TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

DIPLOMARBEIT

A General Version of Hadwiger's Characterization Theorem

Ausgeführt am Institut für

Diskrete Mathematik und Geometrie
der Technischen Universität Wien

unter Anleitung von

Univ. Prof. Dr. Franz Schuster

durch

Irina Hafner, BSc
Wiener Gasse 124-126/2a/1
2380 Perchtoldsdorf

Datum

Unterschrift

Contents

Preface

In this thesis we study valuations on the space $\mathcal{K}(V)$ of convex bodies in an n-dimensional Euclidean vector space V taking values in an Abelian semigroup A, that is, maps $\phi : \mathcal{K}(V) \to A$ with the property that

$$\phi(K \cup L) + \phi(K \cap L) = \phi(K) + \phi(L)$$

whenever $K, L, K \cup L \in \mathcal{K}(V)$. In particular, we analyze the space of continuous, translation invariant, complex valuations, denoted by $\mathbf{Val}$. We will see that this space is a Banach space. The space of i-homogeneous valuations in $\mathbf{Val}$ is denoted by $\mathbf{Val}_i$. There is a natural continuous action of the general linear group $GL(n)$ on $\mathbf{Val}$ and thus $\mathbf{Val}_i$ becomes an $SO(n)$ module. Hence, it is possible to decompose the space $\mathbf{Val}_i$ into a sum of irreducible representations of $SO(n)$. This thesis contains the full proof of this statement and the proof of a reformulation which can be seen as a Hadwiger-type characterization of continuous, translation invariant and $SO(n)$-equivariant tensor valuations of degree i. The proof we present is taken from S. Alesker, A. Bernig and F. Schuster [3].

In Chapter 1, we summarize basic facts about representation theory of compact Lie groups. We introduce the notion of representations and give some useful properties of them. For example, it will be shown that arbitrary representations can be written as a direct sum of irreducible representations.

Afterwards characters of representations are defined and we will see that they completely determine representations. Then the maximal torus of a Lie group is introduced and we also give a short overview of infinite-dimensional representations. For more information on this topic we recommend [18] and [9].

The purpose of Chapter 2 is to classify irreducible representations. Therefore we introduce weights and roots of Lie groups and compute them for the special orthogonal group, $SO(n)$. In the second part of this chapter we define simple roots and Weyl chambers, which will be important in the upcoming chapters. Finally, we discuss a correspondence between those notions.

In Chapter 3, we define highest weights, which will be the crucial tool in the classification process of irreducible representations. We will see that irreducible representations are completely determined by their highest weights and we present a classification of them. There is also a formula to calculate the characters of irreducible representations corresponding to the highest weights. We refer to [18] and recommend [11] for details.

If G is a compact Lie group and H a closed subgroup, it is easy to obtain a representation of H, denoted by $\operatorname{Res}^G_H \Theta$, from a representation Θ of G by restriction. It is also possible to obtain a representation of G from a given representation Γ

of H. This is called an induced representation and is denoted by $\operatorname{Ind}_H^G \Gamma$. The Frobenius Reciprocity Theorem states that if Γ is a representation of H and Θ is a representation of G, thus the following holds

$$\operatorname{Hom}_G(\Theta, \operatorname{Ind}_H^G \Gamma) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G \Theta, \Gamma).$$

In Chapter 4 we collect the background for a proof of this theorem. Then we modify the theorem and get information about the multiplicities of certain representations which will be important in the proof of the main theorem. In this chapter we also state a Branching Theorem for $SO(n)$ and prove it. For further information we recommend [13].

In the first part of Chapter 5, we state basic facts about valuations. Then we discuss the notion of rectifiable sets and explain the concept of integration of differential forms on rectifiable sets. The definition of the normal cycle is given and we will see how it can be interpreted as an integral current. It is possible to show that smooth translation invariant valuations can be obtained by integrating a differential form over the normal cycle,

$$K \mapsto \int\limits_{nc(K)} \omega.$$

Since this correspondence is not injective, we are interested in the kernel of the map

$$\omega \mapsto [K \mapsto \int\limits_{nc(K)} \omega].$$

Therefore, we introduce in the second part of this chapter, contact manifolds and the Rumin operator, which will lead to the Kernel Theorem of A. Bernig and L. Bröcker [8]. We also define a subspace of differential forms, the so called primitive forms, and show that $\mathbf{Val}_i^\infty$ is part of an exact sequence of these primitive forms. For the required background from differential geometry and the Rumin operator, we recommend [5]. For the material on primitive forms confer [7].

Finally in Chapter 6, the main theorem is proved, using all the concepts and important theorems, which were developed in the previous chapters. We also show the equivalence with a Hadwiger-type characterization theorem. The results of this chapter are taken from [3].

Acknowledgements

I would like to express my gratitude to my advisor Prof. Franz Schuster, who first introduced me to the topic of integral geometry. Throughout my master's studies he always ensured that I would be well-prepared to write my master's thesis by suggesting the right courses. I am also grateful for his patience, his motivation and his expertise provided during the writing process.

Apart from my advisor, I would like to thank Astrid Berg, who supported me a lot by explaining her master's thesis and answering arising questions. She also helped me to stay calm during harder times.

My gratitude also goes to my school teacher, Mag. Alexej Tajmel, who sparked my interest in mathematics by his great teaching.

I am also grateful to my partner, Armin Zechmeister, who supported me throughout the course of my studies. I am aware that this has not always been easy, especially on days prior to exams!

Last but not least, I would like to thank my friends and family. Without their cheerfulness, encouragement and patience this thesis would not have been possible.

Irina Hafner

1 Representation Theory

In this chapter we give a short overview of representation theory of compact Lie groups. We begin with the definition of a representation of a Lie group, then we explain how characters of representations are defined and what we are referring to when we talk about maximal tori of Lie groups. At the end of the chapter, we describe how to work with infinite-dimensional representations, since we will need them in the final part of this work.

This chapter is just a short outline of a wide-ranging theory, thus we will give only very few proofs. All definitions and theorems in this chapter are taken from [18]. For readers who are interested in the proofs which have been omitted, we recommend [18] and [9].

1.1 Basics from Representation Theory of Compact Lie Groups

In this section G denotes an n-dimensional compact Lie group and V a finite-dimensional complex vector space. We first consider finite-dimensional representations, but later on we will rework the definitions a little bit to include the infinite-dimensional setting, which is based on a reduction to the finite-dimensional theory.

Definition 1.1.1. *A representation of a Lie group G on a finite-dimensional complex vector space V is a pair (V, ρ), where ρ is a homomorphism $\rho : G \to GL(V)$ from the Lie group G to the Lie group of automorphisms of V.*

Most of the time, if the meaning is clear, we denote a representation (V, ρ) simply by the letter ρ or the letter V. Sometimes we call V a G-module. It is also common to write gv or $g \cdot v$ for $(\rho(g))(v)$. In this section we also write just representation instead of finite-dimensional representation.

Example.
As first example we define a representation of $SO(n)$, the special orthogonal group. To this end, let $P_m(\mathbb{R}^n)$ denote the vector space of complex-valued polynomials on $\mathbb{R}^n$ which are homogeneous of degree m. We define an action of $SO(n)$ on $P_m(\mathbb{R}^n)$ in the following way

$$(g \cdot p)(x) = p(g^{-1}x) \quad \text{for } g \in SO(n), p \in P_m(\mathbb{R}^n), x \in \mathbb{R}^n.$$

Smoothness and invertibility are clear. It remains to check whether the defined action is a homomorphism. This follows from

$$[g_1 \cdot (g_2 \cdot p)](x) = (g_2 \cdot p)(g_1^{-1}x) = p(g_2^{-1}g_1^{-1}x) = p\left((g_1g_2)^{-1}x\right) = [(g_1g_2) \cdot p](x).$$

Every element of $P_m(\mathbb{R}^n)$ is a linear combination of elements of the form $x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}$ where $k_i \in \mathbb{N}$ and $k_1 + k_2 + \cdots + k_n = m$. Thus, $\dim P_m(\mathbb{R}^n) = \binom{m+n-1}{m}$ and the action defined above is an $\binom{m+n-1}{m}$-dimensional representation of $SO(n)$.

Definition 1.1.2. *Let G be a complex Lie group and (V,ρ), (V',ρ') two representations of G.*

i) A function $f \in \mathrm{Hom}(V,V')$ is called G-equivariant if $f(g\cdot v) = g\cdot f(v)$. Here, $\mathrm{Hom}(V,V')$ is the set of all linear maps from V to V'.

ii) The set of all G-equivariant maps is denoted by $\mathrm{Hom}_G(V,V')$

iii) Two representations are called equivalent, $V \cong V'$, if there is a bijective G-equivariant map from V to V'.

It is easy to construct many new representations if at least one is already known. Sometimes this is very useful and therefore we state how a given action carries over to different vector spaces.

Definition 1.1.3. *Let V and W be representations of a Lie group G. Then G acts in the following ways:*

i) on $V \oplus W$ by $g(v,w) = (gv, gw)$;

ii) on $V \otimes W$ by $g\sum v_i \otimes w_j = \sum gv_i \otimes gw_j$;

iii) on $\mathrm{Hom}(V,W)$ by $(gT)(v) = g[T(g^{-1}v)]$;

iv) on $\bigotimes^k V$ by $g\sum v_{i_1}\otimes\cdots\otimes v_{i_k} = \sum (gv_{i_1})\otimes\cdots\otimes(gv_{i_k})$;

v) on $\bigwedge^k V$ by $g\sum v_{i_1}\wedge\cdots\wedge v_{i_k} = \sum (gv_{i_1})\wedge\cdots\wedge(gv_{i_k})$;

vi) on $S^k(V)$ by $g\sum v_{i_1}\cdots v_{i_k} = \sum (gv_{i_1})\cdots(gv_{i_k})$;

vii) on V^* by $(gT)(v) = T(g^{-1}v)$.

Now that we know that there are many representations for a given group, we would like to find somehow the smallest possible representations with which we can build all the others.

Definition 1.1.4. *Let G be a Lie group and V a representation of G. We call a subspace $U \subseteq V$ G-invariant or a submodule, if $g\cdot u \in U$ for all $g \in G, u \in U$. A nonzero representation V is irreducible, if it has only trivial submodules, that is, $\{0\}$ and V. It is called reducible if there is a proper G-invariant subspace of V.*

The next result for irreducible representations is very useful and known as Schur's Lemma.

Theorem 1.1.5 (Schur's Lemma)**.** *Let G be a Lie group and let V and W be irreducible representations. Then the following statements hold.*

i) A G-equivariant map $f: V \to W$ is either zero or an isomorphism.

ii) Every G-equivariant map $f: V \to V$ is of the form $f = \lambda \operatorname{Id}$ for some $\lambda \in \mathbb{C}$.

iii)

$$\dim \operatorname{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{if } V \ncong W. \end{cases}$$

Proof. Since V is irreducible, $\ker f = V$ or $\ker f = \{0\}$. In the first case, $f = 0$. In the second case it follows that f is injective. Thus the image of f is a nonzero submodule of W, and therefore, since W is irreducible, all of W. We conclude that f is an isomorphism, which proves (i).

Next assume, that $f: V \to V$ is nontrivial and let λ be an eigenvalue of f. Denote the corresponding eigenspace by $U = \{u \in V | f(u) = \lambda u\}$. It is obvious that U is a G-invariant submodule, hence $U = V$, which gives (ii).

Now we prove the last statement. We already know from (i) that there exists a nonzero $f \in \operatorname{Hom}_G(V, W)$ if and only if $V \cong W$. So, if $V \cong W$, fix $f_0 \in \operatorname{Hom}_G(V, W)$ and assume that there is another $f \in \operatorname{Hom}_G(V, W)$. Then $f \circ f_0^{-1} \in \operatorname{Hom}_G(V, V)$. From (ii) we know that $f \circ f_0^{-1} = \lambda \operatorname{Id}$ for a $\lambda \in \mathbb{C}$, which finishes the proof. □

Definition 1.1.6. *Let G be a Lie group. We call a representation V of G unitary, if there exists a G-invariant Hermitian inner product $V \times V \to \mathbb{C} : (u, v) \mapsto \langle u, v\rangle$ which is G-invariant, that is, $\langle g \cdot u, g \cdot v\rangle = \langle u, v\rangle$ for all $g \in G$ and $u, v \in V$.*

The next two statements make clear why we work only with compact Lie groups in this thesis: These groups are nicely behaved.

Theorem 1.1.7 ([18, p. 37])**.** *Every representation of a compact Lie group is unitary.*

Definition 1.1.8. *A representation of a Lie group is called completely reducible if it is a direct sum of irreducible submodules.*

Corollary 1.1.9. *Finite-dimensional representations V of compact Lie groups are completely reducible, that is, $V \cong \bigoplus_{i=1}^N n_i V_i$, where every V_i is a different irreducible representation and $n_i V_i$ denotes the direct sum of n_i copies of V_i.*

We will see that a similar statement even holds in the infinite-dimensional setting. By the previous corollary, we just have to study irreducible representations and their multiplicities, to understand any representation of a compact Lie group G. We start with the calculation of the n_i's of the previous corollary. To this end, we need one more definition.

Definition 1.1.10. *Let G be a Lie group. We denote the set of equivalence classes of irreducible representations of G by $\widehat{G}$. For each $[\rho] \in \widehat{G}$ we choose a representative representation (E_ρ, ρ). Let V be any representation of G. For an equivalence class $[\rho] \in \widehat{G}$, we write $V_{[\rho]}$ for the largest subspace of V with $V_{[\rho]} = \bigoplus_i V_i$ where each $V_i \cong E_\rho$. This submodule is called the ρ-isotypic component of V. The multiplicity m_ρ of ρ in V is $\frac{\dim V_{[\rho]}}{\dim E_\rho}$, that is, $V_{[\rho]} \cong m_\rho E_\rho$.*

Note that the direct sum in Definition 1.1.10 is the sum of *all* submodules of V equivalent to E_ρ.

Now we can calculate the n_i's defined in the previous corollary:

Theorem 1.1.11 ([18, p.39], Canonical Decomposition)**.** *Let G be a compact Lie group and V a representation of G. Then*

- *there is a G-equivariant isomorphism*

$$\iota_\rho : \mathrm{Hom}_G(E_\rho, V) \otimes E_\rho \to V_{[\rho]}$$

induced by the mapping $f \otimes v \mapsto f(v)$ for $f \in \mathrm{Hom}_G(E_\rho, V)$ and $v \in V$. In particular, the multiplicity of ρ is $m_\rho = \dim \mathrm{Hom}_G(E_\rho, V)$, and

- *there is a G-equivariant isomorphism*

$$\widetilde{\iota_\rho} : \bigoplus_{[\rho]\in\widehat{G}} \mathrm{Hom}_G(E_\rho, V) \otimes E_\rho \to V = \bigoplus_{[\rho]\in\widehat{G}} V_{[\rho]}.$$

Example.

We want to calculate an irreducible representation of $SO(n)$. Therefore remember the vector space $P_m(\mathbb{R}^n)$ from the first example and denote the subspace of all harmonic polynomials of degree m by $H_m(\mathbb{R}^n)$, that is, $H_m(\mathbb{R}^n) = \{P \in P_m(\mathbb{R}^n) | \Delta P = 0\}$. It follows easily, that the action we defined before, descends to an action of $SO(n)$ on $H_m(\mathbb{R}^n)$. This follows from the fact that for $P \in H_m(\mathbb{R}^n)$ and $g \in SO(n)$ we have $\Delta(g \cdot P) = g \cdot (\Delta P) = 0$, so that $g \cdot P \in H_m(\mathbb{R}^n)$. Thus we constructed a representation of $SO(n)$ on the harmonic polynomials $H_m(\mathbb{R}^n)$. In fact, this representation is irreducible. See [18, p.42] for the calculation.

1.2 Characters

In this section we define the character of a representation and give several properties of this function. We will also see that characters (up to equivalence) completely determine representations.

Definition 1.2.1. *Let G be a Lie group and (V, ρ) a finite-dimensional representation of G. The function $\chi_V : G \to \mathbb{C}$, defined by $\chi_V(g) = \mathrm{tr}(\rho(g))$, is called the character of V.*

Theorem 1.2.2. *Let V, V_i be representations of a compact Lie group G. The characters χ_V and χ_{V_i} have the following properties*

- $\chi_V(e) = \dim V$,
- *If $V_1 \cong V_2$, then $\chi_{V_1} = \chi_{V_2}$,*
- $\chi_V(hgh^{-1}) = \chi_V(g)$ *for $g, h \in G$,*

- $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$,
- $\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$,
- $\chi_{V^*}(g) = \chi_{\overline{V}}(g) = \overline{\chi_V(g)} = \chi_V(g^{-1})$.

The third property shows that the character is a so called class function.
Later we also need matrix coefficients, therefore we define them here.

Definition 1.2.3. *Let G be a Lie group, V a unitary representation, and $\langle \cdot,\cdot \rangle$ the corresponding Hermitian inner product. A matrix coefficient is a smooth map $\alpha : G \to \mathbb{C}$, defined by*

$$\alpha(g) = \langle g \cdot v, w \rangle$$

for $v, w \in V$.

We denote the set of all matrix coefficients by $MC(G)$. Note, that if $\{v_1, \dots, v_n\}$ is an orthonormal basis of V, then the matrix representation of $\alpha(g)$ is given by $(\langle g \cdot v_i, v_j \rangle)_{i,j}$. Moreover, we have $\chi(g) = \sum_i \langle g \cdot v_i, v_i \rangle$.

Definition 1.2.4. *Let V be a finite-dimensional representation of a compact Lie group G. Define V^G as the isotypic component of V corresponding to the trivial representation, that is, $V^G = \{v \in V |\ g \cdot v = v$ for all $g \in G\}$.*

The next theorem is a very important statement in character theory and is a special case of Schur's orthogonality relations for matrix coefficients.

Theorem 1.2.5 ([18, p. 50])**.** *Let V, W be representations of a compact Lie group G. Then*

$$\int_G \chi_V(g)\overline{\chi_W(g)}dg = \dim \mathrm{Hom}_G(V, W),$$

where dg denotes integration with respect to the Haar measure.
In particular, $\int_G \chi_V(g)dg = \dim V^G$, and if V, W are irreducible, we have

$$\int_G \chi_V(g)\overline{\chi_W(g)}dg = \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{if } V \ncong W. \end{cases}$$

Thus, V is up to equivalence determined by its character, that is, $\chi_V = \chi_W$ if and only if $V \cong W$. Moreover, V is irreducible if and only if $\int_G |\chi_V(g)|^2 dg = 1$.

Finally, we state a well known important theorem, the so called Peter-Weyl Theorem, and a nice corollary.

Theorem 1.2.6 ([10], Peter-Weyl Theorem)**.** *Let G be a compact Lie group. The space of matrix coefficients of finite-dimensional representations of G, $MC(G)$, is dense in $C(G)$, the set of continuous functions on G.*

From this theorem one can obtain the following corollary.

Corollary 1.2.7. *Any compact Lie group G is isomorphic to a closed subgroup of a unitary group $U(n)$ for sufficiently large n.*

1.3 Infinite-Dimensional Representations

Sometimes it is not enough to consider finite-dimensional representations. Therefore we modify our previous definitions a little bit, but none of this will effect the finite-dimensional setting. For us, the interesting infinite-dimensional representations will be the unitary representations of Lie groups on Banach spaces. The material for this section is taken from [10].

In the next definition $\mathrm{Hom}(V, V')$ denotes the set of continuous linear transformations form V to V' and $GL(V)$ is the set of invertible elements of $\mathrm{Hom}(V, V)$.

Definition 1.3.1.
- *A representation of a Lie group G on a topological vector space V is a pair (V, ρ), where $\rho : G \to GL(V)$ is a homomorphism and the map $G \times V \to V$ defined by $(g, v) \mapsto \rho(g)v$ is continuous.*
- *If (V, ρ) and (V', ρ') are representations on topological vector spaces, $f \in \mathrm{Hom}(V, V')$ is called a G-equivariant function, if $f \circ \rho = \rho' \circ f$. Again, the set of all G-equivariant functions is denoted by $\mathrm{Hom}_G(V, V')$.*
- *We say V and V' are equivalent if there is a bijective G-equivariant function from V to V'. A subspace $U \subseteq V$ is G-invariant, if $g \cdot u \in U$ for all $g \in G, u \in U$. If U is closed, we call U a submodule.*
- *A nonzero representation V is called irreducible, if the only closed G-invariant subspaces are $\{0\}$ and V. It is called reducible if there is a proper closed G-invariant subspace of V.*

Next we want to find a canonical decomposition similar to Theorem 1.1.11 for unitary representations on Banach spaces. Therefore we need the following definition.

Definition 1.3.2. *Let V be an infinite-dimensional representation of a Lie group G on a complex Banach space.*

- *We call a vector $v \in V$ G-finite if it is contained in a finite-dimensional G-invariant subspace of V. Let V^f denote the vector space spanned by all G-finite vectors.*
- *For a finite-dimensional irreducible complex representation W of G, define the W-isotypic component $V_W \subset V$ as the vector space of all $v \in V$ for which there exists a G-equivariant function $\phi : W \to V$ with $v \in \mathrm{im}(\phi)$.*

It is easy to see that V_W is a linear subspace of V. Fix $v \in V_W$ and consider the corresponding G-equivariant function ϕ. The image of this map is a finite-dimensional G-invariant subspace of V containing v, thus $V_W \subset V^f$. For $v \in V^f$ there exists a finite-dimensional G-invariant subspace W of V which contains v. Since G is a compact Lie group, W is a direct sum of irreducible submodules, by Corollary 1.1.9. This implies that v can be written as a linear combination of vectors contained in some isotypic component. Hence, $V^f = \bigoplus_{W \in \hat{G}} V_W$. As before, $\hat{G}$ denotes the set of equivalence classes of finite-dimensional irreducible representations of G. Hence, the vector space of all G-finite vectors equals the subspace spanned by all finite linear combinations of elements of the subspaces V_W.

Theorem 1.3.3. *Let V be a representation of a compact Lie group G on a complex Banach space. The vector space of all G-finite vectors V^f is a dense subspace of V, that is,* $\mathrm{cl}(V^f) = V$.

Proof. Let $f : G \to \mathbb{C}$ be a continuous function. For a fixed $v \in V$ the map $G \to V$ defined by $h \mapsto f(h)(hv)$ is continuous. Thus, we can define

$$\begin{aligned} T_f : V &\to V \\ v \mapsto T_f(v) &:= \int\limits_G f(h)hv\ dh. \end{aligned}$$

In the following we want to show that $T_f(v) \in V^f$ for all $v \in V$ and that $T_f(v_0)$ is ε-close to $v_0 \in V$.

By the G-invariance of the integral, we obtain

$$g \cdot T_f(v) = \int\limits_G f(h)(gh)v\ dh = \int\limits_G f(g^{-1}h)hv\ dh = T_{g\cdot f}(v).$$

Thus, the subspace $\{T_f(v)|v \in V\} \subset V$ is G-invariant and if we take $f \in C(G)^f$ we obtain $T_f(v) \in V^f$.

