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DISSERTATION

# Stochastic Portfolio Theory from the Point of View of Risk Management

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*To Isabel and Victoria.*

# Deutschsprachige Kurzfassung


Die vorliegende Arbeit befasst sich mit dem Themengebiet der stochastischen Portfolio Theorie und Problemen und Fragestellungen, die in diesem Feld auftreten. Diese Theorie verfolgt einen deskriptiven Ansatz für das Langzeitverhalten von Preisen, der im Gegensatz zu klassischen Ansätzen in der Portfolioselektion steht, wo die explizite Kenntnis von Drifts und Nutzenfunktionen erforderlich ist. Eingangs wird ein Überblick über fundamentale Konzepte und Forschungsergebnisse in der Stochastischen Portfolio Theorie gegeben, die wesentlich durch die wegweisenden Arbeiten von Fernholz und Karatzas geprägt wurden ([43], [44], [46], [47], [48], [49], [50]).

Wir untersuchen in weiterer Folge das bemerkenswert stabile Verhalten zweier ökonometrischer Eigenschaften, nämlich die Kapitalverteilungsstruktur und die Dynamik der Entropie des Marktes sowie deren Reproduktion durch das verwendete Marktmodell. Mit dem Ziel, diese beiden Eigenschaften korrekt abzubilden, entwickeln wir ein Marktmodell ausgehend von einer bestimmten Klasse positiver affiner Prozesse. Unsere Intention ist es, sowohl das Verhalten der genannten Eigenschaften zu reproduzieren, als auch die beobachtbaren Korrelationsstrukturen in Aktienmärkten zu berücksichtigen - ein essentieller Aspekt aus Sicht des Risikomanagements. Im ersten Schritt werden unabhängige quadrierte Brownsche Bewegungen als Ausgangspunkt für ein Aktienmarkt-Modell verwendet, wobei wir zeigen, dass dieser Ansatz die beobachtbare Struktur der Kapitalverteilung sehr gut reproduziert. Im zweiten Schritt erweitern wir unser Modell um eine instantane Kovarianzstruktur auf Basis von GARCH(1,1) Volatilitätsschätzern und der Korrelationsstruktur der Marktdaten. Diese Vorgehensweise liefert eine im Zeitverlauf veränderliche Kovarianzstruktur und vermeidet explizite Elliptizitätsannahmen. Durch die Anwendung von Modellierungsansätzen aus der affinen Kategorie und durch eine adäquate Modellierung der Abhängigkeitsstrukturen zwischen einzelnen Aktien zeigen wir empirisch, dass es gelingt, die Struktur der Kapitalverteilung und die Dynamik der Marktentropie zusammen mit der allgemeinen Dynamik des Aktienmarktes gut zu reproduzieren.

Der entwickelte Modellansatz erweist sich überdies als geeignet für Anwendungen im Bereich des Risikomanagements. Der Vergleich des korrelierten Modells mit einem Modell auf Basis unabhängiger Aktien verdeutlicht, dass die Einbindung einer adäquaten Korrelationsstruktur zu einer deutlichen Verbesserung der Struktur des langfristigen Marktverhaltens führt und dass wir hierdurch ein realistisches Maß an Marktvolatilität erhalten. Diese Aspekte unterstreichen die Eignung des vorgeschlagenen korrelierten Marktmodells für Anwendungen im Risikomanagement, wie beispielsweise die Berechnung von Risikomaßen und längerfristigen Benchmark-Simulationen.

# Abstract

This work is dedicated to the field of stochastic portfolio theory and to a set of problems and questions arising therein. One of the most remarkable aspects of this theoretical setup is that a descriptive approach for long-term price behavior is pursued, in contrast to classical approaches to portfolio selection which depend on the knowledge of drift and utility functions. Initially, we provide an overview of the fundamental concepts of stochastic portfolio theory which are predominantly based on the seminal works by Fernholz and Karatzas ([43], [44], [46], [47], [48], [49], [50]).

Subsequently, we assess the stylized features of two econometric properties, observable in the context of stochastic portfolio theory, namely the reproduction of the capital distribution structure and of the dynamics of market entropy by the utilized market model, which exhibit remarkable stability over time. To the end of adequately accounting for these features, we develop a stock market model based on a special class of positive affine processes. The goal of our approach is to create a model which does not only reproduce the aforementioned stylized facts but which also permits to account for observed correlation structures which is an essential feature from the point of view of risk management. In the first step, we utilize independent squared Brownian Motions as basis for the equity market model and show that this approach indeed replicates the observable structure of the capital distribution very well. In the second step, we enhance our market model by endowing it with an instantaneous covariance structure which is based on GARCH(1,1) estimates for volatilities and on the correlation structure extracted from market data. This approach ensures a time-varying covariance structure and avoids any explicit assumptions on market ellipticity. By applying modeling approaches of the affine category and by adequately capturing inter-stock correlations we show empirically that we attain the goal of satisfyingly reproducing the structure of capital distribution and the dynamics of market entropy together with the general dynamics of the stock market.

The proposed modeling approach furthermore proves well-suited for risk management applications. The comparison of the correlated model with a model based on independent stocks illustrates that the incorporation of an adequate correlation structure clearly improves the long term market behavior and that one obtains a realistic degree of market volatility. These aspects render the proposed model well-suited for risk management applications such as the calculation of risk measures and long-term benchmark simulations.

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Vienna, $11^{th}$ of March, 2013 Florian Clemens Leisch

# Contents

# Chapter 1

# A Semimartingale Model for the Stock Market

## 1.1 Basic Concepts and Motivation

In the course of the last decade, the transfer of theoretical insights, models and trading strategies that stem from the vast field of Stochastic Portfolio Theory has become an important aspect of global financial markets. Notably, the implementation of quantitative trading strategies has changed the face of investment processes and also played a crucial role in the steep rise of the hedge fund industry. However, the turmoils in financial markets since 2007 also showed the limitations of automatic trading strategies, namely when the observed market behavior could not be captured by the implemented models and when the liquidity crunch in many crucial asset classes rendered portfolio rebalancing impossible.

Following classical models in stochastic portfolio theory as e.g. developed by Fernholz [43], [44], Fernholz and Karatzas [46], [47], [48], [49], [50] and Platen and Heath [86], [87] we will introduce a stock market model. Within this model, some approaches to building equity portfolios by means of generating functions will be discussed [44], [48]. In the inspiring works of Fernholz et al.[1] the main focus lay on the implementation and quality of such trading strategies and on the construction of relative arbitrage opportunities. The motivation for applying quantitative trading strategies generally is to create a portfolio which yields as high a performance as possible. Conversely one could also ask, where the risks and weaknesses of the generated portfolio lie, namely:

[1]See [43], [44], [46], [47], [48], [49], [50] and [51]

- What happens, when the model reaches its limits?
- Which risk measure reveals most insight in the different aspects of portfolio risk?
- Which types of portfolios exhibit a beneficial risk-return structure over the long term?

It is this family of questions that we want to draw more attention to and that will lie at the center of the ensuing considerations. But for now, let us start by introducing a logarithmic model for stocks, as it is outlined in [44]. Unless stated otherwise the sketched concepts and properties can be found in further detail in Fernholz [44]. For the following considerations let $(\Omega, \mathfrak{F}, P)$ be a filtered probability space, let furthermore

$$W = \{W(t) = (W_1(t), \dots, W_n(t)),\ n \in \mathbb{N},\ \mathfrak{F}_t,\ 0 \le t \le \infty\}$$

be an $n$-dimensional Brownian Motion (BM). The filtration $\{\mathfrak{F}_t\}_{0 \le t \le \infty}$ is the augmentation of the canonical filtration $\{\mathfrak{F}_t^W = \sigma(W(s));\ 0 \le s \le t\}$ w.r.t. $P$-null sets. Unless stated otherwise, we will use this filtration.

**Definition 1.1:** *Let us define the following basic properties*[2]*:*

1. *A process $\{X(t)\}_{0 \le t \le \infty}$ is called adapted to $\{\mathfrak{F}_t\}_{0 \le t \le \infty}$ if $X_t$ is $\mathfrak{F}_t$-measurable.*
2. *The cross variation process $\{\langle M, N\rangle_t\}_{0 \le t \le \infty}$ for two continuous semimartingales $M(\cdot)$ and $N(\cdot)$ is given by:*
$$\langle M, N\rangle_t = \lim_{n\to\infty} \sum_{i=0}^{2^n-1} \left( M\left(\frac{t(i+1)}{2^n}\right) - M\left(\frac{t(i)}{2^n}\right)\right) \cdot \left( N\left(\frac{t(i+1)}{2^n}\right) - N\left(\frac{t(i)}{2^n}\right)\right)$$
*The cross variation process is adapted, continuous and of bounded variation. Furthermore we denote by $\langle M\rangle_t = \langle M, M\rangle_t$ the quadratic variation of $M$.*
3. *The Brownian Motion process given above is a continuous, square-integrable martingale and the cross variation process equals $\langle W_i, W_j\rangle_t = \delta_{ij} t$, where $\delta_{ij}$ denotes the Dirac Delta.*

**Definition 1.2:** *For $n \in \mathbb{N}$ the stock price process $\{X(t)\}_{0 \le t \le \infty}$ is defined by the following Itô decomposition which is given in terms of the log-price*[3]*:*

$$d \log X(t) = \gamma(t)dt + \sum_{\nu=1}^{n} \xi_\nu(t) dW_\nu(t),\ \ t \in [0, \infty), \tag{1.1}$$

*where:*

[2]See e.g. Karatzas and Shreve [66], Øksendal [80] or Protter [88].
[3]See Fernholz [44], Definition 1.1.1. and Fernholz and Karatzas [48], Section 1.

