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D I P L O M A R B E I T

Asymptotic behavior of the Weyl m-function for one-dimensional Schrödinger operators with measure-valued potentials

unter der Anleitung von

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Abstract

We consider the Sturm-Liouville differential expression with measure-valued coefficients

$$\tau f = \frac{d}{d\rho}\left(-\frac{df}{d\zeta} + \int f\, d\chi\right)$$

and introduce the Weyl m-function for self-adjoint realizations. We further look at the special case of one-dimensional Schrödinger operators with measure-valued potentials. For this case we develop the asymptotic behavior

$$m(z) = -\sqrt{-z} - \int_0^x e^{-2\sqrt{-z}y}\, d\chi(y) + \mathrm{o}\left(\frac{1}{\sqrt{-z}}\right)$$

of the Weyl m-function for large z.

Introduction

The Sturm-Liouville theory, which essentially is the theory of ordinary differential equations of second order, was initiated in the 1830's by Charles François Sturm who soon collaborated with Joseph Liouville. This theory has been a field of interest ever since, with many applications most notably in physics. Especially the modern field of quantum mechanics broadly employs the Sturm-Liouville theory in combination with spectral theory. In the one-dimensional case the Sturm-Liouville eigenvalue problem coincides with the one-dimensional time-independent Schrödinger equation $H\psi(x) = E\psi(x)$. A popular introduction problem for physics students is a model with a potential of the form $V(x) = -\frac{\hbar^2}{m}\delta(x)$ were $\delta(x)$ describes a point charge. This leads to a jump condition for the first derivative of solutions of the Schrödinger equation which is not covered by the classical Sturm-Liouville theory.

In this thesis we look at a generalization of the classical Sturm-Liouville differential expression τ_1 which covers such point charge potentials as well as many other generalized potentials of interest. The derivatives are extended as Radon–Nikodým derivatives to the expression

$$\tau_1 f = \frac{d}{d\varrho}\left(-\frac{df}{d\zeta} + \int f\, d\chi\right),$$

with measures ϱ, ζ and χ not necessarily absolutely continuous with respect to the Lebesgue measure.
Our starting point is the paper of [1], published by Gerald Teschl and Jonathan Eckhardt in 2013. From this paper we explain the basic theory of the differential expression τ_1. This includes an existence and uniqueness result for the initial value problem, categorizing interval endpoint regularities and looking for self-adjoint realizations of τ_1. The methods we use to develop these results are very similar to the classical Sturm-Liouville theory methods with the difference that we have to use the tools of measure theory, i.e. the Radon–Nikodým theorem and integration by parts for Lebesgue-Stieltjes integrals. The main effort lies in finding self-adjoint realizations of τ_1 in a Hilbert space as those correspond to possibly multi-valued linear relations. For certain self-adjoint realizations S of τ_1 we introduce the Weyl m-function, which is a Nevanlinna function containing the spectral information of S.

Equipped with this apparatus of our generalized Sturm-Liouville expression we examine the one-dimensional Schrödinger operator with measure-valued potential. This potential is described by the measure χ resulting in the differential expression

$$\tau_2 f = \left(-f' + \int f\, d\chi\right)'.$$

In this thesis we will develop a new improved estimate for the asymptotic behavior of the Weyl m-function corresponding to τ_2. We do this by first describing a fundamental system of the differential equation $(\tau_2 - z)u = 0$ as solutions of integral equations. These integral equations allow us to extract an asymptotic estimate for their solutions. Then we find an asymptotic estimate for the Weyl m-function with the help of the Weyl circles, as done in the classical case by F. Atkinson in [3]. Combining these two estimates leads to our final estimate for the asymptotic behavior of the Weyl function given as

$$m(z) = -\sqrt{-z} - \int_0^x e^{-2\sqrt{-z}y}\, d\chi(y) + \mathrm{o}\left(\frac{1}{\sqrt{-z}}\right)$$

as $\mathrm{Im}(z) \to +\infty$.
This result is valuable for inverse Sturm-Liouville problems which examines the possibility to

get information about the self-adjoint realization of a differential expression τ (i.e. its potential) by only possessing its spectral information (i.e. its Weyl m-function).

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Contents

Abstract . . . 2
Introduction . . . 3

1 Preliminaries and notations **6**
1.1 Measures . . . 6
1.2 Absolutely continuous functions . . . 8
1.3 Asymptotic behavior of functions . . . 9
1.4 Nevanlinna functions . . . 10

2 Linear relations **11**
2.1 The basics . . . 11
2.2 Self-adjoint linear relations . . . 12
2.3 Self-adjoint extensions of linear relations . . . 14

3 Linear measure differential equations **16**
3.1 Initial value problems . . . 16
3.2 Properties of solutions . . . 22

4 Sturm-Liouville equations with measure-valued coefficients **25**
4.1 Consistency with the classical Sturm-Liouville problem . . . 25
4.2 Generalized cases . . . 26
4.3 Solutions of initial value problems . . . 27
4.4 Regularity of τ . . . 33
4.5 The Sturm-Liouville operator on $L^2((a,b);\varrho)$ . . . 36
4.6 Weyl's alternative and the deficiency index of $\mathrm{T}_{\min}$ . . . 50
4.7 Self-adjoint realizations of τ . . . 54
4.8 The Weyl m-function . . . 55

5 Asymptotics of the Weyl m-function **58**
5.1 The Weyl m-function for S . . . 58
5.2 The one-dimensional Schrödinger operator with measure-valued potential . . . 60

References **74**

1 Preliminaries and notations

We gather some basic definitions and theorems of analysis that will be used throughout this thesis. It might fill some gaps of the knowledge that is needed for this thesis and serve as a standardized reference point for implications later on.

1.1 Measures

We recapitulate some basic measure theory and define a set function which is a complex measure for all compact subsets.

Definition 1.1. Let Ω be a set and $\mathfrak{A}$ be a σ-algebra on Ω. A function $\mu : \mathfrak{A} \to \mathbb{C}$ is called *complex measure* if for all sequences (E_n) with pairwise disjoint $E_n \in \mathfrak{A}$, we have

$$\mu\left(\sum_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Starting from such a complex measure we can construct a positive finite measure.

Definition 1.2. Let $(\Omega, \mathfrak{A})$ be a set equipped with a σ-algebra and let $\mu : \mathfrak{A} \to \mathbb{C}$ be a complex measure on Ω. Then we call

$$|\mu|(A) := \sup\left\{\sum_{j=1}^{\infty} |\mu(A_j)| : A_j \in \mathfrak{A}, \dot{\bigcup}_{k=1}^{\infty} A_k = A\right\}$$

the *variation* of μ.

The variation of a complex measure μ is always a positive finite measure and we have the inequality $|\mu|(A) \leq |\mu(A)|$ for $A \in \mathfrak{A}$.
We want measures which are finite on all compact sets, thus we need the set on which our measure is operating equipped with a topology.

Definition 1.3. Let $(X, \mathcal{T})$ be a topological space. We call the σ-algebra created by $\mathcal{T}$ the *Borel-σ-algebra on X* and denote $\mathcal{B}(X) := A_\sigma(\mathcal{T})$. If $D \subseteq X$ and $\mathcal{T}_D$ the subspace topology we denote $\mathcal{B}(D) := A_\sigma(\mathcal{T}_D)$.

We have the equality $B(D) = A_\sigma(\mathcal{T}) \cap D$

Definition 1.4. Let $(X, \mathcal{T})$ be a topological Hausdorff space. A measure $\mu : \mathcal{B}(X) \to [0, +\infty]$ is called a *positive Borel measure* if $\mu(K) < +\infty$ for all compact $K \subseteq X$.

The property of $(X, \mathcal{T})$ to be Hausdorff is needed in order for compact sets to be closed. Hence $K \in \mathcal{B}(X)$ for all $K \subseteq X$, K compact.
We can extend this definition to complex measures and as those are finite-valued everywhere we only need to specify the underlying σ-algebra.

Definition 1.5. Let $(X, \mathcal{T})$ be a topological space and μ a complex measure on X. We call μ a *complex Borel measure* if it is defined on the σ-algebra $\mathcal{B}(X)$, i.e. $\mu : \mathcal{B}(X) \to \mathbb{C}$.

If $(X, \mathcal{T})$ is a topological Hausdorff space and μ is a complex Borel measure then the mapping $|\mu| : \mathcal{B}(X) \to [0, +\infty)$ is a finite measure, and therefore $|\mu|(K) < \infty$ for all complex $K \subseteq X$, hence $|\mu|$ is a positive Borel measure.

Definition 1.6. Let $(X, \mathcal{T})$ be a local compact Hausdorff space. We say

$$\mu : \bigcup_{\substack{K \subseteq X \\ K \text{compact}}} \mathcal{B}(K) \to \mathbb{C}$$

is a *locally finite complex Borel measure* if for all $K \subseteq X$, K compact, the restricted measure $\mu|_{\mathcal{B}(K)}$ is a complex measure.

Again from the Hausdorff property follows that compact K satisfy $K \in \mathcal{B}(X)$ and thus $\mathcal{B}(K_1) \subseteq \mathcal{B}(K_2)$ for $K_1 \subseteq K_2$ which means $(\mu \mid_{\mathcal{B}(K_2)}) \mid_{\mathcal{B}(K_1)} = \mu \mid_{\mathcal{B}(K_1)}$. Hence the definition above makes sense.
The locally compactness assures that for every point $x \in X$ we find a compact set K_x with $x \in K_x$ which we need for the definition of the support below.
Note that if μ is a locally finite complex Borel measure and K is compact then the measure $|\mu\big|_{\mathcal{B}(K)}|$ is a finite positive Borel measure on K.

Definition 1.7. Let $(X, \mathcal{T})$ be a topological space and $\mu : \mathcal{B}(X) \to \mathbb{C}$ be a complex measure on X. The *support of* μ is defined as

$$\operatorname{supp}(\mu) := \{x \in X \mid \forall N_x \in \mathcal{T} \text{ with } x \in N_x : |\mu|(N_x) > 0\}.$$

If $(X, \mathcal{T})$ is a locally compact Hausdorff space and μ a locally finite complex Borel measure on X, then the *support of* μ is defined as

$$\operatorname{supp}(\mu) := \bigcup_{\substack{K \subseteq X \\ K \text{ compact}}} \operatorname{supp} \mu|_{\mathcal{B}(K)}.$$

Definition 1.8. Let μ, ν be complex measures on $(X, \mathfrak{A})$. We call μ *absolutely continuous with respect to* ν, denoted as $\mu \ll \nu$, if for all $A \in \mathfrak{A}$ with $|\nu|(A) = 0$ follows that $|\mu|(A) = 0$.

Now we will focus our interest on intervals on the real line.

Definition 1.9. Let $[\alpha, \beta]$ be a real finite interval and let μ be a complex Borel measure on $[\alpha, \beta]$. We call a function $f : [\alpha, \beta] \to \mathbb{C}$ satisfying

$$f(c) - f(d) = \mu([c, d)) \quad \text{for all } c, d \in [\alpha, \beta],\ c < d$$

a *distribution function* of μ.

Note that in this definition we have the half-open interval $[c, d)$ with the right endpoint excluded in contrast to most literature were the left endpoint is chosen to be excluded. Our definition leads to the distribution function being left-continuous. Every measure has a distribution function which is unique up to an additive constant.

1.2 Absolutely continuous functions

Combining the theorem of Radon–Nikodým with the distribution function of an absolutely continuous measure leads to a generalization of the fundamental theorem of calculus.

Definition 1.10. Let $[\alpha, \beta]$ be a finite interval and μ a complex Borel measure on $[\alpha, \beta]$. We call a function $f : [\alpha, \beta] \to \mathbb{C}$ *absolutely continuous with respect to* μ if f is a distribution function to some complex Borel measure ν on $[\alpha, \beta]$ satisfying $\nu \ll \mu$. We denote the set of absolutely continuous functions with respect to μ as $AC([\alpha, \beta]; \mu)$.

From the theorem of Radon–Nikodým follows that $f \in AC([\alpha, \beta]; \mu)$ if and only if f can be written as

$$f(x) = f(c) + \int_c^x h \, d\mu, \quad x \in [\alpha, \beta],$$

with $h \in L^1([\alpha, \beta]; \mu)$ and some $c \in [\alpha, \beta]$. The integral is defined as

$$\int_c^x h \, d\mu = \begin{cases} \int_{[c,x)} h \, d\mu & c < x, \\ 0 & c = x, \\ -\int_{[x,c)} h \, d\mu & c > x, \end{cases}$$

corresponding to the left-continuously defined distribution functions. With this notation as the basis, we denote $\int_{c+}^{x} := \int_{(c,x)}$ and $\int_{c+}^{x+} := \int_{(c,x]}$. The function h is the Radon–Nikodým derivative of f with respect to μ and we also write $\frac{df}{d\mu} := h$. It is uniquely defined in $L^1([\alpha, \beta]; \mu)$. From the integral representation of $f \in AC([\alpha, \beta]; \mu)$ follows, that the right-hand limit exists for all $x \in [\alpha, \beta)$. Indeed we have

$$f(x+) := \lim_{\varepsilon \searrow 0} f(x + \varepsilon) = f(x) + \lim_{\varepsilon \searrow 0} \int_x^{x+\varepsilon} h \, d\mu.$$

Now by definition of our integral range this means we integrate over

$$\lim_{\varepsilon \searrow 0} [x, x + \varepsilon) = \bigcap_{\varepsilon \geq 0} [x, x + \varepsilon) = \{x\}$$

and get

$$f(x+) = f(x) + h(x)\mu(\{x\}). \tag{1.1}$$

As f is left-continuous we also see from this identity that f can only be discontinuous at a point x if $\mu(\{x\}) \neq 0$. $|\mu|$ is a finite measure so μ can at most have countable many points with mass, so $f \in AC([\alpha, \beta]; \mu)$ can at most have countable many points of discontinuity in the form of jumps.
Functions $f \in AC([\alpha, \beta]; \mu)$ are of bounded variation, that means if $\mathcal{P}$ is the set of all partitions P of $[\alpha, \beta]$ (with $\alpha = t_1 < t_2, \ldots < t_{n(P)} = \beta$) we have

$$V_\alpha^\beta(f) := \sup_{P \in \mathcal{P}} \sum_{i=1}^{n(P)} |f(t_{i+1}) - f(t_i)| < \infty.$$

As any two points in $[\alpha, \beta]$ are part of some partition, we have for all $y \in [\alpha, \beta]$

$$|f(y)| \leq |f(c)| + |f(y) - f(c)| \leq |f(c)| + V_\alpha^\beta(f) < \infty.$$

Thus any $f \in AC([\alpha,\beta];\mu)$ is bounded in $[\alpha,\beta]$.

We want to extend the idea of absolutely continuous functions to intervals of infinite length.

Definition 1.11. Let (a,b) be an arbitrary interval with $-\infty \le a < b \le +\infty$ and let μ be a locally finite complex Borel measure. A function $f : (a,b) \to \mathbb{C}$ is called *locally absolutely continuous with respect to* μ if for all finite intervals $[\alpha,\beta] \subseteq (a,b)$ with $\alpha < \beta$ the restricted function $f|_{[\alpha,\beta]}$ is absolutely continuous with respect to $\mu|_{\mathcal{B}([\alpha,\beta])}$. In this case we write $f \in AC_{\mathrm{loc}}((a,b);\mu)$.

From the definition and the above stated results follows that f is locally absolutely continuous with respect to μ if and only if we can write f in the form

$$f(x) = f(c) + \int_c^x \frac{df}{d\mu}\, d\mu, \quad x \in (a,b) \tag{1.2}$$

with $\frac{df}{d\mu} \in L^1_{\mathrm{loc}}((a,b);\mu)$ and some $c \in (a,b)$.
For locally absolutely continuous functions we have a variation of the integration by parts formula which will be one of the most prominent tools used in this thesis. For proof and surrounding theory we refer to [17], Section 16.

Lemma 1.12. *Let (a,b) be an arbitrary interval with $-\infty \le a < b \le +\infty$ and let μ and ν be locally finite complex Borel measures on (a,b) with distribution functions F and G respectively. Then we have the identity*

$$\int_\alpha^\beta F(x)\, d\nu(x) = [FG]_\alpha^\beta - \int_\alpha^\beta G(x+)\, d\mu(x), \quad \alpha,\beta \in (a,b) \tag{1.3}$$

where $[FG]_\alpha^\beta := F(\beta)G(\beta) - F(\alpha)G(\alpha)$.

Note that if the measures have the explicit form

$$\mu(B) := \int_B f\, d\lambda \quad \text{and} \quad \nu(B) := \int_B g\, d\lambda$$

for $B \in \mathcal{B}((a,b))$ with $f,g \in L^1_{\mathrm{loc}}((a,b);\lambda)$ we get the classical integration by parts formula. This follows as ν has no point mass since λ has none. Thus G is continuous, i.e. $G(x+) = G(x)$ for all $x \in (a,b)$ and we get

$$\int_\alpha^\beta F(x)g(x)\, d\lambda(x) = [FG]_\alpha^\beta - \int_\beta^\alpha G(x)f(x)\, d\lambda(x) \quad \alpha,\beta \in (a,b).$$

1.3 Asymptotic behavior of functions

Definition 1.13. Let $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$ be two functions. We write

$$f(x) = O(g(x)) \quad \text{as } x \to +\infty$$

if and only if there exist two positive constants x_0 and M such that

$$|f(x)| \le M|g(x)| \quad \text{for all } x > x_0.$$

We write

$$f(x) = o(g(x)) \quad \text{as } x \to +\infty$$

if and only if for every constant $\varepsilon > 0$ there exists some $x_0(\varepsilon) > 0$ such that

$$|f(x)| \leq \varepsilon |g(x)| \quad \text{for all } x > x_0(\varepsilon).$$

We see immediately from the definition that from $f(x) = o(g(x))$ as $x \to +\infty$ it always follows that $f(x) = O(g(x))$ as $x \to +\infty$. If g is a function for which there exists some constant $R > 0$ such that $g(x) \neq 0$ for all $x > R$ we have the equivalences

$$f(x) = O(g(x)) \text{ as } x \to +\infty \quad \Longleftrightarrow \quad \limsup_{x \to +\infty} \frac{f(x)}{g(x)} < +\infty$$

and

$$f(x) = o(g(x)) \text{ as } x \to +\infty \quad \Longleftrightarrow \quad \lim_{x \to +\infty} \frac{f(x)}{g(x)} = 0$$

We can formulate these equivalences in a more practical way as

$$f(x) = O(g(x)) \text{ as } x \to +\infty \quad \Longleftrightarrow \quad f(x) = g(x)D(x)$$

and

$$f(x) = o(g(x)) \text{ as } x \to +\infty \quad \Longleftrightarrow \quad f(x) = g(x)\varepsilon(x)$$

for some function $D : \mathbb{R} \to \mathbb{C}$ for which there exist positive constants x_0 and C such that $|D(x)| \leq C$ for $x > x_0$ and some function $\varepsilon : \mathbb{R} \to \mathbb{C}$ which satisfies $\lim_{x \to +\infty} \varepsilon(x) = 0$.

1.4 Nevanlinna functions

Definition 1.14. We call a function $f : \mathbb{C}^+ \to \mathbb{C}^+$ which is analytic a *Nevanlinna function.*

Theorem 1.15. *Let $f : \mathbb{C}^+ \to \mathbb{C}$ be an analytic function. Then f is a Nevanlinna function if and only if it has the representation*

$$f(z) = c_1 + c_2 z + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\mu(\lambda). \tag{1.4}$$

with $c_1, c_2 \in \mathbb{R}$, $c_2 \geq 0$ and μ a positive Borel measure satisfying

$$\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1 + \lambda^2} < \infty.$$

For proof we refer to [14], Chapter 2, Theorem 2.

Proposition 1.16. *A Nevanlinna function f with representation (1.4) satisfies*

$$\lim_{y \to +\infty} \frac{f(iy)}{iy} = c_2.$$

For proof see [14], Chapter 2, Theorem 3.

2 Linear relations

In this section we will give a short introduction to the theory of linear relations similarly as in [1], Appendix B and state some results that we will need for later use. For a good introduction to linear relations we refer to the manuscript [16] and for further theory we refer to the books [12], [11].

The theory of linear relations is a generalization of the theory of linear operators. One motivation for this theory is the example of operators T for which the element T^* or T^{-1} is multi-valued and thus not an operator. In contrast to linear operators inverting some linear relation T is always possible and convenient and the adjoint T^* always exists. Specifically we will see that the realization of the Sturm-Liouville differential expression in a Hilbert space which we will examine in Section 4 can have a multi-valued part.

2.1 The basics

We start with two linear spaces X and Y over $\mathbb{C}$ and want to look at the cartesian product of those two spaces denoted as $X \times Y$. If X and Y are topological vector spaces we view $X \times Y$ with respect to the product topology. We write the elements as $(x, y) \in X \times Y$ with $x \in X$ and $y \in Y$. We introduce linear relations as subspaces of product spaces of this kind.

Definition 2.1. T is called a *linear relation of* X *into* Y if it is a linear subspace of $X \times Y$. We denote this with $T \in \mathrm{LR}(X, Y)$. We denote $\overline{T}$ as the closure of T in $X \times Y$. A linear relation T is called a *closed linear relation* if it satisfies $\overline{T} = T$.

Linear relations are indeed a generalization of linear operators. To see this we look at a linear subspace $D \subseteq X$ and a linear operator $T : D \to Y$. We can identify the linear operator T with its graph, given as $\mathrm{graph}(T) = \{(x, Tx) \mid x \in D\} \subseteq X \times Y$. Because of the linearity properties of T the graph of T is a linear subspace of $X \times Y$. We see that every linear operator can be identified with a linear relation. As we know that a closed operator is an operator for which its graph is closed in $X \times Y$, a closed operator corresponds to a closed linear relation.

Motivated by the theory of linear operators we make the following definitions, for which the first three are identical with the classical definitions if T is a linear relation resembling the graph of a linear operator.

Definition 2.2. If $T \in \mathrm{LR}(X, Y)$ we define the domain, range, kernel and multi-valued part of T as

$$\begin{aligned} \mathrm{dom}\, T &:= \{x \in X \mid \exists y \in Y : (x, y) \in T\}, \\ \mathrm{ran}\, T &:= \{y \in Y \mid \exists x \in X : (x, y) \in T\}, \\ \ker T &:= \{x \in X \mid (x, 0) \in T\}, \\ \mathrm{mul}\, T &:= \{y \in Y \mid (0, y) \in T\}. \end{aligned}$$

If T is the graph of an operator then $\mathrm{mul}\, T = \{0\}$ has to be true in order for the operator to be well-defined. For $T \in \mathrm{LR}(X, Y)$ with $\mathrm{mul}\, T = \{0\}$ and $(x, y_1), (x, y_2) \in T$ we get $(0, y_1 - y_2) \in T$ and $y_1 = y_2$ follows. We see that T is the graph of an operator if and only if $\mathrm{mul}\, T = \{0\}$.

Now we want to introduce operations between linear relations again motivated by operator theory.

Definition 2.3. For $T, S \in \mathrm{LR}(X, Y)$ and $\lambda \in \mathbb{C}$ we define

$$T + S := \{(x, y) \in X \times Y \mid \exists y_1, y_2 \in Y : (x, y_1) \in S, (x, y_2) \in T, y_1 + y_2 = y\}$$

and

$$\lambda T := \{(x, y) \in X \times Y \mid \exists y_0 \in \mathbb{C} : (x, y_0) \in T, y = \lambda y_0\}.$$

If $T \in \mathrm{LR}(X, Y)$ and $S \in \mathrm{LR}(Y, Z)$ for some linear space Z we define

$$ST := \{(x, z) \in X \times Z \mid \exists y \in Y : (x, y) \in T, (y, z) \in S\}$$

and

$$T^{-1} := \{(y, x) \in Y \times X \mid (x, y) \in T\}.$$

It is easy to check that the above defined subspaces are linear relations in their overlying product spaces. If W is another linear space and $R \in \mathrm{LR}(W, X)$ we have

$$S(TR) = (ST)R \quad \text{and} \quad (ST)^{-1} = T^{-1}S^{-1}.$$

We also have the easily understood identities

$$\mathrm{dom}(T^{-1}) = \mathrm{ran}\, T \quad \text{and} \quad \mathrm{ran}(T^{-1}) = \mathrm{dom}\, T.$$

From the definition of the multi-valued part we see that T^{-1} is the graph of an operator if and only if $\ker T = \{0\}$.

2.2 Self-adjoint linear relations

Now we want to develop a theory for self-adjoint linear relations similarly to the theory of self-adjoint operators. As mentioned before the adjoint of a linear relation always exists even if the domain of the linear relation is not densely defined.

From now on assume X and Y are Hilbert spaces with inner product $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$.

Definition 2.4. If T is a linear relation of X into Y we call

$$T^* := \{(y, x) \in Y \times X \mid \forall (u, v) \in T : \langle x, u \rangle_X = \langle y, v \rangle_Y\}$$

the *adjoint of* T.

The adjoint of a linear relation is a closed linear relation and similar to the adjoint of an operator we have

$$T^{**} = \overline{T}, \quad \ker T^* = (\mathrm{ran}\, T)^\perp \quad \text{and} \quad \mathrm{mul}\, T^* = (\mathrm{dom}\, T)^\perp. \tag{2.1}$$

If $S \in \mathrm{LR}(X, Y)$ is another linear relation we have

$$T \subseteq S \implies S^* \subseteq T^*.$$

The proofs for this and similar properties can be found for example in [16], page 61f or in [11], page 15f.

Spectral properties

From now on let T be a linear relation of X into X. We will only distinguish the terms operator and graph of an operator if not clear in the context. $\mathrm{B}(X,Y)$ will denote the set of bounded linear operators of X into Y and abbriviate $\mathrm{B}(X) := \mathrm{B}(X,X)$.

The definition of spectrum, subsets of the spectrum and resolvent are the same as in the operator theory. Additionally we have the points of regular type.

Definition 2.5. Let T be a closed linear relation. Then we call the set

$$\rho(T) := \{z \in \mathbb{C} \mid (T-z)^{-1} \in \mathrm{B}(X)\}$$

the *resolvent set of* T. We denote $R_z(T) := (T-z)^{-1}$ and call the mapping

$$\begin{aligned} \rho(T) &\to \mathrm{B}(X), \\ z &\mapsto R_z(T) \end{aligned}$$

the resolvent of T. *The spectrum of* T is defined as the complement of $\rho(T)$ in $\mathbb{C}$ and we denote

$$\sigma(T) := \mathbb{C} \backslash \rho(T).$$

The set

$$\mathrm{r}(T) := \{z \in \mathbb{C} \mid (T-z)^{-1} \in \mathrm{B}(\operatorname{ran}(T-z), X)\}$$

is called the *points of regular type of* T.

The inclusion $\rho(T) \subseteq \mathrm{r}(T)$ holds for every closed linear relation T and we have $\mathrm{r}(T) = \mathrm{r}(\overline{T})$.

Theorem 2.6. *Let T be a linear relation on X. Then for every connected $\Omega \subseteq \mathrm{r}(T)$ we have*

$$\dim \operatorname{ran}(T-\lambda)^{\perp} = \textit{const.}$$

for $\lambda \in \Omega$.

For proof see for example [16], Corollary 3.2.20.

Definition 2.7. A linear relation T is said to be *symmetric* provided that $T \subseteq T^*$. A linear relation S is said to be *self-adjoint* provided $S = S^*$ holds.

If T is a symmetric linear relation we have $\mathbb{C}\backslash\mathbb{R} \subseteq \mathrm{r}(T)$. If S is a self-adjoint linear relation it is closed, the spectrum is real and from (2.1) one sees that

$$\operatorname{mul} S = (\operatorname{dom} S)^{\perp} \quad \text{and} \quad \ker S = (\operatorname{ran} S)^{\perp}.$$

In particular we see that if S is densely defined $\operatorname{mul} S = \{0\}$ holds and S is a linear operator. So a self-adjoint linear relation is an operator if and only if it is densely defined. Furthermore for $\mathcal{D} := \overline{\operatorname{dom} S}$ the linear relation

$$S_{\mathcal{D}} := S \cap (\mathcal{D} \times \mathcal{D})$$

is a self-adjoint operator in the Hilbert space $(\mathcal{D}, \langle \cdot, \cdot \rangle|_{\mathcal{D}})$. The following results will show that S and $S_{\mathcal{D}}$ have many spectral properties in common.

Lemma 2.8. *Let S be a self-adjoint linear relation, $S_{\mathcal{D}}$ defined as above and P the orthogonal projection onto $\mathcal{D}$. Then we have*

$$\sigma(S) = \sigma(S_{\mathcal{D}})$$

and

$$R_z f = (S_D - z)^{-1} P f, \quad f \in X, z \in \rho(S).$$

Moreover the eigenvalues and the corresponding eigenspaces are identical.

Proof.
For all $z \in \mathbb{C}$ we have the equality

$$\begin{aligned}(\operatorname{ran}(S - z)) \cap \mathcal{D} &= \{y \in X \mid \exists x \in X : (x, y) \in S - z\} \cap \mathcal{D} \\ &= \{y \in \mathcal{D} \mid \exists x \in \mathcal{D} : (x, y) \in S - z\} \\ &= \operatorname{ran}(S_{\mathcal{D}} - z)\end{aligned}$$

and the orthogonal equivalent

$$\ker(S - z) = \ker(S_{\mathcal{D}} - z).$$

Thus S and $S_{\mathcal{D}}$ have the same spectrum as well as the same point spectrum and eigenspaces. Let $z \in \rho(S)$, $f \in X$ and set $g := (S - z)^{-1} f$, then $(g, f) \in S - z$ and therefore $g \in \mathcal{D}$. If $f \in \mathcal{D}$ we have $(g, f) \in S_{\mathcal{D}} - z$ which means $(S_{\mathcal{D}} - z)^{-1} f = g$. If $f \in \mathcal{D}^{\perp} = \operatorname{dom} S^{\perp} = \operatorname{dom}(S - z)^{\perp} = \operatorname{mul}(S - z)$ then $g = 0$ follows.

□

2.3 Self-adjoint extensions of linear relations

As in the theory of operators on a Hilbert space our goal is to find self-adjoint linear relations as extensions of symmetric linear relations in X.

