + M^n + \int \lambda^{(n)} \, d[M^n], \tag{3.9}$$

where $\int_0^T \left(\lambda_u^{(n)}\right)^2 \, d[M^n]_u < \infty$ $\mathbb{P}$*–a.s. and*

$$M_t^n = \sum_{i=0}^{m_n} \left(M_{\tau_{i+1}^n \wedge t}^{\theta_i^n} - M_{\tau_i^n \wedge t}^{\theta_i^n} \right) \quad \text{and} \quad \lambda_t^{(n)} = \sum_{i=0}^{m_n} \lambda_t^{\theta_i^n} \mathbb{1}_{\left(\tau_i^n, \tau_{i+1}^n\right]}(t). \tag{3.10}$$

2. *If* $\int S(\theta, ds) = \bar{S}_0 + M + A$ *denotes the canonical decomposition of* $\int S(\theta, ds)$*, where* M *is a continuous local martingale, then* $\theta^n \underset{ucp}{\longrightarrow} \theta$ *ensures that*

$$M^n \underset{\mathcal{SM}}{\longrightarrow} M \quad \text{and} \quad [M^n] \underset{\mathcal{SM}}{\longrightarrow} [M] \quad \text{and} \quad \int \lambda^{(n)} \, d[M^n] \underset{\mathcal{SM}}{\longrightarrow} A.$$

3. *Under 2., there exists a subsequence (still indexed by* n*) such that* $\mathbb{P}$*–a.s.*

$$V\left(\int \lambda^{(n)} \, d[M^n] - A\right)_T \longrightarrow 0. \tag{3.11}$$

Proof: Assumption **(NA-II)** and Theorem 3.17 allow us to write the canonical decompositions of $(S(\vartheta, .))_{\vartheta\in\mathbb{R}^d}$ as

$$S(\vartheta, .) = S_0 + M^\vartheta + \int \lambda^\vartheta \, d[M^\vartheta],$$

where $\mathbb{P}(\int_0^T (\lambda_u^\vartheta)^2 \, d[M^\vartheta]_u < \infty) = 1$ for all $\vartheta \in \mathbb{R}^d$. For any sequence $(\theta^n)_{n\geq 1} \subset \mathbf{S}^e\left(\mathbb{R}^d\right)$, Proposition A.14, Theorem 3.17, and Proposition 3.16 ensure that the canonical decompositions of $\left(\int S(\theta^n, ds)\right)_{n\geq 1}$ can be written as

$$\int S(\theta^n, ds) = S_0^n + M^n + \int \lambda^{(n)} \, d[M^n],$$

where $\int_0^T \left(\lambda_u^{(n)}\right)^2 \, d[M^n]_u < \infty$ $\mathbb{P}$–a.s. for all $n \geq 1$. Moreover, for $\theta^n(t) = \theta_{-1}^n + \sum_{i=0}^{m_n} \theta_i^n \mathbb{1}_{\left(\tau_i^n, \tau_{i+1}^n\right]}(t)$ we have

$$M_t^n = \sum_{i=0}^{m_n} \left(M_{\tau_{i+1}^n \wedge t}^{\theta_i^n} - M_{\tau_i^n \wedge t}^{\theta_i^n} \right) \qquad \text{and} \qquad \lambda_t^{(n)} = \sum_{i=0}^{m_n} \lambda_t^{\theta_i^n} \mathbb{1}_{\left(\tau_i^n, \tau_{i+1}^n\right]}(t).$$

Due to Proposition A.7, the claim is proven. □

Remark 3.20: *The lemma is the key tool for checking whether or not the price process* $\int S(\theta, ds)$ *satisfies* **(SC)***. It allows us to use classical results from measure theory that guarantee the convergence of Lebesgue–Stieltjes integrals as in* (3.11) *to a limit of the form* $\int \lambda \, d[M]$*; see e.g.* [9, 4.7.132–4.7.133]*. These results usually assume point–wise convergence of the integrands* $(\lambda^{(n)})_{n\geq 1}$*. However, at this point, we do not know a priori whether or not the ucp–convergence of* $(\theta^n)_{n\geq 1}$ *implies the point–wise convergence of the integrands* $(\lambda^{(n)})_{n\geq 1}$*.*

Sufficient conditions for $\int S(\theta, ds)$ to satisfy (SC)

The following theorem gives a sufficient condition for the price process $\int S(\theta, ds)$ to satisfy the structure condition without assuming point–wise convergence of $(\lambda^{(n)})_{n\geq 1}$. Within the proof, we frequently use the following fact.

Lemma 3.21 ([42, Corollary, p. 40])**:** *Let A be a non–negative, right–continuous, and non–decreasing process and H be a jointly measurable process such that $F = \int H\, dA$ exists and is finite for all $t > 0$ up to indistinguishability. If $V(F)_T$ denotes the total variation of F on the interval $[0, T]$, then*

$$V(F)_T = \int_0^T |H|_u \, dA_u.$$

Theorem 3.22: *If*

$$\liminf_{n\to\infty} \int_0^T \left(\lambda_u^{(n)}\right)^2 \, d[M^n]_u < \infty \tag{3.12}$$

holds $\mathbb{P}$–a.s., then the price process $\int S(\theta, ds)$ satisfies the structure condition **(SC)***.*

Proof: We first prove that there exists a predictable process λ such that

$$A = \int \lambda \, d[M]$$

holds. As $[M^n] \to [M]$ in $\mathcal{SM}$, Proposition A.7 ensures that there exists a subsequence (still indexed by n), such that (recall our convention that equalities between random variables are to be understood $\mathbb{P}$–a.s.)

$$\lim_{n\to\infty} V([M^n] - [M])_T = 0, \tag{3.13}$$

as well as

$$\lim_{n\to\infty} V\left(\int \lambda^{(n)} \, d[M^n] - A\right)_T = 0. \tag{3.14}$$

Therefore, we have

$$V(A)_T \leq \liminf_{n\to\infty} V\left(\int \lambda^{(n)} \, d[M^n]\right)_T. \tag{3.15}$$

Due to Lemma 3.21, we get for $n \geq 1$

$$V\left(\int \lambda^{(n)} \, d[M^n]\right)_T = \int_0^T \left|\lambda_u^{(n)}\right| d[M^n]_u = \int_0^T 1 \left|\lambda_u^{(n)}\right| d[M^n]_u. \tag{3.16}$$

Combining (3.15) and (3.16) and applying Hölder's inequality to the r.h.s. of (3.16) leads to

$$V\left(A\right)_T \leq \liminf_{n\to\infty} \left([M^n]_T\right)^{\frac{1}{2}} \left(\int_0^T \left(\lambda_u^{(n)}\right)^2 \, d[M^n]_u\right)^{\frac{1}{2}}.$$

By assumption (3.12), we have

$$\liminf_{n\to\infty} \int_0^T \left(\lambda_u^{(n)}\right)^2 \, d\left[M^n\right]_u < \infty,$$

and therefore

$$V\left(A\right)_T \leq \left([M]_T\right)^{\frac{1}{2}} \liminf_{n\to\infty} \left(\int_0^T \left(\lambda_u^{(n)}\right)^2 d[M^n]_u\right)^{\frac{1}{2}}$$

holds. Because of the Radon–Nikodym Theorem, see [30, 3.13 Proposition], there exists a predictable process λ such that

$$A = \int \lambda \, d[M]. \tag{3.17}$$

It remains to prove that $\int_0^T \lambda_u^2 \, d[M]_u < \infty$ holds $\mathbb{P}$–a.s. Due to (3.13), there exists a set N^c of measure zero such that the family $([M^n](\omega))_{n\in\mathbb{N}}, [M](\omega)$ is uniformly bounded for all $\omega \in N$. According to Lemma A.21, there exist probability measures $dB(\omega)$ on $[0,T]$ such that

$$\forall\, n \in \mathbb{N} : d[M^n](\omega) \ll dB(\omega), \qquad \text{and} \qquad d[M](\omega) \ll dB(\omega),$$

hold for all $\omega \in N$, where N is the set complement of N^c. Due to Lemma 3.21 and (3.13), we get for all $\omega \in N$

$$\lim_{n\to\infty} V\left([M^n] - [M]\right)_T(\omega) = \lim_{n\to\infty} \int_0^T \left| \frac{d[M^n]}{dB}(\omega,u) - \frac{d[M]}{dB}(\omega,u)\right| \, dB_u(\omega) = 0,$$

and

$$\frac{d[M^n]}{dB}(\omega,.) \longrightarrow \frac{d[M]}{dB}(\omega,.), \qquad \text{in } dB(\omega)\text{–probability.}$$

Combining (3.14), (3.17), and Lemma 3.21, we also have for all $\omega \in N$

$$\int_0^T \left|\lambda_u^{(n)}(\omega)\frac{d[M^n]}{dB}(\omega,u) - \lambda_u(\omega)\frac{d[M]}{dB}(\omega,u)\right| \, dB_u(\omega) \xrightarrow[n\to\infty]{} 0,$$

and

$$\lambda_{\cdot}^{(n)}(\omega)\frac{d[M^n]}{dB}(\omega,.)\xrightarrow[n\to\infty]{}\lambda_{\cdot}(\omega)\frac{d[M]}{dB}(\omega,.),\qquad \text{in } dB(\omega)\text{–probability.}$$

For all $\omega\in N$, we may apply Lemma A.23 and Fatou's Lemma to end up with

$$\begin{aligned}\int_0^T\lambda_u^2(\omega)\,d[M]_u(\omega)&=\int_0^T\mathbb{1}_{\{\frac{d[M]}{dB}(\omega,.)\neq0\}}(u)\lambda_u^2(\omega)\frac{d[M]}{dB}(\omega,.)\,dB_u(\omega)\\&\leq\liminf_{n\to\infty}\int_0^T\mathbb{1}_{\{\frac{d[M]}{dB}\neq0\}}(\omega,.)\left(\lambda_u^{(n)}(\omega)\right)^2\frac{d[M^n]}{dB}(\omega,.)\,dB_u(\omega)\\&\leq\liminf_{n\to\infty}\int_0^T\left(\lambda_u^{(n)}(\omega)\right)^2\,d[M^n]_u(\omega).\end{aligned}$$

Due to assumption (3.12), the claim is proven. □

The next corollary is an immediate consequence of the theorem and Lemma 2.17.

Corollary 3.23: *If there exists a sequence of stopping times $(S_m)_{m\geq1}\uparrow T$ a.s. and a sequence $(C_m)\subset L^2(P)$ such that $\mathbb{P}(S_m=T)\to1$ for $m\to\infty$ and*

$$\liminf_{n\to\infty}\int_0^{S_m}\left(\lambda_u^{(n)}\right)^2\,d\left[M^n\right]_u\leq C_m$$

holds $\mathbb{P}$–a.s., then $\int S(\theta,ds)$ satisfies the structure condition **(SC)**.

Proof: Apply Theorem 3.22 to $\left(\lambda^{(n)}\mathbb{1}_{[0,S_m]}\right)_{n\geq1}$. This ensures that there exists $\lambda\in{}^{p}L^2_{loc}(M)$ such that $A=\int\lambda\,d[M]$ and $E\left[\int_0^T\lambda_u^2\mathbb{1}_{[0,S_m]}(u)\,d[M]_u\right]<\infty$. As $\mathbb{P}(S_m=T)\to1$ for $m\to\infty$, Lemma 2.17 ensures that $\int S(\theta,ds)$ satisfies the structure condition. □

This corollary is tailor–made for our examples.

Example 3.24: Consider the setting of Example 3.15. Let $(\theta^n)_{n\geq1}\subset\mathbf{S}^e(\mathbb{R})$ be a convenient approximating sequence for $\theta\in\mathbb{L}(\mathbb{R})$, where

$$\theta^n(t)=\sum_{i=0}^{m_n}\theta_i^n\mathbb{1}_{\left(\tau_i^n,\tau_{i+1}^n\right]}(t),\qquad\text{for } n\geq1.$$

Denote by $(S_m)_{m\geq1}$ the sequence of stopping times such that the family $\{(\theta^n)^{S_m},\theta^{S_m}\}$ is uniformly bounded and $\mathbb{P}(S_m=T)\to1$ for $m\to\infty$. Recall that

$$\int S(\theta^n,ds)=M^n+\int\lambda^{(n)}\,d\left[M^n\right]$$

is the canonical decomposition of $\int S(\theta^n, ds)$, where M^n and $\lambda^{(n)}$ are defined in (3.10).

