MSc Economics

The Discrete Vickrey Auction and the Role of Ties

A Master's Thesis submitted for the degree of "Master of Science"

supervised by

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Vienna, June 9, 2014

"In our opinion, truths of this kind should be drawn from notions rather than from notations"

— Carl Friedrich Gauss, *Disquisitiones Arithmeticae, Article 76 (1801).*

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Acknowledgments

I am indebted to Klaus Ritzberger for his careful supervision and his rigorous advice throughout the writing process.

I have also received valuable feedback from other faculty members, among them François Forges from the Université Paris-Dauphine and Jamsheed Shorish. I am also very grateful to Jamsheed for allowing me to help him as a teaching assistant for the course "Applied Game Theory" here at the master program in Vienna.

Many of my colleagues from the master program helped me to build some of the ideas presented here. In particular, I appreciate the suggestions from Philipp Hergovich, Karol Mazur, Alexander Satdarov, László Tétényi and Gabriel Ziegler. Besides, I received valuable feedback from the participants of the Joint Vienna Microeconomics Seminar.

I also want to express my gratitude to all the professors that taught me Game Theory or related courses. First, at Guanajuato, I benefited from great introductory courses by Antonio Jiménez-Martínez, Luis Sánchez-Mier and Alejandro Moreno. Later, I took advanced courses in Vienna from Klaus Ritzberger and in Maastricht from Andrés Perea, Christian Bach, and Elias Tsakas. All of them motivated me to study topics where strategic interaction is crucial like auction theory.

During the last two years, I have received financial support from the Mexican Central Bank, which has been incredibly helpful. My sincere gratitude goes to them.

But above all, I could not have made it through the whole master program at Vienna without the permanent support and encouragement of my parents Oliva and Elías. My siblings, Diana and Mario, have also motivated me throughout the last two years. This work, which finishes my master student phase before starting my doctoral studies, is dedicated to them.

MSc Economics

Affidavit

I, Josue Alberto Ortega Sandoval

hereby declare

that I am the sole author of the present Master's Thesis,

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36 pages, bound, and that I have not used any source or tool other than those referenced or any other illicit aid or tool, and that I have not prior to this date submitted this Master's Thesis as an examination paper in any form in Austria or abroad.

Vienna, June 9, 2014

Signature

Contents

List of Figures

Abstract

A typical assumption in the auction literature is that bidders can submit any bid within the non-negative part of the real line. Yet, when auctions are used the set of bids is restricted at least to a countably infinite set due to the discrete nature of currency, and often even to a finite set. We study the effects of explicitly modeling a restricted set of allowed bids in a Vickrey auction within the independent values framework.

We present a characterization of all pure strategy equilibria of the unrestricted Vickrey auction, obtained by Blume and Heidhues (2004), and analyze how this set changes when we restrict ourselves to the Vickrey auction with a restricted set of bids, although we do not provide a complete description of this set. We discuss the consequences of the change of the equilibria on the seller's revenue and the consequences of a set of equilibria that is present in both models.

We examine a tool that is available for the auctioneer to obtain a positive expected revenue, namely a proper specification ex-ante of a dictatorial tie-breaking rule. We also explore the consequences of introducing of a reservation price in our setup, knowing that it yields a unique equilibrium in the non-restricted allowed bids case, as shown by Blume and Heidhues (2001).

Finally, we summarize our contributions and provide a list of the several questions that remain open in the field.

Chapter 1

Introduction

When auctions are used in practice, it is often the case that bids are restricted to a number of amounts previously specified by the auctioneer. Yet, most of the game theoretic analysis of auctions ignores this issue for simplicity, and a consequence of doing so is that specifying a tie-breaking rule becomes irrelevant. However, if we construct a model where bids are restricted to a finite number of points, the specification of the tie-breaking rule becomes important for technical and practical reasons.

The technical reason is that in infinite games the tie-breaking rule may determine the existence of an equilibrium, as shown by Simon and Zame (1990). Although the strategy space becomes finite by allowing for the possibility of ties, the dimension of the type space can still give us an infinite game where the tie-break specification pins down the equilibrium existence, by eliminating the indeterminacies caused by the natural discontinuity of the auction payoffs. The practical reason is that the possibility of ties may affect the behavior of the bidders and thus, the expected auctioneer's revenue. Hence, a natural question for the auctioneer is which is the tie-breaking rule that gives him the highest profit in expectation. We might also suspect that tie-breaking rules can be important in auctions because they do matter in related fields, for example implying a tradeoff between stability and strategy-proofness in matching problems, as shown by Erdil and Ergin (2008), or changing the complexity of voting manipulation, as shown by Obraztsova and Elkind (2011) and Obraztsova et al. (2011).

We study the role of tie-breaking rules using second price auctions, as they provide an interesting framework to work with. They do not only share the interesting properties of Vickrey-Clarke-Groves mechanisms, like strategy-proofness and efficiency in the single item case, but have also been widely and successfully used to sell online advertising, see Edelman et al. (2007). In these auctions, bids represent the price per click, which rarely surpasses 50 dollars. Several bidders try to get one of several ad slots. It is an interesting question how different tie-breaking specifications would modify their behavior.

Still, we restrict ourselves to the one item, sealed-bid second price auctions within the independent value framework to lay the basis upon which further studies could rely. We intend to characterize all equilibria of the one item, sealed bid, second price auction at least in pure strategies in the spirit of Blume and Heidhues (2001) and Blume and Heidhues (2004).

Our first conclusion is that by allowing for ties in the auction (i.e. by restricting the set of allowed bids) the set of equilibria of the auction changes in a non-trivial way, which implies important consequences for the auctioneer's revenue. We also find that the number of participants in the auction determines the classes of equilibria and the equilibrium outcomes that can be achieved through the second price auction in this particular setup.

After fully describing the set of pure Nash equilibria, we analyze whether this set is robust to changes of the specification of the tie-breaking rule. We find that a specific class of tie-breaking rules, which are dictatorial, can restore equilibria that lead to a positive revenue in expectation, something that cannot be achieved under the classical tie-breaking rule or small perturbations of it. We also discuss the effect of another tools that the auctioneer could use to increase his revenue, namely the introduction of a reservation price.

In the end we conclude the paper with a review of the results and a brief discussion of the research opportunities that are still open in the field.

