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Extended BRST Formulation of a Non-Commutative $U_\star(1)$ Gauge Model

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Abstract

Up to now, three of the four fundamental interactions could have been successfully described as quantized gauge theories: the electromagnetism, the weak interaction and the strong interaction, all contained in the standard model with a $SU(3) \times SU(2) \times U(1)$ gauge symmetry. Whereas the further can be formulated as an Abelian $U(1)$ gauge theory (QED), the latter two are non-Abelian Yang-Mills type models. However, the fourth fundamental force, the gravity, which is treated in general relativity as curvature of the space-time, is not covered by the standard model.

There is a strong belief that the space-time has to be quantized as well, at least at the scale of the Planck length. This justifies the introduction of non-commutative coordinates via the so-called Groenewold-Moyal star product. Even though some renormalizable scalar models on non-commutative space-time could have been found, no such gauge theory is yet known. In this thesis we discuss a promising candidate for a non-commutative $U_\star(1)$ gauge model. The gauge symmetry is now a non-Abelian one.

After modifying this $U_\star(1)$ gauge model with a counter term for an arising IR-divergence at one-loop level, a BRST[1] exact formulation via the introduction of BRST doublets is given. Then, two identities are derived, expressing for example the transversality of the two-point vertex graph. Further, relevant considerations and calculations are made with regard to a possible algebraic renormalization procedure. Finally, an equivalent localized model is presented that is formulated BRST exact as well.

[1]Becchi-Rouet-Stora-Tyutin

Kurzfassung

Bislang können drei der vier fundamentalen Wechselwirkungen erfolgreich durch quantisierte Eichtheorien beschrieben werden. Das sind die elektromagnetische, die schwache und die starke Wechselwirkung, alle enthalten im Standardmodell mit einer $SU(3) \times SU(2) \times U(1)$ Eichsymmetrie. Erstere kann als abelsche $U(1)$ Eichtheorie formuliert werden (QED), die letzteren zwei sind nicht-abelsche Yang-Mills Modelle. Wie auch immer, die vierte fundamentale Wechselwirkung, die Gravitation, die in der allgemeinen Relativitätstheorie als Krümmung der Raumzeit behandelt wird, wird vom Standardmodell nicht umfasst.

Man ist davon überzeugt, dass auch die Raumzeit quantisiert sein muss, zumindest auf Distanzen in der Nähe der Planck-Länge. Damit lässt sich das Einführen von nichtkommutativen Koordinaten unter Zuhilfenahme des sogenannten Groenewold-Moyal-Sternprodukts rechtfertigen. Auch wenn renormierbare Skalarmodelle auf nichtkommutativer Raumzeit gefunden werden konnten, ist bis heute kein solches Eichmodell bekannt. In dieser Diplomarbeit wird ein vielversprechendes, nichtkommutatives $U_\star(1)$ Eichmodell behandelt. Die Eichsymmetrie ist nun nicht-abelsch.

Anschließend an die Modifizierung des $U_\star(1)$ Eichmodells durch das Hinzufügen eines Gegenterms aufgrund einer auftauchenden IR-Divergenz auf Ein-Schleifen-Niveau, wird das Modell BRST[2]-exakt formuliert, und zwar durch die Einführung von BRST-Doublets. Weiters werden zwei Identitäten abgeleitet, die unter anderem die Transversalität des Zweipunkt-Vertexgraphen ausdrücken. Außerdem werden für eine allfällige algebraische Renormierung relevante Überlegungen angestellt. Abschließend folgt die Präsentation eines äquivalenten, lokalisierten Modells, dem ebenso eine erweiterte BRST-Symmetrie zugrundeliegt.

[2]Becchi-Rouet-Stora-Tyutin

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Contents

Chapter 1

Introduction to NCQFT

Non-commutative quantum field theories (NCQFT) have become of great interest in the last decades. There are plenty of reasons and hopes that justify such a big effort and I would like to mention some of them now.

To begin with, non-commutativity appears in various domains of physics. A simple example are rotations in a three-dimensional space with their non-commuting generators, the angular momentum operators. Another well known non-commutativity in quantum mechanics is the famous Heisenberg uncertainty relation $\Delta\hat{x}\,\Delta\hat{p} \geq \hbar/2$ and accordingly $[\hat{x}_i, \hat{p}_j] = i\hbar\,\delta_{ij}$. This brings us to the important insight that it is no longer reasonable to talk about "points" in the phase space on the scale of the Planck constant $\hbar$. So, non-commutativity is nothing new in physics.

There is another minimum scale which arises when someone tries to respect gravity, and that's the so called Planck length $l_P = \sqrt{\hbar G/c^3} \approx 1{,}6\,10^{-33}$cm [1, 2]. General relativity deals with the geometry of the spacetime in which everything takes place (like propagation of particles, interactions between fermions via gauge bosons, etc.) and with the influence of the presence of mass and energy on it. Combining this with the Heisenberg uncertainty of a photon or particle with a wavelength small enough for a certain position measurement and living in that spacetime, we can make a short thought experiment [3, 4]: Due to the small wavelength we have a high energy E (or momentum p, since p is related to E by $p = E/c$ for relativistic particles), which alters in turn the underlying geometry due to the gravitational potential connected with E. So, one gets an additional uncertainty in the measured position which is increasing with the energy E. As a consequence, it is not possible to proceed to lengths below l_P. It is this response of the geometry described by general relativity which makes a better position accuracy (by just increasing the momentum of the photon) impossible and, hence, l_P is a fundamental limit of localization. For a detailed discussion of such thought experiments the reader is referred to [5, 3].

So by combining the basic concepts of general relativity with quantum theory, or rather with quantum field theory by additionally considering special relativity, we found a first argument for a spacetime which is not continuous at least at the scale of the Planck length. Already in the year 1946 H. S. Snyder has stated in [6] that for the main proposition of special relativity, the Lorentz invariance of the spacetime length in a 4-dimensional Minkowski space M_4, $x^2 = x_\mu x^\mu = g_{\mu\nu}x^\mu x^\nu = (x_0)^2 - (x_i)^2$, this continuity in spacetime is not necessary. There are a lot of papers concerning Lorentz invariance and its violation. For further details see for example [7, 8, 9].

The implementation of such a minimum length is usually done by replacing the coordinates x by hermitian operators $\hat{x}$ with a non-vanishing commutator

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}\,, \tag{1.1}$$

where $\theta_{\mu\nu}$ is a constant and antisymmetric tensor with mass dimension -2. So we have

$$[\theta^{\mu\nu}, \hat{x}^\rho] = 0 \quad \text{and} \quad \theta^{\mu\nu} = -\theta^{\nu\mu}\,. \tag{1.2}$$

The definition (1.1) is called a canonical structure [10, 11], in contrast to e.g. Lie-algebra structures of the form $[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}C^{\mu\nu}_\rho \hat{x}^\rho$ which are linear in $\hat{x^\rho}$. In this diploma thesis we will work in a 4-dimensional Euclidean space E_4 and further use the following definition:

$$[\hat{x}_\mu, \hat{x}_\nu] = \mathrm{i}\varepsilon\theta_{\mu\nu}\,. \tag{1.3}$$

This results in a dimensionless $\theta_{\mu\nu}$-tensor and a real valued ε with mass dimension -2. It is now not so simple to formulate a field theory (even at the classical level) depending on operators $\hat{x}^\mu$ due to the fact that

$$\phi_i(\hat{x}) \equiv \hat{\phi}_i(\hat{x})\,, \tag{1.4}$$

implying that

$$\hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x}) \neq \hat{\phi}_2(\hat{x})\hat{\phi}_1(\hat{x})\,. \tag{1.5}$$

In order to bypass this difficulty one introduces a deformed product of field variables depending only on ordinary commutative coordinates x_μ. Such a non-commutative structure can be modeled by a replacement of the usual product by the so called star product where the Moyal star product is defined as

$$\phi_1(x) \star \phi_2(x) = e^{\frac{\mathrm{i}}{2}\varepsilon\theta^{\mu\nu}\partial^x_\mu\partial^y_\nu}\phi_1(x)\phi_2(y)\Big|_{x=y}\,. \tag{1.6}$$

$\phi_i(x)$ are any field variables. This gives a representation of Eq. (1.3),

$$[x_\mu \stackrel{\star}{,} x_\nu] = \mathrm{i}\varepsilon\theta_{\mu\nu}\,, \tag{1.7}$$

where the x_μ are the ordinary coordinates. See Sect. 2.1 for the detailed discussion.

The $\theta_{\mu\nu}$-tensor can be regarded as the analogon to $\hbar$ of the Heisenberg uncertainty relation. The corresponding "classical" limits can be achieved by $\theta \to 0$ (in our case $\varepsilon \to 0$) and $\hbar \to 0$, where in the context of non-commutative theories we can also talk about the "commutative" limit. In this limit the star product reduces to the ordinary product as can be seen from the definition. Obviously, an expansion in orders of θ is possible.

Another push for NCQFTs came in the year 1999 by a work of Seiberg and Witten [12], where it has been shown that a non-commutative space-time appears for open strings in a background field B and with their ends fixed on D-branes. Additionally, they illustrated that for large B this situation can be described in a non-commutative Yang-Mills way and that it is just depending on the choice of the regularization scheme whether we get commutativity or non-commutativity.

Hence, it should be possible to find a map between commutative gauge fields a_μ and their non-commutative counterparts A_μ. This is the so called Seiberg-Witten map [12]:

$$A_\nu[a_\mu] + \delta_\Lambda A_\nu[a_\mu] = A_\nu[a_\mu + \delta_\lambda a_\mu]\,, \tag{1.8}$$

For Abelian, commutative fields a_μ one has the usual Abelian gauge transformation

$$a'_\mu = a_\mu + \delta_\lambda a_\mu = a_\mu + \partial_\mu \lambda \,, \tag{1.9}$$

and for non-commutative fields A_μ we have the non-Abelian structure

$$A'_\mu = A_\mu + \delta_\Lambda A_\mu = A_\mu + \partial_\mu \Lambda + \mathrm{i}g \left[\Lambda \stackrel{\star}{,} A_\mu\right] \,. \tag{1.10}$$

The explicit solution up to first order in θ can be written as [13]

$$A_\nu[a_\mu] = a_\nu - \frac{g\theta^{\rho\sigma}}{2}\big(a_\rho \left(\partial_\sigma a_\nu + f_{\sigma\nu}\right)\big) + \mathcal{O}(\theta^2) \,, \tag{1.11}$$

with the Abelian field strength tensor $f_{\sigma\nu} = \partial_\sigma a_\nu - \partial_\nu a_\sigma$.

In this diploma thesis we are dealing with the further development of the $U_\star(1)$ gauge theory presented in [14], and so with non-commutative QED (NCQED). The work presented in [15] leads to the insight that, for example, in the Seiberg-Witten map the photon self-energy is renormalizable to all orders. For a better understanding: the aim of renormalization is to absorb all divergent parts appearing in Feynman integrals by redefinitions of parameters like the mass or the coupling constants, or by redefinitions of the fields.

However, it has been shown in [13] that such a θ-expanded theory leads to the non-renormalizability of a divergence arising in its electron four-point function. So θ-expanded non-commutative QED is non-renormalizable. The non-Abelian equivalent, the non-renormalizability of the θ-expanded non-commutative $U_\star(1)$ Yang-Mills (NCYM) theory is discussed in [16].

Another point that should be mentioned is that space-time non-commutativity leads to an S-matrix which is not unitary (conservation of probability) [17]. The reason is the non-locality in time. So this is not the case if just the space-coordinates aren't commuting, $\theta^{0i} = 0$. For a more detailed discussion and a solution see [18].

A further motivation for NCQFTs and the introduction of a minimum length originally was the appearance of the above mentioned UV-divergences in ordinary QFTs [5]. The hope that one can get rid of these divergences in the Feynman integrals hasn't fullfilled. It somehow gets even worse and somehow not [19, 20, 21]: On the one hand, some of the UV-divergences remain unchanged. On the other hand, other parts of the integrals get UV-convergent due to the fast oszillating phase factor $e^{\mathrm{i}k\theta p}$, but develop IR-divergences for small external momenta p with the same degree as the original UV-divergences. This is the well-known UV/IR-mixing problem.

While the remaining UV-divergences can be treated by ordinary renormalization, the new IR-divergences cannot be absorbed by a redefinition of the parameters or fields and, hence, in general destroys the renormalizability [20].

The first solution of this problem for scalar models was achieved by Grosse and Wulkenhaar [22, 23]: they introduced an additional harmonic oscillator term $\Omega^2/2\,(\tilde{x}_\mu\phi) \star (\tilde{x}^\mu\phi)$ with $\tilde{x}_\mu = 2(\theta^{-1})_{\mu\nu}x^\nu$ into the action to make the model renormalizable. The main critical point on this GW model is that it breaks translation invariance and so Gurau et al. were looking for and found another possibility in the year 2008 [24, 25, 26]:

$$S[\phi] = \int d^4x \left(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{m^2}{2}\phi_\star^2 + \phi \star \frac{a^2}{\theta^2\Box}\phi + \frac{\lambda}{4!}\phi_\star^4\right) \,. \tag{1.12}$$

This so called $1/p^2\ \phi^4_\star$ scalar model alters the propagator IR-finite to

$$G(p) = \frac{1}{p^2 + m^2 + \frac{a^2}{\theta^2 p^2}} \quad \text{with} \quad \lim_{p\to 0} G(p) = 0\,, \tag{1.13}$$

and is perturbatively renormalizable to all orders.

Nevertheless, only non-commutative scalar models with renormalizability but no such gauge models have been found up to now.

The first main challenge for that is to implement a similar damping behaviour as in Eq. (1.13) for small momenta in the gauge propagator $G^{AA}_{\mu\nu}$. This can be done for $U_\star(1)$ gauge models in Euclidean space in the following manner [27, 28]:

$$S[A_\mu] = S_{\text{inv}} + S_{\text{nloc}} = \int d^4x \left(\frac{1}{4} F_{\mu\nu} \star F_{\mu\nu} + \frac{1}{4} F_{\mu\nu} \star \frac{a^2}{D^2 \widetilde{D}^2} F_{\mu\nu} \right)\,, \tag{1.14}$$

with the non-Abelian field strength tensor $F_{\mu\nu}$ and the covariant derivative D_μ, $\widetilde{D}_\mu = \theta_{\mu\nu} D_\nu$ (for further mathematical details see Chap. 3). Its propagator has the desired structure:

$$G^{AA}_{\mu\nu}(p) = \frac{1}{p^2 + \frac{a^2}{\tilde{p}^2}} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \quad \text{with} \quad \lim_{p\to 0} G^{AA}_{\mu\nu}(p) = 0\,. \tag{1.15}$$

This model is suffered from the following main problem [27]: the term $\frac{1}{D^2 \widetilde{D}^2} F_{\mu\nu}$ is a power series in the gauge field A_μ with an infinite number of terms with arbitrary powers and therefore leads to an infinite number of vertices. Furthermore, $D^2 \widetilde{D}^2$ is dimensionless (discussed in [29]).

There is a way out and that is to localize the action (1.14). Two important proposals should be listed here: one of Vilar et al. described in [30], and another ansatz followed up and further developed by Blaschke et al. [31, 32]. Its point of departure reads explicitly:

$$S[A_\mu] = S_{\text{inv}} + S_{\text{loc}} = \int d^4x \left(\frac{1}{4} F_{\mu\nu} \star F_{\mu\nu} + a B_{\mu\nu} \star F_{\mu\nu} - B_{\mu\nu} \star \widetilde{D}^2 D^2 B_{\mu\nu} \right)\,. \tag{1.16}$$

After some troubleshooting of problems arising from this proposal for localization, this will further lead to the model described in [14] which is the theme of this diploma thesis and is described in detail in Chapter 3.

To conclude this introduction, there are plenty of arguments [33, 34] that non-commutativity can also show up much before the Planck scale $l_P \approx 10^{-33}$cm. So far, energy scales up to about 1 TeV could have been reached in collider experiments, and with the new LHC about 10 TeV collisions should be possible [1]. This is approximately one-tenth of the electroweak scale $l_{ew} \approx 10^{-16}$cm (100 GeV) [9] and with the LHC a length $l \approx 10^{-18}$cm should be accessible. There is a strong belief that non-commutativity can have measurable effects in the TeV scale [33, 35].

Chapter 2

Mathematical Background

2.1 The Groenewold-Moyal star product

As described in detail in [10, 21, 11] and [36], one usually introduces the star product via the so called Weyl operator in an n-dimensional Euclidean space

$$\mathcal{W}(\phi) = \hat{\phi}(\hat{x}) = \int \frac{d^n k}{(2\pi)^n}\, e^{\mathrm{i}k_\mu \hat{x}_\mu} \tilde{\phi}(k)\,. \tag{2.1}$$

Additionally, we assume that a function $\phi(x)$ can be described by its Fourier transform

$$\tilde{\phi}(k) = \int d^n x\, e^{-\mathrm{i}k_\mu x_\mu} \phi(x)\,. \tag{2.2}$$

Combining this, a map from a function ϕ to its associated operator $\hat{\phi}$ is given by

$$\mathcal{W}(\phi) = \hat{\phi}(\hat{x}) = \int d^n x\, \phi(x) \int \frac{d^n k}{(2\pi)^n}\, e^{\mathrm{i}k_\mu(\hat{x}_\mu - x_\mu)}\,. \tag{2.3}$$

Now, we have to examine the product of such operators together with its counterpart for the corresponding functions

$$\mathcal{W}(\phi_1)\mathcal{W}(\phi_2) = \mathcal{W}(\phi_1 \star \phi_2)\,, \tag{2.4}$$

the Groenewold-Moyal star product. So we take a closer look at

$$\mathcal{W}(\phi_1)\mathcal{W}(\phi_2) = \int \frac{d^n k}{(2\pi)^n} \int \frac{d^n q}{(2\pi)^n}\, e^{\mathrm{i}k_\mu \hat{x}_\mu} e^{\mathrm{i}q_\nu \hat{x}_\nu} \tilde{\phi}_1(k)\tilde{\phi}_2(q)\,, \tag{2.5}$$

and have to consider that $\hat{x}_\mu$ and $\hat{x}_\nu$ are not commuting, according to (1.3). Hence, we have to use the Baker-Campbell-Hausdorff formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \quad \text{if} \quad [[A,B]\,,A] = 0\,, \tag{2.6}$$

which is fulfilled since $\theta_{\mu\nu}$ has been chosen constant, see (1.2). With this, Eq. (2.5) implies

$$\mathcal{W}(\phi_1)\mathcal{W}(\phi_2) = \int \frac{d^n k}{(2\pi)^n} \int \frac{d^n q}{(2\pi)^n}\, e^{\mathrm{i}(k+q)_\mu \hat{x}_\mu} e^{-\frac{\mathrm{i}}{2}\varepsilon\theta_{\mu\nu}k_\mu q_\nu} \tilde{\phi}_1(k)\tilde{\phi}_2(q)\,. \tag{2.7}$$

Comparing this with

$$\begin{aligned}(\phi_1 \star \phi_2)(x) &= e^{\frac{\mathrm{i}}{2}\varepsilon\theta_{\mu\nu}\partial^x_\mu\partial^y_\nu}\phi(x)\phi(y)\Big|_{x=y} \\ &= \int\frac{d^nk}{(2\pi)^n}\int\frac{d^nq}{(2\pi)^n}\,e^{\mathrm{i}(k+q)_\mu x_\mu}e^{-\frac{\mathrm{i}}{2}\varepsilon\theta_{\mu\nu}k_\mu q_\nu}\tilde{\phi}_1(k)\tilde{\phi}_2(q)\,, \end{aligned} \tag{2.8}$$

where the upper index x of the partial derivative denotes that it is with respect to x, we attain the correspondance in (2.4), as can be seen from

$$\begin{aligned}\mathcal{W}\left(e^{\mathrm{i}(k+q)x}\right) &= \int\frac{d^np}{(2\pi)^n}\,e^{\mathrm{i}p\hat{x}}\int d^nx\,e^{-\mathrm{i}px}e^{\mathrm{i}(k+q)x} \\ &= \int d^np\,e^{\mathrm{i}p\hat{x}}\,\delta^{(n)}(p-k-q) \\ &= e^{\mathrm{i}(k+q)\hat{x}}\,. \end{aligned} \tag{2.9}$$

So we have found in (2.8) the definition of the star product. From now on we use the short notation $kx = k_\mu x_\mu$.

The star product of two exponential functions, which is an often used result in this diploma thesis and, hence, should be stated explicitly, is

$$e^{\mathrm{i}kx}\star e^{\mathrm{i}qx} = e^{\mathrm{i}(k+q)x}e^{-\frac{\mathrm{i}}{2}k\varepsilon\theta q}\,. \tag{2.10}$$

Additionally, there are some useful properties of the star product. At first, we can drop a star under the integral

$$\int d^4x\,(\phi_1\star\phi_2)(x) = \int d^4x\,\phi_1(x)\phi_2(x)\,, \tag{2.11}$$

Then, we have cyclic permutation

$$\int d^4x\,(\phi_1\star\phi_2\star...\star\phi_n)(x) = \pm\int d^4x\,(\phi_2\star...\star\phi_n\star\phi_1)(x)\,, \tag{2.12}$$

where the minus sign just appears if ϕ has a fermionic character. For a bosonic ϕ we have got the plus sign. Furthermore, the star product is associative

$$(\phi_1\star\phi_2)\star\phi_3 = \phi_1\star(\phi_2\star\phi_3)\;, \tag{2.13}$$

and the Fourier transform of a product of n fields can be written as

$$\int d^4x\,(\phi_1\star...\star\phi_n)(x) = \int\frac{d^4k_1}{(2\pi)^4}\cdots\int\frac{d^4k_n}{(2\pi)^4}\,e^{\mathrm{i}\sum_{i=1}^n k_ix}e^{-\frac{\mathrm{i}}{2}\sum_{i<j}^n k_i\varepsilon\theta k_j}\tilde{\phi}(k_1)\ldots\tilde{\phi}(k_n)\,. \tag{2.14}$$

These properties can all be shown quite easily by using the definition (2.8).

In particular, Eq. (2.11) should be pointed out once again because from now on we will neglect the unnecessary stars in the notation.

In addition, we need a definition of partial derivatives [4, 21] which are still commuting:

$$[\partial_\mu \stackrel{\star}{,} x_\nu] = \delta_{\mu\nu} \quad \text{with} \quad [\partial_\mu \stackrel{\star}{,} \partial_\nu] = 0\,. \tag{2.15}$$

2.2 Path integral formalism

The following explanations are based on the references [37, 38, 39] and [40]. In Euclidean space, corresponding to Ref. [14], one defines the generating functional, or also called vacuum-to-vacuum transition amplitude, as

$$Z[j] = \langle 0|e^{-\int d^4x\, j\hat{\phi}}|0\rangle = \mathcal{N}\int \mathcal{D}\phi\, e^{-\int d^4x\,(\mathcal{L}+j\phi)} = \frac{\int \mathcal{D}\phi\, e^{-\left(S+\int d^4x\, j\phi\right)}}{\int \mathcal{D}\phi\, e^{-S}}\,, \tag{2.16}$$

where $\mathcal{L}$ is the Lagranian, $\mathcal{N}$ is the normalization so that $Z[0]=1$ as applied above, and j with $j=(j^A_\mu, j^c, ...)$ is the source of the field operator $\hat{\phi}$ with $\hat{\phi}=(\hat{A}_\mu, \hat{c}, ...)$. Further, $\mathcal{D}\phi$ is the path integral over the fields $\phi=(A_\mu, c, ...)$.

$Z[j]$ generates the Green functions that are in the Euclidean space and in QED the expectation values of products of n gauge field operators $\hat{A}_\mu$, or other field operators like the ghost operators $\hat{c}$,

$$G^{\phi^1\dots\phi^n}_{\mu_1\dots\mu_n}(x_1,\dots,x_n) = \langle 0|\hat{\phi}^1_{\mu_1}(x_1)\dots\hat{\phi}^n_{\mu_n}(x_n)|0\rangle\,. \tag{2.17}$$

They are also called n-point functions and, obviously, can be obtained by

$$G^{\phi^1\dots\phi^n}_{\mu_1\dots\mu_n}(x_1,\dots,x_n) = (-1)^n \left.\frac{\delta^n Z[j]}{\delta j^{\phi^1}_{\mu_1}(x_1)\dots\delta j^{\phi^n}_{\mu_n}(x_n)}\right|_{j=0}\,. \tag{2.18}$$

In addition, the generating functional $Z[j]$ can be written as series expansion

$$Z[j] = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int d^4x_1\dots d^4x_n\, j^{\phi^1}_{\mu_1}(x_1)\dots j^{\phi^n}_{\mu_n}(x_n)\, G^{\phi^1\dots\phi^n}_{\mu_1\dots\mu_n}(x_1,\dots,x_n)\,. \tag{2.19}$$

The summation is over the lower indices μ_i. It should be again mentioned that we have chosen this notation of not differing between upper and lower indices (see also previous section) to emphasize that we are in Euclidean space. One no longer has to distinguish between co- and contravariance. Here, the upper indices just denote the kind of the fields. To shorten the notation we will neglect these upper subscripts from now on.

Now, one can define the generating functional of the connected Green functions via $Z[j]=e^{-Z^c[j]}$ to get rid of disconnected graphs. So one gets the expansion

$$Z^c[j] = \sum_{n=1}^{\infty}\frac{(-1)^{(n-1)}}{n!}\int d^4x_1\dots d^4x_n\, j_{\mu_1}(x_1)\dots j_{\mu_n}(x_n)\, G^c_{\mu_1\dots\mu_n}(x_1,\dots,x_n)\,. \tag{2.20}$$

with the connected Green functions

$$G^c_{\mu_1\dots\mu_n}(x_1,\dots,x_n) = (-1)^{(n-1)}\left.\frac{\delta^n Z^c[j]}{\delta j_{\mu_1}(x_1)\dots\delta j_{\mu_n}(x_n)}\right|_{j=0}\,, \tag{2.21}$$

or also termed as irreducible n-point functions. Additionally, one can introduce classical fields by $\phi^{\rm cl}(x)=\frac{\delta Z^c[j]}{\delta j(x)}$ and with this a Legendre transform

$$\Gamma[\phi^{\rm cl}] = \left.\left(Z^c[j] - \int d^4x\, j(x)\phi^{\rm cl}(x)\right)\right|_{j(x)=j[\phi^{\rm cl}](x)=-\frac{\delta\Gamma[\phi]}{\delta\phi^{\rm cl}(x)}}\,, \tag{2.22}$$

to obtain the so-called generating functional $\Gamma[\phi^{\rm cl}]$ of the vertex functions

$$\Gamma_{\mu_1\dots\mu_n}(x_1,\dots,x_n) = \left.\frac{\delta^n\Gamma[\phi^{\rm cl}]}{\delta\phi^{\rm cl}_{\mu_1}(x_1)\dots\delta\phi^{\rm cl}_{\mu_n}(x_n)}\right|_{\phi^{\rm cl}=0}, \tag{2.23}$$

or one-particle irreducible graphs. Again, we have a series expansion

$$\Gamma[\phi^{\rm cl}] = \sum_{n=2}^{\infty}\frac{1}{n!}\int d^4x_1\dots d^4x_n\,\phi^{\rm cl}_{\mu_1}(x_1)\dots\phi^{\rm cl}_{\mu_n}(x_n)\,\Gamma_{\mu_1\dots\mu_n}(x_1,\dots,x_n)\,. \tag{2.24}$$

It is interesting to mention that these classical fields $\phi^{\rm cl}(x)$ are the vacuum expectation values of the field operators $\hat\phi(x)$ as can be seen from[1]

$$\phi^{\rm cl}(x) = \frac{\delta Z^c[j]}{\delta j(x)} = -\frac{1}{Z[j]}\frac{\delta Z[j]}{\delta j(x)} = \frac{\langle 0|\hat\phi(x)e^{-\int d^4y\, j\hat\phi}|0\rangle}{\langle 0|e^{-\int d^4y\, j\hat\phi}|0\rangle}\,. \tag{2.25}$$

Further, $\Gamma[\phi^{\rm cl}]$ is also termed as effective action. We can make a loop expansion and it turns out that the zeroth loop order $\Gamma^{(0)}[\phi^{\rm cl}]$ is identical to the action $S[\phi^{\rm cl}]$

$$\Gamma[\phi^{\rm cl}] = \sum_{n=0}^{\infty}\hbar^n\Gamma^{(n)}[\phi^{\rm cl}] \quad \text{with} \quad \Gamma^{(0)}[\phi^{\rm cl}] = S[\phi^{\rm cl}]\,, \tag{2.26}$$

where we have dropped the subscript "cl" in the rest of this work. Each term in the action, or in the zeroth order $\Gamma^{(0)}[\phi]$ (tree approximation), leads to an appropriate vertex graph without any loops, whereas the higher orders $\Gamma^{(n)}[\phi]$ with $n > 0$ summarize the contributions of the n-loop vertex graphs.

For a better understanding, we can recapitulate: With Eq. (2.18) we can get all Feynman graphs without restriction of any kind, wheras with Eq. (2.21) we just get all connected graphs, hence, only those where all internal lines are connected. Finally, with Eq. (2.23) we can further exclude those graphs that don't remain connected when someone cuts one internal line. These one-particle irreducible (vertex) graphs can be calculated loop by loop with the action S as initial point. On the other hand, the connected graphs can be reassembled by use of these vertex graphs (and the propagators because vertex graphs don't have external legs).

Now we should say some words about propagators, defined (except prefactors) as the two-point functions $G_{\mu\nu}(x,y)$ (free or dressed). So, we can consider the photon propagator as

$$G^{AA}_{\mu\nu}(x,y) = -\left.\frac{\delta^2 Z^c[j]}{\delta j^A_\mu(x)\delta j^A_\nu(y)}\right|_{j=0} = -\left.\frac{\delta A_\nu(y)}{\delta j^A_\mu(x)}\right|_{j=0}. \tag{2.27}$$

Using $j(x) = -\frac{\delta\Gamma[\phi]}{\delta\phi(x)}$, see the Legendre transformation in (2.22), the appropriate vertex function reads

$$\Gamma^{AA}_{\mu\nu}(x,y) = \left.\frac{\delta^2\Gamma[\phi]}{\delta A_\mu(x)\delta A_\nu(y)}\right|_{\phi=0} = -\left.\frac{\delta j^A_\nu(y)}{\delta A_\mu(x)}\right|_{\phi=0}. \tag{2.28}$$

[1] usually one further has for $j = 0$: $\phi^{\rm cl} = 0$.

One observes immediately that they are inverse to each other, reading explicitly

$$\delta_{\mu\nu}\delta^{(4)}(x-z) = \frac{\delta A_\mu(x)}{\delta A_\nu(z)} = \int d^4y \, \frac{\delta A_\mu(x)}{\delta j^A_\rho(y)} \frac{\delta j^A_\rho(y)}{\delta A_\nu(z)} = \int d^4y \, G^{AA}_{\rho\mu}(y,x) \, \Gamma^{AA}_{\nu\rho}(z,y) \,. \tag{2.29}$$

Usually, one starts with the tree approximation

$$\Gamma^{(0)}[\phi] = S[\phi] = \int d^4x \left(\mathcal{L}_0(x) + \mathcal{L}_{\text{int}}(x)\right) = \int d^4x \left(\frac{1}{2} A_\mu(x) K_{\mu\nu} A_\nu(x) + \mathcal{L}_{\text{int}}(x)\right) \,, \tag{2.30}$$

containing a bilinear and an interaction part. $K_{\mu\nu}$ is a polynomial in partial derivatives, more precisely, of second order for the gauge boson A_μ in QED. So it's easy to derive the two-point vertex function of (2.28) in position space:

$$\Gamma^{AA}_{\mu\nu}(z,y) = \frac{1}{2}\left(K_{\mu\nu} + K_{\nu\mu}\right)\delta^{(4)}(y-z) \overset{\text{symm. in } \mu,\nu}{=} K_{\mu\nu}\delta^{(4)}(y-z) \,, \tag{2.31}$$

where $K_{\mu\nu}$ acts on y. Hence, the challenge in calculating the propagator $G^{AA}_{\mu\nu}$ of (2.27) is to find the inverse of the operator $K_{\mu\nu}$. In gauge theories one needs a gauge fixing to guarantee the invertibility (see Sect. 3.2).

In conclusion, we should point out that the generating functional, see (2.16), $Z = \int \mathcal{D}A \, e^{-S}$, is not finite in gauge theories because a gauge transform $A \to A + \delta_\Lambda A$ can always be applied. This leads to a factor $\int \mathcal{D}\Lambda$ in the above integral that alters Z infinite. In non-Abelian theories one usually treats this problem by the introducton of ghost fields $(c, \bar{c})$, known as the Faddeev-Popov technique. See [41, 37] for further details.

Chapter 3

The Non-Commutative $U_\star(1)$ Gauge Model

The point of departure of this diploma thesis is the model considered in [14]. So we should explain this non-commutative $U_\star(1)$ gauge model in detail and summarize the outcome of [14]. This is done in Sect. 3.3 where we start with Eq. (1.16) of the introduction and give a short summary of how to approach to our model. Further, we discuss the BRST symmetry.

3.1 Some basic definitions

Before we continue to develop the model mentioned in the introduction, we should start with some definitions: the Abelian field strength tensor is well known as

$$f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{3.1}$$

whereas the non-Abelian field strength tensor reads

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g \left[A_\mu \stackrel{\star}{,} A_\nu\right] \,. \tag{3.2}$$

Furthermore, we will need the definition of the covariant derivative

$$D_\mu = \partial_\mu + \mathrm{i}g \left[... \stackrel{\star}{,} A_\mu\right] \,. \tag{3.3}$$

As already mentioned in the introduction, we choose

$$\left[x_\mu \stackrel{\star}{,} x_\nu\right] = \mathrm{i}\varepsilon\theta_{\mu\nu} \tag{3.4}$$

to implement the deformation of our Euclidean space with a real ε of mass dimension -2 and a dimensionless $\theta_{\mu\nu}$-tensor of the form

$$(\theta_{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \,. \tag{3.5}$$

In 4 dimensions it is possible to have this block diagonal form of $\theta_{\mu\nu}$ with rank 4! We use further the abbreviations $\tilde{\psi}_\mu(x) := \theta_{\mu\nu}\psi_\nu(x)$, without ε so that the dimension of $\tilde{\psi}_\mu$ is the same as of ψ_μ, and $\tilde{\psi}(x) := \theta_{\mu\nu}\psi_{\mu\nu}(x)$.

Before we can describe the further development of the localized model in Eq. (1.16) which now reads explicitly by neglecting the star under the integral in the notation

$$S_{\rm inv} + S_{\rm loc} = \int d^4x \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + a B_{\mu\nu} F_{\mu\nu} - B_{\mu\nu} \varepsilon^2 \tilde{D}^2 D^2 B_{\mu\nu} \right) , \tag{3.6}$$

we have to get to know the so-called BRST symmetry.

3.2 BRST symmetry

A gauge transformation is usually given by [42, 43, 44]:

$$A_\mu(x) \to A'_\mu(x) = U(x) \star A_\mu(x) \star U^{-1}(x) + \frac{\mathrm{i}}{g} U(x) \star \partial_\mu U^{-1}(x) . \tag{3.7}$$

For the $U_\star(1)$ gauge group we have $U(x) = e^{\mathrm{i}g\Lambda} = 1 + \mathrm{i}g\Lambda + \mathcal{O}(\Lambda^2)$. This leads us to the infinitesimal gauge transformations of the photon field A_μ

$$A_\mu \to A'_\mu = A_\mu + \delta_\Lambda A_\mu \quad \text{with} \quad \delta_\Lambda A_\mu = \partial_\mu \Lambda + \mathrm{i}g \left[\Lambda \stackrel{\star}{,} A_\mu\right] = D_\mu \Lambda , \tag{3.8}$$

and of the field strength tensor $F_{\mu\nu}$

$$F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu} + \delta_\Lambda F_{\mu\nu} \quad \text{with} \quad \delta_\Lambda F_{\mu\nu} = \mathrm{i}g \left[\Lambda \stackrel{\star}{,} F_{\mu\nu}\right] , \tag{3.9}$$

which can be simply proved by insertion of the above series expansion of $U(x)$. The part $S_{\rm inv} = \int d^4x \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu}\right)$ of the action is invariant under such a gauge transformation, but leads to a transverse two-point vertex function without an inverse. See for this for example the equations (4.27) and (4.28).

Hence, one has to break the ordinary gauge symmetry by the introduction of a gauge fixing part

$$S_{\rm gf} = \int d^4x \left(\frac{\alpha}{2} B^2 + B \partial A \right) , \quad \frac{\delta S_{\rm gf}}{\delta B} = \partial A + \alpha B = 0 , \tag{3.10}$$

to ensure the existence of the photon propagator. $\alpha = 0$ is the so-called Landau gauge.

The need of the BRST procedure [39, 37] is explained by the fact that for a $U_\star(1)$ symmetry the gauge fixing part introduces a non-linear breaking - not a pleasant situation. Therefore, one tries to incorporate the gauge breaking terms as parts of a non-linear symmetry. This is the BRST symmetry (Becchi, Rouet, Stora, Tyutin).

In the ordinary $U(1)$-QED one has

$$\delta A_\mu = \partial_\mu \Lambda . \tag{3.11}$$

This implies for the gauge fixing term the following linear breaking

$$\delta S_{\rm gf}[B, A] = \int d^4x \left(B \partial^2 \Lambda \right) \neq 0 , \tag{3.12}$$

which is linear in the fields.

However, for the case of a $U_\star(1)$ model one has a non-Abelian transformation for the photon field

$$\delta A_\mu = \partial_\mu \Lambda + \mathrm{i}g \left[\Lambda \stackrel{\star}{,} A_\mu\right] . \tag{3.13}$$

This induces a non-linear breaking:

$$\delta S_{\mathrm{gf}}[B,A] = \int d^4x\, B\,\partial_\mu(\partial_\mu\Lambda + \mathrm{i}g\,[\Lambda \stackrel{\star}{,} A_\mu]) \neq 0\,. \tag{3.14}$$

The linear breaking (3.12) can be well understood in the realm of renormalization. However, the non-linear case induces difficulties.