(i) Stochastic Differential Equations. In this setting the local martingales M^n as well as the $\lambda^{(n)}$ are given by

$$M_t^n = \sum_{i=0}^{m_n} \sigma\left(S(\theta_i^n, \tau_i^n)\right)\left(W_{\tau_{i+1}^n \wedge t} - W_{\tau_i^n \wedge t}\right),$$

$$\lambda_t^{(n)} = \sum_{i=0}^{m_n} \lambda_t^{\theta_i^n} \mathbb{1}_{(\tau_i^n, \tau_{i+1}^n]}(t) = \sum_{i=0}^{m_n} \frac{b^{\theta_i^n}\left(S(\theta_i^n, \tau_i^n)\right)}{\sigma^2\left(S(\theta_i^n, \tau_i^n)\right)} \mathbb{1}_{(\tau_i^n, \tau_{i+1}^n]}(t).$$

Since

$$\begin{aligned}\int_0^T \left(\lambda_u^{(n)}\right)^2 d[M^n]_u &= \int_0^T \left(\sum_{i=0}^{m_n} \frac{b^{\theta_i^n}\left(S(\theta_i^n, \tau_i^n)\right)}{\sigma^2\left(S(\theta_i^n, \tau_i^n)\right)} \mathbb{1}_{(\tau_i^n, \tau_{i+1}^n]}(u)\right)^2 d[M^n]_u \\ &= \sum_{i=0}^{m_n} \left(\frac{b^{\theta_i^n}\left(S(\theta_i^n, \tau_i^n)\right)}{\sigma^2\left(S(\theta_i^n, \tau_i^n)\right)}\right)^2 \left(\tau_{i+1}^n - \tau_i^n\right),\end{aligned}$$

the assumptions made in the example and the special choice of $(\theta^n)_{n\geq 1}$ and $(S_m)_{m\geq 1}$ ensure that we can apply Corollary 3.23. Hence, the price process $\int S(\theta, ds)$ satisfies the structure condition **(SC)**.

(ii) Reaction-Diffusion Setting. Similar calculations as in the example above show that

$$M_t^n = \sum_{i=0}^{m_n} \frac{\partial}{\partial x}\psi\left(\tau_i^n, W_{\tau_i^n}, \theta_i^n\right)\left(W_{\tau_{i+1}^n \wedge t} - W_{\tau_i^n \wedge t}\right),$$

$$\lambda_t^{(n)} = \sum_{i=0}^{m_n} \frac{\left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\psi\left(\tau_i^n, W_{\tau_i^n}, \theta_i^n\right)}{\left(\frac{\partial}{\partial x}\psi\left(\tau_i^n, W_{\tau_i}^n, \theta_i^n\right)\right)^2} \mathbb{1}_{(\tau_i^n, \tau_{i+1}^n]}(t),$$

and

$$\int_0^T \left(\lambda_u^{(n)}\right)^2 d[M^n]_u = \sum_{i=0}^{m_n} \left(\frac{\left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\psi\left(\tau_i^n, W_{\tau_i^n}, \theta_i^n\right)}{\frac{\partial}{\partial x}\psi\left(\tau_i^n, W_{\tau_i^n}, \theta_i^n\right)}\right)^2 \left(\tau_{i+1}^n - \tau_i^n\right).$$

For general reaction functions ψ it is likely that one has to impose certain conditions on the fraction above to ensure that the condition of Theorem 3.22 is satisfied. However, for the reaction function $\psi(t, x, \vartheta) = \exp(\sigma x + \kappa\vartheta t)$, where $\sigma, \kappa > 0$, we get

$$\int_0^T \left(\lambda_u^{(n)}\right)^2 d[M^n]_u = \sum_{i=0}^{m_n} \left(\frac{\kappa}{\sigma}\theta_i^n + \frac{1}{2}\sigma\right)^2 \left(\tau_{i+1}^n - \tau_i^n\right).$$

Due to the special choice of $(\theta^n)_{n\geq 1}$ and $(S_m)_{m\geq 1}$, we may apply Corollary 3.23 and conclude that $\int S(\theta, ds)$ satisfies the structure condition **(SC)**.

The following example shows that (3.12) might depend on the choice of the approximating sequence.

Example 3.25: Let W be a standard Brownian motion and set $S(\vartheta, t) = W_t + \vartheta t$. For $n \geq 1$ choose $\theta^n = \lambda^n = n\mathbb{1}_{(0,n^{-3/2}]}$. Clearly $\lambda^n \to 0$ a.s. and in L^1. Therefore,

$$\int S(\theta^n, ds) \xrightarrow[\mathcal{SM}]{} W$$

holds. Although the limit satisfies **(SC)** with $\lambda \equiv 0$, the assumption of Theorem 3.22 does not hold. Indeed, we have $\int_0^T |\lambda_u^n|^2\, du = \sqrt{n}$. On the other hand, $\tilde{\theta}^n = 1/n \to 0$ a.s. and satisfies (3.12).

Sufficient conditions for the existence of an equivalent local martingale measure for $\int S(\theta, ds)$

Due to Theorem 3.17, the structure condition is a necessary condition for the existence of an equivalent local martingale measure for the price process $\int S(\theta, ds)$, where the canonical decomposition is given by

$$\int S(\theta, ds) = M + \int \lambda \, d[M].$$

A possible candidate for a martingale measure is now given via the density process

$$\mathcal{E}\left(-\int \lambda \, dM\right) = \exp\left(-\int \lambda \, dM - \frac{1}{2}\int \lambda^2 \, d[M]\right). \tag{3.18}$$

This stochastic exponential is a strictly positive local $\mathbb{P}$–martingale and therefore a $\mathbb{P}$–supermartingale. It is well known that $E\left[\mathcal{E}\left(-\int \lambda\, dM\right)_T\right] = 1$ if and only if the stochastic exponential is a true martingale. If it is a true martingale, $d\mathbb{Q} = \mathcal{E}\left(-\int \lambda \, dM\right)_T d\mathbb{P}$ defines an equivalent local martingale measure for $\int S(\theta, ds)$. However, in general $\mathcal{E}\left(-\int \lambda \, dM\right)$ is not a true martingale; see [46]. The following proposition gives a sufficient condition for $\mathcal{E}\left(-\int \lambda \, dM\right)$ being a true martingale. Hence, it provides a sufficient condition for the existence of an equivalent local martingale measure for the price process $\int S(\theta, ds)$, if the large trader uses a general strategy $\theta \in \mathbb{L}\left(\mathbb{R}^d\right)$.

Proposition 3.26: *Let $\theta \in \mathbb{L}\left(\mathbb{R}^d\right)$ be a large trader strategy. Moreover, denote by $(\theta^n)_{n\geq 1} \subset \mathbf{S}^e\left(\mathbb{R}^d\right)$ a sequence of strategies such that $\theta^n \to \theta$ in ucp. Recall the notation of Lemma 3.19 and suppose that*

1. *$\mathcal{E}\left(-\int \lambda^n \, dM^n\right)$ are true martingales for all $n \geq 1$;*
2. *the family $\left(\mathcal{E}\left(-\int \lambda^n \, dM^n\right)_T\right)_{n\geq 1}$ of random variables is uniformly integrable;*

3. $\left[\int \lambda^n \, dM^n - \int \lambda \, dM\right]_T \to 0$ *in probability.*

Then, $\mathcal{E}\left(-\int \lambda \, dM\right)$ *is a true martingale. Hence,* $d\mathbb{Q} = \mathcal{E}\left(-\int \lambda \, dM\right)_T d\mathbb{P}$ *defines an equivalent local martingale measure for the price process* $\int S(\theta, ds)$ *affected by the large trader strategy* $\theta \in \mathbb{L}\left(\mathbb{R}^d\right)$*. As soon as all* $\theta \in \mathbb{L}\left(\mathbb{R}^d\right)$ *satisfy these assumptions, the class of all reasonable large trader market models is given by* $(S, F, \mathbb{L}(\mathbb{R}))$ *for arbitrary continuous functions* $F : \mathbb{L}(\mathbb{R}) \longrightarrow \mathbb{L}(\mathbb{R}^d)$*.*

Proof: Due to [35, Proposition 2.7], the third assumption is equivalent to

$$\int \lambda^{(n)} \, dM^n \underset{\mathcal{SM}}{\longrightarrow} \int \lambda \, dM.$$

According to Proposition A.7, the composition of C^2–functions with semimartingales is continuous w.r.t. the semimartingale topology. Therefore,

$$\mathcal{E}\left(-\int \lambda^{(n)} \, dM^n\right) \underset{\mathcal{SM}}{\longrightarrow} \mathcal{E}\left(-\int \lambda \, dM\right)$$

holds. Due to the second item, we have $\mathcal{E}\left(-\int \lambda \, dM\right)_T = 1$. This ensures that $\mathcal{E}\left(-\int \lambda \, dM\right)$ is a true martingale. □

For $\theta \in b\mathbb{L}\left(\mathbb{R}^d\right)$, where $b\mathbb{L}\left(\mathbb{R}^d\right) \subset \mathbb{L}\left(\mathbb{R}^d\right)$ is the subspace of all bounded càglàd processes, our examples admit an equivalent local martingale measure for the price process $\int S(\theta, ds)$ affected by the large trader.

Example 3.27: Consider the setting of Example 3.15. Let $(\theta^n)_{n\geq 1} \subset \mathbf{S}^e(\mathbb{R})$ be a convenient approximating sequence for $\theta \in b\mathbb{L}(\mathbb{R})$ and note that the family $\{(\theta^n)_{n\geq 1}, \theta\}$ is uniformly bounded. Recall that for

$$\theta^n(t) = \sum_{i=0}^{m_n} \theta_i^n \mathbb{1}_{\left(\tau_i^n, \tau_{i+1}^n\right]}(t), \qquad \text{for } n \geq 1,$$

the canonical decomposition of $\int S(\theta^n, ds)$ is given by

$$\int S(\theta^n, ds) = M^n + \int \lambda^{(n)} \, d[M^n],$$

where M^n and $\lambda^{(n)}$ are defined in (3.10).

(i) Stochastic Differential Equations. Due to Example 3.24, we have

$$\int_0^T \lambda_u^2 \, d[M]_u \leq \liminf_{n\to\infty} \sum_{i=0}^{m_n} \left(\frac{b^{\theta_i^n}\left(S(\theta_i^n, \tau_i^n)\right)}{\sigma^2\left(S(\theta_i^n, \tau_i^n)\right)}\right)^2 \left(\tau_{i+1}^n - \tau_i^n\right).$$

As by assumption $\sigma > \epsilon$, the ratio $b^{\vartheta}(x)/\sigma^2(x)$ is again continuous in both arguments and non–decreasing in ϑ. The same arguments as in Lemma A.10 ensure that $b^{\vartheta}(x)/\sigma^2(x)$ is jointly continuous. If, in addition, there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a constant $c > 0$ such that

$$\left| \frac{b^{\vartheta}(x)}{\sigma^2(x)} \right| \leq c f(\vartheta), \qquad \forall\, (\vartheta, x) \in \mathbb{R} \times \mathbb{R},$$

then the fact that the family $\{(\theta^n)_{n\geq 1}, \theta\}$ is uniformly bounded, ensures that there exists $K > 0$ such that

$$\int_0^T \lambda_u^2 \; d[M]_u \leq c^2 \liminf_{n\to\infty} \sum_{i=0}^{m_n} f^2\left(\theta_i^n\right)\left(\tau_{i+1}^n - \tau_i^n\right) \leq c^2 K^2 T.$$

Hence, Novikov's condition ensures that $\mathcal{E}\left(-\int \lambda \; dM\right)$ is a true martingale. Therefore, $d\mathbb{Q} = \mathcal{E}\left(-\int \lambda \; dM\right)_T d\mathbb{P}$ defines an equivalent local martingale measure for the price process $\int S\left(\theta, ds\right)$. Moreover, if f is bounded (S, id, θ) is a reasonable large trader market model for all $\theta \in \mathbb{L}\left(\mathbb{R}\right)$.

(ii) Reaction-Diffusion Setting. Due to Example 3.24, we get for the reaction function $\psi(t, x, \vartheta) = \exp(\sigma x + \kappa\vartheta t)$, where $\sigma, \kappa > 0$,

$$\int_0^T \lambda_u^2 \; d[M]_u \leq \liminf_{n\to\infty} \sum_{i=0}^{m_n} \left(\frac{\kappa}{\sigma}\theta_i^n + \frac{1}{2}\sigma\right)^2 \left(\tau_{i+1}^n - \tau_i^n\right).$$

Since the sequence is uniformly bounded by some constant $K > 0$, we get

$$\int_0^T \lambda_u^2 \; d[M]_u \leq 2T\left(\frac{\kappa^2 K^2}{\sigma^2} + \frac{\sigma^2}{4}\right).$$

Again, Novikov's condition ensures that $\mathcal{E}\left(-\int \lambda \; dM\right)$ is a true martingale and $d\mathbb{Q} = \mathcal{E}\left(-\int \lambda \; dM\right)_T d\mathbb{P}$ defines an equivalent local martingale measure for $\int S\left(\theta, ds\right)$.

Conclusions

In this section, we pointed out that additional regularity assumptions on the primal family $S = (S(\vartheta, .))_{\vartheta\in\mathbb{R}^d}$ ensure that the price process P^{θ} exists in the sense of Definition 3.8 *for all* $\theta \in \mathbb{L}(\mathbb{R})$. Moreover, the regularity Assumption **(P-II)** ensures that the minimal no arbitrage Assumption **(NA-I)** holds as soon as for all *constant* large trader strategies $\theta \in \mathbb{R}$ there exists an equivalent local martingale measure for P^{θ}. In the second part, we provided a sufficient condition that guarantees the existence of an equivalent local martingale measure *for all* large trader strategies

$\theta \in \mathbb{L}(\mathbb{R})$. This in turn implies that for all continuous function $F : \mathbb{L}(\mathbb{R}) \longrightarrow \mathbb{L}(\mathbb{R}^d)$, the class of all reasonable large trader market models in given by $(S, F, \mathbb{L}(\mathbb{R}))$. We achieved this result by extensively using the structure condition **(SC)** and its connection to the **(FTAP)** for *continuous* semimartingales.

