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DISSERTATION

Travelling Around the Harmonic Archipelago

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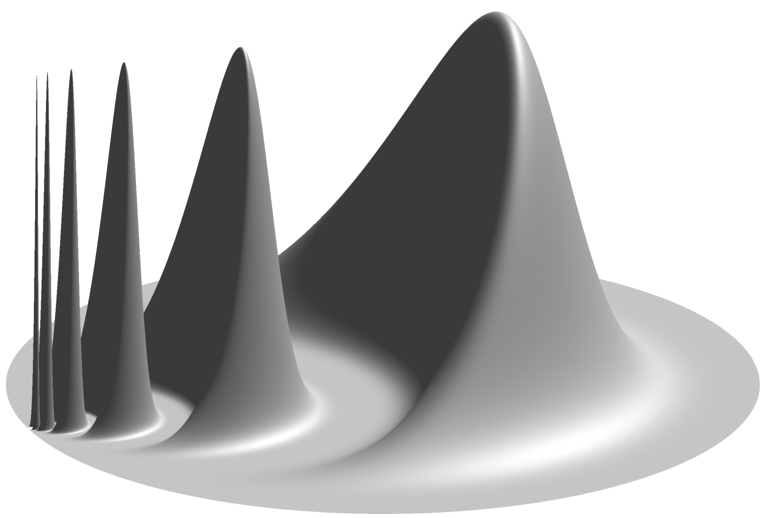

Zusammenfassung

Das harmonische Archipel ist ein Standardbeispiel für einen zwei-dimensionalen topologischen Raum mit ungewöhnlichen und unerwarteten Eigenschaften, vom Standpunkte der algebraischen Topologie aus betrachtet. Der Raum selbst ist mit Ausnahme eines einzigen Punktes homeomorph zu einer Kreisscheibe und kann als reduzierte Einhängung (reduced suspension) des Graphen der Funktion $y = \sin(1/x)$ beschrieben werden. Andererseits hat er auch eine natürliche Darstellung als Abbildungskegel (mapping cone) einer Verklebung (wedge) von Kreisen.

Ersetzt man diese Kreise durch eine beliebige Schar von topologischen Räumen X_i, erhält man den allgemeinen Begriff des Archipelraums (archipelago space). Dessen Fundamentalgruppe ist ein Quotient des Topologen-Produkts (topologist's product) der zugehörigen Fundamentalgruppen $G_i = \pi_1(X_i)$ modulo dem entsprechenden freien Produkt.

Im ersten Kapitel wird das überraschende Ergebnis gezeigt, daß dieser Quotient $\mathscr{A}(G_i)$, für höchstens abzählbare Gruppen G_i ohne Elemente der Ordnung 2, unabhängig von der tatsächlichen Wahl der beteiligten Gruppen ist. Insbesondere ist die Fundamentalgruppe eines Archipels, das aus beliebigen lokal endlichen CW-Komplexen zusammengesetzt ist, immer isomorph zu $\mathscr{A}(\mathbb{Z})$, der des harmonischen Standard-Archipels, oder zu $\mathscr{A}(\mathbb{Z}_2)$, der des aus projektiven Ebenen zusammengesetzen Archipels.

Im zweiten Kapitel wird eine andere bemerkenswerte Eigenschaft gezeigt: jede abzählbare lokal freie Gruppe läßt sich als Untergruppe in $\mathscr{A}(\mathbb{Z})$ einbetten.

Im dritten Kapitel wird das Rekursionskalkül aus dem Beweis dieses Einbettungssatzes auf andere Gruppen erweitert und zeigt sich so als nicht-abelsche Entsprechung der Kotorsions-Eigenschaft. Damit ist es möglich, eine vollständige Beschreibung der ersten Homologiegruppe mancher wilder Räume zu gewinnen. Es erweisen sich insbesondere sämtliche Abelisierungen der Archipelgruppen $\mathscr{A}(G_i)$ als zu einander isomorph, sofern nur alle G_i nicht die Kardinalität des Kontinuums überschreiten.

Abstract

The harmonic archipelago is a standard example of a two-dimensional space with unusual properties, regarding its algebraic topology. The space is homeomorphic to a disc but for a single point and can be described as the reduced suspension of the graph of the topologist's sine curve $y = \sin(1/x)$. On the other hand it also has a natural interpretation as a mapping cone over a wedge of circles. Replacing these circles with an arbitrary family of topological spaces X_i yields the generalized notion of an archipelago space. The fundamental group of such an archipelago is a quotient of the topologist's product of the fundamental groups $G_i = \pi_1(X_i)$ modulo the corresponding free product.

In the first chapter, it is shown that, surprisingly, for countable groups G_i containing no elements of order 2 this quotient $\mathscr{A}(G_i)$ is independent of the actual choice of the constituent groups G_i. In particular, the fundamental group of any archipelago space built of locally finite CW-complexes is isomorphic to either that of the standard harmonic archipelago, $\mathscr{A}(\mathbb{Z})$, or to the one with projective planes instead of circles, $\mathscr{A}(\mathbb{Z}_2)$.

In the second chapter, another remarkable property is shown: that every countable locally free group can be embedded into $\mathscr{A}(\mathbb{Z})$ as a subgroup.

In the third chapter, the recursion technique used in the proof of this embedding theorem is adapted to other groups and identified as a non-abelian analogue of cotorsion. By this it is possible to obtain a complete description of the first singular homology group of some wild spaces. In particular, the abelianizations of the archipelago groups $\mathscr{A}(G_i)$, with the G_i of cardinality less or equal to the continuum, are all isomorphic to each other.

Contents

Preface

Classical concepts in algebraic topology deal first and foremost with spaces that are moderately nice everywhere, such as manifolds and CW-complexes. Then, topological constructions like forming products or gluing spaces together – preferably in an orderly manner! – often yield a wonderfully natural correspondence to operations on groups: sums and products, or the plethora of exact sequences that are a staple of any introductory course in that area. But alas, how quickly do these trusty methods forsake us, if local complexities raise their ugly heads. How little do we know about more complicated wedges of spaces not locally simply connected, be it a simple shrinking wedge of circles, a space known as the Hawaiian earring (in former times also as the clamshell space); or worse yet, a wedge of only two coned copies of that space (Griffiths' double cone).

But there is hope, at least for one-dimensional *wild* spaces some tools have been developed: it is possible to tell spaces apart by their fundamental group, e.g. the earring, the Sierpiński triangle, or the universal Menger sponge; and for Peano continua, the homology has a very complete description. For higher dimensions however, even just for two-dimensional spaces that embed in $\mathbb{R}^3$, much less is known about their algebraic properties. Not, whether the fundamental groups of the mentioned double cone and that of the harmonic archipelago, a space that is homeomorphic to a disc but for a single point, are the same or not; or, if subsets of $\mathbb{R}^3$ have torsion-free fundamental groups. The additional difficulty stems from the greater variability afforded by the extra dimension, as now it is possible that a nontrivial loop might have arbitrary small representatives, thus dooming any metric approach to gain insight into the fundamental group.

Leaving the safe haven of smooth manifolds we will venture on an adventurous journey to the isles of the harmonic archipelago, the quintessential example of a space with nontrivial but small loops. The present thesis is divided into three parts, that, while all analysing aspects of the archipelago, make use of vastly different methods, and can easily be read in any order.

The first chapter is the result of joint work with Greg Conner and Mark Meilstrup. It sprang from discussions at a topology seminar at BYU, where spaces similar to the standard archipelago and double cone were considered. Unexpectedly, it turns out that many archipelagos have the same fundamental group. The proof of this isomorphism, in particular at the most delicate step of the argument, makes use of the notion of infinite words.

As a contrast, the second and third chapters employ a more combinatorial approach in studying certain infinite recursions. These recursions are always solvable in the archipelago group, and they were first used 60 years ago, but seem to have lain dormant ever since. The embedding theorem of the second chapter was first conjectured over breakfast with Greg Conner, at a Viennese Café, naturally. The third and perhaps most purely algebraic part was developed in concert with Wolfgang Herfort; probably at a more expected location, staring at whiteboards and blackboards for endless hours. I am greatly indebted to my advisors for their support and patience.

Wien, October 2013 Wolfram Hojka

CHAPTER 1

Archipelago groups

1. Introduction

In algebraic topology one of the prototypical two-dimensional spaces eliciting unusual properties in its fundamental group is the harmonic archipelago: commonly seen as a shrinking bouquet of circles, with discs glued in between consecutive circles that bulge out to a constant height. Perhaps it should come as no surprise that this space has interesting features, as it is homeomorphic to the reduced suspension of the graph of the topologist's sine curve $y = \sin(1/x)$, a troublemaker well-known from analysis and topology.

For our discussions it will be most useful to switch to a homotopy equivalent construction, namely to the mapping cone of the natural continuous map from a regular wedge of spaces to the same wedge but with a strong topology. We will call this mapping cone A the archipelago space for the sequence of spaces and its fundamental group then turns out to be the penultimate term in the sequence

$$1 \to \pi_1\big(\bigvee_n X_n\big) \to \pi_1\big(\tilde{\bigvee}_n X_n\big) \to \pi_1(A) \to 1.$$

If the spaces X_n are all taken to be circles, the resulting mapping cone is homotopy equivalent to the harmonic archipelago. As another example, consider the archipelago space of a sequence of projective planes. Our main theorem will show that, amazingly, the fundamental group of *any* well-behaved archipelago space, e.g. where each X_n is a locally finite CW-complex, is either trivial or isomorphic to one of these two examples.

If the spaces are locally connected and first-countable at the basepoint (or alternatively, if the wedges are understood as *homotopy* colimits instead of regular ones), the first nontrivial group in the above sequence is isomorphic to the ordinary free product $*_n\, \pi_1 X_n$; the second one can be written as the *topologist's product* $\circledast_n\, \pi_1 X_n$, first studied by Higman and Griffiths in the 50's, [Hig52, Gri56] (and also known as σ-product or free complete product). In some sense the latter could be considered the *correct* nonabelian product, since it allows combinining elements from infinitely many constituent factors in a way that *depends on their order*. It can be intuitively understood by using the concept of infinite words, see [Eda92]. Also, the free product is contained in it as a subgroup, just as the direct sum embeds in the direct product for abelian groups.

Thus it is possible to define an *archipelago group* $\mathscr{A}(G_n)$ of a sequence of groups in purely algebraic terms as the quotient $\circledast_n G_n / *_n G_n$, and in the special case of all groups being $\mathbb{Z}$, this is the fundamental group of the harmonic archipelago.

These groups allow the following remarkable classification.

Theorem 1.1. *Let $(G_n)_{n\in\mathbb{N}}$ be a collection of nontrivial countable (possibly finite) groups. If only finitely many of the groups G_n have elements of order 2, then*

$$\mathscr{A}(G_n) \simeq \mathscr{A}(\mathbb{Z}).$$

If infinitely many of the groups G_n have elements of order 2, then

$$\mathscr{A}(G_n) \simeq \mathscr{A}(\mathbb{Z}_2).$$

That is, any such group is isomorphic to either of two prototypes! In particular, the groups $\mathscr{A}(\mathbb{Z})$, $\mathscr{A}(\mathbb{Q})$, and $\mathscr{A}(\mathbb{Z}_3)$ are all isomorphic. The next theorem collects various properties about this standard case.

Theorem 1.2. *The fundamental group $\mathscr{A}(\mathbb{Z})$ of the harmonic archipelago has the following properties.*

(1) The group $\mathbb{Q}$ of rational numbers embeds as a subgroup in $\mathscr{A}(\mathbb{Z})$;
(2) $\mathscr{A}(\mathbb{Z})$ does not map onto the integers $\mathbb{Z}$; and
(3) $\mathscr{A}(\mathbb{Z})$ is locally free.

The proof of Theorem 1.1 is astonishing in that it only employs arbitrary set theoretic bijections between groups to derive an *isomorphism* between the archipelago groups. This "trick" fails for involution elements, leading to the two possible cases above. However, as we will see later, the group $\mathscr{A}(\mathbb{Z}_2)$ is also torsion-free and it is even possible that the two groups are the same:

Question 1.3. *Is $\mathscr{A}(\mathbb{Z}) \simeq \mathscr{A}(\mathbb{Z}_2)$?*

Indeed, more generally, any archipelago group is completely encoded by a sequence of cardinals (Theorem 1.11); this bears a certain superficial resemblance to the classical situation for abelian groups, as [Hul62], together with standard results in [Kap54], shows the quotient $\prod_n G_n / \bigoplus_n G_n$ to also be depending only on a sequence of cardinal numbers, albeit a quite different one (see section 3).

However, distinct cardinalities do not necessarily lead to different archipelago groups, and it is also conceivable that the following two groups are isomorphic.

Question 1.4. *Is* $\mathscr{A}(\mathbb{Z}) \simeq \mathscr{A}(\mathbb{R})$*?*

2. The Topological Viewpoint

For a collection of spaces X_n with basepoints p_n the wedge $\bigvee_n X_n$ is their coproduct in the category of pointed spaces. As a set, it can be naturally considered as the subset of the product $\prod_n X_n$ consisting of all points $(a_n)_n$ such that $a_n = p_n$ for all but at most one n, the "main axes". If the product is given the box topology this yields the usual weak wedge topology. If however the product is given the standard (Tychonoff) topology, that same subset will be called the *shrinking wedge* $\tilde{\bigvee}_n X_n$.

