INSTITUT FÜR HÖHERE STUDIEN
INSTITUTE FOR ADVANCED STUDIES
Vienna

CONTINUING EDUCATION CENTER

MSc Economics

The Strategic Equivalence of Extensive Form Games Revisited

A Master's Thesis submitted for the degree of
"Master of Science"

supervised by
Klaus Ritzberger

Bernhard Kasberger

0750972

Vienna, 8 June 2013

MSc Economics

Affidavit

I, Bernhard Kasberger

hereby declare

that I am the sole author of the present Master's Thesis,
The Strategic Equivalence of Extensive Form Games Revisited

30 pages, bound, and that I have not used any source or tool other than those referenced or any other illicit aid or tool, and that I have not prior to this date submitted this Master's Thesis as an examination paper in any form in Austria or abroad.

Vienna, 8 June 2013

Signature

Contents

List of Figures

Abstract

Non-cooperative game theory deals with games with complete rules. Most solution concepts in non-cooperative game theory are defined on the normal form. It is ex ante not clear whether a strategic situation is non-cooperative and thus can be analyzed as a normal form game. The means to verify the completeness of the rules is to write the game in extensive form. However, many extensive form games have the same semi-reduced normal form. This paper characterizes the class of finite extensive form games with perfect recall and the same semi-reduced normal form by replicating the work of Elmes and Reny (1994) in a theoretical framework developed by Alós-Ferrer and Ritzberger. The necessary and sufficient conditions are three transformations of the extensive form. These are embedded in the space of extensive form games and an order relation is established.

1 Introduction

Imagine a situation in which there are two cars on a winding road, going in opposing directions and the drivers are unable to communicate with each other and want to avoid a crash. At some point in time, the cars meet and crash if one car is on the right side and the other car is on the left side of the road. How should one model this situation? One way is to write down precisely when a player has to decide between which alternatives and what he knows at this point in time. For example, one could say that Player A starts first, but Player B does not learn A's choice. Or, Player B could start first. We can also imagine a more elaborate description of the situation. Let Player A start first, and let Player B not learn A's choice. After choosing the lane, Player B has the choice to turn on the radio. Of course, Player B is allowed to remember his first choice. Since the crash happens if and only if both drive on the same lane, both players are indifferent between Player B listening to the radio and Player B not listening to the radio. The specific situation analyzed can become quite complicated. But basically, each driver has to decide between the left and the right lane.

Situations like this can be well analyzed with non-cooperative game theory. Non-cooperative game theory is game theory with complete rules. The rules of a game are given in the extensive form. This corresponds to writing down when a player has to decide between which alternatives and what he knows when the decision is made. Together with a payoff function, an extensive form is an extensive form game. Given the rules of a game, players can derive strategies from it. A game in normal form are strategies and a payoff function. In our example, in the first two cases, players A and B have the strategies of 'going right' and 'going left' available. If Player B has the option of turing on the radio, he can go 'left with radio', 'left without radio', 'right with radio' and 'right without radio'. It can be the case, that some strategy combinations of a player are equivalent in the sense that they yield, whatever the other players do, the same payoff for all players. For example, it doesn't matter whether Player B turns on the radio or not. Eliminating all redundant strategies gives rise to the semi-reduced (or pure-strategy-reduced) normal form. One can argue that for rational players, it does only matter which payoff they get in the end. The specific strategy combination does not matter since the preferences over outcomes are represented by the payoff function. Kohlberg and Mertens (1986) go further and argue that the (mixed-strategy) reduced normal form is the correct object for solution theory. Most equilibrium concepts in non-cooperative game theory are defined on the normal form. One example is the Nash equilibrium and refinements of it (e.g. Myerson's

(1978) proper equilibrium). Two related questions are immediate. First, given a strategic situation, how can one verify that the rules are complete and thus that the situation can be analyzed with the tools of non-cooperative game theory? Second, how can one modify the rules of an extensive form game without changing the strategies essentially available to the players?

One can find answers to these questions in the early days of game theory (Krentel, McKinsey, and Quine, 1951; Thompson, 1952; Dalkey, 1953). Back then, the task was mainly to simplify a normal form game derived from an extensive form game. Thompson (1952) proposed four transformations that leave the strategic features of a game unchanged. However, two of his transformations take a game with perfect recall and produce a game without perfect recall. A game has perfect recall if the rules of the game do not force players to forget things they knew or did. Elmes and Reny (1994) take the remaining two transformations and modify Thompson's 'Addition'. Furthermore, they show that two finite extensive form games are strategically equivalent if and only if they differ only by the three transformations. In this paper we replicate their work in a theoretical framework developed by Alós-Ferrer and Ritzberger.

Von Neumann and Morgenstern (1944, Chapter 2) develop a set theoretic representation of extensive form games. However, since Kuhn (1950) extensive forms have usually been formalized as a directed graph with certain desired features. Alós-Ferrer and Ritzberger (2005; 2008; 2011; 2013) develop a framework that formalizes the game in set theoretic terms. Outcomes are seen as the primitive of the game. An outcome, or a play, represents an entire and thus unique history of action in the course of the game. Furthermore, outcomes are the domain of the decision makers's preferences.

The next section collects necessary definitions. Furthermore, the space of extensive form games is introduced and two binary relations are defined on it. The first one is a partial order, the second is an equivalence relation. Section 3 reformulates and generalizes the three transformations of Elmes and Reny (1994). Section 4 provides the main theorem, the last section concludes.

2 Preliminaries

The basic ingredient of an extensive form is a tree. We will work with Definition 3.1 in Ritzberger (2002).

Definition 1. A (game) tree T is a pair $T = (W, N)$, where W is a set of outcomes or plays and N is a collection of nonempty subsets of W, called nodes, such that $\{w\} \in N$ for all $w \in W$, (i.) the collection $M \subseteq N$ is a chain if and only if there is $w \in W$ such that $w \in x$ for all $x \in M$, and

(ii.) every chain in $X = N \setminus \{\{w\}\}_{w \in W}$ has a maximum and either also a minimum or an infimum in $E = \{\{w\}\}_{w \in W}$.

A tree (W, N) is rooted if $W \in N$. The set $W \in N$ is then called the root. Nodes in $E = \{\{w\}\}_{w \in W}$ are called terminal and nodes in $X = N \setminus E$ are called moves.

In the terminology of Alós-Ferrer and Ritzberger (2005), this tree is complete since it contains all singletons of outcomes as nodes, i.e. $\{w\} \in N$ for all $w \in W$. In Proposition 10, Alós-Ferrer and Ritzberger (2005) show that one can remove all infinite terminal nodes without changing the structure of the tree. However, one can also add all infinite terminal nodes. In order to simplify notation, we choose to do that and therefore we consider complete and rooted trees only. Furthermore, the definition of the game tree incorporates discreteness. Discreteness consists of two parts. First, up-discreteness is the existence of a maximum in every chain (Alós-Ferrer and Ritzberger, 2008, p. 235). Second, down-discreteness is that every chain has a minimum or an infimum in the set of terminal nodes (Alós-Ferrer and Ritzberger, 2013, p. 84).

Proposition 3.1(b) in Ritzberger (2002) states that for every rooted tree T, there exists an immediate predecessor function $p : F(N) \to X$ which is surjective, satisfies $p(W) = W$, $x \subset p(x)$ and if $x \subset y \in N$ then $p(x) \subseteq y \subseteq \cup_{t=1}^{\infty} p^t(x)$ for all $x \in F(N) \setminus \{W\}$, where $F(N) = \{x \in N | x \subset \cap_{x \subset y \in N} y\}$ denotes the set of finite elements.[1] Alós-Ferrer and Ritzberger (2013) define the slice $Y_t = \{x \in N | p^{t-1}(x) \subset p^t(x) = W\}$ for all $t = 1, 2, \ldots$ and show that nodes in a slice Y_t are pairwise disjoint.

Ritzberger (2002) introduces the function $W^{-1} : \mathcal{P}(W) \to \mathcal{P}(N)$,

$$W^{-1}(a) = \{x \in N \mid x \subseteq a, \text{ and there is no } y \in N : x \subset y \subseteq a\} \tag{1}$$

for all sets of plays $a \subseteq W$ for a given tree $T = (W, N)$. The function gives the coarsest partition of a set of outcomes a in nodes in N. The function $W : \mathcal{P}(N) \to \mathcal{P}(W)$ maps a collection of nodes into the set of outcomes contained in it, i.e. $W(m) = \{w \in W | \exists x \in m : w \in x\}$. Ritzberger (2002) shows that W^{-1} is the inverse of W under a suitable restriction of the domain.

Alós-Ferrer and Ritzberger (2013, p. 82) define the function

$$P(a) = \{x \in N \mid \exists y \in \downarrow a : \uparrow x = (\uparrow y) \setminus (\downarrow a)\}, \tag{2}$$

[1] Throughout the paper I use the convention $x \subseteq y$ to denote a subset that could be proper and $x \subset y$ to denote a subset that is proper.

where $\uparrow x = \{y \in N | x \subseteq y\}, x \in N$ is the up-set and $\downarrow x = \{y \in N | y \subseteq x\}, x \in N$ is the down-set. The maximal chain $\uparrow \{w\}, w \in W$ is denoted by $\uparrow w$ and called play. Note that there is a bijection between the set of outcomes and the set of plays. In the sequel we will use 'play' and 'outcome' interchangeably. The set $P(a)$ are the immediate predecessors of $a \in \mathcal{P}(W)$. Furthermore, Alós-Ferrer and Ritzberger (2013, p. 84) introduce the function $p^{-1}(y) = \{x \in F(N) | p(x) = y\}$ that gives the set of immediate successors of a move.

Alós-Ferrer and Ritzberger (2013, Definition 6) define an extensive form on a game tree. One should note that their definition is more general than the one given here, since they allow that at least one player has choices available at a given move. However, in this paper it is required that exactly one player has choices available at a given move.

Definition 2. A (discrete) *extensive form* with player set $I = \{1, \ldots, n\}$ is a pair (T, C), where $T = (W, N)$ is a game tree with the set of outcomes W and $C = (C_i)_{i \in I}$ is a system of collections C_i (the sets of players' choices) of nonempty unions of nodes (hence, sets of outcomes) for all $i \in I$, such that

(i.) if $P(c) \cap P(c') \neq \emptyset$ and $c \neq c'$, then $P(c) = P(c')$ and $c \cap c' = \emptyset$, for all $c, c' \in C_i$ for all $i \in I$;

(ii.) $p^{-1}(x) = \{x \cap c | c \in A_i(x)\}$ for all $x \in X_i$ and $i \in I$,

where for all $i \in I$, the set $A_i(x) = \{c \in C_i | x \in P(c)\}$ collects the choices available to i at x and $X_i = \{x \in X | A_i(x) \neq \emptyset\}$ denotes the set of moves at which player i has choices available. Moreover, $\{(X_i)_{i \in I}\}$ forms a partition of X.

The first item says that if one player has two distinct choices jointly available at some moves, then these choices are jointly available at the same moves. Therefore, no player can infer any information from the choices available to him. The choices jointly available are disjoint. The second property formulates that for each move, the set of immediate successors are the nodes induced by choices available to the player at this move. The set $H_i = \{P(c)\}_{c \in C_i}$ is a partition of X_i. An element $h \in H_i$ is an information set for player i. Given a move $x \in X_i$, let $h(x) \in H_i$ denote the information set $h \in H_i$ such that $x \in h$. The function $W^{-1}|_{C_i}$ maps a choice into the set of nodes that can be reached with this choice.