3.4. The minimal large trader market model

As pointed out in the introduction, the driving force behind a large trader model is the motivation of the large trader to achieve a certain *goal*. Most often, the formulation of this goal is strongly related to the large trader real wealth process. This section provides the 'last module' introduced in this chapter, the definition of the large trader real wealth process. As before, we define the wealth process for simple strategies and extend the definition by taking limits in the semimartingale topology. Combining all modules, the 'price module', the 'NA module', and the 'wealth process module', leads to the definition of a *minimal large trader market model*. In the second part, we compare our minimal large trader market model to the large trader model proposed by Bank and Baum; see [4].

3.4.1. The large trader wealth process

Due to the different economic considerations that influence the wealth process of a large trader, this is for sure one of the most delicate tasks in a large trader model. How to model the gains process due to trading? Should we insist on self–financing conditions for the large trader? If the answer to the last question is 'yes', how should we define the self–financing condition? Moreover, how should we incorporate liquidity risks?

As [4] and [11], we only consider self–financing large trader strategies. To incorporate liquidity risk into their large trader model, Bank and Baum introduced a special definition of the bank account as well as an idealized definition of the wealth process, the *asymptotic liquidation proceed* process. In [11], the authors incorporate liquidity risk into their model by using a new definition of self–financing strategies. Using particular integration by parts formulas, both conclude that the *real wealth process* $V(\theta)$ of the large trader is of the form

$$V(\theta) = V_0 + G(\theta) - C(\theta).$$

While $G(\theta)$ represents the gains and losses due to trading, the non–negative process $C(\theta)$ is interpreted as costs due to liquidity risk. We take this representation of a large trader real wealth process for a self–financing strategy as our definition.

Let (S, F, θ) be a reasonable large trader market model. First of all, it is natural to model the accumulated gains and losses $G(\theta)$, caused by trading, by the linear stochastic integral of θ w.r.t. P^θ, i.e.

$$G(\theta) = \int \theta \, dP^\theta. \tag{3.19}$$

Suppose that the price process P^θ, affected by the large trader, can be decomposed into the sum of two semimartingales $\tilde{P}$ and I^θ. On the one hand, $\tilde{P}$ refers to the exogenous part of the price process that is not influenced by the large trader. On the other hand, I^θ models the impact of the large trader on the price process. With this decomposition the accumulated gains and losses (3.19) are given by

$$G(\theta) = \int \theta \, d\tilde{P} + \int \theta \, dI^\theta. \tag{3.20}$$

Note that the Almgren–Chriss model, Example 3.14, features such a decomposition. As in the other modules, we only define the cost function, related to liquidity risk, for simple large trader strategies $\theta \in \mathbf{S}(\mathbb{R})$. More precisely, C is defined as a function

$$C : \mathbf{S}(\mathbb{R}) \to \mathcal{V}^+, \tag{3.21}$$

where $\mathcal{V}^+$ denotes the set of all adapted, non–decreasing càdlàg processes. (Note that we do not consider any kind of transaction costs!) With this notation, we define the large trader real wealth process for $\theta \in \mathbf{S}(\mathbb{R})$ by

$$V(\theta) = V_0 + \int \theta \, dP^\theta - C(\theta). \tag{3.22}$$

Clearly, if P^θ can be decomposed as above, the wealth process is given by

$$V(\theta) = V_0 + \int \theta \, d\tilde{P} + \int \theta \, dI^\theta - C(\theta). \tag{3.23}$$

In general, we think that it is reasonable to add a further component D to the wealth process. This component models payment streams of the large trader. Clearly, such an additional component leads to strategies that are no longer self–financing. As we do not consider such phenomena, we think of (3.22) as a representation of the real wealth process, where $D \equiv 0$.

Let us get back to (3.22). As in the other modules, we define the new component, the cost function C, only for simple strategies. For general strategies θ of a reasonable large trader market model, C is defined by using a limit procedure.

Definition 3.28: *Let $\theta \in \mathbb{L}(\mathbb{R})$, let $(S(\vartheta,.))_{\vartheta\in\mathbb{R}^d}$ be a family of $\mathbb{R}$–valued semimartingales adapted to $\mathbb{F}$. Moreover, let $F : \mathbf{S}(\mathbb{R}) \longrightarrow \mathbf{S}(\mathbb{R}^d)$ and $C : \mathbf{S}(\mathbb{R}) \to \mathcal{V}^+$ be a functions. We say that (S, F, C, θ) is a* minimal large trader market model*, if the following conditions hold:*

1. *(S, F, θ) is a reasonable large trader market model.*

2. *For the same sequence that appears in the definition of the discounted price process P^θ, the sequence $(C(\theta^n))_{n\geq 1}$ converges in the semimartingale topology. Its limit, denoted by $C(\theta)$, is interpreted as the liquidity risk related to the large trader strategy θ.*

We denote by (S, F, C, Ψ_{min}) the class of all minimal large trader market models for (S, F, C). Moreover, we call the large trader strategy θ tame*, if (S, F, C, θ) is a minimal large trader market model and $C(\theta) \equiv 0$.*

3.4.2. Comparison with the large trader model of Bank and Baum

In order to compare our modular model to the large trader model of Bank and Baum [4], we consider a primal family $(S(\vartheta, .))_{\vartheta\in\mathbb{R}}$ of continuous, $\mathbb{R}$–valued semimartingales that are all zero in 0. Furthermore, we assume that, outside of a null set, the functions

$$S(\omega, ., .) : \mathbb{R} \times [0, T] \longrightarrow \mathbb{R}$$

are jointly continuous. In the following, we compare the different modules of both models to each other.

Price process module

For a simple large trader strategy $\theta \in \mathbf{S}(\mathbb{R})$, the evolution of the price process in [4] is modelled as

$$P_{\text{BB}}(\theta_t, t) := S(\theta_{t+}, t).$$

(As Bank and Baum consider strategies having càdlàg paths, we have to use the 'càdlàg version' of our simple strategy $\theta \in \mathbf{S}(\mathbb{R})$.) Note that this price process is in general not continuous. Indeed, if θ is a simple large trader strategy, the jumps of the strategy cause jumps of the price process to 'different levels' of the primal family of price processes. Recall our definition of the price process for a simple large trader strategy $\theta \in \mathbf{S}(\mathbb{R})$. For $F = id$, the price process P^θ is given by

$$P_t^\theta := \int_0^t S(\theta_s, ds) := \sum_{i=0}^{n} \left\{ S\left(\theta_{\tau_i\wedge t+}, \tau_{i+1} \wedge t\right) - S\left(\theta_{\tau_i\wedge t+}, \tau_i \wedge t\right) \right\}.$$

It is clear that our price process has continuous paths. Indeed, due to our definition of the price process, the jumps caused by the simple large trader strategy cancel out. Thus, our definition of the price process can be interpreted as 'gluing together' the different levels of the primal family $(S(\vartheta, .))_{\vartheta\in\mathbb{R}}$ of price processes. The simple large trader strategy θ 'decides' at what point in time what levels are glued together. Nevertheless, for constant large trader strategies the definitions coincide. As a consequence, the interpretation of the primal family $(S(\vartheta,.))_{\vartheta\in\mathbb{R}}$ is the same in both models. Bank and Baum considered $P_{\mathrm{BB}}(\vartheta,.) = P_{.}^{\vartheta} = S(\vartheta,.)$ as the "price fluctuations of the risky asset given that the large trader holds a constant stake of ϑ shares in the asset". At this point, we want to recall our Condition **(O)**:

$$\vartheta \leq \vartheta' \quad \Longrightarrow \quad P^{\vartheta} \leq P^{\vartheta'}.$$

This is exactly [4, Assumption 2]. Bank and Baum argue that this is a necessary condition (in their model) to exclude trivial arbitrage opportunities for the large trader employing an in–and–out strategy. Note that this argument relies on the fact that the in–and–out strategy causes jumps of the price process $P_{\mathrm{BB}}(\theta_{.},.)$. As our price process is continuous, Condition **(O)** is, from this point of view, not necessary for our model to exclude arbitrage opportunities for the large trader. At this point it must be said that in this comparison of both models there exists only liquidity risk due to 'market depth'. Indeed, the market depth refers to the ability of a market to absorb quantities without having a large effect on the prices. Clearly, this 'quantity' is the current position of the large trader in the stock. But be aware of the fact that our model allows us, in a quite general setting, to also incorporate liquidity risks due to 'market resiliency'. In Example 3.14, the Almgren–Chriss model, this is achieved by a proper choice of the function $F : \mathbf{S}(\mathbb{R}) \longrightarrow \mathbf{S}(\mathbb{R}^3)$. Finally, let us consider the definition of the price process for general large trader strategies. Bank and Baum make two technical assumptions. The first one is a regularity assumption on the primal family $(S(\vartheta,.))_{\vartheta\in\mathbb{R}}$. It is significantly stronger than our Assumption **(P-II)**, as it assumes essentially the differentiability of the primal family in the space parameter ϑ. Moreover, Bank and Baum assume that the large trader strategies are (càdlàg) semimartingales. In our opinion, this is a critical assumption from an economic point of view as, in general, semimartingales are *not* predictable processes. Hence, the large trader (in the Bank and Baum model) is, in a certain sense, able to predict the future. Nevertheless, these assumptions allow Bank and Baum to apply the Ito–Wentzell formula. This formula ensures in particular that the price process $P_{\mathrm{BB}}(\theta_{.},.)$ is in fact a semimartingale *for all* large trader strategies $\theta \in \mathcal{S}$. From this perspective, their technical assumptions are similar to our Assumption **(P-II)**. The later also ensures that the price process P^{θ}, affected by a large trader strategy, is a semimartingale *for all* $\theta \in \mathbb{L}(\mathbb{R})$.

Real wealth process module

Bank and Baum define the real wealth process for a *semimartingale strategy* of the large trader as

$$V_{\mathrm{BB}}(\theta,t) := \beta_t^{\theta} + L(\theta_t,t),$$

where the bank account β^θ is defined as

$$\beta_t^\theta = \beta_{0-} - \int_0^t P_{\mathrm{BB}}(\theta_{u-}, u)\; d\theta_u - [P_{\mathrm{BB}}(\theta_., .), \theta]_t$$

The asymptotic liquidation proceed process $L(\theta_., .)$ is defined as

$$L(\theta_t, t) = \int_0^{\theta_t} P_{\mathrm{BB}}(u, t)\; du.$$

The main idea of this definition is the following. In order to avoid turbulences of the price process, the large trader does not sell a huge position of shares en bloc. She rather splits the position into small packages and sells "one after the other in a small time period". As soon as the duration of the liquidation tends to zero, the corresponding proceeds converge to the asymptotic liquidation proceed process. Applying the Ito–Wentzell formula once more, the real wealth process of Bank and Baum can be represented as

$$V_{\mathrm{BB}}(\theta, t) - V_{\mathrm{BB}}(\theta, 0-) = \int_0^t L(\theta_{s-}, ds) - C(\theta),$$

where the cost function due to liquidity risk is given by

$$C(\theta) = \frac{1}{2} \int_0^t P'_{\mathrm{BB}}(\theta_{u-}, u)\; d[\theta]_u^c + \sum_{0 \le s \le t} \int_{\theta_{s-}}^{\theta_s} \left(P_{\mathrm{BB}}(\theta_s, s) - P_{\mathrm{BB}}(u, s)\right)\; du. \tag{3.24}$$

Here, $P'_{\mathrm{BB}}(\vartheta, .)$ denotes the first partial derivative of the price process w.r.t. the space parameter. Due to Condition **(O)**, we know that $P'_{\mathrm{BB}}(\vartheta, .) \ge 0$ for all $\vartheta \in \mathbb{R}$. Hence, the cost function is indeed non–decreasing. Note that the costs of liquidity risk for strategies, whose paths are continuous and of finite variation, are zero. Thus, in the setting of Bank and Baum these strategies are *tame*. Moreover, due to an approximation result for the process $\int L(\theta_{s-}, ds)$, Bank and Baum can conclude that a large trader only employs tame strategies. This process $\int L(\theta_{s-}, ds)$ "accounts for profits or losses from stock price fluctuations due to exogenous shocks". Let us compare the real wealth process of Bank and Baum to our real wealth process. There is one particular nice aspect of the cost function (3.24). It expresses the costs, due to liquidity risk, as a function of the large trader strategy *and* the primal family of price processes. Note that one has to be able to define the quadratic variation of the large trader strategy. As Bank and Baum choose semimartingale strategies this does not cause any problems. However, our large trader strategies are elements of

$\mathbb{L}(\mathbb{R})$. And for those strategies the definition of a quadratic variation is not clear at all.

