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Regularity of the linear Wigner-Fokker-Planck equation

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Chapter 1

The WFP equation and its physical meaning

1.1 The Wigner pseudo distributions

The quantum physical state of a statistical ensemble of particles can be described by a real-valued function of the phase-space $\mathbb{R}^d \times \mathbb{R}^d$, called *Wigner quasiprobability distribution* or *Wigner function*, first introduced by Wigner in [Wig32]. They will be written in this paper $w : \mathbb{R}^{2d} \to \mathbb{R}$.

In the previous paragraph, and in the whole paper, d denotes the dimension of the considered system. Typically, $d = 3$.

This function can be considered as the quantum physical equivalent of the classical phase-space distribution $f : \mathbb{R}^{2d} \to \mathbb{R}^+$ in statistical physics. But, unlike classical particles, quantum particles don't have simultaneously precise positions and velocities (Heisenberg principle), and therefore the Wigner function $w(x, v)$ doesn't describe locally the expected number of particles with position x and velocity v, and can take negative values. The Wigner function can be seen as a probability distribution only at a bigger scale.

1.1 Remark. For a single particle with wave function ψ, the corresponding Wigner function w can be computed by

$$w(x, v) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \psi^*(x + \frac{\hbar\eta}{2m})\, \psi(x - \frac{\hbar\eta}{2m})\, e^{-iv\cdot\eta}\, \mathrm{d}\eta, \tag{1.1}$$

where $\hbar$ is the reduced Planck constant and m the mass of the particle.

More generally, for an ensemble of identical particles with density matrix ρ, the Wigner function is defined by

$$w(x, v) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \rho(x + \frac{\hbar\eta}{2m}, x - \frac{\hbar\eta}{2m})\, e^{-iv\cdot\eta}\, \mathrm{d}\eta. \tag{1.2}$$

The definition of the density matrices and their precise relationship with Wigner functions are out of the scope of this paper. See for instance [PL93, Süd07, Arn08].

The evolution in time of the particle ensemble in the exterior potential V and without interaction between particles is modeled by the following linear partial differential equation acting on the Wigner function and called *Wigner equation*:

$$\partial_t w + v \cdot \nabla_x w - \Theta[V]w = 0 \qquad t \in \mathbb{R},\, x, v \in \mathbb{R}^d, \tag{1.3}$$

where the potential field $V : \mathbb{R}^d \to \mathbb{R}$ appears in the form of the linear operator $\Theta[V]$ which will be studied in more details in Section 1.4.

The Wigner equation is the quantum physical counterpart of the *Liouville equation* for a classical distribution f in phase space

$$\partial_t f + v \cdot \nabla_x f - \frac{1}{m} \nabla_x V \cdot \nabla_v f = 0 \qquad t \in \mathbb{R},\, x, v \in \mathbb{R}^d \tag{1.4}$$

and converges to it in the classical limit $\hbar \to 0$. (See also Remark 1.4 and [PL93].)

1.2 The Wigner-Fokker-Planck equation

In addition to the potential field V, we consider the interaction of the particle ensemble with a heat bath of oscillators. This *quantum Brownian motion* can be modeled by the following *Fokker-Planck operator* $\mathcal{Q}$ acting on the Wigner function:

$$\mathcal{Q}w = \frac{D_{pp}}{m^2}\, \Delta_v w + 2\frac{D_{pq}}{m}\, \mathrm{div}_x(\nabla_v w) + D_{qq}\, \Delta_x w + 2\gamma\, \mathrm{div}_v(wv), \tag{1.5}$$

where $2\gamma\, \mathrm{div}_v(wv)$ models the friction and $D_{pp}\, \Delta_v$, $2D_{pq}\, \mathrm{div}_x \nabla_v$ and $D_{qq}\, \Delta_x$ model the diffusion of the quasiprobability distribution in phase-space.

The combination of the Wigner equation (1.3) with the Fokker-Planck operator gives us the *Wigner-Fokker-Planck equation*:

$$\partial_t w + v \cdot \nabla_x w - \Theta[V]w = \mathcal{Q}w \qquad t \in \mathbb{R}_0^+,\, x, v \in \mathbb{R}^d. \tag{1.6}$$

It is the quantum physical counterpart of the classical *Vlasov-Fokker-Planck equation*:

$$\partial_t f + v \cdot \nabla_x f - \frac{1}{m} \nabla_x V \cdot \nabla_v f = \frac{D_{pp}}{m^2}\, \Delta_v f + 2\gamma\, \mathrm{div}_v(fv). \tag{1.7}$$

1.2 Remark. The constants γ, D_{pp}, D_{pq} and D_{qq} can be expressed in terms of physical magnitudes

$$\gamma = \frac{\eta}{2m}, \quad D_{pp} = \eta k_B T, \quad D_{qq} = \frac{\eta \hbar^2}{12 m^2 k_B T}, \quad D_{pq} = \frac{\eta \Omega \hbar^2}{12 \pi m k_B T}, \tag{1.8}$$

where η is the coupling constant of the heat bath, k_B the Boltzmann constant, T the temperature of the bath and Ω the cut-off frequency of the reservoir oscillators [ALMS04].

Note that in the classical limit $\hbar \to 0$, then $D_{qq} \to 0$ and $D_{pq} \to 0$. The diffusion in space $D_{qq}\, \Delta_x$ and the mixed diffusion $2D_{pq}\, \mathrm{div}_x \nabla_v$ are quantum physical effects and don't appear in the classical Vlasov-Fokker-Planck problem.

1.3 The Lindblad conditions

Some conditions on the coefficients γ, D_{pp}, D_{pq} and D_{qq} are necessary in order to make the model consistent with quantum physics. The *Lindblad conditions* read:

$$D_{pp} > 0 \tag{1.9a}$$

$$D_{pp} D_{qq} - D_{pq}^2 \geq \frac{\hbar^2 \gamma^2}{4} \geq 0 \tag{1.9b}$$

In this paper, we will distinguish the following two cases:

$$D_{pp} D_{qq} - D_{pq}^2 > 0 \tag{1.10a}$$

and

$$D_{pp} D_{qq} - D_{pq}^2 = 0 \quad (\Rightarrow \gamma = 0) \tag{1.10b}$$

that we may call respectively *elliptic case* and *semi-elliptic case*, since the operator $\mathcal{Q}$ is respectively elliptic and semi-elliptic.

The derivation of the Lindblad conditions and their interpretation are out of the scope of this paper. See for instance [ALMS04] for more details.

1.4 The potential operator

The potential V appears in the Wigner equation (1.3) (and the WFP equation (1.6)) in the form of the operator $\Theta[V]$ defined in the following way:

$$\Theta[V]w\,(x,v) = \frac{i}{(2\pi)^d\,\hbar} \iint_{\mathbb{R}^{2d}} \delta V(x,\eta)\, w(x,v')\, e^{i\eta\cdot(v-v')}\, \mathrm{d}v'\mathrm{d}\eta \tag{1.11}$$

where δV is defined by

$$\delta V(x,\eta) = V\left(x + \frac{\hbar}{2m}\eta\right) - V\left(x - \frac{\hbar}{2m}\eta\right). \tag{1.12}$$

Note that $V \mapsto \Theta[V]$ is linear.

The operator $\Theta[V]$ can be re-written [ALMS04, ADM07] in a more compact form as

$$\Theta[V]w = \frac{i}{\hbar}\mathcal{F}^{-1}_{\eta\to v}\left[\delta V\, \mathcal{F}_{v\to\eta}[w]\right] \tag{1.13}$$

where $\mathcal{F}_{v\to\eta}$ is the Fourier transformation with respect to the variable v and $\mathcal{F}^{-1}_{\eta\to v}$ its inverse:

$$\mathcal{F}_{v\to\eta}[f(x,\,.\,)](\eta) = \int_{\mathbb{R}^d} f(x,v)e^{i\eta\cdot v}\mathrm{d}v,$$

$$\mathcal{F}^{-1}_{\eta\to v}[g(x,\,.\,)](v) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} g(x,\eta)e^{-i\eta\cdot v}\mathrm{d}\eta$$

for suitable functions f and g.

Equivalently to (1.13), we can write $\Theta[V]$ as:

$$\Theta[V]w = \frac{i}{\hbar}\left(\mathcal{F}^{-1}_{\eta\to v}[\delta V] *_v w\right) \tag{1.14}$$

where $*_v$ is the partial convolution with respect to the variable v.

1.3 Remark. Despite the appearance of complex numbers in its definition, the operator $\Theta[V]$ maps real-valued functions on real-valued functions.

For a real-valued w in the definition domain of $\Theta[V]$, $\mathcal{F}[w]$ is conjugate symmetric with respect to η (i.e. $\mathcal{F}_{v\to\eta}[w](x,-\eta) = \mathcal{F}_{v\to\eta}[w](x,\eta)^*$, where the star denotes the complex conjugation). Since δV is odd with respect to η, the pointwise product $\delta V \mathcal{F}_{v\to\eta}[w]$ is conjugate antisymmetric and $\mathcal{F}^{-1}_{\eta\to v}\left[\delta V \mathcal{F}_{v\to\eta}[w]\right]$ is then purely imaginary-valued. And finally, $\Theta[V]w$ is real-valued.

1.4 Remark. In the classical limit $\hbar \to 0$,

$$\delta V(x,\eta) \sim \frac{\hbar}{m}\eta \cdot \nabla_x V,$$

and then

$$\Theta[V]w \longrightarrow \frac{1}{m}\nabla_x V \cdot \nabla_v w,$$

which is the potential term in the classical Liouville equation. See also [PL93] for a mathematically more rigorous derivation.

1.5 Remark. For $V(x) = \frac{\omega_0^2}{2}|x|^2$, we have

$$\delta V(x,\eta) = \frac{\hbar\omega_0^2}{m} x \cdot \eta,$$

and subsequently

$$\Theta[V]w = \omega_0^2 x \cdot \nabla_v w. \tag{1.15}$$

In the special case of a quadratic potential, $\Theta[V]$ can easily be expressed and coincides with its classical counterpart.

1.6 Lemma. *If V is a measurable function, $\Theta[V]$ is a closed, densely defined linear operator of $L^2(\mathbb{R}^{2d})$. Furthermore $\Theta[V]$ is bounded if and only if $V \in L^\infty(\mathbb{R}^d)$.*

Proof. With (1.13), we can rewrite $\Theta[V]$ (up to a multiplicative constant) as:

$$\Theta[V] = \mathcal{F}^{-1}_{\eta\to v} \circ M_{\delta V} \circ \mathcal{F}_{v\to\eta}$$

where $M_{\delta V}$ is the multiplication operator with δV.

The Fourier transform is a unitary operator of L^2 and, since δV is measurable, the multiplication operator is closed and densely defined. It follows the closeness and dense definition domain of $\Theta[V]$.

Furthermore, the boundedness of $\Theta[V]$ is equivalent to the boundedness of $M_{\delta V}$, which is equivalent to $\delta V \in L^\infty(\mathbb{R}^{2d}) \Leftrightarrow V \in L^\infty(\mathbb{R}^d)$ □

In the latter case, we have moreover

$$\|\Theta[V]\|_{\mathcal{B}(L^2)} = \frac{1}{\hbar}\|\delta V\|_{L^\infty} \leq \frac{2}{\hbar}\|V\|_{L^\infty}. \tag{1.16}$$

The previous lemma cannot be easily extended to other L^p spaces, since the Fourier transform hasn't the same nice properties in these spaces. However, we can easily prove the following weaker result:

1.7 Lemma. *If $V \in \mathcal{F}[L^1(\mathbb{R}^d)]$, then $\Theta[V]$ is a bounded linear operator of $L^p(\mathbb{R}^{2d})$ for all $1 \leq p \leq \infty$.*

Proof. For the sake of readability, we set every constant to 1. Using (1.14), the Young inequality for convolution with respect to the v variable and the Hölder inequality with respect to the x variable:

$$\begin{aligned}\forall w \in L^p(\mathbb{R}^{2d}) \quad \|\Theta[V]w\|_{L^p} &= \|\mathcal{F}^{-1}_{\eta\to v}[\delta V] *_v w\|_{L^p_{x,v}} \\ &\leq \Big\| \, \|\mathcal{F}^{-1}_{\eta\to v}[\delta V]\|_{L^1_v} \, \|w\|_{L^p_v} \Big\|_{L^p_x} \\ &\leq \|\mathcal{F}^{-1}_{\eta\to v}[\delta V]\|_{L^\infty_x(L^1_v)} \, \|w\|_{L^p_{x,v}}.\end{aligned}$$

Using simple properties of the Fourier transform, we show:

$$\begin{aligned}\mathcal{F}^{-1}_{\eta\to v}\big[\delta V(x,\eta)\big]\,(v) &= \mathcal{F}^{-1}_{\eta\to v}\big[V(x+\eta/2) - V(x-\eta/2)\big]\,(v) \\ &= 2^d e^{-2iv\cdot x}\mathcal{F}^{-1}\big[V\big]\,(-2v) - 2^d e^{2iv\cdot x}\mathcal{F}^{-1}\big[V\big]\,(2v) \\ &= -\,2^{d+1}\, i\,\Im\left(e^{2ivx}\mathcal{F}^{-1}\big[V\big]\,(2v)\right), \qquad (1.17)\end{aligned}$$

since $\mathcal{F}^{-1}\big[V\big]$ is conjugate symmetric.

Then

$$\begin{aligned}\|\mathcal{F}^{-1}_{\eta\to v}[\delta V]\|_{L^\infty_x(L^1_v)} &= 2^{d+1}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\Im\left(e^{2ivx}\,\mathcal{F}^{-1}\big[V\big](2v)\right)\right| \mathrm{d}v \\ &\leq 2^{d+1}\int_{\mathbb{R}^d}\left|\mathcal{F}^{-1}\big[V\big](2v)\right| \mathrm{d}v.\end{aligned}$$

From the presupposed integrability of $\mathcal{F}^{-1}[V]$ follows the boundedness of $\Theta[V]$. □

In the following chapters, we will use the notation:

$$\Theta^{-1}[\mathcal{B}(L^p)] := \left\{V : \mathbb{R}^d \to \mathbb{R} \mid \Theta[V] \text{ is a bounded operator on } L^p(\mathbb{R}^{2d})\right\}. \qquad (1.18)$$

From the previous lemma, we know

$$\Theta^{-1}[\mathcal{B}(L^2)] = L^\infty(\mathbb{R}^d)$$

and for $1 \leq p \leq \infty$

$$\Theta^{-1}[\mathcal{B}(L^p)] \supseteq \mathcal{F}(L^1(\mathbb{R}^d)) \supseteq C^\infty_0(\mathbb{R}^d).$$

1.5 A brief literature review

In the literature, two problems linked to the Wigner-Fokker-Planck equations are mainly studied:

1. The long time behavior of the solutions and the existence of stationary solutions has been studied in [SCDM04], [AGG+12] and [SA13].