Set inclusion defines when a node comes before another. Ritzberger (2002, p. 121) defines the following relation in order to relate choices (and information sets) to each other.

Definition 3. Let $c, c' \in C_i$. We say that the choice c *comes before* c',

$$c <_i c', \text{ if there are } x \in W^{-1}(c) \text{ and } x' \in W^{-1}(c') \text{ such that } x' \subset x.$$

Let $h, h' \in H_i$. The information set h *comes before* h',

$$h <_i h', \text{ if there are } x \in h' \text{ and } x' \in h' \text{ such that } x' \subset x.$$

With the definition of "comes before" at hand, Ritzberger (2002, Definition 3.12) defines perfect recall. A game satisfies perfect recall if the rules of the game do not force players to forget what they have previously known and previously done. Previously refers to a play in which a player decides twice. If a player has to choose twice in the course of a play, then it must be the case that all outcomes he can choose at the later stage must have been available to him at the earlier stage. If this was not the case, then there is an outcome the player deems impossible at the earlier information set, but possible at the later stage.

Definition 4. The extensive form (T, C) satisfies *perfect recall* if for all $i \in I$ and all $c, c' \in C_i$,

$$c <_i c' \text{ implies } c' \subseteq c.$$

Proposition 1. *If an extensive form (T, C) has perfect recall, then for all players $i \in I$ and all choices $c, c' \in C_i$, if $c <_i c'$, then $c' \subset c$ and for all $x' \in W^{-1}(c')$ there is $x \in W^{-1}(c)$ such that $x' \subset x$.*

Proof. Let (T, C) be an extensive form with perfect recall and let $i \in I, c, c' \in C_i$ with $c <_i c'$, i.e. there are $x \in W^{-1}(c), x' \in W^{-1}(c')$ such that $x' \subset x$. Since $(x \setminus x') \neq \emptyset$, it follows that $c' \subset c$. Furthermore, let $x' \in W^{-1}(c')$ and $w \in x'$. Since $c' \subseteq c$, there is $x \in W^{-1}(c)$ with $w \in x$. Hence, $x \cap x' \neq \emptyset$ and by Definition 1, $x' \subseteq x$ or $x \subset x'$. If $x \subset x'$, then $c' <_i c$ and by perfect recall $c \subset c'$, a contradiction. If $x = x'$, then $P(c) \cap P(c') \neq \emptyset$ and by Definition 2, $c \cap c' = \emptyset$, a contradiction. Hence, $x' \subset x$ must be true.

□

Given an extensive form (T, C), a (pure) strategy is a function $s_i : X_i \to C_i$ such that

$$s_i^{-1}(c) = P(c) \text{ for all } c \in s_i(X_i).$$

A strategy assigns the same choice to each decision point in an information set. Strategies of player i are collected in the set S_i, the set of strategy combinations is denoted by $S = \times_{i \in I} S_i$. A pure strategy combination s is an element of this set, i.e. $s \in S$. Alós-Ferrer and Ritzberger (2008) define the correspondence $R_s : W \to W$ for a given $s \in S$ as

$$R_s(w) = \bigcap \{s_i(x) | w \in x \in X_i, i \in I\}.$$

Alós-Ferrer and Ritzberger (2013, Theorem 3) show that in an (discrete) extensive form the following two properties are satisfied:

(A1) For every $s \in S$ there is some $w \in W$ such that $w \in R_s(w)$.

(A2) If for $s \in S$ there is $w \in W$ such that $w \in R_s(w)$, then R_s has no other fixed point and $R_s(w) = \{w\}$.

The interpretation is that every pure strategy combination induces a unique outcome, hence there is a function $\psi : S \to W$ with $\psi(s) \in R_s(w)$. With the definition of an extensive form at hand, one can define an extensive form game.

Definition 5. An *extensive form game* is a triple $G = (T, C, \pi)$, where (T, C) is an extensive form with player set I and $\pi : W \to \mathbb{R}^n$ is the payoff function mapping each outcome into a payoff vector.

The following definition is at the heart of this paper. It defines the notion of strategic equivalence. Recall that every strategy combination induces a unique outcome and that the payoff function assigns a payoff vector to it. For player i, two strategies are said to be strategically equivalent if every strategy combination with these two strategies induce the same payoff vector. Thus, all players are indifferent between the two strategies.

Definition 6. Two strategies $s_i, s_i' \in S_i$ of player i are *strategically equivalent* if $\pi(s_i, s_{-i}) = \pi(s_i', s_{-i})$ for all $s_{-i} \in S_{-i}$.

Let S_i^* denote the set of equivalence classes of player i' pure strategies. Let $S^* = \prod_i S_i^*$. We extend the function π on S^*, so that $\pi(s^*) = \pi(s), s \in s^* \in S^*$.

One can define the space of extensive forms games on a set of possible outcomes $\bar{W}$ and a set of players I.

Definition 7. Let $\bar{W} \neq \emptyset$ be a nonempty set and $I = \{1, \ldots, n\}$, n a positive integer. The space of extensive form games $\Gamma_{\bar{W}}$ on (the set of outcomes) $\bar{W}$ with player set I is the set of all extensive form games such that $G \in \Gamma_{\bar{W}}$ if $W \subseteq \bar{W}$, G an extensive form game and $C_i \neq \emptyset$ for all $i \in I$.

Every game $G \in \Gamma_{\bar{W}}$ is based on a set of plays $W \subseteq \bar{W}$. Note that there can be many distinct games in $\Gamma_{\bar{W}}$ with root $W \subseteq \bar{W}$. Given a game $G \in \Gamma_{\bar{W}}$ and a node $x \in N$, $G_x \in \Gamma_{\bar{W}}$ denotes the game derived from G and x with root x, $N_x = \{y \in N | y \subseteq x\}, C_{x_i} = \{c \cap x, c \in C_i\}$ for all $i \in I$ and $\pi_x = \pi|_x$. For the rest of the paper, we fix a set of universal plays $\bar{W} \neq \emptyset$ and a set of players I with $|I| \geq 2$. We define a binary relation on the set of extensive form games.

Definition 8. For $G, \hat{G} \in \Gamma_{\bar{W}}$, $G \leq \hat{G}$ if

(i.) for all $t \in \mathbb{N}, x \in Y_t$ implies there is a $\hat{x} \in \hat{Y}_t$ such that $x \subseteq \hat{x}$,

(ii.) for all $i \in I$, all $c, c' \in C_i$ with $c \neq c', P(c) = P(c')$, there exist choices $\hat{c}, \hat{c}' \in \hat{C}_i, \hat{c} \neq \hat{c}', \hat{P}(\hat{c}) = \hat{P}(\hat{c}')$ such that $c \subseteq \hat{c}, c' \subseteq \hat{c}'$, and

(iii.) for all $w \in W$, $\pi(w) = \hat{\pi}(w)$.

The first item defines a rough notion of subtree. Since $Y_0 = \{W\}$, plays in the 'smaller' game are possible in the larger game. The second item says that if a player has to decide between two alternatives at an information set in the smaller game, then the same player has to decide between the two alternatives at one information set in the larger game. The last item says that in the larger tree the preferences of the players in the smaller tree are respected.

Proposition 2. *The relation $\leq$ on $\Gamma_{\bar{W}}$ is a partial order.*

Proof. The relation is clearly reflexive. In order to establish transitivity, let $G \leq \hat{G}$ and $\hat{G} \leq \tilde{G}$. Let $t \in \mathbb{N}$ and $x \in Y_t$. There is $\hat{x} \in \hat{Y}_t$ and $\tilde{x} \in \tilde{Y}_t$ with $x \subseteq \hat{x} \subseteq \tilde{x}$.

Let $i \in I$ and let $c, c' \in C_i$ with $c \neq c', P(c) = P(c')$. There are choices $\hat{c}, \hat{c}' \in \hat{C}_i, \hat{c} \neq \hat{c}', \hat{P}(\hat{c}) = \hat{P}(\hat{c}')$ such that $c \subseteq \hat{c}, c' \subseteq \hat{c}'$. Furthermore, there are choices $\tilde{c}, \tilde{c}' \in \tilde{C}_i, \tilde{c} \neq \tilde{c}', \tilde{P}(\tilde{c}) = \tilde{P}(\tilde{c}')$ with $\hat{c} \subseteq \tilde{c}, \hat{c}' \subseteq \tilde{c}'$. Thus $c \subseteq \tilde{c}, c' \subseteq \tilde{c}'$ holds.

For antisymmetry, let $G \leq \hat{G}$ and $\hat{G} \leq G$. Let $t \in \mathbb{N}$ and $x \in Y_t$. There is $\hat{x} \in \hat{Y}_t$ such that $x \subseteq \hat{x}$ and there is $x' \in Y_t$ with $\hat{x} \subseteq x'$. Since nodes in a slice are pairwise disjoint, $x = \hat{x} = x'$ follows.

Let $i \in I$ and let $c, c' \in C_i$ and $\hat{c}, \hat{c}' \in \hat{C}_i$ be such that they satisfy the relevant properties. There are $\tilde{c}, \tilde{c}' \in C_i, \tilde{c} \neq \tilde{c}', P(\tilde{c}) = P(\tilde{c}'), c \subseteq \hat{c} \subseteq \tilde{c}$ and $c' \subseteq \hat{c}' \subseteq \tilde{c}'$. Suppose $c \subset \tilde{c}$. Let $w \in c, x \in P(c)$ with $w \in x$. Then there is $\tilde{x} \in P(\tilde{c})$ with $w \in \tilde{x}$, thus $x \cap \tilde{x} \neq \emptyset$. Then $\tilde{x}$ must be a strict predecessor of x. If player i selects choice $\tilde{c}$ at $\tilde{x}, c, c'$ cannot be jointly available at x, a contradiction. Hence, $c = \tilde{c}$ and $c' = \tilde{c}'$ and furthermore that $G = \hat{G}$.

□

The next definition defines an equivalence relation on the space of extensive form games.

Definition 9. Two extensive form games $G, \hat{G}$ are *isomorphic*, $G \cong \hat{G}$, if

(i.) there is a bijection $b : I \to \hat{I}$,

(ii.) there is a bijection $\omega : W \to \hat{W}$, (iii.) there is a bijection $\nu : N \to \hat{N}$ with the properties

(a) $x \subset y$ iff $\nu(x) \subset \nu(y)$ for all $x, y \in N$,

(b) for all $i \in I$ there is a bijection $\chi_i : C_i \to \hat{C}_{b(i)}, \chi_i(c) = \bigcup_{w \in c} \nu(\{w\})$,

(c) for all $w \in W$, $\pi(w) = \hat{\pi}(\hat{w})$, where $\{\hat{w}\} = \nu(\{w\})$.

3 The Transformations

Thompson (1952) suggests four transformations that leave the semi-reduced normal form unchanged. However, two of them are able to destroy perfect recall. The two innocuous transformations are "Interchange of moves" and "Coalescing of moves". The remaining two are called "Inflation-deflation" and "Addition of superfluous moves". Elmes and Reny (1994) take the two innocuous transformations (INT and COA) and create a new transformation (ADD) by modifying "Addition". Ritzberger (2002) translates the INT and the COA transformation into the set theoretic framework. Whatsoever, he uses a different notion for choices. In his book, choices are collections of nodes, while in the current framework they are unions of these nodes. Therefore, I adjust his INT and COA transformation. However, he does not define the ADD transformation like Elmes and Reny, but gives the inverse of the ADD transformation on the class of extensive form games with perfect recall. Since we do not want to restrict ourselves a priori on games with perfect recall, we translate the ADD transformation in the current framework. But first, we will begin with the INT transformation.

