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Abstract

Justification logic can be seen as an explicit counterpart of modal logic. Instead of saying „A is necessary“ ($\Box A$), it is possible to say „A is necessarily true because of t“ ($t : A$), where the justification term t can even be constructed from more than one justification. The formal way of studying the connection between modal logic and justification logic is called realization, this is a procedure that replaces $\Box$'s in a modal formula with terms, to create a justification formula.

Sergei Artemov proved that the modal logic $\mathsf{S4}$ is realizable into the logic of proofs LP, using a cut-free sequent system for $\mathsf{S4}$. At the moment, it is believed that there exists no cut-free sequent system for $\mathsf{S5}$. Thus to be able to realize $\mathsf{S5}$ into some justification logic syntactically, another method had to be searched.

Using different realizations of the negative introspection axiom, $\mathsf{S5}$ has been realized into some of its justification counterparts by using a hypersequent system or a nested sequent system. In this thesis it will be shown, using hypersequents, that $\mathsf{S5}$ is realizable into any of the ten justification counterparts that can be constructed with these different realizations of the negative introspection axiom and can be found in the literature.

In order to establish the same result using non-modular nested sequents as introduced by Kai Brünnler, too many similar cases have to be considered and, therefore, only some of the less standard realizations will be shown. For the complete list of realizations using nested sequents, the nested sequent calculi introduced by Lutz Straßburger will be employed, these calculi are better suited for this purpose due to their modular nature. This modularity makes it possible to realize each rule independent of the other rules and, accordingly, of the justification operations corresponding to the other rules. Even though the focus lies on the realizations of $\mathsf{S5}$, the modularity of Straßburger's calculi ensures that the given proofs imply fully modular realization theorems for all modal logics in the modal cube between K and $\mathsf{S5}$.

Kurzfassung

Begründung-Logik kann als das explizite Gegenstück von Modallogik gesehen werden. Statt zu sagen „A ist notwendig“ ($\Box A$), ist es hier möglich, zu sagen „A ist notwendigerweise wahr wegen t“ ($t : A$), wobei der Begründungsausdruck t sogar aus mehreren Begründungen geformt werden kann. Die formale Verbindung zwischen Modallogik und Begründung-Logik, ist gegeben durch Realisierung. Das ist eine Prozedur, bei der $\Box$'s durch Begründungsausdrücke ersetzt werden, wodurch eine Begründungsformel entsteht.

Sergei Artemov bewies, dass die Modallogik $\mathsf{S4}$ in die Logic of Proofs LP realisiert werden kann, wobei ein schnittfreier Sequenzenkalkül für $\mathsf{S4}$ verwendet wird. Derzeit wird vermutet, dass es keinen schnittfreien Sequenzenkalkül für $\mathsf{S5}$ gibt. Um $\mathsf{S5}$ syntaktisch in eine Begründung-Logik zu realisieren, muss eine andere Methode gesucht werden.

$\mathsf{S5}$ ist in einige seiner Begründung-Gegenstücke realisiert worden, wobei ein Hypersequenzen-System oder ein verschachtelter Sequenzenkalkül sowie verschiedene Realisierungen des negativen Introspektionsaxioms verwendet wurden. In dieser Arbeit wird mit Hilfe eines Hypersequenzenkalküls gezeigt, dass $\mathsf{S5}$ in alle zehn Begründungs-Logik-Gegenstücke realisiert werden kann, die mit diesen verschiedenen Realisierungen des negativen Introspektionsaxioms konstruiert werden können.

Um das gleiche Resultat zu bekommen, wenn nicht modulare-verschachtelte Sequenzen wie von Kai Brünnler eingeführt verwendet werden, müssen zu viele ähnliche Falle unterschieden werden, und daher wird das nur für einige weniger standard Realisierungen gemacht. Für die komplette Liste mit Realisierungen wobei verschachtelte Sequenzen verwendet werden, werden die verschachtelte Sequenzenkalküle, eingeführt von Lutz Straßburger, verwendet, dieses Kalkül ist auf grund ihter modularen Nature besser geeignet für dieses Ziel. Diese Modularität macht es möglich die Regeln unabhängig von den anderen Regeln und dementsprechend, von dem Begründungsoperationen die mit die anderen Regeln korrespondieren zu realisieren. Obwohl konzentriert wird auf die Realisierungen von $\mathsf{S5}$, gewährleistet die Modularität von Straßburger's Kalkül, dass die gegebenen Beweise vollständig modulare Realisierungs Sätze für alle Modal-Logiken in des Modalkubus zwischen K und $\mathsf{S5}$ impliziert.

Contents

Introduction

Formal reasoning is a crucial part of artificial intelligence, in order to create intelligent entities, one has to understand what intelligence is and how to reason in an intelligent manner. Since human beings assume that they are intelligent, the goal is often to formalize human reasoning. Logic is such a formal way of reasoning, studied since Ancient Greece. Therefore logic is an important part of computational intelligence and more general, of computer science.

Classical propositional and first-order logic and modal logic are taught in bachelor and master curricula, but these are only a rather small tip of the iceberg. Each logic tries to model a specific aspect of (human) reasoning in the best possible way.

In modal logics modalities like $\Box$ and $\Diamond$ are used. Depending on the modal logic, $\Box A$ could mean for example, "A is known" or "A is necessarily true". It is however never possible to change the meaning of this $\Box$ once chosen: each occurrence of a $\Box$ within that specific logic, has the same meaning.

In justification logic, instead of these modalities, justifications are used. Let t be a justification for A, this is denoted by $t : A$ and could mean "A is known because of t". Part of the language of justification logic consists of operators on these justifications. As is the case in real life, one justification to know something is often not enough.

In the literature, justification logic is called an *explicit counterpart of modal logic*. Instead of saying "A is necessary" ($\Box A$), it is possible to say "A is necessarily true because of t" ($t : A$) using justifications and t might even be constructed from more than one justification. The procedure of replacing $\Box$'s in a modal formula with terms, to create a justification formula, is called realization. This might suggest a connection between these two kinds of logics, which is indeed the case.

The logic of proofs, LP as introduced by Sergei Artemov [1, 2], has such a connection with the modal logic $\mathsf{S4}$, more formally: $\mathsf{S4}$ is realizable into LP. The proof of this realization as given by S. Artemov [2], is based on a cut-free sequent system for $\mathsf{S4}$. Since then realization theorems for other connections between modal logic and justification logic have been proven, but $\mathsf{S5}$ seems more difficult.

At the moment, it is believed that there does not exist a cut-free sequent system for $\mathsf{S5}$. Thus to be able to realize $\mathsf{S5}$ into some justification logic syntactically, another method had to be searched. Over the years, different kinds of these methods are introduced to realize $\mathsf{S5}$ into different justification logics:

- Using hypersequents, it has been shown by S. Artemov, E. Kazakov and D. Shapiro [4] that $\mathsf{S5}$ is realizable into $\mathsf{LPS5}$ and $\mathsf{LPS5}_c$, two extensions of the logic LP. Two different realizations of the negative introspection axiom 5, called A5 and $A5_c$, have been used.

- Using nested sequents, it has been shown by R. Goetschi and R. Kuznets [12] that S5 is realizable into the justification logic JT45, an extension of LP as well. Another realization of the negative introspection axiom 5, called j5 was used here.

In addition to the logic JT45, seven other logics were introduced by R. Goeschi and R. Kuznets [12]: JT5, JTB4, JTB5, JDB4, JDB5, JDB45 and JTB45. The axiom j5 is part of the logics JT5, JTB5, JDB5, JDB45 and JTB45, but in JTB4 and in JDB4 none of the aforementioned realizations of 5 are used. The proof of the realization into JT45 applies to the logic JTB45 as well. For the other six logics a translation method, on top of the syntactical method, was used to prove the realization of S5 into these remaining logics.

With the three realizations of the 5-axiom above mentioned, there are ten different justification counterparts of the modal logic S5 that can be found so far in the literature. It has been proven that S5 is realizable into any of these justification counterparts, but using different methods. For three of them a syntactic method of realization can be found, one implying the realization of a forth.

Knowing these results, the goal of the thesis is to use a syntactic method to prove the realizations of the remaining six justification logics and trying to show that any cut-free proof system works for every existing realization. This is done by answering the following questions: is it possible to realize S5 into LPS5 and $\mathsf{LPS5}_c$ using nested sequents,into JT45, JDB45, JTB45, JT5, JTB4, JDB4, JDB5 and JTB5 using hypersequents and into any of these ten justification counterparts using Straßburger's modular nested sequents [15]?

To answer this question hypersequents (in chapter 3), nested sequents (in chapter 4) and Straßburger's modular nested sequents (chapter 5) will be considered. Each sequent system introduces a set of rules that could be used in an S5-derivation. The goal is to prove that, assuming by induction hypothesis that the premise of a rule can be realized, the conclusion of the rule can be realized in the justification counterpart of S5. And this for each rule, for each of these ten justification logics.

The hypersequent system as that was introduced by S. Artemov, E. Kazakov and D. Shapiro [4] is written in a notation that is no longer used. A definition in the standard notation can be found in a paper by A. Avron on hypersequents [5]. The definitions, lemmas and proofs concerning hypersequents will be based on the system that was used to realize S5 but will be written in the standard notation.

In the first chapter a short introduction will be given on modal logic and justification logic, stating the axiom systems and logics that will be used in the thesis. The second chapter covers the general idea of realization theorems and some important notions, such as internalization and substitution.

The third chapter starts with the definition of the hypersequent system for S5 in the standard notation. In this chapter the realization of S5 into JT45, JDB45, JTB45, JT5, JTB4, JDB4, JDB5 and JTB5 using hypersequents is proven. In the forth chapter the nested sequent system for S5 as used by R. Goetschi and R. Kuznets [12] is given, followed by the proofs of the realization of S5 into LPS5 and $\mathsf{LPS5}_c$.

The nested sequent system of S5 consists in addition to the propositional rules of five modal rules. To prove the realization of these five rules into LPS5, LPS5$_c$, JDB45, JT5, JTB4, JDB4, JDB5 and JTB5, forty cases have to be considered. Instead of proving all these cases, the modular nested sequent system, introduced by S. Marin and L. Straßburger [15] will be used. This is an extension of the nested sequent system as it can be found in R. Goetschi and R. Kuznets [12], there are seven new rules. The use of this system makes it possible to prove the realization in a modular manner: each of the rules can be realized independently of the others. Which means that only seven new cases have to be considered, one for each additional rule. The modularity of the realization method in combination with the modularity of Straßburger's calculi implies the modularity theorem for all modal logics in the modal cube between K and S5. The system will be given in chapter five, where the realization of S5 into any of the ten considered justification counterparts will be proven as well.

CHAPTER 1

Modal Logic and Justification Logic

1.1 Modal Logic

When reasoning about necessity, knowledge, certainty or always, classical propositional logic is not enough. This is shown by the following example, as Aristotle pointed out already [9, p. 35-36]:

EXAMPLE 1.1.1. Consider a sea-battle, then there are two possibilities: either it will take place tomorrow or it will not take place tomorrow:

- It is necessary that there will be or will not be a sea-battle tomorrow
- It is not necessary that a sea-battle will take place tomorrow
- It is not necessary that there will not be a sea-battle tomorrow.

Notice that the formulation of these three statements have more or less the same structure. Now, it is necessary that the sea-battle will take place or that it will not take place, but it is not possible to predict the future and conclude that it is necessary that the sea-battle will indeed take place or that it is necessary that the sea-battle will not take place.

In classical propositional logic, it is possible to let atomic formulas denote the three different statements. But it is not possible to reason why the first statement is valid and why the second and third statement are not valid[1]. $\Diamond$

Therefore, with classical propositional logic, it is not possible to reason about necessity, or any other kind of intensional context. Modal logic is a logic that makes it possible to reason with adverbials, like necessary, known to be, permitted, now etc.

Modal logic extends classical propositional logic, by adding modal operators, called modalities. There are many different modalities that form different kinds of modal logics, here only the modalities $\Box$ and $\Diamond$ are considered. Depending on the kind of modal logic these operators have

[1]The original example is more complex and the discussing of it, by many important logicians throughout the ages, is far more extensive. However, here this shorter version is sufficient to show that reasoning about necessity cannot be done in classical propositional logic.

different meanings. Usually, $\Box$ means *necessary* and $\Diamond$ means *possibly*, but when reasoning about knowledge, $\Box$ means *it is known that* and $\Diamond$ means *it is consistent that*[2].

Syntax

The language of modal logic, as considered here, consists of the connectives $\neg$, $\wedge$, $\vee$ and $\supset$, the operators $\Box$ and $\Diamond$, propositional variables $p, q, r, \ldots$, $\bot$, $\top$ and (,). Formulas in modal logic can be defined with the following grammar, let p be a propositional variable:

$$A := p \mid \bot \mid \top \mid \neg A \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A \mid \Diamond A \tag{1.1}$$

Semantics

In modal logic it is not enough to look at some interpretation, as is done in classical propositional logic. *Kripke models* can be used in order to say something about the truth of a modal formula.

Definition 1.1.2. A *Kripke frame* is a tuple $\mathcal{F} = \langle W, R \rangle$. Where W is a non-empty set of possible worlds and $R \subseteq (W \times W)$ a binary relation on W, called the accessibility relation, if vRw then w is accessible from v.

A *Kripke model* is a tuple $\mathcal{M} = \langle W, R, V \rangle$, such that $\langle W, R \rangle$ is a Kripke frame and V is a valuation, a relation that assigns subsets of possible worlds to propositional variables. A proposition p is true in a world $w \in W$ if $w \in V(p)$ and p is false if $w \notin V(p)$.

The Kripke model $\mathcal{M} = \langle W, R, V \rangle$ is based on the frame $\mathcal{F} = \langle W, R \rangle$. $\Diamond$

Definition 1.1.3. Let $\mathcal{M} = \langle W, R, V \rangle$ and $w \in W$, then the truth of a formula in world w is defined as follows:

$$\begin{array}{lcl}
\mathcal{M}, w \not\Vdash \bot & \text{and} & \mathcal{M}, w \Vdash \top \\
\mathcal{M}, w \Vdash p & \Leftrightarrow & w \in V(p) \\
\mathcal{M}, w \Vdash \neg A & \Leftrightarrow & \mathcal{M}, w \not\Vdash A \\
\mathcal{M}, w \Vdash A \wedge B & \Leftrightarrow & \mathcal{M}, w \Vdash A \text{ and } \mathcal{M}, w \Vdash B \\
\mathcal{M}, w \Vdash A \vee B & \Leftrightarrow & \mathcal{M}, w \Vdash A \text{ or } \mathcal{M}, w \Vdash B \\
\mathcal{M}, w \Vdash A \supset B & \Leftrightarrow & \mathcal{M}, w \not\Vdash A \text{ or } \mathcal{M}, w \Vdash B \\
\mathcal{M}, w \Vdash \Box A & \Leftrightarrow & \mathcal{M}, v \Vdash A, \text{ for all } v \text{ such that } wRv \\
\mathcal{M}, w \Vdash \Diamond A & \Leftrightarrow & \mathcal{M}, v \Vdash A, \text{ for some } v \in W, \text{ such that } wRv.
\end{array}$$

$\Diamond$

Definition 1.1.4. ([9]) Let $\mathcal{M} = \langle W, R, V \rangle$ be a Kripke model. A formula A is *valid in a model* $\mathcal{M}$ if and only if for all possible worlds $w \in W$: $\mathcal{M}, w \Vdash A$. A formula A is *valid in a frame* $\mathcal{F} = \langle W, R \rangle$ if and only if A is valid in every modal $\mathcal{M}$ that can be based on this frame.

Let L be a collection of frames, a formula A is *L-valid* if and only if A is valid in every frame in the collection L. $\Diamond$

[2]This section is based on *First-Order Modal Logic*, by M. Fitting and R.L. Mendelsohn [9]

Different Modal Logics

Definition 1.1.5. A formula in the language of classical propositional logic is a tautology if and only if the formula is true for any interpretation of the propositional variables that it contains. $\diamondsuit$

Definition 1.1.6. A tautology in the language of modal logic is a propositional tautology as defined in Definition 1.1.5, where every propositional variable in the tautology can be substituted by a formula built according to Grammar 1.1 on page 6. Within proofs, any instance of a tautology as defined here, will be called a propositional tautology, since these tautologies are based on classical propositional tautologies. $\diamondsuit$

From this definition it follows that, since $p \supset (q \supset p)$ is true for any interpretation of p and q, $\Box A \supset (\Box(A \wedge \Box B) \supset \Box A)$ is a tautology in the modal language.

Definition 1.1.7. The axiom system for the basic modal logic (called K) is defined as follows:

- Tautologies in the modal language as defined in Definition 1.1.6
- Axiom K: $\Box(A \supset B) \supset (\Box A \supset \Box B)$
- Modus Ponens: From A and $A \supset B$ derive B
- Necessitation: From A derive $\Box A$.

By adding one or more of the following axioms to the system, new modal logics occur:

- t: $\Box A \supset A$
- d: $\Box\bot \supset \bot$
- b: $\neg A \supset \Box\neg\Box A$
- $\mathsf{4}$: $\Box A \supset \Box\Box A$
- $\mathsf{5}$: $\neg\Box A \supset \Box\neg\Box A$.

$\diamondsuit$

Based on the axioms that are part of the system, a modal logic has a corresponding accessibility relation R on its frames.

Definition 1.1.8. Let ML be a modal logic, whose axiom system extents the basic modal logic K by zero or more of the above defined axioms and let $\langle W, R\rangle$ be a Kripke frame for ML, then:

- If t is part of the axiom system of ML, then R is *reflexive*, meaning: for all $w \in W$: $(w, w) \in R$.
- If d is part of the axiom system of ML, then R is *serial*, meaning: for all $v \in W$, there is a $w \in W$ such that $(v, w) \in R$.
- If b is part of the axiom system of ML, then R is *symmetric*, meaning: for all $v, w \in W$, if $(v, w) \in R$ then $(w, v) \in R$.
- If $\mathsf{4}$ is part of the axiom system of ML, then R is *transitive*, meaning: for all $u, v, w \in W$, if $(u, v) \in R$ and $(v, w) \in R$ then $(u, w) \in R$.
- If $\mathsf{5}$ is part of the axiom system of ML, then R is *Euclidean*, meaning: for all $u, v, w \in W$, if $(w, u) \in R$ and $(w, v) \in R$ then $(u, v) \in R$.

If the axiom system of ML does not contain any of the additional axioms, meaning ML=K, then there are no restrictions on the accessibility relation R. Kripke frames that do not have any restrictions on R, belong to the collection of frames K. $\diamondsuit$

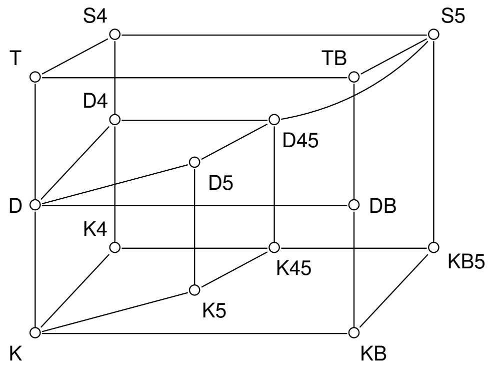

Figure 1.1: The *modal cube*

With the Definition 1.1.4, Definition 1.1.7 and Definition 1.1.8, it is now possible to state soundness and completeness for the basic modal logic K.

Theorem 1.1.9 ([9]). *Consider the basic modal logic K, a formula A is provable in the axiom system of K if and only if A is K-valid. In other words: the basic modal logic K is sound and complete.*

Based on the five axioms that could be added to the basic modal logic K, $2^5 = 32$ different axioms systems can be defined. However, not every axiom system yields a different logic. There are only 15 different modal logics, that are obtained this way. These 15 different modal logics form the so-called *modal cube*, (see Figure 1.1, [10]). The name of a modal logic depends on its axiom system, almost all the names start with K, denoting the basic modal logic. Depending on the additional axioms, the name is extended by the name that denotes the additional axiom. For example, the logic which axiom system extends the axiom system of K with the axioms 4 and 5 has the name K45.

There are two exceptions: the logics S4 and S5, S4 represents two axiom systems and S5 represents thirteen axiom systems:

Definition 1.1.10. The logic S4 has a reflexive, transitive accessibility relation. It is the modal logic that represents the axiom systems KT4 and KTD4.

The logic S5 has a reflexive, symmetric, transitive accessibility relation. It is the modal logic that represents the axiom systems KT45, KT5, KTB5, KTB45, KDB5, KDB45, KDB4, KTB4, KTD5, KTDB4, KTDB5, KTD45 and KTDB45. ♢

Not every modal logic that can be constructed will be considered in this text. The most important modal logics are S4 and S5. When a more general notion of a modal logic is used, only the following well known logics are meant: K, D, T, K4, B, S4 or S5. Using the axioms as defined above, for each of these seven logics, M. Fitting and R.L. Mendelsohn proved the following theorem [9]:

Theorem 1.1.11. *Let L be some modal logic. There is a proof of a formula A in the axiom* *system of L if and only if A is L-valid. This means: the modal logic L is sound and complete.*

1.2 Justification Logic

Plato defined knowledge as *justified, true belief*, until 1963 this was considered to be an adequate definition of knowledge. This would mean that A knows B if and only if: B is true, A believes that B and A is justified in believing B. In 1963, E. Gettier published *Is Justified True Belief Knowledge?* [11] in which he gives two examples that show that this is not a complete definition of knowledge.

EXAMPLE 1.2.1. Jones and Smith apply for a job. Suppose that, for several reasons, Smith has strong evidence that:

(a) Jones is the man who will get the job and Jones has ten coins in his pocket.

From this, Smith concludes that:

(b) The man who will get the job has ten coins in his pocket.

Based on the above observation, Smith is justified in believing that (b) is true. Now suppose that not Jones, but Smith gets the job and that Smith has ten coins in his pocket as well. The conclusion that Smith made from (a) is true, Smith believes in it and he is justified in believing it. But Smith does not know that he has ten coins in his pocket! He bases his belief on the number of coins in Jones' pocket. Does Smith still *know* (b)? $\diamondsuit$

As was described above, modal logic is suitable for reasoning about necessity, certainty and knowledge. However, $\Box A$, is always $\Box A$. In the above example it holds then that $\Box$(b) and hence that (b) is known. In justification logic, it is possible to distinguish between reasons to know something. For Smith (a) was the reason that he knows (b), but what if (a) is not true anymore, does he still know (b)?

Instead of always using the same $\Box$, justification logic uses terms t, such that $t : A$ means: t is a justification for A. There is not just one term, there may be other terms in a derivation or a formula. By using operators, new terms can be constructed[3].

Syntax

The language of any justification logic consists of the connectives $\neg$, $\wedge$, $\vee$ and $\supset$, propositional variables $p, q, r, \ldots$, $\bot$, $\top$, (,), proof variables $x, y, z, \ldots$, proof constants, $a, c, \ldots$ and operators $\cdot$, $+$ and :. The operator $'$:$'$ is used such that: $term : formula$ is a formula of the justification logic. Let x be a proof variable, c a proof constant, a term can be defined using the following grammar:

$$t := x \mid c \mid t \cdot t \mid t + t.$$

Let t be a term as defined above and p a propositional variable, then a formula in justification logic can be defined with the following grammar:

$$A := p \mid \bot \mid \top \mid \neg A \mid A \wedge A \mid A \vee A \mid A \supset A \mid t : A \tag{1.2}$$

[3]This section is based on papers written by S.N. Artemov [2, 3] and by M. Fitting [8]

Semantics

As in modal logic, the semantics of justification logic is defined based on models.

Definition 1.2.2. A justification logic model is a tuple $\mathcal{M} = \langle W, R, \mathcal{E}, V \rangle$. W is a nonempty set of possible worlds, $R \subseteq (W \times W)$ is an accessibility relation. $\mathcal{E}$ is an evidence function on $\langle W, R \rangle$, it is a mapping from terms and formulas to subsets of possible worlds and V is a valuation, a relation that assigns subsets of possible worlds to propositional variables, as in the modal case. $\diamondsuit$

Definition 1.2.3. Let $\mathcal{M} = \langle W, R, \mathcal{E}, V \rangle$ be a model and $w \in W$, then the truth of a formula in world w is defined as follows:

$\mathcal{M}, w \Vdash p$	$\Leftrightarrow$	$w \in V(p)$
$\mathcal{M}, w \nVdash \bot$	and	$\mathcal{M}, w \Vdash \top$
$\mathcal{M}, w \Vdash \neg A$	$\Leftrightarrow$	$\mathcal{M}, w \nVdash A$
$\mathcal{M}, w \Vdash A \wedge B$	$\Leftrightarrow$	$\mathcal{M}, w \Vdash A$ and $\mathcal{M}, w \Vdash B$
$\mathcal{M}, w \Vdash A \vee B$	$\Leftrightarrow$	$\mathcal{M}, w \Vdash A$ or $\mathcal{M}, w \Vdash B$
$\mathcal{M}, w \Vdash A \supset B$	$\Leftrightarrow$	$\mathcal{M}, w \nVdash A$ or $\mathcal{M}, w \Vdash B$
$\mathcal{M}, w \Vdash t : A$	$\Leftrightarrow$	$w \in \mathcal{E}(t, A)$ and for all $v \in W$ such that $(w, v) \in R : \mathcal{M}, v \Vdash A$.