In order to compare the gains and losses processes due to trading, recall our definition of $G(\theta)$. It is given by

$$G(\theta) = \int \theta \, dP^\theta.$$

In our opinion, this process has two advantages. Firstly, the price process P^θ appears in a very natural way in the representation of the gains and losses process. It is the *integrator of the **linear** stochastic integral w.r.t. the large trader strategy* θ. Secondly, suppose that P^θ can be decomposed as in (3.20), i.e.

$$G(\theta) = \int \theta \, d\tilde{P} + \int \theta \, dI^\theta.$$

This is a decomposition of the gains and losses process into two components. One the one hand, the component $\int \theta \, d\tilde{P}$ is only exposed to the *exogenous shock* $\tilde{P}$. On the other hand, $\int \theta \, dI^\theta$ is the part of the gains and losses process which might allow the large trader to use her impact on the price process to her own advantage. In our opinion, it is a priori not clear, why $\int L(\theta_{s-}, ds)$ should *exclusively* account for *exogenous* shocks.

'No arbitrage for the small trader' module

In the last part, we compare the no arbitrage assumptions of our modular model to those in [4]. [4, Assumption 3] supposes that there exists a probability measure, equivalent to the historical measure, such that *all* semimartingales of the primal family are local martingales under this measure. This is quite a strong assumption. Indeed, consider the primal family $(S(\vartheta, .))_{\vartheta\in\mathbb{R}}$ of [4, Example 2.1]. For all $\vartheta \in \mathbb{R}$, the dynamics of $P_{\mathrm{BB}}(\vartheta, .) = S(\vartheta, .)$ are given by the strong solutions of the SDEs

$$dP_{\mathrm{BB}}(\vartheta, t) = P_{\mathrm{BB}}(\vartheta, t) \left(\mu_t^\vartheta \, dt + \sigma_t^\vartheta \, dW_t\right),$$

where W is a standard Brownian motion. Bank and Baum pointed out that [4, Assumption 3] implies that "the market price of risk $(\mu_t^\vartheta - r)/\sigma_t^\vartheta$ associated with the exogenous risk factor dW_t does not depend on the large trader's position ϑ". In comparison, our Assumption **(NA-II)** is much weaker. It only assumes that for all $\vartheta \in \mathbb{R}$ there exists an equivalent local martingale measure for each primal price process $S(\vartheta, .)$. This allows us to consider more general diffusion–type examples; see Example 3.15.

We end this comparison with a short remark on no arbitrage conditions for the large trader. [4, Assumption 3] combined with an admissibility condition for the large trader strategies allow Bank and Baum to derive a no arbitrage condition for the large trader. This condition ensures that there exists a *universal* probability measure, equivalent to the historical measure, such that all large trader real wealth processes of 1–admissible trading strategies are local martingales and supermartingales under this measure. Apart from this particular 'no arbitrage condition' for the large trader, there are other '(no) arbitrage' conditions for the large trader in the literature. For example, Schied [26] discusses 'transaction cost triggered price manipulations' in the classical Almgren–Chriss model. How to incorporate a particular 'no arbitrage module for the large trader' in our model is left for further research.

3.5. Utility maximization

So far, we discussed the existence of an equivalent σ–martingale measure for the price process P^θ, affected by a large trader. This in turn ensures that there are no free lunches with vanishing risk for the small trader. An even more interesting question, which has been risen in the introduction, concerns the motivation of the large trader to trade. Here, we tie the motivation of the large trader to a utility maximization problem.

In order to formulate the utility maximization problem, we first have to provide an admissibility concept for the large trader. On the one hand, we think that this definition should coincide with at least one definition of admissibility in the small trader setting, if the large trader does not influence the price process. In this case, she is in fact a small trader. On the other hand, the most popular admissibility concept, the a–admissibility, where the wealth process is bounded from below by a finite credit line a, is not tractable in our setting. Although this admissibility concept has a clear economic interpretation, it might lead to a prohibition of *all* constant large trader strategies (except $\theta \equiv 0$); see Remark 3.32 below. As simple strategies form the core of *all* our modules, this would be highly unsatisfactory. Here, we use a modified version of the concept suggested by Biagini and Sirbu [7]. These authors suggest (in a small trader setting) to consider those strategies that allow for a loss control of the associated wealth process by a martingale under the historical measure. Taking all these considerations into account, we end up with the following definition of an admissible large trader strategy.

Definition 3.29: *Let (S, F, C, θ) be a minimal large trader market model. The large trader strategy $\theta \in \mathbb{L}(\mathbb{R})$ is called* admissible*, if there exists a strictly positive martingale L^θ under the historical measure $\mathbb{P}$ such that*

$$V(\theta) \geq -L^\theta$$

holds up to indistinguishability. We call L^θ loss control *of the strategy $\theta \in \mathbb{L}(\mathbb{R})$.*

In the following, we highlight several new phenomena arising in a financial market. They arise due to the impact of the large trader on the price process P^θ.

3.5.1. Phenomena arising in a simple large trader setting

In order to formulate the utility maximization problem, we need the following definition. It traces back to [6, 5].

Definition 3.30: *Let $u : \mathbb{R} \to \mathbb{R}$ be a strictly concave, increasing, and twice continuously differentiable function that satisfies the Inada–conditions, i.e.*

$$u'(-\infty) := \lim_{x \to -\infty} u'(x) = -\infty \qquad \textit{and} \qquad u'(+\infty) := \lim_{x \to +\infty} u'(x) = 0.$$

*An admissible large trader strategy $\theta \in \mathbb{L}(\mathbb{R})$ is called u–*compatible *for $\alpha > 0$, if there exists a loss control L^θ that satisfies*

$$E\left[u\left(-\alpha L^\theta_T\right)\right] > -\infty.$$

For $\alpha > 0$, we denote by $\mathcal{H}^\alpha$ the set of all admissible large trader strategies that are u–compatible for α. Furthermore, we set $\mathcal{H} := \bigcap_{\alpha>0} \mathcal{H}^\alpha$.

We analyse the utility maximization problem of a large trader in a basic setting. Despite its rather simple structure, it highlights new phenomena that are not present in the classical utility maximization theory for small traders. At this point, we want to emphasize that these phenomena appear even though the market is arbitrage free for the small trader. And besides, all phenomena appear even though there exist an equivalent local martingale measure for the wealth process of the large trader. Moreover, in some cases the wealth process is even a supermartingale under this measure. These new phenomena are a consequence of the non–linear structure of the value process involved in the problem.

The continuous semimartingales of our primal family $(S(\vartheta, .))_{\vartheta\in\mathbb{R}}$ are given by

$$S(\vartheta, t) := S_0 + \mu(\vartheta)\,[M]_t + \sigma(\vartheta) M_t, \qquad t \in [0, T], \tag{3.25}$$

where M is a continuous and square–integrable martingale starting in zero a.s. and having a deterministic quadratic variation $[M]$. Besides, let $S_0 > 0$, $\mu, \sigma \in C^2(\mathbb{R}, \mathbb{R})$, such that μ vanishes whenever σ is zero. Hence, Assumption **(NA-II)** holds. Due to Theorem A.18, S satisfies Assumption **(P-II)**. Furthermore, let

$$C : \mathbf{S}(\mathbb{R}) \longrightarrow \mathcal{V}^+$$

be a cost function that maps constant strategies to zero. For an admissible large trader strategy $\theta \in \mathbb{L}(\mathbb{R})$, the price process and the large trader wealth process are given by

$$P^\theta = S_0 + \int \mu(\theta_u)\, d\,[M]_u + \int \sigma(\theta_u)\, dM_u$$

and

$$V(\theta) = V_0 + \int \theta\mu(\theta)\, d\,[M] + \int \theta\sigma(\theta)\, dM - C(\theta),$$

respectively. For constant and fixed initial value V_0, we consider the exponential utility maximization problem

$$\sup_{\theta\in\mathcal{H}^1} E\left[u\left(V_T\left(\theta\right)\right)\right], \tag{3.26}$$

where $u(x) = 1 - e^{-\alpha x}$ for $\alpha > 0$. The next lemma gives a partial characterization of $\mathcal{H}^1$.

Lemma 3.31: *Let $(S(\vartheta, .))_{\vartheta\in\mathbb{R}}$ be given by (3.25) and let $u(x) = 1-e^{-\alpha x}$ for $\alpha > 0$. If $\theta \in b\mathbb{L}(\mathbb{R})$ and $C(\theta)$ is bounded then $\theta \in \mathcal{H}$. Moreover, for all $\alpha > 0$ the process*

$$L_t^\theta := E\left[\sup_{s\leq T}|V_s|\middle|\mathcal{F}_t\right]$$

is an u–compatible loss control for θ.

Proof: W.l.o.g. we assume that $C(\theta) \equiv 0$. Note that it suffices to prove

$$E\left[\exp\left(\alpha\sup_{s\leq T}|V_s|\right)\right] < \infty.$$

This implies that L^θ is a u–compatible loss control for θ. As θ is bounded and $[M]$ is deterministic, we know that for all $\alpha > 0$ there exists a constant $K > 0$ such that

$$\begin{aligned}\exp\left(\alpha\sup_{s\leq T}|V_s|\right) &\leq \sup_{s\leq T}\left[\exp\left(\alpha V_s\right)\right] + \sup_{s\leq T}\left[\exp\left(-\alpha V_s\right)\right]\\ &\leq K\sup_{s\leq T}\mathcal{E}\left(-\int \alpha\theta\sigma(\theta)\, dM\right)_s + K\sup_{s\leq T}\mathcal{E}\left(\int \alpha\theta\sigma(\theta)\, dM\right)_s.\end{aligned}$$

Due to Novikov's condition, we can conclude that L^θ is an u–compatible loss control for θ. □

Remark 3.32: *The lemma highlights that the concept of a–admissibility is not suitable for our large trader setting. Indeed, if M is a standard Brownian motion it would prohibit any constant large trader strategy except $\theta = 0$.*

Denote by Φ the set of all bounded, *tame*, admissible strategies, i.e.

$$\Phi := \left\{\theta \in b\mathbb{L}(\mathbb{R})|\ \theta \text{ is a tame admissible large trader strategy.}\right\}.$$

Due to Novikov's condition, we can define probability measures $\mathbb{Q}^\theta \sim \mathbb{P}$ via

$$\frac{d\mathbb{Q}^\theta}{d\mathbb{P}} := \exp\left(-\alpha\int_0^T \theta_u\sigma(\theta_u)\, dM_u - \frac{\alpha^2}{2}\int_0^T \theta_u^2\sigma^2(\theta_u)\, d\,[M]_u\right), \qquad \forall\theta\in\Phi. \tag{3.27}$$

This enables us to rewrite the utility maximization problem (3.26) for all strategies $\theta \in \Phi$ as

$$E\left[u(V_T(\theta))\right] = 1 - \exp\left(-\alpha V_0\right) E_{\mathbb{Q}^\theta}\left[\exp\left(-\alpha \int_0^T p(\theta_u)\, d\left[M\right]_u\right)\right], \tag{3.28}$$

where

$$p(x) := x\mu(x) - \frac{\alpha}{2}x^2\sigma^2(x), \qquad x \in \mathbb{R}. \tag{3.29}$$

After these preliminary observations, we first discuss the following reduced utility maximization problem

$$\sup_{\theta\in\Phi} E\left[u\left(V_T\left(\theta\right)\right)\right]. \tag{3.30}$$

As we will see below, two different scenarios might happen. In the first scenario, the so–called *stable regime*, we can find at least one optimal strategy. These optimal strategies are constant. In the second scenario, the *unstable regime*, the presence of the large trader completely destabilizes the market. This is caused by the fact that, in an unstable regime, it is optimal for the large trader to buy/sell as many shares as possible to maximize her expected utility from terminal wealth.

Remark 3.33: *It will turn out that the existence of an optimal strategy boils down to the existence of a maximum of the function p. The following observations show that the market is stable, iff*[1] *p, defined in* (3.29)*, attains at least one maximum.*

(i) Stable regime: Let us suppose that p has at least one maximum $\vartheta^* \in \mathbb{R}$. Reformulating (3.28), we get for $\theta \in \Phi$

$$\begin{aligned} &E\left[u(V_T(\theta))\right] \\ &\quad = 1 - \exp\left(-\alpha(V_0 + p(\vartheta^*)\left[M\right]_T)\right) E_{\mathbb{Q}^\theta}\left[\exp\left(-\alpha \int_0^T \left(p(\theta_u) - p(\vartheta^*)\right)\, d\left[M\right]_u\right)\right]. \end{aligned}$$

Since

$$\mathbb{P}\left(p(\theta) - p(\vartheta^*) \le 0, \forall\theta \in \Phi\right) = 1,$$

it follows that ϑ^* is the optimal strategy and

$$\sup_{\theta\in\Phi} E\left[u\left(V_T\left(\theta\right)\right)\right] = 1 - \exp\left(-\alpha(V_0 + p(\vartheta^*)\left[M\right]_T)\right).$$

(ii) Unstable regime: Let us suppose that p has no maximum. Due to the continuity of p, we can find a sequence $(\vartheta_n)_{n\in\mathbb{N}} \subset \Phi$ of constant strategies such that

$$\sup_{\vartheta\in\mathbb{R}} p(\vartheta) = \begin{cases} \lim\limits_{n\to\infty} p(\vartheta_n) =: p^* \in \mathbb{R}, & \text{if } p \text{ is bounded from above,} \\ +\infty, & \text{else.} \end{cases}$$

[1] abbr.: if and only if

Keeping this in mind, it follows that

$$\sup_{\theta\in\Phi} E\left[u\left(V_T\left(\theta\right)\right)\right] = \begin{cases} 1-\exp\left(-\alpha(V_0+p^*\left[M\right]_T)\right), & \text{if } p \text{ is bounded from above,} \\ 1, & \text{else.} \end{cases}$$

Since continuous functions attain their extreme points on compact intervals, it is clear that $\vartheta_n \to \pm\infty$. Obviously, $\vartheta_n \to +\infty$ means that the large trader tries to buy as many shares as possible in order to reach her maximal expected utility of terminal wealth. $\vartheta_n \to -\infty$ means that she achieves her goal by short selling. Therefore, there is no optimal strategy $\theta \in \Phi$. Such trading strategies lead to exploding or collapsing prices and therefore destabilize the market.