- *$W = (W_1, \ldots, W_n) \ldots$ is an $n$-dimensional Brownian Motion.*
- *$\gamma$ is measurable and adapted, satisfying $\int_0^T |\gamma(t)|dt < \infty$.*
- *and $\xi_\nu$, $\nu = 1, \ldots, n$, are measurable and adapted, s.t.*
  1. $\int_0^T (\xi_1^2(t) + \ldots + \xi_n^2(t))dt < \infty$, $T \in [0, \infty)$ *a.s.*
  2. $\lim_{t\to\infty} t^{-1}(\xi_1^2(t) + \ldots + \xi_n^2(t)) \log\log t = 0$ *a.s.*
  3. $\xi_1^2(t) + \ldots + \xi_n^2(t) > 0$, $t \in [0, \infty)$ *a.s.*

The process $\gamma = \{\gamma(t)\}_{0\le t\le\infty}$ is the growth rate process of the stock. The processes $\{\xi_\nu(t)\}_{0\le t\le\infty}$ model the volatility direction of the stock price towards the $\nu$-th source of uncertainty $W_\nu$ at time $t$.

Let us furthermore assume that each company has one single share of stock outstanding, i.e. $X(t)$ represents the whole market capitalization of the company at time $t$. Now, integration of Equation 1.1 yields:

$$\log X(t) - \log X(0) = \int_0^t \gamma(s)ds + \int_0^t \sum_{\nu=1}^n \xi_\nu(s)dW_\nu(s), \;\; t \in [0, \infty) \tag{1.2}$$

Applying exp one obtains:

$$X(t) = X(0) \exp\left(\int_0^t \gamma(s)ds + \int_0^t \sum_{\nu=1}^n \xi_\nu(s)dW_\nu(s)\right), \;\; t \in [0, \infty) \tag{1.3}$$

**Remark 1.1:** Point 2 in Definition 1.2 has to be seen in the context of the law of the iterated logarithm[4]. In the version for a real valued local martingale $M$ the law of the iterated logarithm may be stated as:

$$\limsup_{t\to\infty} \frac{M(t)}{\sqrt{2\langle M\rangle_t \log\log\langle M\rangle_t}} = 1 \;\; a.s. \tag{1.4}$$

In our case we look at the local martingale part in the semimartingale characterization of the stock price process $X(t)$ in Equation 1.3, thus obtaining $M(t) = \sum_{\nu=1}^n \int_0^t \xi_\nu(s)dW_\nu(s)$ with quadratic variation $\langle M\rangle_t = \sum_{\nu=1}^n \int_0^t (\xi_\nu(s))^2 ds$. Hence one obtains:

$$\begin{aligned}
\limsup_{t\to\infty} \frac{M(t)}{\sqrt{\sum_{\nu=1}^n \int_0^t (\xi_\nu(s))^2 ds \;\log\log \sum_{\nu=1}^n \int_0^t (\xi_\nu(s))^2 ds}} &= \sqrt{2} \\
\limsup_{t\to\infty} \frac{\frac{M(t)}{t}}{\sqrt{\frac{\sum_{\nu=1}^n \frac{1}{t}\int_0^t (\xi_\nu(s))^2 ds \;\log\log t \sum_{\nu=1}^n \frac{1}{t}\int_0^t (\xi_\nu(s))^2 ds}{t}}} &= \sqrt{2}
\end{aligned}$$

[4]See Karatzas and Shreve [66], Theorem 2.9.23. or Revuz and Yor [89], Chapter II, Theorem 1.9.

Hence, approximating $\log\log\left(t\sum_{\nu=1}^{n}\frac{1}{t}\int_0^t(\xi_\nu(s))^2ds\right)\approx\log\log t$ one obtains:

$$\limsup_{t\to\infty}\frac{\frac{M(t)}{t}}{\sqrt{\frac{\sum_{\nu=1}^{n}\int_0^t(\xi_\nu(s))^2ds}{t^2}\log\log t}}=\sqrt{2} \tag{1.5}$$

And further approximating $\sum_{\nu=1}^{n}\frac{1}{t}\int_0^t(\xi_\nu(s))^2ds\approx\sum_{\nu=1}^{n}(\xi_\nu(s))^2$ yields:

$$\limsup_{t\to\infty}\frac{\frac{M(t)}{t}}{\sqrt{\frac{\sum_{\nu=1}^{n}(\xi_\nu(s))^2}{t}\log\log t}}=\sqrt{2} \tag{1.6}$$

If the denominator on the left hand side of (1.6) vanishes we will need the numerator on the left hand side of (1.6) to vanish as well, in order to satisfy the law of the iterated logarithm which is obtained by the following result in Fernholz[5], stating that for any continuous local martingale $M(t)$ satisfying

$$\lim_{t\to\infty}\frac{\langle M\rangle_t}{t^2}\log\log t=0 \ \ a.s., \tag{1.7}$$

it holds that:

$$\lim_{t\to\infty}\frac{M(t)}{t}=0 \ \ a.s. \tag{1.8}$$

This corresponds to numerator and denominator of the left hand side in Equation 1.5. Summing up, one observes that the long-run behavior of the stock as $t\to\infty$ will be dominated by the average growth rate of the stock whereas the local martingale component of $X(t)$ as given in Equation 1.3 will vanish on average as $t\to\infty$.

**Remark 1.2:** The setup outlined above is defined on $[0,\infty)$. Girsanov's Theorem[6], which would have made a finite time domain $[0,T]$ necessary[7], is not needed in the following results. Furthermore, some asymptotic considerations render the infinite domain necessary. Yet for all non-asymptotic considerations, the finite time domain may be used.

The process defined by Equation 1.1 obviously is a continuous semimartingale with bounded variation component $\gamma(t)dt$ and local martingale component $\sum_{\nu=1}^{n}\xi_\nu(t)dW_\nu(t)$.

Applying Itô's formula[8] for $X(t)=\exp\log X(t)$ and considering that $d\langle\log X,\log X\rangle_t=\sum_{\nu=1}^{n}\xi_\nu^2(t)dt$ one obtains the following SDE for $X(\cdot)$[9]:

[5] See Fernholz, [44], Lemma 1.3.2.
[6] See e.g. Revuz and Yor [89], Chapter VIII, Theorem 1.4.
[7] See Fernholz [44], Definition 1.1.1.
[8] See e.g. Karatzas and Shreve [66], Theorem 3.3.3. or Øksendal [80], Theorem 4.1.2.
[9] See Fernholz [44], Equation 1.1.3.

$$dX(t) = \underbrace{X(t)\left(\gamma(t) + \frac{1}{2}\sum_{\nu=1}^{n}\xi_\nu^2(t)\right)dt}_{(*)} + \underbrace{X(t)\sum_{\nu=1}^{n}\xi_\nu(t)dW_\nu(t)}_{(**)},\ \ t \in [0,\infty) \tag{1.9}$$

It is evident that $\{X_t\}_{t\geq 0}$ is a continuous semimartingale as well with bounded variation component $(*)$ and local martingale component $(**)$.

**Definition 1.3:** *Consequently one can define the rate of return process $\alpha = \{\alpha(t)\}_{0\leq t<\infty}$ as* [10]*:*

$$\alpha(t) = \gamma(t) + \frac{1}{2}\sum_{\nu=1}^{n}\xi_\nu^2(t). \tag{1.10}$$

Analogously to Equation 1.1 we can define a family of $n$ stocks $X_i$, $i = 1,\dots,n$, where each stock is defined by [11]:

$$d\log X_i(t) = \gamma_i(t)dt + \sum_{\nu=1}^{n}\xi_{i,\nu}(t)dW_\nu(t),\ \ t \in [0,\infty). \tag{1.11}$$

**Definition 1.4:** *Let the matrix-valued process $\xi$ be given by $\xi(t) = (\xi_{i,\nu}(t))_{1\leq i,\nu\leq n}$. Then the covariance process $\sigma$ is defined by*[12]*:*

$$\sigma(t) = \xi(t)\xi(t)^T. \tag{1.12}$$

*For $i = 1,\dots,n:\ \ \sigma_{ii}(\cdot)$ is the variance process of stock $i$.*

Since for all

$$x \in \mathbb{R}^n:\ \ x\sigma(t)x^T = x\xi(t)\xi(t)^T x^T = x\xi(t)\left(x\xi(t)\right)^T \geq 0,$$

it follows that $\sigma(t)$ is positive semi-definite for all $t \in [0,\infty)$[13]. Using this setup, the cross-variation process of two stocks $X_i$ and $X_j$ can be calculated following Equation 1.13. The cross-variation process for $\log X_i,\ \log X_j$ is related to $\sigma$ as follows [14]:

$$\sigma_{ij}(t)dt = d\langle \log X_i(t), \log X_j(t)\rangle = \sum_{\nu=1}^{n}\xi_{i,\nu}(t)\xi_{j,\nu}(t)dt\ \ a.s. \tag{1.13}$$

[10] See Fernholz [44], Equation 1.1.4.
[11] See Fernholz [44], Equation 1.1.6.
[12] See Fernholz [44], Section 1.1.
[13] See Fernholz [44], Section 1.1. and Fischer [53], Section 5.4.6.
[14] See Fernholz [44], Equation 1.1.9.