Therefore, one tries to incorporate it as a part of a non-linear symmetry. This leads to the BRST transformation in exchanging the infinitesimal gauge parameter Λ into a fermionic ghost field c with the Faddeev-Popov($\phi\pi$)-charge $+1$,

$$\delta A_\mu \longrightarrow sA_\mu = \partial_\mu c + \mathrm{i}g\,[c \stackrel{\star}{,} A_\mu] = D_\mu c\,, \quad \dim c = 0\,. \tag{3.15}$$

In order to get a symmetry of the whole gauge-fixing action one has to add a piece containing the ghost field c and its anti-ghost field $\bar{c}$. For that purpose, one takes ($\alpha = 0$)

$$S_{\mathrm{gf}} = \int d^4x\,(B\partial A - \bar{c}\partial_\mu sA_\mu)\ , \tag{3.16}$$

and in order to have a symmetry we impose

$$sS_{\mathrm{gf}} = \int d^4x\,\big((sB)\partial A + B\partial_\mu sA_\mu - (s\bar{c})\,\partial_\mu sA_\mu + \bar{c}\,\partial_\mu s^2 A_\mu\big) = 0\,. \tag{3.17}$$

The answer is

$$\begin{aligned} s\bar{c} &= B\,,\\ sB &= 0\,,\\ s^2A_\mu &= 0\,, \end{aligned} \tag{3.18}$$

and from this follows that $\dim\bar{c} = 2$ with $\phi\pi$-charge -1. The full action is $\phi\pi$-neutral, and the last equation gives

$$sc = \mathrm{i}gc\star c\,, \tag{3.19}$$

because to obtain

$$\begin{aligned} s^2A_\mu &= sD_\mu c = s\,(\partial_\mu c + \mathrm{i}g\,[c \stackrel{\star}{,} A_\mu])\\ &= \partial_\mu sc + \mathrm{i}g\,[sc \stackrel{\star}{,} A_\mu] - \mathrm{i}g\,\{c \stackrel{\star}{,} \partial_\mu c + \mathrm{i}g\,[c \stackrel{\star}{,} A_\mu]\} = 0\,, \end{aligned} \tag{3.20}$$

one needs

$$\partial_\mu sc - \mathrm{i}g\,\{c \stackrel{\star}{,} \partial_\mu c\} = 0\,. \tag{3.21}$$

This set of symmetry transformations are the BRST transformations and they have three important properties: they are non-linear, supersymmetric and nilpotent.

A further last comment is the fact that $(\bar{c}, B)$ form a BRST doublet. It is well known in the literature [40, 45] that such doublets are no problem in the renormalization procedure.

Finally, one has also the following shorthand notation:

$$S_{\mathrm{gf}} = \int d^4x\, s\,(\bar{c}\partial A)\ . \tag{3.22}$$

To complete the transformation laws, the matter part of the action has not yet been taken into account:

$$S_{\text{matter}} = \int d^4x\, \bar{\psi} \star (\mathrm{i}\gamma_\mu(\partial_\mu - \mathrm{i}gA_\mu) - m) \star \psi\,, \tag{3.23}$$

where γ_μ are the gamma matrices and ψ stands for the matter fields of the considered theory. Their BRST laws read

$$s\psi = \mathrm{i}gc \star \psi\,, \qquad s\bar{\psi} = \mathrm{i}g\bar{\psi} \star c\,. \tag{3.24}$$

However, we will neglect this matter part for the rest of this diploma thesis, as is customary and fitting to Ref. [14].

3.3 Final approach to the model

At first, the ansatz in (3.6) for the localized part of the action is modified to

$$S_{\text{loc}} = \int d^4x \left[\frac{a}{2}\left(B_{\mu\nu} + \bar{B}_{\mu\nu}\right) F_{\mu\nu} - \bar{B}_{\mu\nu}\varepsilon^2\tilde{D}^2D^2B_{\mu\nu} + \bar{\psi}_{\mu\nu}\varepsilon^2\tilde{D}^2D^2\psi_{\mu\nu}\right]\,, \tag{3.25}$$

see [14, 30, 31], with the antisymmetric (bosonic) auxiliary field $B_{\mu\nu}$ and its complex conjugated field $\bar{B}_{\mu\nu}$. Additionally, we have introduced a pair of (fermionic) ghost fields $(\bar{\psi}_{\mu\nu}, \psi_{\mu\nu})$. The motivation for this structure will get plausible in a few moments since it is needed for a BRST invariant formulation of the action.

This ansatz delivers IR divergences which hinder renormalizability. They are discussed in detail in [32] and their origin is the operator $\tilde{D}^2D^2$ standing in between the new auxiliary fields. So what is done now in [14] is that one part of the operator $(\tilde{D}^2)$ is put into the so-called soft breaking term of the action. Its the part

$$S_{\text{soft}} = \int d^4x \left[\frac{a}{2}\left(B_{\mu\nu} + \bar{B}_{\mu\nu}\right) F_{\mu\nu}\right]\,, \tag{3.26}$$

which, by replacing $\tilde{D}^2$ with $\tilde{\Box}$, gets altered to

$$S_{\text{soft}} = \int d^4x \left[\frac{a'}{2}\left(B_{\mu\nu} + \bar{B}_{\mu\nu}\right)\frac{1}{\tilde{\Box}}F_{\mu\nu}\right]\,. \tag{3.27}$$

This modification also leads to the desired damping in the photon propagator G^{AA} described in the introduction. So the second part of the operator (D^2) is not needed any more, and some further small changes gives [14]:

$$S_{\text{soft}} + S_{\text{aux}} = \int d^4x \left[\frac{\gamma^2}{2}\left(B_{\mu\nu} + \bar{B}_{\mu\nu}\right)\frac{1}{\tilde{\Box}}\left(f_{\mu\nu} + \sigma\frac{\theta_{\mu\nu}}{2}\tilde{f}\right) - \bar{B}_{\mu\nu}B_{\mu\nu} + \bar{\psi}_{\mu\nu}\psi_{\mu\nu}\right]\,. \tag{3.28}$$

Here, for example, the non-Abelian field strength tensor has been replaced by the Abelian one, since for the damping of the gauge propagator just the bilinear part is needed. The parameter γ replaces the old parameter a and has mass dimension 1. While the second part is invariant under the (new) BRST transformations

$$\begin{aligned} s\bar{\psi}_{\mu\nu} &= \bar{B}_{\mu\nu}\,, \qquad & s\bar{B}_{\mu\nu} &= 0\,, \\ sB_{\mu\nu} &= \psi_{\mu\nu}\,, \qquad & s\psi_{\mu\nu} &= 0\,, \end{aligned} \tag{3.29}$$

hence, building BRST doublets, with

$$S_{\text{aux}} = -\int d^4x\, s\left(\bar{\psi}_{\mu\nu} B_{\mu\nu}\right) \,, \tag{3.30}$$

the first part of the action still needs a little more attention. It can be written BRST exact by introduction of two more doublets

$$\begin{aligned} s\bar{Q}_{\mu\nu\alpha\beta} &= \bar{J}_{\mu\nu\alpha\beta}\,, \qquad s\bar{J}_{\mu\nu\alpha\beta} = 0\,, \\ sQ_{\mu\nu\alpha\beta} &= J_{\mu\nu\alpha\beta}\,, \qquad sJ_{\mu\nu\alpha\beta} = 0\,, \end{aligned} \tag{3.31}$$

as

$$S_{\text{soft}} = \int d^4x\, s\left[\left(\bar{Q}_{\mu\nu\alpha\beta} B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu}\right)\frac{1}{\tilde{\square}}\left(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f}\right)\right] \,, \tag{3.32}$$

with their physical values

$$\begin{aligned} \bar{Q}_{\mu\nu\alpha\beta}\big|_{\text{phys}} &= 0\,, \qquad \bar{J}_{\mu\nu\alpha\beta}\big|_{\text{phys}} = \frac{\gamma^2}{4}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right) \,, \\ Q_{\mu\nu\alpha\beta}\big|_{\text{phys}} &= 0\,, \qquad J_{\mu\nu\alpha\beta}\big|_{\text{phys}} = \frac{\gamma^2}{4}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right) \,. \end{aligned} \tag{3.33}$$

The advantage of such BRST doublets is that one can easily modify the IR behaviour of the propagators by a linear coupling of these fields to the original ones [45], as has been done for example through the soft breaking part of Eq. (3.28).

Furthermore, the BRST transformations for the gauge field A_μ (see Eq. (3.18)) and the ghost field c (Eq. (3.19)) are non-linear. This can be treated by introducing BRST invariant external sources Ω^A_μ and Ω^c in the following manner [40]:

$$S_{\text{ext}} = \int d^4x\left(\Omega^A_\mu sA_\mu + \Omega^c sc\right) \,, \qquad sS_{\text{ext}} = 0\,. \tag{3.34}$$

So, we now have found all parts of the action $S = S_{\text{inv}} + S_{\text{gf}} + S_{\text{aux}} + S_{\text{soft}} + S_{\text{ext}}$, given in the equations (3.22), (3.30), (3.32) and (3.34).

The next step in [14] is that all auxiliary fields are integrated out in the path integral formalism. This is equivalent to the use of the unphysical nature of such fields introduced as BRST doublets, meaning that the theory is not depending on $(B_{\mu\nu}, \bar{B}_{\mu\nu})$:

$$\frac{\delta S}{\delta B_{\mu\nu}} = \frac{\gamma^2}{4}\left(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}\right)\frac{1}{\tilde{\square}}\left(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f}\right) - \bar{B}_{\mu\nu} = 0\,, \tag{3.35}$$

and

$$\frac{\delta S}{\delta \bar{B}_{\mu\nu}} = \frac{\gamma^2}{4}\left(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}\right)\frac{1}{\tilde{\square}}\left(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f}\right) - B_{\mu\nu} = 0\,. \tag{3.36}$$

Here, Eq. (3.28) has been used. So, we get

$$B_{\mu\nu} = \bar{B}_{\mu\nu} = \frac{\gamma^2}{2}\frac{1}{\tilde{\square}}\left(f_{\mu\nu} + \sigma\frac{\theta_{\mu\nu}}{2}\tilde{f}\right) \,. \tag{3.37}$$

With the abbreviations $\theta^2 = \theta_{\mu\nu}\theta_{\mu\nu}$, $\rho \equiv \left(2\sigma + \frac{\theta^2}{2}\sigma^2\right)$ and $\tilde{f} = -2\tilde{\partial}A$, one can write the non-localized action, or from now on better termed as the so-called vertex functional in tree approximation, as

$$\Gamma^{(0)} = \int d^4x \left(\frac{1}{4} F_{\mu\nu}F_{\mu\nu} + s(\bar{c}\,\partial A) + \frac{\gamma^4}{4}\left(f_{\mu\nu}\frac{1}{\tilde{\square}^2} f_{\mu\nu} + 2\rho\,\tilde{\partial}A \frac{1}{\tilde{\square}^2}\tilde{\partial}A \right) \right), \tag{3.38}$$

where we have inserted the expressions for $B_{\mu\nu}$ and $\bar{B}_{\mu\nu}$ of Eq. (3.37) into Eq. (3.28). This is the model treated in [14]. Beside the calculation of its propagators and vertices, a one-loop analysis was carried out. These results should be quoted in the next two subsections because they will be needed for the further development of the theory.

3.3.1 Propagators and vertices

So, this (and the next) subsection is based on Ref. [14]. The gauge propagator is in k-space and Landau gauge

$$G^{AA}_{\mu\nu}(k) = \frac{1}{k^2\left(1 + \frac{\gamma^4}{(\tilde{k}^2)^2}\right)} \left\{ \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - \frac{\rho\gamma^4}{\rho\gamma^4 + k^2\left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2}\right)} \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2} \right\}, \tag{3.39}$$

whereas the ghost propagator reads $G^{\bar{c}c}(k) = -\frac{1}{k^2}$. The gauge propagator has the right damping behaviour in the IR-limit $\tilde{k}^2 \to 0$. To obtain the inverse of $G^{AA}_{\mu\nu}$, hence, the two-point vertex function, one has to generalize to an arbitrary gauge $\alpha \neq 0$:

$$\Gamma^{AA}_{\mu\nu}(k) = k^2\mathcal{D}\left\{ \delta_{\mu\nu} + \left(\frac{1}{\alpha\mathcal{D}} - 1\right)\frac{k_\mu k_\nu}{k^2} - \frac{\rho\gamma^4}{k^2\tilde{k}^2\mathcal{D}}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2} \right\}, \quad \text{with} \quad \mathcal{D} = \left(1 + \frac{\gamma^4}{(\tilde{k}^2)^2}\right). \tag{3.40}$$

Additionally, there are three vertices, $\tilde{V}^{3A}_{\rho\sigma\tau}$, $\tilde{V}^{4A}_{\rho\sigma\tau\upsilon}$, and $\tilde{V}^{\bar{c}Ac}_{\rho}$. Their calculation gives:

$$\begin{aligned}\tilde{V}^{3A}_{\rho\sigma\tau}(k_1,k_2,k_3) &= 2\mathrm{i}g\,(2\pi)^4\delta^{(4)}(k_1+k_2+k_3)\,\sin\left(\frac{\varepsilon}{2}k_1\tilde{k}_2\right)\\ &\quad\times\Big\{(k_1-k_3)_\sigma\delta_{\tau\rho} + (k_2-k_1)_\tau\delta_{\rho\sigma} + (k_3-k_2)_\rho\delta_{\sigma\tau}\Big\},\end{aligned} \tag{3.41}$$

$$\begin{aligned}\tilde{V}^{4A}_{\rho\sigma\tau\upsilon}(k_1,k_2,k_3,k_4) &= -4g^2\,(2\pi)^4\delta^{(4)}(k_1+k_2+k_3+k_4)\\ &\quad\times\Big\{\left(\delta_{\rho\tau}\delta_{\sigma\upsilon} - \delta_{\rho\upsilon}\delta_{\sigma\tau}\right)\sin\left(\frac{\varepsilon}{2}k_1\tilde{k}_2\right)\sin\left(\frac{\varepsilon}{2}k_3\tilde{k}_4\right)\\ &\quad + \left(\delta_{\rho\sigma}\delta_{\tau\upsilon} - \delta_{\rho\upsilon}\delta_{\sigma\tau}\right)\sin\left(\frac{\varepsilon}{2}k_1\tilde{k}_3\right)\sin\left(\frac{\varepsilon}{2}k_2\tilde{k}_4\right)\\ &\quad + \left(\delta_{\rho\sigma}\delta_{\tau\upsilon} - \delta_{\rho\tau}\delta_{\sigma\upsilon}\right)\sin\left(\frac{\varepsilon}{2}k_2\tilde{k}_3\right)\sin\left(\frac{\varepsilon}{2}k_1\tilde{k}_4\right)\Big\},\end{aligned} \tag{3.42}$$

and

$$\tilde{V}^{\bar{c}Ac}_{\rho}(q_1,k,q_2) = -2\mathrm{i}g\,(2\pi)^4\delta^{(4)}(q_1+k+q_2)\,q_{2\rho}\,\sin\left(\frac{\varepsilon}{2}q_1\tilde{q}_2\right). \tag{3.43}$$

These are the results in tree approximation. We can now switch over to the one-loop results.

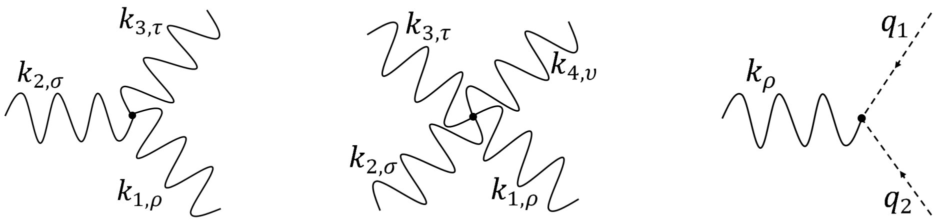

Figure 3.1: The vertices of the action (3.38): $V^{3A}_{\rho\sigma\tau}$, $V^{4A}_{\rho\sigma\tau\upsilon}$, and $V^{\bar{c}Ac}_{\rho}$.

3.3.2 One-loop results

The one-loop corrections to the two-point vertex function (3.40), see [14] for a detailed listing, are proportional to $\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right)$, or to $\frac{\tilde{k}_\mu \tilde{k}_\nu}{(\tilde{k}^2)^2}$. So, they can be absorbed by a redefinition of the parameters ρ and γ. The one-loop part containing $\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right)$ is UV divergent and treated by the introduction of a cutoff parameter Λ.

Further, there are UV divergences arising for the 3A and the 4A vertex, but these corrections can be written proportional to $\ln(\Lambda)\tilde{V}^{3A,\,\mathrm{tree}}_{\rho\sigma\tau}$, or $\ln(\Lambda)\tilde{V}^{4A,\,\mathrm{tree}}_{\rho\sigma\tau\upsilon}$, respectively.

So, there is just one more term left over. It's an IR divergent part contributing to the 3A vertex, reading

$$\Gamma^{3A,\,\mathrm{IR}}_{\rho\sigma\tau}(k_1,k_2,k_3) = -\frac{2\mathrm{i}g^3}{\pi^2}\cos\left(\frac{\varepsilon}{2}k_1\tilde{k}_2\right)\sum_{i=1,2,3}\frac{\tilde{k}_{i,\rho}\tilde{k}_{i,\sigma}\tilde{k}_{i,\tau}}{\varepsilon(\tilde{k}_i^2)^2}\,. \tag{3.44}$$

This term has a complete new structure. There is no way to absorb it into a redefinition of an already existing parameter. We have to add a counter term to our action, which will be the first modification carried out in this diploma thesis (see next chapter), before a further treatment of this model, starting with the BRST exact formulation (Sect. 4.1), and followed by the derivation of two identities. This is done in the sections 4.2 and 4.3.

Chapter 4

Extended BRST Formulation

Here, at first, the action of our $U_\star(1)$ gauge model has been expanded by a counter term corresponding to (3.44), and, subsequently, formulated in a BRST exact manner. Then, two identities for the AA and 3A vertex function has been derived. Moreover, it has been shown that the 3A vertex gets altered in the desired way. This outcome has been published in [46]. To conclude this chapter, the linearized BRST operator has been calculated explicitly since it will be needed for a possible algebraic renormalization of the model.

4.1 BRST exact formulation

We have seen in the previous chapter that by adding the term

$$\frac{\gamma^4}{4}\int d^4x \left(f_{\mu\nu}\frac{1}{\tilde{\square}^2}f_{\mu\nu} + 2\rho\,\tilde{\partial}A\frac{1}{\tilde{\square}^2}\tilde{\partial}A \right) , \tag{4.1}$$

to the usual invariant and the gauge fixing part of the action, the desired damping of the gauge propagator in the IR limit can be achieved.

Furthermore, we now have to add a counter term proportional to the one-loop result in (3.44), concerning the 3A vertex function. It will be seen that

$$\Gamma'_{\text{ext}} = \frac{g'}{2}\int d^4x \left(\{A_\mu \stackrel{\star}{,} A_\nu\}\, \frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\square}^2} A_\rho \right) , \tag{4.2}$$

is the right choice. The part $\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\square}^2}A_\rho$ obviously has the right structure, and later in the Appendix, Eq. (A.72), it has been shown that the first end, the anticommutator, delivers the right cosine. Therefore, we can start with the following action:

$$\begin{aligned}\Gamma^{(0)}_{\text{ext}} = \Gamma'_{\text{ext}} + \Gamma^{(0)} = \int d^4x \Bigg(&\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + s(\bar{c}\,\partial A) + \frac{\gamma^4}{4}\Big(f_{\mu\nu}\frac{1}{\tilde{\square}^2}f_{\mu\nu} + 2\rho\,\tilde{\partial}A\frac{1}{\tilde{\square}^2}\tilde{\partial}A \Big) \\ &+ \frac{g'}{2}\Big(\{A_\mu \stackrel{\star}{,} A_\nu\}\, \frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\square}^2} A_\rho \Big) \Bigg) ,\end{aligned} \tag{4.3}$$

where the parameter γ has mass dimension 1, wheras the new parameter g' is dimensionless, according to the coupling constant g in the non-Abelian field strength tensor. In cases of uncertainty concerning the notation see Sect. 3.1.

This action is not yet invariant under the BRST transformation s,

$$\begin{aligned}
sA_\mu &= \partial_\mu c + \mathrm{i}g\,[c \stackrel{\star}{,} A_\mu] = D_\mu c\,, \\
sc &= \mathrm{i}g c \star c\,, \\
s\bar{c} &= B\,, \\
sB &= 0\,, \\
s^2\phi &= 0\,, \qquad \phi = (A, B, c, \bar{c})\,.
\end{aligned} \tag{4.4}$$

In addition, we have to introduce the external sources Ω^A_μ and Ω^c to treat the non-linearities in the BRST transformations (4.4) for A_μ and c,

$$\Gamma_{\mathrm{ext}} = \int d^4x \left(\Omega^A_\mu sA_\mu + \Omega^c sc\right)\,, \tag{4.5}$$

as already explained above. The BRST invariance at the tree level can usually be denoted via the so-called Slavnov-Taylor identity [40, 39, 37],

$$\mathcal{S}(\Gamma^{(0)}) = \int d^4x \left(\frac{\delta\Gamma^{(0)}}{\delta\Omega^A_\mu} \star \frac{\delta\Gamma^{(0)}}{\delta A_\mu} + \frac{\delta\Gamma^{(0)}}{\delta\Omega^c} \star \frac{\delta\Gamma^{(0)}}{\delta c} + B \star \frac{\delta\Gamma^{(0)}}{\delta\bar{c}}\right) = 0\,, \tag{4.6}$$

but this is not true for our action $\mathcal{S}(\Gamma^{(0)}_{\mathrm{ext}} + \Gamma_{\mathrm{ext}}) \neq 0$ because of the new terms (4.1) and (4.2). Note that, for example, $\frac{\delta\Gamma}{\delta\Omega^A_\mu} = sA_\mu$. Similarly, $\frac{\delta\Gamma}{\delta\Omega^c} = sc$ and $B = s\bar{c}$. The Slavnov-Taylor identity (4.6) is a consequence of the nilpotency of s and of the BRST invariance of some vertex functional Γ.

Hence, we have to extend our BRST transformations (4.4) and introduce further parameters and transformation laws:

$$\begin{aligned}
s\bar{\chi} &= \gamma^4\,, \quad s\gamma^4 = 0\,, \\
s\bar{\delta} &= g'\,, \quad sg' = 0\,.
\end{aligned} \tag{4.7}$$

So, we have installed the new parameter $\bar{\chi}$ as BRST doublet partner of γ^4, and $\bar{\delta}$ as the partner of g'. $\bar{\chi}$ and $\bar{\delta}$ both have ghost number -1 since s is a fermionic operator that increases the ghost number by 1. The mass dimension is identical to the one of the appropriate BRST partner because the operator s leaves it unchanged.

With this we now can write down an extended action as

$$\Gamma^{(0)}_{\mathrm{inv}} = \int d^4x \left(\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + s(\bar{c}\,\partial A) + s(\bar{\chi}\,\mathcal{L}^1_{\mathrm{br}}) + s(\bar{\delta}\,\mathcal{L}^2_{\mathrm{br}}) + \Omega^A_\mu sA_\mu + \Omega^c sc\right)\,, \tag{4.8}$$

with

$$\begin{aligned}
\mathcal{L}^1_{\mathrm{br}} &= \frac{1}{4}\left(f_{\mu\nu} \star \frac{1}{\tilde{\square}^2} f_{\mu\nu} + 2\rho\,\tilde{\partial}A \star \frac{1}{\tilde{\square}^2}\tilde{\partial}A\right)\,, \\
\mathcal{L}^2_{\mathrm{br}} &= \frac{1}{2}\left\{A_\mu \stackrel{\star}{,} A_\nu\right\} \star \frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\square}^2} A_\rho\,,
\end{aligned} \tag{4.9}$$

which is now invariant under the BRST transformations (4.4) and (4.7). We can now express this by the introduction of an extended Slavnov-Taylor operator

$$\mathcal{S}(\Gamma^{(0)}_{\mathrm{inv}}) = \int d^4x \left(\frac{\delta\Gamma^{(0)}_{\mathrm{inv}}}{\delta\Omega^A_\mu} \star \frac{\delta\Gamma^{(0)}_{\mathrm{inv}}}{\delta A_\mu} + \frac{\delta\Gamma^{(0)}_{\mathrm{inv}}}{\delta\Omega^c} \star \frac{\delta\Gamma^{(0)}_{\mathrm{inv}}}{\delta c} + B \star \frac{\delta\Gamma^{(0)}_{\mathrm{inv}}}{\delta\bar{c}}\right) + \gamma^4\frac{\partial\Gamma^{(0)}_{\mathrm{inv}}}{\partial\bar{\chi}} + g'\frac{\partial\Gamma^{(0)}_{\mathrm{inv}}}{\partial\bar{\delta}}\,, \tag{4.10}$$

via the extended Slavnov-Taylor identity

$$\mathcal{S}(\Gamma^{(0)}_{\text{inv}}) = 0\,. \tag{4.11}$$

Note again that now $\gamma^4 = s\bar{\chi}$ and $g' = s\bar{\delta}$. We can go back to the original, physical content of the model by $\bar{\chi} = 0$ and $\bar{\delta} = 0$.

4.2 Identity for the two-point vertex graph

From the extended Slavnov-Taylor identity in (4.11) a few more identities can be derived. Starting with the first term of the extended Slavnov-Taylor operator (4.10)

$$\mathcal{S}(\Gamma^{(0)}_{\text{inv}}) = \int d^4x \left(\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta\Omega^A_\mu} \star \frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta A_\mu} + \text{other terms} \right)\,, \tag{4.12}$$

and inserting the equation of motion for Ω^A_μ

$$\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta\Omega^A_\mu} = sA_\mu = \partial_\mu c + \mathrm{i}g\left[c \stackrel{\star}{,} A_\mu\right]\,, \tag{4.13}$$

we get (setting $g = 1$ for simplification)

$$\mathcal{S}(\Gamma^{(0)}_{\text{inv}}) = \int d^4x\, (\partial_\mu c + \mathrm{i}c \star A_\mu - \mathrm{i}A_\mu \star c) \star \frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta A_\mu} + \text{other terms}\,. \tag{4.14}$$

By using partial integration and cyclic permutation one easily shows that

$$\frac{\delta\mathcal{S}(\Gamma^{(0)}_{\text{inv}})}{\delta c(z)} = -\partial^z_\mu \frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta A_\mu(z)} + \mathrm{i}A_\mu(z) \star \frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta A_\mu(z)} - \mathrm{i}\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta A_\mu(z)} \star A_\mu(z) + \text{other terms}\,. \tag{4.15}$$

We can neglect the "other term", since they won't be important for our first identity. It is shown in (4.38) and below, that these other possible contributions vanish.

Building the functional derivative with respect to $A_\rho(y)$, one gets

$$\frac{\delta^2\mathcal{S}(\Gamma^{(0)}_{\text{inv}})}{\delta A_\rho(y)\delta c(z)} = -\partial^z_\mu \frac{\delta^2\Gamma^{(0)}_{\text{inv}}}{\delta A_\rho(y)\delta A_\mu(z)} + \mathrm{i}\left[\delta^{(4)}(z-y) \stackrel{\star}{,} \frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta A_\rho(z)}\right] + \mathrm{i}\left[A_\mu(z) \stackrel{\star}{,} \frac{\delta^2\Gamma^{(0)}_{\text{inv}}}{\delta A_\rho(y)\delta A_\mu(z)}\right]\,. \tag{4.16}$$

The equation of motion for A_μ has to depend at least linear on the fields since the terms of our action (4.8) are bilinear or of higher order. Therefore, we attain the following identity

$$\left.\frac{\delta^2\mathcal{S}(\Gamma^{(0)}_{\text{inv}})}{\delta A_\rho(y)\delta c(z)}\right|_{\phi=0} = -\partial^z_\mu \left.\frac{\delta^2\Gamma^{(0)}_{\text{inv}}}{\delta A_\rho(y)\delta A_\mu(z)}\right|_{\phi=0} = 0\,, \tag{4.17}$$

where the abbreviation $\phi = (A_\mu, c, \bar{c}, ...)$ has been used. The validity of this identity (4.17) can be shown easily at tree level, which is done now. What we need for this purpose is the two-point vertex graph of this model

$$\Gamma^{AA}_{\rho\mu}(y,z) = \left.\frac{\delta^2\Gamma^{(0)}_{\text{inv}}}{\delta A_\rho(y)\delta A_\mu(z)}\right|_{\phi=0}\,, \tag{4.18}$$

which can be derived either from the inverse of the appropriate connected Green's function, since

$$\int d^4y\, G^{AA}_{\rho\nu}(y,x)\, \Gamma^{AA}_{\mu\rho}(z,y) = \delta_{\mu\nu}\delta^{(4)}(x-z)\,, \tag{4.19}$$

see (2.29), or directly from the bilinear part of the action (4.8). Here, the latter is chosen. The first part of (4.8) that is bilinear in the gauge field reads

$$\Gamma^{(0)}_{\text{inv}} = \int d^4x \left(\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \text{other terms}\right) \xrightarrow{\text{bi}} \Gamma^{(0)}_{\text{bi},1} = \int d^4x \left(\frac{1}{4}f_{\mu\nu}f_{\mu\nu}\right)\,, \tag{4.20}$$

but there is also a second part coming from $\mathcal{L}^1_{\text{br}}$,

$$\Gamma^{(0)}_{\text{bi},2} = \int d^4x \left(\gamma^4\, \mathcal{L}^1_{\text{br}}\right) = \frac{\gamma^4}{4}\int d^4x \left(f_{\mu\nu}\frac{1}{\tilde{\Box}^2}f_{\mu\nu} + 2\rho\,\tilde{\partial}A\frac{1}{\tilde{\Box}^2}\tilde{\partial}A\right)\,. \tag{4.21}$$

To be exact, we have to prove whether a contribution is arising from an arbitrary gauge condition

$$\Gamma^{(0)}_{\text{inv,gf}} = \int d^4x \left(B\partial A\right) \xrightarrow{\alpha\neq 0} \Gamma^{(0)}_{\text{bi,gf}} = \int d^4x \left(B\partial A - \frac{\alpha}{2}B^2\right)\,. \tag{4.22}$$

With the equation of motion for B

$$\frac{\delta\Gamma^{(0)}_{\text{bi,gf}}}{\delta B} = \partial A - \alpha B = -j^B = 0\,, \tag{4.23}$$

we can rewrite (4.22) as

$$\Gamma^{(0)}_{\text{bi,gf}} = \frac{1}{2\alpha}\int d^4x\, (\partial A)^2\,. \tag{4.24}$$

Even if this is also a bilinear part, it should be mentioned, that the identity (4.17) has been derived under the assumption of Landau gauge fixing ($\alpha = 0$). This is fitting to our BRST exact formulation of the action (4.8)

$$\Gamma^{(0)}_{\text{inv}} = \int d^4x \Big(\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + s(\bar{c}\,\partial A) + \text{other terms}\Big)\,. \tag{4.25}$$

Therefore, we can neglect a possible contribution from (4.24) and continue with the calculation for (4.20) and (4.21).

Let us start with the first part (4.20)

$$\Gamma^{(0)}_{\text{bi},1} = \int d^4x \left(\frac{1}{4}f_{\mu\nu}f_{\mu\nu}\right)\,. \tag{4.26}$$

Its contribution is calculated in the Appendix A.1 and is given by Eq. (A.4)

$$\frac{\delta^2\Gamma^{(0)}_{\text{bi},1}}{\delta A_\rho(y)\delta A_\mu(z)} = -\left(\Box^z\delta_{\mu\rho} - \partial^z_\mu\partial^z_\rho\right)\delta^{(4)}(z-y)\,. \tag{4.27}$$

Essentially, this is a so called transverse delta function [38], and, in general, a transverse projection operator has the structure

$$P^T_{\mu\nu} = \delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box} \,, \quad P^T_{\mu\rho} P^T_{\rho\nu} = P^T_{\mu\nu} \,, \quad \partial_\mu P^T_{\mu\nu} = 0 \,, \tag{4.28}$$

and has no inverse. The second contribution to the identity (4.17) comes from (4.21)

$$\Gamma^{(0)}_{\text{bi},2} = \frac{\gamma^4}{4} \int d^4x \left(f_{\mu\nu} \frac{1}{\tilde{\Box}^2} f_{\mu\nu} + 2\rho \, \tilde{\partial} A \frac{1}{\tilde{\Box}^2} \tilde{\partial} A \right) . \tag{4.29}$$

Using the results of Appendix A.2, Eq. (A.15), we get

$$\frac{\delta^2 \Gamma^{(0)}_{\text{bi},2}}{\delta A_\rho(y) \delta A_\mu(z)} = -\frac{\gamma^4}{\tilde{\Box}^2_z} \left(\Box^z \delta_{\mu\rho} - \partial^z_\mu \partial^z_\rho + \rho \, \tilde{\partial}^z_\mu \tilde{\partial}^z_\rho \right) \delta^{(4)}(z-y) \,. \tag{4.30}$$

Summarizing both parts gives

$$\left. \frac{\delta^2 \Gamma^{(0)}_{\text{inv}}}{\delta A_\rho(y) \delta A_\mu(z)} \right|_{\phi=0} = -\left\{ \left(1 + \frac{\gamma^4}{\tilde{\Box}^2_z} \right) \left(\Box^z \delta_{\mu\rho} - \partial^z_\mu \partial^z_\rho \right) + \frac{\rho \gamma^4}{\tilde{\Box}^2_z} \tilde{\partial}^z_\mu \tilde{\partial}^z_\rho \right\} \delta^{(4)}(z-y) \,. \tag{4.31}$$

As a consequence, we have shown that the identity (4.17) is fulfilled at tree level,

$$-\partial^z_\mu \left. \frac{\delta^2 \Gamma^{(0)}_{\text{inv}}}{\delta A_\rho(y) \delta A_\mu(z)} \right|_{\phi=0} = \left\{ \left(1 + \frac{\gamma^4}{\tilde{\Box}^2_z} \right) \partial^z_\mu \left(\Box^z \delta_{\mu\rho} - \partial^z_\mu \partial^z_\rho \right) + \frac{\rho \gamma^4}{\tilde{\Box}^2_z} \partial^z_\mu \tilde{\partial}^z_\mu \tilde{\partial}^z_\rho \right\} \delta^{(4)}(z-y) = 0 \,, \tag{4.32}$$

since the first part in the curly brackets of (4.31) is proportional to the usual transverse projection operator with its properties given in (4.28), and the second part in (4.32) is zero because of the antisymmetric structure of our θ-tensor

$$\partial_\mu \tilde{\partial}_\mu = \partial_\mu \theta_{\mu\nu} \partial_\nu = -\partial_\mu \theta_{\nu\mu} \partial_\nu = -\partial_\nu \theta_{\nu\mu} \partial_\mu = -\partial_\nu \tilde{\partial}_\nu = 0. \tag{4.33}$$

Here, permutability of partial derivatives and the θ-tensor has been used.

So, the identity (4.17) embodies the transversality of the two-point vertex graph and we hope that it holds true even for higher perturbative orders.

Before deriving another identity, we should say a few more words about the gauge fixing. We have already mentioned (see above) that equation (4.17) just holds true for Landau gauge fixing. Otherwise we would additionally gain a contribution from the third term of our extended Slavnov-Taylor operator (4.10)

$$\mathcal{S}(\Gamma^{(0)}_{\text{inv}}) = \int d^4x \left(B \star \frac{\delta \Gamma^{(0)}_{\text{inv}}}{\delta \bar{c}} + \text{other terms} \right) . \tag{4.34}$$

On the one hand we need the equation of motion for $\bar{c}$. Using the action in (4.8) and the BRST transformations (4.4) we get

$$\frac{\delta \Gamma^{(0)}_{\text{inv}}}{\delta \bar{c}} = -s\left(\partial_\mu A_\mu\right) = -\Box c - \mathrm{i} g \partial_\mu \big([c \stackrel{\star}{,} A_\mu] \big) \,. \tag{4.35}$$

On the other hand we can write the B-field by using the equation of motion for B (4.23) as

$$B = \frac{\partial_\mu A_\mu}{\alpha} \,. \tag{4.36}$$

One can easily see that after inserting (4.35) and (4.36) in (4.34) and building the functional derivatives with respect to $c(z)$ and $A(y)$, the equation (4.17) is modified to

$$\left.\frac{\delta^2 \mathcal{S}(\Gamma^{(0)}_{\text{inv}})}{\delta A_\rho(y)\delta c(z)}\right|_{\phi=0} = -\partial^z_\mu \left.\frac{\delta^2 \Gamma^{(0)}_{\text{inv}}}{\delta A_\rho(y)\delta A_\mu(z)}\right|_{\phi=0} - \frac{\partial^z_\rho}{\alpha} \Box^z \delta^{(4)}(z-y) = 0 \,. \tag{4.37}$$

This new term would cancel the contribution that would come from the gauge part (4.24) when inserted into the first term of (4.37). For the detailed derivation see Appendix A.3. We will ignore such considerations from now on and stay in Landau gauge $\alpha = 0$.

