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Directed Search with Saving

A Master's Thesis submitted for the degree of
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Affidavit

I, Bálint Szőke

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47 pages, bound, and that I have not used any source or tool other than those referenced or any other illicit aid or tool, and that I have not prior to this date submitted this Master's Thesis as an examination paper in any form in Austria or abroad.

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Abstract

Using a two-period directed search model with wage-tenure contracts, this paper investigates the role of saving decisions in the determination of labor market outcomes. I find that wealthier unemployed workers apply for jobs that pay higher present value of income and that are harder to obtain than jobs targeted by poorer workers. On the other hand, introducing the possibility of saving brings indeterminacy into the model in the sense that the particular wage path (contract) chosen by the agents is not unique. The reason is that if workers can perfectly smooth their consumption, in effect, they will behave as if they were risk-neutral and will be indifferent to different wage paths yielding the same present value.

In order to address this issue, I introduce two extensions: on-the-job search and borrowing constraint, which work in opposite directions. On-the-job search gives rise to the incentive of firms to offer an increasing wage path (pay wages as late as possible) in order to keep their employees. On the other hand, borrowing constraints prevent workers from smoothing consumption perfectly, and as a result they wish to receive wages as early as possible. The introduction of these two elements eventually makes the equilibrium contract determinate for constrained workers and surprisingly, this contract happens to provide higher present value (conditional on matching) and lower matching probability than the contracts workers would choose if they were unconstrained. In addition, in my two-period model with on-the-job search it turns out that in equilibrium no on-the-job search takes place.

1 Introduction

There are many ways in which labor market decisions and saving behavior can affect each other. It is reasonable to think that the amount of savings affects the type of job to which the worker applies, or the minimum wage that she requires from her prospective employer. One can also claim that it influences search effort, unemployment duration and more generally the option value of staying unemployed. On the other hand, given that one of the most important risks for most workers is the risk of becoming unemployed, the saving generated by this risk can attain considerable amount. Many of the above mentioned relationships were examined in the literature using the random search framework.[1] However, to the best of my knowledge nobody considered these issues in a directed search setup. In this paper I take the first steps to fill this gap.

My results extend the literature on directed search.[2] The distinctive premise of these models is that workers can choose the type of job they are applying to. This feature contrasts sharply with the random search models, where wages are drawn randomly from an offer distribution that is exogenous to the workers. This way in random search models jobseekers have no influence on the wages they are drawing and luck plays an important role in labor market outcomes. Completely identical unemployed workers can get very low wages or jump immediately to the top of the wage distribution. This property of the model is even more unreasonable in case of on-the-job search. It is unlikely that a lowly paid and a highly paid employee have identical probabilities of receiving a particular wage offer. Furthermore, the optimality of agents' behavior is questionable, too: why should workers get in contact with firms offering much lower wages than their current job? By putting the decision of 'where to apply' in the jobseekers' hand, directed search models are able to address these problems.

This requires workers to have perfect information about the conditions of equilibrium vacancies and particularly about the wages firms post *before* meeting. Therefore, market frictions cannot come from imperfect information (the aim of search is not to gather information), rather, they result from coordination problems.[3] Workers cannot coordinate their application behavior, thus in

[1] Just to name a few, see Browning, Crossley, and Smith (2007), Lentz and Tranas (2005), Rendon (2006) and Krusell, Mukoyama, and Şahin (2010).

[2] This literature can be traced back to Montgomery (1991), Moen (1997) and Shimer (1996) and has been developed further by Burdett, Shi, and Wright (2001), Delacroix and Shi (2006), Guerrieri, Shimer, and Wright (2010) and Menzio and Shi (2010).

[3] This does not mean that information asymmetries cannot exist in these models. In fact, Guerrieri (2008) shows that private information about productivity can lead to market inefficiencies. In addition, in a model with adverse selection, Guerrieri et al. (2010) demonstrates that directed search models accompanied by information frictions can be very useful to analyze

equilibrium firms may have multiple applicants or may receive no applicants at all. Nonetheless, since workers prefer jobs with high wages, firms can affect the probability of these events by posting higher wages and attracting more applicants. In fact, the optimal wage-posting strategy balances the trade-off between the costs firm have to pay for attracting workers and the probability of filling its vacancy. In the similar vein the optimal application strategy balances the trade-off between the wage the worker can obtain and the probability of getting it. This coordination problem generates complicated strategic interactions between workers and firms, and the problem easily becomes very involved. In order to avoid unnecessary technical difficulties I will rely on the axiomatic, rather than the strategic approach.[4] This approach takes the matching function as given instead of making it endogenous. As usual, this function gives the flow of new matches as a function of unemployed people and vacancies, so it can be considered as a short-cut for modeling frictions. In this way, I model directed search as in Moen (1997), Acemoglu and Shimer (1999) and Shi (2009).

My paper complements the standard directed search framework by introducing saving decision at the workers' side. To this end, first, one needs risk-averse workers who have at least partial access to asset markets. I assume, however, that markets are incomplete in a sense that workers can save in riskless bond only. There is no aggregate uncertainty in the economy; the only risk workers face is the probability of not getting the job they apply to (unemployment risk). Furthermore, by allowing workers to choose the job to which they apply, I also allow them to influence the magnitude of this risk. Consequently, the degree of risk-aversion naturally affects the type of jobs targeted by the workers, or more precisely the workers' tradeoff between wages and the probability of obtaining it. Assuming decreasing absolute risk-aversion (DARA) preferences, I find that workers with higher savings tend to apply for better paid but less easily attainable jobs than their poorer counterparts. This means that unemployed workers with more wealth have longer expected unemployment duration; a relationship which is relatively well established in the empirical literature (see e.g. Card, Chetty, & Weber, 2007). It is important to note, however, that in my framework, contrarily to the standard random search models, this relationship comes from the varying degree of risk aversion, rather than from differences in reservation wages.

In a closely related paper, Acemoglu and Shimer (1999) find exactly the same relationship between risk-aversion and application decision; however, they focus on a different issue (the optimal degree of unemployment insurance), and use two

'classical information problems'.

[4]For the strategic approach see e.g. Peters (1991), Burdett et al. (2001) and Julien, Kennes, and King (2000).

key assumptions that I relax in this paper. These are constant wage offers and the impossibility of on-the-job search. To relax the first one, I assume that firms, while they take workers' application decisions into account, post *contracts* with full commitment. In order to simplify the analysis, I restrict myself to considering only wage-tenure contracts, still allowing wages to vary over time. This assumption introduces a certain kind of indeterminacy into the model. Since workers, by using capital markets, are able to smooth their consumption path completely, they behave as if they were risk-neutral in a sense that they do not care how income is allocated over time, only the present value provided by the contract makes any difference in their decisions. Even if the particular contract chosen by a worker is indeterminate, the equilibrium can be uniquely characterized in terms of present value and matching probability. After characterizing the equilibrium, comparative statics uncovers a number of interesting results about the relationship between the economic environment and the validity of my finding connecting asset holding and application decisions. In particular, it reveals that market frictions are crucial for my results in a sense that without frictions wealth heterogeneity does not generate any difference in labor market outcomes.

Introducing on-the-job search in this benchmark framework gives rise to a new force, namely the incentive of firms to keep their workers. The reason for this is that search frictions make it hard to refill jobs, so matches provide rent for the firms that they are motivated to keep. As a result of this new force, firms offer only increasing wage paths, hence restricting the set of possible wage contracts. The increasing feature of optimal wage-tenure contracts, given on-the-job search and risk-aversion, is a well-known result in search theory.[5] Two seminal papers, Burdett and Coles (2003) and Shi (2009) show this using a random and directed search setup respectively, though neither of them considers saving decision; they rule it out by the assumption that capital markets are missing. However, even if workers were able to save, within the assumed framework they would have no incentive to do so. This is because (i) they never return to unemployment (the only way matches can separate are quitting or dying), (ii) they have no initial wealth and (iii) since the equilibrium wage contract grows with tenure. In my paper, I relax only the second assumption and show that it is sufficient to render the equilibrium wage contract indeterminate. The reason again is that by allowing for saving, the model allows consumption smoothing for workers and thus they

[5] In a random search model with risk neutral agents and without capital market, Stevens (2004) establishes that the optimal contracts are step wage contracts (up to some point firms pay zero wages and then the marginal product), moreover, in equilibrium, there is a continuum of other contracts, which are payoff equivalent to these. This obviously mitigates the indeterminacy result of my benchmark model.

become indifferent to the particular path of income. Interestingly, however, in the equilibrium of my extended two-period model no job-to-job transition takes place. This follows from the fact that those firms who were successful in period 0 in recruitment can prevent their competitors from hiring employed workers by promising very high period 1 wages.

Since the workers' ability to smooth consumption plays such a crucial role, it is reasonable to think that imposing some restrictions on it leads to different outcome. Assuming some liquidity constraint, e.g. that workers cannot borrow on their future income, renders constrained workers' optimal contracts determinate. It is due to the fact that in their case two forces act in opposite directions. Firms wish to offer an increasing wage path to ensure worker retention, but at the same time constrained workers want to get money as early as possible to be able to smooth their consumption. The resolution of this tension determines the optimal contract. Furthermore, it turns out that somewhat surprisingly this contract provides higher present value and lower matching probability than it would without liquidity constraint. This is followed by the fact that the inability of constrained workers to smooth consumption changes their trade-off between present value and unemployment risk. The above mentioned desire to obtain their wages as soon as possible eventually increases the marginal value of income and thus workers happen to appreciate present value more relative to matching probabilities. For this result, directed search, in particular the fact that application is a decision is obviously crucial. By this assumption, application can serve as a kind of asset to smooth consumption and intuitively, a constrained agent will naturally use it if her other ways are restricted.

One of the main contributions of Shi (2009) was to show that the equilibrium of his model exhibits a certain feature: even if workers are heterogenous (with respect to their current wages), the value and policy functions do not depend on the distributions. This solution concept is termed as a block recursive equilibrium and it follows mainly from the directed search assumption. Indeed, in their seminal paper Menzio and Shi (2010) show that block recursivity is a reoccurring feature of directed search models. They prove that it holds in non-stationary environments, in different contractual frameworks, and also in economies with lots of unobserved heterogeneities. In this paper I show that in some sense block recursivity is robust also to the introduction of saving decision, as the key property of workers' self-separation continues to hold.

The remainder of the paper is structured as follows. In Section 2, I develop a simple two-period model which will serve as a benchmark throughout the paper. I characterize the equilibrium allocation, examine some comparative static results and show a numerical example to study the key relationships and the driving forces behind them. In Section 3, I extend this framework by allowing for on-the-job search and investigate the case when borrowing constraint is imposed on the workers. Section 4 concludes.

2 Benchmark two-period model

In this section my aim is to build a parsimonious model, which nonetheless allows me to analyze saving decision and especially its relation to application decision. The model as such can be considered as a point of departure, which I can contrast with richer specifications later on. It is a two-period production economy with labor as the only factor of production. Firms hire labor input by means of posting wage contracts, which are fully described by the pair of wages, (w_0, w_1). Search is directed since jobseekers observe firms' offers before application.

2.1 Labor market

Labor market exhibits the following frictions. First, applicants cannot coordinate their application decisions (identical workers necessarily employ the same application strategy), which results in the possibility of unmatched agents. In addition, applicants can apply to only one job at a time. As I have mentioned before, for characterizing these frictions I take the short-cut of using an exogenous matching function. This function is a standard tool in the literature and it determines the number of matches as a function of the measure of unemployed workers and vacancies in the market. Assume a constant return to scale, concave and twice-differentiable matching function $\mathcal{M} : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$, which is increasing and strictly concave in both arguments with $\mathcal{M}(0, v) = \mathcal{M}(u, 0) = 0$ for all u, v, where u denotes the measure of unemployed workers, while v represents the measure of vacancies.

As a consequence of the coordination frictions, there may be more competition for some contracts than for others. To capture the degree of this competition, let $q \equiv \frac{u}{v} \in [0, \infty]$ be the ratio of the number of workers applying for a particular contract to the number of firms offering that contract. It is useful to refer to this ratio as the expected queue length of the contract. A key element of the model is the generalization of this concept, namely the function $Q(\cdot, \cdot)$ mapping from the set of possible wage contracts to the associated expected queue length, $Q : \mathbb{R}^2 \mapsto [0, \infty]$. This function is an equilibrium object determined endogenously through the agents' optimal decisions. While they make these decisions, they take it as given, hence it acts as a type of hedonic price that helps to allocate firms and workers across different equilibrium contracts. This price does not exist in random search models, which largely contributes to the fact that in general these models lead to inefficient equilibrium allocations.