Let us now formally define a stock market in the modeling framework outlined above. We will present the definition of a stock market as well as some important properties which will be needed for several further results.

**Definition 1.5:** *A market*[15] *is a family of stocks* $\mathfrak{M} = \{X_1, \ldots, X_n\}$ *defined as before, s.t.* $\sigma(t)$ *is non-singular for all* $t \in [0, \infty)$ *a.s.*

*i.) The market* $\mathfrak{M}$ *is nondegenerate if there exists an* $\epsilon > 0$ *s.t.* $x\sigma(t)x^T \geq \epsilon\|x\|^2$, *for all* $x \in \mathbb{R}^n$, *for all* $t \in [0, \infty)$ *a.s.*

*ii.) The market* $\mathfrak{M}$ *is said to have bounded variance, if there exists an* $M > 0$ *such that* $x\sigma(t)x^T \leq M\|x\|^2$, $x \in \mathbb{R}^n$, $t \in [0, \infty)$ *a.s.*

It may be noted, that point i.) of Definition 1.5 constitutes a uniform constraint on the smallest eigenvalue of $\sigma$ which is a fairly strong assumption. The nondegeneracy condition of the market can also be seen as ellipticity condition on the market $\mathfrak{M}$. Thus we will use the notions nondegenerate market and elliptic market equivalently. Furthermore, an economic interpretation of the non-degeneracy of $\mathfrak{M}$ is that the portfolio variance cannot become smaller than $\epsilon$ for any normed vector of portfolio weights $y = \frac{x}{\|x\|}$ in long and short portfolios. Since in a nondegenerate market, all eigenvalues of $\sigma$ are strictly positive (in fact even bounded from below by $\epsilon$) the matrix $\sigma$ is not just positive semi-definite but positive definite for all $t \in [0, \infty)$.[16]

**Remark 1.3:** Note that in the model introduced in [44], the number of Brownian Motions used is taken to be the same as the number of stocks in the market. Hence a market with $n$ stocks is supposed to depend on $n$ sources of uncertainty driving the evolution of stock prices. This is also a necessary feature in order to obtain the desired ellipticity of the market $\mathfrak{M}$.

**Definition 1.6:** *A portfolio*[17] *in the market* $\mathfrak{M}$ *is a measurable, adapted vector-valued process* $\pi$, $\pi(t) = (\pi_1(t), \ldots, \pi_n(t))$, $t \in [0, \infty)$, *s.t.* $\pi$ *is a.s. bounded on* $[0, \infty)$ *and:*

$$\sum_{i=1}^{n} \pi_i(t) = 1, \ \ t \in [0, \infty) \ \ a.s.$$

$\pi_i \ldots$ *weight of the i-th stock.*

[15] See Fernholz [44], Definition 1.1.2.

[16] See Fernholz [44], Lemma 1.1.3. or Fischer [53], Section 5.7.3.

[17] See e.g. Fernholz [44], Definition 1.1.4 or Fernholz and Karatzas [48], Definition 1.1

Now, let $\pi$ be a portfolio and let $Z_\pi(t) > 0$ be the value of the given portfolio at time $t$. Furthermore using the fact, that the amount invested in the i-th stock is given by $\pi_i(t)Z_\pi(t)$ the instantaneous return of the portfolio can be expressed in terms of the instantaneous stock returns[18], provided, that our portfolio is self-financing. This self-financing condition should be explicitly included in the characterization of portfolios in Definition 1.6 since it is a central aspect of the outlined model.

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \sum_{i=1}^{n} \pi_i(t)\frac{dX_i(t)}{X_i(t)}, \tag{1.14}$$

i.e. the instantaneous portfolio return is the weighted sum of instantaneous stock returns. In Proposition 1.1 we will see, that the SDE for the portfolio value process can be written in a very intuitive way, using the portfolio weights and components.

**Proposition 1.1:** *Let $\pi$ be a self-financing portfolio in $\mathfrak{M}$, then $Z_\pi$ satisfies*[19]*:*

$$d\log Z_\pi(t) = \gamma_\pi(t)dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i,\nu}(t)dW_\nu(t), \tag{1.15}$$

*for $t \in [0, \infty)$ a.s., where:*

$$\gamma_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \underbrace{\frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{ij}(t)\right)}_{=:\gamma_\pi^*}. \tag{1.16}$$

*Whereby $\gamma_\pi(t)$ is called the portfolio growth rate and $\gamma_\pi^*$ is called the excess growth rate.*

The excess growth rate may be interpreted as half of the diversification benefit in terms of variance. It is noteworthy that this aspect connects the two perspectives of portfolio performance as influenced by the growth rate and the rate of return and of portfolio risk given by the standard deviation of portfolio returns (i.e. portfolio volatility). Thereby the given model for $\mathfrak{M}$ incorporates the effect that portfolio diversification will generally lead to a superior risk / return profile than single stock investment, a pattern which is readily observable in the market.

[18] See Fernholz [44], Equation 1.1.12.
[19] See Fernholz [44], Proposition 1.1.5.

*Proof:* We follow the proof as given in [44]. The process $Z_\pi$ is adapted, so we want to show that it satisfies Equation 1.14. By application of Itô's formula to $Z_\pi = \exp(\log(Z_\pi(t)))$, one obtains, that a.s. for $t \in [0, \infty)$:

$$dZ_\pi(t) = Z_\pi(t) d \log Z_\pi(t) + \frac{1}{2} Z_\pi(t) d\langle \log Z_\pi(t)\rangle_t,$$

hence,

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \gamma_\pi(t)dt + \frac{1}{2} d\langle \log Z_\pi\rangle_t + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i,\nu}(t)dW_\nu(t).$$

According to the nature of the cross-variation as depicted in Equation 1.13, we obtain a.s. for $t \in [0, \infty)$ that:

$$d\langle \log Z_\pi\rangle_t = \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{ij}(t)dt. \tag{1.17}$$

Using the Definition of

$$\gamma_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{ij}(t)\right),$$

one obtains:

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \left(\sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t)\right) dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)dW_\nu(t).$$

Furthermore $\sigma_{ii}(t) = \sum_{\nu=1}^{n} \xi_{i\nu}^2(t)$ (see [44]), hence Equation 1.9 implies that for $i = 1, \ldots, n$:

$$dX_i(t) = \underbrace{\left(\gamma_i(t) + \frac{1}{2}\sigma_{ii}(t)\right)}_{=\alpha_i(t)} X_i(t)dt + X_i(t)\sum_{\nu=1}^{n} \xi_{i\nu}(t)dW_\nu(t).$$

Therefore one obtains for $t \in [0, \infty)$, a.s., that:

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \sum_{i=1}^{n} \pi_i(t)\frac{dX_i(t)}{X_i(t)}.$$

□

It is evident, that the right hand side of Equation 1.17 corresponds to the usual definition of portfolio variance. Hence the process $\{\sigma_{\pi\pi}(t)\}_{t\geq 0}$ representing the portfolio variance[20] is given by:

$$\sigma_{\pi\pi}(t) = \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{ij}(t). \tag{1.18}$$

Thus, using Equations 1.17 and 1.18, one obtains that:

$$\langle \log Z_\pi \rangle_t = \int_0^t \sigma_{\pi\pi}(s)ds, \ \ t \in [0, \infty), \ \ a.s. \tag{1.19}$$

Following Equation 1.10 we write the portfolio rate of return as:

$$\alpha_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\alpha_i(t), \ \ t \in [0, \infty). \tag{1.20}$$

And finally, the process $Z_\pi$ can be described by [21]:

$$d \log Z_\pi(t) = \sum_{i=1}^{n} \pi_i(t) d \log X_i(t) + \gamma_\pi^*(t). \tag{1.21}$$

**Remark 1.4:** One may formulate the following relations between the evolution of $X_i(\cdot)$ and $Z_\pi(\cdot)$ and their respective growth rates[22]:

$$\lim_{T\to\infty} \frac{1}{T} \left( \log X_i(T) - \int_0^T \gamma_i(t)dt \right) = 0 \ \text{for all} \ \ X_i \in \mathfrak{M}. \tag{1.22}$$

This is valid in the case, that the stock variances $\sigma_{ii}$ do not increase too quickly (see [48]), e.g. if we have:

$$\lim_{T\to\infty} \left( \frac{\log \log T}{T^2} \int_0^T \sigma_{ii}(t)dt \right) = 0, \ \ a.s. \tag{1.23}$$

[20] See Fernholz [44], 1.1.18.
[21] See Fernholz [44], Corollary 1.1.6.
[22] See Fernholz [44], Prop. 1.3.1. and Corollary 1.3.3 as well as Fernholz and Karatzas [48], (1.6), (1.7), (1.14) and (1.15).