This information can be encoded in the left half of following diagram:

$$\begin{tikzcd} \bigvee_n X_n \arrow[r, hook] \arrow[d, "f"'] & \prod_n^{\mathrm{BOX}} X_n \arrow[d] \arrow[rr, squiggly, "\pi_1"] & & *_n G_n \arrow[r, two heads] \arrow[d, hook] & \sum_n G_n \arrow[d, hook] \\ \tilde{\bigvee}_n X_n \arrow[r, hook] & \prod_n X_n & & \circledast_n G_n \arrow[r, two heads] & \prod_n G_n \end{tikzcd} \tag{1}$$

The horizontal maps are the embeddings, the vertical maps are continuous bijections. Assuming the spaces are *nice* at the basepoint (see below), setting $G_n := \pi_1(X_n)$ and applying the functor π_1 yields the right part of the diagram together with the induced maps between the fundamental groups. On this side, the vertical maps are embeddings, the horizontal ones are onto. The symbol $\sum$ retrieves the set of elements in the product with finite support. For abelian groups we will use the more common $\bigoplus$, denoting the same object.

As an example, recall that the Hawaiian earring is the planar set consisting of circles c_n of radius $1/n$ centered at $(0, 1/n)$. Note that each circle contains the origin, and equivalently this space is the one-point compactification of a sequence of open arcs. It is also a shrinking wedge $\tilde{\bigvee}_{n\in\mathbb{N}} S^1$ of infinitely many circles and its fundamental group turns out to be $\circledast_{n\in\mathbb{N}} \mathbb{Z}$, each copy of $\mathbb{Z}$ corresponding to one circle of the earring.

Algebraically, the most straight-forward method of defining the topologist's product for countable families is simply the following:

$$\circledast_i G_i := \bigcap_i \left(G_i * \varprojlim_n *_{1\le j\le n}^{i\neq j} G_j\right).$$

This group was first implicitly conceived by Higman in [Hig52] while studying subgroups of $\varprojlim_n *_{1\le j\le n} G_j$, a group he called the *unrestricted free product* of the G_i. Griffiths then showed in [Gri56] (with a correction due to Morgan and Morrison in [MM86]) the relation

$$\pi_1(\tilde{\bigvee}_i X_i) = \circledast_i \pi_1(X_i)$$

between the topologist's product and the shrinking wedge of spaces that satisfy some local properties: being locally simply connected and first countable at each basepoint.

Contrast this with the standard fact that

$$\pi_1(\bigvee_i X_i) = *_i \pi_1(X_i)$$

holds under the same assumptions on the X_i.

These local requirements can be avoided by a simple procedure. To each space (X_i, p_i) attach an arc to p_i and shift the basepoint to the arc's other end. Let $(\tilde{X}_i, \tilde{p}_i)$ denote the thus modified space, then we can define a *homotopy shrinking wedge* $\tilde{\bigvee}{}_i^{\mathrm{H}}(X_i, p_i) := \tilde{\bigvee}(\tilde{X}_i, \tilde{p}_i)$. Now, with this notation,

$$\pi_1(\tilde{\bigvee}{}_i^{\mathrm{H}} X_i) = \circledast_i \pi_1(X_i)$$

holds for arbitrary spaces X_i.

Similarly, one can define a *homotopy wedge* by the same method. Note, that the difference between a regular wedge and an homotopy wedge is reflected in the distinction between taking the colimit or the *homotopy* colimit of the diagram of a wedge, i.e. the collection of maps $p_i : \{x\} \to X_i$ that select the basepoint in each space. Those colimits are discussed in detail in [Far04].

Another interesting property of the topologist's product is that it can be interpreted as an infinite word structure.

Definition 1.5. For a sequence $(G_n)_n$ of groups, an *infinite word* is a map $w : L \to \dot{\bigcup} G_n \setminus \{1\}$ from a countable linearly ordered set L to the disjoint union of the non-identity elements of the G_n where the preimage of every G_n is a finite set. Multiplication is simply concatenation, and inverses are given by inverting the order of the word and replacing each element by its inverse in G_n.

Corresponding to the case of finite words, there is a natural notion of cancellation within an infinite word (see [CC00] and [Eda92] for more background on this concept). Cancellation induces an equivalence relation on the set of infinite words, and the classes together with the above operations form the *topologist's product* $\circledast_n G_n$.

We return our attention to the harmonic archipelago. This is usually depicted as an earring with discs D_i attached along each boundary $c_i c_{i+1}^{-1}$ where an interior point of D_i is raised to height $z = 1$ in $\mathbb{R}^3$.

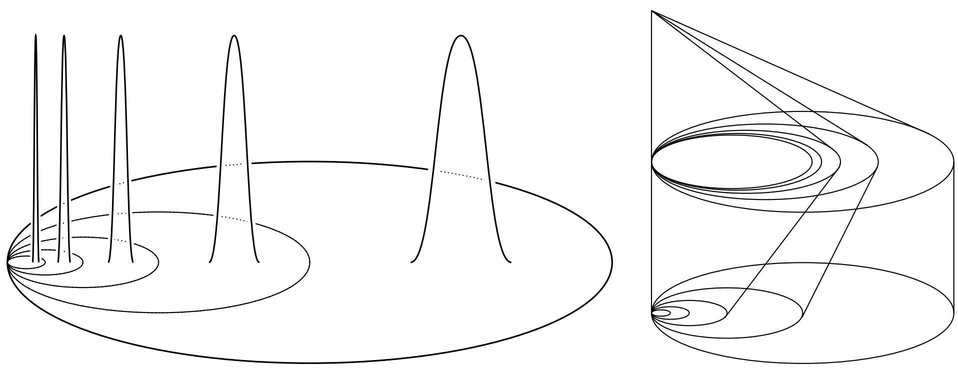

FIGURE 1. (left) The harmonic archipelago. (right) An homotopy equivalent realization as the mapping cone of $f : \bigvee_n S^1 \to \tilde{\bigvee}_n S^1$.

Recall that for a map $f : X \to Y$, the mapping cone C_f is defined as $X \times I \cup Y$ with every $(x, 0)$ identified with $f(x)$ and $X \times \{1\}$ collapsed to a point. Then the archipelago is homotopy equivalent to the mapping cone C_f of the continuous bijective map $f : \bigvee_n S^1 \to \tilde{\bigvee}_n S^1$ from the wedge to the shrinking wedge of circles, corresponding to the leftmost map in (1).

This allows us to generalize the notion of an *archipelago space* A of arbitrary spaces X_n to be the mapping cone C_f for $f : \bigvee_n X_n \to \tilde{\bigvee}_n X_n$.

Theorem 1.6. *The fundamental group of the archipelago space $A = C_f$ is determined by the fundamental groups $G_n := \pi_1(X_n)$ as the cokernel of f^* in the sequence*

$$1 \to \pi_1\left(\bigvee_n X_n\right) = *_n G_n \xrightarrow{f^*} \pi_1\left(\tilde{\bigvee}_n X_n\right) = \circledast_n G_n \to \pi_1(A) \to 1.$$

It can directly be expressed as the quotient

$$\pi_1(A) = \circledast_i G_i / *_i G_i.$$

The free product naturally embeds in the topologist's product, but not as a normal subgroup. To simplify notation we will suppress to write taking first the normal closure of the quotient group. The proof of the theorem is a simple application of van-Kampen's theorem.

In the introduction we have mentioned how this quotient is interesting in purely algebraic terms. The preceding theorem motivates its name:

Definition 1.7. Given a countable collection of groups G_n, the *archipelago group* $\mathscr{A}((G_n)_n)$ is given by the topologist's product of the G_n's, modulo their free product:

$$\mathscr{A}((G_n)_n) := \circledast_n G_n / \ast_n G_n.$$

If each G_n is the same group G, we write $\mathscr{A}(G)$.

3. Further Results

Theorem 1.2 can be extended in various ways. Finding a subgroup isomorphic to $\mathbb{Q}$ is similar in spirit to [BZ12] where the same is shown for Griffiths' double cone space. It is also a corollary to the theorem in the next chapter that *every* countable locally free group embeds in the archipelago (see Proposition 1.18 and Example 2.2).

The second property of Theorem 1.2 is not unique to $\mathscr{A}(\mathbb{Z})$, but holds for any archipelago group.

Proposition 1.8. *For any countable collection $(G_n)_{n\in\mathbb{N}}$ of groups, $\mathscr{A}(G_n)$ does not map onto the integers $\mathbb{Z}$. In other words,* $\mathrm{Hom}(\mathscr{A}(G_n), \mathbb{Z})$ *is trivial.*

Similarly, neither is the third property unique, every archipelago group is locally free as we will see in Chapter 3 (cf. Theorem 3.9). As an intermediate step, here we will be satisfied with the following weaker fact.

Proposition 1.9. *The group $\mathscr{A}(G_n)$ is torsion-free.*

Question 1.3 left as a possibility that every archipelago group composed of countable groups G_n is isomorphic to that of the standard harmonic archipelago. But certainly, not *all* archipelago groups are equal; for reasons of cardinality there is an infinite class of them, all distinct:

Theorem 1.10. *Suppose all groups G_n are of the same cardinality κ, then the archipelago group $\mathscr{A}(G_n)$ has cardinality $\kappa^{\aleph_0}$. Hence there is more than a set's worth of distinct archipelago groups. In particular, the archipelago groups obtained from free groups of $\beth_{\alpha+1}$ many generators are all distinct for $\alpha \geq 0$.*

This however still allows for the two groups from Question 1.4, $\mathscr{A}(\mathbb{Z})$ and $\mathscr{A}(\mathbb{R})$, of cardinality $\leq \mathfrak{c} = \beth_1$ to be isomorphic to each other.

On the other hand there is the following positive result on representing archipelago groups by cardinal numbers.

Theorem 1.11. *Let κ_n denote the cardinality of $G_n \setminus \{1\}$, and λ that of the set of indices i such that G_i contains an involution. Then $\mathscr{A}(G_n)$ is determined by these countably many cardinal numbers in the following way: if $\lambda < \aleph_0$,*

$$\mathscr{A}(G_n) \simeq \mathscr{A}(*_{\kappa_n} \mathbb{Z})$$

and if $\lambda = \aleph_0$,

$$\mathscr{A}(G_n) \simeq \mathscr{A}(*_{\kappa_n} \mathbb{Z}_2).$$

Theorem 1.11 compares to classical results for abelian groups: In [Hul62] Hulanicki shows that the quotient $\prod_i G_i / \bigoplus_i G_i$ is algebraically compact, a slightly generalized statement is in [Fuc63]. Using standard techniques due to Kaplansky [Kap54], the quotient then has a representation as

$$\prod_i G_i / \bigoplus_i G_i \simeq \bigoplus_{\mathfrak{k}} \mathbb{Q} \oplus \prod_{p \text{ prime}} A_p$$

where

$$A_p \simeq p\text{-adic completion of } \left(\bigoplus_{\mathfrak{m}_{p,0}} \mathbb{J}_p \oplus \bigoplus_{k=1}^{\infty} \bigoplus_{\mathfrak{m}_{p,k}} \mathbb{Z}(p^k) \right)$$

that only depends on the countably many cardinals $\mathfrak{k}$, $\mathfrak{m}_{p,k}$ (for p prime, $k \in \mathbb{N}$), with $\mathbb{J}_p$ the p-adic completion of $\mathbb{Z}$.

Similarly, the nonabelian quotient $\mathscr{A}(G_n) = \circledast_n G_n / *_n G_n$ depends only on the cardinalities of the groups and the presence of 2-torsion elements. However there is no direct correspondence between the two sets of cardinal numbers.

As $\circledast_n G_n$ maps onto $\prod_n G_n$, one might wonder if for abelian groups the canonical map $\circledast_n G_n / *_n G_n \to \prod_n G_n / \bigoplus_n G_n$ between the quotients is simply induced by abelianization of the group – something that is suggested by the case of $\mathscr{A}(\mathbb{Z})$: here the abelianization coincides with the first singular homology group of the harmonic archipelago. This homology group was e.g. investigated in [KR12] where it is shown to be torsion-free (which also follows from the stronger result of (3) in Theorem 1.2). It is isomorphic to $\prod_n \mathbb{Z} / \bigoplus_n \mathbb{Z}$. Yet the abelianization map turns out to be more complicated than the canonical map above.

The issue is more readily seen for e.g. finite cyclic groups, as these give rise to torsion in the abelian quotient. To wit, let $\mathbb{Z}_k$ denote the cyclic group of order k, then $(1,1,1,\dots)\in\prod_n \mathbb{Z}_k$ gives an element of order k in $\prod_n \mathbb{Z}_k/\bigoplus_n \mathbb{Z}_k$. But if the order k is coprime to 2, the nonabelian quotient $\circledast_n \mathbb{Z}_k/\ast_n \mathbb{Z}_k$ is isomorphic to $\mathscr{A}(\mathbb{Z})$ by Theorem 1.1 and thus locally free by Theorem 1.2. Hence its abelianization is torsion-free.

In the third chapter it is shown that the abelianizations of $\mathscr{A}(\mathbb{Z})$, $\mathscr{A}(\mathbb{Z}_2)$, and $\mathscr{A}(\mathbb{R})$ are in fact all equal to each other; perhaps offering some support to Questions 1.3 and 1.4.

4. Proofs

A helpful tool in deriving properties for an archipelago is representing it as a limit.

Proposition 1.12. *The archipelago group $\mathscr{A}(G_n)$ is a direct limit:*

$$\mathscr{A}(G_n)=\varinjlim_k \circledast_{i\geq k} G_i,$$

where the bonding maps are the quotient maps

$$\circledast_{i\geq k} G_i \to \big(\circledast_{i\geq k} G_i\big)/G_k \simeq \circledast_{i\geq k+1} G_i.$$

Proof. Note that any element of the free product $\ast_{i\geq 1} G_i$ becomes trivial after the application of finitely many bonding maps. Thus any group mapped to from this directed system of groups must be a factor of $\circledast_n G_n/\ast_n G_n = \mathscr{A}(G_n)$, which then satisfies the universal property of the direct limit. □

In the following lemma, we will use functions between groups that have some of the properties of homomorphisms, but are somewhat less restrictive. We call a function $\varphi : G\to H$ *identity-preserving* if $\varphi(e_G)=e_H$. Similarly, φ is *inverse-preserving* if for all $g\in G$, $\varphi(g^{-1})=\big(\varphi(g)\big)^{-1}$.