2. The so-called *Wigner-Poisson-Fokker-Planck equation* is similar to the WFP equation but considers a potential generated by the interaction between particles and thus dependent of w. The operator $\Theta[V(w)]$ being non-linear, new questions arise. This problem has been studied, for instance in [ADM07] and [ALMS04].

Some of these results (as well as considerations about other quantum physical problems) have been summed up in [Arn08].

Note that similar questions are asked about classical Fokker-Planck problems, for instance in [Bou95] and [Car98].

The purpose of this thesis is to study the existence of solutions and their regularity for the linear Wigner-Fokker-Planck equation with or without potential. In the next chapter, we study the solutions in the special case of a quadratic potential. In Chapter 3, we will considered additionally a bounded potential, at first constant in time and then time dependent.

Chapter 2

Solution for a quadratic potential

From now on, we set $m = 1$ and $\hbar = 1$ for the sake of readability.

In this chapter, we consider the quadratic potential

$$V(x) = \frac{\omega_0^2}{2}|x|^2 + a \cdot x + b, \tag{2.1}$$

for $\omega_0^2 \in \mathbb{R}_0^+$, $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$. In that case, we can express more explicitly the potential term in the WFP equation:

$$\delta V(x,\eta) = (\omega_0^2 x + a) \cdot \eta \quad \Rightarrow \quad \Theta[V]w(x,v) = \big(\omega_0^2 x + a\big) \cdot \nabla_v w(x,v).$$

Without loss of generality, we can shift the problem with respect to x and then set $a = 0$.

The linear partial differential equation that we will study in this chapter is therefore

$$\begin{aligned} \partial_t w = &-v \cdot \nabla_x w + \omega_0^2 x \cdot \nabla_v w + 2\gamma \operatorname{div}_v \cdot (vw) \\ &+ D_{qq}\Delta_x w + 2D_{pq} \operatorname{div}_x \nabla_v w + D_{pp}\Delta_v w, \end{aligned} \tag{2.2}$$

which can be re-written in a more compact form

$$\begin{aligned} \partial_t w &= \nabla^T D \nabla w + P(x,v)^T \nabla w + 2d\gamma w \\ &=: Aw \end{aligned} \tag{2.3}$$

where $\nabla = \begin{pmatrix} \nabla_x \\ \nabla_v \end{pmatrix}$, $P(x,v) = \begin{pmatrix} -v \\ \omega_0^2 x + 2\gamma v \end{pmatrix}$ and $D = \begin{pmatrix} D_{qq}I_d & D_{pq}I_d \\ D_{pq}I_d & D_{pp}I_d \end{pmatrix}$ with I_d the d-dimensional identity matrix.

A solution of the WFP problem with quadratic potential is then a solution of the following Cauchy problem in $L^p(\mathbb{R}^{2d})$:

$$\begin{cases} \partial_t w = Aw & \forall t > 0, \\ w(t=0) = w_0. \end{cases} \tag{2.4}$$

2.1 Existence of a solution

In this section, we show that A is the infinitesimal generator of a strongly continuous semigroup of $L^p(\mathbb{R}^{2d})$. It will follow the existence of a unique solution for the problem (2.4). (See also Appendix A.)

2.1 Theorem. *[Arn08, Thm. 5.2] For $w_0 \in L^2(\mathbb{R}^{2d})$, the problem (2.4) has a unique solution $w \in C(\mathbb{R}_0^+; L^2(\mathbb{R}^{2d}))$.*

If furthermore $w_0 \in \mathcal{D}_{L^2}(A)$, then the solution is a classical solution, i.e. it belongs to $C^1(\mathbb{R}_0^+; L^2(\mathbb{R}^{2d}))$.

The notation $\mathcal{D}_X(A)$ for an operator A and a space X represents the maximal definition domain of A in X,

$$\mathcal{D}_X(A) = \{x \in X \mid Ax \in X\}\,.$$

Proof. One applies the Lumer-Philips Theorem (Theorem A.7) on the dissipative operator $A - d\gamma$. See [Arn08, Thm. 5.2] for more details. ☐

2.2 Remark. A can be seen as an example in the class of the so called *Ornstein Uhlenbeck* operators. These operators have the general form

$$\mathcal{A} = \frac{1}{2}\sum_{i,j=1}^{N} q_{ij}\mathrm{D}_{ij} + \sum_{i,j=1}^{N} b_{ij}x_j\mathrm{D}_i, \tag{2.5}$$

where $Q = (q_{ij})$ is a symmetric and positive definite real matrix and $B = (b_{ij})$ is a non-zero real matrix. These operators play a significant role in the field of stochastic analysis and have been widely studied for instance in [PL93], [Lun97] [Met01], [MPP02] and [MPRS02]. In our framework,

$$Q = D = \begin{pmatrix} D_{qq}I_d & D_{pq}I_d \\ D_{pq}I_d & D_{pp}I_d \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & -I_d \\ \omega_0^2 I_d & 2\gamma I_d \end{pmatrix}. \tag{2.6}$$

2.3 Remark. The (maximal) definition domain of A in $L^p(\mathbb{R}^{2d})$ has been explicitely derived in [MPRS02, Thm. 4.1] as

$$\mathcal{D}_{L^p}(A) = \{u \in W^{2,p}(\mathbb{R}^{2d}) \mid P(x,v)^T\nabla u \in L^p(\mathbb{R}^{2d})\}.$$

for $1 < p < \infty$.

We can extend Theorem 2.1 to L^p for $1 \le p \le \infty$ using for instance known results on Ornstein-Uhlenbeck operators. Note that most of the arguments of Theorem 2.1 also holds in L^p, and a generalization of its proof is also possible.

2.4 Theorem. *For $w_0 \in L^p(\mathbb{R}^{2d})$, $1 \le p \le \infty$, the problem (2.4) has a unique solution $w \in C(\mathbb{R}_0^+; L^p(\mathbb{R}^{2d}))$.*

If furthermore $w_0 \in \mathcal{D}_{L^p}(A)$, then the solution is a classical solution, i.e. it belongs to $C^1(\mathbb{R}_0^+; L^p(\mathbb{R}^{2d}))$.

Proof. A is the generator of a strongly continuous semigroup according to [Met01, Prop. 3.2]. ☐

2.2 Analyticity

In this section, we want to investigate if the strongly continuous semigroup generated by A has the property of being analytic (see also section A.2). It would follow regularity properties on the solution of the WFP problem.

2.2.1 In L^p spaces

2.5 Lemma. *The spectrum of $\nabla^T D\nabla + P(x,v)^T\nabla + 2d\gamma/p$ in $L^p(\mathbb{R}^{2d})$ contains a sub-group of $i\mathbb{R}$ for all $1 \leq p \leq \infty$.*

Proof. We use the results of [Met01]:

- Thm. 3.3: the (boundary) spectrum of $\nabla^T D\nabla + P(x,v)^T\nabla$ contains the spectrum of the drift term $P(x,v)^T\nabla$.

- Depending on P, the spectrum of the drift term is $i\mathbb{R}$ (Thm. 2.3 and 2.5) or a sub-group of $i\mathbb{R}$ (Thm. 2.6).

Note that, in our framework, $\mathrm{tr}(B) = 2d\gamma$. □

2.6 Remark. In the same article [Met01] actually compute the whole spectrum of some Ornstein-Uhlenbeck operators in L^p. In our case, we can derive from it that for $\gamma > 0$ and $\omega_0^2 > 0$ the spectrum is the half-plane defined by

$$\sigma_{L^p}(A) = \{z \in \mathbb{C} \mid \Re(z) \leq -\frac{2\gamma(p-1)}{p}\} \tag{2.7}$$

for all $1 \leq p \leq \infty$.

2.7 Theorem. *The semigroup generated by A is not analytic in $L^p(\mathbb{R}^{2d})$ for any $1 \leq p \leq \infty$.*

Proof. It follows from Lemma 2.5 that A is not sectorial. □

2.2.2 In weighted L^p spaces

Let us now introduce some new function spaces in which A actually generates an analytic semigroup.

2.8 Definition. We denote by L^p_μ the Gaussian weighted Lebesgue space defined by

$$L^p_\mu(\mathbb{R}^N) := L^p(\mathbb{R}^N,\, \mu(x)\mathrm{d}x)$$

for a Gaussian function

$$\mu(x) = \frac{1}{(4\pi)^{\frac{N}{2}}\,(\det Q)^{1/2}} \exp\left(\frac{-\langle Q^{-1}x, x\rangle}{4}\right),$$

where Q is a positive definite matrix and $N \in \mathbb{N}$.

One defines also naturally the Sobolev spaces

$$W^{k,p}_\mu = \{f \in L^p_\mu \mid \forall l \in \mathbb{N}_0^N,\, |l| \leq k,\, \mathrm{D}^l f \in L^p_\mu\}.$$

These spaces are actually the natural study spaces for Ornstein-Uhlenbeck operators [Lun97, MPRS02, MPP02].

In particular, if $\sigma(B) \subset \mathbb{C}^- = \{z \in \mathbb{C} \mid \Re(z) < 0\}$, the semigroup $T(t)$ generated by the Ornstein-Uhlenbeck operator $\mathcal{A}$ defined in (2.5) admits an *invariant measure* of this form, i.e. a Gaussian measure such that

$$\int (T(t)f_0)(x)\,\mu(x)\mathrm{d}x = \int f_0(x)\,\mu(x)\mathrm{d}x.$$

With the help of this measure, one derives the following result:

2.9 Lemma. *[MPP02] Assume $1 < p < \infty$. If Q is positive definite and $\sigma(B) \subset \mathbb{C}^-$, there exists an invariant Gaussian measure μ such that the Ornstein-Uhlenbeck operator $\mathcal{A}$ from* (2.5) *(with domain $W^{2,p}_\mu$) generates an analytic semigroup in L^p_μ.*

2.10 Remark. [MPP02, Thm. 4.1] The previous Lemma doesn't hold for $p = 1$. Although $\mathcal{A}$ also generates a strongly continuous semigroup in L^1_μ, it is not analytic in this space.

In our framework

$$B = \begin{pmatrix} 0 & -I_d \\ \omega_0^2 I_d & 2\gamma I_d \end{pmatrix} \quad \Rightarrow \quad \sigma(B) = \left\{\gamma \pm \sqrt{\gamma^2 - \omega_0^2}\right\} \subset \mathbb{C} \setminus \mathbb{C}^-$$

where the square root has to be understood as $i\sqrt{|.|}$ if its argument is negative. The prerequisite of the previous Lemma are not fulfilled.

To bypass this problem, we consider the adjoint operator of A. Its drift term is the opposite of the drift term of A. Therefore, it satisfies the prerequisite of Lemma 2.9 for $\gamma > 0$ and $\omega_0^2 > 0$.

2.11 Theorem. *Assume $D_{pp}D_{qq} - D_{pq}^2 > 0$, $\gamma > 0$ and $\omega_0^2 > 0$. For all $1 < p < \infty$, there exists a Gaussian measure μ such that A generates an analytic semigroup in L^p_μ.*

Proof. We consider the formal adjoint of A

$$A^* w = \nabla^T D \nabla w - P(x, v)^T \nabla w. \tag{2.8}$$

According to the previous Lemma, $(A^*, W^{2,p'}_\mu)$ generates an analytic semigroup in $L^{p'}_\mu$, for all $1 < p' < \infty$.

According to [Paz83, Cor. 1.10.6], the adjoint of a infinitesimal generator is also the infinitesimal generator of a semigroup (actually, the adjoint semigroup). Since $\sigma(A) = \sigma(A^*)$ and $\|(A-\lambda)^{-1}\| = \|(A^* - \lambda)^{-1}\|$, the adjoint semigroup is analytic in L^p_μ, iff the semigroup is analytic $L^{p'}_\mu$. □

2.12 Remark. Additionally to the L^p_μ spaces, the operator can also be studied in spaces of the form $L^p_{1/\mu}$ where μ is a Gaussian function. Similar results are obtained. See for instance the references in [SA13].

2.2.3 Consequence of the analyticity

2.13 Lemma. *Assume $D_{pp}D_{qq} - D_{pq}^2 > 0$. For all $k \in \mathbb{N}_0$, we have*

$$\mathcal{D}_{L^2_{\mathrm{loc}}}\left(A^k\right) \subseteq H^{2k}_{\mathrm{loc}} \tag{2.9}$$

where the left-hand-side is the maximal definition domain of A^k in L^2_{loc}.

Proof. We will prove this lemma by induction. For $k = 0$, the result holds since

$$\mathcal{D}_{L^2_{\mathrm{loc}}}(I) = L^2_{\mathrm{loc}},$$

where I is the identity operator of L^2_{loc}.

Let us now assume it holds for $k \in \mathbb{N}_0$ and let $u \in \mathcal{D}_{L^2_{\mathrm{loc}}}\left(A^{k+1}\right)$. It follows $Au \in \mathcal{D}_{L^2_{\mathrm{loc}}}\left(A^k\right) \subseteq H^{2k}_{\mathrm{loc}}$, with the induction hypothesis. Therefore, there exists $f \in H^{2k}_{\mathrm{loc}}$ such that $Au = f$. The operator A is elliptic with sufficiently smooth coefficients, so we can apply [Eva98, Thm. 6.3.2]. It follows that $u \in H^{2k+2}_{\mathrm{loc}}$ and the proof is complete. □

2.14 Theorem. *Assume $D_{pp}D_{qq} - D_{pq}^2 > 0$ and A generates an analytic semigroup in a subspace X of $L^2_{\mathrm{loc}}(\mathbb{R}^{2d})$ (for instance, under the prerequisite of Theorem 2.11). The solution w of the WFP problem in this space satisfies:*

$$\forall t > 0 \qquad w(t) \in C^\infty(\mathbb{R}^{2d}). \tag{2.10}$$

Proof. Thanks to Corollary A.12, we know

$$\forall t > 0 \qquad w(t) \in \bigcap_{k\in\mathbb{N}} \mathcal{D}_X\left(A^k\right).$$

We want to show the following inclusions:

$$\bigcap_{k\in\mathbb{N}} \mathcal{D}_X\left(A^k\right) \subseteq \bigcap_{k\in\mathbb{N}} \mathcal{D}_{L^2_{\mathrm{loc}}}\left(A^k\right) \subseteq \bigcap_{k\in\mathbb{N}} H^{2k}_{\mathrm{loc}} \subseteq C^\infty.$$

The first inclusion is obvious, since $X \subseteq L^2_{\mathrm{loc}}$. The second follows from Lemma 2.13.

Finally, for all bounded domains $\Omega \subset \mathbb{R}^{2d}$ and for all $f \in \bigcap_{k\in\mathbb{N}} H^{2k}_{\mathrm{loc}}$, we have $f|_\Omega \in \bigcap_{k\in\mathbb{N}} H^{2k}(\Omega)$. Using the Sobolev embedding, we conclude $f|_\Omega \in C^\infty(\Omega)$. It gives us the last inclusion. □

2.3 Explicit fundamental solution

The unique solution of (2.4) can be expressed explicitly with the help of a fundamental solution, i.e. a distributional solution of the linear equation

$$\begin{cases} \partial_t G(t,x,v,x_0,v_0) = AG(t,x,v,x_0,v_0) & \forall t > 0, \\ \lim_{t\to 0} G(t,x,v,x_0,v_0) = \delta(x - x_0, v - v_0) & \forall (x_0,v_0) \in \mathbb{R}^{2d}, \end{cases} \tag{2.11}$$

where δ is the Dirac distribution.