This thesis is structured as follows. Section 2 provides a review of the literature regarding second price auctions, tie-breaking rules, and auctions with restricted allowed bids in economics as in related fields. Section 3 presents important results for the case with continuous bidding possibilities and formulates our contributions. Section 4 concludes the paper and presents a critical discussion of what is left for future research. Finally, the proof of our main results can be found in the Appendix.

Chapter 2

Literature Review

2.1 On Second Price Auctions

Second price auctions were first studied by Vickrey (1961), who formally proposed the idea of this allocation method. Vickrey noted that introducing the second price rule eliminated strategic behavior as there exists a weakly dominant, or admissible, strategy for each player, which consists of revealing his own valuation. Vickrey restricted his analysis to auctions of a unique indivisible good. He also pointed out that the result of the auction was Pareto optimal and that the introduction of any k-price rule for any k different than 2 would restore strategic behavior, eliminating the truthful revelation of agents' private information.

Vickrey also derived the equilibrium bidding strategy of the sealed-bid first price auction, which he accurately described as unique, and established a revenue equivalence between the first price sealed-bid and the English auction. He also noticed the strategic equivalence between the English and his proposed sealed-bid second price auction.

However, Lucking-Reiley (2000) provided evidence that the origin of the sealed-bid second price auction dated at least 65 years earlier than Vickrey's paper. It turned out that postal stamps auctioneers in the United States were using sealed-bid second price auctions since the late 19$^{\text{th}}$ century, specifically in a stamp auction in 1893 held at Northampton, Massachusetts.

In any case, Vickrey's paper still had an enormous impact on the field and started the systematical study about second price auctions and, more generally, about truth-telling mechanisms. His study of auctions was one of the topics for which he received the Nobel Memorial Prize in Economic Sciences in 1996, jointly with James Mirrlees *"for their fundamental contributions to the economic theory of incentives under asymmetric information"*.[1]

Later, after the work of Clarke (1971) and Groves (1973), the sealed bid second price auction allocation mechanism was generalized to the "Vickrey-Clarke-Groves (VCG) mechanism". This is a pivotal allocation mechanism which achieves strategy-proof implementation of efficient allocations in quasi-linear environments, just as the one in the Vickrey auction. We will not talk about these kinds of mechanisms in general, but it is worth noting that the sealed-bid second price auction is a member of a larger class of allocation mechanisms, which are interested in maximizing social welfare. One could argue that, from the design perspective, the researcher should rather focus on mechanisms that maximize the auctioneer's revenue. This position has been adopted by some authors, see for example Myerson (1981).

Despite the theoretical enthusiasm regarding the properties of the sealed-bid second price auction, its application to solve real life problems was basically nonexistent. Several authors have proposed why second price auctions were so rarely used in practice, like Rothkopf et al. (1990), Ausubel and Milgrom (2006) or Rothkopf (2007). Common arguments in these papers are the existence of multiple equilibria, and the susceptibility of the auction to bidders' collusion.

As they point out, while Vickrey correctly described the truth telling equilibrium, there are many more equilibria in the second price auction (actually, there exists a continuum of equilibria). Some papers have pointed out some other equilibrium bidding functions, but it was not until 10 years ago that Blume and Heidhues (2004) characterized all (measurable) bidding functions that correspond to an equilibrium of the Vickrey auction within the independent values model, an assumption maintained by all previous authors.

The set of equilibria described in their paper can be categorized into three classes: first, there is the truth-telling or full revelation equilibrium, then there is a continuum of hybrid or collusive equilibria, and finally, there are also some threatening equilibria, which are also called bid-rotating in the literature. We will discuss them in detail in the next Chapter.

It is also worth mentioning that the use of second price auctions increased tremendously due to their successful use to sell online advertisement. For example, Google, Yahoo and Facebook make billions of dollars by selling their advertisement slots through either generalized second price or Vickrey auctions, see for example Lucier et al. (2012). Both of these methods are generalizations of

[1] "The Prize in Economic Sciences 1996 - Press Release". Nobelprize.org. Nobel Media AB 2013. Web. 26 May 2014. `http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/1996/press.html`

the simple second price auction to sell multiple goods simultaneously. The main difference while turning to multi-unit auctions is that second price and VCG mechanisms are no longer equivalent, because in the generalized second price auction bidding truthfully is not an equilibrium strategy, as shown by Edelman et al. (2007) and Varian (2007).

In a generalized second price auction (GSP), bidder i pays the bid submitted by player $(i-1)$, assuming we have ordered players by the amount of their bids, where 1 is the bidder with the highest bid. The VCG auction charges each bidder the externality they provoke with their presence to the remaining bidders, i.e. the difference between the aggregate value everyone could receive when the particular bidder is absent and the aggregate value the other bidders receive when the particular player actually participates in the auction.

Lucier et al. (2012) shows that under particular assumptions (convexity in the probability of obtaining the item), the GSP auction maximizes revenue and social welfare at the same time. Without such an assumption there exists a tradeoff between efficiency and revenue maximization. Throughout this paper, we will focus exclusively on single-unit auctions, and hence we will not discuss further GSP and VCG auctions in the multi-unit case. Still, we find it very important to stress that the equivalence between the second price auction and the VCG mechanism in the single unit case is a coincidence and it does not hold for auctions that allocate several items simultaneously.

2.2 On Auctions with Discrete Bidding

Most of the literature concerning the study of auctions considers that bidders can submit any possible bid for simplicity. For example, in his seminal paper, Vickrey assumed continuous bidding, as much of the succeeding literature. He wrote:

> "*For simplicity, we shall assume that the price can vary continuously and that there is no minimum increment between bids.*"

However, in applications bids are usually restricted to a discrete set of bids, that is, at most countably infinite and often even finite. Actually, the first second price mechanisms of which we have knowledge of, described by Lucking-Reiley (2000), explicitly required at least bid increments of 1 to 10 U.S. dollar cents. Other auctions, like those conducted by eBay or by important auction houses as Christie's and Sotheby's, among others, also explicitly require bids that are multiple of some fixed number. Other cases where a finite bid grid was established in practice are described by Yu (1999).

The first paper that studies the role of discrete bids on the auction's outcome was due to Yamey (1972). He realized that in art auctions, the gap among allowed bids played a role in the final price that has to be paid. Almost two decades later, the first formal analysis of the equilibrium strategies in an auction was made by Chwe (1989). He studied a sealed-bid first price auction with finite and evenly spaced bid levels, within the independent private value (IPV) model where valuations are uniformly distributed. He showed that the expected price in the discrete bid case is smaller than in the continuous bid case. He also proved that the equilibrium in the discrete case is unique, symmetric, and converges to the equilibrium of the continuous bid auction as the distance between bid levels goes to zero.