Additionally, there are some possible contributions to our identity in (4.17) from the last two terms of the extended Slavnov-Taylor operator (4.10)

$$\mathcal{S}(\Gamma^{(0)}_{\text{inv}}) = \int d^4x \left((...) + \gamma^4 \frac{\partial \Gamma^{(0)}_{\text{inv}}}{\partial \bar{\chi}} + g' \frac{\partial \Gamma^{(0)}_{\text{inv}}}{\partial \bar{\delta}} \right) \,, \tag{4.38}$$

combined with the following parts of our action (4.8)

$$\Gamma^{(0)}_{\text{inv}} = \int d^4x \left(\text{other terms} + s(\bar{\chi}\,\mathcal{L}^1_{\text{br}}) + s(\bar{\delta}\,\mathcal{L}^2_{\text{br}}) \right) \,, \tag{4.39}$$

and with $\mathcal{L}^1_{\text{br}}$ and $\mathcal{L}^2_{\text{br}}$ given in (4.9). These possible contributions have been neglected above ("other terms" in (4.12) and below) because they are transformed to zero. It is obvious that for vanishing fields $\mathcal{L}^2_{\text{br}}$ can give no additional term to the identity (4.17) because it is trilinear in the fields, and not bilinear. However, this is different for $\mathcal{L}^1_{\text{br}}$. Here, we have to take a closer look at

$$\left.\frac{\delta^2 \mathcal{S}(\Gamma^{(0)}_{\text{inv}})}{\delta A_\rho(y)\delta c(z)}\right|_{\phi=0} = \left.\frac{\delta^2}{\delta A_\rho(y)\delta c(z)} \left(\gamma^4 \frac{\partial \Gamma^{(0)}_{\text{inv}}}{\partial \bar{\chi}} \right)\right|_{\phi=0} . \tag{4.40}$$

It has been shown explicitly in Appendix A.2 that this is zero, Eq. (A.26),

$$\begin{aligned} \left.\frac{\delta^2}{\delta A_\rho(y)\delta c(z)} \left(\gamma^4 \frac{\partial \Gamma^{(0)}_{\text{inv}}}{\partial \bar{\chi}} \right)\right|_{\phi=0} &= -\gamma^4 \frac{\delta^2}{\delta A_\rho(y)\delta c(z)} \left.\int d^4x \left(s\mathcal{L}^1_{\text{br}} \right)\right|_{\phi=0} \\ &= -\gamma^4 \frac{\delta^2}{\delta A_\rho(y)\delta c(z)} \left.\int d^4x \left(c \frac{\partial_\mu}{\tilde{\Box}^2} \left(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu + \rho\tilde{\partial}_\mu\tilde{\partial}_\nu \right) A_\nu \right)\right|_{\phi=0} = 0 \,, \end{aligned} \tag{4.41}$$

where the properties for a transverse projection operator (4.28) and $\partial_\mu \tilde{\partial}_\mu = 0$ have been used. So we have definitely shown that the identity (4.17), embodying the transversality of the two-point vertex graph, is valid at tree level.

4.3 Identity for the 3A vertex

Now, we can start to derive another identity which should tell us something about the behaviour of the 3A vertex. For this purpose, we need the extended Slavnov-Taylor operator in (4.10) again, in particular the following parts:

$$\mathcal{S}(\Gamma^{(0)}_{\text{inv}}) = \int d^4x \left(\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta\Omega^A_\mu} \star \frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta A_\mu} + ... \right) + \gamma^4 \frac{\partial\Gamma^{(0)}_{\text{inv}}}{\partial\bar{\chi}} + g' \frac{\partial\Gamma^{(0)}_{\text{inv}}}{\partial\bar{\delta}} = 0\,. \tag{4.42}$$

Setting in the equation for motion for Ω^A_μ, given in (4.13), leads to

$$\mathcal{S}(\Gamma^{(0)}_{\text{inv}}) = \int d^4x \left((\partial_\mu c + \mathrm{i}g\,[c \stackrel{\star}{,} A_\mu]) \star \frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta A_\mu} + ... \right) + \left(\gamma^4 \frac{\partial}{\partial\bar{\chi}} + g' \frac{\partial}{\partial\bar{\delta}} \right) \Gamma^{(0)}_{\text{inv}} = 0\,. \tag{4.43}$$

Now we cannot neglect the last two terms because they won't vanish. We can build the functional derivatives with respect to $c(z)$ and $A_\rho(y)$ and get, similar to (4.16),

$$\begin{aligned}
\frac{\delta^2\mathcal{S}(\Gamma^{(0)}_{\text{inv}})}{\delta A_\rho(y)\delta c(z)} =& -\partial^z_\mu \frac{\delta^2\Gamma^{(0)}_{\text{inv}}}{\delta A_\rho(y)\delta A_\mu(z)} \\
&+ \mathrm{i}g \left[\delta^{(4)}(z-y) \stackrel{\star}{,} \frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta A_\rho(z)} \right] + \mathrm{i}g \left[A_\mu(z) \stackrel{\star}{,} \frac{\delta^2\Gamma^{(0)}_{\text{inv}}}{\delta A_\rho(y)\delta A_\mu(z)} \right] + ... \\
&+ \left(\gamma^4 \frac{\partial}{\partial\bar{\chi}} + g' \frac{\partial}{\partial\bar{\delta}} \right) \frac{\delta^2\Gamma^{(0)}_{\text{inv}}}{\delta A_\rho(y)\delta c(z)} = 0\,.
\end{aligned} \tag{4.44}$$

By building another functional derivative with respect to $A_\lambda(r)$ we finally arrive at

$$\begin{aligned}
\frac{\delta^3\mathcal{S}(\Gamma^{(0)}_{\text{inv}})}{\delta A_\lambda(r)\delta A_\rho(y)\delta c(z)} =& -\partial^z_\mu \frac{\delta^3\Gamma^{(0)}_{\text{inv}}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)} \\
&+ \mathrm{i}g \left[\delta^{(4)}(z-y) \stackrel{\star}{,} \frac{\delta^2\Gamma^{(0)}_{\text{inv}}}{\delta A_\lambda(r)\delta A_\rho(z)} \right] + \mathrm{i}g \left[\delta^{(4)}(z-r) \stackrel{\star}{,} \frac{\delta^2\Gamma^{(0)}_{\text{inv}}}{\delta A_\rho(y)\delta A_\lambda(z)} \right] + ... \\
&+ \left(\gamma^4 \frac{\partial}{\partial\bar{\chi}} + g' \frac{\partial}{\partial\bar{\delta}} \right) \frac{\delta^3\Gamma^{(0)}_{\text{inv}}}{\delta A_\lambda(r)\delta A_\rho(y)\delta c(z)} = 0\,,
\end{aligned} \tag{4.45}$$

and for vanishing fields

$$\begin{aligned}
\left.\frac{\delta^3\mathcal{S}(\Gamma^{(0)}_{\text{inv}})}{\delta A_\lambda(r)\delta A_\rho(y)\delta c(z)}\right|_{\phi=0} =& -\partial^z_\mu \left.\frac{\delta^3\Gamma^{(0)}_{\text{inv}}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)}\right|_{\phi=0} \\
&+ \mathrm{i}g \left[\delta^{(4)}(z-y) \stackrel{\star}{,} \left.\frac{\delta^2\Gamma^{(0)}_{\text{inv}}}{\delta A_\lambda(r)\delta A_\rho(z)}\right|_{\phi=0} \right] + \mathrm{i}g \left[\delta^{(4)}(z-r) \stackrel{\star}{,} \left.\frac{\delta^2\Gamma^{(0)}_{\text{inv}}}{\delta A_\rho(y)\delta A_\lambda(z)}\right|_{\phi=0} \right] \\
&+ \left(\gamma^4 \frac{\partial}{\partial\bar{\chi}} + g' \frac{\partial}{\partial\bar{\delta}} \right) \left.\frac{\delta^3\Gamma^{(0)}_{\text{inv}}}{\delta A_\lambda(r)\delta A_\rho(y)\delta c(z)}\right|_{\phi=0} \\
&= 0\,.
\end{aligned} \tag{4.46}$$

This is our second identity that we want to discuss a bit closer at tree level. Therefore, one has to derive the partial derivative of the 3A vertex graph

$$\partial^z_\mu \Gamma^{3A}_{\lambda\rho\mu}(r,y,z) = \partial^z_\mu V^{3A}_{\lambda\rho\mu}(r,y,z) = \partial^z_\mu \frac{\delta^3\Gamma^{(0)}_{\text{inv}}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)}\Bigg|_{\phi=0} . \tag{4.47}$$

There are two terms in our action (4.8) which are trilinear in the gauge field A_μ, precisely

$$\begin{aligned}\Gamma^{(0)}_{\text{tri},1} &= \frac{1}{4}\int d^4x \left(F_{\sigma\tau}F_{\sigma\tau}\right)_{\text{tri}} \\ &= -\mathrm{i}g\int d^4x \Big((\partial_\sigma A_\tau)\star(A_\sigma\star A_\tau - A_\tau\star A_\sigma)\Big) .\end{aligned} \tag{4.48}$$

The second trilinear part is coming from our new counter term and reads

$$\Gamma^{(0)}_{\text{tri},2} = \int d^4x \left(g'\mathcal{L}^2_{\text{br}}\right) = \frac{g'}{2}\int d^4x \left(\{A_\sigma \stackrel{\star}{,} A_\nu\}\star\frac{\tilde\partial_\sigma\tilde\partial_\nu\tilde\partial_\tau}{\varepsilon\tilde\square^2}A_\tau\right) , \tag{4.49}$$

with $\mathcal{L}^2_{\text{br}}$ of (4.9). The calculations are done in the Appendices A.4 and A.7, respectively. For the main part (4.48) we get the result (A.42)

$$\begin{aligned}-\partial^z_\mu\frac{\delta^3\Gamma^{(0)}_{\text{tri},1}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)} = -\mathrm{i}g\Big\{ &\Big[\left(\square^z\delta_{\lambda\rho}-\partial^z_\lambda\partial^z_\rho\right)\delta^{(4)}(z-y)\stackrel{\star}{,}\delta^{(4)}(z-r)\Big] \\ &+\Big[\left(\square^z\delta_{\lambda\rho}-\partial^z_\lambda\partial^z_\rho\right)\delta^{(4)}(z-r)\stackrel{\star}{,}\delta^{(4)}(z-y)\Big]\Big\} .\end{aligned} \tag{4.50}$$

For the counter part we arrive at (A.61)

$$\begin{aligned}&-\partial^z_\mu\frac{\delta^3\Gamma^{(0)}_{\text{tri},2}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)} = \\ &-g'\left(\left\{\partial^z_\mu\delta^{(4)}(z-y)\stackrel{\star}{,}\frac{\tilde\partial^z_\mu\tilde\partial^z_\lambda\tilde\partial^z_\rho}{\varepsilon\tilde\square^2_z}\delta^{(4)}(z-r)\right\}+\left\{\partial^z_\mu\delta^{(4)}(z-r)\stackrel{\star}{,}\frac{\tilde\partial^z_\mu\tilde\partial^z_\lambda\tilde\partial^z_\rho}{\varepsilon\tilde\square^2_z}\delta^{(4)}(z-y)\right\}\right) .\end{aligned} \tag{4.51}$$

Furthermore, we have to derive the contributions arising from the second and third term in (4.46). For this, we need all the bilinear parts of our action. They are already stated explicitly in (4.26), which is the main bilinear part, and in (4.29), which implements the damping of the photon propagator. The calculations are done in the Appendices A.6 and A.9. The results (A.53) and (A.78) read

$$\begin{aligned}&\mathrm{i}g\left[\delta^{(4)}(z-y)\stackrel{\star}{,}\frac{\delta^2\Gamma^{(0)}_{\text{bi},1}}{\delta A_\lambda(r)\delta A_\rho(z)}\Bigg|_{\phi=0}\right]+\mathrm{i}g\left[\delta^{(4)}(z-r)\stackrel{\star}{,}\frac{\delta^2\Gamma^{(0)}_{\text{bi},1}}{\delta A_\rho(y)\delta A_\lambda(z)}\Bigg|_{\phi=0}\right] = \\ &= -\mathrm{i}g\Big\{\Big[\delta^{(4)}(z-y)\stackrel{\star}{,}\left(\square^z\delta_{\lambda\rho}-\partial^z_\lambda\partial^z_\rho\right)\delta^{(4)}(z-r)\Big] \\ &\qquad+\Big[\delta^{(4)}(z-r)\stackrel{\star}{,}\left(\square^z\delta_{\lambda\rho}-\partial^z_\lambda\partial^z_\rho\right)\delta^{(4)}(z-y)\Big]\Big\} ,\end{aligned} \tag{4.52}$$

and

$$\mathrm{i}g\left[\delta^{(4)}(z-y)\stackrel{\star}{,}\left.\frac{\delta^2\Gamma^{(0)}_{\mathrm{bi},2}}{\delta A_\lambda(r)\delta A_\rho(z)}\right|_{\phi=0}\right]+\mathrm{i}g\left[\delta^{(4)}(z-r)\stackrel{\star}{,}\left.\frac{\delta^2\Gamma^{(0)}_{\mathrm{bi},2}}{\delta A_\rho(y)\delta A_\lambda(z)}\right|_{\phi=0}\right]=$$

$$\begin{aligned}&=-\mathrm{i}g\gamma^4\Bigg\{\left[\delta^{(4)}(z-y)\stackrel{\star}{,}\frac{1}{\tilde{\Box}_z^2}\left(\Box^z\delta_{\lambda\rho}-\partial^z_\lambda\partial^z_\rho+\rho\,\tilde{\partial}^z_\lambda\tilde{\partial}^z_\rho\right)\delta^{(4)}(z-r)\right]\\&\qquad+\left[\delta^{(4)}(z-r)\stackrel{\star}{,}\frac{1}{\tilde{\Box}_z^2}\left(\Box^z\delta_{\lambda\rho}-\partial^z_\lambda\partial^z_\rho+\rho\,\tilde{\partial}^z_\lambda\tilde{\partial}^z_\rho\right)\delta^{(4)}(z-y)\right]\Bigg\}\,,\end{aligned}\tag{4.53}$$

Now, only the third line of our identity (4.46) still has to be considered more closely. See for this the Appendices A.7 and A.9. The results are given in the Eqs. (A.68) and (A.87), and can be denoted as

$$\begin{aligned}&\frac{\delta^3}{\delta A_\lambda(r)\delta A_\rho(y)\delta c(z)}\left.\left(g'\frac{\partial\Gamma^{(0)}_{\mathrm{inv}}}{\partial\bar{\delta}}\right)\right|_{\phi=0}=\\&=g'\left(\left\{\partial^z_\mu\delta^{(4)}(z-y)\stackrel{\star}{,}\frac{\tilde{\partial}^z_\mu\tilde{\partial}^z_\lambda\tilde{\partial}^z_\rho}{\varepsilon\tilde{\Box}_z^2}\delta^{(4)}(z-r)\right\}+\left\{\partial^z_\mu\delta^{(4)}(z-r)\stackrel{\star}{,}\frac{\tilde{\partial}^z_\mu\tilde{\partial}^z_\lambda\tilde{\partial}^z_\rho}{\varepsilon\tilde{\Box}_z^2}\delta^{(4)}(z-y)\right\}\right)\,,\end{aligned}\tag{4.54}$$

and

$$\begin{aligned}&\frac{\delta^3}{\delta A_\lambda(r)\delta A_\rho(y)\delta c(z)}\left.\left(\gamma^4\frac{\partial\Gamma^{(0)}_{\mathrm{inv}}}{\partial\bar{\chi}}\right)\right|_{\phi=0}=\\&=\mathrm{i}g\gamma^4\Bigg\{\left[\delta^{(4)}(z-r)\stackrel{\star}{,}\frac{1}{\tilde{\Box}_z^2}\left(\Box^z\delta_{\lambda\rho}-\partial^z_\lambda\partial^z_\rho+\rho\tilde{\partial}^z_\lambda\tilde{\partial}^z_\rho\right)\delta^{(4)}(z-y)\right]\\&\qquad+\left[\delta^{(4)}(z-y)\stackrel{\star}{,}\frac{1}{\tilde{\Box}_z^2}\left(\Box^z\delta_{\lambda\rho}-\partial^z_\lambda\partial^z_\rho+\rho\tilde{\partial}^z_\lambda\tilde{\partial}^z_\rho\right)\delta^{(4)}(z-r)\right]\Bigg\}\,.\end{aligned}\tag{4.55}$$

One sees immediately that (4.55) cancels the contribution (4.53) in the equation (4.46). Together with (4.52) and (4.54) we can, hence, rewrite our identity (4.46) as

$$\begin{aligned}&\left.\frac{\delta^3\mathcal{S}(\Gamma^{(0)}_{\mathrm{inv}})}{\delta A_\lambda(r)\delta A_\rho(y)\delta c(z)}\right|_{\phi=0}=-\partial^z_\mu\left.\frac{\delta^3\Gamma^{(0)}_{\mathrm{inv}}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)}\right|_{\phi=0}\\&-\mathrm{i}g\Big[\delta^{(4)}(z-y)\stackrel{\star}{,}\left(\Box^z\delta_{\lambda\rho}-\partial^z_\lambda\partial^z_\rho\right)\delta^{(4)}(z-r)\Big]-\mathrm{i}g\Big[\delta^{(4)}(z-r)\stackrel{\star}{,}\left(\Box^z\delta_{\lambda\rho}-\partial^z_\lambda\partial^z_\rho\right)\delta^{(4)}(z-y)\Big]\\&+g'\left\{\partial^z_\mu\delta^{(4)}(z-y)\stackrel{\star}{,}\frac{\tilde{\partial}^z_\mu\tilde{\partial}^z_\lambda\tilde{\partial}^z_\rho}{\varepsilon\tilde{\Box}_z^2}\delta^{(4)}(z-r)\right\}+g'\left\{\partial^z_\mu\delta^{(4)}(z-r)\stackrel{\star}{,}\frac{\tilde{\partial}^z_\mu\tilde{\partial}^z_\lambda\tilde{\partial}^z_\rho}{\varepsilon\tilde{\Box}_z^2}\delta^{(4)}(z-y)\right\}\\&=0\,.\end{aligned}\tag{4.56}$$

We notice that (4.50) and (4.51) are canceled by the second and third line. Therefore, we have shown the validity of identity (4.46) at tree level and that it can also be depicted in the form (4.56). We hope that it holds true even at higher orders and don't get some anomalies.

4.4 The altered 3A vertex in k-space

While the photon propagator and the 4A vertex are not modified due to the new counter term, the 3A vertex indeed gains an extra term from $\mathcal{L}^2_{\mathrm{br}}$. So this additional contribution comes from the counter part (4.49) of our action and is calculated in the Appendix A.8, Eq. (A.74):

$$\tilde{V}^{3A}_{\lambda\rho\mu}(k_1,k_2,k_3)\big|_{\mathrm{counter}} = 2\mathrm{i}g'\,(2\pi)^4\delta^{(4)}(k_1+k_2+k_3)\,\cos\left(\frac{\varepsilon}{2}k_1\theta k_2\right)\sum_{i=1}^{3}\frac{\tilde{k}_{i,\lambda}\tilde{k}_{i,\rho}\tilde{k}_{i,\mu}}{\varepsilon(\tilde{k}_i^2)^2}\,. \quad (4.57)$$

We have to add this to the usual main part arriving due to (4.48) of our action. In x-space it is derived as by-product of the calculations for the 3A identity, see Eq. (A.40). The fourier transform is carried out in Appendix A.5, Eq. (A.50):

$$\begin{aligned}\tilde{V}^{3A}_{\lambda\rho\mu}(k_1,k_2,k_3)\big|_{\mathrm{main}} &= 2\mathrm{i}g\,(2\pi)^4\delta^{(4)}(k_1+k_2+k_3)\,\sin\left(\frac{\varepsilon}{2}k_1\tilde{k}_2\right)\\ &\times\Big\{(k_1-k_3)_\rho\delta_{\mu\lambda}+(k_2-k_1)_\mu\delta_{\lambda\rho}+(k_3-k_2)_\lambda\delta_{\rho\mu}\Big\}\,.\end{aligned} \quad (4.58)$$

The entire result for the new 3A vertex reads

$$\begin{aligned}\tilde{V}^{3A}_{\lambda\rho\mu}(k_1,k_2,k_3) = 2\mathrm{i}\,(2\pi)^4\delta^{(4)}(k_1+k_2+k_3)\Bigg\{g'\cos\left(\frac{\varepsilon}{2}k_1\theta k_2\right)\sum_{i=1}^{3}\frac{\tilde{k}_{i,\lambda}\tilde{k}_{i,\rho}\tilde{k}_{i,\mu}}{\varepsilon(\tilde{k}_i^2)^2}+\\ +g\sin\left(\frac{\varepsilon}{2}k_1\tilde{k}_2\right)\Big((k_1-k_3)_\rho\delta_{\mu\lambda}+(k_2-k_1)_\mu\delta_{\lambda\rho}+(k_3-k_2)_\lambda\delta_{\rho\mu}\Big)\Bigg\}\,,\end{aligned} \quad (4.59)$$

with the desired extra term of Eq. (3.44). So it proved true that we had to add the term (4.2) to our action.

4.5 The linearized BRST operator b

To start off with, we neglect the star in the notation for simplicity within this section.

For a possible algebraic renormalization (see Sect. 4.6) we still need the explicit form of the extended Slavnov-Taylor operator, more precisely of the linearized BRST operator

$$b = \int d^4x\left(\frac{\delta\Gamma^{(0)}_{\mathrm{inv}}}{\delta\Omega^A_\mu}\frac{\delta}{\delta A_\mu}+\frac{\delta\Gamma^{(0)}_{\mathrm{inv}}}{\delta A_\mu}\frac{\delta}{\delta\Omega^A_\mu}+\frac{\delta\Gamma^{(0)}_{\mathrm{inv}}}{\delta\Omega^c}\frac{\delta}{\delta c}+\frac{\delta\Gamma^{(0)}_{\mathrm{inv}}}{\delta c}\frac{\delta}{\delta\Omega^c}+B\frac{\delta}{\delta\bar{c}}\right)+\gamma^4\frac{\partial}{\partial\bar{\chi}}+g'\frac{\partial}{\partial\bar{\delta}}\,. \quad (4.60)$$

Therefore, we have to calculate the action of b on the fields and the included equations of motion: we can start with the action of b on the gauge field A_μ and on the ghost field c, respectively. Obviously, this is the same as the action of s on these fields. So, this gives the BRST transformations of these fields:

$$bA_\mu = \frac{\delta\Gamma^{(0)}_{\mathrm{inv}}}{\delta\Omega^A_\mu} = D_\mu c = \partial_\mu c + \mathrm{i}g\,[c, A_\mu] = sA_\mu\,, \quad (4.61)$$

$$bc = \frac{\delta\Gamma^{(0)}_{\mathrm{inv}}}{\delta\Omega^c} = \mathrm{i}gcc = sc\,. \quad (4.62)$$

The action of b on Ω^A_μ and Ω^c is much harder to derive. It is not identical to the action of s on these external sources (invariant with respect to s) and delivers the equations of motion for A_μ and for c:

$$b\Omega^c = \frac{\delta\Gamma^{(0)}_{\rm inv}}{\delta c} \neq s\Omega^c \,. \tag{4.63}$$

By quoting the action (4.8) at this point to keep things clear

$$\Gamma^{(0)}_{\rm inv} = \int d^4x \Big(\frac{1}{4} F_{\mu\nu}F_{\mu\nu} + s(\bar c\,\partial A) + s(\bar\chi\,\mathcal{L}^1_{\rm br}) + s(\bar\delta\,\mathcal{L}^2_{\rm br}) + \Omega^A_\mu sA_\mu + \Omega^c sc \Big)\,, \tag{4.64}$$

we obtain the following contributions (see also Appendix B.1)

$$\begin{aligned}
\frac{\delta}{\delta c(y)} \int d^4x\, \Omega^A_\mu sA_\mu &= \left(\partial_\mu\Omega^A_\mu + {\rm i}g\left[\Omega^A_\mu, A_\mu\right]\right)(y) = D_\mu\Omega^A_\mu(y)\,,\\
\frac{\delta}{\delta c(y)} \int d^4x\, \Omega^c sc &= {\rm i}g\left[c,\Omega^c\right](y)\,,\\
\frac{\delta}{\delta c(y)} \int d^4x\, s(\bar c\,\partial A) &= D_\mu\partial_\mu \bar c\,(y)\,.
\end{aligned} \tag{4.65}$$

Furthermore we have to add the parts arising from

$$\begin{aligned}
\frac{\delta}{\delta c(y)} \int d^4x\, s(\bar\chi\,\mathcal{L}^1_{\rm br}) &= -\frac{\delta}{\delta c(y)} \int d^4x\, \bar\chi\,(s\mathcal{L}^1_{\rm br})\,,\\
\frac{\delta}{\delta c(y)} \int d^4x\, s(\bar\delta\,\mathcal{L}^2_{\rm br}) &= -\frac{\delta}{\delta c(y)} \int d^4x\, \bar\delta\,(s\mathcal{L}^2_{\rm br})\,.
\end{aligned} \tag{4.66}$$

The first line gives (see Appendix B.1, Eq. (B.9))

$$-\frac{\delta}{\delta c(y)} \int d^4x\, \bar\chi\,(s\mathcal{L}^1_{\rm br}) = -{\rm i}g\bar\chi \left[A_\mu, \frac{1}{\tilde\Box^2}\left(\partial_\nu f_{\nu\mu} + \rho\,\tilde\partial_\mu\tilde\partial_\nu A_\nu\right)\right](y)\,. \tag{4.67}$$

Similarly, we receive for the second line Eq. (B.10).

$$-\frac{\delta}{\delta c(y)} \int d^4x\, \bar\delta\,(s\mathcal{L}^2_{\rm br}) = -\bar\delta \left(D_\mu \left\{A_\nu, \tfrac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2} A_\rho\right\} + {\rm i}g\left[A_\rho, \tfrac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}(A_\mu A_\nu)\right]\right)(y)\,. \tag{4.68}$$

Summarizing, we achieve for the action of b on Ω^c

$$\begin{aligned}
b\Omega^c = \frac{\delta\Gamma^{(0)}_{\rm inv}}{\delta c} &= D_\mu\Omega^A_\mu + {\rm i}g\left[c,\Omega^c\right] + D_\mu\partial_\mu\bar c\\
&\quad - {\rm i}g\bar\chi\left[A_\mu, \frac{1}{\tilde\Box^2}\left(\partial_\nu f_{\nu\mu} + \rho\,\tilde\partial_\mu\tilde\partial_\nu A_\nu\right)\right]\\
&\quad - \bar\delta\left(D_\mu\left\{A_\nu, \tfrac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}A_\rho\right\} + {\rm i}g\left[A_\rho, \tfrac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}(A_\mu A_\nu)\right]\right).
\end{aligned} \tag{4.69}$$

Now we still need the action of b on Ω^A_μ

$$b\Omega^A_\mu = \frac{\delta\Gamma^{(0)}_{\rm inv}}{\delta A_\mu} \neq s\Omega^A_\mu \,. \tag{4.70}$$

For that we need the parts (see Appendix B.2)

$$\begin{aligned}
\frac{\delta}{\delta A_\mu(y)}\int d^4x\,\Omega^A_\nu sA_\nu &= \mathrm{i}g\left\{\Omega^A_\mu, c\right\}(y)\,,\\
\frac{\delta}{\delta A_\mu(y)}\int d^4x\, s(\bar c\,\partial A) &= (\mathrm{i}g\left\{\partial_\mu\bar c, c\right\} - \partial_\mu B)(y)\,,\\
\frac{\delta}{\delta A_\mu(y)}\int d^4x\,\frac{1}{4}F_{\mu\nu}F_{\mu\nu} &= D_\nu F_{\mu\nu}(y)\,,
\end{aligned}\tag{4.71}$$

and also the contributions of $\mathcal{L}^1_{\mathrm{br}}$ and $\mathcal{L}^2_{\mathrm{br}}$

$$\begin{aligned}
\frac{\delta}{\delta A_\mu(y)}\int d^4x\, s(\bar\chi\,\mathcal{L}^1_{\mathrm{br}}) &= \frac{\delta}{\delta A_\mu(y)}\int d^4x\,\left(\gamma^4\mathcal{L}^1_{\mathrm{br}} - \bar\chi\, s\mathcal{L}^1_{\mathrm{br}}\right)\,,\\
\frac{\delta}{\delta A_\mu(y)}\int d^4x\, s(\bar\delta\,\mathcal{L}^2_{\mathrm{br}}) &= \frac{\delta}{\delta A_\mu(y)}\int d^4x\,\left(g'\mathcal{L}^2_{\mathrm{br}} - \bar\delta\, s\mathcal{L}^2_{\mathrm{br}}\right)\,.
\end{aligned}\tag{4.72}$$

The calculations are done in the Appendix B.2, Eqs. (B.20) and (B.24). Altogether, we achieve (B.26)

$$\begin{aligned}
b\Omega^A_\mu = \frac{\delta\Gamma^{(0)}_{\mathrm{inv}}}{\delta A_\mu} &= -(1+\frac{\gamma^4}{\tilde\Box^2})(\Box\delta_{\mu\nu} - \partial_\nu\partial_\mu)A_\nu - \frac{\rho\gamma^4}{\tilde\Box^2}\tilde\partial_\mu\tilde\partial_\nu A_\nu + \mathrm{i}g\partial_\nu\left[A_\nu, A_\mu\right] + \mathrm{i}g\left[F_{\mu\nu}, A_\nu\right]\\
&+ \mathrm{i}g\left\{\Omega^A_\mu, c\right\} + \mathrm{i}g\left\{\partial_\mu\bar c, c\right\} + \partial_\mu B\\
&+ \mathrm{i}g\bar\chi\left[c, \frac{1}{\tilde\Box^2}\left(\partial_\nu f_{\mu\nu} - \rho\,\tilde\partial_\mu\tilde\partial_\nu A_\nu\right)\right] + \frac{\mathrm{i}g\bar\chi}{\tilde\Box^2}\left(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu + \rho\,\tilde\partial_\mu\tilde\partial_\nu\right)\left[c, A_\nu\right]\\
&+ g'\left\{A_\nu, \frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}A_\rho\right\} - g'\frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}\left(A_\rho A_\nu\right) + \bar\delta\frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}\left\{D_\rho c, A_\nu\right\}\\
&- \bar\delta\left\{D_\rho c, \frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}A_\rho\right\} + \mathrm{i}g\bar\delta\left[c, \left\{A_\nu, \frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}A_\rho\right\}\right]\\
&- \mathrm{i}g\bar\delta\left\{A_\nu, \frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}\left[c, A_\rho\right]\right\} + \mathrm{i}g\bar\delta\left[\frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}(A_\rho A_\nu), c\right]\,.
\end{aligned}\tag{4.73}$$

where the transverse structure of the two-point vertex graph arises again, since the only terms in (4.73) which are linear in the photon field and independent of all other fields are the first two ones.

The action of b on all the other fields and parameters $(\bar c, B, \bar\chi, \gamma^4, \bar\delta, g')$ is identical with the action of s on them:

$$\begin{aligned}
b\bar c = s\bar c &= B\,, & bB = sB &= 0\,,\\
b\bar\chi = s\bar\chi &= \gamma^4\,, & b\gamma^4 = s\gamma^4 &= 0\,,\\
b\bar\delta = s\bar\delta &= g'\,, & bg' = sg' &= 0\,.
\end{aligned}\tag{4.74}$$

To conclude this section, let us note that we can summarize the results as follows: on the one hand, for all the fields ϕ with external sources ρ, hence, for A_μ and c, one gets

$$b\phi = \frac{\delta\Gamma^{(0)}_{\mathrm{inv}}}{\delta\rho} = s\phi \quad\text{and}\quad b\rho = \frac{\delta\Gamma^{(0)}_{\mathrm{inv}}}{\delta\phi} \neq s\rho\,,\tag{4.75}$$

see Eqs. (4.61), (4.62), (4.69) and (4.73). On the other hand, for all the fields ϕ without related external source, as well as for all the parameters λ, one just has $b(\phi,\lambda) = s(\phi,\lambda)$.

4.6 Some remarks on algebraic renormalization and the QAP

Examining this diploma thesis so far, one notices that there is a particular theme that catches one's eye over and over again, and that's "symmetry". The BRST symmetry content of our model is expressed by the extended Slavnov-Taylor operator (4.10) via the extended Slavnov-Taylor identity: $\mathcal{S}(\Gamma^{(0)}_{\rm inv}) = 0$. Both derived identities (4.17) and (4.46) are relying on its validity and we have shown explicitly that they hold true at tree level, hence, for the "classical theory".

The main interest on renormalization is whether such symmetries can be maintained for higher orders meaning that the corresponding identities are fullfilled perturbatively order-by-order. One usually introduces the so-called linearized BRST operator $\mathcal{S}_\Gamma$ [40],

$$\mathcal{S}_\Gamma = \int d^4x \left(\frac{\delta\Gamma}{\delta\Omega^A_\mu}\star\frac{\delta}{\delta A_\mu} + \frac{\delta\Gamma}{\delta A_\mu}\star\frac{\delta}{\delta\Omega^A_\mu} + \frac{\delta\Gamma}{\delta\Omega^c}\star\frac{\delta}{\delta c} + \frac{\delta\Gamma}{\delta c}\star\frac{\delta}{\delta\Omega^c} + B\star\frac{\delta}{\delta\bar{c}} \right) + \gamma^4\frac{\partial}{\partial\bar{\chi}} + g'\frac{\partial}{\partial\bar{\delta}}\,, \tag{4.76}$$

and uses the following relations:

$$\mathcal{S}_\Gamma\mathcal{S}(\Gamma) = 0\,, \quad \forall\,\Gamma\,, \qquad \text{and} \qquad \mathcal{S}_\Gamma^2 = 0\,, \quad \text{if} \quad \mathcal{S}(\Gamma) = 0\,. \tag{4.77}$$

So, one can start with the zeroth loop order $\Gamma^{(0)} = S$ with the definition $b \equiv \mathcal{S}_S$. By using the Slavnov-Taylor identity $\mathcal{S}(\Gamma^{(0)}) = 0$, one sees immediately that next to $s^2 = 0$, we also have a nilpotent b, $b^2 = 0$.

To proceed, we need the so-called quantum action principle (QAP) [39, 40, 47]. What we want is that $\mathcal{S}\big(\Gamma_{(n)}\big) = 0$ holds true even at higher order n:

$$\Gamma_{(n)} = \sum_{l=0}^{n} \hbar^l \Gamma^{(l)}\,. \tag{4.78}$$

As seen, $\mathcal{S}(\Gamma)$ has the following structure:

$$\mathcal{S}(\Gamma) = \int d^4x \left(\frac{\delta\Gamma}{\delta\Omega^A_\mu}\star\frac{\delta\Gamma}{\delta A_\mu} + \frac{\delta\Gamma}{\delta\Omega^c}\star\frac{\delta\Gamma}{\delta c} + B\star\frac{\delta\Gamma}{\delta\bar{c}} \right) + \gamma^4\frac{\partial\Gamma}{\partial\bar{\chi}} + g'\frac{\partial\Gamma}{\partial\bar{\delta}}\,. \tag{4.79}$$

The QAP now tells us how these terms should look like[1]

$$\begin{aligned} &\int d^4x \left(\frac{\delta\Gamma}{\delta\Omega^A_\mu}\frac{\delta\Gamma}{\delta A_\mu} \right) = \Delta^{\Omega A}\cdot\Gamma\,, \quad \int d^4x \left(\frac{\delta\Gamma}{\delta\Omega^c}\frac{\delta\Gamma}{\delta c} \right) = \Delta^{\Omega c}\cdot\Gamma\,, \\ &\int d^4x \left(B\frac{\delta\Gamma}{\delta\bar{c}} \right) = \Delta^{B\bar{c}}\cdot\Gamma\,, \quad \gamma^4\frac{\partial\Gamma}{\partial\bar{\chi}} = \Delta^{\gamma^4\bar{\chi}}\cdot\Gamma\,, \quad g'\frac{\partial\Gamma}{\partial\bar{\delta}} = \Delta^{g'\bar{\delta}}\cdot\Gamma\,, \end{aligned} \tag{4.80}$$

where the breakings Δ are integrated local polynomials in the fields that have ghost number 1 and dimension 4. Further, the QAP tells us that if the Slavnov-Taylor identity is fullfilled at some order $(n-1)$,

$$\mathcal{S}\big(\Gamma_{(n-1)}\big) = \mathcal{O}(\hbar^n)\,, \tag{4.81}$$

[1] without the star product, see below for the discussion.

fields, parameters	A_μ	c	$\bar{c}$	B	Ω_μ^A	Ω_c	$\bar{\chi}$	γ^4	$\bar{\delta}$	g'
mass dimension d	1	0	2	2	3	4	4	4	0	0
ghost number	0	1	-1	0	-1	-2	-1	0	-1	0

Table 4.1: Dimensions and ghost numbers.

which is true for our tree-level action (4.8), we obtain

$$\mathcal{S}\big(\Gamma_{(n-1)}\big) = \Delta \cdot \Gamma = \hbar^n \Delta + \mathcal{O}\Big(\hbar^{(n+1)}\Big) . \tag{4.82}$$

If we now use the left property of (4.77), $\mathcal{S}_\Gamma \mathcal{S}(\Gamma) = 0$, together with $\mathcal{S}_\Gamma = b + \mathcal{O}(\hbar)$, we further get

$$b\left(\mathcal{S}\big(\Gamma_{(n-1)}\big)\right) = b\left(\hbar^n \Delta\right) = 0 . \tag{4.83}$$

This leads us to the so-called consistency condition

$$b\Delta = 0 . \tag{4.84}$$

If this insertion can now be written as

$$\Delta = b\tilde{\Delta} , \tag{4.85}$$

one can add a local counter term to the vertex function

$$\Gamma_{(n)} = \Gamma_{(n-1)} - \hbar^n \tilde{\Delta} + \mathcal{O}(\hbar^{n+1}) , \tag{4.86}$$

so that the BRST identity is fullfilled at order n,

$$\mathcal{S}\big(\Gamma_{(n)}\big) = \mathcal{S}\big(\Gamma_{(n-1)}\big) - \hbar^n \Delta + \mathcal{O}\Big(\hbar^{(n+1)}\Big) = \mathcal{O}\Big(\hbar^{(n+1)}\Big) . \tag{4.87}$$

If that is the case, the theory is free of anomalies and the identity is valid order-by-order[2].