Lemma 1.13. *If for every i there exists an identity-preserving and inverse-preserving function $\varphi_i : G_i\to H_i$ (not necessarily a homomorphism), then the functions φ_i induce a homomorphism $\Phi : \mathscr{A}(G_i)\to\mathscr{A}(H_i)$.*

Remark. Note that we do not require φ_i to preserve multiplication.

Remark. For the technical details of the proof it is necessary to recapitulate certain properties of the word calculus used to describe the topologist's product. Consider a word $w : L\to \dot{\bigcup} G_i \setminus \{1\}$ as in Definition 1.5 and write the concatenation of words u and v as $u\cdot v$. As the preimage of every group G_i is finite, there is a projection $p_n(w)$ of w to a finite word in $*_{i=1}^{n} G_i$ for each n. A word w is called *reduced* if

(i) for each nontrivial subword w' (i.e. $w = u \cdot w' \cdot v$) there exists some projection $p_n(w')$ that is nontrivial; and

(ii) no two consecutive points in the order L are mapped to g_1, g_2 in the same group G_i.

In fact, every element of the topologist's product (seen as a class of words) can be represented by a reduced word, unique up to order isomorphism, and one can naturally identify the elements of the group with the reduced words. Let $R(w)$ denote this reduced representative of a word w.

Of particular interest will be finding $R(u \cdot v)$ for already reduced u and v, which can be accomplished by two simple reduction steps. It is possible to write the two reduced words as concatenations $u = a \cdot g_1 \cdot x$ and $v = y \cdot g_2 \cdot b$, with $y = x^{-1}$ the inverse word of x, and g_1, g_2 in the same G_i (allowing both g_1 and g_2, or x and y to perhaps be trivial). Then $R(u \cdot v) = a \cdot (g\, g') \cdot b$ is reduced as a concatenation.

Proof of the Lemma. The union of the functions φ_i gives a set function φ_0 between the disjoint unions $\dot{\bigcup}_i G_i$ and $\dot{\bigcup}_i H_i$. This function φ_0 will induce a map (not necessarily a homomorphism) φ between the topologist's products, which will in turn induce our desired homomorphism $\Phi : \mathscr{A}(G_i) \to \mathscr{A}(H_i)$, as we describe below.

The set function φ_W is a letter replacement defined by φ_0, in the following manner: an infinite word in the G_n is mapped to an infinite word in the H_n of the same order type, by taking each letter $a \in G_j$ to $\varphi_0(a) = \varphi_j(a)$. For example, $a_1 a_2 a_3 \ldots \mapsto \varphi_0(a_1)\varphi_0(a_2)\varphi_0(a_3)\ldots$. Generally, an infinite word $w : L \to \dot{\bigcup} G_i \setminus \{1\}$ is mapped to

$$\varphi_W(w) := \varphi_0 \circ w \ : \ L \to \dot{\bigcup} H_n \setminus \{1\},$$

an infinite word in the H_n.

By considering group elements as reduced words, φ_W also induces a map on the topologist's product. Define $\varphi : \circledast_n G_n \to \circledast_n H_n$, $\varphi(u) := R \circ \varphi_W(u)$.

This φ will not be a homomorphism; it does however resemble one, at least for reduced concatenations. If a product uv of group elements is equal to its concatenation $u \cdot v$ (or equivalently, $R(u \cdot v) = u \cdot v$), then

$$\varphi(uv) = \varphi(u)\varphi(v). \tag{†}$$

This follows from the chain of equalities $\varphi(uv) = R \circ \varphi_W(u \cdot v) = R(\varphi_W(u) \cdot \varphi_W(v)) = R(\varphi(u) \cdot \varphi(v)) = \varphi(u)\varphi(v)$.

We claim that φ induces an homomorphism Φ on the induced archipelago groups:

$$\begin{array}{ccc} \circledast_n G_n & \xrightarrow{\varphi} & \circledast_n H_n \\ \downarrow & & \downarrow \\ \mathscr{A}(G_n) & \xrightarrow{\Phi} & \mathscr{A}(H_n) \end{array}$$

We will write $w \sim v$ to denote the equivalence between elements in the topologist's product projecting down to the same element in the archipelago group (i.e. in the quotient by the free product). It needs to be shown that Φ is well-defined, in other words, that $w \sim v$ implies $\varphi(w) \sim \varphi(v)$; and that it is actually a homomorphism.

Claim. *For reduced words u and v, $\varphi(R(u \cdot v)) \sim \varphi(u)\varphi(v)$.*

By the remark it is possible to write the reduced words as concatenations $u = a \cdot g_1 \cdot x$ and $v = y \cdot g_2 \cdot b$, with $y = x^{-1}$ the inverse word of x and g_1, g_2 in some G_i Then $R(u \cdot v) = a \cdot (g_1 g_2) \cdot b$ is reduced as a concatenation of reduced words, and thus from (†),

$$\varphi(R(u \cdot v)) = \varphi(a(g_1 g_2)b) = \varphi(a)\varphi(g_1 g_2)\varphi(b).$$

Further note, $h \sim 1$ for $h \in H_i$ and φ preserves inverses, thus

$$\begin{aligned} \varphi(R(u \cdot v)) &= \varphi(a)\varphi(g_1 g_2)\varphi(b) \\ &\sim \varphi(a)\varphi(b) \\ &\sim \varphi(a)\varphi(g_1)\varphi(g_2)\varphi(b) \\ &= \varphi(a)\varphi(g_1)\varphi(x)\varphi(x)^{-1}\varphi(g_2)\varphi(b) \\ &= \varphi(a)\varphi(g_1)\varphi(x)\varphi(x^{-1})\varphi(g_2)\varphi(b) \\ &= \varphi(a g_1 x)\varphi(y g_2 b) \\ &= \varphi(u)\varphi(v), \end{aligned}$$

as claimed.

Now suppose $u \sim v$ for two elements $u, v \in \circledast_n G_n$. Then using the direct limit representation of the archipelago there exists an index j so that $\tau_j(u) = \tau_j(v)$ for the canonical projection $\tau_j : \circledast_n G_n \to \circledast_{n>j} G_n$. By virtue of the free decomposition

$$\circledast_n G_n = (G_1 * \ldots * G_j) * \circledast_{n>j} G_n$$

u can be written as a finite product $u_1 \ldots u_r$, with each u_i either in $G_1 * \ldots * G_j$ or $\circledast_{n>j} G_n$. Let $U_1, \ldots, U_k$ denote the subsequence of factors lying in the latter, then their product $U_1 \dots U_k$ yields precisely $\tau_j(u)$. By recursively applying the claim,

$$\begin{aligned}\varphi(U_1)\dots\varphi(U_k) &\sim \varphi\big(R(U_1 \cdot U_2)\big)\varphi(U_3)\dots\varphi(U_k) \sim \dots \\ &\sim \varphi\big(R(U_1 \cdot \dots \cdot U_k)\big),\end{aligned}$$

and hence

$$\begin{aligned}\varphi(u) &= \varphi(u_1 \dots u_r) \\ &= \varphi(u_1)\dots\varphi(u_r) \\ &\sim \varphi(U_1)\dots\varphi(U_k) \\ &\sim \varphi\big(R(U_1 \cdot \dots \cdot U_k)\big) \\ &= \varphi\big(\tau_j(u)\big).\end{aligned}$$

Repeating the same process for v we arrive at $\varphi(u) \sim \varphi\big(\tau_j(u)\big) = \varphi\big(\tau_j(v)\big) \sim \varphi(v)$, as desired. So Φ is well-defined.

It remains to show that Φ is a homomorphism, i.e.

$$\Phi\big([u][v]\big) = \Phi\big([u]\big)\Phi\big([v]\big)$$

for $u, v \in \circledast_n G_n$. But for this it suffices that $\varphi(uv) = \varphi\big(R(u \cdot v)\big) \sim \varphi(u)\varphi(v)$ holds with respect to their reduced word representations, precisely as stated in the claim. □

Corollary 1.14. *If the maps φ_i are bijections, then Φ is an isomorphism.*

Proof. As the same argument now holds mutatis mutandis for the inverses of the original bijections $G_i \to H_i$, the so defined map φ^{-1} induces a homomorphism that is clearly inverse to Φ. □

From Corollary 1.14, we can find many archipelago groups that are isomorphic. The main factors to consider are the cardinalities of the groups involved, and the number of those groups containing 2-torsion. The groups' cardinalities are essential in order to even construct bijections among them. The 2-torsion is the only obstruction to then constructing *inverse-preserving* bijections.

Before turning to the proofs of the theorems themselves, we mention a few basic properties.

Lemma 1.15. *An archipelago group satisfies the following:*

(1) *$\mathscr{A}(G_n)$ is independent of the ordering of the groups, i.e. for any permutation $f : \mathbb{N} \to \mathbb{N}$, $\mathscr{A}(G_n) \simeq \mathscr{A}(G_{f(n)})$;*

(2) $\mathscr{A}(G_n) \simeq \mathscr{A}(G_n)_{n\geq k}$*; and*
(3) $\mathscr{A}(G_n) \simeq \mathscr{A}(G_{2n-1} * G_{2n})$.

Proof. First, (1) follows from the fact that both, the topologist's and the free product, share this symmetric property: $\circledast_n G_n \simeq \circledast_n G_{f(n)}$ and $*_n G_n \simeq *_n G_{f(n)}$.

Also, finitely many groups can be split off as a free factor from these products, as $\circledast_n G_n \simeq (*_{n\leq k} G_n) * \circledast_{n>k} G_n$. Thus by cancelling the left factors in

$$\begin{aligned}\mathscr{A}(G_n) &= \circledast_n G_n \,/\, *_n G_n \simeq \\ &\simeq \big((*_{n<k} G_n) * \circledast_{n\geq k} G_n\big) \,/\, \big((*_{n<k} G_n) * *_{n\geq k} G_n\big) \simeq \\ &\simeq \circledast_{n\geq k} G_n \,/\, *_{n\geq k} G_n,\end{aligned}$$

property (2) follows.

Lastly, (3) is inherited from $\circledast_n G_n \simeq \circledast_n (G_{2n-1} * G_{2n})$. □

Indeed, these properties can be combined in the following, slightly more general statement, whose proof is left to the reader.

Proposition 1.16. *Suppose* $f : \mathbb{N} \to \mathbb{N} \cup \{o\}$ *is finite-to-one. Then* $\mathscr{A}(G_n) \simeq \mathscr{A}(*_{i\in f^{-1}(n)} G_i)$.

Lemma 1.17. *If G and H are nontrivial, the cardinality of the set of non-involutions of $G * H$ is given by*

$$\big|\{x \in G * H : x^2 \neq 1\}\big| = \max\big\{\aleph_0, |G|, |H|\big\}.$$

If at least one of the two groups contains an involution, then the set of involutions in the product has the same cardinality

$$\big|\{x \in G * H : x^2 = 1\}\big| = \max\big\{\aleph_0, |G|, |H|\big\}.$$

Proof. The cardinality of the free product itself is $\max\big\{\aleph_0, |G|, |H|\big\}$, which is therefore an upper bound. The elements of the form $(gh)^n$ are all distinct in $G * H$ for all choices $g \in G \setminus \{1\}$, $h \in H \setminus \{1\}$, and $n \geq 1$. Since none of them can be involutions, this proves the first claim. For the second part, assume $a \in H \setminus \{1\}$ with $a^2 = 1$. Then the elements $a^{(gh)^n}$ derived by conjugating a with $(gh)^n$ are all distinct and $\big(a^{(gh)^n}\big)^2 = 1$. □

Proof of Theorem 1.11. Suppose $\lambda < \aleph_0$, then by (2) in Lemma 1.15 we may assume no G_n contains an involution. Then neither does any group $G_{2n-1} * G_{2n}$. Now $G_{2n-1} * G_{2n}$ and $*_{\kappa_{2n-1}} \mathbb{Z} * *_{\kappa_{2n}} \mathbb{Z}$ have the same cardinality

$\max\{\aleph_0, \kappa_{2n-1}, \kappa_{2n}\}$. Hence it is possible, to define inverse preserving bijections

$$\varphi_n : G_{2n-1} * G_{2n} \to \ast_{\kappa_{2n-1}} \mathbb{Z} * \ast_{\kappa_{2n}} \mathbb{Z},$$

simply by setting $\varphi(1) := 1$ and taking each pair x, x^{-1} of a nontrivial element together with its inverse to some pair y, y^{-1}. And so, by applying Corollary 1.14 to these maps and using property (3) twice, we see

$$\mathscr{A}(G_n) \simeq \mathscr{A}(G_{2n-1} * G_{2n}) \simeq \mathscr{A}(\ast_{\kappa_{2n-1}} \mathbb{Z} * \ast_{\kappa_{2n}} \mathbb{Z}) \simeq \mathscr{A}(\ast_{\kappa_n} \mathbb{Z}).$$

Now suppose $\lambda = \aleph_0$, then by (1) in Lemma 1.15 the ordering of the groups can be so arranged that every G_{2n} contains an involution. Then by Lemma 1.17 the cardinalities of both, the set of involutions and non-involutions in $G_{2n-1} * G_{2n}$, are $\max\{\aleph_0, \kappa_{2n-1}, \kappa_{2n}\}$. The same holds for the group $\ast_{\kappa_{2n-1}} \mathbb{Z}_2 * \ast_{\kappa_{2n}} \mathbb{Z}_2$. Between these we can again define an identity-preserving map φ_n, this time by mapping involutions to involutions and pairs of non-involutions $x \neq x^{-1}$ to pairs of non-involutions. By the same argument as in the other case, we get

$$\mathscr{A}(G_n) \simeq \mathscr{A}(G_{2n-1} * G_{2n}) \simeq \mathscr{A}(\ast_{\kappa_{2n-1}} \mathbb{Z}_2 * \ast_{\kappa_{2n}} \mathbb{Z}_2) \simeq \mathscr{A}(\ast_{\kappa_n} \mathbb{Z}),$$

as stated. $\square$

Proof of Theorem 1.1. This can be reduced to the case where each group is countably infinite by taking the free product of consecutive pairs of groups G_n, as by (3) in Lemma 1.15 we may write $\mathscr{A}(G_n) = \mathscr{A}(G_{2n-1} * G_{2n})$.

