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Heterogeneous Beliefs, Realtive Wealth Concerns, and Sharpe Ratios

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Heterogeneous Beliefs, Relative Wealth Concerns, and Sharpe Ratios

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Abstract

This work considers a continuous time pure exchange economy where the agents have heterogeneous beliefs about the aggregate dividend process and also care about relative wealth concerns. It is shown, analytically, that in the special case where the agents have logarithmic utility, the relative wealth concerns do not influence the equilibrium. Another special case is considered where the agents' risk aversion parameter is 2 and it is shown that the Sharpe ratios are time-dependent and the final wealth state dependent.

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1 Introduction

> *"Men do not desire merely to be rich, but to be richer than other men. The avaricious or covetous man would find little or no satisfaction in the possession of any amount of wealth, if he were the poorest amongst all his neighbours or fellow-countrymen."*
>
> \- John Stuart Mill (1907)

Standard economic models assume that an individual derives utility from one's own consumption. However, economists have long acknowledged the importance of incorporating relative concerns into individual utility. Veblen (1912) brings to the readers attention the point that as the society becomes wealthier, the amount of consumption of an individual also increases in order to maintain his social standing. Frank (1993) makes a very similar point about incorporating relative wealth concerns in the determination of social status. Abel (1990) was the first to incorporate "catching up with the Joneses" utility specification into a standard asset pricing model.[1] Cole et al. (1992) use the idea of relative wealth concerns in a different setup. They develop a model where the agents compete for mates, and the rate of their success is dependent upon their relative wealth with respect to other competitors. Becker et al. (2005) consider the effects of the assumption that acquiring a higher social position increases marginal utility on the risk taking behaviour and the equilibrium wealth distributions. DeMarzo et al. (2008) explore the idea that relative wealth concerns can explain financial bubbles. They develop a finite horizon overlapping generations model in which the competition on future investment opportunities induces an endogenous concern on relative wealth among the agents. This results in the agents herding towards the risky assets and driving down the expected returns.

[1]Other works in a similar vein include Gali (1994), Abel (1999), and Chan and Kogan (2002)

Another important application of relative wealth concerns can be seen in the fund management industry. The "salaries" of these managers depends on the assets they are holding. If an investor were to walk into one of these fund management offices, her decision would rightly depend on the fund managers' rankings. Empirical evidence to support this has been provided by Sirri and Tufano (1998) and Huang et al. (2007), they establish a positive and convex relationship between fund flows (the net of all cash inflows and outflows in and out of various financial assets) and relative performances. In this present work, the this convexity is modelled as an exogenously given bonus/penalty.

Most asset pricing models build upon a representative-agent framework. While this setup gives us the convenience of getting analytically tractable results, it turns a blind eye towards the fact that the agents in the economy might not be the same and can hold heterogeneous expectations of future economics conditions. Analysing the findings from the Livingston Survey and the Survey of Professional Forecasters, Mankiw et al. (2004) find that interquartile range of inflation expectations among the general public varies from 0% to 5% while among professional economists, it varies from 1.5% to 2.5%. Clearly, the data does not agree with the assumption that the agents in the economy hold homogeneous beliefs about future economic conditions, and as these survey values suggest, the heterogeneity is substantial.

Belief heterogeneity among agents can induce speculative behaviour which can cause endogenous wealth fluctuation. This can be used to account for excess volatility, time varying Sharpe-ratios and high equity premia which cannot be explained by standard representative-agent models with smooth aggregate endowment processes.

In their seminal work, Harrison and Kreps (1978) showed that in a market where agents have heterogeneous beliefs, speculative behaviour can lead to asset price bubbles. In a market in which agents hold heterogeneous beliefs about an asset's fundamental, an asset owner is willing to pay a price higher than her own expectation of the asset's fundamental because she expects to resell the asset to a future optimist at an even higher price. Such speculative behaviour leads to a bubble component in asset prices. They achieved this behaviour using an extremely simple finite-state Markov chain market structure.

Chen and Kohn (2011) achieve similar results using more sophisticated machinery by considering an asset whose dividend follows a mean-reverting stochastic process. The agents in the economy have the same beliefs about the volatility but disagree upon the mean-reversion parameter. They show that with this market structure, there exists a permanent asset price bubble. However their main focus is purely mathematical, to determine the minimum equilibrium price explicitly as a unique solution to a certain differential equation. The takeaway message from an economic perspective is that in such a market structure, there exists a unique minimum equilibrium price and a permanent price bubble, that is the asset is always priced higher than the minimal equilibrium price.

Basak (2000, 2005) establish equilibrium price dynamics in the presence of investor heterogeneity. The model presented in these papers incorporated the heterogeneity as a difference in opinion about the drift parameter of the dividend process and highlight the mechanism through which this disagreement influences asset prices. Kogan et al. (2006) and Yan (2008) analyse a framework in which one of the agents perceives the correct model of the dividend process while the other agent perceives an incorrect model of the dividend process (the correctness of the models is measured by the perception of the drift parameter, i.e. one agent knows the correct value of the drift while the other agent does not). They show that the irrational agent can affect the equilibrium asset prices. Xiong and Yan (2010) present a dynamic equilibrium model where the agent heterogeneity is modelled as different learning models which the agents use to estimate the unobservable inflation target. This difference in opinion leads to speculative behaviour and it is shown that it can help to explain the "excess volatility" of the bond yields and also the failure of the expectation hypothesis.

Allen (2001) points to the importance of institutional investors (such as fund managers, investment bankers etc.) in the modern day financial market. The literature took this cue and has since extensively studied the effect of these institutional investors on the asset prices. Work in this field include, and are not restricted to, Vayanos and Woolley (2013) and He and Krishnamurthy (2013). These papers consider models with one representative investor and thus concerns of relative performance is nugatory. Recently, there has been an increasing amount work which incorporates relative performance into continuous time asset pricing models. The work by Cuoco and Kaniel (2011), and Basak and Pavlova (2013) belong to this strand of literature. These works compare the performance of institutional investors relative to an external benchmark, like the S&P 500. However, the work by Weinbaum (2009) and Kaniel and Kondor (2013) considers effect of the relative performance within a peer group has on the risk-taking behaviour of the investors. To the best of my knowledge, the first paper to consider the effects of relative wealth concerns on the speculative trading caused by investors' heterogeneous beliefs was Huang et al. (2013). They present a model with two types of agents who hold heterogeneous beliefs about the drift parameter and care about relative wealth concern. With this setup, they show that the agents' concern about relative performance affects their stock holdings, and stock prices. Using the same setup, I first provide an analytic proof that if the agents have logarithmic utilities, then the concern on relative performance does not influence the equilibrium. I also look at the Sharpe ratios of the agents and show, analytically, that they are time-varying and the final wealth state dependent.

2 The Economy

The setup of the economy is a standard adaptation of Lucas (1978) type of aggregate dividends, which follow a geometric Brownian motion process. Specifically, consider a pure-exchange, continuous time economy with a finite horizon $[0, T]$. There is only one source of uncertainty in the economy and the agents trade in securities to share risk. The uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ where Ω is the state space and a standard Brownian motion B_t is defined on it. This process, $\{B_t, 0 \leq t \leq T\}$ induces the filtration $\{\mathcal{F}_t, 0 \leq t \leq T\}$, $\mathcal{F}_t$ has the standard interpretation of the set of information available at time t.

There are two groups of risk-averse agents in the economy who allocate their wealth optimally between a risky asset, which is stock in positive net supply, and a risk-free asset, a zero net supply bond. The agents only consume at the end of the period. We also assume that each group has an infinite number of agents that form a continuum with measure 1. We index these groups by i and j.[2]

The stock experiences risk in the form of the standard Brownian motion denoted by B_t. The stock price S_t is assumed to follow :

$$\frac{dS_t}{S_t} = \mu_{S,t} dt + \sigma_{S,t} dB_t \tag{1}$$

with $\sigma_{S,t} > 0$. The mean return μ_S and volatility σ_S are equilibrium objects. The risk-free asset, interpreted as a bond, is in zero net supply and has a constant return r. I assume $r = 0$, which is equivalent to using the bond as the numeraire good. Since the agents in this model only consume at the end of the trading period, there is no intermediate choice which can be used to pin down the interest rate. There is one share of stock, the risky security, which is a claim to a single exogenous dividend D_T at time T, and thus $S_T = D_T$. D_T is the terminal value

[2]With a slight abuse of notation from now on, by agent i, we will mean an agent belonging to group i and by agent j, an agent belonging to group j.

of the process given by

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dB_t \tag{2}$$

where μ_D and $\sigma_D > 0$ are exogenously given constants. Equation Equation 2 implies that D_T is log-normally distributed, this structure is assumed for algebraic convenience.

As mentioned before, there are two groups of risk-averse agents who optimally allocate their wealth between the two assets. Each agent invests a fraction $\theta_{i,t}$ of their wealth in the stock. The wealth process of agent i, $W_{i,t}$ then follows :

$$dW_{i,t} = \theta_{i,t} W_{i,t} \left[\mu_{S,t} dt + \sigma_{S,t} dB_t\right] \tag{3}$$

Assume that all agents start out with the same initial wealth, which means $W_{i,0} = W_{j,0}$.

Agents in the economy are concerned about relative wealth, i.e. they care about how good/bad the other agent is doing. To quantify this notion, define :

$$R_{i,T} = \frac{W_{i,T}/W_{i,0}}{W_{j,T}/W_{j,0}} = \frac{W_{i,T}}{W_{j,T}} \tag{4}$$

as the relative wealth index of agent i with respect to agent j. Here, $W_{i,T}$ denotes the aggregate wealth of agent $i's$ portfolio at time T.

After the trading is closed and before the utilities are realized, each agent gets a transfer from the other agent, which depends on the agents' own wealth and also upon the relative wealth. This transfer is introduced as a bonus/penalty function defined by

$$BP_{i,T} = W_{i,T} \left[R_{i,T}^{\kappa} - 1\right] \tag{5}$$

With this formulation, if $W_{i,T} > W_{j,T}$, then $R_{i,T} > 1$ and hence $BP_{i,T} > 0$ so that agent i receives a bonus, while if $W_{i,T} < W_{j,T}$, then $R_{i,T} < 1$ and hence $BP_{i,T} < 0$ so that agent i incurs a penalty.

The total wealth for agent i is then the sum of investment returns and bonus/penalty :

$$W_{i,T} + BP_{i,T} = W_{i,T} R_{i,T}^{\kappa} \tag{6}$$

The agents in the economy derive utility over terminal wealth and their utility function takes the form

$$U\left(W_{i,T}\right) = \frac{W_{i,T}^{1-\gamma}}{1-\gamma}$$

where the argument is the total wealth for agent i from Equation 6.

The agents in this economy disagree on the drift parameter in the processes. Agent i perceives the expected growth rate of the aggregate dividend process to be μ_i. Hence, agent i lives in a filtered probability space $(\Omega, \mathcal{F}_i, \{\mathcal{F}_{i,t}\}, \mathbb{P}_i)$ in which the dividend process follows :

$$\frac{dD_t}{D_t} = \mu_i dt + \sigma_D dB_{i,t} \tag{7}$$

with constant μ_i. Define $\eta_i \equiv= \frac{\mu_D - \mu_i}{\sigma_D}$ and $dB_{i,t} = dB_t + \eta_i dt$. By Girsanov's theorem, $dB_{i,t}$ is a Brownian motion with respect to agent i's probability measure $\mathbb{P}_i$.