Let $v_0 \in V$ be fixed. By the definition of infinite-dimensional representation, the action of G on V is continuous. This implies that there is a neighborhood U of e in G, such that

$$\|gv_0 - v_0\| < \frac{\varepsilon}{2}$$

for all $g \in U$. Using a partition of unity, we can construct a smooth function $\mu : G \to \mathbb{R}$ with support contained in U and the properties that $\mu(g) \geq 0$ for all $g \in G$ and $\mu(e) > 0$. We can assume that $\mu(g) = \mu(g^{-1})$, thus $\int_G \mu(g)dg > 0$. By rescaling μ we obtain $\int_G \mu(g)dg = 1$. Since the product $\mu(h)hv_0$ vanishes unless $h \in U$, we obtain the following inequality

$$\|\mu(h)hv_0 - \mu(h)v_0\| < \frac{\varepsilon}{2}\mu(h).$$

Integration yields

$$\|T_\mu(v_0) - v_0\| < \frac{\varepsilon}{2}. \tag{1.1}$$

Clearly, every linear combination of matrix coefficients lies in $C(G)^f$. Thus, by the Peter-Weyl Theorem 1.2.6, we find a function $f \in C(G)^f$ arbitrary close to μ. Since, $g \mapsto \|gv_0\|$ is continuous and G is a compact Lie group, there exists $M \in \mathbb{R}$ such that $\|gv_0\| < M$ for all $g \in G$. Now we can choose $f \in C(G)^f$ in a way that $\|f - \mu\| < \varepsilon/(2M)$. Consider the product $f(h)hv_0 - \mu(h)hv_0$. It lies in the ball of radius $\varepsilon/2$ around zero for all $h \in G$. Thus, we obtain by integration

$$\|T_f(v_0) - T_\mu(v_0)\| < \frac{\varepsilon}{2}. \tag{1.2}$$

Combining (1.1) and (1.2) yields $\|T_f(v_0) - v_0\| < \varepsilon$, which finishes the proof. □

Corollary 1.3.4. *Let G be a compact Lie group. Any irreducible representation of G on a Banach space V is finite-dimensional.*

Proof. Assume V is an irreducible and infinite-dimensional representation. Consider a non-zero isotopic component of V. By definition, there exists a G-equivariant function ϕ for which the image is a finite-dimensional, G-invariant subspace of V. This contradicts the irreducibility of V. Hence, for any infinite-dimensional irreducible representation, all isotypic components are zero. Therefore, the space of finite vectors is empty, which is a contradiction to Theorem 1.3.3. □

1.4 Maximal Torus Theorem

In the last section we saw that every compact Lie group G is isomorphic to a Lie subgroup of $U(n)$, therefore it is possible to diagonalize each $g \in G$ using conjugation in $U(n)$. The main theorem in this section shows us that it is even possible to diagonalize each $g \in G$ using conjugation in G. Most of the results are taken from [18]. For more information about maximal tori we recommend [9]. We will start with a few basic definitions and notions.

Recall, that if G is a Lie group, we call it Abelian if $g_1 g_2 = g_2 g_1$ for all $g_i \in G$. A subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(n, \mathbb{C})$ is Abelian if $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$.

Definition 1.4.1. *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. A subgroup $T \subset G$ which is maximal, connected and Abelian is called a maximal torus. A maximal Abelian subalgebra of $\mathfrak{g}$ is called a Cartan subalgebra.*

The next theorem shows that maximal tori and Cartan subalgebras are closely related.

Theorem 1.4.2. *Let G be a compact Lie group, $\mathfrak{g}$ its Lie algebra, T a connected Lie subgroup of G and $\mathfrak{t}$ the corresponding Lie algebra. Then T is a maximal torus if and only if $\mathfrak{t}$ is a Cartan subalgebra.*

Now we want to determine a maximal torus of $SO(n)$.

Example.

First we look at $SO(2n)$. It is well known that the Lie algebra of $SO(2n)$ is given by $\mathfrak{so}(2n) = \{X \in \mathfrak{gl}(2n, \mathbb{R}) | X^t = -X\}$. Observe that the condition $X^t = -X$ implies that $\operatorname{tr} X = 0$. We define T as the set of the following block diagonal matrices, where $\theta_i \in \mathbb{R}$:

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & \\ -\sin\theta_1 & \cos\theta_1 & & & \\ & & \ddots & & \\ & & & \cos\theta_n & \sin\theta_n \\ & & & -\sin\theta_n & \cos\theta_n \end{pmatrix}.$$

Clearly, T is a maximal connected subgroup of G and if we take $T_1, T_2 \in T$ one can easily check that $T_1T_2 = T_2T_1$. It basically follows from the commutativity of multiplication and addition. Thus T is a maximal torus of $SO(2n)$.

Now we define $\mathfrak{t}$ as the set of block diagonal matrices of the following form:

$$\begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta_n \\ & & & -\theta_n & 0 \end{pmatrix}$$

where $\theta_i \in \mathbb{R}$. The set $\mathfrak{t}$ is clearly the Lie algebra corresponding to T and a maximal subalgebra of $\mathfrak{so}(2n)$. If we take $t_1, t_2 \in \mathfrak{t}$ we obtain $[t_1,t_2] = 0$. Hence $\mathfrak{t}$ is a Cartan subalgebra which corresponds to the maximal torus T.

Next we are interested in the maximal torus of the odd-dimensional special orthogonal group, $SO(2n+1)$. As before the corresponding Lie algebra is given by $\mathfrak{so}(2n+1) = \{X \in \mathfrak{gl}(2n+1,\mathbb{R}) | X^t = -X\}$. Let T be the set of block diagonal matrices defined by

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & \\ & & \ddots & & & \\ & & & \cos\theta_n & \sin\theta_n & \\ & & & -\sin\theta_n & \cos\theta_n & \\ & & & & & 1 \end{pmatrix}$$

for $\theta_i \in \mathbb{R}$ and define $\mathfrak{t}$ as the set of block diagonal metrices of the form

$$\begin{pmatrix} 0 & \theta_1 & & & & \\ -\theta_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \theta_n & \\ & & & -\theta_n & 0 & \\ & & & & & 0 \end{pmatrix}$$

where $\theta_i \in \mathbb{R}$. Then by the same argument as above T is a maximal torus and $\mathfrak{t}$ is its corresponding Cartan subalgebra.

The main theorem about maximal tori will be stated without a proof.

Theorem 1.4.3 (Maximal Torus Theorem). *Let G be a compact connected Lie group, $\mathfrak{t}$ a maximal torus of G, and $g_0 \in G$.*

- *There exists $g \in G$ such that $gg_0g^{-1} \in T$.*
- *The exponential map is surjective, that is, $G = \exp(\mathfrak{g})$.*

Since the conjugate of a maximal torus is a maximal torus, the previous theorem shows that every element of G lies in some maximal torus. Moreover, every two maximal tori of a Lie group G are conjugate. Thus, any upcoming construction that depends on a maximal torus T is independent of the choice of T up to an inner automorphism of G.

2 Weight Theory

In the previous chapter we saw that any Banach space representation of a compact Lie group G can be written as the closure of a direct sum of irreducible representations. Now, we want to find a classification of irreducible representations. This chapter provides the theory which, in the end, will give us a great deal of information about irreducible representations and therefore about Lie groups and Lie algebras.

But before we can do all this, we need a little preparation, which happens in the first part of this chapter. In the second part we discuss the important notions of weights and roots. We also calculate them for the special orthogonal group. The end of this chapter refines the weight theory and gives us nice properties of certain roots.

All definitions, theorems, proofs and examples are taken from [18].

2.1 Representations of Lie Algebras

Until now we only considered representations of Lie groups, but we can also define representations of Lie algebras in a similar way.

Definition 2.1.1.
- *A pair (V,ψ) is a representation of a Lie algebra $\mathfrak{g}$ of a Lie subgroup of $GL(n,\mathbb{C})$, if V is a finite-dimensional complex vector space and $\psi : \mathfrak{g} \to \operatorname{End}(V)$ is a linear map, satisfying $\psi([X,Y]) = \psi(X) \circ \psi(Y) - \psi(Y) \circ \psi(X)$ for $X, Y \in \mathfrak{g}$.*
- *We call a representation (V,ψ) irreducible if the only $\psi(\mathfrak{g})$-invariant subspaces of V are $\{0\}$ and V, otherwise we call it reducible.*

As in the Lie group case we often write just ψ or V for the representation (V,ψ) and $X \cdot v$ denotes the action $\psi(X)v$ for every $X \in \mathfrak{g}, v \in V$.

There is a close connection between representations of a Lie group and representations of the corresponding Lie algebra.

Theorem 2.1.2. *Let G be a Lie subgroup of $GL(n,\mathbb{C})$ and (V,ρ) a finite-dimensional representation of G. Then $(V, d\rho)$ is a representation of the corresponding Lie algebra $\mathfrak{g}$, where the differential of ρ is defined as $d\rho(X) = \frac{d}{dt}\rho(e^{tX})|_{t=0}$ for $X \in \mathfrak{g}$. The representation $d\rho$ satisfies $e^{d\rho X} = \rho(e^X)$.*

Now assume that G is connected. Then ρ is completely determined by $d\rho$ and a subspace $W \subseteq V$ is G-invariant if and only if it is $d\rho(\mathfrak{g})$-invariant. Therefore, V is an irreducible representation of the Lie group G if and only if it is an irreducible representation of the Lie algebra $\mathfrak{g}$.

For connected compact G, V is irreducible if and only if just scalar multiples of the identity map are endomorphism of V commuting with all the operators $d\rho(\mathfrak{g})$.

As in the case of representations of Lie groups we can easily construct new representations from given ones. Actually we just take the differentials of the Lie group representations listed in Definition 1.1.3 and obtain the following

Definition 2.1.3. *Let V and W be representations of a Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ acts in the following ways:*

i) *on $V \oplus W$ by $X(v, w) = (Xv, Xw)$;*

ii) *on $V \otimes W$ by $X \sum v_i \otimes w_j = \sum X v_i \otimes w_j + \sum v_i \otimes X w_j$;*

iii) *on $\mathrm{Hom}(V, W)$ by $(XT)(v) = XT(v) - T(XV)$;*

iv) *on $\bigotimes^k V$ by $X \sum v_{i_1} \otimes \cdots \otimes v_{i_k} = \sum (X v_{i_1}) \otimes \cdots \otimes (v_{i_k}) + \cdots + \sum (v_{i_1}) \otimes \cdots \otimes (X v_{i_k})$;*

v) *on $\bigwedge^k V$ by $X \sum v_{i_1} \wedge \cdots \wedge v_{i_k} = \sum (X v_{i_1}) \wedge \cdots \wedge (v_{i_k}) + \cdots + \sum (v_{i_1}) \wedge \cdots \wedge (X v_{i_k})$;*

vi) *on $S^k(V)$ by $X \sum v_{i_1} \cdots v_{i_k} = \sum (X v_{i_1}) \cdots (v_{i_k}) + \cdots + \sum (v_{i_1}) \cdots (X v_{i_k})$;*

vii) *on V^* by $(XT)(v) = -T(Xv)$.*

2.2 Weights and Roots

We want to find an analogue to the character of a Lie group representation, that is, a function which determines a Lie algebra representation. To do this we need a few new notions and definitions, in particular the definition of weights of a representation.

Until now we worked with complex vector spaces V and Lie algebras $\mathfrak{g}$ which were vector spaces over $\mathbb{R}$, but for the upcoming theory we need that the Lie algebras are vector spaces over $\mathbb{C}$. Therefore we need the complexification of a Lie algebra.

Definition 2.2.1. *Let $\mathfrak{g}$ be the Lie Algebra of a Lie subgroup of $GL(n, \mathbb{C})$. The complexification of $\mathfrak{g}$, $\mathfrak{g}_{\mathbb{C}}$, is defined as $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. The Lie bracket on $\mathfrak{g}$ is extended to $\mathfrak{g}_{\mathbb{C}}$ by $\mathbb{C}$-linearity. Extending the domain of the representation ψ of $\mathfrak{g}$ by $\mathbb{C}$-linearity we obtain a representation of $\mathfrak{g}_{\mathbb{C}}$. (V, ψ) is said to be irreducible under $\mathfrak{g}_{\mathbb{C}}$ if there are no proper $\psi(\mathfrak{g}_{\mathbb{C}})$-invariant subspaces of V.*

A matrix $X \in \mathfrak{gl}(n, \mathbb{C})$ can be written as the sum of a skew-Hermitian matrix X_1 and a Hermitian matrix X_2, that is,

$$X = X_1 + X_2 \text{ with } X_1 = \frac{1}{2}(X - X^H), \;\; X_2 = \frac{1}{2}(X + X^H)$$

Remember, X^H denotes the conjugate transpose of X. Because of this we can write $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) \oplus i\mathfrak{u}(n)$. Thus, if $\mathfrak{g}$ is the Lie algebra of a compact Lie group G, we can identify $\mathfrak{g}_{\mathbb{C}}$ with $\mathfrak{g} \oplus i\mathfrak{g}$. The Lie bracket is the standard Lie bracket from $\mathfrak{gl}(n, \mathbb{C})$. Moreover, we have the following identities:

$$\begin{aligned}
\mathfrak{u}(n)_{\mathbb{C}} &= \mathfrak{gl}(n, \mathbb{C}), \\
\mathfrak{su}(u)_{\mathbb{C}} &= \mathfrak{sl}(n,\mathbb{C}), \\
\mathfrak{so}(n)_{\mathbb{C}} &= \mathfrak{so}(n,\mathbb{C}) = \{X \in \mathfrak{sl}(n, \mathbb{C}) | X^t = -X\}.
\end{aligned}$$

A subspace W of a complex vector space V is a complex subspace, therefore W is $\psi(\mathfrak{g})$-invariant if and only if it is $\psi(\mathfrak{g}_{\mathbb{C}})$-invariant and we obtain the following lemma.

Lemma 2.2.2. *Let $\mathfrak{g}$ be the Lie algebra of a Lie subgroup of $GL(n,\mathbb{C})$ and let (V,ψ) be a representation of $\mathfrak{g}$. Then V is irreducible under $\mathfrak{g}$ if and only if it is irreducible under $\mathfrak{g}_{\mathbb{C}}$.*

Let G be a compact Lie group and (V,ρ) a finite-dimensional representation of G. Take a Cartan subalgebra $\mathfrak{t}$ and consider its complexification, $\mathfrak{t}_{\mathbb{C}}$. Recall that every representation of a compact Lie group is unitary, that is, there exists a G-invariant inner product, $\langle\cdot,\cdot\rangle$, on V for which $d\rho$ is skew-Hermitian on $\mathfrak{g}$ and Hermitian on $i\mathfrak{g}$:

$$\begin{aligned} \left\langle\rho(e^{tX})Y_1,\rho(e^{tX})Y_2\right\rangle &= \langle Y_1,Y_2\rangle \qquad \left|\cdot\frac{d}{dt}\right|_{t=0}; \\ \langle d\rho(X)Y_1,Y_2\rangle+\langle Y_1,d\rho(X)Y_2\rangle &= 0. \end{aligned}$$

Thus $\mathfrak{t}_{\mathbb{C}}$ acts on V as a family of commuting normal operators. Since the spectral theorem holds for compact normal operators, all elements of $\mathfrak{t}_{\mathbb{C}}$ are simultaneously diagonalizable.

Definition 2.2.3. *Let G be a compact Lie group, (V,ρ) a finite-dimensional representation of G, and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. There is a finite set $\Delta(V)=\Delta(V,\mathfrak{t}_{\mathbb{C}})\subseteq\mathfrak{t}_{\mathbb{C}}^*$, called the weights of V, such that*

$$V=\bigoplus_{\alpha\in\Delta(V)}V_\alpha,$$

where

$$V_\alpha=\{v\in V|d\rho(H)v=\alpha(H),H\in\mathfrak{t}_{\mathbb{C}}\}$$

is nonzero. The direct sum is called the weight space decomposition of V with respect to $\mathfrak{t}_{\mathbb{C}}$.

There are some useful properties.

Theorem 2.2.4. *Let G be a compact Lie group, (V,ρ) a finite-dimensional representation of G, T a maximal torus of G, and $V=\bigoplus_{\alpha\in\Delta(V)}V_\alpha$ the weight space decomposition.*

- *For each weight $\alpha\in\Delta(V)$, α is purely imaginary on $\mathfrak{t}$ and is real-valued on $i\mathfrak{t}$.*
- *For $t\in T$, choose $H\in\mathfrak{t}$ such that $e^H=t$. Then $tv_\alpha=e^{\alpha(H)}v_\alpha$ for $v_\alpha\in V_\alpha$.*

Proof. Since $d\rho$ is skew-Hermitian on $\mathfrak{t}$ and Hermitian on $i\mathfrak{t}$, the corresponding eigenvalues are purely complex and purely real-valued respectively. So the first property follows. The second part follows from the fact that $\exp\mathfrak{t}=T$ and from $tv_\alpha=e^Hv_\alpha=(\rho(e^H))(v_\alpha)=(e^{d\rho(H)})v_\alpha=e^{\alpha(H)}v_\alpha$. $\square$

We can view α as an element of any of the following dual spaces $\mathfrak{t}_{\mathbb{C}}^*$, $(i\mathfrak{t})^*$ or $\mathfrak{t}^*$ due to the fact that $\alpha \in \Delta(V)$ is completely determined by its restriction to either $\mathfrak{t}$ or $i\mathfrak{t}$, because of $\mathbb{C}$-linearity. In the upcoming examples we will often switch between the dual spaces, since it is sometimes more convenient to work in a dual space different from $(\mathfrak{t}_{\mathbb{C}})^*$.

The weight space decomposition for the adjoint representation plays a special role. Before we give the definition, remember that $(\mathrm{Ad}, \mathfrak{g}_{\mathbb{C}})$ is a representation of $\mathfrak{g}$ with differential given by ad, when extended by $\mathbb{C}$-linearity from $\mathfrak{g}$ to $\mathfrak{g}_{\mathbb{C}}$.

Definition 2.2.5. *Let G be a compact Lie group and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. There is a finite set of nonzero elements $\Delta(\mathfrak{g}_{\mathbb{C}}) = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \in \mathfrak{t}_{\mathbb{C}}^*$, called the roots of $\mathfrak{g}_{\mathbb{C}}$, such that*

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})} \mathfrak{g}_{\alpha},$$

where $\mathfrak{g}_{\alpha} = \{Z \in \mathfrak{g}_{\mathbb{C}} | [H,Z] = \alpha(H)Z, H \in \mathfrak{t}_{\mathbb{C}}\}$ is nonzero. The direct sum is called the root space decomposition of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$.

Note that $\mathfrak{g}_0 = \mathfrak{t}_{\mathbb{C}}$, where $\mathfrak{g}_0 = \{Z \in \mathfrak{g}_{\mathbb{C}} | [H,Z] = 0, h \in \mathfrak{t}_{\mathbb{C}}\}$, since $\mathfrak{t}$ is a maximal Abelian subspace of $\mathfrak{g}$.

Example.

We want to determine the roots of $SO(n) = \{g \in O(n) | \det g = 1\}$. We already know how the corresponding Lie algebra and its Cartan subalgebra look like. Unfortunately it is very messy to work with these blockdiagonalmatrices. Thus we construct an isomorphic Lie group, which has a nice maximal torus.

First we define the following matrices:

$$T_{2m} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_m & I_m \\ iI_m & -iI_m \end{pmatrix}, \qquad E_{2m} = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix},$$

$$T_{2m+1} = \begin{pmatrix} T_{2m} & 0 \\ 0 & 1 \end{pmatrix}, \qquad E_{2m+1} = \begin{pmatrix} E_{2m} & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $E_n = T_n^t T_n$ and $\bar{T}_n = T_n^{-t}$.

Now, consider

$$SO(E_n) := \{g \in SL(n, \mathbb{C}) | \bar{g} = E_n g E_n, g^t E_n g = E_n\}.$$

It is easy to check that $SO(E_n)$ is a compact Lie subgroup of $SU(n)$ with corresponding Lie algebra $\mathfrak{so}(E_n)$ defined by

$$\mathfrak{so}(E_n) := \{X \in \mathfrak{gl}(n, \mathbb{C}) | \bar{X} = E_n X E_n, X^t E_n + E_n X = 0\}.$$

For the calculation of the root-space decomposition, we also need the complexified

Lie algebra

$$\mathfrak{so}(E_n,\mathbb{C}) := \{X \in \mathfrak{gl}(n,\mathbb{C}) | X^t E_n + E_n X = 0\}.$$

The map $\varphi : SO(n) \to SO(E_n); g \mapsto T_n^{-1} g T_n$, is an isomorphism between the Lie groups $SO(n)$ and $SO(E_n)$. Indeed, it is easy to see that φ is bijective and smooth. Because of

$$\varphi(g \cdot h) = T_n^{-1} g h T_n = T_n^{-1} g T_n T_n^{-1} h T_n = \varphi(g) \cdot \varphi(h),$$

it is also a homomorphism.

The map $\psi : SO(n) \to SO(E_n); X \mapsto T_n^{-1} g T_n$, is linear, bijective, smooth and a Lie algebra homomorphism since

$$\begin{aligned}\psi([X,Y]) &= \psi(XY - YX) = T_n^{-1} XY T_n - T_n^{-1} YX T_n \\ &= T_n^{-1} X T_n T_n^{-1} Y T_n - T_n^{-1} Y T_n T_n^{-1} X T_n \\ &= [T_n^1 X T_n, T_n^{-1} Y T_n] = [\psi(X),\psi(Y)].\end{aligned}$$

Thus, ψ is an isomorphism between the Lie algebras $\mathfrak{so}(n)$ and $\mathfrak{so}(E_n)$ and also induces an isomorphism of their complexified versions.

From now on we have to distinguish between the cases where n is even and where n is odd. In the case $n = 2m$, a maximal torus of $SO(2m)$ is given by

$$T = \left\{\operatorname{diag}\left(e^{i\theta_1}, \ldots, e^{i\theta_m}, e^{-i\theta_1}, \ldots, e^{-i\theta_m}\right) | \theta_i \in \mathbb{R}\right\}.$$

The corresponding Cartan subalgebras are the following sets of matrices.

$$\begin{aligned}\mathfrak{t} &= \{\operatorname{diag}(i\theta_1, \ldots, i\theta_m, -i\theta_1, \ldots, -i\theta_m) | \theta_i \in \mathbb{R}\}, \\ \mathfrak{t}_{\mathbb{C}} &= \{\operatorname{diag}(z_1, \ldots, z_m, -z_1, \ldots, -z_m) | z_i \in \mathbb{C}\}.\end{aligned}$$

If we look closer at the condition $X^t E_n + E_n X = 0$ for $X \in \mathfrak{gl}(n,\mathbb{C})$, we see that X has to be a matrix of the form

$$\begin{pmatrix} W & Y \\ Z & -W^t \end{pmatrix},$$

where $W, Y, Z \in \mathfrak{gl}(n,\mathbb{C})$ and $Y^t = -Y$, $Z^t = -Z$. Thus,

$$\mathfrak{so}(E_{2n},\mathbb{C}) = \left\{\begin{pmatrix} W & Y \\ Z & -W^t \end{pmatrix} \middle| W, Y, Z \in \mathfrak{gl}(n,\mathbb{C}), Y^t = -Y,\ Z^t = -Z\right\}.$$

Now we want to know the roots of $SO(E_{2n})$. Therefore we have to solve the following equation

$$[H,Z] \stackrel{!}{=} \alpha(H)Z \quad \text{for } H \in \mathfrak{t}_{\mathbb{C}} \text{ and } Z \in \mathfrak{so}(E_n,\mathbb{C}).$$

It is easy to calculate the set of roots

$$\Delta(\mathfrak{g}_{\mathbb{C}}) = \{\pm(\varepsilon_i \pm \varepsilon_j) | 1 \le i < j \le n\},$$

where ε_i is the functional on $\mathfrak{t}_{\mathbb{C}}$ defined by $\varepsilon_i(\mathrm{diag}(z_1, \ldots, z_n, -z_1 \ldots, z_n)) = z_i$. The corresponding root spaces are given by

$$\begin{aligned} \mathfrak{g}_{\varepsilon_i - \varepsilon_j} &= \mathbb{C}(E_{i,j} - E_{j+n,i+n}), & \mathfrak{g}_{-\varepsilon_i + \varepsilon_j} &= \mathbb{C}(E_{j,i} - E_{i+n,j+n}), \\ \mathfrak{g}_{\varepsilon_i + -\varepsilon_j} &= \mathbb{C}(E_{i,j+n} - E_{j,i+n}), & \mathfrak{g}_{-\varepsilon_i - \varepsilon_j} &= \mathbb{C}(E_{i+n,j} - E_{j+n,i}), \end{aligned}$$

where $\{E_{i,j}\}$ denotes the standard basis for the vector space of all $n \times n$ matrices.