Then applying Theorem 1.11 twice shows

$$\circledast(G_{2n-1} * G_{2n}) \simeq \mathscr{A}(\ast_{\aleph_0} \mathbb{Z}) \simeq \mathscr{A}(\mathbb{Z}),$$

if only finitely many groups G_n contain involutions. Similarly with $\mathbb{Z}_2$ replacing $\mathbb{Z}$, if infinitely many groups have 2-torsion. $\square$

Proof of Theorem 1.10. First note that if $\kappa = 1$, then $\mathscr{A}(G_n)$ is the trivial group. Otherwise we calculate the cardinality of $\mathscr{A}(G_n)$ as follows. The topologist's product consists of words whose domain can be any countable order type. For a fixed order type, there will be at most $(\aleph_0 \cdot \kappa)^{\aleph_0}$ words, mirroring the number of functions from a countable set L to $\bigcup_n G_n$, if each of the countably many groups has order κ. The cardinality of the set of countable order types is $\mathfrak{c} = 2^{\aleph_0}$, thus $|\circledast_n G_n| \leq \mathfrak{c} \cdot \kappa^{\aleph_0} = \kappa^{\aleph_0}$, and that is also an upper bound for the cardinality of the archipelago group.

For simplicity of notation, assume all G_n to be equal to G, and let $g^{(n)}$ denote the instance of a $g \in G$ within G_n. We will define an injective set function (not a homomorphism) from $G^{\mathbb{N}}$ into $\mathscr{A}(G_n)$, thus providing a lower bound, $|G^{\mathbb{N}}| = \kappa^{\aleph_0} \leq |\mathscr{A}(G_n)|$. For a sequence $(g_j)_j \in G^{\mathbb{N}}$ of elements in G define

$$\varepsilon\big((g_j)_j\big) := g_1^{(1)} g_1^{(2)} g_2^{(3)} g_1^{(4)} g_2^{(5)} g_3^{(6)} g_1^{(7)} g_2^{(8)} g_3^{(9)} g_4^{(10)} \ldots,$$

so to each element of $G^{\mathbb{N}}$ we associate an infinite word of order type ω in $\circledast_n G_n$. Now if $\varepsilon((g_j)_j)$ and $\varepsilon((h_j)_j)^{-1}$ are congruent modulo $*_n G_n$ that implies $g_j = h_j$ for all $j \in \mathbb{N}$. Therefore ε composed with the quotient map from $\circledast_n G_n$ to $\mathscr{A}(G_n)$ is injective, and hence $|\mathscr{A}(G_n)| = \kappa^{\aleph_0}$.

Recall that $\beth_0 := \aleph_0$ is the cardinality of the integers and $\beth_{\alpha+1} := 2^{\beth_\alpha}$ is the cardinality of the power set of $\beth_\alpha$. Thus for successor cardinals,

$$(\beth_{\alpha+1})^{\aleph_0} = (2^{\beth_\alpha})^{\aleph_0} = 2^{\beth_\alpha} = \beth_{\alpha+1}.$$

A free group in $\kappa \geq \aleph_0$ many generators has cardinality κ, thus the free groups generated by $\beth_{\alpha+1}$ many generators are all distinct for different $\alpha \geq 0$. $\square$

Next we prove the mapping properties of $\mathscr{A}(\mathbb{Z})$ stated in Theorem 1.2.

Proposition 1.18. *The group of rational numbers $\mathbb{Q}$ is contained as a subgroup in $\mathscr{A}(\mathbb{Z})$.*

Proof. We will content ourselves with the basic idea of finding a subgroup isomorphic to $\mathbb{Q}$, a more general construction can be found in the next chapter. Consider as an element in $\circledast_n \mathbb{Z}$ the infinite word

$$w := a_1(a_2(a_3(a_4(\ldots)^5)^4)^3)^2,$$

then modulo the free product $*_n \mathbb{Z}$, one can remove the symbol a_1 from the word representation and $w \sim (a_2(a_3(a_4(\ldots)^5)^4)^3)^2 =: w_2^2$, so it is a square. Similarly, $w_2 \sim (a_3(a_4(\ldots)^5)^4)^3 =: w_3^3$, and thus $w \sim w_3^6$ is a sixth power. Proceeding in this manner, $w \sim w_n^{n!}$, so w is a divisible element in $\mathscr{A}(\mathbb{Z})$. Thus it is possible to define a homomorphism $\varepsilon : \mathbb{Q} \to \mathscr{A}(\mathbb{Z})$ by setting $\varepsilon(1) := w$, $\varepsilon(1/2) := w_2$, etc. and extending the map to multiples and (additive) inverses of these. $\square$

Remark. Proposition 1.18 also immediately implies that $\mathscr{A}(\mathbb{Z})$ is not $\aleph_1$-free, so (3) in Theorem 1.2 cannot be strengthened in that respect.

Proof of Proposition 1.8. For each G_n choose a connected CW-complex X_n whose fundamental group is G_n. By Theorem 1.6, $G = \mathscr{A}(G_n)$ is the fundamental group of the space C_f where f is the canonical map from the wedge to the shrinking wedge of the X_n, as described in the paragraph preceding Theorem 1.6. By construction, C_f has a countable neighbourhood basis at the wedge point. Let $\varphi \in \mathrm{Hom}(G,\mathbb{Z})$. By [CC06, Theorem 4.4(3)], there is a neighborhood U of the wedge point so that the homotopy class of any loop in U is in the kernel of φ. By construction, C_f has the property that every loop in C_f is homotopic into every neighborhood of the wedge point. Thus φ is trivial. □

Before demonstrating the torsion-freeness of the archipelago groups, it is helpful to characterize the elements of finite order in the topologist's product.

Lemma 1.19. *If $g \in \circledast_n G_n$ has finite order k, then g is conjugate to an element f in some G_i.*

Proof. This relies on the fact that a torsion element in a free product $A * B$ is conjugate to a torsion element in either A or B. Thus using the free decompositions

$$\circledast_n G_n \;=\; G_1 * \ldots * G_j \;*\; \circledast_{n>j} G_n$$

for $j \in \mathbb{N}$, g is either conjugate to an element in some G_i, or can be represented as $g = c_j^{-1} f_j c_j$ with $f_j \in \circledast_{n>j} G_n$ and $c_j \in \circledast_n G_n$, for all j. But in the latter case, the projection of g into $*_{n=1}^{j} G_n$ is trivial for all j. Since the topologist's product embeds in the inverse limit of these groups, that compels g to be trivial. □

Proof of Proposition 1.9. Consider a torsion element $h \in \mathscr{A}(G_n)$. As this group is a direct limit, we know that for some index j there exists a torsion element $g \in \circledast_{n\geq j} G_n$ representing $h = [g]$ in $\mathscr{A}(G_n)$ with respect to the equivalence relation induced by the direct limit. By the preceding lemma, g is conjugate to an element f in G_i (for some $i \geq j$) by $c \in \circledast_{n\geq j} G_n$. Thus

$$h = [g] = [c^{-1} f c] = [c^{-1}][f][c] = [c^{-1}][c] = 1,$$

so the archipelago group is torsion-free. □

In the case where there is no element of order 2 in the individual groups G_n, we can strengthen that, so the archipelago group is not only torsion-free, but also locally free. The general case requires a bit more effort, and is the content of Theorem 3.9.

Proposition 1.20. *$\mathscr{A}(\mathbb{Z})$ is locally free.*

Proof. In this case the groups $G_n = \mathbb{Z}$, so we have the direct limit

$$\mathscr{A}(\mathbb{Z}) = \varinjlim_k \circledast_{i\geq k}\mathbb{Z} = \varinjlim_k \mathrm{HE}_k$$

where $\mathrm{HE}_k = \circledast_{i\geq k}\mathbb{Z}$ is the Hawaiian earring group, using only loops labelled $i \geq k$. Recall that the group HE, being a subgroup of the inverse limit of free groups, is locally free, following the result in [CF59].

So $\mathscr{A}(\mathbb{Z})$ is a direct limit of locally free groups, and thus locally free itself. □

For completeness, we append a proof.

Lemma 1.21. *The direct limit of locally free groups is locally free.*

Proof. Let $(G_i, i \in I, \varphi_{ij})$ be a directed system of locally free groups. As a set, the direct limit group can be represented as a quotient of the disjoint union of the G_i, $G := \varinjlim_i G_i = \coprod_i G_i/\sim$, where $g_i \in G_i$ and $g_j \in G_j$ are equivalent if $\exists k \geq i, j$ such that $\varphi_{ik}(g_i) = \varphi_{jk}(g_j)$.

Let X be a finite subset of G. We may assume X to be a minimal generating set of $\langle X\rangle \leq G$. Then for all $x \in X$ there is $j_x \in I$ and $y_x \in G_{j_x}$, such that $\varphi_{j_x}(y_x) = x$. Take n to be an upper bound of the j_x. Then $K := \langle \varphi_{j_x n}(y_x) : x \in X\rangle$ is a free subgroup of the locally free group G_n, and $\varphi_n(K) = \langle X\rangle$. As the set X was chosen to be minimal, for each $k \geq n$, the group $\varphi_{nk}(K)$ has the same rank as K, and φ_{nk} restricted to K is in fact an isomorphism between free groups. Therefore, also $\langle X\rangle$ is free. □

CHAPTER 2

A universal locally free group

1. Mapping into the harmonic archipelago

The fundamental group of the harmonic archipelago $\mathscr{A}(\mathbb{Z})$ is a locally free group that is universal in the sense that it contains *every* countable locally free group as a subgroup. This group is interesting for multiple reasons.

Firstly, it is the fundamental group of a two-dimensional space that can be embedded in $\mathbb{R}^3$ – in that sense a best possible result as no planar space could share the stated property (in particular, cannot contain divisible elements, which follows from [FZ05]).

Secondly, its description as a quotient $\circledast_i G_i / \ast_i G_i$ is independent of the factor groups G_i, provided they are countable and contain no 2-torsion elements, due to Theorem 1.1. It is thus "universal" in that regard as well.

Lastly, the group was not purposely built for embedding locally free groups (unlike, e.g. the group described in [BC99]), and its behaviour in this regard is rather ancillary. Indeed, the embedding map makes use of a quite ingenious technique devised by Higman in [Hig52]. As a corollary, not only can the rationals $\mathbb{Q}$ be embedded in this group, so also can the perfect grope group.

Theorem 2.1. *Any countable locally free group embeds into the harmonic archipelago group A.*

We mention two particular subgroups whose existence might come as a surprise, judging by the geometric appearance of the archipelago space. Firstly, it contains divisible elements, implying a similar result in [BZ12].

Example 2.2. *The rationals $\mathbb{Q}$ embed in A.*

As another example, consider the fundamental group of a grope, introduced in [Šta70]. For example, the *minimal grope M* is generated by the elements $a_1, a_2, \dots$ together with the relations $a_i = [a_{2i}, a_{2i+1}]$ for all $i \in \mathbb{N}$. Maps from this group to other grope groups were the topic of recent research, e.g. in [CEV13].

Example 2.3. *The grope group M, which is a perfect group, embeds in A.*

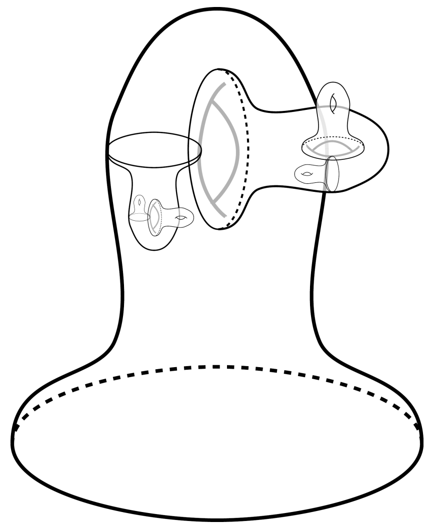

FIGURE 2. The minimal grope space.

2. Proof of the embedding theorem

The first lemma is adapted from Higman's seminal article [Hig52] on unrestricted free products, strengthened to allow the word w_i to access more than only the subsequent h_{i+1}, and adapted in scope to the topologist's product. The basic idea of the proof however remains essentially the same.

Lemma 2.4. *Let $f_1, f_2, \dots$ be a sequence in $\circledast_n H_n$ such that $f_i \in \circledast_{n \geq i} H_n$. Let w_i be a word in $\ell_i + 1$ many arguments. Then there exists a sequence $h_1, h_2, \dots$ in $\circledast_n H_n$ such that $h_i \in \circledast_{n\geq i} H_n$, solving all the equations*

$$h_i = w_i(f_i, h_{i+1}, \dots h_{i+\ell_i}).$$

Proof. We will define the elements h_i by first determining what their projections $p_n(h_i)$ from the inverse limit to $*_{j=1}^n H_j$ ought to be. For $n < i$ set $h_i^{(n)} := 1$; then running through $i = n, n-1, \dots, 1$ recursively define $h_i^{(n)} := w_i(p_n(f_i), h_{i+1}^{(n)}, \dots, h_{\ell_i}^{(n)})$.