$\diamondsuit$

The definition above is almost the same as the definition for modal logic. However, since there is not just one $\Box$, but there are different justifications, the evidence function $\mathcal{E}$ is needed. The condition that $w \in \mathcal{E}(t, A)$ says that in world w, t is a justification for A, based on the defined evidence function and $\mathcal{E}(t, A) \subseteq W$ denotes the set of worlds in which t is a justification for A. The properties of $\mathcal{E}$ depend on the justification logic.

The following definition is based on the definition of validity in modal models and frames, Definition 1.1.4.

Definition 1.2.4. Let $\mathcal{M} = \langle W, R, \mathcal{E}, V \rangle$ be a justification model. A formula A is *valid in a justification model* $\mathcal{M}$ if and only if for all possible worlds $w \in W$: $\mathcal{M}, w \Vdash A$. A formula A is *valid in a frame* $\mathcal{F} = \langle W, R \rangle$ if and only if A is valid in every justification model $\mathcal{M}$ that can be based on this frame.

Let L be a collection of frames, a formula A is *L-valid* if and only if A is valid in every frame in the collection L. $\diamondsuit$

Different Justification Logics

Definition 1.2.5. A tautology in the language of justification logic is a propositional tautology as defined in Definition 1.1.5, where every propositional variable in the tautology can be substituted by a formula built according to Grammar 1.2 on page 9. $\diamondsuit$

From this definition it follows that, since $(p \wedge q) \supset p$ is a classical propositional tautology according to Definition 1.1.5, the formula $(t : A \wedge s : (B \supset C)) \supset t : A$ is a tautology in the language of justification logic. Within proofs, any instance of a tautology as defined here, will be called a propositional tautology, since these tautologies are based on classical propositional tautologies.

Definition 1.2.6. The axiom system for the basic justification logic (called J) is defined as follows:

- taut: Tautologies in the justification language as defined in Definition 1.2.5
- app: $s : (A \supset B) \supset (t : A \supset (s \cdot t) : B)$
- sum: $s : A \supset (s + t) : A$ and $t : A \supset (s + t) : A$
- Modus Ponens (MP): $\dfrac{A \quad A \supset B}{B}$
- Axiom Necessitation (AN): For constants $c_1, \ldots, c_n$: $\dfrac{\text{Axiom } A}{c_n : \ldots : c_1 : A}.$

By adding one or more of the following axioms to the system, new justification logics occur:

- jt: $t : A \supset A$
- jd: $t : \perp \supset \perp$
- jb: $A \supset \bar{?}t : (\neg t : \neg A)$
- j4: $t : A \supset !t : t : A$
- j5: $\neg t : A \supset ?'t : (\neg t : A)$
- A5: $t : (A \supset \neg s : B) \supset (A \supset ?t : (\neg s : B))$
- A5$_c$: $\neg t : A \supset c : \neg t : A$.

$\diamondsuit$

Notice that the language of a logic containing j4, has an operator ! in addition to the already defined operators, a term t in this logic can have the form $!t$ as well. The language of a logic containing jb, j5 or A5, has an operator $\bar{?}$, $?'$ or $?$ respectively, in addition to the already defined operators. A term t in one of these logics can have the form $\bar{?}t$, $?'t$ or $?t$ respectively as well[4].

In accordance with modal logic, the name of a justification logic depends on its axiom system. Instead of the K for the basic modal logic, J is used for the basic justification logic. Notice that maximal one of the axioms j5, A5 or A5$_c$ can be part of an axiom system. Traditionally, there is one exception to this kind of naming, the *Logic of Proofs*, defined by S.N. Artemov. This logic is denoted by LP, its axiom system extends that of J by jt and j4. This exception is also used for the logics LPS5 and LPS5$_c$, to denote the logics with axiom systems extending that of LP with A5 and A5$_c$ respectively. The names as they were introduced by S.N. Artemov, E.L. Kazakov and D. Shapiro in *Logic of knowledge with justifications* [4] are followed.

With these definitions, the most important justification logics for this text can be defined. When in general some justification logic is considered, one of the logics in Table 1.1 are meant, this is denoted by JL. When a (set of) justification logic(s) is/are considered in order to say something about these specific logics, $\mathcal{L}$ will be used to represent these logics.

[4] In some papers terms are called proof polynomials and in the paper by S.N. Artemov, E.L. Kazakov and D. Shapiro [4], they are called extended proof polynomials because of the new $?t$ form of a term, here they are called terms, independent of the form they take.

Name	Additional axioms	Name	Additional axioms
J	None	LP	jt and j4
LPS5	jt, j4 and A5	$\mathsf{LPS5}_c$	jt, j4 and $A5_c$
JT45	jt, j4 and j5	JT5	jt and j5
JTB5	jt, jb and j5	JTB45	jt, jb, j4 and j5
JDB5	jd, jb and j5	JDB45	jd, jb, j4 and j5
JDB4	jd, jb and j4	JTB4	jt, jb and j4

Table 1.1: The justification logics as used here

Evidence Function

Based on the axioms that are part of the axiom system, a justification logic has a corresponding accessibility relation R as was defined for modal logic in Definition 1.1.8. However, now the justification axioms jt, jd, jb, j4 and j5 restrict the accessibility relation R. If the axiom jt is part of the axiom system then R has to be reflexive, the axiom jd means that R has to be serial, the axiom jb means that R is symmetric, an axiom system including j4 has to have a transitive accessibility relation R and an axiom system that includes j5 has to have a Euclidean R.

The restriction of the accessibility relation was enough to create Kripke models for modal logics with the corresponding axioms. This is not the case for justification logics. As was already suggested in Definition 1.2.2, besides the accessibility relation the evidence function is part of a justification logic model.

Based on the axiom system of a justification logic, the evidence function $\mathcal{E}$ has different properties. The following definition states these properties, the properties for app, sum, j4 and j5 are based on the PhD thesis of R. Kuznets [13], the properties for jb, A5 and $A5_c$ are new.

Definition 1.2.7. Consider a justification model $\mathcal{M} = \langle W, R, \mathcal{E}, V\rangle$. A case distinction has to be made based on the axiom system of the justification logic:

- In every justification logic JL, for all terms s and t, for all justification formulas A and B and for all worlds $v, w \in W$, $\mathcal{E}$ has the following two properties, corresponding to app and sum respectively:

$$\text{Application closure: } \mathcal{E}(s, A \supset B) \cap \mathcal{E}(t, A) \subseteq \mathcal{E}(s \cdot t, G)$$
$$\text{Sum closure: } \mathcal{E}(s, A) \cup \mathcal{E}(t, A) \subseteq \mathcal{E}(s + t, A).$$

- A justification logic containing j4 has a transitive accessibility relation and the evidence function has two additional properties:

$$\text{Positive introspection closure: } \mathcal{E}(t, A) \subseteq \mathcal{E}(!t, t : A)$$
$$\text{Monotonicity: If } (v, w) \in R \text{ and } v \in \mathcal{E}(t, A) \text{ then } v \in \mathcal{E}(t, A).$$

- A justification logic containing j5 has a Euclidean accessibility relation. Let $[\Phi]^c$ denote the complement of set Φ. The evidence function in such a logic satisfies the following property:

$$\text{Negative introspection closure: } [\mathcal{E}(t, A)]^c \subseteq \mathcal{E}(?'t, \neg t : A).$$

- If the axiom system of a justification logic contains the axiom A5, the evidence function has to have the following property:

  A5 closure: $\mathcal{M}, w \Vdash A$ and $w \in \mathcal{E}(t, (A \supset \neg s : B))$ implies $w \in \mathcal{E}(?t, \neg s : B)$.

- Models for justification logics which axiom systems contain A5_c, have an evidence function which has at least the following property:

  A5_c closure: $[\mathcal{E}(t, A)]^c \subseteq \mathcal{E}(c, \neg t : A)$.

- A justification logic containing jb has a symmetric accessibility relation and the evidence function has the following property:

  Symmetry closure: $\mathcal{M}, w \Vdash A$ implies $w \in \mathcal{E}(\overline{?}t, \neg t : \neg A)$.

A model for a justification logic that has at least one of the axioms j5, A5, A5_c and/or jb in its axiom system should satisfy the so called *strong evidence property* as well, for any term t, for any justification formula A and a world $w \in W$:

$$\text{If } w \in \mathcal{E}(t, A) \text{ then } \mathcal{M}, w \Vdash t : A.$$

$\diamondsuit$

In order to prove that the given properties for the evidence function are the right ones, soundness and completeness have to be proven:

Theorem 1.2.8. *For any justification logic* JL *a formula A is provable in* JL *if and only if it is* JL*-valid:*

$$\mathsf{JL} \vdash A \Leftrightarrow A \textit{ is } \mathsf{JL}\textit{-valid}.$$

The proof of this theorem can be found in AppendixA.

CHAPTER 2

Realization Theorems

Before proving realizations of $\mathsf{S5}$ into different justification logics, certain concepts, definitions and lemma's require presentation. Furthermore, the general idea of realization will be sketched. When modal or justification axioms, axiom systems and/or logics are mentioned, only those defined in the first chapter are meant.

2.1 Forgetful Projection and Realization Theorems

A proof of realization is usually based on some cut-free sequent calculus. The procedure of transforming a formula from one logic into another is called a *realization procedure* and can be carried out in two directions. The direction in which a justification formula is transformed into a modal formula is the easier direction and is called the *forgetful projection*: every term is replaced with the same $\Box$, which means that some information is lost. The direction in which a modal formula is transformed into a justification formula is more involved, since not all $\Box$'s should be realized into the same justification term.

Definition 2.1.1. Let A be a formula in the justification language (possibly also containing unary term operators '!', '$\bar{?}$', '?'' and '?'), a formula A° in the modal language can be constructed by substitution of $\Box$ for every occurrence of a term as follows:

$$\begin{array}{llll} p^\circ = p, & \top^\circ = \top, & (B_1 \wedge B_2)^\circ = B_1^\circ \wedge B_2^\circ, & (B_1 \supset B_2)^\circ = B_1^\circ \supset B_2^\circ, \\ \bot^\circ = \bot, & (\neg B)^\circ = \neg B^\circ, & (B_1 \vee B_2)^\circ = B_1^\circ \vee B_2^\circ, & (\Box B)^\circ = \Box B^\circ. \end{array}$$

$\diamondsuit$

For the other direction, from modal logic to justification logic, the notion of a realization function is needed:

Definition 2.1.2. A *realization function* on a formula A, is a function that substitutes justification terms for occurrences of $\Box$ in A. The result is a formula in the justification language, denoted by A^r. $\diamondsuit$

Applying first a realization function on a formula from the modal language and then the forgetful projection on the result, gives the original modal formula again:

REMARK. Let A be a formula of the modal language and let r be a realization function on A, then A^r is a formula in the justification language. Applying the forgetful projection to A^r, will give a formula in the modal language again: $(A^r)^\circ$, such that: $(A^r)^\circ = A$.

Counterparts

In the definitions above the possibility of translating between modal and justification formulas has been introduced. Taking this into account and looking at the names of different modal and justification axioms and logics a connection between specific modal and justification axioms and logics can be suspected. Such a connection can be formalized in terms of *counterparts*, which already exists on axiom level:

Definition 2.1.3. Every modal axiom mentioned earlier has at least one *justification counterpart* and every justification axiom has one *modal counterpart*. ♢

These counterparts can be recognized by their names, for example, the justification counterpart of the axiom t is the axiom jt and the modal counterpart of the axiom jb is the axiom b.

Furthermore, the relation of being a counterpart of each other can be extended to the logics:

Definition 2.1.4. Let JL be some justification logic. It is said that JL *realizes* ML, if $\mathsf{JL}^\circ = \mathsf{ML}$. This means: applying the forgetful projection to all of the theorems of JL results in exactly the set of theorems of ML and for each formula in JL there exists a realization function such that appyling this realization function to the formula results in a theorem of ML. If JL realizes ML it is said that ML is the *modal counterpart* of JL and that JL is a *justification counterpart* of ML ♢

The modal counterpart of a justification logic is unique, a modal logic might have more than one justification counterpart. Examples of pairs of justification logics and modal logics that are counterparts of each other are J4 and K4, or J45 and K45, or LP and S4, or any of the justification logics from the set {LPS5, LPS5$_c$, JT45, JT5, JTB5, JTB45, JDB5, JDB45, JDB4, JTB4} and the modal logic S5. In order to prove that two logics are counterparts of each other, a realization theorem has to be proven.

Realization Theorems

Based on the idea described above, S. N. Artemov proved the following theorem for the modal logic S4 and justification logic LP [2]:

Theorem 2.1.5 (Realization of S4).

$$\mathsf{S4} \vdash A \;\Leftrightarrow\; \mathsf{LP} \vdash A^r \text{ for some realization } r.$$

The proof of the "⇒"-direction is based on a cut-free sequent calculus for S4. However, such a calculus is not known for the modal logic S5. In different papers solutions have been suggested, using different axiom systems for justification counterparts of S5. The idea is now, to prove the following theorem, for some of these calculi and axiom systems that are suggested:

Theorem 2.1.6 (Realization of S5).
Let $\mathcal{L} \in \{$LPS5, LPS5$_c$, JT45, JT5, JTB5, JTB45, JDB5, JDB45, JDB4, JTB4$\}$ then:

$$\mathsf{S5} \vdash A \;\Leftrightarrow\; \mathcal{L} \vdash A^r,$$

using a hypersequent calculus based on the calculus from S.N. Artemov, E.L. Kazakov and D. Shapiro [4], the nested sequent calculi from R. Goetschi and R. Kuznets [12] and modular nested sequent calculi from S. Marin and L. Straßburger [15].

2.2 Internalization, Annotation and Substitution

By annotating or labeling □'s during the derivation of a formula, these □'s can be realized by different terms when required and by the same terms if the □'s were really the same. In order to use annotations, some definitions are required. Other lemma's and definitions that are required for every calculus that is used to prove realization will be stated here as well.

Internalization

In the proof of the realization theorem of S4 into LP, a lemma called *The Lifting Lemma* was proven and used. However, the axiom j4 is needed to prove this lemma. R. Goetschi and R. Kuznets [12] used a more general lemma, basically stating the same, but the proof only uses axioms from the axiom system of the basic justification logic J. The lemma is called *Internalization*. This lemma and a corollary that follows from it, are required to prove the realization of S5.

Lemma 2.2.1 (Internalization). *Let JL be some justification logic. If $A_1, \ldots, A_n \vdash_{\mathsf{JL}} B$ then there is a term $t(x_1, \ldots, x_n)$ such that: $s_1 : A_1, \ldots, s_n : A_n \vdash_{\mathsf{JL}} t(s_1, \ldots, s_n) : B$, for any terms $s_1, \ldots, s_n$.*

If there are no premises $A_1, \ldots, A_n$, which means, $n = 0$, then t does not have any variables (it is a ground term) and $\vdash_{\mathsf{JL}} t : B$.

Proof. The proof is by induction on the derivation of $A_1, \ldots, A_n \vdash_{\mathsf{JL}} B$, based on the proofs by S.N. Artemov and R. Goetschi and R. Kuznets [2, 12]:

- Suppose B is one of the axioms of JL, then $\vdash_{\mathsf{JL}} B$. By AN, any constant $c$can be chosen. Let $t := c$, then $\vdash_{\mathsf{JL}} t : B$ and hence for $t(s_1, \ldots, s_n) = t = c$: $s_1 : A_1, \ldots, s_n : A_n \vdash_{\mathsf{JL}} t : B$.
- Suppose B is of the form $c_{n-1} : \ldots : c_1 : C$, infered by AN. By using AN again, any constant c_n can be chosen. Let $t := c_n$, then $\vdash_{\mathsf{JL}} t : B$ and hence for $t(s_1, \ldots, s_n) = t = c$: $s_1 : A_1, \ldots, s_n : A_n \vdash_{\mathsf{JL}} t : B$.
- Let B be one of the premises A_i, then let $t(x_1, \ldots, x_n) = x_i$, from which it follows that $s_i : A_i \vdash_{\mathsf{JL}} t(s_1, \ldots, s_n) : B$ and hence: $s_1 : A_1, \ldots, s_n : A_n \vdash_{\mathsf{JL}} t(s_1, \ldots, s_n) : B$.
- Let B be derived by MP from $C \supset B$ and C. By induction hypotheses, there are terms $t_1(x_1, \ldots, x_n)$ for $C \supset B$ and $t_2(x_1, \ldots, x_n)$ for B. Now $t_1(x_1, \ldots, x_n) : (C \supset B) \supset$

$(t_2(x_1, \ldots, x_n) : C \supset (t_1(x_1, \ldots, x_n) \cdot t_2(x_1, \ldots, x_n)) : B)$ is an instance of app. By MP it follows that for $t := t_1(x_1, \ldots, x_n) \cdot t_2(x_1, \ldots, x_n)$ and arbitrary terms $s_1, \ldots, s_n$: $s_1 : A_1, \ldots, s_n : A_n \vdash_{\mathsf{JL}} t(s_1, \ldots, s_n) : B$.

□

As was already suggested by R. Goetschi and R. Kuznets [12], the following theorem can be proven in any justification logic JL, since the tautologies of classical propositional logic and the rule Modus Ponens are part of the axiom system of JL.

Theorem 2.2.2 (Deduction Theorem). *Let JL be some justification logic and let Γ be a set of formulas in the languages of JL, then: $\Gamma, A \vdash_{\mathsf{JL}} B$ if and only if $\Gamma \vdash_{\mathsf{JL}} A \supset B$.*

Proof. " $\Rightarrow$" Suppose $\Gamma, A \vdash_{\mathsf{JL}} B$, the proof is by induction on the derivation of B from Γ, A:

- Let B be one of the axioms of JL or let B be of the form $c_n : \ldots c_1 : C$, infered from AN. This means: $\vdash_{\mathsf{JL}} B$. Since $B \supset (A \supset B)$ is a propositional tautology, $\vdash_{\mathsf{JL}} B \supset (A \supset B)$. By Modus Ponens it follows that $\vdash_{\mathsf{JL}} A \supset B$. Therefore, $\Gamma \vdash_{\mathsf{JL}} A \supset B$.
- Let $B = A$, then $\Gamma, A \vdash A$. As $A \supset A$ is a propositional tautology, it follows that $\vdash_{\mathsf{JL}} A \supset A$ and hence $\vdash_{\mathsf{JL}} A \supset B$. Therefore, $\Gamma \vdash_{\mathsf{JL}} A \supset B$.
- Let $B \in \Gamma$, then $\Gamma \vdash_{\mathsf{JL}} B$. Since $B \supset (A \supset B)$ is a propositional tautology, $\Gamma \vdash_{\mathsf{JL}} B \supset (A \supset B)$. By Modus Ponens it follows that $\Gamma \vdash_{\mathsf{JL}} A \supset B$.
- Assume that $\Gamma, A \vdash_{\mathsf{JL}} B$ is obtained by applying Modus Ponens to $\Gamma, A \vdash_{\mathsf{JL}} C \supset B$ and $\Gamma, A \vdash_{\mathsf{JL}} C$. By induction hypothesis it follows that $\Gamma \vdash_{\mathsf{JL}} A \supset (C \supset B)$ and $\Gamma \vdash_{\mathsf{JL}} A \supset C$. Since $(A \supset (C \supset B)) \supset ((A \supset C) \supset (A \supset B))$ is a propositional tautology, by applying Modus Ponens twice, it follows that $\Gamma \vdash_{\mathsf{JL}} A \supset B$.

" $\Leftarrow$" Suppose $\Gamma \vdash_{\mathsf{JL}} A \supset B$. It follows immediately that $\Gamma, A \vdash_{\mathsf{JL}} A \supset B$ and since $\Gamma, A \vdash_{\mathsf{JL}} A$, by applying Modus Ponens it follows that $\Gamma, A \vdash_{\mathsf{JL}} B$.

□

The following corollary is based on Internalization and the Deduction Theorem and can be proven by using Lemma 2.2.1 and applying the Deduction Theorem.

Corollary 2.2.3. *Let JL be some justification logic. If $\vdash_{\mathsf{JL}} A_1 \supset \ldots \supset A_n \supset B$, then there is a term $t(x_1, \ldots, x_n)$ such that $\vdash_{\mathsf{JL}} s_1 : A_1 \supset \ldots \supset s_n : A_n \supset t(s_1, \ldots, s_n) : B$, for any terms $s_1, \ldots, s_n$.*

Annotation

The annotation of modal formulas is part of the realization procedure when using a (modular) nested sequent calculus. The idea is to *annotate* occurrences of $\Box$ and $\Diamond$ in a realization, to be able to assign terms. The definitions, facts and lemmas about annotation and substitution, in this and the next subsection, are based on those of R. Goetschi and R. Kuznets [12].

Definition 2.2.4. Let $k, l \in \mathbb{N}^+$, let p be a proposition. *Annotated (modal) formulas* can be defined with the following grammar:

$$A := p \mid \bot \mid \top \mid \neg A \mid A \vee A \mid A \wedge A \mid A \supset A \mid \Box_{2k-1} A \mid \Diamond_{2l} A.$$

Let A be such an annotated formula and let A' be the formula A without the indices, then A is an *annotated version* of A'. If all the indices are unique, no index occurs more than once, the formula is called *properly annotated.* $\diamondsuit$

Part of the realization procedure, is the *pre-realization* of the formulas and objects. With the above definition of annotated formulas, it is possible to define the pre-realization function on annotated formulas.

Definition 2.2.5. Let r be a function that assigns terms to the natural numbers used to annotate a formula, then r is called a *pre-realization function*. If the function is defined on all the given indices, then r is called a *pre-realization function on a given annotated formula*.

The pre-realization function r is called a *realization function* if $r(2l) = x_l$ whenever $r(2l)$ is defined. $\diamondsuit$

Definition 2.2.6. Let A be an annotated formula, r a pre-realization function on A and p a proposition. The table 2.1 shows how by induction, the justification formula A^r can be defined, based on the original modal formula A and the function r.

$(p)^r$	$:=$	p	$(A \vee B)^r$	$:=$	$A^r \vee B^r$	$(\diamondsuit_{2l} A)^r$	$:=$	$\neg r(2l) : \neg A^r$
$(\neg A)^r$	$:=$	$\neg(A^r)$	$(A \wedge B)^r$	$:=$	$A^r \wedge B^r$	$(\Box_{2k-1} A)^r$	$:=$	$r(2k-1) : A^r$
$\bot^r := \bot$		$\top^r := \top$	$(A \supset B)^r$	$:=$	$A^r \supset B^r$			

Table 2.1: Realization of modal formulas, source: [12]

Let B be a justification formula, then there is some properly annotated formula A and a pre-realization function r on A such that $B = A^r$. $\diamondsuit$

It is possible that there exists more than one annotation for a modal formula. If there is a realization function on one of these annotated versions, does this mean that there is a realization function on the other annotated version(s) as well?

Lemma 2.2.7. *Let* JL *be a justification formula. Suppose the modal formula A' has at least two properly annotated versions, call the first two A_1 and A_2. Suppose that r_1 is a realization function on A_1, such that $\vdash_{\mathsf{JL}} (A_1)^{r_1}$. Then there is a realization function r_2 on A_2 such that $\vdash_{\mathsf{JL}} (A_2)^{r_2}$.*

The above lemma states that, as long as the formula is really properly annotated, it does not matter which properly annotated version is chosen: if there is a realization of one of the properly annotated versions of a formula, there is a realization for any of these versions. The proof of the lemma is by induction on the structure of the modal formula A'. It does however require the notion of substitution, which will be introduced in the next subsection.

In order to prove the realization of S5, with a pre-realization based on annotated formulas, some additional notation is needed:

Definition 2.2.8. Let A be an annotated formula and r a pre-realization function. Let f be a partial function then $f \restriction S$ denotes the restriction of f to the set $S \cap \mathrm{dom}(f)$. Then:

$$\begin{aligned}\mathrm{vars}_\Diamond(A) :=& \{x_k \mid \Diamond_{2k} \text{ occurs in } A\}\\ r \restriction A :=& r \restriction \{i \mid i \text{ occurs in } A\}.\end{aligned}$$

$\Diamond$

When realizing $\mathsf{S5}$ using hypersequents, another method of labeling/annotating is used. This method can be defined after firstly defining the language of the hypersequent calculus $\mathsf{LS5}$.

Substitution

Besides internalization and annotation another important concept that should be discussed before defining the different sequent calculi is that of substitution. Starting with a general definition of a substitution.

S.N. Artemov [2] stated already that the substitution lemma holds for the logic LP: if $\Gamma(x, P) \vdash_{\mathsf{LP}} B(x, P)$, then for any term t and any formula F: $\Gamma(x/t, P/F) \vdash_{\mathsf{LP}} B(x/t, P/F)$. This can be generalized for any here considered justification logic JL:

Definition 2.2.9. A *substitution* as it will be used here, is defined to be a total mapping, from variables to terms. Let t be some term, the term $t\sigma$ can be defined inductively:

- Let c be a constant, then $c\sigma := c$.
- Let x be a variable, then $x\sigma := \sigma(x)$.
- Let $*$ be some unary operation, then $(*t)\sigma := *(t\sigma)$.
- Let $\circ$ be some binary operation, then $(t_1 \circ t_2)\sigma := (t_1\sigma) \circ (t_2\sigma)$.