Definition 10 (Interchange)**.** $G \xrightarrow{INT} \hat{G}$ if there is a player $i \in I$ and a player $j \in I \setminus \{i\}$, player i has an information set h such that for $h' \subseteq h$, for all $\bar{x} \in h', p^{-1}(\bar{x}) \subseteq X, p^{-1}(\bar{x}) \cap P(c) \neq \emptyset \Rightarrow p^{-1}(\bar{x}) \subseteq P(c), |p^{-1}(\bar{x})| = |A|$, where $A = \{c \in C_j | p^{-1}(\bar{x}) \subseteq P(c)\}$ and $\hat{G}$ satisfies

(i.) $\hat{W} = W$,

(ii.) $\hat{N} = N \setminus (\bigcup_{\bar{x} \in h'} p^{-1}(\bar{x})) \cup (\bigcup_{\bar{x} \in h'} \bigcup_{c \in A} \{c \cap \bar{x}\})$,

(iii.) $\hat{C}_k = C_k$ for all $k \in I$, and

(iv.) $\hat{\pi}(w) = \pi(w)$ for all $w \in W$.

The interchange transformation states conditions for interchanging the order of decision nodes of two players. First, player i has an information set such that for a subset of this information set the set of immediate successors are moves. Second, the set of immediate successors is a subset of an information set of player

j. Third, the number of immediate successors at player i's information set is equal to the number of choices available at an immediate successor. The trees have the same structure and differ only at the immediate successors of the information set h'. In the new game, player j chooses before player i at some nodes, thus the outcomes that are possible at i's choice are different than when i chooses first. But since player i chooses immediately afterwards, only immediate successors of h' are changed. The choices and preferences of all players remain unchanged.

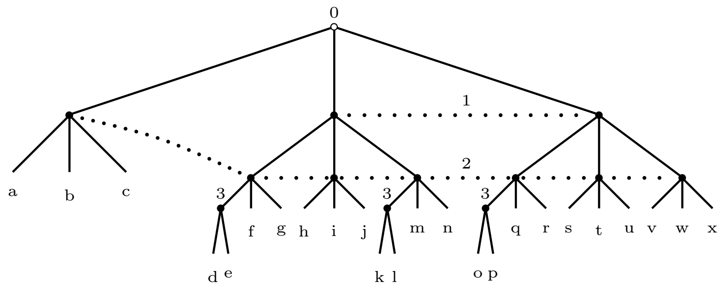
In the four player game depicted in Figure 1a, one can see that the choice of the information set h' is often arbitrary. One has three different possibilities for h'. First, one can take the whole information set of Player 1. This results in Figure 1c. On the other hand, one can take either of the two nodes in Player 1's information set. Taking the node on the right as h', translates the game into the game depicted in Figure 1b. Starting in Figure 1b, we can apply the INT transformation for Player 1 and end up in Figure 1c. One can go back to the original game by flipping the roles of Player 1 and 2 while leaving h' unchanged.

One can interpret the INT transformation as an assignment in the space of extensive form games. If $W \subseteq \bar{W}$, then $G \in \Gamma_{\bar{W}}$ and $\hat{G} \in \Gamma_{\bar{W}}$, thus $\Gamma_{\bar{W}}$ is closed under INT. If two games differ by INT, then they do only differ at immediate successors of the sub-information set h'. If one defines a binary relation such that two games are related if they differ by INT, then this relation is reflexive (since one can take $h' = \emptyset$) and symmetric. However, Figure 2 shows an example where transitivity fails to hold under the current definition. The games on the left and in the middle differ by INT and the games in the middle and on the right differ by INT. However, the games on the left and on the right do not differ by INT, since a single application of INT on one game does not generate the other game.

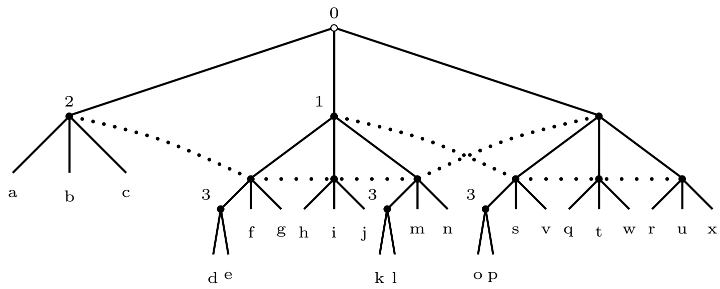
Definition 11 (Coalescing)**.** $G \xrightarrow{COA} \hat{G}$ if for some player $i \in I$ there are two choices $c, c' \in C_i$ with $W^{-1}(c) = P(c')$, i is the only player who has choices available at (any move in) $h' = P(c')$, and $\hat{G}$ satisfies $\hat{T} = (W, N \setminus W^{-1}(c))$, $\hat{C}_i = C_i \setminus \{c\}, \hat{C}_j = C_j$ for $j \in I \setminus \{i\}$ and $\hat{\pi}(w) = \pi(w)$ for all $w \in W$.

If a player has two consecutive choices c, c' and the set of nodes that can be reached with the first choice is equal to the information set at which the second choice is available, one can delete the first choice. Of course, the COA transformation destroys subgames.

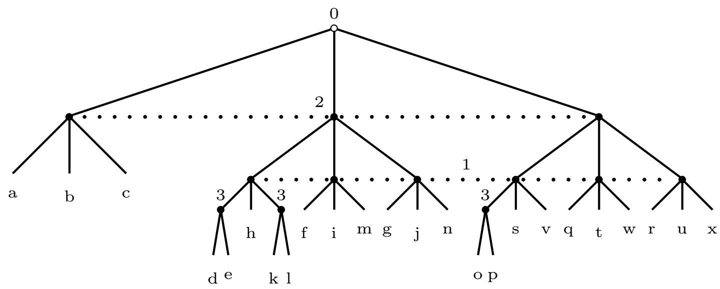
Figure 3a shows an extensive form game in which one can apply the Coalescing transformation twice. Player 1 has two consecutive information sets twice. Figure 3a differs from Figure 3b by COA and Figure 3b differs from Figure 3c by COA.



(a) The initial game in extensive form



(b) The game after the application of the INT transformation



(c) The game after another application of the INT transformation

Figure 1: Interchanging Moves

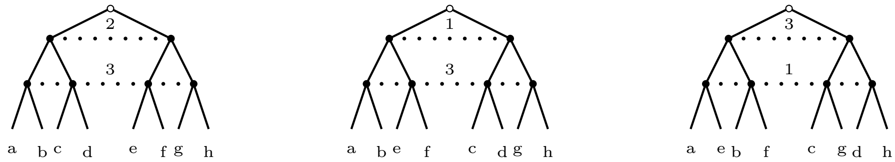
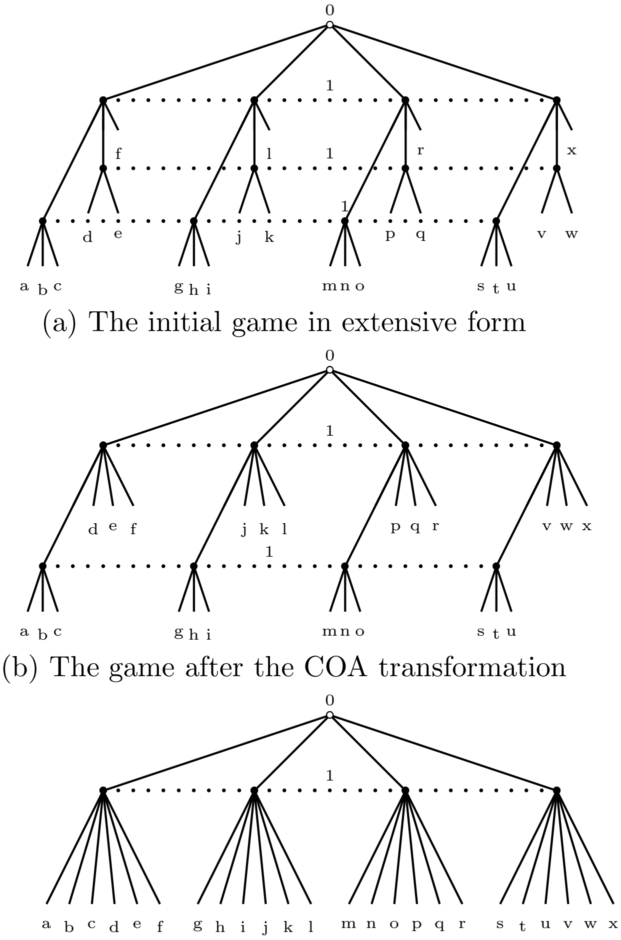


Figure 2: Failure of the Transitivity of the INT Transformation



(a) The initial game in extensive form

(b) The game after the COA transformation

(c) The game after another COA transformation

Figure 3: Coalescing Information Sets

After the transformations, Player 1 has only one information set and two choices less.

Definition 12 (Addition)**.** $\hat{G} \xrightarrow{ADD} G$, if there are sets $\hat{\mathcal{X}} \subseteq \hat{N}, \hat{h} \in \hat{H}_i, \hat{A} = \left\{c \in \hat{C}_i \mid P(c) = \hat{h}\right\}$, and $\mathcal{X} \subseteq \mathcal{P}(\bar{W})$, such that for all $x \in \hat{h}, y \in \hat{\mathcal{X}}, x \cap y = \emptyset$, the elements of $\hat{\mathcal{X}}$ (respectively $\mathcal{X}$) are pairwise disjoint and there is a bijection $\psi : \hat{\mathcal{X}} \to \mathcal{X}$ such that for all $\hat{x} \in \hat{\mathcal{X}}$,

(a) $\hat{x} \subset \psi(\hat{x})$,

(b) $(\psi(\hat{x}) \setminus \hat{x}) \cap \hat{W} = \emptyset$, and

(c) $p^{-1}(\psi(\hat{x}))$ is well defined and $p^{-1}(\psi(\hat{x})) = \{x_{\hat{c}}\}_{\hat{c} \in \hat{A}}$, $\hat{x} = x_{\hat{c}^*}$,

and for all $\hat{c} \in \hat{A} \setminus \{\hat{c}^*\}$, there is a bijection $\omega_{x_{\hat{c}}} : \hat{x} \to x_{\hat{c}}$ with $\pi(\omega_{x_{\hat{c}}}(w)) = \hat{\pi}(w)$ for all $w \in \hat{x}$. Furthermore, the following properties hold:

(i.) for all $\hat{c} \in \hat{C}_i$, if $\hat{W}^{-1}(\hat{c}) \cap N_{c^*} \neq \emptyset$, then $\hat{W}^{-1}(\hat{c}) \subseteq N_{c^*}$, (ii.) for all $c \in C_i, c' \in A$, if $c <_i c'$, then $c' \subseteq c$,