However, there is still something left that has to be examined more closely. The accentuation in “integrated local insertions” is on the word “local”. In the renormalization process one has to start with the canceling of UV-divergences appearing in the Feynman graphs at some order n, for example, by subtracting appropriate terms. These terms can lead to a breaking of the Slavnov-Taylor identity or other symmetries. To restore the identity one can use the above described procedure to add (a finite number of) counter terms. This is relying on the QAP. But, this procedure assumes locality [39, 40].

We have two kinds of non-localities in our model, further considered in Sect. 5.1: firstly, there is the star product. So, one has to show that the QAP stays applicable when the star product is present. Secondly, there are $\frac{1}{\tilde{\Box}^2}$-terms in our action. The further will be concerned in an upcoming paper [48] with strong evidence that the QAP makes sense in the context of the star product. For the latter we will give a possible way of localization in the next chapter.

[2]Furthermore, power-counting renormalizability has to be assumed, see e.g. [39, 40] for details.

Chapter 5

Localization

Firstly, we discuss the non-localities in our model in Sect. 5.1. In Sect. 5.2 a possible way of localization is presented and we show the equivalence of this localized action to our oringinal one (4.8). Further, the BRST symmetry is discussed. From here on the star has been dropped again in the notation. In Sect. 5.3 the gauge propagator is calculated explicitly to illustrate that it hasn't been modified due to the localization. Then, all the new propagators are derived in Sect. 5.4, and the new vertex is deduced in Sect. 5.5.

5.1 The non-localities: the operator $\frac{1}{\tilde{\square}^2}$ and the star product

Looking at the results for the propagators and vertices of our model, see Sect. 3.3.1 and Sect. 4.4, one notices that the gauge and the ghost propagator have not been modified due to the presence of the star product. In general, propagators are identical in the commutative and the associated non-commutative theory. This is a consequence of the property (2.11), telling us that all bilinear terms in the action don't get changed due to the star product. On the other hand, the introduction of the star product indeed leads to the appearance of some additional phase factors in the vertices.

To understand why such phase factors signify the non-locality of the represented interactions, it is usefull to examine what happens if we build the star product of some function $f(x)$ and a plane wave $e^{\mathrm{i}kx}$:

$$\begin{aligned} e^{\mathrm{i}kx} \star f(x) &= \int \frac{d^4q}{(2\pi)^4}\, \tilde{f}(q)\, e^{\mathrm{i}kx} \star e^{\mathrm{i}qx} = \int \frac{d^4q}{(2\pi)^4}\, \tilde{f}(q)\, e^{\mathrm{i}kx} e^{\mathrm{i}qx} e^{-\frac{\mathrm{i}}{2}k\varepsilon\theta q} \\ &= \int \frac{d^4q}{(2\pi)^4}\, \tilde{f}(q)\, e^{\mathrm{i}kx} e^{\mathrm{i}q\left(x+\frac{1}{2}\varepsilon\theta k\right)} = e^{\mathrm{i}kx} \cdot f\Big(x + \frac{\varepsilon}{2}\theta k\Big)\ . \end{aligned} \tag{5.1}$$

Thus, we get a translation $f(x) \to f(x + \Delta x)$ by a vector $\Delta x = \frac{\varepsilon}{2}\theta k$ that is, hence, increasing with the momentum k. Again, one notices that we can go back to the commutative limit by $\varepsilon \to 0$ and obtain the ordinary, commutative multiplication law, above noted by "$\cdot$". Looking at the star product of two functions $f(x)$ and $g(x)$,

$$(f \star g)(x) = \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4}\, \tilde{f}(k)\tilde{g}(q)\, e^{\mathrm{i}kx} \star e^{\mathrm{i}qx} = \int \frac{d^4q}{(2\pi)^4}\, f\Big(x - \frac{\varepsilon}{2}\theta q\Big)\, \tilde{g}(q)\, e^{\mathrm{i}qx}\ , \tag{5.2}$$

it shows that the high momentum parts in the Fourier integral exhibit large nonlocalities in the amplitude function $f(x, \varepsilon\tilde{q})$. Additionally, there is also a representation of the star product through a kernel $K(x;y,z)$ [4, 19]:

$$(f\star g)(x)=\int d^4y\int d^4z\,K(x;y,z)\,f(y)g(z)\,, \tag{5.3}$$

with[1]

$$K(x;y,z)=\delta^{(4)}(x-y)\star\delta^{(4)}(x-z)=\frac{1}{\pi^4|\det(\varepsilon\theta)|}e^{2\mathrm{i}(x-y)(\varepsilon\theta)^{-1}(x-z)}\,, \tag{5.4}$$

where for the last equal sign the θ-tensor has to be invertible which is true in our case of the four-dimensional Euclidean space. Taking our θ-matrix of (3.5), we get $\theta^{-1}_{\mu\nu}=\theta^T_{\mu\nu}$ and $\det(\theta_{\mu\nu})=1$. The integral (5.3) illustrates the nonlocal character of the star product, encoded in its kernel (5.4), in a clear way. It is carried out in detail in [20] that if the width of $f(x)=g(x)$ is some small $\Delta\ll\sqrt{\varepsilon}$, $(f\star f)(x)$ is nonvanishing in a region proportional to $\frac{1}{\Delta}$. So we have a mixture of phenomena at very small scales with those at very large scales. Translating this to energy scales, it gets obvious that this nonlocality thus leads to the already mentioned UV/IR mixing.

The nonlocality due to the star product, together with the QAP, will be discussed in an upcoming paper [48].

The second nonlocality in our model are terms containing the operator $\frac{1}{\tilde{\Box}^2}$, see $\mathcal{L}^1_{\mathrm{br}}$ and $\mathcal{L}^2_{\mathrm{br}}$ in (4.9). To explain why such terms are not local, hence, can be written in the form

$$\int d^4x\int d^4y\,f(x)G(x,y)g(y)\,, \tag{5.5}$$

we insert the operator $1=\Box_x^{-1}\Box_x$ and obtain

$$\int d^4x\int d^4y\,f(x)\,\Box_x^{-1}\Box_x G(x,y)g(y)\,. \tag{5.6}$$

If now $G(x,y)$ is chosen to be the Green function of the operator $\Box_x$,

$$\Box_x G(x,y)=\delta^{(4)}(x-y)\,, \tag{5.7}$$

it follows that

$$\text{Eq. (5.5)}=\int d^4x\int d^4y\,f(x)\,\Box_x^{-1}\delta^{(4)}(x-y)\,g(y)=\int d^4y\,f(y)\,\frac{1}{\Box_y}\,g(y)\,, \tag{5.8}$$

where for the last equal sign we have used partial integration twice. So, operators like $\frac{1}{\Box}$, or $\frac{1}{\tilde{\Box}^2}$, respectively, are nonlocal.

In the following sections we are going to present a way to localize the counter terms $\int d^4x\,s(\bar{\chi}\mathcal{L}^1_{\mathrm{br}})$ and $\int d^4x\,s(\bar{\delta}\mathcal{L}^2_{\mathrm{br}})$ of our action (4.8) without changing the physical or the symmetry content. We thus must find a possibility to do this in a BRST exact manner. Some insights and arising difficulties are considered at the beginning of the next section.

[1] an intermediate step: $\delta^{(4)}(x-y)\star\delta^{(4)}(x-z)=\int\frac{d^4k}{(2\pi)^4}\,e^{\mathrm{i}k(x-y)}\delta^{(4)}(x-z+\frac{\varepsilon}{2}\theta k)$, followed by the solution of the k-integral by use of the delta function.

5.2 Localizing the counter terms

The most obvious way to localize the counter terms is to use an ansatz similar to the proposal already discussed in the introduction (1.16), there for the operator $\frac{1}{D^2\tilde{D}^2}$. We can translate this so that it is fitting to our nonlocal operator $\frac{1}{\tilde{\square}^2}$ by, for example, starting with the first part of $S^1_{\mathrm{br}} = \int d^4x\,\gamma^4\mathcal{L}^1_{\mathrm{br}}$ and propose as first attempt

$$S^{1,1}_{\mathrm{br}} = \frac{\gamma^4}{4}\int d^4x\left(f_{\mu\nu}\frac{1}{\tilde{\square}^2}f_{\mu\nu}\right) \longrightarrow S^{\mathrm{1st}}_{\mathrm{loc}} = \frac{1}{2}\int d^4x\left[B_{\mu\nu}f_{\mu\nu} - \frac{1}{2}B_{\mu\nu}\frac{\tilde{\square}^2}{\gamma^4}B_{\mu\nu}\right]\,. \tag{5.9}$$

The newly introduced auxiliary field $B_{\mu\nu}$ is antisymmetric and has mass dimension 2. Using the equation of motion for $B_{\mu\nu}$ and the fact that the model has to be independent of this new auxiliary field[2],

$$\frac{\delta S^{\mathrm{1st}}_{\mathrm{loc}}}{\delta B_{\mu\nu}} = \frac{1}{2}f_{\mu\nu} - \frac{1}{2}\frac{\tilde{\square}^2}{\gamma^4}B_{\mu\nu} = 0\,, \tag{5.10}$$

one obtains

$$B_{\mu\nu} = \frac{\gamma^4}{\tilde{\square}^2}f_{\mu\nu}\,. \tag{5.11}$$

By reinserting this into the localized action S_{loc} of (5.9) one gets back to the nonlocal action $S^{1,1}_{\mathrm{br}}$:

$$\begin{aligned} S^{\mathrm{1st}}_{\mathrm{loc}} &= \frac{1}{2}\int d^4x\left[B_{\mu\nu}f_{\mu\nu} - \frac{1}{2}B_{\mu\nu}\frac{\tilde{\square}^2}{\gamma^4}B_{\mu\nu}\right] \\ &= \frac{1}{2}\int d^4x\left[\left(\frac{\gamma^4}{\tilde{\square}^2}f_{\mu\nu}\right)f_{\mu\nu} - \frac{1}{2}\left(\frac{\gamma^4}{\tilde{\square}^2}f_{\mu\nu}\right)\frac{\tilde{\square}^2}{\gamma^4}\left(\frac{\gamma^4}{\tilde{\square}^2}f_{\mu\nu}\right)\right] \\ &= \frac{\gamma^4}{4}\int d^4x\left(f_{\mu\nu}\frac{1}{\tilde{\square}^2}f_{\mu\nu}\right) = S^{1,1}_{\mathrm{br}}\,. \end{aligned} \tag{5.12}$$

We could do this procedure in the same manner for the second part of S^1_{br}, but we face some other difficulties: bearing in mind that we have inserted all new counter terms BRST exact into our action (4.8), hence, as $S^1_{\mathrm{br},\bar{\chi}} = \int d^4x\, s(\bar{\chi}\mathcal{L}^1_{\mathrm{br}})$ with $s\bar{\chi} = \gamma^4$, we now would have to restore the BRST symmetry after the above localization.

On the other hand, we can start with the BRST invariant version and try to apply the same method as in (5.9),

$$S_{\mathrm{nloc}} = \int d^4x\, s\left(\frac{\bar{\chi}}{4}f_{\mu\nu}\frac{1}{\tilde{\square}^2}f_{\mu\nu}\right) \longrightarrow S^{\mathrm{2nd}}_{\mathrm{loc}} = \frac{1}{2}\int d^4x\, s\left[c_{\mu\nu}f_{\mu\nu} - \frac{1}{2}c_{\mu\nu}\frac{\tilde{\square}^2}{\bar{\chi}}c_{\mu\nu}\right]\,, \tag{5.13}$$

but $\bar{\chi}$ is a fermionic, Grassmann-valued, and therefore anticommuting parameter with $\bar{\chi}^2 = 0$. So there is no inverse $\frac{1}{\bar{\chi}}$ and for that reason alone the above ansatz fails.

Going back to the localized part of the action (1.16),

$$S_{\mathrm{loc}} = \int d^4x\left(aB_{\mu\nu}F_{\mu\nu} - B_{\mu\nu}\tilde{D}^2D^2B_{\mu\nu}\right)\,, \tag{5.14}$$

[2] for an antisymmetric $B_{\mu\nu}$ one has: $\frac{\delta B_{\mu\nu}}{\delta B_{\sigma\tau}} = \frac{1}{2}\left[\delta_{\mu\sigma}\delta_{\nu\tau} - \delta_{\mu\tau}\delta_{\nu\sigma}\right]$, but $f_{\mu\nu}$ is antisymmetric, too.

one notices that the BRST invariance has been achieved due to the fact that $F_{\mu\nu}$ as well as $B_{\mu\nu}$ transform covariantly[3] [28],

$$sF_{\mu\nu} = \mathrm{i}g\,[c, F_{\mu\nu}]\ , \quad sB_{\mu\nu} = \mathrm{i}g\,[c, B_{\mu\nu}]\ , \tag{5.15}$$

where $B_{\mu\nu} = \frac{a}{2\tilde{D}^2 D^2} F_{\mu\nu}$.

In our case this method is not possible. For the first term, $S^1_{\mathrm{br}} = \int d^4x\, \gamma^4 \mathcal{L}^1_{\mathrm{br}}$, we have dropped this opportunity while replacing the non-Abelian field strength tensor $F_{\mu\nu}$ by the Abelian one $f_{\mu\nu}$, see Sect. 3.3, and argued that we just need the bilinear part to implement the IR-damping into the gauge propagator. Further, this simplifies the action and prevents us from additional contributions to the vertices. For the second part, $S^2_{\mathrm{br}} = \int d^4x\, g' \mathcal{L}^2_{\mathrm{br}}$, see also (4.9), this method can be excluded anyway from the outset.

So, we now want to offer a new way of how to localize the above two counter terms and obtain BRST invariance as well. Firstly, we start with the terms S^1_{br} and S^2_{br}, hence, with the non-invariant versions. Compared with (5.9) we then shift the prefactor γ^4 partly to the second term $\int d^4x\, B_{\mu\nu} f_{\mu\nu}$ so that it is linearly dependent on γ. The ansatz thus reads

$$S^{1,1}_{\mathrm{br}} \longrightarrow S^{1,1}_{\mathrm{loc}} = \int d^4x \left[-B_{\mu\nu} \frac{\tilde{\Box}^2}{\gamma^2} B_{\mu\nu} + \gamma B_{\mu\nu} f_{\mu\nu} \right] , \tag{5.16}$$

where γ still has mass dimension 1, see table (4.1). Looking at the equation of motion,

$$\frac{\delta S^{1,1}_{\mathrm{loc}}}{\delta B_{\mu\nu}} = -2 \frac{\tilde{\Box}^2}{\gamma^2} B_{\mu\nu} + \gamma f_{\mu\nu} = 0\,, \tag{5.17}$$

we also get a 1-dimensional auxiliary field $B_{\mu\nu}$,

$$B_{\mu\nu} = \frac{\gamma^3}{2\tilde{\Box}^2} f_{\mu\nu}\,, \tag{5.18}$$

instead of the 2-dimensional one in the ansatz (5.9), but now the second term of (5.16) can be written BRST exact

$$S^{1,1}_{\mathrm{loc}} = \int d^4x \left[-B_{\mu\nu} \frac{\tilde{\Box}^2}{\gamma^2} B_{\mu\nu} + s\left(\bar{\chi} B_{\mu\nu} f_{\mu\nu}\right) \right] , \tag{5.19}$$

with the new transformation laws $s\bar{\chi} = \gamma$ and $s\gamma = 0$, and the physical value $\bar{\chi}|_{\mathrm{phys}} = 0$. The new BRST doublet $(\bar{\chi}, \gamma)$ replaces the original one $(\bar{\chi}, \gamma^4)$. So the mass dimension of $\bar{\chi}$ changes from 4, see table (4.1), to 1. For a BRST invariant action $S^{1,1}_{\mathrm{loc}}$ we also have to assume BRST invariance of the auxiliary field $B_{\mu\nu}$, $sB_{\mu\nu} = 0$. To show that this assumption is justifiable, one can take a look at the new equation of motion,

$$\frac{\delta S^{1,1}_{\mathrm{loc}}}{\delta B_{\mu\nu}} = -2 \frac{\tilde{\Box}^2}{\gamma^2} B_{\mu\nu} + \gamma f_{\mu\nu} - \bar{\chi} s f_{\mu\nu} = 0\,. \tag{5.20}$$

For a nilpotent operator s, this leads us to

$$B_{\mu\nu} = \frac{\gamma^3}{2\tilde{\Box}^2} f_{\mu\nu} - \frac{\gamma^2 \bar{\chi}}{2\tilde{\Box}^2} s f_{\mu\nu}\,, \quad \text{with} \quad sB_{\mu\nu} = 0\,, \tag{5.21}$$

[3] using cyclic permutation and the covariant transformation laws (5.15) yields $sS_{\mathrm{loc}} = 0$.

and so to $sS^{1,1}_{\text{loc}} = 0$. We can do the same procedure for the second part of S^1_{br},

$$S^{1,2}_{\text{br}} = \frac{\gamma^4}{4} \int d^4x \left(2\rho\, \tilde{\partial} A \frac{1}{\tilde{\Box}^2} \tilde{\partial} A \right) , \tag{5.22}$$

by the introduction of another auxiliary field H,

$$S^{1,2}_{\text{loc}} = \int d^4x \left[-H \frac{\tilde{\Box}^2}{\gamma^2} H + s \left(\bar{\chi} \sqrt{2\rho}\, H(\tilde{\partial} A) \right) \right] , \tag{5.23}$$

where the physical situation is again given by $\bar{\chi}|_{\text{phys}} = 0$. Further, we obtain

$$\frac{\delta S^{1,2}_{\text{loc}}}{\delta H} = -2 \frac{\tilde{\Box}^2}{\gamma^2} H + \gamma \sqrt{2\rho}\, (\tilde{\partial} A) - \bar{\chi} \sqrt{2\rho}\, s(\tilde{\partial} A) = 0 \,. \tag{5.24}$$

So there is again a 1-dimensional and BRST invariant auxiliary field H,

$$H = \frac{\gamma^3 \sqrt{2\rho}}{2\tilde{\Box}^2} (\tilde{\partial} A) - \frac{\gamma^2 \bar{\chi} \sqrt{2\rho}}{2\tilde{\Box}^2} s(\tilde{\partial} A) \,, \quad \text{with} \quad sH = 0 \,. \tag{5.25}$$

The localization of the second counter term

$$S^2_{\text{br}} = \frac{g'}{2} \int d^4x \, \{A_\mu, A_\nu\} \frac{\tilde{\partial}_\mu \tilde{\partial}_\nu \tilde{\partial}_\rho}{\varepsilon \tilde{\Box}^2} A_\rho \,, \tag{5.26}$$

is a little more complicated. Here, we need two new auxiliary fields because we don't have a symmetric structure as in (5.9) and (5.22). Of course there are various possibilities to choose these fields but it turns out that

$$S^2_{\text{loc}} = \int d^4x \left[\kappa^1_\mu \varepsilon \tilde{\Box}^2 \kappa^2_\mu - \kappa^1_\mu \sqrt{g'} \tilde{\partial}_\nu (A_\mu A_\nu) - \kappa^2_\mu \sqrt{g'} \tilde{\partial}_\mu (\tilde{\partial} A) \right] , \tag{5.27}$$

is probably the best one to implement BRST invariance subsequently. At first, the equivalence should be shown. Therefore, the equations of motion for both new fields are needed. For κ^1_μ it reads

$$\frac{\delta S^2_{\text{loc}}}{\delta \kappa^1_\mu} = \varepsilon \tilde{\Box}^2 \kappa^2_\mu - \sqrt{g'} \tilde{\partial}_\nu (A_\mu A_\nu) = 0 \,, \tag{5.28}$$

whereas for κ^2_μ one has

$$\frac{\delta S^2_{\text{loc}}}{\delta \kappa^2_\mu} = \varepsilon \tilde{\Box}^2 \kappa^1_\mu - \sqrt{g'} \tilde{\partial}_\mu (\tilde{\partial} A) = 0 \,. \tag{5.29}$$

With this the new auxiliary fields can be identified as

$$\kappa^1_\mu = \frac{\sqrt{g'}}{\varepsilon \tilde{\Box}^2} \tilde{\partial}_\mu (\tilde{\partial} A) \,, \quad \text{and} \quad \kappa^2_\mu = \frac{\sqrt{g'}}{\varepsilon \tilde{\Box}^2} \tilde{\partial}_\nu (A_\mu A_\nu) \,. \tag{5.30}$$

Thus, both fields are 1-dimensional. Other choices of localization can lead to dimensonless auxiliary fields that should be avoided. Otherwise we would be able to build arbitrary powers of such fields which could spoil the algebraic renormalization process, see also Chap. (4.6). By reinsertion, one obtains

$$\begin{aligned}
S^2_{\rm loc} &= \int d^4x \left[\kappa^1_\mu \varepsilon\tilde{\Box}^2\kappa^2_\mu - \kappa^1_\mu\sqrt{g'}\tilde{\partial}_\nu(A_\mu A_\nu) - \kappa^2_\mu\sqrt{g'}\tilde{\partial}_\mu(\tilde{\partial}A)\right] \\
&= g'\int d^4x \left[\left(\frac{1}{\varepsilon\tilde{\Box}^2}\tilde{\partial}_\mu(\tilde{\partial}A)\right)\varepsilon\tilde{\Box}^2\left(\frac{1}{\varepsilon\tilde{\Box}^2}\tilde{\partial}_\nu(A_\mu A_\nu)\right)\right. \\
&\qquad \left. -\left(\frac{1}{\varepsilon\tilde{\Box}^2}\tilde{\partial}_\mu(\tilde{\partial}A)\right)\tilde{\partial}_\nu(A_\mu A_\nu) - \left(\frac{1}{\varepsilon\tilde{\Box}^2}\tilde{\partial}_\nu(A_\mu A_\nu)\right)\tilde{\partial}_\mu(\tilde{\partial}A)\right] \\
&= g'\int d^4x\left[-\left(\frac{1}{\varepsilon\tilde{\Box}^2}\tilde{\partial}_\nu(A_\mu A_\nu)\right)\tilde{\partial}_\mu\tilde{\partial}_\rho A_\rho\right] \\
&= g'\int d^4x\left[A_\mu A_\nu\,\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho\right] = S^2_{\rm br}\,,
\end{aligned} \tag{5.31}$$

where in the second line the first two terms cancel each other, and from the third to the fourth line we have used partial integration. Now, the BRST invariance can be established in the same manner as for the first two terms by the introduction of a BRST doublet partner for $\sqrt{g'}$,

$$S^2_{\rm loc} = \int d^4x\left[\kappa^1_\mu\varepsilon\tilde{\Box}^2\kappa^2_\mu - s\left(\bar{\delta}\kappa^1_\mu\tilde{\partial}_\nu(A_\mu A_\nu)\right) - s\left(\bar{\delta}\kappa^2_\mu\tilde{\partial}_\mu(\tilde{\partial}A)\right)\right]. \tag{5.32}$$

with $s\bar{\delta} = \sqrt{g'}$ and $s\sqrt{g'} = 0$. The physical situation is obtained by $\bar{\delta}|_{\rm phys} = 0$, and we presume again that $s\kappa^1_\mu = 0$, and that $s\kappa^2_\mu = 0$. This can be legitimated by the equations of motion

$$\frac{\delta S^2_{\rm loc}}{\delta\kappa^1_\mu} = \varepsilon\tilde{\Box}^2\kappa^2_\mu - \sqrt{g'}\tilde{\partial}_\nu(A_\mu A_\nu) + \bar{\delta}\,s\left(\tilde{\partial}_\nu(A_\mu A_\nu)\right) = 0\,, \tag{5.33}$$

and

$$\frac{\delta S^2_{\rm loc}}{\delta\kappa^2_\mu} = \varepsilon\tilde{\Box}^2\kappa^1_\mu - \sqrt{g'}\tilde{\partial}_\mu(\tilde{\partial}A) + \bar{\delta}\,s\left(\tilde{\partial}_\mu(\tilde{\partial}A)\right) = 0\,. \tag{5.34}$$

The new auxiliary fields have the form

$$\kappa^1_\mu = \frac{\sqrt{g'}}{\varepsilon\tilde{\Box}^2}\tilde{\partial}_\mu(\tilde{\partial}A) - \frac{\bar{\delta}}{\varepsilon\tilde{\Box}^2}\,s\left(\tilde{\partial}_\mu(\tilde{\partial}A)\right)\,, \tag{5.35}$$

and

$$\kappa^2_\mu = \frac{\sqrt{g'}}{\varepsilon\tilde{\Box}^2}\tilde{\partial}_\nu(A_\mu A_\nu) - \frac{\bar{\delta}}{\varepsilon\tilde{\Box}^2}\,s\left(\tilde{\partial}_\nu(A_\mu A_\nu)\right)\,, \tag{5.36}$$

where their BRST invariance can be seen immediately. Only setting $\bar{\delta}|_{\rm phys} = 0$ in the end breaks this invariance. It should also be noted explicitly that κ^2_μ is bilinear dependent on the gauge field A_μ in contrast to all the other new auxiliary fields $B_{\mu\nu}$, H, and κ^1_μ that are linear in A_μ.

5.2.1 The localized action and its BRST transformation laws

We now have all parts together, see Eqs. (5.19), (5.23), and (5.32), to write down the localized action which is equivalent to the nonlocal action in (4.8) and formulated BRST exact as well:

$$\Gamma^{(0)}_{\text{inv}} = \int d^4x \left[\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + s(\bar{c}\,\partial A) + \Omega^A_\mu s A_\mu + \Omega^c sc - B_{\mu\nu} \frac{\tilde{\Box}^2}{\gamma^2} B_{\mu\nu} - H \frac{\tilde{\Box}^2}{\gamma^2} H + \kappa^1_\mu \varepsilon \tilde{\Box}^2 \kappa^2_\mu \right.$$
$$\left. + s\left(\bar{\chi} B_{\mu\nu} f_{\mu\nu} + \bar{\chi}\sqrt{2\rho}\, H(\tilde{\partial}A) - \bar{\delta}\kappa^1_\mu \tilde{\partial}_\nu (A_\mu A_\nu) - \bar{\delta}\kappa^2_\mu \tilde{\partial}_\mu (\tilde{\partial}A) \right) \right] . \tag{5.37}$$

We can go back to the original action by setting $\bar{\chi}|_{\text{phys}} = 0$ and $\bar{\delta}|_{\text{phys}} = 0$. Afterwards, one has to use the equations of motion of $B_{\mu\nu}$, H, κ^1_μ, and κ^2_μ, or alternatively integrate out these auxiliary fields in the path integral formalism, $Z = \int \mathcal{D}\phi_i \, e^{-S[\phi_i]}$.

Further, the action (5.37) is invariant with respect to the following usual BRST transformation laws,

$$\begin{aligned} sA_\mu &= \partial_\mu c + \mathrm{i}g\,[c, A_\mu] = D_\mu c\,, \\ sc &= \mathrm{i}gcc\,, \\ s\bar{c} &= B\,, \\ sB &= 0\,, \qquad\qquad s^2 = 0\,, \end{aligned} \tag{5.38}$$

still neglecting the star in the notation, extended by

$$\begin{aligned} s\bar{\chi} &= \gamma\,, & s\gamma &= 0\,, \\ s\bar{\delta} &= \sqrt{g'}\,, & s\sqrt{g'} &= 0\,, \\ sB_{\mu\nu} &= 0\,, & sH &= 0\,, \\ s\kappa^1_\mu &= 0\,, & s\kappa^2_\mu &= 0\,. \end{aligned} \tag{5.39}$$

All new auxiliary fields $B_{\mu\nu}$, H, κ^1_μ, and κ^2_μ, have mass dimension 1. The same is true for the parameters $\bar{\chi}$ and γ. Just $\bar{\delta}$ and $\sqrt{g'}$ are dimensonless but this is matching perfectly to the dimensonless coupling constant $g = e$ of (NC)QED in natural units $[c] = [\hbar] = 0$. Except $\bar{\chi}$ and $\bar{\delta}$ with ghost number -1, all these new fields and parameters carry no ghost charge.

5.2.2 Slavnov-Taylor identity

An extended Slavnov-Taylor identity,

$$\mathcal{S}(\Gamma^{(0)}_{\text{inv}}) = 0\,, \tag{5.40}$$

can now be achived by the introduction of doublet partners for the auxiliary fields $B_{\mu\nu}$, H, κ^1_μ, and κ^2_μ, and by rewriting the action in the following way,

$$\Gamma^{(0)}_{\text{inv}} = \int d^4x \left[\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + s(\bar{c}\,\partial A) - s\left(\bar{B}_{\mu\nu} \frac{\tilde{\Box}^2}{\gamma^2} B_{\mu\nu} - \bar{H} \frac{\tilde{\Box}^2}{\gamma^2} H + \bar{\chi} B_{\mu\nu} f_{\mu\nu} + \bar{\chi}\sqrt{2\rho}\, H(\tilde{\partial}A) \right) \right.$$
$$\left. + s\left(\frac{1}{2}\bar{\kappa}^1_\mu \varepsilon \tilde{\Box}^2 \kappa^2_\mu + \frac{1}{2}\kappa^1_\mu \varepsilon \tilde{\Box}^2 \bar{\kappa}^2_\mu - \bar{\delta}\kappa^1_\mu \tilde{\partial}_\nu (A_\mu A_\nu) - \bar{\delta}\kappa^2_\mu \tilde{\partial}_\mu (\tilde{\partial}A) \right) + \Omega^A_\mu s A_\mu + \Omega^c sc \right], \tag{5.41}$$

where

$$\begin{aligned} s\bar{B}_{\mu\nu} &= B_{\mu\nu}\,, & sB_{\mu\nu} &= 0\,, \\ s\bar{H} &= H\,, & sH &= 0\,, \\ s\bar{\kappa}^1_\mu &= \kappa^1_\mu\,, & s\kappa^1_\mu &= 0\,, \\ s\bar{\kappa}^2_\mu &= \kappa^2_\mu\,, & s\kappa^2_\mu &= 0\,. \end{aligned} \tag{5.42}$$

The appropriate Slavnov-Taylor operator reads

$$\begin{aligned} \mathcal{S}(\Gamma^{(0)}_{\text{inv}}) = \int d^4x \Bigg(& \frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta\Omega^A_\mu}\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta A_\mu} + \frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta\Omega^c}\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta c} + B\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta\bar{c}} + B_{\mu\nu}\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta\bar{B}_{\mu\nu}} + H\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta\bar{H}} \\ & + \kappa^1_\mu\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta\bar{\kappa}^1_\mu} + \kappa^2_\mu\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta\bar{\kappa}^2_\mu}\Bigg) + \gamma\frac{\partial\Gamma^{(0)}_{\text{inv}}}{\partial\bar{\chi}} + \sqrt{g'}\frac{\partial\Gamma^{(0)}_{\text{inv}}}{\partial\bar{\delta}}\,. \end{aligned} \tag{5.43}$$

However, we can work with the localized action (5.37) for the rest of this diploma thesis to keep things simple.

5.3 Gauge field propagator

Here, we want to show that the localized action (5.37) delivers the same gauge propagator as in (3.39), deduced from the original action.

The propagator can be obtained by

$$G^{AA}_{\mu\nu}(x,y) = -\left.\frac{\delta^2 Z^c[j]}{\delta j^A_\mu(x)\delta j^A_\nu(y)}\right|_{j=0} = -\left.\frac{\delta A_\nu(y)}{\delta j^A_\mu(x)}\right|_{j=0}\,. \tag{5.44}$$

So, we have to express the gauge field $A_\mu(x)$ as a function of its source $j^A_\mu(x)$. Further, just all bilinear terms of the action have to be considered since all other parts will be transformed to zero for vanishing sources $j=0$.

We can start with the equation of motion of A_μ for the physical situation $\bar{\chi}|_{\text{phys}}=0$ and $\bar{\delta}|_{\text{phys}}=0$, and collect all linear terms,

$$\left.\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta A_\mu}\right|_{\text{lin}} = -\left(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu\right)A_\nu - \partial_\mu B - 2\gamma\partial_\nu B_{\nu\mu} - \gamma\sqrt{2\rho}\,\tilde{\partial}_\mu H - \sqrt{g'}\tilde{\partial}_\mu\tilde{\partial}_\nu\kappa^2_\nu = -j^A_\mu\,. \tag{5.45}$$

The last term can immediately be eliminated due to the fact that

$$\left.\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta\kappa^1_\mu}\right|_{\text{lin}} = \varepsilon\tilde{\Box}^2\kappa^2_\mu = -j^{\kappa^1}_\mu = 0\,, \tag{5.46}$$

and because we now have to set all sources, except the one for the gauge field $j^A_\mu(x)$, to zero. κ^2_μ is bilinear in the fields, as can be seen from Eq. (5.30), and so it cannot contribute to the gauge propagator.

Furthermore, we have to express the fields B, $B_{\mu\nu}$ and H as functions of A_μ and their respective sources.

For the Lagrange multiplier field B one can start with the gauge fixing part with $\alpha \neq 0$, hence, we are leaving the Landau gauge,

$$S_{\text{gf}} = \int d^4x \left(-\frac{\alpha}{2} B^2 + B\partial_\nu A_\nu \right) . \tag{5.47}$$

Now, its equation of motion,

$$\frac{\delta S_{\text{gf}}}{\delta B} = -\alpha B + \partial_\nu A_\nu = -j^B = 0 \,, \qquad B = \frac{1}{\alpha} \left(\partial_\nu A_\nu + j^B \right) , \tag{5.48}$$

can be used to replace the B-dependent term of (5.45). This gives

$$-j^A_\mu = -\left(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu \right) A_\nu - \partial_\mu \left[\frac{1}{\alpha} \left(\partial_\nu A_\nu + j^B \right) \right] - 2\gamma\partial_\nu B_{\nu\mu} - \gamma\sqrt{2\rho}\,\tilde{\partial}_\mu H \,. \tag{5.49}$$

We can do the same for the auxiliary field $B_{\mu\nu}$,

$$\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta B_{\mu\nu}} = -2\frac{\tilde{\Box}^2}{\gamma^2} B_{\mu\nu} + \gamma f_{\mu\nu} = -j^{B_{\text{aux}}}_{\mu\nu} = 0 \,, \qquad B_{\mu\nu} = \frac{\gamma^2}{2\tilde{\Box}^2} \left(\gamma f_{\mu\nu} + j^{B_{\text{aux}}}_{\mu\nu} \right) . \tag{5.50}$$

Further, we also have to replace the auxiliary field H,

$$\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta H} = -2\frac{\tilde{\Box}^2}{\gamma^2} H + \gamma\sqrt{2\rho}\,(\tilde{\partial}A) = -j^H = 0 \,, \qquad H = \frac{\gamma^2}{2\tilde{\Box}^2} \left(\gamma\sqrt{2\rho}\,(\tilde{\partial}A) + j^H \right) . \tag{5.51}$$

For vanishing sources, except j^A_μ, this implies:

$$\begin{aligned} j^A_\mu &= \left(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu \right) A_\nu + \partial_\mu \left[\frac{1}{\alpha} \left(\partial_\nu A_\nu \right) \right] + 2\gamma\partial_\nu B_{\nu\mu} + \gamma\sqrt{2\rho}\,\tilde{\partial}_\mu H \\ &= \left(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu \right) A_\nu + \partial_\mu \left[\frac{1}{\alpha} \left(\partial_\nu A_\nu \right) \right] + \frac{\gamma^4}{2\tilde{\Box}^2} \left(2\,\partial_\nu f_{\nu\mu} + 2\rho\,\tilde{\partial}_\mu(\tilde{\partial}A) \right) \\ &= \left(1 + \frac{\gamma^4}{\tilde{\Box}^2} \right) \left(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu \right) A_\nu + \partial_\mu \left[\frac{1}{\alpha} \left(\partial_\nu A_\nu \right) \right] + \frac{\gamma^4}{\tilde{\Box}^2} \left(\rho\,\tilde{\partial}_\mu(\tilde{\partial}A) \right) . \end{aligned} \tag{5.52}$$

To write A_μ as a function of j^A_μ, we still need $\partial_\nu A_\nu$. This can be achieved by building the partial derivative of the source,

$$\begin{aligned} \partial_\mu j^A_\mu &= \left(1 + \frac{\gamma^4}{\tilde{\Box}^2} \right) \partial_\mu \left(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu \right) A_\nu + \partial_\mu\partial_\mu \left[\frac{1}{\alpha} \left(\partial_\nu A_\nu \right) \right] + \frac{\gamma^4}{\tilde{\Box}^2} \left(\rho\,\partial_\mu\tilde{\partial}_\mu(\tilde{\partial}A) \right) \\ &= \frac{\Box}{\alpha} \left(\partial_\nu A_\nu \right) , \end{aligned} \tag{5.53}$$

because in the first term the partial derivative is acting on a transverse projection operator, and in the third term one has $\tilde{\partial}_\mu\partial_\mu = 0$. Therefore,

$$\partial_\nu A_\nu = \frac{\alpha}{\Box}\partial_\nu j^A_\nu \,. \tag{5.54}$$

Similarly, an expression for $\tilde{\partial}_\nu A_\nu$ can be obtained by

$$\begin{aligned} \tilde{\partial}_\mu j^A_\mu &= \left(1 + \frac{\gamma^4}{\tilde{\Box}^2} \right) \tilde{\partial}_\mu \left(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu \right) A_\nu + \tilde{\partial}_\mu\partial_\mu \left[\frac{1}{\alpha} \left(\partial_\nu A_\nu \right) \right] + \frac{\gamma^4}{\tilde{\Box}^2} \left(\rho\,\tilde{\partial}_\mu\tilde{\partial}_\mu(\tilde{\partial}A) \right) \\ &= \left[\left(1 + \frac{\gamma^4}{\tilde{\Box}^2} \right) \Box + \rho\gamma^4\frac{1}{\tilde{\Box}} \right] \tilde{\partial}_\nu A_\nu \,. \end{aligned} \tag{5.55}$$

The result is

$$\tilde{\partial}_\nu A_\nu = \frac{\tilde{\partial}_\nu}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4 \frac{1}{\tilde{\Box}}} j^A_\nu \,. \tag{5.56}$$

We can insert these expressions into Eq. (5.52), and receive

$$\begin{aligned} j^A_\mu &= \left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\left(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu\right) A_\nu + \partial_\mu\left[\frac{1}{\alpha}\left(\partial_\nu A_\nu\right)\right] + \frac{\rho\gamma^4}{\tilde{\Box}^2}\left(\tilde{\partial}_\mu(\tilde{\partial}A)\right) \\ &= \left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\left(\Box A_\mu + \frac{\alpha}{\Box}\partial_\mu\partial_\nu j^A_\nu\right) + \frac{\partial_\mu\partial_\nu}{\Box} j^A_\nu + \frac{\rho\gamma^4}{\tilde{\Box}^2}\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde{\Box}}} j^A_\nu \,, \end{aligned} \tag{5.57}$$

or in Landau gauge, $\alpha = 0$,

$$j^A_\mu = \left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box A_\mu + \frac{\partial_\mu\partial_\nu}{\Box} j^A_\nu + \frac{\rho\gamma^4}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde{\Box}}}\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\tilde{\Box}^2} j^A_\nu \,. \tag{5.58}$$

Writing now $A_\mu(x)$ as a function of $j^A_\mu(x)$ gives the desired result

$$A_\mu(x) = \frac{1}{\Box\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)}\left\{ j^A_\mu(x) - \frac{\partial_\mu\partial_\nu}{\Box} j^A_\nu(x) - \frac{\rho\gamma^4}{\rho\gamma^4 + \Box\left(\tilde{\Box} + \frac{\gamma^4}{\tilde{\Box}}\right)}\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\tilde{\Box}} j^A_\nu(x)\right\} \,. \tag{5.59}$$

Calculating the propagator via

$$G^{AA}_{\mu\nu}(x,y) = -\left.\frac{\delta A_\nu(y)}{\delta j^A_\mu(x)}\right|_{j=0} \,, \tag{5.60}$$

obviously leads to

$$G^{AA}_{\mu\nu}(x,y) = -\frac{1}{\Box\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)}\left\{\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box} - \frac{\rho\gamma^4}{\rho\gamma^4 + \Box\left(\tilde{\Box} + \frac{\gamma^4}{\tilde{\Box}}\right)}\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\tilde{\Box}}\right\}\delta^{(4)}(y-x) \,. \tag{5.61}$$

Finally, this can be fourier transformed to

$$G^{AA}_{\mu\nu}(k) = \frac{1}{k^2\left(1+\frac{\gamma^4}{(\tilde{k}^2)^2}\right)}\left\{\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - \frac{\rho\gamma^4}{\rho\gamma^4 + k^2\left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2}\right)}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right\} \,. \tag{5.62}$$

So, we have shown that the gauge propagator hasn't been modified due to the localization procedure. For obvious reasons the same is true for the ghost propagator,

$$G^{\bar{c}c}(k) = -\frac{1}{k^2} \,, \tag{5.63}$$

since we havn't changed the gauge fixing term.