Definition 2.9. S is called an *(n dimensional) extension* to a linear relation T if $T \subseteq S$ (with $S/T = n$). S is said to be a *self-adjoint extension* if it is an extension of T and self-adjoint. For a closed symmetric linear relation the linear relations

$$\mathrm{N}_{\pm}(T) = \{(x, y) \in T^* \mid y = \pm ix\} \subseteq T^*$$

are called *deficiency spaces of T*. The *deficiency indices of T* are defined as

$$\mathrm{n}_{\pm}(T) := \dim \mathrm{N}_{\pm}(T) \in [0, \infty].$$

Note first that because of $(x, \pm ix) \in T^* \implies x \in \ker(T^* \mp i)$ the signs of the deficiency indices are consistent with the classical definition and second that $N_{\pm}(T)$ are operators with

$$\operatorname{dom}(N_{\pm}(T)) = \ker(T^* \mp i) = \operatorname{ran}(T \pm i)^{\perp}.$$

Furthermore one has an analog result of the first von Neumann formula.

Theorem 2.10. *Let T be a closed symmetric linear relation in $X \times X$. Then we have*

$$T^* = T \oplus N_+(T) \oplus N_-(T)$$

where the sums are orthogonal with respect to the usual inner product

$$\langle (f_1, f_2), (g_1, g_2) \rangle_{X \times X} = \langle f_1, g_1 \rangle_X + \langle f_2, g_2 \rangle_X, \quad (f_1, f_2), (g_1, g_2) \in X \times X.$$

For proof we refer to [13] Theorem 6.1. An immediate consequence of the first von Neumann formula is the following corollary.

Corollary 2.11. *If T is a closed symmetric linear relation then T is self-adjoint if and only if $n_+(T) = n_-(T) = 0$.*

As in the operator case there exists a self-adjoint extension of some closed symmetric linear relation T if the deficiency subspaces of T are of the same dimension.

Theorem 2.12. *The closed symmetric linear relation T has a self-adjoint extension if and only if $n_+(T) = n_-(T)$. In this case all self-adjoint extensions S of T are of the form*

$$S = T \oplus (I - V)N_+(T) \tag{2.2}$$

where $V : N_+(T) \to N_-(T)$ is an isometry. Conversely, for each such isometry V the linear relation S given by (2.2) is self-adjoint.

For proof we refer to [13] Theorem 6.2.

Corollary 2.13. *Let T be a closed symmetric linear relation. If $n_-(T) = n_+(T) \in n \in \mathbb{N}$, then the self-adjoint extensions of T are precisely the n-dimensional symmetric extensions of T.*

Proof.
Let S be a self-adjoint extension of T. By Theorem 2.12 we have $S = T \oplus (I - V)N_+(T)$ with V an isometry from $N_+(T)$ onto $N_-(T)$. Since $\dim(I - V)N_+(T) = \dim N_-(T) = n$ the linear relation S is an n-dimensional extension of T.

Conversely, assume that S is an n-dimensional symmetric extension of T, i.e. $S = T \dot{+} N$ for some n-dimensional symmetric subspace N. We show $\dim N_\pm(S) = 0$ and use Corollary 2.11. The linear relation $N \pm i$ is given as the set

$$N \pm i = \{(f, g \pm if) \in X \times X \mid \exists g \in X : (f, g) \in N\}$$

and therefore

$$\operatorname{ran}(N \pm i) = \{(g \pm if) \in X \mid \exists g : (f, g) \in N\}.$$

Since $\pm i \in \mathrm{r}(N)$ we have

$$\{0\} = \operatorname{mul}(N \pm i)^{-1} = \ker(N \pm i)$$

so the mapping

$$\begin{aligned} N &\to \operatorname{ran}(N \pm i) \\ (f, g) &\mapsto (g \pm if) \end{aligned}$$

is bijective and we get

$$\dim \operatorname{ran}(N \pm i) = \dim(N) = n.$$

From $i \in \mathrm{r}(T)$ follows that $\operatorname{ran}(T \pm i)$ is closed and hence

$$n = \dim \operatorname{ran}(T \pm i)^{\perp} = \dim X / \operatorname{ran}(T \pm i).$$

We get

$$\operatorname{ran}(S \pm i) = \operatorname{ran}(T \pm i) \dot{+} \operatorname{ran}(N \pm i) = X.$$

Hence we have $\dim N_{\pm}(S) = 0$ and therefore $S = S^*$ by Corollary 2.11.

□

3 Linear measure differential equations

In this section we introduce the theory for linear measure differential equations as done in the paper [1], Appendix A.

The methods we use to develop the basic theory for linear measure differential equations for locally finite Borel measures is very similar to the classical case where the measure coincides with the Lebesgue measure. As our solutions can now have countable infinitely many jumps, we have to introduce an additional condition to our equation parameters in order to get unique solutions.

3.1 Initial value problems

Let (a, b) be an arbitrary interval in $\mathbb{R}$ with interval endpoints $-\infty \leq a < b \leq +\infty$ and let ω be a positive Borel measure on (a, b). We look at a matrix-valued function $M : (a, b) \to \mathbb{C}^{n \times n}$ with measurable components and a vector-valued function $F : (a, b) \to \mathbb{C}^n$ with measurable components. Additionally we assume the functions M and F to be locally integrable, that means

$$\int_K \|M\| \, d\omega < \infty \quad \text{and} \quad \int_K \|F\| \, d\omega < \infty$$

for all compact sets $K \subseteq (a, b)$. Here $\| \cdot \|$ denotes the norm on $\mathbb{C}^n$ as well as the corresponding operator norm on $\mathbb{C}^{n \times n}$.

Definition 3.1. For $c \in (a, b)$ and $Y_c \in \mathbb{C}^n$ some function $Y : (a, b) \to \mathbb{C}^n$ is called a *solution of the initial value problem*

$$\frac{dY}{d\omega} = MY + F, \quad Y(c) = Y_c, \tag{3.1}$$

if the components Y_i of Y satisfy $Y_i \in AC_{\mathrm{loc}}((a, b); \omega)$ and their Radon–Nikodým derivatives satisfy the differential equation in (3.1) almost everywhere with respect to ω and Y satisfies the given initial value at c.

Lemma 3.2. *A function $Y : (a, b) \to \mathbb{C}^n$ is a solution of the initial value problem (3.1) if and only if it is a solution of the integral equation*

$$Y(x) = Y_c + \int_c^x MY + F \, d\omega, \quad x \in (a, b). \tag{3.2}$$

Proof.
This follows immediately from the calculus for absolute continuous functions (see equation (1.2)). If Y is a solution of the initial value problem (3.1) we can write

$$Y(x) = Y(c) + \int_c^x \frac{dY}{d\omega}\, d\omega = Y_c + \int_c^x MY + F\, d\omega.$$

Conversely a function satisfying (3.1) solves the initial value problem by definition of the integral and the calculus for absolutely continuous functions.

□

Remark 3.3.

- If Y is a solution of the initial value problem (3.1), it's components are absolutely continuous and we can write

  $$Y(x_0+) = \lim_{x\to x_0+} Y(x) = Y(x_0) + \lim_{x\to x_0+} \int_{x_0}^x MY + F\, d\omega.$$

  We get (remember (1.1))

  $$Y(x_0+) = Y(x_0) + (M(x_0)Y(x_0) + F(x_0))\omega(\{x_0\}). \tag{3.3}$$

- Similarly we can take the right-hand limit of the lower integral boundary and get

  $$\begin{aligned} Y(x) &= \lim_{c\to x_0+} Y(c) + \int_c^x MY + F\, d\omega \\ &= Y(x_0+) + \lim_{c\to x_0+} \int_{[c,x)} MY + F\, d\omega = Y(x_0+) + \int_{(x_0,x)} MY + F\, d\omega \end{aligned}$$

  Using the notation of our integral range from the preliminaries this means

  $$Y(x) = Y(x_0+) + \int_{x_0+}^x MY + F\, d\omega. \tag{3.4}$$

In order to get an explicit estimate for the growth of our solutions Y we will introduce a variation of the Gronwall lemma known from the theory of ODEs. For the proof of this lemma we cite a variant of the substitution rule for Lebesgue-Stieltjes integrals from [8].

Lemma 3.4. *Let $H : [c,d] \to \mathbb{R}$ be increasing and let g be a bounded measurable function on the range of H, then we have*

$$\int_c^d g(H(x))\, dH(x) \le \int_{H(c)}^{H(d)} g(y)\, d\lambda(y). \tag{3.5}$$

Lemma 3.5. *Let $c \in (a,b)$ and $v \in L^1_{loc}((a,b);\omega)$ be real-valued, such that*

$$0 \le v(x) \le K + \int_c^x v\, d\omega, \quad x \in [c,b)$$

for some constant $K \geq 0$. Then v can be estimated by

$$v(x) \leq Ke^{\int_c^x d\omega}, \quad x \in [c, b).$$

Similarly for

$$0 \leq v(x) \leq K + \int_{x+}^{c+} v \, d\omega, \quad x \in (a, c]$$

follows the estimate

$$v(x) \leq Ke^{\int_{x+}^{c+} d\omega}.$$

Proof.
Applying Lemma 3.4 by setting $g(x) := x^n$ and the increasing function $H(x) := \int_c^x d\omega$, which means $dH = 1d\omega$, we get the inequality

$$\int_c^x H^n \, d\omega \leq \int_{H(c)}^{H(x)} y^n \, d\lambda(y) = \left[\frac{y^{n+1}}{n+1}\right]_{y=H(c)}^{H(x)} = \frac{H^{n+1}(x)}{n+1} \tag{3.6}$$

for $n \in \mathbb{N}_0$. We will need this inequality to prove that

$$v(x) \leq K \sum_{k=0}^{n} \frac{H(x)^k}{k!} + \frac{H(x)^n}{n!} \int_c^x v \, d\omega$$

for each $n \in \mathbb{N}_0$.
For $n = 0$ this is the assumption of the lemma. Otherwise (with help of inequality (3.6) to get to third and fourth line) we inductively get

$$\begin{aligned}
v(x) \leq & K + \int_c^x v \, d\omega \\
\leq & K + \int_c^x \left(K \sum_{k=0}^{n} \frac{H(t)^k}{k!} + \frac{H(t)^n}{n!} \int_c^t v \, d\omega \right) d\omega(t) \\
\leq & K \left(1 + \sum_{k=0}^{n} \int_c^x \frac{H(t)^k}{k!} \, d\omega(t) \right) + \frac{H(x)^{n+1}}{(n+1)!} \int_c^x v \, d\omega \\
\leq & K \left(1 + \sum_{k=0}^{n} \frac{H(x)^{k+1}}{(k+1)!} \right) + \frac{H(x)^{n+1}}{(n+1)!} \int_c^x v \, d\omega \\
= & K \sum_{k=0}^{n+1} \frac{H(x)^k}{k!} + \frac{H(x)^{n+1}}{(n+1)!} \int_c^x v \, d\omega.
\end{aligned}$$

Taking the limit $n \to \infty$ for fixed $x \in [c, b)$ leads to

$$v(x) \leq Ke^{H(x)} + \lim_{n \to \infty} \frac{H(x)^{n+1}}{(n+1)!} \int_c^x v \, d\omega = Ke^{\int_c^x v \, d\omega},$$

where $\lim_{n\to\infty} \frac{H(x)^{n+1}}{(n+1)!} = 0$ since the partial sums of this sequence converge. Similarly one can start from the estimate for $x \in (a, c]$ and accordingly adjust the proof from above to get the desired result.

□

Now we want to examine the existence and uniqueness of solutions of the initial value problem. In the classical case where $\omega = \lambda$ has no point mass, we know from the theory of ordinary differential equations, that a unique solution is guaranteed for every initial value problem without further conditions. As our measure can have point-mass and therefore a solution of the initial value problem Y can have points of discontinuity we need to add one assumption in order to get uniqueness and existence of Y for the whole interval. The proof will show that this assumption is only needed for the left-hand side of the initial value c of the interval (a, b).

Theorem 3.6. *The initial value problem (3.1) has a unique solution for each $c \in (a, b)$ and $Y_c \in \mathbb{C}^n$ if and only if the matrix*

$$I + \omega(\{x\})M(x) \tag{3.7}$$

is invertible for all $x \in (a, b)$.

Proof.
First we assume that (3.1) has a unique solution for $c \in (a, b), Y_c \in \mathbb{C}^n$. If the matrix (3.7) was not invertible for some $x_0 \in (a, b)$, its columns would be linearly dependent and so we would have two vectors $v_1, v_2 \in \mathbb{C}^n \backslash \{0\}$, $v_1 \neq v_2$ with

$$(I + \omega(\{x_0\}))\, M(x_0)v_1 = (I + \omega(\{x_0\}))M(x_0)v_2.$$

Now by assumption we can find solutions of our differential equation Y_1 and Y_2 going through these vectors v_1 and v_2 respectively, i.e.

$$Y_1(x_0) = v_1 \quad \text{and} \quad Y_2(x_0) = v_2.$$

From $v_1 \neq v_2$ follows that $Y_1 \neq Y_2$. By (3.3) we can write

$$\begin{aligned} Y(x_0+) &= Y(x_0) + (M(x_0)Y(x_0) + F(x_0))\omega(\{x_0\}) \\ &= (I + M(x_0)\omega(\{x_0\}))Y(x_0) + F(x_0)\omega(\{x_0\}). \end{aligned}$$

It follows that

$$\begin{aligned} Y_1(x_0+) - F(x_0)\omega(\{x_0\}) &= (I + M(x_0)\omega(\{x_0\}))Y_1(x_0) \\ &= (I + M(x_0)\omega(\{x_0\}))Y_2(x_0) \\ &= Y_2(x_0+) - F(x_0)\omega(\{x_0\}) \end{aligned}$$

and hence $Y_1(x_0+) = Y_2(x_0+)$. Now using (3.4) for our two solutions Y_1 and Y_2 we get the estimate

$$\begin{aligned} \|Y_1(x) - Y_2(x)\| &= \left\| Y_1(x_0+) + \int_{x_0+}^{x} MY_1 + F\, d\omega - Y_2(x_0+) - \int_{x_0+}^{x} MY_2 + F\, d\omega \right\| \\ &\leq \int_{x_0+}^{x} \|MY_1 - MY_2\|\, d\omega \leq \int_{x_0+}^{x} \|M\|\|Y_1 - Y_2\|\, d\omega. \end{aligned}$$

Applying the Gronwall lemma with $d\omega$ replaced by $\|M\|d\omega$ and with $K = 0$ we get

$$\|Y_1(x) - Y_2(x)\| \leq Ke^{\int_{x_0+}^{x} \|M\|\, d\omega} = 0$$

and hence $Y_1(x) = Y_2(x)$ for $x \in (x_0, b)$. So both functions are solutions for the same initial value problem at some $c \in (x_0, b)$, but since $Y_1 \neq Y_2$ this is a contradiction to our assumption.

For the other implication we assume the matrix (3.7) is invertible for all $x \in (a,b)$ and let $c, \alpha, \beta \in (a,b)$ with $\alpha < c < \beta$.

Uniqueness:
To prove the uniqueness we assume Y is a solution of the homogenous system with $Y_c = 0$. We get

$$\|Y(x)\| \leq \int_c^x \|M(t)\|\|Y(t)\| \, d\omega(t), \quad x \in [c, \beta).$$

The Gronwall lemma implies that $Y(x) = 0$ for all $x \in [c, \beta)$. To the left-hand side of point c we have

$$Y(x) = -\int_x^c MY \, d\omega = -\int_{x+}^c MY \, d\omega - M(x)Y(x)\omega(\{x\})$$

and thus

$$(I + M(x)\omega(\{x\}))Y(x) = -\int_{x+}^c MY \, d\omega.$$

As the matrix on the left side is invertible by assumption, we can write the solution as

$$Y(x) = -(I + M(x)\omega(\{x\}))^{-1} \int_{x+}^c MY \, d\omega$$

for $x \in (\alpha, c)$. Adding the point c to the integration range after performing the triangle inequality leads to

$$\|Y(x)\| \leq \|(I + M(x)\omega(\{x\}))^{-1}\| \int_{x+}^{c+} \|M\|\|Y\| \, d\omega, \quad x \in (\alpha, c). \tag{3.8}$$

Since $\|M\|$ is locally integrable, we have $\|M(x)\omega(\{x\})\| \leq \frac{1}{2}$ for all but finitely many $x \in [\alpha, c]$. For those x we have

$$\begin{aligned} \|(I + M(x)\omega(\{x\}))^{-1}\| &= \left\| \sum_{n=0}^{\infty} (-M(x)\omega(\{x\}))^n \right\| \\ &\leq \sum_{n=0}^{\infty} \|M(x)\|^n \omega(\{x\})^n \leq \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - \frac{1}{2}} = 2 \end{aligned}$$

and we get an estimate $\|(I + M(x)\omega(x))^{-1}\| \leq K$ for all $x \in [\alpha, c)$ with some $K \geq 2$ as only finitely many x can lead to values bigger than 2. Now we can perform the Gronwall lemma to (3.8) and arrive at $Y(x) = 0$ for all $x \in (\alpha, c)$ and the uniqueness is proven for all $x \in (\alpha, \beta)$. This is true for all $\alpha, \beta \in (a,b)$ with $\alpha < c < \beta$. So for a point $x \in (a,b)$ we find an interval (α, β) including x with the properties from before. It follows that we have uniqueness on the whole interval (a,b).

Existence
To prove the existence of a solution we construct the solution through successive approximation. We define

$$Y_0(x) := Y_c + \int_c^x F \, d\omega, \quad x \in [c, \beta)$$

and inductively for each $n \in \mathbb{N}$ through

$$Y_n(x) := \int_c^x MY_{n-1} \, d\omega, \quad x \in [c, \beta).$$

We will show that these functions are bounded by

$$\|Y_n(x)\| \leq \sup_{t\in[c,x)} \|Y_0(t)\| \frac{1}{n!} \left(\int_c^x \|M\| \, d\omega \right)^n, \quad x \in [c, \beta).$$

For $n = 0$ this is true. For $n > 0$ we use (3.6) with $\|M\|d\omega$ as the measure to calculate inductively

$$\begin{aligned}
\|Y_n(x)\| &\leq \int_c^x \|M(t)\| \|Y_{n-1}(t)\| \, d\omega(t) \\
&\leq \sup_{t\in[c,x)} \|Y_0(t)\| \frac{1}{(n-1)!} \int_c^x \left(\int_c^t \|M\| \, d\omega \right)^{n-1} \|M(t)\| \, d\omega(t) \\
&\leq \sup_{t\in[c,x)} \|Y_0(t)\| \frac{1}{(n-1)!} \frac{1}{n} \left(\int_c^x \|M\| \, d\omega \right)^n \\
&= \sup_{t\in[c,x)} \|Y_0(t)\| \frac{1}{n!} \left(\int_c^x \|M\| \, d\omega \right)^n.
\end{aligned}$$

Hence the sum $Y(x) := \sum_{n=0}^{\infty} Y_n(x)$ converges absolutely and uniformly for $x \in [c, \beta)$ and we have

$$Y(x) = Y_0(x) + \sum_{n=1}^{\infty} \int_c^x M Y_{n-1} \, d\omega = Y_c + \int_c^x MY + F \, d\omega$$

for all $x \in [c, \beta)$. Now we will extend the solution to the left of c. Since $\|M\|$ is locally integrable $\omega(\{x\})\|M(x)\| \geq \frac{1}{2}$ is only true for finitely many points. We can divide the interval (α, c) in subintervals with those endpoints and then further divide those subintervals in finitely many so that we get points x_k such that $\alpha = x_{-N} < x_{-N+1} < \ldots < x_0 = c$ with

$$\int_{(x_k, x_{k+1})} \|M\| \, d\omega < \frac{1}{2}.$$

Now we take k such that $-N < k < 0$ and assume Y is a solution on $[x_k, \beta)$. We show that Y can be extended to a solution on $[x_{k-1}, \beta)$. We define

$$Z_0(x) := Y(x_k) + \int_{x_k}^x F \, d\omega, \quad x \in (x_{k-1}, x_k]$$

and inductively

$$Z_n(x) := \int_{x_k}^x M Z_{n-1} \, d\omega, \quad x \in (x_{k-1}, x_k]$$

for $n > 0$. Again one can show inductively that for each $n \in \mathbb{N}$ and $x \in (x_{k-1}, x_k]$ these functions are bounded by

$$\|Z_n(x)\| \leq \left(\|Y(x_k)\| + \int_{[x_{k-1}, x_k)} \|F\| \, d\omega \right) \frac{1}{2^n}. \tag{3.9}$$

Hence we may extend Y onto (x_{k-1}, x_k) by

$$Y(x) := \sum_{n=0}^{\infty} Z_n(x), \quad x \in (x_{k-1}, x_k),$$

where the sum converges uniformly and absolutely. For this reason we can show that Y is a solution of (3.1) in the same way as above for $x \in (x_{k-1}, \beta)$. If we set

$$Y(x_{k-1}) := (I + \omega(\{x_{k-1}\})M(x_{k-1}))^{-1}(Y(x_{k-1}+) - \omega(\{x_{k-1}\})F(x_{k-1}),$$

then one can show that Y satisfies the integral equation (3.2) for all $x \in [x_{k-1}, \beta)$. After finitely many steps we arrive at a solution Y satisfying the integral equation (3.2) for all $x \in (\alpha, \beta)$. If Y_c, F, M are real, all the quantities in the proof are real-valued and we get a real-valued solution. □

3.2 Properties of solutions

We assume M to be dependent on an additional complex variable z and look into the behavior of solutions of our initial value problem with respect to this z. Furthermore we will look into the behavior of the solutions if we assume additional regularity of our equation parameters close to the endpoints.

Corollary 3.7. *Let $M_1, M_2 : (a, b) \to \mathbb{C}^{n\times n}$ be measurable functions on (a, b) such that $\|M_1(\cdot)\|, \|M_2(\cdot)\| \in L^1_{\text{loc}}((a, b); \omega)$. Assume for $z \in \mathbb{C}$*

$$(I + (M_1(x) + zM_2(x))\omega(\{x\}))$$

is invertible for all $x \in (a, b)$. If for $z \in \mathbb{C}$ some function $Y(z, \cdot)$ is the unique solution of the initial value problem

$$\frac{dY}{d\omega} = (M_1 + zM_2)Y + F, \quad Y(c) = Y_c$$

for $c \in (a, b), Y_c \in \mathbb{C}^n$. Then for each $x \in (a, b)$ the function $z \mapsto Y(z, x)$ is analytic.

Proof.
We show that the construction of the solution from Theorem 3.6 yields analytic solutions. For $c, \alpha, \beta \in (a, b)$ with $\alpha < c < \beta$ and for $x \in [c, \beta)$ the functions $z \mapsto Y_n(z, x)$ are polynomial in z for $n \in \mathbb{N}_0$. Furthermore the sum $\sum Y_n(z, \cdot)$ can be estimated with

$$\begin{aligned}\|\sum_{n=0}^{\infty} Y_n(z, x)\| &\le \sup_{t\in[c,x)} \|Y_0(t)\| \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_c^x \|M_1 + zM_2\|\, d\omega\right)^n \\ &\le \sup_{t\in[c,x)} \|Y_0(t)\| e^{\int_c^x \|M_1\|+|z|\|M_2\|\, d\omega}.\end{aligned}$$

It follows that the sum converges locally uniformly in z. This proves that for $x \in [c, \beta)$ the function $z \mapsto Y(z, x)$ is analytic.
For the left side of c we fix $R > 0$. Then there are points x_k as in the proof of Theorem 3.6, such that

$$\int_{(x_k,x_{k+1})} \|M_1 + zM_2\|\, d\omega < \frac{1}{2}, \quad -N \le k \le 0, |z| < R.$$

It is sufficient to prove the following implication: if $z \mapsto Y(z, x_k)$ is analytic $\implies$ the function $z \mapsto Y(z, x)$ is analytic for $x \in [x_{k-1}, x_k)$. With the notation of the proof of Theorem 3.6 it follows that for each $x \in (x_{k-1}, x_k)$ the function $z \mapsto Z_n(z, x)$ is analytic and bounded for z with $|z| < R$. From the bound (3.9) follows that for $x \in (x_{k-1}, x_k)$ the sum $\sum_{n=0}^{\infty} Z_n(z, x)$

converges uniformly for z with $|z| < R$. Hence for those x the function $z \mapsto Y(z,x)$ is analytic. Furthermore from Theorem 3.6 we have the identity

$$Y(z, x_{k-1}) = (I + (M_1(x_{k-1}) + zM_2(x_{k-1}))\omega(\{x_{k-1}\}))^{-1}(Y(z, x_{k-1}+) - F(x_{k-1})\omega(\{x_{k-1}\}))$$

which is analytic in z since $Y(z, x_{k-1}+)$ is the limit of analytic and locally bounded functions. □

Corollary 3.8. *Assume the matrix $(I + M(x)\omega(\{x\}))$ is invertible for all $x \in (a,b)$. Then the initial value problem for $c \in (a,b), Y_c \in \mathbb{C}^n$, stated as*

$$\frac{dY}{d\omega} = MY + F, \quad Y(c+) = Y_c \tag{3.10}$$

has a unique solution. If M, F and Y_c are real, then the solution Y is real.

Proof.
Every solution Y of our differential equation is of the form

$$Y(x) = v + \int_c^x MY + Fd\omega$$

for some $c \in (a,b)$, $v \in \mathbb{C}^n$. From $Y(c+) = Y_c$ follows that

$$v = (I + M(c)\omega(\{c\}))^{-1}(Y_c - F(c)\omega(\{c\})).$$

Because of that, the initial value problem (3.10) stated in this Corollary can be written as an initial value problem with initial value at c

$$\frac{dY}{d\omega} = MY + F, \quad Y(c) = (I + M(c)\omega(\{c\}))^{-1}(Y_c - F(c)\omega(\{c\}))$$

which has a unique solution by Theorem 3.6.

□

Remark 3.9.
Starting from the initial value problem

$$\frac{dY}{d\omega} = MY + F, \quad Y(c) = (I + M(c)\omega(\{c\}))^{-1}(Y_c - F(c)\omega(\{c\})) \tag{3.11}$$

shows that (3.10) and the initial value problem (3.11) are equivalent.

Finally we will show, that we can extend every solution of the initial value problem to the endpoints, in case that M and F are integrable on the whole interval (a,b).

Theorem 3.10. *Assume $\|M(\cdot)\|$ and $\|F(\cdot)\|$ are integrable with respect to ω over (a,c) for some $c \in (a,b)$ and Y is a solution of the initial value problem (3.1). Then the limit*

$$Y(a) := \lim_{x \to a+} Y(x)$$

exists and is finite. A similar result holds for the endpoint b.

Proof.
By assumption there is some $c \in (a,b)$ such that

$$\int_{a+}^{c} \|M\| \, d\omega \leq \frac{1}{2}.$$

Boundedness:
We first prove that $\|Y\|$ is bounded near a. If it was not, there is a monotone sequence $(x_n) \in (a,c), x_n \searrow a$, such that $\|Y(x_n)\| \geq \|Y(x)\|, x \in [x_n, c]$. Since Y is a solution of the integral equation, we get

$$\begin{aligned}\|Y(x_n)\| &\leq \|Y(c)\| + \int_{x_n}^{c} \|M\|\|Y\| \, d\omega + \int_{x_n}^{c} \|F\| \, d\omega \\ &\leq \|Y(c)\| + \|Y(x_n)\| \int_{x_n}^{c} \|M\| \, d\omega + \int_{a+}^{c} \|F\| \, d\omega \\ &\leq \|Y(c)\| + \int_{a+}^{c} \|F\| \, d\omega + \frac{1}{2}\|Y(x_n)\|.\end{aligned}$$

Hence $\|Y(x_n)\| \leq 2\|Y(c)\| + 2\int_{a+}^{c} \|F\| \, d\omega$ which is a contradiction to the assumption that $\|Y(x_n)\|$ is unbounded. It follows that $\|Y(\cdot)\|$ is bounded near a by some constant K.

Cauchy-sequence:
Now it follows that

$$\begin{aligned}\|Y(x) - Y(y)\| &= \left\| \int_{y}^{x} MY + F \, d\omega \right\| \\ &\leq K \int_{y}^{x} \|M\| \, d\omega + \int_{y}^{x} \|F\| \, d\omega\end{aligned}$$

for all $x, y \in (a,c), x < y$, which shows that for all sequences $x_n \to a$ the set $Y(x_n)$ is a Cauchy-sequence.

□

Remark 3.11.

- Under the regularity assumptions of Theorem 3.10 one can show (with almost the same proof as in Theorem 3.6) that there is always a unique solution to the initial value problem

$$\frac{dY}{d\omega} = MY + F, \quad Y(a) = Y_a$$

without additional assumptions.

If $\|M(\cdot)\|$ and $\|F(\cdot)\|$ are integrable near b, then furthermore one has to assume, that the matrix $(I + M(x)\omega(\{x\}))$ is invertible for all $x \in (a,b)$ in order to get a unique solution to the initial value problem

$$\frac{dY}{d\omega} = MY + F, \quad Y(b) = Y_b,$$

in a similar way is in the proof of Theorem 3.6.

- Under the assumptions of Corollary 3.7 we see that $Y(z, x+)$ is analytic for each $x \in (a,b)$. Since $Y(z,x)$ is locally uniformly bounded in x and z this follows from

$$Y(z, x+) = \lim_{\xi \searrow x}, Y(z, \xi) \quad z \in \mathbb{C}.$$

Furthermore the proof of Corollary 3.8 reveals that, if for each $z \in \mathbb{C}$ the function $Y(z,\cdot)$ is the solution of the initial value problem

$$\frac{dY}{d\omega} = (M_1 + zM_2)Y + F, \quad Y(c+) = Y_c,$$

then the function $z \mapsto Y(z,x)$ as well as $z \mapsto Y(z,x+)$ are analytic for each $x \in (a,b)$.

4 Sturm-Liouville equations with measure-valued coefficients

Let (a,b) be an arbitrary interval in $\mathbb{R}$ with interval endpoints $-\infty \leq a < b \leq +\infty$ and let ϱ, ζ, χ be locally finite complex Borel measures on (a,b). Furthermore we assume that $\operatorname{supp}(\zeta) = (a,b)$. We want to look at a linear differential expression which is informally given as

$$\tau f = \frac{d}{d\varrho}\left(-\frac{df}{d\zeta} + \int f\, d\chi\right).$$

To define the maximal domain for which τ makes sense we fix some $c \in (a,b)$ and get

$$\mathcal{D}_\tau := \left\{ f \in AC_{\mathrm{loc}}((a,b);\zeta) \;\middle|\; \left(x \mapsto -\frac{df}{d\zeta}(x) + \int_c^x f\, d\chi\right) \in AC_{\mathrm{loc}}((a,b);\varrho) \right\}$$

We will see below, that the definition of τf is independent of the chosen constant c for $\mathcal{D}_\tau$. Since the expression

$$f_1 := \left(x \mapsto -\frac{df}{d\zeta}(x) + \int_c^x f\, d\chi\right), \quad x \in (a,b)$$

is an equivalence class of functions equal almost everywhere with respect to ζ, the notation $f_1 \in AC_{\mathrm{loc}}((a,b);\varrho)$ has to be understood in the sense, that there exists some representative of f_1, which lies in $AC_{\mathrm{loc}}((a,b);\varrho)$. From the assumption $\operatorname{supp}(\zeta) = (a,b)$ follows, that this representative is unique. We then set $\tau f \in L^1_{\mathrm{loc}}((a,b);\varrho)$ to be the Radon–Nikodým derivative of this function f_1 with respect to ϱ. The definition of τf is independent of the $c \in (a,b)$ set in $\mathcal{D}_\tau$, since the corresponding functions f_1 only differ by an additive constant.