The stock prices, in i's filtered probability space, follow :

$$\frac{dS_t}{S_t} = \mu_{i,t} dt + \sigma_{S,t} dB_{i,t} \tag{8}$$

Define $\bar{\mu} \equiv \frac{\mu_1 - \mu_2}{\sigma_D}$. This captures the agents' disagreement on the drift of the dividend process. $\bar{\mu} > 0$ would imply that the agent 1 is more optimistic than agent 2, and $\bar{\mu} < 0$ would imply that the agent 2 is more optimistic than agent 1. The trading occurs continuously in the two securities, the risky stock and the riskless bond. The market is dynamically complete in the sense that any contingent claims can be replicated. Dynamic completeness and no arbitrage then imply the existence of a unique state price density $\pi_{i,t}$ given by

$$\frac{d\pi_{i,t}}{\pi_{i,t}} = -\omega_{i,t} dB_{i,t} \tag{9}$$

where $\omega_{i,t} = \frac{\mu_{i,t}}{\sigma_{S,t}}$ is agent i's Sharpe ratio. The state price density is the price of a security that pays one unit of consumption good if a given state of nature arises, and zero otherwise. The advantage of this structure of the state price density process is that it follows the martingale property, i.e. $\pi_{i,t} = \mathbb{E}_t[\pi_{i,T}]$ which shall be of use later in the analysis.

3 The Benchmark Case

We first analyse the benchmark case where there are no bonus/penalties, i.e. the agents do not care about relative performance. This happens when $\kappa = 0$ and then the optimization problem for the agent i is

$$\max_{W_{i,T}} \mathbb{E}^i \left[\frac{W_{i,T}^{1-\gamma}}{1-\gamma} \right]$$

subject to the dynamic budget constraint

$$dW_{i,t} = \theta_{i,t} W_{i,t} \left[\mu_{i,t} dt + \sigma_{S,t} dB_{i,t} \right]$$

This dynamic optimization problem can be restated as a static problem using the martingale methodology of Cox and Huang (1989), and Karatzas et al. (1987). The equivalent static problem then is

$$\max_{W_{i,T}} \mathbb{E}^i \left[\frac{W_{i,T}^{1-\gamma}}{1-\gamma} \right]$$

subject to the static constraint

$$\mathbb{E}^i \left[\pi_{i,T} W_{i,T} \right] = W_{i,0}$$

The equivalence of the dynamic optimization problem and the static one is also intuitive. As argued before, the markets are dynamically complete in the presented model. Given the market completeness, we know that any contingent payoff can be perfectly financed by a trading strategy in the existing assets. This means that the agent accounts for the wealth required at the end of the trading period (for all possible realizations of the state) subject only to the affordability constraint which is instantiated by the static budget constraint.

Solving this problem, we get optimal final wealth and state prices at time T which are summarized in the following Theorem.

Theorem 1. *When $\kappa = 0$, the optimal final wealth of the two agents are*

$$
\begin{aligned}
W_{1,T} &= \frac{D_T}{1+\alpha\left(T\right)^{\frac{1}{\gamma}}} \\
W_{2,T} &= \frac{\alpha\left(T\right)^{\frac{1}{\gamma}} D_T}{1+\alpha\left(T\right)^{\frac{1}{\gamma}}}
\end{aligned}
$$

The state prices at time T are

$$
\begin{aligned}
\pi_{1,T} &= \frac{\left(1+\alpha\left(T\right)^{\frac{1}{\gamma}}\right)^{\gamma}}{\lambda_1 D_T^{\gamma}} \\
\pi_{2,T} &= \frac{\left(1+\alpha\left(T\right)^{\frac{1}{\gamma}}\right)^{\gamma}}{\lambda_2 D_T^{\gamma}\alpha\left(T\right)}
\end{aligned}
$$

where

$$
\alpha\left(t\right) = \frac{\lambda_1 \pi_{1,t}}{\lambda_2 \pi_{2,t}} \tag{10}
$$

where λ_i is the Lagrange multiplier in agent i's optimization and $\pi_{i,t}$ is the state price density as perceived by agent i, and $i = 1, 2$.

Proof. The Lagrangian to the static problem is given by

$$
\mathcal{L} = \mathbb{E}^i\left[\frac{W_{i,T}^{1-\gamma}}{1-\gamma}\right] + \lambda_i\left[W_{i,0} - \mathbb{E}^i\left[\pi_{i,T}W_{i,T}\right]\right]
$$

The first order condition with respect to terminal wealth $W_{i,T}$ gives us

$$
\begin{aligned}
W_{i,T}^{-\gamma} &= \lambda_i \pi_{i,T} \\
\Longrightarrow \; W_{i,T} &= \frac{1}{\lambda_i^{\frac{1}{\gamma}}\pi_{i,T}^{\frac{1}{\gamma}}}
\end{aligned} \tag{11}
$$

where the Lagrange multipliers λ_i can be obtained by plugging in the optimal wealth choice into the budget constraint. The market clearing condition in this economy is given by

$$W_{1,T} + W_{2,T} = D_T \tag{12}$$

Combining the first order condition for both agents, we get

$$\frac{W_{1,T}}{W_{2,T}} = \frac{\lambda_2^{\frac{1}{\gamma}} \pi_{2,T}^{\frac{1}{\gamma}}}{\lambda_1^{\frac{1}{\gamma}} \pi_{1,T}^{\frac{1}{\gamma}}}$$

The market clearing condition Equation 12 can be re-written as

$$\frac{W_{1,T}}{W_{2,T}} + 1 = \frac{D_T}{W_{2,T}}$$

Combining the previous two results, we get

$$\begin{aligned} W_{2,T} &= \frac{\lambda_1^{\frac{1}{\gamma}} \pi_{1,T}^{\frac{1}{\gamma}}}{\lambda_1^{\frac{1}{\gamma}} \pi_{1,T}^{\frac{1}{\gamma}} + \lambda_2^{\frac{1}{\gamma}} \pi_{2,T}^{\frac{1}{\gamma}}} D_T \\ W_{1,T} &= \frac{\lambda_2^{\frac{1}{\gamma}} \pi_{2,T}^{\frac{1}{\gamma}}}{\lambda_1^{\frac{1}{\gamma}} \pi_{1,T}^{\frac{1}{\gamma}} + \lambda_2^{\frac{1}{\gamma}} \pi_{2,T}^{\frac{1}{\gamma}}} D_T \end{aligned}$$

Plugging this into the first order condition Equation 11, we get the state price density at time T as

$$\begin{aligned} \pi_{2,T} &= \frac{1}{\lambda_2 D_T^{\gamma}} \left(\frac{\lambda_1^{\frac{1}{\gamma}} \pi_{1,T}^{\frac{1}{\gamma}} + \lambda_2^{\frac{1}{\gamma}} \pi_{2,T}^{\frac{1}{\gamma}}}{\lambda_1^{\frac{1}{\gamma}} \pi_{1,T}^{\frac{1}{\gamma}}} \right)^{\gamma} \\ \pi_{1,T} &= \frac{1}{\lambda_1 D_T^{\gamma}} \left(\frac{\lambda_1^{\frac{1}{\gamma}} \pi_{1,T}^{\frac{1}{\gamma}} + \lambda_2^{\frac{1}{\gamma}} \pi_{2,T}^{\frac{1}{\gamma}}}{\lambda_2^{\frac{1}{\gamma}} \pi_{2,T}^{\frac{1}{\gamma}}} \right)^{\gamma} \end{aligned}$$

Define

$$\alpha\left(t\right) = \frac{\lambda_1 \pi_{1,t}}{\lambda_2 \pi_{2,t}}$$

Then, the final wealth of the two agents are

$$W_{1,T} = \frac{D_T}{1 + \alpha(T)^{\frac{1}{\gamma}}}$$
$$W_{2,T} = \frac{\alpha(T)^{\frac{1}{\gamma}} D_T}{1 + \alpha(T)^{\frac{1}{\gamma}}}$$

and the state prices at time T are

$$\pi_{1,T} = \frac{\left(1 + \alpha(T)^{\frac{1}{\gamma}}\right)^{\gamma}}{\lambda_1 D_T^{\gamma}}$$
$$\pi_{2,T} = \frac{\left(1 + \alpha(T)^{\frac{1}{\gamma}}\right)^{\gamma}}{\lambda_2 D_T^{\gamma} \alpha(T)}$$

□

The above theorem establishes that the agents share the final dividend according to a sharing rule which depends on $\alpha(T)^{\frac{1}{\gamma}}$.

The final dividend sharing rule in this setup is the same as the one demonstrated by Weinbaum (2009). The only difference in the results arise from the fact that in this model, the agents have the same coefficient of risk aversion whereas in Weinbaum (2009), the agents vary in their risk aversion parameters. If the two agents had the same risk aversion parameters, then this result would conform with theirs. This equivalence serves as a good robustness check of the model under consideration here.

To characterize the equilibrium, the explicit expression of the state price density at all intermediate time, i.e, for $\pi_{i,t}$ is needed. Given the structure of the state price process Equation 9, this can be calculated using $\pi_{i,t} = \mathbb{E}_t^i[\pi_{i,T}]$ from its martingale property. However, this task is not trivial because of the term

$\left(1+\alpha\left(T\right)^{\frac{1}{\gamma}}\right)^{\gamma}$. But, if we assume γ to be an integer[3], then,

$$\left(1+\alpha\left(T\right)^{\frac{1}{\gamma}}\right)^{\gamma} = \sum_{i=0}^{\gamma}\binom{\gamma}{i}\alpha\left(T\right)^{\frac{i}{\gamma}}$$

$$\Longrightarrow \pi_{1,T} = \frac{\sum_{i=0}^{\gamma}\binom{\gamma}{i}\alpha\left(T\right)^{\frac{i}{\gamma}}}{\lambda_1 D_T^{\gamma}}$$

$$\pi_{2,T} = \frac{\sum_{i=0}^{\gamma}\binom{\gamma}{i}\alpha\left(T\right)^{\frac{i}{\gamma}}}{\lambda_2 D_T^{\gamma}\alpha\left(T\right)}$$

With this, the state prices can be characterized as in the following Theorem, which can be arrived at after some algebra. Using Itō's Lemma on the state price densities and comparing coefficients with the Equation 9, the Sharpe ratios are also characterized.