As a measure of competition for a particular submarket, Q is closely related to the probability of forming matches at that submarket. Naturally, by offering a contract that attracts more applicants (thus has longer expected queue length), the firm can fill its vacancy faster, while such contract means lower employment probability for the applicants. Using the constant return to scale matching function $\mathcal{M}$ one can formalize this idea by defining the following matching rates (with q as the only argument):

$$\frac{\mathcal{M}(u,v)}{u} = \mathcal{M}\left(1, \frac{1}{q}\right) \equiv \mu(q) \qquad \text{and} \qquad \frac{\mathcal{M}(u,v)}{v} = \mathcal{M}\left(q, 1\right) \equiv \eta(q)$$

One useful feature of these matching rates is that we can take them as model primitives instead of $\mathcal{M}(\cdot,\cdot)$. Here, $\mu(q)$ denotes the probability that a worker who applies to a job with expected queue length q becomes employed, while $\eta(q)$ embodies the job filling probability of a firm operating on submarket q. Given their definitions, one can establish the following relationship between the two rates: $\eta(q) = \mu(q)q$. They obviously inherit the twice differentiability of the matching function and it is also clear that μ is strictly convex and decreasing, while η is strictly concave and increasing in q. Nonetheless, we additionally have to restrict their range to ensure that they are indeed probabilities, hence I assume that $\mu : [0,\infty] \mapsto [0,1]$ and $\eta : [0,\infty] \mapsto [0,1]$ with the boundary conditions $\eta(0) = \mu(\infty) = 0$ and $\eta(\infty) = \mu(0) = 1$.

Within this framework, the set of firms offering identical contracts and the workers applying for that particular contract constitute a kind of separate 'labor market' with a specific value of expected queue length. In other words, the labor market can be considered as a set of submarkets, each of which is associated with an equilibrium contract (w_0, w_1) and a corresponding hedonic price $Q(w_0, w_1)$. Since each contract (or submarket) corresponds to a unique expected queue length q, through the above defined matching functions we can assign employment probability, $\mu(q)$ and job filling probability, $\eta(q)$ to them. Consequently, the agents' decisions can be redefined in the following way. Firms face the trade-off between wages they have to pay and the corresponding job filling probability $\eta(q)$, while workers, observing the offered contracts, face the trade-off between the value of the contracts and the probabilities of obtaining them, $\mu(q)$. A crucial presumption is that every firm can commit fully to the contract they post. In other words, conditional on matching, the firm cannot deviate from the promised wages and cannot terminate the match either. In the benchmark setting, I assume that workers have full commitment as well, i.e. there is no on-the-job search or endogenous separation.

2.2 Preferences and technology

There are two periods ($t = 0, 1$), but labor market opens only once at the beginning of period 0. Workers of total measure one begin their life as unemployed. They are risk-averse and ex ante homogenous except for their initial asset holdings (measured at the beginning of the period), a_0. The assets are distributed according to some initial distribution with a compact support $[\underline{a}, \overline{a}]$, where $\underline{a} > 0$. For sake of brevity, hereafter I will refer to workers with a particular asset level a_0 as a_0-workers. When employed, a worker supplies one unit of labor inelastically per period, thus produces y_t units of output in period $t = 0, 1$. They discount future utility at the rate of β, hence they have the following preferences and constraints:

$$\begin{aligned} U = \mathbb{E}_i \left[u(c_0^i) + \beta u(c_1^i) \right] & \qquad i = E, U \quad \text{where} & \\ c_0^E = a_0 + w_0 - a_1^E/(1+r) & \qquad c_1^E = a_1^E + w_1 & \qquad \text{(if matched)} \\ c_0^U = a_0 - a_1^U/(1+r) & \qquad c_1^U = a_1^U & \qquad \text{(if not matched)} \end{aligned}$$

when they become employed (case E), or stay unemployed (case U), respectively. Regarding preferences I will use the following assumptions about the per period utility function throughout the paper.

Assumption 1. *The function $u(\cdot)$ is C^2, $u'(\cdot) > 0$, $u''(\cdot) < 0$ and $\lim_{c\to 0} u'(c) = \infty$.*

Assumption 2. *Function $u(\cdot)$ is a Decreasing Absolute Risk Aversion (DARA) utility function.*

On the other side of the labor market, firms are identical and risk neutral. They can decide whether to enter the labor market, for which they have to pay the per period vacancy cost $k > 0$. By paying this cost they buy the opportunity of offering a contract, fully described by a pair of wages $(w_0, w_1) \in \mathbb{R}^2$. Each firm can have at most one filled or unfilled vacancy, whose total measure in the economy is pinned down by free-entry. Firms discount future profits by the exogenous interest rate r. One should also notice that firms and workers are separate agents and firms are not owned by consumers.

The timing of the model is the following. At the beginning of period 0 labor market opens. Upon entry, firms decide on the offered contracts, then workers direct their search given the set of offered contracts $\mathcal{W}^2$. After the matching is realized, production takes place, and finally employed and unemployed workers decide on consumption (determining next period asset level). In period 1 no real economic decision takes place.[6]

2.3 Definition of equilibrium

In this economy, an allocation is a tuple $\{\mathcal{W}^2, Q, a_1, \overline{U}\}$, where $\mathcal{W}^2$ is the set of wage contracts offered by firms, Q is the queue length function discussed above, a_1 is the policy function of the worker determining optimal saving and $\overline{U}$ is workers' optimal utility level. Except for Q, all these objects depend on initial asset level, a_0. Given all these considerations the equilibrium can be defined as follows.

Definition 1. *An equilibrium is an allocation* $\{\mathcal{W}^2, Q, a_1, \overline{U}\}$ *with* $\overline{U} : [\underline{a}, \overline{a}] \mapsto \mathbb{R}$, $\mathcal{W}^2 : [\underline{a}, \overline{a}] \rightrightarrows \mathbb{R}^2$, $Q : \mathbb{R}^2 \mapsto [0, \infty]$ *and* $a_1 : [\underline{a}, \overline{a}] \times \mathbb{R}^2 \mapsto \mathbb{R}$ *satisfying:*

1. *Profit maximization and free-entry: for all* $(w_0, w_1) \in \mathbb{R}^2$

$$\eta\left(Q(w_0, w_1)\right)\left[y_0 - w_0 + \frac{1}{1+r}(y_1 - w_1)\right] - k \leqslant 0 \tag{1}$$

   with equality if $(w_0, w_1) \in \mathcal{W}^2\left([\underline{a}, \overline{a}]\right)$.

2. *Optimal saving and application decision: for all* $(w_0, w_1) \in \mathbb{R}^2$ *and* $a_0 \in [\underline{a}, \overline{a}]$

$$\mu\left(Q(w_0, w_1)\right) V(a_0, w_0, w_1) + \left[1 - \mu\left(Q(w_0, w_1)\right)\right] V(a_0, 0, 0) \leqslant \overline{U}(a_0) \tag{2}$$

   with equality if $Q(w_0, w_1) > 0$. *Where, given*

$$U(a_0, w_0, w_1) \equiv \mu\left(Q(w_0, w_1)\right) V(a_0, w_0, w_1) + \left[1 - \mu\left(Q(w_0, w_1)\right)\right] V(a_0, 0, 0)$$

$$\overline{U}(a_0) = \begin{cases} \max\left\{\sup\limits_{\mathcal{W}^2(a_0)} U(a_0, w_0, w_1); V(a_0, 0, 0)\right\} & , \text{ if } \mathcal{W}^2(a_0) \neq \varnothing \\ V(a_0, 0, 0) & , \text{ if } \mathcal{W}^2(a_0) = \varnothing \end{cases}$$

   and

$$V(a_0, w_0, w_1) = \max_{a_1} u\left(a_0 + w_0 - \frac{a_1}{1+r}\right) + \beta u(a_1 + w_1)$$

The possibility of inaction ensures that firms' expected profit is nonnegative, but at the same time free-entry and constant return to scale productivity restrict

[6] I could also assume that labor market reopens in period 1 for unemployed workers only, but qualitatively it would not change any results in this section. I have chosen this specification due to its algebraic simplicity. See also Figure 12 and footnote 25.

obtainable profit to the nonpositive region. Therefore, for wage contracts offered in equilibrium, condition (1) must be satisfied with equality. On the other hand, the second point ensures that workers make their application and saving decisions in a way to maximize their utility. The formulation might look overcomplicated at first glance, but it is necessary to impose a sort of sub-game perfection on equilibrium. Note that inequalities (1) and (2) are required to hold for all (not only for equilibrium) wage contracts. This is because if $(w_0, w_1) \notin \mathcal{W}^2([\underline{a}, \overline{a}])$, the corresponding queue length $Q(w_0, w_1)$ is not actually observed, thus we require it to adjust in a way that agents' conjectures about its off-equilibrium value are consistent with equilibrium behavior. In other words, we would like to rule out situations where firms fail to deviate to a profitable wage contract, since they incorrectly believe that nobody would apply for that contract. For a more detailed discussion about this issue see e.g. Galenianos and Kircher (2012).

Resulting from the timing convention, saving decision is very tractable. As the decision is made after realization of uncertainty (getting a job or staying unemployed), we can separate it from the application decision and define the value function V. Note, however, that with this timing convention we also rule out any precautionary saving motive, thus the only driving force in this respect is the usual life-cycle saving behavior.

2.4 Characterization

Fortunately, the above definition has a very convenient characterization.

Proposition 1. *If the allocation $\{\mathcal{W}^2, Q, a_1, \overline{U}\}$ is an equilibrium, then for any $a_0 \in [\underline{a}, \overline{a}]$, all $(w_0^*, w_1^*) \in \mathcal{W}^2(a_0)$, $q^* = Q(w_0^*, w_1^*)$ and $a_1^* = a_1(a_0, w_0^*, w_1^*)$ solve*

$$\overline{U}(a_0) = \max_{(w_0, w_1), q} \mu(q) V(a_0, w_0, w_1) + [1 - \mu(q)] V(a_0, 0, 0) \tag{3}$$

where

$$V(a_0, w_0, w_1) = \max_{a_1} u\left(a_0 + w_0 - \frac{a_1}{1+r}\right) + \beta u(a_1 + w_1)$$

subject to

$$\eta(q)\left[y_0 - w_0 + \frac{1}{1+r}(y_1 - w_1)\right] = k \tag{4}$$

Conversely, if for all $a_0 \in [\underline{a}, \overline{a}]$ some $\{(w_0^*, w_1^*), q^*, a_1^*\}$ solves the above program, then there exists an equilibrium $\{\mathcal{W}^2, Q, a_1, \overline{U}\}$, such that $(w_0^*, w_1^*) \in \mathcal{W}^2(a_0)$, $q^* = Q(w_0^*, w_1^*)$ and $a_1^* = a_1(a_0, w_0^*, w_1^*)$.

Proof: See Appendix A.

This proposition states that equilibria are exactly those allocations that maximize workers' utility given that free-entry drives firms' profit down to zero. In other words, equilibrium wage contracts can be derived by solving a simple constrained maximization problem with an equality constraint. It is important to note that workers maximize over q and (w_0, w_1) *separately*, even though in Definition 1 queue length is a function of contracts. In fact, this equilibrium relationship is the essence of directed search, since queue length is a hedonic price of submarkets that directs jobseekers' behavior. By choosing q and (w_0, w_1), workers decide in which submarket to look for a job and irrespective of whether they are matched or not, they make their saving decisions optimally.

In addition, the constraint (4) simply states that firms enter the labor market until expected profits reach the vacancy cost k. This condition is essential, as it pins down the equilibrium queue length q for every active submarket. Note that free-entry is a crucial assumption for this. It guarantees that agents do not need to know the exact number of workers and firms apply to each submarket, as they understand that free-entry will always bring queue length to its equilibrium level.[7] As a result, through the free-entry condition (4), the function values of $Q(\cdot,\cdot)$ are well-defined for equilibrium contracts. For contracts not offered in equilibrium (inactive submarkets), the queue length function takes the value of zero by assumption (according to Definition 1, for these contracts, vacancy costs can exceed expected profits). Obtaining function $Q(\cdot,\cdot)$ this way, we can plug it into (3) and get an unconstrained optimization problem only in terms of wage contracts. After solving this maximization problem we come to the following results.

Proposition 2. *There always exists an equilibrium. If $\{(w_0^*, w_1^*), q^*, a_1^*\}$ solves the above program for a given a_0 (it is an equilibrium), then the set of vectors $Z(a_0) \equiv \left\{((w_0, w_1), q^*, a_1) : w_0 + \frac{w_1}{1+r} = w_0^* + \frac{w_1^*}{1+r};\ a_1 = a_1^* + (1+r)(w_0 - w_0^*)\right\}$ solves it as well. Moreover, these are the only solutions, i.e. $\mathcal{W}^2(a_0) = Z(a_0)$.*

Proof: See Appendix A.

What the above proposition states is that the particular offer chosen by a a_0-worker is indeterminate. The present value of wages provided by equilibrium contracts, however, is unique, which means that instead of using single contracts one can redefine submarkets as collections of contracts offering identical present

[7] See also the discussion about block recursivity later in this section.

values. This finding is due to the fact that (i) firms are risk-neutral, and (ii) although workers have concave utility functions, they are able to smooth consumption perfectly. In other words, no matter how wages are allocated over time, through borrowing and/or saving, workers are able to employ their optimal consumption path. At the same time, equation (4), rewriting in the following form

$$\eta(q)\left[y_0 + \frac{y_1}{1+r} - \left(w_0 + \frac{w_1}{1+r}\right)\right] = \eta(q)\,[Y - \omega] = k$$

determines a one-to-one (positive) relationship between the present value offered on a given submarket, $\omega = w_0 + \frac{w_1}{1+r}$ and the expected queue length, q. These observations allow us to write the value function V as a function of present value and initial asset level and the queue length function Q as a function of present value. Formally, $V(a_0, w_0, w_1) = \widetilde{V}(a_0, \omega)$ and $Q(w_0, w_1) = \widetilde{Q}(\omega)$.