Hence the a.s. relationship stated in Equation 1.22 links the growth rates to the respective stock price process. The analogue for $Z_\pi(\cdot)$ is:

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_\pi(T) - \int_0^T \gamma_\pi(t)dt\right) = 0, \ \ a.s. \tag{1.24}$$

which is valid under the analogue of Equation 1.23, namely

$$\lim_{T\to\infty} \left(\frac{\log\log T}{T^2}\int_0^T \|\sigma(t)\|dt\right) = 0, \ \ a.s.; \tag{1.25}$$

This is satisfied if all eigenvalues of the covariance matrix $\sigma$ are bounded away from infinity, which is exactly the case when the market is of bounded variance as specified in Definition 1.5, i.e.

$$x^T\sigma(t)x = x^T\xi(t)\xi(t)^Tx \leq M\|x\|^2, \ \ \forall t\in[0,\infty), \ \ x\in\mathbb{R}^n \ \ and \ \ M\in(0,\infty) \ \ const. \tag{1.26}$$

In Fernholz and Karatzas [48], Equation 1.26 is also called the *uniform boundedness* condition on the volatility structure of $\mathfrak{M}$[23].

**Remark 1.5:** Dividends[24] can be incorporated into the model by defining the measurable, adapted dividend rate process $\delta$ which satisfies:

$$\int_0^t |\delta(s)|\, ds < \infty, \ \ t\in[0,\infty) \ a.s. \tag{1.27}$$

Hence for a given stock $X_i$ we can alternatively consider the total return process $\hat{X}_i$ which is defined as:

$$\hat{X}_i(t) = X(t)\exp\left(\int_0^t \delta(s)ds\right), \ \ t\in[0,\infty) \ a.s. \tag{1.28}$$

Thus $\hat{X}$ represents the value of an investment in $X$ with all dividends fully reinvested.

Any realistic model setup requires us to analyze portfolios w.r.t. a pre-defined benchmark portfolio.

**Definition 1.7:** *The notion of relative returns*[25] *may be defined as follows.*

- *The relative return of $X_i$ versus a portfolio $\eta$ is given by* $\log\left(\frac{X_i(t)}{Z_\eta(t)}\right)$.

[23] See Fernholz and Karatzas [48], (1.16.).
[24] See Fernholz [44], Definition 1.1.9
[25] See Fernholz [44], Section 1.2. and Fernholz and Karatzas [48], (1.19) - (1.21)

- $\sigma_{i\eta}(t) = \sum_{j=1}^{n} \eta_j(t)\sigma_{ij}(t)$; $\sigma_{\eta\eta}(t) := \sum_{i=1}^{n}\sum_{j=1}^{n} \eta_i(t)\sigma_{ij}(t)\eta_j(t)$;
- *The matrix valued relative covariance process* $\tau^{\eta}(t) = \left(\tau^{\eta}_{ij}(t)\right)_{1\le i,j\le n}$ *is given by:*

$$\tau^{\eta}_{ij}(t) = \sigma_{ij}(t) - \sigma_{i\eta}(t) - \sigma_{j\eta}(t) + \sigma_{\eta\eta}(t). \tag{1.29}$$

**Definition 1.8:** *The market portfolio*[26] $\mu = (\mu_1, \ldots, \mu_n)$ *is defined by:*

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \ldots + X_n(t)}, \tag{1.30}$$

*ist value process is given by* $S(t) = X_1(t) + \ldots + X_n(t)$.

The market portfolio is the canonical benchmark for any other asset in the market. It describes the development of the entire investment universe. The market weights $\mu_i(\cdot)$ represent the proportion of the market corresponding to the market capitalization of the $i$-th stock. Analogously to Definition 1.7 the relative return process of two portfolios $\pi$ and $\eta$ may be calculated as:

$$\log\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right), \quad t \in [0,\infty). \tag{1.31}$$

The following Proposition provides a useful result for the relative return process w.r.t. the market portfolio.

**Proposition 1.2:** *Let* $\pi \in \mathfrak{M}$. *Then the relative return process versus the market portfolio* $\mu$ *is given by*[27]*:*

$$d\log\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) = \sum_{i=1}^{n} \pi_i(t)d\log\mu_i(t) + \gamma^*_\pi(t)dt, \;\; a.s. \;\; for \;\; t \in [0,\infty). \tag{1.32}$$

*Proof:* See [44]: One may combine the fact that $\log\mu_i(t) = \log(X_i(t)/Z_\mu(t)),\;\; t \in [0,\infty)$ with

$$d\log\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) = \sum_{i=1}^{n} \pi_i(t)d\log(X_i(t)/Z_\eta(t)) + \gamma^*_\pi(t)dt, \tag{1.33}$$

for two portfolios $\pi$ and $\eta$. Setting $\eta = \mu$, one obtains the result.

□

[26] See e.g. Fernholz and Karatzas [48], Section 2.
[27] See Fernholz [44], Proposition 1.2.5.

The relative variance of a portfolio $\pi$ with respect to a certain other portfolio $\eta$ may be calculated directly by[28]

$$\tau^{\eta}_{\pi\pi}(t) = \pi(t)\tau^{\eta}(t)\pi(t)^T. \tag{1.34}$$

It clearly holds that the roles of $\pi$ and $\eta$ may be switched without changing their relative variance, i.e. $\tau^{\eta}_{\pi\pi}(t) = \tau^{\pi}_{\eta\eta}(t)$.

Now that the elementary properties of the stock market model have been outlined, in the next section we will further investigate the behavior of markets and the construction of portfolios therein.

## 1.2 Market Behavior and Portfolio-Generation

As in the previous section we will follow the model setup as outlined in Fernholz [44] or Fernholz and Karatzas [48]. Similar concepts have also been employed and described in Platen [86] and Platen and Heath [87]. We will now introduce the concepts of market coherence and diversity which will play a central role in the following results.

### 1.2.1 Market Coherence

**Definition 1.9:** *The market $\mathfrak{M}$ is said to be coherent*[29] *if for $i = 1, \ldots, n$*

$$\lim_{t\to\infty} t^{-1} \log \mu_i(t) = 0 \ \ a.s. \tag{1.35}$$

*Hence a market is coherent if none of the stocks declines too rapidly.*

The property of market coherence mandates that the market capitalization (i.e. market weight) of any stock will not decline exponentially or faster. This gives rise to a set of questions concerning the ability of the given model to capture default events. The default of a company would result in its market capitalization declining to virtually zero. Yet historical evidence suggests, that the market capitalization of defaulted companies will still be positive albeit small, reflecting the liquidation value of the company after the creditors have been satisfied. In a coherent market, it would be possible for a stock to vanish slowly. Although this kind of behavior is certainly possible in a classical diffusion-based model, it has to be remarked that a jump to default behavior taking place over a very short period of time may not be accommodated in a coherent setup.

[28]See Fernholz [44], Section 1.2.

[29]See Fernholz [44], Definition 2.1.1. and Fernholz and Karatzas [48], Remark 2.1.

**Proposition 1.3:** *Let $\mathfrak{M} = \{X_1, \dots, X_n\}$ be a market, then the following statements are equivalent*[30]*:*

**i.)** *$\mathfrak{M}$ is coherent.*

**ii.)** *For $i = 1, \dots, n$ $\lim_{T\to\infty} \frac{1}{T}\int_0^T(\gamma_i(t) - \gamma_\mu(t))dt = 0 \quad a.s.$*

**iii.)** *For $i, j = 1, \dots, n$ $\lim_{T\to\infty} \frac{1}{T}\int_0^T(\gamma_i(t) - \gamma_j(t))dt = 0 \;\; a.s.$*

*Proof:* We are following the proof given in [44], showing that i.) $\Rightarrow$ ii.) $\Rightarrow$ iii.) $\Rightarrow$ i.). Let us suppose that $\mathfrak{M}$ is coherent. Then Equation 1.35 states that for $i = 1, \dots, n$:

$$\lim_{T\to\infty} \frac{1}{T}\underbrace{(\log X_i(T) - \log Z_\mu(T))}_{=\log(\mu_i(t))} = 0, \;\; a.s.$$

According to the results stated in Remark 1.4 it holds that

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T) - \int_0^T \gamma_\mu(t)dt\right) = 0, \;\; a.s., \tag{1.36}$$

and

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_i(T) - \int_0^T \gamma_i(t)dt\right) = 0, \;\; a.s., \tag{1.37}$$

hence using the triangle inequality [31] one obtains:

$$\begin{gathered}\lim_{T\to\infty} \tfrac{1}{T}\left(\int_0^T \gamma_i(t)dt - \int_0^T \gamma_\mu(t)dt\right) = \\ \lim_{T\to\infty} \tfrac{1}{T}\left(\log X_i(T) - \log Z_\mu(T) - \log X_i(T) + \int_0^T \gamma_i(t)dt + \log Z_\mu(T) - \int_0^T \gamma_\mu(t)dt\right) \leq \\ \lim_{T\to\infty} \tfrac{1}{T}\left(\log X_i(T) - \log Z_\mu(T)\right) + \lim_{T\to\infty} \tfrac{1}{T}\left(\int_0^T \gamma_i(t)dt - \log X_i(T)\right) + \\ \lim_{T\to\infty} \tfrac{1}{T}\left(\log Z_\mu(T) - \int_0^T \gamma_\mu(t)dt\right) = 0 \;\; a.s.\end{gathered}$$

Conversely, it also holds that:

[30] See Fernholz [44], Proposition 2.1.2.
[31] See e.g. Fischer [53], Section 5.1.2.

$$0 = \lim_{T\to\infty} \tfrac{1}{T}(\log X_i(T) - \log Z_\mu(T)) \leq \lim_{T\to\infty} \tfrac{1}{T}\left(\int_0^T \gamma_i(t)dt - \int_0^T \gamma_\mu(t)dt\right) + \underbrace{\lim_{T\to\infty} \frac{1}{T}\left(\log X_i(T) - \int_0^T \gamma_i(t)dt\right)}_{\underset{T\to\infty}{\longrightarrow 0}} + \underbrace{\lim_{T\to\infty} \frac{1}{T}\left(-\log Z_\mu(T) + \int_0^T \gamma_\mu(t)dt\right)}_{\underset{T\to\infty}{\longrightarrow 0}},$$

and hence condition ii.) (see also [44]).