Let A be some justification formula, the formula that is obtained by simultaneously replacing all the terms t in A by $t\sigma$ is denoted by $A\sigma$. $\Diamond$

With this definition of substitution, some other basic definitions that are required when proving something using substitutions can be given:

Definition 2.2.10. The domain of a substitution σ, denoted by $\mathrm{dom}(\sigma)$, is defined as: $\mathrm{dom}(\sigma) := \{x \mid \sigma(x) \neq x\}$. The variable range of this substitution σ, denoted by $\mathrm{vrange}(\sigma)$, is defined as the set of variables in terms that are part of the set $\{\sigma(x) \mid x \in \mathrm{dom}(\sigma)\}$. $\Diamond$

Definition 2.2.11. Three kinds of compositions with substitutions can be defined. Let x be some variable and σ_1 and σ_2 be some substitutions, then the composition of σ_1 and σ_2 is defined as: $(\sigma_2 \circ \sigma_1)(x) := \sigma_1(x)\sigma_2$.

Let r be some pre-realization function and σ a substitution, then $(\sigma \circ r)(n) := r(n)\sigma$, which is only defined if $r(n)$ is.

Suppose the substitutions σ_1 and σ_2 have disjoint domains, the union of these two substitutions is a substitution as well. Let x be some variable, then:

$$(\sigma_1 \cup \sigma_2)(x) := \begin{cases}\sigma_1(x) & \text{if } x \in \mathrm{dom}(\sigma_1)\\ \sigma_2(x) & \text{if } x \in \mathrm{dom}(\sigma_2)\\ x & \text{otherwise.}\end{cases}$$

$\Diamond$

Definition 2.2.12. Let σ be some substitution and A an annotated formula. Then it is said that σ *lives on* A if $\mathrm{dom}(\sigma) \subseteq \mathrm{vars}_{\Diamond}(A)$ and σ *lives away from* A if $\mathrm{dom}(\sigma) \cap \mathrm{vars}_{\Diamond}(A) = \emptyset$. $\Diamond$

Lemma 2.2.13. *Let* JL *be any justification logic. If* $\vdash_{\mathsf{JL}} A$ *then:*

a) $\vdash_{\mathsf{JL}} A\sigma$ *for any substitution* σ.

b) $\vdash_{\mathsf{JL}} A[P_1 \mapsto B_1, \ldots, P_n \mapsto B_n]$, *where the result of replacing simultaneously all occurrences of the propositions* $P_1, \ldots, P_n$ *in* A *with formulas* $B_1, \ldots, B_n$ *is denoted by* $A[P_1 \mapsto, \ldots, P_n \mapsto B_n]$.

The following facts are directly taken from the paper on nested sequent systems by R. Goetschi and R. Kuznets [12] and are used in many proofs for the nested sequent calculus. Let A be an annotated formula, r a pre-realization function, r', r_1, r_2 realization functions and σ, σ_1, σ_2 be substitutions, then:

(1) $\sigma_2 \circ \sigma_1$ is a substitution with $\mathrm{dom}(\sigma_2 \circ \sigma_1) \subseteq \mathrm{dom}(\sigma_1) \cup \mathrm{dom}(\sigma_2)$ and $\mathrm{vrange}(\sigma_2 \circ \sigma_1) \subseteq \mathrm{vrange}(\sigma_1) \cup \mathrm{vrange}(\sigma_2)$. Moreover $A(\sigma_2 \circ \sigma_1) = (A\sigma_1)\sigma_2$.

(2) If $\mathrm{dom}(\sigma_1) \cap \mathrm{dom}(\sigma_2) = \emptyset$, then $\mathrm{dom}(\sigma_1 \cup \sigma_2) = \mathrm{dom}(\sigma_1) \cup \mathrm{dom}(\sigma_2)$.

(3) If $\mathrm{dom}(\sigma_1) \cap \mathrm{dom}(\sigma_2) = \emptyset$ and $\mathrm{vrange}(\sigma_1) \cap \mathrm{dom}(\sigma_2) = \emptyset$, then $\sigma_1 \cup \sigma_2 = \sigma_2 \circ \sigma_1$.

(4) $\sigma \circ r$ is a pre-realization function with $\mathrm{dom}(\sigma \circ r) = \mathrm{dom}(r)$.

(5) If r is a pre-realizaiton function on A, then so is $\sigma \circ r$ and $A^{\sigma \circ r} = A^r\sigma$.

(6) If r is a (pre-)realization function on A, then so is $r \upharpoonright A$.

(7) If $\mathrm{dom}(r_1) \cap \mathrm{dom}(r_2) \subseteq \{n \mid n \text{ is even}\}$, then $r_1 \cup r_2$ is a realization function.

(8) If $r_1 \cup r_2$ is a realization function, then $\mathrm{dom}(r_1 \cup r_2) = \mathrm{dom}(r_1) \cup \mathrm{dom}(r_2)$.

(9) $\sigma \circ r'$ is a realization function if and only if $x_n \notin \mathrm{dom}(\sigma)$ whenever $r(2n)$ is defined.

Corollary 2.2.14. *Let* A *be an annotated formula and suppose* r *is a realization function on it. If a substitution* σ *lives away from* A *then* $\sigma \circ (r \upharpoonright A)$ *is a realization function on* A.

Theorems in Different Logics

Not every justification logic that is considered in Theorem 2.1.6, is an extension of the logic JT. However, in each of the considered logics, the following is provable: $s : A \supset A$. In a similar way in every of the considered logics, a term *negint*(x) can be constructed such that for any justification formula A and any term s, the following is provable: $\neg s : A \supset \mathit{negint}(s) : (\neg s : A)$. To make proofs short and easy to read, the proofs for these kinds of theorems will be proven here. For this subsection, assume

$$\mathcal{L} \in \{\mathsf{LPS5}, \mathsf{LPS5}_c, \mathsf{JT45}, \mathsf{JT5}, \mathsf{JTB5}, \mathsf{JTB45}, \mathsf{JDB5}, \mathsf{JDB45}, \mathsf{JDB4}, \mathsf{JTB4}\}.$$

First take a look at the jb-axiom, only for the cases in which the axiom system of $\mathcal{L}$ contains this axiom. Because of the many negations in the jb-axiom, the following lemma will be used in proofs:

Lemma 2.2.15. *The* jb*-axiom is defined as follows:* $A \supset \bar{?}t : \neg t : \neg A$*, if* $A = \neg B$*, an instance of this axiom can take the following form:* $\neg B \supset \bar{?}t : \neg t : \neg\neg B$*. From propositional logic it is known that* $\neg\neg B \supset B$*. In any justification logic that contains the* jb*-axiom, there is a term* $qm(x)$ *such that for any formula* B *and arbitrary term* s*,* $\neg B \supset qm(s) : \neg s : B$ *is derivable.*

Proof. Consider only the logics that have the jb-axiom in their axiom system:

$$\mathcal{L} \in \{\mathsf{JTB5}, \mathsf{JTB45}, \mathsf{JDB5}, \mathsf{JDB45}, \mathsf{JDB4}, \mathsf{JTB4}\}.$$

0.	$P \supset \neg\neg P$	Propositional tautology
1.	$x : P \supset t_1(x) : \neg\neg P$	From 0. by Corollary 2.2.3
2.	$\neg t_1(x) : \neg\neg P \supset \neg x : P$	From 1. by prop. reasoning
3.	$\bar{?}t_1(x) : \neg t_1(x) : \neg\neg P \supset t_2(\bar{?}t_1(x)) : \neg x : P$	From 2. by Corollary 2.2.3
4.	$\neg P \supset \bar{?}t_1(x) : \neg t_1(x) : \neg\neg P$	Instance of jb
5.	$\neg P \supset t_2(\bar{?}t_1(x)) : \neg x : P$	From 3. and 4. by prop reasoning

Let $qm(x) := t_2(\bar{?}t_1(x))$. Notice that $qm(x)$ depends neither on s nor on B. From the Substitution Lemma 2.2.13, it follows that $\neg B \supset qm(s) : \neg s : B$ is derivable in $\mathcal{L}$. □

The following lemma is provable for any of the justification counterparts of the modal logic S5, but some case distinctions are needed for different logics. The most important case is the part where $B = s : A$, this case will often occur in proofs.

Lemma 2.2.16. *If* $\vdash_{\mathcal{L}} s : A \supset B$ *there is a term* $p(x)$ *such that* $\vdash_{\mathcal{L}} s : A \supset p(s) : B$*.*

Proof. The proof is by induction on the derivation of B from $s : A$ in $\mathcal{L}$:

- Suppose B is one of the axioms of $\mathcal{L}$, then $\vdash_{\mathcal{L}} B$. By AN, any constant c, such that $\vdash_{\mathcal{L}} c : B$, can be chosen to be $p(s) := c$, then $\vdash_{\mathcal{L}} p(s) : B$. From this it follows that $s : A \vdash_{\mathcal{L}} p(s) : B$ and by the Deduction Theorem 2.2.2: $\vdash_{\mathcal{L}} s : A \supset p(s) : B$.
- Suppose B is of the form $c_{n-1} : \ldots : c_1 : C$, using AN again, any constant c_n such that $\vdash_{\mathcal{L}} c_n : c_{n-1} : \ldots : c_1 : C$ can be used to let $p(s) := c_n$, then $\vdash_{\mathcal{L}} p(s) : B$. Like in the case above, it follows that $s : A \vdash_{\mathcal{L}} p(s) : B$ and by the Deduction Theorem 2.2.2: $\vdash_{\mathcal{L}} s : A \supset p(s) : B$.
- Suppose B is derived by MP from $C \supset B$ and C. By induction hypotheses, there are terms t_1 for $(C \supset B)$ and t_2 for $t_2 : C$. Now $t_1 : (C \supset B) \supset (t_2 : C \supset (t_1 \cdot t_2) : B)$ is an instance of app. By MP it follows that for $p(s) := t_1 \cdot t_2$: $s : A \vdash_{\mathcal{L}} p(s) : B$, by the Deduction Theorem 2.2.2 it then can be concluded that $\vdash_{\mathcal{L}} s : A \supset p(s) : B$.
- Suppose $B = s : A$, the following case distinctions have to be made:
  - Suppose $\mathcal{L} \in \{\mathsf{LPS5}, \mathsf{LPS5}_c, \mathsf{JT45}, \mathsf{JTB45}, \mathsf{JDB45}, \mathsf{JDB4}, \mathsf{JTB4}\}$ by axiom j4, it follows that $\vdash_{\mathcal{L}} s : A \supset !s : s : A$ and since $s : A = B$, $\vdash_{\mathcal{L}} s : A \supset !s : B$, therefore, $p(x) := !x$ and $\vdash_{\mathcal{L}} s : A \supset p(s) : B$ follows.
  - Suppose $\mathcal{L} \in \{\mathsf{JTB5}, \mathsf{JDB5}\}$, then:

0.	$\neg x_1 : P \supset ?'x_1 : \neg x_1 : P$	Instance of j5
1.	$\neg ?'x_1 : \neg x_1 : P \supset x_1 : P$	From 0. by prop. reasoning
2.	$q : (\neg ?'x_1 : \neg x_1 : P \supset x_1 : P)$	From 1. by Lemma 2.2.1
3.	$q : (\neg ?'x_1 : \neg x_1 : P \supset x_1 : P)$ $\supset (\bar{?}(?'x_1) : \neg ?'x_1 : \neg x_1 : P \supset (q \cdot \bar{?}(?'x_1)) : x_1 : P)$	Instance of app

4. $\overline{?}(?'x_1) : \neg ?'x_1 : \neg x_1 : P \supset (q \cdot \overline{?}(?'x_1)) : x_1 : P$ From 2. and 3. by MP
5. $x_1 : P \supset \overline{?}(?'x_1) : \neg ?'x_1 : \neg x_1 : P$ Instance of jb
6. $x_1 : P \supset (q \cdot \overline{?}(?'x_1)) : x_1 : P$ From 4. and 5. by prop. reasoning

Let $p(x) := (q \cdot \overline{?}(?'x))$, notice that $p(x)$ depends neither on s nor on A. By the Substitution Lemma 2.2.13, it then can be concluded that $\vdash_{\mathcal{L}} s : A \supset p(s) : B$.

– Let $\mathcal{L} =$ JT5 and consider the following derivation:

0. $\neg x_2 : P \supset ?'x_2 : \neg x_2 : P$ Instance of j5
1. $\neg ?'x_2 : \neg x_2 : P \supset x_2 : P$ From 0. by prop. reasoning
2. $q : (\neg ?'x_2 : \neg x_2 : P \supset x_2 : P)$ From 1. by Lemma 2.2.1
3. $q : (\neg ?'x_2 : \neg x_2 : P \supset x_2 : P)$
$\supset (?'(?'x_2) : \neg ?'x_2 : \neg x_2 : P \supset (q \cdot ?'(?'x_2)) : x_2 : P)$ Instance of app
4. $?'(?'x_2) : \neg ?'x_2 : \neg x_2 : P \supset (q \cdot ?'(?'x_2)) : x_2 : P$ From 2. and 3. by MP
5. $?'x_2 : \neg x_2 : P \supset \neg x_2 : P$ Instance of jt
6. $x_2 : P \supset \neg ?'x_2 : \neg x_2 : P$ From 5. by prop. reasoning
7. $\neg ?'x_2 : \neg x_2 : P \supset ?'(?'x_2) : \neg ?'x_2 : \neg x_2 : P$ Instance of j5
8. $x_2 : P \supset ?'(?'x_2) : \neg ?'x_2 : \neg x_2 : P$ From 6. and 7. by prop. reasoning
9. $x_2 : P \supset (q \cdot ?'(?'x_2)) : x_2 : P$ From 4. and 8. by prop. reasoning

Let $p(x) := (q \cdot ?'(?'x))$, notice that $p(x)$ depends neither on s nor on A. By the Substitution Lemma 2.2.13, it follows that $\vdash_{\mathcal{L}} s : A \supset p(s) : B$.

□

Now the jt-axiom: $s : A \supset A$, this axiom is not part of the axiom systems of JDB5, JDB45 and JDB4, but it is provable for any $\mathcal{L}$:

Lemma 2.2.17. *Consider any of the possibilities for $\mathcal{L}$, then $\mathcal{L} \vdash s : A \supset A$, for any justification formula A and any term t.*

Proof. Two cases have to be considered, the set of logics that are an extension of the modal logic JT and the logics that do not have the axiom jt in their axiom system:

- Let $\mathcal{L} \in \{$LPS5, LPS5$_c$, JT45, JT5, JTB5, JTB45, JTB4$\}$. Since the jt-axiom is part of the axiom system of $\mathcal{L}$, $\vdash_{\mathcal{L}} s : A \supset A$ follows immediately for any justification formula A and any term t.
- Now the cases without the jt-axiom. Let $\mathcal{L} \in \{$JDB5, JDB45, JDB4$\}$. The proof is based on the proof that can be found in the paper by R. Goetschi and R. Kuznets [12], there a case distinction between JDB4 and JDB5 is made, because of the above lemma, this is not necessary here:

0. $P \supset \neg\neg P$ Propositional tautology
1. $x : P \supset t_1(x) : \neg\neg P$ From 0. by Corollary 2.2.3
2. $\neg t_1(x) : \neg\neg P \supset \neg x : P$ From 1. by prop. reasoning
3. $\overline{?}(t_1(x)) : \neg t_1(x) : \neg\neg P \supset t_2(\overline{?}(t_1(x))) : \neg x : P$ From 2. by Corollary 2.2.3
4. $\neg t_2(\overline{?}(t_1(x))) : \neg x : P \supset \neg \overline{?}(t_1(x)) : \neg t_1(x) : \neg\neg P$ From 3. by prop. reasoning
5. $\neg x : P \supset (x : P \supset \bot)$ Propositional tautology
6. $t_2(\overline{?}(t_1(x))) : \neg x : P \supset t(t_2(\overline{?}(t_1(x)))) : (x : P \supset \bot)$ From 5. by Corollary 2.2.3

Let $s_1 := t_2(\overline{?}t_1(x))$ and $s_2 := t(t_2(\overline{?}(t_1(x))))$, then:

7. $(s_2 \cdot p(x)) : \bot \supset \bot$ Instance of jd
8. $s_2 : (x : P \supset \bot) \supset (p(x) : x : P \supset (s_2 \cdot p(x)) : \bot)$ Instance of app
9. $p(x) : x : P \supset (s_2 : (x : P \supset \bot) \supset (s_2 \cdot p(x)) : \bot)$ From 8. by prop. reasoning
10. $p(x) : x : P \supset (s_2 : (x : P \supset \bot) \supset \bot)$ From 7. and 9. by prop. reas.
11. $(s_2 : (x : P \supset \bot) \supset \bot) \supset \neg s_2 : (x : P \supset \bot)$ Propositional tautology
12. $\neg s_2 : (x : P \supset \bot) \supset \neg s_1 : \neg x : P$ From 6. by prop. reas.
13. $(s_2 : (x : P \supset \bot) \supset \bot) \supset \neg s_1 : \neg x : P$ From 11. and 12. by prop. reasoning
14. $p(x) : x : P \supset \neg s_1 : \neg x : P$ From 10. and 13. by prop. reasoning
15. $p(x) : x : P \supset \neg \bar{?}(t_1(x)) : \neg t_1(x) : \neg\neg P$ From 4. and 14. by prop. reasoning
16. $x : P \supset p(x) : x : P$ From Lemma 2.2.16
17. $x : P \supset \neg \bar{?}(t_1(x)) : \neg t_1(x) : \neg\neg P$ From 15. and 16. by prop. reasoning
18. $\neg P \supset \bar{?}(t_1(x)) : \neg t_1(x) : \neg\neg P$ Instance of jb
19. $\neg \bar{?}(t_1(x)) : \neg t_1(x) : \neg\neg P \supset P$ From 18. by prop. reasoning
20. $x : P \supset P$ From 17. and 19. by prop. reasoning

The result follows from 20. and the Substitution Lemma 2.2.13: $\vdash_{\mathcal{L}} s : A \supset A$.

□

Lemma 2.2.18. *For each $\mathcal{L}$, there is a term negint(x) such that for any term s and any justification formula A: $\vdash_{\mathcal{L}} \neg s : A \supset$ negint$(s) : (\neg s : A)$.*

Proof. Consider every logic separately:

- Start with $\mathcal{L} = \mathsf{LPS5}$:

  0. $\neg x : P \supset \neg x : P$ Propositional tautology
  1. $c : (\neg x : P \supset \neg x : P)$ From 0. by Axiom Necessitation
  2. $c : (\neg x : P \supset \neg x : P) \supset (\neg x : P \supset ?c : (\neg x : P))$ Instance of A5
  3. $\neg x : P \supset ?c : (\neg x : P)$ From 1. and 2. by Modus Ponens

  Let *negint*$(x) := ?c$, then indeed, by the Substitution Lemma 2.2.13, for any term s and any justification formula A: $\vdash_{\mathsf{LPS5}} \neg s : A \supset$ *negint*$(s) : (\neg s : A)$.
- Now let $\mathcal{L} = \mathsf{LPS5}_c$. Notice that $\neg x : P \supset c : \neg x : P$ is an instance of the axiom A5_c, let *negint*$(x) := c$, then, by the Substitution Lemma 2.2.13, for any term s and any justification formula A: $\vdash_{\mathsf{LPS5}_c} \neg s : A \supset$ *negint* $: (\neg s : A)$ follows.
- Let $\mathcal{L} \in \{\mathsf{JT45}, \mathsf{JT5}, \mathsf{JTB5}, \mathsf{JTB45}, \mathsf{JDB5}, \mathsf{JDB45}\}$, since the axiom j5 is part of the axiom system of $\mathcal{L}$: $\vdash_{\mathcal{L}} \neg x : P \supset ?'x : (\neg x : P)$. Let *negint*$(x) := ?'x$, by the Substitution Lemma 2.2.13 for any term s and any justification formula A it then follows that $\vdash_{\mathcal{L}} \neg s : A \supset$ *negint*$(s) : (\neg s : A)$.
- Suppose $\mathcal{L} \in \mathsf{JTB4}, \mathsf{JDB4}$, since the jt-axiom and the jd-axiom are not part of the proof, the proof is the same for both logics.

  0. $\neg x : P \supset qm(!x) : \neg !x : x : P$ By Lemma 2.2.15
  1. $x : P \supset !x : x : P$ Instance of j4

2. $\neg !x : x : P \supset \neg x : P$ From 1. by prop. reasoning
3. $q : (\neg !x : x : P \supset \neg x : P)$ From 2. by Lemma 2.2.1
4. $q : (\neg !x : x : P \supset \neg x : P)$
   $\supset (qm(!x) : \neg !x : x : P \supset (q \cdot qm(!x)) : \neg x : P)$ Instance of app
5. $qm(!x) : \neg !x : x : P \supset (q \cdot qm(!x)) : \neg x : P$ From 3. and 4. by MP
6. $\neg x : P \supset (q \cdot qm(!x)) : \neg x : P$ From 0. and 5. by prop. reasoning

Let $negint(x) := (q \cdot qm(!x))$, then, by the Substitution Lemma 2.2.13 for any term s and any justification formula A: $\vdash_{\mathcal{L}} \neg s : A \supset negint(s) : \neg s : A$ for $\mathcal{L} = \mathsf{JTB4}$ and for $\mathcal{L} = \mathsf{JDB4}$. Notice that the constructed term $negint(x)$ depends neither on s nor on A.

$\square$

CHAPTER 3

Hypersequent Method

To be able to realize S5 into its justification counterparts, a cut-free sequent system is needed. Such a system is LS5, as was already presented by G. Mints [16] and used by S. N. Artemov, E.L. Kazakov and D. Shapiro [4]. The system as it was introduced in both papers, is written in a notation that is no longer used. Therefore, the definitions, rules and theorems concerning the system LS5 in the next section will be formulated in what is called the *standard notation* as A. Avron described this in his paper on hypersequents [5]. Any of the definitions in this chapter are taken from the paper by S. N. Artemov, E.L. Kazakov and D. Shapiro [4], where necessary translated into the standard notation.

3.1 The System LS5

The system LS5, as used here, is defined syntactically as follows:

Definition 3.1.1. The language of LS5 contains propositional variables $p_1, \dots, p_k, \dots$, propositional constant $\bot$, connective $\supset$, modal operator $\Box$ and (,), $|$. The system has the following syntactic objects:

- A formula A of LS5 is constructed using the following grammar:

$$A := p_i \mid \bot \mid A \supset A \mid \Box A.$$

  Where p_i is a propositional variable.
- A sequent exists of two sequences of formulas, separated by "$\Rightarrow$":

$$B_1, \dots, B_m \Rightarrow A_1, \dots, A_n.$$

  In general the notation of sequents is shortened to $\Gamma \Rightarrow \Delta$, with indices when needed. If Γ stands for $B_1, \dots, B_m$, then $\Box\Gamma$ stands for $\Box B_1, \dots, \Box B_m$, in the same way, if Δ stands for $A_1, \dots, A_n$, then $\Box\Delta$ denotes $\Box A_1, \dots, \Box A_n$.

- A hypersequent consists of a finite sequence of sequents:

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_k \Rightarrow \Delta_k.$$

  Each $\Gamma_i \Rightarrow \Delta_i$ is called a component of the hypersequent, they are sequents as defined above. Hypersequents are denoted by G and H, if needed with indices.

$\diamondsuit$

Definition 3.1.2. A modal translation of a syntactic object X of the system LS5, as defined above, is a modal formula denoted by X^t constructed as follows:
- The modal translation of a sequent: $(B_1, \ldots, B_m \Rightarrow A_1, \ldots, A_n)^t := B_1 \wedge \ldots \wedge B_m \supset A_1 \vee \ldots \vee A_n$.
- The modal translation of a hypersequent: $(\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_k \Rightarrow \Delta_k)^t := \Box(\Gamma_1 \Rightarrow \Delta_1)^t \vee \ldots \vee \Box(\Gamma_k \Rightarrow \Delta_k)^t$.

$\diamondsuit$

With these definitions, it is now possible to define the rules of the system LS5 that were used by S.N. Artemov, E.L. Kazakov and D. Shapiro [4], in the notation used by A. Avron in his paper [5]. See Table 3.1.

Axioms	$A \Rightarrow A$	$\bot \Rightarrow$
Contraction	$\dfrac{G \mid A, A, \Gamma \Rightarrow \Delta \mid H}{G \mid A, \Gamma \Rightarrow \Delta \mid H}$ **CL**	$\dfrac{G \mid \Gamma \Rightarrow \Delta, A, A \mid H}{G \mid \Gamma \Rightarrow \Delta, A \mid H}$ **CR**
	$\dfrac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \mid H}{G \mid \Gamma \Rightarrow \Delta \mid H}$ **CE**	
Weakening	$\dfrac{G \mid \Gamma \Rightarrow \Delta \mid H}{G \mid A, \Gamma \Rightarrow \Delta \mid H}$ **WL**	$\dfrac{G \mid \Gamma \Rightarrow \Delta \mid H}{G \mid \Gamma \Rightarrow \Delta, A \mid H}$ **WR**
	$\dfrac{G \mid \Gamma \Rightarrow \Delta \mid H}{G \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta' \mid H}$ **WE**	
Exchange	$\dfrac{G \mid A, B, \Gamma \Rightarrow \Delta \mid H}{G \mid B, A, \Gamma \Rightarrow \Delta \mid H}$ **EL**	$\dfrac{G \mid \Gamma \Rightarrow \Delta, A, B \mid H}{G \mid \Gamma \Rightarrow \Delta, B, A \mid H}$ **ER**
	$\dfrac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta' \mid H}{G \mid \Gamma' \Rightarrow \Delta' \mid \Gamma \Rightarrow \Delta \mid H}$ **EE**	
Cut	$\dfrac{G_1 \mid \Gamma_1 \Rightarrow \Delta_1, A \mid H_1 \quad G_2 \mid A, \Gamma_2 \Rightarrow \Delta_2 \mid H_2}{G_1 \mid G_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \mid H_1 \mid H_2}$	
Implication	$\dfrac{G \mid \Gamma \Rightarrow \Delta, A \mid H \quad G \mid B, \Gamma \Rightarrow \Delta \mid H}{G \mid A \supset B, \Gamma \Rightarrow \Delta \mid H} \supset\Rightarrow$	$\dfrac{G \mid A, \Gamma \Rightarrow \Delta, B \mid H}{G \mid \Gamma \Rightarrow \Delta, A \supset B \mid H} \Rightarrow\supset$
Modal rules	$\dfrac{G \mid A, \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta' \mid H}{G \mid \Gamma \Rightarrow \Delta \mid \Box A, \Gamma' \Rightarrow \Delta' \mid H} \Box\Rightarrow$	$\dfrac{G \mid \Gamma \Rightarrow \Delta \mid \Rightarrow A \mid H}{G \mid \Gamma \Rightarrow \Delta, \Box A \mid H} \Rightarrow\Box$

Table 3.1: Rules of the system LS5.