Theorem 2. *Under the assumption that γ is an integer, the state prices at time t are given by*

$$\pi_{1,t} = \frac{1}{\lambda_1 D_t^{\gamma}}\sum_{i=0}^{\gamma}\binom{\gamma}{i}\alpha\left(t\right)^{\frac{i}{\gamma}} e^{\left(-\frac{\bar{\mu}^2}{2}\frac{i}{\gamma}-\gamma\left(\mu_1-\frac{\sigma_D^2}{2}\right)+\frac{1}{2}\left[\frac{i}{\gamma}\bar{\mu}+\gamma\sigma_D\right]^2\right)(T-t)}$$

$$\pi_{2,t} = \frac{1}{\lambda_2 D_t^{\gamma}}\sum_{i=0}^{\gamma}\binom{\gamma}{i}\alpha\left(t\right)^{\frac{i-\gamma}{\gamma}} e^{\left(\frac{\bar{\mu}^2}{2}\frac{i-\gamma}{\gamma}-\gamma\left(\mu_2-\frac{\sigma_D^2}{2}\right)+\frac{1}{2}\left[\frac{i-\gamma}{\gamma}\bar{\mu}+\gamma\sigma_D\right]^2\right)(T-t)}$$

The Sharpe ratios for the two agents are given by

$$\omega_{1,t} = \gamma\sigma_D + \frac{\sum_{i=0}^{\gamma}\binom{\gamma}{i}\alpha\left(t\right)^{\frac{i}{\gamma}} e^{\left(-\frac{\bar{\mu}^2}{2}\frac{i}{\gamma}-\gamma\left(\mu_1-\frac{\sigma_D^2}{2}\right)+\frac{1}{2}\left[\frac{i}{\gamma}\bar{\mu}+\gamma\sigma_D\right]^2\right)(T-t)}\frac{i}{\gamma}}{\sum_{i=0}^{\gamma}\binom{\gamma}{i}\alpha\left(t\right)^{\frac{i}{\gamma}} e^{\left(-\frac{\bar{\mu}^2}{2}\frac{i}{\gamma}-\gamma\left(\mu_1-\frac{\sigma_D^2}{2}\right)+\frac{1}{2}\left[\frac{i}{\gamma}\bar{\mu}+\gamma\sigma_D\right]^2\right)(T-t)}}$$

$$\omega_{2,t} = \gamma\sigma_D + \frac{\sum_{i=0}^{\gamma}\binom{\gamma}{i}\alpha\left(t\right)^{\frac{i-\gamma}{\gamma}} e^{\left(-\frac{\bar{\mu}^2}{2}\frac{i-\gamma}{\gamma}-\gamma\left(\mu_2-\frac{\sigma_D^2}{2}\right)+\frac{1}{2}\left[\frac{i-\gamma}{\gamma}\bar{\mu}+\gamma\sigma_D\right]^2\right)(T-t)}\frac{i-\gamma}{\gamma}}{\sum_{i=0}^{\gamma}\binom{\gamma}{i}\alpha\left(t\right)^{\frac{i-\gamma}{\gamma}} e^{\left(-\frac{\bar{\mu}^2}{2}\frac{i-\gamma}{\gamma}-\gamma\left(\mu_2-\frac{\sigma_D^2}{2}\right)+\frac{1}{2}\left[\frac{i-\gamma}{\gamma}\bar{\mu}+\gamma\sigma_D\right]^2\right)(T-t)}}$$

[3] This approach is similar to the one used by Yan (2008).

The two previous theorems establish the benchmark case where the agents do not have concerns about relative wealth, this shall be used to gauge the next two special cases.

4 A Special Case

In this section, we now introduce relative wealth concerns in our model. Solving the the general model where the agents have CRRA utility is algebraically cumbersome, so we focus on a special case where the agents have logarithmic utility, i.e. $\gamma = 1$.

In this special case where the agents have log utility and care about relative performance ($\kappa \neq 0$), we solve the optimization problem to get the final wealth distribution and state price density at time T in the following Theorem.

Theorem 3. *In the special case when the agents have log utility* ($\gamma = 1$) *and have concerns over relative performance* ($\kappa \neq 0$)*, the optimal final wealth of the two agents are*

$$W_{1,T} = \frac{D_T}{1 + \alpha(T)}$$
$$W_{2,T} = \frac{\alpha(T) D_T}{1 + \alpha(T)}$$

The state prices at time T are

$$\pi_{1,T} = \frac{1 + \alpha(T)}{\lambda_1 D_T}$$
$$\pi_{2,T} = \frac{1 + \alpha(T)}{\lambda_2 D_T \alpha(T)}$$

Proof. Then, agent i's maximization problem now becomes

$$\max_{W_{i,T}} \mathbb{E}^i \left[\log\left(W_{i,T} R_{i,T}^{\kappa}\right)\right]$$

subject to

$$\mathbb{E}^i \left[\pi_{i,T} W_{i,T}\right] = W_{i,0}$$

The Lagrangian now is given by

$$\mathcal{L} = \mathbb{E}^i \left[\log\left(W_{i,T} R_{i,T}^{\kappa}\right)\right] + \lambda_i \left[W_{i,0} - \mathbb{E}^i \left[\pi_{i,T} W_{i,T}\right]\right]$$

Using the definition of $R_{i,T}$ from equation Equation 4, the Lagrangian can be rewritten as

$$\begin{aligned}
\mathcal{L} &= \mathbb{E}^i \left[\log\left(W_{i,T} \frac{W_{i,T}^{\kappa}}{W_{j,T}^{\kappa}}\right)\right] + \lambda_i \left[W_{i,0} - \mathbb{E}^i \left[\pi_{i,T} W_{i,T}\right]\right] \\
&= \mathbb{E}^i \left[\log\left(W_{i,T}^{\kappa+1}\right) - \log\left(W_{j,T}^{\kappa}\right)\right] + \lambda_i \left[W_{i,0} - \mathbb{E}^i \left[\pi_{i,T} W_{i,T}\right]\right] \\
&= \mathbb{E}^i \left[(\kappa+1)\log\left(W_{i,T}\right) - \kappa \log\left(W_{j,T}\right)\right] + \lambda_i \left[W_{i,0} - \mathbb{E}^i \left[\pi_{i,T} W_{i,T}\right]\right]
\end{aligned}$$

The FOC now gives us

$$\frac{\kappa+1}{W_{i,T}} = \lambda_i \pi_{i,T} \tag{13}$$

The same market clearing condition Equation 12 holds

$$W_{1,T} + W_{2,T} = D_T$$

Since the FOC holds for both agents, we get

$$\frac{W_{1,T}}{W_{2,T}} = \frac{\lambda_2 \pi_{2,T}}{\lambda_1 \pi_{1,T}}$$

We therefore get the wealth distribution as

$$\begin{aligned}
W_{2,T} &= \frac{\lambda_1 \pi_{1,T}}{\lambda_1 \pi_{1,T} + \lambda_2 \pi_{2,T}} D_T \\
W_{1,T} &= \frac{\lambda_2 \pi_{2,T}}{\lambda_1 \pi_{1,T} + \lambda_2 \pi_{2,T}} D_T
\end{aligned}$$

Plugging the optimal wealth back into the first order condition Equation 13 we get the state prices at time T as

$$\pi_{2,T} = \frac{\kappa + 1}{\lambda_2 D_T} \left(\frac{\lambda_1 \pi_{1,T} + \lambda_2 \pi_{2,T}}{\lambda_1 \pi_{1,T}} \right)$$
$$\pi_{1,T} = \frac{\kappa + 1}{\lambda_1 D_T} \left(\frac{\lambda_1 \pi_{1,T} + \lambda_2 \pi_{2,T}}{\lambda_2 \pi_{2,T}} \right)$$

Using

$$\alpha(t) = \frac{\lambda_1 \pi_{1,t}}{\lambda_2 \pi_{2,t}}$$

the final wealth of the two agents are

$$W_{1,T} = \frac{D_T}{1 + \alpha(T)}$$
$$W_{2,T} = \frac{\alpha(T) D_T}{1 + \alpha(T)}$$

and the state prices at time T are

$$\pi_{1,T} = \frac{1 + \alpha(T)}{\lambda_1 D_T}$$
$$\pi_{2,T} = \frac{1 + \alpha(T)}{\lambda_2 D_T \alpha(T)}$$

$\square$

The structure of the state price density at time T in this special case conforms perfectly with the previous section where the characterization of the state price density at intermediate time t was possible when γ is an integer (because here we have $\gamma = 1$). Therefore, we find an explicit solution for the state price densities at all intermediate times t and also a closed form expression for the Sharpe ratios as below :

Theorem 4. *In the special case when the agents have log utility* ($\gamma = 1$) *and have concerns over relative performance* ($\kappa \neq 0$)*, the agents' state price density for all* *intermediate time t is given by*

$$\pi_{1,t} = \frac{\kappa+1}{\lambda_1 D_t} \cdot \left\{ e^{\left(\sigma_D^2-\mu_1\right)(T-t)} + \alpha\left(t\right) e^{\left(\sigma_D^2-\mu_1+\bar{\mu}\sigma_D\right)(T-t)} \right\}$$
$$\pi_{2,t} = \frac{\kappa+1}{\lambda_2 D_t} \cdot \left\{ \frac{1}{\alpha\left(t\right)} e^{\left(\sigma_D^2+\bar{\mu}^2-\mu_2-\sigma_D\bar{\mu}\right)(T-t)} + e^{\left(\sigma_D^2-\mu_2\right)(T-t)} \right\}$$

The Sharpe ratios for the two agents are given by

$$\omega_{1,t} = \sigma_D + \bar{\mu}\frac{\alpha\left(t\right)}{\alpha\left(t\right) + e^{-\bar{\mu}\sigma_D(T-t)}}$$
$$\omega_{2,t} = \sigma_D - \bar{\mu}\frac{1}{1 + \alpha\left(t\right) e^{-\bar{\mu}\sigma_D(T-t)}}$$

Proof. Consider first agent 1. With the definition of $\alpha\left(T\right)$ as in equation Equation 10, for the special case we have

$$W_{1,T} = \frac{D_T}{1+\alpha\left(T\right)}$$

Together with the first order condition, this gives us

$$\pi_{1,T} = \frac{\kappa+1}{\lambda_1} \cdot \frac{1+\alpha\left(T\right)}{D_T} \tag{14}$$

Using Itō's lemma on Equation 10, we get the following dynamics of $\alpha\left(\cdot\right)$:

$$\begin{aligned} \frac{d\alpha\left(t\right)}{\alpha\left(t\right)} &= -\bar{\mu} dB_{1,t} \\ \implies \ln\left(\frac{\alpha\left(T\right)}{\alpha\left(t\right)}\right) &= -\frac{\bar{\mu}^2}{2}\left(T-t\right) - \bar{\mu}\left(B_{1,T}-B_{1,t}\right) \\ \implies \alpha\left(T\right) &= \alpha\left(t\right) e^{-\frac{\bar{\mu}^2}{2}(T-t)-\bar{\mu}\left(B_{1,T}-B_{1,t}\right)} \end{aligned} \tag{15}$$

The dividend process, under agent 1's subjective measure looks like

$$\begin{aligned} \frac{dD_t}{D_t} &= \mu_{1,D} dt + \sigma_D dB_{1,t} \\ \implies \ln\left(\frac{D_T}{D_t}\right) &= \left(\mu_1 - \frac{\sigma_D^2}{2}\right)\left(T-t\right) + \sigma_D\left(B_{1,T}-B_{1,t}\right) \end{aligned}$$

$$\implies D_T = D_t e^{\left(\mu_1 - \frac{\sigma_D^2}{2}\right)(T-t)+\sigma_D\left(B_{1,T}-B_{1,t}\right)} \tag{16}$$

Plugging in Equation 15 and Equation 16 into Equation 14, we get

$$\begin{aligned}
\pi_{1,T} =& \frac{\kappa+1}{\lambda_1 D_t} \cdot \frac{1+\alpha(t)\, e^{-\frac{\bar{\mu}^2}{2}(T-t)-\bar{\mu}\left(B_{1,T}-B_{1,t}\right)}}{e^{\left(\mu_1-\frac{\sigma_D^2}{2}\right)(T-t)+\sigma_D\left(B_{1,T}-B_{1,t}\right)}} \\
=& \frac{\kappa+1}{\lambda_1 D_t} \cdot \Bigg\{ e^{\left(\frac{\sigma_D^2}{2}-\mu_1\right)(T-t)-\sigma_D\left(B_{1,T}-B_{1,t}\right)} + \\
& \qquad \alpha(t)\, e^{\left(\frac{\sigma_D^2}{2}-\frac{\bar{\mu}^2}{2}-\mu_1\right)(T-t)-(\bar{\mu}+\sigma_D)\left(B_{1,T}-B_{1,t}\right)} \Bigg\}
\end{aligned}$$