In her Ph.D. thesis, Yu (1999) presents an extensive justification of why discrete bidding should be included in theoretical models. She also analyses the four common auction institutions (English, Dutch, first and second price) under the discreteness assumption. She establishes existence of equilibrium strategies directly from Kakutani's fixed point theorem, and she also describe some properties of the equilibrium bidding function. She imposes no restrictions on the distribution of valuations, but she restricts the bidding grid to be evenly spaced. While most of her results make intuitive sense, some of her conclusions seem preliminary and exhibit some mistakes, like using a fixed point theorem for finite spaces in an infinite game. The game is infinite because the support of bidders' valuations is assumed to be the whole unit interval.

Later, Mathews and Sengupta (2008) relaxed the restriction of evenly spaced bids, and at the same time limited to only auctions with two bidders. In this particular setup, they showed the existence of a symmetric Nash equilibrium in which players bid the nearest point in the grid to their valuations. They recognized that this trivially allows for a possibility of inefficiency, which increases as the number of bidding levels gets smaller.

In a somewhat different spirit to the previous papers, Rothkopf and Harstad (1994) and David et al. (2007) study the importance of optimal bid levels in internet auctions and how such optimal bid levels should be constructed by the auctioneer to maximize his revenue.

None of the mentioned papers covers a general case with arbitrary bid levels, distributions of valuations and number of bidders. Moreover, no paper describes all the equilibria that exists in their particular setting.

2.3 On Tie-Breaking Rules

There exists a fair amount of literature about tie-breaking rules in auctions as well as in other related allocation mechanisms. Most of them are concerned with the existence of equilibrium in auctions that are infinite games, either because of the existence of an infinite number of strategies or underlying types. An infinite number of bidders seems unrealistic and hence is not traditionally considered by the literature.

Maskin and Riley (2000) studied existence of equilibrium strategies in sealed first-price auctions, where players submit their bids privately and the highest bidder pays his bid. They established the equilibrium existence for cases when valuations can be dependent and asymmetrically distributed. They search for a monotonic equilibrium in which the equilibrium strategy is a function of the valuation. They use a Vickrey auction as a tie-breaking rule, with which they can prove existence for an auction with finitely many types where preferences satisfy a couple of regularity conditions.

Another approach is taken by Simon and Zame (1990). They tackled the question of equilibrium existence in infinite games with discontinuous payoffs in a different way as their predecessors. They propose to understand the tie-breaking rule as an endogenous feature of the model, rather than an exogenous description of it. Under such a condition, the game is guaranteed to have one equilibrium as the indeterminacies provoked by discontinuities disappear. They use the term sharing rule to be more general but such sharing rule is identical to a tie-breaking rule in any auction. Their paper was generalized to games with incomplete information by Jackson et al. (2002) under the presence of incentive compatible communication.

A critique to the approach of viewing the tie-breaking rule as an endogenous element of the solution of a game has been raised by Reny (1999). He expresses two main points: first, while the endogenous tie-break approach may be useful in particular games, it is inconvenient to analyze environments where discontinuities are introduced on purpose, like auctions. In those cases, players must know ex ante the consequences of their actions. The second point is that this approach restores equilibrium existence only in mixed strategies.

It is also worth pointing out that the paper by Jackson (2009) provides a proof of nonexistence of equilibrium in second price (and English) auctions where bidders' valuations are composed both by private and public information. Still, he argues that a type-dependent tie-breaking rule or a finite bidding grid may restore the existence of equilibrium in such an auction.

Finally, we will briefly discuss the literature related to tie-breaking rules that is not mainly concerned with equilibrium existence problems. In school choice matching, Erdil and Ergin (2008) show that the tie-break specification causes a tradeoff between the stability and the efficiency of the allocation mechanism. Particularly, the application of the deferred acceptance mechanism (as proposed by Gale and Shapley (1962)) after exogenous tie-breaking imposes artificial stability constraints that reduce the students' welfare.

The effect of tie-breaking rules have also been explored in computational social choice. Obraztsova et al. (2011) and Obraztsova and Elkind (2011) explore the effects of tie-breaking specifications on the complexity of manipulating voting rules. The first paper questions the role of an earlier assumption in social choice that ties are resolved in favor of the manipulator candidate, an assumption which allows to construct a results which states that most voting rules are easily manipulable. They show that the voting rules are still manipulable under a weaker condition, which imposes that ties are always broken by using a fixed ranking.

In the second paper, they explore the consequences of substituting a lexicographic for a random tie-breaking rule. They find that all scoring rules remain easy to manipulate, but the manipulation of both Copeland and Maximin voting rules becomes NP-hard. The interested reader is encouraged to look into the respective papers for additional details.

Chapter 3

The Model

3.1 General Setup of the Model

In this Section we will outline the setup of the model that was used by Blume and Heidhues (2004) to fully describe the set of equilibria in the sealed-bid second price auction. We will use almost the same setting to characterize all the equilibria of the sealed-bid second price auction with discrete bids. For brevity, we will call these auctions the Vickrey and the discrete Vickrey auction respectively. In both cases, we will restrict ourselves to measurable bidding functions, following the original paper, to ensure well-defined payoffs.

We will assume that the bidders' valuations have positive densities in the common support [0,1], which is a normalization for any closed interval. The valuations' distribution is assumed to be atomless and independent between players.[1]

The number of bidders is denoted by $\mathbf{N}$. The valuations are denoted by $\mathbf{v}$. Henceforth, we will say that a property is satisfied almost everywhere (AE) if the set of elements for which the property does not hold is a set of measure zero. Unless otherwise stated, we will only consider pure strategies, as Blume and Heidhues did.

3.2 All the Equilibria of the Vickrey Auction

In this section, we will explain the results from Blume and Heidhues (2004) because later we will compare the differences of equilibria between the continuous and the discrete bid case.

They impose no further conditions on equilibrium strategies aside from measurability. Their characterization of equilibria depends crucially on the number

[1] The original paper allows for atoms at 0.

of bidders. There is a clear distinction between the form of equilibria in auctions with only two bidders and those with at least three. Still, there are classes of equilibria that are present in both cases, as we will see. Only the working paper (Blume and Heidhues (2001)) presents the results of the particular case where $\mathbf{N} = \mathbf{2}$.