Definition 4.1. We denote the Radon–Nikodým derivative with respect to ζ of some function $f \in \mathcal{D}_\tau$ by

$$f^{[1]} := \frac{df}{d\zeta} \in L^1_{\mathrm{loc}}((a,b);\zeta).$$

The function $f^{[1]}$ is called the *first quasi-derivative of* f.

4.1 Consistency with the classical Sturm-Liouville problem

We show that our differential expression is consistent with the classical Sturm-Liouville problem stated by the expression

$$\tau_{\mathrm{classic}} f(x) := \frac{1}{r(x)}\left(-(p(x)f'(x))' + f(x)q(x)\right), \quad x \in (a,b),$$

with the assumptions $\frac{1}{p}, q, r \in L^1_{\mathrm{loc}}((a,b);\lambda)$ and $p > 0$, $r > 0$. The maximal domain of functions for which this expression makes sense is

$$\mathcal{D}_{\mathrm{classic}} := \{f \in AC_{\mathrm{loc}}((a,b);\lambda) \mid pf' \in AC_{\mathrm{loc}}((a,b);\lambda)\}.$$

We set the measures of τ as

$$\varrho(B) := \int_B r\,d\lambda, \quad \zeta(B) := \int_B \tfrac{1}{p}\,d\lambda, \quad \chi(B) := \int_B q\,d\lambda, \quad B \in \mathcal{B}((a,b)).$$

We see that $\varrho, \zeta, \chi \ll \lambda$ and it is also true that $\lambda \ll \varrho, \zeta$. Indeed if we take $K \subset (a,b)$, compact and look at $B \in \mathcal{B}(K)$ for which $\zeta(B) = 0$, we get

$$0 = \zeta(B) = \int_B \frac{1}{p(x)}\,d\lambda(x) \geq \inf_K \frac{1}{p}\lambda(B) \geq 0.$$

Hence $\lambda(B) = 0$ and the same argument works for ϱ. Because of this we can write

$$\frac{d\lambda}{d\varrho}(x) = \left(\frac{d\varrho}{d\lambda}\right)^{-1}(x) = \frac{1}{r(x)}, \qquad \frac{d\lambda}{d\zeta}(x) = \left(\frac{d\zeta}{d\lambda}\right)^{-1}(x) = p(x).$$

Now we can write

$$f_1(x) = -\frac{df}{d\lambda}(x)\frac{d\lambda}{d\zeta}(x) + \int_c^x fq\,d\lambda = -f'(x)p(x) + \int_c^x fq\,d\lambda$$

and we see that $f_1 \in AC_{\mathrm{loc}}((a,b);\varrho) \iff pf' \in AC_{\mathrm{loc}}((a,b);\lambda)$. From $\zeta \ll \lambda \ll \zeta$ follows $f \in AC_{\mathrm{loc}}((a,b);\zeta) \iff f \in AC_{\mathrm{loc}}((a,b);\lambda)$. This means $\mathcal{D}_\tau = \mathcal{D}_{\mathrm{classic}}$ and for $f \in \mathcal{D}_\tau$ we arrive at

$$\begin{aligned}(\tau f)(x) &= \frac{d}{d\varrho}\left(t \mapsto -\frac{df}{d\zeta}(t) + \int_c^t f\,d\chi\right)(x)\\ &= \frac{1}{r(x)}(-(p(x)f'(x))' + f(x)q(x))\\ &= \tau_{\mathrm{classic}} f(x).\end{aligned}$$

4.2 Generalized cases

Now we want to look at some generalized cases by modifying the classical case from above through adding Dirac measures centered at a point $c \in (a,b)$ denoted by δ_c.

- We add a point mass α to ζ from the classical case

  $$\varrho(B) := \int_B r\,d\lambda, \quad \zeta(B) := \int_B \tfrac{1}{p}\,d\lambda + \alpha\delta_c, \quad \chi(B) := \int_B q\,d\lambda, \quad B \in \mathcal{B}((a,b)).$$

  For $f \in \mathcal{D}_\tau$ follows $f \in AC_{\mathrm{loc}}((a,b);\zeta)$ and this means we can write

  $$\begin{aligned}f(x) &= f(c) + \int_c^x \frac{df}{d\zeta}\,d\zeta = f(c) + \int_c^x f^{[1]}\frac{1}{p}\,d\lambda + \alpha\int_c^x f^{[1]}\,d\delta_c\\ &= f(c) + \int_c^x f^{[1]}\frac{1}{p}\,d\lambda + \begin{cases}\alpha f^{[1]}(c) & x > c,\\ 0 & x \leq c.\end{cases}\end{aligned}$$

  It follows, that we get the jump condition

  $$f(c+) - f(c) = \alpha f^{[1]}(c).$$

- Similarly we can add a point mass α to χ

$$\varrho(B) := \int_B r\, d\lambda, \quad \zeta(B) := \int_B \tfrac{1}{p}\, d\lambda, \quad \chi(B) := \int_B q\, d\lambda + \alpha\delta_c, \quad B \in \mathcal{B}((a,b)).$$

In this case f_1 has the form

$$\begin{aligned} f_1(x) &= -\frac{df}{d\zeta}(x) + \int_c^x fq\, d\lambda + \alpha \int_c^x f\, d\delta_c(x) \\ &= -\frac{df}{d\zeta}(x) + \int_c^x fq\, d\lambda + \begin{cases} \alpha f(c) & x > c, \\ 0 & x \le c. \end{cases} \end{aligned}$$

Since we need f for which $f_1 \in AC_{\mathrm{loc}}((a,b);\varrho)$, there needs to be a continuous representative of f_1, which leads to the jump condition

$$\alpha f(c) = f^{[1]}(c+) - f^{[1]}(c).$$

4.3 Solutions of initial value problems

For the results for initial value problems of our Sturm-Liouville equation we can rewrite this one-dimensional equation of second order into a two-dimensional differential equation of first order and use Theorem 3.6 with $\omega = |\varrho| + |\zeta| + |\chi|$. First we define, what we consider a solution of our Sturm-Liouville initial value problem similar to Section 1.

Definition 4.2. For $g \in L^1_{\mathrm{loc}}((a,b);\varrho)$, $c \in (a,b)$ and $z, d_1, d_2 \in \mathbb{C}$ some function $f : (a,b) \to \mathbb{C}$ is called a solution of the initial value problem

$$(\tau - z)f = g \quad \text{with} \quad f(c) = d_1,\ f^{[1]}(c) = d_2 \tag{4.1}$$

if $f \in \mathcal{D}_\tau$, the differential equation is satisfied almost everywhere with respect to ϱ and the given initial values at c are satisfied.

Theorem 4.3. *For every function $g \in L^1_{\mathrm{loc}}((a,b);\varrho)$ there exists a unique solution f of the initial value problem*

$$(\tau - z)f = g \quad \textit{with} \quad f(c) = d_1,\ f^{[1]}(c) = d_2$$

for each $z \in \mathbb{C}$, $c \in (a,b)$ and $d_1, d_2 \in \mathbb{C}$, if and only if

$$\varrho(\{x\})\zeta(\{x\}) = 0 \quad \textit{and} \quad \chi(\{x\})\zeta(\{x\}) \neq 1 \tag{4.2}$$

for all $x \in (a,b)$. If in addition all measures as well as g, d_1, d_2 and z are real, then the solution is real.

Proof.
For a function $f \in \mathcal{D}_\tau$ which satisfies the initial values from (4.1), the following equivalence holds

$$\begin{aligned} ((\tau - z)f)(x) &= g(x) \\ \frac{d}{d\varrho}\left(t \mapsto -f^{[1]}(t) + \int_c^t f\, d\chi\right)(x) &= zf(x) + g(x) \\ -f^{[1]}(x) + \underbrace{f^{[1]}(c)}_{d_2} + \int_c^x f\, d\chi &= \int_c^x (zf + g)\, d\varrho \\ f^{[1]}(x) &= d_2 + \int_c^x f\, d\chi - \int_c^x (zf + g)\, d\varrho. \end{aligned}$$

Therefore some function $f \in \mathcal{D}_\tau$ is a solution of (4.1) if and only if for each $x \in (a,b)$

$$f(x) = d_1 + \int_c^x f^{[1]}\, d\zeta,$$
$$f^{[1]}(x) = d_2 + \int_c^x f\, d\chi - \int_c^x (zf+g)\, d\varrho.$$

Now we set $\omega := |\varrho| + |\zeta| + |\chi|$. Hence $\varrho, \zeta, \chi \ll \omega$ holds and we set $m_{12} := \frac{d\zeta}{d\omega}, m_{21} := \frac{d(\chi - z\varrho)}{d\omega}$ and $f_2 := -\frac{d(g\varrho)}{d\omega}$. Then the above equations can be written as

$$\begin{pmatrix} f(x) \\ f^{[1]}(x) \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \int_c^x \underbrace{\begin{pmatrix} 0 & m_{12} \\ m_{21} & 0 \end{pmatrix}}_{=:M} \begin{pmatrix} f \\ f^{[1]} \end{pmatrix} d\omega + \int_c^x \begin{pmatrix} 0 \\ f_2 \end{pmatrix} d\omega.$$

Applying Theorem 3.6 leads to a unique solution of (4.1), if and only if

$$I + \omega(\{x\}) \begin{pmatrix} 0 & m_{12} \\ m_{21} & 0 \end{pmatrix}(x) = \begin{pmatrix} 1 & \zeta(\{x\}) \\ \chi(\{x\}) - z\varrho(\{x\}) & 1 \end{pmatrix}$$

is invertible for all $x \in (a,b)$. From

$$\det(I + \omega(\{x\})M(x)) = 1 - \zeta(\{x\})\chi(\{x\}) + z\zeta(\{x\})\varrho(\{x\}) \neq 0$$

for all $z \in \mathbb{C}$ follow the conditions (4.2).

□

Remark 4.4.
Note that if $g \in L^1_{\mathrm{loc}}((a,b);\varrho)$ and (4.2) holds for all $x \in (a,b)$, then there is also a unique solution of the initial value problem

$$(\tau - z)f = g; \quad \text{with} \quad f(c+) = d_1, f^{[1]}(c+) = d_2$$

for every $z \in \mathbb{C}, c \in (a,b), d_1, d_2 \in \mathbb{C}$ by Corollary 3.8.

In the following we will always assume that

$$\zeta(\{x\})\varrho(\{x\}) = \zeta(\{x\})\chi(\{x\}) = 0 \tag{4.3}$$

for all $x \in (a,b)$. This is stronger than needed for Theorem 4.3, but will be neccessary for the Lagrange identity below.

Remark 4.5.
From assumption (4.3) follows that for $f \in \mathcal{D}_\tau$ we have

$$\int_\alpha^\beta f_1(t+)\, d\zeta(t) = \int_\alpha^\beta f_1(t)\, d\zeta(t), \tag{4.4}$$
$$\int_\alpha^\beta f(t+)\, d\chi(t) = \int_\alpha^\beta f(t)\, d\chi(t), \tag{4.5}$$

for $\alpha, \beta \in (a,b)$, $\alpha < \beta$.

To show this we define

$$S := \{x \in (\alpha,\beta) : \varrho(\{x\}) \neq 0\}.$$

$f \in \mathcal{D}_\tau$ implies $f_1 \in AC_{\mathrm{loc}}((a,b);\varrho)$ and therefore f_1 can only have jumps in points x with $\varrho(\{x\}) \neq 0$, i.e. $\{x \in (\alpha,\beta) : f_1(x+) \neq f_1(x)\} \subseteq S$. Now since ϱ is a locally finite complex Borel measure S can have at most countable infinitely many elements. From assumption (4.3) follows that $\zeta(S) = 0$, which means

$$0 = \int_S f_1(t+)\, d\zeta(t) = \int_S f_1(t)\, d\zeta(t)$$

and (4.4) follows. A similar argument follows for (4.5) with f and χ.

Definition 4.6. For $f, g \in \mathcal{D}_\tau$ we define the *Wronski determinant*

$$W(f,g)(x) := f(x)g^{[1]}(x) - f^{[1]}(x)g(x)$$

for $x \in (a,b)$.

The Wronski determinant is well defined as it is absolutely continuous with respect to ϱ by the next Remark 4.8.

Proposition 4.7. *For each $f, g \in \mathcal{D}_\tau$ and $\alpha, \beta \in (a,b)$ the* Lagrange identity

$$\int_\alpha^\beta (g(x)\tau f(x) - f(x)\tau g(x))\, d\varrho(x) = W(f,g)(\beta) - W(f,g)(\alpha) \tag{4.6}$$

holds.

Proof.
Let $f, g \in \mathcal{D}_\tau$ and $\alpha, \beta \in (a,b)$ be fixed. By definition of $\mathcal{D}_\tau$ the function g is a distribution function of $g^{[1]}\zeta$. The function

$$f_1(x) = -f^{[1]}(x) + \int_\alpha^x f\, d\chi, \quad x \in (a,b)$$

is a distribution function of $(\tau f)\varrho$. Hence by applying the integration by parts formula (1.3) setting $F = g, d\mu = g^{[1]}d\zeta$ and $G = f_1, d\nu = \tau f d\varrho$, we get

$$\int_\alpha^\beta g(t)\tau f(t)\, d\varrho(t) = [f_1(t)g(t)]_{t=\alpha}^\beta - \int_\alpha^\beta f_1(t+)g^{[1]}(t)\, d\zeta(t).$$

We use (4.4) from the above remark to replace the $f_1(t+)$ with $f_1(t)$ on the right hand side. By performing integration by parts using formula (1.3) with $F = \int_\alpha^t f\, d\chi, d\mu = f d\chi$ and $G = g$, $d\nu = g^{[1]}d\zeta$ and then using (4.5) from the last Remark we get

$$\begin{aligned}
\int_\alpha^\beta f_1(t)g^{[1]}(t)\, d\zeta(t) &= \int_\alpha^\beta \int_\alpha^t f\, d\chi\, g^{[1]}(t)\, d\zeta(t) - \int_\alpha^\beta f^{[1]}(t)g^{[1]}(t)\, d\zeta(t) \\
&= \left[g(t)\int_\alpha^t f\, d\chi\right]_{t=\alpha}^\beta - \int_\alpha^\beta g(t+)f(t)\, d\chi - \int_\alpha^\beta f^{[1]}(t)g^{[1]}(t)\, d\zeta(t) \\
&= g(\beta)\int_\alpha^\beta f\, d\chi - \int_\alpha^\beta g(t)f(t)\, d\chi - \int_\alpha^\beta f^{[1]}g^{[1]}(t)\, d\zeta(t).
\end{aligned}$$

Note that the last two integrals are symmetric with respect to f and g. By inserting the identities from above it follows that

$$\begin{aligned}
\int_\alpha^\beta g\tau f - f\tau g \, d\varrho &= [f_1(t)g(t)]_{t=\alpha}^\beta - [g_1(t)f(t)]_{t=\alpha}^\beta - g(\beta)\int_\alpha^\beta f\, d\chi + f(\beta)\int_\alpha^\beta g\, d\chi \\
&= [-f^{[1]}(t)g(t)]_{t=\alpha}^\beta + \left[\int_\alpha^t f\, d\chi\, g(t)\right]_{t=\alpha}^\beta - [-g^{[1]}(t)f(t)]_{t=\alpha}^\beta - \\
&\quad - \left[\int_\alpha^t g\, d\chi f(t)\right]_{t=\alpha}^\beta - g(\beta)\int_\alpha^\beta f\, d\chi + f(\beta)\int_\alpha^\beta g\, d\chi \\
&= [-f^{[1]}(t)g(t)]_{t=\alpha}^\beta + \int_\alpha^\beta f\, d\chi\, g(\beta) - [-g^{[1]}(t)f(t)]_{t=\alpha}^\beta - \\
&\quad - \int_\alpha^\beta g\, d\chi\, f(\beta) - g(\beta)\int_\alpha^\beta f\, d\chi + f(\beta)\int_\alpha^\beta g\, d\chi \\
&= W(f,g)(\beta) - W(f,g)(\alpha).
\end{aligned}$$

□

Remark 4.8.

- As a consequence of the Lagrange identity we see, that for $f, g \in \mathcal{D}_\tau$ the function $W(f,g)(x)$ is absolutely continuous with respect to ϱ, with

$$\frac{dW(f,g)}{d\varrho} = g\tau f - f\tau g.$$

- Furthermore if $u_1, u_2 \in \mathcal{D}_\tau$ are two solutions of $(\tau - z)u = 0$ the Lagrange identity shows, that $W(u_1, u_2)$ is constant for all $x \in (a,b)$. We have

$$W(u_1,u_2) \neq 0 \iff u_1, u_2 \text{ are linearly independent.}$$

  If $u_1 = cu_2$ the Wronskian $W(u_1,u_2)$ vanishes obviously. Conversly $W(u_1,u_2) = 0$ means, that the vectors

$$\begin{pmatrix} u_1(x) \\ u_1^{[1]}(x) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_2(x) \\ u_2^{[1]}(x) \end{pmatrix}$$

  are linearly dependent for each $x \in (a,b)$. So we have some complex valued function h with

$$\begin{pmatrix} u_1(x) \\ u_1^{[1]}(x) \end{pmatrix} = h(x) \begin{pmatrix} u_2(x) \\ u_2^{[1]}(x) \end{pmatrix} \quad x \in (a,b).$$

  The functions $u_2 h$ and $u_2 h(c)$ both satisfy the initial value problem with initial values $d_1 = u_1(c)$ and $d_2 = u_1^{[1]}(c)$ at c. Now since the initial value problem has a unique solution $h(x) = h(c)$ for all $x \in (a,b)$ follows. This means the solutions u_1 and u_2 are linearly dependent if $W(u_1,u_2) \neq 0$.

Definition 4.9. For every $z \in \mathbb{C}$ we call two linearly independent solutions of $(\tau - z)u = 0$ a *fundamental system* of $(\tau - z)u = 0$.

From Theorem 4.3 follows, that there exist solutions for two linearly independent initial values. Because the Wronskian is constant for all $x \in (a,b)$, a fundamental system always exists.

Proposition 4.10. *Let $z \in \mathbb{C}$ and u_1, u_2 be a fundamental system of $(\tau - z)u = 0$. Furthermore, let $c \in (a,b)$, $d_1, d_2 \in \mathbb{C}$, $g \in L^1_{\mathrm{loc}}((a,b);\varrho)$. Then there exist $c_1, c_2 \in \mathbb{C}$, such that the solution f of*

$$(\tau - z)f = g \quad \text{with} \quad f(c) = d_1,\ f^{[1]}(c) = d_2$$

is given by

$$f(x) = c_1 u_1(x) + c_2 u_2(x) + \int_c^x \frac{u_1(x)u_2(t) - u_2(x)u_1(t)}{W(u_1,u_2)} g(t)\, d\varrho(t),$$
$$f^{[1]}(x) = c_1 u_1^{[1]}(x) + c_2 u_2^{[1]}(x) + \int_c^x \frac{u_1^{[1]}(x)u_2(t) - u_2^{[1]}(x)u_1(t)}{W(u_1,u_2)} g(t)\, d\varrho(t)$$

for each $x \in (a,b)$. If u_1, u_2 is the fundamental system with

$$u_1(c) = 1,\ u_1^{[1]}(c) = 0 \quad \text{and} \quad u_2(c) = 0,\ u_2^{[1]}(c) = 1$$

then $c_1 = d_1$ and $c_2 = d_2$.

Proof.
For $c \in (a,b)$ we set

$$h_1(x) := u_1(x) \int_c^x u_2 g\, d\varrho, \quad h_2(x) := u_2(x) \int_c^x u_1 g\, d\varrho, \quad h(x) := h_1(x) - h_2(x)$$

with $x \in (a,b)$. We show that $(\tau - z)h = W(u_1,u_2)g$ by showing that the integrated equation

$$h^{[1]}(\alpha) - h^{[1]}(\beta) + \int_\alpha^\beta h\, d\chi - z \int_\alpha^\beta h\, d\varrho = W(u_1,u_2) \int_\alpha^\beta g\, d\varrho$$

is satisfied with the help of integration by parts.

First integration by parts shows that

$$\begin{aligned}
&\int_\alpha^\beta \left(u_1^{[1]}(x) \int_c^x u_2 g\, d\varrho - u_2^{[1]}(x) \int_c^x u_1 g\, d\varrho \right) d\zeta = \\
&\quad = \left[u_1(x) \int_c^x u_2 g\, d\varrho - u_2(x) \int_c^x u_1 g\, d\varrho \right]_{x=\alpha}^{\beta} - \underbrace{\int_\alpha^\beta u_1(x)u_2(x)g(x) - u_2(x)u_1(x)g(x)\, d\varrho}_{=0} \\
&\quad = h(\beta) - h(\alpha)
\end{aligned}$$

for all $\alpha, \beta \in (a,b)$ with $\alpha < \beta$. This means h is a distribution function with ζ-density

$$h^{[1]}(x) = u_1^{[1]}(x) \int_c^x u_2 g\, d\varrho - u_2^{[1]}(x) \int_c^x u_1 g\, d\varrho, \quad x \in (a,b). \tag{4.7}$$

From $(\tau - z)u_1 = 0$ follows the identity

$$u_1^{[1]}(x) - u_1^{[1]}(c) = \int_c^x u_1\, d\chi - z \int_c^x u_1\, d\varrho. \tag{4.8}$$

By first using integration by parts and then (4.8) twice, we calculate for h_1

$$\begin{aligned}
&\int_\alpha^\beta h_1\,d\chi - z\int_\alpha^\beta h_1\,d\varrho = \\
&\quad= \int_\alpha^\beta u_1(x)\int_c^x u_2 g\,d\varrho\,d\chi(x) - z\int_\alpha^\beta u_1(x)\int_c^x u_2 g\,d\varrho\,d\varrho(x) \\
&\quad= \left[\int_c^x u_1\,d\chi\int_c^x u_2 g\,d\varrho - z\int_c^x u_1\,d\varrho\int_c^x u_2 g\,d\varrho\right]_{x=\alpha}^\beta - \\
&\qquad\quad - \int_\alpha^\beta\left(\int_c^x u_1\,d\chi\,u_2(x)g(x) - z\int_c^x u_1\,d\varrho\,u_2(x)g(x)\right)d\varrho(x) \\
&\quad= \left[\left(\int_c^x u_1\,d\chi - z\int_c^x u_1\,d\varrho\right)\int_c^x u_2 g\,d\varrho\right]_{x=\alpha}^\beta - \\
&\qquad\quad - \int_\alpha^\beta\left(\int_c^x u_1\,d\chi - z\int_c^x u_1\,d\varrho\right)u_2(x)g(x)\,d\varrho(x) \\
&\quad= \left[\left(u_1^{[1]}(x) - u_1^{[1]}(c)\right)\int_c^x u_2 g\,d\varrho\right]_{x=\alpha}^\beta - \int_\alpha^\beta (u_1^{[1]}(x) - u_1^{[1]}(c))u_2(x)g(x)\,d\varrho \\
&\quad= u_1^{[1]}(\beta)\int_c^\beta u_2 g\,d\varrho - u_1^{[1]}(\alpha)\int_c^\alpha u_2 g\,d\varrho - \int_\alpha^\beta u_2 u_1^{[1]} g\,d\varrho
\end{aligned}$$

for all $\alpha,\beta\in(a,b)$ with $\alpha<\beta$. The same is true for h_2 with u_1 and u_2 changing places. Hence for h it follows with (4.7) that

$$\int_\alpha^\beta h\,d\chi - z\int_\alpha^\beta h\,d\varrho = h^{[1]}(\beta) - h^{[1]}(\alpha) + W(u_1,u_2)\int_\alpha^\beta g\,d\varrho.$$

Hence h is a solution of $(\tau-z)h = W(u_1,u_2)g$ and therefore the function f, given in the claim, is a solution of $(\tau-z)f=g$.

To show that we can find $c_1,c_2\in\mathbb{C}$ such that for some $d_1,d_2\in\mathbb{C}$ the initial values $f(c)=d_1$ and $f^{[1]}(c)=d_2$ are satisfied, we set

$$c_1 := \frac{d_1u_2^{[1]}(c) - d_2u_2(c)}{W(u_1,u_2)} \quad\text{and}\quad c_2 := \frac{u_1(c)d_2 - u_1^{[1]}(c)d_1}{W(u_1,u_2)}.$$

We calculate

$$\begin{aligned}
f(c) &= c_1u_1(c) + c_2u_2(c) \\
&= \frac{1}{W(u_1,u_2)}\left[\left(d_1u_2^{[1]}(c) - d_2u_2(c)\right)u_1(c) + \left(u_1(c)d_2 - u_1^{[1]}(c)d_1\right)u_2(c)\right] \\
&= \frac{1}{W(u_1,u_2)}d_1\left(\underbrace{u_2^{[1]}(c)u_1(c) - u_1^{[1]}(c)u_2(c)}_{=W(u_1,u_2)}\right) + \frac{1}{W(u_1,u_2)}d_2\left(\underbrace{u_1(c)u_2(c) - u_2(c)u_1(c)}_{=0}\right) \\
&= d_1
\end{aligned}$$

and a similiar calculation holds for $f^{[1]}(c)=d_2$.

□

The following Proposition will be needed later in Section 5.

Proposition 4.11. *For all functions $f_1, f_2, f_3, f_4 \in \mathcal{D}_\tau$ we have the* Plücker identity

$$W(f_1, f_2)W(f_3, f_4) + W(f_1, f_3)W(f_2, f_4) + W(f_1, f_4)W(f_2, f_3) = 0.$$

Proof.
The left-hand side is equal to the determinant of the clearly not invertible matrix

$$\frac{1}{2}\begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ f_1^{[1]} & f_2^{[1]} & f_3^{[1]} & f_4^{[1]} \\ f_1 & f_2 & f_3 & f_4 \\ f_1^{[1]} & f_2^{[1]} & f_3^{[1]} & f_4^{[1]} \end{pmatrix}.$$

□

4.4 Regularity of τ

Definition 4.12. We say τ *is regular at* a, if $|\varrho|((a,c])$, $|\zeta|((a,c])$ and $|\chi|((a,c])$ are finite for one (and hence for all[1]) $c \in (a,b)$. Similarly one defines regularity at the endpoint b. Finally we say that τ is regular, if τ is regular at both endpoints, i.e. if $|\varrho|, |\zeta|$ and $|\chi|$ are finite.

Theorem 4.13. *Let τ be regular at a, $z \in \mathbb{C}$ and $g \in L^1((a,c);\varrho)$ for each $c \in (a,b)$. Then for every solution f of $(\tau - z)f = g$ the limits*

$$f(a) := \lim_{x \to a+} f(x) \quad \text{and} \quad f^{[1]}(a) := \lim_{x \to a+} f^{[1]}(x)$$

exist and are finite. For each $d_1, d_2 \in \mathbb{C}$ there exists a unique solution of

$$(\tau - z)f = g \quad \text{with} \quad f(a) = d_1,\ f^{[1]}(a) = d_2.$$

Furthermore, if all measures as well as g, d_1, d_2 and z are real, then the solution is real. Similar results hold for the right endpoint b.

Proof.
The first part of the proof is an immediate consequence of Theorem 3.10 applied to the two-dimensional integral equation of the proof of Theorem 4.3. From Proposition 4.10 we infer, that all solutions of $(\tau - z)f = g$ are given by

$$f(x) = c_1 u_1(x) + c_2 u_2(x) + f_0(x), \quad x \in (a,b),$$

where $c_1, c_2 \in \mathbb{C}$, u_1, u_2 are a fundamental system of $(\tau - z)u = 0$ and f_0 is some solution of $(\tau - z)f = g$. Now since $W(u_1, u_2)$ is constant and doesn't vanish for $x \in (a,b)$ we get $W(u_1, u_2)(a) \neq 0$. It follows that

$$\begin{pmatrix} u_1(a) \\ u_1^{[1]}(a) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_2(a) \\ u_2^{[1]}(a) \end{pmatrix}$$

[1] since the measures are finite on all $[c_1, c_2]$ with $c_1, c_2 \in (a,b)$

are linearly independent and there is exactly one choice for the coefficients c_1, c_2, such that the solution f satisfies the initial value at a.
If g, d_1, d_2 and z are real then u_1, u_2 and f_0 can be chosen real and hence c_1 and c_2 are real.
□

Remark 4.14.
Under the assumptions of Theorem 4.13 one can see that Proposition 4.10 remains valid even in the case, when $c = a$ (respectively $c = b$) with essentially the same proof.

Theorem 4.15. *Let $g \in L^1_{\mathrm{loc}}((a,b);\varrho)$, $c \in (a,b)$, $d_1, d_2 \in \mathbb{C}$ and for each $z \in \mathbb{C}$, let $f(z,\cdot)$ be the unique solution of*

$$(\tau - z)f = g, \quad \text{with} \quad f(c) = d_1,\ f^{[1]}(c) = d_2.$$

Then for each point $x \in (a,b)$ the functions $z \mapsto f(z,x)$ and $z \mapsto f^{[1]}(z,x)$ are entire functions of order at most $\frac{1}{2}$ in z. Moreover, for each $\alpha, \beta \in (a,b)$ with $\alpha < \beta$ there are constants $C, B \in \mathbb{R}$ such that

$$|f(z,x)| + |f^{[1]}(z,x)| \le Ce^{B\sqrt{|z|}}, \quad x \in [\alpha,\beta], z \in \mathbb{C}. \tag{4.9}$$

Proof.
As done in the proof of Theorem 4.3, we can write the initial value problem as a two dimensional integral equation with $\omega := |\varrho| + |\zeta| + |\chi|$ of the form

$$\begin{pmatrix} f(x) \\ f^{[1]}(x) \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \int_c^x \left(\begin{pmatrix} 0 & \frac{d\zeta}{d\omega} \\ \frac{d\chi}{d\omega} & 0 \end{pmatrix} + z \begin{pmatrix} 0 & 0 \\ -\frac{d\varrho}{d\omega} & 0 \end{pmatrix} \right) \begin{pmatrix} f(x) \\ f^{[1]}(x) \end{pmatrix} d\omega + \int_c^x \begin{pmatrix} 0 \\ \frac{d(g\varrho)}{d\omega} \end{pmatrix} d\omega.$$

Since all the matrix coefficients are locally integrable with respect to ω and (3.7) is satisfied by the general assumption (4.3) we can apply Theorem 3.7. Therefore the functions $z \mapsto f(z,x)$ and $z \mapsto f^{[1]}(z,x)$ are analytic for all $z \in \mathbb{C}$.