(iii.) $W = \hat{W} \cup \bigcup_{x \in \mathcal{X}} x$,

(iv.) $N = \hat{N} \setminus \left\{ \hat{p}^t(\hat{x}) \mid \hat{x} \in \hat{\mathcal{X}}, t \in \mathbb{N} \right\} \cup \mathcal{X} \cup \bigcup_{c \in A} N_c \cup$
$\cup \bigcup_{\hat{x} \in \hat{\mathcal{X}}} \bigcup_{t \in \mathbb{N}} \left\{ \hat{p}^t(\hat{x}) \cup \bigcup_{\hat{p}^t(\hat{x}) \supset \hat{x}' \in \hat{\mathcal{X}}} \psi(\hat{x}') \right\}$,

(v.) $C_j = \bigcup_{\hat{c} \in \hat{C}_j} \left\{ \hat{c} \cup \bigcup_{\hat{y} \in \hat{W}^{-1}(\hat{c})} \bigcup_{\hat{y} \supseteq \hat{x} \in \hat{\mathcal{X}}} \psi(\hat{x}) \cup \bigcup_{\hat{z} \in \hat{W}^{-1}(\hat{c}) \cap N_{c^*}} \bigcup_{c \in A \setminus c^*} \phi_{c^* c}(\hat{z}) \right\}$

(vi.) $C_i = A \cup \bigcup_{\hat{c} \in \hat{C}_i \setminus \hat{A}} \left\{ \hat{c} \cup \bigcup_{\hat{y} \in \hat{W}^{-1}(\hat{c})} \bigcup_{\hat{y} \supseteq \hat{x} \in \hat{\mathcal{X}}} \psi(\hat{x}) \right\} \cup$
$\cup \bigcup_{c \in A \setminus \{c^*\}} \left\{ \bigcup_{\hat{z} \in \hat{W}^{-1}(\hat{c}) \cap N_{c^*}} \phi_{c^* c}(\hat{z}) \right\}$,

(vii.) $\pi(w) = \hat{\pi}(w)$ for all $w \in \hat{W}$,

(viii.) $A = \{c^*\} \cup \bigcup_{\hat{c} \in \hat{A} \setminus \{\hat{c}^*\}} \hat{c} \cup \bigcup_{\hat{x} \in \hat{\mathcal{X}}} \bigcup_{w \in \hat{x}} \{\omega_{\hat{c}}(w)\}$,

where $c^* = \hat{c}^* \cup \bigcup_{\hat{x} \in \hat{\mathcal{X}}} \hat{x}, \hat{c}^* \in \hat{A}$, for all $c \in A$, $N_c = \left\{ x \in N | x \subseteq c \cap \bigcup_{x \in \mathcal{X}} x \right\}$, there is a bijection $\phi_{c^* c} : N_{c^*} \to N_c$ with $y \subset x$ iff $\phi_{c^* c}(y) \subset \phi_{c^* c}(x)$ for all $x, y \in N_{c^*}$ and $\hat{\pi}(w) = \pi(\phi_{c^* c}(w))$, where $\{\phi_{c^* c}(w)\} = \phi_{c^* c}(\{w\})$ for all $w \in \bigcup_{\hat{x} \in \hat{\mathcal{X}}} \hat{x}$.

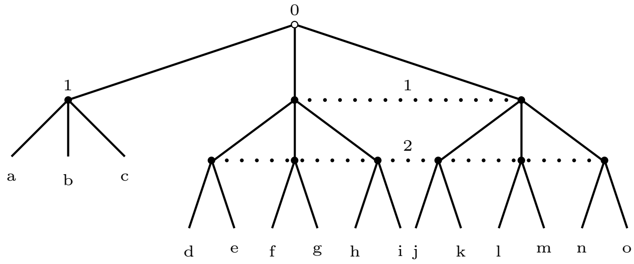
The elements in $\mathcal{X}$ are superfluous moves added to the information set $\hat{h}$, while the elements in $\hat{\mathcal{X}}$ are the maximal nodes being copied. Note that $N_{c^*} = \left\{ \hat{y} \in \hat{N} | \hat{y} \subseteq \hat{x} \in \hat{\mathcal{X}} \right\}$ and that there is an information set $h = \hat{h} \cup \mathcal{X}$ in the new game. In the sequel, we consider $\phi_{c^* c^*}$ as the identity.

Figure 4a shows a game in which one can employ the ADD transformation. Moves are added to the information set $\hat{h} = \{\{a, b, c\}\}$. The added moves are superfluous in the sense that however Player 1 decides, the payoff materialized depends entirely on other choices. The other information set of Player 1 gets copied. It appears three times in the new game, each copy refers to a choice available at $\hat{h}$. The information set gets copied such that perfect recall is preserved. The other player's choices and information remain essentially the same. Player 2 does not get to know at which copy he can choose.

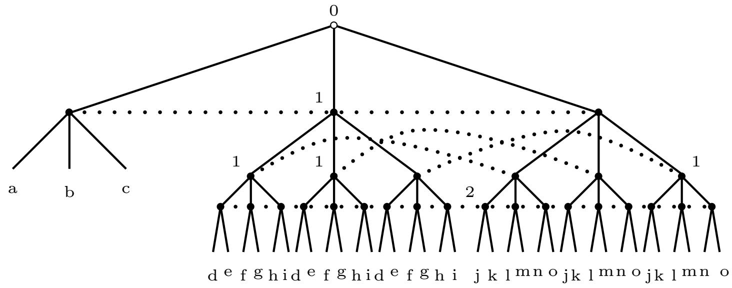
The INT and COA transformations are local transformations in the sense that they leave the set of outcomes unchanged. The ADD transformation on the other hand increases the set of outcomes. If $\bar{W}$ is sufficiently large, $\Gamma_{\bar{W}}$ is closed under ADD. One can take $\bar{W} = \mathbb{N}$ in the case of countable games and $\bar{W} = \mathbb{R}$ for uncountable games.

Proposition 3. *If* $\hat{G} \xrightarrow{ADD} G$, *then* $\hat{G} \leq G$.

Proof. Let $t \in \mathbb{N}$ and $\hat{x} \in \hat{Y}_t$. There are four exhaustive cases how $x \in Y_t, \hat{x} \subseteq x$ can look like. First, if $\hat{x} \cap \bar{x} = \emptyset, \bar{x} \in \hat{\mathcal{X}}$, then $x = \hat{x}$. Second, if $\hat{x} \in \hat{\mathcal{X}}$, then



(a) The initial game in extensive form



(b) The game after the ADD transformation

Figure 4: Addition of Superfluous Moves

$x = \psi(\hat{x})$. Third, if $\hat{x} \subset \bar{x}, \bar{x} \in \hat{\mathcal{X}}$, then $x = \hat{p}(\hat{x})$. Finally, if $\bar{x} \subset \hat{x}, \bar{x} \in \hat{\mathcal{X}}$, then $x = \hat{x} \cup \bigcup_{\hat{x} \supset \bar{x}'} \psi(\bar{x}')$.

If two choices are jointly available for a player in $\hat{G}$, then there are larger corresponding choices jointly available in G. Moreover, $\hat{\pi}(w) = \pi(w)$ for all $w \in \hat{W}$. $\square$

Ritzberger (2002) defines the inverse of the ADD-Transformation on the class of extensive form games with perfect recall. The following two definitions translate his definition into the current setup.

Definition 13 (Irrelevant Choices)**.** Let G be an extensive form game with perfect recall and suppose that, for some player $i \in I$ the subset $A \subseteq C_i$ of choices satisfies $P(c) = P(c') = h$ for all $c, c' \in A$ and $|A| \geq 2$. Let $h' \subseteq h$ be a subset of $h \in H_i$ and define

$$N_c = \{x \in N | x \subseteq c \cap W(h')\}$$

as the nodes met after c along plays passing through h' for all $c \in A$. The choices in A are called *irrelevant at* (any move in) h' if i is the only player who has choices available at (any move in) h', $W^{-1}(c) \cap N_a \neq \emptyset \Rightarrow c \subseteq a$ for all $c \in C_i$ and all $a \in A$ and for all $c, c' \in A$ the function $\psi_{cc'} : N_c \to N_{c'}$ which satisfies

(i.) $y \subset x$ if and only if $\psi_{cc'}(y) \subset \psi_{cc'}(x)$ for all $x, y \in N_c$,

(ii.) $x \subseteq \tilde{c}$ if and only if $\psi_{cc'}(x) \subseteq \tilde{c}$ for all $\tilde{c} \in C_j$, all $x \in N_c$, and all $j \neq i$,

(iii.) $W^{-1}(\tilde{c}) \subseteq N_c$ if and only if $\psi_{cc'}(W^{-1}(\tilde{c})) \subseteq N_{c'}$ for all $\tilde{c} \in C_i$, and (iv.) $\pi(w) = \pi(\psi_{cc'}(w))$ where $\psi_{cc'}(w) \in c \cap W(h')$ denotes the outcome that ends at the image of the terminal node $\{w\} \in N_c$ under $\psi_{cc'}$, for all $\{w\} \in N_c$,

is a bijection.

Definition 14 (Deletion of irrelevant choices). $G \xrightarrow{DEL} \hat{G}$ if for some player $i \in I$ there is a set of choices $A \subseteq C_i$, $|A| \geq 2$ such that

(i.) at any move in $h = P(c) = P(c')$ for all $c, c' \in A$ player i is the only player who has nontrivial choices available and the choices in A are the only choices available to i at h,

(ii.) the choices in A are irrelevant at the proper subset h' of h and

(iii.) $\hat{G}$ satisfies

$$\begin{aligned}
\hat{W} &= W \setminus (\cup_{a \in A \setminus \{c^*\}} a \cap W(h')), \\
\hat{N} &= \left\{ x \cap \hat{W} \mid x \in N \setminus (\cup_{a \in A \setminus \{c^*\}} N_a \cup h') \right\}, \\
\hat{C}_j &= \left\{ c \cap \hat{W} \mid c \in C_j \right\}, \text{ for all } j \neq i, \\
\hat{C}_i &= \left\{ c \cap \hat{W} \mid c \in C_i \setminus \{c^*\} \right\} \cup \{c^* \setminus W(h')\},
\end{aligned}$$

and $\hat{\pi}(w) = \pi(w)$ for all $w \in \hat{W}$.

Proposition 4. *If $G, \hat{G}$ differ by one of the three transformations INT, COA or ADD, then G has perfect recall if and only if $\hat{G}$ has perfect recall.*

Proof. Let $G \xrightarrow{INT} \hat{G}, k \in I$, $c, c' \in C_k = \hat{C}_k$, with $\hat{W}^{-1}(c) = W^{-1}(c), \hat{W}^{-1}(c') = W^{-1}(c')$ and $x' \subset x$. This holds true for all choices of all players in $I \setminus \{i, j\}$ and for some choice combinations of players i and j eventually. Then $c <_k c'$ if and only if $c \mathrel{\hat{<}_k} c'$. Therefore, one game has perfect recall if and only if the other game has perfect recall.