We collect the above results in the following proposition.

Proposition 3.34: *Under the assumptions made above, we have*

$$\sup_{\theta\in\Phi} E\left[u\left(V_T\left(\theta\right)\right)\right] = \sup_{\theta\in\mathcal{H}^1} E\left[u\left(V_T\left(\theta\right)\right)\right].$$

Furthermore, either of the following statements hold:

1. *(Stable regime): There exists at least one solution to the utility maximization problem (3.26), iff the function p defined in (3.29) attains at least one maximum. The set of optimal strategies contains only constant strategies and (if considered as subset of $\mathbb{R}$) coincides with the set of maxima of the function p. If $\vartheta^* \in \mathbb{R}$ is an optimal strategy, the value of the utility maximization problem (3.26) is given by*

$$\sup_{\theta\in\Phi} E\left[u\left(V_T\left(\theta\right)\right)\right] = 1-\exp\left(-\alpha(V_0+p(\vartheta^*)\left[M\right]_T)\right).$$

2. *(Unstable regime): There is no optimal trading strategy in $\mathcal{H}^1$. Moreover, by maximizing the expected utility of terminal wealth, the large trader destabilizes the market, since the prices either explode or collapse. Here, the utility maximization efforts of the large trader lead to*

$$\sup_{\theta\in\Phi} E\left[u\left(V_T\left(\theta\right)\right)\right] = \begin{cases} 1-\exp\left(-\alpha(V_0+p^*\left[M\right]_T)\right), & p \text{ is bounded from above,} \\ 1, & \text{else.} \end{cases}$$

Remark 3.35:

1. *Due to Proposition 3.34, it is easy to find an example, in which the necessary condition is not sufficient. Choose for example $\mu(x) = x$ and $\sigma(x) = x^2$. Then $\vartheta^* = 0$ satisfies the necessary condition $p'(\vartheta^*) = 0$. As $p''(\vartheta^*) = 2$, ϑ^* is a local minimum of p. Due to Proposition 3.34, the constant trading strategy is not optimal.*

2. *Note that in the stable regime, there exists an equivalent local martingale measure for the large trader wealth process. Moreover, the 'destabilization' of the market in the unstable regime can be achieved by a sequence of large trader strategies such that for all of these wealth processes there exists an equivalent local martingale measure. Finally, if M is a Brownian motion, the corresponding wealth processes are also supermartingale unter the equivalent local martingale measure.*

3.5.2. Case study: An illiquid Bachelier model

Here, we discuss a concrete example in detail. It shows that, despite the absence of arbitrage, the presence of a large trader may destabilise a financial market. Therefore, absence of arbitrage alone is not enough to rule out unrealistic features when modelling an illiquid financial market.

Consider a modified Bachelier model where the drift of the asset price is positively influenced by the engagement of a large investor. One can think of this positive influence being caused by momentum traders who react to the signal given by the large investor increasing her stake. A similar Bachelier model (which contains in addition a term modelling temporary impact) has been studied in the context of illiquid markets e.g. in Schied and Schöneborn [48]. We would like to point out that these illiquid Bachelier models are random field models. Therefore, they are much more complex than the classical Bachelier model, in particular since the spatial parameter will get replaced by dynamic trading strategies. The primitive family $(S(\vartheta, .))_{\vartheta\in\mathbb{R}}$ is given as

$$S(\vartheta, t) = S_0 + (\mu + \kappa\vartheta)\, t + \sigma W_t,$$

or, in differential notation,

$$S(\vartheta, dt) = (\mu + \kappa\vartheta)\ dt + \sigma\ dW_t,$$

where W is a Brownian motion and μ, κ, σ are positive parameters. The filtration is supposed to be the smallest one fulfilling the usual conditions and containing the one generated by W.

Remark 3.36: *This model is not included in the model class studied by Kühn [36] whose assumption 2.1 ('Largeness is not favourable') implies that the drift is non-increasing in ϑ.*

The dynamics of the discounted price process in the illiquid Bachelier model are given by

$$P^{\theta} = \tilde{P} + I^{\theta},$$

where

$$\tilde{P} = S_0 + \mu t + \int \sigma dW_u,$$
$$I^\theta = \kappa \int \theta \, du.$$

Moreover, we have for $\theta \in \Phi$

$$V_T(\theta) = V_0 + \int_0^T \left(\mu \, \theta_u + \kappa \theta_u^2\right) \, du + \int_0^T \sigma \theta_u \, dW_u.$$

To analyse the model, we apply the same arguments as in Proposition 3.34. The function p defined in (3.29) is a polynomial of order 2. In particular, we find that

$$-\alpha p(x) = \alpha \frac{\alpha\sigma^2 - 2\kappa}{2} \left(x - \frac{\mu}{\alpha\sigma^2 - 2\kappa}\right)^2 - \frac{\alpha\mu^2}{2(\alpha\sigma^2 - 2\kappa)}.$$

Clearly,

$$\vartheta^* = \frac{\mu}{\alpha\sigma^2 - 2\kappa}$$

satisfies the necessary condition $p'(\vartheta^*) = 0$. If we define $\mathbb{Q}^\theta \sim \mathbb{P}$ as in (3.27), we find that (for given initial wealth 0)

$$1 - \sup_{\theta \in \Phi} E\left[u\left(V_T(\theta)\right)\right] = -\sup_{\theta \in \Phi} \left\{ -E_{\mathbb{Q}^\theta} \left[\exp\left(\alpha \int_0^T \left(\frac{\alpha\sigma^2 - 2\kappa}{2} \theta_u^2 - \mu\theta_u \right) \, du \right) \right] \right\} \tag{3.31}$$

$$= \exp\left(- \frac{\alpha\mu^2 T}{2\left(\alpha\sigma^2 - 2\kappa\right)} \right) \inf_{\theta \in \Phi} E_{\mathbb{Q}^\theta} \left[\exp\left(\alpha \frac{\alpha\sigma^2 - 2\kappa}{2} \int_0^T \left(\theta_u - \frac{\mu}{\alpha\sigma^2 - 2\kappa} \right)^2 \, du \right) \right].$$

Therefore, utility maximization relies on the so–called *stability condition*

$$2\kappa < \alpha\sigma^2. \tag{3.32}$$

(i) Unstable regime $2\kappa > \alpha\sigma^2$: In this case the strategy ϑ^* performs worst among all strategies, while the expected utility grows with $|\vartheta|$ up to the maximum value. This can be interpreted in a way that the impact of the strategy on the drift is so substantial that the large investor buys as many shares as possible.

(ii) Stable regime $2\kappa < \alpha\sigma^2$: The strategy ϑ^* performs best under all strategies. Moreover, it is the only strategy among all admissible ones which fulfils the necessary optimality condition. In case the large trader chooses the candidate strategy ϑ^*, she gains an expected utility of

$$1 - \exp\left(- \frac{\alpha\mu^2 T}{2\left(\alpha\sigma^2 - 2\kappa\right)} \right).$$

We now want to compare the expected utility of the large trader with the optimal utility in the classical Merton problem in this stable regime. Consider the Merton problem where we face a hypothetical small investor with the same utility function and initial wealth 0, and with given price process $S(\theta,.)$. We can calculate the optimal strategy by substituting 0 for κ, and $\mu+\kappa\theta$ for μ in the above calculations. Given that the large trader is present in the market and behaves rationally, i.e. chooses the constant strategy ϑ^*, it results that the small trader would choose a constant strategy as well, namely

$$\psi = \psi\left(\vartheta^*\right) = \frac{\mu+\kappa\vartheta^*}{\alpha\sigma^2}.$$

His expected utility in that case would be

$$1-\exp\left(-\frac{\alpha^2\psi^2\left(\vartheta^*\right)\sigma^2 T}{2}\right) = 1-\exp\left(-\frac{\alpha\mu^2 T}{2\left(\alpha\sigma^2-2\kappa\right)}\left(1+\frac{\kappa^2}{\alpha\sigma^2\left(\alpha\sigma^2-2\kappa\right)}\right)\right).$$

Therefore, the small investor would achieve a higher expected utility. If there was no large trader around, which corresponds to the case $\theta=0$, the small trader would hold an optimal portfolio of

$$\psi(0) = \frac{\mu}{\alpha\sigma^2}$$

stocks, and his expected utility in that case would be

$$1-\exp\left(-\frac{\mu^2 T}{2\sigma^2}\right) = 1-\exp\left(-\frac{\alpha\mu^2 T}{2\left(\alpha\sigma^2-2\kappa\right)}\frac{\left(\alpha\sigma^2-2\kappa\right)}{\alpha\sigma^2}\right).$$

Hence, the small investor purchases less stocks than a large investor would do, and he achieves, in contrast to the case studied in Bank and Baum [4], only a lower expected utility, compared to the large investor. The only exception would be the case $\mu=0$. However, in this case a large trader could not benefit from the fact that her actions could enlarge the drift of the price process and thereby change a martingale into a submartingale.

(iii) Critical case $2\kappa=\alpha\sigma^2$: it follows from (3.31) that the result depends on μ. In case $\mu=0$, all strategies perform equally as the investor always gets the expected utility of the zero strategy. For $\mu\neq 0$ she can, like in the unstable regime, achieve expected utility arbitrarily close to the maximum value of one. Yet, now her stake has to have the right sign, depending on the sign of μ.

A. Appendix

A.1. Strong non–linear integrators and the non–linear stochastic integral

The purpose of this section is to give an overview of strong non–linear integrators and the non–linear stochastic integral. For a detailed and complete introduction of these concepts, we refer to Carmona and Nualart [10]. We consider a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions. Moreover, T is a finite constant and $\mathcal{F}_0$ is trivial apart from null sets. Results from the literature formulated for an infinite time horizon are used by applying the corresponding result to the stopped process.

A.1.1. Simple integrands and the semimartingale topology

As in the case of the linear stochastic integral, the main ingredients to define the stochastic integral are the simple integrands and the semimartingale topology. Once the non–linear stochastic integral is defined for simple integrands, the non–linear stochastic integral for general integrands is defined via a limit procedure.

Simple integrands

There are different ways to define simple integrands. This version is given in [10].

Definition A.1: *Let θ be a predictable process with representation*

$$\theta(t) = \theta_{-1}\mathbb{1}_{\{0\}} + \sum_{i=0}^{n} \theta_i \mathbb{1}_{(\tau_i,\tau_{i+1}]}(t), \tag{A.1}$$

where $0 = \tau_0 \leq \tau_1 \leq ... \leq \tau_{n+1} = T$ is a finite sequence of $(\mathcal{F}_t)$–stopping times, $\theta_{-1} \in \mathbb{R}^d$, and θ_i is for each $i \in \{0,\dots,n\}$ a bounded, $\mathcal{F}_{\tau_i}$–measurable, $\mathbb{R}^d$–valued random variable. We call θ an extended simple integrand, *if the random variables θ_i take only finitely many values. An extended simple integrand is called* simple integrand, *if the stopping times τ_i take only finitely many values. We denote the space of all $\mathbb{R}^d$–valued, extended simple integrands by $\mathbf{S}^e(\mathbb{R}^d)$. The space of all $\mathbb{R}^d$–valued simple integrands is denoted by $\mathbf{S}(\mathbb{R}^d)$. For $d = 1$ the set $\mathbf{S}_1 \subset \mathbf{S}(\mathbb{R})$ is the set of all simple integrands bounded by 1.*

The next result is well known. A proof can be found in [10, Proposition II.1.1].

Lemma A.2: *Let $\mathbb{L}\left(\mathbb{R}^d\right)$ be the space of $\mathbb{R}^d$–valued, càglàd, adapted processes and denote by $b\mathbb{L}\left(\mathbb{R}^d\right) \subset \mathbb{L}\left(\mathbb{R}^d\right)$ the subspace of bounded, $\mathbb{R}^d$–valued, càglàd, adapted processes. The closure of $\mathbf{S}\left(\mathbb{R}^d\right)$ w.r.t. the ucp–convergence is the space $\mathbb{L}\left(\mathbb{R}^d\right)$.*

The following lemma collects several results about càglàd, adapted processes.