Now we claim $p_{nm}(h_i^{(n)}) = h_i^{(m)}$ holds for $n > m$. For $n < i$ this is obvious as then both are equal to 1, in the other case, again proceed by induction counting i down from n,

$$\begin{aligned} p_{nm}(h_i^{(n)}) &= w_i\big(p_m(f_i), p_{nm}(h_{i+1}^{(n)}), \ldots, p_{nm}(h_{\ell_i}^{(n)})\big) \\ &\overset{\text{by ind.}}{=} w_i(p_m(f_i), h_{i+1}^{(m)}, \ldots, h_{\ell_i}^{(m)}) = h_i^{(m)}. \end{aligned}$$

Consequently, the sequence $(h_i^{(n)})_n$ defines for each i an element h_i in the inverse limit $\varprojlim_n *_{m=1}^n H_m$, and by construction these satisfy all the equations $h_i = w_i(f_i, h_{i+1}, \ldots h_{i+\ell_i})$.

It remains to show that h_i is in $\circledast_{n\geq i} H_n$. Beginning with $h_n^{(n)} = w_n(p_n(f_n), 1, \ldots, 1) \in H_n$, another simple induction argument shows $h_i^{(n)} \in *_{m=i}^n H_m$. Thus h_i is certainly at least in the limit $\varprojlim_n *_{m=i}^n H_m$. By recursively unravelling the word representations of the h_i up to an index $j > i$, one arrives at a word $w_i^{(j)}$ such that

$$h_i = w_i^{(j)}(f_i, \ldots, f_j, h_{j+1}, \ldots, h_{\max\{m+\ell_m : i\leq m\leq j\}}).$$

Here the $f_i, \ldots, f_j$ are in $\circledast_{n\geq i} H_n$ while the $h_{j+1}, \ldots$ are in $\varprojlim_n *_{m=j+1}^n H_m$. Hence $h_i \in (\circledast_{n\geq i} H_n) * (\varprojlim_n *_{m=j}^n H_m)$, considered as an internal free product within $\varprojlim_n *_{m=1}^n H_m$, holds for all j, and so by the definition of the topologist's product, $h_i \in \circledast_{n\geq i} H_n$. □

Now, turning to the proof of Theorem 2.1, let $G = \{g_1, g_2, g_3, \ldots\}$ be a countable locally free group. Then for each n there exists a free basis $b_{n,1}, b_{n,2}, \ldots, b_{n,r}$ with $r \leq n$ the rank of the group G_n generated by the elements $g_1, g_2, \ldots, g_n$. To simplify notation set $b_{n,i} := 1$ for $i > r$. Then for each n, g_n can be expressed as a word w_n in these generators

$$g_n = w_n(b_{n,1}, \ldots, b_{n,n}).$$

Similarly, as $G_n < G_{n+1}$, there exist words $w_{n,i}$ for $1 \leq i \leq n$, with

$$b_{n,i} = w_{n,i}(b_{n+1,1}, \ldots, b_{n+1,n+1}),$$

corresponding to the representation of an element $b_{n,i}$ as written in the basis for G_{n+1}. We may require all these words to only involve the basis elements (and not the $b_{n+1,i}$ equal to 1), which forces such a word to be unique.

Define $\alpha : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ by $\alpha(n,i) := n(n-1)/2 + i$ and let a_i denote the generator of the i-th $\mathbb{Z}$-factor in the topologist's product $H = \circledast_i \mathbb{Z}$. The map α^{-1} confers the order of the natural numbers onto the indices of the basis elements $b_{n,i}$, as in the sequence

$$b_{1,1}, b_{2,1}, b_{2,2}, b_{3,1}, b_{3,2}, b_{3,3}, \dots;$$

so by writing briefly $h_{n,i}$ for $h_{\alpha(n,i)}$ and $a_{n,i}$ for $a_{\alpha(n,i)}$, Lemma 2.4 then shows that the equations

$$h_{n,i} = a_{n,i} \cdot w_{n,i}(h_{n+1,1}, \dots, h_{n+1,n+1}) \tag{$*$}$$

can be solved simultaneously for all the unknowns $h_{n,i} \in H$. Now define a function $\varphi : G \to H$ by

$$\varphi(g_n) := w_n(h_{n,1}, \dots, h_{n,n}).$$

This φ is not a homomorphism to H but we claim it will induce one, once factored to the archipelago group A. Let "$\sim$" denote the equivalence in H brought about by this quotient to A, factoring out the normal closure of the free group $\langle a_1, a_2, \dots \rangle$. The key property is the following.

Claim 2.5. *For $n \geq k$ and a word w, the relation $g_k = w(b_{n,1}, \dots, b_{n,n})$ implies $\varphi(g_k) \sim w(h_{n,1}, \dots, h_{n,n})$.*

Proof of Claim. First observe that for $1 \leq i \leq k \leq n$ there exists a unique word $w_{k,i}^{(n)}$ satisfying the relation $b_{k,i} = w_{k,i}^{(n)}(b_{n,1}, \dots, b_{n,n})$, this time corresponding to the representation of an element $b_{k,i}$ as written in the basis for G_n. Evidently then, the identity

$$w_{k,i}^{(n)}(b_{n,1}, \dots, b_{n,n}) = w_{k,i}\big(w_{k+1,1}^{(n)}(b_{n,1}, \dots, b_{n,n}), \dots, w_{k+1,k+1}^{(n)}(b_{n,1}, \dots, b_{n,n})\big)$$

holds for $n > k$, and due to the uniqueness of the presentation with respect to a free basis, the identity holds for any evaluation of these words.

Equipped with this, we will now prove $h_{k,i} \sim w_{k,i}^{(n)}(h_{n,1}, \dots, h_{n,n})$ by induction, counting k downwards from $n-1$. To begin, for $k = n-1$ this follows directly from $w_{k,i} = w_{k,i}^{(n)}$ and $a_{n,i} \sim 1$ in $(*)$. For $k < n-1$,

$$\begin{aligned} h_{k,i} &\overset{\text{by } (*)}{\sim} w_{k,i}(h_{k+1,1}, \dots, h_{k+1,k+1}) \\ &\overset{\text{by ind.}}{\sim} w_{k,i}\big(w_{k+1,1}^{(n)}(h_{n,1}, \dots, h_{n,n}), \dots, w_{k+1,1}^{(n)}(h_{n,1}, \dots, h_{n,n})\big) \\ &= w_{k,i}^{(n)}(h_{n,1}, \dots, h_{n,n}). \end{aligned}$$

We turn our attention back to the claim. If $n = k$, the statement is simply the definition of φ with $w = w_n$. For $n > k$, one arrives at

$$\begin{aligned} g_k = w_k(b_{k,1}, \dots, b_{k,k}) &= w_k\big(w_{k,1}^{(n)}(b_{n,1}, \dots, b_{n,n}), \dots, w_{k,k}^{(n)}(b_{n,1}, \dots, b_{n,n})\big) \\ &= w(b_{n,1}, \dots, b_{n,n}), \end{aligned}$$

where the last equality again holds for arbitrary evaluations. Hence

$$\begin{aligned}\varphi(g_k) &= w_k(h_{k,1},\dots,h_{k,k}) \\ &\sim w_k\big(w^{(n)}_{k,1}(h_{n,1},\dots,h_{n,n}),\dots,w^{(n)}_{k,k}(h_{n,1},\dots,h_{n,n})\big) \\ &= w(h_{n,1},\dots,h_{n,n}),\end{aligned}$$

precisely as claimed. □

Now to show the homomorphism property of φ, suppose $g_k g_l = g_m$ and set $n \geq k, l, m$. Then all three elements have a unique representation $g_i = w'_i(b_{n,1},\dots,b_{n,n})$ in G_n, whence $w'_k(b_{n,1},\dots,b_{n,n})w'_l(b_{n,1},\dots,b_{n,n}) = w'_m(b_{n,1},\dots,b_{n,n})$ follows. Thus

$$\begin{aligned}\varphi(g_k)\varphi(g_l) &\sim w'_k(h_{n,1},\dots,h_{n,n})w'_l(h_{n,1},\dots,h_{n,n}) \\ &= w'_m(h_{n,1},\dots,h_{n,n}) \\ &\sim \varphi(g_m) = \varphi(g_k g_l)\end{aligned}$$

holds for the images, as desired.

To show that φ induces an embedding, the next property is quite useful. Let τ_n denote the projection of $\circledast_i H_i$ to the free factor $H_{\alpha(n,1)} * \dots * H_{\alpha(n,n)}$. Then the representation of an element g_k with respect to the basis chosen for G_n can be read off from such a projection:

Claim 2.6. *For $n \geq k$ and a word w, the relation $g_k = w(b_{n,1},\dots,b_{n,n})$ implies $\tau_n(\varphi(g_k)) = w(a_{n,1},\dots,a_{n,n})$.*

Proof of Claim. Let $\tau_{\geq n}$ denote the canonical projection from $\circledast_i H_i$ to $\circledast_{i\geq\alpha(n,1)} H_i$; the argument in the proof of Claim 2.5 does in fact show that $\tau_{\geq n}(\varphi(g_k)) = \tau_{\geq n}(w(h_{n,1},\dots,h_{n,n}))$, as every invoked equivalence "$\sim$" is only concerned with dropping some a_i for $i < \alpha(n,1)$.

But on the other hand, $\tau_n(h_{n+1,i}) = 1$ by Lemma 2.4, so $\tau_n(h_{n,i}) = \tau_n(a_{n,i})w_{n,i}(\tau_n(h_{n+1,1}),\dots,\tau_n(h_{n+1,n+1})) = a_{n,i}$ by $(*)$. Hence, $\tau_n(\varphi(g_k)) = \tau_n(w(h_{n,1},\dots,h_{n,n})) = w(a_{n,1},\dots,a_{n,n})$. □

Now suppose $\varphi(g_k) \sim 1$. This implies that for some j the projection of $\varphi(g_k)$ to $\circledast_{i>j}\mathbb{Z}$ is trivial. Choose n such that $\alpha(n,1) \geq j$ and let w be the word, such that the relation $g_k = w(b_{n,1},\dots,b_{n,n})$ holds. Then $\tau_n(\varphi(g_k)) = w(a_{n,1},\dots,a_{n,n}) = 1$ which can only be satisfied if w is trivial; this in turn implies $g_k = 1$. Hence, the composed map $G \to H \to A$ is injective, and the proof is complete.

CHAPTER 3

Cotorsion and wild homology

1. Introduction

In the study of infinite abelian groups, two classes of groups are closely related. Firstly, a group is *cotorsion*, if it is a direct summand of every extension by a torsion-free group. Secondly, a group is *algebraically compact*, if it is a direct summand of every extension where it embeds as a pure subgroup. The former are known to be precisely the epimorphic images of the latter, [Fuc70, 54.1]. Algebraically compact groups can be characterized in various ways; one property of such a group is, that a system of equations is solvable if every finite subsystem is.

This chapter provides a similar vantage point for cotorsion groups: they are precisely those groups where every recursively defined system of equations $x_i = w_i(f_i, x_{i+1})$ is solvable for any sequence of elements $f_1, f_2, \ldots$ and equations $w_1, w_2, \ldots$ (Theorem 3.3). These recursions can easily be applied also to the study of nonabelian groups, and indeed, they first appeared in Higman's article [Hig52] in investigating maps from an unrestricted free product. Accordingly, we will call a not necessarily abelian group *Higman-complete*, if every such recursive system is solvable.

Turning to topology, this property holds for the fundamental group of any space with arbitrarily small representatives. This class of spaces contains some intensly studied examples of wild 2-dimensional spaces, such as Griffiths' double cone and the harmonic archipelago. By virtue of this technique we obtain a description of the first singular homology of spaces of this type (Corollary 3.5); in particular, it is shown that the three archipelago groups, $\mathscr{A}(\mathbb{Z})$, $\mathscr{A}(\mathbb{Z}_2)$, and $\mathscr{A}(\mathbb{R})$, whose mutual identity is an open question (cf. Section 1 of the first chapter), at least agree on their abelianization (Corollary 3.6).

Generally, cotorsion of homotopy and homology groups seems to play an important role in the topological behaviour of spaces; independently, a characterization of cotorsion-freeness in terms of homomorphisms to fundamental groups of Peano continua was recently introduced by Eda and Fischer in [EF13], where that notion is also extended to nonabelian groups. It remains open, in how far the notions of Higman-completeness (interpreted as n-cotorsion) and their n-cotorsion-freeness match up beyond the abelian case.

Ending on an algebraic note, we give a more tangible and geometric approach to the embedding of the p-adically complete subgroups arising in the homology of the Hawaiian earring and other spaces in Section 4.

2. Cotorsion and Higman-Completeness

Definition 3.1. A group G is called *Higman-complete* if for any sequence $f_1, f_2, \ldots \in G$, and for a given sequence of words $w_1, w_2, \ldots$, there exists a sequence $h_1, h_2, \ldots \in G$ such that all equations

$$h_i = w_i(f_i, h_{i+1})$$

hold simultaneously.

Lemma 3.2. *If G is Higman-complete then so is every epimorphic image. In particular,* $\mathrm{Ab}(G)$ *is Higman-complete.*

Proof. Let N be a normal subgroup of G and $h_i = w_i(f_i, h_{i+1})$ be any inverse recurrence for elements in G/N as in the definition. Every constant $f_i \in G/N$ can be lifted to a $\tilde{f}_i \in G$, and by assumption the inverse recurrence $\tilde{h}_i = w_i(\tilde{f}_i, \tilde{h}_{i+1})$ admits a sequence of $\tilde{h}_i$ as a solution, whose images $h_i := \tilde{h}_i N/N$ form a solution sequence of our given recurrence in G/N. Hence G/N is Higman-complete.