Condition iii.) may be obtained directly from condition ii.):

$$\lim_{T\to\infty} \tfrac{1}{T}\left(\int_0^T \gamma_i(t) - \gamma_j(t)dt\right) \leq \underbrace{\lim_{T\to\infty} \frac{1}{T}\left(\int_0^T \gamma_i(t) - \gamma_\mu(t)dt\right)}_{\underset{T\to\infty}{\longrightarrow 0}} + \underbrace{\lim_{T\to\infty} \frac{1}{T}\left(\int_0^T \gamma_\mu(t) - \gamma_j(t)dt\right)}_{\underset{T\to\infty}{\longrightarrow 0}} = 0,$$

and further:

$$\lim_{T\to\infty} \tfrac{1}{T}\left(\int_0^T \gamma_i(t) - \gamma_j(t)dt\right) \geq \underbrace{\lim_{T\to\infty} \frac{1}{T}\left(\int_0^T \gamma_i(t)dt - \log X_i(T)\right)}_{\underset{T\to\infty}{\longrightarrow 0}} + \underbrace{\lim_{T\to\infty} \frac{1}{T}\left(\log X_j(T) - \int_0^T \gamma_j(t)dt\right)}_{\underset{T\to\infty}{\longrightarrow 0}} = 0;$$

again using the triangle inequality [53] and Equation 1.37, hence one obtains:

$$\lim_{T\to\infty} \frac{1}{T}\left(\int_0^T \gamma_i(t)dt - \int_0^T \gamma_j(t)dt\right) = 0 \;\; a.s.$$

Now, let us suppose that iii.) holds. Equation 1.37 and condition iii.) imply that there is a subset $\Omega' \subset \Omega$ with $P(\Omega') = 1$ s.t. for $\omega \in \Omega'$:

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_i(T,\omega) - \int_0^T \gamma_i(t,\omega)dt\right) = 0, \tag{1.38}$$

and

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T (\gamma_i(t,\omega) - \gamma_j(t,\omega))dt = 0, \tag{1.39}$$

for $i, j = 1, \ldots, n$.

Choose $\omega \in \Omega'$. Then Equations 1.38 and 1.39 with $j = 1$ imply that for all $i$:

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_i(T,\omega) - \int_0^T \gamma_1(t,\omega)dt\right) = 0. \tag{1.40}$$

Hence, for $i = 1, \ldots, n$,

$$\lim_{T\to\infty} \frac{1}{T}\left(\max_{1\le i\le n}(\log X_i(T,\omega)) - \int_0^T \gamma_1(t,\omega)dt\right) = 0,$$

which is equivalent to

$$\lim_{T\to\infty} \frac{1}{T}\left(\log(\max_{1\le i\le n} X_i(T,\omega)) - \int_0^T \gamma_1(t,\omega)dt\right) = 0, \tag{1.41}$$

for $i = 1, \ldots, n$. Now, for $t \in [0,\infty)$,

$$X_1(t,\omega) \le \underbrace{X_1(t,\omega) + \ldots + X_n(t,\omega)}_{=Z_\mu(t,\omega)} \le n \max_{1\le i\le n} X_i(t,\omega),$$

hence, for $t \in [0,\infty)$,

$$\log X_1((t,\omega) \le \log Z_\mu(t,\omega) \le \log n + \log(\max_{1\le i\le n} X_i(t,\omega)). \tag{1.42}$$

Since

$$\lim_{T\to\infty} \frac{1}{T}\log n = 0,$$

it follows from Equations 1.40, 1.41 and 1.42 that:

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T,\omega) - \int_0^T \gamma_1(t,\omega)dt\right) = 0. \tag{1.43}$$

Through 1.40 and 1.43 one obtains

$$\lim_{T\to\infty} \frac{1}{T}(\log X_i(T,\omega) - \log Z_\mu(T,\omega)) = 0.$$

This holds for any $\omega \in \Omega'$, thus $\mathfrak{M}$ is coherent.

$\square$

The concept of market coherence may further be used to prove the fairly remarkable result stated below.

**Proposition 1.4:** [32] *Suppose that the market $\mathfrak{M}$ is nondegenerate and coherent, and that $\pi$ is a portfolio with constant weights, at least two of which are nonnegative. Then:*

$$\liminf_{T\to\infty} \frac{1}{T} \log\left(\frac{Z_\pi(T)}{Z_\mu(T)}\right) > 0, \quad a.s. \tag{1.44}$$

*Proof:* For the proof we refer to Fernholz [44]. The proof utilizes the coherence and non-degeneracy (ellipticity) of $\mathfrak{M}$ as well as several auxiliary Lemmata. □

According to Proposition 1.4 any constantly weighted portfolio satisfying the specified criteria in a coherent market will outperform the market portfolio as $T \to \infty$. This is a remarkable result which is especially astonishing considering its generality. The proof depends on the coherence of the market, hence in a non-coherent market it is not possible to maintain the result in this generality.

### 1.2.2 Market Diversity

**Definition 1.10:** *The market $\mathfrak{M}$ is diverse*[33] *if there exists a $\delta > 0$ s.t.:*

$$\mu_{max}(t) \leq 1 - \delta, \ \ t \in [0, \infty) \ \ a.s. \tag{1.45}$$

*$\mathfrak{M}$ is weakly diverse on $[0, T]$ if there exists a $\delta > 0$ s.t.:*

$$\frac{1}{T}\int_0^T \mu_{max}(t)dt \leq 1 - \delta \ \ a.s. \tag{1.46}$$

**Proposition 1.5:** [34] *Let $\mathfrak{M}$ be a non-degenerate and diverse market, then there exists some $\delta > 0$ such that the excess growth rate of the market portfolio may be bounded from below.*

$$\gamma_\mu^*(t) \geq \delta, \ \ t \in [0, \infty). \tag{1.47}$$

[32] See Fernholz, [44], Proposition 2.1.9.

[33] See Fernholz [44], Definition 2.2.1, Fernholz and Karatzas [48], Section 5., Fernholz, Karatzas and Kardaras [51], Section 4; the concept of market diversity is introduced in Fernholz [43] together with some of the results also given in [44].

[34] See Fernholz [44], Proposition 2.2.2.

*Proof:* We follow the proof as given in Fernholz [44]. Since $\mathfrak{M}$ is non-degenerate and diverse, there exists a $\delta > 0$ such that

$$\mu_{max}(t) \leq 1 - \delta, \; for\, all\, t \in [0, \infty), \; a.s.$$

Furthermore, by the non-degeneracy of $\mathfrak{M}$, Lemma 2.1.7. in Fernholz [44] implies that

$$\gamma_\mu^*(t) \geq \epsilon (1 - \mu_{max}(t))^2, \; for\, all\, t \in [0, \infty), \; a.s.$$

By these two notions, one directly obtains (1.47). □

The diversity of a stock market may be discussed using classical measures for the disorder of systems. One approach outlined in Fernholz [44] is to use the entropy[35] function as a canonical measure for market diversity. The entropy function (aka. Shannon entropy) was introduced by Shannon[36] in 1948 to measure the uniformity of the distribution of signal transmission probabilities. The properties of the entropy function render it an ideal candidate both to make statements about the dispersion of market weights and to approach our core topic of functional generation of portfolios. The entropy function is given by:

$$\mathbf{S}(x) = - \sum_{i=1}^{n} x_i \log x_i, \tag{1.48}$$

for all $x \in \Delta^n = \{x \in \mathbb{R}^n : x_1 + \ldots + x_n = 1; 0 < x_i < 1, i = 1, \ldots, n\}$.