We illustrate the calculation for the case $SO(E_4)$ and purely real-valued matrices. So, define

$$H = \mathrm{diag}(z_1, z_2, -z_1, -z_2), \quad \text{and} \quad Z = \begin{pmatrix} a & b & 0 & f \\ c & d & -f & 0 \\ 0 & -e & -a & -c \\ e & 0 & -b & -d \end{pmatrix},$$

for $z_i, a, b, c, d, e, f \in \mathbb{R}$. We want to find functionals α on $\mathfrak{t}_{\mathbb{C}}$ such that

$$[H, Z] = \alpha(H) Z.$$

If we calculate the Lie bracket of H and Z, the previous equation has the following form:

$$\begin{pmatrix} 0 & b(z_1 - z_2) & 0 & f(z_1 + z_2) \\ c(-z_1 + z_2) & 0 & -f(z_1 + z_2) & 0 \\ 0 & -e(-z_1 - z_2) & 0 & -c(-z_1 + z_2) \\ e(-z_1 - z_2) & 0 & -b(z_1 - z_2) & 0 \end{pmatrix} = \alpha(H) \begin{pmatrix} a & b & 0 & f \\ c & d & -f & 0 \\ 0 & -e & -a & -c \\ e & 0 & -b & -d \end{pmatrix}$$

Thus, it makes sense to define the four functionals, that is, roots,

$$\begin{aligned} &(\varepsilon_1 - \varepsilon_2), \quad (-\varepsilon_1 + \varepsilon_2), \\ &(\varepsilon_1 + \varepsilon_2), \quad (-\varepsilon_1 - \varepsilon_2), \end{aligned}$$

where ε_i is defined as above. The corresponding root spaces are

$$\mathfrak{g}_{\varepsilon_1 - \varepsilon_2} = \mathbb{C} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathfrak{g}_{-\varepsilon_1 + \varepsilon_2} = \mathbb{C} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathfrak{g}_{\varepsilon_1 + \varepsilon_2} = \mathbb{C} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{g}_{-\varepsilon_1 - \varepsilon_2} = \mathbb{C} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Now, we consider the second case, where $n = 2m + 1$. In this case everything works almost the same as before. Note that we have a different maximal torus and consequently the Cartan subalgebra has to change too:

$$\begin{aligned} T &= \left\{\operatorname{diag}\left(e^{i\theta_1},\dots,e^{i\theta_m},e^{-i\theta_1},\dots,e^{-i\theta_m},1\right)|\theta_i \in \mathbb{R}\right\}, \\ \mathfrak{t} &= \{\operatorname{diag}(i\theta_1,\dots,i\theta_m,-i\theta_1\dots,-i\theta_m,0)\,|\theta_i \in \mathbb{R}\}, \\ \mathfrak{t}_{\mathbb{C}} &= \{\operatorname{diag}(z_1,\dots,z_m,-z_1,\dots,-z_m,0)\,|z_i \in \mathbb{C}\}. \end{aligned}$$

Thus, the set of roots is

$$\Delta(\mathfrak{g}_{\mathbb{C}}) = \{\pm(\varepsilon_i \pm \varepsilon_j)|1 \le i < j \le n\} \cup \{\pm\varepsilon_i|1 \le i \le n\}.$$

The corresponding root spaces are the same as in the even case plus the following two

$$\mathfrak{g}_{\varepsilon_i} = \mathbb{C}(E_{i,2n+1} - E_{2n+1,i+n}), \quad \mathfrak{g}_{-\varepsilon_i} = \mathbb{C}(E_{i+n,2n+1} - E_{2n+1,i}).$$

2.3 Killing Form

For the upcoming material we will need a special linear form, the so-called Killing form.

Definition 2.3.1. *Let $\mathfrak{g}$ be the Lie algebra of a Lie subgroup of $GL(n,\mathbb{C})$. The Killing form is the symmetric complex bilinear form $B(\cdot,\cdot)$ on $\mathfrak{g}_{\mathbb{C}}$, defined by*

$$B(X,Y) = \operatorname{tr}(\operatorname{ad}X \circ \operatorname{ad}Y)$$

for $X, Y \in \mathfrak{g}_{\mathbb{C}}$.

The Killing form has several useful properties.

Theorem 2.3.2. *Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group G.*

i) For $X, Y \in \mathfrak{g}, B(X,Y) = \operatorname{tr}(\operatorname{ad}X \circ \operatorname{ad}Y)$ on $\mathfrak{g}$.

ii) B is Ad-invariant, that is, $B(X,Y) = B(\operatorname{Ad}(g)X, \operatorname{Ad}(g)Y)$ for $g \in G$ and $X, Y \in \mathfrak{g}_{\mathbb{C}}$.

iii) B is skew ad-invariant, that is, $B(\operatorname{ad}(Z)X, Y) = -B(X, \operatorname{ad}(Z)Y)$ for $Z, X, Y, \in \mathfrak{g}_{\mathbb{C}}$.

iv) B restricted to $\mathfrak{g}' \times \mathfrak{g}'$ is negative definite. Here, $\mathfrak{g}'$ denotes the ideal of $\mathfrak{g}$ spanned by $[\mathfrak{g},\mathfrak{g}]$.

v) B restricted to $\mathfrak{g}_\alpha \times \mathfrak{g}_\beta$ is zero when $\alpha + \beta \neq 0$ for $\alpha, \beta \in \Delta(\mathfrak{g}_{\mathbb{C}}) \cup \{0\}$.

vi) B is non-singular on $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$. If $\mathfrak{g}$ is semisimple with a Cartan subalgebra $\mathfrak{t}$, then B is also nonsingular on $\mathfrak{t}_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}}$.

vii) The radical of B, $\operatorname{rad}B = \{X \in \mathfrak{g}|B(X,\mathfrak{g}_{\mathbb{C}}) = 0\}$, is the center of $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{z}(\mathfrak{g}_{\mathbb{C}})$.

viii) Let $\mathfrak{g}$ be simple and choose a linear realization of G, such that $\mathfrak{g} \subseteq \mathfrak{u}(n)$. Then there exists a positive $c \in \mathbb{R}$, such that $B(X,Y) = c\operatorname{tr}(XY)$ for $X, Y \in \mathfrak{g}_{\mathbb{C}}$.

When the Lie algebra $\mathfrak{g}$ of a Lie group G is semisimple, we saw in the previous theorem that B is negative definite on $\mathfrak{t}$. So B restricts to a real inner product on the real vector space $i\mathfrak{t}$. Therefore B induces an isomorphism between $i\mathfrak{t}$ and $(i\mathfrak{t})^*$ as follows.

Definition 2.3.3.
- *Let G be a compact Lie group with a semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\alpha \in (i\mathfrak{t})^*$. Let $u_\alpha \in i\mathfrak{t}$ be the uniquely determined element in the following equation*

$$\alpha(H) = B(H, u_\alpha)$$

for all $H \in i\mathfrak{t}$ and, when $\alpha \neq 0$, define

$$h_\alpha = \frac{2u_\alpha}{B(u_\alpha,u_\alpha)}.$$

- *In case $\mathfrak{g}$ is not semisimple, define $u_\alpha \in i\mathfrak{t}' \subseteq i\mathfrak{t} \subseteq \mathfrak{t}$ by first restricting B to $i\mathfrak{t}'$. For $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$, recall an earlier statement that α is completely determined by its restriction to $i\mathfrak{t}$, where α is a real-valued linear functional. Viewing α as an element of $(i\mathfrak{t})^*$, define u_α and h_α via the above definition. Note that the equation $\alpha(H) = B(H, u_\alpha)$ now holds for all $H \in \mathfrak{t}_\mathbb{C}$ by $\mathbb{C}$-linear extension.*
The elements h_α are also called dual roots and we write for the set of all dual roots

$$\Delta(\mathfrak{g}_\mathbb{C})^\vee = \{h_\alpha | \alpha \in \Delta(\mathfrak{g}_\mathbb{C})\}.$$

- *When $\mathfrak{g} \subseteq \mathfrak{u}(n)$ is simple, we know from the previous theorem that there exists a positive $c \in \mathbb{R}$, such that $B(X,Y) = c\,\mathrm{tr}(XY)$ for $X,Y \in \mathfrak{g}_\mathbb{C}$. Thus if $\alpha \in (i\mathfrak{t})^*$ and $u'_\alpha, h'_\alpha \in i\mathfrak{t}$ are determined by the equations $\alpha(H) = \mathrm{tr}(Hu'_\alpha)$ and $h'_\alpha = \frac{2u'_\alpha}{\mathrm{tr}(u'_\alpha,u'_\alpha)}$, it follows that $u'_\alpha = cu_\alpha$ but that $h'_\alpha = h_\alpha$. In particular, h_α can be computed with respect to the trace form instead of the Killing form.*

Example.
We compute one h_α of $SO(E_4)$ to illustrate how the calculation works. Let $\alpha = \varepsilon_1 + \varepsilon_2$ and $H = \mathrm{diag}(z_1,z_2,-z_1,-z_2)$. We are looking for an element $u'_\alpha \in i\mathfrak{t}$ such that

$$z_1 + z_2 = \mathrm{tr}(Hu'_\alpha).$$

If we multiply

$$H \cdot u'_\alpha = \mathrm{diag}(z_1,z_2,-z_1,-z_2) \cdot \mathrm{diag}(u_1,u_2,-u_1,-u_2) = z_1(u_1+u_1) + z_2(u_2+u_2),$$

we see that

$$2u_1 = 1, \quad 2u_2 = 1.$$

Thus the desired elements u_α and h_α are given by

$$\begin{aligned} u'_\alpha &= \operatorname{diag}\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right), \\ h'_\alpha &= \frac{2u'_\alpha}{\operatorname{tr}(u_\alpha,u_\alpha)} \\ &= \operatorname{diag}(1,1,-1,-1)\cdot \operatorname{tr}\left(\operatorname{diag}\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)\right) \\ &= \operatorname{diag}(1,1,-1,-1). \end{aligned}$$

If we set $E_i = \operatorname{diag}(0,\dots,0,1,0,\dots,0)$, where 1 is in the ith position, then we can write $h_\alpha = E_1 + E_2 - (E_3 + E_4)$.

In general the dual roots of $SO(E_{2n})$ are given by

$$\begin{aligned} h_{\varepsilon_i-\varepsilon_j} &= (E_i - E_j) - (E_{i+n} - E_{j+n}), \\ h_{\varepsilon_i+\varepsilon_j} &= (E_i + E_j) - (E_{i+n} + E_{j+n}). \end{aligned}$$

Note that $h_{-\alpha} = -h_\alpha$.

In the case $SO(E_{2n+1})$, one obtains the same dual roots as above plus one additional dual root

$$h_{\varepsilon_i} = 2E_i - 2E_{i+n}.$$

Next we state a few properties of roots and dual roots.

Corollary 2.3.4. *Let G be a compact Lie group, $\mathfrak{g}$ the corresponding Lie algebra, $\mathfrak{t}$ its Cartan subalgebra and $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$.*

i) The only multiples of α in $\Delta(\mathfrak{g}_\mathbb{C})$ are $\pm\alpha$.

ii) $\dim \mathfrak{g}_\alpha = 1$.

iii) If $\beta \in \Delta(\mathfrak{g}_\mathbb{C})$, then $\alpha(h_\beta) \in \pm\{0,1,2,3\}$.

iv) If (V,ρ) is a representation of G and $\lambda \in \Delta(V)$, then $\lambda(h_\alpha) \in \mathbb{Z}$.

Let G be a compact Lie group, $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$ and $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$, note that $\alpha \in (i\mathfrak{t})^*$. We defined the Killing form on $i\mathfrak{t}$, but can transfer it to $(i\mathfrak{t})^*$ by setting

$$B(\lambda_1,\lambda_2) := B(u_{\lambda_1},u_{\lambda_2})$$

for $\lambda_1,\lambda_2 \in (i\mathfrak{t})^*$. The elements u_{λ_i} are uniquely defined by $\lambda_i(H) = B(H,u_{\lambda_i})$ for all $H \in (i\mathfrak{t})$. In particular, for $\lambda \in (i\mathfrak{t})^*$ we obtain

$$\lambda(h_\alpha) = B(h_\alpha,u_\lambda) = B\left(\frac{2u_\alpha}{B(u_\alpha,u_\alpha)},u_\alpha\right) = \frac{2B(u_\alpha,u_\lambda)}{B(u_\alpha,u_\alpha)} = \frac{2B(\alpha,\lambda)}{B(\alpha,\alpha)}.$$

Also note that

$$\alpha(H) = \frac{2B(H,h_\alpha)}{B(h_\alpha,h_\alpha)}$$

for $H \in i\mathfrak{t}$.

Now it is possible to define new weight spaces. Some of these spaces will be important in the next chapter.

Definition 2.3.5. *Let G be a compact Lie group and T a maximal torus of G with corresponding Cartan subalgebra $\mathfrak{t}$.*

i) The root lattice, $R = R(\mathfrak{t})$, is the lattice in $(i\mathfrak{t})^*$ given by

$$R = \operatorname{span}_{\mathbb{Z}}\{\alpha | \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})\}.$$

ii) The set of algebraically integral weights, $P = P(\mathfrak{t})$, is the lattice in $(i\mathfrak{t})^*$ given by

$$P = \{\lambda \in (i\mathfrak{t})^* | \lambda(h_\alpha) \in \mathbb{Z} \text{ for } \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})\},$$

where $\lambda \in (i\mathfrak{t})^*$ is extended to an element of $(\mathfrak{t}_{\mathbb{C}})^*$ by $\mathbb{C}$-linearity.

iii) The set of analytically integral weights, $A = A(T)$, is the lattice in $(i\mathfrak{t})^*$ given by

$$A = \{\lambda \in (i\mathfrak{t})^* | \lambda(H) \in 2\pi i\mathbb{Z} \text{ whenever } \exp H = I \text{ for } H \in \mathfrak{t}\}.$$

We could also define the corresponding dual spaces, but we will not need them and therefore skip it.

Example.
The root lattice of $SO(E_{2n})$ is given by

$$\begin{aligned} R &= \operatorname{span}_{\mathbb{Z}}\{\alpha | \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})\} = \operatorname{span}_{\mathbb{Z}}\{\pm(\varepsilon_i \pm \varepsilon_j) | 1 \le i < j \le n\} \\ &= \left\{ \sum_{i=1}^n \lambda_i \varepsilon_i | \lambda_i \in \mathbb{Z}, \sum_i^n \lambda_i \in 2\mathbb{Z} \right\}. \end{aligned}$$

The condition on the sum of the $\lambda_i' s$ comes from the fact that the roots of $SO(E_{2n})$ come in pairs.

A general element of $(i\mathfrak{t})^*$ has the form $\sum_i^n \lambda_i \varepsilon_i$. Thus the algebraically integral weights of $SO(E_{2n})$ are given by

$$\begin{aligned} P &= \{\lambda \in (i\mathfrak{t})^* | \lambda(h_\alpha) \in \mathbb{Z}, \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})\} \\ &= \left\{ \sum_{i=1}^n \left(\lambda_i + \frac{\lambda_0}{2}\right) \varepsilon_i | \lambda_i \in \mathbb{Z} \right\}. \end{aligned}$$

We have to add $\frac{\lambda_0}{2}$ because of the special form of h_α.

The set of analytically integral weights of $SO(E_{2n})$ is given by

$$A = \left\{ \sum_{i=1}^{n} \lambda_i \varepsilon_i | \lambda_i \in \mathbb{Z} \right\}.$$

Remember that an element in $\mathfrak{t}$ has the form $\operatorname{diag}(i\theta_1, \ldots, i\theta_n, -i\theta_1, \ldots, -i\theta_n)$. The condition $\exp H = I$ for $H \in \mathfrak{t}$ tells us that $i\theta_i \in 2\pi i\mathbb{Z}$. Thus we obtain the previous set.

For $SO(E_{2n+1})$ we have

$$\begin{aligned} R &= \left\{ \sum_{i=1}^{n} \lambda_i \varepsilon_i | \lambda_i \in \mathbb{Z} \right\}, \\ A &= R, \\ P &= \left\{ \sum_{i=1}^{n} \left(\lambda_i + \frac{\lambda_0}{2} \right) \varepsilon_i | \lambda_i \in \mathbb{Z} \right\}. \end{aligned}$$

In order to state the next important theorem, we need some preparation.

Lemma 2.3.6. *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{t}$. For $H \in \mathfrak{t}$, $\exp H$ is an element of the center $Z(G)$ of G, where $Z(G) = \{h \in G | hg = gh \ \ \forall g \in G\}$, if and only if $\alpha(H) \in 2\pi i\mathbb{Z}$ for all $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$.*

Proof. The proof is straightforward. Let $g = \exp H$ and remember that $g \in Z(G)$ if and only if $\operatorname{Ad}(g)X = X$ for all $X \in \mathfrak{g}$. Now for $\alpha \in \Delta \cup \{0\}$ and $X \in \mathfrak{g}_\alpha$ the following equation holds

$$\operatorname{Ad}(g)X = \operatorname{Ad}(\exp H)X = e^{\operatorname{ad} H}X = e^{\alpha(H)}X.$$

The second equation is a property of the exponential map and the root decomposition finishes the proof. □

For the main theorem of this section, we need one last definition.

Definition 2.3.7. *Let G be a compact Lie group and T a maximal torus. The set of all Lie group homomorphisms $\xi : T \to \mathbb{C}\backslash\{0\}$ is called the character group of T and denoted by $\chi(T)$.*

Now we can show that there is a connection between the analytically integral weights and the character group of T:

Theorem 2.3.8 ([18, p. 131])**.** *Let G be a compact Lie group with a maximal torus T. Then the following statements are true.*

- *The weight spaces are related in the following way: $R \subseteq A \subseteq P$.*
- *Let $\lambda \in (i\mathfrak{t})$. λ is an analytically integral weight, that is, $\lambda \in A$, if and only if there exists $\xi_\lambda \in \chi(T)$ satisfying*

$$\xi_\lambda(\exp H) = e^{\lambda(H)}$$

for $H \in \mathfrak{t}$, where $\lambda \in (i\mathfrak{t})^*$ is extended to an element of $(\mathfrak{t}_{\mathbb{C}})^*$ by $\mathbb{C}$-linearity. The map $\lambda \to \xi_\lambda$ establishes a bijection

$$A \leftrightarrow \chi(T).$$

- *For semisimple $\mathfrak{g}$, $|P/R|$ is finite.*

2.4 Weyl Group

The Weyl group of a Lie group will be very important in the following chapters. In this section we give an analytical definition, but later on we will see that this group is actually isomorphic to an algebraically defined Weyl group.

Definition 2.4.1. *Let G be a compact connected Lie group with maximal torus T. Let $N = N(T) = \{g \in G |\ gTg^{-1} = T\}$ be the normalizer in G of T. We call $W = N/T$ the Weyl group of G and denote it by $W = W(G) = W(G,T)$.*

We already know that $c_g T = T'$ is another maximal torus T'. This shows that $c_g N(T) = N(T')$ so that the Weyl group is up to isomorphism, independent of the choice of the maximal torus, that is, $W(G,T) \cong W(G,T')$.

For given $w \in N, H \in \mathfrak{t}$ and $\lambda \in \mathfrak{t}^*$, we define an action of N on $\mathfrak{t}$ and $\mathfrak{t}^*$ by

$$\begin{aligned} w(H) &= \mathrm{Ad}(w)H, \\ [w(\lambda)]\,(H) &= \lambda(w^{-1}(H)) = \lambda(\mathrm{Ad}(w^{-1}H)). \end{aligned}$$

2.5 Simple Roots and Weyl Chambers

At the beginning of this section we work with a special subset of the roots and introduce the notion of a Weyl chamber. In the second part we give the already mentioned algebraic definition of a Weyl group and show that both notions of Weyl groups are equivalent.

Definition 2.5.1. *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$. Write $\mathfrak{t}' = \mathfrak{g}' \cap \mathfrak{t}$, where $\mathfrak{g}'$ is the ideal of $\mathfrak{g}$ spanned by $[\mathfrak{g},\mathfrak{g}]$.*

- *A system of simple roots, $\Pi = \Pi(\mathfrak{g}_{\mathbb{C}})$, is a subset of $\Delta(\mathfrak{g}_{\mathbb{C}})$ that is a basis of $(i\mathfrak{t})^*$ and satisfies the property that any $\beta \in \Delta(\mathfrak{g}_{\mathbb{C}})$ can be written as*

$$\beta = \sum_{\alpha \in \Pi} k_\alpha \alpha,$$

where $\{k_\alpha | \alpha \in \Pi\} \subseteq \mathbb{Z}_{\geq 0}$ or $\{k_\alpha | \alpha \in \Pi\} \subseteq \mathbb{Z}_{\leq 0}$. Here, $\mathbb{Z}_{\geq 0} = \{k \in \mathbb{Z} | k \geq 0\}$ and $\mathbb{Z}_{\leq 0} = \{k \in \mathbb{Z} | k \leq 0\}$. The elements of Π are called simple roots.

- *Given a system of simple roots Π, the set of positive roots with respect to Π is given by*

$$\Delta^+(\mathfrak{g}_{\mathbb{C}}) = \left\{ \beta \in \Delta(\mathfrak{g}_{\mathbb{C}}) | \beta = \sum_{\alpha \in \Pi} k_\alpha \alpha \text{ with } k_\alpha \in \mathbb{Z}_{\geq 0} \right\},$$

and the set of negative roots with respect to Π is given by

$$\Delta^-(\mathfrak{g}_{\mathbb{C}}) = \left\{ \beta \in \Delta(\mathfrak{g}_{\mathbb{C}}) | \beta = \sum_{\alpha \in \Pi} k_\alpha \alpha \text{ with } k_\alpha \in \mathbb{Z}_{\leq 0} \right\},$$

such that $\Delta(\mathfrak{g}_{\mathbb{C}}) = \Delta^+(\mathfrak{g}_{\mathbb{C}}) \dot{\cup} \Delta^-(\mathfrak{g}_{\mathbb{C}})$ and $\Delta^-(\mathfrak{g}_{\mathbb{C}}) = -\Delta^+(\mathfrak{g}_{\mathbb{C}})$.

So far we do not know if systems of simple roots exist. But we will state a lemma in this section which guarantees this existence. For this lemma we need the next definition.