By choosing $W(u_1,u_2) = 1$ we know from Proposition 4.10 that a solution $f(z,\cdot)$ of $(\tau - z)f = g$ is given by

$$\begin{aligned} f(z,x) &= u_1(z,x)\left(c_1 + \int_c^x u_2(z,\cdot)g\, d\varrho\right) - u_2(z,x)\left(c_2 + \int_c^x u_1(z,\cdot)g\, d\varrho\right) \\ &\le |u_1(z,x)|\left(|c_1| + \int_c^x |g|\, d|\varrho| \sup_{t\in[c,x)} |u_2(z,t)|\right) + \\ &\qquad + |u_2(z,x)|\left(|c_2| + \int_c^x |g|\, d|\varrho| \sup_{t\in[c,x)} |u_1(z,t)|\right). \end{aligned}$$

Because $\int_c^x |g|\, d|\varrho|$ is a constant in z, it is sufficient to look at solutions of the homogeneous system for the asymptotics. Using the integration by parts formula (1.3) with $F = f(z,\cdot), d\mu = f^{[1]}(z,\cdot)d\zeta$ and the complex conjugate $G = f(z,\cdot)^*, d\nu = f^{[1]}(z,\cdot)^* d\zeta$, we get

$$f(z,x)f(z,x)^* = f(z,c)f(z,c)^* + \int_c^x \left(f(z,\cdot)f^{[1]}(z,\cdot)^* + f^{[1]}(z,\cdot)f(z,\cdot)^*\right) d\zeta.$$

As we assume $g = 0$, the differential equation implies

$$f^{[1]}(z,x) = f^{[1]}(z,c) + \int_c^x f(z,\cdot)\, d\chi - z\int_c^x f(z,\cdot)\, d\varrho = f^{[1]}(z,c) + \int_c^x f(z,\cdot)\, d\kappa$$

with $\kappa := \chi - z\varrho$. With this identity we get

$$\begin{aligned} f^{[1]}(z,x)f^{[1]}(z,x)^* &= \left(f^{[1]}(z,c) + \int_c^x f(z,\cdot)\,d\kappa\right)\left(f^{[1]}(z,c) + \int_c^x f(z,\cdot)\,d\kappa\right)^* \\ &= f^{[1]}(z,c)f^{[1]}(z,c)^* + \int_c^x \left(f^{[1]}(z,c) + \int_c^t f(z,\cdot)\,d\kappa\right) f(z,t)^*\,d\kappa(t) + \\ &\quad + \int_c^x f(z,t)\left(f^{[1]}(z,c) + \int_c^x f(z,\cdot)\,d\kappa\right)^* d\kappa(t) \\ &= |f^{[1]}(z,c)|^2 + \int_c^x f^{[1]}(z,t)f(z,t)^* + f(z,t)f^{[1]*}(z,t)\,d\kappa(t). \end{aligned}$$

Putting all the parts together we compute for $|z| \geq 1$ and $\omega = |\varrho| + |\zeta| + |\chi|$ as above

$$\begin{aligned} v(z,x) &:= |z||f(z,x)|^2 + |f^{[1]}(z,x)|^2 \\ &= v(z,c) + |z|\int_c^x (f(z,\cdot)f^{[1]*}(z,\cdot) + f(z,\cdot)^* f^{[1]}(z,\cdot))\,d\zeta + \\ &\quad + \int_c^x f^{[1]}(z,\cdot)f(z,\cdot)^* + f(z,\cdot)f^{[1]}(z,\cdot)^*\,d\chi - z\int_c^x f^{[1]}(z,\cdot)f(z,\cdot)^* + f(z,\cdot)f^{[1]}(z,\cdot)^*\,d\zeta \\ &\leq v(z,c) + |z|\int_c^x 2\operatorname{Re}(f(z,\cdot)^* f^{[1]}(z,\cdot))\,d\omega. \end{aligned}$$

With the fact $(a-b)^2 \geq 0$, $a, b \in \mathbb{R}$ we get the estimate

$$2\operatorname{Re}(f(z,x)^* f^{[1]}(z,x))| \leq \frac{|z||f(z,x)|^2 + |f^{[1]}(z,x)|^2}{\sqrt{|z|}} = \frac{v(z,x)}{\sqrt{|z|}}$$

which gives us the upper bound for $v(z,\cdot)$

$$v(z,x) \leq v(z,c) + \int_c^x v(z,\cdot)\sqrt{|z|}\,d\omega \quad x \in [c,b).$$

Now an application of the Gronwall Lemma 3.5 leads to

$$v(z,x) \leq v(z,c)e^{\int_c^x \sqrt{|z|}\,d\omega} \quad x \in [c,b).$$

For $x \in (a,c)$ we have

$$v(z,x) \leq v(z,c) + \int_{x+}^c v(z,\cdot)\sqrt{|z|}\,d\omega$$

and hence again by the Gronwall Lemma we get

$$v(z,x) \leq v(z,c)e^{\int_{x+}^c \sqrt{|z|}\,d\omega}, \quad x \in (a,c).$$

So with

$$B_1 := \begin{cases} \int_c^x d\omega & x \geq c \\ \int_{x+}^c d\omega & x < c \end{cases}$$

we get for each $x \in (a,b)$

$$v(z,x) \leq v(z,c)e^{B_1\sqrt{|z|}} = (|z||d_1|^2 + |d_2|^2)e^{B_1\sqrt{|z|}} \leq (|d_1|^2 + |d_2|^2)e^{(B_1+1)\sqrt{|z|}}$$

and we see, that the claim of the theorem is proven.

□

The following Corollary extends the analytic properties of solutions of the initial value problem for right-hand limits as well as interval endpoints (mind the included point a in the second case of initial values), if we assume regularity at the endpoints.

Corollary 4.16. *Let the assumptions of Theorem 4.15 be satisfied. Additionally let τ be regular at a and $g \in L^1((a,c);\varrho)$ for $c \in (a,b)$. Let $f(z,\cdot)$ be the solution of the initial value problem*

$$(\tau - z)f = g$$

with initial values

$$f(z,c) = d_1,\ f^{[1]}(z,c) = d_2 \quad c \in (a,b)$$

or

$$f(z,c+) = d_1,\ f^{[1]}(z,c+) = d_2 \quad c \in [a,b).$$

Then for each $x \in (a,b)$ the functions $z \mapsto f(z,x)$, $z \mapsto f^{[1]}(z,x)$ and for each $x \in [a,b)$ the functions $z \mapsto f(z,x+)$, $z \mapsto f^{[1]}(z,x+)$ are entire functions. They are of order at most $\frac{1}{2}$ and satisfy the asymptotics (4.9) where α can also be set as a.

Proof.
The right-hand limits of $f(z,x)$ and $f^{[1]}(z,x)$ for $x \in (a,b)$ are entire functions as mentioned in Remark 3.11. Since the upper bound of $f(z,\cdot)$ holds for all $x \in [\alpha,\beta)$, this upper bound remains true for the right-hand limits, so the asymptotic behavior is the same. With the regularity assumptions for τ and g the proof of Theorem 4.15 can be extended for $\alpha = a$ and we get the asymptotic behavior for $[a,\beta)$ and therefore also the right-hand limits $f(z,a)$ and $f^{[1]}(z,a)$ are analytic.

□

4.5 The Sturm-Liouville operator on $L^2((a,b);\varrho)$

Now we want to introduce linear operators in the Hilbert space $L^2((a,b);\varrho)$ as realizations of the differential expression τ. To this end we gather the assumptions on the measures we already put into action and introduce some new ones, which will be needed for further results.

Definition 4.17. We say that some interval (α,β) is a gap of $\operatorname{supp}(\varrho)$ if $(\alpha,\beta) \subseteq (a,b)\setminus \operatorname{supp}(\varrho)$ and $\alpha,\beta \in \operatorname{supp}(\varrho)$. We introduce the abbreviations

$$\alpha_\varrho := \inf \operatorname{supp}(\varrho) \quad \text{and} \quad \beta_\varrho := \sup \operatorname{supp}(\varrho).$$

Hypothesis 4.18. *The following requirements for the measures shall be fulfilled for the rest of the paper*

(A1) The measure ϱ is non-negative-valued.

(A2) The measure χ is real-valued.

(A3) The measure ζ is real-valued and supported on the whole interval;

$$\operatorname{supp}(\zeta) = (a,b).$$

(A4) The measure ζ has no point mass in common with ϱ or χ;

$$\zeta(\{x\})\varrho(\{x\}) = 0 \quad \text{and} \quad \zeta(\{x\})\chi(\{x\}) = 0, \quad x \in (a,b).$$

(A5) *The measures ϱ, χ, ζ have the following property: For each gap (α, β) of* $\operatorname{supp}(\varrho)$ *and every function $f \in \mathcal{D}_\tau$ with the outer limits $f(\alpha-) = f(\beta+) = 0$ it follows that $f(x) = 0$, $x \in (\alpha, \beta)$.*

(A6) *The measure ϱ is supported on more than one point.*

The assumptions (A3) and (A4) have already been introduced above. We need the non-negativity of ϱ, in order to be able to get a positive definite inner product for $L^2((a,b);\varrho)$ later. The real-valuedness of the measures imply a real-valued differential expression, i.e. $(\tau f)^* = \tau f^*$, $f \in \mathcal{D}_\tau$. Assumption (A5) is important for Proposition 4.22 and Proposition 4.23 to hold. As it is a rather implicit requirement for the measures it is not immediately clear if this requirement is satisfied for a usefull class of measures. But the following lemma will show that this assumption is satisfied for a large class of measures. (A6) is needed as otherwise $L^1((a,b);\varrho)$ would only be one-dimensional and hence the Proposition 4.22 does not hold in this case as all solutions are linearly dependent. We refer to [1], Appendix C for the one dimensional case.

Lemma 4.19. *If for each gap (α, β) of* $\operatorname{supp}(\varrho)$ *the measures $\zeta|_{(\alpha,\beta)}$ and $\chi|_{(\alpha,\beta)}$ are either both non-negative valued or both non-positive valued, then (A5) is satisfied.*

Proof.
Let (α, β) be a gap of $\operatorname{supp}(\varrho)$ and $f \in \mathcal{D}_\tau$ satisfy $f(\alpha-) = f(\beta+) = 0$. Since f is left-continuous we have

$$\int_\alpha^\beta f^* \tau f \, d\varrho = 0$$

and we calculate similarly to the first two calculations in the proof of Proposition 4.7 with $g = f^*$

$$\begin{aligned} f(\beta)^* \tau f(\beta) \varrho(\{\beta\}) &= \int_\alpha^{\beta+} f^* \tau f \, d\varrho \\ &= \underbrace{[f_1(t) f(t)^*]_{t=\alpha}^{\beta+}}_{=0} - \int_\alpha^{\beta+} f_1 f^{[1]*} \, d\zeta \\ &= \int_\alpha^{\beta+} |f|^2 \, d\chi + \int_\alpha^{\beta+} |f^{[1]}|^2 \, d\zeta. \end{aligned}$$

The left-hand side vanishes, since either $\varrho(\{\beta\}) = 0$ or f is continuous at β by $(A4)$. So if the measures share the same sign on (α, β), this means that $f^{[1]} = 0$ $\zeta-$a.e. and because of the left-continuity of $f^{[1]}$ and the condition (A3), this means $f^{[1]}(x) = 0$, $x \in (\alpha, \beta)$. Therefore $f(d) - f(c) = \int_c^d f^{[1]} \, d\zeta = 0$ for $c, d \in (\alpha, \beta)$, i.e. f is constant in (α, β). Now from $f(\beta) + f^{[1]}(\beta)\zeta(\{\beta\}) = f(\beta+) = 0$ follows, that f vanishes in (α, β).

□

Now we start constructing our linear operator on $L^2((a,b);\varrho)$. Thus we have to embed functions $f \in \mathcal{D}_\tau$ into $L^2((a,b);\varrho)$. Functions $f \in \mathcal{D}_\tau$ are left-continuous with at most countable-infinitely many jumps (since $\mathcal{D}_\tau \subseteq AC_{\mathrm{loc}}((a,b);\zeta)$, see Section 1, Equation (1.1)), but the support of ϱ can be very small and because of that the equivalence classes of functions equal almost everywhere with respect to ϱ, which we denote as $[f]_\varrho$, can be very big. Thus an injective embedding $\mathcal{D}_\tau \to L^2((a,b);\varrho)$ is hardly possible. Nonetheless below we will see that there is an embedding

$\mathcal{D}_\tau \to L^2((a,b);\varrho) \times L^2((a,b);\varrho)$ which is injective. To emphasize the difference between elements of the two sets $\mathcal{D}_\tau$ and $L^2((a,b);\varrho)$, we will use the equivalence class notation for elements of $L^2((a,b);\varrho)$ from now on.

We begin by defining a linear relation on $L^1_{\mathrm{loc}}((a,b);\varrho)$ which we will restrict later in order to get a linear relation on $L^2((a,b);\varrho)$. This linear relation is given as

$$\mathrm{T}_{\mathrm{loc}} := \{([f]_\varrho, [\tau f]_\varrho) \mid f \in \mathcal{D}_\tau\}.$$

Note that this linear relation is in fact a subset of $L^1_{\mathrm{loc}}((a,b);\varrho) \times L^1_{\mathrm{loc}}((a,b);\varrho)$ as $f \in \mathcal{D}_\tau$ is locally bounded (see Section 1, Subsection 1.2) and $[\tau f]_\varrho \in L^1_{\mathrm{loc}}((a,b);\varrho)$ by the definition of the Radon–Nikodým derivative of f_1 with respect to ϱ.

We will show that the map $\mathcal{D}_\tau \to \mathrm{T}_{\mathrm{loc}}, f \mapsto ([f]_\varrho, [\tau f]_\varrho)$ is bijective and for that we need some lemmas first.

Lemma 4.20. *Let $f : (a,b) \to \mathbb{C}$ satisfy $f = 0$ almost everywhere with respect to ϱ, i.e. $f \in [0]_\varrho$. If f is left-continuous, then $f(x) = 0$ for all [2] $x \in \mathrm{supp}(\varrho)^\circ$. If $f \in \mathcal{D}_\tau$ and (α,β) is a gap of $\mathrm{supp}(\varrho)$, then $f(\alpha-) = f(\beta+) = 0$.*

Proof.
Let $x \in \mathrm{supp}(\varrho)^\circ$ and assume $f(x) \neq 0$. Since f is left-continuous, there exists some $d < x$ such that $f(y) \neq 0$ for all $y \in (d,x]$. As $f(x) = 0$ a.e. with respect to ϱ, this interval can have no mass with respect to ϱ, i.e. $\varrho((d,x]) = 0$. Thus $(d,x) \subseteq \mathrm{supp}(\varrho)^c$ which contradicts $x \in \mathrm{supp}(\varrho)^\circ$ as this means we can find some $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq \mathrm{supp}(\varrho)^\circ$.

If $f \in [0]_\varrho$ and additionally $f \in \mathcal{D}_\tau$, then f is left-continuous and $f(\alpha-) = f(\alpha)$. Assume $f(\alpha) \neq 0$ then there exists some $d < \alpha$ such that $f(y) \neq 0$ for $y \in (d,\alpha]$. As above this means $(d,\alpha) \subseteq \mathrm{supp}(\varrho)^c$ follows. Since (α,β) is a gap we have $\alpha \in \mathrm{supp}(\varrho)$ with $(d,\alpha) \cup (\alpha,\beta) \subseteq \mathrm{supp}(\varrho)^c$ and hence $\varrho(\{\alpha\}) \neq 0$ which contradicts $f \in [0]_\varrho$ because we assumed $f(\alpha) \neq 0$.

Now assume $f(\beta+) \neq 0$, then there exists some $d > \beta$, such that $f(y) \neq 0$, $y \in (\beta,d)$, which means $(\beta,d) \subseteq \mathrm{supp}(\varrho)^c$, so ϱ has point mass at β and therefore $f(\beta) = 0$ as $f \in [0]_\varrho$. Now by (A4) we get that ζ can have no point mass at β. As $f \in \mathcal{D}_\tau$ implies $f \in AC_{\mathrm{loc}}((a,b);\zeta)$ this would mean f is continuous at β, i.e. $f(\beta+) = f(\beta) = 0$ which is a contradiction.

□

In the following lemma we show which elements of $\mathcal{D}_\tau$ lie in the equivalence class $[0]_\varrho$, which we will need to prove the injectivity of our bijection between $\mathcal{D}_\tau$ and $\mathrm{T}_{\mathrm{loc}}$.

Lemma 4.21. *Let $f \in \mathcal{D}_\tau$. Then $f \in [0]_\varrho$, if and only if there exist $c_a, c_b \in \mathbb{C}$ such that*

$$f(x) = \begin{cases} c_a u_a(x) & x \in (a, \alpha_\varrho], \\ 0 & x \in (\alpha_\varrho, \beta_\varrho], \\ c_b u_b(x) & x \in (\beta_\varrho, b), \end{cases} \tag{4.10}$$

for $u_a, u_b \in \mathcal{D}_\tau$ solutions of the initial value problem

$$\begin{aligned} \tau u = 0 \quad & \text{with} \quad u_a(\alpha_\varrho-) = 0,\ u_a^{[1]}(\alpha_\varrho-) = 1 \\ & \text{and} \quad u_b(\beta_\varrho+) = 0,\ u_b^{[1]}(\beta_\varrho+) = 1. \end{aligned}$$

[2] $\mathrm{supp}(\varrho)^\circ$ is the biggest open set contained in $\mathrm{supp}(\varrho)$.

In this case the identity

$$[\tau f]_\varrho = [\mathbb{1}_{\{\alpha_\varrho\}} c_a - \mathbb{1}_{\{\beta_\varrho\}} c_b]_\varrho$$

holds.

Proof.
The property $f \in [0]_\varrho$ means $f = 0$ a.e. with respect to ϱ. Obviously $f(x) = 0$ for all x with point mass and with Lemma 4.20 we also have $f(x) = 0$ for $x \in \operatorname{supp}(\varrho)^\circ$. If (α, β) is a gap of $\operatorname{supp}(\varrho)$, Lemma 4.20 yields $f(\alpha-) = f(\beta+) = 0$ and therefore $f(x) = 0$ for all $x \in [\alpha, \beta]$ by (A5).
It follows, that for all remaining boundary points of $\operatorname{supp}(\varrho)$ with $f(x) \neq 0$, $x \in (\alpha_\varrho, \beta_\varrho)$, there exists a monotonous sequence $x_n^- \nearrow x$ with $(x_n^-) \in \operatorname{supp}(\varrho)$. If $x_n^- \in \operatorname{supp}(\varrho)^\circ$, again with Lemma 4.20, $f(x_n^-) = 0$ follows. If $x_n^- \notin \operatorname{supp}(\varrho)^\circ$, there exists a gap of one side of x_n^- and hence either $f(x_n^-) = 0$ or $f(x_n^-+) = 0$. So for every $n \in \mathbb{N}$ there is some $t_n < x$ such that $|x_n^- - t_n| < \frac{1}{n}$ and $|f(t_n)| < \frac{1}{n}$. With the triangle inequality follows that $t_n \to x$ hence there exists some monotonous subsequence $(t_{n_k}) \nearrow x$ satisfying

$$f(x-) = \lim_{k\to\infty} f(t_{n_k}) = 0.$$

A similar argument for $(x_n^+) \in \operatorname{supp}(\varrho)$, $x_n^+ \searrow x$ yields $f(x+) = 0$. We arrive at $f(x) = f(x+) = 0$ for $x \in (\alpha_\varrho, \beta_\varrho)$.

Outside of $[\alpha_\varrho, \beta_\varrho]$ we have $\int_c^d \tau f \, d\varrho = \int_c^d 0 \, d\varrho = 0$ for $c, d < \alpha_\varrho$ or $c, d > \beta_\varrho$ and therefore f is a solution of $\tau f = 0$ outside $[\alpha_\varrho, \beta_\varrho]$. And according to Proposition 4.10 f is of the form

$$f(x) = \begin{cases} c_a u_a(x) & x \in (a, \alpha_\varrho), \\ c_b u_b(x) & x \in (\beta_\varrho, b), \end{cases}$$

for some $c_a, c_b \in \mathbb{C}$, $u_a, u_b \in \mathcal{D}_\tau$ solutions of $\tau u = 0$. The initial values follow similar to Lemma 4.20.
It remains to show, that $f(\alpha_\varrho) = f(\beta_\varrho) = 0$. For that assume f is not continuous in α_ϱ, i.e. $\zeta(\{\alpha_\varrho\}) \neq 0$. Then from (A4) it follows that $f^{[1]}$ is continuous at α_ϱ. Hence $f^{[1]}(\alpha_\varrho) = 0$. But this yields

$$f(\alpha_\varrho-) = f(\alpha_\varrho+) - \int_{\alpha_\varrho}^{\alpha_\varrho+} f^{[1]} \, d\zeta = \underbrace{f(\alpha_\varrho+)}_{=0} - \underbrace{f^{[1]}(\alpha_\varrho)}_{=0} \zeta(\{\alpha_\varrho\}) = 0,$$

which means $f(\alpha_\varrho) = 0$. Similarly one shows that $f(\beta_\varrho) = 0$ and we are finished with the first part of the claim.

Now we want to compute τf for $f \in \mathcal{D}_\tau$ with $f = 0$ a.e. with respect to ϱ. From integrating the differential expression, we get the identity

$$\begin{aligned} \tau f(\alpha_\varrho) \varrho(\{\alpha_\varrho\}) &= f^{[1]}(\alpha_\varrho) - \underbrace{f^{[1]}(\alpha_\varrho+)}_{=0} + \underbrace{f(\alpha_\varrho)}_{=0} \chi(\{\alpha_\varrho\}) \\ &= f^{[1]}(\alpha_\varrho). \end{aligned}$$

If ϱ has mass in α_ϱ, we have

$$\tau f(\alpha_\varrho) = \frac{f^{[1]}(\alpha_\varrho)}{\varrho(\{\alpha_\varrho\})} = \frac{u_a^{[1]}(\alpha_\varrho-) c_a}{\varrho(\{\alpha_\varrho\})} = \frac{c_a}{\varrho(\{\alpha_\varrho\})}.$$

A similar argument leads to $\tau f(\beta_\varrho) = -\frac{c_b}{\varrho(\{\beta_\varrho\})}$ and we arrive at

$$\tau f = c_a \mathbb{1}_{\{\alpha_\varrho\}} - c_b \mathbb{1}_{\{\beta_\varrho\}}$$

a.e. with respect to ϱ.

□

Now we arrive at the main result which will allow us to construct an operator on $L^2((a,b);\varrho)$ later.

Proposition 4.22. *The linear map*

$$\begin{aligned} \mathcal{D}_\tau &\to \mathrm{T}_{\mathrm{loc}} \\ f &\mapsto ([f]_\varrho, [\tau f]_\varrho) \end{aligned}$$

is bijective.

Proof.
The mapping is linear as τ is linear and surjective by definition of $\mathrm{T}_{\mathrm{loc}}$.
To show injectivity, let $f \in \mathcal{D}_\tau$ be such, that $f \in [0]_\varrho$ and $[\tau f]_\varrho = [0]_\varrho$. By Lemma 4.21 we know, that for every $c \in (\alpha_\varrho, \beta_\varrho)$, we have $f(c) = f^{[1]}(c) = 0$. Now since the initial value problem for $\tau f = 0$ a.e. with respect to ϱ, with $f(c) = f^{[1]}(c) = 0$ has a unique solution by Theorem 4.3, we get $f = 0$ as an element of $\mathcal{D}_\tau$.

□

By the above proposition, every element $\mathfrak{f} \in \mathrm{T}_{\mathrm{loc}}$ has the form $([f]_\varrho, [\tau f]_\varrho)$ for a uniquely defined $f \in \mathcal{D}_\tau$ and we can identify $\mathfrak{f}$ with f.

If $\alpha_\varrho = a$ we view the condition that ϱ has no mass at α_ϱ from the next proposition trivially satisfied, similarly for b.

Proposition 4.23. *The multi-valued part of* $\mathrm{T}_{\mathrm{loc}}$ *is given as*

$$\mathrm{mul}(\mathrm{T}_{\mathrm{loc}}) = \mathrm{span}\left\{[\mathbb{1}_{\{\alpha_\varrho\}}]_\varrho, [\mathbb{1}_{\{\beta_\varrho\}}]_\varrho\right\}.$$

In particular

$$\dim \mathrm{mul}(\mathrm{T}_{\mathrm{loc}}) = \begin{cases} 0, & \text{if } \varrho \text{ has neither mass in } \alpha_\varrho \text{ nor } \beta_\varrho, \\ 1, & \text{if } \varrho \text{ has either mass in } \alpha_\varrho \text{ or } \beta_\varrho, \\ 2, & \text{if } \varrho \text{ has mass in } \alpha_\varrho \text{ and } \beta_\varrho. \end{cases}$$

Hence $\mathrm{T}_{\mathrm{loc}}$ *is an operator, if and only if* ϱ *has neither mass in* α_ϱ *nor* β_ϱ.

Proof.
Let $[g]_\varrho \in \mathrm{mul}(\mathrm{T}_{\mathrm{loc}})$. This implies $([0]_\varrho, [g]_\varrho) \in \mathrm{T}_{\mathrm{loc}}$ by definition and from the above remark, we know that there exists a unique representative $f \in \mathcal{D}_\tau$, such that $([0]_\varrho, [g]_\varrho) = ([f]_\varrho, [\tau f]_\varrho)$. From Lemma 4.21 we know, that the possible $f \in \mathcal{D}_\tau$, which satisfy this, are of the form (4.10). By the second result of Lemma 4.21, we get the inclusion

$$[\tau f]_\varrho \subseteq \mathrm{span}\left\{[\mathbb{1}_{\{\alpha_\varrho\}}]_\varrho, [\mathbb{1}_{\{\beta_\varrho\}}]_\varrho\right\}.$$

To show the opposite inclusion, consider the function

$$f(x) = \begin{cases} u_a(x) & x \in (a, \alpha_\varrho], \\ 0 & x \in (\alpha_\varrho, b), \end{cases}$$

with u_a as in Lemma 4.21. We can write f as

$$f(x) = -\int_{\alpha_\varrho}^{x} u_a^{[1]}(t)\mathbb{1}_{(a,\alpha_\varrho]}(t)\,d\zeta(t)$$

and it is easy to see, that $f \in \mathcal{D}_\tau$. Therefore $f \in [0]_\varrho$ and $[\tau f]_\varrho = [\mathbb{1}_{\{\alpha_\varrho\}}]_\varrho$ by Lemma 4.21, which means $([0]_\varrho, [\mathbb{1}_{\{\alpha_\varrho\}}]_\varrho) \in \mathrm{T}_{\mathrm{loc}}$ and therefore $[\mathbb{1}_{\{\alpha_\varrho\}}]_\varrho \in \mathrm{mul}(\mathrm{T}_{\mathrm{loc}})$.

Similarly one shows that $[\mathbb{1}_{\{\beta_\varrho\}}]_\varrho \in \mathrm{mul}(\mathrm{T}_{\mathrm{loc}})$. Note that if ϱ has no mass at α_ϱ, we have $[\mathbb{1}_{\{\alpha_\varrho\}}]_\varrho = [0]_\varrho$ and the same for β_ϱ, which finishes the proof.

□

Definition 4.24. For $\mathfrak{f}, \mathfrak{g} \in \mathrm{T}_{\mathrm{loc}}$ with $\mathfrak{f} = ([f]_\varrho, [\tau f]_\varrho), \mathfrak{g} = ([g]_\varrho, [\tau g]_\varrho)$ we define the *Wronskian of* $\mathrm{T}_{\mathrm{loc}}$ as

$$W(\mathfrak{f}, \mathfrak{g})(x) := f(x)g^{[1]}(x) - f^{[1]}(x)g(x), \quad x \in (a, b).$$

Remark 4.25.

- The Lagrange identity (Proposition 4.7) of $\mathrm{T}_{\mathrm{loc}}$ then takes the form

$$\begin{aligned} W(\mathfrak{f}, \mathfrak{g})(\beta) - W(\mathfrak{f}, \mathfrak{g})(\alpha) &= W(f, g)(\beta) - W(f, g)(\alpha) \\ &= \int_\alpha^\beta (g\tau f - f\tau g)\, d\varrho \end{aligned}$$

  by (4.6).

- By applying Theorem 4.3 to get to the last equality, we have

$$\begin{aligned} \mathrm{ran}(\mathrm{T}_{\mathrm{loc}} - z) &= \left\{[g]_\varrho \in L^1_{\mathrm{loc}}((a,b);\varrho) \mid \exists [h]_\varrho \in L^1_{\mathrm{loc}}((a,b);\varrho) : ([h]_\varrho, [g]_\varrho) \in \mathrm{T}_{\mathrm{loc}} - z\right\} \\ &= \left\{[g]_\varrho \in L^1_{\mathrm{loc}}((a,b);\varrho) \mid \exists f \in \mathcal{D}_\tau : ([f]_\varrho, [(\tau - z)f]_\varrho) = ([f]_\varrho, [g]_\varrho)\right\} \\ &= \left\{[g]_\varrho \in L^1_{\mathrm{loc}}((a,b);\varrho) \mid \exists f \in \mathcal{D}_\tau : [(\tau - z)f]_\varrho = [g]_\varrho\right\} \\ &= L^1_{\mathrm{loc}}((a,b);\varrho) \end{aligned}$$

  for each $z \in \mathbb{C}$.

- We also have

$$\begin{aligned} \dim\ker(\mathrm{T}_{\mathrm{loc}} - z) &= \dim\left\{[f]_\varrho \in L^1_{\mathrm{loc}}((a,b);\varrho) \mid ([f]_\varrho, [0]_\varrho) \in \mathrm{T}_{\mathrm{loc}} - z\right\} \\ &= \dim\left\{f \in \mathcal{D}_\tau \mid [(\tau - z)f]_\varrho = [0]_\varrho\right\} = 2 \end{aligned}$$

  for each $z \in \mathbb{C}$ by Proposition 4.10.