Furthermore, Proposition 2 and the above considerations allow us to depict an equilibrium graphically in the (q, ω)-space. Observe that by using the zero-profit constraint we can express ω explicitly as a function of expected queue length in the following way: $\omega = Y - \frac{k}{\eta(q)}$. Given that by assumption $\eta(\cdot)$ is strictly concave, the zero-profit condition (ZP) is a strictly concave and increasing curve in the (q, ω)-space as shown in Figure 1. In the similar vein, the trade-off faced by the workers can be visualized as well in the (q, ω)-space by means of indifference curves (ICs). These curves depend on the asset level, thus for clarity reasons fix a particular asset level a_0. The workers' problem is characterized by equation (3), from which the slope of indifference curves can be derived:[8]

$$\frac{d\omega}{dq} = -\frac{\mu'(q)}{\mu(q)} \frac{\tilde{V}(a_0, \omega) - \tilde{V}(a_0, 0)}{\partial \tilde{V}(a_0, \omega)/\partial \omega} \tag{5}$$

Due to the properties of $\mu(\cdot)$, these indifference curves are positively sloped and smooth. However, for convexity (and thus for uniqueness) we need to assume that $1 \geqslant \frac{\mu(q)\mu''(q)}{\mu^2(q)}$ as well. Doing this yields the family of indifference curves portrayed in the figure. An equilibrium (q^*, ω^*) is a point of tangency between ZP and the IC associated with the value $\overline{U}(a_0)$. The slopes of these curves have intuitive economic meaning, namely they capture the agents' valuation of a marginal increase of queue length in terms of present value. A marginal increase in q means higher job filling probability for a firm for which it is willing to pay $\frac{\eta'(q^*)k}{\eta^2(q^*)}$ in present value, while for a similar change, which implies lower employment probability for the worker, she demands by $\left|\frac{\mu'(q^*)}{\mu(q^*)}\right| \frac{\tilde{V}(a_0, \omega^*) - \tilde{V}(a_0, 0)}{\partial \tilde{V}(a_0, \omega^*)/\partial \omega}$ higher present

[8] Differentiability of the value function comes from Assumption 1 and in particular from the C^2 property of the utility function $u(\cdot)$. See also Crouzeix (1983) and the proof of Proposition 3 in the Appendix.

value.

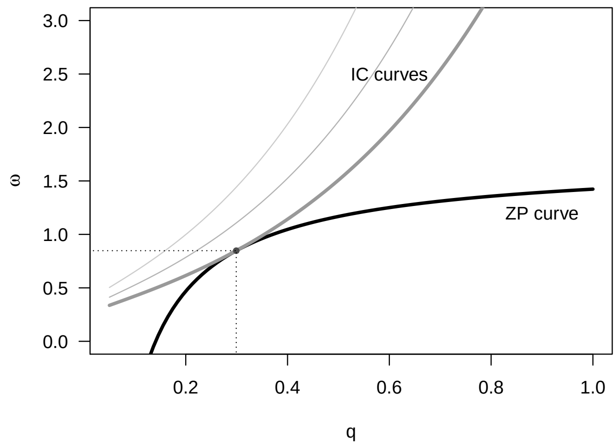

Figure 1: Indifference curves, the zero-profit condition and the equilibrium allocation (at the point of tangency) in the (q, ω)-space for a given a_0

In equilibrium the two valuations have to be consistent, which happens only at the tangency point. Note also that from the convenient convexity properties of ZP and IC it follows that every a_0-worker chooses a unique submarket, or in other words, each worker type can be found at most on one submarket. The following proposition states that even more is true, namely that on each submarket we can find at most one type of worker.

Proposition 3. *Self-Separation: Each submarket attracts at most one type of worker, i.e.* $\mathcal{W}^2(a_0) \cap \mathcal{W}^2(a_0') = \emptyset$ *if* $a_0 \neq a_0'$. *Moreover, if the utility function satisfies Assumption 2 (it is DARA), then* $a_0' > a_0$ *implies* $\omega(a_0') > \omega(a_0)$.

Proof: See Appendix A.

The self-separation result embodies one of the key differences between random and directed search models and leads to the applicability of a very useful equilibrium concept, the so-called block recursive equilibrium (Shi, 2009, Menzio & Shi, 2010). In a nutshell, this concept can be captured by the property that the value and policy functions do not depend on the entire distribution over agents' state variables. As a result, the solution of such models is significantly easier than that of the models where distributions appear as (infinite dimensional) state variables. Random search models belong to the latter group, because they do not feature self-separation. In random search models (with wage posting), different asset levels generate different reservation wages that firms have to take into account when they choose their offer because reservation wages naturally affect their job filling probability. Consequently, in order for firms to be able to calculate their expected profit, they have to know the distribution of workers across different asset levels (or more precisely, across different reservation wages).

However, in the directed search model discussed in this paper, firms do not need to know the entire distribution because workers separate themselves into different submarkets. When they choose which submarket to enter, firms know exactly which type of worker they will encounter there and that this type is unique. In the current context, the independence of value and policy functions from the asset distribution is embodied by the fact that neither $\mathcal{W}^2$, nor $\overline{U}$, nor a_1 depend on the whole asset distribution; they are functions of a specific asset level only.

The second part of the proposition characterizes the relationship between the asset at hand and the present value of contracts for which a worker applies. It captures a very intuitive idea. If the utility function is DARA, wealthier workers are less risk-averse and thus they tend to take higher risk in the form of seeking jobs with higher offered present value and lower probability. This result is exactly the same as that of Acemoglu and Shimer (1999), which comes from the fact that due to its simple specification, the above model can be mapped to their static framework. Even though similar relationships can be proven for CARA and IARA utility functions as well, DARA is not a necessary (only a sufficient) condition as the utility function can behave very differently on different parts of its domain.

Lower probability also implies longer unemployment duration, thus the model can reproduce the empirical finding that *ceteris paribus* unemployed workers with higher wealth level stay unemployed longer.[9] Still, the story behind this is different from the usual random search argument, namely that unemployment duration increases with asset level because wealthier workers' outside option and reservation wages are higher. In the current setting the positive relationship comes from workers' decreasing risk-aversion.

The positive relationship between asset level and the target present value suggests that directed search enhances the extent of workers' inequality.[10] However, as the higher present value also means lower probability of getting the job, the total effect is ambiguous. In addition, due to the indeterminacy of the equilibrium contract, we cannot draw meaningful conclusions about the dynamics of wealth inequality; the best we can do is to examine consumption inequality. This is because consumption levels are determined by the worker's lifetime income, which is unique. Nonetheless, in order to investigate this aspect of the model we need

[9]Direct empirical support for this phenomenon is provided by Card et al. (2007)

[10]This relationship is a feature not existing in random search models unless we assume an offer distribution differing with asset level, or somehow we connect search intensity and the value of jobs.

to specify some of its objects, in particular the utility and matching functions, therefore I postpone this analysis to Section 2.6.

2.5 Comparative statics

The effect of savings on application decision and more generally on equilibrium allocation can be represented graphically by means of graphs similar to Figure 1. It is clear that the ZP curve is not affected by wealth level. Looking at the slope of indifference curves in (5), however, reveals that it is the steepness and convexity of these curves (i.e. the determinants of the degree of risk-aversion) through which wealth level matters. In fact, one can show that with DARA higher asset level makes workers' indifference curves flatter, thus the tangency point must shift to the right (see the first panel on Figure 2). Recall that the slope of these curves represents the workers' trade-off between unemployment risk and wages, so the possible interpretation of the flattened curves is that workers appreciate shorter queue lengths (hence higher matching rates) relatively less. Put it differently, workers with more wealth apply to jobs with higher present value and lower employment probability, the same result that we had in Proposition 3. It is instructive to consider the risk-neutral case as well, because, with DARA utility, as wealth level limits to infinity, workers behave more and more as if they were risk-neutral. Figure 2 shows this case and confirms the above intuition.

On the firms' side, productivity and vacancy cost are the two parameters that can affect equilibrium allocation. The third and fourth panels on Figure 2 depict the corresponding cases. Regarding productivity, as workers can commit to stay with the firm during the entire contract period, it is sufficient to consider only the present value of output $Y = y_0 + \frac{y_1}{1+r}$. Using the zero-profit constraint, it follows that Y increases ω for any given queue length by the same amount, that is it shifts the whole ZP curve upwards. As a result, the equilibrium present value increases, while in most cases the equilibrium queue length becomes shorter.[11] This comes from the fact that due to higher productivity, matches become more profitable for firms, thus (i) more vacancies are opened (this decreases q) and (ii) firms are willing to pay more to guarantee to fill these vacancies. On the other hand, vacancy cost k affects the *slope* of ZP as well. The formula for this slope is $\frac{\eta'(q)k}{\eta^2(q)}$, hence it follows that at any given q, k unambiguously increases the steepness of ZP. Consequently, q must rise, while ω must fall in equilibrium when vacancy cost goes up. The intuition behind these changes is that higher costs discourage firms from entering the labor market, so the measure of vacancies falls

[11] One exception for the latter effect is when the indifference curves exhibit a quasi-linear fashion with respect to ω. In this case the equilibrium q does not change.

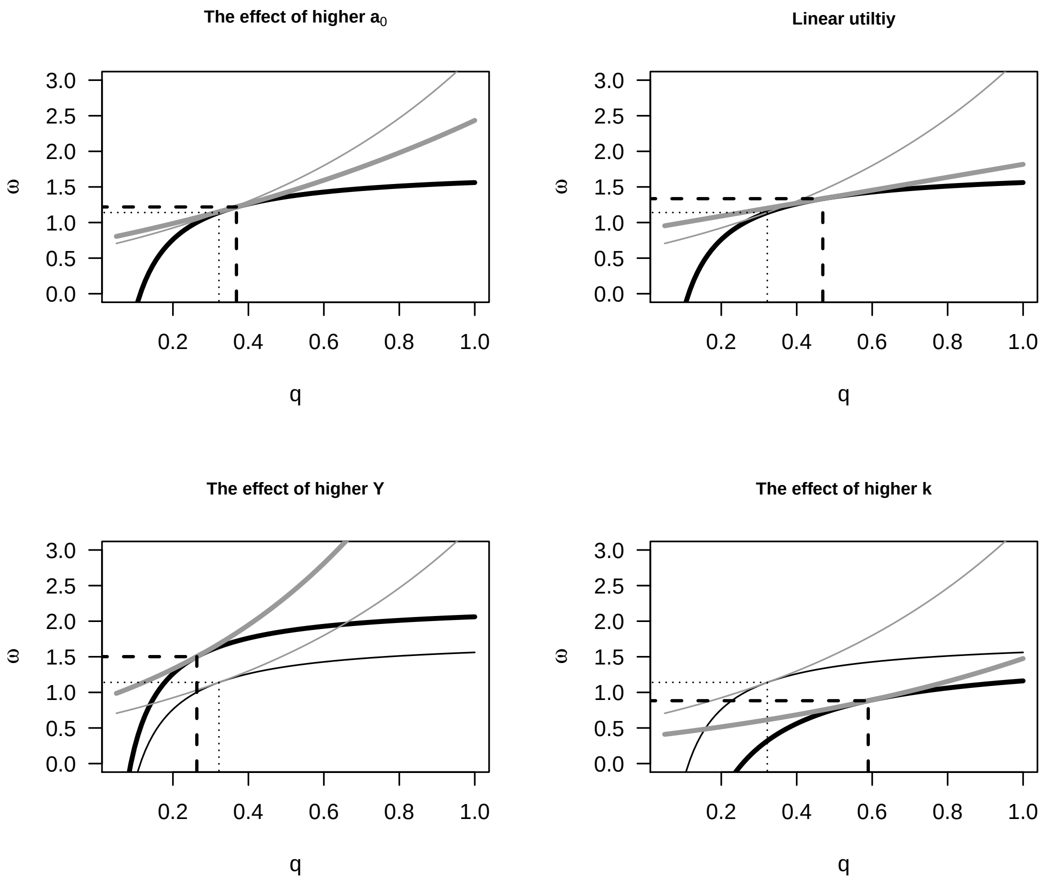

Figure 2: Comparative statics – the effect of some parameters on equilibrium
Thin curves and dotted lines: initial state; Thick curves and dashed lines: new state

in equilibrium. This leads to higher queue lengths and less pressure on firms who entered despite the worsened conditions, to pay high present value for their employees.