Definition 2.5.2. *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{t}$.*

- *The connected components of $(i\mathfrak{t}')^* \backslash (\bigcup_{\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})} \alpha^\perp)$ are called the (open) Weyl chambers of $(i\mathfrak{t})^*$. If K is a Weyl chamber, the subsets $\bar{K} \cap \alpha^\perp$ are called the Weyl chamber walls.*
- *The connected components of $i\mathfrak{t}' \backslash (\bigcup_{\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})} h_\alpha^\perp)$ are called the (open) Weyl chambers of $i\mathfrak{t}$.*
- *If C is a Weyl chamber of $(i\mathfrak{t})^*$, $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$ is called C-positive if $B(C, \alpha) > 0$ and C-negative, if $B(C, \alpha) < 0$.*
- *If α is C-positive, it is called decomposable with respect to C, if there exist C-positive $\beta, \gamma \in \Delta(\mathfrak{g}_{\mathbb{C}})$, such that $\alpha = \beta + \gamma$. Otherwise α is called indecomposable with respect to C.*
- *Analogous notions can be defined for the Weyl chamber $C^\vee$ of $(i\mathfrak{t})$.*
- *If C is a Weyl chamber of $(i\mathfrak{t})^*$, let*

$$\Pi(C) = \{\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}) | \alpha \text{ is } C\text{-positive and indecomposable}\}.$$

If $C^\vee$ is a Weyl chamber of $i\mathfrak{t}$, let

$$\Pi(C^\vee) = \{\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}) | \alpha \text{ is } C^\vee\text{-positive and indecomposable}\}.$$

- *If Π is a system of simple roots, the associated Weyl chamber of $(i\mathfrak{t})^*$ is*

$$C(\Pi) = \{\lambda \in (i\mathfrak{t})^* | B(\lambda, \alpha) > 0 \text{ for } \alpha \in \Pi\}$$

and the associated Weyl chamber of $i\mathfrak{t}$ *is*

$$C^\vee(\Pi) = \{H \in i\mathfrak{t} | \alpha(H) > 0 \textit{ for } \alpha \in \Pi\}.$$

If we have a system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_l\}$, we can define a basis $\{\pi_1, \ldots, \pi_l\}$ of $(i\mathfrak{t})^*$ by $2\frac{B(\pi_i, \alpha_i)}{B(\alpha_i, \alpha_i)} = \delta_{i,j}$ and call these elements of the basis the fundamental weights. We also define $\rho = \rho(\Pi) \in (i\mathfrak{t})^*$ as $\rho = \sum_i \pi_i$.

Later we will see, that there is a one-to-one correspondence between Weyl chambers and systems of simple roots.

Example.

For $SO(E_{2n})$ with $\mathfrak{t} = \{\mathrm{diag}(i\theta_1, \ldots, i\theta_n, -i\theta_1, \ldots, -i\theta_n) | \theta_i \in \mathbb{R}\}$ we have the following sets

$$\begin{aligned} \Pi &= \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} | 1 \leq i \leq n-1\} \cup \{\alpha_n = \varepsilon_{n-1} + \varepsilon_n\}, \\ C^\vee &= \{\mathrm{diag}(\theta_1, \ldots, \theta_n, -\theta_1, \ldots, -\theta_n) \mid \theta_i > \theta_{i+1}, \theta_{n-1} > |\theta_n|, \theta_i \in \mathbb{R}\}. \end{aligned}$$

The conditions on the θ_i's come from the condition $\alpha(H) > 0$ for all $\alpha \in \Pi$. This is equivalent to $B(H, h_\alpha) > 0$ and since $SO(E_{2n})$ has a simple Lie algebra, the Killing form is just the trace form. Thus the inequality $\mathrm{tr}(Hh_\alpha) > 0$ gives us the above mentioned conditions.

The sum of the fundamental weights of $SO(E_{2n})$ is

$$\rho = n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + \varepsilon_{n-1}.$$

For $SO(E_{2n+1})$ with $\mathfrak{t} = \{\mathrm{diag}(i\theta_1, \ldots, i\theta_n, -i\theta_1, \ldots, -i\theta_n, 0) | \theta_i \in \mathbb{R}\}$ we have

$$\begin{aligned} \Pi &= \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} | 1 \leq i \leq n-1\} \cup \{\alpha_n = \varepsilon_n\}, \\ C^\vee &= \{\mathrm{diag}(\theta_1, \ldots, \theta_n, -\theta_1, \ldots, -\theta_n, 0) \mid \theta_i > \theta_{i+1} > 0, \theta_i \in \mathbb{R}\}, \\ \rho &= \frac{1}{2}\left((2n-1)\varepsilon_1 + (2n-3)\varepsilon_2 + \cdots + \varepsilon_n\right). \end{aligned}$$

Every hyperplane $\alpha^\perp$ determines a reflection r_α of $(i\mathfrak{t})^*$, which leaves $\alpha^\perp$ pointwise fixed. Thus the following definition is justified.

Definition 2.5.3. *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$.*

- *For $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$, define the reflection $r_\alpha : (i\mathfrak{t})^* \to (i\mathfrak{t})^*$ by*

$$r_\alpha(\lambda) = \lambda - 2\frac{B(\lambda, \alpha)}{B(\alpha, \alpha)}\alpha = \lambda - \lambda(h_\alpha)\alpha,$$

and the dual reflection $r_{h_\alpha} : i\mathfrak{t} \to i\mathfrak{t}$ by

$$r_{h_\alpha}(H) = H - 2\frac{B(H, h_\alpha)}{B(h_\alpha, h_\alpha)}h_\alpha = H - \alpha(H)h_\alpha.$$

- *The set* $\{r_\alpha | \alpha \in \Delta(\mathfrak{g}_\mathbb{C})\}$ *generates a reflection group, which is denoted by* $W(\Delta(\mathfrak{g}_\mathbb{C}))$ *and we denote the group generated by* $\{r_{h_\alpha} | \alpha \in \Delta(\mathfrak{g}_\mathbb{C})\}$ *by* $W(\Delta(\mathfrak{g}_\mathbb{C})^\wedge)$.

The next lemma shows that the reflections defined above are elements of the Weyl group W, defined in the previous section.

Lemma 2.5.4. *Let G be a compact Lie group with a maximal torus T.*

- *For $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$, there exists an element $w_\alpha \in N(T)$, such that r_α describes the action of w_α on $(i\mathfrak{t})^*$ and r_{h_α} the action of w_α on $(i\mathfrak{t})$.*
- *For $\alpha, \beta \in \Delta(\mathfrak{g}_\mathbb{C})$ the following properties hold:*

$$r_\alpha(\beta) \in \Delta(\mathfrak{g}_\mathbb{C}) \text{ and } r_{h_\alpha}(h_\beta) = h_{r_\alpha(\beta)}.$$

Lemma 2.5.5 ([18, p. 144])**.** *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$.*

i) There is a one-to-one correspondence between

$$\{\text{systems of simple roots}\} \longleftrightarrow \{\text{Weyl chambers of } (i\mathfrak{t})^*\}.$$

The bijection maps a system of simple roots Π to the Weyl chamber $C(\Pi)$ and maps a Weyl chamber C to the system of simple roots $\Pi(C)$.

ii) There is a one-to-one correspondence between

$$\{\text{systems of simple roots}\} \longleftrightarrow \{\text{Weyl chambers of } i\mathfrak{t}\}.$$

The bijection maps a system of simple roots Π to the Weyl chamber $C^\vee(\Pi)$ and maps a Weyl chamber $C^\vee$ to the system of simple roots $\Pi(C^\vee)$.

iii) If Π is a system of simple roots with $\alpha, \beta \in \Pi$, then $B(\alpha, \beta) \leq 0$.

The final result of the chapter shows that the generated reflection group $W(\Delta(\mathfrak{g}_\mathbb{C}))$ is actually the whole Weyl group W, not just a subgroup.

Theorem 2.5.6 ([18, p. 145])**.** *Let G be a compact Lie group with maximal torus T.*

- *The action of $W(G)$ on $i\mathfrak{t}$ establishes an isomorphism $W(G) \cong W(\Delta(\mathfrak{g}_\mathbb{C})^\vee)$.*
- *The action of $W(G)$ on $(i\mathfrak{t})^*$ establishes an isomorphism $W(G) \cong W(\Delta(\mathfrak{g}_\mathbb{C}))$.*
- *$W(G)$ acts simply transitively on the set of Weyl chambers.*

3 Highest Weight Theory

In this chapter we discuss the highest weight theory. First we define highest weights, then we state the highest weight theorem which states that irreducible representations are completely determined by their highest weights. At the end of this chapter we parametrize all possible highest weights of irreducible representations, in particular we give a highest weight classification. On the way we also prove the Weyl Character Formula and the Second Determinantal Formula, which will be needed in the proof of the general version of Hadwiger's Characterization Theorem.

Most of the time we follow [18], except for the part on the Second Determinantal Formula. There the corresponding material was taken from [3] and we recommend [11] for further readings.

3.1 Highest Weights

Our goal is to describe all irreducible representations of a compact Lie group. To this end, we need the notion of highest weights.

In this section, G will always denote a compact Lie group, T a maximal torus, and $\Delta^+(\mathfrak{g}_\mathbb{C})$ a system of positive roots with corresponding system of simple roots $\Pi(\mathfrak{g}_\mathbb{C})$. Define

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})} \mathfrak{g}_\alpha,$$
$$\mathfrak{n}^- = \bigoplus_{\alpha\in\Delta^-(\mathfrak{g}_\mathbb{C})} \mathfrak{g}_\alpha,$$

so that

$$\mathfrak{g}_\mathbb{C} = \mathfrak{n}^- \oplus \mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}^+,$$

by the root space decomposition.

Definition 3.1.1. *Let V be a representation of $\mathfrak{g}$ with weight space decomposition $V = \bigoplus_{\lambda\in\Delta(V)} V_\lambda$.*

- *A nonzero $v \in V_{\lambda_0}$ is called a highest weight vector of weight λ_0 with respect to $\Delta^+(\mathfrak{g}_\mathbb{C})$ if $\mathfrak{n}^+v = 0$, that is, if $Xv = 0$ for all $X \in \mathfrak{n}^+$. In this case, λ_0 is called a highest weight of V.*

- *A weight λ is said to be dominant if $B(\lambda, \alpha) \geq 0$ for all $\alpha \in \Pi(\mathfrak{g}_\mathbb{C})$, that is, if λ lies in the closed Weyl chamber corresponding to $\Delta^+(\mathfrak{g}_\mathbb{C})$.*

Remark.

If we have a Lie group G with corresponding Lie algebra $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{t}$, we can fix a basis of $(i\mathfrak{t})^*$. With respect to this basis, we can define a lexicographic order by saying $\alpha > \beta$ if the first nonzero coordinate of $\alpha - \beta$ is positive. Now we can define $\Pi = \{\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}) | \alpha > 0, \alpha \neq \beta_1 + \beta_2 \text{ for any } \beta_i \in \Delta(\mathfrak{g}_{\mathbb{C}}) \text{ with } \beta_i > 0\}$. Clearly, Π is a system of simple roots. Then the highest weight is the largest weight in the lexicographic ordering.

The next theorem shows that highest weights uniquely determine irreducible representations. Later on we will also see that there is a one-to-one correspondence between the set of irreducible representations of G and the set of analytically integral weights.

Theorem 3.1.2 ([18, p. 152], Theorem of Highest Weights)**.** *Let G be a connected Lie group and V an irreducible representation of G.*

i) V has a unique highest weight λ_0.

ii) The highest weight λ_0 is dominant and analytically integral, that is, $\lambda_0 \in A(T)$.

iii) Up to nonzero scalar multiplication, there is a unique highest weight vector.

iv) Any weight $\lambda \in \Delta(V)$ is of the form

$$\lambda = \lambda_0 - \sum_{\alpha_i \in \Pi(\mathfrak{g}_{\mathbb{C}})} n_i \alpha_i$$

for $n_i \in \mathbb{Z}^{\geq 0}$.

v) For $w \in W$, $wV_\lambda = V_{w\lambda}$, so that $\dim V_\lambda = \dim V_{w\lambda}$. Here $W(G)$ is identified with $W(\Delta(\mathfrak{g}_{\mathbb{C}}))$, via the Ad*-action.*

vi) Using the norm induced by the Killing form, $\|\lambda\| \leq \|\lambda_0\|$ with equality if and only if $\lambda = w\lambda_0$ for $w \in W(\mathfrak{g}_{\mathbb{C}})$.

vii) Up to isomorphism, V is uniquely determined by λ_0.

Since V is uniquely determined by λ, we write V_λ for V and χ_λ for its character.

In the example on page 20, we saw that the analytically integral weights of $SO(E_{2n})$ and $SO(E_{2n+1})$ are given by the set $\{\sum_{i=1}^{n} \lambda_i \varepsilon_i | \lambda_i \in \mathbb{Z}\}$. Thus, we can identify an analytically integral weight with a tuple of integers. By Theorem 3.1.2, highest weights are analytically integral and dominant, therefore they correspond to a tuple of positive integers. Together with the remark above we obtain the following condition for highest weights λ of irreducible $SO(n)$ representations. The highest weight λ is a tuple of integers $(\lambda_1, \lambda_2, \ldots, \lambda_{n/2})$ such that

$$\begin{cases} \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n/2} \geq 0 & \text{for } n \text{ odd}, \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n/2-1} \geq |\lambda_{n/2}| & \text{for } n \text{ even}. \end{cases} \tag{3.1}$$

In the following, if we work with any isomorphic copy of a representation of $SO(E_n)$ or $SO(n)$ with highest weight $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_{n/2})$ we will denote it by Γ_λ.

Examples.

- Let Γ be the trivial representation of $SO(n)$, then the corresponding highest weight is $\lambda = (0,\dots,0)$.
- The standard representation of $SO(n)$ on $\Gamma = \mathbb{R}^n_{\mathbb{C}}$ has the highest weight $\lambda = (1,0,\dots,0)$ and, thus, $\Gamma_{(1,0,\dots,0)}$ is the corresponding $SO(n)$ module.
- Let $\Gamma_{\mathbb{C}}$ denote the complexification of Γ. We already know that if Γ is a representation of $SO(n)$, then also $\Lambda^k\Gamma_{\mathbb{C}}$ is a representation.
  In the case where $0 \le k \le \lfloor \frac{n}{2} \rfloor - 1$, $\Lambda^k\Gamma_{\mathbb{C}}$ is an irreducible $SO(n)$ module with highest weight $\lambda = (1,\dots,1,0,\dots,0)$, where 1 appears k times.Thus $\Lambda^k\Gamma_{\mathbb{C}} = \Gamma_{(1,\dots,1,0,\dots,0)}$. This also holds in the case $n = 2k+1$.
  But if $n = 2k$, $\Lambda^k\Gamma_{\mathbb{C}}$ is a direct sum of two irreducible representations. More precisely, we get $\Lambda^k\Gamma_{\mathbb{C}} = \Gamma_{(1,\dots,1)} \oplus \Gamma_{(1,\dots,1,-1)}$. Note that for every $i \in \{0,\dots,n\}$, there is a natural isomorphism
  $$\Lambda^i\Gamma_{\mathbb{C}} \cong \Lambda^{n-i}\Gamma_{\mathbb{C}}. \tag{3.2}$$
- For $k \ge 2$, the space of symmetric tensors $\operatorname{Sym}^k \Gamma_{\mathbb{C}}$ is not an irreducible $SO(n)$ module. Its decomposition into a direct sum of irreducible submodules is given by
  $$\operatorname{Sym}^k \Gamma_{\mathbb{C}} = \bigoplus_{j=0}^{\lfloor k/2 \rfloor} \Gamma_{(k-2j,0,\dots,0)}. \tag{3.3}$$

The previous theorem shows that irreducible representations are characterized by their highest weights. So, to fully understand irreducible representations, it remains to parametrize all possible highest weights of irreducible representations. This will be done in the Highest Weight Classification Theorem in this chapter. We will see that there is a bijection between the set of dominant analytically integral weights and irreducible representations of G.

3.2 Regular Elements

On the way to our characterization of irreducible representations, we need the notion of regular elements. Let G be again a compact Lie group, T a maximal torus and $X \in \mathfrak{g}$. X is called a regular element of $\mathfrak{g}$ if $\mathfrak{z}_{\mathfrak{g}}(X) = \{Y \in \mathfrak{g} \mid [Y,X] = 0\}$ is a Cartan subalgebra. Recall the bijection between the set of analytically integral weights, $A(T)$, and the character group, $\chi(T)$, that maps $\lambda \in A(T)$ to $\xi_\lambda \in \chi(T)$ and satisfies $\xi_\lambda(\exp H) = e^{\lambda(H)}$ for $H \in T$.

We can also define regular elements of a Lie group.

Definition 3.2.1. *We say an element $g \in G$ is regular if $Z_G(g)^0$ is a maximal torus. Recall that $Z_G(g)$ denotes the centralizer of g in G and the superscript 0 means that we take the connected component of $Z_G(g)$ containing e.*
For the set of regular elements in G we write G^{reg} and $\mathfrak{g}^{reg}$ denotes the set of regular elements in $\mathfrak{g}$.
For $t \in T$, define $d(t) = \prod_{\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})}(1 - \xi_{-\alpha}(t))$.

These subsets of G and $\mathfrak{g}$ have a few nice properties.

Theorem 3.2.2 ([18, p. 156])**.** *Let G be a compact connected Lie group. The following properties hold:*

- *$\mathfrak{g}^{reg}$ is open dense in $\mathfrak{g}$.*
- *G^{reg} is open dense in G.*
- *If T is a maximal torus and $t \in T$, then $t \in T^{reg}$ if and only if $d(t) \neq 0$.*
- *For $H \in \mathfrak{t}$, e^H is regular if and only if $H \in \Xi = \{H \in \mathfrak{t} \mid \alpha(H) \notin 2\pi i\mathbb{Z}, \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})\}$.*
- *$G^{reg} = \bigcup_{g \in G}(gT^{reg}g^{-1})$.*

For the proof of the Weyl Character Formula we will need the following formula.

Theorem 3.2.3 ([18, p.161], Weyl Integration Formula)**.** *Let G be a compact connected Lie group, T a maximal torus, and $f \in C(G)$. Then*

$$\int_G f(g)dg = \frac{1}{|W(G)|}\int_T d(t) \int_{G/T} f(gtg^{-1})dgdt,$$

where $d(t) = \prod_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}})} |1 - \xi_{-\alpha}(t)|^2$ for $t \in T$.

3.3 Weyl Character Formula

In an earlier section we saw that we can map analytically integral weights bijective to the character group. We can define a similar concept for more general functions on $\mathfrak{t}$.

Definition 3.3.1. *Let G be a compact Lie group with maximal torus T.*

- *Let $f : \mathfrak{t} \to \mathbb{C}$ be a function. We say f descends to T if $f(H + Z) = f(H)$ for $H, Z \in \mathfrak{t}$ with $Z \in \ker(\exp)$. In that case, write $F : T \to \mathbb{C}$ for the function given by $F(e^H) := f(H)$.*
- *If $f : \mathfrak{t} \to \mathbb{C}$ satisfies $f(wH) = f(H)$ for $w \in W(\Delta(\mathfrak{g}_{\mathbb{C}})^{\vee})$, f is called W-invariant.*
- *If $F : T \to \mathbb{C}$ satisfies $F(c_w t) = F(t)$ for $w \in N(T)$, F is called W-invariant.*

- *If $f : \mathfrak{t} \to \mathbb{C}$ satisfies $f(wH) = \det(w)f(H)$ for $w \in W(\Delta(\mathfrak{g}_\mathbb{C}))$, f is called skew-W-invariant.*
- *If $F : T \to \mathbb{C}$ satisfies $F(c_w t) = \det(\mathrm{Ad}(w)|_\mathfrak{t})F(t)$ for $w \in N(T)$, F is called skew-W-invariant.*

In particular, for $\lambda \in A(T)$, the function $H \to e^{\lambda(h)}$ on $\mathfrak{t}$ descends to the function ξ_λ on T. Also note that $\det w \in \{\pm 1\}$ since w is a product of reflections.

We already know that the set of irreducible characters $\{\chi_\lambda\}$ is an orthonormal basis for the set of L^2 class functions on G. Assume G is simply connected. We will choose a skew-W-invariant function Δ defined on T, so that $|\Delta(t)|^2 = d(t)$. It easily follows from the Weyl Integration Formula that $\{\Delta\chi_\lambda|_T\}$ is therefore an orthonormal basis for the set of L^2 skew-W-invariant functions on T with respect to the measure $|W(G)|^{-1}dt$. On the other hand, by looking at alternating sums over the Weyl group of certain characters on T, we can write down a second basis for L^2 skew-W-invariant functions on T. But we will see that those bases are the same and that this yields an explicit formula for χ_λ, called the Weyl Character Formula.

Definition 3.3.2. *For a compact Lie group G with a maximal torus T, let $\Delta : \mathfrak{t} \to \mathbb{C}$ be given by*

$$\Delta(H) = \prod_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})} \left(e^{\alpha(H)/2} - e^{-\alpha(H)/2}\right)$$

for $H \in \mathfrak{t}$.

Lemma 3.3.3. *The previous function Δ has the following properties:*

- *It is skew-symmetric on $\mathfrak{t}$.*
- *The function descends to T if and only if the function $H \to e^{\rho(H)}$ descends to T, where $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})} \alpha$.*
- *The function $|\Delta|^2$ always descends to T and there $|\Delta(t)|^2 = d(t), t \in T$.*

With this lemma and the previous definition we can rewrite the Weyl Integration Formula in the following way:

$$\int_G f(g)dg = \frac{1}{|W(G)|} \int_T |\Delta(t)|^2 \int_{G/T} f(gtg^{-1})dgdt \tag{3.4}$$

For the next definition we need to know, that $\Xi = \{H \in \mathfrak{t} | \alpha(H) \notin 2\pi i \mathbb{Z} \text{ for all roots } \alpha\}$ is open dense in $\mathfrak{t}$ by the Baire Category Theorem and that $\exp \Xi = T^{reg}$.

Definition 3.3.4. *Let G be a compact Lie group with a maximal torus T. Fix an* *analytically integral weight $\lambda \in A(T)$. Define $\Theta_\lambda : \Xi \to \mathbb{C}$ by*

$$\Theta_\lambda(H) = \frac{\sum_{w\in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w) e^{[w(\lambda+\rho)](H)}}{\Delta(H)} = \frac{\sum_{w\in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w) e^{[w(\lambda+\rho)-\rho](H)}}{\prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})}(1-e^{-\alpha(H)})}$$

for $H \in \Xi$.

The next lemma tells us that the above defined function uniquely extends to a smooth class function on G^{reg}.

Lemma 3.3.5 ([18, p.165])**.** *Let G be a compact Lie group with a maximal torus T. Fix an analytically integral weight $\lambda \in A(T)$. The function Θ_λ descends to a smooth W-invariant function on G^{reg}.*

For the proof of the Weyl Character Formula we also need the following fact about elements of the character group $\chi(T)$. Take $\lambda, \lambda' \in A(T)$. Then the functions $\xi_\lambda, \xi_{\lambda'} : T \to T$ are 1-dimensional representations of T. These functions are equivalent if and only if they are the same, that is, when $\lambda = \lambda'$. Hence,

$$\int_T \xi_\lambda(t)\xi_{-\lambda'}(t)dt = \begin{cases} 1 & \text{if } \lambda = \lambda', \\ 0 & \text{if } \lambda \neq \lambda'. \end{cases} \tag{3.5}$$

The next theorem tells us how to calculate explicitly the character of an irreducible representation.