Let $\hat{G}$ have perfect recall and consider player i. Let $c, c' \in C_i$ and $c <_i c'$, i.e. there is $x \in W^{-1}(c)$ and $x' \in W^{-1}(c')$ with $x' \subset x$. Either $\hat{W}^{-1}(c) \not\subseteq \hat{N}$, or $\hat{W}^{-1}(c') \not\subseteq \hat{N}$, or $\hat{W}^{-1}(c) \cup W^{-1}(c') \subseteq \hat{N}$. Note that the last case was discussed above. Suppose $P(c) = h$ and $x \in \hat{N}$. Then $c \mathrel{\hat{<}_i} c'$. Suppose $P(c) = h$ and $x \notin \hat{N}$. Then there is $\hat{x} \in \hat{W}^{-1}(c)$, with $x' \subset \hat{x} \subset x$ and thus $c \mathrel{\hat{<}_i} c'$. Now suppose $P(c') = h$ and $x' \in \hat{N}$. It is again obvious that $c \mathrel{\hat{<}_i} c'$. If $P(c') = h$ and $x' \notin \hat{N}$, then there is $\hat{x}' \in \hat{W}^{-1}(c')$ with $\hat{x}' \subset x' \subset x$, hence $c \mathrel{\hat{<}_i} c'$. In each case, we obtain $c \mathrel{\hat{<}_i} c'$ and perfect recall in $\hat{G}$ implies $c' \subseteq c$.

Now consider player j. Let $c, c' \in C_j$ and let $c <_j c'$, i.e. there is $x \in W^{-1}(c)$ and $x' \in W^{-1}(c')$ such that $x' \subset x$. At most one of the choices can be in the set

A. Let $c \in A$. If $x \in \hat{W}^{-1}(c)$, then $c \mathrel{\hat{<}_j} c'$. On the other hand, if $x \notin \hat{W}^{-1}(c)$, then there is $\hat{x} \in \hat{W}^{-1}(c)$ with $x' \subset x \subseteq \hat{x}$, consequently $c \mathrel{\hat{<}_j} c'$. Now let $c' \in A$ and suppose $x' \in \hat{W}^{-1}(c')$. Then we have $c \mathrel{\hat{<}_j} c'$. If $x' \notin \hat{W}^{-1}(c')$, then there is $\bar{x} \in h'$ with $x' \subset \bar{x} \subset x$. There exists $\hat{x}' \in \hat{W}^{-1}(c')$ with $x' \subset \hat{x}' = \bar{x} \cap c' \subset \bar{x} \subset x$, thus $c \mathrel{\hat{<}_j} c'$. In each case, $c <_j c'$ implies $c \mathrel{\hat{<}_j} c'$ and perfect recall in $\hat{G}$ guarantees $c' \subseteq c$.

The same arguments hold if G satisfies perfect recall. Note that one can apply INT on $\hat{G}$ with the same h', but flipping the roles of i and j.

Let $G \xrightarrow{COA} \hat{G}$ and let G have perfect recall. Let $k \in I$ and $\hat{c}, \hat{c}' \in \hat{C}_k$ with $\hat{c} \mathrel{\hat{<}_k} \hat{c}'$. Then it follows that $\hat{c}, \hat{c}' \in C_k$ and thus $\hat{c} <_k \hat{c}'$. Perfect recall implies $\hat{c}' \subseteq \hat{c}$.

Suppose that $\hat{G}$ has perfect recall. Let $k \in I$ and $c, c' \in C_k$ with $c <_k c'$ and $c, c' \in \hat{C}_k$. The last requirement holds true for all choices of player $k \neq i$ and eventually for some of player i's choices. Since $c, c' \in \hat{C}_k$, we have $c \mathrel{\hat{<}_k} c'$. Perfect recall in $\hat{G}$ gives $c' \subseteq c$.

Now, consider player i. Let $c, c' \in C_i$ and $c <_i c'$. Suppose $c \notin \hat{C}_i$ and $W^{-1}(c) = P(c')$. This equality holds only if $c' \subseteq c$. Now let $c \notin \hat{C}_i$ and let $W^{-1}(c) \neq P(c')$. Then there is a choice $\bar{c} \in C_i$ with $W^{-1}(c) = P(\bar{c})$ and $\bar{c} <_i c'$. From the case above, we know that $\bar{c} \subseteq c$. Furthermore, $\bar{c}, c' \in \hat{C}_i$ and $\bar{c} \mathrel{\hat{<}_i} c'$. Perfect recall in $\hat{G}$ implies $c' \subseteq \bar{c} \subseteq c$.

Suppose $c' \notin \hat{C}_i$. Consider the set $\mathcal{X} = \{\bar{c} \in C_i | W^{-1}(c') = P(\bar{c})\}$. Recall there are $x \in W^{-1}(c),\ x' \in W^{-1}(c')$ such that $x' \subset x$. Since $W^{-1}(c') = P(\bar{c})$, there is a $\bar{x} \in W^{-1}(\bar{c})$ with $\bar{x} \subset x' \subset x$. Thus we have that $c <_i \bar{c}$, and $c, \bar{c} \in \hat{C}_i$, thus $\bar{c} \subseteq c$. Since $\bar{c} \subseteq c$ for all $\bar{c} \in \mathcal{X}$ and $c' = \cup_{\bar{c} \in \mathcal{X}} \bar{c}$, it follows that $c' \subseteq c$.

Let $\hat{G} \xrightarrow{ADD} G$. Suppose G has perfect recall. Consider player $j \in I \setminus \{i\}$ and $\hat{c}, \hat{c}' \in \hat{C}_j$, $\hat{c} \mathrel{\hat{<}_j} \hat{c}'$, i.e. there is $\hat{x} \in \hat{W}^{-1}(\hat{c}), \hat{x}' \in \hat{W}^{-1}(\hat{c}')$ with $\hat{x}' \subset \hat{x}$. There are choices $c, c' \in C_j$ in the larger game G such that

$$c = \hat{c} \cup \bigcup_{\hat{z} \in \hat{W}^{-1}(\hat{c}) \cap N_{c^*}} \bigcup_{c \in A \setminus c^*} \phi_{c^* c}(\hat{z}) \cup \bigcup_{\hat{y} \in \hat{W}^{-1}(\hat{c})} \bigcup_{\hat{y} \supseteq \hat{x} \in \hat{\mathcal{X}}} \psi(\hat{x})$$

and

$$c' = \hat{c}' \cup \bigcup_{\hat{z} \in \hat{W}^{-1}(\hat{c}') \cap N_{c^*}} \bigcup_{c \in A \setminus c^*} \phi_{c^* c}(\hat{z}) \cup \bigcup_{\hat{y} \in \hat{W}^{-1}(\hat{c})'} \bigcup_{\hat{y} \supseteq \hat{x} \in \hat{\mathcal{X}}} \psi(\hat{x}).$$

For $\hat{x} \in \hat{W}^{-1}(\hat{c})$, either $\hat{x} \in W^{-1}(c)$ or $\hat{x} \notin W^{-1}(c)$. In the first case, it must be that $\hat{x}' \in W^{-1}(c')$ and therefore $c <_i c'$. Perfect recall implies $c' \subseteq c$ and thus $\hat{c}' = c' \cap \hat{W} \subseteq c \cap \hat{W} = \hat{c}$.

Let $\hat{x} \notin W^{-1}(c)$, i.e. there is $\bar{x} \in \hat{\mathcal{X}}$ with $\hat{x} \supseteq \bar{x}$. Then there is $x \in W^{-1}(c)$ such that $x = \hat{x} \cup \bigcup_{\hat{x} \supseteq \bar{x}' \in \hat{\mathcal{X}}} \psi(\bar{x}')$. If $\hat{x}' \in \hat{W}^{-1}(c')$, then $\hat{x}' \subset x$ and thus $c <_j c'$.

If $\hat{x}' \notin \hat{W}^{-1}(c')$, then $\hat{x}' \supseteq \bar{x}' \in \hat{\mathcal{X}}, x' = \hat{x}' \cup \bigcup_{\hat{x}' \supseteq \bar{x}'' \in \hat{\mathcal{X}}} \psi(\bar{x}''), x' \subset x$ and therefore $c <_j c'$.

Now consider player i. Let $\hat{c}, \hat{c}' \in \hat{C}_i, \hat{c} \mathrel{\hat{<}_i} \hat{c}'$, i.e. there are $\hat{x} \in \hat{W}^{-1}(\hat{c}), \hat{x}' \in \hat{W}^{-1}(\hat{c}')$ with $\hat{x}' \subset \hat{x}$. Again, there are corresponding choices in the larger game denoted by c, c'. If $\hat{x} \in W^{-1}(c)$, then also $\hat{x}' \in W^{-1}(c')$ and hence $c <_i c'$ and perfect recall implies $c' \subseteq c$. If none of the choices is in A, then $\hat{c}' = c' \cap \hat{W} \subseteq c \cap \hat{W} = c$. If $c' \in A$, then $\hat{c}' \subseteq c' \cap \hat{W} \subseteq c \cap \hat{W} = \hat{c}$. If $c \in A$, then $\hat{c} \in \hat{A}$ and $\hat{c}' = c'$. If follows that $\hat{c}' = c' \cap \hat{W} \setminus W(\hat{\mathcal{X}}) \subseteq c \cap \hat{W} \setminus W(\hat{\mathcal{X}}) = \hat{c}$. The first equality comes from the fact that $\hat{W}^{-1}(\hat{c}') \cap N_{c^*} = \emptyset$.

Let there be $\bar{x} \in \hat{\mathcal{X}}$ with $\bar{x} \subseteq \hat{x} \in \hat{W}^{-1}(\hat{c})$. Then there is $x \in W^{-1}(c), x = \hat{x} \cup \bigcup_{\hat{x} \supseteq \bar{x}' \in \hat{\mathcal{X}}} \psi(\bar{x}')$. If $\hat{x}'$ is not a predecessor of $\hat{\mathcal{X}}$, then $\hat{x}' \in W^{-1}(c')$ and thus $c <_i c'$. If $c' \in A$, then $\hat{c}' = c' \cap \hat{W} \setminus W(\hat{\mathcal{X}}) \subseteq c \cap \hat{W} = \hat{c}$. If $c' \notin A$, then $\hat{c}' = c' \cap \hat{W} \subseteq c \cap \hat{W} = \hat{c}$.

If $\hat{x}'$ is a predecessor of $\hat{\mathcal{X}}$, then we do the same as for player j. Therefore, we have shown that for all $k \in I$ and all choices $\hat{c}, \hat{c}' \in \hat{C}_k, \hat{c} \mathrel{\hat{<}_k} \hat{c}'$ implies $\hat{c}' \subseteq \hat{c}$, thus $\hat{G}$ satisfies perfect recall.

Now let $\hat{G}$ have perfect recall and consider player $j \neq i$ first. Let $c, c' \in C_j$ such that $c <_j c'$, i.e. there are $x \in W^{-1}(c), x' \in W^{-1}(c')$ with $x' \subset x$. There are corresponding choices $\hat{c}, \hat{c}' \in \hat{C}_j$ such that

$$c = \hat{c} \cup \bigcup_{\hat{z} \in \hat{W}^{-1}(\hat{c}) \cap N_{c^*}} \bigcup_{c \in A \setminus c^*} \phi_{c^* c}(\hat{z}) \cup \bigcup_{\hat{y} \in \hat{W}^{-1}(\hat{c})} \bigcup_{\hat{y} \supseteq \hat{x} \in \hat{\mathcal{X}}} \psi(\hat{x}).$$

First, we show $\hat{c} \mathrel{\hat{<}_j} \hat{c}'$ and second that $c' \subseteq c$.