To establish the equilibrium, we need to find an explicit expression of $\pi_{1,t}$, which is calculated as $\pi_{1,t} = \mathbb{E}_t^1\left[\pi_{1,T}\right]$ from its martingale property :

$$\begin{aligned}
\pi_{1,t} &= \mathbb{E}_t^1\left[\pi_{1,T}\right] \\
&= \frac{\kappa+1}{\lambda_1 D_t} \cdot \Bigg\{ e^{\left(\frac{\sigma_D^2}{2}-\mu_1+\frac{\sigma_D^2}{2}\right)(T-t)} + \\
& \qquad \alpha(t)\, e^{\left(\frac{\sigma_D^2}{2}-\frac{\bar{\mu}^2}{2}-\mu_1+\frac{(\bar{\mu}+\sigma_D)^2}{2}\right)(T-t)} \Bigg\} \\
&= \frac{\kappa+1}{\lambda_1 D_t} \cdot \left\{ e^{\left(\sigma_D^2-\mu_1\right)(T-t)} + \alpha(t)\, e^{\left(\sigma_D^2-\mu_1+\bar{\mu}\sigma_D\right)(T-t)} \right\}
\end{aligned} \tag{17}$$

Uing Itō's lemma on this, we get

$$\begin{aligned}
d\pi_{1,t} =& \frac{\kappa+1}{\lambda_1} \cdot \left(-\frac{1}{D_t^2} dD_t + \frac{1}{2} \cdot \frac{2}{D_t^3} \cdot D_t^2 \sigma_D^2 dt\right) \cdot e^{\left(\sigma_D^2-\mu_1\right)(T-t)} \\
& + \frac{\kappa+1}{\lambda_1} \left(\frac{1}{D_t} \cdot -\alpha(t)\, \bar{\mu} dB_{1,t} + \alpha(t) \left(-\frac{1}{D_t^2} dD_t + \frac{1}{2} \cdot \frac{2}{D_t^3} \cdot D_t^2 \sigma_D^2 dt\right)\right) \\
& e^{\left(\sigma_D^2-\mu_1+\bar{\mu}\sigma_D\right)(T-t)} \\
=& \frac{\kappa+1}{\lambda_1 D_t} \left(-\mu_{1,D} dt - \sigma_D dB_{1,t} + \sigma_D^2 dt\right) \cdot e^{\left(\sigma_D^2-\mu_1\right)(T-t)} \\
& + \frac{\kappa+1}{\lambda_1 D_t} \left(-\alpha(t)\, \bar{\mu} dB_{1,t} - \alpha(t)\, \mu_{1,D} dt - \alpha(t)\, \sigma_D dB_{1,t} + \alpha(t)\, \sigma_D^2 dt\right) \\
& e^{\left(\sigma_D^2-\mu_1+\bar{\mu}\sigma_D\right)(T-t)} \\
\implies d\pi_{1,t} =& \frac{\kappa+1}{\lambda_1 D_t} \cdot \Big\{ \left(\left(\sigma_D^2-\mu_1\right) dt - \sigma_D dB_{1,t}\right) \cdot e^{\left(\sigma_D^2-\mu_1\right)(T-t)}
\end{aligned}$$

$$+\left(\alpha\left(t\right)\left(\sigma_D^2-\mu_1\right)dt-\alpha\left(t\right)\left(\sigma_D+\bar{\mu}\right)dB_{1,t}\right)\cdot e^{\left(\sigma_D^2-\mu_1+\bar{\mu}\sigma_D\right)(T-t)}\Big\}$$

Dividing this by Equation 17, and comparing the coefficients with equation Equation 9, we get agent 1's Sharpe ratio :

$$\begin{aligned}\omega_{1,t}&=\sigma_D+\frac{\alpha\left(t\right)e^{\left(\sigma_D^2-\mu_1+\bar{\mu}\sigma_D\right)(T-t)}}{e^{\left(\sigma_D^2-\mu_1\right)(T-t)}+\alpha\left(t\right)e^{\left(\sigma_D^2-\mu_1+\bar{\mu}\sigma_D\right)(T-t)}}\bar{\mu}\\&=\sigma_D+\bar{\mu}\frac{\alpha\left(t\right)}{\alpha\left(t\right)+e^{-\bar{\mu}\sigma_D(T-t)}}\end{aligned}$$

Now consider agent 2. With the definition of $\alpha\left(T\right)$ as in equation Equation 10, for the special case we have

$$W_{2,T}=\frac{\alpha\left(T\right)D_T}{1+\alpha\left(T\right)}$$

Together with the first order condition, this gives us

$$\pi_{2,T}=\frac{\kappa+1}{\lambda_2}\cdot\frac{1+\alpha\left(T\right)}{\alpha\left(T\right)D_T}\tag{18}$$

Using Itō's lemma on Equation 10, we get the following dynamics of $\alpha\left(\cdot\right)$:

$$\begin{aligned}d\frac{1}{\alpha\left(t\right)}&=\frac{1}{\alpha\left(t\right)}\bar{\mu}dB_{2,t}\\\ln\left(\frac{\alpha\left(T\right)}{\alpha\left(t\right)}\right)&=-\frac{\bar{\mu}^2}{2}\left(T-t\right)-\bar{\mu}\left(B_{2,T}-B_{2,t}\right)\\\implies\alpha\left(T\right)&=\alpha\left(t\right)e^{-\frac{\bar{\mu}^2}{2}(T-t)-\bar{\mu}\left(B_{2,T}-B_{2,t}\right)}\end{aligned}\tag{19}$$

The dividend process, under agent 2's subjective measure looks like

$$\begin{aligned}\frac{dD_t}{D_t}&=\mu_{2,D}dt+\sigma_DdB_{2,t}\\\implies\ln\left(\frac{D_T}{D_t}\right)&=\left(\mu_2-\frac{\sigma_D^2}{2}\right)\left(T-t\right)+\sigma_D\left(B_{2,T}-B_{2,t}\right)\\\implies D_T&=D_te^{\left(\mu_2-\frac{\sigma_D^2}{2}\right)(T-t)+\sigma_D\left(B_{2,T}-B_{2,t}\right)}\end{aligned}\tag{20}$$

Plugging in Equation 19 and Equation 20 into Equation 18, we get

$$\begin{aligned}\pi_{2,T} =& \frac{\kappa+1}{\lambda_2 D_t} \cdot \frac{1+\alpha\left(t\right)e^{-\frac{\bar{\mu}^2}{2}(T-t)-\bar{\mu}\left(B_{2,T}-B_{2,t}\right)}}{\alpha\left(t\right)e^{\left(\mu_1-\frac{\sigma_D^2}{2}-\frac{\bar{\mu}^2}{2}\right)(T-t)+(\sigma_D-\bar{\mu})\left(B_{2,T}-B_{2,t}\right)}} \\ =& \frac{\kappa+1}{\lambda_2 D_t} \cdot \left\{\frac{1}{\alpha\left(t\right)}e^{\left(\frac{\sigma_D^2}{2}+\frac{\bar{\mu}^2}{2}-\mu_2\right)(T-t)-(\sigma_D-\bar{\mu})\left(B_{2,T}-B_{2,t}\right)}+ \right. \\ & \left. e^{\left(\frac{\sigma_D^2}{2}-\mu_2\right)(T-t)-\sigma_D\left(B_{2,T}-B_{2,t}\right)}\right\}\end{aligned}$$

Again, to completely characterize the equilibrium, we need to find an explicit expression of $\pi_{2,t}$, which is calculated as $\pi_{2,t} = \mathbb{E}_t^2\left[\pi_{2,T}\right]$ from its martingale property :

$$\begin{aligned}\pi_{2,t} &= \mathbb{E}_t^2\left[\pi_{2,T}\right] \\ &= \frac{\kappa+1}{\lambda_2 D_t} \cdot \left\{\frac{1}{\alpha\left(t\right)}e^{\left(\frac{\sigma_D^2}{2}+\frac{\bar{\mu}^2}{2}-\mu_2+\frac{(\sigma_D-\bar{\mu})^2}{2}\right)(T-t)}+ \right. \\ & \left. e^{\left(\frac{\sigma_D^2}{2}-\mu_2+\frac{\sigma_D^2}{2}\right)(T-t)}\right\} \\ &= \frac{\kappa+1}{\lambda_2 D_t} \cdot \left\{\frac{1}{\alpha\left(t\right)}e^{\left(\sigma_D^2+\bar{\mu}^2-\mu_2-\sigma_D\bar{\mu}\right)(T-t)}+e^{\left(\sigma_D^2-\mu_2\right)(T-t)}\right\}\end{aligned} \tag{21}$$

Uing Itō's lemma on this, we get

$$\begin{aligned}d\pi_{2,t} =& \frac{\kappa+1}{\lambda_2} \cdot \left(-\frac{1}{D_t^2}dD_t + \frac{1}{2}\cdot\frac{2}{D_t^3}\cdot D_t^2\sigma_D^2 dt\right)\cdot e^{\left(\sigma_D^2-\mu_2\right)(T-t)} \\ &+ \frac{\kappa+1}{\lambda_2}\left(\frac{1}{D_t}\cdot\frac{1}{\alpha\left(t\right)}\bar{\mu}dB_{2,t}+\frac{1}{\alpha\left(t\right)}\left(-\frac{1}{D_t^2}dD_t+\frac{1}{2}\cdot\frac{2}{D_t^3}\cdot D_t^2\sigma_D^2 dt\right)\right) \\ & e^{\left(\sigma_D^2+\bar{\mu}^2-\mu_2+\bar{\mu}\sigma_D\right)(T-t)} \\ =& \frac{\kappa+1}{\lambda_2 D_t}\left[-\mu_2 dt-\sigma_D dB_{2,t}+\sigma_D^2 dt\right]\cdot e^{\left(\sigma_D^2-\mu_2\right)(T-t)} \\ &+ \frac{\kappa+1}{\lambda_2 D_t}\left(\frac{1}{\alpha\left(t\right)}\bar{\mu}dB_{2,t}-\frac{1}{\alpha\left(t\right)}\mu_2 dt-\frac{1}{\alpha\left(t\right)}\sigma_D dB_{2,t}+\frac{1}{\alpha\left(t\right)}\sigma_D^2 dt\right) \\ & e^{\left(\sigma_D^2+\bar{\mu}^2-\mu_2+\bar{\mu}\sigma_D\right)(T-t)} \\ \Longrightarrow \quad d\pi_{2,t} =& \frac{\kappa+1}{\lambda_2 D_t}\cdot\left\{\left(\left(\sigma_D^2-\mu_2\right)dt-\sigma_D dB_{2,t}\right)\cdot e^{\left(\sigma_D^2-\mu_2\right)(T-t)}\right.\end{aligned}$$

$$+\left(\frac{1}{\alpha(t)}\left(\sigma_D^2-\mu_2\right)dt-\frac{1}{\alpha(t)}\left(\sigma_D-\bar{\mu}\right)dB_{2,t}\right)\cdot e^{\left(\sigma_D^2+\bar{\mu}^2-\mu_2+\bar{\mu}\sigma_D\right)(T-t)}\Bigg\}$$

Dividing this by Equation 21, and comparing the coefficients with equation Equation 9, we get agent 2's Sharpe ratio :

$$\begin{aligned}\omega_{2,t} &= \sigma_D-\frac{\frac{1}{\alpha(t)}e^{\left(\sigma_D^2+\bar{\mu}^2-\mu_2-\sigma_D\bar{\mu}\right)(T-t)}}{\frac{1}{\alpha(t)}e^{\left(\sigma_D^2+\bar{\mu}^2-\mu_2-\sigma_D\bar{\mu}\right)(T-t)}+e^{\left(\sigma_D^2-\mu_2\right)(T-t)}}\bar{\mu}\\&=\sigma_D-\bar{\mu}\frac{1}{1+\alpha(t)\,e^{-\bar{\mu}\sigma_D(T-t)}}\end{aligned}$$

□

One can easily see that the final wealth distribution in this special case does not depend on the relative wealth parameter κ and is the same as the benchmark case as discussed in the previous section (with $\gamma=1$). However, one may be inclined to conclude that the state price densities are not the same any more as there is an explicit dependence on κ. But, from Equation 13, we see that the Lagrange multipliers are linearly dependent on κ and thus the effect of κ on the state prices vanishes once the Lagrange multipliers are substituted out for.