Lemma A.3: *Let $(\theta^n)_{n\geq 1} \subset \mathbb{L}\left(\mathbb{R}^d\right)$ and $\theta \in \mathbb{L}\left(\mathbb{R}^d\right)$.*

1. *If $\theta^n \to \theta$ in ucp, then there exists a subsequence (still indexed by n) such that $(\theta^n - \theta)^* \to 0$[1] a.s. for $n \to \infty$.*
2. *The sequence $(\tau_m)_{m\geq 1}$ of stopping times, where*
$$\tau_m := \inf\left\{t > 0 : \|\theta_t\|_{\mathbb{R}^d} \geq m\right\} \wedge T, \qquad m \geq 1,$$
converges a.s. to T and satisfies $P\left(\tau_m < T\right) \to 0$ for $m \to \infty$.
3. *Let $c > 0$ and $(\theta^n)_{n\geq 1} \subset \mathbf{S}\left(\mathbb{R}^d\right)$. For $m \geq 1$ define $(\theta^{m,n})_{n\geq 1} \subset \mathbf{S}^e\left(\mathbb{R}^d\right)$ for all $i \leq d$ via*
$$\left(\theta_t^{m,n}\right)_i := \begin{cases} \min\left\{m + c, \left(\theta^n_{\tau_m\wedge t}\right)_i\right\}, & \text{on } \{(\theta^n)_i \geq 0\}, \\ \max\left\{-m - c, \left(\theta^n_{\tau_m\wedge t}\right)_i\right\}, & \text{on } \{(\theta^n)_i < 0\}. \end{cases}$$
Then, for all $m \geq 1$, the family $\{(\theta^{m,n})_{n\geq 1}, \theta^{\tau_m}\}$ is uniformly bounded by $m+c$. Furthermore, if $\theta^n \to \theta$ in ucp, then
$$\theta^{m,n} \xrightarrow[ucp]{n\to\infty} \theta^{\tau_m}, \qquad \text{for all } m \geq 1.$$
4. *There exists a sequence of extended simple integrands $(\theta_n) \subset \mathbf{S}^e\left(\mathbb{R}^d\right)$, a sequence $(a_m)_{m\geq 1} \subset \mathbb{N}$, and a sequence of stopping times $(\tau_m)_{m\geq 1} \uparrow T$ a.s. such that the following conditions hold:*
   a) *$(\theta_n - \theta)^* \to 0$ a.s. for $n \to \infty$.*
   b) *$P\left(\tau_m = T\right) \to 1$ for $m \to \infty$.*
   c) *The family $\{(\theta_n^{\tau_m})_{n\geq 1}, \theta^{\tau_m}\}$ is uniformly bounded by a_m.*

Proof: The first statement is well known. It implies in particular that almost all paths of $\theta \in \mathbb{L}\left(\mathbb{R}^d\right)$ are bounded. This fact clearly implies the second statement. The first part of the third statement holds by definition. As for $m, n \geq 1$ and $0 < \epsilon \leq \frac{c}{2}$

$$E\left[1 \wedge \sup_{s\leq T} \|\theta_s^{m,n} - \theta_s^{\tau_m}\|_{\mathbb{R}^d}\right] \leq P\left(\sup_{s\leq T} \|\theta_s^n - \theta_s\|_{\mathbb{R}^d} > \epsilon\right) + E\left[1 \wedge \sup_{s\leq T} \|\theta_s^n - \theta_s\|_{\mathbb{R}^d}\right],$$

[1] For a measurable function f on $[0, T]$ we set $f^* := \sup_{t\leq T} \|f(t)\|_{\mathbb{R}^d}$.

the assumption $\theta^n \to \theta$ in *ucp* ensures that the third statement holds. To prove the last statement, note that Lemma A.2 ensures that there exists a sequence $(\tilde{\theta}^n)_{n\geq 1} \subset \mathbf{S}\left(\mathbb{R}^d\right)$ such that $\tilde{\theta}^n \to \theta$ in *ucp*. Using the notation of item *3.*, we define for $c > 0$ the sequence $(\theta_n)_{n\geq 1} \subset \mathbf{S}^e\left(\mathbb{R}^d\right)$ of extended simple integrands by $\theta_n := \tilde{\theta}^{n,n}$. Due to item *1.*, there exists a subsequence of $(\theta_n)_{n\geq 1} \subset \mathbf{S}^e\left(\mathbb{R}^d\right)$ that satisfies the desired properties. □

Definition A.4: *Let $\theta \in \mathbb{L}\left(\mathbb{R}^d\right)$. A sequence $(\theta^n)_{n\geq 1} \subset \mathbf{S}^e\left(\mathbb{R}^d\right)$ of extended simple integrands that satisfies all properties of Lemma A.3 4. is called* convenient approximating sequence of *θ.*

Semimartingale topology

The semimartingale topology is induced by the metric

$$\begin{aligned} d_{\mathcal{SM}} &: \mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{R}_{\geq 0} \\ (X,Y) &\longmapsto \sup_{H \in \mathbf{S}_1} E\left[1 \wedge \left(\sup_{t\leq T}\left|\int_0^t H_u \, d\left(X-Y\right)_u\right|\right)\right], \end{aligned}$$

where $\mathcal{S}$ denotes the space of $\mathbb{R}$–valued semimartingales. We say that a sequence $(X^n)_{n\geq 1}$ of semimartingales converges to X in the semimartingale topology (notation: $X^n \underset{\mathcal{SM}}{\longrightarrow} X$), if $d_{\mathcal{SM}}(X^n, X) \to 0$ for $n \to \infty$.

Theorem A.5: *$(\mathcal{S}, d_{\mathcal{SM}})$ is a complete metric space. In particular, the set of continuous semimartingales $\mathcal{S}_c$ is a closed subset in $(\mathcal{S}, d_{\mathcal{SM}})$, i.e. $(\mathcal{S}_c, d_{\mathcal{SM}})$ is a complete metric space.*

Proof: The first statement is [39, II.7 Théorème]. The second statement follows from the fact that convergence in the semimartingale topology implies convergence in *ucp*. □

Denote by $\mathcal{A}_{loc}$ the space of all càdlàg adapted processes, whose total variation process is locally integrable. The next result is [39, IV.7 Théorème].

Proposition A.6: *The space $\mathcal{A}_{loc}$ is closed in $\mathcal{S}$. Moreover, for $A \in \mathcal{A}_{loc}$ we have $d_{\mathcal{SM}}(0, A) = E\left[1 \wedge V(A)_T\right]$, where $V(A)_T$ denotes the total variation of A on $[0,T]$.*

In the following proposition, we collect several results on convergence in the semimartingale topology.

Proposition A.7: *Let $(M^n)_{n\geq 0}$ be a sequence of local martingales and let $(A^n)_{n\geq 0}$ be a sequence of processes of finite variation. Define*

$$S^n := M^n + A^n, \qquad n \geq 0.$$

If $(A^n)_{n\geq 0} \subset \mathcal{A}_{loc}$*, then* $A^n \underset{\mathcal{SM}}{\longrightarrow} A^0$ *if and only if the total variation process* $V(A^n - A^0)_T$ *converges to zero in probability. Moreover, if* $S^n \underset{\mathcal{SM}}{\longrightarrow} S^0$ *then the following statements hold:*

1. *If* $(S^n)_{n\geq 0} \subset \mathcal{S}_c$ *then* $M^n \underset{\mathcal{SM}}{\longrightarrow} M^0$ *and* $A^n \underset{\mathcal{SM}}{\longrightarrow} A^0$.
2. *If* $f : \mathbb{R} \to \mathbb{R}$ *is twice continuously differentiable then* $f(S^n) \underset{\mathcal{SM}}{\longrightarrow} f(S^0)$.
3. *If* $(X^n)_{n\geq 0} \subset \mathcal{S}$ *and* $X^n \underset{\mathcal{SM}}{\longrightarrow} X^0$ *then* $X^n S^n \underset{\mathcal{SM}}{\longrightarrow} X^0 S^0$.
4. *If* $(X^n)_{n\geq 0} \subset \mathcal{S}$ *and* $X^n \underset{\mathcal{SM}}{\longrightarrow} X^0$ *then* $[X^n, S^n] \underset{\mathcal{SM}}{\longrightarrow} [X^0, S^0]$.

Proof: W.l.o.g. we assume that all processes are a.s. zero in 0. The first statement follows from Proposition A.6. The second and third statement are [39, Remarque IV.3] and [23, Proposition 4], respectively. Since

$$X^n S^n = \frac{1}{4}\left((X^n + S^n)^2 - (X^n - S^n)^2\right)$$

holds, item *2.* implies the third item. Hence, it remains to prove the last item. Due to the product rule, we have

$$[X^n, S^n] = X^n S^n - \int S^n_- \, dX^n - \int X^n_- \, dS^n.$$

As convergence in the semimartingale topology implies *ucp*–convergence, we know that

$$X^n_- \underset{ucp}{\longrightarrow} X^0_- \qquad \text{and} \qquad S^n_- \underset{ucp}{\longrightarrow} S^0_-.$$

Due to *3.* and [39, III.13 Théorème], the claim is proven. □

A.1.2. Strong non–linear integrators

Here, we give a short overview of strong non–linear integrators. A detailed discussion can be found in [10]. Throughout this subsection, we consider a family $S = (S(\vartheta, \cdot))_{\vartheta\in\mathbb{R}^d}$ of $\mathbb{R}$–valued semimartingales on $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

In the linear stochastic integration theory the semimartingales build the biggest class of 'good integrators'. For the non–linear stochastic integral, the strong non–linear integrators form the class of 'good integrators'. In order to give a definition of a strong non–linear integrator, we need the following notation. Let $L\left(\mathbb{R}^d\right)$ denote the set of all deterministic, $\mathbb{R}^d$–valued, càglàd functions on the interval $[0, T]$. For $h \in L\left(\mathbb{R}^d\right)$ and $t \in (0, T)$ we set $h(t+) := \lim_{t_n\downarrow t} h(t_n)$ and $h(t-) := \lim_{t_n\uparrow t} h(t_n)$. Furthermore, the *elementary non–linear stochastic integral* of $\theta \in \mathbf{S}\left(\mathbb{R}^d\right)$ w.r.t.

$S = (S(\vartheta,\cdot))_{\vartheta\in\mathbb{R}^d}$, where the representation of $\theta \in \mathbf{S}\left(\mathbb{R}^d\right)$ is given in (A.1), is defined as

$$\int_0^t S(\theta_s, ds) := S(\theta_{-1}, 0) + \sum_{i=0}^{n} \left\{ S\left(\theta_{\tau_i\wedge t+}, \tau_{i+1}\wedge t\right) - S\left(\theta_{\tau_i\wedge t+}, \tau_i\wedge t\right)\right\}.$$

Definition A.8 ([10, Proposition II.3.1])**:** *Let $S = (S(\vartheta,\cdot))_{\vartheta\in\mathbb{R}^d}$ be a family of semimartingales on $(\Omega, \mathcal{F}, \mathbb{F}, P)$. S is a strong non–linear integrator if the following conditions hold:*

1. *For all $t \in (0,T]$ and all $h, h' \in L(\mathbb{R}^d)$ we have*

   $$S(h,t) = S(h',t)$$

   outside of a P–null set (possibly depending upon h, h' and t) whenever:

   $$h(s) = h'(s) \quad \text{for all } s \leq t \text{ and } \quad h(t+) = h'(t+).$$

2. *For all fixed $t \in (0,T]$ and $K > 0$, the set of random variables $\left(\eta\cdot\int S(\theta, ds)\right)_t$ with $\theta \in \mathbf{S}\left(\mathbb{R}^d\right)$, $\eta \in \mathbf{S}$ and $\max\{\theta^*, \eta^*\} \leq K$ is bounded in probability.*

3. *For fixed $t \in (0,T]$, the mapping $\theta \mapsto \int S^t(\theta, ds)$ is locally uniformly continuous from $\mathbf{S}\left(\mathbb{R}^d\right)$, endowed with the ucp–topology, into $\mathcal{S}$, endowed with the semimartingale topology.*

Remark A.9: *This definition is one way to define strong non–linear integrators; see* [10, Proposition II.3.1]*. For our purposes, the last item is the most important one. It allows us to define the non–linear stochastic integral for càglàd–processes.*

Note that Definition A.8 *1.* is a regularity property of the family of semimartingales $S = (S(\vartheta,\cdot))_{\vartheta\in\mathbb{R}^d}$. The following order condition is introduced in [4]:

Condition (O): $\vartheta \leq \vartheta'$ implies $S(\vartheta,\cdot) \leq S(\vartheta',\cdot)$.

Due to the next lemma, condition **(O)** is sufficient for S to satisfy Definition A.8 *1.*.