Let us first introduce the characteristic flow generated by the first order terms of (2.11) [SCDM04, Lem. 3.1]

2.15 Lemma. *The solution of the ordinary differential equation*

$$\begin{cases} \ddot{X} + 2\gamma\dot{X} + \omega_0^2 X = 0 \\ X(t=0) = x \\ \dot{X}(t=0) = v \end{cases} \tag{2.12}$$

can be expressed by:

a. For $\omega_0 > \gamma \geq 0$, we set $\omega = \sqrt{\omega_0^2 - \gamma^2}$ and

$$X(t,x,v) = \frac{e^{-\gamma t}}{\omega}\left((\omega\cos(\omega t) + \gamma\sin(\omega t))\,x + \sin(\omega t)v\right). \tag{2.13a}$$

b. For $\gamma > \omega_0 \geq 0$, we set $\omega = \sqrt{\gamma^2 - \omega_0^2}$ and

$$X(t,x,v) = \frac{e^{-\gamma t}}{\omega}\left((\omega\cosh(\omega t) + \gamma\sinh(\omega t))\,x + \sinh(\omega t)v\right). \tag{2.13b}$$

c. For $\gamma = \omega_0$, we have

$$X(t,x,v) = e^{-\gamma t}\left((\gamma t + 1)x + tv\right). \tag{2.13c}$$

Proof. Straightforward calculations. □

Further, we define the coefficients λ, μ and ν:

$$\lambda(t) := \frac{1}{d}\int_0^t \begin{pmatrix}\alpha(s) & \beta(s)\end{pmatrix} D \begin{pmatrix}\alpha(s) \\ \beta(s)\end{pmatrix} \mathrm{d}s, \tag{2.14a}$$

$$\nu(t) := \frac{1}{d}\int_0^t \begin{pmatrix}\dot\alpha(s) & \dot\beta(s)\end{pmatrix} D \begin{pmatrix}\dot\alpha(s) \\ \dot\beta(s)\end{pmatrix} \mathrm{d}s, \tag{2.14b}$$

$$\mu(t) := -\frac{2}{d}\int_0^t \begin{pmatrix}\dot\alpha(s) & \dot\beta(s)\end{pmatrix} D \begin{pmatrix}\alpha(s) \\ \beta(s)\end{pmatrix} \mathrm{d}s, \tag{2.14c}$$

where α and β are such that

$$X(-t,x,v) = \alpha(t)x + \beta(t)v$$

and we recall that

$$D = \begin{pmatrix} D_{qq}I_d & D_{pq}I_d \\ D_{pq}I_d & D_{pp}I_d \end{pmatrix}.$$

2.16 Lemma. *For all $t > 0$, we have*

$$\lambda(t) > 0, \tag{2.15a}$$

$$\nu(t) > 0, \tag{2.15b}$$

$$4\lambda(t)\nu(t) - \mu(t)^2 > 0. \tag{2.15c}$$

Proof. In case $\gamma > 0$, the matrix D is positive definite. Since $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \not\equiv 0$ and $\begin{pmatrix} \dot\alpha \\ \dot\beta \end{pmatrix} \not\equiv 0$, we have the two first inequalities.

The third inequality follows directly from Cauchy-Schwarz (the s variable is omitted inside the integrals for the sake of readability):

$$
\begin{aligned}
\mu(t)^2 &= \frac{4}{d^2} \left(\int_0^t (\dot\alpha\ \dot\beta)\, D \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mathrm{d}s \right)^2 \\
&< \frac{4}{d^2} \left(\int_0^t \sqrt{(\alpha\ \beta)\, D \begin{pmatrix} \alpha \\ \beta \end{pmatrix}} \sqrt{(\dot\alpha\ \dot\beta)\, D \begin{pmatrix} \dot\alpha \\ \dot\beta \end{pmatrix}} \mathrm{d}s \right)^2 \\
&\leq \frac{4}{d^2} \int_0^t (\alpha\ \beta)\, D \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mathrm{d}s \int_0^t (\dot\alpha\ \dot\beta)\, D \begin{pmatrix} \dot\alpha \\ \dot\beta \end{pmatrix} \mathrm{d}s \\
&= 4\lambda(t)\nu(t),
\end{aligned}
$$

where the first inequality is strict, since $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}(s)$ and $\begin{pmatrix} \dot\alpha \\ \dot\beta \end{pmatrix}(s)$ are not collinear, for almost all $s \in (0, t)$.

In case $\gamma = 0$, it can be checked by the explicit computation of λ, μ and ν with the expression of X from (2.13a):

$$
X(t, x, v) = \frac{1}{\omega_0} \left(\omega_0 \cos(\omega_0 t) x + \sin(\omega_0 t) v \right).
$$

See also [Süd07, Lem. 3.4]. □

Finally we introduce the function G defined by

$$
G(t, x, v, x_0, v_0) = e^{2d\gamma t} g(t, X(-t, x, v) - x_0, \dot X(-t, x, v) - v_0), \tag{2.16}
$$

where

$$
g(t, x, v) = \frac{\exp\left(-\frac{\nu(t)|x|^2 + \mu(t)(x \cdot v) + \lambda(t)|v|^2}{4\lambda(t)\nu(t) - \mu^2(t)} \right)}{(2\pi)^d (4\lambda(t)\nu(t) - \mu^2(t))^{d/2}} \tag{2.17}
$$

a the $2d$-dimensional Gaussian function with covariance matrix

$$
\Lambda(t) = \begin{pmatrix} 2\lambda(t)\, I_d & -\mu(t)\, I_d \\ -\mu(t)\, I_d & 2\nu(t)\, I_d \end{pmatrix}.
$$

According to Lemma 2.16, this matrix is positive definite.

2.17 Theorem. *[SCDM04, Prop. 3.1] G is the fundamental solution associated to* (2.4).

Proof. By replacing G with (2.16) in (2.11), the first order terms disappear. The function g is the fundamental solution of the PDE

$$
\partial_t g(t, x, v) = \left(\frac{d\lambda}{dt}(t) \Delta_x + \frac{d\mu}{dt}(t) \operatorname{div}_x \nabla_v + \frac{d\nu}{dt}(t) \Delta_v \right) g(t, x, v),
$$

which can be solved in Fourier space. (See [SCDM04] and [Süd07] for more details.) □

Thanks to this fundamental solution, we can back up the previous regularity result with the following corollary:

2.18 Corollary. *[SCDM04, Cor. 3.1] For every initial condition $w_0 \in L^p(\mathbb{R}^{2d})$ ($1 \le p < \infty$), the unique classical solution w of the problem* (2.4) *is expressed by*

$$w(t) = \iint_{\mathbb{R}^{2d}} G(t, x, v, x_0, v_0)\, w_0(x_0, v_0)\, \mathrm{d}x_0\, \mathrm{d}v_0, \tag{2.18}$$

and therefore belongs to $C(\mathbb{R}_0^+; L^p(\mathbb{R}^{2d})) \cap C^1(\mathbb{R}^+; C^\infty(\mathbb{R}^{2d}))$.

2.19 Remark. In the framework of Ornstein-Uhlenbeck operators, already mentioned in Remark 2.6, the generated semigroup can be written as [MPP02, Eq. 1.2]

$$(T(t)f)(x) = \frac{1}{(4\pi)^{N/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^N} e^{-\langle Q_t^{-1}y, y\rangle/4} f(e^{tB}x - y)\, \mathrm{d}y, \tag{2.19}$$

where

$$Q_t = \int_0^t e^{sB} Q e^{sB^*}\, \mathrm{d}s.$$

We recognize the convolution of a Gaussian function with the shifted initial state f. The shift e^{tB} coincides with the function X in our framework.

2.4 Small time estimates

In this section, our objective is to derive small time estimates of the derivatives of the function g defined in the previous section, and subsequently for the solution w of the WFP problem. Our main result will be Corollary 2.27.

2.20 Remark. In the function g the variables x_i and x_j, x_i and v_j, v_i and v_j are uncorrelated for $i \neq j$. In other words, we can write the $2d$-dimensional Gaussian function as a product of 2-dimensional Gaussian:

$$g_{2d}(t, x_1, \ldots, x_d, v_1, \ldots, v_d) = \prod_{i=1}^d g_2(t, x_i, v_i)$$

where g_{2d} denotes the $2d$-dimensional Gaussian function of covariance matrix $\begin{pmatrix} 2\lambda I_d & -\mu I_d \\ -\mu I_d & 2\nu I_d \end{pmatrix}$, for all $d \in \mathbb{N}$.

Using Fubini's theorem, it follows that, for all multiindices l and m:

$$\left\| \mathrm{D}_x^l \mathrm{D}_v^m g_{2d}(t) \right\|_{L^1(\mathbb{R}^{2d})} = \prod_{i=1}^d \left\| \partial_x^{l_i} \partial_v^{m_i} g_2(t) \right\|_{L^1(\mathbb{R}^2)}.$$

Therefore, it is sufficient to study the two dimensional function g_2.

In the following paragraphs, g denotes the 2-dimensional Gaussian function of covariance matrix

$$\begin{pmatrix} 2\lambda(t) & -\mu(t) \\ -\mu(t) & 2\nu(t) \end{pmatrix}.$$

All results can be extended from $\mathbb{R}^2$ to $\mathbb{R}^{2d}$ using the previous remark. Note that we may omit the t variable for the sake of readability.

2.21 Lemma. *For $l, m \in \mathbb{N}_0$, the derivatives of the Gaussian function g can be estimated by a sum of terms of the form:*

$$\left\|\partial_x^l \partial_v^m g\right\|_{L^1(\mathbb{R}^2)} \leq \sum C \frac{|\lambda|^{n_\lambda} |\nu|^{n_\nu} |\mu|^{n_\mu}}{|4\lambda\nu - \mu^2|^k}, \tag{2.20}$$

where $C \in \mathbb{N}$ and for each term of the sum the indices n_λ, n_ν, n_μ and k respect the following relationships:

$$2k = 2n_\lambda + n_\mu + l, \tag{2.21a}$$
$$2k = 2n_\nu + n_\mu + m. \tag{2.21b}$$

Proof. It will be proved in two steps.

Step 1 First, we show by a trivial induction on l and m that the derivatives of the Gaussian function g have the following form:

$$\partial_x^l \partial_v^m g(x, v) = \sum C x^{n_x} v^{n_v} \frac{\lambda^{n_\lambda} \nu^{n_\nu} \mu^{n_\mu}}{(4\lambda\nu - \mu^2)^{n_\lambda + n_\mu + n_\nu}} g(x, v), \tag{2.22}$$

where $C \in \mathbb{Z}$ and for each term of the sum the indices n_λ, n_ν, n_μ, n_x and n_v respect the following relationships:

$$n_x + l = 2n_\nu + n_\mu, \tag{2.23a}$$
$$n_v + m = 2n_\lambda + n_\mu. \tag{2.23b}$$

Step 2 The estimate (2.20) follows from the previous equality by computing the following integrals as function of λ, μ and ν:

$$\iint_{\mathbb{R}^2} |x|^{n_x} \, |v|^{n_v} \, g(x, v) \, \mathrm{d}x \, \mathrm{d}v = 4 \int_0^\infty \int_0^\infty x^{n_x} \, v^{n_v} \, g(x, v) \, \mathrm{d}x \, \mathrm{d}v$$

□

2.22 Remark. Equations (2.21) (respectively (2.23)) characterize the equality of the physical dimensions on the two sides of (2.20) (respectively (2.22)).

If the physical dimension of a variable a is written $[a]$, then the left-hand-side of (2.20) has dimension

$$[x]^{-l} [v]^{-m}$$

and the right-hand-side has dimension

$$\frac{[x]^{2n_\lambda} ([x][v])^{n_\mu} [v]^{2n_\nu}}{([x][v])^{2k}}.$$

Equations (2.21) assures the equality of the dimensions.

Now that we are able to estimate the derivatives of $g(t)$ with respect to the coefficients of its covariance matrix (for all t), we need an approximation of these coefficients.

2.23 Lemma. *For $t \to 0$, the expression* (2.13) *can be approximated by*

$$X(t,x,v) = \left(1 - \frac{\omega_0^2}{2}t^2 + \frac{\gamma\omega_0^2}{3}t^3 + o\left(t^3\right)\right) x + \left(t - \gamma t^2 + \frac{2\gamma^2 - \omega_0^2}{3}t^3 + o\left(t^3\right)\right) v. \tag{2.24}$$

Therefore, the coefficients λ, μ and ν defined in (2.14) *can be approximated by:*

$$\lambda(t) = D_{qq}t - D_{pq}t^2 + \left(D_{pp} - \omega_0^2 D_{qq} - 2\gamma D_{pq}\right)\frac{t^3}{3} + o\left(t^3\right), \tag{2.25a}$$

$$\nu(t) = D_{pp}t + o\left(t\right), \tag{2.25b}$$

$$\frac{1}{2}\mu(t) = -2D_{pq}t + \left(-\frac{D_{qq}\omega_0^2}{2} + \frac{D_{pp}}{2} - 2\gamma D_{pq}\right)t^2 + o\left(t^2\right). \tag{2.25c}$$

Proof. Straightforward calculations. □

2.24 Remark. Let f and $g : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous with

$$f(t) = g(t) + o(g(t)) \qquad \text{for } t \to 0.$$

Since

$$\frac{f(t)}{g(t)} = 1 + o(1) \qquad \text{for } t \to 0,$$

it exists ϵ such that the ratio is bounded (above and below) on $(0,\epsilon]$. Since it is continuous, it is also bounded (above and below) on all segments $[\epsilon, T]$. Then for all $T > 0$, there exists c_T and C_T such that

$$\forall t \in (0,T] \qquad c_T\, g(t) \leq f(t) \leq C_T\, g(t).$$

2.25 Theorem. *Assume $D_{qq}D_{pp} - D_{pq}^2 > 0$. For all $T > 0$, the derivatives of the fundamental solution g with respect to the multi-indices l and $m \in \mathbb{N}_0^d$ can be estimated by:*

$$\forall t \in (0,T] \qquad \left\|\mathrm{D}_x^l \mathrm{D}_v^m g(t)\right\|_{L^1} \leq C_T\, t^{-|l+m|/2} \tag{2.26}$$

where C_T is a real constant depending on T.

Proof. Since $D_{qq}D_{pp} - D_{pq}^2 > 0$, we have necessarily $D_{qq} > 0$ and $D_{pp} > 0$.