**Definition 1.11:** *Let $\mu$ be the market portfolio, then the market entropy*[37] *process $\boldsymbol{S}(\mu)$ is defined by:*

$$\boldsymbol{S}(\mu(t)) = - \sum_{i=1}^{n} \mu_i(t) \log(\mu_i(t)), \;\; t \in [0, T]. \tag{1.49}$$

$\mathbf{S}(\mu(\cdot))$ is a continuous semimartingale and $0 < \mathbf{S}(\mu(t)) \leq \log n$ for all $t \in [0, T]$ *a.s.*

**Proposition 1.6:** *The market $\mathfrak{M}$ is diverse if and only if there is an $\epsilon > 0$ s.t.*[38]*:*

$$\boldsymbol{S}(\mu(t)) > \epsilon, \;\; t \in [0, T] \;\; a.s. \tag{1.50}$$

*Proof:* The idea of the proof given in [43] and [44] depends on the continuity of $\mathbf{S}$ on the closure of $\Delta^n$:

$$\overline{\Delta}^n = \{x \in \mathbb{R}^n : x_1 + \ldots + x_n = 1; 0 \leq x_i \leq 1, i = 1, \ldots, n\}.$$

[35] See Fernholz [44], Section 2.3.
[36] See Shannon [92], Theorem 2.
[37] See Fernholz [43], Definition 4.1 or [44], Definition 2.3.1.
[38] See Fernholz [43], Proposition 4.1. or [44], Proposition 2.3.2.

The function $\mathbf{S}$ is nonnegative on the compact set $\overline{\Delta}^n$, and $\mathbf{S}(x) = 0$ only on the vertices. If a neighborhood of the vertices in $\overline{\Delta}^n$ is deleted, then $\mathbf{S}$ is bounded away from 0 on the rest of $\overline{\Delta}^n$.

□

The market entropy function may be used directly to build a portfolio $\pi$ which also has some fairly remarkable properties relative to the market portfolio. We will introduce the entropy-weighted portfolio at this point and revisit it in Example 1.1. We shall call the portfolio $\pi$ *entropy-weighted portfolio*[39] if its portfolio weights are given by:

$$\pi_i(t) = \frac{-\mu_i(t)\log\mu_i(t)}{\mathbf{S}(\mu(t))}, \quad t \in [0, T]. \tag{1.51}$$

It is straightforward to see that $\sum_{i=1}^{n} \pi_i(t) = 1$ and furthermore that the $\pi_i$ are bounded if the market is diverse. This holds since the numerator of (1.51) is bounded from above by $e^{-1}$ which is the maximum of the function $f(x) = -x \log x$ and the denominator of (1.51) is bounded from below by some $\epsilon > 0$ if the market is diverse according to Proposition 1.6. Moreover it is noteworthy that the ratio $\pi_i(t)/\mu_i(t)$ is decreasing in $\mu_i(t)$, hence the entropy-weighted portfolio overweighs small cap stocks and underweighs large cap stocks relative to the market. By using a technical result[40] provided in Fernholz [44] one obtains that the following limit vanishes:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left( \gamma_\pi(t) - \gamma_\mu(t) - \frac{\gamma_\mu^*(t)}{\mathbf{S}(\mu(t))} \right) dt = 0, \quad a.s. \tag{1.52}$$

Thus according to Equation 1.52 the average growth rate of the entropy-weighted portfolio $\gamma_\pi$ will be larger than the average growth rate of the market portfolio $\gamma_\mu$ as $T \to \infty$, provided that the market is diverse. In fact an even stronger result is stated as a Corollary in Fernholz [44] which we shall restate as a Lemma at this point.

**Lemma 1.1:** [41] *Let $\mu$ be the market portfolio and $\pi$ be the entropy-weighted portfolio in a diverse and non-degenerate (elliptic) market $\mathfrak{M}$. Then for a sufficiently large number $T$:*

$$\frac{Z_\pi(T)}{Z_\pi(0)} > \frac{Z_\mu(T)}{Z_\mu(0)}, \quad a.s. \tag{1.53}$$

[39] See Fernholz [44], Definition 2.3.3.
[40] See Fernholz [44], Theorem 2.3.4.
[41] See Fernholz [44], Corollary 2.3.5.

*Proof:* The proof in [44] uses the technical result mentioned before[42]. By this result one obtains:

$$\begin{aligned}\log\left(\frac{Z_\pi(T)}{Z_\pi(0)}\right) &= \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right)+\log\left(\frac{\mathbf{S}(\mu(T))}{\mathbf{S}(\mu(0))}\right)+\int_0^T\frac{\gamma_\mu^*(t)}{\mathbf{S}(\mu(t))}dt\\ &> \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right)+\underbrace{\log\delta_1-\log\log n+\frac{T\delta_2}{\log n}}_{(*)}.\end{aligned}$$

This holds because:

- $\mathbf{S}(\mu(T)) > \delta_1$ by Proposition 1.6.
- $\mathbf{S}(\mu(t)) \leq \log n$ by the definition of the entropy function, $t \in [0,T]$.
- $\gamma_\mu^*(t) \geq \delta_2$ by Inequality 1.47.

We are looking for a bound $T$ after which the entropy-weighted portfolio outperforms the market, hence we want $(*)$ to be positive. By this one obtains the desired lower bound on the time $T$:

$$T > \frac{1}{\delta_2}\log n\left(\log\log n-\log\delta_1\right). \tag{1.54}$$

□

The essential question raised by Lemma 1.1 clearly is how large this time bound $T$ is going to be, a result which may be obtained directly from Inequality 1.54. From the proof of Inequality 1.47 given in Fernholz[43] one obtains that $\delta_2 = \epsilon\delta^2$ whereby $\epsilon$ is the lower bound on the smallest eigenvalue of $\sigma$ as mandated by the non-degeneracy of $\mathfrak{M}$ and $\delta > 0$ is the bound used in Definition 1.10 for market diversity. The value for $n$ is clearly given by the number of stocks in the market which is known. So the only remaining question is to find a suitable $\delta_1 > 0$ as lower bound for the market entropy.

We will assess the behavior of the entropy on the closure of the simplex $\overline{\Delta}^n = \{x \in \mathbb{R}^n | \sum_{i=1}^n x_i = 1;\ 0 \leq x_i \leq 1;\ i = 1,\dots,n\}$ in a diverse market $\mathfrak{M}$. If we consider the entropy function in two dimensions it can be seen, that the highest degree of concentration (i.e. the lowest values for $\mathbf{S}$ will be obtained for $x = 1-\delta$ and $y = \delta$.

[42]See Fernholz [44], Theorem 2.3.4. This Theorem states that for the market portfolio and the entropy portfolio it holds a.s. for $t \in [0,T]$ that: $d\log\mathbf{S}(\mu(t)) = d\log(Z_\pi/Z_\mu(t)) - \frac{\gamma_\mu^*(t)}{\mathbf{S}(\mu(t))}dt$.

[43]See Fernholz [44], Proof of Proposition 2.2.2.

Hence the portfolio entropy in this case would be:

$$\mathbf{S}(x, y) = -\delta \log \delta - (1 - \delta) \log(1 - \delta) \geq -\delta \log \delta.$$

Since both terms in the above equation are positive and $-(1 - \delta) \log(1 - \delta)$ will be small for small $\delta$ we may discard this term and the inequality is actually strict. Hence the idea is to use:

$$\mathbf{S}(\mu(t)) \geq -\delta \log \delta = \delta_1. \tag{1.55}$$

We have seen that this works in two dimensions, thus we shall use an induction[44] argument in order to extend this notion to $n$ dimensions. Let us suppose that (1.55) holds for $n - 1$ dimensions. We are only interested in those points where the highest amount of concentration is observed, i.e. where $\mu_k = 1 - \delta$, $\mu_l = \delta$ and $\mu_i = 0$ for all $i \neq k, l$. Hence one may distinguish the following cases:

***(i.)*** $\mu_n = 0$: hence the $n$-dimensional entropy function $\mathbf{S}_n(\mu_1, \ldots, \mu_{n-1}, 0)$ is equal to the $n-1$-dimensional entropy $\mathbf{S}_{n-1}(\mu_1, \ldots, \mu_{n-1})$ by the definition of the entropy function and $\lim_{x \to 0} -x \log x = 0$.

***(ii.)*** $\mu_n = 1 - \delta$ and $\mu_n = \delta$: in this case we can exploit the fact that the entropy function is symmetric under re-ordering of the components of the vector $\mu$; hence we may simply reorder $\mu$ in a way that the $n$-th component is again 0 and reduce this case to the one stated in *(i.)*.

Consequently we will use the bound $\delta_1 = -\delta \log \delta$ for $\delta > 0$ is determined by the degree of requested market diversity. The nature of the numbers involved in Inequality 1.54 suggest, that the resulting time $T$ will be fairly large. Figure 1.1 illustrates the behavior of the timebound depending on $\delta$ in a market consisting of $n = 4$ stocks in which the smallest eigenvalue of the covariance matrix is of order $10^{-5}$.

[44] See e.g. Walter [100], Section A.2.2.