Theorem 3.3.6 (Weyl Character Formula)**.** *Let G be a compact connected Lie group with a maximal torus T. If V_λ is an irreducible representation of G with highest weight λ, then the character χ_λ of V_λ, satisfies*

$$\chi_\lambda(g) = \Theta_\lambda(g)$$

for $g \in G^{reg}$.

Proof. By Theorem 3.2.2 it is sufficient to prove the theorem only for $g = e^H$ where $H \in \Xi$. For arbitrary $\gamma \in A(T)$ define the following skew-symmetric function $D_\gamma : \mathfrak{t} \to \mathbb{C}$ by

$$D_\gamma(H) = \sum_{w\in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w) e^{(w\gamma)(H)}.$$

Now we have to show that

$$\xi_\lambda(e^H)\Delta(H) = D_{\lambda+\rho}(H)$$

for $H \in \mathfrak{t}$, where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})} \alpha$.

The weight space decomposition of $V(\lambda)$ and the correspondence between $A(T)$ and $\chi(T)$ allow us to write $\chi_\lambda = \sum_{\gamma_i\in A(T)} n_j \xi_{\gamma_j}$. The sum is finite and $n_j \in \mathbb{Z}^{\geq 0}$.

Thus,

$$\begin{aligned}\xi_\lambda(e^H)\Delta(H) &= \left(\sum_{\gamma_j\in A(T)} n_j e^{\gamma_j(H)}\right)\prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})}\left(e^{\alpha(H)/2}-e^{-\alpha(H)/2}\right)\\ &= \left(\sum_{\gamma_j\in A(T)} n_j e^{\gamma_j(H)}\right)e^{\rho(H)}\prod_{\alpha\in\Delta^+(\mathfrak{g}_\mathbb{C})}\left(1-e^{-\alpha(H)}\right)\\ &= \sum_{\gamma_j\in A(T)} m_j e^{(\gamma_i+\rho)(H)}\end{aligned}$$

for some $m_j\in\mathbb{Z}$. By definition χ_λ is a symmetric function. Since Δ is skew-symmetric, compare Theorem 3.3.3, we get that $\chi_\lambda(e^H)\Delta(H)$ is also skew-symmetric. This means that $\chi_\lambda\Delta\circ r_{h_\alpha}=-\chi_\lambda\Delta$ for $\alpha\in\Delta^+$. This property, together with the fact that the set $\{e^{\gamma_i+\rho}\mid\gamma_i\in A(T)\}$ is linearly independent, determines the value of m_j, if $r_\alpha(\gamma_i+\rho)=\gamma_i+\rho$. More precisely, $m_j=0$ whenever $\gamma_i+\rho$ lies on a Weyl chamber wall. Because of Theorem 2.5.6, we know that the Weyl group acts simply transitively on the open Weyl chambers. Thus for two elements γ_i,γ_j in the same Weyl chamber, there exists an element w of the Weyl group, such that $w\gamma_i=\gamma_j$. If we analyze the Weyl group orbits of $A(T)+\rho$ and keep the skew-symmetry in mind, we get

$$\chi_\lambda(e^H)\Delta(H)=\sum_{\gamma_i\in A(T),\gamma_i+\rho\text{ strictly dominant}} m_jD_{\gamma_j+\rho}(H).$$

The restriction to strictly dominant means that $\gamma_j+\rho$ has to lie in the open Weyl chamber, since it cannot lie on the Weyl chamber wall.

In the next step we calculate the m_j's. Therefore we remember from character theory that $\int_G|\chi_\lambda|^2dg=1$ and apply the Weyl Integration Formula (3.4) to $f=|\chi_\lambda|^2$:

$$1=\int\limits_G|\chi_\lambda|^2dg=\frac{1}{|W(G)|}\int\limits_T|\Delta|^2|\chi_\lambda|^2dt \tag{3.6}$$

$$=\frac{1}{|W(G)|}\int\limits_T\left|\sum_{\gamma_j\in A(T),\gamma_j+\rho\text{ strictly dominant}} m_jD_{\gamma_j+\rho}\right|^2dt. \tag{3.7}$$

Note that D_γ is defined on $\mathfrak{t}$, nevertheless we can calculate the integral over T. This is due to the fact that $\left|\sum_{\gamma_j\in A(T),\gamma_j+\rho\text{ strictly dominant}} m_jD_{\gamma_j+\rho}\right|^2$ descends to T since $|\Delta|^2|\chi_\lambda|^2$ descends to T, compare Lemma 3.3.3. In fact, even the function $H\to e^{-\rho(H)}D_{\gamma_j+\rho}$ descends to T. Thus, also the function $D_{\gamma_j+\rho}\overline{D_{\gamma_{j'}+\rho}}$ inherits this property since it is equivalent to $(e^{-\rho}D_{\gamma_j+\rho})\overline{(e^{-\rho}D_{\gamma_{j'}+\rho})}$.
In particular, we obtain

$$\frac{1}{|W(G)|}\int\limits_T D_{\gamma_j+\rho}\overline{D_{\gamma_{j'}+\rho}}dt=\frac{1}{|W(G)|}\sum_{w,w'\in W(\Delta(\mathfrak{g}_\mathbb{C}))}\det(ww')\int\limits_T\xi_{w(\gamma_j+\rho)}\xi_{-w'(\gamma_{j'}+\rho)}dt.$$

Now we use formula (3.5),

$$\int_T \xi_{w(\gamma_j+\rho)}\xi_{-w'(\gamma_{j'}+\rho)}dt = \begin{cases} 1 & \text{if } w(\gamma_j+\rho) = w'(\gamma_{j'}+\rho), \\ 0 & \text{otherwise .} \end{cases}$$

Since $w(\gamma_j+\rho) = w'(\gamma_{j'}+\rho)$ if and only if $w = w'$ and $j = j'$, we compute

$$\frac{1}{|W(G)|}\int_T D_{\gamma_j+\rho}\overline{D_{\gamma_{j'}+\rho}}dt = \begin{cases} \frac{1}{|W(G)|}\cdot|W(G)|\cdot 1 = 1 & \text{if } j = j', \\ 0 & \text{if } j \neq j'. \end{cases} \tag{3.8}$$

If we compare (3.6) and (3.8), then we obtain the following equation

$$1 = \sum_{\gamma_j\in A(T),\gamma_j+\rho \text{ strictly dominant}} m_j^2.$$

Since all m_j's are integers, the previous equation shows that all but one of the m_j's are zero. Thus there is an analytically integral weight γ with $\gamma+\rho$ strictly dominant such that

$$\chi_\lambda(e^H)\Delta(H) = \pm D_{\gamma+\rho}(H).$$

To finish the proof we have to calculate the right sign and show that $\gamma = \lambda$. Because of the weight space decomposition and the fact that λ is a highest weight, we can write

$$\chi_\lambda(e^H) = e^{\lambda(H)} + e^{\lambda_1(H)} + e^{\lambda_2(H)} + \cdots ,$$

where $\lambda_i \in A(T)$ and $\lambda > \lambda_i$. This yields

$$\begin{aligned}\chi_\lambda(e^H)\Delta(H) &= e^{\rho(H)}\chi_\lambda(e^H)e^{-\rho(H)}\Delta(H) \\ &= \left(e^{(\lambda+\rho)(H)} + e^{(\lambda_1+\rho)(H)} + \cdots\right)\prod_{\alpha\in\delta^+(\mathfrak{g}_\mathbb{C})}\left(1-e^{-\alpha(H)}\right).\end{aligned}$$

If we expand this product, then we see that $e^{\lambda+\rho}$ appears with the coefficient 1.

Now we look at $\pm D_{\gamma+\rho}$. If $\gamma_j+\rho$ has to be dominant, then the only term of the form $e^{\gamma_i+\rho}$ appearing in $\pm D_{\gamma+\rho}$ is $\pm e^{\gamma+\rho}$. By comparing these two statements, we obtain that $\gamma = \lambda$ and the right sign is $+$, which completes the proof. □

There is a consequence of the Weyl Character Formula, called the Second Determinantal Formula, which will be very useful in the proof of the general version of Hadwiger's Characterization Theorem. The formula states that $\chi_{\bar{\Gamma}_\lambda}$ can be calculated by multiplying characters F_i of the fundamental representations $\Lambda^i V_\mathbb{C}$ in a certain way. Note that $F_0 = F_n = 1$ and we set $F_i = 0$ for $i < 0$ and $i > n$.

Let λ be a highest weight with non-negative components satisfying (3.1) and define a new representation of $SO(n)$ by

$$\bar{\Gamma}_\lambda := \begin{cases} \Gamma_\lambda \oplus \Gamma_{\lambda'} & \text{if } n \text{ is even and } \lambda_{n/2} \neq 0, \\ \Gamma_\lambda & \text{otherwise.} \end{cases} \tag{3.9}$$

In this definition λ' denotes the tuple $(\lambda_1, \dots, \lambda_{n/2-1}, -\lambda_{n/2})$.

Define μ, the conjugate of λ, by the tuple $(\mu_1, \dots, \mu_s)$ where $s := \lambda_1$ and μ_j is the number of λ_i's in λ such that $\lambda_i \geq j$. We write $\#(\lambda, j)$ for the number of λ_i's that are equal to j. Sometimes we allow s to be greater than λ_1, but this just adds more zeros to the end of the conjugate tuple and does not change the following determinant.

Theorem 3.3.7 ([11, p.411], Second Determinantal Formula)**.** *Let λ be a highest weight with non-negative components satisfying (3.1) and let μ be the conjugate of λ defined as above. The character of $\bar{\Gamma}_\lambda$ equals the determinant of the $s \times s$-matrix whose ith row is given by*

$$\begin{pmatrix} F_{\mu_i-i+1} & F_{\mu_i-i+2} + F_{\mu_i-i} & \cdots & F_{\mu_i-i+s} + F_{\mu_i-i-s+2} \end{pmatrix} \tag{3.10}$$

As a consequence of the Second Determinantal Formula we have the following corollary.

Corollary 3.3.8. *If $i,j \in \mathbb{N}$ are such that $n/2 \leq i \leq n$ and $i + j \leq n$, then*

$$\sum_\lambda \chi_{\bar{\Gamma}_\lambda} = F_i F_j - F_{i-1} F_{j-1},$$

where the sum ranges over all $\lfloor n/2 \rfloor$-tuples of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$ satisfying (3.1) and

$$\lambda_1 \leq 2, \quad \#(\lambda,1) = n - i - j, \quad \#(\lambda,2) \leq j. \tag{3.11}$$

Proof. Let λ be a tuple of non-negative integers satisfying (3.1) and (3.11). Since $\lambda_1 \leq 2$, the conjugate μ of λ is of the form, $\mu = (\mu_1, \mu_2)$. From (3.11) we also get that there are no components of λ which are greater than 2, therefore $\mu_2 = \#(\lambda,2) \leq j$. A similar observation yields $\mu_1 - \mu_2 = \#(\lambda,1) = n - i - j$.
Now we use the Second Determinantal Formula:

$$\chi_{\bar{\Gamma}_\lambda} = \det \begin{pmatrix} F_{\mu_1} & F_{\mu_1+1} + F_{\mu_1-1} \\ F_{\mu_2-1} & F_{\mu_2} + F_{\mu_2-2} \end{pmatrix}.$$

If we set $k = n - i - j$ and use $\mu_1 = \mu_2 + k$, we get the following equation:

$$\begin{aligned} \sum_\lambda \chi_{\bar{\Gamma}_\lambda} &= \sum_{\mu_2=0}^{j} (F_{\mu_2+k}(F_{\mu_2} + F_{\mu_2-2}) - F_{\mu_2-1}(F_{\mu_2+k+1} + F_{\mu_2+k-1})) \\ &= F_{n-i}F_j - F_{n-(i-1)}F_{j-1} \\ &= F_i F_j - F_{i-1} F_{j-1}. \end{aligned}$$

The last equation follows from (3.2) and the fact that isomorphic representations have the same characters. □

The last statement in this chapter will be the following highest weight classification.

Theorem 3.3.9. *For a connected compact Lie group G with a maximal torus T, there is a one-to-one correspondence between irreducible representations and dominant analytically integral weights given by the mapping $V(\lambda) \to \lambda$ for dominant $\lambda \in A(T)$.*

4 Induced Representations and Branching Theorem

In this chapter we state and prove two theorems, which will be important in the proof of the Hadwiger-type Characterization Theorem.

4.1 Frobenius Reciprocity Theorem

Let G be a compact Lie group and H a closed subgroup. Let (Θ, π) be an arbitrary representation of G. It is obvious that we obtain a representation $(\operatorname{Res}_H^G \Theta, \operatorname{Res}_H^G(\pi))$ of H by restriction. We also want to know how we obtain from a given representation (Γ, ρ) of H a representation of G. This works with the following construction, which was taken from [3].

Let $C^\infty(G;\Gamma)$ be the space of all smooth functions from G to Γ. Define the following subspace $\operatorname{Ind}_H^G \Gamma \subseteq C^\infty(G;\Gamma)$ by

$$\operatorname{Ind}_H^G \Gamma := \left\{ f \in C^\infty(G;\Gamma) \mid f(gh) = \rho^{-1}(h)f(g) \text{ for all } g \in G, h \in H \right\}.$$

The corresponding induced representation of G on $\operatorname{Ind}_H^G \Gamma$ is given by the left translation action of G on $\operatorname{Ind}_H^G \Gamma$:

$$L(g)f(x) = f(g^{-1}x), \quad g, x \in G.$$

Now we can state the fundamental result on induced representations.

Theorem 4.1.1 (Frobenius Reciprocity)**.** *Let G be a Lie group and H a closed subgroup of G. If Γ is a representation of H and Θ is a representation of G, then as vector spaces*

$$\operatorname{Hom}_G(\Theta, \operatorname{Ind}_H^G \Gamma) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G \Theta, \Gamma).$$

Proof. First take $T \in \operatorname{Hom}_G(\Theta, \operatorname{Ind}_H^G \Gamma)$ and define a function $S_T \in \operatorname{Hom}_H(\operatorname{Res}_H^G \Theta, \Gamma)$ by evaluating T on the identity element, that is, $S_T(w) = (T(w))(e)$ for $w \in \Theta$. It follows that S_T is H-equivariant:

$$\begin{aligned} S_T(\pi(h)w) &= T(\pi(h)w)(e) = (L(h)T(w))(e) \\ &= T(w)(h^{-1}) = \rho(h)T(w)(e) = \rho(h)S_T(w), \end{aligned}$$

for $h \in H$ and $w \in \Gamma$. Here, the first equation is just the definition of S_T and the second equation follows from the G-equivariance of T. The first term in the second row comes from the definition of the left translation action. Since $T(w) \in \operatorname{Ind}_H^G \Gamma$ we get the final equation by the definition of $\operatorname{Ind}_H^G \Gamma$.

Next take $S \in \operatorname{Hom}_H(\operatorname{Res}_H^G \Theta, \Gamma)$ and define a function $T_S \in \operatorname{Hom}_G(\Theta, \operatorname{Ind}_H^G \Gamma)$ by $(T_S(u))(g) = S(\pi(g)^{-1}u)$ for $u \in \Theta$ and $g \in G$. We see, that $T_S(u)(g)$ is infact an element of $\operatorname{Ind}_H^G \Gamma$:

$$(T_S(u))(gh) = S(\pi(h)^{-1}\pi(g)^{-1}u) = \rho(h)^{-1}S(\pi(g)^{-1}u) = \rho(h)^{-1}(T_S(u))(g).$$

Moreover, for $g, g' \in G$,

$$(L(g)T_S(u))(g') = T_S(g^{-1}g') = S(\pi(g')^{-1}\pi(g)u) = T_S(\pi(g)u)(g'),$$

which shows that T_S is G-equivariant.

It remains to prove that the maps $T \mapsto S_T$ and $S \mapsto T_S$ are inverse. Take $w \in \Gamma$, since $(T_S(u))(e) = S(u)$ we have $S_{T_S} = S$. For the other direction we have the Γ-valued function $T_{S_T}w$:

$$g \mapsto S_T(\pi(h)^{-1}w) = T(\pi(g)^{-1}w)(e) = (L(g^{-1})T(w))(e) = (T(w))(g).$$

Thus $T_{S_T} = T$, which finishes the proof. □

The Frobenius Reciprocity Theorem implies an equality of multiplicities of representations. Recall that if Θ is an irreducible representation of G, the multiplicity $m(\Psi, \Theta)$ of Θ in an arbitrary representation Ψ of G is given by

$$m(\Psi, \Theta) = \dim \operatorname{Hom}_G(\Psi, \Theta) = \dim \operatorname{Hom}_G(\Theta, \Psi).$$

This follows from Schur's Lemma. If we combine this statement with the Frobenius Reciprocity Theorem, we get that if Θ and Γ are irreducible, then

$$m(\operatorname{Ind}_H^G \Gamma, \Theta) = m(\operatorname{Res}_H^G \Theta, \Gamma). \tag{4.1}$$

We will apply this statement in the situation, where $G = SO(n)$ and $H = SO(n-1)$. Since the restricted representation $\operatorname{Res}_{SO(n-1)}^{SO(n)} \Gamma$ of an irreducible representation Γ is not irreducible, we need to know how to decompose $\operatorname{Res}_{SO(n-1)}^{SO(n)} \Gamma$ into irreducible representations of $SO(n-1)$. This will be the purpose of the next section.

4.2 Branching Theorem for $SO(n)$

A theorem about computing multiplicities for an irreducible representation after restriction to a closed subgroup is called a branching theorem. In this section we will state the branching theorem for $SO(n)$ and the closed subgroup $SO(n-1)$. The proof was taken from [13]. We use the standard embedding of the subgroup $SO(n-1)$ "in the upper left block of $SO(n)$" and we parametrize irreducible representations of $SO(n)$ and $SO(n-1)$ as usual by their highest weights, that is, by a tuple of integers $\lambda = (\lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor})$ satisfying (3.1).

Now we can state the following theorem.

Theorem 4.2.1 (Branching Theorem for $SO(n)$)**.** *Let Γ_λ be an irreducible representation of $SO(n)$ with highest weight $\lambda = (\lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor})$ satisfying (3.1), then the restricted representation decompose with multiplicity 1 under $SO(n-1)$ and we get*

$$\mathrm{Res}^{SO(n)}_{SO(n-1)} \Gamma_\lambda = \bigoplus_\mu \Gamma_\mu,$$

where the sum ranges over all $\mu = (\mu_1, \ldots, \mu_k)$ with $k := \lfloor (n-1)/2 \rfloor$ and

$$\begin{cases} \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{k-1} \geq \lambda_{\lfloor n/2 \rfloor} \geq |\mu_k| & \text{for odd } n, \\ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_k \geq |\lambda_{n/2}| & \text{for even } n. \end{cases} \tag{4.2}$$

The proof of this statement, which is taken from [13], uses Kostant's Branching Theorem. We will use this theorem without giving a proof. But before we do so, we have to fix some notation. Let G be a connected compact Lie group with a maximal torus T and H a connected closed subgroup with maximal torus S. We also require that $Z_G(S)$ is a maximal torus of G. As usual we denote the corresponding complexified Lie algebras by $\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C}, \mathfrak{h}_\mathbb{C}$ and $\mathfrak{s}_\mathbb{C}$. In the following theorem $\bar{\cdot}$ denotes the restriction from $(\mathfrak{t}_\mathbb{C})^*$ to $(\mathfrak{s}_\mathbb{C})^*$. δ_G is the half of the sum of the members of $\Delta^+(\mathfrak{g}_\mathbb{C})$. The elements of $\Delta^+(\mathfrak{g}_\mathbb{C})$ restricted to $\mathfrak{s}_\mathbb{C}$, repeated according to their multiplicities, are the nonzero positive roots of $\mathfrak{s}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$. If we delete from this set the elements from $\Delta^+(\mathfrak{h}_\mathbb{C})$, each with multiplicity 1, we obtain the set of positive weights of $\mathfrak{s}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}/\mathfrak{h}_\mathbb{C}$, repeated according to multiplicities. We denote this set by Σ. Finally, we also need the Kostant partition function $\mathcal{P}$ on $(\mathfrak{s}_\mathbb{C})^*$. It is defined in the following way: $\mathcal{P}(\nu)$ is the number of ways that $\nu \in (\mathfrak{s}_\mathbb{C})^*$ can be written as a sum of elements of Σ. Note that every multiple of a member of Σ is counted as a distinct element.

Now we can state the following theorem.

Theorem 4.2.2 ([13, p. 573], Kostant's Branching Theorem)**.** *Let G be a compact connected Lie group with maximal torus T, let H be a closed connected subgroup with maximal torus S and suppose that the centralizer $Z_G(S)$ is abelian, which implies that it is a maximal torus of G. Let $\lambda \in (\mathfrak{t}_\mathbb{C})^*$ be the highest weight of an irreducible representation π of G, and let $\mu \in (\mathfrak{s}_\mathbb{C})^*$ be the highest weight of an irreducible representation ρ of H. Then the multiplicity of ρ in the restriction of π to H is given by*

$$m_\lambda(\mu) = \sum_{w \in W(G)} \det(w) \mathcal{P}(\overline{w(\lambda + \delta_G) - \delta_G} - \mu). \tag{4.3}$$

The assumption on the centralizer is satisfied if either $H = T$ or if the subgroup H is the identity component of the set of fixed points of an involution of G. We can use Kostant's Branching Theorem to prove branching from $G = SO(n)$ to $H = SO(n-1)$, since H is the identity component of the set of fixed points of the involution of G given by conjugation by the $n \times n$-matrix $\mathrm{diag}(1, \ldots, 1, -1)$. Therefore we can prove

Theorem 4.2.1 by using (4.3), but we have to look at the cases where n is even and where n is odd separately.

Proof of Branching Theorem from SO(2n+1) to SO(2n). We already know that the highest weight of $SO(2n+1)$ is $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ and the highest weight for $SO(2n)$ is $\mu = (\mu_1, \dots, \mu_n)$ satisfying $\mu_1 \geq \dots \geq \mu_{n-1} \geq |\mu_n|$.

In this case (4.3) simplifies to

$$m_\lambda(\mu) = \sum_{w \in W(G)} \det(w) \mathcal{P}(w(\lambda + \delta) - (\mu + \delta)). \tag{4.4}$$

If we compute the elements w of the Weyl group of $SO(2n+1)$, we see that they are of the form $w = sp$, where s is a sign change and p a permutation. Also note that Σ is the set of ε_i's with $1 \leq i \leq n$.

For $\mathcal{P}$ and δ we get the following expressions:

$$\mathcal{P}(\nu) = \begin{cases} 1 & \text{if } \langle \nu, \varepsilon_j \rangle \geq 0 \text{ for all } j \leq n, \\ 0 & \text{otherwise,} \end{cases}$$
$$\delta = \left(n + \frac{1}{2}\right) \varepsilon_1 + \left(n - \frac{1}{2}\right) \varepsilon_2 + \dots + \frac{1}{2} \varepsilon_n.$$

Now we have to show that $m_\lambda(\mu) = 1$ if (4.2) holds and $m_\lambda(\mu) = 0$ otherwise.

We will do this in four steps. First, we find a necessary condition for $m_\lambda(\mu)$ to be nonzero. Second, we compute the value of the wth term of the sum in (4.4). Afterward, we state two auxiliary results, which will help us to conclude the proof.