For the system LS5 G. Mints [16] proved already the following two theorems, concerning the equivalence between S5 and LS5 and cut-elimination.

Theorem 3.1.3. *Let G be a hypersequent and let G^t be its modal translation constructed according to Definition 3.1.2. Then:*

$$\vdash_{LS5} G \Leftrightarrow \vdash_{S5} G^t.$$

Theorem 3.1.4. *Any derivation in* LS5 *can be made cut-free: the cut-rule can be eliminated from it.*

Let $\mathsf{LS5}^-$ denote the hypersequent system that consists of all of the rules of LS5, but without the cut-rule. From now on, the system $\mathsf{LS5}^-$ will be used, instead of LS5.

3.2 Auxiliary Definitions

In order to prove the realization theorem, the following more general definitions are needed.

Definition 3.2.1. Two formulas in a rule, one in the premise and one in the conclusion, are called *related*, if they occur at the corresponding places of the rule and are denoted by the same letter. The relation of being *related* is extended by transitivity.

In a similar way, occurrences of the $\Box$ are *related* if they occur in related formulas at the corresponding place. The class of related occurrences of the $\Box$ is called a *family*. $\diamondsuit$

Definition 3.2.2. Every subformula A of a given formula Φ has a *positive polarity* or a *negative polarity*. Let B and C be some formulas, then:

- Let $A := \Phi$, then A has a positive polarity.
- Let $A := \neg B$, then B has a positive polarity if A has a negative polarity in Φ and B has a negative polarity if A has a positive polarity in Φ.
- Let $A := B \wedge C$ or $A := B \vee C$, then B and C have a positive polarity if A has a positive polarity in Φ and they have a negative polarity if A has a negative polarity in Φ.
- Let $A := B \supset C$, then B has a negative and C has a positive polarity if A has a positive polarity in Φ and if A has a negative polarity in Φ, then B has a positive polarity and C has a negative polarity.
- Let $A := \Box B$, then B has positive polarity if A has positive polarity and B has a negative polarity if A has a negative polarity.
- The polarity of $\Box$ is the polarity of the minimal formula that contains this occurrence of $\Box$.

$\diamondsuit$

Definition 3.2.3. If all $\Box$'s in a family have a positive polarity, it is said that the family is *positive*. Otherwise the family is called *negative*.

A family that contains $\Box$'s introduced by the rule $(\Rightarrow \Box)$ is called *essential*, such essential families have a positive polarity. $\diamondsuit$

REMARK. Every $\Box$ in a negative family has a negative polarity, because the cut-rule is not part of $\mathsf{LS5}^-$.

To be able to assume that the axioms in a derivation are all atomic, the following lemma has to be proven.

Lemma 3.2.4. *Any non-atomic axiom can be derived from atomic axioms.*

Proof. There are two forms of axioms in $\mathsf{LS5}^-$: $A \Rightarrow A$ and $\bot \Rightarrow$, the latter is already atomic. For A any $\mathsf{LS5}^-$-formula as defined in Definition 3.1.1 can be substituted. The proof will be by induction on the structure of A. Let B and C be $\mathsf{LS5}^-$-formulas and let p be a propositional variable, consider the following cases:

- Suppose $A = p$ or $A = \bot$, then $A \Rightarrow A$ is already an atomic axiom.
- Suppose $A = B \supset C$, consider the following derivation using the $\mathsf{LS5}^-$-sequent rules:

$$\dfrac{\dfrac{\dfrac{B \Rightarrow B}{B \Rightarrow C, B}\,\mathbf{WR} \quad \dfrac{C \Rightarrow C}{C, B \Rightarrow C}\,\mathbf{WL}}{B, (B \supset C) \Rightarrow C}\,{\supset}{\Rightarrow}}{(B \supset C) \Rightarrow (B \supset C)}\,{\Rightarrow}{\supset}$$

  By induction hypothesis, the axioms $C \Rightarrow C$ and $B \Rightarrow B$ can be derived from atomic axioms.
- Suppose $A = \Box B$ and consider the $\mathsf{LS5}^-$-derivation:

$$\dfrac{\dfrac{\dfrac{B \Rightarrow B}{B \Rightarrow B \mid \ \Rightarrow}\,\mathbf{WE}}{\Box B \Rightarrow \mid \Rightarrow B}\,\Box\Rightarrow}{\Box B \Rightarrow \Box B}\,\Rightarrow\Box$$

  By induction hypothesis, the axiom $B \Rightarrow B$ can be derived from atomic axioms.

By Definition 3.1.1, these three cases cover all possible $\mathsf{LS5}^-$-formulas. From this it follows that it can be assumed that the axioms in a derivation are atomic. $\Box$

In Chapter 2 on realization in general the definition of a realization function on an annotated formula has already been introduced in Definition 2.2.6. Here the definition of an annotated formula has to be extended to an annotated hypersequent and it has to be defined what a realization function on an annotated hypersequent is.

Definition 3.2.5. An *annotated hypersequent* is a hypersequent in which only annotated formulas occur, each occurrence of a $\Box$ and each component of a hypersequent is annotated by an odd index. A hypersequent is *properly annotated* if no index occurs twice. $\diamondsuit$

With this definition, it is possible to define a realization function on syntactic objects X of $\mathsf{LS5}^-$. The syntactic objects of $\mathsf{LS5}^-$have been defined in Definition 3.1.1

Definition 3.2.6 ([4]). Let X be a properly annotated syntactic object and p a proposition. A realization function r applied on X results in a justification formula, constructed by induction according to Table 3.2. $\diamondsuit$

p^r	$:=$	p	$(\Box_{2p-1}A)^r$	$:=$	$r(2p-1) : A^r$
$\bot^r$	$:=$	$\bot$	$(A \supset B)^r$	$:=$	$A^r \supset B^r$
$(B_1, \ldots, B_m \Rightarrow A_1, \ldots, A_n)^r$		$:=$	$B_1^r \wedge \ldots \wedge B_m^r \supset A_1^r \vee \ldots \vee A_n^r$		
$((\Gamma_1 \Rightarrow \Delta_1)_{2l_1-1} \mid \ldots \mid (\Gamma_k \Rightarrow \Delta_k)_{2l_k-1})^r$		$:=$			
$r(2l_1-1) : (\Gamma_1 \Rightarrow \Delta_1)^r$		$\vee \ldots \vee$	$r(2l_k-1) : (\Gamma_k \Rightarrow \Delta_k)^r$		

Table 3.2: Realization of properly annotated syntactic objects of $\mathsf{LS5}^-$, source: [4]

Definition 3.2.7. The realization $r(\mathcal{D})$ of an $\mathsf{LS5}^-$-derivation $\mathcal{D}$, is the result of applying realization r to the hypersequents of $\mathcal{D}$.

A realization r of an $\mathsf{LS5}^-$-derivation $\mathcal{D}$ is called *normal* if all negative occurrences of $\Box$ in in the endsequent of $\mathcal{D}$ are realized by distinct proof variables. $\diamondsuit$

3.3 The Realization of LS5

Theorem 3.3.1. *Let*

$$\mathcal{L} \in \{\mathit{LPS5}, \mathit{LPS5}_c, \mathit{JT45}, \mathit{JT5}, \mathit{JTB5}, \mathit{JTB45}, \mathit{JDB5}, \mathit{JDB45}, \mathit{JDB4}, \mathit{JTB4}\}.$$

For any $\mathsf{LS5}^-$-derivation $\mathcal{D}$, a normal realization r can be constructed, such that for every hypersequent G of $\mathcal{D}$, the image $(G)^r$ of G under r, is derivable in $\mathcal{L}$.

Proof. The idea of the proof is based on the proof given by S. N. Artemov, E. L. Kazakov and D. Shapiro [4]. The proofs here are written for the standard notation sequents and with the many lemma's from chapter 2, the proofs are short and the same for any of the considered logics.

Assume, without loss of generality, that the axioms of $\mathcal{D}$ are atomic (this can be assumed because of Lemma 3.2.4). An $\mathcal{L}$-derivation $\mathcal{D}'$ and a realization r will be constructed as follows:

- Every non-essential family (the negative and the non-essential positive) is realized by a fresh proof variable.
- Let f be an essential family. Enumerate all the occurrences of the rule $(\Rightarrow \Box)$ that introduce $\Box$'s to this family, let k be the total number. Realize all the $\Box$'s of f with $\widetilde{u} = (u_1 + \ldots + u_k)$, where the u_i's are fresh proof variables (called *provisional variables*).

The derivation $\mathcal{D}'$ and the realization r will be constructed by induction on $\mathcal{D}$. At the start, $\mathcal{D}'$ is empty and r is constructed for families as described. The $\Box$'s of negative families are realized by fresh proof variables and the provisional variables will be replaced by terms, during the construction of r, from which it follows that r is normal.

During the construction of $\mathcal{D}'$ and r an initially empty substitution σ is built. This idea is based on the paper by V. Brezhnev and R. Kuznets [6]. At the end, σ assigns a certain term to each provisional variable.

To be able to substitute the provisional variables u_i by terms t_i, it is sufficient to check that the t_i's are not containing u_j's, for $j \neq i$. Because:

- If the sequents of the premise are annotated by terms that do not contain any u_i's, then the sequents of the conclusion will be annotated by terms that do not contain any of the u_i's either. Therefore, all the sequents are annotated by terms that do not contain any of the u_i's.
- When a term t is is substituted for some u_i's, t does not contain any u_j's.

Start with proving the realization of the axioms into the basic justification logic J.

Axiom (1): $A \Rightarrow A$, by Definition 3.1.2, the axiom can be translated into the modal formula $A \supset A$. Since it has been assumed that the axioms are atomic, A is a propositional variable or $A = \bot$. From the Table 3.2 of realization functions on syntactic $\mathsf{LS5}^-$-objects, it is known that $p^r := p$, for any propositional variable and $\bot^r := \bot$. Hence: $(A \supset A)^r := A^r \supset A^r$, for any realization function r on $A \Rightarrow A$, translates to $A \supset A$, which is a justification tautology as well.

Axiom (2): $\perp \Rightarrow$, by Definition 3.1.2, the axiom can be translated into the modal formula $\perp \supset \perp$. From the definition of a realization function on a syntactic $\mathsf{LS5}^{-}$-object in Table 3.2, it follows that $(\perp \supset \perp)^r = \perp^r \supset \perp^r = \perp \supset \perp$. Hence $(\perp \supset \perp)^r$ can be realized into the justification formula $\perp \supset \perp$, which is a justification tautology as well.

Now consider the rules *contraction*, *weakening* and *implication*. By Definition 3.1.2 the premise(s) and the conclusion of each of these rules can be translated into a modal formula. Assume by induction hypothesis that there is a realization function r, that realizes the premise of the rule into a formula of the justification logic $\mathcal{L}$. The goal is to prove that the conclusion of the rule can be realized into $\mathcal{L}$ as well, using the realization of the premise(s).

In general assume that $A^r = K$, $B^r = L$, $\Gamma^r = C$ and $\Delta^r = D$. Now consider each of the eight rules in turn:

Contraction left (CL): Consider the first contraction rule:

$$\frac{G \mid A, A, \Gamma \Rightarrow \Delta \mid H}{G \mid A, \Gamma \Rightarrow \Delta \mid H} \;.$$

By induction hypothesis, there is a realization function r that realizes the premise into $\mathcal{L}$: $G^r \vee s : ((K \wedge K \wedge C) \supset D) \vee H^r$. It has to be proven that the conclusion of the rule can be realized into $\mathcal{L}$ as well:

0.	$G^r \vee s : ((K \wedge K \wedge C) \supset D) \vee H^r$	By induction hypothesis
1.	$(K \wedge C) \supset (K \wedge K \wedge C)$	Propositional tautology
2.	$((K \wedge K \wedge C) \supset D) \supset ((K \wedge C) \supset D)$	From 1. by prop. reasoning
3.	$s : ((K \wedge K \wedge C) \supset D) \supset t(s) : ((K \wedge C) \supset D)$	From 2. by Corollary 2.2.3
4.	$G^r \vee t(s) : ((K \wedge C) \supset D) \vee H^r$	From 0. and 3. by prop. reasoning.

Let $s' := t(s)$, then $G^r \vee s' : ((K \wedge C) \supset D) \vee H^r$ is a realization of the contraction rule to the left.

Contraction right (CR): Consider the second contraction rule:

$$\frac{G \mid \Gamma \Rightarrow \Delta, A, A \mid H}{G \mid \Gamma \Rightarrow \Delta, A \mid H} \;.$$

By induction hypothesis, there is a realization function r that realizes the premise into $\mathcal{L}$: $G^r \vee s : (C \Rightarrow (D \vee K \vee K)) \vee H^r$. It has to be proven that the conclusion of the rule can be realized into $\mathcal{L}$ as well:

0.	$G^r \vee s : (C \Rightarrow (D \vee K \vee K)) \vee H^r$	By induction hypothesis
1.	$(D \vee K \vee K) \supset (D \vee K)$	Propositional tautology
2.	$(C \supset (D \vee K \vee K)) \supset (C \supset (D \vee K))$	From 1. by prop. reasoning
3.	$s : (C \supset (D \vee K \vee K)) \supset t(s) : (C \supset (D \vee K))$	From 2. by Corollary 2.2.3
4.	$G^r \vee t(s) : (C \supset (D \vee K)) \vee H^r$	From 0. and 3. by prop. reasoning.

Let $s' := t(s)$, then $G^r \vee s' : (C \supset (D \vee K)) \vee H^r$ is a realization of the conclusion of contraction rule to the right.

External contraction (CE): Consider the external contraction rule:

$$\frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \mid H}{G \mid \Gamma \Rightarrow \Delta \mid H} .$$

By induction hypothesis, there is a realization function r that realizes the premise into $\mathcal{L}$: $G^r \vee s : (C \supset D) \vee t : (C \supset D) \vee H^r$. It has to be proven that the conclusion of the rule can be realized into $\mathcal{L}$ as well:

0.	$G^r \vee s : (C \supset D) \vee t : (C \supset D) \vee H^r$	By induction hypothesis
1.	$s : (C \supset D) \supset (s + t) : (C \supset D)$	Instance of sum
2.	$t : (C \supset D) \supset (s + t) : (C \supset D)$	Instance of sum
3.	$((s + t) : (C \supset D) \vee (s + t) : (C \supset D)) \supset (s + t) : (C \supset D)$	Prop. tautology
4.	$G^r \vee (s + t) : (C \supset D) \vee (s + t) : (C \supset D) \vee H^r$	From 0., 1. and 2. by prop. reasoning
5.	$G^r \vee (s + t) : (C \supset D) \vee H^r$	From 3. and 4. by prop. reasoning.

From this it follows that $G^r \vee (s + t) : (C \supset D) \vee H^r$ is a realization of the conclusion of the external contraction rule.

Weakening left (WL): Consider the first weakening rule:

$$\frac{G \mid \Gamma \Rightarrow \Delta \mid H}{G \mid A, \Gamma \Rightarrow \Delta \mid H.}$$

By induction hypothesis, there is a realization function r that realizes the premise into $\mathcal{L}$: $G^r \vee s : (C \supset D) \vee H^r$. It has to be proven that the conclusion of the rule can be realized into $\mathcal{L}$ as well:

0.	$G^r \vee s : (C \supset D) \vee H^r$	By induction hypothesis
1.	$(K \wedge C) \supset C$	Propositional tautology
2.	$(C \supset D) \supset ((K \wedge C) \supset D)$	From 1. by prop. reasoning
3.	$s : (C \supset D) \supset t(s) : ((K \wedge C) \supset D)$	From 2. by Corollary 2.2.3
4.	$G^r \vee t(s) : ((K \wedge C) \supset D) \vee H^r$	From 0. and 3. by prop. reasoning.

Let $s' := t(s)$, then $G^r \vee s' : ((K \wedge C) \supset D) \vee H^r$ is a realization of the conclusion of the weakening rule to the left.

Weakening right (WR): Consider the second weakening rule:

$$\frac{G \mid \Gamma \Rightarrow \Delta \mid H}{G \mid \Gamma \Rightarrow \Delta, A \mid H.}$$

By induction hypothesis, there is a realization function r that realizes the premise into $\mathcal{L}$: $G^r \vee s : (C \supset D) \vee H^r$. It has to be proven that the conclusion of the rule can be realized into $\mathcal{L}$ as well:

0.	$G^r \vee s : (C \supset D) \vee H^r$	By induction hypothesis
1.	$D \supset (D \vee K)$	Propositional tautology
2.	$(C \supset D) \supset (C \supset (D \vee K))$	From 1. by prop. reasoning

3. $s : (C \supset D) \supset t(s) : (C \supset (D \vee K))$ From 2. by Corollary 2.2.3
4. $G^r \vee t(s) : (C \supset (D \vee K)) \vee H^r$ From 0. and 3. by prop. reasoning.

Let $s' := t(s)$, then $G^r \vee s' : (C \supset (D \vee K)) \vee H^r$ is a realization of the weakening rule to the right.

External weakening: Consider the external weakening rule:

$$\frac{G \mid \Gamma \Rightarrow \Delta \mid H}{G \mid \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta' \mid H}.$$

By induction hypothesis, there is a realization function r that realizes the premise into $\mathcal{L}$: $G^r \vee s : (C \supset D) \vee H^r$. It has to be proven that the conclusion of the rule can be realized into $\mathcal{L}$ as well:

0. $G^r \vee s : (C \supset D) \vee H^r$ By induction hypothesis
1. $s : (C \supset D) \supset (s : (C \supset D) \vee s' : (C' \supset D'))$ Propositional tautology
2. $G^r \vee s : (C \supset D) \vee s' : (C' \supset D') \vee H^r$ From 0. and 1. by prop. reasoning.

From this it follows that $G^r \vee s : (C \supset D) \vee s' : (C' \supset D') \vee H^r$ is a realization of the external weakening rule.

Implication to the left ($\supset\Rightarrow$): Consider the rule for implication to the left:

$$\frac{G \mid \Gamma \Rightarrow \Delta, A \mid H \quad G \mid B, \Gamma \Rightarrow \Delta \mid H}{G \mid A \supset B, \Gamma \Rightarrow \Delta \mid H}.$$

By induction hypothesis, there is a realization function r that realizes the premise into $\mathcal{L}$: $G^r \vee s_1 : (C \supset (D \vee K)) \wedge s_2 : ((L \wedge C) \supset D) \vee H^r$. It has to be proven that the conclusion of the rule can be realized into $\mathcal{L}$ as well:

0. $G^r \vee s_1 : (C \supset (D \vee K)) \wedge s_2 : ((L \wedge C) \supset D) \vee H^r.$ By induction hypothesis
1. $(C \supset (D \vee K)) \wedge ((L \wedge C) \supset D) \supset (((K \supset L) \wedge C) \supset D)$ Propositional tautology
2. $(C \supset (D \vee K)) \supset (((L \wedge C) \supset D) \supset (((K \supset L) \wedge C) \supset D))$ From 1. by prop. reas.
3. $s_1 : (C \supset (D \vee K)) \supset (s_2 : ((L \wedge C) \supset D) \supset t(s_1, s_2) : (((K \supset L) \wedge C) \supset D))$ From 2. by Corollary 2.2.3
4. $s_1 : (C \supset (D \vee K)) \wedge s_2 : ((L \wedge C) \supset D) \supset t(s_1, s_2) : (((K \supset L) \wedge C) \supset D)$ From 3. by prop. reasoning
5. $G^r \vee t(s_1, s_2) : (((K \supset L) \wedge C) \supset D) \vee H^r$ From 0. and 4. by prop. reasoning

Let $s := t(s_1, s_2)$, then $G^r \vee s : (((K \supset L) \wedge C) \supset D) \vee H^r$ is a realization of the rule for implication to the left.

Implication to the right ($\Rightarrow\supset$): Consider the rule for implication to the right:

$$\frac{G \mid A, \Gamma \Rightarrow \Delta, B \mid H}{G \mid \Gamma \Rightarrow \Delta, A \supset B \mid H}.$$

By induction hypothesis, there is a realization function r that realizes the premise into $\mathcal{L}$: $G^r \vee s : ((K \wedge C) \supset (D \vee L)) \vee H^r$. It has to be proven that the conclusion of the rule can be realized into $\mathcal{L}$ as well:

0. $G^r \vee s : ((K \wedge C) \supset (D \vee L)) \vee H^r$ By induction hypothesis
1. $((K \wedge C) \supset (D \vee L)) \supset (C \supset (D \vee (K \supset L)))$ Propositional tautology
2. $s : ((K \wedge C) \supset (D \vee L)) \supset t(s) : (C \supset (D \vee (K \supset L)))$ From 1. by Corollary 2.2.3
3. $G^r \vee t(s) : (C \supset (D \vee (K \supset L))) \vee H^r$ From 0. and 2. by prop. reasoning.

Let $s' := t(s)$, then $G^r \vee s' : (C \supset (D \vee (K \supset L))) \vee H^r$ is a realization of the conclusion of implication to the right rule.

Modal rule ($\Rightarrow \Box$): Consider the first modal rule:

$$\frac{G \mid \Gamma \Rightarrow \Delta \mid\Rightarrow A \mid H}{G \mid \Gamma \Rightarrow \Delta, \Box A \mid H} \; .$$

Assume that this is the j-th introduction of a $\Box$ using the ($\Rightarrow \Box$)-rule into this family of $\Box$'s and suppose that the $\Box$ in $\Box A$ has been annotated with $(v_1 + \ldots + v_k)$, as described at the beginning of the proof. Possibly some of the v_i's, $(i \neq j)$ are already substituted in an earlier stage of the construction of the derivation, therefore, instead of considering $(v_1 + \ldots + v_k)$, the annotation $(v_1\sigma, \ldots, v_k\sigma)$ will be considered. However, v_j is definitely a provisional variable. In the entire derivation v_j has to be replaced with s_2.

By induction hypothesis, the premise of the rule can be realized into the justification logic $\mathcal{L}$: $G^r \vee s_1 : (C \supset D) \vee s_2 : K \vee H^r$. Let $\widetilde{v} = v_1\sigma + \ldots + v_{j-1}\sigma + s_2 + v_{j+1}\sigma + \ldots + v_k\sigma$.

0. $G^r \vee s_1 : (C \supset D) \vee s_2 : K \vee H^r$ By induction hypothesis
1. $D \supset (D \vee \widetilde{v} : K)$ Propositional tautology
2. $(C \supset D) \supset (C \supset (D \vee \widetilde{v} : K)$ From 1. by prop. reasoning
3. $s_1 : (C \supset D) \supset t'(s_1) : (C \supset D \vee \widetilde{v} : K)$ By Corollary 2.2.3
4. $s_2 : K \supset \widetilde{v} : K$ Several times sum
5. $(D \vee s_2 : K) \supset (D \vee \widetilde{v} : K)$ From 4. by prop. reasoning
6. $s_2 : K \supset (C \supset D \vee s_2 : K)$ Propositional tautology
7. $s_2 : K \supset (C \supset D \vee \widetilde{v} : K)$ From 5. and 6. by prop. reasoning
8. $s_2 : K \supset p_2(s_2) : (C \supset D \vee \widetilde{v} : K)$ From 7. by Lemma 2.2.16
9. $t'(s_1) : (C \supset D \vee \widetilde{v} : K) \vee p_2(s_2) : (C \supset D \vee \widetilde{v} : K)$
   $\supset (t(s_1) + p_2(s_2)) : (C \supset D \vee \widetilde{v} : K)$ From 3. and 8. by sum and prop. reasoning
10. $G^r \vee (t'(s_1) + p_2(s_2)) : (C \supset D \vee \widetilde{v} : K) \vee H^r$ From 0. and 9. by prop. reason.

Let $t(s_1, s_2) := (t'(s_1) + p_2(s_2))$, then $G^r \vee t(s_1, s_2) : (C \supset D \vee \widetilde{v} : K) \vee H^r$ is a realization of the conclusion of the first modal rule. Since Lemma 2.2.16 has been proven for all possibilities of $\mathcal{L}$, the proof of the realization for this rule holds for any of the logics that are considered here.

It is left to prove that s_2 does not contain any of the v_i's (the provisional variables), where $i \neq j$. Since s_2 is part of the realization of the premise, by induction hypothesis s_2 does not contain any of the provisional variables v_i, $i \neq j$.