3.2.1 Equilibria with only 2 bidders

The following theorem characterizes the set of equilibria in the Vickrey auction with continuous bidding in the 2 bidders case. The proof can be found in the original paper.

Theorem 1 (Blume and Heidhues, 2001) *In the Vickrey auction (as defined in Section 3.1) with only two bidders, any equilibrium has the form that there exists a (possibly empty) collection* $\mathbf{C}$ *of nonempty disjoint subintervals of* $[0, 1]$ *such that:*

1. *on each interval* $(\mathbf{a}, \mathbf{b}) \in \mathbf{C}$ *for which* $\mathbf{b} < \mathbf{1}$*, one player bids* $\mathbf{a}$ *and the other bids* $\mathbf{b}$ *AE in* $(\mathbf{a}, \mathbf{b})$*,*

2. *if there exists an interval* $(\mathbf{a}, \mathbf{b}) \in \mathbf{C}$ *for which* $\mathbf{b} = \mathbf{1}$*, one player bids* $\mathbf{a}$ *and the other at or above* $\mathbf{b}$ *AE in* $(\mathbf{a}, \mathbf{b})$*,*

3. *and players bid their value AE in* $[\mathbf{0}, \mathbf{1}] \setminus \bigcup \mathbf{C}$

Conversely, for any collection $\mathbf{C}$ *of nonempty disjoint subintervals of* $[0, 1]$*, there exists an equilibrium of this form. Moreover, for any isolated element of* $\mathbf{C}$*, any described placement of mass points is part of an equilibrium. In adjoining elements of* $\mathbf{C}$ *the placement of mass points is restricted by the requirement that two bidders cannot have identical mass points.*

This practical construction allows us to see that when $\mathbf{C} = \emptyset$, we have the classical truth-telling equilibrium. This is the first class of equilibria and, as Vickrey pointed out, we will have it also when the number of bidders is greater than two.

The second class of equilibria can be seen when $\mathbf{C} = [0,1]$. In this case, we have $\mathbf{N}$ different equilibria (without accounting for deviations in measure zero points of $\mathbf{v}$). In both of them, one player always bids $\mathbf{0}$ and one player always bids at or above $\mathbf{1}$. It is interesting that in this class of equilibria, the actual realization of bidders' valuation does not matter for determining who obtains the auctioned item. We call this type of equilibrium a threatening equilibrium, because one player threats the other by always bidding a higher price than any valuation possible. Hence, the remaining player always has to bid 0. If he would move, then the threatening player could no longer sustain to always bid at or above the upper bound of the valuations' support as he would win paying a prohibitive price. This class of equilibria is also known as bid-rotating equilibria because, as we have discussed, the identity of the players does not play a role.

Of course, the biggest class of equilibria consists of those where $\mathbf{C}$ is non-trivial. Moreover, the number of subintervals contained in $\mathbf{C}$ is completely arbitrary. Any equilibrium that is an element of this last class will be called hybrid. Note that, in a sense, this kind of equilibria arises from a mixture of the threatening equilibria for a set of subintervals in [0,1], and where truth-telling holds for the remaining parts of [0,1].

We can rank these three classes of equilibria according to the corresponding auctioneer's revenue. In the truth-telling case, the expected revenue is the second order statistic. It is not hard to see that the revenue will necessarily be smaller as the cardinality of $\bigcup \mathbf{C}$ increases, and will eventually reach 0, i.e. the lower bound of the support of the distribution of valuations when $\mathbf{C}$ covers the whole unit interval. Hence, the truth-telling equilibrium is the outcome that is most preferred by the auctioneer, followed by any hybrid equilibrium, and finally the threatening equilibrium is the worst for the seller as it guarantees a revenue of zero. These properties will be useful for posterior analysis.

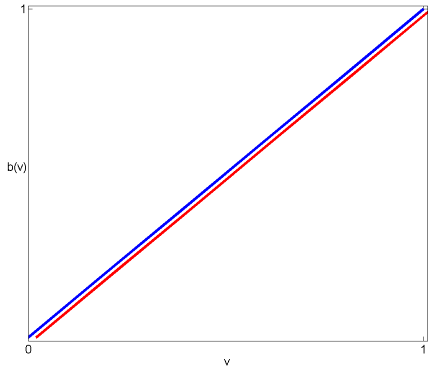
Figure 3.1 illustrates typical examples of the three kinds of equilibria in the two bidders case. In all the subfigures the red and the blue lines represent the bid functions of the two corresponding red and blue players.

3.2.2 Equilibria with at least three bidders

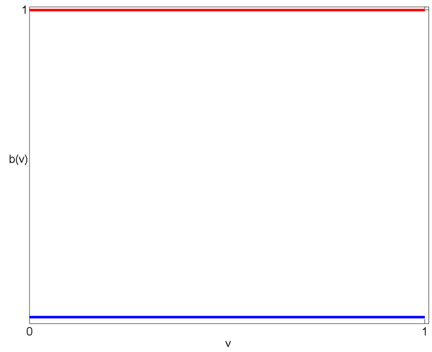
The characterization of the equilibria with more than two players is similar to the one we just presented, but still rules out many hybrid equilibria that exists for the two bidder case. The proof again can be found in any of the mentioned papers.

Theorem 2 (Blume and Heidhues, 2001 & 2004) *In the Vickrey auction (as defined in Section 3.1) with at least three bidders, a strategy profile is a Nash equilibrium if it there exists a $\hat{\mathbf{b}}$ such that:*

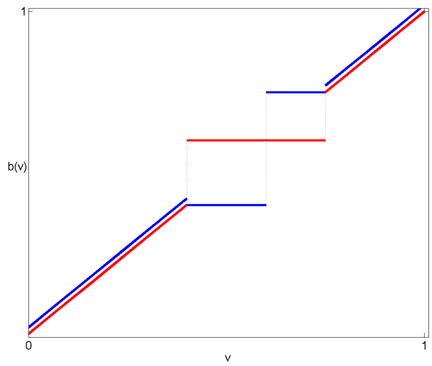
1. *any player with valuation $\mathbf{v} > \hat{\mathbf{b}}$ bids his valuation,*
2. *if $\hat{\mathbf{b}} < \mathbf{1}$, then there is one player who bids at $\hat{\mathbf{b}}$ whenever his valuation $\mathbf{v}$ satisfies $\mathbf{v} < \hat{\mathbf{b}}$, and if $\hat{\mathbf{b}} \geq \mathbf{1}$, then there is one player who bids at or above $\hat{\mathbf{b}}$ for any valuation $\mathbf{v}$,*