Lemma A.10: *Let $d = 1$. $S = (S(\vartheta,\cdot))$ satisfies the first item of Definition A.8, if at least one of the following conditions hold for almost all $\omega \in \Omega$:*

1. *The mapping $S(.,.,\omega) : \mathbb{R}\times[0,T] \to \mathbb{R}$ is jointly continuous.*

2. *S is continuous in both arguments and satisfies condition* **(O)***.*

Proof: The first statement is clear. To prove the second statement, we show that condition **(O)** implies joint continuity. We oppress the ω in the following proof. Let

$(\overline{\vartheta},\overline{t}) \in \mathbb{R} \times [0,T]$. Since $S(.,\overline{t})$ is continuous, for all $\epsilon > 0$ we can choose an open interval $(\vartheta_{\min},\vartheta_{\max}) \ni \overline{\vartheta}$ such that

$$\left|S\left(\vartheta_{\min},\overline{t}\right) - S\left(\vartheta_{\max},\overline{t}\right)\right| < \frac{\epsilon}{3}. \tag{A.2}$$

Besides, $S(\vartheta_{\min},.)$ and $S(\vartheta_{\max},.)$ are continuous, too. Therefore, for all $\epsilon > 0$ we can find $\delta > 0$ such that for all $t \in (\overline{t}-\delta,\overline{t}+\delta) \cap [0,T]$

$$\left|S(\vartheta_{\min},t) - S\left(\vartheta_{\min},\overline{t}\right)\right| < \frac{\epsilon}{3} \qquad \text{as well as} \qquad \left|S(\vartheta_{\max},t) - S\left(\vartheta_{\max},\overline{t}\right)\right| < \frac{\epsilon}{3}.$$

By the triangle inequality, we get for $(\vartheta,t) \in (\vartheta_{\min},\vartheta_{\max}) \times (\overline{t}-\delta,\overline{t}+\delta) \cap [0,T]$

$$\left|S\left(\overline{\vartheta},\overline{t}\right) - S(\vartheta,t)\right| \le \left|S\left(\overline{\vartheta},\overline{t}\right) - S\left(\vartheta,\overline{t}\right)\right| + \left|S\left(\vartheta,\overline{t}\right) - S(\vartheta,t)\right|.$$

Due to (A.2) and the monotonicity of $S(.,\overline{t})$, the first term in the above equation is less than $\frac{\epsilon}{3}$ for all $\vartheta \in (\vartheta_{\min},\vartheta_{\max})$. By monotonicity, the second term is less than $\frac{2\epsilon}{3}$ for all $\vartheta \in (\vartheta_{\min},\vartheta_{\max})$ and for all $t \in (\overline{t}-\delta,\overline{t}+\delta) \cap [0,T]$. Indeed, suppose

$$\left|S(\vartheta,t) - S\left(\vartheta,\overline{t}\right)\right| = S(\vartheta,t) - S\left(\vartheta,\overline{t}\right).$$

Due to monotonicity, we get

$$\begin{aligned}
-\frac{2\epsilon}{3} &\le S(\vartheta_{\min},t) - S\left(\vartheta_{\min},\overline{t}\right) + S\left(\vartheta_{\min},\overline{t}\right) - S\left(\vartheta_{\max},\overline{t}\right) \\
&\le S(\vartheta,t) - S\left(\vartheta,\overline{t}\right) \\
&\le S(\vartheta_{\max},t) - S\left(\vartheta_{\max},\overline{t}\right) + S\left(\vartheta_{\max},\overline{t}\right) - S\left(\vartheta_{\min},\overline{t}\right) \le \frac{2\epsilon}{3}.
\end{aligned}$$

□

A.1.3. Non–linear stochastic integral

Let $\theta \in \mathbf{S}\left(\mathbb{R}^d\right)$ with representation (A.1). In [10], the elementary non–linear stochastic integral of θ w.r.t. a strong non–linear integrator $S = (S(\vartheta,.))_{\vartheta\in\mathbb{R}^d}$ is defined by

$$\int_0^t S(\theta_s,ds) := S(\theta_{-1},0) + \sum_{i=0}^{n} \left\{S\left(\theta_{\tau_i\wedge t+},\tau_{i+1}\wedge t\right) - S\left(\theta_{\tau_i\wedge t+},\tau_i\wedge t\right)\right\}.$$

Remark A.11: *Let $d = 1$. If $S(\vartheta,.) = \vartheta S(0,.)$ for all $\vartheta \in \mathbb{R}$, the definition coincides with the classical linear stochastic integral.*

For strong non–linear integrators S, the non–linear stochastic integral of $\theta \in b\mathbb{L}\left(\mathbb{R}^d\right)$ is defined as the limit in the semimartingale topology.

Definition A.12: *Let $S = (S(\vartheta, .))_{\vartheta \in \mathbb{R}^d}$ be a strong non–linear integrator, $\theta \in b\mathbb{L}\left(\mathbb{R}^d\right)$ and let $(\theta^n)_{n\geq 1} \subset \mathbf{S}\left(\mathbb{R}^d\right)$ denote a uniformly bounded sequence of simple integrands such that*

$$\theta^n \underset{ucp}{\longrightarrow} \theta.$$

The non–linear stochastic integral $\int S(\theta, ds)$ of $\theta \in b\mathbb{L}\left(\mathbb{R}^d\right)$ w.r.t. S is the limit of $\left(\int S(\theta^n, ds)\right)_{n\geq 1}$ in the semimartingale topology, i.e.

$$\int S(\theta^n, ds) \underset{\mathcal{SM}}{\longrightarrow} \int S(\theta, ds).$$

Remark A.13: *Since S is a strong non–linear integrator, the limit in the above definition exists.*

The next proposition provides an explicit expression for the non–linear stochastic integral of an extended simple integrand.

Proposition A.14: *Let $S = (S(\vartheta, .))_{\vartheta \in \mathbb{R}^d}$ be a strong non–linear integrator and let $\theta(t) = \theta_{-1}\mathbb{1}_{\{0\}} + \sum_{i=0}^n \theta_i \mathbb{1}_{(\tau_i, \tau_{i+1}]}(t)$ be an $\mathbb{R}^d$–valued extended simple integrand. Then $\int S(\theta, ds)$ is given by*

$$\int_0^t S(\theta_s, ds) = S(\theta_{-1}, 0) + \sum_{i=0}^n \left\{S\left(\theta_{\tau_i \wedge t+}, \tau_{i+1} \wedge t\right) - S\left(\theta_{\tau_i \wedge t+}, \tau_i \wedge t\right)\right\}.$$

Proof: To prove the statement, we construct a uniformly bounded sequence of simple integrands $(\theta^m)_{m\geq 1} \subset \mathbf{S}\left(\mathbb{R}^d\right)$ that converges to θ in *ucp*. Define for $i \in \{0, \ldots, n\}$ the sequence $(\tau_i^m)_{m\geq 1}$ via

$$\tau_i^m := \frac{\lfloor 2^m \tau_i \rfloor + 1}{2^m}.$$

By definition $\tau_i^m \downarrow \tau_i$ a.s. for $i \in \{0, \ldots, n\}$. Furthermore, $(\tau_i^m)_{m\geq 1}$ is a sequence of stopping times. Indeed, let $k \geq 0$ such that $r \in [\frac{k}{2^m}, \frac{k+1}{2^m}) \subset [0, T)$ holds. Then

$$\{\tau_i^m < r\} = \left\{\tau_i < \frac{k}{2^m}\right\} \in \mathcal{F}_r.$$

Since, by assumption, the filtration is right–continuous, we have for $t \in [0, T)$

$$\{\tau_i^m \leq t\} = \bigcap_{r>t} \{\tau_i^m < r\} \in \mathcal{F}_t.$$

Define $(\theta^m)_{m\geq 1} \subset \mathbf{S}\left(\mathbb{R}^d\right)$ via

$$\theta^m(t) = \theta_{-1}\mathbb{1}_{\{0\}} + \sum_{i=0}^n \theta_i \mathbb{1}_{\left(\tau_i^m, \tau_{i+1}^m\right]}(t).$$

By definition $\theta^m \to \theta$ in *ucp*. Since S is a strong non–linear integrator, the sequence $\int S(\theta^m_s, ds)$ converges to $\int S(\theta_s, ds)$ in the semimartingale topology . Due to the fact that the limit in the semimartingale topology and the limit in *ucp* are indistinguishable, the following computation proves the statement:

$$\begin{aligned}\int_0^t S(\theta^m_s, ds) - S(\theta_{-1}, 0) &= \sum_{i=0}^n \left\{S\left(\theta_{\tau_i^m \wedge t+}, \tau^m_{i+1} \wedge t\right) - S\left(\theta_{\tau_i^m \wedge t+}, \tau^m_i \wedge t\right)\right\}\\ &= \sum_{i=0}^n \left\{S\left(\theta_i, \tau^m_{i+1} \wedge t\right) - S\left(\theta_i, \tau^m_i \wedge t\right)\right\} \xrightarrow[ucp]{} \sum_{i=0}^n \left\{S\left(\theta_i, \tau_{i+1} \wedge t\right) - S\left(\theta_i, \tau_i \wedge t\right)\right\}\\ &= \sum_{i=0}^n \left\{S\left(\theta_{\tau_i \wedge t+}, \tau_{i+1} \wedge t\right) - S\left(\theta_{\tau_i \wedge t+}, \tau_i \wedge t\right)\right\}\\ &= \int_0^t S(\theta_s, ds) - S(\theta_{-1}, 0).\end{aligned}$$

□

Proposition A.14 leads to the following rules for stopping times and non–linear stochastic integrals.

Lemma A.15: *Let $S = (S(\vartheta, .))_{\vartheta \in \mathbb{R}^d}$ be a strong non–linear integrator, let τ be a stopping time, and $\theta \in b\mathbb{L}\left(\mathbb{R}^d\right)$. Then*

$$\begin{aligned}\int_0^{\tau\wedge t} S(\theta_s, ds) &= \int_0^t S^\tau(\theta_s, ds) = \int_0^t S^\tau(\theta^\tau_s, ds)\\ &= \int_0^{\tau\wedge t} S(\theta^\tau_s, ds) = \int_0^{\tau\wedge t} S^\tau(\theta^\tau_s, ds).\end{aligned}$$

Proof: Due to Proposition A.14, it is straightforward to check that the statement holds for all extended simple integrands. Since S is a strong non–linear integrator, the result holds for $\theta \in b\mathbb{L}\left(\mathbb{R}^d\right)$ by an approximation argument. □

Now we have all tools at hand to prove the existence of the non–linear stochastic integral for $\theta \in \mathbb{L}\left(\mathbb{R}^d\right)$.

Theorem A.16: *Let $S = (S(\vartheta, .))_{\vartheta \in \mathbb{R}^d}$ be a strong non–linear integrator, $(\theta^n)_{n\geq 1} \subset$ $\mathbf{S}\left(\mathbb{R}^d\right)$, $\theta \in \mathbb{L}\left(\mathbb{R}^d\right)$ and*

$$\theta^n \xrightarrow[ucp]{} \theta, \quad n \to \infty.$$

The sequence $\left(\int S(\theta^n, ds)\right)_{n\geq 1}$ is a Cauchy–sequence in the semimartingale topology. Furthermore, there exists a semimartingale $\int S(\theta, ds)$ such that

$$\int S(\theta^n, ds) \xrightarrow[\mathcal{SM}]{} \int S(\theta, ds). \tag{A.3}$$

Moreover, $\int S(\theta, ds)$ is independent of the approximating sequence.

Proof: Let $\theta^n \to \theta$ in *ucp*. We have to prove that $\left(\int S(\theta^n, ds)\right)_{n\geq 1}$ is a Cauchy–sequence in the semimartingale topology. Then Theorem A.5 ensures that there exists a limit, denoted by $\int S(\theta, ds)$, such that (A.3) holds. This limit is independent of the approximating sequence due to the third item of Definition A.8. Let $N, n, m \geq 1$, $c > 0$ and recall the notation of Lemma A.3. Since $\theta^n = \theta^{N,n}$ and $\theta^m = \theta^{N,m}$ on

$$\{\tau_N = T\} \cap \left\{\sup_{s\leq T} \|\theta^n_s - \theta^m_s\|_{\mathbb{R}^d} \leq \frac{c}{2}\right\},$$

Proposition A.14 guarantees that on this set

$$\int S(\theta^n, ds) = \int S\left(\theta^{N,n}, ds\right) \quad \text{and} \quad \int S(\theta^m, ds) = \int S\left(\theta^{N,m}, ds\right)$$

holds. Hence, we get

$$\begin{aligned} d_{\mathcal{SM}}\left(\int S(\theta^n, ds), \int S(\theta^m, ds)\right) &\leq P(\tau_N < T) + \\ &+ P\left(\sup_{s\leq T}\|\theta^n_s - \theta^m_s\|_{\mathbb{R}^d} > \frac{c}{2}\right) + d_{\mathcal{SM}}\left(\int S\left(\theta^{N,n}, ds\right), \int S\left(\theta^{N,m}, ds\right)\right). \end{aligned}$$

As $(\theta^n)_{n\geq 1} \subset \mathbf{S}\left(\mathbb{R}^d\right)$ is a Cauchy–sequence in *ucp*, Lemma A.3 and Definition A.12 ensure that the sequence $\left(\int S(\theta^n, ds)\right)_{n\geq 1}$ is a Cauchy–sequence in the semimartingale topology for all $N \geq 1$. □

Definition A.17: *Let $S = (S(\vartheta, .))_{\vartheta\in\mathbb{R}^d}$ be a strong non–linear integrator, $\theta \in \mathbb{L}\left(\mathbb{R}^d\right)$, $(\theta^n)_{n\geq 1} \subset \mathbf{S}\left(\mathbb{R}^d\right)$, and*

$$\theta^n \underset{ucp}{\longrightarrow} \theta, \quad n \to \infty.$$

The semimartingale $\int S(\theta, ds)$, defined in Theorem A.16, is called the non–linear stochastic integral of $\theta \in \mathbb{L}\left(\mathbb{R}^d\right)$ w.r.t. S.