Now we want to restrict the differential relation $\mathrm{T}_{\mathrm{loc}}$ in order to get a linear relation on the Hilbert space $L^2((a,b);\varrho)$ with the scalar product

$$\langle f, g\rangle := \int_a^b f g^*\, d\varrho.$$

To this end we define the maximal linear relation $\mathrm{T}_{\max}$ on $L^2((a,b);\varrho)$ by

$$\mathrm{T}_{\max} := \left\{([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\mathrm{loc}} \mid [f]_\varrho, [\tau f]_\varrho \in L^2((a,b);\varrho)\right\}.$$

In general $\mathrm{T}_{\max}$ doesn't have to be an operator because all elements of

$$\mathrm{mul}(\mathrm{T}_{\mathrm{loc}}) = \mathrm{span}\left\{[\mathbb{1}_{\{\alpha_\varrho\}}]_\varrho, [\mathbb{1}_{\{\beta_\varrho\}}]_\varrho\right\}$$

are square-integrable and this means

$$\mathrm{mul}(\mathrm{T}_{\max}) = \mathrm{mul}(\mathrm{T}_{\mathrm{loc}}).$$

If $f \in \mathcal{D}_\tau$ has compact support, then f is of bounded variation and therefore bounded (see Section 1, Subsection 1.2) and $[f]_\varrho \in L^2((a,b);\varrho)$ follows.

Now let T_0 be the restriction of $\mathrm{T}_{\max}$ defined as

$$\begin{aligned} T_0 &:= \{([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\mathrm{loc}} \mid f \in \mathcal{D}_\tau, \mathrm{supp}(f) \text{ is compact in } (a,b)\} \\ &\subseteq L^2((a,b);\varrho) \times L^2((a,b);\varrho). \end{aligned}$$

We will prove later that T_0 is an operator, i.e. $\mathrm{mul}(T_0) = 0$. Because the differential expression τ satisfies $(\tau f)^* = \tau f^*$ by Hypothesis 4.18 the elements of the relations T_0 and $\mathrm{T}_{\max}$ satisfy $([f]_\varrho, [\tau f]_\varrho)^* := ([f]_\varrho^*, [\tau f]_\varrho^*) = ([f^*]_\varrho, [\tau f^*]_\varrho)$. This means that for every $\mathfrak{f} \in \mathrm{T}_{\max}$ follows $\mathfrak{f}^* \in \mathrm{T}_{\max}$ and for every $\mathfrak{f} \in T_0$ follows $\mathfrak{f}^* \in T_0$.

Definition 4.26. We say that some ϱ-measureable function $[f]_\varrho$ *lies in* $L^2((a,b);\varrho)$ *near* a if $[f|_{(a,c)}]_\varrho \in L^2((a,c);\varrho)$ for all $c \in (a,b)$. We say some $([f]_\varrho, [\tau f]_\varrho)$ *lies in* $\mathrm{T}_{\max}$ near a if both $[f]_\varrho$ and $[\tau f]_\varrho$ lie in $L^2((a,b);\varrho)$ near a. Similarly we define $[f]_\varrho$ *lies in* $L^2((a,b);\varrho)$ *near* b and $([f]_\varrho, [\tau f]_\varrho)$ *lies in* $\mathrm{T}_{\max}$ near b.

Before we look at the adjoints of the introduced operators, we collect similar results as for $\mathcal{D}_\tau$: extension of the solutions to the endpoints with additional regularity requirements, Wronski-determinant and Lagrange-identity.

Proposition 4.27. *Let τ be regular at a and $([f]_\varrho, [\tau f]_\varrho)$ lie in $\mathrm{T}_{\max}$ near a. Then both limits*

$$f(a) := \lim_{x \to a} f(x) \quad \textit{and} \quad f^{[1]}(a) := \lim_{x \to a} f^{[1]}(x)$$

exist and are finite. A similar result holds at b.

Proof.
If $([f]_\varrho, [\tau f]_\varrho)$ lies in $\mathrm{T}_{\max}$ near a, we have $[\tau f|_{(a,c)}]_\varrho \in L^2((a,c);\varrho)$. Since τ is regular the measure $\varrho|_{(a,c)}$ is a finite measure and therefore $[\tau f|_{(a,c)}]_\varrho \in L^1((a,c);\varrho)$ for each $c \in (a,b)$. Hence we have $[(\tau - z)f|_{(a,c)}]_\varrho \in L^1((a,c);\varrho)$ and the limits exist by Theorem 4.13.

□

Definition 4.28. Let $\mathfrak{f}, \mathfrak{g} \in \mathrm{T}_{\mathrm{loc}}$. For convinience we denote

$$W(\mathfrak{f},\mathfrak{g})(a) := \lim_{\alpha \to a} W(\mathfrak{f},\mathfrak{g})(\alpha) \quad \text{and} \quad W(\mathfrak{f},\mathfrak{g})(b) := \lim_{\beta \to b} W(\mathfrak{f},\mathfrak{g})(\beta)$$

and for $c, d \in [a,b]$ we denote

$$W_c^d(\mathfrak{f},\mathfrak{g}) := W(\mathfrak{f},\mathfrak{g})(d) - W(\mathfrak{f},\mathfrak{g})(c).$$

Lemma 4.29. *Let* $\mathfrak{f} = ([f]_\varrho, [\tau f]_\varrho), \mathfrak{g} = ([g]_\varrho, [\tau g]_\varrho) \in \mathrm{T}_{\mathrm{loc}}$ *both lie in* $\mathrm{T}_{\max}$ *near* a*. Then the limit* $W(\mathfrak{f}, \mathfrak{g}^*)(a)$ *exists and is finite. If additionally* τ *is regular at* a *we have*

$$W(\mathfrak{f}, \mathfrak{g}^*)(a) = f(a)g^{[1]}(a)^* - f^{[1]}(a)g(a)^*.$$

A similar result holds at b*. Furthermore if* $\mathfrak{f} = ([f]_\varrho, [\tau f]_\varrho), \mathfrak{g} = ([g]_\varrho, [\tau g]_\varrho) \in \mathrm{T}_{\max}$ *then we have the identity*

$$\langle [\tau f]_\varrho, [g]_\varrho \rangle - \langle [f]_\varrho, [\tau g]_\varrho \rangle = W_a^b(\mathfrak{f}, \mathfrak{g}^*). \tag{4.11}$$

Proof.
By Proposition 4.7 we have for all $\mathfrak{f} = ([f]_\varrho, [\tau f]_\varrho), \mathfrak{g} = ([g]_\varrho, [\tau g]_\varrho) \in \mathrm{T}_{\mathrm{loc}}$ and all $\alpha, \beta \in (a, b)$ the Lagrange identity

$$W_\alpha^\beta(\mathfrak{f}, \mathfrak{g}^*) = W(f, g^*)(\beta) - W(f, g^*)(\alpha) = \int_\alpha^\beta g^* \tau f - f(\tau g)^* \, d\varrho. \tag{4.12}$$

If $\mathfrak{f}, \mathfrak{g}$ lie in $\mathrm{T}_{\max}$ near a this means that the right-hand side of (4.11) is finite for all $\alpha < \beta$ in (a, b) and that the limit $\alpha \to a$ on the right-hand side of (4.11) exists and is finite for all $\beta \in (a, b)$. It follows that $W(f, g^*)(a)$ exists and is finite. Combining the first result and Proposition 4.27 leads to the second claim.
Similarly one can see that the limit $\beta \to b$ exists and is finite. Now taking the limits $\alpha \to a$ and $\beta \to b$ one after the other leads to the third claim.

□

In order to determine the adjoint of T_0 defined as

$$T_0^* = \{([f]_\varrho, [g]_\varrho) \in L^2((a, b); \varrho) \times L^2((a, b); \varrho) \mid \forall ([u]_\varrho, [v]_\varrho) \in T_0 : \langle f, v \rangle = \langle g, u \rangle\}$$

we need the following basic lemma from linear algebra.

Lemma 4.30. *Let* V *be a vector space over* $\mathbb{C}$ *and* $F_1, \dots, F_n, F \in V^*$*. Then*

$$F \in \operatorname{span}\{F_1, \dots, F_n\} \iff \bigcap_{i=1}^n \ker F_i \subseteq \ker F.$$

Proof.
If $F = \sum_{j=1}^n \alpha_j F_j$ for some $\alpha_j \in \mathbb{C}$ we clearly have $\bigcap_{i=1}^n \ker F_i \subseteq \ker F$.

Conversely we define $\tilde{F}(x) := (F_1(x), \dots, F_n(x)) \in \mathbb{C}^n$. We have $\ker \tilde{F} = \bigcap_{i=1}^n \ker F_i$. Therefore from $\tilde{F}(x_1) = \tilde{F}(x_2)$ follows that $F(x_1) = F(x_2)$. Hence the function

$$\begin{aligned} g_0 &: \operatorname{ran} \tilde{F} \to \mathbb{C} \\ g_0(\tilde{F}(x)) &:= F(x) \end{aligned}$$

is well defined. We extend g_0 linearly to a function $g : \mathbb{C}^n \to \mathbb{C}$ which has the form $g(y_1, \dots, y_n) = \sum_{j=1}^n \alpha_j y_j$ for some $\alpha_j \in \mathbb{C}$. We arrive at

$$F(x) = g_0(\tilde{F}(x)) = g(\tilde{F}(x)) = \sum_{j=1}^n \alpha_j F_j(x)$$

for all $x \in V$.

□

Theorem 4.31. *T_0 is symmetric and the adjoint of T_0 is $\mathrm{T}_{\max}$.*

Proof.
First we show the inclusion $\mathrm{T}_{\max} \subseteq T_0^*$. Let $([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\max}$. From Lemma 4.29 we get for all $([g]_\varrho, [\tau g]_\varrho) \in T_0$ that

$$\begin{aligned}\langle [\tau f]_\varrho, [g]_\varrho\rangle - \langle [f]_\varrho, [\tau g]_\varrho\rangle &= \lim_{\beta\to b} W(([f]_\varrho, [\tau f]_\varrho), ([g]_\varrho, [\tau g]_\varrho)^*)(\beta) - \\ &\quad - \lim_{\alpha\to a} W(([f]_\varrho, [\tau f]_\varrho), ([g]_\varrho, [\tau g]_\varrho)^*)(\alpha) \\ &= \lim_{\beta\to b}\left(f(\beta)g^{[1]*}(\beta) - f^{[1]}(\beta)g(\beta)^*\right) - \\ &\quad - \lim_{\alpha\to a}\left(f(\alpha)g^{[1]*}(\alpha) - f^{[1]}(\alpha)g(\alpha)^*\right).\end{aligned}$$

The right-hand side of this equation is zero since f and g have compact support in (a,b). As an immediate consequence we get $([f]_\varrho, [\tau f]_\varrho) \in T_0^*$.

For the other inclusion $T_0^* \subseteq \mathrm{T}_{\max}$ let $([f]_\varrho, [f_2]_\varrho) \in T_0^*$ and $\tilde{f} \in \mathcal{D}_\tau$ be the solution of $\tau\tilde{f} = f_2$. We know that $([\tilde{f}]_\varrho, [f_2]_\varrho) = ([\tilde{f}]_\varrho, [\tau\tilde{f}]_\varrho) \in \mathrm{T}_{\mathrm{loc}}$ by Proposition 4.22. Now we show that $([f]_\varrho, [f_2]_\varrho) \in \mathrm{T}_{\mathrm{loc}}$. For that we define the subspace

$$L_c^2((a,b);\varrho) := \{[f]_\varrho \in L^2((a,b);\varrho) \mid \exists f \in [f]_\varrho : \operatorname{supp}(f) \text{ is compact in } (a,b)\}.$$

On this subspace of $L^2((a,b);\varrho)$ we define the following linear functionals

$$\begin{aligned} l: \quad & L_c^2((a,b);\varrho) \to \mathbb{C} \\ & l(g) := \int_a^b (f-\tilde{f})^* g \, d\varrho \end{aligned}$$

and for u_1, u_2 solutions of $\tau u = 0$ with $W(u_1,u_2) = 1$

$$\begin{aligned} l_i: \quad & L_c^2((a,b);\varrho) \to \mathbb{C} \\ & l_i(g) := \int_a^b u_i^* g \, d\varrho, \quad i = 1,2. \end{aligned}$$

To apply Lemma 4.30 we want to show that $\ker l_1 \cap \ker l_2 \subseteq \ker l$. For that take $[g]_\varrho \in \ker l_1 \cap \ker l_2$. Then there exists an interval $[\alpha,\beta] \subseteq (a,b)$ such that

$$0 = l_i(g) = \int_a^b \mathbb{1}_{[\alpha,\beta]} g u_i \, d\varrho \quad \text{for } i = 1,2.$$

We define the function

$$\begin{aligned} u(x) &:= u_1(x)\int_a^x u_2 g \, d\varrho + u_2(x)\int_x^b u_1 g \, d\varrho \\ &= u_1(x)\int_a^x \mathbb{1}_{[\alpha,\beta]} u_2 g \, d\varrho + u_2(x)\int_x^t \mathbb{1}_{[\alpha,\beta]} u_1 g \, d\varrho. \end{aligned}$$

From the definition of u we see that u is a solution of $\tau u = g$ by Proposition 4.10 and from the second line and $g \in \ker l_1 \cap \ker l_2$ follows that u has compact support. Hence $([u]_\varrho, [\tau u]_\varrho) \in T_0$.

Applying the Lagrange identity for u with compact support and using the definition of the adjoint we get

$$\begin{aligned}\int_a^b (f-\tilde{f})^* g \, d\varrho &= \int_a^b (f-\tilde{f})^* \tau u \, d\varrho \\ &= \langle \tau u, f\rangle - \langle \tau u, \tilde{f}\rangle \\ &= \langle u, f_2\rangle - \langle u, \tau\tilde{f}\rangle = \langle u, f_2\rangle - \langle u, f_2\rangle = 0.\end{aligned}$$

It follows that $g \in \ker l$. Now applying Lemma 4.30 there are $c_1, c_2 \in \mathbb{C}$ such that

$$\int_a^b (f - \tilde{f} + c_1u_1 + c_2u_2)^* g \, d\varrho = 0$$

for each function $g \in L^2_c((a,b);\varrho)$ which leads to $[f]_\varrho = [\tilde{f}+c_1u_1+c_2u_2]_\varrho$. From $\tilde{f}+c_1u_1+c_2u_2 \in \mathcal{D}_\tau$ and $[\tau(\tilde{f}+c_1u_1+c_2u_2)]_\varrho = [f_2]_\varrho$ follows $([\tilde{f}+c_1u_1+c_2u_2]_\varrho, [f_2]_\varrho) = ([f]_\varrho, [f_2]_\varrho) \in \mathrm{T}_{\mathrm{loc}}$ by the definition of $\mathrm{T}_{\mathrm{loc}}$ and $([f]_\varrho, [f_2]_\varrho) \in \mathrm{T}_{\max}$ follows . Hence $T_0^* = \mathrm{T}_{\max}$ and since $T_0 \subseteq \mathrm{T}_{\max}$ by definition we see that T_0 is symmetric.

□

Since T_0 is symmetric by the last theorem we can define the closure of T_0 as

$$\mathrm{T}_{\min} := \overline{T_0} = T_0^{**} = T_{\max}^*.$$

Since every adjoint linear relation is closed we know that $\mathrm{T}_{\max}$ is closed and $T_0 \subseteq \mathrm{T}_{\max}$ yields $\mathrm{T}_{\min} = T_{\max}^* \subseteq \mathrm{T}_{\max}$ and $T_{\min}^* = T_{\max}^{**} = \mathrm{T}_{\max}$ so $\mathrm{T}_{\min}$ is symmetric again. In order to determine $\mathrm{T}_{\min}$ we need the following lemma.

Lemma 4.32. *Let $([f_a]_\varrho, [\tau f_a]_\varrho)$ lie in $\mathrm{T}_{\max}$ near a and $([f_b]_\varrho, [\tau f_b]_\varrho)$ lie in $\mathrm{T}_{\max}$ near b. Then there exists a function $f \in \mathcal{D}_\tau$ with $([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\max}$ such that $f = f_a$ near a and $f = f_b$ near b.*

Proof.
To connect f_a and f_b we will fix an initial value problem close to a and vary g so that we get the desired values of our solution close to b. To achieve that we will use Proposition 4.10 to show that there are two linearly independent functionals dependent on g determining our solution.

Let u_1, u_2 be a fundamental system of $\tau u = 0$ with $W(u_1, u_2) = 1$. We define on $L^2((a,b);\varrho)$ the linear functionals

$$F_j^{\alpha,\beta}(g) := \int_\alpha^\beta u_j g \, d\varrho \quad \text{for } j = 1,2 \text{ and } \alpha < \beta, \alpha, \beta \in (a,b).$$

Assume there exists some $c \in \mathbb{C}$ such that for all $\alpha < \beta$, $\alpha, \beta \in (a,b)$ we have $F_1^{\alpha,\beta}(g) = cF_2^{\alpha,\beta}(g)$ for all $g \in L^2((a,b);\varrho)$, i.e.

$$\int_\alpha^\beta (u_1 - cu_2) g \, d\varrho = 0 \quad \text{for } \alpha < \beta, \alpha, \beta \in (a,b), g \in L^2((a,b);\varrho).$$

Therefore $[u_1]_\varrho = [cu_2]_\varrho$ follows and hence $([u_1]_\varrho, [\tau u_1]_\varrho) = ([u_1]_\varrho, [0]_\varrho) = ([cu_2]_\varrho, [0]_\varrho) = ([cu_2]_\varrho, [\tau cu_2]_\varrho)$. As the representation in $\mathcal{D}_\tau$ is unique this means $u_1(x) = cu_2(x)$ for all $x \in (a,b)$. This is a contradiction to $W(u_1, u_2) = 1$.

It follows that there exists $\alpha_0 < \beta_0$, $\alpha_0, \beta_0 \in (a,b)$ such that $F_1 := F_1^{\alpha_0,\beta_0}$ and $F_2 := F_2^{\alpha_0,\beta_0}$ are linearly independent, i.e. there exist $g_1, g_2 \in L^2((a,b);\varrho)$ such that $F_1(g_1) = 1$, $F_2(g_1) = 0$ and $F_1(g_2) = 0$, $F_2(g_2) = 1$. Now we want to show that there exists some $u \in \mathcal{D}_\tau$ which satisfies

$$u(\alpha_0) = f_a(\alpha_0),\ u^{[1]}(\alpha_0) = f_a^{[1]}(\alpha_0),\ u(\beta_0) = f_b(\beta_0),\ u^{[1]}(\beta_0) = f_b^{[1]}(\beta_0).$$

From Proposition 4.10 we know that for every $g \in L^2((a,b);\varrho) \subseteq L^1_{\mathrm{loc}}((a,b);\varrho)$ there exists a solution $u \in \mathcal{D}_\tau$ of $\tau u = g$ with the initial values $u(\alpha_0) = f_a(\alpha_0)$, $u^{[1]}(\alpha_0) = f_a^{[1]}(\alpha_0)$ of the form

$$\begin{aligned} u(x) &= c_1 u_1(x) + c_2 u_2(x) + u_1(x) F_2^{\alpha_0,x}(g) - u_2(x) F_1^{\alpha_0,x}(g) \\ u^{[1]}(x) &= c_1 u_1^{[1]}(x) + c_2 u_2^{[1]}(x) + u_1^{[1]} F_2^{\alpha_0,x}(g) - u_2^{[1]}(x) F_1^{\alpha_0,x}(g) \end{aligned}$$

with $c_1, c_2 \in \mathbb{C}$ and u_1, u_2 as stated in Proposition 4.10. In order for u, $u^{[1]}$ to satisfy the values of β_0 we need a solution $(F_2(g), F_1(g))$ of the linear equation system

$$\begin{pmatrix} u_1(\beta_0) & -u_2(\beta_0) \\ u_1^{[1]}(\beta_0) & -u_2^{[1]}(\beta_0) \end{pmatrix} \begin{pmatrix} F_2(g) \\ F_1(g) \end{pmatrix} = \begin{pmatrix} f_b(\beta_0) - c_1 u_1(\beta_0) - c_2 u_2(\beta_0) \\ f_b^{[1]}(\beta_0) - c_1 u_1^{[1]}(\beta_0) - c_2 u_2^{[1]}(\beta_0). \end{pmatrix}$$

Now since u_1, u_2 satisfy $W(u_1, u_2) = 1$ the matrix on the left-hand side is invertible and since F_1 and F_2 are linearly independent it is possible to find a function $g \in L^2((a,b);\varrho)$ such that $(F_2(g), F_1(g))$ is a solution of the linear equation system. Hence $u \in \mathcal{D}_\tau$ with the desired properties exists. Now the function f defined by

$$f(x) = \begin{cases} f_a(x) & x \in (a, \alpha_0], \\ u(x) & x \in (\alpha_0, \beta_0], \\ f_b(x) & x \in (\beta_0, b), \end{cases}$$

has the claimed properties.

□

Theorem 4.33. *The minimal relation* $\mathrm{T}_{\min}$ *is given by*

$$\mathrm{T}_{\min} = \{\mathfrak{f} \in \mathrm{T}_{\max} \mid \forall \mathfrak{g} \in \mathrm{T}_{\max} : W(\mathfrak{f}, \mathfrak{g})(a) = W(\mathfrak{f}, \mathfrak{g})(b) = 0\}.$$

Furthermore, $\mathrm{T}_{\min}$ *is an operator, i.e.*

$$\dim \operatorname{mul}(\mathrm{T}_{\min}) = 0.$$

Proof.
We define $M := \{\mathfrak{f} \in \mathrm{T}_{\max} \mid \forall \mathfrak{g} \in \mathrm{T}_{\max} : W(\mathfrak{f}, \mathfrak{g})(a) = W(\mathfrak{f}, \mathfrak{g})(b) = 0\}$ and show $\mathrm{T}_{\min} \subseteq M$ first.

Let $([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\min}$. As $\mathrm{T}_{\min} = T^*_{\max}$ we have the identity

$$\langle [\tau f]_\varrho, [g]_\varrho \rangle = \langle [f]_\varrho, [\tau g]_\varrho \rangle$$

for all $([g]_\varrho, [\tau g]_\varrho) \in \mathrm{T}_{\max}$. As $\mathrm{T}_{\min} = T^*_{\max} \subseteq \mathrm{T}_{\max}$ the Lagrange identity (4.11) holds for $([f]_\varrho, [\tau f]_\varrho)$ and all $([g]_\varrho, [\tau g]_\varrho) \in \mathrm{T}_{\max}$. We get

$$0 = \langle [\tau f]_\varrho, [g]_\varrho \rangle - \langle [f]_\varrho, [\tau g]_\varrho \rangle = W(f, g^*)(b) - W(f, g^*)(a)$$

for all $([g]_\varrho, [\tau g]_\varrho) \in \mathrm{T}_{\max}$. Now we want to apply Lemma 4.32 to some $([g]_\varrho, [\tau g]_\varrho) \in \mathrm{T}_{\max}$ by setting $f_a := g^*$ and $f_b := 0$ as elements of $\mathcal{D}_\tau$. As for these f_a and f_b the corresponding elements in $\mathrm{T}_{\mathrm{loc}}$ lie in $\mathrm{T}_{\max}$, we have some $([\tilde{g}]_\varrho, [\tau\tilde{g}]_\varrho) \in \mathrm{T}_{\max}$ which satisfies $\tilde{g}^* = g$ near a and $\tilde{g} = 0$ near b. Therefore we get

$$\begin{aligned} -W(([f]_\varrho, [\tau f]_\varrho), ([g]_\varrho, [\tau g]_\varrho))(a) &= -W(f, g)(a) \\ &= W(f, 0)(b) - W(f, g)(a) \\ &= W(f, \tilde{g}^*)(b) - W(f, \tilde{g}^*)(a) \\ &= 0. \end{aligned}$$

Similarly one sees that $W(([f]_\varrho, [\tau f]_\varrho), ([g]_\varrho, [\tau g]_\varrho))(b) = 0$ for all $([g]_\varrho, [\tau g]_\varrho) \in \mathrm{T}_{\max}$ and we arrive at $\mathrm{T}_{\min} \subseteq M$.

Conversely if $([f]_\varrho, [\tau f]_\varrho) \in M \subseteq \mathrm{T}_{\max}$ the Lagrange-identity yields for each $([g]_\varrho, [\tau g]_\varrho) \in \mathrm{T}_{\max}$

$$\langle [\tau f]_\varrho, [g]_\varrho \rangle - \langle [f]_\varrho, [\tau g]_\varrho \rangle = W(f, g^*)(b) - W(f, g^*)(a) = 0$$

and hence $([f]_\varrho, [\tau f]_\varrho) \in T^*_{\max} = \mathrm{T}_{\min}$ and we arrive at $\mathrm{T}_{\max} = M$.

To show that $\mathrm{T}_{\min}$ is an operator let $f \in \mathcal{D}_\tau$ such that $f \in [0]_\varrho$ and $([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\max}$. From Lemma 4.21 we know that f is of the form

$$f(x) = \begin{cases} c_a u_a(x) & x \in (a, \alpha_\varrho], \\ 0 & x \in (\alpha_\varrho, \beta_\varrho], \\ c_b u_b(x) & x \in (\beta_\varrho, b). \end{cases}$$

If $\alpha_\varrho = a$ or if $\varrho(\{\alpha_\varrho\}) = 0$ we have $([f]_\varrho, [\tau f]_\varrho) = ([0]_\varrho, [0]_\varrho)$ left of β_ϱ and therefore $f(x) = 0$ left of β_ϱ. So let $\alpha_\varrho > a$ and $\varrho(\{\alpha_\varrho\}) \neq 0$. Since $([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\max}$ we know from the first part of the proof that for all fundamental systems $\{u_1, u_2\}$ of $\tau u = 0$ we have $W(f, u_j)(a) = 0$. Since $W(u, \tilde{u})$ is constant for all solutions of $\tau u = 0$ we have

$$W(f, u_j)(x) = W(c_a u_a, u_j)(x) = \text{const.}$$

for all $x \in (a, \alpha_\varrho)$. Thus we get

$$0 = \lim_{x \to a} W(f, u_j)(x) = \lim_{x \to \alpha_\varrho -} W(f, u_j)(x) = \lim_{x \to \alpha_\varrho -} f(x) u_j^{[1]}(x) - f^{[1]} x u_j(x) = -c_a u_j(\alpha_\varrho).$$

Since this can only be true for all fundamental systems of $\tau u = 0$ if $c_a = 0$ we get $f(x) = 0$ for all $x \in (a, \alpha_\varrho]$. Similarly one proves that $f(x) = 0$ for all $x \in (\beta_\varrho, b)$ and therefore $f(x) = 0$ for all $x \in (a, b)$. So we arrive at $\mathrm{mul}(\mathrm{T}_{\min}) = \{0\}$.

□

Lemma 4.34. *If τ is regular at a and $c_1, c_2 \in \mathbb{C}$ then there exists some $([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\max}$ such that $f(a) = c_1$ and $f^{[1]}(a) = c_2$. A similar result holds at b.*

Proof.
If τ is regular at a and $z \in \mathbb{C}$, we can find some solution $\tilde{f} \in \mathcal{D}_\tau$ of the initial value problem of $(\tau - z)f = 0$ with $\tilde{f}(a) = c_1$ and $\tilde{f}^{[1]}(a) = c_2$. Since $\varrho|_{(a,c)}$ is a finite measure and $\tilde{f}$ is locally bounded on $[a, c)$ as it is of bounded variation (see Preliminaries) it follows that $\tilde{f}$ lies in $L^2((a, b); \varrho)$ near a. Since $\tau\tilde{f} = z\tilde{f}$ we have $([\tilde{f}]_\varrho, [\tau\tilde{f}]_\varrho) \in \mathrm{T}_{\max}$ near a. Now applying Lemma

4.32 by setting $f_a = \tilde{f}$ and $f_b = 0$ yields some $([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\max}$ such that $f(a) = c_1$ and $f^{[1]}(a) = c_2$.

□

Corollary 4.35. *If τ is regular at a and $([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\max}$ then we have*

$$f(a) = f^{[1]}(a) = 0 \iff \forall ([g]_\varrho, [\tau g]_\varrho) \in \mathrm{T}_{\max} : W(f,g)(a) = 0.$$

A similar result holds for b.

Proof.
If τ is regular at a we have

$$W(f,g)(a) = f(a)g^{[1]}(a) - f^{[1]}(a)g(a)$$

so the implication $\implies$ follows immediately.

Conversely let $c_1, c_2 \in \mathbb{C}$. As τ is regular we can find some $([g]_\varrho, [\tau g]_\varrho) \in \mathrm{T}_{\max}$ such that $g(a) = c_1$ and $g^{[1]}(a) = c_2$ by Lemma 4.34. It follows that

$$0 = W(f,g)(a) = f(a)c_2 - f^{[1]}(a)c_1.$$

As $c_1, c_2 \in \mathbb{C}$ was arbitrary this means means $f(a) = f^{[1]}(a) = 0$. Similarly one shows the result for b.

□

Corollary 4.36. *If $\alpha_\varrho > a$ and $([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\max}$ then we have*

$$f(\alpha_\varrho -) = f^{[1]}(\alpha_\varrho -) = 0 \iff \forall ([g]_\varrho, [\tau g]_\varrho) \in \mathrm{T}_{\max} : W(f,g)(a) = 0$$

In this case $f(x) = f^{[1]}(x) = 0$ for all $x \in (a, \alpha_\varrho]$. A similar result holds for b.

Proof.
If $\alpha_\varrho > a$ and two elements $([f]_\varrho, [\tau f]_\varrho), ([g]_\varrho, [\tau g]_\varrho)$ lie in $\mathrm{T}_{\max}$ near a, we have for all $x \in (a, \alpha_\varrho)$

$$W_x^b(f,g) = \int_x^b (g\tau f - f\tau g)\, d\varrho = \int_a^b (g\tau f - f\tau g)\, d\varrho = W_a^b(f,g).$$

Therefore $W(f,g)(x)$ is constant for all $x \in (a, \alpha_\varrho)$ and we have

$$W(f,g)(a) = \lim_{x \to \alpha_\varrho -} f(x)g^{[1]}(x) - f^{[1]}(x)g(x)$$

and the implication $\implies$ follows immediately.

For the other implication note that the assumption $\alpha_\varrho > a$ means, that for every solution u of $\tau u = 0$ the elment $([u]_\varrho, [\tau u]_\varrho)$ lies in $\mathrm{T}_{\max}$ near a. A similar argument as in Corollary 4.35, now with initial values at $x \in (a, \alpha_\varrho)$, leads to $f(x) = f^{[1]}(x) = 0$ for each $x \in (a, \alpha_\varrho)$ and therefore also for the left-hand side limit at α_ϱ.