Let's first assume $D_{pq} > 0$. Using Lemma 2.23, we have for all $t \in (0,T]$

$$|\lambda(t)| \leq C_T\, D_{qq}\, t,$$
$$|\nu(t)| \leq C_T\, D_{pp}\, t,$$
$$|\mu(t)| \leq C_T\, D_{pq}\, t,$$

and

$$|4\lambda(t)\nu(t) - \mu(t)^2| \geq c_T\, 4(D_{qq}D_{pp} - D_{pq}^2)\, t^2 > 0.$$

If we replace these estimates in (2.20), we have

$$\|\partial_x^l \partial_v^m g\|_{L^1(\mathbb{R}^2)} \leq C_T\, t^{n_\lambda + n_\mu + n_\nu - 2k}.$$

By considering the relations (2.21), we get the wanted result.

In the case $D_{pq} = 0$, we have $|\mu(t)| \leq o(t) \leq C_T t$, so the estimate still holds. It can easily be checked that we have $n_\mu = 0$ in a least one term of (2.20). For that reason, the estimate cannot be improved.

$\square$

2.26 Theorem. *Assume $D_{pp} > 0$ and $D_{qq}D_{pp} - D_{pq}^2 = 0$. For all $T > 0$, the derivatives of the fundamental solution g with respect to the multi-indices l and $m \in \mathbb{N}_0^d$ can be estimated by:*

$$\forall t \in (0, T] \qquad \left\|\mathrm{D}_x^l \mathrm{D}_v^m g(t)\right\|_{L^1} \leq C_T\, t^{-|3l+m|/2} \tag{2.27}$$

where C_T is a real constant depending on T.

Proof. Since $D_{qq}D_{pp} - D_{pq}^2 = 0$, we have necessarily $D_{qq} = 0$ and $D_{pq} = 0$. Using Lemma 2.23, we have for all $t \in (0, T]$

$$\begin{aligned}
|\lambda(t)| &\leq C_T\, \frac{D_{pp}}{3}\, t^3, \\
|\nu(t)| &\leq C_T\, D_{pp}\, t, \\
|\mu(t)| &\leq C_T\, \frac{D_{pp}}{2}\, t^2,
\end{aligned}$$

and

$$|4\lambda(t)\nu(t) - \mu(t)^2| \geq c_T\, (4\frac{D_{pp}}{3} D_{pp} - D_{pp}^2)\, t^4 > 0.$$

If we replace these estimates in (2.20), we have

$$\|\partial_x^l \partial_v^m g\|_{L^1(\mathbb{R}^2)} \leq C_T\, t^{3n_\lambda + 2n_\mu + n_\nu - 4k}, \tag{2.28}$$

which gives the wanted estimates by considering the relations (2.21). $\square$

2.27 Corollary. *For all $T > 0$, the derivatives of the solution w with respect to the multi-indices l and $m \in \mathbb{N}_0^d$ can be estimated for all $1 \leq p \leq \infty$ by:*

$$\left\|\mathrm{D}_x^l \mathrm{D}_v^m w(t)\right\|_{L^p} \leq C_T \|w_0\|_{L^p} \begin{cases} t^{-|l+m|/2} & \text{if } D_{qq}D_{pp} - D_{pq}^2 > 0, \\ t^{-|3l+m|/2} & \text{if } D_{qq}D_{pp} - D_{pq}^2 = 0, \end{cases} \tag{2.29}$$

for all $t \in (0, T]$.

Proof. We consider the coordinates $X_t := X(-t, x, v)$, $V_t := \dot{X}(-t, x, v)$, previously introduced in Section 2.3. We define α_t and β_t such that $X_t = \alpha_t x + \beta_t v$, $V_t = -\dot{\alpha}_t x - \dot{\beta}_t v$.

We use (2.18), replace G with the help of (2.16) and change the derivatives in (x,v) into derivatives in (X_t, V_t)

$$
\begin{aligned}
&\mathrm{D}_x^l \mathrm{D}_v^m w(t,x,v) \\
&\quad = \iint_{\mathbb{R}^{2d}} \mathrm{D}_x^l \mathrm{D}_v^m G(t,x,x_0,v,v_0)\, w_0(x_0,v_0)\, \mathrm{d}x_0\, \mathrm{d}v_0 \\
&\quad = e^{2d\gamma t} \iint \mathrm{D}_x^l \mathrm{D}_v^m g(t, X_t - x_0, V_t - v_0)\, w_0(x_0,v_0)\, \mathrm{d}x_0\, \mathrm{d}v_0 \\
&\quad = e^{2d\gamma t} \iint (\alpha_t \mathrm{D}_{X_t} - \dot{\alpha}_t \mathrm{D}_{V_t})^l\, (\beta_t \mathrm{D}_{X_t} - \dot{\beta}_t \mathrm{D}_{V_t})^m \\
&\qquad\qquad\qquad\qquad g(t, X_t - x_0, V_t - v_0)\, w_0(x_0,v_0)\, \mathrm{d}x_0\, \mathrm{d}v_0 \\
&\quad = e^{2d\gamma t} \Big((\alpha_t \mathrm{D}_{X_t} - \dot{\alpha}_t \mathrm{D}_{V_t})^l\, (\beta_t \mathrm{D}_{X_t} - \dot{\beta}_t \mathrm{D}_{V_t})^m\, g(t) *_{x,v} w_0 \Big)(X_t, V_t) \\
&\quad = e^{2d\gamma t} \sum_{\substack{l_1+l_2=l \\ m_1+m_2=m}} \alpha_t^{|l_1|}\, \dot{\alpha}_t^{|l_2|}\, \beta_t^{|m_1|}\, \dot{\beta}_t^{|m_2|}\, \big(\mathrm{D}_{X_t}^{l_1+m_1}\, \mathrm{D}_{V_t}^{l_2+m_2} g(t) *_{x,v} w_0\big)(X_t, V_t).
\end{aligned}
$$

We can now compute the norm. The relationship between the integration with respect to (x,v) and (X_t, V_t) is:

$$
\det \left(\frac{\partial(X_t, V_t)}{\partial(x,v)} \right) = \exp(-2d\gamma t) \quad \Rightarrow \quad \|.\|_{L^p_{x,v}} = e^{-2d\gamma t/p} \|.\|_{L^p_{X_t,V_t}}
$$

It follows, by using the Young inequality for convolutions, the estimates (2.26) (respectively (2.27)) for g and the relations (2.24) for α and β:

$$
\begin{aligned}
&\|\mathrm{D}_x^l \mathrm{D}_v^m w(t)\|_{L^p} \\
&\quad \le C_T \sum_{\substack{l_1+l_2=l \\ m_1+m_2=m}} |\alpha_t|^{|l_1|}\, |\dot{\alpha}_t|^{|l_2|}\, |\beta_t|^{|m_1|}\, |\dot{\beta}_t|^{|m_2|}\, \|\mathrm{D}_{X_t}^{l_1+m_1}\, \mathrm{D}_{V_t}^{l_2+m_2} g(t)\|_{L^1} \|w_0\|_{L^p} \\
&\quad \le C_T \|w_0\|_{L^p} \sum_{\substack{l_1+l_2=l \\ m_1+m_2=m}} t^{|l_2+m_1|}\, t^{-|\epsilon l_1 + \epsilon m_1 + l_2 + m_2|/2}
\end{aligned}
$$

where $\epsilon = 1$ if $D_{qq}D_{pp} - D_{pq}^2 > 0$ and $\epsilon = 3$ if $D_{qq}D_{pp} - D_{pq}^2 = 0$. Finally, since the dominant term (in the neighborhood of 0) in the sum is reached for $l_1 = l$ and $m_2 = m$, we have

$$
\|\mathrm{D}_x^l \mathrm{D}_v^m w(t)\|_{L^p} \le C_T \|w_0\|_{L^p}\, t^{-|\epsilon l + m|/2}.
$$

□

2.28 Remark. Note that in case $D_{qq}D_{pp} - D_{pq}^2 = 0$, our equation is similar to the classical Vlasov-Fokker-Planck system. We can check that our estimates in this case coincide with those derived by Carpio in the classical case [Car98, Lemma 1.(ii)].

2.29 Remark. Similar estimates have been proved in L^p_μ spaces in the more general context of Ornstein-Uhlenbeck operators. For instance, one derives from [MPP02, Lem. 2.2]:

$$
\forall t \in (0,1) \qquad \left\| \mathrm{D}_x^l \mathrm{D}_v^m w(t) \right\|_{L^p_\mu} \le C \left\| w_0 \right\|_{L^p_\mu} t^{-|l+m|/2} \tag{2.30}
$$

for all $1 < p < \infty$, for $D_{qq}D_{pp} - D_{pq}^2 > 0$.

Chapter 3

Quadratic potential with bounded perturbation

In this chapter, we consider the potential

$$V(x) = \frac{\omega_0^2}{2}|x|^2 + V_0(x). \tag{3.1}$$

With this potential, the Wigner-Fokker-Planck problem takes the form

$$\begin{cases} \partial_t w = Aw + \Theta[V_0]w & \forall t > 0, \\ w(t=0) = w_0, \end{cases} \tag{3.2}$$

where A is the same operator as in the previous chapter, namely

$$Aw := \nabla^T D \nabla w + P(x,v)^T \nabla w + 2d\gamma w.$$

In the following sections, we will study the existence of a solution and its regularity in the case of a bounded $\Theta[V_0]$.

3.1 Theorem. *For $1 \leq p \leq \infty$, assume $V_0 \in \Theta^{-1}[\mathcal{B}(L^p)]$ and $w_0 \in L^p(\mathbb{R}^{2d})$, then (3.2) has a unique solution in $C(\mathbb{R}_0^+, L^p(\mathbb{R}^{2d}))$.*

If additionally, $w_0 \in \mathcal{D}(A)$ then the solution is actually in $C^1(\mathbb{R}_0^+, L^p(\mathbb{R}^{2d}))$.

Proof. Since $\Theta[V_0]$ is a bounded operator, we can use Theorem A.13. □

3.1 Analyticity

Theorem A.13 tells us that if A generates an analytic semigroup and $\Theta[V]$ is bounded, then $A + \Theta[V]$ generates also an analytic semigroup.

3.1.1 In Gaussian weighted L^p spaces

We already know from Theorem 2.11 that A generates an analytic semigroup in the weighted spaces L^p_μ. We can then derive that $A + \Theta[V_0]$ generates an analytic semigroup in those spaces for all $V_0 \in \Theta^{-1}[\mathcal{B}(L^p_\mu)]$. However, as we will see in the next Lemma, the set $\Theta^{-1}[\mathcal{B}(L^p_\mu)]$ is trivial.

3.2 Lemma. *Assume $\mathcal{F}[V] \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\Theta[V]$ bounded in $L^p_\mu(\mathbb{R}^{2d})$ for $1 < p < \infty$. Then $V = 0$.*

In particular, $L^1(\mathbb{R}^d) \cap \Theta^{-1}[\mathcal{B}(L^p_\mu)] = \{0\}$.

Proof. For the sake of readability we present only $d = 1$ and $Q = I_2$. The other cases can be similarly derived. Furthermore, we write $\theta := \mathcal{F}^{-1}_{\eta \to v}[\delta V] \in L^1_{loc}(\mathbb{R}^{2d})$ and $\|.\|$ represents the L^p_μ norm.

Let $a \in \mathbb{R}$, $b > 0$. We apply $\Theta[V]$ on the function

$$L^p_\mu(\mathbb{R}^{2d}) \ni \phi : (x, v) \mapsto \mathbb{1}_{[a,a+b]}(v)$$

where $\mathbb{1}_I$ denotes the indicator function on the interval I.

$$\begin{aligned}
\|\Theta[V]\phi\|^p &= \|\theta *_v \phi\|^p \\
&= \left\| \int \theta(x, v - v_0)\, \mathbb{1}_{[a,a+b]}(v_0)\, \mathrm{d}v_0 \right\|^p \\
&= \left\| \int_a^{a+b} \theta(x, v - v_0)\, \mathrm{d}v_0 \right\|^p \\
&= \iint_{\mathbb{R}^2} \left| \int_a^{a+b} \theta(x, v - v_0)\, \mathrm{d}v_0 \right|^p e^{-(x^2+v^2)}\, \mathrm{d}x\, \mathrm{d}v \\
&= e^{-a^2} \iint_{\mathbb{R}^2} \left| \int_0^b \theta(x, \xi - \xi_0)\, \mathrm{d}\xi_0 \right|^p e^{-(x^2+\xi^2+2a\xi)}\, \mathrm{d}x\, \mathrm{d}\xi
\end{aligned}$$

where we apply the changes of variable $\xi = v - a$ and $\xi_0 = v_0 - a$.

Furthermore, we have

$$\begin{aligned}
\|\phi\|^p &= \iint_{\mathbb{R}^2} \left| \mathbb{1}_{[a,a+b]}(v) \right|^p e^{-(x^2+v^2)}\, \mathrm{d}x\, \mathrm{d}v \\
&= \int_{\mathbb{R}} e^{-x^2} \mathrm{d}x \int_a^{a+b} e^{-v^2} \mathrm{d}v \\
&= e^{-a^2} \int_{\mathbb{R}} e^{-x^2} \mathrm{d}x \int_0^b e^{-(\xi^2+2a\xi)} \mathrm{d}\xi
\end{aligned}$$

with the same change of variables $\xi = v - a$.

Since $\Theta[V]$ is bounded, we have

$$\|\Theta[V]\phi\| \le C\|\phi\|$$

and then

$$\iint_{\mathbb{R}^2} \left| \int_0^b \theta(x, \xi - \xi_0)\, \mathrm{d}\xi_0 \right|^p e^{-(x^2+\xi^2+a\xi)}\, \mathrm{d}x\, \mathrm{d}\xi \le C \int_0^b e^{-(\xi^2+a\xi)}\, \mathrm{d}\xi$$

Since the right hand side converges to 0 for $a \to +\infty$, the left hand side integrand converges almost everywhere to 0.

$$\left| \int_0^b \theta(x, \xi - \xi_0)\, \mathrm{d}\xi_0 \right|^p e^{-(x^2+\xi^2+2a\xi)} \overset{a \to +\infty}{\longrightarrow} 0 \qquad \text{for } x, \xi \text{ a.e.}$$

The exponential diverges to $+\infty$ for $\xi < 0$. Thus, necessarily

$$\left| \int_0^b \theta(x, \xi - \xi_0) \,\mathrm{d}\xi_0 \right| = 0 \qquad \text{for } x, \xi \text{ a.e.}$$

Since $\theta = \mathcal{F}^{-1}_{\eta \to v}[\delta V]$ is conjugate symmetric, it holds also for $\xi > 0$. With the help of (1.17), one can conclude from $\mathcal{F}^{-1}_{\eta \to v}[\delta V] = 0$ a.e that $V = 0$. $\square$

3.3 Remark. One checks easily that $\Theta[V]$ is also unbounded in the case of a potential of the form $\mathcal{F}[V](\xi) \propto \delta(\xi - k)$ for $k \neq 0$. One could extend the previous Lemma to a larger class than $\mathcal{F}^{-1}[L^1_{\mathrm{loc}}]$.