On the other hand, there are a lot of new propagators, and, furthermore, a new vertex. We are going to deduce them in the next two sections.

5.4 Calculation of the new propagators

There are ten new propagators delivered by the action (5.37):

$$G^{\kappa^1\kappa^1}_{\mu\nu},\ G^{\kappa^1 A}_{\mu\nu},\ G^{H\kappa^1}_{\mu},\ G^{\kappa^1 B_{\rm aux}}_{\rho,\mu\nu},\ G^{\kappa^1\kappa^2}_{\mu\nu},\ G^{HH},\ G^{HA}_{\mu},\ G^{HB_{\rm aux}}_{\mu\nu},\ G^{B_{\rm aux}B_{\rm aux}}_{\mu\nu,\rho\sigma} \text{ and } G^{B_{\rm aux}A}_{\mu\nu,\rho}.$$

In particular the first two ones, $G^{\kappa^1\kappa^1}_{\mu\nu}$ and $G^{\kappa^1 A}_{\mu\nu}$, are of special interest because there is also a new vertex $V^{\kappa^1 AA}_{\rho\sigma\tau}$, see Sect. 5.5. So, one can build loop corrections to the photon propagator by using these two propagators together with $V^{\kappa^1 AA}_{\rho\sigma\tau}$.

5.4.1 The propagator $G^{\kappa^1\kappa^1}_{\mu\nu}$

The first new propagator is $G^{\kappa^1\kappa^1}_{\mu\nu}(k)$:

$$G^{\kappa^1\kappa^1}_{\mu\nu}(x,y) = -\left.\frac{\delta^2 Z^c[j]}{\delta j^{\kappa^1}_\mu(x)\,\delta j^{\kappa^1}_\nu(y)}\right|_{j=0} = -\left.\frac{\delta\kappa^1_\nu(y)}{\delta j^{\kappa^1}_\mu(x)}\right|_{j=0}. \tag{5.64}$$

Therefore, one needs κ^1 as a function of its source. All other sources disappear again. Lets start with the equation of motion of κ^2_ν,

$$\frac{\delta\Gamma^{(0)}_{\rm inv}}{\delta\kappa^2_\nu} = \varepsilon\tilde{\Box}^2\kappa^1_\nu - \sqrt{g'}\tilde{\partial}_\nu(\tilde{\partial}A) = -j^{\kappa^2}_\nu\,, \qquad \kappa^1_\nu = -\frac{1}{\varepsilon\tilde{\Box}^2}j^{\kappa^2}_\nu + \frac{\sqrt{g'}}{\varepsilon\tilde{\Box}^2}\tilde{\partial}_\nu(\tilde{\partial}A)\,. \tag{5.65}$$

Again, we need an expression for $\tilde{\partial}_\nu A_\nu$, but now its $j^{\kappa^1}_\nu$-dependent part.

So, perhaps it's better to derive the full equation for $\tilde{\partial}_\nu A_\nu$ that is still containing all sources. We can continue with the ansatz of Eq. (5.45). Applying the same procedure as in the previous chapter for the gauge fixing part, see Eq. (5.47) and below, gives

$$j^A_\mu = (\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)\,A_\nu + \frac{\partial_\mu}{\alpha}\,(\partial_\nu A_\nu) + 2\gamma\partial_\nu B_{\nu\mu} + \gamma\sqrt{2\rho}\,\tilde{\partial}_\mu H + \sqrt{g'}\tilde{\partial}_\mu\tilde{\partial}_\nu\kappa^2_\nu\,. \tag{5.66}$$

Then, we need the Eqs. (5.50) and (5.51),

$$B_{\mu\nu} = \frac{\gamma^2}{2\tilde{\Box}^2}\left(\gamma f_{\mu\nu} + j^{B_{\rm aux}}_{\mu\nu}\right)\,, \qquad H = \frac{\gamma^2}{2\tilde{\Box}^2}\left(\gamma\sqrt{2\rho}\,(\tilde{\partial}A) + j^H\right)\,, \tag{5.67}$$

without setting the respective sources to zero. Furthermore, we need only the linear part of κ^2_ν,

$$\left.\frac{\delta\Gamma^{(0)}_{\rm inv}}{\delta\kappa^1_\nu}\right|_{\rm lin} = \varepsilon\tilde{\Box}^2\kappa^2_\nu = -j^{\kappa^1}_\nu\,, \qquad \left.\kappa^2_\nu\right|_{\rm lin} = -\frac{1}{\varepsilon\tilde{\Box}^2}j^{\kappa^1}_\nu\,. \tag{5.68}$$

Inserting all these expressions for the auxiliary fields gives

$$\begin{aligned} j^A_\mu = {}& (\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)\,A_\nu + \frac{\partial_\mu}{\alpha}\,(\partial_\nu A_\nu) + \gamma\partial_\nu\frac{\gamma^2}{\tilde{\Box}^2}\left(\gamma f_{\nu\mu} + j^{B_{\rm aux}}_{\nu\mu}\right) \\ & + \gamma\sqrt{2\rho}\,\tilde{\partial}_\mu\frac{\gamma^2}{2\tilde{\Box}^2}\left(\gamma\sqrt{2\rho}\,(\tilde{\partial}A) + j^H\right) - \sqrt{g'}\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\varepsilon\tilde{\Box}^2}j^{\kappa^1}_\nu\,. \end{aligned} \tag{5.69}$$

Building the partial derivative $\partial_\mu j^A_\mu$ also leads to the result we have already obtained in the previous section (5.54),

$$\partial_\nu A_\nu = \frac{\alpha}{\Box}\partial_\nu j^A_\nu\,. \tag{5.70}$$

Here, the antisymmetry of $j^{B_{\rm aux}}_{\mu\nu}$ has been used, as well as the properties of a transverse projection operator (4.28), and the fact that $\tilde\partial_\mu\partial_\mu = 0$. With this, the source of (5.69) reads in Landau gauge

$$\begin{aligned} j^A_\mu =& \left(1+\frac{\gamma^4}{\tilde\Box^2}\right)\Box A_\mu + \frac{\partial_\mu\partial_\nu}{\Box}j^A_\nu + \gamma^3\frac{\partial_\nu}{\tilde\Box^2}j^{B_{\rm aux}}_{\mu\nu} + \gamma\sqrt{2\rho}\,\tilde\partial_\mu\frac{\gamma^2}{2\tilde\Box^2}\left(\gamma\sqrt{2\rho}\,(\tilde\partial A) + j^H\right) \\ & - \sqrt{g'}\frac{\tilde\partial_\mu\tilde\partial_\nu}{\varepsilon\tilde\Box^2}j^{\kappa^1}_\nu\,. \end{aligned} \tag{5.71}$$

This implies

$$\tilde\partial_\mu j^A_\mu = \left(1+\frac{\gamma^4}{\tilde\Box^2}\right)\Box\tilde\partial_\mu A_\mu + \gamma^3\frac{\tilde\partial_\mu\partial_\nu}{\tilde\Box^2}j^{B_{\rm aux}}_{\mu\nu} + \gamma\sqrt{2\rho}\,\frac{\gamma^2}{2\tilde\Box}\left(\gamma\sqrt{2\rho}\,(\tilde\partial A) + j^H\right) - \sqrt{g'}\frac{\tilde\partial_\nu}{\varepsilon\tilde\Box}j^{\kappa^1}_\nu\,, \tag{5.72}$$

and finally one arrives at

$$\tilde\partial_\mu A_\mu = \frac{1}{\left(1+\frac{\gamma^4}{\tilde\Box^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde\Box}}\left(\tilde\partial_\mu j^A_\mu - \frac{\gamma^3}{\tilde\Box^2}\tilde\partial_\mu\partial_\nu j^{B_{\rm aux}}_{\mu\nu} - \frac{\gamma^3}{2\tilde\Box}\sqrt{2\rho}\,j^H + \frac{\sqrt{g'}}{\varepsilon\tilde\Box}\tilde\partial_\nu j^{\kappa^1}_\nu\right)\,. \tag{5.73}$$

Therefore, we now can finish the calculations for the propagator $G^{\kappa^1\kappa^1}_{\mu\nu}(x,y)$ by combining the Eqs. (5.64), (5.65), and (5.73):

$$\begin{aligned} G^{\kappa^1\kappa^1}_{\mu\nu}(x,y) &= -\frac{\delta\kappa^1_\nu(y)}{\delta j^{\kappa^1}_\mu(x)}\bigg|_{j=0} = -\frac{\delta}{\delta j^{\kappa^1}_\mu(x)}\left(\frac{\sqrt{g'}}{\varepsilon\tilde\Box^2}\tilde\partial_\nu(\tilde\partial A)\right)\bigg|_{j=0} \\ &= -\frac{g'}{\left(\tilde\Box+\frac{\gamma^4}{\tilde\Box}\right)\Box + \rho\gamma^4}\frac{\tilde\partial_\mu\tilde\partial_\nu}{\varepsilon^2\tilde\Box^2}\,\delta^{(4)}(y-x)\,. \end{aligned} \tag{5.74}$$

The fourier transformed propagator[4], hence, reads

$$G^{\kappa^1\kappa^1}_{\mu\nu}(k) = \frac{g'}{\rho\gamma^4 + k^2\left(\tilde k^2+\frac{\gamma^4}{\tilde k^2}\right)}\frac{\tilde k_\mu\tilde k_\nu}{\varepsilon^2(\tilde k^2)^2}\,. \tag{5.75}$$

5.4.2 The propagator $G^{\kappa^1 A}_{\mu\nu}$

The propagator $G^{A\kappa^1}_{\mu\nu}(k)$,

$$G^{A\kappa^1}_{\mu\nu}(x,y) = -\frac{\delta\kappa^1_\nu(y)}{\delta j^A_\mu(x)}\bigg|_{j=0}\,, \tag{5.76}$$

[4]explanatory note: $\tilde\partial_\mu\tilde\partial_\nu\delta^{(4)}(z) = \int\frac{d^4k}{(2\pi)^4}\,(\mathrm{i}\tilde k_\mu)(\mathrm{i}\tilde k_\nu)\,e^{\mathrm{i}kz} = -\int\frac{d^4k}{(2\pi)^4}\,\tilde k_\mu\tilde k_\nu\,e^{\mathrm{i}kz}$

can easily be derived by use of Eq. (5.65),

$$\kappa^1_\mu = -\frac{1}{\varepsilon\tilde{\Box}^2} j^{\kappa^2}_\mu + \frac{\sqrt{g'}}{\varepsilon\tilde{\Box}^2}\tilde{\partial}_\mu(\tilde{\partial}A)\,, \tag{5.77}$$

and the appropriate part of Eq. (5.73), containing the source of A_μ,

$$\tilde{\partial}A = \frac{1}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde{\Box}}}\left(\tilde{\partial}_\mu j^A_\mu + ...\right)\,. \tag{5.78}$$

Combining this, one sees immediately that the propagator in k-space is

$$G^{A\kappa^1}_{\mu\nu}(k) = \frac{-\sqrt{g'}}{\rho\gamma^4 + k^2\left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2}\right)}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\varepsilon\tilde{k}^2}\,. \tag{5.79}$$

For the propagator

$$G^{\kappa^1 A}_{\mu\nu}(k) = -\left.\frac{\delta A_\nu(y)}{\delta j^{\kappa^1}_\mu(x)}\right|_{j=0}, \tag{5.80}$$

and also for other propagators, one needs $A_\mu = A_\mu[j]$. Starting with Eq. (5.71) and inserting $\tilde{\partial}A$ of (5.73) delivers

$$\begin{aligned}
j^A_\mu &= \left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box A_\mu + \frac{\partial_\mu\partial_\nu}{\Box}j^A_\nu + \gamma^3\frac{\partial_\nu}{\tilde{\Box}^2}j^{B_{\text{aux}}}_{\mu\nu} + \gamma\sqrt{2\rho}\,\tilde{\partial}_\mu\frac{\gamma^2}{2\tilde{\Box}^2}\left(\gamma\sqrt{2\rho}\,(\tilde{\partial}A) + j^H\right) \\
&\quad - \sqrt{g'}\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\varepsilon\tilde{\Box}^2}j^{\kappa^1}_\nu \\
&= \left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box A_\mu + \frac{\partial_\mu\partial_\nu}{\Box}j^A_\nu + \gamma^3\frac{\partial_\nu}{\tilde{\Box}^2}j^{B_{\text{aux}}}_{\mu\nu} + \gamma\sqrt{2\rho}\,\tilde{\partial}_\mu\frac{\gamma^2}{2\tilde{\Box}^2}j^H - \sqrt{g'}\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\varepsilon\tilde{\Box}^2}j^{\kappa^1}_\nu \\
&\quad + \frac{\frac{\rho\gamma^4}{\tilde{\Box}^2}\tilde{\partial}_\mu}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde{\Box}}}\left(\tilde{\partial}_\nu j^A_\nu - \frac{\gamma^3}{\tilde{\Box}^2}\tilde{\partial}_\rho\partial_\nu j^{B_{\text{aux}}}_{\rho\nu} - \frac{\gamma^3}{2\tilde{\Box}}\sqrt{2\rho}\,j^H + \frac{\sqrt{g'}}{\varepsilon\tilde{\Box}}\tilde{\partial}_\nu j^{\kappa^1}_\nu\right)\,.
\end{aligned} \tag{5.81}$$

Expressing $A_\mu[j]$ leads to

$$\begin{aligned}
A_\mu[j] &= \frac{1}{\Box\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)}\Bigg\{j^A_\mu - \frac{\partial_\mu\partial_\nu}{\Box}j^A_\nu - \gamma^3\frac{\partial_\nu}{\tilde{\Box}^2}j^{B_{\text{aux}}}_{\mu\nu} - \gamma\sqrt{2\rho}\,\tilde{\partial}_\mu\frac{\gamma^2}{2\tilde{\Box}^2}j^H + \sqrt{g'}\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\varepsilon\tilde{\Box}^2}j^{\kappa^1}_\nu \\
&\quad - \frac{\frac{\rho\gamma^4}{\tilde{\Box}^2}\tilde{\partial}_\mu}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde{\Box}}}\left(\tilde{\partial}_\nu j^A_\nu - \frac{\gamma^3}{\tilde{\Box}^2}\tilde{\partial}_\rho\partial_\nu j^{B_{\text{aux}}}_{\rho\nu} - \frac{\gamma^3}{2\tilde{\Box}}\sqrt{2\rho}\,j^H + \frac{\sqrt{g'}}{\varepsilon\tilde{\Box}}\tilde{\partial}_\nu j^{\kappa^1}_\nu\right)\Bigg\}\,.
\end{aligned} \tag{5.82}$$

Its $j^{\kappa^1}_\nu$-dependent part reads

$$\begin{aligned}
A_\mu\left[j^{\kappa^1}_\nu, ...\right] &= \frac{1}{\Box\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)}\left\{\sqrt{g'}\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\varepsilon\tilde{\Box}^2}j^{\kappa^1}_\nu - \frac{\frac{\rho\gamma^4}{\tilde{\Box}^2}\tilde{\partial}_\mu}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde{\Box}}}\left(\frac{\sqrt{g'}}{\varepsilon\tilde{\Box}}\tilde{\partial}_\nu j^{\kappa^1}_\nu\right) + ...\right\} \\
&= \frac{\sqrt{g'}}{\left(\tilde{\Box}+\frac{\gamma^4}{\tilde{\Box}}\right)\Box + \rho\gamma^4}\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\varepsilon\tilde{\Box}}j^{\kappa^1}_\nu + ...
\end{aligned} \tag{5.83}$$

Hence, we can summarize

$$G^{A\kappa^1}_{\mu\nu}(k) = G^{\kappa^1 A}_{\mu\nu}(k) = \frac{-\sqrt{g'}}{\rho\gamma^4 + k^2\left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2}\right)} \frac{\tilde{k}_\mu \tilde{k}_\nu}{\varepsilon \tilde{k}^2}\,. \tag{5.84}$$

5.4.3 The propagators $G^{H\kappa^1}_{\mu}$, $G^{B_{\mathbf{aux}}\kappa^1}_{\rho,\mu\nu}$, and $G^{\kappa^1\kappa^2}_{\mu\nu}$

In this subsection, we want to derive all remaining propagators containing the auxiliary field κ^1_μ: for the propagator $G^{\kappa^1 H}_{\mu}(k)$,

$$G^{\kappa^1 H}_{\mu}(x,y) = -\frac{\delta H(y)}{\delta j^{\kappa^1}_\mu(x)}\bigg|_{j=0}\,, \tag{5.85}$$

one needs Eq. (5.67),

$$H = \frac{\gamma^2}{2\tilde{\Box}^2} j^H + \frac{\gamma^3}{2\tilde{\Box}^2}\sqrt{2\rho}\,(\tilde{\partial}A)\,, \tag{5.86}$$

and the following term of Eq. (5.73),

$$\tilde{\partial}A = \frac{1}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde{\Box}}}\left(\frac{\sqrt{g'}}{\varepsilon\tilde{\Box}}\tilde{\partial}_\nu j^{\kappa^1}_\nu + \ldots\right)\,. \tag{5.87}$$

Further, fourier transforming finally gives

$$G^{\kappa^1 H}_{\mu}(k) = -\frac{\mathrm{i}\gamma^3\sqrt{\rho g'}}{\rho\gamma^4 + k^2\left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2}\right)}\frac{\tilde{k}_\mu}{\sqrt{2}\varepsilon(\tilde{k}^2)^2}\,. \tag{5.88}$$

Calculating $G^{H\kappa^1}_{\mu}(k)$ via

$$G^{H\kappa^1}_{\mu}(x,y) = -\frac{\delta \kappa^1_\mu(y)}{\delta j^H(x)}\bigg|_{j=0} \tag{5.89}$$

requires $\kappa^1_\mu\left[j^H\right]$. For that purpose, we can use Eq. (5.65),

$$\kappa^1_\nu = -\frac{1}{\varepsilon\tilde{\Box}^2} j^{\kappa^2}_\nu + \frac{\sqrt{g'}}{\varepsilon\tilde{\Box}^2}\tilde{\partial}_\nu(\tilde{\partial}A)\,, \tag{5.90}$$

together with Eq. (5.73),

$$\tilde{\partial}A = \frac{1}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde{\Box}}}\left(-\frac{\gamma^3}{2\tilde{\Box}}\sqrt{2\rho}\,j^H + \ldots\right)\,. \tag{5.91}$$

All in all, we now gain

$$G^{\kappa^1 H}_{\mu}(k) = -G^{H\kappa^1}_{\mu}(k) = -\frac{\mathrm{i}\gamma^3\sqrt{\rho g'}}{\rho\gamma^4 + k^2\left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2}\right)}\frac{\tilde{k}_\mu}{\sqrt{2}\varepsilon(\tilde{k}^2)^2}\,. \tag{5.92}$$

The same method can be applied for $G^{B_\mathrm{aux}\kappa^1}_{\rho,\mu\nu}$:

$$G^{B_\mathrm{aux}\kappa^1}_{\mu\nu,\rho}(x,y) = -\left.\frac{\delta\kappa^1_\rho(y)}{\delta j^{B_\mathrm{aux}}_{\mu\nu}(x)}\right|_{j=0}. \tag{5.93}$$

By use of Eq. (5.67),

$$\kappa^1_\rho = -\frac{1}{\varepsilon\tilde{\Box}^2} j^{\kappa^2}_\rho + \frac{\sqrt{g'}}{\varepsilon\tilde{\Box}^2}\tilde{\partial}_\rho(\tilde{\partial}A)\,, \tag{5.94}$$

and the following part of Eq. (5.73),

$$\tilde{\partial}A = \frac{1}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde{\Box}}}\left(-\frac{\gamma^3}{\tilde{\Box}^2}\tilde{\partial}_\mu\partial_\nu j^{B_\mathrm{aux}}_{\nu\mu} + ...\right)\,, \tag{5.95}$$

where we have to take into account that $j^{B_\mathrm{aux}}_{\nu\mu}$ is antisymmetric in the indices, we arrive at

$$G^{B_\mathrm{aux}\kappa^1}_{\mu\nu,\rho}(k) = \frac{\mathrm{i}\gamma^3\sqrt{g'}\,\tilde{k}_\rho}{\rho\gamma^4 + k^2\left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2}\right)}\frac{k_\mu\tilde{k}_\nu - k_\nu\tilde{k}_\mu}{2\varepsilon(\tilde{k}^2)^3}\,. \tag{5.96}$$

Calculating $G^{\kappa^1 B_\mathrm{aux}}_{\rho,\mu\nu}(k)$,

$$G^{\kappa^1 B_\mathrm{aux}}_{\rho,\mu\nu}(x,y) = -\left.\frac{\delta B_{\mu\nu}(y)}{\delta j^{\kappa^1}_\rho(x)}\right|_{j=0}, \tag{5.97}$$

requires $B_{\mu\nu}\left[j^{\kappa^1}_\rho\right]$. We again derive the full dependency on all the sources, $B_{\mu\nu}[j]$, since it will be needed below. We have to combine Eq. (5.67),

$$B_{\mu\nu}[j] = \frac{\gamma^2}{2\tilde{\Box}^2}\left(\gamma f_{\mu\nu}[j] + j^{B_\mathrm{aux}}_{\mu\nu}\right) = \frac{\gamma^2}{2\tilde{\Box}^2}\left(\gamma\left(\partial_\mu A_\nu[j] - \partial_\nu A_\mu[j]\right) + j^{B_\mathrm{aux}}_{\mu\nu}\right)\,, \tag{5.98}$$

with Eq. (5.82),

$$\begin{aligned} A_\mu[j] = \frac{1}{\Box\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)}\Bigg\{ & j^A_\mu - \frac{\partial_\mu\partial_\nu}{\Box}j^A_\nu - \gamma^3\frac{\partial_\nu}{\tilde{\Box}^2}j^{B_\mathrm{aux}}_{\mu\nu} - \gamma\sqrt{2\rho}\,\tilde{\partial}_\mu\frac{\gamma^2}{2\tilde{\Box}^2}j^H + \sqrt{g'}\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\varepsilon\tilde{\Box}^2}j^{\kappa^1}_\nu \\ & - \frac{\frac{\rho\gamma^4}{\tilde{\Box}^2}\tilde{\partial}_\mu}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde{\Box}}}\left(\tilde{\partial}_\nu j^A_\nu - \frac{\gamma^3}{\tilde{\Box}^2}\tilde{\partial}_\rho\partial_\nu j^{B_\mathrm{aux}}_{\rho\nu} - \frac{\gamma^3}{2\tilde{\Box}}\sqrt{2\rho}\,j^H + \frac{\sqrt{g'}}{\varepsilon\tilde{\Box}}\tilde{\partial}_\nu j^{\kappa^1}_\nu\right)\Bigg\}\,. \end{aligned} \tag{5.99}$$

That's a really awful looking expression. However, it is sufficient to take out the adequate terms. So, one obtains

$$\begin{aligned} A_\mu\left[j^{\kappa^1}_\rho\right] &= \frac{1}{\Box\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)}\left\{\sqrt{g'}\frac{\tilde{\partial}_\mu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}j^{\kappa^1}_\rho - \frac{\frac{\rho\gamma^4}{\tilde{\Box}^2}\tilde{\partial}_\mu}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde{\Box}}}\frac{\sqrt{g'}}{\varepsilon\tilde{\Box}}\tilde{\partial}_\rho j^{\kappa^1}_\rho\right\} \\ &= \frac{\sqrt{g'}}{\left(\tilde{\Box}+\frac{\gamma^4}{\tilde{\Box}}\right)\Box + \rho\gamma^4}\frac{\tilde{\partial}_\mu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}}j^{\kappa^1}_\rho\,, \end{aligned} \tag{5.100}$$

and

$$B_{\mu\nu}\left[j_\rho^{\kappa^1}\right] = \frac{\gamma^3}{2\tilde{\Box}^2}\left(\partial_\mu A_\nu\left[j_\rho^{\kappa^1}\right] - \partial_\nu A_\mu\left[j_\rho^{\kappa^1}\right]\right), \tag{5.101}$$

what finally leads to

$$G_{\mu\nu,\rho}^{B_{\rm aux}\kappa^1}(k) = -G_{\rho,\mu\nu}^{\kappa^1 B_{\rm aux}}(k) = \frac{\mathrm{i}\gamma^3\sqrt{g'}\,\tilde{k}_\rho}{\rho\gamma^4 + k^2\left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2}\right)}\frac{k_\mu\tilde{k}_\nu - k_\nu\tilde{k}_\mu}{2\varepsilon(\tilde{k}^2)^3}\,. \tag{5.102}$$

The third propagator of this subsection, $G_{\mu\nu}^{\kappa^1\kappa^2}(k)$, follows directly from

$$G_{\mu\nu}^{\kappa^2\kappa^1}(x,y) = -\left.\frac{\delta\kappa_\nu^1(y)}{\delta j_\mu^{\kappa^2}(x)}\right|_{j=0}, \tag{5.103}$$

the Eq. (5.65),

$$\kappa_\mu^1 = -\frac{1}{\varepsilon\tilde{\Box}^2}j_\mu^{\kappa^2} + \frac{\sqrt{g'}}{\varepsilon\tilde{\Box}^2}\tilde{\partial}_\mu(\tilde{\partial}A)\,, \tag{5.104}$$

and the fact that $\tilde{\partial}A$ is independent of $j_\nu^{\kappa^2}$, see Eq. (5.73). So, we get

$$G_{\mu\nu}^{\kappa^2\kappa^1}(k) = G_{\mu\nu}^{\kappa^1\kappa^2}(k) = \frac{\delta_{\mu\nu}}{\varepsilon(\tilde{k}^2)^2}\,. \tag{5.105}$$

5.4.4 The propagators G^{HH}, G_μ^{HA}, and $G_{\mu\nu}^{HB_{\rm aux}}$

In this subsection, we care about all remaining propagators containing the field H: we start with $G^{HH}(k)$,

$$G^{HH}(x,y) = -\left.\frac{\delta H(y)}{\delta j^H(x)}\right|_{j=0}. \tag{5.106}$$

Taking Eq. (5.67),

$$H = \frac{\gamma^2}{2\tilde{\Box}^2}\left(\gamma\sqrt{2\rho}\,(\tilde{\partial}A) + j^H\right), \tag{5.107}$$

together with Eq. (5.73),

$$\tilde{\partial}A = \frac{1}{\left(1 + \frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde{\Box}}}\left(-\frac{\gamma^3}{2\tilde{\Box}}\sqrt{2\rho}\,j^H + ...\right), \tag{5.108}$$

immediately leads to

$$G^{HH}(k) = \frac{-\gamma^2}{2(\tilde{k}^2)^2}\left(1 - \frac{\rho\gamma^4}{\rho\gamma^4 + k^2\left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2}\right)}\right). \tag{5.109}$$

The propagator $G^{HA}_\mu(k)$,

$$G^{HA}_\mu(x,y) = -\frac{\delta A_\mu(y)}{\delta j^H(x)}\bigg|_{j=0}, \tag{5.110}$$

can directly be obtained from the appropriate part of Eq. (5.82),

$$\begin{aligned} A_\mu[j^H] &= \frac{1}{\Box\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)}\left\{-\gamma\sqrt{2\rho}\,\tilde{\partial}_\mu\frac{\gamma^2}{2\tilde{\Box}^2}j^H + \frac{\frac{\rho\gamma^4}{\tilde{\Box}^2}\tilde{\partial}_\mu}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box+\rho\gamma^4\frac{1}{\tilde{\Box}}}\frac{\gamma^3}{2\tilde{\Box}}\sqrt{2\rho}\,j^H\right\} \\ &= -\frac{\gamma^3\sqrt{\rho}}{\left(\tilde{\Box}+\frac{\gamma^4}{\tilde{\Box}}\right)\Box+\rho\gamma^4}\frac{\tilde{\partial}_\mu}{\sqrt{2}\tilde{\Box}}j^H\,. \end{aligned} \tag{5.111}$$

Hence, we have

$$G^{HA}_\mu(k) = -\frac{\mathrm{i}\gamma^3\sqrt{\rho}}{\rho\gamma^4+k^2\left(\tilde{k}^2+\frac{\gamma^4}{\tilde{k}^2}\right)}\frac{\tilde{k}_\mu}{\sqrt{2}\,\tilde{k}^2}\,. \tag{5.112}$$

For propagator $G^{AH}_\mu(k)$,

$$G^{AH}_\mu(x,y) = -\frac{\delta H(y)}{\delta j^A_\mu(x)}\bigg|_{j=0}, \tag{5.113}$$

one needs Eq. (5.67),

$$H = \frac{\gamma^2}{2\tilde{\Box}^2}\left(\gamma\sqrt{2\rho}\,(\tilde{\partial}A)+j^H\right)\,, \tag{5.114}$$

and again Eq. (5.73),

$$\tilde{\partial}A = \frac{1}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box+\rho\gamma^4\frac{1}{\tilde{\Box}}}\left(\tilde{\partial}_\mu j^A_\mu+...\right)\,. \tag{5.115}$$

The fourier transform now leads to

$$G^{AH}_\mu(k) = -G^{HA}_\mu(k) = \frac{\mathrm{i}\gamma^3\sqrt{\rho}}{\rho\gamma^4+k^2\left(\tilde{k}^2+\frac{\gamma^4}{\tilde{k}^2}\right)}\frac{\tilde{k}_\mu}{\sqrt{2}\,\tilde{k}^2}\,. \tag{5.116}$$

Now, we still have to derive the propagator $G^{B_{\mathrm{aux}}H}_{\mu\nu}(k)$,

$$G^{B_{\mathrm{aux}}H}_{\mu\nu}(x,y) = -\frac{\delta H(y)}{\delta j^{B_{\mathrm{aux}}}_{\mu\nu}(x)}\bigg|_{j=0}. \tag{5.117}$$

Using Eq. (5.67),

$$H = \frac{\gamma^2}{2\tilde{\Box}^2}\left(\gamma\sqrt{2\rho}\,(\tilde{\partial}A)+j^H\right)\,, \tag{5.118}$$

and Eq. (5.73),

$$\tilde{\partial} A = \frac{1}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde{\Box}}}\left(-\frac{\gamma^3}{\tilde{\Box}^2}\tilde{\partial}_\mu\partial_\nu j^{B_{\rm aux}}_{\mu\nu} + ...\right) , \tag{5.119}$$

results in

$$G^{B_{\rm aux}H}_{\mu\nu}(k) = \frac{\gamma^6\sqrt{\rho}}{\rho\gamma^4 + k^2\left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2}\right)}\frac{k_\mu\tilde{k}_\nu - k_\nu\tilde{k}_\mu}{2\sqrt{2}\,(\tilde{k}^2)^3} . \tag{5.120}$$

Finally, obtaining $G^{HB_{\rm aux}}_{\mu\nu}(k)$ via

$$G^{HB_{\rm aux}}_{\mu\nu}(x,y) = -\left.\frac{\delta B_{\mu\nu}(y)}{\delta j^H(x)}\right|_{j=0} , \tag{5.121}$$

requires Eq. (5.101), but now depending on j^H,

$$B_{\mu\nu}\left[j^H\right] = \frac{\gamma^3}{2\tilde{\Box}^2}\left(\partial_\mu A_\nu\left[j^H\right] - \partial_\nu A_\mu\left[j^H\right]\right) , \tag{5.122}$$

and Eq. (5.82),

$$\begin{aligned} A_\mu\left[j^H\right] &= \frac{1}{\Box\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)}\left\{-\gamma\sqrt{2\rho}\,\tilde{\partial}_\mu\frac{\gamma^2}{2\tilde{\Box}^2}j^H + \frac{\frac{\rho\gamma^4}{\tilde{\Box}^2}\tilde{\partial}_\mu}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde{\Box}}}\frac{\gamma^3}{2\tilde{\Box}}\sqrt{2\rho}\,j^H\right\} \\ &= -\frac{\gamma^3\sqrt{\rho}}{\left(\tilde{\Box} + \frac{\gamma^4}{\tilde{\Box}}\right)\Box + \rho\gamma^4}\frac{\tilde{\partial}_\mu}{\sqrt{2}\,\tilde{\Box}}j^H . \end{aligned} \tag{5.123}$$

The result, hence, reads

$$G^{B_{\rm aux}H}_{\mu\nu}(k) = G^{HB_{\rm aux}}_{\mu\nu}(k) = \frac{\gamma^6\sqrt{\rho}}{\rho\gamma^4 + k^2\left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2}\right)}\frac{k_\mu\tilde{k}_\nu - k_\nu\tilde{k}_\mu}{2\sqrt{2}\,(\tilde{k}^2)^3} . \tag{5.124}$$

5.4.5 The propagators $G^{B_{\rm aux}B_{\rm aux}}_{\mu\nu,\rho\sigma}$ and $G^{B_{\rm aux}A}_{\mu\nu,\rho}$

There are still two more propagators left for the auxiliary field $B_{\mu\nu}$: the first one is

$$G^{B_{\rm aux}B_{\rm aux}}_{\rho\sigma,\mu\nu}(x,y) = -\left.\frac{\delta B_{\mu\nu}(y)}{\delta j^{B_{\rm aux}}_{\rho\sigma}(x)}\right|_{j=0} . \tag{5.125}$$

We rewrite Eq. (5.67) to

$$B_{\mu\nu}\left[j^{B_{\rm aux}}_{\rho\sigma}\right] = \frac{\gamma^2}{2\tilde{\Box}^2}\left(\gamma\partial_\mu A_\nu\left[j^{B_{\rm aux}}_{\rho\sigma}\right] - \gamma\partial_\nu A_\mu\left[j^{B_{\rm aux}}_{\rho\sigma}\right] + j^{B_{\rm aux}}_{\mu\nu}\right) , \tag{5.126}$$

and recall the $j^{B_{\rm aux}}_{\rho\sigma}$-dependence of Eq. (5.82),

$$A_\mu\left[j^{B_{\rm aux}}_{\rho\sigma}\right] = \frac{1}{\Box\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)}\left\{-\gamma^3\frac{\partial_\sigma}{\tilde{\Box}^2}j^{B_{\rm aux}}_{\mu\sigma} - \frac{\frac{\rho\gamma^4}{\tilde{\Box}^2}\tilde{\partial}_\mu}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde{\Box}}}\frac{\gamma^3}{\tilde{\Box}^2}\tilde{\partial}_\rho\partial_\nu j^{B_{\rm aux}}_{\rho\nu}\right\} . \tag{5.127}$$