Let $x \in W^{-1}(c) \cap N_{\tilde{c}}, \tilde{c} \in A$. Note that $\phi^{-1}_{c^*\tilde{c}}(x) \in N_{c^*}, \phi^{-1}_{c^*\tilde{c}}(x) \in W^{-1}(c)$ and $\phi^{-1}_{c^*\tilde{c}}(x) \in \hat{W}^{-1}(\hat{c})$. Since $x' \subset x$, $\phi^{-1}_{c^*\tilde{c}}(x') \subset \phi^{-1}_{c^*\tilde{c}}(x)$ and therefore $\phi^{-1}_{c^*\tilde{c}}(x') \in \hat{W}^{-1}(\hat{c}')$, thus $\hat{c} \mathrel{\hat{<}_j} \hat{c}'$.

Let $x \supseteq \tilde{x} \in \mathcal{X}$. By construction of x, for all $\bar{x} \in \hat{\mathcal{X}}, \bar{x} \subseteq x$, it is true that $\psi(\bar{x}) \subseteq x$. Thus there is $\hat{x} \in \hat{W}^{-1}(\hat{c})$ with $\bar{x} \subseteq \hat{x}$ if and only if $\bar{x} \subseteq x, \bar{x} \in \hat{\mathcal{X}}$. If $x' \in N_{\tilde{c}}, \tilde{c} \in A$, then there is $\tilde{x} \in \mathcal{X}$ with $x' \subset \tilde{x} \subseteq x$ and $\hat{x}' \in \hat{W}^{-1}(\hat{c}')$ such that $\hat{x}' = \phi^{-1}_{c^*\tilde{c}}(x') \subset \hat{x}$. If $x' \supseteq \tilde{x} \in \mathcal{X}$, then for all $\bar{x} \in \hat{\mathcal{X}}$ with $\bar{x} \subseteq x'$, it holds that $\bar{x} \subseteq x$. Thus $\hat{x}' \subset \hat{x}$. If for all $\tilde{x} \in \mathcal{X}$ we have $x' \cap \tilde{x} = \emptyset$, then $x' = \hat{x}' \in \hat{W}^{-1}(\hat{c}')$ and thus $\hat{x}' \subset \hat{x}$.

If $x \in W^{-1}(c)$ is such that for all $\tilde{x} \in \mathcal{X}, x \cap \tilde{x} = \emptyset$, then $x \in \hat{W}^{-1}(\hat{c})$ and since $x' \subset x$, it follows that $x' \in \hat{W}^{-1}(\hat{c}')$. In every case we get that $\hat{c} \mathrel{\hat{<}_j} \hat{c}'$ and perfect recall in $\hat{G}$ implies $\hat{c}' \subseteq \hat{c}$.

Now we show $c' \subseteq c$. First, note that $\hat{c} \subseteq c$ and $\hat{c}' \subseteq c'$. Let $w \in c'$. If $w \in \hat{c}'$, then $w \in \hat{c} \subseteq c$. Let $w \in c' \setminus \hat{c}'$. Clearly, there is $y' \in W^{-1}(c')$ such that $w \in y'$. If $y' \in N_{\tilde{c}}, \tilde{c} \in A$, then $\phi^{-1}_{c^*\tilde{c}}(y') \in W^{-1}(c') \cap N_{c^*}$. Denote $\hat{y}' = \phi^{-1}_{c^*\tilde{c}}(y')$

and observe $\hat{y}' \in \hat{W}^{-1}(\hat{c}')$. Since $\hat{c}' \subseteq \hat{c}$, by Proposition 1 there is $\hat{y} \in \hat{W}^{-1}(\hat{c})$ with $\hat{y}' \subset \hat{y}$. If $\hat{y} \in N_{c^*}$, then $\phi_{c^*\tilde{c}}(\hat{y}) \in W^{-1}(c), \phi_{c^*\tilde{c}}(\hat{y}') \subset \phi_{c^*\tilde{c}}(\bar{y})$ and $w \in c$. If $\hat{y} \supset \bar{y} \supset \hat{y}', \bar{y} \in \hat{\mathcal{X}}$, then $\phi_{c^*\tilde{c}}(\hat{y}) \subset y$ and thus $w \in c$.

If $y' \supseteq \bar{y} \in \hat{\mathcal{X}}$, then there is $\tilde{c} \in A$ such that $w \in \phi_{c^*\tilde{c}}(\bar{y}) \subset y'$. Hence, there is $\hat{y}' \in \hat{W}^{-1}(\hat{c}')$ with $\bar{y} \subseteq \hat{y}' \subset y'$. Furthermore, there is $\hat{y} \in \hat{W}^{-1}(\hat{c})$ such that $\hat{y}' \subset \hat{y}$ and $w \in \hat{y} \cup \phi_{c^*\tilde{c}}(\bar{y}) \subseteq y$, thus $w \in c$.

Now, consider player i. Let $c, c' \in C_i \setminus A$ and let $c <_i c'$, i.e. there are $x \in W^{-1}(c), x' \in W^{-1}(c')$ with $x' \subset x$. First, we deal with the case $x \in W^{-1}(c) \cap N_{\tilde{c}}, \tilde{c} \in A$. From Definition 12 we know that $W^{-1}(c) \subseteq N_{\tilde{c}}$ and since $x' \subset x$, also $W^{-1}(c') \subseteq N_{\tilde{c}}$. Therefore, there are choices $\hat{c}, \hat{c}' \in \hat{C}_i$ such that $\hat{x} = \phi^{-1}_{c^*\tilde{c}}(x) \in \hat{W}^{-1}(\hat{c}), \hat{x}' = \phi^{-1}_{c^*\tilde{c}}(x') \in \hat{W}^{-1}(\hat{c}')$. By the properties of $\phi_{c^*\tilde{c}}, \hat{x}' \subset \hat{x}$, thus $\hat{c} \mathrel{\hat{<}_i} \hat{c}'$ and by perfect recall in $\hat{G}, \hat{c}' \subseteq \hat{c}$. Before we show $c' \subseteq c$, we consider a second case.

Second, let there be $\tilde{x} \in \mathcal{X}$ and $\tilde{c} \in A$ with $x' \subset \tilde{x} \subseteq x$, i.e. $x' \in N_{\tilde{c}} \cap W^{-1}(c')$. By construction of $c, c' \in C_i$, there are corresponding choices $\hat{c}, \hat{c}' \in \hat{C}_i$ such that

$$c = \hat{c} \cup \bigcup_{\hat{y} \in \hat{W}^{-1}(\hat{c})} \bigcup_{\hat{y} \supseteq \hat{x} \in \hat{\mathcal{X}}} \psi(\hat{x})$$

and

$$c' = \bigcup_{\hat{z} \in \hat{W}^{-1}(\hat{c}') \cap N_{c^*}} \phi_{c^*\tilde{c}}(\hat{z}).$$

There are $\hat{x} \in \hat{W}^{-1}(\hat{c}), \hat{x}' \in \hat{W}^{-1}(\hat{c}')$ with $x = \hat{x} \cup \bigcup_{\hat{x} \supset \bar{x}' \in \hat{\mathcal{X}}} \psi(\bar{x}')$ and $\hat{x}' = \phi^{-1}_{c^*\tilde{c}}(x')$. Since $\hat{x}' \subset \tilde{x}$, there is $\bar{x} \in \hat{\mathcal{X}}, \hat{x}' \subset \bar{x} \subseteq \hat{x}$, thus $\hat{x}' \subset x$.Consequently, $\hat{c} \mathrel{\hat{<}_i} \hat{c}'$ and by perfect recall $\hat{c}' \subseteq \hat{c}$. Let $w \in c'$. There is $y' \in W^{-1}(c')$ with $w \in y'$.

For both cases, c' is constructed such that there is $\hat{y}' = \phi^{-1}_{c^*\tilde{c}}(y') \in \hat{W}^{-1}(\hat{c}')$. By Proposition 1 there is $\hat{y} \in \hat{W}^{-1}(\hat{c})$ with $\hat{y}' \subset \hat{y}$. Furthermore, there is $\bar{y} \in \hat{\mathcal{X}}$ with $\hat{y}' \subset \bar{y}$. If $\hat{y} \subset \bar{y}$, then $w \in \phi_{c^*\tilde{c}}(\hat{y}) \subseteq c$. If $\bar{y} \subseteq \hat{y}$, then $w \in \phi_{c^*\tilde{c}}(\bar{y}) \subseteq c$

Now, we consider cases in which $W^{-1}(c') \cap N_{\tilde{c}} = \emptyset$ for all $\tilde{c} \in A$. There are again corresponding choices $\hat{c}, \hat{c}'$ and $\hat{x} \in \hat{W}^{-1}(\hat{c}), \hat{x}' \in \hat{W}^{-1}(\hat{c}')$ nodes corresponding to x, x'. We consider three different cases and for each of the cases we show $c <_i c'$ implies $\hat{c} \mathrel{\hat{<}_i} \hat{c}'$. Then we show that this implies $c' \subseteq c$.

Let there be $\tilde{x} \in \mathcal{X}$ with $\tilde{x} \subseteq x' \subset x$. There is $\bar{x} \in \hat{\mathcal{X}}$ such that $\bar{x} \subseteq \tilde{x}$, thus $\bar{x} \subseteq \hat{x}' \subset \hat{x}$ holds. If $\tilde{x} \subseteq x$, but $\tilde{x}' \cap x' = \emptyset$ for all $\tilde{x}' \in \mathcal{X}$, then $x' \in \hat{W}^{-1}(\hat{c}')$ and $\hat{x} = x \cap \hat{W} \supset x' \cap \hat{W} = x'$. If $x \cap \tilde{x} = \emptyset$ for all $\tilde{x} \in \mathcal{X}$, then $x \in \hat{W}^{-1}(\hat{c}), x' \in \hat{W}^{-1}(\hat{c}')$ and thus $\hat{c} \mathrel{\hat{<}_i} \hat{c}'$ and $\hat{c}' \subseteq \hat{c}$.

Let $w \in c'$. If $w \in \hat{c}'$, then $w \in c$. If $w \notin \hat{c}'$, then there is $y' \in W^{-1}(c')$ with $w \in y'$ and $\bar{y} \in \hat{\mathcal{X}}, \bar{y} \subset y', \tilde{c} \in A$ such that $w \in \phi_{c^*\tilde{c}}(\bar{y})$. There is $\hat{y}' \in \hat{W}^{-1}(\hat{c}')$ with

$\bar{y} \subseteq \hat{y}' \subseteq y'$, by Proposition 1 there is $\hat{y} \in \hat{W}^{-1}(\hat{c})$ with $\hat{y}' \subset \hat{y}$ and $y \in W^{-1}(c)$ with $\hat{y} \subseteq y$. Thus $w \in \phi_{c^*\tilde{c}} \subseteq c$.

Now, let $c' \in A, c \in C_i, c <_i c'$. By Definition 12 we know that $c' \subseteq c$. Let $c \in A, c' \in C_i, c <_i c'$. If $W^{-1}(c') \cap N_c \neq \emptyset$, then $W^{-1}(c') \subseteq N_c$ and thus $c' \subseteq c$. If $W^{-1}(c') \cap N_c = \emptyset$, then $x' \in \hat{W}^{-1}(c'), c' \in \hat{C}_i$ and there is $\hat{c} \in \hat{A}, \hat{c} \subseteq c, x \in \hat{W}^{-1}(\hat{c})$. Thus $\hat{c} \mathrel{\hat{<}_i} c'$, by perfect recall $c' \subseteq \hat{c} \subseteq c$. $\square$

4 The Theorem

Definition 15. Two extensive form games $G, \hat{G}$ are said to have the same *semi-reduced normal form* if there is

(i.) a bijection $b : I \to \hat{I}$,

(ii.) for each $i \in I$, a bijection $b_i : S_i^* \to \hat{S}_{b(i)}^*$,

(iii.) for all $s^* \in S^*$, $\pi(s^*) = \hat{\pi}(\hat{s}^*)$, where $\hat{s}_{b(i)}^* = b_i(s_i^*)$ for all $i \in I$.