(a) Truth-telling Equilibrium



(b) Threatening Equilibrium



(c) Hybrid Equilibrium

Figure 3.1: Classes of Equilibria for the Vickrey Auction with $\mathbf{N} = \mathbf{2}$

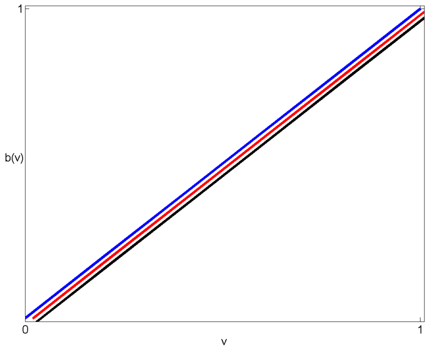
3. *all other players bid 0 whenever their valuation* $\mathbf{v}$ *is in* $[\mathbf{0}, \hat{\mathbf{b}})$.

Conversely, every equilibrium satisfies these conditions up to changes of the bid functions on a set of measure zero of buyers' valuations.

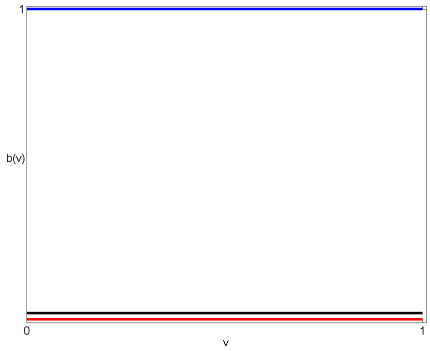
Theorem 2 gives us a more restricted set than the one we obtained with Theorem 1. Note that truth-telling and threatening equilibria are particular cases when $\mathbf{b} = \mathbf{0}$ and $\mathbf{b} \geq \mathbf{1}$ respectively, but we have fewer hybrid equilibria than in the two players case. Note that, for example, we cannot have an interval $(\mathbf{a}, \mathbf{b})$ where $(\mathbf{N} - \mathbf{1})$ players bid $\mathbf{a} \neq \mathbf{0}$ and the remaining player bids $\mathbf{b}$, which was something allowed in the case with only two bidders. In particular, the collusive situation where $(\mathbf{N} - \mathbf{1})$ players bid $\mathbf{0}$ can only occur in the interval $[\mathbf{0}, \hat{\mathbf{b}})$, i.e. when $\mathbf{a} = \mathbf{0}$. We cannot have this collusive behavior for other subintervals because there always exists a third bidder that will always prefer to bid inside the interval $(\mathbf{a}, \mathbf{b})$.

The ranking of equilibria with respect to the auctioneer's revenue they generate remains the same. The truth-telling equilibrium is better for the auctioneer than any hybrid equilibrium, and any hybrid equilibrium yields an expected revenue greater than zero, which is the revenue obtained when a threatening equilibrium realizes.

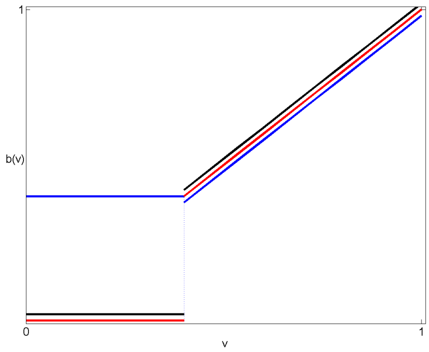
Figure 3.2 illustrates a typical nontrivial equilibrium of the Vickrey auction with at least three bidders. As in the two players case, the black, blue and red lines represent the bidding function of the black, blue and red players.



(a) Truth-telling Equilibrium



(b) Threatening Equilibrium



(c) Hybrid Equilibrium

Figure 3.2: Classes of Equilibria for the Vickrey Auction with $\mathbf{N} > \mathbf{3}$

3.2.3 Differences between Equilibrium Classes

We have already discussed the implications of the particular equilibrium realization on auctioneer's revenue. Now we will focus on the question of why we should consider all the equilibria, when it is clear that the truth-telling equilibrium yields a higher revenue. Moreover, the truth-telling equilibrium is the only one completely efficient, in the sense that the bidder with the highest valuation always obtains the auctioned item. Whenever the hybrid or the threatening equilibria realizes, in all intervals $(\mathbf{a}, \mathbf{b})$ where players do not bid their true valuations there exists a possibility of an inefficient allocation. This possibility is maximized when the set of intervals with non-truthful behavior covers the whole unit interval. In the last case, the allocation result does not depends at all on the bidders' valuation, and hence the auction is as inefficient as an allocation mechanism can be.

Another reason to discard any hybrid or threatening equilibria relies on an argument raised by Milgrom (1981). He argues that any equilibrium which is not in admissible strategies would fail to survive refinement concepts such as trembling hand perfection, as proposed by Selten (1975). Still, one should notice that the concept of trembling hand perfection is unsuitable for infinite games, as the one we face here due to our type space.

An additional argument that supports hybrid and collusive equilibria is that similar outcomes have been observed after the implementation of second price auctions, like in the case of New Zealand spectrum auction, where some second highest bids were almost zero, as reported by McMillan (1994).

Another justification to consider all the equilibria is that it is not the case that the payoff generated by one of the equilibria always dominate all of the others: the payoff structure depends crucially on the assumed distribution of valuations. Blume and Heidhues (2001) show that there are specific distributions for which each class of equilibria payoff dominates the others.

The last reason we will argue in this section has to do with completeness. Having the whole set of equilibrium bidding functions described allows us to compare its changes after modifications of the standard setup, in order to observe which equilibria survive and which do not. In a sense, this allows us to perform some kind of robustness analysis of the set of equilibria after changes on the underlying structure of the auction. We will do such comparission after describing the set of equilibria in the discrete Vickrey auction.

3.3 All the Equilibria of the Discrete Vickrey Auction

From now on, we consider the Vickrey auction where bids are restricted to a finite set $\mathbf{S}$, which is publicly specified by the seller before the auctions starts. For simplicity, we assume that both 0 and 1 are elements of $\mathbf{S}$, although this assumption does not drive the following results. The elements of $\mathbf{S}$ are denoted by $\mathbf{b}$.