3.4 Remark. The proof of the previous Lemma can easily be adapted to the spaces $L^p_{1/\mu}$.

3.1.2 In exponentially weighted L^p spaces

Let us briefly summarize the previous results:

- In the usual Lebesgue spaces L^p, the potential operator is bounded for a large class of potentials (see Section 1.4), but the semigroup solution of the WFP equation is not analytic (see Section 2.2.1).

- In Gaussian (resp. inverse Gaussian) weighted spaces L^p_μ, the semigroup is analytic (see Section 2.2.2), but the potential operator is not bounded (see Section 3.1.1).

It motivates the study of some "intermediate" spaces between L^p and L^p_μ. This is done in [SA13]: the authors study the behavior of a WFP-type equation in $L^2_\omega(\mathbb{R}^N) := L^2(\mathbb{R}^N, \omega(x)\mathrm{d}x)$, where $\omega(x) = \cosh(\beta|x|)$ for a $\beta > 0$. In this space, the semigroup is analytic and the operator $f \mapsto \theta * f$ is bounded for a large class of function θ.

We consider $L^2_{\omega(v)\mu(x)}(\mathbb{R}^{2d}) := L^2_\omega(\mathbb{R}^d, L^2_\mu(\mathbb{R}^d))$.

3.5 Lemma. *Assume V is analytically extendable and essentially bounded on $\Omega_\beta := \{z \in \mathbb{C} \mid |\Im(z)| < \beta\}$, then $\Theta[V]$ is a bounded operator of $L^2_{\omega(v)\mu(x)}$.*

Proof. With this prerequisite, $\delta V(x, .)$ can be (for all x) analytically extended on $\Omega_{\beta/2}$ and is essentially bounded on that domain. According to [SA13, Cor. 3.3 and p. 18], the operator is then bounded in $L^2_{\omega(v)}$.

V is in particular bounded on $\mathbb{R}^d$. It follows the boundedness of the operator in $L^2_{\mu(x)}$ $\square$

3.1.3 Consequence of the analyticity

Let us assume we have a space in which the semigroup is analytic. We want to use this property to derive some regularity result, similarly to Theorem 2.14. Let us study the behavior of $\Theta[V]$ in order to deal with $\mathcal{D}\left((A + \Theta[V])^k\right)$.

3.6 Lemma. *The following computation rules hold:*

$$\partial_{x_j}(\Theta[V]w) = \Theta[\partial_{x_j}V]w + \Theta[V]\partial_{x_j}w, \tag{3.3a}$$
$$\partial_{v_j}(\Theta[V]w) = \Theta[V]\partial_{v_j}w, \tag{3.3b}$$
$$x_j(\Theta[V]w) = \Theta[V](x_j w), \tag{3.3c}$$
$$v_j(\Theta[V]w) = \frac{1}{2i}\tilde{\Theta}[\partial_{x_j}V]w + \Theta[V](v_j w), \tag{3.3d}$$

for all $j \in \{1 \dots d\}$.

$\tilde{\Theta}$ *is defined similarly to* Θ *with, instead of* δV,

$$\tilde{\delta}V(x,\eta) = V\left(x + \frac{\eta}{2}\right) + V\left(x - \frac{\eta}{2}\right). \tag{3.4}$$

Proof. We consider the expression (1.13) of $\Theta[V]$:

$$\begin{aligned}
\partial_{x_j}\big(\Theta[V]w\big) &= \partial_{x_j}\Big(i\mathcal{F}^{-1}_{\eta\to v}\big[\delta V\,\mathcal{F}_{v\to\eta}[w]\big]\Big) \\
&= i\,\mathcal{F}^{-1}_{\eta\to v}\Big[\partial_{x_j}\big(\delta V\,\mathcal{F}_{v\to\eta}[w]\big)\Big] \\
&= i\,\mathcal{F}^{-1}_{\eta\to v}\Big[\big(\partial_{x_j}\delta V\big)\,\mathcal{F}_{v\to\eta}[w] + \delta V\,\big(\partial_{x_j}\mathcal{F}_{v\to\eta}[w]\big)\Big] \\
&= i\,\mathcal{F}^{-1}_{\eta\to v}\Big[\delta\big(\partial_{x_j}V\big)\,\mathcal{F}_{v\to\eta}[w]\Big] + i\mathcal{F}^{-1}_{\eta\to v}\Big[\delta V\,\mathcal{F}_{v\to\eta}[\partial_{x_j}w]\Big] \\
&= \Theta[\partial_{x_j}V]w + \Theta[V]\partial_{x_j}w
\end{aligned}$$

since ∂_{x_j} commutes with $\mathcal{F}_{v\to\eta}$, $\mathcal{F}^{-1}_{\eta\to v}$ and δ (as defined in Section 1.4).

Similarly,

$$\begin{aligned}
\partial_{v_j}\big(\Theta[V]w\big) &= i\,\mathcal{F}^{-1}_{\eta\to v}\big[i\eta_j\,\delta V\,\mathcal{F}_{v\to\eta}[w]\big] \\
&= i\,\mathcal{F}^{-1}_{\eta\to v}\big[\delta V\,\mathcal{F}_{v\to\eta}[\partial_{v_j}w]\big] \\
&= \Theta[V]\partial_{v_j}w
\end{aligned}$$

by noticing that the Fourier transform changes ∂_{v_j} into a multiplication $i\eta_j$ (and conversely).

The third equality is trivial since x_j and $\mathcal{F}_{v\to\eta}$ (resp. $\mathcal{F}^{-1}_{\eta\to v}$) commute.

Finally,

$$\begin{aligned}
v_j(\Theta[V]w) &= \mathcal{F}^{-1}_{\eta\to v}\Big[\partial_{\eta_j}\big(\delta V\,\mathcal{F}_{v\to\eta}[w]\big)\Big] \\
&= \mathcal{F}^{-1}_{\eta\to v}\Big[\big(\partial_{\eta_j}\delta V\big)\,\mathcal{F}_{v\to\eta}[w] + \delta V\,\big(\partial_{\eta_j}\mathcal{F}_{v\to\eta}[w]\big)\Big] \\
&= \mathcal{F}^{-1}_{\eta\to v}\Big[\frac{1}{2}\,\tilde{\delta}\big(\partial_{x_j}V\big)\,\mathcal{F}_{v\to\eta}[w]\Big] + i\,\mathcal{F}^{-1}_{\eta\to v}\Big[\delta V\,\mathcal{F}_{v\to\eta}[v_j w]\Big] \\
&= \frac{1}{2i}\tilde{\Theta}[\partial_{x_j}V]w + \Theta[V](v_j w)
\end{aligned}$$

because

$$\begin{aligned}
\partial_{\eta_j}\delta V &= \partial_{\eta_j}\left(V\left(x + \frac{\eta}{2}\right) - V\left(x - \frac{\eta}{2}\right)\right) \\
&= \frac{1}{2}\,\tilde{\delta}(\partial_{x_j}V).
\end{aligned}$$

□

3.7 Remark. The results about $\Theta[V]$ in Section 1.4, as well as the previous Lemma, can easily be extended to $\tilde{\Theta}[V]$.

3.8 Lemma. *For all $k \in \mathbb{N}$, the commutator of A^k and $\Theta[V]$ has the following form:*

$$A^k \Theta[V] - \Theta[V] A^k = \sum_i C_i \, \Theta_i \big[\mathrm{D}_x^{l_i} V \big] \, p_i(x, v, \nabla_x, \nabla_v) \tag{3.5}$$

where Θ_i is Θ or $\tilde{\Theta}$, C_i is a constant, l_i is a multiindex in $\mathbb{N}_0^d$ and p_i is a polynomial of degree $2k - |l_i|$.

Proof. For $k = 1$, one computes with the help of Lemma 3.6,

$$\begin{aligned} A\Theta[V] - \Theta[V]A = {} & D_{qq}\Big(\Theta[\Delta V] + 2\Theta[\nabla V] \cdot \nabla_x\Big) + 2D_{pq}\,\Theta[\nabla V] \cdot \nabla_v \\ & - \left(\frac{1}{2i}\,\tilde{\Theta}[\Delta V] + \Theta[\nabla V] \cdot v + \frac{1}{2i}\,\tilde{\Theta}[\nabla V] \cdot \nabla_x\right) \\ & + \frac{\gamma}{i}\,\tilde{\Theta}[\nabla V] \cdot \nabla_v, \end{aligned} \tag{3.6}$$

where $\Theta[\nabla V]$ denotes the vector $(\Theta[\partial_{x_i} V])_{i=1\ldots d}$.

Equation (3.5) can be proved by induction with Lemma 3.6 and (3.6). $\square$

3.9 Lemma. *Assume $\phi \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ has a compact support. Then the convolution $f \mapsto \phi * f$ maps functions in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ on functions in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, for any $1 \le p \le \infty$.*

Proof. Let $f \in L^p_{\mathrm{loc}}$ and $K \subset \mathbb{R}^N$ any compact subset. We want to show that $g := \phi * f$ is p-integrable on K. Let $r > 0$ such that the compact support of ϕ is contained in $B_r(0)$ (the open ball of radius r and center $0 \in \mathbb{R}^N$).

Since ϕ is zero outside of B_r, we have

$$\begin{aligned} \forall x \in K, \quad g(x) &= \int \phi(y)\, f(x-y)\,\mathrm{d}y \\ &= \int \phi(y)\, \mathbb{1}_{K+B_r}(x-y)\, f(x-y)\,\mathrm{d}y \\ &= \big(\phi * (\mathbb{1}_{K+B_r} f)\big)(x) \end{aligned}$$

where the indicator function $\mathbb{1}_{K+B_r}$ is such that

$$\mathbb{1}_{K+B_r}(\xi) = \begin{cases} 1 & \text{if } d(\xi, K) < r, \\ 0 & \text{else.} \end{cases}$$

Since $\phi \in L^1$ and $\mathbb{1}_{K+B_r} f \in L^p$ are (globally) integrable on $\mathbb{R}^N$, their convolution is also p-integrable on $\mathbb{R}^N$, according to Young's inequality for convolution. On K, the function g coincides with a p-integrable function, so g is p-integrable on K. $\square$

3.10 Lemma. *Assume $\mathcal{F}[V]$ has a compact support, then we have*

$$\mathcal{D}_{L^2_{\mathrm{loc}}}\left((A + \Theta[V])^k\right) \subseteq \mathcal{D}_{L^2_{\mathrm{loc}}}\left(A^k\right) \tag{3.7}$$

for all $k \in \mathbb{N}_0$.

Proof. We will prove this lemma by induction. For $k = 0$, we have obviously $(A + \Theta[V])^k = A^k$.

Let us assume it holds for all $0 \leq k < N$ and let $f \in \mathcal{D}_{L^2_{\text{loc}}}\left((A+\Theta[V])^N\right) \subseteq L^2_{\text{loc}}$ With the induction hypothesis and Lemma 2.13, we have

$$f \in \mathcal{D}_{L^2_{\text{loc}}}\left((A+\Theta[V])^{N-1}\right) \subseteq \mathcal{D}_{L^2_{\text{loc}}}\left(A^{N-1}\right) \subseteq H^{2N-2}_{\text{loc}}.$$

Furthermore, with the help of Lemma 3.8, we can write

$$(A+\Theta[V])^N f = A^N f + \sum_i C_i \, \Theta_i[\mathrm{D}_x^{l_i} V] \, p_i(x, v, \nabla_x, \nabla_v) \tag{3.8}$$

where p_i is a polynomial of degree smaller or equal $2N-2$.

For all multiindices l, $\mathcal{F}[\mathrm{D}_x^l V] \propto x^l \mathcal{F}[V]$ has a compact support. According to (1.17) the same holds for $\mathcal{F}^{-1}_{\eta \to v}[\delta(\mathrm{D}_x^l V)]$. With Lemma 3.9 the operator $\Theta[\mathrm{D}_x^l V] = f \mapsto \mathcal{F}^{-1}_{\eta \to v}[\delta(\mathrm{D}_x^l V)] *_v f$ maps functions in L^2_{loc} on functions in L^2_{loc}.

Then the left-hand-side of (3.8) and the sum on the right-hand-side belong to L^2_{loc}. It follows $A^N f \in L^2_{\text{loc}}$. $\square$

3.11 Theorem. *Assume that $D_{pp}D_{qq} - D_{pq}^2 > 0$, that A generates an analytic semigroup in X a subspace of $L^2_{\text{loc}}(\mathbb{R}^{2d})$, that $\Theta[V]$ is bounded in this space and that $\mathcal{F}[V]$ has a compact support. Then the solution w to the WFP problem in X respects:*

$$\forall t > 0 \qquad w(t) \in C^\infty(\mathbb{R}^{2d}) \tag{3.9}$$

Proof. With the help of Corollary A.12, we know

$$\forall t > 0 \qquad w(t) \in \bigcap_{k \in \mathbb{N}} \mathcal{D}_X\left((A+\Theta[V])^k\right)$$

It is sufficient to show:

$$\bigcap_{k \in \mathbb{N}} \mathcal{D}_X\left((A+\Theta[V])^k\right) \subseteq \bigcap_{k \in \mathbb{N}} \mathcal{D}_{L^2_{\text{loc}}}\left((A+\Theta[V])^k\right) \subseteq \bigcap_{k \in \mathbb{N}} \mathcal{D}_{L^2_{\text{loc}}}\left(A^k\right) \subseteq C^\infty$$

The first inclusion is obvious since $X \subseteq L^2_{\text{loc}}$, the second inclusion comes from Lemma 3.10, and finally the last one comes from Lemma 2.13 and Theorem 2.14. $\square$

3.2 Small time estimates

In this section, we want to extend the result of Section 2.4 to the case of the bounded perturbation $\Theta[V]$. We will start with a few introducing lemmas. The main results will be presented in Theorem 3.17 and Corollary 3.18. Finally, some remarks will present variants and extensions of the result.

3.2.1 Preliminaries

3.12 Lemma. *The solution w of* (3.2) *respects*

$$\begin{aligned} w(t,x,v) = &\iint_{\mathbb{R}^{2d}} G(t,x,x_0,v,v_0)\, w_0(x_0,v_0)\,\mathrm{d}x_0\,\mathrm{d}v_0 \\ &+ \int_0^t \iint_{\mathbb{R}^{2d}} G(s,x,x_0,v,v_0)\,(\Theta[V]w)(t-s,x_0,v_0)\,\mathrm{d}x_0\,\mathrm{d}v_0\,\mathrm{d}s \end{aligned} \tag{3.10}$$

for all $t > 0$, $x, v \in \mathbb{R}^d$, where G is the fundamental solution of the non-perturbed system defined in Section 2.3.

Proof. We apply Lemma A.14 for the bounded perturbation $\Theta[V]$. □

3.13 Lemma. *For multiindices l and $m \in \mathbb{N}_0^d$ and sufficiently smooth w and V, we have:*

$$\mathrm{D}_x^l \mathrm{D}_v^m(\Theta[V]w) = \sum_{\tilde{l}\le l} \binom{l}{\tilde{l}}\, \Theta\big[\mathrm{D}_x^{l-\tilde{l}} V\big]\big(\mathrm{D}_x^{\tilde{l}} \mathrm{D}_v^m w\big) \tag{3.11}$$

where $\binom{l}{\tilde{l}}$ is the binomial coefficient for multiindices.