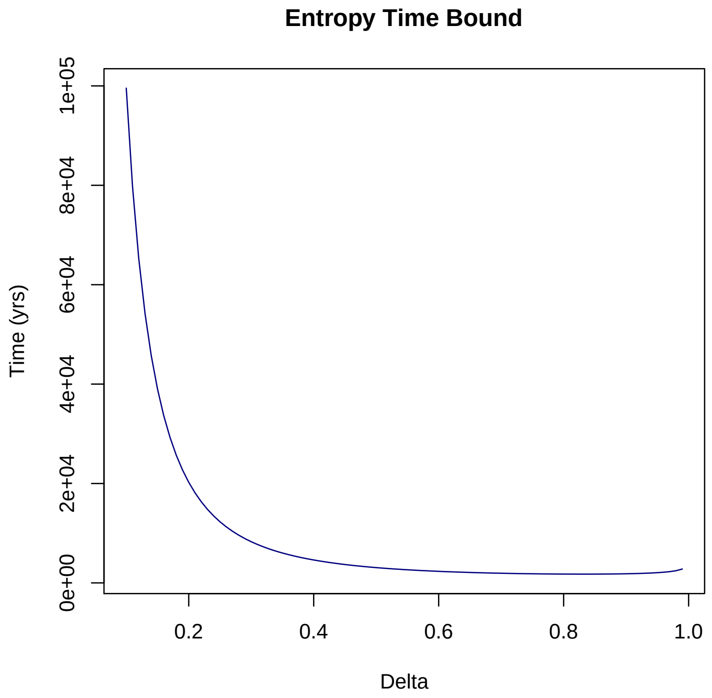


Figure 1.1: Timebound of the Entropy-Weighted Portfolio dependant on $\delta$ (Yrs.).

It is evident that the obtained finite time bound is immensely large since it is dominated by the term $1/\delta_2$, and in fact for most values of $\delta$ the time bound is in excess of 2000 years. Nonetheless other simulations indicate, that the entropy-weighted portfolio is still an attractive investment alternative relative to the market portfolio. In Figure 1.2 we show the relative returns of the entropy-weighted portfolio versus the market portfolio for 5000 simulated paths with investment horizon one year (i.e. 250 trading days). We see that in slightly more than half of the simulations, the entropy-weighted portfolio outperforms the market with - at least partly - handsome returns.

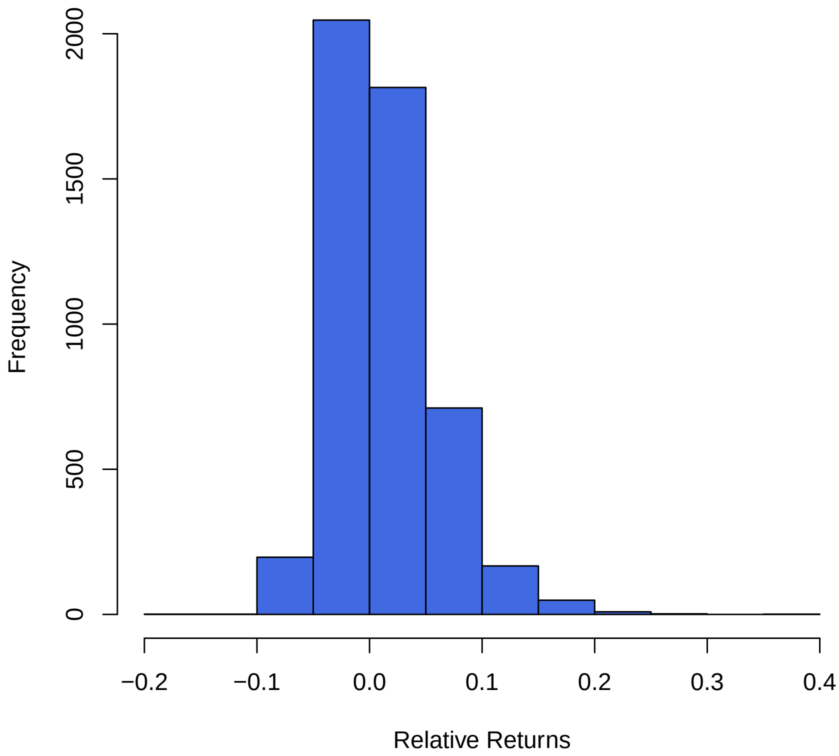


Figure 1.2: Relative Returns of the Entropy-Weighted Portfolio vs. the Market.

The concept of utilizing the entropy function for the generation of portfolios will be discussed in further detail in the following Section 1.2.3 together with some practical examples.

### 1.2.3 Functionally Generated Portfolios

The central idea which will be presented in this section is to use certain, real-valued functions on $\Delta^n$(or $\overline{\Delta}^n$) to generate portfolios. Through the tool of portfolio-generating functions it is possible to mathematically formulate various patterns of investment strategies which can also be found in the classical, discretionary investment process. Hence, what renders the concept of functionally-generated portfolios so attractive is the fact, that it permits us to formalize and quantify many investment concepts. The entropy-weighted portfolio (see also Example 1.1) and its behavior w.r.t. the market portfolio is discussed in some detail in [43]. Portfolio-generating functions are comprehensively covered in [44] and [48].

**Definition 1.12:** [45] *Let $\boldsymbol{S}$ be a positive continuous function on $\Delta^n$ and let $\pi$ be a portfolio. Then $\boldsymbol{S}$ generates $\pi$ if there exists a measurable process of bounded variation $\Theta$ s.t.:*

$$\log \frac{Z_\pi(t)}{Z_\mu(t)} = \log \boldsymbol{S}(\mu(t)) + \Theta(t), \quad t \in [0,T]. \tag{1.56}$$

The process $\Theta$ is the drift process corresponding to $\mathbf{S}$. Since $\log \frac{Z_\pi(t)}{Z_\mu(t)}$ and $\log \mathbf{S}(\mu(t))$ are continuous and adapted, hence $\Theta$ is continuous and adapted. Since $\Theta$ is of bounded variation, $\log \mathbf{S}(\mu)$ is a continuous semi-martingale, as a consequence one can write Equation 1.56 in differential form as:

$$d \log \left( \frac{Z_\pi(t)}{Z_\mu(t)} \right) = d \log \mathbf{S}(\mu(t)) + d\Theta(t), \quad t \in [0,T]. \tag{1.57}$$

**Example 1.1:** *The entropy-weighted portfolio*[46] *is given via the generating function:*

$$\boldsymbol{S}(x) = -\sum_{i=1}^{n} x_i \log x_i.$$

*The entropy-weighted portfolio has the portfolio weights:*

$$\pi_i(t) = -\frac{\mu_i(t) \log(\mu_i(t))}{\boldsymbol{S}(\mu(t))}.$$

*The drift process satisfies:*

$$d\Theta(t) = \frac{\gamma_\mu^*(t)}{\boldsymbol{S}(\mu(t))} dt.$$

The growth rates of $\pi$, $\mu$ are related to the drift process $\Theta$ by the relation stated in the Proposition below.

[45] See Fernholz [44], Definition 3.1.1. or Fernholz and Karatzas [48], Section 11.

[46] See Fernholz [43], Section 2.3. and [44], Example 3.1.2.

**Proposition 1.7:** *Let $\boldsymbol{S}$ generate portfolio $\pi$ with drift $\Theta$ and suppose that*[47]*:*

$$\lim_{t\to\infty}\frac{1}{t}\log \boldsymbol{S}(\mu(t))=0 \ \ a.s. \tag{1.58}$$

*Then*

$$\lim_{T\to\infty}\frac{1}{T}\left(\int_0^T\gamma_\pi(t)dt-\int_0^T\gamma_\mu(t)dt-\Theta(T)\right)=0 \ \ a.s. \tag{1.59}$$

*Proof:* From Equation 1.56 one obtains [44]: $\log\mathbf{S}(\mu(T))+\Theta(T)=\log Z_\pi(T)-\log Z_\mu(T)$, thus through Equation 1.15 in Proposition 1.1 we obtain the following representation:

$$\log\mathbf{S}(\mu(T))+\Theta(T)=\int_0^T(\gamma_\pi(t)-\gamma_\mu(t))dt+\int_0^T\sum_{i,\nu=1}^n(\pi_i(t)-\mu_i(t))\xi_{i\nu}(t)dW_\nu(t). \tag{1.60}$$

If we apply $\lim_{T\to\infty}T^{-1}$ to both sides of Equation 1.60, $\log\mathbf{S}(\mu(T))$ vanishes by Equation 1.58 and so does the second integral on the right hand side[48]. This yields the result stated above.

□

**Theorem 1.1:** *Let $\boldsymbol{S}$ be a positive $C^2$ function defined on a neighborhood $U$ of $\Delta^n$ s.t. for all $i=1,\dots,n$ $x_i\frac{\partial}{\partial x_i}\log\boldsymbol{S}(x)$ is bounded on $\Delta^n$. Then $\boldsymbol{S}$ generates the portfolio $\pi$ with weights*[49]*:*

$$\pi_i(t)=\left(\frac{\partial}{\partial\mu_i}\log\boldsymbol{S}(\mu(t))+1-\sum_{j=1}^n\mu_j(t)\frac{\partial}{\partial\mu_j}\log\boldsymbol{S}(\mu(t))\right)\mu_i(t), \tag{1.61}$$

*for $t\in[0,T]$ and $i=1,\dots,n$ and with a drift process $\Theta$ s.t. a.s. for $t\in[0,T]$:*

$$d\Theta(t)=\frac{-1}{2\boldsymbol{S}(\mu(t))}\sum_{i,j=1}^n\frac{\partial^2}{\partial\mu_i\partial\mu_j}\boldsymbol{S}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)dt. \tag{1.62}$$

[47] See Fernholz [44], Proposition 3.1.3.
[48] See Fernholz [44], Lemma 1.3.2.: if $M$ is a continuous local martingale s.t. $\lim_{t\to\infty}t^{-2}\langle M\rangle_t\log\log t=0$ a.s., then it holds that $\lim_{t\to\infty}t^{-1}M(t)=0$ a.s.
[49] See Fernholz [44], Theorem 3.1.5 and Fernholz and Karatzas [48], Section 11.