2.6 Numerical example

In order to develop some intuition about the mechanism of the model, consider a simple numerical example. Of course, due to its two-period fashion, the model is too unrealistic, so this example can serve at most as an illustration. Nonetheless I try to employ reasonable parametrization to obtain interpretable results. First of all, we need to specify the utility and the matching function. Let the former be CRRA with $\gamma = 1$, i.e. $u(c) = \log(c)$, while the latter is the matching function $\mathcal{M}(u,v) = \frac{uv}{(u^l+v^l)^{1/l}}$, suggested by DenHaan, Ramey, and Watson (2000) with parameter value $l = 1.2$. The two matching rates are then the following

$$\mu(q) = \frac{1/q}{\left(1 + (1/q)^l\right)^{\frac{1}{l}}} \qquad\qquad \eta(q) = \frac{q}{(1 + q^l)^{\frac{1}{l}}}$$

Beside these, let the interest rate be $r = 0.04$, the subjective discount factor, $\beta = 0.97$, the labor productivities $y_0 = y_1 = 1$ and the per period vacancy cost $k = 0.2$.

Using these specifications and solving the constraint maximization problem defined by (3) and (4), we obtain Figure 3, which portrays some indifference curves of the a_0-workers with $a_0 = 0.2$, derived by expressing q from (4) and plugging it into (3). Note that I restricted the domain to the positive orthant only in order to make the results more visible.

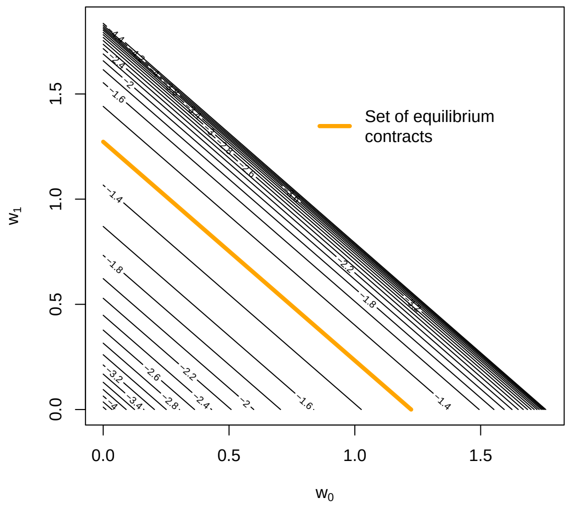

Figure 3: Some indifference curves and the uncountable set of equilibrium wage contracts providing the same present value

Benchmark two-period model, with $a_0 = 0.2$

The orange line represents the uncountable (offered) wage contracts to which a_0-workers apply with equal probability. All points on this line provide the same expected utility for the worker, while they ensure zero profit for the firms. The slope of the line is $-(1+r)$, hence all points deliver the same present value, which, through the zero-profit condition, implies a unique corresponding queue length. In other words, the orange line embodies the unique equilibrium submarket populated by a_0-workers only. Recall that the reason for the observed indeterminacy is that risk-averse workers' can perfectly smooth their consumption, so they do not care about the timing of their payoffs, only the present value.

One objective of the introduction of asset decision was to see how different variables depend on wealth level, therefore Figure 4 portrays some of these dependencies. In accordance with Proposition 3, the present value for which the worker applies depends positively on the asset level. Queue length also increases with a_0, which determines, in the anticipated way, both η and μ, the two matching probabilities. Still, the shape of the q curve is interesting, in particular its average level and the range of its values. Regarding the first feature note that its values are always smaller than one, which means that there is a relatively large amount of vacancies in the market. This is because the cost of vacancy is small relative to the expected gain from opening it: while $k = 0.2$, the obtainable output $Y = 1 + 1/1.04 = 1.96$, thus vacancy cost is around one tenth of the total output. Although one should consider the job filling probability as well, this difference obviously constitutes a huge incentive for firms to enter the market.[12] Considering the range of equilibrium q values, it is notable how limited it is. This can be traced back to the shape of the ZP curve on Figure 1, and more generally on the matching function parameter l. The lower value is set to l, the less concave the ZP curve is and the larger the change in q resulting from varying asset level. This suggests that, beside vacancy costs, the shape of the matching function (parameterized by l) can also have significant effect on the results.

This statement is supported by the fact that the curvature visible in case of all variables also comes from the concavity of matching function (and to a lesser extent from workers' risk-aversion). In particular, note that contrary to the usual story, the curvature of the employed workers' consumption function is not the result of precautionary saving (recall that I ruled this kind of saving motive out), but the consequence of the extremely steep present value curve at very small asset levels that is determined by the shape of the matching function. In short, the higher the value that l takes the less curvature PV and the other variables have.

Fortunately, this parameter has a very intuitive interpretation: it measures the magnitude of frictions on the labor market. In order to understand this, look how the matching probabilities behave if $l \to \infty$.

$$\lim_{l\to\infty} \mu(q) = \min\left\{\frac{1}{q}, 1\right\} \qquad\qquad \lim_{l\to\infty} \eta(q) = \min\{q, 1\}$$

These functions can be interpreted as the matching rates for a "frictionless" economy, in a sense that with these, the shorter side of the market is always fully employed. If the measure of vacancies exceeds the mass of unemployed workers,

[12] This can also be seen on the fourth panel of Figure 2, where the increase in k has a significant impact on equilibrium queue length.

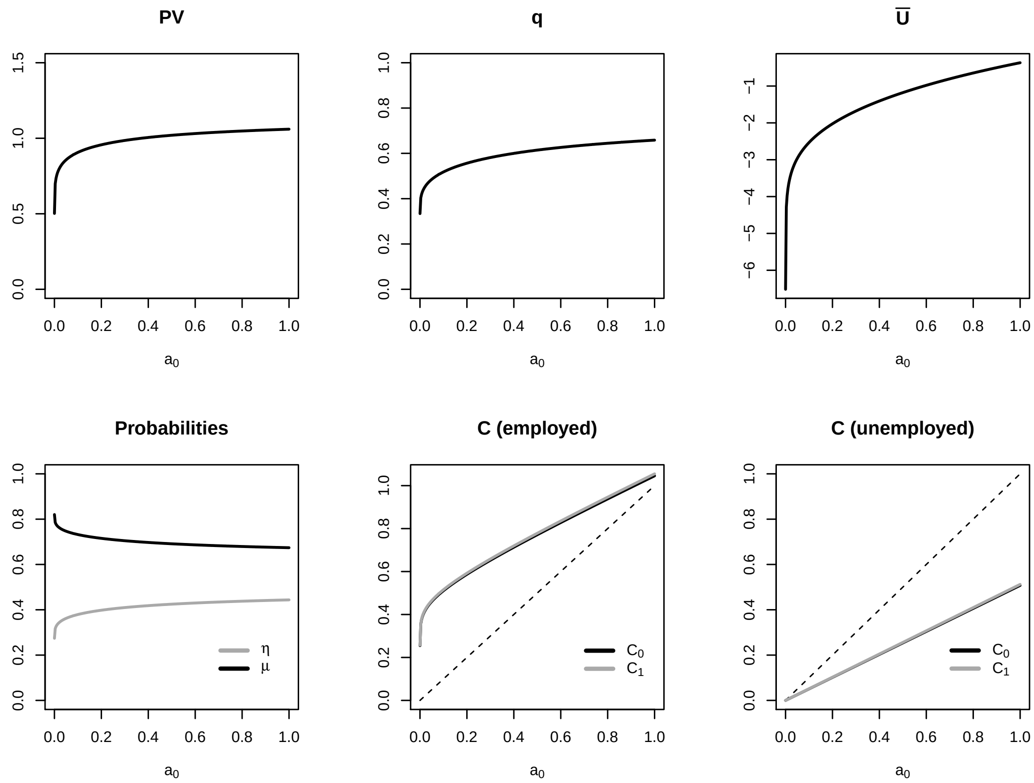

Figure 4: Optimal values of some variables as a function of initial asset level
Benchmark two-period model

the employment probability is one, while the job filling probability is given by the queue length itself, and the other way around if the measure of vacancies falls short of the measure of unemployed workers. This means that unemployment and vacancies cannot exist at the same time in this economy (the coordination friction vanished) and in fact, it turns out that in equilibrium q is always one and $u = v = 0$.[13] Of course, this also implies the independence of the equilibrium allocation on asset level, demonstrating that frictions are crucial to my findings. On the other hand, if $l \to 0$, both $\mu(\cdot)$ and $\eta(\cdot)$ go pointwise to the zero functions, that is, regardless of the given queue length, the matching probabilities are jointly equal to zero. One can interpret this as a result of the vast extent of frictions and therefore l indeed can serve as a measure for the magnitude of these frictions.

How does this parameter affect my results? This question can be answered through the examination of how l affects q, since the equilibrium queue length is one of the most important driving forces in the model. To this end, consider the first two panels of Figure 4 from a different angle: fix a particular a_0 and plot the equilibrium level of q and PV as a function of l as shown in Figure 5.

[13] Without going into details, one can see this using the ZP and IC curves of Figure 1, because with the above matching rates both curves have a kink at $q = 1$, so the equilibrium allocation is always determined by this point.

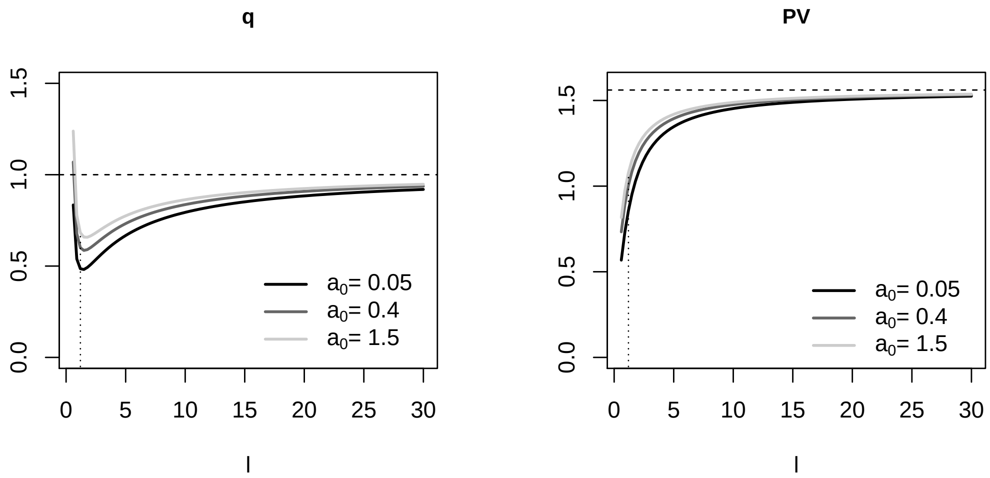

Figure 5: Equilibrium queue length and present value as a function of the matching function parameter l for different asset levels

This confirms that as $l \to \infty$ both q and PV go to their frictionless limit which is independent of a_0 and that the ranges of q and PV (as a function of a_0) are shrinking with l.[14] Regarding the average value of q, however, one can see that it is non-monotonic in the matching parameter. The reason is that the rising l (i.e. vanishing frictions) affects the matching rate $\eta(\cdot)$ in two different ways: (i) increases its value for each $q > 0$, (ii) enhances its slope for $q < 1$. The first effect, through the growing incentive of firms to enter and post more vacancies, reduces q, while the second effect expands it because the higher slopes represent more valuable marginal q. In short, the tension resulting from these two effects is the reason for the observed non-monotonicity. Given this finding, it apparently matters which value one sets to l. Following DenHaan et al. (2000), I use $l = 1.2$ (represented by the dotted segment on the figure).[15,16]

2.6.1 Consumption inequality

I have mentioned before that the positive relationship between asset level and present value might also expand workers' inequality. In accordance with my

[14]In other words, the sensitivity of q and PV with respect to a_0 diminishes.

[15]As I do not calibrate the model, it makes no sense pushing this issue too far; nevertheless, it might be interesting to consider some values frequently found in the literature. As I have just mentioned, DenHaan et al. (2000) calibrated its value to $l = 1.25$. On the other hand, Hagedorn and Manovskii (2008) and Trigari (2009) found much lower values, 0.407 and 0.55, respectively. Note that these low values would largely exaggregate the sensitivity of q and PV with respect to a_0 (as it is also visible in the figure).

[16]Note that the value of 1.2 still betokens a considerable extent of frictions. The theoretical matching function, for instance, arising endogenously from the aforementioned coordination frictions (see Burdett et al., 2001) takes the form of $\eta(q) = 1 - \exp(q)$, which is by and large comparable to the value $l = 1.9$.

former argument about the limitations of such analysis, let me consider how directed search alters the distribution of consumption among workers. In order to do so, one needs to specify the initial distribution of asset level $G(\cdot)$ and derive the distribution of consumption $F(\cdot)$ from it by calculating optimal consumption for each asset value as if all workers would remain unemployed, that is if they would have to survive on their initial asset level only. Due to the CRRA utility specification the unemployed workers' consumption function is linear, thus the transformation between the distribution of a_0 and c is straightforward in this case. On the other hand, the introduction of search generates employed workers as well, for whom one can derive the following relationship between the densities.[17]

$$f\left(\bar{C}\right) = \frac{\mu\left(c_E^{-1}(\bar{C})\right)}{c_E'\left(c_E^{-1}(\bar{C})\right)} g\left(c_E^{-1}(\bar{C})\right) + \frac{\left[1-\mu\left(c_U^{-1}(\bar{C})\right)\right]}{c_U'\left(c_U^{-1}(\bar{C})\right)} g\left(c_U^{-1}(\bar{C})\right) \tag{6}$$

where $g(\cdot)$ is the density function of a_0, $c_i : [\underline{a}, \overline{a}] \to (0, \infty)$ denotes the consumption function of the worker with $i = E, U$ and the prime is for the first derivate. Let asset be distributed equally over the interval $[0, 0.5]$, i.e. $g(a_0) = 2 \cdot \mathbb{1}_{[0,.5]}(a_0)$. Figure 6 then displays the densities of consumption corresponding to the two scenarios: the one where workers deterministically remain unemployed (without search), and the one where they are allowed to apply for a job (with search).