Unless otherwise specified, we will consider that the tie-breaking rule is random and fair, i.e. the winner is randomly selected among highest bids and all highest bids have the same probability of winning. This is the classical tie-breaking rule used in the literature. The price to be paid in a tie is also the second price, i.e. equals the winning bid.

As we have mentioned, most authors have ignored that bids are restricted to at most a countable set, and instead modeled the strategy set as all non-negative real numbers. We would expect that restricting the set of allowed bids would not affect substantially the outcomes of the Vickrey auction.

It is straightforward that truth-telling behavior cannot directly survive when we restrict the set of allowed bids (because the bidding functions cannot longer be continuous). Still, a natural guess would be that the truth telling behavior could be approximated by a bidding function that takes the closest allowed bid for any valuation in the unit interval, or even the closest allowed bid such that the valuation is always bigger than the bid or vice versa. Now we will formally state this idea. Its proof follows from the fact that: i) there is no subinterval $[\mathbf{a}, \mathbf{b})$ of the unit interval where every player bids zero, and ii) if there exists a player bidding at or above 1, he is the only one doing so. To state this as a Proposition, first we introduce some notation.

Definition 1 *Let* $\mathbf{U_i^+(v_i) = \{b \in S | b \geq v_i\}}$ *and* $\mathbf{U_i^-(v_i) = \{b \in S | b \leq v_i\}}$ $\forall \mathbf{i}$ *be the upper and lower contour sets. Then define the ceiling and floor functions as:* $\lceil \mathbf{v_i} \rceil = \min(\mathbf{U_i^+})$ *and* $\lfloor \mathbf{v_i} \rfloor = \max(\mathbf{U_i^-})$ $\forall \mathbf{i}$*. Moreover, let* $]\mathbf{v_i}[$ *be the function that returns the closest bid for every valuation, i.e.* $]\mathbf{v_i}[= \arg\min_{\mathbf{b} \in \mathbf{S}} |\mathbf{v_i} - \mathbf{b}|$.

Let $\lceil \mathbf{v} \rceil = (\lceil \mathbf{v_1} \rceil, \ldots, \lceil \mathbf{v_N} \rceil)$*,* $\lfloor \mathbf{v} \rfloor = (\lfloor \mathbf{v_1} \rfloor, \ldots, \lfloor \mathbf{v_N} \rfloor)$ *and* $]\mathbf{v}[= (]\mathbf{v_1}[, \ldots,]\mathbf{v_N}[)$ *denote the respective symmetric strategy profiles.*

Now we state the first Proposition, which is proven in the Appendix.

Proposition 1 *The symmetric strategy profiles* $\lfloor \mathbf{v} \rfloor$*,* $\lceil \mathbf{v} \rceil$ *and* $]\mathbf{v}[$ *are never a Nash equilibrium of the discrete Vickrey auction (as defined in section 3.1 and 3.5).*

Now we know something about which strategy profiles are not a Nash Equilibrium, but we still have not presented one of them. Proposition 2 states the existence of a class of equilibria that exists in the continuous case, namely the threatening one. But before, we formalize what a threatening strategy profile is.

Definition 2 (Threatening Strategy Profile) *A strategy profile is a specification of a bidding function for each bidder, which maps from the unit interval to the set of allowed bids* $\mathbf{S}$*, i.e.* $\mathbf{b_i(v_i)} : [\mathbf{0}, \mathbf{1}] \to \mathbf{S}$, $\forall \mathbf{i}$*. A strategy profile is said to be threatening if one player (say* $\mathbf{i}$*) bids AE at or above 1 while the remaining* $(\mathbf{N-1})$ *players bid AE 0, i.e.* $\mathbf{b_i(v_i) = x \geq 1, x \in S}$, $\forall \mathbf{v_i}$ *and* $\mathbf{b_j(v_j) = 0}$, $\forall \mathbf{v_j}$*, and for* $\mathbf{j = 1, \ldots, i-1, i+1, \ldots, N}$.

With this definition, we can now state the Proposition as follows:

Proposition 2 *In a discrete Vickrey auction (as defined in section 3.1 and 3.5) with at least three bidders and a fair and random tie-breaking rule, a strategy profile is an equilibrium if is a threatening strategy profile.*

The proof is relegated to the Appendix, but we will discuss the idea of the proof below. We should state that we actually suspect that Proposition 2 could be strengthened by including an *only if* condition, however while we have a rough intuition of why this must be true we were not able to finish a correct proof, and this remains as work in progress. We present this as a first conjecture as follows:

Conjecture 1 *In a discrete Vickrey auction (as defined in section 3.1 and 3.5) with at least three bidders and a fair and random tie-breaking rule, a strategy profile is an equilibrium if and only if is a threatening strategy profile.*

To see that the *if* direction is true, notice that the threatening bidder has no incentive whatsoever to deviate, because he wins the auction always without paying. Any deviation to any bid different than zero for any subinterval of valuations will yield exactly the same payoffs to him. Any deviation to bidding zero in a subinterval will give him a lower payoff. We conclude that the threatening bidder has no incentives to deviate.

For any threatened bidder, deviating to any positive amount below 1 for a subinterval of the unit interval gives him the same payoff, so he does not have strict incentives to deviate. Bidding at or more than one for any interval will give him a possibility of winning the auction paying a price higher than any possible valuation, and hence he would have a negative payoff. Hence, any threatened bidder also does not have incentives to deviate, and we conclude that any threatening profile of bidding functions is a Nash Equilibrium. Still, it should be clear from the previous discussion that the best replies are not unique in general neither for the threatening nor for the threatened bidders. Consequently, none of these equilibria are strict, in the sense of Harsanyi (1973).

As we have mentioned, the *only if* direction is far from trivial and we have not finished it yet, so the reader should be cautious about the statements that follow. Our idea relies on describing properties that must be satisfied for every equilibrium strategy profile in five steps. First, we expect that in equilibrium, there is no subinterval $[\mathbf{0}, \mathbf{v}^*)$ of $[\mathbf{0}, \mathbf{1}]$ where every player bids $\mathbf{0}$ AE or where every player bids always a positive amount. Then we would proceed by trying to show that there is a subinterval $[\mathbf{0}, \mathbf{v}^*)$ where $(\mathbf{N}-\mathbf{1})$ players actually bid $\mathbf{0}$ and the remaining player bids at or above a positive amount b^*, just as we observed in the equilibria characterization of the original Vickrey auction by Blume and Heidhues (2004).