□

By the last Corollary we can characterize the minimal Operator $\mathrm{T}_{\min}$ by restricting $\mathrm{T}_{\max}$ to those elements which are zero at the boundary points if τ is regular.

Corollary 4.37. *If τ is regular or if $\alpha_\varrho > a$ and $\beta_\varrho < b$ then we have*

$$\mathrm{T}_{\min} = \{([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\max} \mid f(a) = f^{[1]}(a) = f(b) = f^{[1]}(b) = 0\}.$$

Proof.
The characterization follows immediately from Theorem 4.33 in combination with 4.35 and 4.36. □

Remark 4.38.

- Note that all functions $f \in \mathcal{D}_\tau$ with $([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\min}$ vanish outside of $(\alpha_\varrho, \beta_\varrho)$ by Corollary 4.36.
- $\mathrm{dom}(\mathrm{T}_{\min})$ is dense in $L^2((a,b);\varrho)$ if and only if $\varrho(\{\alpha_\varrho\}) = \varrho(\{\beta_\varrho\}) = 0$. This is true since we have

$$\mathrm{dom}(\mathrm{T}_{\min})^\perp = \mathrm{mul}(\mathrm{T}_{\min}{}^*) = \mathrm{mul}(T_{\max}^{**}) = \mathrm{mul}(\mathrm{T}_{\max}) = \mathrm{span}\{[\mathbb{1}_{\{\alpha_\varrho\}}]_\varrho, [\mathbb{1}_{\{\beta_\varrho\}}]_\varrho\}.$$

- $\mathrm{dom}(\mathrm{T}_{\max})$ is dense in $L^2((a,b);\varrho)$. This follows from

$$\mathrm{dom}(\mathrm{T}_{\max})^\perp = \mathrm{mul}(T_{\max}^*) = \mathrm{mul}(\mathrm{T}_{\min}) = \{0\}.$$

The next theorem shows that $\mathrm{T}_{\min}$ always has self-adjoint extensions.

Theorem 4.39. *The deficiency indices of the minimal relation $\mathrm{T}_{\min}$ are equal and at most two, i.e.*

$$\mathrm{n}(\mathrm{T}_{\min}) := \dim \mathrm{ran}(\mathrm{T}_{\min} - i)^\perp = \dim \mathrm{ran}(\mathrm{T}_{\min} + i)^\perp \leq 2.$$

Proof.
We have the inclusion

$$\mathrm{ran}(\mathrm{T}_{\min} \pm i)^\perp = \ker(\mathrm{T}_{\max} \mp i) \subseteq \ker(\mathrm{T}_{\mathrm{loc}} \mp i)$$

and from Remark 4.25 we know that $\dim \ker(\mathrm{T}_{\mathrm{loc}} \mp i) = 2$. To show the equality we remember that we have

$$([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\max} \implies ([f^*]_\varrho, [\tau f^*]_\varrho) = ([f]_\varrho, [\tau f]_\varrho)^* \in \mathrm{T}_{\max}$$

as already noted. Hence if $([f]_\varrho, [0]_\varrho) \in \mathrm{T}_{\min} \pm i$ it follows that $([f^*]_\varrho, [0]_\varrho) = ([f]_\varrho, [0]_\varrho)^* \in \mathrm{T}_{\min} \mp i$. Therefore the mapping

$$\begin{aligned} * : \ker(\mathrm{T}_{\min} + i) &\to \ker(\mathrm{T}_{\min} - i) \\ [f]_\varrho &\to [f^*]_\varrho \end{aligned}$$

is a conjugate-linear isometry and hence the deficiency indices are equal.

□

4.6 Weyl's alternative and the deficiency index of $\mathrm{T}_{\min}$

In this section we want to categorize the different cases of endpoint regularity of solutions of $(\tau - z)u = 0$ for $z \in \mathbb{C}$. We will see that if τ is regular at one endpoint the solutions of $(\tau - z)u = 0$ lie in $L^2((a,b);\varrho)$ near this endpoint for all $z \in \mathbb{C}$.

By the following theorem we see that it is sufficient to look at one $z_0 \in \mathbb{C}$ to know the regularity of solutions of $(\tau - z)u = 0$ for all $z \in \mathbb{C}$.

Theorem 4.40. *If there exists some $z_0 \in \mathbb{C}$ such that all solutions of $(\tau - z_0)u = 0$ lie in $L^2((a,b);\varrho)$ near a then for all $z \in \mathbb{C}$ all solutions of $(\tau - z)u = 0$ lie in $L^2((a,b);\varrho)$ near a. A similar result holds at the endpoint b.*

Proof.
If $\{u_1, u_2\}$ is a fundamental system of $(\tau - z_0)u = 0$ with $W(u_1, u_2) = 1$, then u_1 and u_2 lie in $L^2((a,b);\varrho)$ near a by assumption. Thus for each $z \in \mathbb{C}$ there is some $c \in (a,b)$ such that the function $v := |u_1| + |u_2|$ satisfies

$$|z - z_0| \int_a^c v^2 \, d\varrho \leq \frac{1}{2}. \tag{4.13}$$

Now let $z \in \mathbb{C}$ and u be a solution of $(\tau - z)u = 0$. Then we get $(\tau - z_0)u = (z - z_0)u$ with $(z - z_0)u \in L^1_{\mathrm{loc}}((a,b);\varrho)$. By Proposition 4.10 with g set as $g = (z - z_0)u$ we get for each $x \in (a,b)$

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + (z - z_0) \int_c^x (u_1(x)u_2(t) - u_2(x)u_1(t))u(t) \, d\varrho(t)$$

for some $c_1, c_2 \in \mathbb{C}$ and $c \in (a,b)$ such that (4.13) holds. It follows that

$$\begin{aligned}
|u(x)| &\leq \underbrace{\max\{|c_1|, |c_2|\}}_{K:=} v(x) + |z - z_0| \int_x^c (|u_1(x)u_2(t)| + |u_2(x)u_1(t)|)|u(t)| d\varrho(t) \\
&\leq Kv(x) + |z - z_0| \int_x^c (|u_1(x)u_2(t)| + |u_1(t)u_2(x)| + |u_1(x)u_1(t)| + |u_2(x)u_2(t)|)|u(t)| \, d\varrho(t) \\
&\leq Kv(x) + |z - z_0| v(x) \int_x^c v(t)|u(t)| \, d\varrho(t)
\end{aligned}$$

for $x \in (a,c)$. Furthermore with the fact that $(a+b)^2 \leq 2a^2 + 2b^2$ for $a, b \in \mathbb{R}$ and the Cauchy-Schwarz inequality we get

$$\begin{aligned}
|u(x)|^2 &\leq 2Kv(x)^2 + 2|z - z_0|^2 v(x)^2 \left(\int_x^c v(t)|u(t)| \, d\varrho(t) \right)^2 \\
&\leq 2Kv(x)^2 + 2|z - z_0|^2 v(x)^2 \left(\int_x^c v^2 \, d\varrho \right) \left(\int_x^c |u|^2 \, d\varrho \right)
\end{aligned}$$

for $x \in (a,c)$. Now an integration yields for each $s \in (a,c)$

$$\begin{aligned}
\int_s^c |u|^2 \, d\varrho &\leq 2K^2 \int_a^c v^2 \, d\varrho + 2|z - z_0|^2 \int_s^c \left[v(x)^2 \left(\int_x^c v^2 \, d\varrho \right) \left(\int_x^c |u|^2 \, d\varrho \right) \right] d\varrho(x) \\
&\leq 2K^2 \int_a^c v^2 \, d\varrho + 2|z - z_0|^2 \int_s^c \left[v(x)^2 \left(\int_s^c v^2 \, d\varrho \right) \left(\int_s^c |u|^2 d\varrho \right) \right] d\varrho(x) \\
&\leq 2K^2 \int_a^c v^2 \, d\varrho + 2|z - z_0|^2 \left(\int_s^c v^2 \, d\varrho \right)^2 \left(\int_s^c |u|^2 \, d\varrho \right).
\end{aligned}$$

Now we use the estimate (4.13) and get

$$\int_s^c |u|^2 \, d\varrho \leq 2K^2 \int_a^c v^2 \, d\varrho + \frac{1}{2} \int_s^c |u|^2 \, d\varrho.$$

Rearranging this leads to

$$\int_s^c |u|^2 \, d\varrho \leq 4K^2 \int_s^c v^2 \, d\varrho \leq 4K^2 \int_a^c v^2 \, d\varrho < +\infty$$

Since $s \in (a, c)$ was arbitrary the claim follows.

□

The last theorem suggests to distinguish between two possible cases of regularity with respect to all $z \in \mathbb{C}$.

Definition 4.41. We say τ is in the *limit-circle (l.c.) case at* a, if for each $z \in \mathbb{C}$ all solutions of $(\tau - z) = 0$ lie in $L^2((a, b); \varrho)$ near a. Furthermore we say τ is in the *limit-point (l.p.) case at* a, if for each $z \in \mathbb{C}$ there is some solution of $(\tau - z)u = 0$ which does not lie in $L^2((a, b); \varrho)$ near a.

The motivation for the terms "limit-point" and "limit-circle" stem from the original proof from H. Weyl for the Weyl alternative (see [2] section 13.3 or Lemma 5.9 of this thesis). He showed that near an endpoint all solution parameters of a self-adjoint realization lie on a circle which either stays a circle or shrinks to a point as the endpoint is approached.

As an immediate consequence of Theorem 4.40 we get

Corollary 4.42. (Weyl's alternative) *Each boundary point τ is either in the l.c. case or in the l.p. case.*

Notice that the case in which a specific τ is at an endpoint is dependent on the properties of the measures ϱ, ζ, χ. We will now develop some possibilities to check in what case τ is at an endpoint and further examine the l.p. case.

Proposition 4.43. *If τ is regular at a or if ϱ has no weight near a, i.e. $\alpha_\varrho > a$ then τ is in the l.c. case at a. A similar result holds at b.*

Proof.
If τ is regular at a each solution $f \in \mathcal{D}_\tau$ can be continuously extended to the endpoint a by Theorem 4.13. Since $\varrho|_{(a,c)}$ is a finite measure and f is locally bounded (as it is locally of bounded variation) it follows that f lies in $L^2((a, b); \varrho)$ near a. If $\alpha_\varrho > a$ for each $c \in (a, b)$ and some $d \in (a, \alpha_\varrho)$ we get

$$\int_a^c |u|^2 \, d\varrho = \underbrace{\int_a^d |u|^2 \, d\varrho}_{=0} + \int_d^c |u|^2 \, d\varrho \leq \varrho([d, c]) \sup_{t \in [d,c]} |u(t)| < \infty.$$

This means that u lies in $L^2((a, b); \varrho)$ near a.

□

Remember (see Section 2) that the set $\mathrm{r}(\mathrm{T}_{\min})$ of points of regular type of $\mathrm{T}_{\min}$ consists of all $z \in \mathbb{C}$ such that $(\mathrm{T}_{\min} - z)^{-1} : \operatorname{ran}(\mathrm{T}_{\min} - z) \to L^2((a,b);\varrho)$ is a bounded operator with $\operatorname{ran}(\mathrm{T}_{\min} - z) \subseteq L^2((a,b);\varrho)$ not neccessarily defined on the whole space $L^2((a,b);\varrho)$. Since $\mathrm{T}_{\min}$ is symmetric we have $\mathbb{C}\backslash\mathbb{R} \subseteq \mathrm{r}(\mathrm{T}_{\min})$. Recall from Section 2, that $\dim \operatorname{ran}(\mathrm{T}_{\min} - z)^{\perp}$ is constant on every connected component of $\mathrm{r}(\mathrm{T}_{\min})$ and thus

$$\dim \ker(\mathrm{T}_{\max} - z^*) = \dim \operatorname{ran}(\mathrm{T}_{\min} - z)^{\perp} = \mathrm{n}(\mathrm{T}_{\min})$$

for every $z \in \mathrm{r}(\mathrm{T}_{\min})$.

Lemma 4.44. *For each $z \in \mathrm{r}(\mathrm{T}_{\min})$ there is a non-trivial solution of the equation $(\tau - z)u = 0$ which lies in $L^2((a,b);\varrho)$ near a. A similar result holds at b.*

Proof.
Let $z \in \mathrm{r}(\mathrm{T}_{\min})$. First we prove the case in which we assume that τ *is regular at* b: Assume that τ has no non-trivial solution of $(\tau - z)u = 0$ which lies in $L^2((a,b);\varrho)$ near the other endpoint a. Thus there is certainly no non-trivial solution f for which $([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\max}$. And we get

$$0 = \dim \ker(\mathrm{T}_{\max} - z) = \dim \operatorname{ran}(\mathrm{T}_{\min} - z^*)^{\perp} = \mathrm{n}(\mathrm{T}_{\min}).$$

Hence $\mathrm{T}_{\min}$ is self-adjoint by Corollary 2.11, i.e.

$$\mathrm{T}_{\min} = T^*_{\min} = \mathrm{T}_{\max}\,.$$

Now we can use Lemma 4.34 with $c_1 = 1$ and $c_2 = 0$ and we get some $([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\max}$ which satisfies $f(b) = 1$ and $f^{[1]}(b) = 0$. As $([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\min}$ Theorem 4.33 yields

$$W(([f]_\varrho, [\tau f]_\varrho), ([g]_\varrho, [\tau g]_\varrho))(b) = 0 \quad \forall ([g]_\varrho, [\tau g]_\varrho) \in \mathrm{T}_{\max}\,.$$

But by characterization of $\mathrm{T}_{\max}$ of Corollary 4.35 it follows that $f(b) = f^{[1]}(b) = 0$ which is a contradiction to our specific f. Therefore there always exists some non-trivial solution of $(\tau - z)f = 0$ for which $([f]_\varrho, [\tau f]_\varrho)$ lies in $\mathrm{T}_{\max}$ near a if τ is regular at b.

Now the *general case*: We take some $c \in (a,b)$ and consider the minimal relation T_c in $L^2((a,c);\varrho)$ induced by $\tau|_{(a,c)}$ and show that $z \in \mathrm{r}(\mathrm{T}_{\min})$ implies $z \in \mathrm{r}(T_c)$. Since $\tau|_{(a,c)}$ is regular at c, each $([f_c]_\varrho, [\tau f_c]_\varrho) \in T_c$ satisfies $f_c(c) = f_c^{[1]}(c) = 0$. Because of that we can extend each $f_c \in \mathcal{D}_{\tau_c}$ with $([f_c]_\varrho, [\tau f_c]_\varrho) \in T_c$ continuously with zero to some function $f \in \mathcal{D}_\tau$, i.e. $[f_c]_\varrho \in \operatorname{dom} T_c$ implies $[f]_\varrho \in \operatorname{dom} \mathrm{T}_{\min}$. We will use the operator notation for $\mathrm{T}_{\min}$ and T_c (which is possible as they have trivial multi-valued part by Theorem 4.33) and denote the Hilbert space norm on $L^2((a,c);\varrho)$ as $\|\cdot\|_c$. As $z \in \mathrm{r}(\mathrm{T}_{\min})$ we get

$$\|(T_c - z)[f_c]_\varrho\|_c = \|(\mathrm{T}_{\min} - z)[f]_\varrho\| \leq K\|[f]_\varrho\| = K\|[f_c]_\varrho\|_c$$

for some constant $K > 0$. Now since the solutions of the equations $(\tau|_{(a,c)} - z)u = 0$ for $z \in \mathbb{C}$ are exactly the solutions of $(\tau - z)u = 0$ restricted to (a,c) (remember the construction of solutions from the proof of Theorem 3.6) the claim follows from the first part.

□

Corollary 4.45. *If $z \in \mathrm{r}(\mathrm{T}_{\min})$ and τ is in the l.p. case at a, then there is a unique non-trivial solution of $(\tau - z)u = 0$ (up to scalar multiple), which lies in $L^2((a,b);\varrho)$ near a. A similar result holds at b.*

Proof.
If there were two linearly independent solutions in $L^2((a,b);\varrho)$ near a, then τ would already be in the l.c. case at a.

□

Lemma 4.46. *τ is in the l.p. case at a if and only if*

$$W(\mathfrak{f},\mathfrak{g})(a) = 0 \quad \forall \mathfrak{f},\mathfrak{g} \in \mathrm{T}_{\max}.$$

τ is in the l.c. case at a if and only if there exists $\mathfrak{f},\mathfrak{g} \in \mathrm{T}_{\max}$ such that

$$W(\mathfrak{f},\mathfrak{f}^*)(a) = 0 \text{ and } W(\mathfrak{f},\mathfrak{g})(a) \neq 0.$$

A similar result holds at b.

Proof.
Let τ be in the l.c. case at a and $\{u_1,u_2\}$ be a fundamental system of $\tau u = 0$ with $W(u_1,u_2) = 1$. As all variables are real-valued the fundamental system is real-valued. Then both $([u_1]_\varrho,[\tau u_1]_\varrho)$ and $([u_2]_\varrho,[\tau u_2]_\varrho)$ lie in $\mathrm{T}_{\max}$ near a. Applying Lemma 4.32 yields $([f]_\varrho,[\tau f]_\varrho),([g]_\varrho,[\tau g]_\varrho) \in \mathrm{T}_{\max}$ such that $f = u_1$ near a, $f = 0$ near b and $g = u_2$ near a, $g = 0$ near b. Thus we have

$$W(f,g)(a) = W(u_1,u_2) = 1$$

and since u_1 is real

$$W(f,f^*)(a) = W(u_1,u_1^*)(a) = W(u_1,u_1) = 0.$$

We have shown the right-hand implication of the l.c. case and the left-hand implication of the l.p. case.

Now assume τ is in the l.p. case at a and regular at b. By Corollary 4.45 we have $\dim\ker(\mathrm{T}_{\min} - i) = 1$ and by the first von Neumann formula (Theorem 2.10)

$$\mathrm{T}_{\max} = T_{\min}^* = \mathrm{T}_{\min} \oplus N_+(\mathrm{T}_{\min}) \oplus N_-(\mathrm{T}_{\min}).$$

Hence $\mathrm{T}_{\max}$ is a two-dimensional extension of $\mathrm{T}_{\min}$. By Lemma 4.34 we can find $([\tilde f]_\varrho,[\tau\tilde f]_\varrho) \in \mathrm{T}_{\max}$ with $\tilde f(b) = c_1$ and $\tilde f^{[1]}(b) = c_2$. Now with Lemma 4.32 we can find $([f]_\varrho,[\tau f]_\varrho) \in \mathrm{T}_{\max}$ with $f = 0$ near a and $f = \tilde f$ near b. This way we can find $([v]_\varrho,[\tau v]_\varrho),([w]_\varrho,[\tau w]_\varrho) \in \mathrm{T}_{\max}$ such that $v = w = 0$ near a and $v(b) = 1$, $v^{[1]}(b) = 0$ and $w(b) = 0$, $w^{[1]}(b) = 1$. By Corollary 4.37 v and w are linearly independent modulo $\mathrm{T}_{\min}$ and do not lie in $\mathrm{T}_{\min}$. Hence we can write

$$\mathrm{T}_{\max} = \mathrm{T}_{\min} + \operatorname{span}\{v,w\}.$$

Now that means for each $([f]_\varrho,[\tau f]_\varrho),([g]_\varrho,[\tau g]_\varrho) \in \mathrm{T}_{\max}$ we have some $([f_0]_\varrho,[\tau f_0]_\varrho),([g_0]_\varrho,[\tau g_0]_\varrho) \in \mathrm{T}_{\min}$ such that

$$f = f_0 + c_1 v + c_2 w \quad \text{and} \quad g = g_0 + d_1 v + d_2 w.$$

Hence $f = f_0$ near a and $g = g_0$ near a and therefore

$$W(f,g)(a) = W(f_0,g_0)(a) = 0.$$

Now if τ is not regular at b we take some $c \in (a,b)$. Then for each $([f]_\varrho,[\tau f]_\varrho) \in \mathrm{T}_{\max}$ the function $f|_{(a,c)}$ lies in the maximal relation induced by $\tau|_{(a,c)}$. Since $\tau|_{(a,c)}$ is regular at c we can apply the first part of the proof. This proves the right-hand implication of the l.p. case and the left-hand implication of the l.c. case.

□

Lemma 4.47. *Let τ be in the l.p. case at both endpoints and $z \in \mathbb{C}\backslash\mathbb{R}$. Then there is no non-trivial solution u of $(\tau - z)u = 0$ such that $([u]_\varrho, [\tau u]_\varrho) \in \mathrm{T}_{\max}$.*

Proof.
If $[u]_\varrho \in L^2((a,b);\varrho)$ such that u is a solution of $(\tau - z)u = 0$ then u^* is a solution of $(\tau - z^*)u = 0$ and both $([u]_\varrho, [\tau u]_\varrho)$ and $([u^*]_\varrho, [\tau u^*]_\varrho)$ lie in $\mathrm{T}_{\max}$. Now the Lagrange identity yields for each $\alpha, \beta \in (a,b)$, $\alpha < \beta$

$$\begin{aligned} W(u,u^*)(\beta) - W(u,u^*)(\alpha) &= \int_\alpha^\beta (u^*\tau u - u\tau u^*)\, d\varrho \\ &= \int_\alpha^\beta (zuu^* - uz^*u^*)\, d\varrho \\ &= 2i\,\mathrm{Im}(z) \int_\alpha^\beta |u|^2\, d\varrho. \end{aligned}$$

As $\alpha \to a$ and $\beta \to b$ the left-hand side converges to zero by Lemma 4.46 and the right-hand side converges to $2i\,\mathrm{Im}(z)\|u\|^2$, hence $\|u\| = 0$.

□

Now we can collect the results for the different endpoint scenarios of τ by looking at the deficiency index of our minimal relation $\mathrm{T}_{\min}$.

Theorem 4.48. *The deficiency index of the minimal relation is given by*

$$\mathrm{n}(\mathrm{T}_{\min}) = \begin{cases} 0 & \text{if } \tau \text{ is in the l.c. case at no boundary point,} \\ 1 & \text{if } \tau \text{ is in the l.c. case at exactly one boundary point,} \\ 2 & \text{if } \tau \text{ is in the l.c. case at both boundary points.} \end{cases}$$

Proof.
If τ is in the l.c. case at both endpoints for all solutions u of $(\tau - i)u = 0$ the elements $([u]_\varrho, [\tau u]_\varrho)$ lie in $\mathrm{T}_{\max}$ by definition of the l.c. cases and the fact that all solutions are locally integrable. It follows that $\mathrm{n}(\mathrm{T}_{\min}) = \dim\ker(\mathrm{T}_{\max} - i) = 2$.

If τ is in the l.c. case at exactly one endpoint there is exactly one non-trivial solution u of $(\tau - i)u = 0$ for which $([u]_\varrho, [\tau u]_\varrho) \in \mathrm{T}_{\max}$ by Corollary 4.45.

If τ is in the l.p. case at both endpoints, we have $\ker(\mathrm{T}_{\max} - i) = \{0\}$ by Lemma 4.47 and hence $\mathrm{n}(\mathrm{T}_{\min}) = 0$.

□

4.7 Self-adjoint realizations of τ

As we have seen in Proposition 4.22 we can always identify an element $\mathfrak{f} \in \mathrm{T}_{\mathrm{loc}}$ with some element $f \in \mathcal{D}_\tau$. As it is now mostly clear from the context what element we mean we will no longer differentiate the two elements in notation and write $f \in \mathrm{T}_{\mathrm{loc}}$ for some element $([f]_\varrho, [\tau f]_\varrho) \in \mathrm{T}_{\mathrm{loc}}$ for convenience and simplicity. We use the same simplified notation for all the restricted linear relations of $\mathrm{T}_{\mathrm{loc}}$.

Theorem 4.49. *Let $S \subseteq \mathrm{T}_{\max}$ be a linear relation. Then S is self-adjoint if and only if it is of the form*

$$S = \{f \in \mathrm{T}_{\max} \mid \forall g \in S : W_a^b(f, g^*) = 0\}.$$

Proof.
We define $S_0 := \{f \in \mathrm{T}_{\max} \mid \forall g \in S : W_a^b(f, g^*) = 0\}$ and first assume that S is self-adjoint: Take $f \in S$ then we have

$$\langle \tau f, g\rangle = \langle f, \tau g\rangle, \quad \forall g \in S.$$

Since $S \subseteq \mathrm{T}_{\max}$ the Lagrange identity (4.11) holds and therefore

$$0 = \langle \tau f, g\rangle - \langle f, \tau g\rangle = W_a^b(f, g^*)$$

for all $g \in S$. It follows that $f \in S_0$, so $S \subseteq S_0$. If $f \in S_0$ the definition of S_0 and the Lagrange identity ($S_0 \subseteq \mathrm{T}_{\max}$) yields

$$0 = W_a^b(f, g^*) = \langle \tau f, g\rangle - \langle f, \tau g\rangle$$

for all $g \in S$. By the definition of the adjoint this means $f \in S^*$ and as S is self-adjoint by assumption we have $f \in S$ and arrive at $S = S_0$.

Conversely, we assume that $S = S_0$. Again from the Lagrange identity follows that for $f \in S$ we have

$$\langle \tau f, g\rangle = \langle f, \tau g\rangle$$

for all $g \in S$, hence S is symmetric. From $T_0 \subseteq S_0 = S$ follows $S^* \subseteq T_0^* = \mathrm{T}_{\max}$ so for $f \in S^*$ we get

$$0 = \langle \tau f, g\rangle - \langle f, \tau g\rangle = W_a^b(f, g^*)$$

for all $g \in S$, hence $f \in S_0 = S$ and we arrive at $S^* = S$.

□

From now on we look at the specific case in which τ is regular at a and in the l.p. case at b. The restriction of τ to be in the l.p. case at b grants more convenient boundary conditions for the self-adjoint realizations and is better suited to understand the concept. We refer to [1], Section 6f for the case where τ is in the l.c. case at b, which works very similarly but with more parameters for the boundary conditions.

Theorem 4.50. *Let τ be regular at a and in the l.p. case at b and let S be a linear relation with $S \subseteq \mathrm{T}_{\max}$. Then S is self-adjoint if and only if*

$$S = \{f \in \mathrm{T}_{\max} \mid f(a)\cos\varphi - f^{[1]}(a)\sin\varphi = 0\}$$

for some $\varphi \in [0, \pi)$.

For proof we refer to [1], Proposition 7.1 and Theorem 7.3.

4.8 The Weyl m-function

General assumption: *From now on let τ be regular at a and in the l.p. case at b. Furthermore we denote S as the self-adjoint realization of τ with Dirichlet boundary conditions, i.e. $S := \{f \in \mathrm{T}_{\max} \mid f(a) = 0\}$*

For $z \in \mathbb{C}$ we denote $\theta(z,\cdot)$ and $\phi(z,\cdot)$ as the solutions of $(\tau - z)u = 0$ with the initial values

$$\theta(z,a) = 1,\ \theta^{[1]}(z,a) = 0 \quad \text{and} \quad \phi(z,a) = 0,\ \phi^{[1]}(z,a) = 1.$$

For all $z \in \mathbb{C}$ the functions $\{\theta(z,\cdot), \phi(z,\cdot)\}$ are a fundamental system of $(\tau - z)u = 0$ satisfying $W(\theta(z,\cdot), \phi(z,\cdot)) = 1$ and $\phi(z,\cdot)$ satisfies the boundary condition of the self-adjoint realization S of τ, i.e. $\phi(z,\cdot)$ lies in S near a. Note that for $\lambda \in \mathbb{R}$ the solutions $\theta(\lambda,\cdot)$ and $\phi(\lambda,\cdot)$ are real-valued by Theorem 4.3 as all parameters are real.

Definition 4.51. The function $m(z) : \rho(S) \to \mathbb{C}$ is defined such that the solutions

$$\psi(z,x) = \theta(z,x) + m(z)\phi(z,x), \quad z \in \rho(S),\ x \in (a,b)$$

lie in $L^2((a,b);\varrho)$ and is called *Weyl m-function*. For $z \in \rho(S)$ we call $\psi(z,\cdot)$ the *Weyl solutions* of S.

Remark 4.52.

- From Theorem 4.48 follows, that $m(z)$ and thus $\psi(z,x)$ are well defined, as for our specific τ there is always a solution in $\mathrm{T}_{\max}$ and therefore in $L^2((a,b);\varrho)$. As mentioned $\phi(z,\cdot)$ satisfies the boundary conditions at a, i.e. lies in S near a and now the Weyl solution $\psi(z,\cdot)$ satisfies $L^2((a,b);\varrho)$ near b, i.e. $\psi(z,\cdot)$ lies in S near b.
- The solutions $\psi(z,\cdot)$ and $\phi(z,\cdot)$ are linearly independent: Assume they are not, then $\psi(z,a) = h(z)\phi(z,a) = 0$ for some function $h : \rho(S) \to \mathbb{C}$ and $\psi(z,\cdot)$ would be an eigenfunction for $z \in \rho(S)$ which is a contradiction to the definition of $\rho(S)$.
- $W(\phi(z,\cdot), \psi(z,\cdot)) \neq 0$ as the two functions are linear independent solutions of $(\tau - z)u = 0$.
- As our initial values for θ and ϕ are real-valued and the equation $(\tau - z)\theta(z,x) = 0$ implies $(\tau - z^*)\theta(z,x)^* = 0$ we have $\theta(z,x)^* = \theta(z^*,x)$ and similarly $\phi(z,x)^* = \phi(z^*,x)$ and $\psi(z,x)^* = \psi(z^*,x)$.
- If $\varphi \in (0,\pi)$ and a self-adjoint realization of τ is given as the linear relation $S_\varphi := \{f \in \mathrm{T}_{\max} \mid f(a)\cos\varphi - f^{[1]}(a)\sin\varphi\}$ (see Theorem 4.50) then we can define the Weyl m-function of $S_\varphi$with the solutions of $(\tau - z)u = 0$ satisfying

  $$\theta(z,a) = \cos\varphi,\ \theta^{[1]}(z,a) = -\sin\varphi \quad \text{and} \quad \phi(z,a) = \sin\varphi,\ \phi^{[1]}(z,a) = \cos\varphi$$

  for $z \in \mathbb{C}$.
- For $z \in \mathbb{C}\backslash\mathbb{R}$ the solution $\phi(z,\cdot)$ does not vanish for $x > a$.
  To show this suppose there exists some $z \in \mathbb{C}\backslash\mathbb{R}$ and $x > 0$ such that $\phi(z,x) = 0$. We look at the restricted differential expression $\tau|_{[\alpha,x]}$ which is regular at both endpoints. There exists a self-adjoint realization $S_{[\alpha,x]}$ of τ with Dirichlet boundary conditions (see [1], Theorem 7.6), i.e. $S_{[\alpha,x]} := \{f \in T_{\max,[\alpha,x]} \mid f(a) = 0,\ f(x) = 0\}$. Since $\phi(z,\cdot)$ satisfies both boundary conditions of $S_{[\alpha,x]}$ we have $\phi(z,\cdot)|_{[\alpha,x]} \in S_{[\alpha,x]}$. The function $\phi(z,\cdot)$ is a solution of $(\tau|_{[\alpha,x]} - z)u = 0$ which means it is an eigenfunction to the eigenvalue z. As $z \in \mathbb{C}\backslash\mathbb{R}$ this is a contradiction to the self-adjointness of $S_{[\alpha,x]}$.