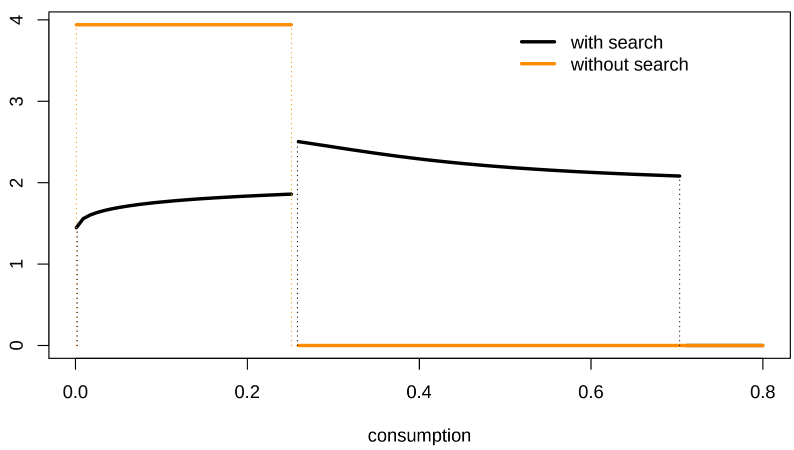

Figure 6: Analytic densities of period 0 consumption, with and without search

The level and dispersion of consumption are obviously altered by search, but as the figure indicates, the shape of the densities changes as well. As consumption decision is made after the realization of uncertainty, the two peaks of the black density are attributable to the fact that workers become heterogenous ac-

[17]The details of the derivation can be found in Appendix B.

cording to their luck during the application process. The left peak is associated with the remaining unemployed people, while the right mass represents those who found jobs.[18] Regarding the latter, two interesting features are worth mentioning. Namely, (i) it is right-skewed and (ii) its dispersion is similar to that of the initial asset distribution, so that the dispersion of asset holding is not magnified, it is transformed almost one-to-one to the dispersion of consumption. Naturally, these properties depend on the factor preceding the initial asset density, $\mu\left(c_E^{-1}(\bar{C})\right)\left[c_E'\left(c_E^{-1}(\bar{C})\right)\right]^{-1}$. In this factor, the second component represents the fact that workers with higher wealth level apply for more profitable jobs (inequality enhancing force), while the first term is responsible for the fact that these jobs are associated with lower employment probabilities (inequality reducing force).

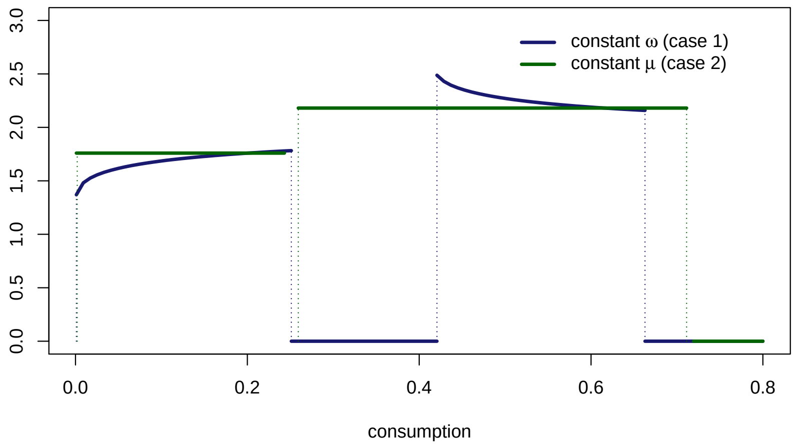

Figure 7: Effect of two forces: while the positive a_0-PV relationship (case 2) increases the dispersion, the decreasing μ (case 1) rather affects the curvature of consumption density.

In order to see how these two forces affect the density, consider two counterfactual cases in which we cut off these channels separately. First, assume that everybody, irrespective of their savings, applies for jobs providing the same present value, say, $\bar{\omega} = \mathbb{E}[\omega]$,[19] but obtains those jobs with the true probability $\mu(q(a))$. This scenario (case 1) corresponds to the first component of the above factor (inequality reducing force). On the other hand, as a second case, assume that each worker applies for the present value that her savings imply, but she gets it with a common counterfactual probability. This case represents the second, inequality enhancing component. Figure 7 depicts the two counterfactual

[18]Of course, it is not necessary that the two peaks are just separated. They can overlap or be located far from each other.

[19]This amount is given by the expression $\bar{\omega} = \frac{\int_0^{0.5}\mu(q(a))\omega(a)da}{\int_0^{0.5}\mu(q(a))da}$.

"densities". From this, it is clear that the dispersion of consumption is determined mainly by the present value-asset level relationship (case 2), in particular, that they are positively related. At the same time, varying probability (case 1) is responsible for the slope of the density, i.e. that the density increases for unemployed and decreases for employed workers.

This model specification, of course, is far from being realistic and since we do not have any reasonable benchmark to which we could compare the resulted density, we cannot say too much about the overall effect of search on consumption inequality. Nonetheless, it is notable that directed search, through the application decision and its dependence on the wealth level, can generate a force that alleviates inequality in a sense that it makes the density function more steeply decreasing at the upper tail.

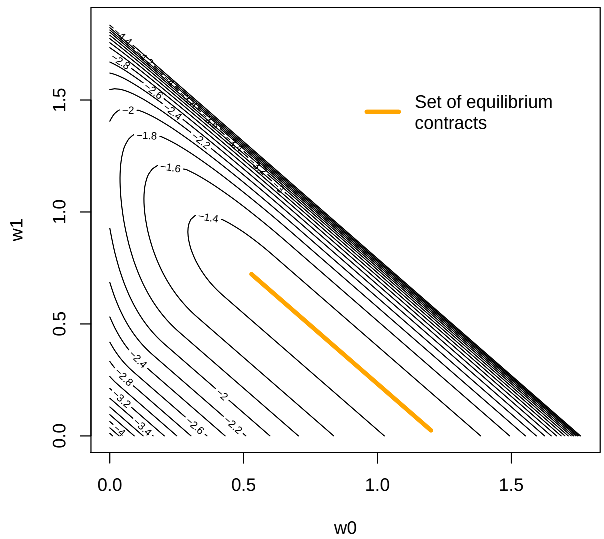

Figure 8: Some indifference curves and the reduced set of equilibrium contracts for a constrained agent resulting from her incentive to require relatively low w_1 LC-Benchmark two-period model with $a_0 = 0.2$

2.6.2 Liquidity constraint

In the last section I claimed that the feature of the model that the equilibrium contract is indeterminate is attributable to the ability of workers to perfectly smooth their consumption. For this reason, it might be reasonable to consider the case when they are unable to do so, in other words, when they are liquidity constrained. For instance, assume that they cannot borrow and restrict the choice space to $a_1 \geqslant a_{min} = 0$. Figure 8 portrays the resulting indifference curves and equilibrium wage contracts (orange line). The solution is still indeterminate, yet offered wage contracts now are biased toward the ones providing less steep wage paths. The intuition is obvious: since constrained workers cannot borrow on their future income, they prefer contracts with high w_0 and low w_1. Of course, the magnitude of this preference depends (negatively) on asset level, or more precisely, on the extent to which the borrowing constraint is binding. Beside these, the present value associated with the job to which they apply is the same as that without liquidity constraint. That is workers apply to the same present value, but the composition of the contract now matters since it can enable them to optimally allocate their consumption over time.[20]

In short, workers use the choice of the composition of wage contract as a substitute tool for smoothing consumption in place of borrowing or lending at the capital market. They are able to do this because (i) firms are risk-neutral and (ii) there is no risk of match separation, so employers do not care in which terms do they pay the same amount to their workers. In the next section I relax the second condition by allowing for on-the-job search, thus introducing the possibility of endogenous separation at the workers' side.

3 Two-period model with on-the-job search

In this section I now turn to models which incorporate on-the-job search as well. This can be reached by relaxing the assumption that workers can fully commit to employment, which brings a new force into the picture, namely the intention of firms to retain senior workers. Firms care about worker retention since they cannot immediately refill vacant jobs from the pool of unemployed (due to search frictions), so matches are valuable for them as a source of economic rent. Analyzing this mechanism requires us to enrich the benchmark two-period model from the previous chapter. The major changes are the following.

In period 1 labor market reopens. Firms, in exchange for paying k, are again allowed to offer a contract, but now each of these contracts contain only one specified wage level $\hat{w}_1$.[21] As in period 0, free-entry drives profit down to zero in period 1 as well. Most importantly, in addition to the unemployed, employed workers can also apply for a job in period 1. Let $s(a_1, w_1)$ denote the probability of quitting depending on the employee's savings at the beginning of period 1, a_1 and her current wage w_1. The applicants (no matter which pool they come from)

[20] The limiting case of liquidity constraint is if we completely drop the assumption of an operating capital market and assert that workers always have to consume their current wages. Of course, in this case some positive amount of goods (e.g. $b > 0$) is required to be given for the worker in both periods, in particular in period 1, otherwise the problem would cease to be well-defined for those who do not find a job. In this case, the solution is determinate and can be derived using the optimal present value ω and the Euler equation $u'(a_0 + w_0) = \beta u'(w_1)$.

[21] Hereafter, all period 1 variables will be denoted by a hat.

together with the opened vacancies generate a particular level of queue length for each equilibrium contract, giving rise to the function $\widehat{Q}(\hat{w}_1)$.

The economy in period 1 unfolds as follows. At the beginning of the period firms have the option to set up a vacancy by offering a wage contract. Then, unemployed and employed workers make their application decision, where I use a tie-breaking assumption, namely, if the value of search for an employed worker is zero, she does not search at all. This means that workers engage in on-the-job search only if there is a strictly positive gain from this activity.[22] After deciding where to apply, workers consume all their disposable income and the economy closes. There is nothing in this enriched model that prevents us from considering equilibrium allocations as the solution set for a particular constrained maximization problem. The particular problem, however, is now a bit more complex.

Proposition 4. *The equilibrium is given by the solution* $\{(w_0, w_1), \hat{w}_1, q, \hat{q}, a_1\}$ *for the following program:*

$$\overline{U}(a_0) = \max_{(w_0,w_1),q} \mu(q)V(a_0, w_0, w_1) + [1 - \mu(q)]V(a_0, 0, 0) \tag{7}$$

subject to

$$\eta(q)\left[y_0 - w_0 + \frac{1 - s(a_1, w_1)}{1 + r}(y_1 - w_1)\right] = k \tag{8}$$

and

$$\eta(\hat{q})\left[y_1 - \hat{w}_1\right] = k \tag{9}$$

where

$$\begin{aligned} V(a_0, w_0, w_1) &= \max_{a_1} u\left(a_0 + w_0 - \frac{a_1}{1 + r}\right) + \beta W(a_1, w_1) && (10) \\ W(a_1, w_1) &= \max_{\hat{w}_1,\hat{q}} \mu(\hat{q})\, u(a_1 + \hat{w}_1) + [1 - \mu(\hat{q})]\, u(a_1 + w_1) && (11) \end{aligned}$$

and

$$s(a_1, w_1) = \mu\left(\hat{q}(a_1, w_1)\right)$$

The worker's problem remains the same, except that the value function V now embeds the value of period 1 application decision W irrespective of the employment status. Given their asset level a_1 and current wage w_1 (for unemployed it is equal to zero) workers choose to apply for another job or to stay at their current firm. This situation is convenient for the worker because her current job is certain, it serves as a backup for her. Nevertheless, the aforementioned tie-breaking

[22]Possibly since there is a small ε cost of search and we consider the limiting case 0.

assumption ensures that they don't always choose to apply for another job, they decide to leave only if the expected gain is strictly positive. Another important assumption is that I rule out the possibility of counteroffer by the current firm, thus a new match automatically leads to a quit.[23]

What the introduction of on-the-job search really brings in is the function $s(\cdot,\cdot)$ capturing the probability of (endogenous) separation. Since in period 1 the value of search is strictly decreasing in the current wage, w_1,[24] firms, in period 0 are able to mitigate this separation probability by offering contracts with relatively high w_1. In other words, $s(\cdot,\cdot)$ is strictly decreasing in w_1, giving rise to a force that narrows the set of equilibrium contracts, namely that firms are eager to offer high w_1 and low w_0 combinations. The following proposition shows that this force is so prominent that it completely rules out the possibility of equilibria including on-the-job search.