From this it follows that v_j can be replaced with s_2 in the entire derivation and $\sigma := \sigma \cup \{v_j \mapsto s_2\}$.

Modal rule ($\Box \Rightarrow$): Consider the second modal rule:

$$\frac{G \mid A, \Gamma \Rightarrow \Delta \mid \Gamma' \Rightarrow \Delta' \mid H}{G \mid \Gamma \Rightarrow \Delta \mid \Box A, \Gamma' \Rightarrow \Delta' \mid H.}$$

As described at the beginning of the proof, the $\Box$ in $\Box A$ has to be realized by a fresh proof variable. Assume without loss of generality, that q is such a fresh proof variable.

By induction hypothesis, the premise of the rule can be realized into the justification logic $\mathcal{L}$ as the following formula: $G^r \vee s_1 : ((K \wedge C) \supset D) \vee s_2 : (C' \supset D') \vee H^r$. It is left to prove that the conclusion of the ($\Box \Rightarrow$)-rule is realizable into $\mathcal{L}$ as well.

0. $G^r \vee s_1 : ((K \wedge C) \supset D) \vee s_2 : (C' \supset D') \vee H^r$ — By induction hypothesis
1. $((K \wedge C) \supset D) \supset (K \supset (C \supset D))$ — Propositional tautology
2. $s_1 : ((K \wedge C) \supset D) \supset t(s_1) : (K \supset (C \supset D))$ — From 1. by Corollary 2.2.3
3. $t(s_1) : (K \supset (C \supset D)) \supset (q : K \supset (t(s_1) \cdot q) : (C \supset D))$ — Instance of app
4. $\neg q : K \supset (\neg q : K \vee (C' \supset D'))$ — Propositional tautology
5. $(\neg q : K \vee (C' \supset D')) \supset ((q : K \wedge C') \supset D')$ — Propositional tautology
6. $p_1 : (\neg q : K \supset ((q : K \wedge C') \supset D'))$ — From 4. and 5. by Lemma 2.2.1 and prop. reas.
7. $\neg q : K \supset \mathit{negint}(q) : \neg q : K$ — By Lemma 2.2.18
8. $p_1 : (\neg q : K \supset ((q : K \wedge C') \supset D'))$
   $\supset (\mathit{negint}(q) : \neg q : K \supset (p_1 \cdot \mathit{negint}(q)) : ((q : K \wedge C') \supset D'))$ — Instance of app
9. $\neg q : K \supset (p_1 \cdot \mathit{negint}(q)) : ((q : K \wedge C') \supset D')$ From 6., 7. and 8. by MP and prop. reas.
10. $D' \supset (D' \vee \neg q : K)$ — Propositional tautology
11. $(C' \supset D') \supset (C' \supset (D' \vee \neg q : K))$ — From 10. by prop. reasoning
12. $s_2 : (C' \supset D') \supset t'(s_2) : (C' \supset D' \vee \neg q : K)$ — From 11. By Corollary 2.2.3
13. $t'(s_2) : (C' \supset D' \vee \neg q : K)$
    $\supset p_4(s_2) : ((q : K \wedge C') \supset D')$ — By prop. reasoning and Corollary 2.2.3
14. $(p_1 \cdot \mathit{negint}(q)) : ((q : K \wedge C') \supset D') \vee p_4(s_2) : ((q : K \wedge C') \supset D')$
    $\supset \quad ((p_1 \cdot \mathit{negint}(q)) + p_4(s_2)) : ((q : K \wedge C') \supset D')$
    From 9. and 13. by sum and prop. reason.

From these derivations it follows that:
- $s_1 : ((K \wedge C) \supset D) \supset (q : K \supset (t(s_1) \cdot q) : (C \supset D))$
- $s_2 : (C' \supset D') \supset p_4(s_2) : ((q : K \wedge C') \supset D')$

Since $q : K \vee \neg q : K$ is a propositional tautology and $\neg q : K \supset (p_1 \cdot \mathit{negint}(q)) : ((q : K \wedge C') \supset D')$, it can be derived that:

$$\mathcal{L} \vdash (s_1 : ((K \wedge C) \supset D) \vee s_2 : (C' \supset D')) \supset$$
$$(t(s_1) \cdot q) : (C \supset D) \vee (p_1 \cdot \mathit{negint}(q)) : ((q : K \wedge C') \supset D') \vee p_4(s_2) : ((q : K \wedge C') \supset D').$$

Let $t_1 := (t(s_1) \cdot q)$ and $t_2 := ((p_1 \cdot \mathit{negint}(q)) + p_4(s_2))$, then from 0., 14. and the above result, by propositional reasoning it follows that: $G^r \vee t_1 : (C \supset D) \vee t_2 : ((q : K \wedge C') \supset D') \vee H^r$. This is a realization of the conclusion of the modal rule that introduces a $\Box$ to the left.

Now that it has been proven that any of the rules of $\mathsf{LS5}^-$is realizable into $\mathcal{L}$, it can be concluded that for any $\mathsf{LS5}^-$-derivation $\mathcal{D}$, a normal realization r can be constructed such that for every hypersequent G of $\mathcal{D}$, the image $(G)^r$ of G under r is derivable in $\mathcal{L}$, where:

$$\mathcal{L} \in \{\mathsf{LPS5}, \mathsf{LPS5}_c, \mathsf{JT45}, \mathsf{JT5}, \mathsf{JTB5}, \mathsf{JTB45}, \mathsf{JDB5}, \mathsf{JDB45}, \mathsf{JDB4}, \mathsf{JTB4}\}.$$

$\Box$

Based on this theorem, Theorem 3.1.3 and Theorem 3.1.4, the following corollary states the realization of S5 into its justification counterparts.

Corollary 3.3.2.
Let $\mathcal{L} \in \{$LPS5, LPS5$_c$, JT45, JT5, JTB5, JTB45, JDB5, JDB45, JDB4, JTB4$\}$. Let A be a modal formula, if there exists an S5-derivation of A, a normal derivation r can be constructed, such that $\mathcal{L} \vdash A^r$. This means: S5 is realizable into any of the ten considered justification logics, using hypersequents.

CHAPTER 4

Nested Sequents Method

In this chapter a cut-free nested sequent system, that was used by R. Goetschi and R. Kuznets in their paper on realization [12] and introduced by Kai Brünnler [7], is considered. With this nested sequent system the realization of many modal logics into their justification counterparts can be proven, as was done by R. Goetschi and R. Kuznets [12].

However, here only the realization of $\mathsf{S5}$ into justification logic is considered. In the paper, the realization of $\mathsf{S5}$ into $\mathsf{JT45}$ was directly proven using nested sequents. The realization of $\mathsf{S5}$ into $\mathsf{JT5}$, $\mathsf{JTB5}$, $\mathsf{JDB5}$, $\mathsf{JDB4}$, $\mathsf{JTB4}$, $\mathsf{JTB45}$ and $\mathsf{JDB45}$ was proven by using in addition to nested sequents the notions of embedding and operation translation. This method requires additional restrictions on the axiom system of the justification logic. Here the interest lies in a direct realization of $\mathsf{S5}$ into its justification counterparts using nested sequents.

Because of the direct realization of $\mathsf{S5}$ into $\mathsf{JT45}$ using nested sequents, it follows that $\mathsf{S5}$ is directly realizable into $\mathsf{JTB45}$ using nested sequents as well. Below the realization of $\mathsf{S5}$ into $\mathsf{LPS5}$ and $\mathsf{LPS5}_c$ will be discussed. For the direct realization of $\mathsf{S5}$ into the logics $\mathsf{JT5}$, $\mathsf{JTB5}$, $\mathsf{JTB4}$, $\mathsf{JDB5}$, $\mathsf{JDB4}$ and $\mathsf{JDB45}$, another system will be considered, which can be found the next chapter.

4.1 A Nested Sequent System

The nested sequents as used here, are based on the paper by R. Goetschi and R. Kuznets [12]. Unless stated otherwise, the definitions, facts, lemmas and theorem, for this and the next section, are taken from this paper as well. The proofs can be found there.

Definition 4.1.1. A *nested sequent* can be defined inductively:

- $\emptyset$, the empty sequence, is a nested sequent
- Γ, A and $\Gamma, [\Delta]$ are nested sequents, where Γ and Δ are nested sequents and A is a modal formula

The comma denotes concatenation, the brackets in $[\Delta]$ are called *structural box*. (Nested) sequents will be denoted by Greek uppercase letters. $\Diamond$

Definition 4.1.2. The *corresponding formula* of a sequent Γ, denoted by $\underline{\Gamma}$ can inductively be defined as follows:

$$\underline{\emptyset} := \bot; \qquad \underline{\Gamma, A} := \begin{cases} \underline{\Gamma} \vee A & \text{if } \Gamma \neq \emptyset \\ A & \text{otherwise} \end{cases} \qquad \underline{\Gamma, [\Delta]} := \begin{cases} \underline{\Gamma} \vee \Box\underline{\Delta} & \text{if } \Gamma \neq \emptyset \\ \Box\underline{\Delta} & \text{otherwise} \end{cases}$$

$\diamondsuit$

Because of an ambiguous grammar in the definition of a context in the papers by K. Brünnler [7] and R. Goetschi and R. Kuznets [12], the following definition will be used here and can be found in the paper by R. Kuznets and M. Fitting [14].

Definition 4.1.3. A *(sequent) context*, is a sequent with exactly one *hole*: the occurrence of $\{\}$. These contexts are denoted by $\Gamma\{\}$. Let Δ be some sequent and let $\Sigma\{\}$ be a context, a context can inductively be defined as follows:

- $\Delta, \{\}$ is a context
- $[\Sigma\{\}]$ and $\Sigma\{\}, \Delta$ are contexts.

Let $\Gamma\{\}$ be a context and Π a sequent, then $\Gamma\{\Pi\}$ can be defined based on the above cases:

- If $\Gamma\{\} = \Delta, \{\}$, then $\Gamma\{\Pi\} = \Delta, \Pi$.
- If $\Gamma\{\} = [\Sigma\{\}]$, then $\Gamma\{\Pi\} = [\Sigma\{\Pi\}]$.
- If $\Gamma\{\} = \Sigma\{\}, \Delta$, then $\Gamma\{\Pi\} = \Sigma\{\Pi\}, \Delta$.

$\diamondsuit$

Definition 4.1.4. A nested sequent-rule, denoted by ρ, is a set of instances that have the following form $\dfrac{\Gamma_1 \ \ldots \ \Gamma_n}{\Gamma}$, for $n \geq 0$, where $\Gamma, \Gamma_1, \ldots, \Gamma_n$ are sequents. $\Gamma_1, \ldots, \Gamma_n$ is the premise of rule ρ and Γ is the conclusion of ρ.

A nested sequent-rule ρ may be *context-preserving*, this means that ρ is a set of instances of the following form: $\dfrac{\Gamma\{S_1\} \ \ldots \ \Gamma\{S_n\}}{\Gamma\{S\}}\,\rho$, for $n \geq 0$ and some set of tuples of sequents $(S_1, \ldots, S_n, S)$. Here $\Gamma\{\}$ is an arbitrary context that is common for all premises and for the conclusion of ρ.

Instances of context-preserving nested sequent-rules have a *shallow version*, denoted with $\mathsf{sh}\text{-}\rho$, for rule ρ. This is the same instance of the rule, but with an empty common context: $\Gamma\{\} = \{\}$. Let $S, S_1, \ldots, S_n$ be sequents, for $n \geq 0$, then:

$$\frac{\Gamma\{S_1\} \ \ldots \ \Gamma\{S_n\}}{\Gamma\{S\}}\,\rho \qquad\qquad \frac{S_1 \ \ldots \ S_n}{S}\,\mathsf{sh}\text{-}\rho$$

$\diamondsuit$

With these definitions, it is now possible to define the nested sequent rules, as they will be used here.

Definition 4.1.5.

$$\frac{}{\Gamma\{a,\overline{a}\}}\ \mathsf{id} \qquad \frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}}\ \vee \qquad \frac{\Gamma\{A\} \qquad \Gamma\{B\}}{\Gamma\{A\wedge B\}}\ \wedge$$

$$\frac{\Gamma\{A,A\}}{\Gamma\{A\}}\ \mathsf{ctr} \qquad \frac{\Gamma\{\Delta,\Sigma\}}{\Gamma\{\Sigma,\Delta\}}\ \mathsf{exch} \qquad \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}}\ \Box \qquad \frac{\Gamma\{[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}\ \mathsf{k}$$

$$\frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}}\ \mathsf{d} \qquad \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}}\ \mathsf{t} \qquad \frac{\Gamma\{[\Delta],A\}}{\Gamma\{[\Delta,\Diamond A]\}}\ \mathsf{b} \qquad \frac{\Gamma\{[\Diamond A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}\ \mathsf{4}$$

$$\frac{\Gamma\{[\Delta],\Diamond A\}}{\Gamma\{[\Delta,\Diamond A]\}}\ \mathsf{5a} \qquad \frac{\Gamma\{[\Delta],[\Pi,\Diamond A]\}}{\Gamma\{[\Delta,\Diamond A],[\Pi]\}}\ \mathsf{5b} \qquad \frac{\Gamma\{[\Delta,[\Pi,\Diamond A]]\}}{\Gamma\{[\Delta,\Diamond A,[\Pi]]\}}\ \mathsf{5c}$$

Notice that each of these rules is context-preserving, from which it follows that, for each of these rules a shallow version exists. $\Diamond$

4.2 Auxiliary Definitions and Lemma's

Definition 4.2.1. A *subsequent* of a sequent Γ is any sequent Δ, such that $\Gamma = \Sigma\{\Delta\}$ and $\Sigma\{\}$ is some context. $\Diamond$

Definition 4.2.2. An *annotated sequent (context)* is a sequent (context) with only annotated formulas, all the structural boxes are annotated by odd indices, a sequent (context) is *properly annotated* if no index occurs twice. The *corresponding formula* of an annotated sequent is defined above in Definition 4.1.2, except for the last case, which is defined as:

$$\underline{\Sigma,[\Delta]_k} := \begin{cases} \underline{\Sigma} \vee \Box_k\underline{\Delta} & \text{if } \Sigma \neq \emptyset \\ \Box_k\underline{\Delta} & \text{otherwise} \end{cases}$$

Let Γ be some annotated sequent, if the sequent Γ' is obtained by removing all indices, it is said that Γ is an *annotated version* of Γ'. This idea also holds for a context: let $\Gamma\{\}$ be some annotated context, if the context $\Gamma'\{\}$ is obtained by removing all indices, it is said that $\Gamma\{\}$ is an *annotated* version of the context $\Gamma'\{\}$. $\Diamond$

Definition 4.2.3. Given an instance of a nested rule, with common context $\Gamma'\{\}$:

$$\frac{\Gamma'\{\Lambda'_1\}\ \ldots\ \Gamma'\{\Lambda'_n\}}{\Gamma'\{\Lambda'\}}$$

An *annotated version* of this rule-instance, has the form:

$$\frac{\Gamma\{\Lambda_1\}\ \ldots\ \Gamma\{\Lambda_n\}}{\Gamma\{\Lambda\}}$$

Where $\Gamma\{\}, \Lambda_1, \ldots, \Lambda_n, \Lambda$ are annotated versions of $\Gamma'\{\}, \Lambda'_1, \ldots, \Lambda'_n, \Lambda'$ respectively. The sequents $\Gamma\{\Lambda_1\}, \ldots, \Gamma\{\Lambda_n\}, \Gamma\{\Lambda\}$ are properly annotated and no index occurs in both Λ_i and Λ_j for any $1 \leq i < j \leq n$. The annotated context $\Gamma\{\}$ is the same for all premises and the conclusion. $\diamondsuit$

The definition of a realization function on an annotated formula can be found in Definition 2.2.6. The realization function on an annotated nested sequent is defined as the realization function on the corresponding formula, as defined in Definition 4.2.2. Taking into account that a rule consists of sequents, the realization of a nested sequent-rule into a justification logic JL can be defined.

REMARK. If r is a realization function on an annotated sequent $\Gamma\{\Delta\}$, then r is a realization function on the subsequent Δ as well.

Definition 4.2.4. A rule is called *realizable* in a justification logic JL if all the instances of the rule are realizable in JL.

- An instance $\dfrac{}{\Gamma'\{\Lambda'\}}$ of a 0-premise nested rule is called *realizable* in a justification logic JL, if there is an annotated version $\dfrac{}{\Gamma\{\Lambda\}}$ of it and a realization function r on its conclusion $\Gamma\{\Lambda\}$ such that $\Gamma\{\Lambda\}^r$ is provable in JL.
- An instance of a context-preserving n-premise nested rule, with $n > 0$ and with common context $\Gamma'\{\}$:

$$\frac{\Gamma'\{\Lambda'_1\} \ \ldots \ \Gamma'\{\Lambda'_n\}}{\Gamma'\{\Lambda'\}}$$

  is called *realizable* in a justification logic JL if there exists an annotated version:

$$\frac{\Gamma\{\Lambda_1\} \ \ldots \ \Gamma\{\Lambda_n\}}{\Gamma\{\Lambda\}}$$

  of this rule, such that for any realization functions $r_1, \ldots, r_n$ on the premises $\Gamma\{\Lambda_1\}, \ldots, \Gamma\{\Lambda_n\}$ respectively, there exists a realization function r on the conclusion $\Gamma\{\Lambda\}$ and a substitution σ that lives on each of the $\Gamma\{\Lambda_i\}$, $i = 1, \ldots, n$ such that

$$\mathsf{JL} \vdash \Gamma\{\Lambda_1\}^{r_1}\sigma \supset \ldots \supset \Gamma\{\Lambda_n\}^{r_n}\sigma \supset \Gamma\{\Lambda\}^r.$$

$\diamondsuit$

Lemma 4.2.5. *For any nested rule ρ, if its shallow version sh-ρ is realizable in a justification logic JL, then ρ itself is realizable into JL.*

Theorem 4.2.6. *Let S be a system of nested rules, whose shallow versions are realizable in a justification logic JL. Then, for every sequent Γ' that is provable in S, there is a properly annotated version Γ of it and a realization function r on this Γ, such that Γ^r is provable in JL.*

Using this theorem, the realization of $\mathsf{S5}$ into $\mathsf{LPS5}$ and $\mathsf{LPS5}_c$ can be proven, by proving that sh-ρ, for $\rho \in \{\mathsf{id}, \vee, \wedge, \mathsf{ctr}, \mathsf{exch}, \Box, \mathsf{k}, \mathsf{t}, \mathsf{4}, \mathsf{5a}, \mathsf{5b}, \mathsf{5c}\}$, is provable in $\mathsf{LPS5}$ and $\mathsf{LPS5}_c$. Most of the axioms in $\mathsf{LPS5}$ and $\mathsf{LPS5}_c$ are the same as those of $\mathsf{JT45}$. That is why the proof of the following two lemma's, is completely based on the proof that is given by R. Goetschi and R. Kuznets [12]:

Lemma 4.2.7. *Let $\rho \in \{$id$, \vee, \wedge,$ ctr, exch, $\Box$, k, t, 4$\}$. The shallow version of ρ is realizable in LPS5 and in LPS5$_c$.*

Proof. Consider three cases, based on the justification logic into which the shallow version of ρ is provable:

- Let $\rho \in \{$id$, \vee, \wedge,$ ctr, exch, $\Box$, k$\}$. By R. Goetschi and R. Kuznets [12] it has been proven that sh-ρ is provable into the based justification logic J.
- Let $\rho =$ t, it has been proven by R. Goetschi and R. Kuznets [12], that sh-ρ is realizable into JT and therefore into any extension of JT as well.
- Let $\rho =$ 4, in the same paper, it has been proven that sh-ρ is realizable into J4 and hence into any of the justification logics that extend J4.

Since LPS5 and LPS5$_c$ are extensions of J, JT and of J4, it follows that sh-ρ is realizable into LPS5 and LPS5$_c$ for any of the considered rules. $\Box$

It is left to prove for both LPS5 and LPS5$_c$ that the rules 5a, 5b and 5c are realizable.

4.3 Realization into LPS5 and LPS5$_c$ using Nested Sequents

In the realization into LPS5 an axiom that can be derived from A5 will be used, instead of the axiom A5 itself. Therefore the proofs of the realization of the shallow versions of 5a, 5b and 5c are very similar for LPS5 and LPS5$_c$. In the paper by R. Goetschi and R. Kuznets [12] some auxiliary lemma's were needed to prove the realization of these three shallow versions into J5. The axiom systems of LPS5 and LPS5$_c$ contain, in addition to the axioms A5 and A5$_c$ respectively, the axioms jt and j4. Because of these two lateer axioms the auxiliary lemma's are not needed here.

Realization into LPS5

The axiom system of LPS5 consists of propositional tautologies, jt, app, sum, j4 and A5: $t : (A \supset \neg s : B) \supset (A \supset ?t : (\neg s : B))$. To prove the realization of the shallow versions of 5a, 5b and 5c, another version of A5 is considered, call it A5': $\neg t : A \supset ?c : \neg t : A$, where c is some constant. This axiom can be derived from A5 as follows:

0. $\neg t : A \supset \neg t : A$	Propositional tautology
1. $c : (\neg t : A \supset \neg t : A)$	From 0. by Axiom Necessitation
2. $c : (\neg t : A \supset \neg t : A) \supset (\neg t : A \supset ?c : \neg t : A)$	Instance of A5
3. $\neg t : A \supset ?c : \neg t : A$	From 1. and 2. by Modus Ponens

Notice that a similar proof was given for Lemma 2.2.18, but to keep as close as possible to the logic LPS5, the axiom A5' will be used here, instead of the more general theorem that was introduced by Lemma 2.2.18.

With this new axiom A5', the realization of 5a, 5b and 5c can be proven for the logic LPS5.