Theorem 4.53. *The Weyl m-function $m : \rho(S) \to \mathbb{C}$ is a Nevanlinna function on $\mathbb{C}^+$ satisfying*

$$m(z^*) = m(z)^*, \quad z \in \rho(S).$$

Proof.
For the proof of analyticity we refer to [1], Theorem 9.2.

The property $m(z^*) = m(z)^*$ for $z \in \rho(S)$: With the last Remark we have

$$\psi(z^*, x) = \psi(z, x)^* = (\theta(z, x) + m(z)\phi(z, x))^* = \theta(z^*, x) + m(z)^*\phi(z^*, x).$$

Since the Weyl solution and m are uniquely defined, $m(z)^* = m(z^*)$ must hold true.

To show that m is a Nevanlinna function, we show that

$$\frac{\operatorname{Im}(m(z))}{\operatorname{Im}(z)} = \|\psi(z, \cdot)\|^2 > 0 \quad \text{for } z \in \mathbb{C}\backslash\mathbb{R}.$$

First we compute for $z, z_1, z_2 \in \mathbb{C}\backslash\mathbb{R}$

$$\begin{aligned}\psi(z, a) &= \theta(z, a) + m(z)\phi(z, a) = 1,\\ \psi^{[1]}(z, a) &= \theta^{[1]}(z, a) + m(z)\phi^{[1]}(z, a) = m(z).\end{aligned}$$

and therefore

$$W(\psi(z_1, \cdot), \psi(z_2, \cdot))(a) = m(z_2) - m(z_1).$$

Since $\psi(z, \cdot)$ lies in $\mathrm{T}_{\max}$ near b, Lemma 4.46 yields

$$W(\psi(z_1, \cdot), \psi(z_2, \cdot))(b) = 0.$$

With the Lagrange identity (see equation (4.11)) we get for $z_1, z_2 \in \mathbb{C}\backslash\mathbb{R}$ that

$$\begin{aligned}(z_1 - z_2)\int_a^b \psi(z_1, \cdot)\psi(z_2, \cdot)\, d\varrho &= \int_a^b \psi(z_2, \cdot)\tau\psi(z_1, \cdot) - \psi(z_1, \cdot)\tau\psi(z_2)\, d\varrho =\\ &= W_a^b(\psi(z_1, \cdot), \psi(z_2, \cdot)) = -W(\psi(z_1, \cdot), \psi(z_2, \cdot))(a) = m(z_1) - m(z_2).\end{aligned}$$

Now using the above identity by setting $z_1 = z$ and $z_2 = z^*$ and the properties $\psi(z, \cdot)^* = \psi(z^*, \cdot)$ and $m(z)^* = m(z^*)$ as already proven, we get

$$\|\psi(z, \cdot)\|^2 = \int_a^b \psi(z, \cdot)\psi(z^*, \cdot)\, d\varrho = \frac{m(z) - m(z^*)}{z - z^*} = \frac{\operatorname{Im}(m(z))}{\operatorname{Im}(z)}.$$

Since $\psi(z, \cdot)$ is a non-trivial solution of $(\tau - z)u = 0$ for all $z \in \mathbb{C}\backslash\mathbb{R}$ we arrive at $\frac{\operatorname{Im}(m(z))}{\operatorname{Im}(z)} > 0$, hence the function m is a Nevanlinna function.

□

5 Asymptotics of the Weyl m-function

In this section we examine the behavior of the Weyl m-function for $\mathrm{Im}(z) \to +\infty$. This can be helpful in order to get information about the measure χ by only looking at the Weyl m-function of S. We will start with describing the asymptotics in terms of the fundamental system $\{\theta, \phi\}$. Then develop further estimates for the asymptotic behavior with help of integral equations for $\{\theta, \phi\}$ and the Weyl circles.

5.1 The Weyl m-function for S

Before we focus on further results for the Weyl m-function we need some tools that we will quote from [1].

Definition 5.1. With the same notation as above we call the function

$$G(z,x,y) := \begin{cases} \phi(z,y)\psi(z,x) & y \le x, \\ \phi(z,x)\psi(z,y) & y > x, \end{cases}$$

for $x, y \in (a,b)$ and $z \in \rho(S)$ the *Green function of* S.

From the definition we see that for $x, y \in (a,b)$ the function $z \mapsto G(z,x,y)$ is analytic in $\rho(S)$ and that $G(z,x,\cdot) \in \operatorname{dom} S$. The Green function is of general interest as it is the core function of the resolvent R_z. For $z \in \rho(S)$ we have the identity

$$R_z g(x) = \int_a^b G(z,x,y) g(y)\, d\varrho(y), \quad x \in (a,b),\ g \in L^2((a,b);\varrho).$$

For proof see [1], Theorem 8.3.

We give a quick summary of the Fourier transform associated with the self-adjoint realization S of τ. For the basic concept we refer to [4], Section 9.3. and a detailed discussion in regard of our measure-valued differential expression τ can be found in [1], Section 10.

As the Weyl m-function is a Nevanlinna function by Theorem 4.53 there exists a unique positive Borel measure μ on $\mathbb{R}$ so that the Weyl m-function has the representation

$$m(z) = c_1 + z c_2 + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) d\mu(\lambda)$$

for some $c_1, c_2 \in \mathbb{R}$ with $c_2 \ge 0$. From a variation of the Stieltjes-inversion formula (see [1], Lemma 10.1) follows that this positive Borel measure μ can be extracted from the Weyl m-function via

$$\mu((\lambda_1, \lambda_2]) = \lim_{\delta \searrow 0} \lim_{\varepsilon \searrow 0} \frac{1}{\pi} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} \mathrm{Im}(m(t + i\varepsilon))\, d\lambda(t)$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$. Denoting $\mathcal{D} := \overline{\operatorname{dom} S}$ (from Section 2 we know that $S \cap (\mathcal{D} \times \mathcal{D})$ is an operator) and $\phi(\lambda, \cdot)$ as above we can construct an isometric linear operator $\mathcal{F}$ by

$$\mathcal{F} : \mathcal{D} \to L^2(\mathbb{R}; \mu)$$
$$\mathcal{F} f(\lambda) := \lim_{\alpha \to a} \lim_{\beta \to b} \int_\alpha^\beta \phi(\lambda, x) f(x)\, d\varrho(x).$$

This Fourier transformation $\mathcal{F}$ (in the simple case $\tau = \frac{d^2}{dx^2}$ with $(a,b) = (0,+\infty)$ we get the classical one-dimensional Fourier transformation) with respect to S satisfies

$$\mathcal{F}G(z,x,\cdot)(\lambda) = \frac{\phi(\lambda,x)}{\lambda - z} \in L^2(\mathbb{R};\mu) \tag{5.1}$$

and as $\mathcal{F}$ is an isometry we have the equality

$$\|G(z,x,\cdot)\|^2 = \left\| \left(\lambda \mapsto \frac{\phi(\lambda,x)}{\lambda - z} \right) \right\|^2_{L^2(\mathbb{R};\mu)} \tag{5.2}$$

for $x \in (a,b)$, $z \in \mathbb{C}\backslash\mathbb{R}$.

With the help of these tools we can now calculate the first estimate for the asymptotic behavior of the Weyl m-function as an expression of the fundamental system $\{\theta, \phi\}$. The proof is done in a similar manner as the proof in the paper [6], Lemma A.1. which covers distributional potentials.

Theorem 5.2. *With the notation from above the Weyl m-function satisfies the asymptotic behavior*

$$m(z) = -\frac{\theta(z,x)}{\phi(z,x)} + \mathrm{o}\left(\frac{z}{\phi(z,x)^2}\right), \quad x \in (a,b)$$

as $\mathrm{Im}(z) \to +\infty$.

Proof.
We first show that $z \mapsto G(z,x,x)$ is a Nevanlinna function (see Theorem 1.15) with $c_2 = 0$.

Let $z \in \mathbb{C}\backslash\mathbb{R}$ and u be a solution of $\tau u = zu$ then u^* is a solution of $\tau u = z^* u$ and the Lagrange identity (see equation (4.11)) yields

$$-2i\,\mathrm{Im}(z) \int_a^x |u|^2 \, d\varrho = W_a^x(u,u^*). \tag{5.3}$$

Note that for $u \in S$ we have $W(u,u^*)(a) = W(u,u^*)(b) = 0$ by Lemma 4.46. With help of equation (5.3) we calculate for $z \in \mathbb{C}\backslash\mathbb{R}, x \in (a,b)$ (with abbreviating $\phi(x) := \phi(z,x)$ and $\psi(x) := \psi(z,x)$)

$$\begin{aligned}
&\mathrm{Im}(z) \int_a^b |G(z,x,y)|^2 \, d\varrho(y) = \\
&= |\psi(z,x)|^2 \,\mathrm{Im}(z) \int_a^x |\phi(z,y)|^2 \, d\varrho(y) + |\phi(z,x)|^2 \,\mathrm{Im}(z) \int_x^b |\psi(x,y)|^2 \, d\varrho(y) \\
&= |\psi(z,x)|^2 \left(-\frac{1}{2i} W_a^x(\phi,\phi^*) \right) + |\phi(z,x)|^2 \left(-\frac{1}{2i} W_x^b(\psi,\psi^*) \right) \\
&= \frac{1}{2i} \left[-\psi(x)^*\psi(x) \left(\phi(x)\phi^{[1]}(x)^* - \phi^{[1]}(x)\phi(x)^* \right) + \phi(x)^*\phi(x) \left(\psi(x)\psi^{[1]*}(x) - \psi^{[1]}(x)\psi(x)^* \right) \right] \\
&= \frac{1}{2i} \Big[-G(z,x,x)\psi^*\phi^{[1]*} + G^*(z,x,x)\psi(x)\phi^{[1]}(x) + \\
&\qquad + G(z,x,x)\phi(x)^*\psi^{[1]}(x)^* - G(z,x,x)^*\phi(x)\psi^{[1]}(x) \Big] \\
&= \frac{1}{2i} \left[G(z,x,x) \underbrace{W(\phi^*,\psi^*)(x)}_{=1} - G(z,x,x)^* \underbrace{W(\phi,\psi)(x)}_{=1} \right] \\
&= \mathrm{Im}\, G(z,x,x).
\end{aligned}$$

This result together with the equation (5.1) for the second identity yields

$$\begin{aligned}
&\operatorname{Im}(G(z,x,x)) = \operatorname{Im}(z)\int_a^b |G(z,x,y)|^2\, d\varrho(y) = \operatorname{Im}(z)\int_{\mathbb{R}} \left|\frac{\phi(\lambda,x)}{\lambda - z}\right|^2 d\mu(\lambda) = \\
&= \int_{\mathbb{R}} \phi(\lambda,x)^2 \frac{\operatorname{Im}(\lambda - z^*)}{|\lambda - z|^2}\, d\mu(\lambda) = \operatorname{Im}\int_{\mathbb{R}} \phi(\lambda,x)^2 \frac{1}{\lambda - z}\, d\mu(\lambda) = \\
&= \operatorname{Im}\int_{\mathbb{R}} \phi(\lambda,x)^2 \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) d\mu(\lambda).
\end{aligned}$$

As the left and right-hand side are imaginary parts of analytic functions in z which coincide, the functions can only differ by an additive term $c_1(x) \in \mathbb{R}$ independent of z, i.e.

$$G(z,x,x) = c_1(x) + \int_{\mathbb{R}} \phi(\lambda,x)^2 \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) d\mu(\lambda).$$

Hence for $x \in (a,b)$ the map $z \mapsto G(z,x,x)$ is a Nevanlinna function with representation (1.4) with $c_2 = 0$ and Proposition 1.16 yields $\lim_{t\to+\infty} \frac{G(it,x,x)}{it} = c_2 = 0$, and formulated with the little-o notation this means

$$\psi(z,x)\phi(z,x) = G(z,x,x) = o(z) \tag{5.4}$$

as $\operatorname{Im}(z) \to +\infty$. Solving the expression of $\psi(z,x)$ for $m(z)$ and inserting (5.4) leads to

$$m(z) = -\frac{\theta(z,x)}{\phi(z,x)} + \frac{\psi(z,x)\phi(z,x)}{\phi(z,x)^2} = -\frac{\theta(z,x)}{\phi(z,x)} + \mathrm{o}\left(\frac{z}{\phi(z,x)^2}\right)$$

as $\operatorname{Im}(z) \to +\infty$.

□

5.2 The one-dimensional Schrödinger operator with measure-valued potential

General assumption: *Let τ be the differential expression on $(0,+\infty)$ and let τ be regular at 0 and in the l.p. case at $+\infty$ and let $\varrho = \zeta = \lambda$.*

In this subsection we look at the even more specific case of our differential expression τ on the interval $(0,+\infty)$ with the measures ϱ and ζ set as the Lebesgue measure λ. The motivation to set the left endpoint to zero is merely to simplify the notation, the following results stay true for τ with arbitrary regular left endpoint $a \in \mathbb{R}$. As our left endpoint 0 is regular we know from Theorem 4.13, that we can extend the interval to $[0,+\infty)$. This leads to the differential expression with the usual derivatives

$$\tau f = \frac{d}{d\lambda}\left(-\frac{df}{d\lambda} + \int f\, d\chi\right) = \left(-f' + \int f\, d\chi\right)'$$

and choosing some $c \in [0,+\infty)$ the maximum domain of functions for which this differential expression makes sense is given as

$$\mathcal{D}_\tau = \left\{ f \in AC_{\mathrm{loc}}([0,+\infty);\lambda) \mid \left(x \mapsto -f'(x) + \int_c^x f\, d\chi\right) \in AC_{\mathrm{loc}}([0,+\infty);\lambda) \right\}.$$

Hence our solutions are continuous and as deduced in the example of generalized cases in Section 4, the first derivative can have jumps if the measure χ has point mass. As we assume the

differential expression τ to be regular at the endpoint 0 the measure $|\chi|$ satisfies $|\chi|([0,d]) < +\infty$ for all $d \in (0, +\infty)$.

We can interpret the equation $(\tau - z)u = 0$ physically: In this case the measure χ represents a physical potential. By describing the potential through the mathematical class of measures we are able to describe a very broad range of physical potentials. The equation $(\tau - z)u = 0$ then represents the one-dimensional time-independent Schrödinger equation well known with the notation of quantum mechanics as $H\psi = E\psi$.

In the following we are going to improve the accuracy for estimating the asymptotic behavior of the Weyl m-function for the self-adjoint realization S of τ on $[0, +\infty)$ with the goal to get

$$m(z) = -\sqrt{-z} - \int_0^x e^{-2\sqrt{-z}y}\, d\chi(y) + \mathrm{o}\left(\frac{1}{\sqrt{-z}}\right)$$

as $\mathrm{Im}(z) \to +\infty$.
The asymptotic behavior of the Weyl m-function is important for inverse spectral theory (see [4] Section 9.4 for an introduction) since it holds all the spectral information of the corresponding self-adjoint realization S. Knowing the behavior of the Weyl m-function for large values can give us information about the corresponding potential represented as the measure χ (see [4], Theorem 9.22 for the basic idea).

Remark 5.3.
If we look at the example $\chi \equiv 0$ the equation $(\tau - z)u = 0$ is simply given as $-u''(x) = zu(x)$. The fundamental system $\{\theta, \phi\}$ (as defined in the last section with the canonical initial values at a) is then given as $\theta(z,x) = \cosh(\sqrt{-z}x)$ and $\phi(z,x) = \frac{1}{\sqrt{-z}}\sinh(\sqrt{-z}x)$ for $z \in \mathbb{C}\backslash\mathbb{R}$ taking the standard branch of the square root with $(-\infty, 0]$ cut out.

In order to determine the asymptotic behavior of a fundamental system for $(\tau - z)u = 0$ we rewrite the solutions as integral equations. We choose the notation $c(z,x) := \theta(z,x)$ and $s(z,x) := \phi(z,x)$ in the next lemma motivated by the cosine and sine solution of the basic case stated in the last remark.

In the following we use $\sqrt{\cdot}$ as the square root of the standard branch with $(-\infty, 0]$ cut out and abbreviate $k := \sqrt{-z}$ for $z \in \mathbb{C}\backslash\mathbb{R}$. Note that for every $z \in \mathbb{C}\backslash\mathbb{R}$ we always have $\mathrm{Re}(k) > 0$. If we fix the real part of z and look at the limit $\mathrm{Im}(z) \to +\infty$ we have $\mathrm{Re}(k) \to +\infty$ as well as $\mathrm{Im}(k) \to -\infty$.

Lemma 5.4. *The solutions $c(z,x)$ and $s(z,x)$ of the equation $(\tau - z)u = 0$ for $z \in \mathbb{C}\backslash\mathbb{R}$ with the initial values*

$$c(z,0) = 1,\ c'(z,0) = 0 \quad \textit{and} \quad s(z,0) = 0,\ s'(z,0) = 1$$

are given as the solutions of the integral equations

$$c(z,x) = \cosh(\sqrt{-z}x) + \frac{1}{\sqrt{-z}}\int_0^x \sinh(\sqrt{-z}(x-y))c(z,y)\, d\chi(y),$$
$$s(z,x) = \frac{1}{\sqrt{-z}}\sinh(\sqrt{-z}x) + \frac{1}{\sqrt{-z}}\int_0^x \sinh(\sqrt{-z}(x-y))s(z,y)\, d\chi(y).$$

The derivatives are given as the the solutions of the integral equations

$$c'(z,x) = \sqrt{-z}\sinh(\sqrt{-z}x) + \int_0^x \cosh(\sqrt{-z}(x-y))c(z,y)\,d\chi(y),$$
$$s'(z,x) = \cosh(\sqrt{-z}x) + \int_0^x \cosh(\sqrt{-z}(x-y))s(z,y)\,d\chi(y).$$

Proof.
We only prove the statement for $c(z,x)$ as the calculations for $s(z,x)$ are done in a similar way. Since $c(z,x)$ is a solution of $(\tau - z)u = 0$ we can integrate this equation for c and get

$$\int_0^x c\,d\chi = c'(z,x) + z\int_0^x c\,d\lambda. \tag{5.5}$$

We start on the right-hand side of the integral equation and show that this side can be transformed into the left-hand side $c(z,x)$. We first use the integration by parts formula (1.3), setting $F(y) = \sinh(k(x-y))$, $d\mu = -k\cosh(k(x-y))d\lambda$ and $G(y) = \int_0^y c\,d\chi$, $d\nu = c(y)\,d\chi$, then insert (5.5) and then another integration by parts:

$$\begin{aligned}
&\cosh(kx) + \frac{1}{k}\int_0^x \sinh(k(x-y))c(z,y)\,d\chi(y) = \\
&\cosh(kx) + \frac{1}{k}\Bigg(\underbrace{\left[\int_0^y c\,d\chi\sinh(k(x-y))\right]_{y=0}^x}_{=0} - \\
&\qquad\qquad - \int_0^x\int_0^y c\,d\chi\cosh(k(x-y))(-k)d\lambda(y)\Bigg) = \\
&\cosh(kx) + \int_0^x \left(c'(z,y) - k^2\int_0^y c\,d\lambda\right)\cosh(k(x-y))\,d\lambda(y) = \\
&\cosh(kx) + [c(z,y)\cosh(k(x-y))]_{y=0}^x - \int_0^x c(z,y)\sinh(k(x-y))(-k)\,d\lambda(y) - \\
&\qquad - k^2\left(\underbrace{\left[\frac{\sinh(k(x-y))}{-k}\int_0^y c\,d\lambda\right]_{y=0}^x}_{=0} - \int_0^x \frac{\sinh(k(x-y))}{-k}c(z,y)\,d\lambda(y)\right) = \\
&= c(z,x).
\end{aligned}$$

For $c'(z,x)$ the same method leads to

$$
\begin{aligned}
&k\sinh(kx) + \int_0^x \cosh(k(x-y))c(z,y)\,d\chi(y) = \\
&k\sinh(kx) + \left(\left[\int_0^y c\,d\chi\cosh(k(x-y))\right]_{y=0}^x - \int_0^x\int_0^y c\,d\chi\sinh(k(x-y))(-k)\,d\lambda(y)\right) = \\
&k\sinh(kx) + \int_0^x c\,d\chi + k\int_0^x \left(c'(z,y) - k^2\int_0^y c\,d\lambda\right)\sinh(k(x-y))\,d\lambda(y) = \\
&k\sinh(kx) + \int_0^x c\,d\chi + \\
&\qquad + k\left(\underbrace{[c(z,y)\sinh(k(x-y))]_{y=0}^x}_{=-\sinh(kx)} - \int_0^x c(z,y)\cosh(k(x-y))(-k)\,d\lambda(y)\right) - \\
&\qquad - k^3\left(\left[\frac{\cosh(k(x-y))}{-k}\int_0^y c\,d\lambda\right]_{y=0}^x - \int_0^x \frac{\cosh(k(x-y))}{-k}c(z,y)\,d\lambda(y)\right) = \\
&\int_0^x c\,d\chi + k^2\int_0^x c\,d\lambda = c'(z,x).
\end{aligned}
$$

□

With the help of the integral equations we can extract an estimate of the asymptotic behavior for our fundamental system $\{c,s\}$ as $\operatorname{Im}(z)\to+\infty$. We remember (see Definition 1.13) that for some function f satisfying $f(z)=\mathrm{O}(g(z))$ as $\operatorname{Im} z\to+\infty$, we can write $f(z)=g(z)D(z)$ with some function $D(z)$ for which there exists some $C>0$ such that $|D(z)|<C$ for large enough $\operatorname{Im}(z)$.

Now we consider that the function f is also dependent on another variable x. We say *the function $f(z,x)$ satisfies $f(z,x)=\mathrm{O}(g(z,x))$ locally uniformly in x* if $f(z,x)=g(z,x)D(z,x)$ with some error function $D(z,x)$ for which there exists $C(x_0)>0$ such that $\sup_{x\in[0,x_0]}|D(z,x)|\le C(x_0)$ for large enough $\operatorname{Im}(z)$.

To simplify notation we denote

$$
\chi(x) := \begin{cases} \chi([0,x)) & x\in(0,+\infty), \\ 0 & x=0. \end{cases}
$$

Lemma 5.5. *The function $c(z,x)$ and its derivative $c'(z,x)$ can be written as*

$$
\begin{aligned}
c(z,x) &= \cosh(\sqrt{-z}x) + \frac{1}{2\sqrt{-z}}\sinh(\sqrt{-z}x)\chi(x) + \\
&\quad + \frac{e^{\sqrt{-z}x}}{4\sqrt{-z}}\left(\int_0^x e^{-2\sqrt{-z}y}\,d\chi(y) - \int_0^x e^{-2\sqrt{-z}(x-y)}\,d\chi(y)\right) - \frac{e^{\sqrt{-z}x}}{z}D_1(z,x), \\
c'(z,x) &= \sqrt{-z}\sinh(\sqrt{-z}x) + \frac{1}{2}\cosh(\sqrt{-z}x)\chi(x) + \\
&\quad + \frac{e^{\sqrt{-z}x}}{4}\left(\int_0^x e^{-2\sqrt{-z}y}\,d\chi(y) + \int_0^x e^{-2\sqrt{-z}(x-y)}\,d\chi(y)\right) + \frac{e^{\sqrt{-z}x}}{\sqrt{-z}}D_2(z,x),
\end{aligned}
$$

with error functions $D_i(z,x)$ satisfying $D_i(z,x) = \mathrm{O}(1)$ as $\mathrm{Im}(z) \to +\infty$ locally uniform in x for $i = 1, 2$.

The function $s(z,x)$ and its derivative $s'(z,x)$ can be written as

$$\begin{aligned} s(z,x) &= \frac{1}{\sqrt{-z}} \sinh(\sqrt{-z}x) - \frac{1}{2z} \cosh(\sqrt{-z}x)\chi(x) + \\ &\quad + \frac{e^{\sqrt{-z}x}}{4z} \left(\int_0^x e^{-2\sqrt{-z}y} \, d\chi(y) + \int_0^x e^{-2\sqrt{-z}(x-y)} \, d\chi(y) \right) + \frac{e^{\sqrt{-z}x}}{\sqrt{-z}^3} D_3(z,x), \\ s'(z,x) &= \cosh(\sqrt{-z}x) + \frac{1}{2\sqrt{-z}} \sinh(\sqrt{-z}x)\chi(x) - \\ &\quad - \frac{e^{\sqrt{-z}x}}{4\sqrt{-z}} \left(\int_0^x e^{-2\sqrt{-z}y} \, d\chi(y) - \int_0^x e^{-2\sqrt{-z}(x-y)} \, d\chi(y) \right) - \frac{e^{\sqrt{-z}x}}{z} D_4(z,x), \end{aligned}$$

with error functions $D_i(z,x)$ satisfying $D_i(z,x) = \mathrm{O}(1)$ as $\mathrm{Im}(z) \to +\infty$ locally uniform in x for $i = 3, 4$.

Proof.
Let $x_0 > 0$. First we define $\tilde{c}(z,x) := e^{-kx} c(z,x)$ and show that there exists some $C_0(x_0) \geq 0$ such that

$$|\tilde{c}(z,x)| \leq C_0(x_0), \quad z \in \mathbb{C} \backslash [0, +\infty), x \in [0, x_0].$$

For that we use the integral equation for c from the last lemma (Lemma 5.4) to write

$$\begin{aligned} \tilde{c}(z,x) &= e^{-kx} c(z,x) \\ &= e^{-kx} \cosh(kx) + \frac{1}{k} \int_0^x e^{-kx} \sinh(k(x-y)) c(z,y) \, d\chi(y) \\ &= \frac{1 + e^{-2kx}}{2} + \int_0^x \frac{1 - e^{-2k(x-y)}}{2k} \tilde{c}(z,y) \, d\chi(y). \end{aligned} \tag{5.6}$$

We show that the solution of this integral equation is bounded for all $z \in \mathbb{C} \backslash [0, +\infty), x \in [0, x_0]$. We define inductively

$$\varphi_0(z,x) = \frac{1 + e^{-2kx}}{2} \quad \text{and} \quad \varphi_{n+1}(z,x) = \int_0^x \frac{1 - e^{-2k(x-y)}}{2k} \varphi_n(z,y) \, d\chi(y)$$

and claim

$$|\varphi_n(z,x)| \leq \frac{x^n}{n!} \left(\int_0^x d|\chi| \right)^n, \quad n \geq 0.$$

For $n = 0$ this is true and with the estimate

$$\left| \frac{1 - e^{-2kx}}{2k} \right| = \left| \int_0^x e^{-2kt} \, dt \right| \leq x$$

as well as the substitution rule from Lemma 3.4 (similarly as in (3.6)) we inductively have

$$\begin{aligned}
|\varphi_{n+1}(z,x)| &\le \int_0^x \left|\frac{1-e^{-2k(x-y)}}{2k}\right| \frac{y^n}{n!}\left(\int_0^y d|\chi|\right)^n d|\chi|(y) \\
&\le \int_0^x (x-y)\frac{x^n}{n!}\left(\int_0^y d|\chi|\right)^n d|\chi|(y) \\
&\le \frac{x^{n+1}}{n!}\int_0^x \left(\int_0^y d|\chi|\right)^n d|\chi|(y) \\
&\le \frac{x^{n+1}}{(n+1)!}\left(\int_0^x d|\chi|\right)^{n+1}.
\end{aligned}$$

It follows that

$$\sum_{n=0}^N |\varphi_n(z,x)| \le \sum_{n=0}^N \frac{x^n}{n!}\left(\int_0^x d|\chi|\right)^n \le e^{x\int_0^x d|\chi|} \le e^{x_0\int_0^{x_0} d|\chi|} =: C_0(x_0) \tag{5.7}$$

for all $N \in \mathbb{N}$ and therefore the series $\varphi(z,x) := \sum_{n=0}^\infty \varphi_n(z,x)$ converges uniformly for $z \in \mathbb{C}\backslash[0,+\infty)\,, x \in [0,x_0]$. Going back to the definition of φ_n we have

$$\begin{aligned}
\sum_{n=0}^N \varphi_n(z,x) &= \frac{1+e^{-2kx}}{2} + \sum_{n=1}^N \int_0^x \frac{1-e^{-2k(x-y)}}{2k}\varphi_{n-1}(z,y)\,d\chi(y) \\
&= \frac{1+e^{-2kx}}{2} + \int_0^x \frac{1-e^{-2k(x-y)}}{2k}\sum_{n=0}^{N-1}\varphi_n(z,y)\,d\chi(y).
\end{aligned}$$

Taking the limit $N \to +\infty$ which commutes with the integral due to the uniform convergence leads to

$$\varphi(z,x) = \frac{1+e^{-2kx}}{2} + \int_0^x \frac{1-e^{-2k(x-y)}}{2k}\varphi(z,y)\,d\chi(y).$$

Since the solution of this integral equation is unique by the Gronwall lemma (Lemma 3.5) we have $\tilde{c}(z,x) = \varphi(z,x)$ and we see that the function $\tilde{c}(z,x)$ satisfies

$$|\tilde{c}(z,x)| \le C_0(x_0), \quad z \in \mathbb{C}\backslash[0,+\infty),\ x \in [0,x_0].$$

Next we show that the function $c(z,x)$ has the asymptotic behavior stated in the lemma.