After that we would like to show that, for any equilibria, the property defined in Part 3 holds for an arbitrary interval of valuations where $\mathbf{v}^* = \mathbf{b}^*$. And finally, we should be able to show that in equilibrium, $\mathbf{v}^* = \mathbf{b}^* = \mathbf{1}$. The proof of the *only if* then would follow directly from the last step and how we got there by construction.

The *only if* direction applied to Proposition 2 would imply that by restricting the allowed bid set we lose two out of three equilibrium classes, namely the truth-telling equilibrium and all the hybrid equilibria. As we pointed out, it was straightforward that the truth-telling equilibrium would disappear, but losing all the hybrid equilibria is far from trivial. The reader may argue that in a sense missing the hybrid equilibria is foreseeable as it is obtained as a combination of truth-telling and threatening equilibria, but this is not exactly true. Indeed, Theorem 2 would allow for hybrid equilibria that is a combination of several threatening equilibria for several intervals. To exemplify our point, we will illustrate one of those equilibrium in Figure 3.3.

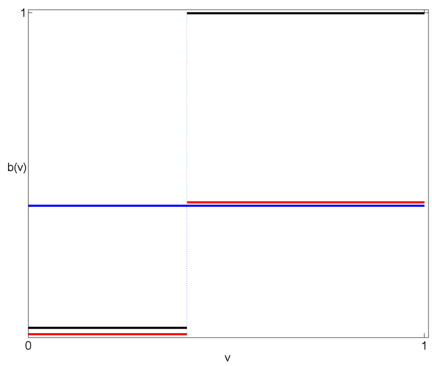


Figure 3.3: An Equilibrium of the Vickrey but not of the Discrete Vickrey Auction (in general) with $\mathbf{N} > \mathbf{2}$

Of course, we assume that the bid functions depicted in Figure 3.3 only cover allowed bid levels. The strategy profile shown has never a component of truth-telling behavior. Such a strategy profile would be an equilibrium in the continuous bid case, but it can be clearly seen that it cannot be an equilibrium of the discrete Vickrey auction for those cases when the valuation's distribution of the blue bidder puts enough probability on the first part of the interval.

3.3.1 Properties of the Threatening Equilibria

Again, in the threatening equilibria not only the auctioneer obtains the lowest possible revenue, but the allocation outcome is in general inefficient, as the winner determination does not depend on the valuations. Note also that the existence of this kind of equilibria does not depend on whether we have continuous or discrete bidding, on the particular number of allowed bids (we do not even require the existence of a non trivial one).

It also does not depend on the distance between allowed bid levels, nor on the specific distribution or the valuations' distribution. In a way, this equilibrium is really robust to how we set up the model and it is hard to get rid of it theoretically. As the reader can guess, the auction designer can still avoid such outcome by setting an effective reservation price, but this possibility will be explored latter.

3.3.2 The Role of Ties

Consider again the strategy profile described in Figure 3.3, in which the red player bids $\mathbf{0}$ in a subinterval $[\mathbf{0}, \mathbf{x})$ and then bids $\mathbf{x}$ in the remaining part of the unit interval $[\mathbf{x}, \mathbf{1}]$, the blue player bids $\mathbf{x}$ in the whole unit interval and the black player bids $\mathbf{0}$ in $[\mathbf{0}, \mathbf{x})$ and $\mathbf{1}$ in $(\mathbf{x}, \mathbf{1}]$.

Now assume that instead of considering the random and fair tie-breaking rule, the auctioneer decides to implement a dictatorial one, which always awards the auctioned item to the red bidder whenever he is involved in a tie among winning bids, leaving the random and fair tie-break for the remaining cases. The reader can check that under this new specification the profile of bidding functions depicted in Figure 3.3 becomes an equilibrium of the discrete Vickrey auction (under the new dictatorial tie-breaking rule) for every possible distribution of bidders' valuations.

Why is this true? Roughly speaking, it is because the red player does not need to bid above $\mathbf{x}$ to obtain the auctioned item when the valuation of blue and black players is in $[\mathbf{0}, \mathbf{x})$. One may think at first that given that red has the dictatorial tie-breaking rule in his favor, it is not rational that he bids $\mathbf{x}$ in the subinterval $[\mathbf{x}, \mathbf{1}]$, because by bidding $\mathbf{1}$ he will obtain the auctioned item with probability one. Still, this is never a good idea when black is bidding $\mathbf{1}$, because then he would win the auctioned item paying always a price higher than his valuation.

Note that since the tie-breaking rule is defined exogenously by the seller, he is always better off setting a dictatorial tie-breaking specification as the one we just mentioned, as otherwise he guarantees himself the lowest possible revenue. By setting a dictatorial rule, he obtains a revenue of $\mathbf{x}$ with a probability that depends on the particular distribution of the bidders' valuations.

3.3.3 The Role of N > 2

An example of why our conjecture about the *only if* part of Proposition 2 fails to hold with only two bidders is illustrated in Figure 3.4. This hybrid strategy profile is definitely an equilibrium of the discrete Vickrey auction, as it is straightforward that no bidder has incentives to deviate in any subinterval.

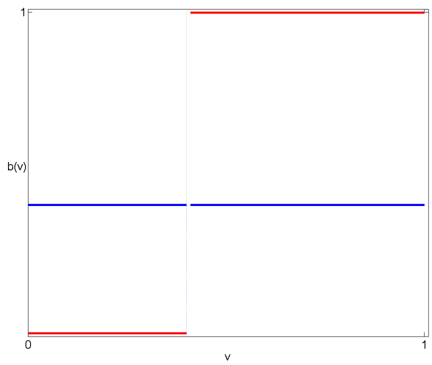


Figure 3.4: An Example of a Violation of Proposition 2 when $\mathbf{N} = \mathbf{2}$

Hence, we observe that some of the hybrid equilibria survive in the discrete auction with only two players. The reason why these equilibria disappear when we include one more bidder is because the third bidder would prefer to play a mixed strategy in the second subinterval that allows him to win without ties with some probability, which is determined by the specific distribution of bidders' valuations.