The second statement follows by letting N be the commutator subgroup of G. □

Theorem 3.3. *An abelian group A is Higman-complete if and only if it is cotorsion.*

Proof. Suppose first that A is Higman-complete. It suffices to show that any exact sequence $0 \to A \to G \to \mathbb{Q} \to 0$ of abelian groups splits. Consider A embedded as a subgroup of G. $\mathbb{Q}$ possesses a presentation generated by the countably many $x_i := \frac{1}{i!}$ and with the relations $x_i - (i+1)x_{i+1}$, for $i \geq 1$. Lift the x_i to elements ξ_i in G. The relations in $\mathbb{Q}$ translate into

$$\xi_i = (i+1)\xi_{i+1} + a_i$$

for suitable elements a_i in A. Since A is by assumption Higman-complete the infinite system of equations

$$h_i = (i+1)h_{i+1} + a_i$$

admits a solution sequence h_i in A. The elements $z_i := \xi_i - h_i \in G$ satisfy the relations

$$z_i = (i+1)z_{i+1}$$

The $\mathbb{Z}$-module, say Q_0, generated by $Z := \{z_1, z_2, \dots\}$ projects modulo A onto $\mathbb{Q}$. We still need to show that $A \cap Q_0 = 0$. Any $q_0 \in A \cap Q_0$ can be presented in the form $q_0 = \lambda z_j$ for some $\lambda \in \mathbb{Z}$ and $j \in \mathbb{N}$, due to the relations among the elements in Z. Modulo A this tells us that $\lambda x_j = 0$ and, since $\mathbb{Q}$ is torsion-free, we must have that $\lambda = 0$, i.e., $q_0 = 0$. Hence $Q_0 \simeq \mathbb{Q}$ and thus the extension splits, as claimed.

Conversely, assume now that A is cotorsion. In the abelian group A, any system of equations as in Definition 3.1 is of the type of an inverse linear recurrence $h_i = d_i h_{i+1} + f_i$ with $d_i \in \mathbb{Z}$ and $f_i \in A$.

There is an algebraically compact group G such that $A \simeq G/N$ for a suitable subgroup N of G. Lift the elements f_i to elements $\tilde{f}_i \in G$. Since every finite subsystem of the inverse recurrence $\tilde{h}_i = d_i \tilde{h}_{i+1} + \tilde{f}_i$ admits a solution, the algebraic compactness of G implies the existence of a sequence of $\tilde{h}_i$ in G solving all equations; the sequence of their images $h_i \in A$ constitute a solution sequence of the original recurrence $h_i = d_i h_{i+1} + f_i$ in A. Hence A is Higman complete. □

In the following, we will consider spaces, where elements in the fundamental group can be represented by arbitrarily small loops. A basepoint independent variation of this property is analysed in [Vir10].

Theorem 3.4. *Let X be a first-countable space. Suppose that for each neighbourhood U of x every element g in the fundamental group $G := \pi_1(X, x)$ has a representative loop included in U. Then G is Higman-complete.*

Proof. Let $f_1, f_2, \dots$ be a sequence of elements in G and $w_1, w_2, \dots$ one of words. We will inductively construct paths $\eta_i : I \to X$ such that for the classes $h_i := [\eta_i] \in G$ all equations $h_i = w_i(f_i, h_{i+1})$ hold. The definition of each η_i will be built up by slowly exhausting the domain I.

X is first-countable at x, so let $U_1 \supseteq U_2 \supseteq \dots$ be a neighbourhood basis of x. By assumption, we can pick for each f_i, a representing path γ_i whose image is enclosed in U_i. Let $\overline{\gamma_i}$ denote the reversed path corresponding to the group element f_i^{-1}.

Begin with the first equation: $h_1 = w_1(f_1, h_2)$. The word w_1 has finite length, so subdivide I, as the domain of η_1, into accordingly many pieces (intervals) of equal size. For each place in w_1 occupied by a f_1 (or f_1^{-1}) define

η_1, restricted to the corresponding piece of I, equal to an appropriately scaled copy of γ_1 (or $\overline{\gamma_1}$), while leaving the other pieces undefined for the moment.

Now proceed in this manner for the second word w_2, again splitting I into as many pieces as is the length of w_2, then setting η_2 equal to a scaled copy of γ_2 (or $\overline{\gamma_2}$) for each piece corresponding to f_2 (or f_2^{-1}). After that, fill the thus partially defined η_2 into the pieces of the domain of γ_1 that correspond to h_2 in the word w_1 (and the reversed $\overline{\eta_2}$ for h_2^{-1}).

Going forward, each time the partial definition of η_i is extended, it is reinserted in the definition of η_{i-1}, and recursively all the way to η_1. Note, that the definitions of the prospective paths are in fact only *extended*, but never changed in this process. Further, each endpoint of an interval is always mapped to the basepoint by the γ_i, so bordering definitions do match up properly. After running through all infinitely many steps of this construction, each η_i is defined everywhere on I but for a closed, totally disconnected set of limit points; set η_i constant to the base point on this set. Thus η_i is well-defined.

Next, we show these paths are continuous. Clearly, η_i is continuous restricted to the interior of a piece corresponding to some f_n (or f_n^{-1}), with $n \geq i$. For $t \in I$ not inside such a piece, t is mapped to the basepoint and the left (resp. right) continuity follows either from that of the bordering piece, or, in the absence of one, from the fact that the pieces converging to t have their image enclosed in eventually smaller and smaller neighbourhoods U_k.

In summary, we have constructed a sequence of loops η_i, each a concatenation according to the word w_i of the loops γ_i and η_{i+1}. Hence, the equations $h_i = w_i(f_i, h_{i+1})$ of the corresponding elements $f_i, h_i \in G$ all hold, as desired. □

3. Wild Homology

Cotorsion or Higman-completeness and, in the torsion-free case, algebraic compactness has a strong impact on the algebraic structure of the homology of a space.

Corollary 3.5. *Let X be as in Theorem 3.4. If additionally its first singular homology group $H := H_1(X,x)$ is torsion-free, it has the form*

$$H \simeq \bigoplus_{\mathfrak{k}} \mathbb{Q} \oplus \prod_{p\ prime} \widehat{\bigoplus_{\mathfrak{m}_p} \mathbb{J}_p}^{p},$$

depending only on the sequence of cardinal numbers $\mathfrak{k}$, $\mathfrak{m}_2, \mathfrak{m}_3, \ldots$ (where "$\widehat{\ }^p$" denotes the p-adic completion and $\mathbb{J}_p$ the p-adic integers).

Proof. By Theorem 3.4, the fundamental group $\pi_1(X,x)$ is Higman-complete, and by Lemma 3.2 so also is its abelianization $H_1(X,x)$. The latter is cotorsion by Theorem 3.3, and a torsion-free cotorsion group is algebraically compact, [Fuc70, 54.5]. The given presentation now follows from applying a series of known facts about abelian groups. First, H splits into a direct sum of a divisible group D and a reduced group C (ibid. 21.3). D is torsion-free, hence a direct sum of $\mathbb{Q}$'s (23.1), while $C \simeq H/D$, due to a result by Kaplansky, is isomorphic to a product $\prod_p A_p$ over all primes with each A_p complete in the p-adic topology (40.1). Again using the torsion-freeness, each A_p is isomorphic to $\widehat{\bigoplus_{\mathfrak{m}_p} \mathbb{J}_p}^{\,p}$ (remark after 40.2), arriving at the claimed decomposition. □

Indeed, it was recently shown in [EF13] and [KR12] that two well-known spaces in this class have isomorphic first homology groups: Griffiths' double cone space (first defined in [Gri54]) and the standard harmonic archipelago. The second space has a natural interpretation as a mapping cone from a weak wedge of circles to the same set given a strong topology (as discussed in Section 1.2).

Let us review some notation: Consider a sequence of pointed spaces (X_i, x_i) (good at the base point) and set $G_i := \pi_1(X_i, x_i)$. Then the fundamental group of their *shrinking wedge* is the *topologist's product* $\circledast_{i\geq 1} G_i$ (see Definition 1.5 and the discussion preceeding it). The fundamental group of the *archipelago space* over the X_i is given by the group $\mathscr{A}(G_i) = \circledast_{i\geq 1} G_i / (*_{i\geq 1} G_i)$, where the normal closure of the embedded free product is factored out (Theorem 1.6 and Definition 1.7). Remarkably, if in the standard archipelago the circles $X_i = S^1$ are replaced by any other sequence of locally finite CW-complexes, the fundamental group of the resulting archipelago space has to be equal to one of two prototypes: to the standard $\mathscr{A}(\mathbb{Z})$ – or to $\mathscr{A}(\mathbb{Z}_2)$, induced by each X_i being a projective plane (that is Theorem 1.1, a fact we will make use of again later).

However, it is unknown, whether or not the two groups $\mathscr{A}(\mathbb{Z})$ and $\mathscr{A}(\mathbb{Z}_2)$ are isomorphic. Similarly, increasing the cardinality of the groups to continuum, i.e. as in $\mathscr{A}(\mathbb{R})$, may or may not alter the group (Questions 1.3 and 1.4). Not so on the other hand, for their abelianizations; the first singular homology groups of the underlying archipelago spaces happen to be all isomorphic to each other.

Corollary 3.6. $\mathrm{Ab}(\mathscr{A}(\mathbb{Z}))$, $\mathrm{Ab}(\mathscr{A}(\mathbb{Z}_2))$, *and* $\mathrm{Ab}(\mathscr{A}(\mathbb{R}))$ *are all isomorphic to each other and to*

$$\prod_{\mathbb{N}} \mathbb{Z} / \bigoplus_{\mathbb{N}} \mathbb{Z} \simeq \bigoplus_{\mathfrak{c}} \mathbb{Q} \oplus \prod_{p \text{ prime}} \widehat{\bigoplus_{\mathfrak{c}} \mathbb{J}_p}^{p}. \tag{†}$$

We have to show that the assumptions of Corollary 3.5 can be satisfied and then that the claimed cardinalities are attained. This will be accomplished in a couple of steps. The first lemma is a slight generalization of Lemma 1.19.

Lemma 3.7. *Every finite subgroup of* $\circledast_i G_i$ *is conjugate into some factor* G_j.

Proof. Let $I_0 \subset I$ be finite then G is the free product $G = G_{I_0} * G_{I_0'}$ with $G_{I_0} := *_{i \in I_0} G_i$ and $G_{I_0'} := \circledast_{i \in I_0'} G_i$. By the conjugacy theorem for free products, up to conjugacy, either $H \leq G_{I_0}$ or $H \leq G_{I_0'}$. In the first case we can conclude our assertion from the conjugacy theorem applied to the free product G_{I_0}. Otherwise there is $g_{I_0} \in G$ and $H \leq G_{I_0'}^{g_{I_0}}$. Hence, for proving the assertion, it suffices to show that

$$\bigcap_{I_0 \Subset I} \left(\circledast_{i \in I \setminus I_0} G_i \right)^{g_{I_0}}$$

cannot contain torsion, for any choice of elements $g_{I_0} \in G$. Now observe that $\bigcap_{I_0 \Subset I} \left(\circledast_{i \in I \setminus I_0} G_i \right)^{g_{I_0}} \leq \bigcap_{I_0 \Subset I} \varprojlim_{J \Subset I, I_0 \cap J = \emptyset} *_{i \in J} G_i$. The group on the r.h.s. is trivial whence the result follows. □

We will make use of embedding the topologist's product in a group introduced in [Hig52]; thus we have by chance again returned to the cradle of the recurrences used in characterizing cotorsion.

Consider a sequence of groups G_i and the inverse system defined by the canonical epimorphisms $*_{i=1}^{n+1} G_i \to *_{i=1}^{n} G_i$ with the normal closure of G_{n+1} as its kernel. The inverse limit $\varprojlim_n *_{i=1}^{n} G_i$ is called the *unrestricted free product*, and, imparted by the projection maps from $G := \circledast_i G_i$ to finite products, G embeds as a subgroup in $\varprojlim_n *_{i=1}^{n} G_i$.

Lemma 3.8. *The exact sequences* $1 \to K_n \to *_{i=1}^{n} G_i \to \prod_{i=1}^{n} G_i \to 1$ *give rise to an exact sequence*

$$1 \to \varprojlim_n K_n \to \varprojlim_n \left(*_{i=1}^{n} G_i \right) \to \prod_{n=1}^{\infty} G_n \to 1.$$

Proof. This follows for the most part from the general fact that the inverse limit is "left-exact" in the category of groups. Namely, let (A_n, α_{nm}),

(B_n, β_{nm}) and (C_n, γ_{nm}) be inverse systems of groups and denote the respective inverse limits by A, B, and C. Suppose that the sequences

$$1 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n$$

are all exact and that the maps commute with the bonding maps (i.e. $f_n \circ \beta_{nm} = \alpha_{nm} \circ f_m$ and $g_n \circ \gamma_{nm} = \beta_{nm} \circ g_m$). Then the induced sequence

$$1 \to A \xrightarrow{f} B \xrightarrow{g} C$$

of inverse limits is exact.

The only thing left to show is that the last term of the sequence in the statement is exact. However, given $y := (g_1, g_2, \ldots) \in \prod_{i=1}^{\infty} G_i = \varprojlim_n (\prod_{i=1}^{n} G_i)$ then the element $(g_1, g_1 g_2, \ldots)$ evidently maps to y. $\square$

Theorem 3.9. *Given arbitrary groups G_i, the archipelago group $\mathscr{A}(G_i)$ is locally free.*

Proof. Let $G := \circledast_i G_i$, and let K denote the kernel of the canonical epimorphism onto $\prod_i G_i$. Set $N := \left(\bigcup_i G_i\right)^G$, then $\mathscr{A}(G_i) = G/N$.

Claim 1. Both, K and KN/N, are locally free.