To this end we first write

$$\begin{aligned}
c(z,x) &= e^{kx}\tilde{c}(z,x) \\
&= \cosh(kx) + \frac{e^{kx}}{k}\underbrace{\int_0^x \frac{1-e^{-2k(x-y)}}{2}\tilde{c}(z,y)\,d\chi(y)}_{\tilde{D}(z,x):=}.
\end{aligned}$$

The function $\tilde{D}(z,x)$ satisfies

$$\begin{aligned}
\sup_{x\in[0,x_0]}|\tilde{D}(z,x)| &= \sup_{x\in[0,x_0]}\left|\int_0^x \frac{1-e^{-2k(x-y)}}{2}\tilde{c}(z,y)\,d\chi(y)\right| \\
&\le e^{x_0\int_0^{x_0} d|\chi|}\int_0^{x_0} d|\chi| =: \tilde{C}(x_0)
\end{aligned} \tag{5.8}$$

for all $z \in \mathbb{C}\backslash[0,+\infty)$. Reinserting this into our integral equation for $c(z,x)$ leads to

$$\begin{aligned}
c(z,x) &= \cosh(kx) + \frac{1}{k}\int_0^x \sinh(k(x-y))\left(\cosh(ky) + \frac{e^{ky}}{k}\tilde{D}(z,y)\right) d\chi(y) \\
&= \cosh(kx) + \frac{e^{kx}}{4k}\int_0^x (1-e^{-2k(x-y)})(1+e^{-2ky})\, d\chi(y) + \\
&\qquad\qquad + \frac{e^{kx}}{k^2}\underbrace{\int_0^x \frac{1-e^{-2k(x-y)}}{2}\tilde{D}(z,y)\, d\chi(y)}_{D_1(z,x):=} \\
&= \cosh(kx) + \frac{e^{kx}}{4k}\int_0^x (1 - e^{-2kx} + e^{-2ky} - e^{-2k(x-y)})\, d\chi(y) + \frac{e^{kx}}{k^2}D_1(z,x) \\
&= \cosh(kx) + \frac{1}{2k}\sinh(kx)\chi(x) + \\
&\qquad + \frac{e^{kx}}{4k}\left(\int_0^x e^{-2ky}\, d\chi(y) - \int_0^x e^{-2k(x-y)}\, d\chi(y)\right) + \frac{e^{kx}}{k^2}D_1(z,x).
\end{aligned}$$

The function $D_1(z,x)$ satisfies

$$\begin{aligned}
\sup_{x\in[0,x_0]} |D_1(z,x)| &= \sup_{x\in[0,x_0]} \left|\int_0^x \frac{1-e^{-2k(x-y)}}{2}\tilde{D}(z,y)\, d\chi(y)\right| \\
&\leq \tilde{C}(x_0)\int_0^{x_0} d|\chi| := C_1(x_0)
\end{aligned}$$

for all $z \in \mathbb{C}\backslash[0,+\infty)$ and we arrive at the desired asymptotic for $c(z,x)$. The other asymptotics are calculated similarly.

□

Remark 5.6.
Note that the integral terms of last lemma

$$I_1(z,x) := \int_0^x e^{-2ky}\, d\chi(y) \quad \text{and} \quad I_2(z,x) := \int_0^x e^{-2k(x-y)}\, d\chi(y)$$

both are uniformly bounded in $z \in \mathbb{C}\backslash[0,+\infty)$. Applying the dominated convergence theorem to $I_1(z,x)$ for $x \in (0,+\infty)$ leads to

$$I_1(z,x) = \int_{\{0\}} 1\, d\chi(y) + \int_{(0,x)} e^{-2ky}\, d\chi(y) = \chi(\{0\}) + \mathrm{o}(1) \tag{5.9}$$

as $\mathrm{Im}(z) \to +\infty$ as well as

$$I_2(z,x) = \int_0^x e^{-2k(x-y)}\, d\chi(y) = \mathrm{o}(1)$$

as $\mathrm{Im}(z) \to +\infty$.

We will calculate better estimates for the error functions of Lemma 5.5 with the help of the dominated convergence theorem.

Lemma 5.7. *For $x \in (0,+\infty)$ the error functions of Lemma 5.5 can be further estimated as*

$$D_i(z,x) = \frac{1}{8}\int_{(0,x)} \big(\chi(y) + \chi(\{0\})\big)\, d\chi(y) + \mathrm{o}(1), \quad i = 1,2,$$
$$D_i(z,x) = \frac{1}{8}\int_{(0,x)} \big(\chi(y) - \chi(\{0\})\big)\, d\chi(y) + \mathrm{o}(1), \quad i = 3,4,$$

as $\mathrm{Im}(z) \to +\infty$.

Proof.
Let $x > 0$ be fixed. We continue with the notation introduced in the proof of Lemma 5.5 and calculate the estimate for the error function $D_1(z,x)$ of the solution $c(z,x)$.

First note that we have

$$\tilde{c}(z,x) = \frac{1+e^{-2kx}}{2} + \frac{1}{k}\tilde{D}(z,x)$$

where $\tilde{D}(z,x)$ satisfies $\sup_{x\in[0,x_0]} |\tilde{D}(z,x)| \le \tilde{C}(x_0)$ as $\mathrm{Im}(z) \to +\infty$ for $x_0 \in (0,+\infty)$ by (5.8). Reinserting this into the definition of the function $\tilde{D}(z,x)$ leads to

$$\begin{aligned}\tilde{D}(z,x) &= \int_0^x \frac{1-e^{-2k(x-y)}}{2}\tilde{c}(z,y)\, d\chi(y)\\ &= \frac{1}{4}\int_0^x \underbrace{(1-e^{-2k(x-y)})(1+e^{-2ky})}_{f(z,y):=}\, d\chi(y) + \frac{1}{2k}\underbrace{\int_0^x (1-e^{-2k(x-y)})\tilde{D}(z,y)\, d\chi(y)}_{F(z,x):=}.\end{aligned}$$

Now as the function $f(z,y)$ is locally uniformly bounded and since $f(z,y) \to 1 + \mathbb{1}_{\{0\}}(y)$ as $\mathrm{Im}(z) \to +\infty$ we can apply the dominated convergence theorem and get

$$\tilde{D}(z,x) = \begin{cases} \frac{1}{4}\big(\chi(x) + \chi(\{0\})\big) + \frac{1}{2k}F(z,x) & x > 0\\ 0 & x = 0\end{cases}$$

where the integral $F(z,x)$ satisfies $F(z,x) = \mathrm{O}(1)$ locally uniformly in x as $\mathrm{Im}(z) \to +\infty$. From the proof of Lemma 5.5 we know that the error function $D_1(z,x)$ is defined as

$$D_1(z,x) = \frac{1}{2}\int_0^x (1-e^{-2k(x-y)})\tilde{D}(z,y)\, d\chi(y).$$

If we now insert the calculation for $\tilde{D}(z,x)$ from above and apply the dominated convergence theorem again we get

$$\begin{aligned}D_1(z,x) &= \frac{1}{2}\int_{(0,x)} (1-e^{-2k(x-y)})\left(\frac{1}{4}(\chi(y) + \chi(\{0\}) + \frac{1}{2k}F(z,y)\right) d\chi(y)\\ &= \frac{1}{8}\int_{(0,x)} \big(\chi(y) + \chi(\{0\})\big)\, d\chi(y) + \mathrm{o}(1)\end{aligned}$$

as $\mathrm{Im}(z) \to +\infty$.

The estimates for the error functions of $c'(z,x)$, $s(z,x)$ and $s'(z,x)$ are calculated in a similar manner.

□

We introduce circles discovered by Weyl which he used for the original proof of the Weyl alternative. Given a self-adjoint realization S of the differential expression τ, we know the solutions of the equation $(\tau - z)u = 0$, which lie in the domain of S, satisfy certain boundary conditions (see Theorem 4.50). This solutions can be parameterized on circles called *Weyl circles*.

Definition 5.8. For $x \in (0, +\infty)$ we denote

$$q(z,x) := -\frac{W(c(z), s(z)^*)(x)}{W(s(z), s(z)^*)(x)}, \qquad r(z,x) := \frac{1}{|W(s(z), s(z)^*)(x)|}.$$

This functions are called *Weyl-center* and *Weyl-radius* respectively.

We denote the open disk centered at $q \in \mathbb{C}$ with radius $r > 0$ as $U_r(q) := \{z \in \mathbb{C} \mid |z - q| < r\}$. The next Lemma is based on the original proof of the Weyl alternative, which can be found for example in [2] after Theorem 13.18.

Lemma 5.9. *The Weyl-function satisfies $m(z) \in U_{r(z,x)}(q(z,x))$ for all $x \in (0, +\infty)$. For $z \in \mathbb{C}^+$ and $x \in (0, +\infty)$ some $m \in \mathbb{C}$ satisfies the equivalences*

$$\begin{aligned} m \in U_{r(z,x)}(q(z,x)) &\iff \frac{c'(z,x) + ms'(z,x)}{c(z,x) + ms(z,x)} \in \mathbb{C}^+ \\ m \in \left(\overline{U_{r(z,x)}(q(z,x))}\right)^c &\iff \frac{c'(z,x) + ms'(z,x)}{c(z,x) + ms(z,x)} \in \mathbb{C}^-. \end{aligned}$$

Proof.
For fixed $z \in \mathbb{C}^+$ we define

$$\{f, g\}(x) := \frac{1}{z^* - z} W(f^*, g)(x).$$

If u is a solution of $(\tau - z)u = 0$ the Lagrange identity (see equation (4.11)) yields $\{u,u\}(x) - \{u,u\}(c) = \int_c^x |u|^2 \, d\lambda$. Hence the function $x \mapsto \{u,u\}(x)$ is increasing. Since $\{c,c\}(0) = \{s,s\}(0) = 0$, the functions $x \mapsto \{c,c\}(x)$ and $x \mapsto \{s,s\}(x)$ are positive and increasing for $x \in (0, +\infty)$. For an arbitrary solution u of $(\tau - z)u = 0$ we have the identity

$$\{u,u\}(x) = \frac{1}{2\operatorname{Im}(z^*)} W(u^*, u)(z) = \frac{\operatorname{Im}(u^* u')}{\operatorname{Im}(z^*)} = \frac{|u(x)|^2}{\operatorname{Im}(z^*)} \operatorname{Im}\left(\frac{u'(x)}{u(x)}\right). \tag{5.10}$$

For $m \in \mathbb{C}$ we have

$$\begin{aligned} &\{c + ms, c + ms\}(x) = \{c,c\}(x) + m^*\{s,c\}(x) + m\{c,s\}(x) + |m|^2\{s,s\}(x) = \\ &= \{s,s\}(x)\left(|m|^2 + \frac{(m\{c,s\}(x))^* + m\{c,s\}(x)}{\{s,s\}(x)} + \frac{|\{c,s\}(x)|^2}{|\{s,s\}(x)|^2} - \right. \\ &\qquad \left. - \underbrace{\frac{\{c,c\}(x)^2\{s,s\}(x) - \{c,c\}(x)\{s,s\}(x)}{\{s,s\}(x)^2}}_{=:h(z,x)}\right) \\ &= \{s,s\}(x)\left(\left|m - \left(\underbrace{-\frac{\{c,s\}(x)}{\{s,s\}(x)}}_{=q(z,x)}\right)\right|^2 - h(z,x)\right) \\ &= \{s,s\}(x)(|m - q(z,x)|^2 - h(z,x)). \end{aligned}$$

After some computing with help of the Plücker identity from Proposition 4.11 (for details see [2], after Theorem 13.18.) we see that $h(z,x)$ coincides with the squared Weyl-radius as defined above, i.e. $h(z,x) = r(z,x)^2$ and we get the equation

$$\{c+ms, c+ms\} = \{s,s\}(x)(|m-q(z,x)|^2 - r(z,x)^2). \tag{5.11}$$

So for a fixed $x > 0$ the $m \in \mathbb{C}$ for which $\{c+ms, c+ms\} = 0$ is satisfied lie on a circle[3] with radius $r(z,x)$ centered at $q(z,x)$. From this and with the identity (5.10) follows the equivalence

$$\begin{aligned} m \in U_{r(z,x)}(q(z,x)) &\iff \{c+ms, c+ms\}(x) < 0 \\ &\iff \frac{|c(x)+ms(x)|^2}{\underbrace{\operatorname{Im}(z^*)}_{<0}} \operatorname{Im}\left(\frac{c'(x)+ms'(x)}{c(x)+ms(x)}\right) < 0 \\ &\iff \operatorname{Im}\left(\frac{c'(x)+ms'(x)}{c(x)+ms(x)}\right) > 0 \\ &\iff \frac{c'(x)+ms'(x)}{c(x)+ms(x)} \in \mathbb{C}^+. \end{aligned}$$

And similarly it follows that

$$m \in \left(\overline{U_{r(z,x)}(q(z,x))}\right)^c \iff \frac{c'(x)+ms'(x)}{c(x)+ms(x)} \in \mathbb{C}^-.$$

Now we show that $m(z) \in U_{r(z,x)}(q(z,x))$. As $m(z)$ is a Nevanlinna function $m(z) \in \mathbb{C}^+$ and we have

$$\{\psi,\psi\}(0) = \frac{\operatorname{Im}(m(z))}{\operatorname{Im}(z^*)} < 0$$

by definition of the Weyl solution ψ of the self-adjoint realization S. Now as ψ and ψ^* lie in S near b, Lemma 4.46 yields

$$\lim_{x\to\infty} \{\psi,\psi\}(x) = \frac{1}{2\operatorname{Im}(z^*)} \lim_{x\to\infty} W(\psi,\psi^*)(x) = 0.$$

We know from above that for solutions of $(\tau - z)u = 0$ the function $x \mapsto \{u,u\}(x)$ is increasing so this is true for $x \mapsto \{\psi,\psi\}(x)$ and as $\{\psi,\psi\}(x)$ is negative-valued at 0 and increasing to 0 for infinity, we have $0 > \{\psi,\psi\}(x) = \{c+ms, c+ms\}(x)$ for $x \in [0,+\infty)$. With the first part of the proof this is equivalent to $m(z) \in U_{r(z,x)}(q(z,x))$ for $x \in [0,+\infty)$.

□

The following lemma is based on the idea of Lemma 1 in [3].

Lemma 5.10. *Let $x \in (0,+\infty)$ and $v(z,x)$ be the solution of the initial value problem $(\tau - z)u = 0$ with $v(z,x) = 1$, $v'(z,x) = -\sqrt{-z}$. Then the asymptotic behavior of the Weyl m-function is given as*

$$m(z) = \frac{v'(z,0)}{v(z,0)} + \mathrm{o}(e^{-2\sqrt{-z}x}z) \tag{5.12}$$

as $\operatorname{Im}(z) \to +\infty$.

[3] It can be shown that all the Weyl circles are nested for $x \to +\infty$ and that the Weyl function is the only element in the intersection of all Weyl circles, hence the name limit point case at $+\infty$.

Proof.
Let $x \in (0, +\infty)$ be fixed. From the last lemma we know that $m(z) \in U_{r(z,x)}(q(z,x))$ and now we show that we also have $\frac{v'(z,0)}{v(z,0)} \in U_{r(z,x)}(q(z,x))$ which lets us estimate the distance between $m(z)$ and $\frac{v'(z,0)}{v(z,0)}$ through the Weyl radius at x. As the Wronskian is constant for two solutions of $(\tau - z)u = 0$ we get

$$\frac{v'(z,0)}{v(z,0)} = \frac{W(c,v)(0)}{W(v,s)(0)} = \frac{W(c,v)(x)}{W(v,s)(x)} = \frac{c(x)v'(x) - c'(x)v(x)}{v(x)s'(x) - v'(x)s(x)} = \frac{-c(x)k - c'(x)}{s'(x) + ks(x)}. \tag{5.13}$$

Now an easy computation shows that if we use the above identity for $\frac{v'(z,0)}{v(z,0)}$ we get

$$\frac{c'(x) + s'(x)\frac{v'(0)}{v(0)}}{c(x) + s(x)\frac{v'(0)}{v(0)}} = -k \in \mathbb{C}^+$$

and by the last lemma (Lemma 5.9) this is equivalent to $\frac{v'(z,0)}{v(z,0)} \in U_{r(z,x)}(q(z,x))$. It follows that we have

$$\left| m(z) - \frac{v'(z,0)}{v(z,0)} \right| \leq 2r(z,x). \tag{5.14}$$

Now we will use Lemma 5.5 to calculate the asymptotic behavior for $r(z,x)$. We abbreviate

$$I_1(z,x) := \int_0^x e^{-2ky}\, d\chi(y) \quad \text{and} \quad I_2(z,x) := \int_0^x e^{-2k(x-y)}\, d\chi(y)$$

and note that the integrals $I_1(z,x)$ and $I_2(z,x)$ are uniformly bounded in $z \in \mathbb{C}\backslash[0,+\infty)$. We have

$$\begin{aligned} s(z,x) &= \frac{e^{kx}}{k} \Bigg(\underbrace{\frac{1}{2}(1 - e^{-2kx}) + \frac{1}{2k}(1 + e^{-2kx})\chi(x) - \frac{1}{4k}(I_1(z,x) - I_2(z,x)) + \mathrm{O}(\frac{1}{k^2})}_{B_1(z,x) :=} \Bigg) \\ &= \frac{e^{kx}}{k} B_1(z,x) \end{aligned}$$

with a function $B_1(z,x) \to \frac{1}{2}$ as $\mathrm{Im}(z) \to +\infty$. Similarly for $s'(z,x)$ we have

$$\begin{aligned} s'(z,x)^* &= e^{k^*x} \Bigg(\underbrace{\frac{1}{2}(1 + e^{-2kx}) + \frac{1}{2k}(1 - e^{-2kx})\chi(x) - \frac{1}{4k}(I_1(z,x) - I_2(z,x)) + \mathrm{O}(\frac{1}{k^2})}_{B_2(z,x) :=} \Bigg)^* \\ &= e^{k^*x} B_2(z,x)^* \end{aligned}$$

with a function $B_2(z,x) \to \frac{1}{2}$ as $\mathrm{Im}(z) \to +\infty$. Inserting this into the definition of the Weyl radius leads to

$$\begin{aligned} r(z,x) &= \frac{1}{W(s,s^*)(x)} = \frac{1}{2|\operatorname{Im}(s(x)s'(x)^*)|} = \frac{1}{2|\operatorname{Im}(\frac{1}{k}e^{kx}B_1(z,x)e^{k^*x}B_2(z,x)^*)|} = \\ &= \frac{|k|^2 e^{-2\operatorname{Re}(k)x}}{2|\operatorname{Im}(k^* B_1(z,x) B_2(z,x)^*)|} = \frac{e^{-2\operatorname{Re}(k)x}|k|^2}{2|\operatorname{Im}(k^* B_1(z,x) B_2(z,x)^*)|}. \end{aligned}$$

Looking at the limit $\operatorname{Im}(z) \to +\infty$ we know $\operatorname{Im}(k) \to -\infty$ and since $B_1(z,x)B_2(z,x)^* \to \frac{1}{4}$ we also know that $\operatorname{Im}(B_1(z,x)B_2(z,x)^*) \to 0$. It follows that $|\operatorname{Im}(k^* B_1(z,x)B_2(z,x)^*)| \to +\infty$ and we arrive at

$$r(z,x) = \frac{e^{-2\operatorname{Re}(k)x}|k|^2}{2|\operatorname{Im}(k^* B_1(z,x)B_2(z,x))|} = \mathrm{o}(e^{-2\operatorname{Re}(k)x}z)$$

as $\operatorname{Im}(z) \to +\infty$. Now by (5.14) this means we are finished.

□

We arrive at the final theorem of this thesis, which combines the asymptotic behavior of the fundamental system for $\{s(z,x), c(z,x)\}$ from Lemma 5.5 with the asymptotic behavior for the Weyl m-function from Lemma 5.10.

Theorem 5.11. *For every $x \in (0,+\infty)$ the Weyl m-function has the asymptotic behavior*

$$m(z) = -\sqrt{-z} - \int_0^x e^{-2\sqrt{-z}y}\, d\chi(y) + \mathrm{o}\left(\frac{1}{\sqrt{-z}}\right) \tag{5.15}$$

as $\operatorname{Im}(z) \to +\infty$.

Proof.
Let $x \in (0,+\infty)$ be fixed. Combining the asymptotic behavior of the Weyl function $m(z)$ from (5.12) with the equation (5.13) leads to

$$m(z) = \frac{v'(z,0)}{v(z,0)} + \mathrm{o}(e^{-2kx}z) = \frac{-kc(x) - c'(x)}{s'(x) + ks(x)} + \mathrm{o}(e^{-2kx}z) \tag{5.16}$$

as $\operatorname{Im}(z) \to +\infty$. Now we can use Lemma 5.5 to calculate the asymptotics of $\frac{-kc(x)-c'(x)}{s'(x)+ks(x)}$ with an error term of the order $\mathrm{o}\left(\frac{1}{k}\right)$. To do this we prepare some calculations first.

We simplify the numerator and denominator of the fraction $\frac{-c(x)k-c'(x)}{s'(x)+ks(x)}$ separately:
For the numerator we get

$$\begin{aligned}
-kc(x) - c'(x) &= -k\cosh(kx) - \frac{1}{2}\sinh(kx)\chi(x) - \frac{e^{kx}}{4}I_1(z,x) + \frac{e^{kx}}{4}I_2(z,x) - \frac{e^{kx}}{k}D_1(z,x) - \\
&\quad - k\sinh(kx) - \frac{1}{2}\cosh(kx)\chi(x) - \frac{e^{kx}}{4}I_1(z,x) - \frac{e^{kx}}{4}I_2(z,x) - \frac{e^{kx}}{k}D_2(z,x) \\
&= -ke^{kx} - \frac{1}{2}\chi(x)e^{kx} - \frac{1}{2}e^{kx}I_1(z,x) - \frac{e^{kx}}{k}(D_1(z,x) + D_2(z,x)) \\
&= -ke^{kx}\left(1 + \frac{1}{2k}(\chi(x) + I_1(z,x)) + \frac{1}{k^2}(D_1(z,x) + D_2(z,x))\right).
\end{aligned} \tag{5.17}$$

For the denominator we calculate in a similar way

$$\begin{aligned}
s'(x)+ks(x) &= \cosh(kx)+\frac{1}{2k}\sinh(kx)\chi(x)-\frac{e^{kx}}{4k}I_1(z,x)+\frac{e^{kx}}{4k}I_2(z,x)+\frac{e^{kx}}{k^2}D_3(z,x)+\\
&\quad+\sinh(kx)+\frac{1}{2k}\cosh(kx)\chi(x)-\frac{e^{kx}}{4k}I_1(z,x)-\frac{e^{kx}}{4k}I_2(z,x)+\frac{e^{kx}}{k^2}D_4(z,x)\\
&= e^{kx}+\frac{e^{kx}}{2k}\chi(x)-\frac{e^{kx}}{2k}I_1(z,x)+\frac{e^{kx}}{k^2}(D_3(z,x)+D_4(z,x))\\
&= e^{kx}\left(1+\underbrace{\frac{1}{2k}(\chi(x)-I_1(z,x))+\frac{1}{k^2}(D_3(z,x)+D_4(z,x))}_{M(z,x):=}\right)\\
&= e^{kx}(1+M(z,x)). \qquad (5.18)
\end{aligned}$$

As $I_1(z,x)$ as well as $D_i(z,x)$, $i=3,4$ are bounded as $\mathrm{Im}(z)\to+\infty$, the term $|M(z,x)|$ gets arbitrary small for large enough $\mathrm{Im}(z)$. Hence we can use the identity $\frac{1}{1+x}=\sum_{n\geq 0}(-x)^n$ for $|x|<1$ for the term $M(z,x)$ which leads to

$$\begin{aligned}
\frac{1}{1+M(z,x)} &= 1-M(z,x)+M(z,x)^2+\mathrm{o}\left(\frac{1}{k^2}\right)\\
&= 1-\frac{1}{2k}(\chi(x)-I_1)-\frac{1}{k^2}(D_3+D_4)+M(z,x)^2+\mathrm{o}\left(\frac{1}{k^2}\right)\\
&= 1-\frac{1}{2k}(\chi(x)-I_1)-\frac{1}{k^2}(D_3+D_4)+\frac{1}{4k^2}\big(\chi(x)^2+I_1^2-2\chi(x)I_1\big)+\mathrm{o}\left(\frac{1}{k^2}\right) \qquad (5.19)
\end{aligned}$$

as $\mathrm{Im}(z)\to+\infty$.

By Lemma 5.7 we have

$$\begin{aligned}
D_1(z,x)+D_2(z,x)-D_3(z,x)-D_4(z,x) &=\\
&= \frac{1}{4}\int_{(0,x)}\big(\chi(y)+\chi(\{0\})\big)\,d\chi(y)-\frac{1}{4}\int_{(0,x)}\big(\chi(y)-\chi(\{0\})\big)\,d\chi(y)+\mathrm{o}(1)\\
&= \frac{1}{2}\chi(\{0\})\int_{(0,x)}d\chi(y)+\mathrm{o}(1)\\
&= \frac{1}{2}\chi(\{0\})\big(\chi(x)-\chi(\{0\})\big)+\mathrm{o}(1) \qquad (5.20)
\end{aligned}$$

as $\mathrm{Im}(z)\to+\infty$.

We will now use the above calculations to determine the asymptotic behavior of $\frac{-kc(x)-c'(x)}{s'(x)+ks(x)}$ with accuracy of order $\mathrm{o}\left(\frac{1}{k}\right)$ as $\mathrm{Im}(z)\to+\infty$.

For the first equal sign in the next calculation we insert the identities (5.17) and (5.18) and see that the exponential factors cancel each other. For the second identity we use (5.19) and then sort the terms by order of k. For the third equal sign we use the equation (5.20) as well as the

equation (5.9) from Remark 5.6:

$$\begin{aligned}
\frac{-kc(x)-c'(x)}{s'(x)+ks(x)} &= -k\left(1+\frac{1}{2k}(\chi(x)+I_1)+\frac{1}{k^2}(D_1+D_2)\right)\left(\frac{1}{1+M(z,x)}\right)\\
&= -k-I_1-\frac{1}{k}\left(D_1+D_2-D_3-D_4\right)+\frac{1}{2k}\left(\chi(x)I_1-I_1^2\right)+\mathrm{o}\left(\frac{1}{k}\right)\\
&= -k-I_1-\frac{1}{2k}\left(\chi(\{0\})\big(\chi(x)-\chi(\{0\})\big)-\chi(x)\chi(\{0\})+\chi(\{0\})^2\right)+\mathrm{o}\left(\frac{1}{k}\right)\\
&= -k-I_1-\frac{1}{2k}\left(\chi(\{0\})\chi(x)-\chi(\{0\})^2-\chi(x)\chi(\{0\})+\chi(\{0\})^2\right)+\mathrm{o}\left(\frac{1}{k}\right)\\
&= -k-I_1+\mathrm{o}\left(\frac{1}{k}\right)\\
&= -k-\int_0^x e^{-2ky}\,d\chi(y)+\mathrm{o}\left(\frac{1}{k}\right)
\end{aligned}$$

as $\mathrm{Im}(z)\to+\infty$.

Finally we can plug this into the equation (5.16) and get the desired result

$$\begin{aligned}
m(z) &= \frac{-kc(x)-c'(x)}{s'(x)+ks(x)}+\mathrm{o}(e^{-2kx}z)\\
&= -k-\int_0^x e^{-2ky}\,d\chi(y)+\mathrm{o}\left(\frac{1}{k}\right)
\end{aligned}$$

as $\mathrm{Im}(z)\to+\infty$.

□

We will now look at an easy example to check the result of Theorem 5.11. We extend the domain of the differential expression τ to the inverval $(-\infty,+\infty)$ and look at the measure $\chi=\alpha\delta_0$ with some constant $\alpha\in\mathbb{C}$. This leads to the differential expression τ for a function $f\in\mathcal{D}(\tau)$ given as

$$(\tau f)(x)=\left(t\mapsto -f'(t)+\alpha\int_0^t f\,d\delta_0\right)'(x),\quad x\in\mathbb{R}.$$

Applying Theorem 5.11 immediately yields a Weyl function estimate given as

$$m(z)=-\sqrt{-z}-\alpha+\mathrm{o}(\frac{1}{\sqrt{-z}})$$

as $\mathrm{Im}(z)\to+\infty$. We will check this result by calculating the Weyl solution $\psi(z,x)$ which has to satisfy the conditions (see Definition 4.51)

- $\psi(z,\cdot)\in\mathcal{D}(\tau)$,
- $\psi(z,\cdot)$ is solution of the differential equation $(\tau-z)u=0$, with
- $\psi(z,0)=1$ and $\psi(z,\cdot)\in L^2((c,+\infty);\lambda)$, $c\in\mathbb{R}$

for $z\in\mathbb{C}\backslash\mathbb{R}$.

From the first condition $\psi(z,\cdot)\in\mathcal{D}(\tau)$ follows that both the function $x\mapsto\psi(z,x)$ as well as the function $x\mapsto\psi_1(z,x):=-\psi'(z,x)+\alpha\int_0^x\psi(z,\cdot)\,d\delta_0$ have to be continuous on $\mathbb{R}$ for every

$z \in \mathbb{C}\backslash\mathbb{R}$. This means the function $x \mapsto \psi'(z,x)$ has to be continuous for $x \in (-\infty,0)\cup(0,+\infty)$ and at the point $x = 0$ has to satisfy the jump condition $\psi'(z,0+) - \psi'(z,0-) = \alpha\psi(z,0)$ for $z \in \mathbb{C}\backslash\mathbb{R}$.

Now in this example the differential equation $(\tau - z)\psi(z,\cdot) = 0$ takes the form $\psi''(z,\cdot) = -z\psi(z,\cdot)$ which leads to a solution of the form

$$\psi(z,x) = \begin{cases} c_1 e^{-kx} + c_2 e^{kx} & \text{for } x > 0 \\ 1 & x = 0 \\ d_1 e^{-kx} + d_2 e^{kx} & \text{for } x < 0 \end{cases}$$

with constants $c_1, c_2, d_1, d_2 \in \mathbb{C}$. Because of the conditions $\psi(z,0) = 1$ and $\psi(z,\cdot) \in L^2((c,+\infty);\lambda)$ with $c \in \mathbb{R}$ follows that $c_1 = 1$ and $c_2 = 0$.
Since $\psi(z,\cdot)$ has to be continuous it follows that $d_1 + d_2 = 1$. The jump condition leads to

$$\begin{aligned} \psi'(z,0+) - \psi'(z,0-) &= \alpha\psi(z,0) \\ -k - (-kd_1 + kd_2) &= \alpha \\ -k + k(1-d_2) - kd_2 &= \alpha \\ d_2 &= -\frac{\alpha}{2k} \end{aligned}$$

and it follows that $d_1 = 1 - d_2 = 1 + \frac{\alpha}{2k}$. Inserting this constants leads to the Weyl solution given as

$$\psi(z,x) = \begin{cases} e^{-kx} & x > 0 \\ (1+\frac{\alpha}{2k})e^{-kx} - \frac{\alpha}{2k}e^{kx} & x \leq 0. \end{cases}$$

Now from Definition 4.51 follows that

$$m(z) = \psi'(z,0) = -k - \alpha = -\sqrt{-z} - \alpha$$

which shows that the estimate of Theorem 5.11 is valid.

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