Proof. It can easily be proved by induction with the help of Lemma 3.6 similarly to the usual Leibniz rule for derivation of product. □

We recall that $L_v^n(L_x^m)$ denotes the Lebesgue space $L^n(\mathbb{R}^d, L^m(\mathbb{R}^d))$ considered with the norm

$$\|\phi\|_{L_v^n(L_x^m)} := \big\|\|\phi\|_{L_x^m}\big\|_{L_v^n} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |\phi(x,v)|^m \mathrm{d}x\right)^{n/m} \mathrm{d}v\right)^{1/n}$$

for $\phi \in L^n(\mathbb{R}^d, L^m(\mathbb{R}^d))$.

We also write $L^k(\mathbb{R}^{2d}) =: L_{v,x}^k = L_v^k(L_x^k)$.

3.14 Lemma. *The following estimates hold:*

1. *For $f \in L_v^2(L_x^q)$ and $g \in L_v^1(L_x^r)$,*

$$\|g *_{x,v} f\|_{L_{v,x}^2} \le \|g\|_{L_v^1(L_x^r)}\, \|f\|_{L_v^2(L_x^q)} \tag{3.12}$$

   where $1/r + 1/q = 3/2$.

2. *For $U \in L^p(\mathbb{R}^d)$ and $z \in L^2(\mathbb{R}^{2d})$,*

$$\|\Theta[U]z\|_{L_v^2(L_x^q)} \le 2\|U\|_{L_x^p}\|z\|_{L_{v,x}^2} \tag{3.13}$$

   where $1/p + 1/2 = 1/q$.

Proof. 1. The first inequality is an application of the Young inequality used separately on the two variables. We rewrite the convolution as

$$\big(g *_{x,v} f\big)(x,v) = \int \big(f(.,v-v_0) *_x g(.,v_0)\big)(x)\,\mathrm{d}v_0$$

and then use the Young inequality for the convolution with respect to x

$$\begin{aligned}
\|g *_{x,v} f\|_{L^2_x}(v) &\le \int \|f(.,v-v_0) *_x g(.,v_0)\|_{L^2_x} \,\mathrm{d}v_0 \\
&\le \int \|f\|_{L^q_x}(v-v_0)\,\|g\|_{L^r_x}(v_0)\,\mathrm{d}v_0 \\
&= \big(\|f\|_{L^q_x} *_v \|g\|_{L^r_x}\big)(v)
\end{aligned}$$

We get inequality (3.12) by using a second time the Young inequality on the convolution with respect to v.

2. Since $\mathcal{F}_{v\to\eta}$ is isometric in L^2_v, we have, using (1.13) and the Hölder inequality:

$$\begin{aligned}
\|\Theta[U]z\|_{L^2_v(L^q_x)} &= \|\delta U \mathcal{F}_{v\to\eta}[z]\|_{L^2_\eta(L^q_x)} \\
&\le \big\|\|\delta U\|_{L^p_x}\|\mathcal{F}_{v\to\eta}[z]\|_{L^2_x}\big\|_{L^2_\eta}
\end{aligned}$$

for $1/p+1/2=1/q$. Moreover, for all $\eta \in \mathbb{R}^d$,

$$\|\delta U\|_{L^p_x}(\eta) = \Big\|U(.+\frac{\eta}{2}) - U(.-\frac{\eta}{2})\Big\|_{L^p_x} \le 2\|U\|_{L^p_x}$$

which is independent of η. It follows

$$\begin{aligned}
\|\Theta[U]z\|_{L^2_v(L^q_x)} &\le 2\|U\|_{L^p_x}\|\mathcal{F}_{v\to\eta}[z]\|_{L^2_{x,\eta}} \\
&= 2\|U\|_{L^p_x}\|z\|_{L^2_{x,v}}.
\end{aligned}$$

□

Note that the second inequality of Lemma 3.14 coincides with Lemma 1.6 and (1.16) for $q=2$ ($\Rightarrow p=\infty$).

3.15 Lemma. *For $T>0$ and $1 \le r \le infty$, the following estimate holds:*

$$\forall t \in (0,T] \qquad \|g(t)\|_{L^1_v(L^r_x)} \le C_T\, t^{-\epsilon d(1-1/r)/2}, \tag{3.14}$$

where $\epsilon = 1$ if $D_{qq}D_{pp} - D_{pq}^2 > 0$ and $\epsilon = 3$ if $D_{qq}D_{pp} - D_{pq}^2 = 0$.

Proof. We use the Gagliardo-Nirenberg inequality to show

$$\|g(t)\|_{L^1_v(L^r_x)} \le C\,\|\mathrm{D}_x g(t)\|^{\theta}_{L^1_{x,v}}\,\|g(t)\|^{1-\theta}_{L^1_{x,v}}$$

with $\theta = d\left(1-\frac{1}{r}\right)$. The right hand side can be estimated with the help of Theorem 2.25 (resp. 2.26). □

3.16 Remark. By combining the results of Lemmas 3.15 and 3.14, we can estimate the second term of (3.10).

$$\begin{aligned}
\|g(s) *_{x,v} \Theta[U]\,z(t-s)\|_{L^2} &\le \|g(s)\|_{L^1_v(L^r_x)}\,\|\Theta[U]z(t-s)\|_{L^2_v(L^q_x)} \\
&\le C_T\, s^{-\epsilon d(1-1/r)/2}\,\|U\|_{L^p}\,\|z(t-s)\|_{L^2}
\end{aligned}$$

where $1/p+1/2=1/q$ and $1/r+1/q=3/2$, and therefore $1-1/r=1/p$,

$$\|g(s) *_{x,v} \Theta[U]\,z(t-s)\|_{L^2} \le C_T\, s^{-\epsilon d/2p}\,\|U\|_{L^p}\,\|z(t-s)\|_{L^2}$$

We will use this relation for $U = \mathrm{D}^{l'}_x V_0$ and $z = \mathrm{D}^l_x \mathrm{D}^m_v w$ in the proof of the following theorem.

3.2.2 Main result

With the previous lemmas, we now have enough tools to prove the main result of this section, inspired by [ADM07, Thm. 5.1].

3.17 Theorem. *Assume $D_{pp}D_{qq} - D_{pq}^2 > 0$ and $V_0 \in L^\infty \cap W^{k,p}(\mathbb{R}^d)$ for some $k \in \mathbb{N}$ and $1 \le p \le \infty$. Then the solution w of the WFP problem satisfies for all $T > 0$, $l, m \in \mathbb{N}_0^d$ with $|l+m| \le k$*

$$\forall t \in (0,T] \qquad \left\|\mathrm{D}_x^l \mathrm{D}_v^m w(t)\right\|_{L^2} \le C_T\, t^{-\kappa|l+m|/2} \tag{3.15}$$

where $\kappa = \max\{d/p - 2, 1\}$.

Note that the constant C_T depends on T, $\|w\|_{C([0,T],L^2)}$, $\|V\|_{L^\infty}$ and $\|V\|_{W^{k,p}}$ among other.

In particular, for $V_0 \in W^{k,\infty}$, we have

$$\forall t \in (0,T] \qquad \left\|\mathrm{D}_x^l \mathrm{D}_v^m w(t)\right\|_{L^2} \le C_T\, t^{-|l+m|/2} \tag{3.16}$$

From the previous estimates of the derivatives of w, we can conclude on the regularity of w, with the help of the Sobolev embedding theorem.

3.18 Corollary. *Under the assumptions of Theorem 3.17, the solution w of the WFP problem satisfies*

$$\forall t \in (0,\infty) \qquad w(t) \in H^k(\mathbb{R}^{2d}) \hookrightarrow C_B^{k-d-1}(\mathbb{R}^{2d}) \tag{3.17}$$

In particular, if $V_0 \in C_B^\infty(\mathbb{R}^d)$ then $w(t) \in C_B^\infty(\mathbb{R}^{2d})$.

Proof of Theorem 3.17.

We will prove the result by induction. (3.15) holds for $l = m = 0$ (See Theorem 3.1). Let us assume it holds for every l' and m' such that $|l'+m'| < k_0$ and consider l and m such that $|l+m| = k_0$.

Let w be the solution of the WFP problem, w satisfies (3.10). We separate the right hand side of (3.10) into three terms, such that $w = I_1 + I_2 + I_3$:

$$\begin{aligned}
I_1(t,x,v) &= \iint_{\mathbb{R}^{2d}} G(t,x,x_0,v,v_0)\, w_0(x_0,v_0)\,\mathrm{d}x_0\,\mathrm{d}v_0 \\
I_2(t,x,v) &= \int_{t/2}^{t} \iint_{\mathbb{R}^{2d}} G(s,x,x_0,v,v_0)\,(\Theta[V_0]w)(t-s,x_0,v_0)\,\mathrm{d}x_0\,\mathrm{d}v_0\,\mathrm{d}s \\
I_3(t,x,v) &= \int_{0}^{t/2} \iint_{\mathbb{R}^{2d}} G(s,x,x_0,v,v_0)\,(\Theta[V_0]w)(t-s,x_0,v_0)\,\mathrm{d}x_0\,\mathrm{d}v_0\,\mathrm{d}s
\end{aligned}$$

The derivatives of G and $\Theta[V]w$ may become infinite for $t \to 0$, and therefore may not be integrable. To avoid this problem, we separate the time integral in I_2 and I_3 and we will apply the derivative on G in I_2 and on $\Theta[V]w$ in I_3.

We consider the coordinates $X_t := X(-t,x,v)$, $V_t := \dot{X}(-t,x,v)$, previously used in the proof of Corollary 2.27.

Step 1 The first estimate follows from Corollary 2.27.

$$\left\|\mathrm{D}_x^l \mathrm{D}_v^m I_1(t)\right\|_{L^2} \leq C_T\, t^{-|l+m|/2}.$$

Step 2 Let us now deal with I_2. Using the same arguments as in Corollary 2.27, we have:

$$\mathrm{D}_x^l \mathrm{D}_v^m I_2(t) = \int_{t/2}^{t} e^{2d\gamma s} \sum_{\substack{l_1+l_2=l\\ m_1+m_2=m}} \alpha_s^{|l_1|}\, \dot{\alpha}_s^{|l_2|}\, \beta_s^{|m_1|}\, \dot{\beta}_s^{|m_2|} \Big(\big(\mathrm{D}_{X_t}^{l_1+m_1}\, \mathrm{D}_{V_t}^{l_2+m_2} g(s)\big) *_{x,v} \Theta[V_0] w(t-s) \Big)(X_t, V_t)\,\mathrm{d}s$$

then

$$\begin{aligned}
&\left\|\mathrm{D}_x^l \mathrm{D}_v^m I_2(t)\right\|_{L^2}\\
&\quad\leq C_T \sum_{\substack{l_1+l_2=l\\ m_1+m_2=m}} \int_{t/2}^{t} s^{|l_2|+|m_1|} \left\|\mathrm{D}_{X_t}^{l_1+m_1}\, \mathrm{D}_{V_t}^{l_2+m_2} g(s)\right\|_{L^1} \|\Theta[V_0] w(t-s)\|_{L^2}\,\mathrm{d}s\\
&\quad\leq C_T \int_{t/2}^{t} s^{-|l+m|/2} \|\Theta[V_0]\|_{\mathcal{B}(L^2)} \|w(t-s)\|_{L^2}\,\mathrm{d}s\\
&\quad\leq C_T \|V_0\|_{L^\infty} \|w\|_{C([0,T],L^2)} \int_{t/2}^{t} s^{-|l+m|/2}\,\mathrm{d}s
\end{aligned}$$

and finally
$$\left\|\mathrm{D}_x^l \mathrm{D}_v^m I_2(t)\right\|_{L^2} \leq C_T\, t^{-|l+m|/2+1}$$

Step 3 Finally we deal with I_3. We apply the derivative on $\Theta[V]w$, use Lemma 3.13 and Remark 3.16. Note that we estimate $s^{|l_2|+|m_1|}$ with a constant.

$$\begin{aligned}
&\left\|\mathrm{D}_x^l \mathrm{D}_v^m I_3(t)\right\|_{L^2}\\
&\leq C_T \sum_{\substack{l_1+l_2=l\\ m_1+m_2=m}} \int_0^{t/2} \|g(s) *_{x,v} \big(\mathrm{D}_x^{l_1+m_1}\, \mathrm{D}_v^{l_2+m_2} (\Theta[V_0] w)\big)(t-s)\|_{L^2}\,\mathrm{d}s\\
&\leq C_T \sum_{\substack{l_1+l_2=l\\ m_1+m_2=m}} \sum_{|l'|=0}^{l_1+m_1} \int_0^{t/2} \|g(s) *_{x,v} \Theta[\mathrm{D}_x^{l'} V_0]\, (\mathrm{D}_x^{l_1+m_1-l'}\, \mathrm{D}_v^{l_2+m_2} w)\,(t-s)\|_{L^2}\,\mathrm{d}s\\
&\leq C_T \sum_{\substack{l_1+l_2=l\\ m_1+m_2=m}} \sum_{|l'|=1}^{l_1+m_1} \int_0^{t/2} \left\|g(s)\right\|_{L_v^1(L_x^r)} \left\|\mathrm{D}_x^{l'} V_0\right\|_{L^p} \left\|\mathrm{D}_x^{l_1+m_1-l'}\, \mathrm{D}_v^{l_2+m_2} w(t-s)\right\|_{L^2}\,\mathrm{d}s\\
&\quad + K_T \sum_{\substack{l_1+l_2=l\\ m_1+m_2=m}} \int_0^{t/2} \|g(s)\|_{L^1} \|V_0\|_{L^\infty} \|\mathrm{D}_x^{l_1+m_1}\, \mathrm{D}_v^{l_2+m_2} w(t-s)\|_{L^2}\,\mathrm{d}s
\end{aligned}$$

We have separated the sum into two parts:

- In the first part ($|l'|>0$), we can use the induction hypothesis to estimate $\|(\mathrm{D}_x^{l_1+m_1-l'}\,\mathrm{D}_v^{l_2+m_2}w)\,(t-s)\|_2$ and Corollary 3.15 to estimate $\|g(s)\|_{L^1_v L^r_x}$.
- The second part ($l'=0$) can't be estimated, we keep it as it is.

$$\begin{aligned}
&\left\|\mathrm{D}_x^l \mathrm{D}_v^m I_3(t)\right\|_{L^2} \\
&\quad\le C_T \sum_{\substack{l_1+l_2=l\\ m_1+m_2=m}} \sum_{|l'|=1}^{l_1+m_1} \int_0^{t/2} s^{-d/2p}\,(t-s)^{-\kappa|l_1+m_1-l'+l_2+m_2|/2}\mathrm{d}s \\
&\qquad + K_T \sum_{\substack{l_1+l_2=l\\ m_1+m_2=m}} \int_0^{t/2} \left\|\mathrm{D}_{X_t}^{l_1+m_1}\,\mathrm{D}_{V_t}^{l_2+m_2} w(t-s)\right\|_{L^2}\mathrm{d}s \\
&\quad\le C_T \int_0^{t/2} s^{-d/2p}\,(t-s)^{-\kappa(|l+m|-1)/2}\mathrm{d}s \\
&\qquad + K_T \sum_{|l'+m'|=k_0} \int_{t/2}^{t} \left\|\mathrm{D}_x^{l'}\,\mathrm{D}_v^{m'} w(s)\right\|_{L^2}\mathrm{d}s \\
&\quad\le C_T\, t^{-d/2p-\kappa(|l+m|-1)/2+1} + K_T \sum_{|l'+m'|=k_0} \int_{t/2}^{t} \left\|\mathrm{D}_x^{l'}\,\mathrm{D}_v^{m'} w(s)\right\|_{L^2}\mathrm{d}s \\
&\quad= C_T\, t^{-\kappa|l+m|/2+(\kappa-(d/p-2))/2} + K_T \sum_{|l'+m'|=k_0} \int_{t/2}^{t} \left\|\mathrm{D}_x^{l'}\,\mathrm{D}_v^{m'} w(s)\right\|_{L^2}\mathrm{d}s
\end{aligned}$$

Step 4 By combining the estimates of the three parts, we get:

$$\begin{aligned}
\left\|\mathrm{D}_x^l \mathrm{D}_v^m w(t)\right\|_{L^2} \le C_T &\left(t^{-|l+m|/2} + t^{-|l+m|/2+1} + t^{-\kappa|l+m|/2+(\kappa-(d/p-2))/2}\right) \\
&+ K_T \sum_{|l'+m'|=k_0} \int_{t/2}^{t} \left\|\mathrm{D}_x^{l'}\,\mathrm{D}_v^{m'} w(s)\right\|_{L^2}\mathrm{d}s
\end{aligned}$$

We recall that $\kappa = \max\{d/p-2, 1\}$. It follows $\kappa-(d/p-2)\ge 0$. We can consider two distinct cases:

- If $\kappa = 1$, the first term $t^{-|l+m|/2}$ is dominant.
- If $\kappa = \frac{d}{p}-2 > 1$, the third term $t^{-\kappa|l+m|/2+(\kappa-(d/p-2))/2} = t^{-\kappa|l+m|/2}$ is dominant.