As mentioned before Lemma 3.8, G embeds naturally in $\varprojlim_n *_{i=1}^{n} G_i$. The lemma then implies that the kernel of the epimorphism from the latter group onto $\prod_{i=1}^{\infty} G_i$ agrees with the inverse limit $\varprojlim_n K_n$. Each K_n is a free group by Kurosh's subgroup theorem, and thus their inverse limit is locally free by [CF59, Theorem 1]. Therefore, the subgroup $K = \left(\varprojlim_n K_n\right) \cap G$ is locally free.

Next observe that $G/N = \varinjlim_n P_n$ where $P_n := G/\left(\bigcup_{i=1}^{n} G_i\right)^G$. Then $P_n \simeq \circledast_{i>n} G_i$ and so, using the first part, find that $K/K \cap \left(\bigcup_{i=1}^{n} G_i\right) G$ is locally free. Now $KN/N \simeq K/K \cap N$ is the direct limit of the locally free groups $K/K \cap \left(\bigcup_{i=1}^{n} G_i\right)^G$ and, as that property is preserved under direct limits (by Lemma 1.21), it is locally free as well.

Claim 2. G/N is torsion free.

Suppose that some $g \in G$ has finite order modulo N, i.e. $g^k \in N$. Observing that $N = \bigcup_{n \geq 1} \left(\bigcup_{i=1}^{n} G_i\right)^G = \bigcup_{n \geq 1} \left(*_{i=1}^{n} G_i\right)^G$ is an ascending chain of normal subgroups, it follows that for some n we must have that $g^k \in M := \left(*_{i=1}^{n} G_i\right)^G$. Therefore, as g maps onto a torsion element in G/M, and since $G/M \simeq \circledast_{i>n} G_i$, an application of Lemma 3.7 shows that gM/M is conjugate to an element in some G_jM/M. Since G_jM is a subgroup of N, we conclude that $g \in N$.

Claim 3. All groups G_i can be assumed to be of the same finite exponent $e \in \mathbb{N}$.

Let κ_i denote the cardinality of G_i. By Theorem 1.11, the group $\mathscr{A}(G_i)$ is either isomorphic to $\mathscr{A}(\bigoplus_{\kappa_i} \mathbb{Z}_3)$ if only finitely many G_i contain elements of order 2, or to $\mathscr{A}(\bigoplus_{\kappa_i} \mathbb{Z}_2)$ otherwise. The exponent of every G_i is then either 2 or 3.

Returning to the statement of the theorem, let $\bar{K} := KN/N$, and suppose H is a finitely generated subgroup of G/N. We want to show, H is free. The abelian quotient group G/K has exponent e and hence $H\bar{K}/\bar{K} \simeq H/(H \cap \bar{K})$ has finite exponent. Therefore $H \cap \bar{K}$ has finite index in the finitely generated group H, so $H \cap \bar{K}$ itself is finitely generated. Since $H \cap \bar{K}$ is a subgroup of $\bar{K} = KN/N$, it is free by Claim 1.

Now H is torsion free by Claim 2 and an extension of finite index of a free group, hence, by Stallings' celebrated result [Sta68, Theorem 3], H is free. □

Showing that the cardinalities in Corollary 3.6 are actually attained is most directly accomplished by finding a subgroup with such decomposition. For the larger group $H := \mathrm{Ab}(\mathscr{A}(G_i))$ to then inherit these cardinalities, it is necessary that the embedded subgroup A is *pure* in H, i.e. if an element $a \in A$ is divisible in H by $k \in \mathbb{N}$, then it is already divisible by k within A. This guarantees that the p-adic factors of A translate into those of H, as otherwise they might simply be contained in the divisible part of the group.

The simplest candidate for a suitable subgroup would be $\prod_i \mathbb{Z}$, an approach pursued in [Eda92]. Instead we will describe an embedding of the whole homology group of the Hawaiian earring into that of an archipelago space. As a preparation, a better understanding of the combinatorial structure of infinite words representing group elements in $\circledast_i G_i$ is required. Recall from Definition 1.5 that for a sequence $(G_i)_i$ of groups, an *infinite word* is a map $w : L \to \dot{\bigcup}_i G_i \setminus \{1\}$ from a countable linearly ordered set L to the disjoint union of the non-identity elements of the G_i where the preimage of each G_i is a finite set. Multiplication is simply concatenation, and inverses are given by inverting the order of the word and replacing each element by its inverse in G_i.

Cancellation induces an equivalence relation on the set of infinite words, and these equivalence classes form the *topologist's product* $\circledast_i G_i$.

Suppose $g \in \circledast_i G_i$ is in the kernel of the abelianization map, then it can be written as a finite product $[a_1, b_2] \dots [a_k, b_k]$ of commutators. As a word, we may consider that as a finite concatenation of reduced words $w_1 \cdot \ldots \cdot w_{4k}$, together with a pairing taking each w_i to some $w_j = w_i^{-1}$. This idea generalizes to the concept of *commutator forms* and, controlling for divisibility, to *n-forms*, both introduced in [Eda92, Definition 4.10] for $\circledast_i \mathbb{Z}$. We will make use of the following variation.

Definition 3.10. Suppose all groups G_i are abelian. A word $w : L \to \dot{\bigcup}_i G_i$ is of *n-form* if it can be written as a finite concatenation $w = w_1 \cdot \ldots \cdot w_\ell$ of reduced words and there exists a partition of $\{1, \ldots, \ell\}$ into disjoint sets A, B, C, and permutations β of B and γ of C, respectively, such that:

(1) for $i \in A$, w_i is a finite word, and the finite word concatenation of only these w_i, projected to $\bigoplus_i G_i$, is divisible by n;
(2) all permutation cycles of β have length 2, and for $i \in B$, $w_{\beta(i)} = w_i^{-1}$; and
(3) all permutation cycles of γ have length n, and for $i \in C$, $w_{\gamma(i)} = w_i$.

If, in addition, $w = w_1 \cdot \ldots \cdot w_\ell$ is reduced, it is of *canonical n-form*. If $n = 0$ (and thus the set C is empty), $w = w_1 \cdot \ldots \cdot w_\ell$ is a *commutator form*.

Lemma 3.11. *Let x be in $\circledast_i G_i$, with all G_i abelian. Suppose* $\mathrm{Ab}(x)$ *is divisible by $n \in \mathbb{N}$, then x has a canonical n-form.*

The proof of this lemma is quite the same as that of Lemma 4.11 in [Eda92]. If $\mathrm{Ab}(x)$ is divisible by n, a word representing x can be written as a finite concatenation $y^n[a_1, b_2] \dots [a_k, b_k]$ with all a_i, b_i, and y reduced. A reduced representation of x can then be obtained by finitely many times removing cancelling blocks $X \cdot X^{-1}$ and modifying β and γ on the corresponding blocks mapped into X or X^{-1} by these functions. The details are left to the reader.

Lemma 3.12. *The homology group $H_1(\mathrm{HE})$ of the Hawaiian earring embeds as a pure subgroup in the abelianization of any archipelago group $\mathscr{A}(G)$, with G abelian and nontrivial.*

Proof. Consider the archipelago group $\mathscr{A}(G_i)$, and for each $i \in \mathbb{N}$ choose a nontrivial element $g_i \in G_i \simeq G$. Let $r : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection, then define a homomorphism $\varphi : \pi_1(\mathrm{HE}) = \circledast_i \mathbb{Z} \to \circledast_i G_i$, first by mapping a generator a_i of the i-th copy of $\mathbb{Z}$ to an infinite word of order type ω

$$\varphi : a_i \to g_{r(i,1)} g_{r(i,2)} g_{r(i,3)} \cdots,$$

then by uniquely extending this map to all elements of $\circledast_i \mathbb{Z}$. Let q denote the canonical epimorphism mapping $\circledast_i G_i$ to its quotient $\mathscr{A}(G_i)$. The functoriality of the abelianization induces homomorphisms $\overline{\varphi}$ and $\overline{q}$, fitting into the commutative diagram:

$$\begin{tikzcd}
\circledast_i \mathbb{Z} \arrow[r, "\varphi"] \arrow[d, "\mathrm{Ab}"'] & \circledast_i G_i \arrow[r, "q"] \arrow[d, "\mathrm{Ab}"'] & \mathscr{A}(G_i) \arrow[d, "\mathrm{Ab}"'] \\
H_1(\mathrm{HE}) \arrow[r, "\overline{\varphi}"'] & \mathrm{Ab}\circledast_i G_i \arrow[r, "\overline{q}"'] & \mathrm{Ab}\,\mathscr{A}(G_i)
\end{tikzcd}$$

Similarly, it is possible to define an endomorphism σ on $\circledast_i G_i$ that maps $G_{r(i,1)}$ to 1 and shifts $G_{r(i,n+1)}$ to $G_{r(i,n)}$, for $i, n \in \mathbb{N}$. The significance of these maps lies in the following properties.

(1) $x \cdot y$ is reduced, if and only if $\varphi(x) \cdot \varphi(y)$ is reduced;
(2) if $\varphi(x)$ is of n-form, then so is x;
(3) $\sigma(\varphi(x)) = \varphi(x)$; and
(4) for all $z \in \ker q$ there exists an exponent $e \in \mathbb{N}$, such that $\sigma^e(z) = 1$.

Here, (1) is evident from the definition of φ, which in turn implies (2): any n-form on $\varphi(x)$ gives rise to a partition A, B, C and maps β, γ, as in Definition 3.10; the restriction of the words to letters in the groups $\{G_{r(i,1)} : i \in \mathbb{N}\}$ allows the same partition pattern, and by replacing each $g_{r(i,1)}$ with a_i, the partition and the maps β, γ immediately translates into an n-form on x.

Property (3) is inherited from $\sigma(\varphi(a_i)) = \sigma(g_{r(i,1)} g_{r(i,2)} g_{r(i,3)} \cdots) = 1 g_{r(i,1)} g_{r(i,2)} \cdots = \varphi(a_i)$. On the other hand, $z \in \ker q$ can be written as $z = z_1^{c_1} \dots z_n^{c_n}$, a product of conjugates with each z_j in some G_k. Then choosing e so large that all these indices k are contained in $\{r(i,m) : i \in \mathbb{N}, m \le e\}$, property (4) follows.

We will show that $\overline{q} \circ \overline{\varphi}$ is the claimed embedding of a pure subgroup. For $x \in \circledast_i \mathbb{Z}$, suppose n divides $(\mathrm{Ab} \circ q \circ \varphi)(x) = (\overline{q} \circ \overline{\varphi} \circ \mathrm{Ab})(x)$. Using the surjectivity of q, there exists $y \in \circledast_i G_i$ such that $n(\mathrm{Ab} \circ q)(y) = (\mathrm{Ab} \circ q \circ \varphi)(x)$, and hence we may write $y^n \varphi(x)^{-1} = z \prod_{i=1}^k [a_i, b_i]$, with $z \in \ker q$

and the $[a_i, b_i]$ some commutators in $\circledast_i G_i$. Using e as in (4) yields

$$\sigma^e(y^n\varphi(x)^{-1}) = (\sigma^e(y))^n\varphi(x)^{-1} = \prod_{i=1}^{k}[\sigma(a_i), \sigma(b_i)].$$

Consequently, n also divides $\mathrm{Ab}\circ\varphi(x)$, and thus $\varphi(x)$ has a (canonical) n-form by Lemma 3.11. By (2), then so does x, which means n divides $\mathrm{Ab}(x)$ as well. We have thus found an element $w \in \circledast_i \mathbb{Z}$ with $n\,\mathrm{Ab}(w) = \mathrm{Ab}(x)$, and finally applying the homomorphisms $\overline{q}$ and $\overline{\varphi}$,

$$n(\overline{q}\circ\overline{\varphi}\circ\mathrm{Ab})(w) = (\overline{q}\circ\overline{\varphi}\circ\mathrm{Ab})(x).$$

Since $(\overline{q}\circ\overline{\varphi}\circ\mathrm{Ab})(w)$ is in the image of $\overline{q}\circ\overline{\varphi}$, we have shown that this image is a pure subgroup of $\mathrm{Ab}(\mathscr{A}(G_i))$. In particular, setting $n := 0$ shows $(\overline{q}\circ\overline{\varphi})(x) = 0$ implies $x = 0$, so the map is injective. $\square$

Proof of Corollary 3.6. Consider the archipelago spaces built respectively over circles S^1, projective planes $\mathbb{P}^2$, or Hawaiian earrings HE, corresponding to their fundamental groups $\mathscr{A}(\mathbb{Z})$, $\mathscr{A}(\mathbb{Z}_2)$, and $\mathscr{A}(\mathbb{R})$; in each one of them, each based loop has one homotopic to it, arbitrarily close to the base point, so they each satisfy the condition of Theorem 3.4, and hence their fundamental groups are Higman-complete. As they are also locally free by Theorem 3.9, the homology groups as the corresponding abelianizations are torsion-free, so the requirements of Theorem 3.5 are satisfied. We thus have a decomposition $\bigoplus_{\mathfrak{k}}\mathbb{Q} \oplus \prod_{p\text{ prime}}\widehat{\bigoplus_{\mathfrak{m}_p}\mathbb{J}_p}^p$ for each of them.

The only thing left to show is that the involved cardinalities are all equal to that of the continuum $\mathfrak{c}$. This can be accomplished by embedding a group of sufficient size. By Lemma 3.12, $H_1(\mathrm{HE})$ embeds in each as a pure subgroup, and that group is isomorphic to

$$\mathbb{Z}^{\mathbb{N}} \oplus \bigoplus_{\mathfrak{c}}\mathbb{Q} \oplus \prod_{p\text{ prime}}\widehat{\bigoplus_{\mathfrak{c}}\mathbb{J}_p}^p,$$

due to a result of Eda and Kawamura in [EK00]. This provides us with a lower bound for the cardinalities.