Definition 16. An extensive form game G is in *reduced form* if

(i.) for all $i \in I$, we have $H_i = \{X_i\}$,

(ii.) for all $w \in W, i \in I, \uparrow w \cap X_i \neq \emptyset$,

(iii.) for all $i, j \in I, i \neq j$, either for all $x \in X_i$ there is $y \in X_j$ with $x \subset y$ or for all $y \in X_j$ there is $x \in X_i$ with $y \subset x$,

(iv.) for all $i \in I, x \in X_i, y, z \in p^{-1}(x)$, there exist $x' \in X_i, y', z' \in p^{-1}(x')$ such that there are choices $c, c' \in C_i$ with $y, y' \in W^{-1}(c), z, z' \in W^{-1}(c')$, and $G_{y'}$ is not isomorphic to $G_{z'}$.

Thompson (1952) proves the following result, but we will provide a new proof.

Lemma 1. *If G and $\hat{G}$ are in reduced form, then they have the same semi-reduced normal form if and only if they are isomorphic.*

Before we proof the lemma, we proof a related lemma.

Lemma 2. *If G is an extensive form game in reduced form, then no player has strategically equivalent strategies.*

Proof. Since every player has only one information set, every strategy corresponds to one choice. Since for all players each play intersects nontrivially with the information set, there is a choice that selects the play. Let $i \in I$. We show that player i has no strategically equivalent strategies. We apply the INT transformation such that player i is the last to choose. Let $x \in X_i$ and $y, z \in p^{-1}(x)$. There are nodes $x' \in X_i, y', z' \in p^{-1}(x')$ and choices $c, c' \in C_i$ with $y, y' \in W^{-1}(c), z, z' \in W^{-1}(c')$. Furthermore, $G_{y'}$ is not isomorphic to $G_{z'}$. Player i is the last to choose, $y', z' \in E$, thus the only way the subgames cannot be isomorphic is through a violation to the preference requirement. We slightly abuse notation and say that $\pi_i(y') \neq \pi_i(z')$. Let $s_i(x) = c, s_i'(x') = c'$. Let the other players play a strategy combinations s_{-i} such that (s_i, s_{-i}) induces y' and (s_i', s_{-i}) induces z'. Thus there is a strategy profile s_{-i} with $\pi(s_i, s_{-i}) \neq \pi(s_i', s_{-i})$, therefore s_i, s_i' are not strategically equivalent. Since we can choose c, c' arbitrarily, no strategies of player i are strategically equivalent. $\square$

Now we prove Lemma 1.

Proof of Lemma 1. Let $G, \hat{G}$ be in reduced form. Let $G \cong \hat{G}$. Since no player has strategically equivalent strategies, there are bijections for each player $\beta_i : C_i \to S_i^*$ and $\hat{\beta}_i : \hat{C}_i \to \hat{S}_i^*$ such that $c = s_i(x) \in \beta_i(c)$ and $\hat{c} = \hat{s}_i(x) \in \hat{\beta}_i(\hat{c})$. Since G and $\hat{G}$ are isomorphic, there is a bijection $b : I \to \hat{I}$ and a bijection for every player $i \in I$ that preserves the preferences $b_i : S_i^* \to \hat{S}_i^*, b_i = \hat{\beta}_{b(i)} \circ \chi_i \circ \beta_i^{-1}$. Hence, the games have the same semi-reduced normal form.

Let $G, \hat{G}$ have the same semi-reduced normal form. From the definition we get the bijection $b : I \to \hat{I}$. Since the games are in reduced form, each equivalence class contains exactly one strategy and this strategy corresponds to one choice. Each strategy profile induces a unique outcome, thus there is a bijection $\omega : W \to \hat{W}$ and $\pi(w) = \hat{\pi}(\omega(w))$. Furthermore, we define $\nu(\{w\}) = \{\omega(w)\}$ and for $y \in N, \nu(y) = \bigcup_{w' \in W} \{\omega(w')\}$. This construction yields a bijection. The bijection for the choices is $\chi_i = \hat{\beta}_{b(i)}^{-1} \circ b_i \circ \beta_i$. $\square$

Lemma 3. *If G and $\hat{G}$ differ by one of INT, COA or ADD, then G and $\hat{G}$ have the same semi-reduced normal form.*

Proof. See Thompson (1952) and Elmes and Reny (1994). $\square$

For a given outcome $w \in W$, we can enumerate the up-set, that is the play $\uparrow w = \{y_0, y_1, y_2, \dots, y_N\}$, where $W = y_0 \supset y_1 \supset \dots \supseteq y_M = \{w\}$. Since the game is finite, $M \in \mathbb{N}$.

Definition 17. An information set $h \in H_i$ is *maximal* if it satisfies the following condition for all $w \in W$: If $\uparrow w \cap h = \emptyset$, then there is $y_t \in X_i \cap \uparrow w$ and $x \in h$ such that $y_t \supset x$, but $y_{t+1} \not\supseteq x$, for all $x \in h$.

We call $\uparrow w$ a path and if h is not maximal. A path that violates the required condition is a test path.

Lemma 4. *Let G be a game with perfect recall, $h \in H_i$ player i's first non-maximal information set (i.e. $\forall h' \in H_i$, if $h' <_i h$, then h' maximal) and for all $x \in X_i, |p^{-1}(x)| = 2$. Then there is a sequence of games with perfect recall $G_1, G_2, \ldots, G_L$ such that $G_1 = G$; consecutive games in the sequence differ by ADD; and the image of h in G_L is maximal.*

Proof. Since $h \in H_i$ not maximal, there is a test path $P = \uparrow w = \{y_0, y_1, \ldots, y_M\}$, $M \in \mathbb{N}$, with $y_0 = W$, $y_M = \{w\}$. Thus, $P \cap h = \emptyset$ and for all $t \in \{0, \ldots, M-1\}$ $X_i \ni y_t \supset x, x \in h$ implies $y_{t+1} \supseteq x$. Let $\mathcal{P}(h)$ denote the set of test paths for information set h in G.

Define J so that there is $x \in h$ with $y_J \supset x$ and $y_{J+1} \not\supset x'$, for all $x' \in h$. Furthermore, define the two sets:

$$A = \{t | y_t \in X_i, t > J\}$$

and

$$B = \{t \in A | \exists y \in h(y_t) \exists x \in h : x \subset y\}.$$

We consider all possible cases.

1. Let $A = \emptyset$. The game G_1 is constructed by $G \xrightarrow{ADD} G_1$ with $\hat{h} = h, \hat{\mathcal{X}} = \{y_{J+1}\}, \mathcal{X} = \{x_1\}$. This construction is compatible with perfect recall. To see this, let $c_1, c_1' \in C_{1_i}, c_1' <_{1_i} c_1, P_1(c_1) = h_1 = h \cup \mathcal{X}, P_1(c_1') = h'$. There are corresponding choices $c, c' \in C_i, P(c) = h, P(c') = h' \in H_i, h' <_i h$ and thus h' maximal. Recall there is $x \in h, x \subset y_J$. Let $\bar{x} \in W^{-1}(c), \bar{x} \subset x$. Then $\bar{x} \subset y_J \subseteq y_{N+1} \subseteq y_N \in h' \cap \uparrow w$. The intersection is nonempty since h' maximal. Furthermore, $y_K \subseteq y_{N+1}$, thus h and y_K follow from the same choice c'. Perfect recall in G implies $c \subseteq c'$ and hence $c_1 \subseteq c_1'$.

2. Let $A \neq \emptyset, B = \emptyset$. Let $K = \min A$ and $m = |h(y_K)|$. Then G_1 can be constructed so that $G \xrightarrow{ADD} G_1$, with $\hat{h} = h, \hat{\mathcal{X}} = h(y_k), \mathcal{X} = \{x_1, \ldots, x_m\}$. In order that G_1 satisfies perfect recall, we must show that $h' <_i h(y_K)$ if and only if $h' <_i h$ and that there is a choice $c' \in C_i$ with $P(c') = h'$ that selects $h(y_K)$ and h. Let $h' <_i h$. We know h' is maximal, thus $P \cap h' = \{y_N\}, y_K \subset y_N$ and hence $h' <_i h(y_K)$. Let $h' <_i h(y_K)$, i.e. there is $x' \in h', \tilde{x} \in h(y_K), \tilde{x} \subset x'$. By the definition of K and $B = \emptyset$ there is $y_N \in h', y_K \subset y_N$. Since $x \subset y_J \subset y_N, h' <_i h$.

Let $h' <_i h$ and $h' <_i h(y_K)$. There is a choice $c' \in C_i, P(c') = h', y_{N+1} \in W^{-1}(c')$. For all choices $c \in C_i, P(c) = h$, we have $c' <_i c$ and for all $\tilde{c} \in C_i, P(\tilde{c}) = h(y_K)$ we have $c' <_i \tilde{c}$. Perfect recall implies $c \cup \tilde{c} \subseteq c'$.

3. Let $B \neq \emptyset$. Let $K = \max B, h^* = \{x' \in h(y_K) \mid \exists x \in h : x \subset x'\}$ and $m = |h(y_K) \setminus h^*|$. Then there is a choice $c' \in C_i, P(c') = h(y_K)$ and $W(h) \subseteq c'$. The game G_1 can be constructed so that $G \xrightarrow{ADD} G_1$ with $\hat{h} = h, \hat{\mathcal{X}} = W^{-1}(c') \cap \bigcup_{x' \in h(y_K) \setminus h^*} p^{-1}(x'), \mathcal{X} = \{x_1, \ldots, x_m\}$. That this construction is feasible follows the same kind of reasoning as in the other two cases.

Note that in every case $P \cap (h \cup \mathcal{X}) \neq \emptyset$, thus the path corresponding to P in G_1 is not a test path for $h_1 = h \cup \mathcal{X}$, the information set corresponding to h in G_1.

If h_1 is maximal, the sequence of games is complete. Suppose h_1 is not maximal. Then there is a mapping μ that maps test paths for the information set h_1 in G_1 to the set of test paths for the information set h in G, i.e. $\mu : \mathcal{P}_1(h_1) \to \mathcal{P}(h)$. By construction of G_1, $P_1 \in \mathcal{P}_1(h_1)$ implies $P_1 \in \mathcal{P}(h)$, but $P \notin \mu(\mathcal{P}_1(h_1))$, hence μ is not surjective and thus $|\mathcal{P}_1(h_1)| \leq |\mathcal{P}(h)| - 1$. Since the game G is finite, also the set $|\mathcal{P}(h)|$ is finite. Thus we can construct a sequence of games $G_1 \xrightarrow{ADD} G_2 \xrightarrow{ADD} \cdots \xrightarrow{ADD} G_L$, such that $h_k \subseteq h_{k+1}$ and $|\mathcal{P}_{k+1}(h_{k+1})| \leq |\mathcal{P}_k(h_k)| - 1, k = 1, \ldots, L-1$. For h_L, the image of h in G_L, it holds that $|\mathcal{P}_L(h_L)| \leq |\mathcal{P}(h)| - L = 0$, thus h_L is maximal in G_L. By Proposition 4 we know that all games have perfect recall, thus we have found the desired sequence of games.