Lemma. The wth term can contribute to $m_\lambda(\mu)$ only if $s = 1$ or if $s = r_{\varepsilon_n}$, where r_α denotes the reflection defined in Definition 2.5.3.

Since $\mathcal{P}(\nu)$ equals 1 if and only if $\langle \nu, \varepsilon_j \rangle \geq 0$, we consider

$$\begin{aligned} &\langle w(\lambda + \delta) - (\mu + \delta), \varepsilon_j \rangle \geq 0 \\ &\qquad \Leftrightarrow \langle w(\lambda + \delta), \varepsilon_j \rangle \geq \langle (\mu + \delta), \varepsilon_j \rangle, \end{aligned}$$

for $j < n$. Since $\langle (\mu + \delta), \varepsilon_j \rangle > 0$, we must have $\langle w(\lambda + \delta), \varepsilon_j \rangle > 0$ otherwise the corresponding summand in (4.4) would be zero. Therefore $w^{-1}\varepsilon_j > 0$ for $j < n$, and $p^{-1}s^{-1}\varepsilon_j > 0$ for $j < n$, because of the form of w, which yields $s^{-1}\varepsilon_j > 0$. Thus $s = 1$. In the case $j = n$, we get $s = r_{\varepsilon_n}$.

Lemma. Fix $i < n$, and suppose that $\mu_j \geq \lambda_{j+1}$ for $j \leq i$. Then the only case where $\mathcal{P}(w(\lambda + \delta) - (\mu + \delta))$ is nonzero, is if $w\varepsilon_j = \varepsilon_j$ for $j \leq i$.

Fix $l \leq i$ and choose $r = r(l)$ such that $w\varepsilon_r = \varepsilon_l$. Note that the previous lemma implies that we do not have to consider the case $w\varepsilon_r = -\varepsilon_l$. Thus, consider

$$\begin{aligned}\langle w(\lambda+\delta)-(\mu+\delta),\varepsilon_l\rangle &= \langle \lambda+\delta,\varepsilon_r\rangle - \langle \mu+\delta,\varepsilon_l\rangle \\ &= \lambda_r + n + \frac{1}{2} - r - \mu_l - n - \frac{1}{2} + l \\ &= (\lambda_r - \mu_l) - (r - l).\end{aligned}$$

This equation has to be ≥ 0 if the wth term in (4.4) should be nonzero. Hence, we get

$$\lambda_r \geq \mu_l + (r-l) \geq \lambda_{l+1} + (r-l).$$

The second inequality follows from our assumption. Now we use induction:

- Let $l = 1$. We get $\lambda_r \geq \lambda_2 + (r-1)$. If $r \geq 2$ then $\lambda_2 \geq \lambda_r \geq \lambda_2 + (r-1)$, which is a contradiction. The first inequality follows from the form of the highest weight λ. Thus if $l = 1$, r has to be 1 and $w\varepsilon_1 = \varepsilon_1$.
- Let $1 < l \leq i$ and suppose that $w\varepsilon_j = \varepsilon_j$ for $j < l$.
- Suppose $r(l) \geq l$ and $w\varepsilon_{r(l)} = \varepsilon_l$. We know from above that

  $$\lambda_{r(l)} \geq \lambda_{l+1} + (r(l) - l).$$

  If $r(l) > l$, then

  $$\lambda_{l+1} \geq \lambda_{r(l)} \geq \lambda_{l+1} + (r(l) - l) > \lambda_{l+1}$$

  which is a contradiction. Thus $r(l) = l$, and the induction is complete.

Lemma. If $\mu_1 \geq \lambda_2, \mu_2 \geq \lambda_3, \ldots, \mu_{n-1} \geq \lambda_n$ hold, then

$$m_\lambda(\mu) = \begin{cases} 1 & \text{if } \lambda_i \geq \mu_i \text{ for } 1 \leq i \leq n-1 \text{ and } \lambda_n \geq |\mu_n|, \\ 0 & \text{otherwise .}\end{cases}$$

The wth term can contribute to $m_\lambda(\mu)$ only if $w\varepsilon_j = \varepsilon_j$ for $j \leq n-1$. Thus we have to consider the terms where $w = 1$ and $w = r_{\varepsilon_n}$. Note that

$$\mathcal{P}(1(\lambda+\delta) - (\mu+\delta)) = \mathcal{P}(\lambda - \mu).$$

From the first lemma we obtain that this equals 1 if and only if $\lambda_i - \mu_i \geq 0$ for $i < n$ and $\lambda_n - |\mu_n| \geq 0$. For $w = r_{\varepsilon_n}$, we get

$$\mathcal{P}(r_{\varepsilon_n}(\lambda+\delta) - (\mu+\delta)) = 0.$$

Lemma. If one or more of the inequalities $\mu_1 \geq \lambda_2, \mu_2 \geq \lambda_3, \ldots, \mu_{n-1} \geq \lambda_n$ do not hold, then $m_\lambda(\mu) = 0$.

We choose the smallest i such that $\mu_i < \lambda_{i+1}$. Because of our previous considerations we only have to look at terms where $w\varepsilon_j = \varepsilon_j$. We shall show that the

wth term in the sum cancels with the wpth term, where p is the reflection in the root $\varepsilon_i - \varepsilon_{i+1}$. Since $\det(w) = 1$ and $\det(wp) = -1$ we only have to show that the corresponding values of $\mathcal{P}$ are equal.

Define k and l by $w\varepsilon_i = \pm\varepsilon_k$ and $w\varepsilon_{i+1} = \pm\varepsilon_l$. Since we already know that $w\varepsilon_j = \varepsilon_j$ if $j < i$, we get that $k \geq i$ and $l \geq i$. We have to carry the minus sign for the cases where $k = n$ or $l = n$. We have

$$wp(\lambda + \delta) - (\mu + \delta) = w(\lambda + \delta) - (\mu + \delta) - (\lambda_i - \lambda_{i+1} + 1)w(\varepsilon_i - \varepsilon_{i+1}).$$

Thus the arguments of $\mathcal{P}$ for wp and w coincide except for the kth and lth component. But an easy calculation shows that

$$\langle wp(\lambda + \delta) - (\mu + \delta), \varepsilon_k \rangle \geq 0,$$
$$\langle w(\lambda + \delta) - (\mu + \delta), \varepsilon_k \rangle \geq 0,$$

if $w\varepsilon_i = +\varepsilon_k$. For $k = n$, if $w\varepsilon_i = -\varepsilon_n$, then the above terms are both < 0. Thus the arguments coincide also in the kth component. It remains to show the same for the lth component. We see again

$$\langle wp(\lambda + \delta) - (\mu + \delta), \varepsilon_l \rangle \geq 0,$$
$$\langle w(\lambda + \delta) - (\mu + \delta), \varepsilon_l \rangle \geq 0,$$

if $w\varepsilon_{i+1} = +\varepsilon_l$. As in the previous case we can compute, that the terms above are < 0 if $w\varepsilon_{i+1} = -\varepsilon_l$ for $l = n$.

We proved that the arguments of $\mathcal{P}$ for w and wp always have the same sign and therefore

$$\mathcal{P}(wp(\lambda + \delta) - (\mu + \delta)) = \mathcal{P}(w(\lambda + \delta) - (\mu + \delta)),$$

which proves the lemma and at the same time finishes the proof of the branching theorem for $SO(2n + 1)$.

The proof of the branching theorem from $SO(2n)$ to $SO(2n - 1)$ works almost the same. We therefore refer to [13, p.582] for details. $\square$

5 Valuations and Normal Cycles

In this chapter we first recall the notion of valuations, give a definition of the normal cone and use it to define an integral current, the normal cycle. Afterward we are able to connect special valuations to this current. We follow [5], which is based on [2] and [4]. In the following sections let V be an n-dimensional Euclidean vector space, $\mathcal{K}(V)$ the set of all convex, compact, non-empty subsets of V, and A an Abelian semi-group.

5.1 Valuations

Definition 5.1.1. *A function $\phi : \mathcal{K}(V) \to A$ is called a valuation, if*

$$\phi(K \cup L) + \phi(K \cap L) = \phi(K) + \phi(L)$$

whenever $K, L, K \cup L \in \mathcal{K}(V)$.

For us, the cases $A = \mathbb{C}$ (complex-valued valuations) and $A = \operatorname{Sym}^k V$ (tensor valuations) are of particular interest.

Definition 5.1.2. *A valuation ϕ is called translation-invariant if $\phi(K + v) = \phi(K)$ for all $v \in V$ and $K \in \mathcal{K}(V)$. It is said to have degree i if $\phi(tK) = t^i \phi(K)$ for all $K \in \mathcal{K}(V)$ and $t > 0$. A valuation ϕ is called even if $\phi(-K) = \phi(K)$ and odd if $\phi(-K) = -\phi(K)$ for all $K \in \mathcal{K}(V)$. It is called continuous if it is continuous with respect to the Hausdorff metric on $\mathcal{K}(V)$.*

We denote the vector space of all continuous, translation-invariant, complex-valued valuations by $\mathbf{Val}$ and write $\mathbf{Val}_i^{\pm}$ for its subspaces of all i-homogeneous valuations with even/odd parity. Note that $\mathbf{Val}_0$ and $\mathbf{Val}_n$ are one-dimensional. The first statement is obvious and the second non-trivial statement for n-homogeneous valuations was proved by Hadwiger [12].

A very important result by McMullen [17] states that there exists the following decomposition:

$$\mathbf{Val} = \bigoplus_{i=0}^{n} (\mathbf{Val}_i^+ \oplus \mathbf{Val}_i^-). \tag{5.1}$$

The following corollary of McMullen's decomposition shows that there is a Banach space structure on $\mathbf{Val}$:

Corollary 5.1.3. *Let $C \in \mathcal{K}(V)$ be a fixed convex body with non-empty interior. The space $\mathbf{Val}$ becomes a Banach space when endowed with the norm*

$$\|\phi\| := \sup\{|\phi(K)| : K \subseteq C\}.$$

The Banach space structure is unique, since another choice of C gives an equivalent norm.

The natural continuous action of the general linear group $GL(n)$ on the Banach space **Val** is defined by

$$g\phi(K) = \phi(g^{-1}K), \quad g \in GL(n), \phi \in \mathbf{Val}, K \in \mathcal{K}(V).$$

Therefore, it is easy to check that with this action **Val** becomes an $SO(n)$ module.

The next fundamental theorem was shown by S. Alesker [1] and is known as the *Irreducibility Theorem.*

Theorem 5.1.4. *The natural action of $GL(n)$ on the space of even/odd translation-invariant continuous valuations with given degree of homogeneity is irreducible.*

There are two important dense subsets of **Val**:

Definition 5.1.5. *A valuation $\phi \in \mathbf{Val}$ is called smooth if the map*

$$\begin{aligned} GL(n) &\to \mathbf{Val}, \\ g &\mapsto g\phi, \end{aligned}$$

is infinitely differentiable. We denote the set of all smooth translation-invariant valuations by $\mathbf{Val}^\infty$. A valuation $\phi \in \mathbf{Val}$ is called $O(n)$-finite if

$$\dim\left(\operatorname{span}\left\{\vartheta\phi | \vartheta \in O(n)\right\}\right) < \infty.$$

$\mathbf{Val}^f$ *is the set of all continuous translation-invariant and $O(n)$-finite valuations.*

We will also need the subspaces of even or odd, $O(n)$-finite and smooth valuations of degree i, for which we write $\mathbf{Val}_i^{\pm,f}$ and $\mathbf{Val}_i^{\pm,\infty}$. The set of $O(n)$-finite valuations $\mathbf{Val}_i^{\pm,f}$ is a dense $O(n)$-invariant subspace of $\mathbf{Val}_i^{\pm}$ and the set of smooth valuations is a dense $GL(n)$-invariant subspace of $\mathbf{Val}_i^{\pm}$. For more details see [9, p. 141]. One can also show that $\mathbf{Val}^f \subseteq \mathbf{Val}^\infty$. From (5.1) it is easily derived that the following holds:

$$\begin{aligned} \mathbf{Val}^f &= \bigoplus_{i=0}^{n} (\mathbf{Val}_i^{+,f} \oplus \mathbf{Val}_i^{-,f}), \\ \mathbf{Val}^\infty &= \bigoplus_{i=0}^{n} (\mathbf{Val}_i^{+,\infty} \oplus \mathbf{Val}_i^{-,\infty}). \end{aligned}$$

This will be important in the proof of our main theorem.

5.2 Integration of Differential Forms on Rectifiable Sets

The goal of this section is to give a description of smooth translation-invariant valuations in terms of differential forms. But before we can do this, we need some background about integration of differential forms over rectifiable sets and the notion of the normal cycle. Therefore this section will be a short summary of this material. We expect a certain familiarity with measure theory, especially the definition of the Hausdorff measure $\mathcal{H}^m$. For further reading on the topics discussed we refer to [14] and [16].

First we recall the definition of rectifiable sets. One can think of them as generalized submanifolds.

Definition 5.2.1. *An $\mathcal{H}^m$-measurable set $M \subseteq \mathbb{R}^n$ with $\mathcal{H}^m(M) < \infty$ is called countably m-rectifiable if*

$$M \subseteq \bigcup_{j=0}^{\infty} N_j, \tag{5.2}$$

where $\mathcal{H}^m(N_0) = 0$ holds and every N_j for $j \geq 1$ is an m-dimensional embedded C^1-submanifold of $\mathbb{R}^n$.

The following statement shows that a rectifiable set consists of *nice* sets.

Lemma 5.2.2. *If the set $M \subseteq \mathbb{R}^n$ is measurable and countably m-rectifiable, $m \geq 1$, then there are*

- *countably many m-dimensional, embedded C^1-submanifolds N_j, $j > 1$, of $\mathbb{R}^n$,*
- *compact subsets $M_j \subseteq N_j$ and*
- *a set M_0 with $\mathcal{H}^m(M_0) = 0$,*

such that $\{M_j\}_{i\in\mathbb{N}}$ is a partition of M, that is, $M = \bigcup_{j=0}^{\infty} M_j$ and $M_i \cap M_j = \emptyset$, if $i \neq j$.

This lemma will be helpful to define integration over rectifiable sets similar to integration over manifolds.

Definition 5.2.3. *Let $M \subseteq \mathbb{R}^n$ be an $\mathcal{H}^m$-measurable subset with $\mathcal{H}^m(M) < \infty$, $x \in M$ and P an m-dimensional subspace of $\mathbb{R}^n$. Then P is called an approximate tangent space of M in $x \in \mathbb{R}^n$ if for every $f \in C_c^\infty(\mathbb{R}^n)$ the following equation holds:*

$$\lim_{\lambda\to 0^+} \int_{\eta_{x,\lambda}(M)} f(y)d\mathcal{H}^m(y) = \int_P f(y)d\mathcal{H}^m(y), \tag{5.3}$$

where $\eta_{x,\lambda}(y) := \frac{y-x}{\lambda}$. If the approximate tangent space P exists, it is unique and we denote it by $\mathcal{T}_x M$.

Note that if M is a C^1-submanifold, the approximate tangent space corresponds to the *usual* tangent space.

Theorem 5.2.4 ([6, p.39])**.** *Let $M \subseteq \mathbb{R}^n$ be m-rectifiable. The approximate tangent space $\mathcal{T}_x M$ exists for $\mathcal{H}^m$-almost every $x \in M$.*

Next we want to define integration over rectifiable sets. Therefore we recommend to keep the well-known theory of integration on manifolds in mind. See [15] for the details. We just want to recall the definition of differential forms, mostly to set the notation for the rest of this work. This definition is taken from [15].

Definition 5.2.5. *Let M be a smooth manifold. The bundle of covariant k-tensors on M is defined by*

$$T^k M = \coprod_{p \in M} T^k(T_p M).$$

The subset of $T^k M$ of alternating k-tensors is denoted by $\Lambda^k M$, that is,

$$\Lambda^k M = \coprod_{p \in M} \Lambda^k(T_p M).$$

We call a map $\omega : M \to \Lambda^k M$ such that for k smooth tensor fields $X_1, \ldots, X_k$ the map $p \mapsto \omega_p(X_1(p), \ldots, X_k(p))$ is smooth, a k-differential form. So it is basically just a tensor field which gives you at every point of M an alternating tensor. The space of all k-differential forms is denoted by $\Omega^k(M)$.

For the integration of differential forms on manifolds we need oriented manifolds. Since integration on rectifiable sets is very similar, we require oriented rectifiable sets.

Definition 5.2.6. *Let $M \in \mathbb{R}^n$ be a k-rectifiable set. We say M is oriented by $\xi : M \to \Delta_n \mathbb{R}^n$ if ξ is $\mathcal{H}^k$-measurable and for $\mathcal{H}^k$-almost every x we have $|\xi(x)| = 1$ and $\xi(x)$ is the $\wedge$-product of k tensors of a basis of $T_x M$.*

Finally, we can define integration over rectifiable sets.

Definition 5.2.7. *Let $M \in \mathbb{R}^n$ be a k-rectifiable set oriented by $\xi : M \to \Lambda_n \mathbb{R}^n$. The integral of a k-form ω on M is defined by*

$$\int_M \omega = \int_M \langle \xi(x), \omega(x) \rangle \, d\mathcal{H}^k(x). \tag{5.4}$$

Now we use Lemma 5.2.2 to rewrite the integral above in familiar terms. Take Lipschitz functions f_i such that $f_i(N_i)$ are pairwise disjoint, then we can define

$$\int_M \omega = \sum_{i=1}^{\infty} \int_{N_i} f_i^* \omega. \tag{5.5}$$

The Lipschitz functions take the place of the partition of unity, since the images are disjoint.

5.3 Currents

The following material is taken from [6].

Definition 5.3.1. *Let $U \in \mathbb{R}^n$ be an open set and $0 \leq k \leq n$. A k-dimensional current on U is a continuous linear functional on $\mathcal{D}^k(U)$, the space of infinitely differential k-forms with compact support in U. The space of all k-dimensional currents is denoted by $\mathcal{D}_k(U)$.*

Definition 5.3.2. *Let $T \in \mathcal{D}_k(U)$. The boundary of T is defined as the $(k-1)$-current*

$$\partial T(\omega) := T(d\omega), \quad \omega \in \mathcal{D}^{k-1}(U),$$

where $\partial T = 0$, if $k = 0$.

Since we are interested in integral currents, we need the following definition.

Definition 5.3.3. *Let U be an open set in $\mathbb{R}^n$, $1 \leq k \leq n$ and $T \in \mathcal{D}_k(U)$. We say that T is an integer multiplicity rectifiable k-current if there are*

- *a countable k-rectifiable set $M \in U$ which is $\mathcal{H}^k$-measurable,*
- *an $\mathcal{H}^k$-measurable and locally $\mathcal{H}^k$-integrable nonnegative function $\theta : M \to \mathbb{R}$,*
- *an $\mathcal{H}^k$-measurable map $\xi : M \to \Lambda_k\mathbb{R}^n$ such that for $\mathcal{H}^k$ almost every point $x \in M$, $\xi(x)$ is a simple unit k-tensor in T_xM such that*

$$T(\omega) = \int_M \langle \xi(x), \omega(x) \rangle \, \theta(x) d\mathcal{H}^k(x).$$

We will denote the class of integer multiplicity rectifiable k-currents in U by $\mathcal{R}_k(U)$. Let $K \subseteq U$ be a compact set. We write $\mathcal{R}_{k,K}$ for the class of integer multiplicity rectifiable k-currents with support in K

$$\mathcal{R}_{k,K}(U) := \{T \in \mathcal{R}_k(U) : \operatorname{spt} T \in K\}.$$

The set of integral k-currents with support in K is defined by

$$I_{k,K}(U) := \{T \in \mathcal{R}_{k,K}(U) : \partial T \in \mathcal{R}_{k-1,K}(U)\},$$

and $\mathfrak{I}_k(U)$ denotes the class of all integral k-currents in U.

Later on we want to talk about continuous functions from the space of convex sets $\mathcal{K}(V)$ to the space of integral currents, therefore we need a topology on that space.

Definition 5.3.4. *The flat seminorm of a k-current $\phi \in \mathcal{D}_k(U)$ relative to K is defined by*

$$\|\phi\|_K^\flat := \max\left\{ \sup_{x \in K} \|\phi(x)\|, \sup_{x \in K} \|d\phi(x)\| \right\}.$$

The flat norm of $T \in \mathcal{D}_k(U)$ *relative to* K *is defined by*

$$\|T\|_\flat^K := \sup\left\{T(\phi)|\phi \in \mathcal{D}^k(U), \|\phi\|_K^\flat \leq 1\right\}.$$

The local flat topology on $\mathcal{D}_k(U)$ *is determined by the condition that* $T_j \to T$ *if and only if*

$$\|T - T_j\|_\flat^K \to 0$$

for every compact $K \subseteq U$.

5.4 Translation Invariant Smooth Valuations and Differential Forms

In this section we will first introduce the normal cycle of a convex body and explain how it can be identified with an integral current. Afterward we will see how one can connect valuations on convex bodies with integration of differential forms over rectifiable sets. Also in this section we will not give proofs, but recommend [2] and [4] for details.

Definition 5.4.1. *Let* K *be a convex body. The normal cycle* $\mathrm{Nc}(K)$ *is the integral current in* $\mathfrak{I}_{n-1}(SV)$ *defined by*

$$\mathrm{Nc}(K)(\omega) := \int\limits_{\mathrm{nc}(K)} \omega,$$

where $\mathrm{nc}(K)$ *is the* $(n-1)$*-rectifiable set defined by*

$$\mathrm{nc}(K) = \{(x,u) \in SV | x \in \partial K, u \in N(K,x)\}.$$

Here $N(K,x)$ *denotes the normal cone of* K *at* x *and* $SV = V \times S^{n-1}$ *is the tangent sphere bundle.*

The normal cycle has several nice properties, which we will state without giving the proofs.

Theorem 5.4.2.
- Nc *is a continuous injection from* $\mathcal{K}(V)$ *to* $\mathfrak{I}_{n-1}(SV)$ *with the Hausdorff topology on* $\mathcal{K}(V)$ *and the local flat topology on* $\mathfrak{I}_{n-1}(SV)$.
- Nc *is a valuation, that is,* $\mathrm{Nc}(A \cup B) + \mathrm{Nc}(A \cap B) = \mathrm{Nc}(A) + \mathrm{Nc}(B)$ *if* $A, B, A \cup B, A \cap B \in \mathcal{K}(V)$.

Denote the space of all C^1-smooth differential forms of degree $n-1$ with finite flat norm on SV by $C^1_\flat(SV)$. Then we obtain the following conclusion. For $\omega \in C^1_\flat(SV)$ the map

$$\begin{aligned}\mathcal{K}(V) &\to \mathbb{R},\\ K &\mapsto \int\limits_{nc(K)} \omega,\end{aligned}$$

defines a continuous valuation on $\mathcal{K}(V)$. We also get the following theorem.

Theorem 5.4.3. *The map given by*

$$\begin{aligned} \mathcal{K}(V) \times C^1_\flat(SV) &\to \mathbb{R}, \\ (K, \omega) &\mapsto \mathrm{Nc}(K)(\omega), \end{aligned}$$

is continuous.

Therefore we see that the map

$$\begin{aligned} \mathcal{K}(V) \times \mathbb{R} \times C^1_\flat(SV) &\to \mathbb{R}, \\ (K, a, \omega) &\mapsto a \cdot Vol_V(K) + \int\limits_{\mathrm{nc}(K)} \omega, \end{aligned}$$

is continuous.