In both cases, the dominant term can be written as $t^{-\kappa|l+m|/2}$. So we have,

$$\|\mathrm{D}_x^l \mathrm{D}_v^m w(t)\|_2 \le C_T\, t^{-\kappa|l+m|/2} + K_T \int_{t/2}^{t} \sum_{|l'+m'|=k_0} \|\mathrm{D}_x^{l'}\,\mathrm{D}_v^{m'} w(s)\|_2\,\mathrm{d}s$$

If we sum the previous inequality over all multiindices with order k_0,

$$\sum_{|l'+m'|=k_0} \|\mathrm{D}_x^{l'}\,\mathrm{D}_v^{m'} w(t)\|_2 \le C_T\, t^{-\kappa k_0/2} + K_T \int_{t/2}^{t} \sum_{|l'+m'|=k_0} \|\mathrm{D}_x^{l'}\,\mathrm{D}_v^{m'} w(s)\|_2\,\mathrm{d}s$$

we can apply the Gronwall lemma to get the result. □

3.2.3 Extensions and remarks

Let us first study the semi-elliptic case ($D_{pp}D_{qq} - D_{pq}^2 = 0$) that we omitted in the previous paragraphs.

3.19 Theorem. *Assume $D_{pp}D_{qq} - D_{pq}^2 = 0$ and $V_0 \in L^\infty \cap W^{k,p}(\mathbb{R}^d)$ for $k \in \mathbb{N}$ and $1 \le p \le \infty$. Then the solution of the WFP problem w satisfies for all $T > 0$, $l, m \in \mathbb{N}_0^d$ with $|l+m| \le k$*

$$\forall t \in (0,T] \qquad \left\|\mathrm{D}_x^l \mathrm{D}_v^m w(t)\right\|_{L^2} \le C_T\, t^{-|3\kappa l + (3\kappa-2)m|/2} \tag{3.18}$$

where $\kappa = \max\{d/p - 2/3, 1\}$.

In particular, for $p = \infty$, we have

$$\forall t \in (0,T] \qquad \left\|\mathrm{D}_x^l \mathrm{D}_v^m w(t)\right\|_{L^2} \le C_T\, t^{-|3l+m|/2} \tag{3.19}$$

We can extend the Corollary 3.18 to the semi-elliptic case.

Proof. Most of the proof is the same as in Corollary 2.27 and Theorem 3.17. The main difference appears in the C_T term of Step 3 in the proof of Theorem 3.17. For the sake of readability, the unchanged K_T term is not re-written here. Note that unlike previously, we didn't estimate the term $s^{|l_2+m_1|}$ (coming from the α and β) with a constant.

$$\begin{aligned}
&\left\|\mathrm{D}_x^l \mathrm{D}_v^m I_3(t)\right\|_{L^2} \\
&\le C_T \sum_{\substack{l_1+l_2=l\\ m_1+m_2=m}} \sum_{|l'|=0}^{l_1+m_1} \int_0^{t/2} s^{|l_2+m_1|} \left\|g * \Theta\big[\mathrm{D}_x^{l'} V_0\big]\left(\mathrm{D}_x^{l_1+m_1-l'}\,\mathrm{D}_v^{l_2+m_2} w\right)\right\|_{L^2} \mathrm{d}s \\
&\le C_T \sum_{\substack{l_1+l_2=l\\ m_1+m_2=m}} \sum_{|l'|=1}^{l_1+m_1} \int_0^{t/2} s^{|l_2+m_1|}\, s^{-3d/2p}\,(t-s)^{-|3\kappa(l_1+m_1-l') + (3\kappa-2)(l_2+m_2)|/2}\,\mathrm{d}s \\
&\qquad\qquad + K_T[...] \\
&\le C_T \sum_{\substack{l_1+l_2=l\\ m_1+m_2=m}} \sum_{|l'|=1}^{l_1+m_1} t^{-|3\kappa l_1 + (3\kappa-2)m_1 + (3\kappa-4)l_2 + (3\kappa-2)m_2|/2 \,+\, (3\kappa l' - 3d/p + 2)/2} \\
&\qquad\qquad + K_T[...] \\
&\le C_T\, t^{-|3\kappa l + (3\kappa-2)m|/2 \,+\, 3\,(\kappa - (d/p-2/3))/2} + K_T[...]
\end{aligned}$$

The proof can be concluded by the same arguments.

□

3.20 Remark. The assumption of Theorem 3.17 (and Corollary 3.18) can be slightly weakened: we assume $V \in L^\infty(\mathbb{R}^d)$ and

$$\forall l \in \mathbb{N}_0^d,\ |l| \le k \ \ \exists p_l,\ 1 \le p_l \le \infty \text{ such that } \mathrm{D}_x^l V \in L^{p_l}(\mathbb{R}^d).$$

We set

$$\kappa = \max\{\sup_{\substack{l\in\mathbb{N}_0^d\\ |l|\le k}}\{d/p_l - 2\},\ 1\}. \tag{3.20}$$

Note that $\kappa < \infty$ since $p_l \geq 1$. Under these assumptions, the same result (3.15) holds (and therefore also Corollary 3.18).

A similar result holds for Theorem 3.19.

Proof. We will check it in the proof of Theorem 3.17, the same arguments hold in the case of Theorem 3.19. The only difference appears in the C_T term of the third step (for the sake of readability, the K_T term is not re-written here):

$$\begin{aligned}
&\left\|\mathrm{D}_x^l \mathrm{D}_v^m I_3(t)\right\|_{L^2} \\
&\leq C_T \sum_{\substack{l_1+l_2=l \\ m_1+m_2=m}} \sum_{|l'|=0}^{l_1+m_1} \int_0^{t/2} \left\|g(s) *_{x,v} \Theta\left[\mathrm{D}_x^{l'} V_0\right] \left(\mathrm{D}_x^{l_1+m_1-l'} \mathrm{D}_v^{l_2+m_2} w\right)(t-s)\right\|_{L^2} \mathrm{d}s \\
&\leq C_T \sum_{\substack{l_1+l_2=l \\ m_1+m_2=m}} \sum_{|l'|=1}^{l_1+m_1} \int_0^{t/2} \|g\|_{L_v^1(L_x^{r_{l'}})} \left\|\mathrm{D}_x^{l'} V_0\right\|_{L^{p_{l'}}} \left\|\mathrm{D}_x^{l_1+m_1-l'} \mathrm{D}_v^{l_2+m_2} w(t-s)\right\|_{L^2} \mathrm{d}s \\
&\qquad\qquad + K_T[...] \\
&\leq C_T \sum_{\substack{l_1+l_2=l \\ m_1+m_2=m}} \sum_{|l'|=1}^{l_1+m_1} \int_0^{t/2} s^{-d/2p_{l'}} (t-s)^{-\kappa|l_1+m_1-l'+l_2+m_2|/2} \mathrm{d}s + K_T[...] \\
&\leq C_T \int_0^{t/2} s^{-\sup\{d/2p_l\}} (t-s)^{-\kappa(|l+m|-1)/2} \mathrm{d}s \; + \; K_T[...] \\
&\leq C_T \, t^{-\kappa|l+m|/2 \,+\, (\kappa \,-\, \sup\{d/p_l\}-2)/2} \; + \; K_T[...]
\end{aligned} \tag{3.21}$$

The proof can be concluded by the same arguments. □

Nevertheless, note that in this case the κ given in (3.20) is not optimal and the fourth inequality hereabove can be sharpened.

Finally, we'd like to generalize the previous results to estimate the L^n norm of $\mathrm{D}_x^l \mathrm{D}_v^m w(t)$ for $1 \leq n \leq \infty$. Since we can't easily generalize Lemma 3.14 for other norms than L^2, we will settle for the following result[1].

3.21 Theorem. *Let $k \in \mathbb{N}$ and $1 \leq n \leq \infty$. Assume $\forall l \in \mathbb{N}_0^d$, $|l| \leq k$, that $\mathrm{D}_x^l V_0 \in \Theta^{-1}[\mathcal{B}(L^n)]$. Then the solution of the WFP problem w satisfies for all $T > 0$, $l, m \in \mathbb{N}_0^d$ with $|l+m| \leq k$*

$$\forall t \in (0,T] \quad \|\mathrm{D}_x^l \mathrm{D}_v^m w(t)\|_n \leq C_T \begin{cases} t^{-|l+m|/2} & \text{if } D_{pp}D_{qq} - D_{pq}^2 > 0, \\ t^{-|3l+m|/2} & \text{if } D_{pp}D_{qq} - D_{pq}^2 = 0. \end{cases} \tag{3.22}$$

Proof. The proof is similar to Theorems 3.17 and 3.19. We use

$$\|\Theta[U]z\|_{L^n} \leq \|\Theta[U]\|_{\mathcal{B}(L^n)} \|z\|_{L^n}$$

instead of Lemma 3.14 in Remark 3.16. □

[1] This theorem is actually the generalization of the estimates (3.16) and (3.19), which can be proved without Lemma 3.14.

3.22 Remark. It is pertinent to ask if the κ which appears in most of theorems of this section have a physical meaning or is only an unwanted artifact from the proof.

If we have for a fixed k

$$\sup\{p \mid 1 \leq p \leq \infty,\, V_0 \in W^{k,p}\} \geq \frac{d}{3}$$

then, we can use the sharpest estimate ($\kappa = 1$), according to Theorem 3.17. This case is actually very common for physical (smooth) potentials in physically meaningful dimensions ($d \leq 3$).

However, we present in the next paragraphs an example of a potential such that

$$\sup\{p \mid 1 \leq p \leq \infty,\, V_0 \in W^{k,p}\} < \frac{d}{3}$$

and therefore we cannot estimate with a better coefficient than $\kappa > 1$ in (3.15).

Let us define $V_0 : \mathbb{R}^d \to \mathbb{R}$ such that

$$V_0(x) = \begin{cases} \sqrt{|x|} & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 2 \end{cases} \tag{3.23}$$

and smoothly extended in-between ($|x|$ denotes here the euclidean norm in $\mathbb{R}^d$). Obviously, $V_0 \in L^\infty(\mathbb{R}^d)$.

The derivatives of V_0 have a singularity in $|x| = 0$. Namely, for a multiindex $l \in \mathbb{N}_0^d$,

$$|\mathrm{D}_x^l V_0(x)| = \begin{cases} C_l\, |x|^{1/2-|l|} & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 2 \end{cases}$$

and bounded in-between. Using simple analysis results, we show that

$$\mathrm{D}_x^l V_0 \in L^p(\mathbb{R}^d) \iff \left(|x|^{1/2-|l|}\right)^p = o(|x|^{-d}) \iff p\left(\frac{1}{2} - |l|\right) > -d$$

and therefore

$$V_0 \in W^{k,p} \iff p < \frac{d}{k - \frac{1}{2}}.$$

Considering V_0 as element of $W^{d,p}$ for a $1 \leq p < \frac{d}{d-\frac{1}{2}}$, we have

$$d \geq 4 \iff \frac{d}{p} - 2 > 1 \iff \kappa > 1$$

For $d \geq 4$, the best estimates that we can derive from Theorem 3.17 is

$$\forall t \in (0, T] \qquad \left\|\mathrm{D}_x^l \mathrm{D}_v^m w(t)\right\|_{L^2} \leq C_T\, t^{-(d-7/2)|l+m|/2} \tag{3.24}$$

for l and m such that $|l + m| = d$.

With an analytical or numerical solution of the problem with this potential, we could test if this estimate is really the best possible. It would be a first step to study the meaningfulness of κ.

3.3 Time dependent potential

In this last section, we will slightly extend the previous result by considering a time dependent potential of the form

$$V(t,x) = \frac{\omega_0^2}{2}|x|^2 + V_0(t,x). \tag{3.25}$$

That means, the differential system we study has the form

$$\begin{cases} \partial_t w = Aw + \Theta[V_0(t)]w & \forall t > 0 \\ w(t=0) = w_0 \in L^2(\mathbb{R}^{2d}) \end{cases} \tag{3.26}$$

Let's see some conditions on $V_0 : \mathbb{R}_0^+ \times \mathbb{R}^d \to \mathbb{R}$ to ensure the existence and regularity of a solution.

3.23 Theorem. *Let $V_0 \in C(\mathbb{R}_0^+, L^\infty(\mathbb{R}^d))$, then* (3.26) *has a unique solution in $C(\mathbb{R}_0^+, L^2(\mathbb{R}^{2d}))$.*

If additionally, th potential V_0 is Lipschitz continuous with respect to t (i.e. $V_0 \in \mathrm{Lip}(\mathbb{R}_0^+, L^\infty(\mathbb{R}^d))$) and $w_0 \in \mathcal{D}(A)$, then the unique solution belongs to $C^1(\mathbb{R}_0^+, L^2(\mathbb{R}^{2d}))$.