Hence, one gets

$$G^{B_{\rm aux}B_{\rm aux}}_{\rho\sigma,\mu\nu}(x,y) = -\frac{\delta}{\delta j^{B_{\rm aux}}_{\rho\sigma}(x)}\frac{\gamma^2}{2\tilde{\Box}^2}\left\{ j^{B_{\rm aux}}_{\mu\nu} - \frac{\gamma^4}{\Box\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)}\left(\frac{\partial_\mu\partial_\tau}{\tilde{\Box}^2}j^{B_{\rm aux}}_{\nu\tau} - \frac{\partial_\nu\partial_\tau}{\tilde{\Box}^2}j^{B_{\rm aux}}_{\mu\tau}\right)\right.$$
$$\left. -\frac{\gamma^8\rho}{\Box\left(\tilde{\Box}+\frac{\gamma^4}{\tilde{\Box}}\right)\tilde{\Box}^2\left[\left(\tilde{\Box}+\frac{\gamma^4}{\tilde{\Box}}\right)\Box+\rho\gamma^4\right]}\left(\partial_\mu\tilde{\partial}_\nu - \partial_\nu\tilde{\partial}_\mu\right)\tilde{\partial}_\alpha\partial_\tau j^{B_{\rm aux}}_{\alpha\tau}\right\}(y)\,. \tag{5.128}$$

Bearing in mind that $j^{B_{\rm aux}}_{\rho\nu}$ is antisymmetric, one obtains

$$G^{B_{\rm aux}B_{\rm aux}}_{\rho\sigma,\mu\nu}(k) = \frac{-\gamma^2}{4(\tilde{k}^2)^2}\left\{\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} - \frac{\gamma^4\left(k_\mu k_\rho\delta_{\nu\sigma} - k_\mu k_\sigma\delta_{\nu\rho} - k_\nu k_\rho\delta_{\mu\sigma} + k_\nu k_\sigma\delta_{\mu\rho}\right)}{k^2\tilde{k}^2\left(\tilde{k}^2+\frac{\gamma^4}{\tilde{k}^2}\right)}\right.$$
$$\left. +\frac{\gamma^8\rho\left(k_\mu\tilde{k}_\nu k_\rho\tilde{k}_\sigma - k_\mu\tilde{k}_\nu k_\sigma\tilde{k}_\rho - k_\nu\tilde{k}_\mu k_\rho\tilde{k}_\sigma + k_\nu\tilde{k}_\mu k_\sigma\tilde{k}_\rho\right)}{k^2(\tilde{k}^2)^2\left(\tilde{k}^2+\frac{\gamma^4}{\tilde{k}^2}\right)\left(\rho\gamma^4 + k^2\left(\tilde{k}^2+\frac{\gamma^4}{\tilde{k}^2}\right)\right)}\right\}\,. \tag{5.129}$$

The propagator $G^{AB_{\rm aux}}_{\rho,\mu\nu}(k)$,

$$G^{AB_{\rm aux}}_{\rho,\mu\nu}(x,y) = -\left.\frac{\delta B_{\mu\nu}(y)}{\delta j^A_\rho(x)}\right|_{j=0}, \tag{5.130}$$

requires the use of Eq. (5.67),

$$B_{\mu\nu}[j^A_\rho] = \frac{\gamma^3}{2\tilde{\Box}^2}\left(\partial_\mu A_\nu[j^A_\rho] - \partial_\nu A_\mu[j^A_\rho]\right)\,, \tag{5.131}$$

and the j^A_ρ-dependence of Eq. (5.82),

$$A_\mu[j^A_\rho] = \frac{1}{\Box\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)}\left\{ j^A_\mu - \frac{\partial_\mu\partial_\nu}{\Box}j^A_\nu - \frac{\frac{\rho\gamma^4}{\tilde{\Box}^2}\tilde{\partial}_\mu}{\left(1+\frac{\gamma^4}{\tilde{\Box}^2}\right)\Box + \rho\gamma^4\frac{1}{\tilde{\Box}}}\tilde{\partial}_\nu j^A_\nu\right\}\,. \tag{5.132}$$

This leads to

$$G^{AB_{\rm aux}}_{\rho,\mu\nu}(k) = \frac{{\rm i}\gamma^3}{2k^2\tilde{k}^2\left(\tilde{k}^2+\frac{\gamma^4}{\tilde{k}^2}\right)}\left\{k_\mu\delta_{\nu\rho} - k_\nu\delta_{\mu\rho} - \frac{\rho\gamma^4}{\rho\gamma^4 + k^2\left(\tilde{k}^2+\frac{\gamma^4}{\tilde{k}^2}\right)}\frac{k_\mu\tilde{k}_\nu\tilde{k}_\rho - k_\nu\tilde{k}_\mu\tilde{k}_\rho}{\tilde{k}^2}\right\}\,. \tag{5.133}$$

Finally, the propagator $G^{B_{\rm aux}A}_{\mu\nu,\rho}(k)$,

$$G^{B_{\rm aux}A}_{\mu\nu,\rho}(x,y) = -\left.\frac{\delta A_\rho(y)}{\delta j^{B_{\rm aux}}_{\mu\nu}(x)}\right|_{j=0}, \tag{5.134}$$

can directly be derived from the $j^{B_{\rm aux}}_{\mu\nu}$-dependence of Eq. (5.82),

$$
\begin{aligned}
A_\rho[j^{B_{\rm aux}}_{\mu\nu}] &= \frac{1}{\Box\left(1+\frac{\gamma^4}{\tilde\Box^2}\right)}\left\{-\gamma^3\frac{\partial_\nu}{\tilde\Box^2}j^{B_{\rm aux}}_{\rho\nu} - \frac{\frac{\rho\gamma^4}{\tilde\Box^2}\tilde\partial_\rho}{\left(1+\frac{\gamma^4}{\tilde\Box^2}\right)\Box+\rho\gamma^4\frac{1}{\tilde\Box}}\left(-\frac{\gamma^3}{\tilde\Box^2}\tilde\partial_\tau\partial_\nu j^{B_{\rm aux}}_{\tau\nu}\right)\right\} \\
&= \frac{-\gamma^3}{\Box\left(1+\frac{\gamma^4}{\tilde\Box^2}\right)\tilde\Box^2}\left\{\partial_\nu j^{B_{\rm aux}}_{\rho\nu} - \frac{\rho\gamma^4}{\left(\tilde\Box+\frac{\gamma^4}{\tilde\Box}\right)\Box+\rho\gamma^4}\left(\frac{\tilde\partial_\rho\tilde\partial_\tau\partial_\nu}{\tilde\Box}j^{B_{\rm aux}}_{\tau\nu}\right)\right\}. \qquad (5.135)
\end{aligned}
$$

So, we now have

$$
G^{B_{\rm aux}A}_{\mu\nu,\rho}(k) = \frac{-\mathrm{i}\gamma^3}{2k^2\tilde k^2\left(\tilde k^2+\frac{\gamma^4}{\tilde k^2}\right)}\left\{k_\mu\delta_{\nu\rho}-k_\nu\delta_{\mu\rho}-\frac{\rho\gamma^4}{\rho\gamma^4+k^2\left(\tilde k^2+\frac{\gamma^4}{\tilde k^2}\right)}\frac{k_\mu\tilde k_\nu\tilde k_\rho-k_\nu\tilde k_\mu\tilde k_\rho}{\tilde k^2}\right\}, \tag{5.136}
$$

with

$$
G^{AB_{\rm aux}}_{\rho,\mu\nu}(k) = -G^{B_{\rm aux}A}_{\mu\nu,\rho}(k). \tag{5.137}
$$

5.5 The new vertex $V^{\kappa^1AA}_{\rho\sigma\tau}$

The localized action (5.37) contains not just terms that are tri- and quadrilinear in the photon field A_μ. There is also a part that is bilinear in A_μ and, in addition, linear in κ^1_μ, as can be seen from

$$
\Gamma^{(0)}_{\rm inv} = \int d^4x\left[\frac{1}{4}F_{\mu\nu}F_{\mu\nu}+...-s\left(\bar\delta\kappa^1_\mu\tilde\partial_\nu(A_\mu A_\nu)\right)\right]. \tag{5.138}
$$

So, there is another vertex $V^{\kappa^1AA}_{\rho\sigma\tau}$. We, now, derive it directly in the k-space by use of

$$
\tilde V^{\kappa^1AA}_{\rho\sigma\tau}(q,k_1,k_2) = -(2\pi)^{12}\left.\frac{\delta^3\Gamma^{(0)}_{\rm inv}}{\delta\kappa^1_\rho(-q)\,\delta A_\sigma(-k_1)\,\delta A_\tau(-k_2)}\right|_{\phi=0}. \tag{5.139}
$$

Firstly, we have to remember the BRST transformation of the parameter $\bar\delta$: $s\bar\delta=\sqrt{g'}$, see (5.39). Don't forget that we are in non-commutative Euclidean space. So, there are always star products in-between that deliver phase factors. Let's start with the fourier transformation of the relevant part of the localized action,

$$
\begin{aligned}
\Gamma^{(0)}_{\rm inv} &= -\sqrt{g'}\int d^4x\left[\kappa^1_\mu\tilde\partial_\nu(A_\mu A_\nu)\right] = \sqrt{g'}\int d^4x\left[(\tilde\partial_\nu\kappa^1_\mu)A_\mu A_\nu\right] \\
&= \sqrt{g'}\int d^4x\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\left[\mathrm{i}\tilde p_{1,\nu}\,\kappa^1_\mu(p_1)A_\mu(p_2)A_\nu(p_3)\right] \\
&\qquad\times\, e^{\mathrm{i}(p_1+p_2+p_3)x}\,e^{-\frac{\mathrm{i}}{2}\varepsilon p_1\tilde p_2}\,e^{-\frac{\mathrm{i}}{2}\varepsilon p_1\tilde p_3}\,e^{-\frac{\mathrm{i}}{2}\varepsilon p_2\tilde p_3} \\
&= \frac{\mathrm{i}\sqrt{g'}}{(2\pi)^8}\int d^4p_1\int d^4p_2\int d^4p_3\left[\tilde p_{1,\nu}\,\kappa^1_\mu(p_1)A_\mu(p_2)A_\nu(p_3)\right] \\
&\qquad\times\,\delta^{(4)}(p_1+p_2+p_3)\,e^{\frac{\mathrm{i}}{2}\varepsilon p_1\tilde p_3}, \qquad (5.140)
\end{aligned}
$$

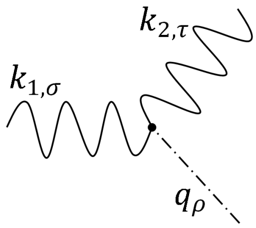

Figure 5.1: The new vertex of the localized action (5.37): $V^{\kappa^1 AA}_{\rho\sigma\tau}$.

because the delta function $\delta^{(4)}(p_1 + p_2 + p_3)$ yields $p_2 = -p_1 - p_3$. Inserting this into Eq. (5.139), gives

$$\begin{aligned}
\tilde{V}^{\kappa^1 AA}_{\rho\sigma\tau}(q, k_1, k_2) &= -(2\pi)^{12} \frac{\delta^3 \Gamma^{(0)}_{\text{inv}}}{\delta\kappa^1_\rho(-q)\,\delta A_\sigma(-k_1)\,\delta A_\tau(-k_2)}\Bigg|_{\phi=0} \\
&= -\sqrt{g'}(2\pi)^4 \mathrm{i} \frac{\delta^3}{\delta\kappa^1_\rho(-q)\,\delta A_\sigma(-k_1)\,\delta A_\tau(-k_2)} \int d^4p_1 \int d^4p_2 \int d^4p_3 \\
&\quad \times \left[\tilde{p}_{1,\nu}\, \kappa^1_\mu(p_1) A_\mu(p_2) A_\nu(p_3)\right] \delta^{(4)}(p_1 + p_2 + p_3)\, e^{\frac{\mathrm{i}}{2}\varepsilon p_1 \tilde{p}_3} \\
&= \sqrt{g'}(2\pi)^4 \mathrm{i}\tilde{q}_\nu \frac{\delta^2}{\delta A_\sigma(-k_1)\,\delta A_\tau(-k_2)} \int d^4p_2 \int d^4p_3 \\
&\quad \times \left[A_\rho(p_2) A_\nu(p_3)\right] \delta^{(4)}(-q + p_2 + p_3)\, e^{-\frac{\mathrm{i}}{2}\varepsilon q \tilde{p}_3} \\
&= \sqrt{g'}(2\pi)^4 \mathrm{i}\tilde{q}_\nu \frac{\delta}{\delta A_\sigma(-k_1)} \int d^4p_2 \int d^4p_3\, \delta^{(4)}(-q + p_2 + p_3)\, e^{-\frac{\mathrm{i}}{2}\varepsilon q \tilde{p}_3} \\
&\quad \times \left[\delta_{\rho\tau}\delta^{(4)}(p_2 + k_2) A_\nu(p_3) + A_\rho(p_2)\delta_{\nu\tau}\delta^{(4)}(p_3 + k_2)\right] \\
&= \sqrt{g'}(2\pi)^4 \mathrm{i}\tilde{q}_\nu \int d^4p_2 \int d^4p_3\, \delta^{(4)}(-q + p_2 + p_3)\, e^{-\frac{\mathrm{i}}{2}\varepsilon q \tilde{p}_3} \\
&\quad \times \left[\delta_{\rho\tau}\delta^{(4)}(p_2 + k_2)\delta_{\nu\sigma}\delta^{(4)}(p_3 + k_1) + \delta_{\rho\sigma}\delta^{(4)}(p_2 + k_1)\delta_{\nu\tau}\delta^{(4)}(p_3 + k_2)\right]. \qquad (5.141)
\end{aligned}$$

Solving the remaining integrals by use of the delta functions, leads to

$$\tilde{V}^{\kappa^1 AA}_{\rho\sigma\tau}(q, k_1, k_2) = \sqrt{g'}(2\pi)^4 \delta^4(q + k_1 + k_2) \left\{\mathrm{i}\tilde{q}_\sigma \delta_{\rho\tau}\, e^{\frac{\mathrm{i}}{2}\varepsilon q \tilde{k}_1} + \mathrm{i}\tilde{q}_\tau \delta_{\rho\sigma}\, e^{\frac{\mathrm{i}}{2}\varepsilon q \tilde{k}_2}\right\}. \qquad (5.142)$$

This can be rewritten in the following manner,

$$\tilde{V}^{\kappa^1 AA}_{\rho\sigma\tau}(q, k_1, k_2) = \sqrt{g'}(2\pi)^4 \delta^4(q + k_1 + k_2) \left\{\mathrm{i}\tilde{q}_\sigma \delta_{\rho\tau}\, e^{\frac{\mathrm{i}}{2}\varepsilon k_1 \tilde{k}_2} + \mathrm{i}\tilde{q}_\tau \delta_{\rho\sigma}\, e^{-\frac{\mathrm{i}}{2}\varepsilon k_1 \tilde{k}_2}\right\}, \qquad (5.143)$$

since $q = -k_1 - k_2$.

So, all new propagators (see previous section) as well as the new vertex (see above) have now been calculated explicitly. Even if some of these propagators deliver horrible looking expressions, we can take comfort in the fact that most of them can not be relevant because of the absence of fitting vertices.

5.6 Recapitulation of the localized model

Driven by the appearance of non-localities in the BRST-exact action (4.8), more precisely of the operator $\frac{1}{\square^2}$ in the counter terms, we were looking for and found a way to localize them. The main feature is that this localized action (5.37) is formulated BRST-invariant as well.

Furthermore, the photon propagator $G^{AA}_{\mu\nu}$ (5.62) and the ghost propagator $G^{\bar{c}c}$ (5.63) still have the same structure as for the original action. However, we obtained a wealth of new propagators: $G^{\kappa^1\kappa^1}_{\mu\nu}$, $G^{\kappa^1 A}_{\mu\nu}$, $G^{H\kappa^1}_{\mu}$, $G^{\kappa^1 B_{\text{aux}}}_{\rho,\mu\nu}$, $G^{\kappa^1\kappa^2}_{\mu\nu}$, G^{HH}, G^{HA}_{μ}, $G^{HB_{\text{aux}}}_{\mu\nu}$, $G^{B_{\text{aux}}B_{\text{aux}}}_{\mu\nu,\rho\sigma}$ and $G^{B_{\text{aux}}A}_{\mu\nu,\rho}$.

There is also a new vertex, $V^{\kappa^1 AA}_{\rho\sigma\tau}$, in addition to the usual vertices: $V^{3A}_{\rho\sigma\tau}$, $V^{4A}_{\rho\sigma\tau\upsilon}$, and $V^{\bar{c}Ac}_{\rho}$.

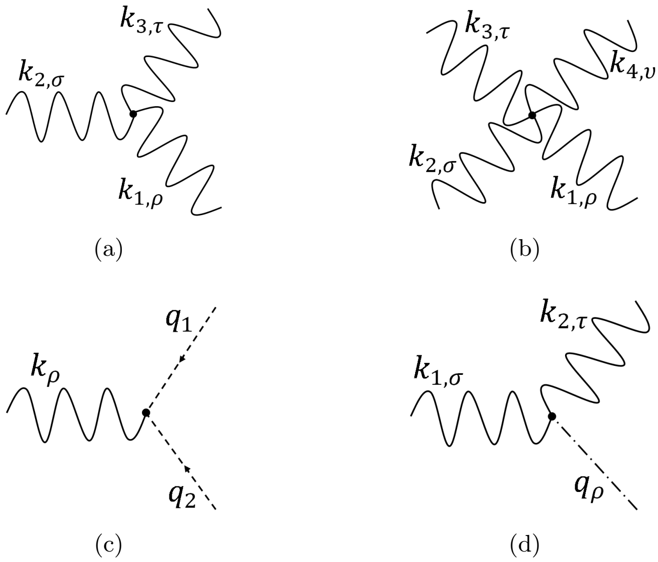

Figure 5.2: The vertices of the localized action (5.37): $V^{3A}_{\rho\sigma\tau}$, $V^{4A}_{\rho\sigma\tau\upsilon}$, $V^{\bar{c}Ac}_{\rho}$ and $V^{\kappa^1 AA}_{\rho\sigma\tau}$.

The main interest is on the new vertex $V^{\kappa^1 AA}_{\rho\sigma\tau}$, together with the two new propagators $G^{\kappa^1\kappa^1}_{\mu\nu}$ and $G^{\kappa^1 A}_{\mu\nu}$. As a consequence, there are further contributions to the loop-corrections of the gauge propagator that would have to be considered in, for example, a possible one-loop analysis.

Chapter 6

Conclusion and Outlook

This diploma thesis dealt with the further development of the non-commutative $U_\star(1)$ gauge model presented in [14].

At first, an introduction to non-commutative field theories was given in Chap. 1. Particularly, a brief historical overview, ranging from some first motivations to implement non-commutativity in space-time right up to the first renormalizable non-commutative scalar models, was presented. Especially, the one of Gurau et al. [24] has to be pointed out. Up to now, it is probably the most important milestone on the long and arduous journey to a consistent non-commutative field-theoretical formulation of the fundamental interactions. However, it is "only" a scalar model - a fully renormalizable non-commutative QED, for example, is still missing.

After discussing some mathematical details in Chap. 2, and adjacent explanations on the BRST symmetry at the beginning of Chap. 3, a short summary of how to obtain the model in Ref. [14] was given. In non-commutative gauge theories there is a quadratic IR-singularity arising in the two-point vertex graph at one-loop. Therefore, an appropriate counter term has been introduced in [14] that, further, provides the IR-damping of the gauge propagator. However, an additional IR-divergence was arising in the 3A vertex function at one-loop:

$$\Gamma^{3A,\,\mathrm{IR}}_{\rho\sigma\tau}(k_1,k_2,k_3) \propto \cos\left(\frac{\varepsilon}{2}k_1\tilde{k}_2\right)\sum_{i=1,2,3}\frac{\tilde{k}_{i,\rho}\tilde{k}_{i,\sigma}\tilde{k}_{i,\tau}}{\varepsilon(\tilde{k}_i^2)^2}\,. \tag{6.1}$$

It couldn't be absorbed by a redefinition of the parameters or the fields of the model because of its complete new look.

Hence, the first thing to do in Chap. 4 was to add an appropriate (non-local) counter term into the action. This, of course, also leads to a further parameter g' in the model. Afterwards, we proposed an extended BRST formulation of the action by the introduction of doublet partners for the new parameter g', as well as for the already existing parameter γ^4 of the IR-damping term, resulting in two new BRST doublets: $(\bar{\chi},\gamma^4)$ and $(\bar{\delta},g')$. Both non-local counter terms were, thus, formulated BRST invariant. As such, they should not spoil an algebraic renormalization process. Further, we derived some identities (e.g. expressing the transversality of the two-point vertex graph), and showed their validity at tree level. The hope is that they hold true even at higher loop order.

Finally, a localized model (except the non-locality due to the star product), also formulated BRST exact, was presented in Chap. 5, including the calculation of all new propagators and the new vertex.

There are now two main difficulties in applying the algebraic renormalization procedure since it relies on the quantum action principle (QAP) and on locality:

- It is questionable whether the QAP is applicable in the presence of the star product. This will be treated in a forthcoming paper [48].
- It is not clear whether one should continue with the more complicated localized theory, or whether someone can find good arguments for applying the algebraic renormalization procedure on the non-localized model.

Appendix A

Calculations for the Identities

A.1 Main bilinear contribution to the first identity

The main bilinear contribution to the identities is coming from the first term of our action

$$\Gamma^{(0)}_{\text{inv}} = \int d^4x \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \text{other terms} \right) , \tag{A.1}$$

more precisely from its Abelian part, which is obviously

$$\begin{aligned}
\Gamma^{(0)}_{\text{bi},1} &= \frac{1}{4} \int d^4x \Big(f_{\mu\nu} f_{\mu\nu} \Big) \\
&= \frac{1}{4} \int d^4x \Big(\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \Big) \\
&= \frac{1}{2} \int d^4x \Big(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu \Big) \\
&= -\frac{1}{2} \int d^4x \Big(A_\mu \left(\partial^2 \delta_{\mu\nu} - \partial_\mu \partial_\nu \right) A_\nu \Big) .
\end{aligned} \tag{A.2}$$

Building the functional derivative with respect to $A_\mu(z)$ gives

$$\frac{\delta \Gamma^{(0)}_{\text{bi},1}}{\delta A_\mu(z)} = - \left(\Box^z \delta_{\mu\nu} - \partial^z_\mu \partial^z_\nu \right) A_\nu(z) \, , \tag{A.3}$$

and finally we get

$$\frac{\delta^2 \Gamma^{(0)}_{\text{bi},1}}{\delta A_\rho(y) \delta A_\mu(z)} = - \left(\Box^z \delta_{\mu\rho} - \partial^z_\mu \partial^z_\rho \right) \delta^{(4)}(z-y) \, . \tag{A.4}$$

This is the usual two-point vertex graph appearing in QED.

Explanatory note: In the above derivation we have used two mathematical operations we will need now all the time. These are the functional derivative

$$\frac{\delta A_\mu(x)}{\delta A_\nu(y)} = \delta_{\mu\nu} \delta^{(4)}(x-y) \, , \tag{A.5}$$

and partial integration

$$\int d^4x \left((\partial_\mu \phi) \star A_\nu \right)(x) = - \int d^4x \left(\phi \star \partial_\mu A_\nu \right)(x) \, , \tag{A.6}$$

where $\phi(x)$ can be any polynomial in the fields and their partial derivatives.

A.2 Damping-term contributions to the first identity

Here, we take a look at the contributions coming from

$$\mathcal{L}^1_{\mathrm{br}} = \frac{1}{4}\left(f_{\mu\nu} \star \frac{1}{\tilde{\square}^2} f_{\mu\nu} + 2\rho\,\tilde{\partial}A \star \frac{1}{\tilde{\square}^2}\tilde{\partial}A\right)\,, \tag{A.7}$$

which is also bilinear in the gauge field. The corresponding action reads

$$\Gamma^{(0)}_{\mathrm{bi},2} = \int d^4x \left(\gamma^4\,\mathcal{L}^1_{\mathrm{br}}\right) = \frac{\gamma^4}{4}\int d^4x \left(f_{\mu\nu}\frac{1}{\tilde{\square}^2} f_{\mu\nu} + 2\rho\,\tilde{\partial}A\frac{1}{\tilde{\square}^2}\tilde{\partial}A\right)\,. \tag{A.8}$$

The first part can be rewritten as

$$\begin{aligned}
\Gamma^{(0)}_{\mathrm{bi},2(1)} &= \frac{\gamma^4}{4}\int d^4x \left(f_{\mu\nu}\frac{1}{\tilde{\square}^2} f_{\mu\nu}\right) \\
&= \frac{\gamma^4}{4}\int d^4x \left(\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\frac{1}{\tilde{\square}^2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\right) \\
&= \frac{\gamma^4}{2}\int d^4x \left(\partial_\mu A_\nu \frac{1}{\tilde{\square}^2}\partial_\mu A_\nu - \partial_\mu A_\nu \frac{1}{\tilde{\square}^2}\partial_\nu A_\mu\right) \\
&= -\frac{\gamma^4}{2}\int d^4x \left(A_\mu \frac{1}{\tilde{\square}^2}\left(\partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu\right) A_\nu\right)\,,
\end{aligned} \tag{A.9}$$

and for the second part one gets

$$\begin{aligned}
\Gamma^{(0)}_{\mathrm{bi},2(2)} &= \frac{\rho\gamma^4}{2}\int d^4x \left(\tilde{\partial}A\frac{1}{\tilde{\square}^2}\tilde{\partial}A\right) \\
&= -\frac{\rho\gamma^4}{2}\int d^4x \left(A_\mu \frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\tilde{\square}^2} A_\nu\right)\,.
\end{aligned} \tag{A.10}$$

Hence, one can derive

$$\frac{\delta\Gamma^{(0)}_{\mathrm{bi},2(1)}}{\delta A_\mu(z)} = -\frac{\gamma^4}{\tilde{\square}^2_z}\left(\square^z\delta_{\mu\nu} - \partial^z_\mu\partial^z_\nu\right) A_\nu(z)\,, \tag{A.11}$$

and

$$\frac{\delta^2\Gamma^{(0)}_{\mathrm{bi},2(1)}}{\delta A_\rho(y)\delta A_\mu(z)} = -\frac{\gamma^4}{\tilde{\square}^2_z}\left(\square^z\delta_{\mu\rho} - \partial^z_\mu\partial^z_\rho\right)\delta^{(4)}(z-y)\,. \tag{A.12}$$

For the second part one gets

$$\frac{\delta\Gamma^{(0)}_{\mathrm{bi},2(2)}}{\delta A_\mu(z)} = -\rho\gamma^4\frac{\tilde{\partial}^z_\mu\tilde{\partial}^z_\nu}{\tilde{\square}^2_z} A_\nu(z)\,, \tag{A.13}$$

and

$$\frac{\delta^2\Gamma^{(0)}_{\mathrm{bi},2(2)}}{\delta A_\rho(y)\delta A_\mu(z)} = -\rho\gamma^4\frac{\tilde{\partial}^z_\mu\tilde{\partial}^z_\rho}{\tilde{\square}^2_z}\delta^{(4)}(z-y)\,. \tag{A.14}$$

Summarizing brings

$$\frac{\delta^2\Gamma^{(0)}_{\mathrm{bi},2}}{\delta A_\rho(y)\delta A_\mu(z)} = -\frac{\gamma^4}{\tilde{\square}^2_z}\left(\square^z\delta_{\mu\rho} - \partial^z_\mu\partial^z_\rho + \rho\,\tilde{\partial}^z_\mu\tilde{\partial}^z_\rho\right)\delta^{(4)}(z-y)\,. \tag{A.15}$$

Furthermore, we have to calculate the contribution from

$$\frac{\delta^2}{\delta A_\rho(y)\delta c(z)}\left(\gamma^4\frac{\partial\Gamma^{(0)}_{\rm inv}}{\partial\bar\chi}\right)\Bigg|_{\phi=0} = -\gamma^4\frac{\delta^2}{\delta A_\rho(y)\delta c(z)}\int d^4x\,\left(s\mathcal{L}^1_{\rm br}\right)\Bigg|_{\phi=0}. \tag{A.16}$$

Inserting $\mathcal{L}^1_{\rm br}$ brings

$$(A.16) = -\frac{\gamma^4}{4}\frac{\delta^2}{\delta A_\rho(y)\delta c(z)}\int d^4x\,s\left(f_{\mu\nu}\star\frac{1}{\tilde\Box^2}f_{\mu\nu}+2\rho\,\tilde\partial A\star\frac{1}{\tilde\Box^2}\tilde\partial A\right)\Bigg|_{\phi=0}. \tag{A.17}$$

It consists of the two parts

$$-\frac{\gamma^4}{4}\frac{\delta^2}{\delta A_\rho(y)\delta c(z)}\int d^4x\,s\left(f_{\mu\nu}\star\frac{1}{\tilde\Box^2}f_{\mu\nu}\right)\Bigg|_{\phi=0}, \tag{A.18}$$

and

$$-\frac{\rho\gamma^4}{2}\frac{\delta^2}{\delta A_\rho(y)\delta c(z)}\int d^4x\,s\left(\tilde\partial A\star\frac{1}{\tilde\Box^2}\tilde\partial A\right)\Bigg|_{\phi=0}. \tag{A.19}$$

These can be rewritten similar to (A.9) and (A.10) as

$$(A.18) = \frac{\gamma^4}{2}\frac{\delta^2}{\delta A_\rho(y)\delta c(z)}\int d^4x\,s\left(A_\mu\frac{1}{\tilde\Box^2}\left(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu\right)A_\nu\right)\Bigg|_{\phi=0}, \tag{A.20}$$

and

$$(A.19) = \frac{\rho\gamma^4}{2}\frac{\delta^2}{\delta A_\rho(y)\delta c(z)}\int d^4x\,s\left(A_\mu\frac{\tilde\partial_\mu\tilde\partial_\nu}{\tilde\Box^2}A_\nu\right)\Bigg|_{\phi=0}. \tag{A.21}$$

Now we use the BRST transformation (4.4) for A_μ, in particular its linear part,

$$sA_\mu = \partial_\mu c + \mathrm{i}g\left[c \stackrel{\star}{,} A_\mu\right] = \partial_\mu c + \ldots\,. \tag{A.22}$$

This shows that both parts are zero: The first one because of

$$\partial_\mu\left(\partial^2\delta_{\mu\nu}-\partial_\mu\partial_\nu\right) = 0\,, \tag{A.23}$$

and the second one due to $\partial_\mu\tilde\partial_\mu = 0$. Explicitly this reads

$$\begin{aligned}(A.20) &= \frac{\gamma^4}{2}\frac{\delta^2}{\delta A_\rho(y)\delta c(z)}\int d^4x\left\{\partial_\mu c, \frac{1}{\tilde\Box^2}\left(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu\right)A_\nu\right\}\Bigg|_{\phi=0}\\ &= -\frac{\gamma^4}{2}\frac{\delta^2}{\delta A_\rho(y)\delta c(z)}\int d^4x\left\{c, \frac{\partial_\mu}{\tilde\Box^2}\left(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu\right)A_\nu\right\}\Bigg|_{\phi=0} = 0\,,\end{aligned} \tag{A.24}$$

and

$$
\begin{aligned}
(\text{A.21}) &= \frac{\rho\gamma^4}{2} \frac{\delta^2}{\delta A_\rho(y)\delta c(z)} \int d^4x \left\{ \partial_\mu c, \frac{\tilde{\partial}_\mu \tilde{\partial}_\nu}{\tilde{\Box}^2} A_\nu \right\} \Bigg|_{\phi=0} \\
&= -\frac{\rho\gamma^4}{2} \frac{\delta^2}{\delta A_\rho(y)\delta c(z)} \int d^4x \left\{ c, \partial_\mu \frac{\tilde{\partial}_\mu \tilde{\partial}_\nu}{\tilde{\Box}^2} A_\nu \right\} \Bigg|_{\phi=0} = 0\,,
\end{aligned} \tag{A.25}
$$

where the symmetry in the indices μ and ν and partial integration has been used. Hence, we can close with the result

$$
\begin{aligned}
\frac{\delta^2}{\delta A_\rho(y)\delta c(z)} \left(\gamma^4 \frac{\partial \Gamma^{(0)}_{\text{inv}}}{\partial \bar{\chi}} \right) \Bigg|_{\phi=0} &= -\gamma^4 \frac{\delta^2}{\delta A_\rho(y)\delta c(z)} \int d^4x \left(s\mathcal{L}^1_{\text{br}} \right) \Bigg|_{\phi=0} \\
&= -\frac{\gamma^4}{2} \frac{\delta^2}{\delta A_\rho(y)\delta c(z)} \int d^4x \left\{ c, \frac{\partial_\mu}{\tilde{\Box}^2} \left(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu + \rho\tilde{\partial}_\mu\tilde{\partial}_\nu \right) A_\nu \right\} \Bigg|_{\phi=0} \\
&= -\gamma^4 \frac{\delta^2}{\delta A_\rho(y)\delta c(z)} \int d^4x \left(c\, \frac{\partial_\mu}{\tilde{\Box}^2} \left(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu + \rho\tilde{\partial}_\mu\tilde{\partial}_\nu \right) A_\nu \right) \Bigg|_{\phi=0} = 0\,,
\end{aligned} \tag{A.26}
$$

and so it is not contributing to the identity (4.17).

A.3 Calculations for the gauge part in the first identity

In this section, we want to study the dependence of our first identity (4.17) on an arbitrary gauge parameter α. For that we would have to change our action in the following manner

$$
\Gamma^{(0)}_{\text{inv,gf}} = \int d^4x \Big(B\partial A \Big) \overset{\alpha\neq 0}{\longrightarrow} \Gamma^{(0)}_{\text{bi,gf}} = \int d^4x \left(B\partial A - \frac{\alpha}{2} B^2 \right) . \tag{A.27}
$$

This is bilinear in A for $\alpha \neq 0$ because with the equation of motion for B

$$
\frac{\delta \Gamma^{(0)}_{\text{bi,gf}}}{\delta B} = \partial A - \alpha B = -j^B = 0\,, \quad \text{where} \quad B = \frac{\partial A}{\alpha}\,, \tag{A.28}
$$

we can rewrite this as

$$
\Gamma^{(0)}_{\text{bi,gf}} = \frac{1}{2\alpha} \int d^4x \, (\partial A)^2 \,. \tag{A.29}
$$

So, we would obtain the following additional contribution

$$
-\partial^z_\mu \frac{\delta^2 \Gamma^{(0)}_{\text{bi,gf}}}{\delta A_\rho(y)\delta A_\mu(z)} = \frac{\partial^z_\rho}{\alpha} \Box^z \delta^{(4)}(z-y)\,. \tag{A.30}
$$

in the identity (4.17). On the other hand, the extended Slavnov-Taylor operator (4.10)

$$
\mathcal{S}(\Gamma^{(0)}_{\text{inv}}) = \int d^4x \left(B \star \frac{\delta \Gamma^{(0)}_{\text{inv}}}{\delta \bar{c}} + \text{other terms} \right) . \tag{A.31}
$$

would lead to an additional term in (4.17). By use of the equation of motion for $\bar{c}$, considering for this the BRST transformations (4.4),

$$\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta\bar{c}} = -s\left(\partial_\mu A_\mu\right) = -\Box c - \mathrm{i}g\partial_\mu\left[c \stackrel{\star}{,} A_\mu\right] \,, \tag{A.32}$$

we get from (A.31), neglecting the "other terms",

$$\frac{\delta^2}{\delta A_\rho(y)\delta c(z)}\int d^4x\left(B\star\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta\bar{c}}\right)\Bigg|_{\phi=0} = -\frac{\delta^2}{\delta A_\rho(y)\delta c(z)}\int d^4x\left(B\star\Box c\right) . \tag{A.33}$$

With the equation of motion for B (A.28) we arrive at

$$\begin{aligned}\frac{\delta^2}{\delta A_\rho(y)\delta c(z)}\int d^4x\left(B\star\frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta\bar{c}}\right)\Bigg|_{\phi=0} &= -\frac{1}{\alpha}\frac{\delta^2}{\delta A_\rho(y)\delta c(z)}\int d^4x\left\{(\partial A)\star\Box c\right\}\\ &= -\frac{1}{\alpha}\frac{\delta}{\delta A_\rho(y)}\int d^4x\left\{\delta^{(4)}(x-z)\star\Box(\partial A)\right\}\\ &= -\frac{1}{\alpha}\frac{\delta}{\delta A_\rho(y)}\left\{\Box^z\partial^z_\mu A_\mu(z)\right\}\\ &= -\frac{1}{\alpha}\Box^z\partial^z_\rho\delta^{(4)}(z-y)\,.\end{aligned} \tag{A.34}$$

This would alter our identity (4.17) to

$$\frac{\delta^2\mathcal{S}(\Gamma^{(0)}_{\text{inv}})}{\delta A_\rho(y)\delta c(z)}\Bigg|_{\phi=0} = -\partial^z_\mu\frac{\delta^2\Gamma^{(0)}_{\text{inv}}}{\delta A_\rho(y)\delta A_\mu(z)}\Bigg|_{\phi=0} - \frac{\partial^z_\rho}{\alpha}\Box^z\delta^{(4)}(z-y) = 0\,, \tag{A.35}$$

where the new term would cancel the contribution (A.30).