Lemma 4.3.1. *Let $\rho \in \{$5a, 5b, 5c$\}$. The shallow version of ρ is realizable in LPS5.*

Proof. Consider an arbitrary instance of sh-ρ for each rule ρ in turn:

Case $\rho = \mathsf{5a}$: Let $\dfrac{[\Delta'], \Diamond A'}{[\Delta', \Diamond A']}$ be an arbitrary instance of $\mathsf{sh\text{-}5a}$, let $[\Delta]_k, \Diamond_{2m} A$ and $[\Delta, \Diamond_{2m} A]_i$ be properly annotated versions of its premise and conclusion respectively. Then $\dfrac{[\Delta]_k, \Diamond_{2m} A}{[\Delta, \Diamond_{2m} A]_i}$ is an annotated version of this instance. Consider an arbitrary realization function r_1 on the premise, then:

0. $\Delta^{r_1} \supset \Delta^{r_1} \vee \neg x_m : \neg A^{r_1}$ — Propositional tautology
1. $r_1(k) : \Delta^{r_1} \supset t_1(r_1(k)) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1})$ — From 0. by Corollary 2.2.3
2. $\neg x_m : \neg A^{r_1} \supset (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1})$ — Propositional tautology
3. $?c : \neg x_m : \neg A^{r_1} \supset t_2(?c) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1})$ — From 2. by Corollary 2.2.3
4. $\neg x_m : \neg A^{r_1} \supset ?c : \neg x_m : \neg A^{r_1}$ — Instance of A5'
5. $\neg x_m : \neg A^{r_1} \supset t_2(?c) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1})$ — From 3. and 4. by prop. reasoning
6. $r_1(k) : \Delta^{r_1} \vee \neg x_m : \neg A^{r_1} \supset (t_1(r_1(k)) + t_2(?c)) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1})$ — From 1. and 5. by sum and prop. reasoning

Let $t := (t_1(r_1(k)) + t_2(?c))$, then it follows that

$$\mathsf{LPS5} \vdash r_1(k) : \Delta^{r_1} \vee \neg x_m : \neg A^{r_1} \supset t : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1}).$$

The index i does not occur in either Δ or $\Diamond_{2m} A$, since $[\Delta, \Diamond_{2m} A]_i$ is properly annotated. Hence: the realization r, formulated as $r := (r_1 \upharpoonright \Delta, \Diamond_{2m} A) \cup \{i \mapsto t\}$ is a realization function on $[\Delta, \Diamond_{2m} A]_i$. For the identity substitution σ and r:

$$\mathsf{LPS5} \vdash ([\Delta]_k, \Diamond_{2m} A)^{r_1}\sigma \supset ([\Delta, \Diamond_{2m} A]_i)^r.$$

Case $\rho = \mathsf{5b}$: Let $\dfrac{[\Delta'], [\Pi', \Diamond A']}{[\Delta', \Diamond A'], [\Pi']}$ be an arbitrary instance of $\mathsf{sh\text{-}5b}$, let $[\Delta]_k, [\Pi, \Diamond_{2m} A]_i$ and $[\Delta, \Diamond_{2m} A]_l, [\Pi]_j$ be properly annotated version of the premise respectively conclusion. Then the following is an annotated version of this instance: $\dfrac{[\Delta]_k, [\Pi, \Diamond_{2m} A]_i}{[\Delta, \Diamond_{2m} A]_l, [\Pi]_j}$. Consider an arbitrary realization function r_1 on the premise, then:

0. $x_m : \neg A^{r_1} \supset !x_m : x_m : \neg A^{r_1}$ — Instance of $\mathsf{j4}$
1. $\neg !x_m : x_m : \neg A^{r_1} \supset \neg x_m : \neg A^{r_1}$ — From 0. by prop. reasoning
2. $?c : \neg !x_m : x_m : \neg A^{r_1} \supset t_1(?c) : \neg x_m : \neg A^{r_1}$ — From 1. by Corollary 2.2.3
3. $\neg !x_m : x_m : \neg A^{r_1} \supset ?c : \neg !x_m : x_m : \neg A^{r_1}$ — Instance of A5'
4. $\neg !x_m : x_m : \neg A^{r_1} \supset t_1(?c) : \neg x_m : \neg A^{r_1}$ — From 2. and 3. by prop. reasoning
5. $x_m : \neg A^{r_1} \supset \Pi^{r_1} \vee \neg x_m : \neg A^{r_1} \supset \Pi^{r_1}$ — Prop. tautology
6. $!x_m : x_m : \neg A^{r_1} \supset r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1}) \supset t_2(!x_m, r_1(i)) : \Pi^{r_1}$ — From 5. by Corollary 2.2.3
7. $r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1}) \supset \neg !x_m : x_m : \neg A^{r_1} \vee t_2(!x_m, r_1(i)) : \Pi^{r_1}$ — From 6. by prop. reasoning
8. $r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1}) \supset t_1(?c) : \neg x_m : \neg A^{r_1} \vee t_2(!x_m, r_1(i)) : \Pi^{r_1}$ — From 4. and 7. by prop. reasoning

9. $\neg x_m : \neg A^{r_1} \supset \Delta^{r_1} \vee \neg x_m : \neg A^{r_1}$ Propositioanl tautology
10. $t_1(?c) : \neg x_m : \neg A^{r_1} \supset t_3(t_1(?c)) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1})$ From 9, by Corollary 2.2.3
11. $r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1}) \supset t_3(t_1(?c)) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1}) \vee t_2(!x_m, r_1(k)) : \Pi^{r_1}$ From 8. and 10. by prop. reasoning
12. $\Delta^{r_1} \supset \Delta^{r_1} \vee \neg x_m : \neg A^{r_1}$ Propositional tautology
13. $r_1(k) : \Delta^{r_1} \supset t_4(r_1(k)) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1})$ From 12. by Corollary 2.2.3
14. $t_3(t_1(?c)) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1}) \vee t_4(r_1(k)) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1})$ $\supset (t_3(t_1(?c)) + t_4(r_1(k))) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1})$ By prop. reasoning and sum

Let $s := t_2(!x_m, r_1(i))$ and $t := t_3(t_1(?c)) + t_4(r_1(k))$, then from 11., 13. and 14, by propositional reasoning it follows that:

$$\mathsf{LPS5} \vdash r_1(k) : \Delta^{r_1} \vee r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1}) \supset t : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1}) \vee s : \Pi^{r_1}.$$

The indices l and j do not occur in Δ, Π or $\Diamond_{2m} A$, since $[\Delta, \Diamond_{2m} A]_l, [\Pi]_j$ is properly annotated. Therefore $r := (r_1 \upharpoonright \Delta, \Diamond_{2m} A, \Pi) \cup \{l \mapsto t, j \mapsto s\}$ is a realization on the conclusion $[\Delta, \Diamond_{2m} A]_l, [\Pi]_j$. For the identity substitution σ and r it follows that:

$$\mathsf{LPS5} \vdash ([\Delta]_k, [\Pi, \Diamond_{2m} A]_i)^{r_1} \sigma \supset ([\Delta, \Diamond_{2m} A]_l, [\Pi]_j)^r.$$

Case $\rho = \mathsf{5c}$: Let $\dfrac{[\Delta', [\Pi', \Diamond A']]}{[\Delta', \Diamond A', [\Pi']]}$ be an arbitrary instance of $\mathsf{sh\text{-}5c}$ and let $[\Delta, [\Pi, \Diamond_{2m} A]_i]_k$ and $[\Delta, \Diamond_{2m} A, [\Pi]_j]_l$ be properly annotated versions of the premise respectively conclusion. Then $\dfrac{[\Delta, [\Pi, \Diamond_{2m} A]_i]_k}{[\Delta, \Diamond_{2m} A, [\Pi]_j]_l}$ is an annotated version of this instance. Consider an arbitrary realization function r_1 on the premise, as in the case of $\rho = \mathsf{5b}$, the following is derivable:

$$r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1}) \supset t_1(?c) : \neg x_m : \neg A^{r_1} \vee t_2(!x_m, r_1(i)) : \Pi^{r_1}.$$

Notice that this is the same as 8. in the proof of case $\rho = \mathsf{5b}$. From here the proof of $\rho = \mathsf{5c}$ differs. Let $s := t_2(!x_m, r_1(i))$, by propositional reasoning it follows that:

$$\mathsf{LPS5} \vdash \Delta^{r_1} \vee r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1}) \supset \Delta^{r_1} \vee t_1(?c) : \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1}$$

By Corollary 2.2.3, there is a term $t_5(x_5)$ such that:

$$\begin{aligned} \mathsf{LPS5} \vdash r_1(k) :&(\Delta^{r_1} \vee r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1})) \\ &\supset t_5(r_1(k)) : (\Delta^{r_1} \vee t_1(?c) : \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1}) \end{aligned}$$

Call this Result A. Since $t_1(?c) : \neg x_m : \neg A^{r_1} \supset \neg x_m : \neg A^{r_1}$ is an instance of the jt-axiom, by Axiom Necessitation it follows that, for any constant c':

$$\mathsf{LPS5} \vdash c' : (t_1(?c) : \neg x_m : \neg A^{r_1} \supset \neg x_m : \neg A^{r_1}).$$

Consider the following propositional tautology:

$$\begin{aligned} \mathsf{LPS5} \vdash &(t_1(?c) : \neg x_m : \neg A^{r_1} \supset \neg x_m : \neg A^{r_1}) \\ &\supset ((\Delta^{r_1} \vee t_1(?c) : \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1}) \supset (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1})). \end{aligned}$$

By Corollary 2.2.3, there is a term $t_6(x_6^1, x_6^2)$ such that:

$$\begin{aligned}\mathsf{LPS5} \vdash c' :&(t_1(?c) : \neg x_m : \neg A^{r_1} \supset \neg x_m : \neg A^{r_1}) \\ &\supset t_5(r_1(k)) : (\Delta^{r_1} \vee t_1(?c) : \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1}) \\ &\quad \supset t_6(c', t_5(r_1(k))) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1}).\end{aligned}$$

Applying Modus Ponens gives:

$$\begin{aligned}\mathsf{LPS5} \vdash t_5(r_1(k)) :&(\Delta^{r_1} \vee t_1(?c) : \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1}) \\ &\supset t_6(c', t_5(r_1(k))) : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1}).\end{aligned}$$

Define $t := t_6(c', t_5(r_1(k)))$, then from Result A, the above result and propositional reasoning it then follows that:

$$\mathsf{LPS5} \vdash r_1(k) : (\Delta^{r_1} \vee r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1})) \supset t : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1})$$

Since $[\Delta, \Diamond_{2m}A, [\Pi]_j]_l$ is properly annotated, the indices l and j do not occur in Δ, Π or $\Diamond_{2m}A$. Therefore, $r := (r_1 \upharpoonright \Delta, \Diamond_{2m}A, \Pi) \cup \{j \mapsto s, l \mapsto t\}$ is a realization function on the conclusion $[\Delta, \Diamond_{2m}A[\Pi]_j]_l$. For the identity substitution σ and r it follows that

$$\mathsf{LPS5} \vdash ([\Delta, [\Pi, \Diamond_{2m}A]_i]_k)^{r_1}\sigma \supset ([\Delta, \Diamond_{2m}A, [\Pi]_j]_l)^r.$$

□

Realization into $\mathsf{LPS5}_c$

The proofs for $\rho \in \{\mathsf{5a}, \mathsf{5b}, \mathsf{5c}\}$ are almost the same in $\mathsf{LPS5}_c$ as in $\mathsf{LPS5}$. The idea is basically to substitute c for all occurrences of $?c$ in the proof of Lemma 4.3.1.

Lemma 4.3.2. *Let $\rho \in \{\mathsf{5a}, \mathsf{5b}, \mathsf{5c}\}$. The shallow version of ρ is realizable in $\mathsf{LPS5}_c$.*

Proof. Consider an arbitrary instance of $\mathsf{sh}\text{-}\rho$ for each rule ρ in turn:

Case $\rho = \mathsf{5a}$: Let $\dfrac{[\Delta'], \Diamond A'}{[\Delta', \Diamond A']}$ be an arbitrary instance of $\mathsf{sh}\text{-}\mathsf{5a}$, let $[\Delta]_k, \Diamond_{2m}A$ and $[\Delta, \Diamond_{2m}A]_i$ be properly annotated versions of its premise and conclusion respectively Then $\dfrac{[\Delta]_k, \Diamond_{2m}A}{[\Delta, \Diamond_{2m}A]_i}$ is an annotated version of this instance.

By a derivation analogous to the derivation of case $\rho = \mathsf{5a}$ in Lemma 4.3.1 a term $t := (t_1(r_1(k)) + t_2(c))$ can be constructed sucht that:

$$\mathsf{LPS5}_c \vdash r_1(k) : \Delta^{r_1} \vee \neg x_m : \neg A^{r_1} \supset t : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1}).$$

The index i does not occur in either Δ or $\Diamond_{2m}A$, since $[\Delta, \Diamond_{2m}A]_i$ is properly annotated. Hence: the realization r, formulated as $r := (r_1 \upharpoonright \Delta, \Diamond_{2m}A) \cup \{i \mapsto t\}$ is a realization function on $[\Delta, \Diamond_{2m}A]_i$. For the identity substitution σ and r:

$$\mathsf{LPS5}_c \vdash ([\Delta]_k, \Diamond_{2m}A)^{r_1}\sigma \supset ([\Delta, \Diamond_{2m}A]_i)^r.$$

Case $\rho = 5b$: Let $\dfrac{[\Delta'], [\Pi', \Diamond A']}{[\Delta', \Diamond A'], [\Pi']}$ be an arbitrary instance of sh-5b, let $[\Delta]_k, [\Pi, \Diamond_{2m} A]_i$ and $[\Delta, \Diamond_{2m} A]_l, [\Pi]_j$ be properly annotated versions of the premise respectively conclusion. Then the following is an annotated version of this instance $\dfrac{[\Delta]_k, [\Pi, \Diamond_{2m} A]_i}{[\Delta, \Diamond_{2m} A]_l, [\Pi]_j}$.

A derivation analogous to the derivation in Lemma 4.3.1 can be given, where terms $s := t_2(!x_m, r_1(i))$ and $t := t_3(t_1(c)) + t_4(r_1(k))$ are constructed such that:

$$\mathsf{LPS5}_c \vdash r_1(k) : \Delta^{r_1} \vee r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1}) \supset t : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1}) \vee s : \Pi^{r_1}.$$

The indices l and j do not occur in Δ, Π or $\Diamond_{2m} A$, since $[\Delta, \Diamond_{2m} A]_l, [\Pi]_j$ is properly annotated. Therefore $r := (r_1 \restriction \Delta, \Diamond_{2m} A, \Pi) \cup \{l \mapsto t, j \mapsto s\}$ is a realization on the conclusion $[\Delta, \Diamond_{2m} A]_l, [\Pi]_j$. For the identity substitution σ and r it follows that:

$$\mathsf{LPS5}_c \vdash ([\Delta]_k, [\Pi, \Diamond_{2m} A]_i)^{r_1}\sigma \supset ([\Delta, \Diamond_{2m} A]_l, [\Pi]_j)^r.$$

Case $\rho = 5c$: Let $\dfrac{[\Delta', [\Pi', \Diamond A']]}{[\Delta', \Diamond A', [\Pi']]}$ be an arbitrary instance of sh-5c and let $[\Delta, [\Pi, \Diamond_{2m} A]_i]_k$ and $[\Delta, \Diamond_{2m} A, [\Pi]_j]_l$ be properly annotated versions of the premise respectively conclusion. Then $\dfrac{[\Delta, [\Pi, \Diamond_{2m} A]_i]_k}{[\Delta, \Diamond_{2m} A, [\Pi]_j]_l}$ is an annotated version of this instance.

Based on part of the derivation of case $\rho = 5b$ and analogous to the derivation of case $\rho = 5c$ in Lemma 4.3.1, terms $s := t_2(!x_m, r_1(i))$ and $t := t_6(c', t_5(r_1(k)))$ can be constructed such that:

$$\mathsf{LPS5}_c \vdash r_1(k) : (\Delta^{r_1} \vee r_1(i) : (\Pi^{r_1} \vee \neg x_m : \neg A^{r_1})) \supset t : (\Delta^{r_1} \vee \neg x_m : \neg A^{r_1} \vee s : \Pi^{r_1})$$

Since $[\Delta, \Diamond_{2m} A, [\Pi]_j]_l$ is properly annotated, the indices l and j do not occur in Δ, Π or $\Diamond_{2m} A$. Therefore, $r := (r_1 \restriction \Delta, \Diamond_{2m} A, \Pi) \cup \{j \mapsto s, l \mapsto t\}$ is a realization function on the conclusion $[\Delta, \Diamond_{2m} A, [\Pi]_j]_l$. For the identity substitution σ and r it follows that

$$\mathsf{LPS5}_c \vdash ([\Delta, [\Pi, \Diamond_{2m} A]_i]_k)^{r_1}\sigma \supset ([\Delta, \Diamond_{2m} A, [\Pi]_j]_l)^r.$$

□

4.4 The Realization of S5

To be able to prove the realization of S5, the following theorem is required:

Theorem 4.4.1. *The nested sequent system that belongs to the modal logic S5 is sound and complete with respect to S5.*

R. Goetschi and R. Kuznets [12] have proven that S5 is realizable into JT45, using the realization of the 5-axiom j5. Therefore, S5 is realizable into JTB45 as well. In the section above it has been proven that the shallow versions of the nested sequent rules 5a, 5b and 5c are realizable into the logics LPS5 and $\mathsf{LPS5}_c$. With these proofs it is possible to prove the following theorem, a slightly changed version of the realization theorem of S5 into JT45:

Theorem 4.4.2. *Let $\mathcal{L} \in \{\mathsf{LPS5}, \mathsf{LPS5}_c\}$, then $\mathcal{L}^\circ = \mathsf{S5}$. Moreover, for each $A' \in \mathsf{S5}$, there exists a properly annotated version A of it and a realization function r on A such that $\mathcal{L} \vdash A^r$.*

Proof. As often, to be able to prove the equality $\mathcal{L}^\circ = \mathsf{S5}$, two inclusions will be proven.

The inclusion $\mathcal{L}^\circ \subseteq \mathsf{S5}$ is the easiest inclusion to prove, it is based on the forgetful projection of the rules in $\mathcal{L}$. Since it is possible to derive the forgetful projection of any of the axioms of $\mathcal{L}$ and the forgetful projections of all the rules of $\mathcal{L}$ are derivable in $\mathsf{S5}$, it follows that $\mathcal{L}^\circ \subseteq \mathsf{S5}$.

The other inclusion, $\mathcal{L}^\circ \supseteq \mathsf{S5}$, is harder to prove, as was already mentioned in Chapter 2 on realizations, since here the $\Box$'s have to be realized by terms that may be different from each other. From Lemma 4.2.7 it follows that for $\rho \in \{\mathsf{id}, \vee, \wedge, \mathsf{ctr}, \mathsf{exch}, \Box, \mathsf{k}, \mathsf{t}, 4\}$, $\mathsf{sh}\text{-}\rho$ is realizable in $\mathsf{LPS5}$ and $\mathsf{LPS5}_c$. If $\mathcal{L} = \mathsf{LPS5}$, then it follows from Lemma 4.3.1 that for $\rho \in \{\mathsf{5a}, \mathsf{5b}, \mathsf{5c}\}$, $\mathsf{sh}\text{-}\rho$ is realizable in $\mathcal{L}$. In the same way for $\mathcal{L} = \mathsf{LPS5}_c$, from Lemma 4.3.2, for $\rho \in \{\mathsf{5a}, \mathsf{5b}, \mathsf{5c}\}$, it follows that $\mathsf{sh}\text{-}\rho$ is realizable in $\mathcal{L}$. Therefore, the shallow versions of all the rules of the nested sequent system of $\mathsf{S5}$ are realizable in $\mathcal{L}$.

Now let $A' \in \mathsf{S5}$ be some modal formula, then $\mathsf{S5} \vdash A'$. By Theorem 4.4.1 (completeness) it follows that there is a derivation of A' in the nested sequent calculus that belongs to the modal logic $\mathsf{S5}$. Then, by Theorem 4.2.6, there is a properly annotated version A of A' and a realization function r on this A such that $\mathcal{L} \vdash A^r$. Notice that $(A^r)^\circ = A'$, from which it follows that $A' \in \mathcal{L}^\circ$. $\Box$

CHAPTER 5

Straßburger's Modular Nested Sequents Method

The nested sequent calculus, as it was presented in the previous chapter, is sufficient to prove the realization of S5 into JT45 and hence into JTB45, as was proven by R. Goetschi and R. Kuznets [12] and into LPS5 and $\mathsf{LPS5}_c$. In addition S5 is also realizable into other justification logics, for example when taking into account the sequent calculi that were presented by S. Marin and L. Straßburger [15].

The logic S5 has different axiom systems, it can be obtained by adding axioms b and 4 or by adding axioms t and 5 to the basic modal logic K. The modular nested sequent calculi that are presented by S. Marin and L. Straßburger [15] and will be used here, are such that for any axiom system for modal logic S5, the corresponding calculus is cut-free. With this modular system, it is possible to realize S5 into JT5, JTB5, JTB4, JDB5, JDB4 and JDB45, the proof of which will be discussed here. The definitions given in this chapter are based on the definitions in the original paper by S. Marin and L. Straßburger [15].

5.1 Modular Nested Sequent System

In the system that was presented in the previous chapter (chapter 4), only *unary* contexts were used. The depth of such unary contexts can be defined as follows:

Definition 5.1.1. The depth of a unary context can be defined inductively as:

- $\text{depth}(\Delta, \{\ \}) = 0$
- $\text{depth}(\Gamma\{\ \}, \Delta) = \text{depth}(\Gamma\{\ \})$
- $\text{depth}([\Gamma\{\ \}]) = \text{depth}(\Gamma\{\ \}) + 1$.

$\diamondsuit$

Now the 5-rule consists of a *binary* context in its premise and conclusion. In this case, there are exactly two *holes*.

EXAMPLE 5.1.2 ([15]). Let $\Gamma\{\}\{\} = A, [B, \{\}, [\{\}], C]$, for any sequents Δ_1 and Δ_2 it follows that: $\Gamma\{\Delta_1\}\{\Delta_2\} = A, [B, \Delta_1, [\Delta_2], C]$. And if one of the sequents is the empty sequent then: $\Gamma\{\emptyset\}\{\Delta_2\} = A, [B, [\Delta_2], C]$ and $\Gamma\{\Delta_1\}\{\emptyset\} = A, [B, \Delta_1, [\emptyset], C]$.

The depth can be computed as follows: depth$(\Gamma\{\}\{\Delta\}) = 1$ and depth$(\Gamma\{\Delta\}\{\}) = 2$. $\Diamond$

Nested Sequents

Taking into account the definitions from the chapter on the nested sequents method, the system NK can be defined as follows:

Definition 5.1.3.

$$\frac{}{\Gamma\{a, \overline{a}\}}\ \mathsf{id} \qquad \frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}\ \vee \qquad \frac{\Gamma\{A\} \qquad \Gamma\{B\}}{\Gamma\{A \wedge B\}}\ \wedge$$

$$\frac{\Gamma\{A, A\}}{\Gamma\{A\}}\ \mathsf{ctr} \qquad \frac{\Gamma\{\Delta, \Sigma\}}{\Gamma\{\Sigma, \Delta\}}\ \mathsf{exch} \qquad \frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}}\ \Box \qquad \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}\ \mathsf{k}$$

Notice that this system is the same as the first seven rules defined in Definition 4.1.5. $\Diamond$

REMARK. The system NK as that was defined by S. Marin and L. Straßburger [15] did not contain the exch-rule. However, since nested sequents are defined using sequences, the system had to be modified analogous to the modifications that R. Goetschi and R. Kuznets [12] made to the system of K. Brünnler [7] for the same purpose.

The modal $\Diamond$-rules for axioms d, t, b, 4 and 5 are defined as follows:

Definition 5.1.4.

$$\frac{\Gamma\{[A]\}}{\Gamma\{\Diamond A\}}\ \mathsf{d}^\Diamond \qquad \frac{\Gamma\{A\}}{\Gamma\{\Diamond A\}}\ \mathsf{t}^\Diamond \qquad \frac{\Gamma\{[\Delta], A\}}{\Gamma\{[\Delta, \Diamond A]\}}\ \mathsf{b}^\Diamond$$

$$\frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}\ \mathsf{4}^\Diamond \qquad \frac{\Gamma\{\emptyset\}\{\Diamond A\}}{\Gamma\{\Diamond A\}\{\emptyset\}}\ \mathsf{5}^\Diamond\ \text{depth}(\Gamma\{\}\{[\emptyset]\}) \geq 1$$

Except for the rule for axiom $\mathsf{5}^\Diamond$, the rules are the same as the rules defined in Definition 4.1.5. The names have been changed, to avoid confusion with the rules that are defined in the next section. Therefore, it follows from the paper by R. Goetschi and R. Kuznets [12], that for any of these rules (except $\mathsf{5}^\Diamond$), its shallow version is realizable into the justification logics containing its corresponding axiom. For example, the here defined rule $\mathsf{t}^\Diamond$ is realizable into the justification logic JT.

Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, then $\mathsf{X}^\Diamond$ denotes the corresponding subset of the here defined rules: $\{\mathsf{d}^\Diamond, \mathsf{t}^\Diamond, \mathsf{b}^\Diamond, \mathsf{4}^\Diamond, \mathsf{5}^\Diamond\}$. $\Diamond$

Structural Modal Rules

Besides the above given nested sequent rules, structural modal rules are defined below, to give a system that is cut-free for any of the axiomatizations of S5 and eventually to prove the realization into justification logics.

Definition 5.1.5.

$$\frac{\Gamma\{[\emptyset]\}}{\Gamma\{\emptyset\}}\ \mathsf{d}^{\Box} \qquad \frac{\Gamma\{[\Delta]\}}{\Gamma\{\Delta\}}\ \mathsf{t}^{\Box} \qquad \frac{\Gamma\{[\Sigma,[\Delta]]\}}{\Gamma\{[\Sigma],\Delta\}}\ \mathsf{b}^{\Box}$$

$$\frac{\Gamma\{[\Delta],[\Sigma]\}}{\Gamma\{[[\Delta],\Sigma]\}}\ \mathsf{4}^{\Box} \qquad \frac{\Gamma\{[\Delta]\}\{\emptyset\}}{\Gamma\{\emptyset\}\{[\Delta]\}}\ \mathsf{5}^{\Box}\ \mathrm{depth}(\Gamma\{\}\{[\Delta]\}) \geq 1$$

Let $\mathsf{X} \subseteq \{\mathsf{d}, \mathsf{t}, \mathsf{b}, \mathsf{4}, \mathsf{5}\}$, then $\mathsf{X}^{\Box}$ denotes the corresponding subset of the here defined rules: $\{\mathsf{d}^{\Box}, \mathsf{t}^{\Box}, \mathsf{b}^{\Box}, \mathsf{4}^{\Box}, \mathsf{5}^{\Box}\}$. $\Diamond$

Theorem 5.1.6. *([15]) Let $X \subseteq \{d,t,b,4,5\}$. A formula A is a theorem of K+X if and only if there is a derivation of A in the nested sequent system $NK \cup X^{\Diamond} \cup X^{\Box}$.*

The proof of this theorem can be found in the paper by S. Marin and L. Straßburger [15].

5.2 Auxiliary Lemma's

Lemma 5.2.1. *([7]) The following three rules are each subrules of the rule $5^{\Diamond}$, in terms of instances:*

$$\frac{\Gamma\{[\Delta],\Diamond A\}}{\Gamma\{[\Delta,\Diamond A]\}}\ 5a^{\Diamond} \qquad \frac{\Gamma\{[\Delta],[\Pi,\Diamond A]\}}{\Gamma\{[\Delta,\Diamond A],[\Pi]\}}\ 5b^{\Diamond} \qquad \frac{\Gamma\{[\Delta,[\Pi,\Diamond A]]\}}{\Gamma\{[\Delta,\Diamond A,[\Pi]]\}}\ 5c^{\Diamond}$$

R. Goetschi and R. Kuznets have shown that these three rules are realizable into any justification logic containing the j5 axiom [12]. From chapter 4 on the nested sequents method it follows that these rules are also realizable into the justification logics LPS5 and $\mathsf{LPS5}_c$.

Lemma 5.2.2. *([15]) The following three rules are each subrules of the rule $5^{\Box}$, in terms of instances:*

$$\frac{\Gamma\{[\Pi,[\Delta]]\}}{\Gamma\{[\Pi],[\Delta]\}}\ 5a^{\Box} \qquad \frac{\Gamma\{[\Pi,[\Delta]],[\Sigma]\}}{\Gamma\{[\Pi],[[\Delta],\Sigma]\}}\ 5b^{\Box} \qquad \frac{\Gamma\{[\Pi,[\Delta],[\Sigma]]\}}{\Gamma\{[\Pi,[[\Delta],\Sigma]]\}}\ 5c^{\Box}$$

As was already suggested above, for $\rho \in \{\mathsf{id}, \vee, \wedge, \mathsf{ctr}, \Box, \mathsf{k}, \mathsf{d}^{\Diamond}, \mathsf{t}^{\Diamond}, \mathsf{b}^{\Diamond}, \mathsf{4}^{\Diamond}, \mathsf{5a}^{\Diamond}, \mathsf{5b}^{\Diamond}, \mathsf{5c}^{\Diamond}\}$, its shallow version sh-ρ is realizable into the justification logic, which has the corresponding axiom in its system. By Lemma 4.2.5 it then follows that ρ is realizable into the justification logic as well. What about the structural modal rules?