Proposition 5. *There always exists an equilibrium. If $\{(w_0^*, w_1^*), \hat{w}_1^*, q^*, \hat{q}^*, a_1^*\}$ solves the above program for a given a_0, then the set of vectors*

$$Z(a_0) \equiv \left\{ \begin{array}{ll} ((w_0, w_1), \hat{w}_1^*, q^*, \hat{q}^*, a_1): & w_0 + \frac{w_1}{1+r} = w_0^* + \frac{w_1^*}{1+r}; \\ & w_1 \geqslant y_1 - k; \;\; a_1 = a_1^* + (1+r)(w_0 - w_0^*) \end{array} \right\}$$

solve it as well. Moreover, these are the only solutions, i.e. $\mathcal{W}^2(a_0) = Z(a_0)$.

Proof: See Appendix A.

A natural question is how the particular threshold value in Proposition 5 is determined. To answer this one needs to look at the condition which characterizes the one-to-one relationship between $\hat{w}_1$ and $\hat{q}$, that is the zero profit condition (9). Since η can take only values from the unit interval, the newly offered wage in period 1, $\hat{w}_1$ is bounded from above. Precisely

$$\eta(\hat{q})\left[y_1 - \hat{w}_1\right] = k \leqslant y_1 - \hat{w}_1 \qquad \Rightarrow \qquad \hat{w}_1 \leqslant y_1 - k$$

This means that if firms in period 0 offer contracts with higher period 1 wage than this particular threshold, $w_1 \geqslant y_1 - k$, they can bind their employees to themselves during the entire contract period and eliminate equilibrium on-the-job search. This last statement can be shown as follows. Plug the value $y_1 - k$ in place of $\hat{w}_1$ in condition (9), then, given that $k > 0$, $\eta(\hat{q})k = k$ implies $\hat{q} = \infty$

[23]For a random search model with counteroffer see e.g. Postel-Vinay and Robin (2002).

[24]This can be seen by simply using the Envelope theorem. Define the search value as $S(a_1, w_1) \equiv W(a_1, w_1) - u(a_1 + w_1)$, then its first derivative w.r.t w_1 is $\partial S/\partial w_1 = -\mu(\hat{q})u'(a_1 + w_1) < 0$, whenever $\mu(\hat{q}) > 0$ and so the claim follows.

and thus $\mu(\hat{q}) = s(a_1, w_1) = 0$. The zero probability of getting the job means that the value of search is at most zero, so by our tie-breaking assumption no on-the-job search happens if $w_1 \geqslant y_1 - k$. In this region of the contract space, $W(a_1, w_1) = u(a_1 + w_1)$ and $s(a_1, w_1) = 0$, thus the problem collapses exactly into the benchmark two-period model.

The reason for this phenomenon is the firms' heterogeneity generated by their success during the matching process in period 0. Firms become heterogenous in a sense that they consider different contract horizons. In short, employers who are unable to fill their jobs in period 0 cannot attain the same expected match surplus, because they have only one period to produce. This matters since on-the-job search generates competition not only among firms in a given period, but also among firms in different periods. The fact that matched firms have more to offer means that they have competitive advantage over their future competitors and can prevent them from entering markets that include employed workers as well. Apparently, this feature of the model is an artifact of our two-period assumption and it would disappear if we turned to a stationary infinite horizon environment, where both the matched and unmatched firms would face identical problems.

The lack of equilibrium on-the-job search has implications for workers' self-separation as well. *Ex ante* we might think that the result that each job attracts only one type of worker is violated if we allow employees to search. This conjecture can be justified by the fact that what really matters for the application decision is not the asset level but the value of cash-on-hand. Consider for example an unemployed worker in period 1 with asset level a_1 and an employed worker with asset level a'_1 and wage w'_1, such that $a'_1 + w'_1 = a_1$. Then we can reasonably anticipate that they will apply for the same jobs thus breaking down self-separation and block recursivity. However, as we have seen above, in equilibrium there is no on-the-job search and therefore workers' self-separation still holds.

3.1 Numerical example (continued)

We continue the numerical example presented in section 2.6 to examine what happens if we embed on-the-job search into the model. With the values set to the critical parameters we get the threshold value of $y_1 - k = 1 - 0.2 = 0.8$, above which the model is identical to the benchmark two-period framework. This can be seen on Figure 9 as well, where in the corresponding region the indifference curves are parallel straight lines (see also Figure 3).

The solution set is again uncountable, but it is now narrowed by the force mentioned in the introduction of this section. Because of endogenous separation,

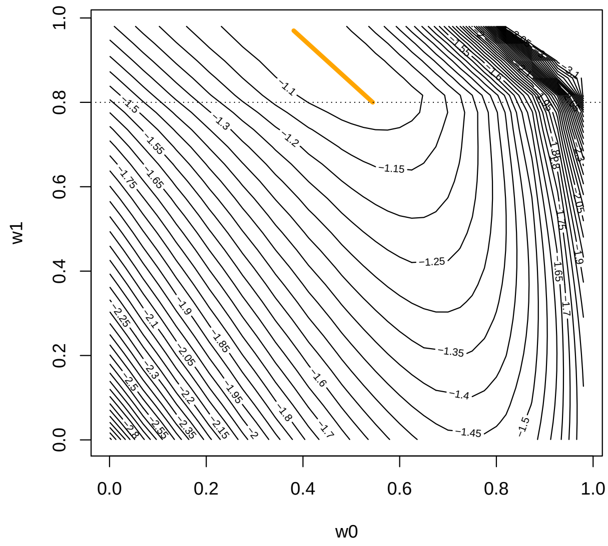

Figure 9: Some indifference curves and the reduced set of equilibrium contracts that comes from firms' incentive to pay relatively high w_1.
Two-period model with on-the-job search; $a_0 = 0.2$

firms have an incentive to offer contract with high w_1 and low w_0. As the firms that are able to operate for two periods can afford to pay higher period 1 wage than the maximum offered by their competitors operating only in period 1, they can completely eliminate on-the-job search by paying sufficiently high w_1. This mechanism gives rise to equilibrium contracts characterized by an increasing wage path. That the wage path is increasing with tenure is compatible with the results of Burdett and Coles (2003) and Shi (2009). Notwithstanding, the current specification shows that this feature still exists in models with saving decision, though the particular optimal contract is indeterminate.

3.2 Liquidity constraint with on-the-job search

We have seen before that if capital markets are imperfect and liquidity constraint is imposed, a restriction is being put on the possible set of equilibrium wage contracts for workers with binding constraint (see Figure 4). Constrained workers are no longer indifferent to the allocation of wages over time, they prefer contracts with high w_0 and low w_1. On the other hand, we have also seen that on-the-job search generates a similar incentive on the firms' side, though with opposite sign. Firms tend to offer contracts with high w_1 and low w_0. These considerations lead to the conjecture that if the two conflicting incentives are at work together in the model, the solution might become determinate.

This conjecture turns out to be true for those workers whose required w_1 is lower than $y_1 - k$, i.e. for whom the borrowing constraint is binding. Figure 10 illustrates this case for an a_0-worker with $a_0 = 0.2$. Since these workers have sufficiently low initial asset level (in the sense of being liquidity constrained), the two opposing forces determine a unique wage contract. Yet for this result it is necessary to consider a sufficiently poor worker. If a_0 exceeds a particular asset level, the solution set becomes uncountable again. This is not surprising. As I have noted in Section 2.6.2, the degree of preference for high w_0 depends negatively on the strictness of the liquidity constraint and the same relationship is operating here.

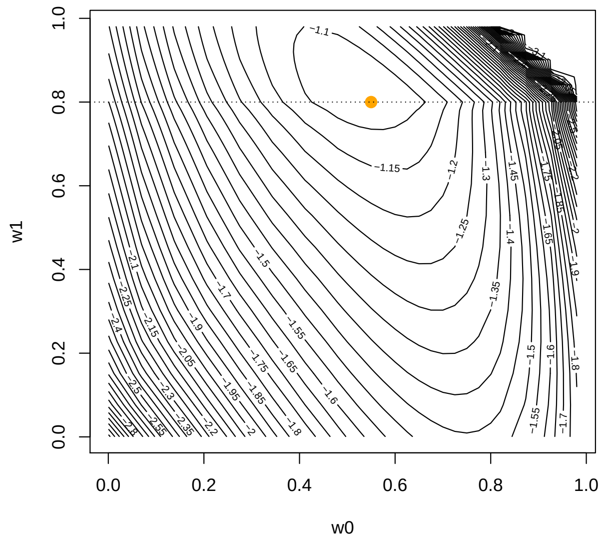

Figure 10: Some indifference curves and the determinate wage contract (orange point) for a constrained worker
Two-period model with on-the-job search and liquidity constraint; $a_0 = 0.2$

Surprisingly, however, liquidity constrained unemployed who cannot commit to stay with the employer apply for jobs with higher present value and lower matching probability than their counterparts living in an economy without liquidity constraint. The differences between the two scenarios are visible on Figure 11, where the black and grey lines represent the two-period model with on-the-job search only, while the orange lines the model where additionally the no-borrowing constraint is imposed. Note that on-the-job search in itself does not change the qualitative features of the benchmark two-period model.[25] The reason for this, of course, is that in equilibrium no on-the-job search takes place as was shown

[25]The only exception is the unemployed workers' consumption function. Nevertheless, this is not the effect of on-the-job search either, but the assumption that labor market reopens in period 1 and workers who failed to get a job in period 0 can reapply. This generates a new kind of saving motive, namely saving in order to obtain a relatively large a_1 which helps them to attain higher utility in period 1. In the figure, the grey line, in effect represents the amount of saving in period 0, which is no longer a straight line. This is because of the shape of value function W and the appearance of this new saving motive.

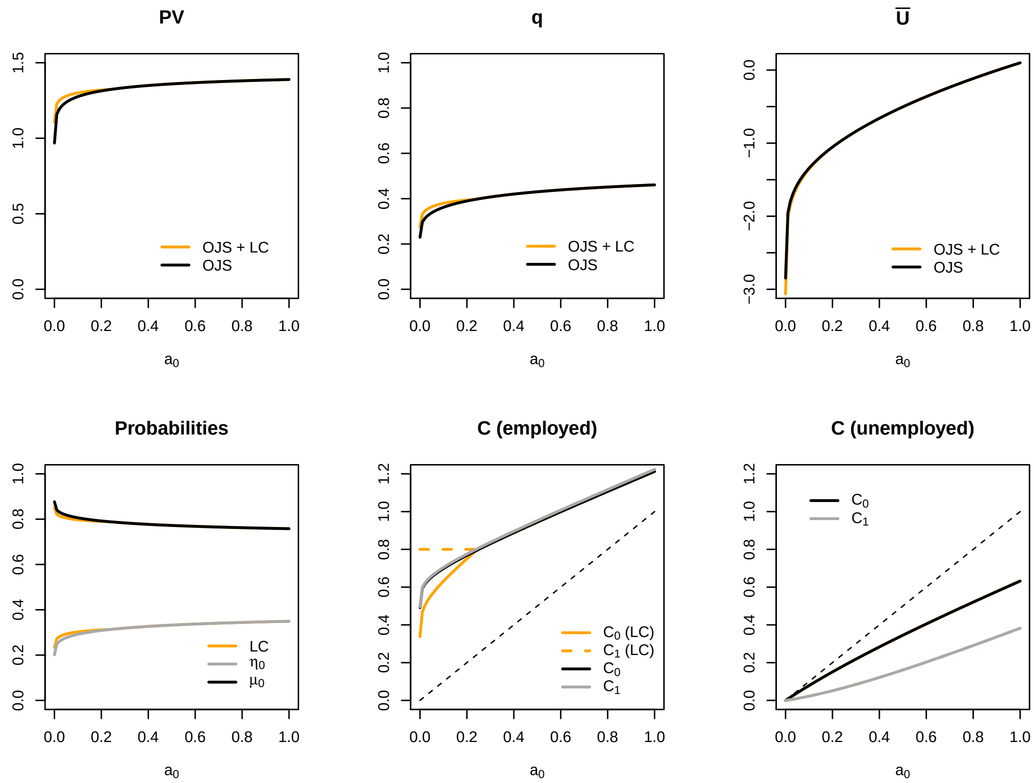

Figure 11: Optimal values of some variables as a function of initial asset level
Two-period model with on-the-job search (black and grey), and accompanied with borrowing constraint (orange)

in Proposition 5. Still one can see that with borrowing limit there exists a critical asset level (around 0.23) below which agents are constrained and this makes them to apply for jobs which pay better but are harder to obtain, thus getting a slightly lower utility. The explanation of this phenomenon is that for constrained agents the trade-off between present value and unemployment risk is different compared to their unconstrained counterparts. These workers appreciate present value relatively more in a sense that they are willing to accept lower matching probability to obtain a marginal increase in the offered present value. As the following proposition shows, this phenomenon is a lot more general than this simple numerical example suggests.

Proposition 6. *In the two-period on-the-job search model if an unemployed a_0-worker is liquidity constrained, she applies for higher present value and lower matching probability than she would without the constraint. That is $\omega^{LC}(a_0) > \omega^{UC}(a_0)$ and $q^{LC}(a_0) > q^{UC}(a_0)$.*[26]

Proof:

[26] Where the superscripts LC and UC denote whether the function corresponds to a liquidity constrained or an unconstrained worker, respectively.