We denote the space of continuous valuations in V equipped with the topology of uniform convergence on compact subsets of $\mathcal{K}(V)$ by $CV(V)$. If we put all the previous facts together, we obtain the next corollary.

Corollary 5.4.4. *The map*

$$\begin{aligned} \mathbb{R} \times C^1_\flat(SV) &\to CV(V), \\ (a, \omega) &\mapsto \left[K \mapsto a \cdot Vol_V(K) + \int\limits_{\mathrm{nc}(K)} \omega \right], \end{aligned}$$

is continuous.

We already know that integration of a differential form over the normal cycle provides a continuous valuation, but our main interest lies in smooth translation-invariant valuations. The next lemma gives an answer to the question when we get a smooth valuation via integration over normal cycles. Recall that a differential form ω is called translation-invariant if for every $t \in V$ and for the translation map $\tau_t : (x, y) \mapsto (x + t, y)$ the following equation holds: $\tau_t^* \omega = \omega$. Let $\Omega^{n-1}(SV)$ denote the space of smooth translation-invariant $n-1$-forms on SV.

Lemma 5.4.5. *For any $\omega \in \Omega^{n-1}(SV)$ the valuation*

$$[K \mapsto \int\limits_{\mathrm{nc}(K)} \omega] \tag{5.6}$$

is smooth and translation-invariant, that is, it belongs to $\mathbf{Val}^\infty(V)$.

For $0 \leq i \leq n-1$, let $\Omega^{i,n-i-1}$ denote the space of smooth translation-invariant differential forms of bi-degree $(i, n-i-1)$ on SV with the finite flat norm.

For the proof of our main theorem we need a special case of the previous statement.

Corollary 5.4.6 ([2]). *For $0 \leq i \leq n-1$ and any $\omega \in \Omega^{i,n-i-1}$ the valuation*

$$[K \mapsto \int_{\mathrm{nc}(K)} \omega]$$

is smooth, translation-invariant and i-homogeneous, that is, it belongs to $\mathbf{Val}_i^\infty(V)$.

It would be very nice if there were a one-to-one correspondence between translation-invariant differential forms and i-homogeneous valuations. But we only have the following statement:

Theorem 5.4.7. *If $0 \leq i \leq n-1$, the map defined by*

$$\Omega^{i,n-i-1} \to \mathbf{Val}_i^\infty(V),$$
$$\omega \mapsto [K \mapsto \int_{\mathrm{nc}(K)} \omega],$$

is surjective.

So we know that a continuous, translation-invariant valuation ν is in $\mathbf{Val}_i^\infty$ if and only if it can be written in the form $\nu(K) = \int_{nc(K)} \omega$ where $\omega \in \Omega^{i,n-i-1}$.
But we do not know how ω is defined, since it could be the case that

$$\nu(K) = \int_{\mathrm{nc}(K)} \omega_1 = \int_{\mathrm{nc}(K)} \omega_2$$

with $\omega_1 \neq \omega_2$. We would like to know under which restrictions $\int_{\mathrm{nc}(K)} \omega_1 - \omega_2 = 0$. Therefore we are interested in the kernel of the map from Theorem 5.4.7. This will be the topic of the next section.

But before we deal with the Kernel Theorem we give another very important statement. Theorem 5.4.7 was the main tool on the way to establish the Hard Lefschetz Theorem. It is quite hard work to prove this theorem, therefore we just state a consequence which we need and refer to [5] for more details and the proof.

Theorem 5.4.8. *For every $i \in \{0, \dots, n\}$,*

$$\mathbf{Val}_i^\infty \cong \mathbf{Val}_{n-i}^\infty$$

as $SO(n)$ modules.

5.5 The Rumin-de Rham Complex

In order to state the so called Kernel Theorem, we need a few definitions from contact geometry. This will be the first part of this section. We follow [8]. The second part of this section deals with primitive forms and the fact that $\mathbf{Val}_i^\infty$ fits into an exact sequence of spaces of primitive forms. The theorems and proofs regarding the second topic are taken from [7] and [3].

Let M be an n-dimensional manifold, TM its tangent bundle and $H \subset TM$ a smooth field of tangent hyperplanes. Locally, $H : p \mapsto H_p \subset T_pM$ determines a 1-form $\alpha_p \in T_p^*M \backslash 0$ via $H_p = \ker \alpha_p$ up to multiplication by a smooth, non-vanishing function $f : M \to \mathbb{R}$.

Definition 5.5.1. *Let M be a smooth manifold of dimension $2n-1$. A contact structure on M is a smooth field of tangent hyperplanes H such that the $(2n-1)$-form $\alpha \wedge (d\alpha)^n \neq 0$ for any locally defining 1-form. The pair (M, H) is then called a contact manifold and α is called a local contact form.*

Note that the condition $\alpha \wedge (d\alpha)^n \neq 0$ is independent of the choice of α, since $(f\alpha) \wedge d(f\alpha)^n = f^{n+1}\alpha \wedge (d\alpha)^n$. Moreover, $\alpha \wedge (d\alpha)^n \neq 0$ is equivalent to the fact that $d\alpha|_H$ is non-degenerate, that is, $d\alpha|_{H_p}$ is a symplectic form.

We are interested in the case where $M = V \times S^{n-1}$. This becomes a $(2n-1)$-dimensional contact manifold if we define the canonical contact form α by

$$\alpha|_{(x,u)}(w) = \left\langle u, d_{(x,u)}\pi(w) \right\rangle, \quad w \in T_{(x,u)}SV, \tag{5.7}$$

where $\pi : SV \to V$ is the canonical projection.

We can rewrite the contact form in coordinates. Therefore if $x_1, \ldots, x_n, y_1, \ldots, y_n$ are coordinates on $V \times V$ with respect to a given basis, we can write the contact form α in the following way:

$$\alpha = \sum_{i=1}^{n} y_i dx_i,$$

where $y_1^2 + \cdots + y_n^2 = 1$. As usual $dx_1, \ldots, dx_n$ are the dual elements of the first n elements of a base $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_{n-1}}$ of $T(V \times S^{n-1}) = TV \times TS^{n-1}$.

The kernel of the contact form α defined in (5.7) is denoted by $Q := \ker \alpha$ and is called the contact distribution. As we saw before the restriction of $d\alpha$ to Q is a symplectic form, that is, a non-degenerate two form. So, each $Q_{(x,u)}$ becomes a symplectic vector space.

On each contact manifold with given contact form α there is a unique vector field R, the so called Reeb vector field, satisfying $\alpha(R) = 1$ and $i_R d\alpha = 0$, where i_R denotes the interior product. The Reeb vector field on SV is given by $R = \sum_{i=1}^n y_i \frac{\partial}{\partial x_i}$, where $y_i, \frac{\partial}{\partial x_i}$ for $i = 1, \ldots, n$ are defined as above. A coordinate free way to define R is given by $R_{(x,u)} = (u, 0)$.

Since $\alpha|_{(x,u)}$ is defined as the dual product of elements who live in T_uS^{n-1}, $Q_{(x,u)}$ is the orthogonal sum of two copies of T_uS^{n-1} and thus we have the following splitting of the tangent space of SV:

$$T_{(x,u)}SV = \operatorname{span}_{\mathbb{R}} R_{(x,u)} \oplus T_uS^{n-1} \oplus T_uS^{n-1}. \tag{5.8}$$

The critical tool to state and prove the Kernel Theorem is the Rumin operator, which is a differential operator on a contact manifold. In particular, there is a Rumin operator on SV.

Lemma 5.5.2. *Let M be a smooth $2n-1$-dimensional contact manifold with global* *contact form α. If $\omega \in \Omega^{n-1}(M)$, then there is a unique differential form $D\omega \in \Omega^n(M)$ such that $D\omega$ annihilates the contact distribution and such that there exists $\xi \in \Omega^{n-2}(M)$ with $D\omega = d(\omega + \alpha \wedge \xi)$. Also $D\omega \wedge \alpha = 0$. D is called the Rumin operator.*

Now we can state a special case of the Kernel Theorem.

Theorem 5.5.3 ([8]). *For $0 \leq i \leq n-1$, let $\nu : \Omega^{i,n-i-1} \to \mathbf{Val}_i^\infty(V)$ be defined by $\omega \mapsto \nu(\omega)(K) = \int_{\mathrm{nc}(K)} \omega$. Then $\omega \in \ker \nu$ if and only if $D\omega = 0$ and $\pi_* \omega = 0$.*

Next we want to show that $\mathbf{Val}_i^\infty$ is part of an exact sequence. The following material and especially the proof of the main theorem about the exact sequence was taken from [7].

First consider the vector space $\Omega^*(SV)$ of complex-valued smooth differential forms. Since $SV = V \times S^{n-1}$ has product structure, we can write $\Omega^*(SV)$ in the following way

$$\Omega^*(SV) = \bigoplus_{i,j} \Omega^{i,j}(SV),$$

where $\Omega^{i,j}(SV)$ denotes the subspace of forms of bidegree (i,j). Remember that $\Omega^{i,j} \subseteq \Omega^{i,j}(SV)$ denotes the subspace of translation-invariant forms, as defined before. We now define the following subspaces (here α still denotes the contact form of SV):

$$\begin{aligned}
\mathcal{I}^{i,j} &:= \{\omega \in \Omega^{i,j} : \omega = \alpha \wedge \xi + d\alpha \wedge \phi, \xi \in \Omega^{i-1,j}, \phi \in \Omega^{i-1,j-1}\},\\
\Omega_v^{i,j} &:= \{\omega \in \Omega^{i,j} : \alpha \wedge \omega = 0\},\\
\Omega_h^{i,j} &:= \Omega^{i,j} / \Omega_v^{i,j},\\
\Omega_p^{i,j} &:= \Omega^{i,j} / \mathcal{I}^{i,j}.
\end{aligned}$$

The wedge product with $-d\alpha$ induces an operator $L : \Omega_h^{i,j} \to \Omega_h^{i+1,j+1}$. Let $\omega \in \Omega^{i,j}$ and $\tilde{\omega} \in \Omega_v^{i,j}$, then $(\omega + \tilde{\omega}) \in \Omega_h^{i,j}$. Thus, we obtain

$$L(\omega + \tilde{\omega}) = -d\alpha \wedge (\omega + \tilde{\omega}) = -d\alpha \wedge \omega - d\alpha \wedge \tilde{\omega} = L\omega - d\alpha \wedge \tilde{\omega},$$

where $-d\alpha \wedge \tilde{\omega}$ is vertical, which follows from $\tilde{\omega} \in \Omega_v^{i,j}$ and

$$\begin{aligned}
\alpha \wedge \tilde{\omega} &= 0,\\
d\alpha \wedge \tilde{\omega} - \alpha \wedge d\tilde{\omega} &= 0,\\
-d\alpha \wedge \tilde{\omega} &= -\alpha \wedge d\tilde{\omega}.
\end{aligned}$$

The right hand side of the equation is a vertical form since $\alpha \wedge \alpha = 0$, thus the left hand side has to be vertical, too. The operator L is an injection for $i - j \leq n-2$ and because of the definition of $\Omega_p^{i,j}$ the following spaces are isomorphic

$$\Omega_p^{i,j} \cong \Omega_h^{i,j} / L\Omega_h^{i-1,j-1}.$$

The exterior derivative induces an operator $d_Q : \Omega_p^{i,j} \to \Omega_p^{i,j+1}$. Note that the Rumin operator vanishes on $\mathcal{I}^{i,n-j-1}$, and therefore induces an operator

$$D : \Omega_p^{i,n-i-1} \to \Omega_v^{i,n-i}.$$

For the proof of Theorem 5.5.5 below, we need the following lemma, which was first established by A. Bernig in [7]. The lemma and the corresponding proof were taken from there.

Lemma 5.5.4. *Let $\omega \in \Omega^{i,j}$ with $d\omega = 0$. Then*

i) there exists $\phi \in \Omega^{i,j-1}$ with $d\phi = \omega$, if $0 < j < n-1$,

ii) $\omega \in \Lambda^i V^* \otimes \mathbb{C}$, if $j = 0$ and

iii) in the case $i = 0, j = n-1$, there exists $\phi \in \Omega^{0,n-2}$ with $d\phi = \omega$ provided that $\pi_* \omega = 0$.

Proof. If $\omega \in \Omega^{i,j}$, then we can write the differential form in the following way

$$\omega = \sum_{k=1}^{q} c_k \kappa_k \wedge \theta_k,$$

where $\kappa_1, \ldots, \kappa_q$ is a basis of $\Lambda^i V^*$ and $\theta_1, \ldots, \theta_q$ are j-forms on the unit sphere S^{n-1}. Because of the assumption that $d\omega = 0$ and the fact that ω is translation-invariant, we get

$$d\omega = (-1)^i \sum_{k=1}^{q} c_k \kappa_k \wedge d\theta_k = 0,$$

which shows that all θ_k are closed.

If $0 < j < n-1$, then the jth de Rham cohomology group of S^{n-1} is empty, that is, $H^j_{dR}(S^{n-1}) = 0$, and thus closed differential forms on S^{n-1} are also exact. So there is a $\rho_k \in \Omega^{j-1}(S^{n-1})$ with $d\rho_k = \theta_k$. Then $\phi := (-1)^i \sum_{k=1}^q c_k dx_k \wedge \rho_k$ satisfies $d\phi = \omega$. If $l = 0$, then all θ_k are constant and hence $\omega \in \Lambda^i V^* \otimes \mathbb{C}$.

The statement for the case $i = 0, j = n-1$ follows from the fact that $\Omega^{0,n-1} = \Omega^{n-1}(S^{n-1})$ and that $H^{n-1}_{dR}(S^{n-1})$ is one-dimensional. ☐

Now we gathered all the statements and definitions we need to state the following theorem, which was proved by A. Bernig in [7]. The proof was also taken from that paper.

Theorem 5.5.5. *Let $0 \leq i \leq n$. There is an exact sequence*

$$0 \to \Lambda^i V^* \otimes \mathbb{C} \hookrightarrow \Omega_p^{i,0} \overset{d_Q}{\to} \Omega_p^{i,1} \overset{d_Q}{\to} \cdots \overset{d_Q}{\to} \Omega_p^{i,n-i-1} \overset{\nu}{\to} \mathbf{Val}_i \to 0. \tag{5.9}$$

Proof. In the case $i = n$ there is an exact sequence

$$0 \to \Lambda^n V^* \otimes \mathbb{C} \to \mathbf{Val}_n \to 0$$

This is true, since $\mathbf{Val}_n$ is spanned by the Lebesgue measure.

Suppose $i < n$. The sequence is closed, since $d_Q \circ d_Q(\omega) = 0$. Therefore it follows that $\mathrm{im}(d_Q) \subseteq \ker(d_Q)$. It remains to show that it is exact, that is, $\mathrm{im}(d_Q) = \ker(d_Q)$. The second arrow on the left hand side is trivial, since $\Omega_p^{i,0} = \Omega^{i,0}$.

Consider $0 \leq i < n-i-1$ and $\omega \in \Omega^{i,j}$ with the additional condition $d\omega = (\alpha \wedge \xi + d\alpha \wedge \psi) \in \mathcal{I}^{i,j+1}$. Define $\omega' := \omega - \alpha \wedge \psi \in \Omega^{i,j}$ and $\xi' := \xi + d\psi$. Then

$$\begin{aligned} d\omega' = d(\omega - \alpha \wedge \psi) &= d\omega - d\alpha \wedge \psi - (-1)\alpha \wedge d\psi \\ &= \alpha \wedge \xi + d\alpha \wedge \psi - d\alpha \wedge \psi + \alpha \wedge d\psi \\ &= \alpha \wedge \xi + \alpha \wedge d\psi = \alpha \wedge (\xi + d\psi) = \alpha \wedge \xi'. \end{aligned}$$

Differentiation yields

$$0 = d(d\omega') = d\alpha \wedge \xi' - \alpha \wedge d\xi'.$$

Hence, if we look at $d\alpha \wedge \xi' = \alpha \wedge d\xi'$, then we obtain that $d\alpha \wedge \xi'$ is vertical since $\alpha \wedge d\xi'$ is vertical. A differential form is vertical if and only if it vanishes on the contact distribution Q. Thus we get $d\alpha \wedge \xi'|_Q = 0$ or in a different notation, $L(\xi'|_Q) = 0$. Since the operator L is injective for degrees $i + j < n - 1$ we get $\xi'|_Q = 0$, which implies, together with the equation above, that $d\omega' = 0$.

If $j > 0$, we can use Lemma 5.5.4. There exists $\phi \in \Omega^{i,j-1}$ with $d\phi = \omega'$. Hence the following equations for equivalence classes hold: $[\omega] = [\omega'] = [d\phi] = d_Q[\phi]$. This shows that $[\omega]$ is d_Q-exact.

If $j = 0$, then ω' is a translation-invariant i-form on V, hence $[\omega] = [\omega']$ is in the image of $\Lambda^i V^* \otimes \mathbb{C}$.

Now we look at the right hand side. Note that the map ν induces a linear map, denoted by the same letter, $\nu : \Omega_p^{i,n-i-1} \to \mathbf{Val}_i$. The map ν is surjective by Theorem 5.4.7. Let $[\omega] \in \Omega_p^{i,n-i-1}$ be an element of the kernel of ν. Then we know from Theorem 5.5.3, that $D[\omega] = d(\omega + \xi) = 0$ for some vertical form $\xi \in \Omega_v^{i,n-i-1}$. It follows that $\omega' := \omega + \xi$ is a closed translation-invariant form of bidegree $(i,n-i-1)$.

Consider $i > 0$, then by Lemma 5.5.4, there exists $\phi \in \Omega^{i,n-i-2}$ with $d\phi = \omega'$. We obtain, again, that $[\omega] = [\omega'] = [d\phi] = d_Q[\phi]$ is d_Q-exact.

It remains the case $i = 0$. By Theorem 5.5.3 we obtain $\pi_* \omega' = \pi_* \omega = 0$. Hence we can apply Theorem 5.5.4 to get $\phi \in \Omega^{0,n-2}$ with $d\phi = \omega'$. So also in this case $[\omega]$ is d_Q-exact. □

Note that the vector space $\Omega^{i,j}$ is an $SO(n)$ module with the continuous action

$$\vartheta\omega = l^*_{\vartheta^{-1}}\omega, \quad \vartheta \in SO(n), \omega \in \Omega^{i,j},$$

where l_ϑ denotes the natural smooth left action of $SO(n)$ on SV defined by

$$l_\vartheta(x,u) := (\vartheta x, \vartheta u), \quad \vartheta \in SO(n), (x,u) \in SV.$$

Since the above defined operator L is $SO(n)$-equivariant and d_Q and ν are also $SO(n)$-equivariant, the following corollary holds:

Corollary 5.5.6. *If $0 \leq i \leq n$, then there is an exact $SO(n)$-equivariant sequence of $SO(n)$ modules*

$$0 \to \Lambda^i V^* \otimes \mathbb{C} \hookrightarrow \Omega_p^{i,0} \stackrel{d_Q}{\to} \Omega_p^{i,1} \stackrel{d_Q}{\to} \dots \stackrel{d_Q}{\to} \Omega_p^{i,n-i-1} \stackrel{\nu}{\to} \mathbf{Val}_i^\infty \to 0. \tag{5.10}$$

For the main proof we need one more statement about isomorphic $SO(n)$ modules. By (5.8) and the definition of $\Omega_h^{i,j}$, $\omega \in \Omega^{i,j}$ is horizontal if and only if

$$\omega|_{(x,u)} \in \Lambda^i T_u S^{n-1} \otimes \Lambda^j T_u S^{n-1} \otimes \mathbb{C},$$

for all $x \in V, u \in S^{n-1}$. Since, this is independent of x, we write $\omega|_u$ instead of $\omega|_{(x,u)}$. Fix a point $u_0 \in SO(n)$, the stabilizer of $SO(n)$ at u_0 is $SO(n-1)$ and write W_0 for the complexification of the tangent space of S^{n-1} at u_0, that is, $W_0 := T_{u_0} S^{n-1} \otimes \mathbb{C}$. Then we have the following isomorphism of $SO(n)$ modules.

Lemma 5.5.7. *For i,j $\in \mathbb{N}$, there is an isomorphism of $SO(n)$ modules*

$$\Omega_h^{i,j} \cong \mathrm{Ind}_{SO(n-1)}^{SO(n)} (\Lambda^i W_0 \otimes \Lambda^j W_0).$$

Proof. This proof is taken from [3]. First define, for every $\vartheta \in SO(n)$, the bijective rotation map $r_\vartheta : S^{n-1} \to S^{n-1}$ by $u \mapsto \vartheta u$. If we take the differential of this map and pass to the adjoint operator, this map induces a linear isomorphism on the exterior power of W_u

$$(d_{u_0} r_\vartheta)^* : \Lambda^i W_{\vartheta u_0}^* \otimes \Lambda^j W_{\vartheta u_0}^* \to \Lambda^i W_0^* \otimes \Lambda^j W_0^*.$$

We will denote this isomorphism by $\widehat{d_{u_0} r_\vartheta}$. It gives us a natural representation ρ of $SO(n-1)$ on $\Lambda^i W_0^* \otimes \Lambda^j W_0^*$ by $\rho(\eta) := \widehat{d_{u_0} r_{\eta^{-1}}}$.

Now, let $\omega \in \Omega_h^{i,j}$ and remember the definition of $\mathrm{Ind}_{SO(n-)}^{SO(n)} (\Lambda^i W_0^* \otimes \Lambda^j W_0^*)$. It is the set of all smooth functions f from $SO(n)$ to $\Lambda^i W_0^* \otimes \Lambda^j W_0^*$, satisfying $f(\vartheta\eta) = \eta^{-1} f(\vartheta)$ for $\vartheta, \eta \in SO(n)$. Thus, we define $f_\omega : SO(n) \to \Lambda^i W_0^* \otimes \Lambda^j W_0^*$ by

$$f_\omega(\vartheta) = \widehat{d_{u_0} r_\vartheta}(\omega|_{\vartheta u_0}).$$

The definition of the representation on $\Lambda^i W_0^* \otimes \Lambda^j W_0^*$ yields

$$\begin{aligned} \eta^{-1} f_\omega(\vartheta) &= \widehat{d_{u_0} r_\eta}(f_\omega(\vartheta)) \\ &= \widehat{d_{u_0} r_\eta}\left(\widehat{d_{u_0} r_\vartheta}(\omega|_{\vartheta u_0})\right) \\ &= \widehat{d_{u_0} r_{\vartheta\eta}}(\omega|_{\vartheta\eta u_0}) \\ &= f_\omega(\vartheta\eta), \end{aligned}$$

for $\eta \in SO(n-1)$ and every $\vartheta \in SO(n)$. Note that this works, since $SO(n-1)$ is the stabilizer of $SO(n)$ at u_0. Therefore, we get that $f_\omega \in \mathrm{Ind}_{SO(n-1)}^{SO(n)} \left(\Lambda^i W_0^* \otimes \Lambda^j W_0^*\right)$.