Then, combining the above discussion and the previous theorems, we get the following result.

Theorem 5. *If the agents have logarithmic utility, then the relative wealth concerns are inconsequential.*

5 Next Step Forward

Now that it has been established that the case when the agents have logarithmic utility the relative wealth concerns do not play a role in equilibrium, we move on and take $\gamma = 2$.

The results of the agents' optimization problem are summarized in Theorem 6 below.

Theorem 6. *When $\gamma = 2$ and $\kappa \neq 0$, the optimal final wealth sharing rule among the agents is given by*

$$W_{1,T} = \frac{1}{1 + \alpha(T)^{\frac{1}{2\kappa+2}}} D_T$$
$$W_{2,T} = \frac{\alpha(T)^{\frac{1}{2\kappa+2}}}{1 + \alpha(T)^{\frac{1}{2\kappa+2}}} D_T$$

The state prices at time T are

$$\pi_{1,T} = \frac{\kappa+1}{\lambda_2} \frac{\left(1 + \alpha(T)^{\frac{1}{2\kappa+2}}\right)^2}{D_T^2} \alpha(T)^{\frac{\kappa}{2\kappa+2}}$$
$$\pi_{2,T} = \frac{\kappa+1}{\lambda_2} \frac{\left(1 + \alpha(T)^{\frac{1}{2\kappa+2}}\right)^2}{D_T^2} \alpha(T)^{\frac{-(\kappa+2)}{2\kappa+2}}$$

Proof. In this case, agent i's optimization problem is

$$\max_{W_{i,T}} \mathbb{E}^i \left[\frac{-1}{W_{i,T} R_{i,T}^{\kappa}} \right]$$

subject to

$$\mathbb{E}^i \left[\pi_{i,T} W_{i,T} \right] = W_{i,0}$$

The Lagrangian to this static problem is given by

$$\mathcal{L} = \mathbb{E}^i \left[\frac{-1}{W_{i,T} R_{i,T}^{\kappa}} \right] + \lambda_i \left[W_{i,0} - \mathbb{E}^i \left[\pi_{i,T} W_{i,T} \right] \right]$$

$$= \mathbb{E}^i \left[\frac{-W_{j,T}^{\kappa}}{W_{i,T}^{\kappa+1}} \right] + \lambda_i \left[W_{i,0} - \mathbb{E}^i \left[\pi_{i,T} W_{i,T} \right] \right]$$

The FOC w.r.t $W_{i,T}$ gives us

$$(\kappa+1) \frac{W_{j,T}^{\kappa}}{W_{i,T}^{\kappa+2}} = \lambda_i \pi_{i,T}$$

The same market clearing condition Equation 12 holds

$$W_{1,T} + W_{2,T} = D_T$$

Since the FOC holds for both agents, we get

$$\alpha(T) = \frac{W_{2,T}^{2\kappa+2}}{W_{1,T}^{2\kappa+2}} \tag{22}$$

We therefore get the wealth distribution as

$$W_{1,T} = \frac{D_T}{1 + \alpha(T)^{\frac{1}{2\kappa+2}}}$$
$$W_{2,T} = \frac{\alpha(T)^{\frac{1}{2\kappa+2}} D_T}{1 + \alpha(T)^{\frac{1}{2\kappa+2}}}$$

Plugging the optimal wealth back into the first order condition Equation 22 we get the state prices at time T as

$$\pi_{1,T} = \frac{\kappa+1}{\lambda_2} \frac{\left(1 + \alpha(T)^{\frac{1}{2\kappa+2}}\right)^2}{D_T^2} \alpha(T)^{\frac{\kappa}{2\kappa+2}}$$
$$\pi_{2,T} = \frac{\kappa+1}{\lambda_1} \frac{\left(1 + \alpha(T)^{\frac{1}{2\kappa+2}}\right)^2}{D_T^2} \alpha(T)^{\frac{-(\kappa+2)}{2\kappa+2}}$$

□

In this case, to find an explicit expression for the state price densities, we expand the square terms and get the following result.

Theorem 7. *The state price density at all intermediate times is given by*

$$\pi_{1,t} = \frac{\kappa+1}{\lambda_1 D_t^2}\left({}^1\beta_1 + {}^1\beta_2 + {}^1\beta_3\right)$$
$$\pi_{2,t} = \frac{\kappa+1}{\lambda_2 D_t^2}\left({}^2\beta_1 + {}^2\beta_2 + {}^2\beta_3\right)$$

where ${}^1\beta_1, {}^1\beta_2, {}^1\beta_3, {}^2\beta_1, {}^2\beta_2, {}^2\beta_3$ *are functions of* $\alpha(t)$.

The Sharpe ratio of the agents is given by

$$\omega_{1,t} = 2\sigma_D + \bar{\mu}\frac{{}^1\beta_2 \cdot \left(\frac{\kappa+1}{2\kappa+2} + \frac{1}{2}\left(\frac{\kappa+1}{2\kappa+2}\right)^2\bar{\mu}\right) + {}^1\beta_1 \cdot \left(\frac{\kappa}{2\kappa+2} + \frac{1}{2}\frac{\kappa(\kappa+2)}{(2\kappa+2)^2}\bar{\mu}\right) + {}^1\beta_3 \cdot \left(\frac{\kappa+2}{2\kappa+2} + \frac{1}{2}\frac{\kappa(\kappa+2)}{(2\kappa+2)^2}\bar{\mu}\right)}{{}^1\beta_1 + {}^1\beta_2 + {}^1\beta_3}$$

$$\omega_{2,t} = 2\sigma_D - \bar{\mu}\frac{{}^2\beta_2 \cdot \left(\frac{\kappa+1}{2\kappa+2} - \frac{1}{2}\left(\frac{\kappa+1}{2\kappa+2}\right)^2\bar{\mu}\right) + {}^2\beta_1 \cdot \left(\frac{\kappa}{2\kappa+2} - \frac{1}{2}\frac{\kappa(\kappa+2)}{(2\kappa+2)^2}\bar{\mu}\right) - {}^2\beta_3 \cdot \left(\frac{\kappa+2}{2\kappa+2} + \frac{1}{2}\frac{\kappa(\kappa+2)}{(2\kappa+2)^2}\bar{\mu}\right)}{{}^2\beta_1 + {}^2\beta_2 + {}^2\beta_3}$$

Proof. Consider first agent 1. Expanding the square terms we get

$$\begin{aligned}
\pi_{1,T} &= \frac{\kappa+1}{\lambda_1}\frac{1}{D_T^2}\alpha(T)^{\frac{\kappa}{2\kappa+2}}\left[1 + 2\alpha(T)^{\frac{1}{2\kappa+2}} + \alpha(T)^{\frac{2}{2\kappa+2}}\right] \\
&= \frac{\kappa+1}{\lambda_1}\frac{1}{D_T^2}\left[\alpha(T)^{\frac{\kappa}{2\kappa+2}} + 2\alpha(T)^{\frac{\kappa+1}{2\kappa+2}} + \alpha(T)^{\frac{\kappa+2}{2\kappa+2}}\right]
\end{aligned}$$

Using Equation 16 and Equation 15,

$$\begin{aligned}
\pi_{1,T} &= \frac{\kappa+1}{\lambda_1}\frac{\left[\alpha(T)^{\frac{\kappa}{2\kappa+2}} + 2\alpha(T)^{\frac{\kappa+1}{2\kappa+2}} + \alpha(T)^{\frac{\kappa+2}{2\kappa+2}}\right]}{D_T^2} \\
&= \frac{\kappa+1}{\lambda_1}\left[\frac{\alpha(T)^{\frac{\kappa}{2\kappa+2}}}{D_T^2} + \frac{2\alpha(T)^{\frac{\kappa+1}{2\kappa+2}}}{D_T^2} + \frac{\alpha(T)^{\frac{\kappa+2}{2\kappa+2}}}{D_T^2}\right] \\
&= \frac{\kappa+1}{\lambda_1}\left[\frac{\alpha(t)^{\frac{\kappa}{2\kappa+2}}e^{-\frac{\kappa\bar{\mu}^2}{2(2\kappa+2)}(T-t) - \frac{\kappa\bar{\mu}}{2\kappa+2}\left(B_{1,T}-B_{1,t}\right)}}{D_t^2 e^{\left(2\mu_1-\sigma_D^2\right)(T-t)+2\sigma_D\left(B_{1,T}-B_{1,t}\right)}} + \right. \\
&\quad \frac{2\alpha(t)^{\frac{\kappa+1}{2\kappa+2}}e^{-\frac{(\kappa+1)\bar{\mu}^2}{2(2\kappa+2)}(T-t) - \frac{(\kappa+1)\bar{\mu}}{2\kappa+2}\left(B_{1,T}-B_{1,t}\right)}}{D_t^2 e^{\left(2\mu_1-\sigma_D^2\right)(T-t)+2\sigma_D\left(B_{1,T}-B_{1,t}\right)}} +
\end{aligned}$$

$$\left.\frac{\alpha(t)^{\frac{\kappa+2}{2\kappa+2}} e^{-\frac{(\kappa+2)\bar{\mu}^2}{2(2\kappa+2)}(T-t)-\frac{(\kappa+2)\bar{\mu}}{2\kappa+2}\left(B_{1,T}-B_{1,t}\right)}}{D_t^2 e^{\left(2\mu_1-\sigma_D^2\right)(T-t)+2\sigma_D\left(B_{1,T}-B_{1,t}\right)}}\right]$$

$$= \frac{\kappa+1}{\lambda_1 D_t^2}\left[\alpha(t)^{\frac{\kappa}{2\kappa+2}} e^{\left(\sigma_D^2-2\mu_1-\frac{\kappa\bar{\mu}^2}{2(2\kappa+2)}\right)(T-t)-\left(2\sigma_D+\frac{\kappa\bar{\mu}}{2\kappa+2}\right)\left(B_{1,T}-B_{1,t}\right)}+\right.$$

$$2\alpha(t)^{\frac{\kappa+1}{2\kappa+2}} e^{\left(\sigma_D^2-2\mu_1-\frac{(\kappa+1)\bar{\mu}^2}{2(2\kappa+2)}\right)(T-t)-\left(2\sigma_D+\frac{(\kappa+1)\bar{\mu}}{2\kappa+2}\right)\left(B_{1,T}-B_{1,t}\right)}+$$

$$\left.\alpha(t)^{\frac{\kappa+2}{2\kappa+2}} e^{\left(\sigma_D^2-2\mu_1-\frac{(\kappa+2)\bar{\mu}^2}{2(2\kappa+2)}\right)(T-t)-\left(2\sigma_D+\frac{(\kappa+2)\bar{\mu}}{2\kappa+2}\right)\left(B_{1,T}-B_{1,t}\right)}\right]$$