Proof. We apply Theorem A.15. The (Lipschitz-)continuity of $t \mapsto \Theta[V_0(t)]$ from $\mathbb{R}_0^+$ to $\mathcal{B}(L^2)$ follows from the (Lipschitz-)continuity of $t \mapsto V_0(t)$, since (1.16) gives us:

$$\|\Theta[V_0(s)] - \Theta[V_0(t)]\|_{\mathcal{B}(L^2)} \leq 2\,\|V_0(s) - V_0(t)\|_{L^\infty}$$

for all $t, s \in \mathbb{R}_0^+$. □

3.24 Remark. One can easily generalize Lemma 3.12 to the time dependent case:

$$\begin{aligned} w(t,x,v) = &\iint_{\mathbb{R}^{2d}} G(t,x,x_0,v,v_0) w_0(x_0,v_0)\,\mathrm{d}x_0\,\mathrm{d}v_0 \\ &+ \int_0^t \iint_{\mathbb{R}^{2d}} G(s,x,x_0,v,v_0)(\Theta[V(t-s)]w)(t-s,x_0,v_0)\,\mathrm{d}x_0\,\mathrm{d}v_0\,\mathrm{d}s \end{aligned} \tag{3.27}$$

See also Section A.4.

To study the regularity with respect to x and v, we can show these extensions of Theorem 3.17 and Corollary 3.18.

3.25 Theorem. *Assume $D_{pp}D_{qq} - D_{pq}^2 > 0$ and $V_0 \in C(\mathbb{R}_0^+, L^\infty(\mathbb{R}^d)) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_0^+, W^{k,p}(\mathbb{R}^d))$ for $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. Then the solution of the WFP problem w satisfies for all $T > 0$, $l, m \in \mathbb{N}_0^d$ with $|l+m| \leq k$,*

$$\forall t \in (0,T] \qquad \|\mathrm{D}_x^l \mathrm{D}_v^m w(t)\|_{L^2} \leq C_T\, t^{-\kappa|l+m|/2} \tag{3.28}$$

where $\kappa = \max\{d/p - 2, 1\}$.

Proof. The proof is the same as in Theorem 3.17, with the help of the following inequality:

$$\int_0^t \|\mathrm{D}_x^l V(t-s)\|_{L^p}\, \phi(t,s)\,\mathrm{d}s \le \|V\|_{L^\infty([0,T],W^{k,p})} \int_0^t \phi(t,s)\,\mathrm{d}s.$$

□

It follows the corollary:

3.26 Corollary. *Assume* $D_{pp}D_{qq} - D_{pq}^2 > 0$ *and* $V_0 \in C(\mathbb{R}_0^+, L^\infty(\mathbb{R}^d)) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_0^+, W^{k,p}(\mathbb{R}^d))$ *for* $k \in \mathbb{N}$ *and* $1 \le p \le \infty$. *Then the solution of the WFP problem* w *satisfies for all* $T > 0$, $l, m \in \mathbb{N}_0^d$. *Then the solution* w *satisfies*

$$\forall t \in (0,\infty) \qquad w(t) \in H^k(\mathbb{R}^{2d}) \hookrightarrow C^{k-d-1}(\mathbb{R}^{2d}) \tag{3.29}$$

Similar results holds for $D_{pp}D_{qq} - D_{pq}^2 = 0$.

Appendix A

Some results on strongly continuous semigroups

In this chapter, we briefly recall some useful definitions and theorems from the theory of strongly continuous semigroups. We refer to [Paz83] and [EN00] for more details and proofs.

A.1 Semigroups, generators and generation theorems

Let first define what a strongly continuous semigroup of operators is.

A.1 Definition. Let X be a Banach space. A one parameter family $\{T(t)\}_{t\in\mathbb{R}_0^+}$ of bounded linear operators from X into X is called *semigroup of bounded linear operators on X* if

i. $T(0) = I$

ii. $T(t+s) = T(t)T(s) \quad \forall t, s \geq 0$

If additionally

$$\forall x \in X \qquad \lim_{t\to 0^+} T(t)x = x$$

then the semigroup is called *strongly continuous.*

Note that a strongly continuous semigroup is not only right continuous in $t = 0$, but also in any $t \geq 0$, thanks to the semigroup property $T(t+s) = T(t)T(s)$.

The most important notion linked to the strongly continuous semigroups is the notion of infinitesimal generator.

A.2 Definition. The linear operator A defined by

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t} \tag{A.1}$$

for all x such that the limit exists, is called the *infinitesimal generator* of the semigroup $T(.)$.

A.3 Lemma. *[Paz83, Cor. 1.2.5] A is a closed densely defined linear operator.*

As its name suggests, the infinitesimal generator defines entirely the semigroup:

A.4 Lemma. *[Paz83, Thm. 1.2.6] Let $T(.)$ and $S(.)$ be strongly continuous semigroups with infinitesimal generators A and B respectively. If $A = B$ then $T(.) = S(.)$.*

The semigroup can be seen as the general solution of the Cauchy problem

$$\begin{cases} \partial_t x = Ax & \forall t > 0 \\ x(t=0) = x_0 \end{cases} \tag{A.2}$$

A.5 Theorem. *[Paz83, Chap. 4] For all $x_0 \in \mathcal{D}(A)$, the function $x(t) = T(t)x_0$ is the unique classical solution of* (A.2).

In order to prove the existence of solutions for Cauchy problems similar to (A.2), we have to determine if A generates a strongly continuous semigroup. It can be done with the help of *generation theorem*, such as the *Hille-Yoshida theorem* (not presented in this paper) or the following *Lumer-Phillips theorem.*

A.6 Definition. A linear operator A is *dissipative* if for every $x \in \mathcal{D}(A)$ there is an $x^* \in \left\{x^* \in X^* \mid \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\right\}$ such that $\Re\langle Ax, x^*\rangle \leq 0$. ($X^*$ denotes here the topological dual space to X.)

If X is a Hilbert space, it is equivalent to $\Re\langle Ax, x\rangle \leq 0 \quad \forall x \in \mathcal{D}(A)$.

A.7 Theorem. *[Paz83, Cor. 1.4.4] Let A be a densely defined closed linear operator. If A and its adjoint A^* are dissipative, then A is the infinitesimal generator of a strongly continuous semigroup on X.*

A.2 Analytic semigroups

A special class of strongly continuous semigroups is the class of *analytic semigroups.* They present some useful regularity properties.

A.8 Definition. ([Paz83, Def. 2.5.1] and [EN00, Def. II.4.5.]) A semigroup $T(.)$ is called *analytic* (of angle δ) if it has an analytic continuation on a domain $\Delta_\delta := \{z \in \mathbb{C} \mid |\arg(z)| < \delta\} \cup \{0\}$ for a given $\delta \in (0, \frac{\pi}{2}]$, with

i. $T(0) = I$ and $\lim_{\Delta_\delta \ni z \to 0} T(z)x = x$ for every $x \in X$.

ii. $T(z + w) = T(z)T(w) \quad \forall z, w \in \Delta_\delta$

It is called *bounded analytic semigroup* if, moreover, it is bounded on every $\Delta_{\delta'}$ for $0 < \delta' < \delta$.

The analyticity of a semigroup can be inferred from properties of its infinitesimal generator:

A.9 Definition. [EN00, Def. II.4.1.] A closed densely defined operator A is called *sectorial* (of angle θ), if there exists $\theta \in (0, \frac{\pi}{2}]$ and $M > 0$ such that

$$\rho(A) \supset \Delta_{\theta+\frac{\pi}{2}} := \{z \in \mathbb{C} \mid |\arg(z)| < \theta + \frac{\pi}{2}\} \tag{A.3}$$

and if for each $\epsilon \in (0, \delta)$, there exists $M_\epsilon \geq 1$ such that

$$\|(A-\lambda)^{-1}\| \leq \frac{M_\epsilon}{|\lambda|} \qquad \forall \lambda \in \Delta_{\theta+\frac{\pi}{2}-\epsilon} \setminus \{0\}. \tag{A.4}$$

A.10 Theorem. *[Paz83, Thm. 2.5.2] The strongly continuous semigroup $T(.)$ is bounded analytic of angle δ, iff its infinitesimal generator is sectorial of angle $\frac{\pi}{2} + \delta$.*

A useful regularity property of the analytic semigroups is the following:

A.11 Theorem. *[Paz83, Thm. 2.5.2] The strongly continuous semigroup $T(.)$ with infinitesimal generator A is bounded analytic, iff $\forall t > 0$, $ran(T(t)) \subset \mathcal{D}(A)$ and there exists a constant C such that*

$$\forall t > 0 \qquad \|AT(t)\| \leq \frac{C}{t} \tag{A.5}$$

A.12 Corollary. *Let $T(.)$ be bounded analytic semigroup and A its infinitesimal generator. We have*

$$\forall x_0 \in X, \forall k \in \mathbb{N}, \forall t > 0 \qquad T(t)x_0 \in \mathcal{D}(A^k) \tag{A.6}$$

Proof. For all $k \in \mathbb{N}$ and all $t > 0$, we have

$$A^k T(t) = \left(AT\left(\frac{t}{k}\right)\right)^k$$

According to Theorem A.11, $AT\,(t/k)$ is bounded. It follows that $A^kT(t)$ is also bounded. One concludes that $\operatorname{ran}(T(t)) \subset \mathcal{D}(A^k)$. □

A.3 Bounded perturbation

A.13 Theorem. *[Paz83, Thm. 3.1.1 and 3.2.1] If A is the infinitesimal generator of a strongly continuous semigroup and B is a bounded linear operator then $A + B$ is the infinitesimal generator of a strongly continuous semigroup. Moreover if A generates an analytic semigroup, then $A + B$ generates an analytic semigroup.*

With the help of the previous theorem, on can show the existence of a unique solution for the following differential problem

$$\begin{cases} \partial_t x = Ax + Bx & \forall t > 0 \\ x(t=0) = x_0 \end{cases} \tag{A.7}$$

where A is the infinitesimal generator of a strongly continuous semigroup and B is a bounded linear operator.

A.14 Lemma. *[EN00, Cor. II.1.7.] If $T(.)$ is the semigroup generated by A and $S(.)$ the semigroup generated by $A+B$ where B is a bounded perturbation of A, then for all $x_0 \in X$ and $t \geq 0$, it holds*

$$S(t)x_0 = T(t)x_0 + \int_0^t T(t-s)BS(s)x_0\,ds \tag{A.8}$$

Proof. For a fixed $t > 0$, we define $H(s) := T(t-s)S(s)$ for all $0 \leq s \leq t$. The derivation of H reads

$$\begin{aligned} H'(s) &= -AT(t-s)S(s) + T(t-s)(A+B)S(s) \\ &= T(t-s)BS(s) \end{aligned}$$

since A commute with $T(.)$. By inserting $H(s) = T(t-s)S(s)$ by S and T in

$$H(t) - H(0) = \int_0^t H'(s)ds$$

we get the equation (A.8). $\square$

A.4 Time dependent perturbation

We consider now the following problem:

$$\begin{cases} \partial_t x = Ax + B(t)x & \forall t > 0 \\ x(t=0) = x_0 \end{cases} \tag{A.9}$$

With the same arguments as in Lemma A.14, every classical solution of (A.9) satisfies the integral equation

$$x(t) = T(t)x_0 + \int_0^t T(t-s)B(s)x(s)\,\mathrm{d}s \tag{A.10}$$

where $T(.)$ is the semigroup generated by A.

A.15 Theorem. *[Paz83, Thm. 6.1.2 and 6.1.5] Let A be the infinitesimal generator of a strongly continuous semigroup and $B : [0,T] \to \mathcal{B}(X)$ continuous. Then the problem* (A.9) *has a unique (mild) solution $x \in C([0,T], X)$.*

If subsequently B is Lipschitz-continuous and $w_0 \in \mathcal{D}(A)$, then the solution actually belongs to $C^1([0,T], X)$.

Bibliography

[ADM07] A. Arnold, E. Dhamo, and C. Manzini. Dispersive effects in quantum kinetic equations. *Indiana University mathematics journal*, 56(3):1299–1332, 2007.

[AGG+12] A. Arnold, I.M. Gamba, M.P. Gualdani, S. Mischler, C. Mouhot, and C. Sparber. The Wigner-Fokker-Planck equation: Stationary states and large time behavior. *Mathematical Models and Methods in Applied Sciences*, 22(11), 2012.

[ALMS04] A. Arnold, J. L López, P.A. Markowich, and J. Soler. An analysis of quantum Fokker-Planck models: A Wigner function approach. *Revista Matemática Iberoamericana*, 20(3):771–814, 2004.

[Arn08] Anton Arnold. Mathematical properties of quantum evolution equations. In *Quantum Transport*, pages 45–109. Springer, 2008.

[Bou95] François Bouchut. Smoothing effect for the non-linear Vlasov-Poisson-Fokker-Planck system. *Journal of differential equations*, 122(2):225–238, 1995.

[Car98] Ana Carpio. Long-time behavior for solutions of the Vlasov-Poisson-Fokker-Planck equation. *Mathematical methods in the applied sciences*, 21(11):985–1014, 1998.

[EN00] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*, volume 194. Springer Verlag, 2000.

[Eva98] Lawrence C. Evans. *Partial Differential Equations*. Graduate Studies in Mathematics. American Mathematical Society, 1998.

[Lun97] Alessandra Lunardi. On the Ornstein-Uhlenbeck operator in L^2 spaces with respect to invariant measures. *Transactions of the American Mathematical Society*, 349(1):155–169, 1997.

[Met01] Giorgio Metafune. L^p-spectrum of Ornstein-Uhlenbeck operators. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 30(1):97–124, 2001.

[MPP02] G. Metafune, D. Pallara, and E. Priola. Spectrum of Ornstein-Uhlenbeck operators in L^p spaces with respect to invariant measures. *Journal of Functional Analysis*, 196(1):40–60, 2002.

[MPRS02] G. Metafune, J. Prüss, A. Rhandi, and R. Schnaubelt. The domain of the Ornstein-Uhlenbeck operator on an L^p-space with invariant measure. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 1(2):471–485, 2002.

[Paz83] Amnon Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Number vol. 44 in Applied Mathematical Sciences. Springer, 1983.

[PL93] Thierry Paul and Pierre-Louis Lions. Sur les mesures de Wigner. *Revista Matemática Iberoamericana*, 9(3):553–618, 1993.

[SA13] Dominik Stürzer and Anton Arnold. Spectral analysis and long-time behavior of a Fokker-Planck equation with a non-local perturbation. 2013.

[SCDM04] C. Sparber, J.A. Carrillo, J. Dolbeault, and P.A. Markowich. On the long-time behavior of the quantum Fokker-Planck equation. *Monatshefte für Mathematik*, 141(3):237–257, 2004.

[Süd07] Bastian Südfeld. Die lineare Wigner-Fokker-Planck Gleichung. Master's thesis, Westfalische Wilhelms-Universität Münster, 2007.

[Wig32] Eugene Wigner. On the quantum correction for thermodynamic equilibrium. *Physical Review*, 40(5):749, 1932.