As the cardinality of each of the groups $\mathscr{A}(\mathbb{Z})$, $\mathscr{A}(\mathbb{Z}_2)$, and $\mathscr{A}(\mathbb{R})$ is itself only continuum, we have proven the isomorphism of their abelianiziations to the decomposition on the right in (†). The isomorphism of that group to $\prod_{\mathbb{N}}\mathbb{Z}/\bigoplus_{\mathbb{N}}\mathbb{Z}$ on the left is a theorem in [Bal59] by Balcerzyk, and the proof is complete. $\square$

4. p-adic Embeddings

Finally, we would like to shed some more light on how the summands in the algebraic decomposition appear geometrically in the homology of these spaces. For the divisible part $\bigoplus_{\mathfrak{c}} \mathbb{Q}$, this is done abstractly in [Eda92, Theorem 4.14] for the earring, and in [BZ12, Corollary 9.2] in more concrete terms for the harmonic archipelago. The principal idea is that a recursively defined element $h_1(h_2(h_3(\dots)^4)^3)^2$, with each $h_i := [g_{2i}, g_{2i+1}]$ a commutator, give rise to a $\mathbb{Q}$-subgroup in $H_1(\mathrm{HE})$. By varying the index sequences involved, one can get continuum many linearly independent copies of $\mathbb{Q}$.

A similar, but technically more involved presentation of the p-adic part in terms of infinite words in the earring group is explained below. The core idea is to ascertain specific divisibility properties whilst avoiding intersection with the Ulm subgroup. Additional difficulty is caused by having to resort to a non-constructive basis argument in a vector space.

Proposition 3.13. *$\prod_{p\ prime} \widehat{\bigoplus_{\mathfrak{c}} \mathbb{J}_p}^{p}$ embeds as a pure subgroup in $H_1(\mathrm{HE})$.*

Proof. First some notation: Let G denote $\circledast_{i\geq 1} \mathbb{Z} = \pi_1(\mathrm{HE})$ the fundamental group of the Hawaiian earring, and $H := H_1(\mathrm{HE})$ its abelianization. For $i \in \mathbb{N}$, let the symbol g_i represent the generator of the i-th $\mathbb{Z}$-factor in this product (topologically, the path class in G running along the i-th circle of the earring). We will lay out the embedding first for each $\mathbb{J}_p$ individually, and then gradually proceed towards the full complexity of embedding the whole product. To lower the notational clutter, writing brackets for function arguments will be omitted when possible.

Let $h_i := [g_{2i}, g_{2i+1}]$, a commutator of generators. Let $q : \mathbb{N} \to \mathbb{N}$ be the map that runs through all integers not divisible by p. Consider $\mathbb{J}_p$ as the set $\{0, \dots, p-1\}^{\mathbb{N}}$ of sequences corresponding to the coefficients in the formal power series representation, $a = \sum_{k\geq 1} a_k p^{k-1}$. Then

$$\begin{aligned} &\varphi : (a_k)_{k\in\mathbb{N}} \mapsto b_1^{a_1}(b_2^{a_2}(b_3^{a_3}(\dots)^{pq3})^{pq2})^{pq1}, \text{ where} \\ &b_n := h_n(h_{n+1}(h_{n+2}(\dots)^{q(n+2)})^{q(n+1)})^{qn}, \end{aligned}$$

is a mapping from $\mathbb{J}_p$ into G. If we denote passing to the abelianization by "$\overline{}$", then $\overline{\varphi} := \mathrm{Ab} \circ \varphi$ is also injective. By construction, each $\overline{b}_n$ is divisible by any prime other than p, and since

$$\varphi((a_k)_{k\in\mathbb{N}}) \equiv b_1^{a_1} b_2^{a_2 \cdot p \cdot q1} \dots b_n^{a_n \cdot p^{n-1} \cdot q1 \cdots q(n-1)} (\dots)^{p^n \cdot q1 \cdots qn},$$

modulo the commutator subgroup, $\overline{\varphi}((a_k)_{k\in\mathbb{N}})$ is similarly divisible.

On the other hand, $p \nmid \overline{b}_n$ and therefore $p^n \mid \overline{\varphi}((a_k)_{k\in\mathbb{N}})$ if and only if $a_k = 0$ for $k = 1, \ldots, n$. Since $\overline{b}_n = qn \cdot \overline{b}_{n+1}$ (in the abelian group H), the exponents take on the role of the coefficients in the power series, and as a consequence,

$$k \mid z_1 + \ldots + z_n \;\text{ if and only if }\; k \mid \overline{\varphi}z_1 + \ldots + \overline{\varphi}z_n \tag{‡}$$

for arbitrary $k \in \mathbb{Z}$ and $z_i \in \mathbb{J}_p$. However, $\overline{\varphi}$ is not a group homomorphism, as adding the elements $\frac{1}{1-p^2} = (1,0,1,0,1,\ldots)$, $\frac{p}{1-p^2} = (0,1,0,1,0\ldots) \in \mathbb{J}_p$ illuminates.

Let Λ be a basis of the p-adic numbers $\widehat{\mathbb{Q}}_p$ as a vector space over $\mathbb{Q}$, such that Λ is already included as a subset in $\mathbb{J}_p$. We point out two important properties of $\widehat{\mathbb{Q}}_p$ and H: *(i)* Both groups are torsion-free; hence for $k \in \mathbb{Z}\setminus\{0\}$, $kx = ky$ implies $x = y$, i.e. if there exists a k-th root of an element, it is unique. *(ii)* Every $x \in \mathbb{J}_p$ can be written as $x = \frac{1}{k}y$ with $y \in \bigoplus_{\lambda\in\Lambda}\mathbb{Z}\lambda$ and $k \in \mathbb{N}$. Thus the map φ restricted to Λ can be naturally extended, first to a homomorphism on $\bigoplus_{\lambda\in\Lambda}\mathbb{Z}\lambda$, then to all of $\mathbb{J}_p$ by choosing the uniquely determined k-th root in H – which exists according to (‡).

This extended map $\psi : \mathbb{J}_p \to H$, $\frac{1}{k}(m_1\lambda_1 + \ldots + m_n\lambda_n) \mapsto \frac{1}{k}(m_1\overline{\varphi}\lambda_1 + \ldots + m_n\overline{\varphi}\lambda_n)$ *is a well-defined injective group homomorphism, and its image is a pure subgroup of* H.

Concerning the injectivity, consider $z \in \mathbb{J}_p$ with $\psi(z) = 0$. Using the representation as $z = \frac{1}{k}(m_1\lambda_1 + \ldots + m_n\lambda_n)$ with respect to the basis Λ, we have $p^r k \mid m_1\overline{\varphi}\lambda_1 + \ldots + m_n\overline{\varphi}\lambda_n$ for every $r \in \mathbb{N}$. Thus by (‡), $p^r k \mid m_1\lambda_1 + \ldots + m_n\lambda_n$, and z must be the 0-element in $\mathbb{J}_p$, whence the injectivity follows.

The purity is yet another consequence of (‡), as $\psi(z) = kg$ for $g \in H$ shows $k \mid z$ and thus $\psi(z) = k\psi(\frac{1}{k}z)$. In particular, it follows that the image of ψ has trivial intersection with the Ulm subgroup $U(H)$. Note that ψ and $\overline{\varphi}$ will disagree on many elements in $\mathbb{J}_p$. However again employing (‡), $k \mid \overline{\varphi}(z) + \overline{\varphi}(-z)$ for all $k \in \mathbb{Z}$ shows $\overline{\varphi}(z) \equiv -\overline{\varphi}(-z)$ modulo the Ulm group, and similarly, from $k \mid \overline{\varphi}(z_1 + z_2) - \overline{\varphi}(z_1) - \overline{\varphi}(z_2)$ for all $k \in \mathbb{Z}$, it follows that $\overline{\varphi}(z_1 + z_2) \equiv \overline{\varphi}(z_1) + \overline{\varphi}(z_2)$. We have thus salvaged the following embedding, that does not rely on a vector basis argument:

Let υ be the canonical projection passing to the quotient modulo the Ulm group. The composed map $\upsilon \circ \mathrm{Ab} \circ \varphi : \mathbb{J}_p \mapsto H/U(H)$ is an injective group homomorphism.

As another intermediate step, we turn our attention to the group $\prod_{p \text{ prime}} \mathbb{J}_p$. Now the coefficients need to be encoded for all primes at the same time. Let $\mathbb{P}$ be the set of primes, then there are maps $\pi : \mathbb{N} \to \mathbb{P}$, $\iota : \mathbb{N} \to \mathbb{N}$, such that $(\pi, \iota) : \mathbb{N} \to \mathbb{P} \times \mathbb{N}$ is a bijection, and ι restricted to $\pi^{-1} p$ is always monotonic for each $p \in \mathbb{P}$. Let $\rho_p n := \pi n$ if $p \neq \pi n$, else $\rho_p n := 1$. Then define a mapping $\varphi : \prod_{p \text{ prime}} \mathbb{J}_p \to H$ by

$$\begin{aligned}&\big((a_{p,k})_{k\in\mathbb{N}}\big)_{p\in\mathbb{P}} \mapsto c_1^{a_{\pi 1,\iota 1}}(c_2^{a_{\pi 2,\iota 2}}(c_3^{a_{\pi 3,\iota 3}}(\ldots)^{\pi 3})^{\pi 2})^{\pi 1}, \text{ where}\\ &c_n := h_n(h_{n+1}(h_{n+2}(\ldots)^{\rho_{\pi n}(n+2)})^{\rho_{\pi n}(n+1)})^{\rho_{\pi n} n}.\end{aligned}$$

Note that p divides $\overline{c}_n$ if and only if $p \neq \pi n$, and $p_0^n \mid \overline{\varphi}\big(((a_{p,k})_{k\in\mathbb{N}})_{p\in\mathbb{P}}\big)$ if and only if $a_{p_0,k} = 0$ for $k = 1, \ldots, n$. Thus again, $k \mid z_1 + \ldots + z_n$ if and only if $k \mid \overline{\varphi} z_1 + \ldots + \overline{\varphi} z_n$. Now $\prod_{p\in\mathbb{P}} \widehat{\mathbb{Q}}_p$ is a vector space over $\mathbb{Q}$, and there exists a basis Λ, included in $\prod_{p\in\mathbb{P}} \mathbb{J}_p$, spanning the subspace generated by that product. As above, $\overline{\varphi}$ restricted to this basis can be extended to the whole product of p-adic integers.

The extended map $\psi : \prod_{p\in\mathbb{P}} \mathbb{J}_p \to H$ is an injective group homomorphism, and its image is a pure subgroup of H.

For the general case, observe that $\widehat{\bigoplus_{\mathfrak{c}} \mathbb{J}_p}^p \simeq \widehat{\bigoplus_{\mathfrak{c}} \mathbb{Z}}^p \simeq A_p := \big\{((a_{f,k})_{k\in\mathbb{N}})_{f\in\mathfrak{c}} : \forall k \tilde{\forall} f : a_{f,k} = 0\big\} \subseteq \prod_{\mathfrak{c}} \mathbb{J}_p$, where $\mathfrak{c} = 2^{\mathbb{N}}$ and $\tilde{\forall}$ denotes "for all but finitely many". To allow for the higher variability afforded by the greater cardinality, a larger family of infinite commutator words is required. Let $\theta : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be a map with the property that $(\theta f)n = (\theta g)n$ holds for almost all $n \in \mathbb{N}$ if and only if $f = g$, and $(\theta f)n \geq n$. Choose some order on $\mathfrak{c}$, then the mapping $\varphi : \prod_{p\in\mathbb{P}} A_p \to H$,

$$\begin{aligned}&(a_{p,f,k})_{p\in\mathbb{P}, f\in\mathfrak{c}, k\in\mathbb{N}} \mapsto \prod_{f\in\mathfrak{c}} d_{1,f}^{a_{\pi 1,f,\iota 1}}(\prod_{f\in\mathfrak{c}} d_{2,f}^{a_{\pi 2,f,\iota 2}}(\prod_{f\in\mathfrak{c}} d_{3,f}^{a_{\pi 3,f,\iota 3}}(\ldots)^{\pi 3})^{\pi 2})^{\pi 1},\\ &\text{where } d_{n,f} := h_{(\theta f)n}(h_{(\theta f)(n+1)}(h_{(\theta f)(n+2)}(\ldots)^{\rho_{\pi n}(n+2)})^{\rho_{\pi n}(n+1)})^{\rho_{\pi n} n},\end{aligned}$$

is well-defined, as, by definition of A_p, each of the products over the $f \in \mathfrak{c}$ is in fact only finite and, with the order inherited from $\mathfrak{c}$, well-defined. The words $d_{n,f}$ share the same divisibility properties as the c_n, i.e. $p \mid \overline{d}_{n,f}$ if and only if $p \neq \pi n$; and $p_0^n \mid \overline{\varphi}((a_{p,f,k})_{p\in\mathbb{P}, f\in\mathfrak{c}, k\in\mathbb{N}})$ if and only if $a_{p_0,f,k} = 0$ for $k = 1, \ldots, n$ and all $f \in \mathfrak{c}$.

Let Λ be a basis of the subspace spanned by $\prod_{p\in\mathbb{P}} A_p$ in the vector space $\prod_{p\in\mathbb{P}}\prod_{\mathfrak{c}}\widehat{\mathbb{Q}}_p$. Finally, $\overline{\varphi}$ restricted to this Λ can be extended as above:

The extended map $\psi : \prod_{p\in\mathbb{P}}\widehat{\bigoplus_{\mathfrak{c}}\mathbb{Z}}^p \to H$ *is an injective group homomorphism, and its image is a pure subgroup of* H. $\square$

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