Conversely, take $f \in \operatorname{Ind}_{SO(n-1)}^{SO(n)} \left(\Lambda^i W_0^* \otimes \Lambda^j W_0^*\right)$ and define

$$\omega_f|_{\vartheta u_0} = \widehat{d_{u_0} r_\vartheta}^{-1}(f(\vartheta)).$$

By definition, this is a horizontal form, that is, $\omega_f \in \Omega_h^{i,j}$. We see that if $\vartheta u_0 = \vartheta' u_0$, then for $\vartheta, \vartheta' \in SO(n)$, the maps $\widehat{d_{u_0} r_\vartheta}^{-1}(\cdot)$ and $\widehat{d_{u_0} r_{\vartheta'}}^{-1}(\cdot)$ map to the same space, namely $\Lambda^i W_{\vartheta u_0}^* \otimes \Lambda^j W_{\vartheta u_0}^*$. Hence, $\omega|_{\vartheta u_0} = \omega|_{\vartheta' u_0}$, which shows that ω is well defined.

To conclude the proof, we have to show that the $SO(n)$-equivariant maps

- $i : \Omega_h^{i,j} \to \operatorname{Ind}_{SO(n-1)}^{SO(n)}(\Lambda^i W_0 \otimes \Lambda^j W_0)$, defined by $i(\omega) = f_\omega$, and
- $l : \operatorname{Ind}_{SO(n-1)}^{SO(n)}(\Lambda^i W_0 \otimes \Lambda^j W_0) \to \Omega_h^{i,j}$, defined by $l(f) = \omega_f$,

are inverse to each other. To this end we compute

$$\begin{aligned} i\left(l(f(\vartheta))\right) &= i\left(\omega_f|_{\vartheta u_0}\right) = f_{\omega_f|_{\vartheta u_0}}(\vartheta) \\ &= \widehat{d_{u_0} r_\vartheta}\left(\omega_f|_{\vartheta u_0}\right) = \widehat{d_{u_0} r_\vartheta}\left(\widehat{d_{u_0} r_\vartheta}^{-1}(f(\vartheta))\right) \\ &= f(\vartheta), \end{aligned}$$

and

$$\begin{aligned} l\left(i(\omega|_{\vartheta u_0})\right) &= l(f_\omega(\vartheta)) = l\left(\widehat{d_{u_0} r_\vartheta}(\omega|_{\vartheta u_0})\right) \\ &= \widehat{d_{u_0} r_\vartheta}^{-1}\left(\widehat{d_{u_0} r_\vartheta}(\omega|_{\vartheta u_0})\right) \\ &= \omega|_{\vartheta u_0}. \end{aligned}$$

So, i and l are inverse functions, which finishes the proof. $\square$

Remember that by definition $\Omega_p^{i,j} = \Omega_h^{i,j}/L\Omega_h^{i-1,j-1}$ for i, j suitable. Thus we obtain the following corollary.

Corollary 5.5.8. *If $i, j \in \mathbb{N}$ are such that $i + j < n - 1$, then there is an isomorphism of $SO(n)$ modules*

$$\Omega_p^{i,j} \oplus \operatorname{Ind}_{SO(n-1)}^{SO(n)}(\Lambda^{i-1} W_0 \otimes \Lambda^{j-1} W_0) \cong \operatorname{Ind}_{SO(n-1)}^{SO(n)}(\Lambda^i W_0 \otimes \Lambda^j W_0).$$

6 A General Version of Hadwiger's Characterization Theorem

Now we are able to prove the decomposition of **Val** into irreducible $SO(n)$ modules. The statement and the proof presented here were given by S. Alesker, A. Bernig and F. Schuster. Everything in this chapter was taken from their paper [3].

6.1 Statement and Proof

Theorem 6.1.1. *Let $0 \leq i \leq n$. The space of i-homogeneous, continuous, translation-invariant, complex-valued valuations $\mathbf{Val}_i$ is the direct sum of irreducible representations of $SO(n)$ with highest weights $\lambda = (\lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor})$ precisely satisfying the following conditions:*

i) $\lambda_j = 0$ *for* $j > \min\{i, n-i\}$,

ii) $|\lambda_j| \neq 1$ *for* $1 \leq j \leq \lfloor n/2 \rfloor$,

iii) $|\lambda_2| \leq 2$.

In particular, under the action of $SO(n)$ the space $\mathbf{Val}_i$ is multiplicity free.

We will denote the set of highest weights of $SO(n)$ satisfying conditions i)-iii) by S. Since $\mathbf{Val}_i$ is a Banach space, the previous theorem means that

$$\mathbf{Val}_i = \mathrm{cl}(\mathbf{Val}_i^f) = \mathrm{cl}(\oplus_{\lambda \in S} \Gamma_\lambda).$$

From the previous theorem it is not much work to obtain the following Hadwiger-type characterization theorem for continuous, translation-invariant, $SO(n)$-equivariant, i-homogeneous valuations with values in an $SO(n)$ irreducible space Γ.

Theorem 6.1.2. *Let (Γ, ρ) be an irreducible $SO(n)$ representation and let $0 \leq i \leq n$. There exists a non-trivial continuous, translation-invariant and $SO(n)$-equivariant valuation of degree i with values in Γ if and only if the highest weight of Γ satisfies the conditions i)-iii) from Theorem 6.1.1. This valuation is unique up to scaling.*

We begin with the proof of Theorem 6.1.1.

Proof of Theorem 6.1.1. We know from representation theory for infinite-dimensional representations that $\mathbf{Val}_i^f = \oplus m_\lambda \Gamma_\lambda$, where Γ_λ are irreducible $SO(n)$ modules. Here $m_\lambda = m(\mathbf{Val}_i, \lambda)$ denotes the multiplicity of Γ_λ in the $SO(n)$ module $\mathbf{Val}_i$. Thus we have to show that $m(\mathbf{Val}_i, \lambda) = 1$ if $\lambda \in S$ and $m(\mathbf{Val}_i, \lambda) = 0$ otherwise. The cases

$i = 0$ and $i = n$ are trivial. By Theorem 5.4.8, we just have to consider the cases where $n/2 \leq i < n$.

Let Γ_λ be an arbitrary irreducible $SO(n)$ module with highest weight $\lambda = (\lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor})$. It follows from Corollary 5.5.8 that the multiplicity of Γ_λ in the spaces $\Omega_p^{i,j}$ of primitive forms is finite. Since there is the exact sequence (5.10), which means that the spaces $\mathbf{Val}_i^\infty$ are quotients of $\Omega_p^{i,n-i-1}$, we get that the multiplicity of Γ_λ in $\mathbf{Val}_i^f \subseteq \mathbf{Val}_i^\infty$ is finite, too. Thus, by Theorem 5.5.6, we have

$$m(\mathbf{Val}_i, \lambda) = (-1)^{n-i} m(\Lambda^i V_\mathbb{C}, \lambda) + \sum_{j=0}^{n-i-1} (-1)^{n-1-i-j} m(\Omega_p^{i,j}, \lambda). \tag{6.1}$$

Let W denote the complex standard representation of $SO(n-1)$ and note that in this case $W \cong W^*$. By Corollary 5.5.8, the property $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$ and the fact that $V = \bigoplus m_i V_i$ implies $\chi_V = \sum m_i \chi_{V_i}$, we get

$$m(\Omega_p^{i,j}, \lambda) = m\left(Ind_{SO(n-1)}^{SO(n)}(\Lambda^i W \otimes \Lambda^j W), \lambda\right) - m\left(Ind_{SO(n-1)}^{SO(n)}(\Lambda^{i-1} W \otimes \Lambda^{j-1} W), \lambda\right).$$

If we look at the right hand side of this equation we see that it has exactly the form of the right hand side of the equation in Corollary 3.3.8, with n replaced by $n-1$ and $0 \leq j \leq n-1-i$. Thus,

$$m(\Omega_p^{i,j}, \lambda) = \sum_\sigma m(\mathrm{Ind}_{SO(n-1)}^{SO(n)} \bar{\Gamma}_\sigma, \lambda). \tag{6.2}$$

Note that $\bar{\Gamma}_\sigma$ is defined by (3.9) and the sum ranges over all $k := \lfloor (n-1)/2 \rfloor$-tuples of highest weights $\sigma = (\sigma_1, \ldots, \sigma_k)$ of irreducible $SO(n-1)$ modules such that

$$\sigma_1 \leq 2, \quad \#(\sigma, 1) = n-1-i-j, \quad \#(\sigma, 2) \leq j.$$

Now we take the union over j of these highest weights and denote it by P_i. If we use (6.1) and (6.2), we obtain

$$m(\mathbf{Val}_i, \lambda) = (-1)^{n-i} m(\Lambda^i V_\mathbb{C}, \lambda) + \sum_{\sigma \in P_i} (-1)^{|\sigma|} m\left(\mathrm{Ind}_{SO(n-1)}^{SO(n)} \bar{\Gamma}_\sigma, \lambda\right). \tag{6.3}$$

Here $|\sigma|$ stands for the sum of all integers of the integer tupel σ.

We want to use the Frobenius Reciprocity Theorem at this point. Thus instead of calculating the multiplicity of Γ_λ in $\mathrm{Ind}_{SO(n-1)}^{SO(n)} \bar{\Gamma}_\sigma$ we compute the multiplicity of $\bar{\Gamma}_\sigma$ in $\mathrm{Res}_{SO(n-1)}^{SO(n)} \Gamma_{\lambda^*}$, where $\lambda^* = (\lambda_1^*, \ldots, \lambda_{\lfloor n/2 \rfloor}^*)$ is defined by $\lambda_1^* := \min\{\lambda_1, 2\}$ and $\lambda_j^* := |\lambda_j|$ for every $1 < j \leq \lfloor n/2 \rfloor$. Now, by the Branching Theorem from $SO(n)$ to $SO(n-1)$, we get that

$$\mathrm{Res}_{SO(n-1)}^{SO(n)} \Gamma_{\lambda^*} = \bigoplus_\mu \Gamma_\mu,$$

where $\mu = (\mu_1, \dots, \mu_k)$ with $\mu_{n-i} = 0$ and

$$\begin{cases} \lambda_1^* \geq \mu_1 \geq \lambda_2^* \geq \mu_2 \geq \cdots \geq \mu_{k-1} \geq \lambda_{\lfloor n/2 \rfloor}^* \geq |\mu_k| & \text{for odd } n, \\ \lambda_1^* \geq \mu_1 \geq \lambda_2^* \geq \mu_2 \geq \cdots \geq \mu_k \geq \lambda_{n/2}^* & \text{for even } n. \end{cases} \tag{6.4}$$

This yields

$$\begin{aligned} \sum_{\sigma \in P_i} (-1)^{|\sigma|} m \left(\operatorname{Ind}_{SO(n-1)}^{SO(n)} \bar{\varGamma}_\sigma, \lambda \right) &= \sum_{\sigma \in P_i} (-1)^{|\sigma|} m \left(\bar{\varGamma}_\sigma, \operatorname{Res}_{SO(n-1)}^{SO(n)} \varGamma_{\lambda^*} \right) \\ &= \sum_{\mu} (-1)^{|\mu|}. \end{aligned}$$

If we look again at (6.4), we see that there is no such sequence if $\lambda_{n-i+1}^* > 0$. (If $\lambda_{n-i+1}^* > 0$, then this leads to the contradiction $0 = \mu_{n-i} \geq \lambda_{n-i+1} > 0$.) Thus $\lambda_{n-i+1}^* = 0$ and we obtain

$$\sum_{\mu} (-1)^{|\mu|} = \prod_{j=1}^{n-i-1} \sum_{\mu_j = \lambda_{j+1}^*}^{\lambda_j^*} (-1)^{\mu_j}.$$

This product is zero if the λ_j^*, $j = 1, \dots, n-i$, have different parity. So we consider only the case where the λ_j's have the same parity. In this case we get

$$\sum_{\mu} (-1)^{|\mu|} = \prod_{j=1}^{n-i-1} \sum_{\mu_j = \lambda_{j+1}^*}^{\lambda_j^*} (-1)^{\mu_j} = (-1)^{(n-i-1)\lambda_1^*}.$$

Before we continue the calculation, I want to recall the restrictions we had to make on λ^* so far. First $\lambda_1^* := \min\{\lambda_1, 2\}$ and $\lambda_j := |\lambda_j|$, second $\lambda_{n-i+1}^* = 0$ and third all $\left(\lambda_j^*\right)_{j=1}^{n-i}$ have to be of the same parity. These restrictions also yield conditions for λ. Thus we obtain, for $i > n/2$,

$$(-1)^{(n-i-1)\lambda_1^*} = \begin{cases} (-1)^{n-i-1} & \text{if } \varGamma_\lambda \cong \varLambda^{n-i} V_{\mathbb{C}}, \\ 1 & \text{if } \lambda \in S, \\ 0 & \text{otherwise}, \end{cases}$$

and for $i = n/2$, where n has to be even,

$$(-1)^{(n-i-1)\lambda_1^*} = \begin{cases} (-1)^{i-1} & \text{if } \lambda = (1, \dots, 1, \pm 1), \\ 1 & \text{if } \lambda \in S, \\ 0 & \text{otherwise}. \end{cases}$$

Now we compute $m(\mathbf{Val}_i, \lambda)$ for these three possibilities of λ if $i > n/2$.

- $\lambda = (1,\dots,1,0,\dots,0)$, that is, $\Gamma_\lambda \cong \Lambda^i V_{\mathbb{C}} \cong \Lambda^{n-i} V_{\mathbb{C}}$:

$$m(\mathbf{Val}_i, \lambda) = (-1)^{n-i} \underbrace{m(\Lambda^i V_{\mathbb{C}}, \lambda)}_{=1} + (-1)^{n-i-1} = 0.$$

- $\lambda \in S$:

$$m(\mathbf{Val}_i, \lambda) = (-1)^{n-i} \underbrace{m(\Lambda^i V_{\mathbb{C}}, \lambda)}_{=0} + 1 = 1.$$

- for all other λ:

$$m(\mathbf{Val}_i, \lambda) = (-1)^{n-i} \underbrace{m(\Lambda^i V_{\mathbb{C}}, \lambda)}_{=0} + 0 = 0.$$

To finish the proof, it remains to compute $m(\mathbf{Val}_{n/2}, \lambda)$ for the different λ's. Thus, consider

- $\lambda = (1,\dots,1,\pm 1)$, that is, $\bar{\Gamma}_\lambda \cong \Lambda^{n/2} V_{\mathbb{C}} \cong \Gamma_{(1,\dots,1)} \oplus \Gamma_{(1,\dots,-1)}$:

$$m(\mathbf{Val}_{n/2}, \lambda) = (-1)^{n/2} \underbrace{m(\Lambda^{n/2} V_{\mathbb{C}}, \lambda)}_{=1} + (-1)^{n/2-1} = 0.$$

- $\lambda \in S$:

$$m(\mathbf{Val}_{n/2}, \lambda) = (-1)^{n/2} \underbrace{m(\Lambda^{n/2} V_{\mathbb{C}}, \lambda)}_{=0} + 1 = 1.$$

- for all other λ:

$$m(\mathbf{Val}_{n/2}, \lambda) = (-1)^{n/2} \underbrace{m(\Lambda^{n/2} V_{\mathbb{C}}, \lambda)}_{=0} + 0 = 0.$$

Thus, for $0 \le i \le n$, $m(\mathbf{Val}_i, \lambda) = 1$ if $\lambda \in S$ and $\mathbf{Val}_i^f = \oplus_{\lambda \in S} \Gamma_\lambda$, which completes the proof. $\square$

To proof Theorem 6.1.2 we actually show its equivalence to Theorem 6.1.1:

Proof. Let $\Gamma = \Gamma_\mu$ be an irreducible $SO(n)$ module.
First we show that the linear map $\iota : \mathbf{Val}^f \otimes \Gamma_\lambda \to \Gamma \mathbf{Val}^f$ defined by

$$\phi \otimes v \mapsto \phi \cdot v$$

is an isomorphism. It is injective, since the kernel of this map consists only of the zero vector. To show that ι is onto, take $\Phi \in \Gamma \mathbf{Val}$, which can be written as

$$\Phi = \langle \Phi, e_1 \rangle\, e_1 + \cdots + \langle \Phi, e_{\dim \Gamma} \rangle\, e_{\dim \Gamma}$$

where $\{e_i\}_{i=1}^{\dim \Gamma}$ is a basis of Γ.

By a Theorem of McMullen we also get the statement for the i-homogeneous case, $\Gamma\mathbf{Val}_i^f \cong \mathbf{Val}_i^f \otimes \Gamma$. From Theorem 6.1.1 we know that $\mathbf{Val}_i^f = \oplus_{\lambda\in S}\Gamma_\lambda$. Therefore we obtain

$$\begin{aligned}\dim \Gamma_\mu \mathbf{Val}_i^{SO(n)} &\overset{(i)}{=} \dim(\mathbf{Val}_i \otimes \Gamma_\mu)^{SO(n)} \overset{(ii)}{=} \dim(\mathbf{Val}_i^f \otimes \Gamma_\mu)^{SO(n)} \\ &\overset{(iii)}{=} \dim(\sum_{\lambda\in S} \Gamma_\lambda \otimes \Gamma_\mu)^{SO(n)} = \sum_{\lambda\in S} \dim(\Gamma_\lambda \otimes \Gamma_\mu)^{SO(n)} \\ &\overset{(iv)}{=} \sum_{\lambda\in S} \dim \operatorname{Hom}_{SO(n)}(\Gamma_\lambda^*, \Gamma_\mu) = (\star).\end{aligned}$$

The first equation, (i), follows from the isomorphism-property described above. Note that the superscript $SO(n)$ denotes the subspaces of $SO(n)$-invariant elements. We obtain this since the map ι changes $SO(n)$ equivariance on $\Gamma\mathbf{Val}^f$ to $SO(n)$ invariance on $\mathbf{Val}^f \otimes \Gamma$. Equation (ii) is due to the fact that $\mathbf{Val}_i^f$ is a dense $O(n)$-invariant subspace of $\mathbf{Val}_i$. The following equation is precisely the statement of Theorem 6.1.1. And the equation (iv) follows from the basic fact that $V^* \otimes W \cong \operatorname{Hom}(V,W)$. This isomorphism changes $SO(n)$ equivariance back to $SO(n)$ invariance. Therefore we get $\operatorname{Hom}_{SO(n)}$, the space of continuous linear $SO(n)$-equivariant maps.

Recall that the dual representation is defined on the dual space Γ^* by

$$(\vartheta u^*)(v) = u^*(\vartheta^{-1}v), \qquad \vartheta \in SO(n),\ u^* \in \Gamma^*,\ v \in \Gamma.$$

Note that Γ_λ with $\lambda \in S$ is not necessarily self-dual (that is, $\Gamma_\lambda \cong \Gamma_\lambda^*$). It can happen that the dual of Γ_λ is $\Gamma_{\lambda'} = (\Gamma_\lambda^*)$ with $\lambda' = (\lambda_1, \ldots, \lambda_{n/2-1}, -\lambda_{n/2})$. However, if $\lambda \in S$, then also $\lambda' \in S$. Thus, we get

$$(\star) = \sum_{\lambda\in S} \dim \operatorname{Hom}_{SO(n)}(\Gamma_\lambda, \Gamma_\mu) = \begin{cases} 1 & \text{if} \quad \mu \in S, \\ 0 & \text{otherwise.}\end{cases}$$

The equation on the right hand side follows from Schur's Lemma, which finishes the proof. □

Examples.

- Consider the trivial representation of $SO(n)$. Remember that this representation is given by $\Gamma_{(0,\ldots,0)} \cong \mathbb{C}$. Since $\lambda = (0,\ldots,0)$ is an element of S, Theorem 6.1.2 shows that there exists a non-trivial continuous, rigid-motion invariant, i-homogeneous valuation, which is unique up to scaling. Thus, Theorem 6.1.2 gives us the well known Hadwiger characterization theorem.

- Now take the standard representation of $SO(n)$, namely $\Gamma_{(1,0,\ldots,0)} \cong V_{\mathbb{C}}$. The highest weight $\lambda = (1,0,\ldots,0)$ does not satisfy the conditions in Theorem 6.1.2. Therefore, there is no continuous, translation-invariant and $SO(n)$-equivariant valuation of degree i with values in $V_{\mathbb{C}}$.

- We get the same result for the space $\Lambda^2 V_{\mathbb{C}}$, which is an $SO(n)$ module with highest weight $\lambda = (1,1,0,\dots,0)$.
- Let $\Gamma = \operatorname{Sym}^k V_{\mathbb{C}}$. This space has a decomposition into irreducible submodules, compare (3.3), and by Theorem 6.1.2, we have

$$\dim(\mathbf{Val}_i \times \operatorname{Sym}^k V_{\mathbb{C}})^{SO(n)} = \begin{cases} k/2+1 & \text{if } k \text{ is even,} \\ (k-1)/2 & \text{if } k \text{ is odd,} \end{cases}$$

  for $1 \leq i \leq n-1$. In particular, there exist two non-trivial, continuous, translation-invariant and $SO(n)$-equivariant $\operatorname{Sym}^2 V_{\mathbb{C}}$-valued valuations of a given degree $1 \leq i \leq n-1$.

Bibliography

[1] Semyon Alesker, *Description of translation invariant valuations on convex sets with solutions of P.McMullen's conjecture*, Geom. Funct. Anal. 11 (2001), 244-272.

[2] Semyon Alesker, *Theory of valuations on manifolds, I. Linear spaces*, Israel J. Math. 156 (2006), 311-339.

[3] Seymon Alesker, Andreas Bernig and Franz Schuster, *Harmonic Analysis of translation invariant Valuations*, Geom. Funct. Anal 21 (2011), 751-773

[4] Semyon Alesker and Joseph Fu, *Theory of valuations on manifolds, III. Multiplicative structure in the general case*, Trans. Amer. Math. Soc. 360 (2008), no. 4, 1951-1981.

[5] Astrid Berg, *The Hard Lefschetz Theorem*, Diplomarbeit, Wien (2012).

[6] Andreas Bernig, *Geometrische Masstheorie*, Course material, Zürich, 2003.

[7] Andreas Bernig, *Invariant valuations on quaternionic vector spaces*, J. Inst. de Math. de Jussieu 11 (2012), 467-499.

[8] Andreas Bernig and Ludwig Bröcker, *Valuations on manifolds and Rumin cohomology*, J. Differential Geom. 75 (2007), no. 3, 433-457

[9] Theodor Bröcker, Tammo tom Dieck *Representations of Compact Lie Groups. Graduate Texts in Mathematics 98*, Springer-Verlag, New York (1985).

[10] Andreas Cap, *Lie Groups*, Course material, Wien, 2013.

[11] William Fulton, Joe Harris, *Representation Theory. A First Course. Graduate texts in Mathematics 129*, Springer-Verlag, New York (1991).

[12] Hugo Hadwiger, *Vorlesung über Inhalt, Oberfläche und Isoperimetrie*, Springer Berlin, 1957.

[13] Anthony W. Knapp, *Lie Groups. Beyond an Introduction. Second Edition.*, Birkhäuser, Boston (2005).

[14] Steven G. Krantz and Harold R. Parks, *Geometric Integration Theory*, Birkhäuser, Boston, 2008.

[15] John M. Lee, *Introduction to Smooth Manifolds*, Springer-Verlag, New York, 2003

[16] Fanghua Lin and Xiaoping Yang, *Geometric Measure Theory - An Introduction*, Science Press, Beijing/New York, 2002.

[17] Peter McMullen, *Valuations and Euler-type relations on certain classes of convex polytopes*, Proc. London Math. Soc. 35 (1977), 113–135.

[18] Mark R. Sepanski, *Compact Lie Groups*, Springer-Verlag, New York (2007).