Again, as before, to completely characterize the equilibrium, we need to find an explicit expression of $\pi_{1,t}$, which is calculated as $\pi_{1,t} = \mathbb{E}_t^1\left[\pi_{1,T}\right]$ from its martingale property :

$$\begin{aligned}
\pi_{1,t} &= \mathbb{E}_t^1\left[\pi_{1,T}\right] \\
&= \frac{\kappa+1}{\lambda_1 D_t^2}\left[\alpha(t)^{\frac{\kappa}{2\kappa+2}} e^{\left(\sigma_D^2-2\mu_1-\frac{\kappa\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(2\sigma_D+\frac{\kappa\bar{\mu}}{2\kappa+2}\right)^2\right)(T-t)}+\right. \\
&\quad 2\alpha(t)^{\frac{\kappa+1}{2\kappa+2}} e^{\left(\sigma_D^2-2\mu_1-\frac{(\kappa+1)\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(2\sigma_D+\frac{(\kappa+1)\bar{\mu}}{2\kappa+2}\right)^2\right)(T-t)}+ \\
&\quad \left.\alpha(t)^{\frac{\kappa+2}{2\kappa+2}} e^{\left(\sigma_D^2-2\mu_1-\frac{(\kappa+2)\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(2\sigma_D+\frac{(\kappa+2)\bar{\mu}}{2\kappa+2}\right)^2\right)(T-t)}\right]
\end{aligned} \tag{23}$$

To find the Sharpe ratio, we use Itō's lemma on Equation 23 :

$$\begin{aligned}
d\pi_{1,t} = \frac{\kappa+1}{\lambda_1}&\left[e^{\left(\sigma_D^2-2\mu_1-\frac{\kappa\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(2\sigma_D+\frac{\kappa\bar{\mu}}{2\kappa+2}\right)^2\right)(T-t)}\cdot\right. \\
&\left[\alpha(t)^{\frac{\kappa}{2\kappa+2}}\left(\frac{-2}{D_t^3}dD_t+\frac{1}{2}\frac{6}{D_t^2}\sigma_D^2 dt\right)\right. \\
&\left.+\frac{1}{D_t^2}\left(\frac{\kappa}{2\kappa+2}\alpha(t)^{\frac{-(\kappa+2)}{2\kappa+2}}d\alpha(t)-\frac{1}{2}\frac{\kappa(\kappa+2)}{(2\kappa+2)^2}\alpha(t)^{\frac{\kappa}{2\kappa+2}}\bar{\mu}^2 dB_{1,t}\right)\right]+ \\
&2e^{\left(\sigma_D^2-2\mu_1-\frac{(\kappa+1)\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(2\sigma_D+\frac{(\kappa+1)\bar{\mu}}{2\kappa+2}\right)^2\right)(T-t)}\cdot \\
&\left[\alpha(t)^{\frac{\kappa+1}{2\kappa+2}}\left(\frac{-2}{D_t^3}dD_t+\frac{1}{2}\frac{6}{D_t^2}\sigma_D^2 dt\right)\right. \\
&\left.+\frac{1}{D_t^2}\left(\frac{\kappa+1}{2\kappa+2}\alpha(t)^{\frac{-(\kappa+1)}{2\kappa+2}}d\alpha(t)-\frac{1}{2}\left(\frac{\kappa+1}{2\kappa+2}\right)^2\alpha(t)^{\frac{\kappa+1}{2\kappa+2}}\bar{\mu}^2 dB_{1,t}\right)\right]+ \\
&e^{\left(\sigma_D^2-2\mu_1-\frac{(\kappa+2)\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(2\sigma_D+\frac{(\kappa+2)\bar{\mu}}{2\kappa+2}\right)^2\right)(T-t)}\cdot
\end{aligned}$$

$$\left[\alpha\left(t\right)^{\frac{\kappa+2}{2\kappa+2}}\left(\frac{-2}{D_t^3}dD_t+\frac{1}{2}\frac{6}{D_t^2}\sigma_D^2dt\right)\right.$$

$$\left.\left.+\frac{1}{D_t^2}\left(\frac{\kappa+2}{2\kappa+2}\alpha\left(t\right)^{\frac{-\kappa}{2\kappa+2}}d\alpha\left(t\right)-\frac{1}{2}\frac{\kappa\left(\kappa+2\right)}{\left(2\kappa+2\right)^2}\alpha\left(t\right)^{\frac{\kappa+2}{2\kappa+2}}\bar{\mu}^2dB_{1,t}\right)\right]\right]$$

$$\Longrightarrow\ d\pi_{1,t}=\frac{\kappa+1}{\lambda_1D_t^2}\left[e^{\left(\sigma_D^2-2\mu_1-\frac{\kappa\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(2\sigma_D+\frac{\kappa\bar{\mu}}{2\kappa+2}\right)^2\right)(T-t)}\cdot\right.$$

$$\left[\alpha\left(t\right)^{\frac{\kappa}{2\kappa+2}}\left(\left(3\sigma_D^2-2\mu_{1,D}\right)dt-2\sigma_DdB_{1,t}\right)\right.$$

$$\left.-\left(\frac{\kappa}{2\kappa+2}\alpha\left(t\right)^{\frac{\kappa}{2\kappa+2}}\bar{\mu}dB_{1,t}+\frac{1}{2}\frac{\kappa\left(\kappa+2\right)}{\left(2\kappa+2\right)^2}\alpha\left(t\right)^{\frac{\kappa}{2\kappa+2}}\bar{\mu}^2dB_{1,t}\right)\right]+$$

$$2e^{\left(\sigma_D^2-2\mu_1-\frac{(\kappa+1)\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(2\sigma_D+\frac{(\kappa+1)\bar{\mu}}{2\kappa+2}\right)^2\right)(T-t)}\cdot$$

$$\left[\alpha\left(t\right)^{\frac{\kappa+1}{2\kappa+2}}\left(\left(3\sigma_D^2-2\mu_{1,D}\right)dt-2\sigma_DdB_{1,t}\right)\right.$$

$$\left.-\left(\frac{\kappa+1}{2\kappa+2}\alpha\left(t\right)^{\frac{\kappa+1}{2\kappa+2}}\bar{\mu}dB_{1,t}+\frac{1}{2}\left(\frac{\kappa+1}{2\kappa+2}\right)^2\alpha\left(t\right)^{\frac{\kappa+1}{2\kappa+2}}\bar{\mu}^2dB_{1,t}\right)\right]+$$

$$e^{\left(\sigma_D^2-2\mu_1-\frac{(\kappa+2)\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(2\sigma_D+\frac{(\kappa+2)\bar{\mu}}{2\kappa+2}\right)^2\right)(T-t)}\cdot$$

$$\left[\alpha\left(t\right)^{\frac{\kappa+2}{2\kappa+2}}\left(\left(3\sigma_D^2-2\mu_{1,D}\right)dt-2\sigma_DdB_{1,t}\right)\right.$$

$$\left.\left.-\left(\frac{\kappa+2}{2\kappa+2}\alpha\left(t\right)^{\frac{\kappa+2}{2\kappa+2}}\bar{\mu}dB_{1,t}+\frac{1}{2}\frac{\kappa\left(\kappa+2\right)}{\left(2\kappa+2\right)^2}\alpha\left(t\right)^{\frac{\kappa+2}{2\kappa+2}}\bar{\mu}^2dB_{1,t}\right)\right]\right]$$

$$\Longrightarrow\ d\pi_{1,t}=\frac{\kappa+1}{\lambda_1D_t^2}\left[\alpha\left(t\right)^{\frac{\kappa}{2\kappa+2}}e^{\left(\sigma_D^2-2\mu_1-\frac{\kappa\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(2\sigma_D+\frac{\kappa\bar{\mu}}{2\kappa+2}\right)^2\right)(T-t)}\cdot\right.$$

$$\left[\left(3\sigma_D^2-2\mu_{1,D}\right)dt\right.$$

$$\left.-\left(\frac{\kappa}{2\kappa+2}\bar{\mu}+\frac{1}{2}\frac{\kappa\left(\kappa+2\right)}{\left(2\kappa+2\right)^2}\bar{\mu}^2+2\sigma_D\right)dB_{1,t}\right]+$$

$$2\alpha\left(t\right)^{\frac{\kappa+1}{2\kappa+2}}e^{\left(\sigma_D^2-2\mu_1-\frac{(\kappa+1)\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(2\sigma_D+\frac{(\kappa+1)\bar{\mu}}{2\kappa+2}\right)^2\right)(T-t)}\cdot$$

$$\left[\left(3\sigma_D^2-2\mu_{1,D}\right)dt\right.$$

$$\left.-\left(\frac{\kappa+1}{2\kappa+2}\bar{\mu}+\frac{1}{2}\left(\frac{\kappa+1}{2\kappa+2}\right)^2\bar{\mu}^2+2\sigma_D\right)dB_{1,t}\right]+$$

$$\alpha\left(t\right)^{\frac{\kappa+2}{2\kappa+2}}e^{\left(\sigma_D^2-2\mu_1-\frac{(\kappa+2)\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(2\sigma_D+\frac{(\kappa+2)\bar{\mu}}{2\kappa+2}\right)^2\right)(T-t)}\cdot$$

$$
\left[\left(3\sigma_D^2 - 2\mu_{1,D}\right) dt \right.
$$

$$
\left.\left. - \left(\frac{\kappa+2}{2\kappa+2}\bar{\mu} + \frac{1}{2}\frac{\kappa(\kappa+2)}{(2\kappa+2)^2}\bar{\mu}^2 + 2\sigma_D\right) dB_{1,t}\right]\right]
$$

Define

$$
\begin{aligned}
{}^1\beta_1 &= \alpha(t)^{\frac{\kappa}{2\kappa+2}}\, e^{\left(\sigma_D^2 - 2\mu_1 - \frac{\kappa\bar{\mu}^2}{2(2\kappa+2)} + \frac{1}{2}\left(2\sigma_D + \frac{\kappa\bar{\mu}}{2\kappa+2}\right)^2\right)(T-t)} \\
{}^1\beta_2 &= 2\alpha(t)^{\frac{\kappa+1}{2\kappa+2}}\, e^{\left(\sigma_D^2 - 2\mu_1 - \frac{(\kappa+1)\bar{\mu}^2}{2(2\kappa+2)} + \frac{1}{2}\left(2\sigma_D + \frac{(\kappa+1)\bar{\mu}}{2\kappa+2}\right)^2\right)(T-t)} \\
{}^1\beta_3 &= \alpha(t)^{\frac{\kappa+2}{2\kappa+2}}\, e^{\left(\sigma_D^2 - 2\mu_1 - \frac{(\kappa+2)\bar{\mu}^2}{2(2\kappa+2)} + \frac{1}{2}\left(2\sigma_D + \frac{(\kappa+2)\bar{\mu}}{2\kappa+2}\right)^2\right)(T-t)}
\end{aligned}
$$

Dividing this by Equation 23, and then comparing the coefficients with Equation 9, we get agent 1's Sharpe ratio as

$$
\omega_{1,t} = 2\sigma_D + \bar{\mu}\frac{\begin{array}{c} {}^1\beta_2 \cdot \left(\frac{\kappa+1}{2\kappa+2} + \frac{1}{2}\left(\frac{\kappa+1}{2\kappa+2}\right)^2\bar{\mu}\right) \\ + {}^1\beta_1 \cdot \left(\frac{\kappa}{2\kappa+2} + \frac{1}{2}\frac{\kappa(\kappa+2)}{(2\kappa+2)^2}\bar{\mu}\right) + {}^1\beta_3 \cdot \left(\frac{\kappa+2}{2\kappa+2} + \frac{1}{2}\frac{\kappa(\kappa+2)}{(2\kappa+2)^2}\bar{\mu}\right)\end{array}}{{}^1\beta_1 + {}^1\beta_2 + {}^1\beta_3}
$$