The last theorem ensures that most of the examples considered in Chapter 3 are indeed strong non–linear integrators.

Theorem A.18: *Let A be a non–decreasing, continuous, adapted process, M be a continuous local martingale with $E\left[\sqrt{[M]_T}\right] < \infty$, and $\mu, \sigma : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be jointly continuous functions. Let $S = (S(\vartheta, .))_{\vartheta\in\mathbb{R}^d}$ be given by*

$$S(\vartheta, t) = S^\vartheta_0 + \int_0^t \mu(\vartheta, S(\vartheta, u))\, dA_u + \int_0^t \sigma(\vartheta, S(\vartheta, u))\, dM_u,$$

and suppose that for almost all $\omega \in \Omega$ the mapping $S(., ., \omega) : \mathbb{R}^d \times [0, T] \to \mathbb{R}$ is jointly continuous. Then $(S(\vartheta, \cdot))_{\vartheta\in\mathbb{R}^d}$ is a strong non–linear integrator.

Proof: W.l.o.g. we assume that $S_0^\vartheta = 0$ for all $\vartheta \in \mathbb{R}^d$. Due to Lemma A.10, the first item of Definition A.8 holds. To prove the second item, let $K > 0$ and define the sequence of stopping times $(\tau_n^K)_{n\geq 1}$ via

$$\tau_n^K := \inf\left\{t > 0 : \sup_{\|\vartheta\|_{\mathbb{R}^d}\leq K} |\mu(\vartheta, S(\vartheta,t))| \vee \sup_{\|\vartheta\|_{\mathbb{R}^d}\leq K} |\sigma(\vartheta, S(\vartheta,t))| \geq n\right\} \wedge T. \quad \text{(A.4)}$$

For all $K > 0$ the joint continuity ensures that $P(\tau_n^K < T) \to 0$ as $n \to \infty$. Let $\theta \in \mathbf{S}(\mathbb{R}^d)$, $\eta \in \mathbf{S}(\mathbb{R})$, and $\max\{\eta^*, \theta^*\} \leq K$. W.l.o.g. we may and do assume that

$$\theta(t) = \sum_{i=0}^{n} \theta_i \mathbb{1}_{(\tau_i,\tau_{i+1}]}(t) \qquad \text{and} \qquad \eta(t) = \sum_{i=0}^{n} \eta_i \mathbb{1}_{(\tau_i,\tau_{i+1}]}(t).$$

Then $\eta \cdot \int S(\theta, ds)$ is given by

$$\left(\eta \cdot \int S(\theta, ds)\right)_t =$$
$$= \sum_{i=0}^{n} \eta_i \mu(\theta_i, S(\theta_i,\tau_i)) \left(A_{\tau_{i+1}\wedge t} - A_{\tau_i\wedge t}\right) + \sum_{i=0}^{n} \eta_i \sigma(\theta_i, S(\theta_i,\tau_i)) \left(M_{\tau_{i+1}\wedge t} - M_{\tau_i\wedge t}\right).$$

If we set

$$\tilde{\mu}_u(\eta,\theta) := \sum_{i=0}^{n} \eta_i \mu(\theta_i, S(\theta_i,\tau_i)) \mathbb{1}_{(\tau_i,\tau_{i+1}]}(u),$$
$$\tilde{\sigma}_u(\eta,\theta) := \sum_{i=0}^{n} \eta_i \sigma(\theta_i, S(\theta_i,\tau_i)) \mathbb{1}_{(\tau_i,\tau_{i+1}]}(u),$$

we get

$$\left(\eta \cdot \int S(\theta, ds)\right)_t = \int_0^t \tilde{\mu}_u(\eta,\theta)\, dA_u + \int_0^t \tilde{\sigma}_u(\eta,\theta)\, dM_u.$$

Let $C > 0$ and $n \in \mathbb{N}$. A straightforward computation leads to

$$P\left(\sup_{t\leq T}\left|\left(\eta \cdot \int S(\theta, ds)\right)_t\right| > C\right)$$
$$\leq P\left(\tau_n^K < T\right) +$$
$$+ P\left(\sup_{t\leq T}\left|\int_0^{\tau_n^K\wedge t} \tilde{\mu}_u(\eta,\theta)\, dA_u\right| > \frac{C}{2}\right) + P\left(\sup_{t\leq T}\left|\int_0^{\tau_n^K\wedge t} \tilde{\sigma}_u(\eta,\theta)\, dM_u\right| > \frac{C}{2}\right).$$

Furthermore, by the definition of τ_n^K, Chebyshev's inequality and the Burkholder–Davis–Gundy–inequality, we can conclude that

$$P\left(\sup_{t\leq T}\left|\int_0^{\tau_n^K\wedge t}\tilde{\mu}_u(\eta,\theta)\,dA_u\right|>\frac{C}{2}\right)\leq P\left(A_T>\frac{C}{2nK}\right),$$

$$P\left(\sup_{t\leq T}\left|\int_0^{\tau_n^K\wedge t}\tilde{\sigma}_u(\eta,\theta)\,dM_u\right|>\frac{C}{2}\right)\leq \tilde{C}\frac{2nK}{C}E\left[\sqrt{[M]_T}\right].$$

Combining these results, we get

$$\begin{aligned}P&\left(\sup_{t\leq T}\left|\left(\eta\cdot\int S(\theta,ds)\right)_t\right|>C\right)\\&\leq P\left(\tau_n^K<T\right)+P\left(A_T>\frac{C}{2nK}\right)+\tilde{C}\frac{2nK}{C}E\left[\sqrt{[M]_T}\right].\end{aligned}$$

Since the r.h.s. of the inequality is independent of $\theta\in\mathbf{S}(\mathbb{R}^d)$, $\eta\in\mathbf{S}(\mathbb{R})$, and $P(\tau_n^K<T)\to 0$ as $n\to\infty$, the second item of Definition A.8 holds (for $c\to\infty$). Hence, it remains to prove the last item of Definition A.8. Let $(\theta^n)\subset\mathbf{S}(\mathbb{R}^d)$ be a Cauchy–sequence w.r.t. *ucp*–convergence that is uniformly bounded by K and set $\tau_m:=\tau_m^K$. For $N,n,m\in\mathbb{N}$ we have

$$\begin{aligned}d_{\mathcal{SM}}&\left(\int S\left(\theta^n,ds\right),\int S\left(\theta^m,ds\right)\right)\\&\leq P\left(\tau_N<T\right)+d_{\mathcal{SM}}\left(\left(\int S\left(\theta^n,ds\right)\right)^{\tau_N},\left(\int S\left(\theta^m,ds\right)\right)^{\tau_N}\right).\end{aligned}$$

As $P(\tau_N<T)\to 0$ for $N\to\infty$, it remains to prove that $((\int S\left(\theta^n,ds\right))^{\tau_N})_{n\geq 1}$ is a Cauchy–sequence in the semimartingale topology for all $N\geq 1$. To see this, note that $((\theta^n)^{\tau_N})_{n\geq 1}$ is a Cauchy–sequence in the *ucp*–topology and

$$\left(\int S\left(\theta^n,ds\right)\right)^{\tau_N}=\int S^{\tau_N}\left((\theta^n)^{\tau_N},ds\right),\qquad\forall\,n\geq 1,$$

for all $N\geq 1$. Due to the joint continuity of μ,σ, and $S(\vartheta,t)$, and the dominated convergence theorem for semimartingales, [39, III.13 Théorème], we can conclude that $(\int S^{\tau_N}\left((\theta^n)^{\tau_N},ds\right))_{n\geq 1}$ is a Cauchy–sequence in the semimartingale topology for all $N\geq 1$. Hence, the theorem is proven. □

A.2. Miscellaneous results

Definition A.19: *A function $f:\mathbb{R}_{\geq 0}\longrightarrow\mathbb{R}_{\geq 0}$ is called* subadditive*, if for all $x,y\in\mathbb{R}_{\geq 0}$*

$$f\left(x+y\right)\leq f\left(x\right)+f\left(y\right)$$

holds.

Lemma A.20: *If $f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is a concave function such that $f(0) = 0$, then f is subadditive.*

Proof: Let $x, y \in \mathbb{R}_{\geq 0}$ and w.l.o.g. $x > 0$. Due to the concavity of f, we get

$$\begin{aligned} f(x) &\geq \frac{y}{x+y} f(0) + \frac{x}{x+y} f(x+y), \\ f(y) &\geq \frac{x}{x+y} f(0) + \frac{y}{x+y} f(x+y). \end{aligned}$$

Since $f(0) = 0$, adding up the two inequalities gives the desired result. $\square$

Lemma A.21: *Let $(\mu_n)_{n\geq 1}$ be a bounded sequence of measures on $(\Omega, \mathcal{F})$. Then there exists a probability measure P on $(\Omega, \mathcal{F})$ such that*

$$\forall n \in \mathbb{N}: \ \mu_n \ll P.$$

Proof: Since all measures dominate the zero–measure, we assume w.l.o.g. that $\mu_n(\Omega) > 0$ for all $n \in \mathbb{N}$. Define

$$\tilde{\mu}_n := \frac{1}{\mu_n(\Omega)} \mu_n, \qquad n \in \mathbb{N}.$$

As $\tilde{\mu}_n \sim \mu_n$ for all $n \in \mathbb{N}$, it suffices to prove the lemma for the sequence $(\tilde{\mu}_n)_{n\geq 1}$ of probability measures. Choose $(\alpha_n)_{n\geq 1} \subset \mathbb{R}$ such that

$$\forall n \in \mathbb{N}: \ \alpha_n > 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = 1.$$

By construction the measure

$$P := \sum_{n=1}^{\infty} \alpha_n \tilde{\mu}_n$$

satisfies the desired properties. $\square$

Theorem A.22 (Continuous Mapping Theorem)**:** *Let (M_1, d_1), (M_2, d_2) be complete metric spaces and $(\Omega, \mathcal{F}, P)$ be a probability space. Further, let*

$$\begin{aligned} X, X_n : (\Omega, \mathcal{F}, P) &\longrightarrow (M_1, d_1), \qquad n \in \mathbb{N}, \\ h : (M_1, d_1) &\longrightarrow (M_2, d_2) \end{aligned}$$

be measurable functions such that $P(X \in D_h) = 0$, where $D_h \subset M_1$ denotes the set of discontinuity points of h. Then

$$X_n \to X, \text{ in } P\text{–probability} \quad \Longrightarrow \quad h(X_n) \to h(X), \text{ in } P\text{–probability}.$$

Proof: The proof of [53, 2.3 Theorem] translates directly in this slightly more general setting. □

Lemma A.23: *For all $n \in \mathbb{N}$ let g_n, g, f_n, f be measurable functions on the probability space $(\Omega, \mathcal{F}, P)$. Further, let $g_n, g \geq 0$ P–a.s. for all $n \in \mathbb{N}$,*

$$f_n g_n \to fg, \text{ in } P\text{–probability} \quad \text{and} \quad g_n \to g, \text{ in } P\text{–probability}.$$

Then

$$\mathbf{1}_{\{g \neq 0\}} f_n^2 g_n \to \mathbf{1}_{\{g \neq 0\}} f^2 g, \text{ in } P\text{–probability}.$$

Proof: Due to Theorem A.22 and the assumptions, we have

$$f_n^2 g_n^2 \to f^2 g^2, \text{ in } P\text{–probability} \quad \text{and} \quad \frac{1}{g_n + \mathbf{1}_{\{g=0\}}} \to \frac{1}{g + \mathbf{1}_{\{g=0\}}}, \text{ in } P\text{–probability}.$$

Applying Theorem A.22 again, we get

$$\frac{f_n^2 g_n^2}{g_n + \mathbf{1}_{\{g=0\}}} \to \frac{f^2 g^2}{g + \mathbf{1}_{\{g=0\}}}, \text{ in } P\text{–probability}.$$

This implies the desired result

$$\mathbf{1}_{\{g \neq 0\}} f_n^2 g_n = \mathbf{1}_{\{g \neq 0\}} \frac{f_n^2 g_n^2}{g_n + \mathbf{1}_{\{g=0\}}} \to \frac{f^2 g^2}{g + \mathbf{1}_{\{g=0\}}} \mathbf{1}_{\{g \neq 0\}} = \mathbf{1}_{\{g \neq 0\}} f^2 g, \text{ in } P\text{–probability}.$$

□

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Other Symbols

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Work Experience

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2003 – 2004 **Military Service**.

Scientific Talks

April 23rd 2015 **Oberseminar Finanzmathematik und Numerik**, *Martingale Decomposition Theorems and the Structure of No Arbitrage*, Christian–Albrechts–Universität zu Kiel, Germany.

Oct. 14th 2011 **PRisMa Day 2011**, *Brownian Moving Averages and Applications Towards Interest Rate Modelling*, Vienna University of Technology, Austria.