The proof of Lemma 4.2.5 as that was given by R. Goetschi and R. Kuznets in their paper on nested sequents [12], is based on the inductive definition of a context, Definition 4.1.3. This definition does not change for the structural modal nested rules. Therefore, the proof still holds for the structural modal nested rules with one hole and for the rules $\mathsf{5a}^{\Box}$, $\mathsf{5b}^{\Box}$ and $\mathsf{5c}^{\Box}$. The following lemma can be stated for any of the rules as defined in Definitions 5.1.3, 5.1.4 and 5.1.5:

Lemma 5.2.3. *For any nested rule ρ, if its shallow version sh-ρ is realizable in a justification logic JL, then ρ itself is also realizable in JL.*

In the paper by R. Goetschi and R. Kuznets some lemma's were necessary to keep the realization of the $\mathsf{5}$-rules short. Those lemma's are used here as well, for the same reason. Notice that due to the presence of the axioms jt and $\mathsf{j4}$ in the logics $\mathsf{LPS5}$ and $\mathsf{LPS5}_c$, these lemma's were not required in the proofs of the realizations into $\mathsf{LPS5}$ and $\mathsf{LPS5}_c$. The realization of the rules of Lemma 5.2.2 has to be proven for each of the three realizations of the modal $\mathsf{5}$-axiom: $\mathsf{j5}$, A5 and A5$_c$.

Instead of considering each of these realizations, Lemma 2.2.18 will be used. The proof of this Lemma, for logics containing $\mathsf{j5}$, A5 or A5$_c$, is based only on the basic justification logic J and the realization of the $\mathsf{5}$-axiom. This means, if the realization of the shallow versions of the rules of Lemma 5.2.2 can be accomplished by considering only axioms and rules of J and Lemma 2.2.18, then the rules are realizable into J + $\mathsf{j5}$, J + A5 and J + A5$_c$.

Let $\mathcal{L} \in \{\mathsf{J} + \mathsf{j5}, \mathsf{J} + \mathrm{A5}, \mathsf{J} + \mathrm{A5}_c\}$, consider the following lemma's

Lemma 5.2.4 (Syllogism, [12]). *There exists a term $syl(x_1, x_2)$ such that for arbitrary terms t_1 and t_2 and for arbitrary justification formulas A, B and C:*

$$\mathsf{J} \vdash t_1 : (A \supset B) \supset t_2 : (B \supset C) \supset syl(t_1, t_2) : (A \supset C).$$

Lemma 5.2.5 (Internalized Factivity). *There exists a term $fact(x)$ such that for any term s and any justification formula A:*

$$\mathcal{L} \vdash fact(s) : (s : A \supset A).$$

Proof.

0. $P \supset x : P \supset P$	Propositional tautology
1. $x : P \supset t_1(x) : (x : P \supset P)$	From 0. by Corollary 2.2.3
2. $\neg x : P \supset x : P \supset P$	Propositional tautology
3. $negint(x) : \neg x : P \supset t_2(negint(x)) : (x : P \supset P)$	From 2. by Corollary 2.2.3
4. $\neg x : P \supset negint(x) : \neg x : P$	By Lemma 2.2.18
5. $\neg x : P \supset t_2(negint(x)) : (x : P \supset P)$	From 3. and 4. by prop. reasoning
6. $x : P \vee \neg x : P \supset (t_1(x) + t_2(negint(x))) : (x : P \supset P)$	From 1. and 5., using sum and prop reasoning

Let $fact(x) := (t_1(x) + t_2(negint(x)))$, then by propositional reasoning and the Substitution Lemma 2.2.13 the result follows. Notice that $fact(x)$ does neither depend on s nor on A. $\Box$

Lemma 5.2.6 (Inverse to Negative Introspection, Internalized). *There exists a term $invnegint(x)$ such that, for arbitrary terms s and t and for any justification formula A:*

$$\mathcal{L} \vdash s : \neg negint(t) : \neg t : A \supset invnegint(s) : t : A.$$

Proof.

0. $(\neg x_2 : P_1 \supset \textit{negint}(x_2) : \neg x_2 : P_1) \supset \neg\textit{negint}(x_2) : \neg x_2 : P_1 \supset x_2 : P_1$ Prop. tautology
1. $c : ((\neg x_2 : P_1 \supset \textit{negint}(x_2) : \neg x_2 : P_1) \supset \neg\textit{negint}(x_2) : \neg x_2 : P_1 \supset x_2 : P_1)$ From 0. by AN
2. $\neg x_2 : P_1 \supset \textit{negint}(x_2) : \neg x_2 : P_1$ By Lemma 2.2.18
3. $p : (\neg x_2 : P_1 \supset \textit{negint}(x_2) : \neg x_2 : P_1)$ From 2. by Lemma 2.2.1
4. $(c \cdot p) : (\neg\textit{negint}(x_2) : \neg x_2 : P_1 \supset x_2 : P_1)$ From 1. and 3. by app and MP
5. $x_1 : \neg\textit{negint}(x_2) : \neg x_2 : P_1 \supset (c \cdot p \cdot x_1) : x_2 : P_1$ From 4. by app and MP
6. $s : \neg\textit{negint}(t) : \neg t : A \supset (c \cdot p \cdot s) : t : A$ From 5. by the Substitution Lemma 2.2.13

Let $\textit{invnegint}(x) = (c \cdot p \cdot x)$, then the result follows from substituting $\textit{invnegint}(x)$. Notice that the constructed term $\textit{invnegint}(x)$ does neither depend on s, nor on t, nor on A. □

Lemma 5.2.7 (Internalized Positive Introspection). *There exist terms $\textit{posint}(x_1)$ and $t_!(x_1)$ such that for any term s and any justification formula A:*

$$\mathcal{L} \vdash \textit{posint}(s) : (s : A \supset t_!(s) : s : A).$$

Proof. Start with proving that there is a term $s(x_1)$ such that

$$\mathcal{L} \vdash s(x_1) : (P_1 \supset \textit{negint}(x_1) : \neg x_1 : \neg P_1) :$$

0. $\textit{fact}(x_1) : (x_1 : \neg P_1 \supset \neg P_1)$ Instance of Lemma 5.2.5
1. $p : ((x_1 : \neg P_1 \supset \neg P_1) \supset P_1 \supset \neg x_1 : \neg P_1)$ From prop. reasoning and Lemma 2.2.1
2. $(p \cdot \textit{fact}(x_1)) : (P_1 \supset \neg x_1 : \neg P_1)$ From 0. and 1. by app and MP
3. $t_1 : (\neg x_1 : \neg P_1 \supset \textit{negint}(x_1) : \neg x_1 : \neg P_1)$ By Lemma 2.2.18 and Lemma 2.2.1
4. $\textit{syl}(p \cdot \textit{fact}(x_1), t_1) : (P_1 \supset \textit{negint}(x_1) : \neg x_1 : \neg P_1)$ From 2. and 3. by Lemma 5.2.4

Let $s(x_1) := \textit{syl}(p \cdot \textit{fact}(x_1), t_1)$, then $s(x_1) : (P_1 \supset \textit{negint}(x_1) : \neg x_1 : \neg P_1)$ follows. With this term $s(x_1)$, the proof is then:

5. $\textit{negint}(\textit{negint}(x)) : \neg\textit{negint}(x) : \neg x : P \supset \textit{invnegint}(\textit{negint}(\textit{negint}(x))) : x : P$ Instance of Lemma 5.2.6
6. $p : (\textit{negint}(\textit{negint}(x)) : \neg\textit{negint}(x) : \neg x : P \supset \textit{invnegint}(\textit{negint}(\textit{negint}(x))) : x : P)$ From 5. by Lemma 2.2.1
7. $s(\textit{negint}(x)) : (x : P \supset \textit{negint}(\textit{negint}(x)) : \neg\textit{negint}(x) : \neg x : P)$ By derivation above
8. $\textit{syl}(s(\textit{negint}(x)), p) : (x : P \supset \textit{invnegint}(\textit{negint}(\textit{negint}(x))) : x : P)$ From 1. and 2. by Lemma 5.2.4

Let $\textit{posint}(x) := \textit{syl}(s(\textit{negint}(x)), p)$ and $t_!(x) := \textit{invnegint}(\textit{negint}(\textit{negint}(x)))$, the result follows by the Substitution Lemma 2.2.13. The constructed terms $\textit{posint}(x)$ and $t_1(x)$ neither depend on s nor on A. □

Proof of Shallow Versions

From the paper by R. Goetschi and R. Kuznets [12] it is known that the rules of Definition 5.1.3 are realizable into the basic justification logic J. From the same paper it is known that the rules of Definition 5.1.4 are realizable into justification logics, of which the axiom systems contain the axiom corresponding to the rule. Instead of realizing the $5^{\diamond}$ rule, the rules as stated in Lemma 5.2.1 are realized, using the j5-axiom. In chapter 4 it was proven that the same rules can be realized into the logics LPS5 and LPS5$_c$ as well.

It is left to prove that the rules as stated in Definition 5.1.5 can be realized into justification logics, of which the axiom systems contain the axiom that corresponds to the rule. Instead of the $5^{\Box}$ rule, the three rules as given in Lemma 5.2.2 will be realized, for each of the three realizations of the 5 axiom: j5, A5 and A5$_c$. In the realization of these three rules the Lemma's 2.2.18, 5.2.5, 5.2.6 and 5.2.7 will be used.

Lemma 5.2.8. *Let $\rho \in \{d^{\Box}, t^{\Box}, b^{\Box}, 4^{\Box}, 5a^{\Box}, 5b^{\Box}, 5c^{\Box}\}$. The shallow version of ρ is realizable in Jρ, by J$d^{\Box}$ the justification logic JD is meant and by Jρ for $\rho \in \{5a^{\Box}, 5b^{\Box}, 5c^{\Box}\}$ the logics J5, J + A5 and J + A5$_c$ are meant.*

Proof. Consider an arbitrary instance for each rule ρ in turn.

Case $\rho = \mathsf{d}^{\Box}$: Let $\dfrac{[\emptyset]}{\emptyset}$ be an arbitrary instance of sh-d$^{\Box}$, let $[\emptyset]_k$ be a properly annotated version of the premise. Then $\dfrac{[\emptyset]_k}{\emptyset}$ is an annotated version of this instance. Let r_1 be a realization function on the premise. Notice that $\emptyset^{r^*} = \bot$ for any realization r^*. Then $r_1(k) : \emptyset^{r_1} \supset \emptyset^{r_1}$ is an instance of jd and $r := r_1$ is a realization on the conclusion. For the identity substitution σ and r:

$$\mathsf{JD} \vdash ([\emptyset]_k)^{r_1}\sigma \supset (\emptyset)^r.$$

Case $\rho = \mathsf{t}^{\Box}$: Let $\dfrac{[\Delta']}{\Delta'}$ be an arbitrary instance of sh-t$^{\Box}$, let $[\Delta]_k$ and Δ be properly annotated versions of the premise and conclusion respectively. Then $\dfrac{[\Delta]_k}{\Delta}$ is an annotated version of this instance. Let r_1 be a realization function on the premise.

Consider the jt-instance $r_1(k) : \Delta^{r_1} \supset \Delta^{r_1}$. Then, $r := r_1$ is a realization on the conclusion. For the identity substitution σ and r:

$$\mathsf{JT} \vdash ([\Delta]_k)^{r_1}\sigma \supset (\Delta)^r.$$

Case $\rho = \mathsf{b}^{\Box}$: Let $\dfrac{[\Sigma', [\Delta']]}{[\Sigma'], \Delta'}$ be an arbitrary instance of sh-b$^{\Box}$, let $[\Sigma, [\Delta]_i]_k$ and $[\Sigma]_l, \Delta$ be properly annotated versions of the premise and conclusion respectively. Then $\dfrac{[\Sigma, [\Delta]_i]_k}{[\Sigma]_l, \Delta}$ is an annotated version of this instance. Let r_1 be a realization function on the premise.

0. $\Sigma^{r_1} \vee r_1(i) : \Delta^{r_1} \supset \neg r_1(i) : \Delta^{r_1} \supset \Sigma^{r_1}$ — Propositional tautology
1. $r_1(k) : (\Sigma^{r_1} \vee r_1(i) : \Delta^{r_1}) \supset$
   $qm(r_1(i)) : \neg r_1(i) : \Delta^{r_1} \supset t(r_1(k), qm(r_1(i))) : \Sigma^{r_1}$ — From 0. by Corollary 2.2.3

2. $r_1(k) : (\Sigma^{r_1} \vee r_1(i) : \Delta^{r_1}) \supset (\neg(qm(r_1(i)) : \neg r_1(i) : \Delta^{r_1}) \vee t(r_1(k), qm(r_1(i))) : \Sigma^{r_1})$
From 1. by prop. reasoning
3. $\neg\Delta^{r_1} \supset qm(r_1(i)) : \neg r_1(i) : \Delta^{r_1}$ By Lemma 2.2.15
4. $\neg qm(r_1(i)) : \neg r_1(i) : \Delta^{r_1} \supset \Delta^{r_1}$ From 3. by prop reasoning
5. $r_1(k) : (\Sigma^{r_1} \vee r_1(i) : \Delta^{r_1})$
$(t(r_1(k), qm(r_1(i))) : \Sigma^{r_1} \vee \Delta^{r_1})$ From 2. and 4. by prop. reasoning

Let $s := t(r_1(k), qm(r_1(i)))$, then it follows that

$$\mathsf{JB} \vdash r_1(k) : (\Sigma^{r_1} \vee r_1(i) : \Delta^{r_1}) \supset (s : \Sigma^{r_1} \vee \Delta^{r_1}).$$

The index l does not occur in either Σ or Δ, since $[\Sigma]_l, \Delta$ is properly annotated. Hence, the realization r, formulated as $r := (r_1 \upharpoonright \Sigma, \Delta) \cup \{l \mapsto s\}$ is a realization on $[\Sigma]_l, \Delta$. For the identity substitution σ and r:

$$\mathsf{JB} \vdash ([\Sigma, [\Delta]_i]_k)^{r_1}\sigma \supset ([\Sigma]_l, \Delta)^r.$$

Case $\rho = \mathsf{4}^{\Box}$: Let $\dfrac{[\Delta'], [\Sigma']}{[[\Delta'], \Sigma']}$ be an arbitrary instance of $\mathsf{sh}\text{-}\mathsf{4}^{\Box}$, let $[\Delta]_i, [\Sigma]_k$ and $[[\Delta]_i, \Sigma]_l$ be properly annotated versions of the premise and conclusion respectively. Then $\dfrac{[\Delta]_i, [\Sigma]_k}{[[\Delta]_i, \Sigma]_l}$ is an annotated version of this instance. Let r_1 be a realization function on the premise.

0. $r_1(i) : \Delta^{r_1} \supset r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}$ Propositional tautology
1. $!r_1(i) : r_1(i) : \Delta^{r_1} \supset t_1(!r_1(i)) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ From 0. by Corollary 2.2.3
2. $r_1(i) : \Delta^{r_1} \supset !r_1(i) : r_1(i) : \Delta^{r_1}$ Instance of $\mathsf{j4}$
3. $r_1(i) : \Delta^{r_1} \supset t_1(!r_1(i)) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ From 1. and 2. by prop. reasoning
4. $\Sigma^{r_1} \supset r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}$ Propositional tautology
5. $r_1(k) : \Sigma^{r_1} \supset t_2(r_1(k)) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ From 4. by Corollary 2.2.3
6. $r_1(i) : \Delta^{r_1} \vee r_1(k) : \Sigma^{r_1} \supset (t_1(!r_1(i)) + t_2(r_1(k))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$
From 3. and 5., using sum and prop. reasoning

Let $t := (t_1(!r_1(i)) + t_2(r_1(k)))$. Then it follows that:

$$\mathsf{J4} \vdash r_1(i) : \Delta^{r_1} \vee r_1(k) : \Sigma^{r_1} \supset t : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}).$$

The indices i and l do not occur in either $[\Delta]_i$ or Σ since $[[\Delta]_i, \Sigma]_l$ is properly annotated. Hence: the realization r, formulated as $r := (r_1 \upharpoonright [\Delta]_i, \Sigma) \cup \{l \mapsto t\}$ is a realization on $[[\Delta]_i, \Sigma]_l$. For the identity substitution σ and r:

$$\mathsf{J4} \vdash ([\Delta]_i, [\Sigma]_k)^{r_1}\sigma \supset ([[\Delta]_i, \Sigma]_l)^r.$$

For the last three cases let $\mathcal{L} \in \{\mathsf{J} + \mathsf{j5}, \mathsf{J} + \mathsf{A5}, \mathsf{J} + \mathsf{A5}_c\}$.

Case $\rho = \mathsf{5a}^{\Box}$: Let $\dfrac{[\Pi', [\Delta']]}{[\Pi'], [\Delta']}$ be an arbitrary instance of sh-$\mathsf{5a}^{\Box}$, let $[\Pi, [\Delta]_i]_k$ and $[\Pi]_l, [\Delta]_i$ be properly annotated versions of the premise and conclusion respectively. Then $\dfrac{[\Pi, [\Delta]_i]_k}{[\Pi]_l, [\Delta]_i}$ is an annotated version of this instance. Let r_1 be a realization function on the premise.

0. $(\Pi^{r_1} \vee r_1(i) : \Delta^{r_1}) \supset (\neg r_1(i) : \Delta^{r_1} \supset \Pi^{r_1})$ — Propositional tautology

1. $r_1(k) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1}) \supset \textit{negint}(r_1(i)) : \neg r_1(i) : \Delta^{r_1}$
   $\supset t(r_1(k), \textit{negint}(r_1(i))) : \Pi^{r_1}$ — From 0. by Corollary 2.2.3

2. $r_1(k) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1}) \supset (\neg(\textit{negint}(r_1(i)) : \neg r_1(i) : \Delta^{r_1})$
   $\vee\, t(r_1(k), \textit{negint}(r_1(i))) : \Pi^{r_1})$ — From 1. by prop. reasoning

3. $\neg r_1(i) : \Delta^{r_1} \supset \textit{negint}(r_1(i)) : \neg r_1(i) : \Delta^{r_1}$ — By Lemma 2.2.18

4. $\neg \textit{negint}(r_1(i)) : \neg r_1(i) : \Delta^{r_1} \supset r_1(i) : \Delta^{r_1}$ — From 3. by prop. reasoning

5. $(\neg \textit{negint}(r_1(i)) : \neg r_1(i) : \Delta^{r_1} \vee t(r_1(k), \textit{negint}(r_1(i))) : \Pi^{r_1})$
   $\supset (r_1(i) : \Delta^{r_1} \vee t(r_1(k), \textit{negint}(r_1(i))) : \Pi^{r_1})$ — From 4. by prop. reasoning

6. $r_1(k) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1})$
   $\supset (r_1(i) : \Delta^{r_1} \vee t(r_1(k), \textit{negint}(r_1(i))) : \Pi^{r_1})$ — From 2. and 5. by prop. reasoning

Let $s := t(r_1(k), \textit{negint}(r_1(i)))$. Then it follows that:

$$\mathcal{L} \vdash r_1(k) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1}) \supset (r_1(i) : \Delta^{r_1} \vee s : \Pi^{r_1}).$$

The index l does not occur in either $[\Delta]_i$ or Π, since $[\Pi]_l, [\Delta]_i$ is properly annotated. This means, the realization r: $r := (r_1 \restriction \Pi, [\Delta]_i) \cup \{l \mapsto s\}$ is a realization on $[\Pi]_l, [\Delta]_i$. For the identity substitution σ and r:

$$\mathcal{L} \vdash ([\Pi, [\Delta]_i]_k)^{r_1}\sigma \supset ([\Pi]_l, [\Delta]_i)^r.$$

Case $\rho = \mathsf{5b}^{\Box}$: Let $\dfrac{[\Pi', [\Delta']], [\Sigma']}{[\Pi'], [[\Delta'], \Sigma]'}$ be an arbitrary instance of sh-$\mathsf{5b}^{\Box}$, let $[\Pi, [\Delta]_i]_j, [\Sigma]_h$ and $[\Pi]_k, [[\Delta]_i, \Sigma]_l$ be properly annotated versions of the premise and conclusion respectively. Then $\dfrac{[\Pi, [\Delta]_i]_j, [\Sigma]_h}{[\Pi]_k, [[\Delta]_i, \Sigma]_l}$ is an annotated version of this instance. Let r_1 be a realization function on the premise.

0. $\neg r_1(i) : \Delta^{r_1} \supset \textit{negint}(r_1(i)) : \neg r_1(i) : \Delta^{r_1}$ — By Lemma 2.2.18

1. $\neg \textit{negint}(r_1(i)) : \neg r_1(i) : \Delta^{r_1} \supset \neg\neg r_1(i) : \Delta^{r_1}$ — From 0. by prop. reasoning

2. $\neg\neg r_1(i) : \Delta^{r_1} \supset r_1(i) : \Delta^{r_1}$ — Propositional tautology

3. $\neg \textit{negint}(r_1(i)) : \neg r_1(i) : \Delta^{r_1} \supset r_1(i) : \Delta^{r_1}$ — From 1. and 2. by prop. reasoning

4. $p : (\neg \textit{negint}(r_1(i)) : \neg r_1(i) : \Delta^{r_1} \supset r_1(i) : \Delta^{r_1})$ — From 3. by Lemma 2.2.1

5. $\neg \textit{negint}(r_1(i)) : \neg r_1(i) : \Delta^{r_1} \supset \textit{negint}(\textit{negint}(r_1(i))) : \neg \textit{negint}(r_1(i)) : \neg r_1(i) : \Delta^{r_1}$
   — By Lemma 2.2.18

6. $\textit{negint}(\textit{negint}(r_1(i))) : \neg \textit{negint}(r_1(i)) : \neg r_1(i) : \Delta^{r_1}$
   $\supset p \cdot \textit{negint}(\textit{negint}(r_1(i))) : r_1(i) : \Delta^{r_1}$ — From 4. by app and Modus Ponens

7. $\neg \textit{negint}(r_1(i)) : \neg r_1(i) : \Delta^{r_1} \supset p \cdot \textit{negint}(\textit{negint}(r_1(i))) : r_1(i) : \Delta^{r_1}$
   — From 5. and 6. by prop. reasoning

8. $\neg r_1(i) : \Delta^{r_1} \supset \Pi^{r_1} \vee r_1(i) : \Delta^{r_1} \supset \Pi^{r_1}$ — Propositional tautology

9. $\textit{negint}(r_1(i)) : \neg r_1(i) : \Delta^{r_1} \supset r_1(j) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1})$
   $\supset t_3(\textit{negint}(r_1(i)), r_1(j)) : \Pi^{r_1}$ — From 8. by Corollary 2.2.3

10. $r_1(j) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1})$
   $\supset \neg \textit{negint}(r_1(i)) : \neg r_1(i) : \Delta^{r_1} \vee t_3(\textit{negint}(r_1(i)), r_1(j)) : \Pi^{r_1}$ — From 9. by pr. reas.

Let $s' := p \cdot \textit{negint}(\textit{negint}(r_1(i)))$ and $s := t_3(\textit{negint}(r_1(i)), r_1(j))$, then:

11. $r_1(j) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1}) \supset s' : r_1(i) : \Delta^{r_1} \vee s : \Pi^{r_1}$ — From 7. and 10. by prop. reasoning
12. $r_1(i) : \Delta^{r_1} \supset (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ — Propositional tautology
13. $s' : r_1(i) : \Delta^{r_1} \supset t_1(s') : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ — From 12. by Corollary 2.2.3
14. $r_1(j) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1}) \supset t_1(s') : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}) \vee s : \Pi^{r_1}$ — From 11. and 13. by prop. reasoning
15. $\Sigma^{r_1} \supset (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ — Propositional tautology
16. $r_1(h) : \Sigma^{r_1} \supset t_2(r_1(h)) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ — From 15. by Corollary 2.2.3
17. $r_1(j) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1}) \vee r_1(h) : \Sigma^{r_1} \supset s : \Pi^{r_1} \vee t_1(s') : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}) \vee t_2(r_1(h)) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ — From 14. and 16. by prop. reasoning
18. $t_1(s') : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}) \vee t_2(r_1(h)) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}) \supset (t_1(s') + t_2(r_1(h))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ — By sum and prop. reasoning
19. $r_1(j) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1}) \vee r_1(h) : \Sigma^{r_1} \supset s : \Pi^{r_1} \vee (t_1(s') + t_2(r_1(h))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ — From 17. and 18. by prop. reas.