Consider now agent 2. Expanding the square terms we get

$$
\begin{aligned}
\pi_{2,T} &= \frac{\kappa+1}{\lambda_2}\frac{1}{D_T^2}\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+2}{2\kappa+2}}\left[1 + 2\alpha(T)^{\frac{1}{2\kappa+2}} + \alpha(T)^{\frac{2}{2\kappa+2}}\right] \\
&= \frac{\kappa+1}{\lambda_1}\frac{1}{D_T^2}\left[\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+2}{2\kappa+2}} + 2\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+1}{2\kappa+2}} + \left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa}{2\kappa+2}}\right]
\end{aligned}
$$

Using Equation 20 and Equation 19,

$$
\begin{aligned}
\pi_{2,T} &= \frac{\kappa+1}{\lambda_2}\frac{\left[\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+2}{2\kappa+2}} + 2\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+1}{2\kappa+2}} + \left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa}{2\kappa+2}}\right]}{D_T^2} \\
&= \frac{\kappa+1}{\lambda_2}\left[\frac{\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+2}{2\kappa+2}}}{D_T^2} + \frac{2\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+1}{2\kappa+2}}}{D_T^2} + \frac{\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa}{2\kappa+2}}}{D_T^2}\right] \\
&= \frac{\kappa+1}{\lambda_2}\left[\frac{\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+2}{2\kappa+2}} e^{\frac{(\kappa+2)\bar{\mu}^2}{2(2\kappa+2)}(T-t) + \frac{(\kappa+2)\bar{\mu}}{2\kappa+2}\left(B_{2,T}-B_{2,t}\right)}}{D_t^2 e^{\left(2\mu_2 - \sigma_D^2\right)(T-t) + 2\sigma_D\left(B_{2,T}-B_{2,t}\right)}} + \right.
\end{aligned}
$$

$$\frac{2\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+1}{2\kappa+2}} e^{\frac{(\kappa+1)\bar{\mu}^2}{2(2\kappa+2)}(T-t)+\frac{(\kappa+1)\bar{\mu}}{2\kappa+2}\left(B_{2,T}-B_{2,t}\right)}}{D_t^2 e^{\left(2\mu_2-\sigma_D^2\right)(T-t)+2\sigma_D\left(B_{2,T}-B_{2,t}\right)}}+$$

$$\left.\frac{\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa}{2\kappa+2}} e^{\frac{\kappa\bar{\mu}^2}{2(2\kappa+2)}(T-t)+\frac{\kappa\bar{\mu}}{2\kappa+2}\left(B_{2,T}-B_{2,t}\right)}}{D_t^2 e^{\left(2\mu_2-\sigma_D^2\right)(T-t)+2\sigma_D\left(B_{2,T}-B_{2,t}\right)}}\right]$$

$$= \frac{\kappa+1}{\lambda_2 D_t^2}\left[\left(\frac{1}{\alpha\left(T\right)}\right)^{\frac{\kappa+2}{2\kappa+2}} e^{\left(\sigma_D^2-2\mu_2+\frac{(\kappa+2)\bar{\mu}^2}{2(2\kappa+2)}\right)(T-t)+\left(\frac{(\kappa+2)\bar{\mu}}{2\kappa+2}-2\sigma_D\right)\left(B_{2,T}-B_{2,t}\right)}+\right.$$

$$2\left(\frac{1}{\alpha\left(T\right)}\right)^{\frac{\kappa+1}{2\kappa+2}} e^{\left(\sigma_D^2-2\mu_2+\frac{(\kappa+1)\bar{\mu}^2}{2(2\kappa+2)}\right)(T-t)+\left(\frac{(\kappa+1)\bar{\mu}}{2\kappa+2}-2\sigma_D\right)\left(B_{2,T}-B_{2,t}\right)}+$$

$$\left.\left(\frac{1}{\alpha\left(T\right)}\right)^{\frac{\kappa}{2\kappa+2}} e^{\left(\sigma_D^2-2\mu_2+\frac{\kappa\bar{\mu}^2}{2(2\kappa+2)}\right)(T-t)+\left(\frac{\kappa\bar{\mu}}{2\kappa+2}-2\sigma_D\right)\left(B_{2,T}-B_{2,t}\right)}\right]$$

Again, as before, to completely characterize the equilibrium, we need to find an explicit expression of $\pi_{2,t}$, which is calculated as $\pi_{2,t} = \mathbb{E}_t^2\left[\pi_{2,T}\right]$ from its martingale property :

$$\begin{aligned}\pi_{2,t} &= \mathbb{E}_t^2\left[\pi_{2,T}\right]\\ &= \frac{\kappa+1}{\lambda_2 D_t^2}\left[\left(\frac{1}{\alpha\left(T\right)}\right)^{\frac{\kappa+2}{2\kappa+2}} e^{\left(\sigma_D^2-2\mu_2+\frac{(\kappa+2)\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(\frac{(\kappa+2)\bar{\mu}}{2\kappa+2}-2\sigma_D\right)^2\right)(T-t)}+\right.\\ &\qquad 2\left(\frac{1}{\alpha\left(T\right)}\right)^{\frac{\kappa+1}{2\kappa+2}} e^{\left(\sigma_D^2-2\mu_2+\frac{(\kappa+1)\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(\frac{(\kappa+1)\bar{\mu}}{2\kappa+2}-2\sigma_D\right)^2\right)(T-t)}+\\ &\qquad \left.\left(\frac{1}{\alpha\left(T\right)}\right)^{\frac{\kappa}{2\kappa+2}} e^{\left(\sigma_D^2-2\mu_2+\frac{\kappa\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(\frac{\kappa\bar{\mu}}{2\kappa+2}-2\sigma_D\right)^2\right)(T-t)}\right]\end{aligned} \tag{24}$$

To find the Sharpe ratio, we use Itō's lemma on Equation 24 :

$$d\pi_{2,t} = \frac{\kappa+1}{\lambda_2}\left[e^{\left(\sigma_D^2-2\mu_2+\frac{(\kappa+2)\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(\frac{(\kappa+2)\bar{\mu}}{2\kappa+2}-2\sigma_D\right)^2\right)(T-t)}\cdot\right.$$

$$\left[\left(\frac{1}{\alpha\left(T\right)}\right)^{\frac{\kappa+2}{2\kappa+2}}\left(\frac{-2}{D_t^3}dD_t+\frac{1}{2}\frac{6}{D_t^2}\sigma_D^2 dt\right)\right.$$

$$\left.\left.+\frac{1}{D_t^2}\left(\frac{\kappa+2}{2\kappa+2}\left(\frac{1}{\alpha\left(T\right)}\right)^{\frac{-\kappa}{2\kappa+2}} d\left(\frac{1}{\alpha\left(t\right)}\right)-\frac{1}{2}\frac{\kappa\left(\kappa+2\right)}{\left(2\kappa+2\right)^2}\left(\frac{1}{\alpha\left(T\right)}\right)^{\frac{\kappa+2}{2\kappa+2}}\bar{\mu}^2 dB_{2,t}\right)\right]+\right.$$

$$2e^{\left(\sigma_D^2-2\mu_2+\frac{(\kappa+1)\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(\frac{(\kappa+1)\bar{\mu}}{2\kappa+2}-2\sigma_D\right)^2\right)(T-t)}\cdot$$

$$
\left[\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+1}{2\kappa+2}}\left(\frac{-2}{D_t^3}dD_t+\frac{1}{2}\frac{6}{D_t^2}\sigma_D^2dt\right)\right.
$$

$$
\left.+\frac{1}{D_t^2}\left(\frac{\kappa+1}{2\kappa+2}\left(\frac{1}{\alpha(T)}\right)^{\frac{-(\kappa+1)}{2\kappa+2}}d\left(\frac{1}{\alpha(T)}\right)-\frac{1}{2}\left(\frac{\kappa+1}{2\kappa+2}\right)^2\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+1}{2\kappa+2}}\bar{\mu}^2dB_{2,t}\right)\right]+
$$

$$
e^{\left(\sigma_D^2-2\mu_2+\frac{\kappa\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(\frac{\kappa\bar{\mu}}{2\kappa+2}-2\sigma_D\right)^2\right)(T-t)}\cdot
$$

$$
\left[\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa}{2\kappa+2}}\left(\frac{-2}{D_t^3}dD_t+\frac{1}{2}\frac{6}{D_t^2}\sigma_D^2dt\right)\right.
$$

$$
\left.\left.+\frac{1}{D_t^2}\left(\frac{\kappa}{2\kappa+2}\left(\frac{1}{\alpha(T)}\right)^{\frac{-(\kappa+2)}{2\kappa+2}}d\left(\frac{1}{\alpha(T)}\right)-\frac{1}{2}\frac{\kappa(\kappa+2)}{(2\kappa+2)^2}\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa}{2\kappa+2}}\bar{\mu}^2dB_{2,t}\right)\right]\right]
$$

$$
\implies d\pi_{2,t}=\frac{\kappa+1}{\lambda_2D_t^2}\left[e^{\left(\sigma_D^2-2\mu_2+\frac{(\kappa+2)\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(\frac{(\kappa+2)\bar{\mu}}{2\kappa+2}-2\sigma_D\right)^2\right)(T-t)}\cdot\right.
$$

$$
\left[\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+2}{2\kappa+2}}\left(\left(3\sigma_D^2-2\mu_{2,D}\right)dt-2\sigma_DdB_{2,t}\right)\right.
$$

$$
\left.+\left(\frac{\kappa+2}{2\kappa+2}\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+2}{2\kappa+2}}\bar{\mu}dB_{2,t}-\frac{1}{2}\frac{\kappa(\kappa+2)}{(2\kappa+2)^2}\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+2}{2\kappa+2}}\bar{\mu}^2dB_{2,t}\right)\right]+
$$

$$
2e^{\left(\sigma_D^2-2\mu_2+\frac{(\kappa+1)\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(\frac{(\kappa+1)\bar{\mu}}{2\kappa+2}-2\sigma_D\right)^2\right)(T-t)}\cdot
$$

$$
\left[\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+1}{2\kappa+2}}\left(\left(3\sigma_D^2-2\mu_{2,D}\right)dt-2\sigma_DdB_{2,t}\right)\right.
$$

$$
\left.+\left(\frac{\kappa+1}{2\kappa+2}\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+1}{2\kappa+2}}\bar{\mu}dB_{2,t}-\frac{1}{2}\left(\frac{\kappa+1}{2\kappa+2}\right)^2\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+1}{2\kappa+2}}\bar{\mu}^2dB_{2,t}\right)\right]+
$$

$$
e^{\left(\sigma_D^2-2\mu_2+\frac{\kappa\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(\frac{\kappa\bar{\mu}}{2\kappa+2}-2\sigma_D\right)^2\right)(T-t)}\cdot
$$

$$
\left[\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa}{2\kappa+2}}\left(\left(3\sigma_D^2-2\mu_{2,D}\right)dt-2\sigma_DdB_{2,t}\right)\right.
$$

$$
\left.\left.+\left(\frac{\kappa}{2\kappa+2}\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa}{2\kappa+2}}\bar{\mu}dB_{2,t}-\frac{1}{2}\frac{\kappa(\kappa+2)}{(2\kappa+2)^2}\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa}{2\kappa+2}}\bar{\mu}^2dB_{2,t}\right)\right]\right]
$$