A.4 Main trilinear contribution to the second identity

Here, we need the trilinear part of (A.1), which is (neglecting the star in the notation)

$$\begin{aligned}\Gamma^{(0)}_{\text{tri},1} &= \frac{1}{4}\int d^4x\Big(F_{\sigma\tau}F_{\sigma\tau}\Big)_{\text{tri}}\\ &= -\frac{\mathrm{i}g}{4}\int d^4x\Big(\left(\partial_\sigma A_\tau - \partial_\tau A_\sigma\right)\left(A_\sigma A_\tau - A_\tau A_\sigma\right) + \left(A_\sigma A_\tau - A_\tau A_\sigma\right)\left(\partial_\sigma A_\tau - \partial_\tau A_\sigma\right)\Big)\\ &= -\frac{\mathrm{i}g}{2}\int d^4x\Big(\left(\partial_\sigma A_\tau - \partial_\tau A_\sigma\right)\left(A_\sigma A_\tau - A_\tau A_\sigma\right)\Big)\\ &= -\mathrm{i}g\int d^4x\Big(\left(\partial_\sigma A_\tau\right)\left(A_\sigma A_\tau - A_\tau A_\sigma\right)\Big)\,,\end{aligned} \tag{A.36}$$

where the defintion (3.2) for the non-Abelian field strength tensor has been used. It is important to note here that there are always star products in between, where just one can be dropped. So what we have got is

$$\Gamma^{(0)}_{\text{tri},1} = -\mathrm{i}g\int d^4x\Big(\left(\partial_\sigma A_\tau\right)\star\left(A_\sigma\star A_\tau - A_\tau\star A_\sigma\right)\Big)\,. \tag{A.37}$$

Building a first functional derivative leads to

$$
\begin{aligned}
\frac{\delta\Gamma^{(0)}_{\text{tri},1}}{\delta A_\mu(z)} &= \mathrm{i}g\int d^4x\,\Big\{\delta^{(4)}(x-z)\delta_{\tau\mu}\star\partial_\sigma\left(A_\sigma\star A_\tau - A_\tau\star A_\sigma\right)\\
&\quad -(\partial_\sigma A_\tau)\star\Big(\delta^{(4)}(x-z)\delta_{\sigma\mu}\star A_\tau + A_\sigma\star\delta^{(4)}(x-z)\delta_{\tau\mu}\Big)\\
&\quad +(\partial_\sigma A_\tau)\star\Big(\delta^{(4)}(x-z)\delta_{\tau\mu}\star A_\sigma + A_\tau\star\delta^{(4)}(x-z)\delta_{\sigma\mu}\Big)\Big\}\\
&= \mathrm{i}g\Big\{\partial^z_\sigma\left(A_\sigma\star A_\mu - A_\mu\star A_\sigma\right) - A_\tau\star\left(\partial^z_\mu A_\tau\right)\\
&\quad -(\partial^z_\sigma A_\mu)\star A_\sigma + A_\sigma\star(\partial^z_\sigma A_\mu) + \left(\partial^z_\mu A_\tau\right)\star A_\tau\Big\}\\
&= \mathrm{i}g\Big\{(\partial^z_\sigma A_\sigma)\star A_\mu - A_\mu\star(\partial^z_\sigma A_\sigma) - A_\tau\star\left(\partial^z_\mu A_\tau\right)\\
&\quad -2\,(\partial^z_\sigma A_\mu)\star A_\sigma + 2A_\sigma\star(\partial^z_\sigma A_\mu) + \left(\partial^z_\mu A_\tau\right)\star A_\tau\Big\}\\
&= \mathrm{i}g\Big\{[(\partial^z_\sigma A_\sigma)\stackrel{\star}{,} A_\mu] + \left[\left(\partial^z_\mu A_\tau\right)\stackrel{\star}{,} A_\tau\right] - 2\,[(\partial^z_\sigma A_\mu)\stackrel{\star}{,} A_\sigma]\Big\}, \qquad \text{(A.38)}
\end{aligned}
$$

where A_μ is now a function of z because in general $\int d^4x\,\delta^{(4)}(x-z)\,f(x) = f(z)$. Furthermore, we get

$$
\begin{aligned}
\frac{\delta^2\Gamma^{(0)}_{\text{tri},1}}{\delta A_\rho(y)\delta A_\mu(z)} &= \mathrm{i}g\Big\{\left[\left(\partial^z_\rho\delta^{(4)}(z-y)\right)\stackrel{\star}{,} A_\mu\right] - \left[(\partial^z_\sigma A_\sigma)\stackrel{\star}{,}\delta^{(4)}(z-y)\right]\delta_{\mu\rho}\\
&\quad + \left[\left(\partial^z_\mu\delta^{(4)}(z-y)\right)\stackrel{\star}{,} A_\rho\right] + \left[\left(\partial^z_\mu A_\rho\right)\stackrel{\star}{,}\delta^{(4)}(z-y)\right]\\
&\quad -2\left[\left(\partial^z_\sigma\delta^{(4)}(z-y)\right)\stackrel{\star}{,} A_\sigma\right]\delta_{\mu\rho} + 2\left[\left(\partial^z_\rho A_\mu\right)\stackrel{\star}{,}\delta^{(4)}(z-y)\right]\Big\}, \qquad \text{(A.39)}
\end{aligned}
$$

and finally

$$
\begin{aligned}
&\frac{\delta^3\Gamma^{(0)}_{\text{tri},1}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)} =\\
&\mathrm{i}g\Big\{\left[\left(\partial^z_\rho\delta^{(4)}(z-y)\right)\stackrel{\star}{,}\delta^{(4)}(z-r)\right]\delta_{\mu\lambda} + \left[\left(\partial^z_\lambda\delta^{(4)}(z-r)\right)\stackrel{\star}{,}\delta^{(4)}(z-y)\right]\delta_{\mu\rho}\\
&\quad + \left[\left(\partial^z_\mu\delta^{(4)}(z-y)\right)\stackrel{\star}{,}\delta^{(4)}(z-r)\right]\delta_{\rho\lambda} + \left[\left(\partial^z_\mu\delta^{(4)}(z-r)\right)\stackrel{\star}{,}\delta^{(4)}(z-y)\right]\delta_{\rho\lambda}\\
&\quad -2\left[\left(\partial^z_\lambda\delta^{(4)}(z-y)\right)\stackrel{\star}{,}\delta^{(4)}(z-r)\right]\delta_{\mu\rho} - 2\left[\left(\partial^z_\rho\delta^{(4)}(z-r)\right)\stackrel{\star}{,}\delta^{(4)}(z-y)\right]\delta_{\mu\lambda}\Big\}. \qquad \text{(A.40)}
\end{aligned}
$$

This looks a little bit complicated in the spacetime notation, but in k-space it can be simplified to the usual result for the 3A vertex. However, what we need at first is its partial derivative

$$\partial^z_\mu \frac{\delta^3\Gamma^{(0)}_{\text{tri},1}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)} =$$
$$\mathrm{i}g\Big\{ \Big[\Big(\partial^z_\mu\partial^z_\rho\delta^{(4)}(z-y)\Big) \stackrel{\star}{,} \delta^{(4)}(z-r)\Big]\,\delta_{\mu\lambda} + \Big[\Big(\partial^z_\mu\partial^z_\lambda\delta^{(4)}(z-r)\Big) \stackrel{\star}{,} \delta^{(4)}(z-y)\Big]\,\delta_{\mu\rho}$$
$$+ \Big[\Big(\partial^z_\mu\partial^z_\mu\delta^{(4)}(z-y)\Big) \stackrel{\star}{,} \delta^{(4)}(z-r)\Big]\,\delta_{\rho\lambda} + \Big[\Big(\partial^z_\mu\partial^z_\mu\delta^{(4)}(z-r)\Big) \stackrel{\star}{,} \delta^{(4)}(z-y)\Big]\,\delta_{\rho\lambda}$$
$$- 2\Big[\Big(\partial^z_\mu\partial^z_\lambda\delta^{(4)}(z-y)\Big) \stackrel{\star}{,} \delta^{(4)}(z-r)\Big]\,\delta_{\mu\rho} - 2\Big[\Big(\partial^z_\mu\partial^z_\rho\delta^{(4)}(z-r)\Big) \stackrel{\star}{,} \delta^{(4)}(z-y)\Big]\,\delta_{\mu\lambda}\Big\}\,. \tag{A.41}$$

Here, all the commutators where ∂^z_μ has been acting on the right delta function have canceled and not written down. Rewriting brings

$$\partial^z_\mu \frac{\delta^3\Gamma^{(0)}_{\text{tri},1}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)} = \mathrm{i}g\Big\{ \Big[\Big(\big(\Box^z\delta_{\lambda\rho} - \partial^z_\lambda\partial^z_\rho\big)\,\delta^{(4)}(z-y)\Big) \stackrel{\star}{,} \delta^{(4)}(z-r)\Big]$$
$$+ \Big[\Big(\big(\Box^z\delta_{\lambda\rho} - \partial^z_\lambda\partial^z_\rho\big)\,\delta^{(4)}(z-r)\Big) \stackrel{\star}{,} \delta^{(4)}(z-y)\Big]\Big\}\,. \tag{A.42}$$

A.5 Fourier transform and the 3A vertex

In (A.40) we have already calculated the 3A vertex in spacetime coordinates. We can use this result to gain the 3A vertex in k-space via Fourier transform

$$V^{3A}_{\lambda\rho\mu}(r,y,z) = \prod_{i=1}^{3}\int \frac{d^4k_i}{(2\pi)^4}\, e^{\mathrm{i}(k_1r+k_2y+k_3z)}\,\tilde{V}^{3A}_{\lambda\rho\mu}(k_1,k_2,k_3)\,. \tag{A.43}$$

Additionally, we have to consider that in (A.38) the x-integral has been solved by use of the delta function $\delta^{(4)}(x-z)$, which will be reindroduced now. So, combining (A.40) and (A.38), we get

$$\frac{\delta^3\Gamma^{(0)}_{\text{tri},1}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)} = \mathrm{i}g\int d^4x\,\delta^{(4)}(x-z)\times$$
$$\times\Big\{ \Big[\Big(\partial^x_\rho\delta^{(4)}(x-y)\Big) \stackrel{\star}{,} \delta^{(4)}(x-r)\Big]\,\delta_{\mu\lambda} + \Big[\Big(\partial^x_\lambda\delta^{(4)}(x-r)\Big) \stackrel{\star}{,} \delta^{(4)}(x-y)\Big]\,\delta_{\mu\rho}$$
$$+ \Big[\Big(\partial^x_\mu\delta^{(4)}(x-y)\Big) \stackrel{\star}{,} \delta^{(4)}(x-r)\Big]\,\delta_{\rho\lambda} + \Big[\Big(\partial^x_\mu\delta^{(4)}(x-r)\Big) \stackrel{\star}{,} \delta^{(4)}(x-y)\Big]\,\delta_{\rho\lambda}$$
$$- 2\Big[\Big(\partial^x_\lambda\delta^{(4)}(x-y)\Big) \stackrel{\star}{,} \delta^{(4)}(x-r)\Big]\,\delta_{\mu\rho} - 2\Big[\Big(\partial^x_\rho\delta^{(4)}(x-r)\Big) \stackrel{\star}{,} \delta^{(4)}(x-y)\Big]\,\delta_{\mu\lambda}\Big\}\,. \tag{A.44}$$

Now, we need the definition of the delta function

$$\delta^{(4)}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-\mathrm{i}k(x-y)}\,. \tag{A.45}$$

Using (A.45) in (A.44) we arrive at

$$\frac{\delta^3\Gamma^{(0)}_{\text{tri},1}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)} = g\int d^4x \iiint \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4} e^{-\mathrm{i}(k_1+k_2+k_3)x} e^{\mathrm{i}(k_1r+k_2y+k_3z)}$$
$$\times\, 2\mathrm{i}\sin\left(\frac{\varepsilon}{2}k_1\theta k_2\right)\left\{k_{2,\rho}\delta_{\mu\lambda} - k_{1,\lambda}\delta_{\mu\rho} + k_{2,\mu}\delta_{\rho\lambda} - k_{1,\mu}\delta_{\rho\lambda} - 2k_{2,\lambda}\delta_{\mu\rho} + 2k_{1,\rho}\delta_{\mu\lambda}\right\}, \tag{A.46}$$

since

$$\begin{aligned}
&\left[\left(\partial^x_\rho\delta^{(4)}(x-y)\right) \stackrel{\star}{,} \delta^{(4)}(x-r)\right] \\
&\quad= \iint \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}(-\mathrm{i}k_{2,\rho})\left(e^{-\frac{\mathrm{i}}{2}k_2\varepsilon\theta k_1} - e^{-\frac{\mathrm{i}}{2}k_1\varepsilon\theta k_2}\right)e^{-\mathrm{i}(k_1+k_2)x}e^{\mathrm{i}(k_1r+k_2y)} \\
&\quad= \iint \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}(-\mathrm{i}k_{2,\rho})\left(e^{+\frac{\mathrm{i}}{2}k_1\varepsilon\theta k_2} - e^{-\frac{\mathrm{i}}{2}k_1\varepsilon\theta k_2}\right)e^{-\mathrm{i}(k_1+k_2)x}e^{\mathrm{i}(k_1r+k_2y)} \\
&\quad= \iint \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}(-\mathrm{i}k_{2,\rho})\, 2\mathrm{i}\sin\left(\frac{\varepsilon}{2}k_1\theta k_2\right)e^{-\mathrm{i}(k_1+k_2)x}e^{\mathrm{i}(k_1r+k_2y)}\,.
\end{aligned} \tag{A.47}$$

The spacetime integral gives a delta function

$$\int d^4x\, e^{-\mathrm{i}(k_1+k_2+k_3)x} = (2\pi)^4\delta^{(4)}(k_1+k_2+k_3)\,, \tag{A.48}$$

meaning that $k_1 = -(k_2+k_3)$. With these insights, one can rewrite (A.46) as

$$\frac{\delta^3\Gamma^{(0)}_{\text{tri},1}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)} = g\frac{1}{(2\pi)^8}\iiint d^4k_1\, d^4k_2\, d^4k_3\, \delta^{(4)}(k_1+k_2+k_3)e^{\mathrm{i}(k_1r+k_2y+k_3z)}$$
$$\times\, 2\mathrm{i}\sin\left(\frac{\varepsilon}{2}k_1\theta k_2\right)\left\{(k_1-k_3)_\rho\delta_{\mu\lambda} + (k_2-k_1)_\mu\delta_{\lambda\rho} + (k_3-k_2)_\lambda\delta_{\rho\mu}\right\}. \tag{A.49}$$

Using (A.43) we get

$$\tilde{V}^{3A}_{\lambda\rho\mu}(k_1,k_2,k_3)\big|_{\text{main}} = 2\mathrm{i}g\,(2\pi)^4\delta^{(4)}(k_1+k_2+k_3)\,\sin\left(\frac{\varepsilon}{2}k_1\tilde{k}_2\right)$$
$$\times\left\{(k_1-k_3)_\rho\delta_{\mu\lambda} + (k_2-k_1)_\mu\delta_{\lambda\rho} + (k_3-k_2)_\lambda\delta_{\rho\mu}\right\}. \tag{A.50}$$

This usual result for the 3A vertex gains an additional term from the new counter term, resp. $\mathcal{L}^2_{\text{br}}$ (see Appendix, Sect. A.8).

A.6 Main bilinear contributions to the second identity

The main bilinear part contribution in our identity (4.46) comes from

$$\mathrm{i}g\left[\delta^{(4)}(z-y) \stackrel{\star}{,} \left.\frac{\delta^2\Gamma^{(0)}_{\text{bi},1}}{\delta A_\lambda(r)\delta A_\rho(z)}\right|_{\phi=0}\right] + \mathrm{i}g\left[\delta^{(4)}(z-r) \stackrel{\star}{,} \left.\frac{\delta^2\Gamma^{(0)}_{\text{bi},1}}{\delta A_\rho(y)\delta A_\lambda(z)}\right|_{\phi=0}\right], \tag{A.51}$$

precisely from the bilinear part of $F_{\mu\nu}F_{\mu\nu}$ (see Appendix, Sect A.1). In (A.4)

$$\frac{\delta^2\Gamma^{(0)}_{\mathrm{bi},1}}{\delta A_\rho(y)\delta A_\mu(z)} = -\left(\Box^z\delta_{\mu\rho} - \partial^z_\mu\partial^z_\rho\right)\delta^{(4)}(z-y)\,, \tag{A.52}$$

we have already calculated the result. Inserting this is into (A.51) brings

$$\begin{aligned}(\mathrm{A.51}) = -\mathrm{i}g\Big\{ &\left[\delta^{(4)}(z-y) \stackrel{\star}{,} \left(\Box^z\delta_{\lambda\rho} - \partial^z_\lambda\partial^z_\rho\right)\delta^{(4)}(z-r)\right] \\ &+ \left[\delta^{(4)}(z-r) \stackrel{\star}{,} \left(\Box^z\delta_{\lambda\rho} - \partial^z_\lambda\partial^z_\rho\right)\delta^{(4)}(z-y)\right]\Big\}\,.\end{aligned} \tag{A.53}$$

A.7 Counter-term contributions to the second identity

Now we take a look at the following part of the action (4.8)

$$\Gamma^{(0)}_{\mathrm{tri},2} = \int d^4x\left(g'\mathcal{L}^2_{\mathrm{br}}\right)\,, \tag{A.54}$$

which is trilinear in the gauge field A_μ. Explicitly, this reads

$$\begin{aligned}\Gamma^{(0)}_{\mathrm{tri},2} &= \frac{g'}{2}\int d^4x\left(\{A_\sigma \stackrel{\star}{,} A_\nu\}\star\frac{\tilde\partial_\sigma\tilde\partial_\nu\tilde\partial_\tau}{\varepsilon\tilde\Box^2}A_\tau\right) \\ &= g'\int d^4x\left(A_\sigma\star A_\nu\star\frac{\tilde\partial_\sigma\tilde\partial_\nu\tilde\partial_\tau}{\varepsilon\tilde\Box^2}A_\tau\right)\,.\end{aligned} \tag{A.55}$$

Again, building functional derivatives brings

$$\begin{aligned}\frac{\delta\Gamma^{(0)}_{\mathrm{tri},2}}{\delta A_\mu(z)} = g'\int d^4x\Big(&\delta^{(4)}(x-z)\star A_\nu\star\frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\tau}{\varepsilon\tilde\Box^2}A_\tau + A_\sigma\star\delta^{(4)}(x-z)\star\frac{\tilde\partial_\sigma\tilde\partial_\mu\tilde\partial_\tau}{\varepsilon\tilde\Box^2}A_\tau \\ &+ A_\sigma\star A_\nu\star\frac{\tilde\partial_\sigma\tilde\partial_\nu\tilde\partial_\mu}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-z)\Big)\,,\end{aligned} \tag{A.56}$$

and

$$\begin{aligned}&\frac{\delta^2\Gamma^{(0)}_{\mathrm{tri},2}}{\delta A_\rho(y)\delta A_\mu(z)} = \\ &g'\int d^4x\Big(\delta^{(4)}(x-z)\star\delta^{(4)}(x-y)\star\frac{\tilde\partial_\mu\tilde\partial_\rho\tilde\partial_\tau}{\varepsilon\tilde\Box^2}A_\tau + \delta^{(4)}(x-z)\star A_\nu\star\frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-y) \\ &+\delta^{(4)}(x-y)\star\delta^{(4)}(x-z)\star\frac{\tilde\partial_\rho\tilde\partial_\mu\tilde\partial_\tau}{\varepsilon\tilde\Box^2}A_\tau + A_\sigma\star\delta^{(4)}(x-z)\star\frac{\tilde\partial_\sigma\tilde\partial_\mu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-y) \\ &+\delta^{(4)}(x-y)\star A_\nu\star\frac{\tilde\partial_\rho\tilde\partial_\nu\tilde\partial_\mu}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-z) + A_\sigma\star\delta^{(4)}(x-y)\star\frac{\tilde\partial_\sigma\tilde\partial_\rho\tilde\partial_\mu}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-z)\Big)\,.\end{aligned} \tag{A.57}$$

Finally, we get

$$\begin{aligned}\frac{\delta^3\Gamma^{(0)}_{\text{tri},2}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)} &= g'\int d^4x\,\Bigg(\delta^{(4)}(x-z)\star\delta^{(4)}(x-y)\star\frac{\tilde\partial_\mu\tilde\partial_\rho\tilde\partial_\lambda}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-r)\\ &+\delta^{(4)}(x-z)\star\delta^{(4)}(x-r)\star\frac{\tilde\partial_\mu\tilde\partial_\lambda\tilde\partial_\rho}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-y)\\ &+\delta^{(4)}(x-y)\star\delta^{(4)}(x-z)\star\frac{\tilde\partial_\rho\tilde\partial_\mu\tilde\partial_\lambda}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-r)\\ &+\delta^{(4)}(x-r)\star\delta^{(4)}(x-z)\star\frac{\tilde\partial_\lambda\tilde\partial_\mu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-y)\\ &+\delta^{(4)}(x-y)\star\delta^{(4)}(x-r)\star\frac{\tilde\partial_\rho\tilde\partial_\lambda\tilde\partial_\mu}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-z)\\ &+\delta^{(4)}(x-r)\star\delta^{(4)}(x-y)\star\frac{\tilde\partial_\lambda\tilde\partial_\rho\tilde\partial_\mu}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-z)\Bigg)\,. \end{aligned}\tag{A.58}$$

Due to the permutability of $\tilde\partial_\rho$ and $\tilde\partial_\lambda$, $\left[\tilde\partial_\rho,\tilde\partial_\lambda\right]=0$, the order of $\tilde\partial_\lambda$, $\tilde\partial_\mu$ and $\tilde\partial_\rho$ is not important. Hence, it is always the same operator, but it is acting on different delta functions.

We can write (A.58) in a more compact way as

$$\begin{aligned}\frac{\delta^3\Gamma^{(0)}_{\text{tri},2}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)} &= g'\int d^4x\,\Bigg(\left\{\delta^{(4)}(x-z)\overset{\star}{,}\,\delta^{(4)}(x-y)\right\}\star\frac{\tilde\partial_\lambda\tilde\partial_\rho\tilde\partial_\mu}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-r)\\ &+\left\{\delta^{(4)}(x-z)\overset{\star}{,}\,\delta^{(4)}(x-r)\right\}\star\frac{\tilde\partial_\lambda\tilde\partial_\rho\tilde\partial_\mu}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-y)\\ &+\left\{\delta^{(4)}(x-r)\overset{\star}{,}\,\delta^{(4)}(x-y)\right\}\star\frac{\tilde\partial_\lambda\tilde\partial_\rho\tilde\partial_\mu}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-z)\Bigg)\,. \end{aligned}\tag{A.59}$$

However, using (A.58) and calculating its partial derivative brings

$$\begin{aligned}\partial^z_\mu\frac{\delta^3\Gamma^{(0)}_{\text{tri},2}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)} &= g'\partial^z_\mu\Bigg\{\delta^{(4)}(z-y)\star\frac{\tilde\partial^z_\mu\tilde\partial^z_\rho\tilde\partial^z_\lambda}{\varepsilon\tilde\Box^2_z}\delta^{(4)}(z-r)\\ &+\delta^{(4)}(z-r)\star\frac{\tilde\partial^z_\mu\tilde\partial^z_\lambda\tilde\partial^z_\rho}{\varepsilon\tilde\Box^2_z}\delta^{(4)}(z-y)\\ &+\left(\frac{\tilde\partial^z_\rho\tilde\partial^z_\mu\tilde\partial^z_\lambda}{\varepsilon\tilde\Box^2_z}\delta^{(4)}(z-r)\right)\star\delta^{(4)}(z-y)\\ &+\left(\frac{\tilde\partial^z_\lambda\tilde\partial^z_\mu\tilde\partial^z_\rho}{\varepsilon\tilde\Box^2_z}\delta^{(4)}(z-y)\right)\star\delta^{(4)}(z-r)\\ &-\frac{\tilde\partial^z_\rho\tilde\partial^z_\lambda\tilde\partial^z_\mu}{\varepsilon\tilde\Box^2_z}\left(\delta^{(4)}(z-y)\star\delta^{(4)}(z-r)\right)\\ &-\frac{\tilde\partial^z_\lambda\tilde\partial^z_\rho\tilde\partial^z_\mu}{\varepsilon\tilde\Box^2_z}\left(\delta^{(4)}(z-r)\star\delta^{(4)}(z-y)\right)\Bigg\}\,. \end{aligned}\tag{A.60}$$

The last two terms are zero because of $\partial_\mu \tilde{\partial}_\mu = 0$. So, we get

$$
\begin{aligned}
&- \partial^z_\mu \frac{\delta^3 \Gamma^{(0)}_{\text{tri},2}}{\delta A_\lambda(r) \delta A_\rho(y) \delta A_\mu(z)} = \\
&- g' \left(\left\{ \partial^z_\mu \delta^{(4)}(z-y) \stackrel{\star}{,} \frac{\tilde{\partial}^z_\mu \tilde{\partial}^z_\lambda \tilde{\partial}^z_\rho}{\varepsilon \tilde{\square}^2_z} \delta^{(4)}(z-r) \right\} + \left\{ \partial^z_\mu \delta^{(4)}(z-r) \stackrel{\star}{,} \frac{\tilde{\partial}^z_\mu \tilde{\partial}^z_\lambda \tilde{\partial}^z_\rho}{\varepsilon \tilde{\square}^2_z} \delta^{(4)}(z-y) \right\} \right) .
\end{aligned}
\tag{A.61}
$$

This contribution of our second identity (4.46) is canceled by the term

$$
\frac{\delta^3}{\delta A_\lambda(r) \delta A_\rho(y) \delta c(z)} \left(g' \frac{\partial \Gamma^{(0)}_{\text{inv}}}{\partial \bar{\delta}} \right) \Bigg|_{\phi=0} = -g' \frac{\delta^3}{\delta A_\lambda(r) \delta A_\rho(y) \delta c(z)} \int d^4x \left(s\mathcal{L}^2_{\text{br}} \right) \Bigg|_{\phi=0} , \tag{A.62}
$$

as shown now. Inserting $\mathcal{L}^2_{\text{br}}$ (4.9) gives

$$
\begin{aligned}
\text{(A.62)} &= -\frac{g'}{2} \frac{\delta^3}{\delta A_\lambda(r) \delta A_\rho(y) \delta c(z)} \int d^4x\, s \left(\{A_\sigma \stackrel{\star}{,} A_\nu\} \star \frac{\tilde{\partial}_\sigma \tilde{\partial}_\nu \tilde{\partial}_\tau}{\varepsilon \tilde{\square}^2} A_\tau \right) \Bigg|_{\phi=0} \\
&= -g' \frac{\delta^3}{\delta A_\lambda(r) \delta A_\rho(y) \delta c(z)} \int d^4x\, s \left(A_\sigma \star A_\nu \star \frac{\tilde{\partial}_\sigma \tilde{\partial}_\nu \tilde{\partial}_\tau}{\varepsilon \tilde{\square}^2} A_\tau \right) \Bigg|_{\phi=0} .
\end{aligned}
\tag{A.63}
$$

Now, we need the trilinear part which is bilinear in A_μ and linear in c. For that we use the BRST transformation (4.4) for A_μ

$$
sA_\mu = \partial_\mu c + \mathrm{i}g \, [c \stackrel{\star}{,} A_\mu] = \partial_\mu c + ... , \tag{A.64}
$$

and neglect all quadrilinear terms of (A.63), since they vanish for $\phi = 0$. Hence, we get

$$
\begin{aligned}
\text{(A.63)} = -g' \frac{\delta^3}{\delta A_\lambda(r) \delta A_\rho(y) \delta c(z)} \int d^4x \bigg\{ &(\partial_\sigma c) \star A_\nu \star \frac{\tilde{\partial}_\sigma \tilde{\partial}_\nu \tilde{\partial}_\tau}{\varepsilon \tilde{\square}^2} A_\tau \\
&+ A_\sigma \star (\partial_\nu c) \star \frac{\tilde{\partial}_\sigma \tilde{\partial}_\nu \tilde{\partial}_\tau}{\varepsilon \tilde{\square}^2} A_\tau + A_\sigma \star A_\nu \star \frac{\tilde{\partial}_\sigma \tilde{\partial}_\nu \tilde{\partial}_\tau}{\varepsilon \tilde{\square}^2} (\partial_\tau c) \bigg\} \bigg|_{\phi=0} .
\end{aligned}
\tag{A.65}
$$

The last term is zero due to $\partial_\tau \tilde{\partial}_\tau = 0$. The rest can be evaluated by additional use of partial integration and cyclic permutation under the integral:

$$
\begin{aligned}
\text{(A.65)} &= g' \frac{\delta^3}{\delta A_\lambda(r) \delta A_\rho(y) \delta c(z)} \int d^4x \left\{ c \star (\partial_\sigma A_\nu) \star \frac{\tilde{\partial}_\sigma \tilde{\partial}_\nu \tilde{\partial}_\tau}{\varepsilon \tilde{\square}^2} A_\tau + (\partial_\nu A_\sigma) \star c \star \frac{\tilde{\partial}_\sigma \tilde{\partial}_\nu \tilde{\partial}_\tau}{\varepsilon \tilde{\square}^2} A_\tau \right\} \\
&= g' \frac{\delta^2}{\delta A_\lambda(r) \delta A_\rho(y)} \int d^4x \, \delta^{(4)}(x-z) \\
&\quad \times \left\{ (\partial_\sigma A_\nu) \star \left(\frac{\tilde{\partial}_\sigma \tilde{\partial}_\nu \tilde{\partial}_\tau}{\varepsilon \tilde{\square}^2} A_\tau \right) + \left(\frac{\tilde{\partial}_\sigma \tilde{\partial}_\nu \tilde{\partial}_\tau}{\varepsilon \tilde{\square}^2} A_\tau \right) \star (\partial_\nu A_\sigma) \right\} \\
&= g' \frac{\delta^2}{\delta A_\lambda(r) \delta A_\rho(y)} \left\{ (\partial^z_\sigma A_\nu) \star \left(\frac{\tilde{\partial}^z_\sigma \tilde{\partial}^z_\nu \tilde{\partial}^z_\tau}{\varepsilon \tilde{\square}^2_z} A_\tau \right) + \left(\frac{\tilde{\partial}^z_\sigma \tilde{\partial}^z_\nu \tilde{\partial}^z_\tau}{\varepsilon \tilde{\square}^2_z} A_\tau \right) \star (\partial^z_\nu A_\sigma) \right\} .
\end{aligned}
\tag{A.66}
$$

In the second line the star after the delta function has been dropped to solve the integral. Further, we can derive

$$
\begin{aligned}
(\text{A.66}) = g'\frac{\delta}{\delta A_\lambda(r)}\Bigg\{&(\partial^z_\sigma\delta^{(4)}(z-y))\star\left(\frac{\tilde\partial^z_\sigma\tilde\partial^z_\rho\tilde\partial^z_\tau}{\varepsilon\tilde\Box^2_z}A_\tau\right)+(\partial^z_\sigma A_\nu)\star\left(\frac{\tilde\partial^z_\sigma\tilde\partial^z_\nu\tilde\partial^z_\rho}{\varepsilon\tilde\Box^2_z}\delta^{(4)}(z-y)\right)\\
&+\left(\frac{\tilde\partial^z_\sigma\tilde\partial^z_\nu\tilde\partial^z_\rho}{\varepsilon\tilde\Box^2_z}\delta^{(4)}(z-y)\right)\star(\partial^z_\nu A_\sigma)+\left(\frac{\tilde\partial^z_\rho\tilde\partial^z_\nu\tilde\partial^z_\tau}{\varepsilon\tilde\Box^2_z}A_\tau\right)\star(\partial^z_\nu\delta^{(4)}(z-y))\Bigg\}.
\end{aligned}
\tag{A.67}
$$

Renaming σ or ν, respectively, to μ and building the last functional derivative leads to

$$
\begin{aligned}
&\frac{\delta^3}{\delta A_\lambda(r)\delta A_\rho(y)\delta c(z)}\left(g'\frac{\partial\Gamma^{(0)}_{\text{inv}}}{\partial\bar\delta}\right)\Bigg|_{\phi=0}=\\
&=g'\left(\left\{\partial^z_\mu\delta^{(4)}(z-y)\stackrel{\star}{,}\frac{\tilde\partial^z_\mu\tilde\partial^z_\lambda\tilde\partial^z_\rho}{\varepsilon\tilde\Box^2_z}\delta^{(4)}(z-r)\right\}+\left\{\partial^z_\mu\delta^{(4)}(z-r)\stackrel{\star}{,}\frac{\tilde\partial^z_\mu\tilde\partial^z_\lambda\tilde\partial^z_\rho}{\varepsilon\tilde\Box^2_z}\delta^{(4)}(z-y)\right\}\right).
\end{aligned}
\tag{A.68}
$$

Again, the fact that the order of $\tilde\partial^z_\mu$, $\tilde\partial^z_\lambda$ and $\tilde\partial^z_\rho$ is not important has been used. This contribution (A.68) cancels the part coming from (A.61) in our vertex identity (4.46).

A.8 Fourier transform of counter-term contributions to the 3A vertex

Here, we want to calculate the Fourier transform of (A.59) and show that the 3A vertex in k-space gets an additional term from $\mathcal{L}^2_{\text{br}}$, with

$$
\Gamma^{(0)}_{\text{tri},2}=\int d^4x\,\left(g'\mathcal{L}^2_{\text{br}}\right)\,,
\tag{A.69}
$$

see (A.54). So we start with (A.59)

$$
\begin{aligned}
\frac{\delta^3\Gamma^{(0)}_{\text{tri},2}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)}=g'\int d^4x\Bigg(&\left\{\delta^{(4)}(x-z)\stackrel{\star}{,}\delta^{(4)}(x-y)\right\}\star\frac{\tilde\partial_\lambda\tilde\partial_\rho\tilde\partial_\mu}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-r)\\
&+\left\{\delta^{(4)}(x-z)\stackrel{\star}{,}\delta^{(4)}(x-r)\right\}\star\frac{\tilde\partial_\lambda\tilde\partial_\rho\tilde\partial_\mu}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-y)\\
&+\left\{\delta^{(4)}(x-r)\stackrel{\star}{,}\delta^{(4)}(x-y)\right\}\star\frac{\tilde\partial_\lambda\tilde\partial_\rho\tilde\partial_\mu}{\varepsilon\tilde\Box^2}\delta^{(4)}(x-z)\Bigg)\,,
\end{aligned}
\tag{A.70}
$$

and use (A.45) for our delta function. Additionally, we can drop one star under the x-integral and get

$$
\begin{aligned}
&\frac{\delta^3\Gamma^{(0)}_{\text{tri},2}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)}=-\frac{2g'}{\mathrm{i}}\int d^4x\iiint\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}e^{-\mathrm{i}(k_1+k_2+k_3)x}e^{\mathrm{i}(k_1r+k_2y+k_3z)}\\
&\times\left\{\cos\left(\frac{\varepsilon}{2}k_2\theta k_3\right)\frac{\tilde k_{1,\lambda}\tilde k_{1,\rho}\tilde k_{1,\mu}}{\varepsilon(\tilde k_1^2)^2}+\cos\left(\frac{\varepsilon}{2}k_1\theta k_3\right)\frac{\tilde k_{2,\lambda}\tilde k_{2,\rho}\tilde k_{2,\mu}}{\varepsilon(\tilde k_2^2)^2}+\cos\left(\frac{\varepsilon}{2}k_1\theta k_2\right)\frac{\tilde k_{3,\lambda}\tilde k_{3,\rho}\tilde k_{3,\mu}}{\varepsilon(\tilde k_3^2)^2}\right\},
\end{aligned}
\tag{A.71}
$$

since

$$
\begin{aligned}
&\Big\{\delta^{(4)}(x-y) \stackrel{\star}{,} \delta^{(4)}(x-r)\Big\} \\
&\quad = \iint \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\left(e^{-\frac{\mathrm{i}}{2}k_2\varepsilon\theta k_1} + e^{-\frac{\mathrm{i}}{2}k_1\varepsilon\theta k_2}\right) e^{-\mathrm{i}(k_1+k_2)x} e^{\mathrm{i}(k_1 r + k_2 y)} \\
&\quad = \iint \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\left(e^{+\frac{\mathrm{i}}{2}k_1\varepsilon\theta k_2} + e^{-\frac{\mathrm{i}}{2}k_1\varepsilon\theta k_2}\right) e^{-\mathrm{i}(k_1+k_2)x} e^{\mathrm{i}(k_1 r + k_2 y)} \\
&\quad = \iint \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\, 2\cos\left(\frac{\varepsilon}{2}k_1\theta k_2\right) e^{-\mathrm{i}(k_1+k_2)x} e^{\mathrm{i}(k_1 r + k_2 y)} \,. \qquad \text{(A.72)}
\end{aligned}
$$

We can solve the spacetime integral using (A.48). Therefore, (A.71) yields

$$
\begin{aligned}
\frac{\delta^3\Gamma^{(0)}_{\mathrm{tri},2}}{\delta A_\lambda(r)\delta A_\rho(y)\delta A_\mu(z)} &= 2\mathrm{i}g'\frac{1}{(2\pi)^8}\iiint d^4k_1\, d^4k_2\, d^4k_3\, \delta^{(4)}(k_1+k_2+k_3)\, e^{\mathrm{i}(k_1 r + k_2 y + k_3 z)} \\
&\quad \times \cos\left(\frac{\varepsilon}{2}k_1\theta k_2\right)\sum_{i=1}^{3}\frac{\tilde{k}_{i,\lambda}\tilde{k}_{i,\rho}\tilde{k}_{i,\mu}}{\varepsilon(\tilde{k}_i^2)^2} \,. \qquad \text{(A.73)}
\end{aligned}
$$

With the Fourier transform given in (A.43) we can conclude with

$$
\tilde{V}^{3A}_{\lambda\rho\mu}(k_1,k_2,k_3)\Big|_{\mathrm{counter}} = 2\mathrm{i}g'\,(2\pi)^4\delta^{(4)}(k_1+k_2+k_3)\,\cos\left(\frac{\varepsilon}{2}k_1\theta k_2\right)\sum_{i=1}^{3}\frac{\tilde{k}_{i,\lambda}\tilde{k}_{i,\rho}\tilde{k}_{i,\mu}}{\varepsilon(\tilde{k}_i^2)^2} \,. \qquad \text{(A.74)}
$$

Eq. (A.50) and (A.74) are representing the whole result for the 3A vertex of our model.