This can be seen by looking at the slope of the IC curve (5), which, recall, represents the trade-off between present value and employment probability. Let $\breve{a}_0$ denote the asset value at which the agent who applies for a contract with $\bar{w}_1 = y_1 - k$ is just constrained, in other words, the value which induces the saving decision $a_1(\breve{a}_0, w_0(\breve{a}_0), \bar{w}_1) = 0$. This is without loss in generality, because constrained agents will always apply for such contracts, while the unconstrained agents can smooth consumption perfectly so we can fix freely one member of the wage pair for which they apply. This implies that effectively workers' application decision can be characterized as a choice about w_0. According to Proposition 3, the optimal present value is strictly increasing in asset level, thus $\breve{a}_0$ is associated with a unique present value $\breve{\omega}$ and wage level $\breve{w}_0$.

Consider then a a'_0-worker such that $a'_0 < \breve{a}_0$. By revealed preference the following relation has to hold: $\tilde{V}\left(a'_0, \omega^{LC}(a'_0)\right) \leqslant \tilde{V}\left(a'_0, \omega^{UC}(a'_0)\right)$. Moreover, based on the above argument the partial derivative of the value function w.r.t ω is equal to the derivative w.r.t. w_0. It follows that for all $w_0 < \breve{w}_0$

$$\frac{\partial V^{LC}}{\partial w_0} = u'(a'_0 + w_0) > u'\left(a'_0 + w_0 - \frac{a_1(a'_0, w_0, \bar{w}_1)}{1+r}\right) = \frac{\partial V^{UC}}{\partial w_0} \quad \text{since} \quad a_1 < 0$$

and so the following always holds for all $w_0 < \breve{w}_0$

$$\frac{d\omega^{LC}}{dq} < \frac{d\omega^{UC}}{dq}$$

Since each constrained a'_0-worker can choose only from wages $w_0 < \breve{w}_0$ (otherwise she would not be constrained) and since the ZP curve is strictly concave, the worker's IC curve can tangent it only at a point with relatively higher q and ω. $\square$

In other words, since workers are risk-averse they wish to have a smooth consumption stream, but they can finance it only if they are not liquidity constrained. This is because workers cannot commit to stay with the firm in the long run. Consequently, firms do not want to offer them contracts that would urge them to leave, i.e. contracts providing relatively low w_1, which on the other hand would be necessary for constrained workers to smooth their consumption. For this reason, for constrained workers the desire for a smooth consumption path makes each additional unit of w_0 more valuable and thus they will apply for higher present value even at the cost of lower matching probability.

This story also has an intuitive graphical representation in the familiar (q, ω)-space (see Figure 12). One can see that the trade-off between present value and

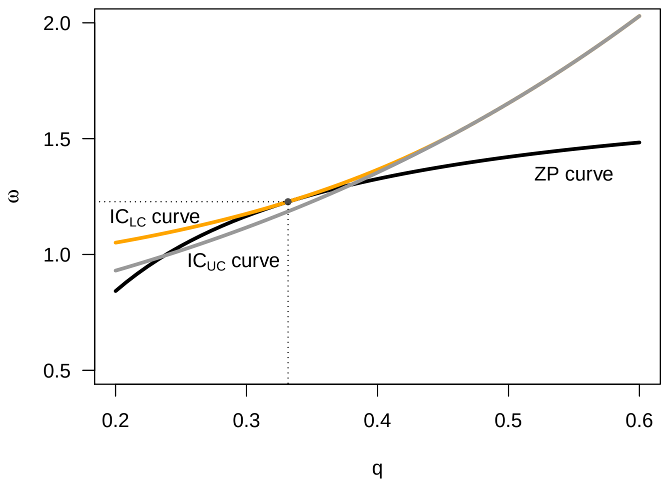

Figure 12: Indifference curves for constrained (LC) and unconstrained agents (UC) evaluated at the optimal value of the LC worker's problem.

matching probability, i.e. the slope of the IC curve, is different for constrained workers if $\omega < \breve{\omega}$. In particular, their slope is lower, which means that they are willing to sacrifice less present value for each additional reduction in q and thus the corresponding IC curve can tangent ZP only at a relatively large q and ω. Moreover, since the utility level is increasing in the north-east direction, the figure also suggests that even if they apply for better paying jobs, in terms of welfare, constrained agents attain less desirable outcomes. Recall that for these results we need two forces at the same time working in opposite directions, (i) a constrained worker's desire to a smooth consumption stream and (ii) employers' incentive to reduce quits.

Note also that this effect is sensitive to the directed search assumption, more precisely to the assumption that workers can choose where to apply. As I have mentioned before, within this framework, for risk-averse and liquidity constrained agents application decision have another function, namely it serves as a vehicle for consumption smoothing. In random search models this device is not available and thus the relationship between optimal contracts and the degree of smoothing ability does not appear either.

However, two remarks are in order. First, even if this finding is qualitatively interesting, quantitatively does not seem to be significant in the numerical exercises discussed above. Second, the empirical plausibility of the result that liquidity constrained workers have relatively longer expected unemployment duration (implied by the lower matching probability) is hard to justify. For instance, Chetty (2008) finds that in case of liquidity constrained workers unemployment benefit has a significant, positive impact on unemployment duration, two-third of which is attributable to "liquidity effect", i.e. to that the benefit ensures them better positions to consumption smoothing. In other words, in the data lessening liquidity constraint for constrained unemployed seems to lengthen the unemployment duration and not shorten it. Contrarily, in our model unemployment benefit would help constrained workers to smooth consumption, as a result of which they would apply for more easily attainable jobs and shorten expected unemployment duration. Of course, Chetty's finding does not falsify our model implications directly, but at least it points to some of their possible weaknesses.

4 Concluding Remarks

This paper explored the consequences and possible difficulties of incorporating saving decisions in a two-period directed search model. I found that savings, through the effect of varying degree of risk-aversion, affect the application decision in a way that wealthier unemployed workers apply for better paid and less easily attainable jobs. Nonetheless, introducing saving decision in an unconstrained way brings indeterminacy of the optimal wage path into the model. Intuitively, since workers can smooth their consumption perfectly, they do not care about the particular timing of wages, only the present value matters. One way to overcome (at least partially) this difficulty is to allow for on-the-job search and impose borrowing constraint at the same time. This generates a tension of two opposing forces, the resolution of which determines a unique optimal contract for constrained workers. In short, due to on-the-job search firms are willing to offer increasing wage paths, but at the same time borrowing limit forces constrained workers to demand as high current wage as possible. Surprisingly, this latter force turns out to be so strong that it changes the workers' tradeoff between present value and unemployment risk in a way that it may lead them to choose better paid but more risky jobs than they would without the constraint.

Throughout the paper, in order to keep the exposition as simple as possible, I used two critical assumptions: (i) the time horizon is finite, thus the economy is non-stationary, (ii) upon matching, employees face no risk, i.e. they never return to unemployment. These assumptions, however, are not innocuous regarding my findings. First of all, by assuming finite horizon we automatically generate heterogeneity across firms with respect to their opportunities. In short, the sooner the firm can match with a worker and start producing, the longer horizon it is able to calculate with and the more it can pay for its employee, thus keeping its future competitors out from the "employee market". As a result, in equilibrium no job-to-job transition takes place, which has consequences for the self-separation property of the on-the-job search model as well. In particular, it rules out the possibility of its violation, which shows that in order to seriously investigate the possible limitations of the concept of block recursive equilibrium we definitely need an infinite horizon framework. Finally, considering the lack of exogenous separation, one should notice that this assumption, in addition to excluding precautionary saving motives, deactivates a force similar to that of liquidity constraint, namely the workers' desire to obtain their income as early as possible, which might help us to uniquely determine the optimal wage contract. The relaxation of these assumptions is left for future research.

Appendices

Appendix A – Proofs

A.1 Proof of Proposition 1

if Let $\{\mathcal{W}^2, Q, a_1, \overline{U}\}$ be an equilibrium and for any fixed $a_0 \in [\underline{a}, \overline{a}]$ let $(w_0^*, w_1^*) \in \mathcal{W}^2$, $q^* = Q(w_0^*, w_1^*)$ and $a_1^* = a_1(a_0, w_0^*, w_1^*)$. Since $(w_0^*, w_1^*) \in \mathcal{W}^2$, profit maximization and free entry ensure that the constraint (4) holds. Moreover, a_1^* is trivially the maximizer of the corresponding objective, since in the definition of the equilibrium exactly the same program appears. That is, all we need to show is that $\{(w_0^*, w_1^*), q^*, a_1^*\}$ maximizes (3), i.e.

$$\overline{U}(a_0) = \mu(q^*)V(a_0, w_0^*, w_1^*) + [1 - \mu(q^*)]\, V(a_0, 0, 0)$$

Suppose not, i.e. there exists another allocation $\{(w_0^{'}, w_1^{'}), q^{'}, a_1^{'}\}$ which provides higher utility

$$\overline{U}(a_0) < \mu(q^{'})V(a_0, w_0^{'}, w_1^{'}) + \left[1 - \mu(q^{'})\right] V(a_0, 0, 0)$$

Since $\{\mathcal{W}^2, Q, a_1, \overline{U}\}$ is an equilibrium, an optimal saving and application decision requirement in the definition implies that

$$\overline{U}(a_0) \geqslant \mu(Q(w_0^{'}, w_1^{'}))V(a_0, w_0^{'}, w_1^{'}) + \left[1 - \mu(Q(w_0^{'}, w_1^{'}))\right] V(a_0, 0, 0)$$

The above two inequalities imply $\mu(q^{'}) > \mu(Q(w_0^{'}, w_1^{'}))$, thus $q^{'} < Q(w_0^{'}, w_1^{'})$. However, this violates the zero profit constraint, since

$$\eta\left(q^{'}\right)\left[y_0 - w_0^{'} + \frac{y_1 - w_1^{'}}{1+r}\right] - k < \eta\left(Q(w_0^{'}, w_1^{'})\right)\left[y_0 - w_0^{'} + \frac{y_1 - w_1^{'}}{1+r}\right] - k \leqslant 0$$

therefore $\left\{(w_0^{'}, w_1^{'}), q^{'}, a_1^{'}\right\}$ is not in the constraint set. Contradiction.

only if Suppose for any fixed $a_0 \in [\underline{a}, \overline{a}]$ that $\{(w_0^*, w_1^*), q^*, a_1^*\}$ solves the program (3)-(4). I would like to prove that the constructed set $\{\mathcal{W}^2, Q, a_1, \overline{U}\}$ with $\mathcal{W}^2(a_0) = \{(w_0^*, w_1^*)\}$, $Q(w_0^*, w_1^*) = q^*$ and $a_1(a_0, w_0^*, w_1^*) = a_1^*$ is an equilibrium.

First, let $\overline{U}(a_0)$ be defined by

$$\overline{U}(a_0) = \mu(q^*)V(a_0, w_0^*, w_1^*) + [1 - \mu(q^*)]\, V(a_0, 0, 0)$$

and let $Q(w_0, w_1)$ satisfy

$$\overline{U}(a_0) = \mu(Q(w_0, w_1))V(w_0, w_1, a_0) + [1 - \mu(Q(w_0, w_1))]\, V(a_0, 0, 0)$$

or let $Q(w_0, w_1) = 0$ if there is no solution to the equation. Then, $\{\mathcal{W}^2, Q, a_1, \overline{U}\}$ trivially satisfies the optimal saving and application decision.
However, it also satisfies profit maximization. Suppose not, i.e. there is some $\{(w_0^{'}, w_1^{'}),\, Q(w_0^{'}, w_1^{'}), a_1^{'}\}$, such that

$$\eta\left(Q(w_0^{'}, w_1^{'})\right)\left[y_0 - w_0^{'} + \frac{1}{1+r}(y_1 - w_1^{'})\right] > k$$

This implies that $Q(w_0^{'}, w_1^{'}) > 0$, so $\exists q^{'} < Q(w_0^{'}, w_1^{'})$, s.t.

$$\eta\left(q^{'}\right)\left[y_0 - w_0^{'} + \frac{1}{1+r}(y_1 - w_1^{'})\right] = k$$

Moreover, by construction of $Q(\cdot, \cdot)$, $q^{'} < Q(w_0^{'}, w_1^{'})$ indicates that

$$\overline{U}(a_0) < \mu(q^{'})V(a_0, w_0^{'}, w_1^{'}) + \left[1 - \mu(q^{'})\right] V(a_0, 0, 0)$$

which means that $\left\{(w_0^{'}, w_1^{'}), q^{'}, a_1^{'}\right\}$ satisfies the zero profit constraint (4) and yields higher utility than $\{(w_0^*, w_1^*), q^*, a_1^*\}$, contradiction. □

A.2 Proof of Proposition 2

1. *Existence*
   First, I show that the saving decision always has a solution. The nonnegativity requirement of consumption in the two periods generates for every given (w_0, w_1) and a_0 a compact feasible set for a_1. Namely,

   $$a_1(a_0, w_0, w_1) \in [-w_1, (1+r)(a_0 + w_0)]$$

   Given the properties of the utility function (see Assumption 1), for every tuple (a_0, w_0, w_1) there exists a unique a_1 which solves the maximization problem. Moreover, by Berge's Maximum Theorem and the uniqueness of the solution, the policy function, $a_1(a_0, w_0, w_1)$, as well as the value function, $V(a_0, w_0, w_1)$ are all at least C^0.