$$
\implies d\pi_{2,t}=\frac{\kappa+1}{\lambda_2D_t^2}\left[\left(\frac{1}{\alpha(T)}\right)^{\frac{\kappa+2}{2\kappa+2}}e^{\left(\sigma_D^2-2\mu_2+\frac{(\kappa+2)\bar{\mu}^2}{2(2\kappa+2)}+\frac{1}{2}\left(\frac{(\kappa+2)\bar{\mu}}{2\kappa+2}-2\sigma_D\right)^2\right)(T-t)}\cdot\right.
$$

$$
\begin{aligned}
&\left[\left(3\sigma_D^2 - 2\mu_{2,D}\right) dt\right.\\
&\left.+\left(\frac{\kappa}{2\kappa+2}\bar{\mu} - \frac{1}{2}\frac{\kappa(\kappa+2)}{(2\kappa+2)^2}\bar{\mu}^2 - 2\sigma_D\right) dB_{2,t}\right] +\\
&2\left(\frac{1}{\alpha(t)}\right)^{\frac{\kappa+1}{2\kappa+2}} e^{\left(\sigma_D^2 - 2\mu_2 + \frac{(\kappa+1)\bar{\mu}^2}{2(2\kappa+2)} + \frac{1}{2}\left(\frac{(\kappa+1)\bar{\mu}}{2\kappa+2} - 2\sigma_D\right)^2\right)(T-t)}\cdot\\
&\left[\left(3\sigma_D^2 - 2\mu_{2,D}\right) dt\right.\\
&\left.+\left(\frac{\kappa+1}{2\kappa+2}\bar{\mu} - \frac{1}{2}\left(\frac{\kappa+1}{2\kappa+2}\right)^2\bar{\mu}^2 - 2\sigma_D\right) dB_{2,t}\right] +\\
&\left(\frac{1}{\alpha(t)}\right)^{\frac{\kappa+2}{2\kappa+2}} e^{\left(\sigma_D^2 - 2\mu_2 + \frac{\kappa\bar{\mu}^2}{2(2\kappa+2)} + \frac{1}{2}\left(\frac{\kappa\bar{\mu}}{2\kappa+2} - 2\sigma_D\right)^2\right)(T-t)}\cdot\\
&\left[\left(3\sigma_D^2 - 2\mu_{2,D}\right) dt\right.\\
&\left.+\left(\frac{\kappa}{2\kappa+2}\bar{\mu} - \frac{1}{2}\frac{\kappa(\kappa+2)}{(2\kappa+2)^2}\bar{\mu}^2 - 2\sigma_D\right) dB_{2,t}\right]\Bigg]
\end{aligned}
$$

Define

$$
\begin{aligned}
{}^2\beta_1 &= \left(\frac{1}{\alpha(t)}\right)^{\frac{\kappa}{2\kappa+2}} e^{\left(\sigma_D^2 - 2\mu_2 + \frac{(\kappa+2)\bar{\mu}^2}{2(2\kappa+2)} + \frac{1}{2}\left(\frac{(\kappa+2)\bar{\mu}}{2\kappa+2} - 2\sigma_D\right)^2\right)(T-t)}\\
{}^2\beta_2 &= 2\left(\frac{1}{\alpha(t)}\right)^{\frac{\kappa+1}{2\kappa+2}} e^{\left(\sigma_D^2 - 2\mu_2 + \frac{(\kappa+1)\bar{\mu}^2}{2(2\kappa+2)} + \frac{1}{2}\left(\frac{(\kappa+1)\bar{\mu}}{2\kappa+2} - 2\sigma_D\right)^2\right)(T-t)}\\
{}^2\beta_3 &= \left(\frac{1}{\alpha(t)}\right)^{\frac{\kappa+2}{2\kappa+2}} e^{\left(\sigma_D^2 - 2\mu_2 + \frac{\kappa\bar{\mu}^2}{2(2\kappa+2)} + \frac{1}{2}\left(\frac{\kappa\bar{\mu}}{2\kappa+2} - 2\sigma_D\right)^2\right)(T-t)}
\end{aligned}
$$

Dividing this by Equation 24, and then comparing the coefficients with Equation 9, we get agent 2's Sharpe ratio as

$$
\omega_{2,t} = 2\sigma_D - \bar{\mu}\frac{\begin{aligned}&{}^2\beta_2\cdot\left(\tfrac{\kappa+1}{2\kappa+2} - \tfrac{1}{2}\left(\tfrac{\kappa+1}{2\kappa+2}\right)^2\bar{\mu}\right)\\ &+{}^2\beta_1\cdot\left(\tfrac{\kappa}{2\kappa+2} - \tfrac{1}{2}\tfrac{\kappa(\kappa+2)}{(2\kappa+2)^2}\bar{\mu}\right) + {}^2\beta_3\cdot\left(\tfrac{\kappa+2}{2\kappa+2} - \tfrac{1}{2}\tfrac{\kappa(\kappa+2)}{(2\kappa+2)^2}\bar{\mu}\right)\end{aligned}}{{}^2\beta_1 + {}^2\beta_2 + {}^2\beta_3}
$$

□

As the previous two results demonstrate, the final wealth sharing rule, state price densities, and Sharpe ratios are affected by the relative wealth concerns.

Comparing these results to the benchamrk case, if α_T is sufficiently small, the final wealth of the optimistic group is less than what they had in the benchmark case and the pessimistic group gets more wealth than what they would have had in the benchmark case. The opposite holds when α_T is sufficiently small.

Observe that the final dividend sharing rule depends on $\alpha_T^{\frac{1}{\gamma}}$. Thus, the agents' share, the final wealth, varies with the realized state. For example, the optimistic group receives a larger portion of the final dividend than the pessimistic group when α_T is small.

Looking at the Sharpe ratios of the agents, it is easy to observe that the pessimistic agents shift some risk to the optimistic agents, because $\omega_{1,t} - \omega_{2,t} = \bar{\mu}$. This is a standard result in literature and it is shown that it still holds in this expanded set of models with relative wealth concerns. If the economy is good, i.e. the optimistic agents expect to receive a greater portion of the final dividend, then the Sharpe ratio is higher when there are relative wealth concerns than when there are no such concerns. Also, the Sharpe ratios increase with an increase in the relative wealth concern. A similar result holds when the economy is bad, the Sharpe ratio is lower in the presence of relative wealth concerns and they are a decreasing function of relative wealth concerns.

From an economic perspective, in a good state of the economy, the optimists have less wealth when they care about relative wealth than when they do not. Although the optimists still dominate the market, the stock is less overvalued and hence the market price of risk is higher in the presence of relative wealth concerns than in the case when there are none. Similarly, in the bad state of the economy, the Sharpe ratios are lower in the case when there are relative wealth concerns.

6 Conclusions

This work considers a model where agents have heterogeneous beliefs about the drift parameter of the aggregate dividend process and also have concerns over relative wealth. The aggregate dividend process is modelled as a geometric Brownian motion while the relative wealth concerns are modelled as an exogenously given convex function. This works provides an analytic proof of a special case that when the agents have logarithmic utility, the relative wealth concerns do not affect the equilibrium at all. The other focus of this work was on the perceived Sharpe ratios of the agents. A closed form expression for the Sharpe ratios for both agents is characterized in two cases, one when the agents have logarithmic utility and one when the agents' risk aversion parameter is 2. In both cases, the conclusion is that the Sharpe ratios are time-varying and the final wealth state dependent. This work also demonstrates that even in broader set of models where the agents have relative wealth concerns, the expected results that the Sharpe ratios are time-varying and final wealth state dependent still holds.

Avenues where this research could go further are numerous. It would be interesting to check whether the results are robust to a change in the structure of the dividend process. For example, if the dividend process were to be modelled as a mean-reverting process instead of a geometric Brownian motion, and the heterogeneity as a disagreement on the mean reversion parameter as in Chen and Kohn (2011), the effect on Sharpe ratios and the asset prices and the mechanism through which this effect occurs will be interesting to know. Also, the introduction of learning into this model, as implemented by Xiong and Yan (2010), might have some implications on the trading behaviour similar to what they achieved in the model without relative wealth concerns.

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7 Appendix

Some Mathematical Definitions

Definition 1. *A* ***filtration*** $\{\mathcal{F}_t\}_{t\geq 0}$ *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a collection of sub-sigma-algebras of* $\mathcal{F}$ *satisfying* $\mathcal{F}_s \subseteq \mathcal{F}_t$ *whenever* $s \leq t$. *The probability space taken with the filtration* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *is called a* ***filtered probability space****.*

Definition 2. *A stochastic process* $X = \{X_t : 0 \leq t \leq T\}$ *is said to be* ***adapted*** *to a filtration* $\{\mathcal{F}_t\}_{t\geq 0}$*, if* X_t *is* $\mathcal{F}_t$*-measurable for every* $t \geq 0$.

Definition 3. *Consider a complete probability space* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *where the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *satisfies the following properties*

1. $\mathcal{F}_0$ *contains all the* $\mathbb{P}$*-negligible sets (and so does every* $\mathcal{F}_t$*).*

2. *The filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *is right-continuous, i.e.,* $\mathcal{F}_t = \mathcal{F}_{t+} = \cap_{s>t}\mathcal{F}_s$.

Then, an adapted process $X = (X_t, \mathcal{F}_t)$ *is said to be a* ***martingale*** *with respect to the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *if it is integrable, and*

$$\mathbb{E}\left[X_t|\mathcal{F}_s\right] = X_s$$

almost surely, for all $0 \leq s \leq t$.

Definition 4. *A stochastic process* $X = \{X_t : 0 \leq t \leq T\}$ *given on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is called a* ***Brownian motion*** *if it satisfies the following properties :*

1. $X_0 = 0$ $(\mathbb{P} - a.s.)$

2. *X has stationary independent increments. That is, for* $s, t \in [0, \infty)$ *with* $s < t$*, the distribution of* $X_t - X_s$ *has the same distribution as* X_{t-s}.

3. *Increments $X_t - X_s$ have a Gaussian normal distribution with*

$$\mathbb{E}\left[X_t - X_s\right] = 0, \quad \mathbb{V}\left[X_t - X_s\right] = \sigma^2 |t - s|$$

4. *For almost all $\omega \in \Omega$, the functions $X_t = X_t(\omega)$ are continuous on $0 \leq t \leq T$.*

In the case $\sigma^2 = 1$, the process X is called the standard Brownian motion process.

Definition 5. *A stochastic process X_t is said to follow a **geometric Brownian motion** if it satisfies the following stochastic differential equation :*

$$\frac{dX_t}{X_t} = \mu dt + \sigma dB_t$$

where B_t is a Brownian motion and μ ('the drift') and σ ('the volatility') are constants.

Theorem (Itō's Lemma)**.** *Suppose you are given an Itō process*

$$dX_t = \mu_t dt + \sigma_t dB_t$$

where μ_t is the drift parameter, σ_t is the diffusion parameter, and B_t is a Brownian motion. Also, suppose that you have a twice continuously differentiable scalar function $f(X, t)$ of the real variables t and the Itō process X described above. Then, $f(X)$ is also an Itō process satisfying

$$df(X_t) = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)\, \sigma_t^2 dt$$

Example 8. *As a demonstration of Itō's Lemma, consider a geometric Brownian motion as above*

$$\frac{dX_t}{X_t} = \mu dt + \sigma dB_t$$

Applying Itō's lemma with $f(X) = \log X$ *we get*

$$\begin{aligned} d\log X_t &= f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)\, \sigma_t^2 dt \\ &= \frac{1}{X_t}\left(\mu X_t dt + \sigma X_t dB_t\right) - \frac{1}{\sigma^2} dt \\ &= \sigma dB + \left(\mu - \frac{\sigma^2}{2}\right) dt \end{aligned}$$