A.9 Damping-term contributions to the second identity

There are two contributions to the identity (4.46) we have to consider with respect to $\mathcal{L}^1_{\mathrm{br}}$. The first one is coming from the second and third term of our identity (4.46)

$$
\mathrm{i}g\left[\delta^{(4)}(z-y) \stackrel{\star}{,} \left.\frac{\delta^2\Gamma^{(0)}_{\mathrm{inv}}}{\delta A_\lambda(r)\delta A_\rho(z)}\right|_{\phi=0}\right] + \mathrm{i}g\left[\delta^{(4)}(z-r) \stackrel{\star}{,} \left.\frac{\delta^2\Gamma^{(0)}_{\mathrm{inv}}}{\delta A_\rho(y)\delta A_\lambda(z)}\right|_{\phi=0}\right] , \qquad \text{(A.75)}
$$

where we can use the result of (A.15)

$$
\frac{\delta^2\Gamma^{(0)}_{\mathrm{bi},2}}{\delta A_\rho(y)\delta A_\mu(z)} = -\frac{\gamma^4}{\tilde{\Box}^2_z}\left(\Box^z\delta_{\mu\rho} - \partial^z_\mu\partial^z_\rho + \rho\,\tilde{\partial}^z_\mu\tilde{\partial}^z_\rho\right)\delta^{(4)}(z-y) \,, \qquad \text{(A.76)}
$$

with (A.8)

$$
\Gamma^{(0)}_{\mathrm{bi},2} = \int d^4x\Big(\gamma^4\,\mathcal{L}^1_{\mathrm{br}}\Big) = \frac{\gamma^4}{4}\int d^4x\left(f_{\mu\nu}\frac{1}{\tilde{\Box}^2}f_{\mu\nu} + 2\rho\,\tilde{\partial}A\frac{1}{\tilde{\Box}^2}\tilde{\partial}A\right) . \qquad \text{(A.77)}
$$

Inserting (A.76) in (A.75) brings

$$\mathrm{i}g\left[\delta^{(4)}(z-y)\stackrel{\star}{,}\frac{\delta^2\Gamma^{(0)}_{\mathrm{bi},2}}{\delta A_\lambda(r)\delta A_\rho(z)}\bigg|_{\phi=0}\right]+\mathrm{i}g\left[\delta^{(4)}(z-r)\stackrel{\star}{,}\frac{\delta^2\Gamma^{(0)}_{\mathrm{bi},2}}{\delta A_\rho(y)\delta A_\lambda(z)}\bigg|_{\phi=0}\right]=$$
$$=-\mathrm{i}g\gamma^4\bigg\{\left[\delta^{(4)}(z-y)\stackrel{\star}{,}\frac{1}{\tilde{\Box}^2_z}\left(\Box^z\delta_{\lambda\rho}-\partial^z_\lambda\partial^z_\rho+\rho\,\tilde{\partial}^z_\lambda\tilde{\partial}^z_\rho\right)\delta^{(4)}(z-r)\right]$$
$$+\left[\delta^{(4)}(z-r)\stackrel{\star}{,}\frac{1}{\tilde{\Box}^2_z}\left(\Box^z\delta_{\lambda\rho}-\partial^z_\lambda\partial^z_\rho+\rho\,\tilde{\partial}^z_\lambda\tilde{\partial}^z_\rho\right)\delta^{(4)}(z-y)\right]\bigg\}. \tag{A.78}$$

This is the first result, but the above contributions are canceled by the part coming from

$$\frac{\delta^3}{\delta A_\lambda(r)\delta A_\rho(y)\delta c(z)}\left(\gamma^4\frac{\partial\Gamma^{(0)}_{\mathrm{inv}}}{\partial\bar{\chi}}\right)\bigg|_{\phi=0}=-\gamma^4\frac{\delta^3}{\delta A_\lambda(r)\delta A_\rho(y)\delta c(z)}\int d^4x\left(s\mathcal{L}^1_{\mathrm{br}}\right)\bigg|_{\phi=0} \tag{A.79}$$

in our identity (4.46), which is shown now. We can devide

$$(\text{A.79})=-\frac{\gamma^4}{4}\frac{\delta^3}{\delta A_\lambda(r)\delta A_\rho(y)\delta c(z)}\int d^4x\,s\left(f_{\mu\nu}\star\frac{1}{\tilde{\Box}^2}f_{\mu\nu}+2\rho\,\tilde{\partial}A\star\frac{1}{\tilde{\Box}^2}\tilde{\partial}A\right)\bigg|_{\phi=0} \tag{A.80}$$

into two parts and rewrite them in similar manner as in (A.17) and below. As a result, we get for the first part

$$\frac{\gamma^4}{2}\frac{\delta^3}{\delta A_\lambda(r)\delta A_\rho(y)\delta c(z)}\int d^4x\,s\left(A_\mu\star\frac{1}{\tilde{\Box}^2}\left(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu\right)A_\nu\right)\bigg|_{\phi=0}, \tag{A.81}$$

and for the second part

$$\frac{\rho\gamma^4}{2}\frac{\delta^3}{\delta A_\lambda(r)\delta A_\rho(y)\delta c(z)}\int d^4x\,s\left(A_\mu\star\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\tilde{\Box}^2}A_\nu\right)\bigg|_{\phi=0}. \tag{A.82}$$

We again use the BRST transformation (4.4) for A_μ, but now its bilinear part

$$sA_\mu=\partial_\mu c+\mathrm{i}g\left[c\stackrel{\star}{,}A_\mu\right]=...+\mathrm{i}g\left(c\star A_\mu-A_\mu\star c\right). \tag{A.83}$$

Starting with the first part, using partial integration and cylic permutation, one gets

$$(\text{A.81})=\mathrm{i}g\gamma^4\frac{\delta^3}{\delta A_\lambda(r)\delta A_\rho(y)\delta c(z)}\int d^4x\left(\left(c\star A_\mu-A_\mu\star c\right)\star\frac{1}{\tilde{\Box}^2}\left(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu\right)A_\nu\right)$$
$$=\mathrm{i}g\gamma^4\frac{\delta^2}{\delta A_\lambda(r)\delta A_\rho(y)}\int d^4x\;\delta^{(4)}(x-z)\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu}{\tilde{\Box}^2}A_\nu\left[A_\mu\stackrel{\star}{,}\frac{1}{\tilde{\Box}^2}\left(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu\right)A_\nu\right]$$
$$=\mathrm{i}g\gamma^4\frac{\delta^2}{\delta A_\lambda(r)\delta A_\rho(y)}\left[A_\mu\stackrel{\star}{,}\frac{1}{\tilde{\Box}^2_z}\left(\Box^z\delta_{\mu\nu}-\partial^z_\mu\partial^z_\nu\right)A_\nu\right]. \tag{A.84}$$

The same procedure gives for the second part

$$
\begin{aligned}
(\mathrm{A.82}) &= \mathrm{i}g\rho\gamma^4 \frac{\delta^3}{\delta A_\lambda(r)\delta A_\rho(y)\delta c(z)} \int d^4x \left((c \star A_\mu - A_\mu \star c) \star \frac{\tilde{\partial}_\mu \tilde{\partial}_\nu}{\tilde{\square}^2} A_\nu \right) \\
&= \mathrm{i}g\rho\gamma^4 \frac{\delta^2}{\delta A_\lambda(r)\delta A_\rho(y)} \int d^4x \; \delta^{(4)}(x-z) \star \left[A_\mu \overset{\star}{,} \frac{\tilde{\partial}_\mu \tilde{\partial}_\nu}{\tilde{\square}^2} A_\nu \right] \\
&= \mathrm{i}g\rho\gamma^4 \frac{\delta^2}{\delta A_\lambda(r)\delta A_\rho(y)} \left[A_\mu \overset{\star}{,} \frac{\tilde{\partial}^z_\mu \tilde{\partial}^z_\nu}{\tilde{\square}^2_z} A_\nu \right] . \qquad (\mathrm{A.85})
\end{aligned}
$$

Summarizing the results of (A.84) and (A.85) brings

$$
\begin{aligned}
(\mathrm{A.79}) &= \mathrm{i}g\gamma^4 \frac{\delta^2}{\delta A_\lambda(r)\delta A_\rho(y)} \left[A_\mu \overset{\star}{,} \frac{1}{\tilde{\square}^2_z} \left(\square^z \delta_{\mu\nu} - \partial^z_\mu \partial^z_\nu + \rho \tilde{\partial}^z_\mu \tilde{\partial}^z_\nu \right) A_\nu \right] \\
&= \mathrm{i}g\gamma^4 \frac{\delta}{\delta A_\lambda(r)} \Bigg\{ \left[\delta^{(4)}(z-r) \overset{\star}{,} \frac{1}{\tilde{\square}^2_z} \left(\square^z \delta_{\rho\nu} - \partial^z_\rho \partial^z_\nu + \rho \tilde{\partial}^z_\rho \tilde{\partial}^z_\nu \right) A_\nu \right] \\
&\qquad + \left[A_\mu \overset{\star}{,} \frac{1}{\tilde{\square}^2_z} \left(\square^z \delta_{\mu\rho} - \partial^z_\mu \partial^z_\rho + \rho \tilde{\partial}^z_\mu \tilde{\partial}^z_\rho \right) \delta^{(4)}(z-r) \right] \Bigg\} , \qquad (\mathrm{A.86})
\end{aligned}
$$

and, finally, the result

$$
\begin{aligned}
&\frac{\delta^3}{\delta A_\lambda(r)\delta A_\rho(y)\delta c(z)} \left(\gamma^4 \frac{\partial \Gamma^{(0)}_{\mathrm{inv}}}{\partial \bar{\chi}} \right) \Bigg|_{\phi=0} = \\
&= \mathrm{i}g\gamma^4 \Bigg\{ \left[\delta^{(4)}(z-r) \overset{\star}{,} \frac{1}{\tilde{\square}^2_z} \left(\square^z \delta_{\lambda\rho} - \partial^z_\lambda \partial^z_\rho + \rho \tilde{\partial}^z_\lambda \tilde{\partial}^z_\rho \right) \delta^{(4)}(z-y) \right] \\
&\qquad + \left[\delta^{(4)}(z-y) \overset{\star}{,} \frac{1}{\tilde{\square}^2_z} \left(\square^z \delta_{\lambda\rho} - \partial^z_\lambda \partial^z_\rho + \rho \tilde{\partial}^z_\lambda \tilde{\partial}^z_\rho \right) \delta^{(4)}(z-r) \right] \Bigg\} . \qquad (\mathrm{A.87})
\end{aligned}
$$

This is the same as in (A.78) but with opposite sign. So, the contribution given in (A.87) cancels the one in (A.78) in the vertex identity (4.46).

Appendix B

Calculations for the Linearized BRST Operator b

In this chapter we neglect the star product in the notation for simplification. Nonetheless, do not forget that it is always present and that we can always apply the properties of the star product, see Sect. (2.1), in our calculations.

Additionally, one has to take care of the fermionic character of some fields (e.g. c, $\bar{c}$ or Ω^A_μ), which always gives a factor (-1) when changing the order of two such fermionic fields.

B.1 The equation of motion for c

We start with the action (4.8)

$$\Gamma^{(0)}_{\text{inv}} = \int d^4x \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + s(\bar{c}\,\partial A) + s(\bar{\chi}\,\mathcal{L}^1_{\text{br}}) + s(\bar{\delta}\,\mathcal{L}^2_{\text{br}}) + \Omega^A_\mu s A_\mu + \Omega^c sc \right) , \tag{B.1}$$

and look for all the parts contributing to

$$b\Omega^c = \frac{\delta \Gamma^{(0)}_{\text{inv}}}{\delta c} , \tag{B.2}$$

where b of (4.60)

$$b = \int d^4x \left(\frac{\delta\Gamma}{\delta\Omega^A_\mu} \frac{\delta}{\delta A_\mu} + \frac{\delta\Gamma}{\delta A_\mu} \frac{\delta}{\delta\Omega^A_\mu} + \frac{\delta\Gamma}{\delta\Omega^c} \frac{\delta}{\delta c} + \frac{\delta\Gamma}{\delta c} \frac{\delta}{\delta\Omega^c} + B \frac{\delta}{\delta\bar{c}} \right) + \gamma^4 \frac{\partial}{\partial\bar{\chi}} + g' \frac{\partial}{\partial\bar{\delta}} , \tag{B.3}$$

has been used. The first term of (B.1) is just depending on A_μ and not on c. The second, fifth and sixth term contain the ghost field c and we can treat them in the following manner:

$$\begin{aligned} \frac{\delta}{\delta c(y)} \int d^4x\, s\,(\bar{c}\,\partial A) &= -\frac{\delta}{\delta c(y)} \int d^4x\,(\bar{c}\,\partial_\mu s A_\mu) = -\frac{\delta}{\delta c(y)} \int d^4x\,(\bar{c}\,\partial_\mu D_\mu c) \\ &= \frac{\delta}{\delta c(y)} \int d^4x\;(\partial_\mu \bar{c})\,(\partial_\mu c + \mathrm{i}g\,[c, A_\mu]) = (\partial_\mu(\partial_\mu \bar{c}) + \mathrm{i}g\,[(\partial_\mu \bar{c}), A_\mu])\,(y) = D_\mu \partial_\mu \bar{c}\,(y) , \end{aligned} \tag{B.4}$$

The above (and below) applied BRST transformations are given in (4.4) and (4.7). An important insight is that we can treat the covariant derivative D_μ in similar manner to a partial derivative ∂_μ where one just gets a factor -1 in partial integration.
The fifth and sixth term give

$$\frac{\delta}{\delta c(y)}\int d^4x\,\Omega^A_\mu sA_\mu = \frac{\delta}{\delta c(y)}\int d^4x\,\Omega^A_\mu D_\mu c = D_\mu\Omega^A_\mu(y) = \left(\partial_\mu\Omega^A_\mu + \mathrm{i}g\left[\Omega^A_\mu, A_\mu\right]\right)(y)\,, \quad \text{(B.5)}$$

$$\frac{\delta}{\delta c(y)}\int d^4x\,\Omega^c sc = \mathrm{i}g\frac{\delta}{\delta c(y)}\int d^4x\,\Omega^c cc = \mathrm{i}g\left[c,\Omega^c\right](y)\,. \quad \text{(B.6)}$$

Furthermore, we get contributions from the third and fourth term of the action (B.1):

$$\frac{\delta}{\delta c(y)}\int d^4x\, s\left(\bar\chi\mathcal{L}^1_{\mathrm{br}}\right) \quad \text{and} \quad \frac{\delta}{\delta c(y)}\int d^4x\, s\left(\bar\delta\mathcal{L}^2_{\mathrm{br}}\right)\,. \quad \text{(B.7)}$$

The left term gives

$$\begin{aligned}
\frac{\delta}{\delta c(y)}\int d^4x\, s\left(\bar\chi\mathcal{L}^1_{\mathrm{br}}\right) &= -\frac{\delta}{\delta c(y)}\int d^4x\,\bar\chi\left(s\mathcal{L}^1_{\mathrm{br}}\right)\\
&= \bar\chi\,\frac{\delta}{\delta c(y)}\int d^4x\, s\left(\frac{1}{4}f_{\mu\nu}\frac{1}{\tilde\Box^2}f_{\mu\nu} + \frac{\rho}{2}\,\tilde\partial A\frac{1}{\tilde\Box^2}\tilde\partial A\right)\\
&= \bar\chi\,\frac{\delta}{\delta c(y)}\int d^4x\, s\left(-\frac{1}{2}A_\mu\frac{1}{\tilde\Box^2}(\partial^2\delta_{\mu\nu}-\partial_\mu\partial_\nu)A_\nu - \frac{\rho}{2}A_\mu\frac{\tilde\partial_\mu\tilde\partial_\nu}{\tilde\Box^2}A_\nu\right)\\
&= -\frac{\bar\chi}{2}\,\frac{\delta}{\delta c(y)}\int d^4x\left\{D_\mu c, \frac{1}{\tilde\Box^2}\left(\partial^2\delta_{\mu\nu}-\partial_\mu\partial_\nu - \rho\,\tilde\partial_\mu\tilde\partial_\nu\right)A_\nu\right\}\\
&= \bar\chi\,\frac{\delta}{\delta c(y)}\int d^4x\, c\,\frac{D_\mu}{\tilde\Box^2}\left(\partial^2\delta_{\mu\nu}-\partial_\mu\partial_\nu - \rho\,\tilde\partial_\mu\tilde\partial_\nu\right)A_\nu\\
&= \bar\chi\frac{D_\mu}{\tilde\Box^2}\left(\partial^2\delta_{\mu\nu}-\partial_\mu\partial_\nu - \rho\,\tilde\partial_\mu\tilde\partial_\nu\right)A_\nu(y)\,.
\end{aligned} \quad \text{(B.8)}$$

In the first and second line we got a minus sign due to the fermionic character of c, $\bar\chi$ and the s operator. In the third line we used partial integration, as well as in the fourth line after letting s act on every A_μ, which brought the anticommutator. Due to cyclic permutation, this cancels the factor $\frac{1}{2}$.

Now, we can further simplify this result in the following manner

$$\text{(B.8)} = \bar\chi\frac{\partial_\mu}{\tilde\Box^2}\left(\partial^2\delta_{\mu\nu}-\partial_\mu\partial_\nu - \rho\,\tilde\partial_\mu\tilde\partial_\nu\right)A_\nu(y) + \mathrm{i}g\bar\chi\left[\frac{1}{\tilde\Box^2}\left(\partial^2\delta_{\mu\nu}-\partial_\mu\partial_\nu - \rho\,\tilde\partial_\mu\tilde\partial_\nu\right)A_\nu, A_\mu\right](y),$$

where the first part is zero due to the transversality of $\left(\partial^2\delta_{\mu\nu}-\partial_\mu\partial_\nu\right)$, and because of $\partial_\mu\tilde\partial_\mu = 0$. So we arrive at

$$\begin{aligned}
\frac{\delta}{\delta c(y)}\int d^4x\, s\left(\bar\chi\,\mathcal{L}^1_{\mathrm{br}}\right) &= -\mathrm{i}g\bar\chi\left[A_\mu, \frac{1}{\tilde\Box^2}\left(\partial^2\delta_{\mu\nu}-\partial_\mu\partial_\nu - \rho\tilde\partial_\mu\tilde\partial_\nu\right)A_\nu\right](y)\\
&= -\mathrm{i}g\bar\chi\left[A_\mu, \frac{1}{\tilde\Box^2}\left(\partial_\nu f_{\nu\mu} - \rho\tilde\partial_\mu\tilde\partial_\nu A_\nu\right)\right](y)\,.
\end{aligned} \quad \text{(B.9)}$$

The right term of (B.7) can be treated as follows:

$$\begin{aligned}
\frac{\delta}{\delta c(y)}\int d^4x\, s\left(\bar\delta\,\mathcal{L}^2_{\text{br}}\right) &= -\frac{\delta}{\delta c(y)}\int d^4x\,\bar\delta\left(s\mathcal{L}^2_{\text{br}}\right)\\
&= \bar\delta\,\frac{\delta}{\delta c(y)}\int d^4x\, s\left(\frac{1}{2}\left\{A_\mu, A_\nu\right\}\frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}A_\rho\right)\\
&= \bar\delta\,\frac{\delta}{\delta c(y)}\int d^4x\, s\left(A_\mu A_\nu\frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}A_\rho\right)\\
&= \bar\delta\,\frac{\delta}{\delta c(y)}\int d^4x\left(\left\{D_\mu c, A_\nu\right\}\frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}A_\rho + A_\mu A_\nu\frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}D_\rho c\right)\\
&= -\bar\delta\,\frac{\delta}{\delta c(y)}\int d^4x\left(c\,D_\mu\left\{A_\nu, \frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}A_\rho\right\} - A_\mu A_\nu\frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}\mathrm{i}g\left[c, A_\rho\right]\right)\\
&= -\bar\delta\,\frac{\delta}{\delta c(y)}\int d^4x\, c\left(D_\mu\left\{A_\nu, \frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}A_\rho\right\} + \mathrm{i}g\left[A_\rho, \frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}\left(A_\mu A_\nu\right)\right]\right)\\
&= -\bar\delta\left(D_\mu\left\{A_\nu, \frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}A_\rho\right\} + \mathrm{i}g\left[A_\rho, \frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}\left(A_\mu A_\nu\right)\right]\right)(y)\,. \qquad \text{(B.10)}
\end{aligned}$$

Hence, we can summarize all these results to

$$\begin{aligned}
b\Omega^c = \frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta c} &= D_\mu\partial_\mu\bar c + D_\mu\Omega^A_\mu + \mathrm{i}g\left[c, \Omega^c\right] - \mathrm{i}g\bar\chi\left[A_\mu, \frac{1}{\tilde\Box^2}\left(\partial_\nu f_{\nu\mu} - \rho\,\tilde\partial_\mu\tilde\partial_\nu A_\nu\right)\right]\\
&\quad - \bar\delta\left(D_\mu\left\{A_\nu, \frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}A_\rho\right\} + \mathrm{i}g\left[A_\rho, \frac{\tilde\partial_\mu\tilde\partial_\nu\tilde\partial_\rho}{\varepsilon\tilde\Box^2}\left(A_\mu A_\nu\right)\right]\right)\,. \qquad \text{(B.11)}
\end{aligned}$$

B.2 The equation of motion for A_μ

Here, we have to scan the action (B.1) for all parts contributing to

$$b\Omega^A_\mu = \frac{\delta\Gamma^{(0)}_{\text{inv}}}{\delta A_\mu}\,, \qquad \text{(B.12)}$$

and notice that all terms except the last one have to be considered. We start with the easy ones:

$$\frac{\delta}{\delta A_\mu(y)}\int d^4x\,\Omega^A_\nu sA_\nu = \mathrm{i}g\frac{\delta}{\delta A_\mu(y)}\int d^4x\,\Omega^A_\nu\left[c, A_\nu\right] = \mathrm{i}g\left\{\Omega^A_\mu, c\right\}(y)\,, \qquad \text{(B.13)}$$

where one has to respect the fermionic character of Ω^A_ν and c, which leads to the anticommutator. It is similar for the next part

$$\begin{aligned}
\frac{\delta}{\delta A_\mu(y)}\int d^4x\, s\left(\bar c\,\partial A\right) &= \frac{\delta}{\delta A_\mu(y)}\int d^4x\left(B\partial A - \bar c\,\partial_\nu D_\nu c\right)\\
&= -\partial_\mu B(y) + \mathrm{i}g\frac{\delta}{\delta A_\mu(y)}\int d^4x\left(\partial_\nu\bar c\right)\left[c, A_\nu\right] = \mathrm{i}g\left\{\partial_\mu\bar c, c\right\}(y) - \partial_\mu B(y)\,, \qquad \text{(B.14)}
\end{aligned}$$

where c and $\bar{c}$ are fermionic fields. Further, we have to derive (neglecting the argument y in the notation)

$$
\begin{aligned}
&\frac{\delta}{\delta A_\mu}\int d^4x\,\frac{1}{4}F_{\sigma\nu}F_{\sigma\nu} = \frac{\delta}{\delta A_\mu}\int d^4x\,\frac{1}{4}F_{\sigma\nu}\left(\partial_\sigma A_\nu - \partial_\nu A_\sigma + \mathrm{i}g\,[A_\nu, A_\sigma]\right)\\
&= -\frac{\delta}{\delta A_\mu}\int d^4x\,\frac{1}{4}\left((\partial_\sigma F_{\sigma\nu})A_\nu - (\partial_\nu F_{\sigma\nu})A_\sigma - \mathrm{i}g\,F_{\sigma\nu}\,[A_\nu, A_\sigma]\right)\\
&= \frac{\delta}{\delta A_\mu}\int d^4x\,\frac{1}{2}\left(A_\sigma(\partial_\nu F_{\sigma\nu}) + \mathrm{i}g\,A_\sigma A_\nu F_{\nu\sigma}\right)\\
&= \frac{\delta}{\delta A_\mu}\int d^4x\,\frac{1}{2}A_\sigma\left(\partial_\nu\partial_\sigma A_\nu - \partial^2 A_\sigma + \mathrm{i}g\,\partial_\nu\,[A_\nu, A_\sigma] + \mathrm{i}g\,A_\nu\left(\partial_\nu A_\sigma - \partial_\sigma A_\nu - \mathrm{i}g\,[A_\nu, A_\sigma]\right)\right)\\
&= \left(\partial_\nu\partial_\mu - \partial^2\delta_{\mu\nu}\right)A_\nu + \mathrm{i}g\left(\partial_\nu\,[A_\nu, A_\mu] + [\partial_\mu A_\nu, A_\nu] - [\partial_\nu A_\mu, A_\nu]\right) + (\mathrm{i}g)^2\left([[A_\nu, A_\mu]\,, A_\nu]\right)\\
&= \partial_\nu F_{\mu\nu} + \mathrm{i}g\,[F_{\mu\nu}, A_\nu] = D_\nu F_{\mu\nu}\,, \qquad \text{(B.15)}
\end{aligned}
$$

which is the expected result, analogous to $\partial_\nu f_{\mu\nu}$ in the case of an Abelian gauge field in commutative theories. Additionally, we need

$$
\frac{\delta}{\delta A_\mu(y)}\int d^4x\,s\left(\bar{\chi}\mathcal{L}^1_{\mathrm{br}}\right) \quad \text{and} \quad \frac{\delta}{\delta A_\mu(y)}\int d^4x\,s\left(\bar{\delta}\mathcal{L}^2_{\mathrm{br}}\right)\,. \qquad \text{(B.16)}
$$

The left term leads to

$$
\frac{\delta}{\delta A_\mu(y)}\int d^4x\,s\left(\bar{\chi}\mathcal{L}^1_{\mathrm{br}}\right) = \frac{\delta}{\delta A_\mu(y)}\int d^4x\left(\gamma^4\mathcal{L}^1_{\mathrm{br}} - \bar{\chi}s\mathcal{L}^1_{\mathrm{br}}\right)\,, \qquad \text{(B.17)}
$$

where we have to calculate

$$
\begin{aligned}
\frac{\delta}{\delta A_\mu(y)}\int d^4x\left(\gamma^4\mathcal{L}^1_{\mathrm{br}}\right) &= \gamma^4\frac{\delta}{\delta A_\mu(y)}\int d^4x\left(\frac{1}{4}f_{\sigma\nu}\frac{1}{\tilde{\square}^2}f_{\sigma\nu} + \frac{\rho}{2}\,\tilde{\partial}A\frac{1}{\tilde{\square}^2}\tilde{\partial}A\right)\\
&= \frac{\gamma^4}{\tilde{\square}^2}\left(\partial_\nu f_{\mu\nu} - \rho\,\tilde{\partial}_\mu\tilde{\partial}_\nu A_\nu\right)(y)\,, \qquad \text{(B.18)}
\end{aligned}
$$

and

$$
\begin{aligned}
-\frac{\delta}{\delta A_\mu(y)}\int d^4x\left(\bar{\chi}s\mathcal{L}^1_{\mathrm{br}}\right) &= -\bar{\chi}\,\frac{\delta}{\delta A_\mu(y)}\int d^4x\,s\left(\frac{1}{4}f_{\sigma\nu}\frac{1}{\tilde{\square}^2}f_{\sigma\nu} + \frac{\rho}{2}\,\tilde{\partial}A\frac{1}{\tilde{\square}^2}\tilde{\partial}A\right)\\
&= -\bar{\chi}\,\frac{\delta}{\delta A_\mu(y)}\int d^4x\,s\left(\frac{1}{2}A_\sigma\frac{1}{\tilde{\square}^2}\partial_\nu f_{\sigma\nu} - \frac{\rho}{2}\,A_\sigma\frac{\tilde{\partial}_\sigma\tilde{\partial}_\nu}{\tilde{\square}^2}A_\nu\right)\\
&= -\mathrm{i}g\bar{\chi}\,\frac{\delta}{\delta A_\mu(y)}\int d^4x\left([c, A_\sigma]\,\frac{1}{\tilde{\square}^2}\partial_\nu f_{\sigma\nu} - \rho\,[c, A_\sigma]\,\frac{\tilde{\partial}_\sigma\tilde{\partial}_\nu}{\tilde{\square}^2}A_\nu\right)\\
&= \mathrm{i}g\bar{\chi}\left[c, \frac{1}{\tilde{\square}^2}\left(\partial_\nu f_{\mu\nu} - \rho\,\tilde{\partial}_\mu\tilde{\partial}_\nu A_\nu\right)\right](y) + \frac{\mathrm{i}g\bar{\chi}}{\tilde{\square}^2}\left(\square\delta_{\mu\nu} - \partial_\mu\partial_\nu + \rho\,\tilde{\partial}_\mu\tilde{\partial}_\nu\right)[c, A_\nu]\,(y)\,, \qquad \text{(B.19)}
\end{aligned}
$$

where from the second to the third line we used $\partial_\mu\partial_\nu f_{\mu\nu} = 0$ (because of the antisymmetry of $f_{\mu\nu}$ and the permutability of ∂_μ and ∂_ν), or stated explicitly: $\partial_\mu\big(\partial_\mu\partial_\nu - \partial^2\delta_{\mu\nu}\big)A_\nu = 0$ (transverse projection operator with $\partial P_\perp = 0$). So we get for the left term of (B.16)

$$
\begin{aligned}
\frac{\delta}{\delta A_\mu(y)}\int d^4x\,s\left(\bar{\chi}\mathcal{L}^1_{\mathrm{br}}\right) = \frac{\gamma^4}{\tilde{\square}^2}\left(\partial_\nu f_{\mu\nu} - \rho\,\tilde{\partial}_\mu\tilde{\partial}_\nu A_\nu\right)(y) + \mathrm{i}g\bar{\chi}\left[c, \frac{1}{\tilde{\square}^2}\left(\partial_\nu f_{\mu\nu} - \rho\,\tilde{\partial}_\mu\tilde{\partial}_\nu A_\nu\right)\right](y)&\\
+ \frac{\mathrm{i}g\bar{\chi}}{\tilde{\square}^2}\left(\square\delta_{\mu\nu} - \partial_\mu\partial_\nu + \rho\,\tilde{\partial}_\mu\tilde{\partial}_\nu\right)[c, A_\nu]\,(y)\,.& \qquad \text{(B.20)}
\end{aligned}
$$

The right term of (B.16) reads

$$\frac{\delta}{\delta A_\mu(y)}\int d^4x\, s\left(\bar{\delta}\mathcal{L}^2_{\mathrm{br}}\right) = \frac{\delta}{\delta A_\mu(y)}\int d^4x\left(g'\mathcal{L}^2_{\mathrm{br}} - \bar{\delta}s\mathcal{L}^2_{\mathrm{br}}\right) . \tag{B.21}$$

Here, we need

$$\begin{aligned}\frac{\delta}{\delta A_\mu(y)}\int d^4x\left(g'\mathcal{L}^2_{\mathrm{br}}\right) &= g'\frac{\delta}{\delta A_\mu(y)}\int d^4x\left(A_\sigma A_\nu\frac{\tilde{\partial}_\sigma\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho\right)\\ &= g'\left\{A_\nu,\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho\right\}(y) - g'\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}\left(A_\rho A_\nu\right)(y)\,,\end{aligned} \tag{B.22}$$

and

$$\begin{aligned}&-\frac{\delta}{\delta A_\mu(y)}\int d^4x\left(\bar{\delta}s\mathcal{L}^2_{\mathrm{br}}\right) = -\bar{\delta}\,\frac{\delta}{\delta A_\mu(y)}\int d^4x\, s\left(A_\sigma A_\nu\frac{\tilde{\partial}_\sigma\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho\right)\\ &\quad= -\bar{\delta}\,\frac{\delta}{\delta A_\mu(y)}\int d^4x\left(\{D_\sigma c, A_\nu\}\,\frac{\tilde{\partial}_\sigma\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho + \mathrm{i}g\,A_\sigma A_\nu\frac{\tilde{\partial}_\sigma\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}\left[c, A_\rho\right]\right)\\ &\quad= -\bar{\delta}\,\frac{\delta}{\delta A_\mu(y)}\int d^4x\left(\{\partial_\sigma c + \mathrm{i}g\left[c, A_\sigma\right], A_\nu\}\,\frac{\tilde{\partial}_\sigma\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho + \mathrm{i}g\,A_\sigma A_\nu\frac{\tilde{\partial}_\sigma\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}\left[c, A_\rho\right]\right)\\ &\quad= \bar{\delta}\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}\left\{D_\rho c, A_\nu\right\}(y) - \bar{\delta}\left\{D_\rho c,\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho\right\}(y) + \mathrm{i}g\bar{\delta}\left[c,\left\{A_\nu,\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho\right\}\right](y)\\ &\quad\quad- \mathrm{i}g\bar{\delta}\left\{A_\nu,\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}\left[c, A_\rho\right]\right\}(y) + \mathrm{i}g\bar{\delta}\left[\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}(A_\rho A_\nu), c\right](y)\,,\end{aligned} \tag{B.23}$$

where the first term of the result is obtained by letting the functional derivative $\frac{\delta}{\delta A_\mu}$ act on A_ρ of the first term in the second or third line, respectively (after partial integration), followed by a renaming of σ to ρ. The second term comes from $\frac{\delta A_\nu}{\delta A_\mu}$ and the third one from the functional derivative $\frac{\delta(D_\sigma c)}{\delta A_\mu} = \mathrm{i}g\frac{\delta[c,A_\sigma]}{\delta A_\mu}$ in the anticommutator.

So, all in all, we achieve for this right term of (B.16)

$$\begin{aligned}\frac{\delta}{\delta A_\mu(y)}\int d^4x\, s\left(\bar{\delta}\mathcal{L}^2_{\mathrm{br}}\right) &= g'\left\{A_\nu,\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho\right\}(y) - g'\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}\left(A_\rho A_\nu\right)(y)\\ &+ \bar{\delta}\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}\left\{D_\rho c, A_\nu\right\}(y) - \bar{\delta}\left\{D_\rho c,\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho\right\}(y) + \mathrm{i}g\bar{\delta}\left[c,\left\{A_\nu,\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho\right\}\right](y)\\ &- \mathrm{i}g\bar{\delta}\left\{A_\nu,\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}\left[c, A_\rho\right]\right\}(y) + \mathrm{i}g\bar{\delta}\left[\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}(A_\rho A_\nu), c\right](y)\,.\end{aligned} \tag{B.24}$$

In our shorthand notation we always neglect the argument y for simplification.

Now, we can summarize all the terms contributing to the equation of motion for A_μ as

$$\begin{aligned}
b\Omega^A_\mu = \frac{\delta\Gamma^{(0)}_{\rm inv}}{\delta A_\mu} &= D_\nu F_{\mu\nu} + \mathrm{i}g\left\{\Omega^A_\mu, c\right\} + \mathrm{i}g\left\{\partial_\mu\bar{c}, c\right\} - \partial_\mu B + \frac{\gamma^4}{\tilde{\Box}^2}\left(\partial_\nu f_{\mu\nu} - \rho\,\tilde{\partial}_\mu\tilde{\partial}_\nu A_\nu\right) \\
&+ \mathrm{i}g\bar{\chi}\left[c, \frac{1}{\tilde{\Box}^2}\left(\partial_\nu f_{\mu\nu} - \rho\,\tilde{\partial}_\mu\tilde{\partial}_\nu A_\nu\right)\right] + \frac{\mathrm{i}g\bar{\chi}}{\tilde{\Box}^2}\left(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu + \rho\,\tilde{\partial}_\mu\tilde{\partial}_\nu\right)[c, A_\nu] \\
&+ g'\left\{A_\nu, \frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho\right\} - g'\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}\left(A_\rho A_\nu\right) + \bar{\delta}\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}\left\{D_\rho c, A_\nu\right\} \\
&- \bar{\delta}\left\{D_\rho c, \frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho\right\} + \mathrm{i}g\bar{\delta}\left[c, \left\{A_\nu, \frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho\right\}\right] \\
&- \mathrm{i}g\bar{\delta}\left\{A_\nu, \frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}[c, A_\rho]\right\} + \mathrm{i}g\bar{\delta}\left[\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}(A_\rho A_\nu), c\right]\,. \qquad \text{(B.25)}
\end{aligned}$$

This can be rewritten in the following way

$$\begin{aligned}
b\Omega^A_\mu = \frac{\delta\Gamma^{(0)}_{\rm inv}}{\delta A_\mu} &= -(1+\frac{\gamma^4}{\tilde{\Box}^2})(\Box\delta_{\mu\nu} - \partial_\nu\partial_\mu)A_\nu - \frac{\rho\gamma^4}{\tilde{\Box}^2}\tilde{\partial}_\mu\tilde{\partial}_\nu A_\nu + \mathrm{i}g\partial_\nu\left[A_\nu, A_\mu\right] + \mathrm{i}g\left[F_{\mu\nu}, A_\nu\right] \\
&+ \mathrm{i}g\left\{\Omega^A_\mu, c\right\} + \mathrm{i}g\left\{\partial_\mu\bar{c}, c\right\} + \partial_\mu B \\
&+ \mathrm{i}g\bar{\chi}\left[c, \frac{1}{\tilde{\Box}^2}\left(\partial_\nu f_{\mu\nu} - \rho\,\tilde{\partial}_\mu\tilde{\partial}_\nu A_\nu\right)\right] + \frac{\mathrm{i}g\bar{\chi}}{\tilde{\Box}^2}\left(\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu + \rho\,\tilde{\partial}_\mu\tilde{\partial}_\nu\right)[c, A_\nu] \\
&+ g'\left\{A_\nu, \frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho\right\} - g'\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}\left(A_\rho A_\nu\right) + \bar{\delta}\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}\left\{D_\rho c, A_\nu\right\} \\
&- \bar{\delta}\left\{D_\rho c, \frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho\right\} + \mathrm{i}g\bar{\delta}\left[c, \left\{A_\nu, \frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}A_\rho\right\}\right] \\
&- \mathrm{i}g\bar{\delta}\left\{A_\nu, \frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}[c, A_\rho]\right\} + \mathrm{i}g\bar{\delta}\left[\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\varepsilon\tilde{\Box}^2}(A_\rho A_\nu), c\right]\,. \qquad \text{(B.26)}
\end{aligned}$$

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