3.4 Introducing a Reservation Price

Going back to the setup with the random and fair tie-breaking rule, Blume and Heidhues (2001) showed that if a reservation price (in the open unit interval) is included in the Vickrey auction with at least three players, then the equilibrium is unqiue and simple: every player with a valuation below the reservation price bids $\mathbf{0}$ and every player with a valuation above the reserve price bids his own valuation. In a sense, this results just emerges almost as a particular case of Theorem 2 when $\hat{\mathbf{b}}$ is exogenously specified by the auctioneer as the reserve price (there exists a slight difference when all the valuations are below $\hat{\mathbf{b}}$). Hence, they conclude that the introduction of an effective (again, this means inside the open unit interval) reservation price can be a powerful tool that the auctioneer can use to rule out all the collusive (namely hybrid and threatening) equilibria as the auction outcomes.

To obtain this result, it is crucial that in the auction there are at least 3 bidders, as otherwise the uniqueness of equilibrium disappear. To see this, note for example that one hybrid equilibrium with $\mathbf{N} = \mathbf{2}$ is that both players bid 0 for all valuations below the reservation price, and for the remaining valuations one player bids the reservation price while the other bids at or above 1.

In our setup where the set of allowed bids is restricted to be finite, we can see that any threatening equilibrium can no longer be supported after the introduction of a nontrivial reservation price, because the threatening bidder cannot anymore make a credible threat that he will always bid above the upper bound of the valuation' support, because he will make losses with positive probability. This implies that if our conjecture is correct, we would conclude that there is no Nash equilibrium in the discrete Vickrey auction with a nontrivial reservation price.

Conjecture 2 *The discrete Vickrey auction (as defined in Sections 3.1 and 3.3) with a nontrivial reservation price does not have a Nash equilibrium in pure strategies.*

Chapter 4

Conclusions

4.1 Contributions

We examine the Vickrey auction in which the set of allowed bids is restricted to a finite set and present some equilibria that survive from the continuous to the discrete bidding case. We describe the properties of those equilibria that survive: in any of them the auctioneer obtains no revenue, as the second price is zero. Moreover, the allocation obtained through those strategy profiles in pure strategies is inefficient, as the winner determination is independent of bidders' valuations, provided that there are at least three bidders.

We show that, by adopting a dictatorial tie-breaking rule, the seller can avoid the no revenue scenarios. We present one example of this. We also show that whenever a non-trivial reservation price is introduced, any threatening equilibrium cannot longer be supported.

4.2 Future Research

The first element of this section is of course to prove our Conjectures 1 and 2 (actually, 2 will immediatly follow from the proof of 1). Another clear opportunity for future research is to fully characterize the set of mixed strategy equilibria, which is far from trivial (it can be seen from very simple examples that it has a complicated structure). It should be stressed that there is a whole literature that analyzes what it means to choose to randomize between available choices in equilibrium, in which there is not yet a full consensus (for an excellent review of the literature on this area, see Chapter 6 of Gintis (2009)).

Another opportunity is to generalize this work by allowing for multi-unit auctions to see the differences in the VCG and GSP allocation methods. Another idea that could be explored is to expand the work presented here to other auction institutions. However, we believe that another approach should be used because in many cases the equilibrium is unique (as in the sealed-bid first price auction). Still, the assumption of restricted allowed bids is easier to justify with open outcry auctions.

Anyway, these are just suggestions of a much broader list of topics in the research agenda of auction theory and design, and more broadly of the study of mechanisms for resource allocation. In our view, the possibilities for innovative research in the field are still ample and fascinating from both a theoretical and a design perspective.

Appendix A

Proof of Proposition 1

First, we will show that $\lceil \mathbf{v} \rceil$ is not a Nash equilibrium. Remember that $\lceil \mathbf{v} \rceil$ assigns to each valuation the closest available bid from the right (i.e. the closes larger allowed bid) for every player. In this case, there is an subinterval $[\mathbf{0}, \mathbf{x}] \subseteq [\mathbf{0}, \mathbf{1}]$ in which every player bids $\mathbf{x} > \mathbf{0}$. Any player would be better off bidding $\mathbf{0}$ in the interval $[\mathbf{0}, \mathbf{x}]$ as otherwise they win (with a probability only, due to the tie) paying a price strictly higher than its valuation whenever their valuation is in $[\mathbf{0}, \mathbf{x}]$, leaving his payoff unchanged when their valuation is in $(\mathbf{x}, \mathbf{1}]$. That $\mathbf{x} \in \mathbf{S}$ follows from the definition of $\lceil \mathbf{v} \rceil$. We conclude that $\lceil \mathbf{v} \rceil$ is not a Nash equilibrium.

Now we consider the strategy profile $\lfloor \mathbf{v} \rfloor$. Recall that this function assigns to every valuation the closets allowed bid from the left, for every player. In this case, there is a subinterval $(\mathbf{x}, \mathbf{1}] \subseteq [\mathbf{0}, \mathbf{1}]$ where every player bids $\mathbf{x} < \mathbf{1}$ (Again, that $\mathbf{x} \in \mathbf{S}$ follows from the definition of $\lfloor \mathbf{v} \rfloor$). Any player would be better off bidding $\mathbf{1}$ in such interval, as he would win when his valuation is in the interval $(\mathbf{x}, \mathbf{1}]$ and paying $\mathbf{x}$ at most whenever his valuation is in $(\mathbf{x}, \mathbf{1}]$, while his payoff remains the same whenever his valuation is in $[\mathbf{0}, \mathbf{x}]$. We conclude that $\lfloor \mathbf{v} \rfloor$ is not an equilibrium.

Finally, consider the strategy profile $]\mathbf{v}[$, which assigns to every valuation the closest allowed bid, for every player. In this case, both arguments above apply (there are intervals where every player bids $\mathbf{0}$ and $\mathbf{1}$, and any player would find profitable to deviate to the closet interior allowed bid in such intervals) and it follows directly that such strategy profile can never be a Nash equilibrium.

Appendix B

Proof of Proposition 2

For the player who bids at or above $\mathbf{1}$, any deviation to a positive bid does not affect his payoff. If he deviates to $\mathbf{0}$ his expected payoff decreases, as he will win the auction only $\frac{\mathbf{1}}{\mathbf{N}}$ times without paying, and before he was always winning the auction without paying too. For any of the players bidding $\mathbf{0}$, deviating to any value below the winning bid does not affect his payoff and deviating to a bid higher or equal than the winning one makes him win the auction paying a price strictly higher than his own valuation. Since no player has incentives to deviate, we conclude that the strategy profile described is a Nash equilibrium.

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