Let $t := (t_1(s') + t_2(r_1(h)))$, then, from 19. and this substitution:

$$\mathcal{L} \vdash r_1(j) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1}) \vee r_1(h) : \Sigma^{r_1} \supset s : \Pi^{r_1} \vee t : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}).$$

The indices k and l do not occur in Π, $[\Delta]_i$ or Σ, since $[\Pi]_k, [[\Delta]_i, \Sigma]_l$ is properly annotated. This means, the realization $r := (r_1 \restriction \Pi, [\Delta]_i, \Sigma) \cup \{k \mapsto s, l \mapsto t\}$ is a realization on $[\Pi]_k, [[\Delta]_i, \Sigma]_l$. For the identity substitution σ and r:

$$\mathcal{L} \vdash ([\Pi, [\Delta]_i]_j, [\Sigma]_h)^{r_1}\sigma \supset ([\Pi]_k, [[\Delta]_i, \Sigma]_l)^r.$$

Case $\rho = \mathsf{5c}^{[]}$: Let $\dfrac{[\Pi', [\Delta'], [\Sigma']]}{[\Pi', [[\Delta'], \Sigma']]}$ be an arbitrary instance of sh-$\mathsf{5c}^{[]}$, let $[\Pi, [\Delta]_i, [\Sigma]_j]_h$ and $[\Pi, [[\Delta]_i, \Sigma]_k]_l$ be properly annotated versions of the premise and conclusion respectively. Then $\dfrac{[\Pi, [\Delta]_i, [\Sigma]_j]_h}{[\Pi, [[\Delta]_i, \Sigma]_k]_l}$ is an annotated version of this instance. Let r_1 be a realization function on the premise.

0. $\Sigma^{r_1} \supset r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}$ — Propositional tautology
1. $r_1(j) : \Sigma^{r_1} \supset t_1(r_1(j)) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ — From 0. by Corollary 2.2.3
2. $\textit{posint}(r_1(i)) : (r_1(i) : \Delta^{r_1} \supset t_!(r_1(i)) : r_1(i) : \Delta^{r_1})$ — By Lemma 5.2.7
3. $r_1(i) : \Delta^{r_1} \supset r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}$ — Propositional tautology
4. $t_!(r_1(i)) : r_1(i) : \Delta^{r_1} \supset t_4(t_!(r_1(i))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ — From 3. by Corollary 2.2.3
5. $(r_1(i) : \Delta^{r_1} \supset t_!(r_1(i)) : r_1(i) : \Delta^{r_1}) \supset (r_1(i) : \Delta^{r_1} \supset t_4(t_!(r_1(i))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}))$ — From 4. by prop. reasoning
6. $\textit{posint}(r_1(i)) : (r_1(i) : \Delta^{r_1} \supset t_!(r_1(i)) : r_1(i) : \Delta^{r_1}) \supset t_5(\textit{posint}(r_1(i))) : (r_1(i) : \Delta^{r_1} \supset t_4(t_!(r_1(i))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}))$ — From 5. by Corollary 2.2.3
7. $t_5(\textit{posint}(r_1(i))) : (r_1(i) : \Delta^{r_1} \supset t_4(t_!(r_1(i))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}))$ — From 2. and 6. by MP

8. $(r_1(i) : \Delta^{r_1} \supset t_4(t_!(r_1(i))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})) \supset (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1} \vee r_1(j) : \Sigma^{r_1} \supset \Pi^{r_1} \vee t_4(t_!(r_1(i))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}) \vee t_1(r_1(j)) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}))$
Propositional tautology, from 1.
9. $t_4(t_!(r_1(i))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}) \vee t_1(r_1(j)) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}) \supset (t_4(t_!(r_1(i))) + t_1(r_1(j))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})$ By sum and prop. reasoning
10. $(r_1(i) : \Delta^{r_1} \supset t_4(t_!(r_1(i))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})) \supset (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1} \vee r_1(j) : \Sigma^{r_1} \supset \Pi^{r_1} \vee (t_4(t_!(r_1(i))) + t_1(r_1(j))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}))$ From 8. and 9. by prop. reasoning
11. $t_5(\mathit{posint}(r_1(i))) : (r_1(i) : \Delta^{r_1} \supset t_4(t_!(r_1(i))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})) \supset t_6(t_5(\mathit{posint}(r_1(i)))) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1} \vee r_1(j) : \Sigma^{r_1} \supset (\Pi^{r_1} \vee (t_4(t_!(r_1(i))) + t_1(r_1(j))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})))$ From 10. by Corollary 2.2.3
12. $t_6(t_5(\mathit{posint}(r_1(i)))) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1} \vee r_1(j) : \Sigma^{r_1} \supset \Pi^{r_1} \vee (t_4(t_!(r_1(i))) + t_1(r_1(j))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}))$ From 7. and 11. by MP
13. $r_1(h) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1} \vee r_1(j) : \Sigma^{r_1}) \supset (t_6(t_5(\mathit{posint}(r_1(i)))) \cdot r_1(h)) : (\Pi^{r_1} \vee (t_4(t_!(r_1(i))) + t_1(r_1(j))) : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1}))$ From 12. by app and MP

Let $s := t_4(t_!(r_1(i))) + t_1(r_1(j))$ and $t := t_6(t_5(\mathit{posint}(r_1(i)))) \cdot r_1(h)$, from 13. and these substitutions it then follows that:

$$\mathcal{L} \vdash r_1(h) : (\Pi^{r_1} \vee r_1(i) : \Delta^{r_1} \vee r_1(j) : \Sigma^{r_1}) \supset t : (\Pi^{r_1} \vee s : (r_1(i) : \Delta^{r_1} \vee \Sigma^{r_1})).$$

The indices k and l do not occur in Π, $[\Delta]_i$ or Σ, since $[\Pi, [[\Delta]_i, \Sigma]_k]_l$ is properly annotated. The realization r, defined as: $r := (r_1 \upharpoonright \Pi, [\Delta]_i, \Sigma) \cup \{k \mapsto s, l \mapsto t\}$, is a realization on the conclusion: $[\Pi, [[\Delta]_i, \Sigma]_k]_l$. For the identity substitution σ and r:

$$\mathcal{L} \vdash ([\Pi, [\Delta]_i, [\Sigma]_j]_h)^{r_1}\sigma \supset ([\Pi, [[\Delta]_i, \Sigma]_k]_l)^r.$$

This concludes the proof that the shallow versions of the rules of Definition 5.1.5 are realizable into the justification logics containing the corresponding axiom. □

The following two corollaries are based this lemma. The first uses Lemma 5.2.3 as well. The second combines the lemma above with the results that can be found in the paper by R. Goetschi and R. Kuznets [12] and in Chapter 4.

Corollary 5.2.9. *Any of the rules that can be found in Definition 5.1.5 can be realized into the justification logics whose axiom systems contain the corresponding axiom(s).*

Corollary 5.2.10.

- *Any of the shallow versions of the rules from Definition 5.1.3 can be realized into the basic justification logic J.*
- *Any of the shallow versions of the rules from Definition 5.1.4 can be realized into the justification logic of which the axiom system contains the corresponding axiom. Furthermore, the $5^{\diamond}$-rule is also realizable into LPS5 and $LPS5_c$.*
- *Any of the shallow versions of the rules from Definition 5.1.5 can be realized into the justification logic of which the axiom system contains the corresponding axiom. Furthermore, the rule $5^{\square}$ is realizable into (any extensions of) J+j5, J+A5 and J + $A5_c$.*

5.3 The Realization Theorem

Based on Theorem 5.1.6 and Corollary5.2.10, the realization of S5 into any of the ten considered justification counterparts can be proven. The proof is very similar to the proof of Theorem 4.4.2.

Theorem 5.3.1. *Let $\mathcal{L} \in \{$LPS5, LPS5$_c$, JT45, JT5, JTB4, JDB4, JDB5, JTB5, JTB45, JDB45$\}$, then $\mathcal{L}^\circ =$ S5. Moreover, for each $A' \in$ S5 there exists a properly annotated version A of it and a realization function r on A such that $\mathcal{L} \vdash A^r$.*

Proof. Since it is known that any of the considered sets K+X is an axiomatization of S5 and by Theorem 5.1.6, it follows that each of the considered modular nested sequent systems NK$\cup$X$^{\Diamond}\cup$X$^{\Box}$ is complete for S5.

As often, to be able to prove the equality $\mathcal{L}^\circ =$ S5, two inclusions will be proven. The inclusion $\mathcal{L}^\circ \subseteq$ S5 is the easiest inclusion to prove, it is based on the forgetful projection of the rules in $\mathcal{L}$. Since it is possible to derive the forgetful projection of any of the axioms of $\mathcal{L}$ and the forgetful projections of all the rules of $\mathcal{L}$ are derivable in S5, it follows that $\mathcal{L}^\circ \subseteq$ S5.

Now let $A' \in$ S5 be some modal formula, then S5 $\vdash A'$. By the above mentioned completeness, it follows that there is a derivation of A' in the modular nested sequent calculus NK$\cup$X$^{\Diamond}\cup$X$^{\Box}$ that belongs to the corresponding axiomatization of the modal logic S5. Then, by Corollary 5.2.10 and Theorem 4.2.6, there is a properly annotated version A of A' and a realization function r on this A such that $\mathcal{L} \vdash A^r$. Notice that $(A^r)^\circ = A'$, from which it follows that $A' \in \mathcal{L}^\circ$.

Therefore, for any $\mathcal{L} \in \{$LPS5, LPS5$_c$, JT45, JT5, JTB4, JDB4, JDB5, JTB5, JTB45, JDB45$\}$, S5 is realizable into $\mathcal{L}$. $\square$

Based on the proofs that can be found in the paper by R. Goetschi and R. Kuznets [12] and in this chapter a more general theorem can be proven as well.

Theorem 5.3.2. *Any modal logic that can be found in the modal cube, can be realized into its justification counterpart(s) using Straßburger's modular nested sequent calculi. Moreover, there is a direct and constructive method to realize any modal axiomatization into any potential justification counterpart.*

The proof is based on the combination of the Completeness Theorem 5.1.6, Theorem 4.2.6 and the fact that the shallow version of any of the rules that are part of Straßburger's modular nested sequent calculi, are realizable into any justification logic of which the axiom system contains the corresponding axiom.

Conclusion

In the main part of this thesis ten justification logics were considered: LPS5, LPS5$_c$, JT45, JT5, JTB4, JTB5, JDB4, JDB5, JTB45 and JDB45 and three different realizations of the negative introspection axiom. In different papers, using different syntactic methods, it was already shown that S5 is realizable into four of these logics. Part of the goal of the thesis, as stated in the introduction, was to use a syntactic method to prove the realization of S5 into the remaining six justification logics. As was already expected, the following theorem has been proven in this thesis:

Theorem *Let A be a formula and let r be a realization function on A, then:*

$$\mathsf{S5} \vdash A \Leftrightarrow \mathcal{L} \vdash A^r$$

by:

- *using the cut-free hypersequent system LS5, for each $\mathcal{L} \in \{$LPS5, LPS5$_c$, JT45, JT5, JTB4, JTB5, JDB4, JDB5, JTB45, JDB45$\}$;*
- *using nested sequent systems, for $\mathcal{L} \in \{$LPS5, LSP5$_c$, JT45, JTB45$\}$;*
- *using Straßburger's modular nested sequents, for each $\mathcal{L} \in \{$ LPS5, LPS5$_c$, JT45, JT5, JTB4, JTB5, JDB4, JDB5, JTB45, JDB45$\}$.*

In fact, not only this theorem has been proven, but the realization into some other justification logics as well. Let $5^* \in \{$j5, A5, A5$_c\}$ then, because of the proof of Lemma 2.2.18, using the hypersequent system LS5 and Straßburger's modular nested sequents, S5 is realizable into logics with one of the following axiom systems: J+jt+j4+5*, J+jt+5*, J+jt+jb+j4, J+jt+jb+5*, J+jd+jb+j4, J+jd+jb+5*, J+jt+jb+j4+5* and J+jd+jb+j4+5*. This means that, using these two methods, S5 can be realized into twenty justification counterparts.

Another goal of the thesis was showing that any cut-free proof system works for every existing realization. This part of the goal has not been achieved. As was already described in the introduction, to be able to prove realization into any of the ten considered justification logics using the nested sequent calculi that were used by R. Goetschi and R. Kuznets [12], the realization of each of the five modal rules into each of the ten logics has to be proven. For two logics this is already done, but that still left forty cases to prove. Because a similar amount of cases had to be proven for the realization proof using the hypersequent system LS5, only the logics with another realization of the modal axiom 5 were considered. We are confident that applying this method to the realization into the other justification counterparts is straightforward.

On the other hand, the rules of the nested sequent calculi of Chapter 4 have been used in Straßburger's modular nested sequent calculi. With these calculi a modular realization of S5 into any of the ten considered justification logics could be proven, something that was not possible before.

Even more can be concluded, the fully modular realization using Straßburger's calculi in Chapter 5, completes the project of realizing modal logics from the modal cube. A direct and constructive method of realizing any axiomatization into any potential justification counterpart has been provided this way.

REMARK. Although it is not truly related to realization, the first chapter of the thesis includes the first semantics for justification logics that contain the jb axiom. Soundness and completeness have been established in the appendix.

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APPENDIX A

Soundness and Completeness

For the whole appendix, assume that JL is one of the considered logics in Table 1.1.

Theorem 1.2.8. *For any justification logic* JL *a formula* A *is provable in* JL *if and only if it is* JL*-valid:*

$$\mathsf{JL} \vdash A \Leftrightarrow A \textit{ is } \mathsf{JL}\textit{-valid}.$$

A proof of this theorem for J and for justification logics that extend J with one or more of the axioms jt, jd, j4 and j5 can be found in the PhD thesis of R. Kuznets [13]. The soundness and completeness for these logics will be assumed and only the axioms jb, A5 and A5$_c$ will be considered. The proof will follow the structure of the proof in [13] and will sometimes omit details and proofs of smaller lemma's and theorems, that can be found there.

A.1 Auxiliary Definitions and Lemma's

To prove completeness some auxiliary definitions and lemma's based on the PhD thesis of R. Kuznets [13], are required. Proofs of lemma's that are not proven here, can be found in the same thesis. One of the important notions in this proof is that of consistency:

Theorem A.1.1. *Any of the considered logics* JL *is consistent.*

The proof of this theorem requires the realization of modal logic into JL. Since this is being established syntactically for any of the not yet considered JL's in this thesis, the proof as given by R. Kuznets [13] applies here as well.

Definition A.1.2. A set of justification formulas Γ is called *JL-consistent* if, for any finite subset $\{A_1, \ldots, A_n\} \subseteq \Gamma$, $\neg(A_1 \wedge \ldots \wedge A_n) \notin \mathsf{JL}$.

Such a set Γ is called maximal JL-consistent if Γ is JL-consistent and there is no set Δ such that $\Delta \supsetneq \Gamma$ ♢

Some properties of maximal consistent sets that are important in proving the completeness are listed in the following lemma:

Lemma A.1.3. *Let Γ be an arbitrary maximal* JL*-consistent set.*

1. *For any formula A, exactly one of the formulas A and $\neg A$ is part of Γ.*
2. *The set Γ is closed under Modus Ponens.*
3. $\mathsf{JL} \subseteq \Gamma$.
4. *If $A \notin \mathsf{JL}$, then the set $\{\neg A\}$ is* JL*-consistent.*
5. *For each* JL*-consistent set Δ, there exists a maximal* JL*-consistent set Δ' sucht that $\Delta' \supseteq \Delta$.*

Definition A.1.4. Let Γ be a set of justification formulas, then:

$$\Gamma^{\#} = \{A \mid t : A \in \Gamma \text{ for some term } t\}.$$

$\diamondsuit$

With this definition the canonical justification model can be defined:

Definition A.1.5. As with a justification model, the canonical justification model is defined as a tuple $\mathcal{M} = \langle W, R, \mathcal{E}, V \rangle$. Let Γ and Δ be sets of justification formulas, the canonical justification model has the following properties:

- $W = \{\Gamma \mid \Gamma \text{ is a maximal } \mathsf{JL}\text{-consistent set}\}$
- $\Gamma R \Delta$ if and only if $\Gamma^{\#} \subseteq \Delta$
- $V(p) = \{\Gamma \in W \mid p \in \Gamma\}$
- $\mathcal{E}(t, A) = \{\Gamma \in W \mid t : A \in \Gamma\}$

$\diamondsuit$

The following lemma can be proven for any justification logic, therefore, the proof is omitted here.

Lemma A.1.6. *[Truth Lemma] In canonical justification models as defined above in definition A.1.5, the following holds for any world $w \in W$ and any justification formula A:*

$$\mathcal{M}, w \Vdash A \textit{ if and only if } A \in w.$$

It has to be proven that the constructed canonical justification model is a justification model, this is done in the proof of the following lemma:

Lemma A.1.7. *The canonical justification model as defined in Definition A.1.5 is a justification model as defined in Definition 1.2.2 for* JL*.*

The proof of the lemma consists of many cases each of which proves some property of the model. Most of these cases are already considered by R. Kuznets [13]. Here it will only be proven that the accessibility relation R, as defined in Definition A.1.5, is symmetric in the presence of the jb axiom and that $\mathcal{E}$, as defined in the same definition, is an evidence function when one or more of the axioms jb, A5 and A5$_c$ are part of the axiom system.

Proof. Let $\mathcal{M} = \langle W, R, \mathcal{E}, V \rangle$ be a canonical justification model.

First it has to be proven that the accessibility relation R as defined in Definition A.1.5 is symmetric in the presence of the jb-axiom. This means: if $\Gamma R \Delta$, then $\Delta R \Gamma$ as well for any maximal JL-consistent sets Γ and Δ.

Assume without loss of generality that $\Gamma R \Delta$ and assume towards a contradiction that $\Delta \Vdash t : A$, but that $\Gamma \Vdash \neg A$. The, by the Truth Lemma A.1.6, it follows that $t : A \in \Delta$ and $\neg A \in \Gamma$. By propositional reasoning and Corollary 2.2.3 it follows that there is a term t' such that: $\mathsf{JL} \vdash t : A \supset t' : \neg\neg A$. Since Δ is maximal JL-consistent it follows that $t : A \supset t' : \neg\neg A \in \Delta$ and by the Modus Ponens closure as stated in Lemma A.1.3: $t' : \neg\neg A \in \Delta$.

Since the jb-axiom is present, it follows that $\mathsf{JL} \Vdash \neg A \supset \bar{?}t : \neg t : \neg\neg A$ for any term t, thus also for the term t'. By Lemma A.1.3: $\neg A \supset \bar{?}t' : \neg t' : \neg\neg A \in \Gamma$, since $\neg A \in \Gamma$, by the Modus Ponens closure by Lemma A.1.3: $\bar{?}t' : \neg t' : \neg\neg A \in \Gamma$. Since $\Gamma^{\#} \subseteq \Delta$, it follows that $\neg t' : \neg\neg A \in \Delta$, which is in contradiction with the maximal JL-consistency of Δ and the above established result that $t' : \neg\neg A \in \Delta$. From which it follows that $A \in \Gamma$ and hence that $\Delta R \Gamma$ as well.

For each of the new defined properties on the evidence function $\mathcal{E}$ it has to be proven that $\mathcal{E}$, as defined in Definition A.1.5, is indeed an evidence function.

- The A5 closure property: $\mathcal{M}, w \Vdash A$ and $w \in \mathcal{E}(t, (A \supset \neg s : B))$ implies $w \in \mathcal{E}(?t, \neg s : B)$.
  Let $\Gamma \Vdash A$ and $\Gamma \in \mathcal{E}(t, A \supset \neg s : B)$, then by the Truth Lemma A.1.6 it follows that $A \in \Gamma$ and by Definition A.1.5 it follows that $t : (A \supset \neg s : B) \in \Gamma$. The axiom A5 is part of any logic for which this property is considered, hence by Lemma A.1.3 $t : (A \supset \neg s : B) \supset (A \supset ?t : \neg s : B) \in \Gamma$. Applying twice that Γ is closed by Modus Ponens (Lemma A.1.3), it follows that $?t : \neg s : B \in \Gamma$. From Definition A.1.5 it then follows that $\Gamma \in \mathcal{E}(?t, \neg s : B)$.
- The A5$_c$ closure property: $[\mathcal{E}(t, A)]^c \subseteq \mathcal{E}(c, \neg t : A)$.
  Assume $\Gamma \notin \mathcal{E}(t, A)$. Then by Definition A.1.5, it follows that $t : A \notin \Gamma$ and since Γ is maximal JL-consistent, by Lemma A.1.3: $\neg t : A \in \Gamma$. Because the axiom A5$_c$ is part of the axiom system of any of the logics for which this property of the evidence function applies, it follows by Lemma A.1.3 that $\neg t : A \supset c : \neg t : A \in \Gamma$. Applying Modus Ponens, by Lemma A.1.3 it follows that $c : \neg t : A \in \Gamma$. By Definition A.1.5 it can then be concluded that $\Gamma \in \mathcal{E}(c, \neg t : A)$.
- The symmetry closure property: $\mathcal{M}, w \Vdash A$ implies $w \in \mathcal{E}(\bar{?}t, \neg t : \neg A)$.
  Assume $\Gamma \Vdash A$, then by the Truth Lemma A.1.6 it follows that $A \in \Gamma$. Since the axiom jb is part of the axiom system of any logic for which this property is considered, by Lemma A.1.3, $A \supset \bar{?}t : \neg t : \neg A \in \Gamma$ for any term t. Applying Modus Ponens (by Lemma A.1.3), it follows that $\bar{?}t : \neg t : \neg A \in \Gamma$. By Definition A.1.5, it follows that $\Gamma \in \mathcal{E}(\bar{?}t, \neg t : \neg A)$.

That the strong evidence property holds for any canonical justification model has been proven by R. Kuznets [13]. □

A.2 Proof of Theorem 1.2.8

With these lemma's, soundness and completeness (Theorem 1.2.8) can be proven:

Proof. Without loss of generality, assume $\mathcal{M} = \langle W, R, \mathcal{E}, V \rangle$ is a justification model. Both directions of the theorem statement have to be considered.

- $\Rightarrow$ Soundness can be proven by induction on the derivation in the logic JL. From the PhD thesis of R. Kuznets [13] it can already be concluded that each of the axioms of the basic justification logic J and the axioms jt, jd, $\mathsf{j4}$ and $\mathsf{j5}$ are valid for the justification logics whose axiom systems contain the corresponding axioms and that the Modus Ponens and Axiom Necessitation rules are admissible in any justification logic JL. It is left to prove that the axioms A5, A5$_c$ and jb are valid in justification logics which axiom systems contain these axioms:
  - The axiom A5. To prove validity of the A5 axiom, assume $\mathcal{M}, w \Vdash A$ and $\mathcal{M}, w \Vdash t : (A \supset \neg s : B)$, to show that $\mathcal{M}, w \Vdash ?t : \neg s : B$. Any model for the logics of which the axiom system contains the A5 axiom, satisfies the A5 closure property and the strong evidence property.
    From the assumption and Definition 1.2.3 it follows that $\mathcal{M}, w \Vdash A$ and $w \in \mathcal{E}(t, (A \supset \neg s : B))$. By the A5 closure property: $w \in \mathcal{E}(?t, \neg s : B)$. Using the strong evidence property it follows that $\mathcal{M}, w \Vdash ?t : \neg s : B$.
  - The axiom A5$_c$. Let $\mathcal{M}, w \Vdash \neg t : A$, this means that $\mathcal{M}, w \nVdash t : A$. It has to be shown that $\mathcal{M}, w \Vdash c : \neg t : A$. Any model for the logics of which the axiom system contains the A5$_c$ axiom, satisfies the A5$_c$ closure property and the strong evidence property.
    By the strong evidence property it follows from the assumption that $w \notin \mathcal{E}(t, A)$. By the A5$_c$ closure property: $w \in \mathcal{E}(c, \neg t : A)$. Using the strong evidence property again it follows that $\mathcal{M}, w \Vdash c : \neg t : A$.
  - The axiom jb. Let $\mathcal{M}, w \Vdash A$, to show validity, it has to be shown that $\mathcal{M}, w \Vdash \bar{?}t : \neg t : \neg A$, for any term t. Any model for the logics of which the axiom system contains the jb axiom, satisfies the symmetry closure property and the strong evidence property.
    By symmetry it follows that for any $v \in W$ such that wRv: $\mathcal{M}, v \Vdash \neg t : \neg A$, for any term t. By the symmetry closure property it follows, from the assumption, that $w \in \mathcal{E}(\bar{?}t, \neg t : \neg A)$. Since for all $v \in W$ such that wRv it holds that $\mathcal{M}, v \Vdash \neg t : \neg A$ and $w \in \mathcal{E}(\bar{?}t, \neg t : \neg A)$ for any term t, it follows by Definition 1.2.3 that $\mathcal{M}, w \Vdash \bar{?}t : \neg t : \neg A$.
- $\Leftarrow$ The canonical justification model $\mathcal{M} = \langle W, R, \mathcal{E}, V \rangle$ as constructed in Definition A.1.5 for logic JL is sufficient to refute all justification formulas A such that $\mathsf{JL} \nvdash A$. From Lemma A.1.7 it is known that $\mathcal{M}$ is a justification model for JL.

  Consider any justification formula A such that $\mathsf{JL} \nvdash A$. By Theorem A.1.1 it follows that JL is consistent. Then by Lemma A.1.3, it follows that the set $\{\neg A\}$ is JL-consistent. By Lemma A.1.3 it follows that this set can be extended to a maximal JL-consistent set Δ, such that $\neg A \in \Delta$. Then by the Truth Lemma A.1.6: $\mathcal{M}, \Delta \Vdash \neg A$ and hence $\mathcal{M}, \Delta \nVdash A$.

Both directions have been proven, from which it follows that Theorem 1.2.8 has been proven. $\square$