   Next, I show that for any $a_0 \in [\underline{a}, \overline{a}]$, the optimal application decision has a solution which completes the proof. Since the LHS of (4) is a continuous function of w_0, w_1 and q, while the RHS is a singleton (and so its preimage is closed), the constraint set is closed. Moreover, $\eta(\cdot) \leqslant 1$ indicates that $y_0 + \frac{y_1}{1+r} - \left(w_0 + \frac{w_1}{1+r}\right) \geqslant k$, i.e. $(w_0, w_1, q) \in [0, Y-k] \times [0, (1+r)(Y-k)] \times [0, \infty]$, where $Y = y_0 + y_1/(1+r)$. All these intervals are compact, so by Tychnoff's Theorem their product is also compact. Given that the objective function (3) is obviously continuous (recall that the value function from the savings problem is at least C^0), there exists a solution, i.e. $\mathcal{W}^2(a_0) \neq \varnothing$, $\forall a_0 \in [\underline{a}, \overline{a}]$. In fact, we also know that $\overline{U}$ is continuous, while $\mathcal{W}^2$ is upper hemicontinuous.

2. *Continuum of solution*

   We have seen above that the optimal saving decision always has a unique, interior solution, which is determined by the necessary and sufficient FOC.

$$\beta(1+r)u'(a_1^* + w_1^*) = u'\left(a_0 + w_1^* - \frac{a_1^*}{1+r}\right)$$

   Let $\Delta w_0 = w_0 - w_0^*$, then by the condition given in the definition of $Z(a_0)$, $\Delta w_0 = \frac{-\Delta w_1}{1+r}$ and $\Delta a_1 = (1+r)\Delta w_0$ which means that

$$\beta(1+r)u'(a_1^* + \Delta a_1 + w_1^* + \Delta w_1) = u'\left(a_0 + w_1^* + \Delta w_1 - \frac{a_1^* + \Delta a_1}{1+r}\right)$$

   That is, neither c_0^*, nor c_1^* change and since the allocations in $Z(a_0)$ obviously satisfy the feasibility constraint, they indeed solve the program. The q is determined through (4):

$$q = \eta^{-1}\left(\frac{k}{Y - w_0 - \frac{w_1}{1+r}}\right)$$

   and since the present value of wages is unique in $Z(a_0)$, q^* does not change.

3. $\mathcal{W}^2(a_0) = Z(a_0)$

   Simply calculating the fraction $\frac{\partial \overline{U}/\partial w_0}{\partial \overline{U}/\partial w_1}$ yields the value $-(1+r)$, which depends only on a parameter r. This means that identical function values of $\overline{U}$ lie on a line with a slope of $-(1+r)$ in the $w_0 - w_1$ coordinate-system. Suppose that (w_0, w_1) is not an element of $Z(a_0)$, that is $w_0 + \frac{w_1}{1+r} \neq w_0^* + \frac{w_1^*}{1+r}$, or in other words, (w_0, w_1) is not on the line having the slope $-(1+r)$ and going through (w_0^*, w_1^*). Due to the above considerations this implies that $\overline{U} \neq \overline{U}^*$, so $\overline{U} < \overline{U}^*$ and also $(w_0, w_1) \notin \mathcal{W}^2(a_0)$. □

A.3 Proof of Proposition 3

Define $O(a_0,\omega) \equiv \mu(\widetilde{Q}(\omega))\widetilde{V}(a_0,\omega) + [1-\mu(\widetilde{Q}(\omega))]\widetilde{V}(a_0,0)$. Then $O(a_0,0) = O(a_0,Y-k) = \widetilde{V}(a_0,0)$ and $O(\cdot,\cdot) > \widetilde{V}(a_0,0)$ if $\omega \in (0,Y-k)$. Since O is at least C^0, this indicates an interior solution which can be characterized by the FOC:

$$\frac{\widetilde{V}(\omega(a_0),a_0) - \widetilde{V}(0,a_0)}{\frac{\partial \widetilde{V}(\omega(a_0),a_0)}{\partial \omega}} = -\frac{\mu(\omega(a_0))}{\mu'(\omega(a_0))}$$

where $\widetilde{V}$ is differentiable due to Assumption 1 (recall $u(\cdot)$ is C^2). Suppose now $a_0' \neq a_0$, but $\omega(a_0) = \omega(a_0')$, which is, using Proposition 2, equivalent to the claim $\mathcal{W}^2(a_0) \cap \mathcal{W}^2(a_0') \neq \varnothing$. This leads to two different values on the LHS of the above equation. Since the RHS is a function of $\omega(\cdot)$, it cannot take two different values, contradiction. That is $\mathcal{W}^2(a_0) \cap \mathcal{W}^2(a_0') = \varnothing$ if $a_0 \neq a_0'$ which strengthens the usual revealed preference relation (which states only $\geqslant$) to $U(\omega(a_0),a_0) > U(\omega(a_0'),a_0)$, for every $a_0 \neq a_0'$.

The utility function is DARA iff for every $a_0 < a_0'$ there exists a concave transformation $h(\cdot)$, s.t. $u(a_0+b) = h\left(u(a_0'+b)\right)$ (see Mas-Colell, Prop.6.C.3.(ii)) This assumption together with Assumption 1 buys us the property that the transformation is even strictly concave and C^2. After these considerations, let $a_0 < a_0'$, $\omega = \omega(a_0)$, $\omega' = \omega(a_0')$ and the corresponding queue lengths, q and q', respectively. Using the above revealed preference relations

$$\begin{aligned} \mu(q)\widetilde{V}(a_0,\omega) + [1-\mu(q)]\widetilde{V}(a_0,0) &> \mu(q)\widetilde{V}(a_0,\omega') + [1-\mu(q)]\widetilde{V}(a_0,0) \\ \mu(q')\widetilde{V}(a_0',\omega') + [1-\mu(q')]\widetilde{V}(a_0',0) &> \mu(q)\widetilde{V}(a_0',\omega) + [1-\mu(q)]\widetilde{V}(a_0',0) \end{aligned}$$

Rearranging terms yields (of course, $\omega,\omega' > 0$)

$$[\widetilde{V}(a_0,\omega)-\widetilde{V}(a_0,0)][\widetilde{V}(a_0',\omega')-\widetilde{V}(a_0',0)] > [\widetilde{V}(a_0',\omega)-\widetilde{V}(a_0',0)][\widetilde{V}(a_0,\omega')-\widetilde{V}(a_0,0)] \tag{12}$$

Now, suppose (by contradiction) that $\omega > \omega'$. Since $\widetilde{V}(a_0,\cdot)$ is increasing and continuous, by intermediate value theorem $\exists\alpha \in (0,1)$, s.t.

$$\widetilde{V}(a_0',\omega) = \alpha\widetilde{V}(a_0',\omega') + (1-\alpha)\widetilde{V}(a_0',0)$$

Applying the transformation $h(\cdot)$ on both sides and exploiting its strict concavity lead to

$$\widetilde{V}(a_0,\omega) > \alpha\widetilde{V}(a_0,\omega') + (1-\alpha)\widetilde{V}(a_0,0)$$

where I used that $\widetilde{V}$ inherits the DARA property of $u(\cdot)$. Using these two expressions (12) becomes

$$[\widetilde{V}(a_0,\omega')-\widetilde{V}(a_0,0)][\widetilde{V}(a_0',\omega')-\widetilde{V}(a_0',0)] > [\widetilde{V}(a_0,\omega')-\widetilde{V}(a_0,0)][\widetilde{V}(a_0',\omega')-\widetilde{V}(a_0',0)]$$

Contradiction. Therefore $\omega' \geqslant \omega$, which according to the first part of this proof means $\omega' > \omega$. Moreover, from (4) we also know $q' > q$. □

A.4 Proof of Proposition 5

If $w_1 \geqslant y_1 - k$, then $\mu(\widehat{Q}(\hat{w}_1)) = s(a_1, w_1) = 0$, so in this region the problem collapses into the model without on the job search, therefore the results derived there hold here as well. In other words what we need to show is that $w_1 < y_1 - k$ cannot be part of any optimal wage contract.

Suppose not, i.e. consider a wage policy (w_0, w_1) with $w_1 < y_1 - k$ and let $\omega = w_0 + \frac{w_1}{1+r}$. The zero profit condition can be written as

$$\eta(q)\left[y_0 - w_0 + \frac{1-s(a_1,w_1)}{1+r}(y_1 - w_1)\right] = \eta(q)P = k$$

then the gradient of P wrt wages is the following

$$\begin{aligned}\left[\frac{\partial P}{w_0}, \frac{\partial P}{w_1}\right] &= \left[-1, \frac{d-1}{1+r}\right] \\ \text{where} \quad d &= s(a_1,w_1) - \frac{\partial s}{\partial w_1}(y_1 - w_1)\end{aligned}$$

where $d = 0$ if $w_1 \geqslant y_1 - k$, otherwise $d > 0$. Moreover, it can be easily established that $\lim_{w_1 \to (y_1-k)^-} d = \frac{y_1 - w_1}{2k} > 0$. This means that with the vector $\left(\frac{-\Delta w_1}{1+r}, \Delta w_1\right)$ where $\Delta w_1 = w_1' - w_1$ we can keep the present value ω constant and at the same time increase P. Then, the zero profit condition requires the queue length to fall, i.e. $q' < q$.

On the workers' side, given that they can perfectly smooth their consumption, the same value of ω implies

$$u\left(a_0 + w_0 - \frac{a_1}{1+r}\right) + \beta u(w_1 + a_1) = u\left(a_0 + w_0' - \frac{a_1'}{1+r}\right) + \beta u(w_1' + a_1') \leqslant \tilde{V}(a_0, \omega)$$

where the inequality comes from the definition of value function W. Since $q' < q$ implies $\mu(q') > \mu(q)$, these mean that with wage policy (w_0', w_1') higher utility can be attained, i.e. $w_1 < y_1 - k$ cannot be part of an optimal wage contract. □

Appendix B – Derivation of equation (6)

Let $G(\cdot)$ be a continuous distribution function of initial asset value and derive the corresponding distribution function $F(\cdot)$ of consumption from it. To this end, consider an infinitesimal interval of consumption $[\bar{C}, \bar{C}+\Delta]$. There are two types of agents who can reach this interval after the matching process. The a_0'-workers, who got a job and as a result consume the amount of $c_E(a_0') \in [\bar{C}, \bar{C}+\Delta]$ and the ones (possibly with different asset level, call it a_0''), who remain unemployed but consume $c_U(a_0'') \in [\bar{C}, \bar{C}+\Delta]$, where $c_i(\cdot)$ represents the policy function of consumption for $i = E, U$. Let $\hat{\mu}(a_0) = \mu\left(q(a_0)\right)$. It follows that the probability that a a_0-worker moves into the interval of interest is

$$\mathbb{P}\left(c_E^{-1}(\bar{C}) \leqslant a_0 \leqslant c_E^{-1}(\bar{C}+\Delta)\right)\hat{\mu}(a_0) + \mathbb{P}\left(c_U^{-1}(\bar{C}) \leqslant a_0 \leqslant c_U^{-1}(\bar{C}+\Delta)\right)[1-\hat{\mu}(a_0)]$$

Dividing by Δ and taking the limit $\Delta \to 0$ leads to

$$\begin{aligned}
\lim_{\Delta\to 0}\frac{\mathbb{P}\left(c_E^{-1}(\bar{C}) \leqslant a_0 \leqslant c_E^{-1}(\bar{C}+\Delta)\right)}{\Delta}\hat{\mu}(a_0) &= \mu\left(c_E^{-1}(\bar{C})\right)\frac{\partial G(c_E^{-1}(\bar{C}))/\partial \bar{C}}{c_E'(c_E^{-1}(\bar{C}))} \\
\lim_{\Delta\to 0}\frac{\mathbb{P}\left(c_U^{-1}(\bar{C}) \leqslant a_0 \leqslant c_U^{-1}(\bar{C}+\Delta)\right)}{\Delta}[1-\hat{\mu}(a_0)] &= \left[1-\mu\left(c_U^{-1}(\bar{C})\right)\right]\frac{\frac{\partial G(c_U^{-1}(\bar{C}))}{\partial \bar{C}}}{c_U'(c_U^{-1}(\bar{C}))}
\end{aligned}$$

and therefore the probability density function of consumption at $\bar{C}$, $f(\bar{C})$ is given by

$$f(\bar{C}) = \mu\left(c_E^{-1}(\bar{C})\right)g(c_E^{-1}(\bar{C}))\frac{1}{c_E'(c_E^{-1}(\bar{C}))} + \left[1-\mu\left(c_U^{-1}(\bar{C})\right)\right]g(c_U^{-1}(\bar{C}))\frac{1}{c_U'(c_U^{-1}(\bar{C}))}$$

which is equation (6).

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