□

Lemma 5. *If G is a game with perfect recall, then there exists a sequence of games with perfect recall $G_1, \ldots, G_L$, such that $G_1 = G$; consecutive games in the sequences differ by one of INT, COA, or ADD; and G_L is in reduced form.*

Proof. We state the algorithm of Elmes and Reny (1994) for constructing the relevant sequence of games.

Step 1. As in Thompson (1952) use COA and/or ADD (finitely often) to reduce or increase the number of immediate successors of each move to two.

Step 2. Use ADD (finitely often) to render each information set maximal as outlined in Steps 2.0, 2.1,... below.

Step 2.0. Let $i \in I$. Let $\lambda : H_i \to \{1, 2, \ldots, K\}, K \in \mathbb{N}, \lambda$ bijective and $h <_i h'$ implies $\lambda(h) < \lambda(h')$.

Step 2.1. Find the unique non-maximal information set h with the lowest label. Use the construction in the proof of Lemma 4 to render h maximal. This requires finitely many applications of ADD. Note that under 'Addition', an information set in the original game becomes either cloned, added to, left unchanged, or made larger in the sense that predecessors of the copied nodes are enlarged by the added nodes. We refer to the last two cases as "left unchanged." Maximal information sets in the original game that are unchanged or added to are maximal in the new game. Assign after every application of ADD the previous label to these information sets. Furthermore, copies of maximal information sets are maximal. Assign to each copied information set and each copy thereof the same label. Hence, the maximum label remains K.

Step 2.k. ($k \geq 2$). Render every $h \in H_i$ with $\lambda(h) = k$ maximal. After every application of ADD, label i's information sets according to the procedure outlined in Step 2.1. If one can render each $h \in H_i, \lambda(h) = k$ maximal with finitely many applications of ADD, then there are only non-maximal information sets with labels in $\{k+1, \ldots, K\}$ left. Therefore, the Steps 2.1-2.k are sufficient to make all information sets of player i maximal. It remains to show that Step 2.k. can be done in finitely many steps.

If $\lambda(h) = k$ and h is rendered maximal, then after relabeling, the number of information sets with label k is left unchanged. This means that the number of non-maximal information sets with k is strictly lower. To see this, note that two information sets with the same label must come from different choices at a maximal information set. Hence, no test path intersects the clone of a cloned information set. The construction in Lemma 5 shows that the only information sets that get cloned are the ones that intersect with the test path.

Maximal information sets of the other players remain maximal throughout the procedure, thus it suffices to look at one player at the time.

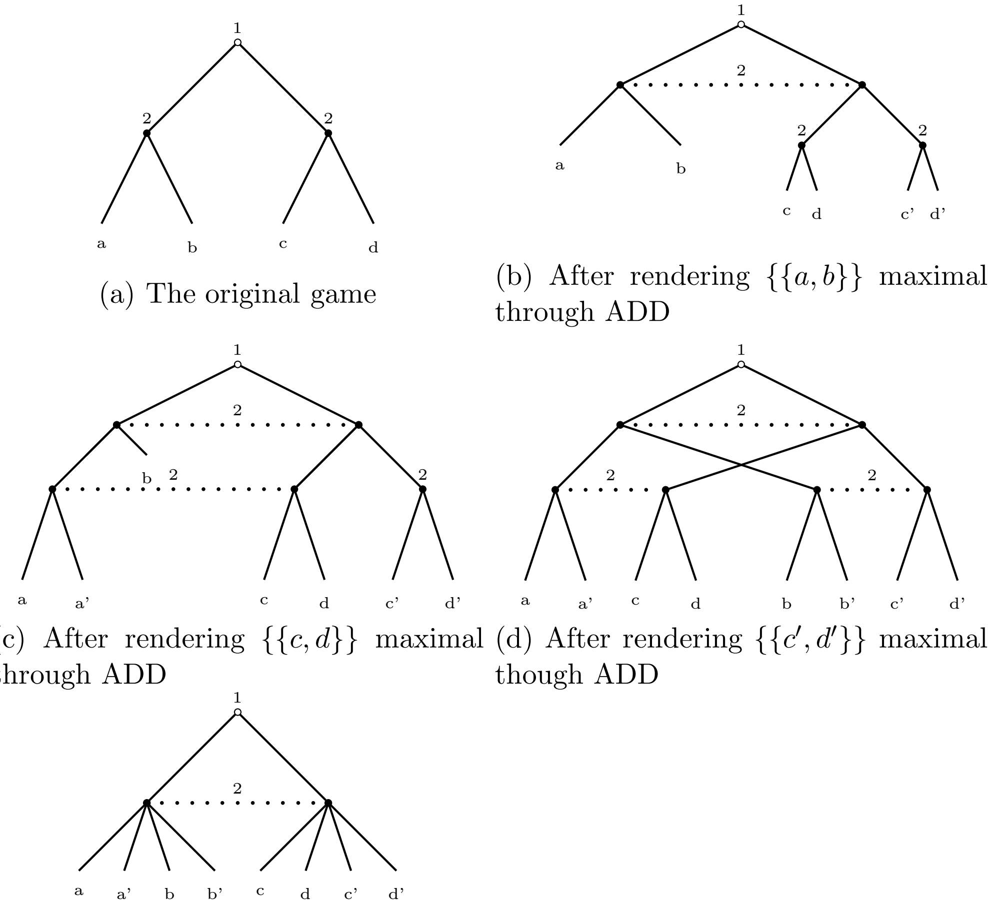
Step 3. Use COA and/or INT to produce a game in which every player has a single information set which intersects every play and in which each pair of information sets is ordered. This can be done in finitely many steps since all information sets are maximal and COA and INT preserve maximality.

Step 4. If the game is not in reduced form, then (iv) in Definition 16 must be violated. Use ADD and/or COA to remove y' (or z'). Finitely many such applications will yield a game in reduced form. □

Theorem 1. *If G and $\hat{G}$ are games with perfect recall and have the same semi-reduced normal form, then there exists a finite sequence of games with perfect recall $G_1, \ldots, G_L$ such that $G_1 = G, G_L = \hat{G}$, and consecutive games in the sequence differ by one of the transformations INT, COA, or ADD.*

Proof. The theorem follows from Lemmata 1,3, and 5. □

If two finite extensive form games differ by one of the Thompson transformations, then they have the same semi-reduced normal form. On the other hand,



(a) The original game

(b) After rendering $\{\{a, b\}\}$ maximal through ADD

(c) After rendering $\{\{c, d\}\}$ maximal through ADD

(d) After rendering $\{\{c', d'\}\}$ maximal though ADD

(e) Game in reduced form after applying COA twice

Figure 5: Example for the Algorithm

if two games have the same semi-reduced normal form, then they differ by the Thompson transformations.

Figure 5 provides a simple example for the algorithm. In Figure 5a, Player 1 has one information set that is maximal. Player 2 has two unordered information sets. The only way to order these two unordered information sets by using the ADD transformation. Let $\lambda(\{\{a, b\}\}) = 1, \lambda(\{\{c, d\}\}) = 2$. The information set $\{\{a, b\}\}$ is non-maximal and has the lowest label. A test path is $\uparrow c$. Figure 5b shows the game after the application of ADD, rendering $\{\{a, b\}\}$ maximal, while using test path $\uparrow c$. Note that the information set $\{\{c, d\}\}$ gets cloned. The information set and its copy get the same label assigned. A test path for $\{\{c, d\}\}$ is $\uparrow a$. Figure 5c shows the game after rendering $\{\{c, d\}\}$ maximal, using test path $\uparrow a$. After another application of ADD, all information sets are maximal (Figure 5d). Now we coalesce the information sets of player 2 to a single information set. The game in Figure 5e is in reduced form.

5 Conclusion

Non-cooperative game theory is game theory with complete rules. The rules are complete if the game can be written as extensive form game. In game theory, most solution concepts and most prominently the Nash equilibrium, are defined on the normal form. Kohlberg and Mertens (1986) argue that for rational players, the normal form is the correct object for strategic analysis. The normal form is derived from an extensive form and the semi-reduced normal form is the partition induced by the binary relation "strategic equivalence." Elmes and Reny (1994) state necessary and sufficient conditions for two extensive form games having the same semi-reduced normal form. The work of Elmes and Reny (1994) goes back to the early days of game theory. It turns out that two extensive forms games have the same semi-reduced normal form if and only if they differ by the Thompson transformations. These transformations allow under certain conditions the interchanging of moves, coalescing of information sets and the addition of superfluous decision nodes. This is done by preserving perfect recall.

The idea of the proof of the theorem is an algorithm constructed by Elmes and Reny (1994). This algorithm uses the Thompson transformations in order to bring any finite extensive form game into reduced form. Note that no transformation changes the the semi-reduced normal form game derived from the extensive form. Since one can do that with any finite extensive form and since two games in reduced form are isomorphic if and only if they have the same semi-reduced form, the theorem follows.

However, one might ask whether this theorem holds for large extensive form games with perfect recall. Finiteness is important in the construction of the algorithm. A label is assigned to every information set. The finiteness is used for the convergence of the algorithm. Wether and how the theorem holds for large games is open for research.

References

Alós-Ferrer, Carlos, Johannes Kern, and Klaus Ritzberger (2011), "Comment on "trees and extensive forms"." *Journal of Economic Theory*, 146, 2165–2168.

Alós-Ferrer, Carlos and Klaus Ritzberger (2005), "Trees and decisions." *Economic Theory*, 25, 763–798.

Alós-Ferrer, Carlos and Klaus Ritzberger (2008), "Trees and extensive forms." *Journal of Economic Theory*, 143, 216–250.

Alós-Ferrer, Carlos and Klaus Ritzberger (2013), "Large extensive form games." *Economic Theory*, 52, 75–102.

Dalkey, N. (1953), "Equivalence of information patterns and essentially determinate games." In *Contributions to the Theory of Games*, volume 2, 217–245, Princeton University Press.

Elmes, Susan and Philip J. Reny (1994), "On the strategic equivalence of extensive form games." *Journal of Economic Theory*, 62, 1–13.

Kohlberg, Elon and Jean-Francois Mertens (1986), "On the strategic stability of equilibria." *Econometrica*, 54, 1003 – 1037.

Krentel, W.D., J.C.C. McKinsey, and W.V. Quine (1951), "A simplification of games in extensive form." *Duke Mathematical Journal*, 18, 885–900.

Kuhn, Harold (1950), "Extensive games." *Proceeding of the National Academy of Sciences of the United States of America*, 36, 570–576.

Myerson, Roger B. (1978), "Refinements of the nash equilibrium concept." *International Journal of Game Theory*, 7, 73–80.

Ritzberger, Klaus (2002), *Foundations of Non-Cooperative Game Theory*. Oxford University Press.

Thompson, F.B. (1952), "Equivalence of games in extensive form." *RAND Research Memorandum*, RM-759.

von Neumann, John and Oskar Morgenstern (1944), *Theory of Games and Economic Behavior*. Princeton University Press.