*Proof:* A proof for this theorem can e.g. be found in [44].

□

In the central theorem stated above, we consider generating functions living on an open neighborhood of $\Delta^n$, $U \subset \mathbb{R}^n$, and $\mathbf{S} \in C^2$.

**N.b.**: The weights $\pi_i$ only depend on the market weights $\mu_i$, the covariance structure enters only in the relative covariance term $\tau_{ij}(t)$ in the expression for $d\Theta(t)$.

**Example 1.2: *Price-to-Book Ratio*[50]**

*Let $b_i > 0$ be the book value of the i-th company, $b_i$ constant. Then the price-to-book ratio at time $t$ is $\frac{X_i(t)}{b_i}$. In Fernholz [44] this ratio is substituted by $\frac{\mu_i(t)}{b_i}$ which shows a comparable behavior and is more amenable to analysis under the introduced framework.*
*This ratio distinguishes growth stocks from value stocks. The weighted average PtB ratio is given by:*

$$\sum_{i=1}^{n} \frac{\mu_i^2(t)}{b_i}, \;\; t \in [0, T].$$

*The function:*

$$\boldsymbol{S}(x) = \left( \sum_{i=1}^{n} \frac{x_i^2}{b_i} \right)^{1/2}$$

*generates $\pi$ with weights:*

$$\pi_i(t) = \frac{\mu_i(t)^2}{b_i \boldsymbol{S}^2(\mu(t))}, \;\; t \in [0, T].$$

*Relative to the market, the portfolio $\pi$ overweighs growth stocks and underweighs value stocks. The drift process follows :*

$$d\Theta(t) = -\gamma_\pi^*(t), \;\; t \in [0, T].$$

Generating functions may also be combined as follows [44]: Let $\mathbf{S}_1, \ldots, \mathbf{S}_k$ generate portfolios $\pi_1, \ldots, \pi_k$ respectively. Then for constants $p_1, \ldots, p_k$ such that $p_1 + \ldots + p_k = 1$ the function:

$$\mathbf{S} = \mathbf{S}_1^{p_1} \mathbf{S}_2^{p_2} \cdots \mathbf{S}_k^{p_k} \tag{1.63}$$

[50] See Fernholz [44], Example 3.1.10

generates a portfolio with weights:

$$\pi_i = p_1\pi_{1i} + \ldots + p_k\pi_{ki}, \;\; i = 1, \ldots, n, \tag{1.64}$$

whereby $\pi_{hi}$ gives the weight of the i-th stock $x_i$ in the h-th portfolio $\pi_h$.

**Definition 1.13:** *A positive $C^2$ function defined on an open neighborhood of $\Delta^n$ is a measure of diversity*[51] *if it is symmetric and concave. A portfolio generated by a measure of diversity is called a diversity-weighted portfolio.*

The following functions are (among others) used to create portfolios in [44] where various simulation results are outlined. In order to examine the quality of results in a real world setting, we examined some generating function when applied to the real development of stocks. As market we use the universe of the MSCI European Monetary Union (EMU) equity index and we consider the movements of the featured stocks between August 2006 and September 2007. N.b. in the following plots, portfolio generating functions are applied to real time-series data and not to paths simulated via the Itô decomposition given in (1.1).

1. Entropy Function $\mathbf{S}(x) = -\sum_{i=1}^n x_i \log x_i$.
2. For $0 < p < 1$ let $D_p(x) = (\sum_{i=1}^n x_i^p)^{1/p}$. This generates a portfolio $\pi$ with: $\pi_i(t) = \frac{\mu_i^p(t)}{(D_p(\mu(t)))^p}, \;\; t \in [0, T]$ for $i = 1, \ldots, n$; for $p \to 1$, $\pi$ approaches the market portfolio. The (increasing) drift process is given by $d\Theta(t) = (1-p)\gamma_\pi^*(t)dt, \;\; t \in [0, T]$.
3. A 'normalized' version of $D_p$ is given by $\tilde{D}_p(x) = (n^{p-1} \sum_{i=1}^n x_i^p)^{1/p}$.

The charts shown in Figure 1.3 illustrate the performance of portfolios generated by $D_p$ with parameters $p = 0.2, 0.4, 0.6$ and $0.8$ versus the market portfolio (black).

[51] See Fernholz [44], Definition 3.4.1.

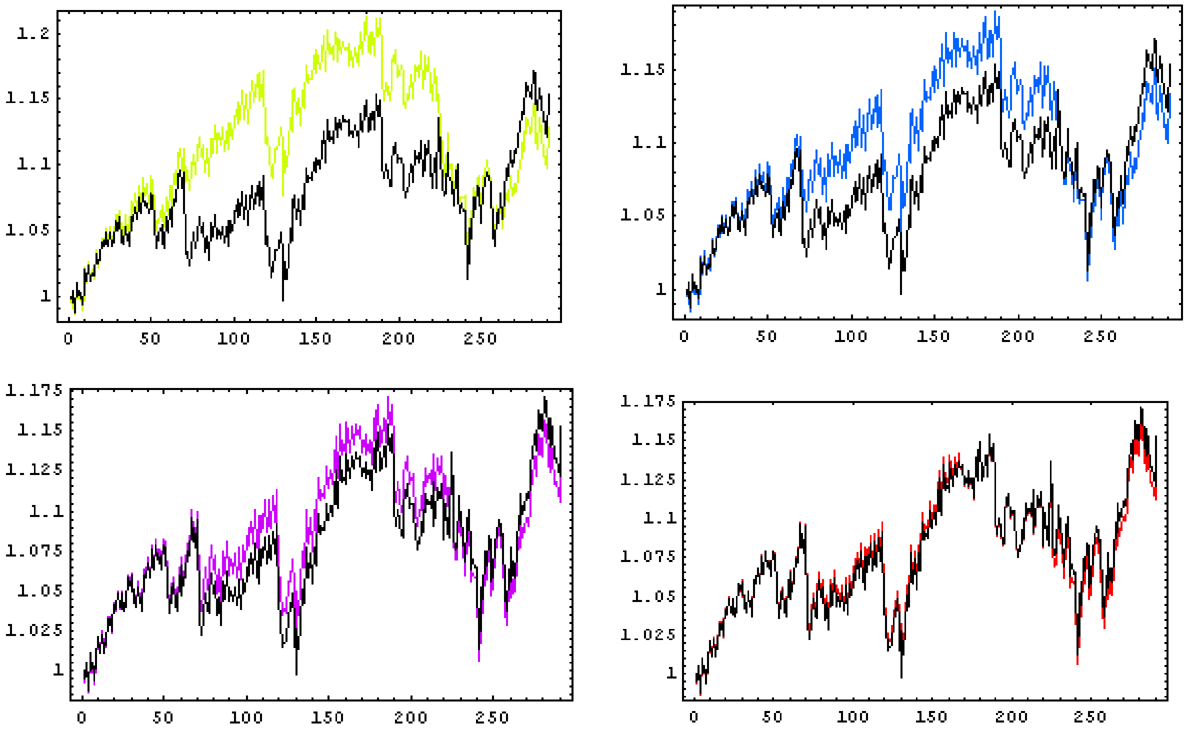


Figure 1.3: Diversity-Weighted Portfolios vs. Market.

Figure 1.3 illustrates, how the $D_p$-generated portfolios converge towards the market portfolio as $p \to 1$. By means of this strategy one may construct portfolios which are overweighted in small cap stocks and underweighted in large cap stocks to an extent which is predefined by the parameter $p$. The merits and drawbacks of this approach are quite evident. Small cap stocks tend to outperform large caps in certain, exceptionally bullish market conditions, however this chance is generally coupled with higher volatility and downside risks. Furthermore, the relative outperformance of small cap stocks versus large cap stocks partly reflects the higher market liquidity risk which a small cap investment bears. The implied liquidity premium for equities has been analyzed in some detail in Fernholz and Karatzas [47]. This effect is illustrated quite well by the utilized data set which comprises pre-crisis market data from 2006 as well as data from Autumn 2007 when some stress at least in niches of the financial market was already observable. This period in the data set corresponds to the rightmost part of the charts in Figure 1.3, when the diversity-weighted portfolios start to underperform the market.

A similar pattern can be observed in the behavior of the entropy-weighted portfolio versus the market. This little example illustrates one of the most attractive features of stochastic portfolio theory, namely that it is possible to obtain portfolio weights and investment decisions based on clearly defined rules without having to undergo any optimization routines. The fact that at least some of the portfolio generating functions presented in Fernholz [44] have been successfully applied to construct institutional investment products underlines the merits of this approach. Another attractive feature of the theoretic framework developed by Fernholz and Karatzas[52] is the possibility to mathematically formalize classical investment approaches like for instance taking decisions based on the price-to-book ratio as outlined in Example 1.2, thus rendering these portfolio strategies amenable to mathematical analysis. In Chapters 4 and 5 we will add to these results some further considerations from the perspective of risk management and related applications.

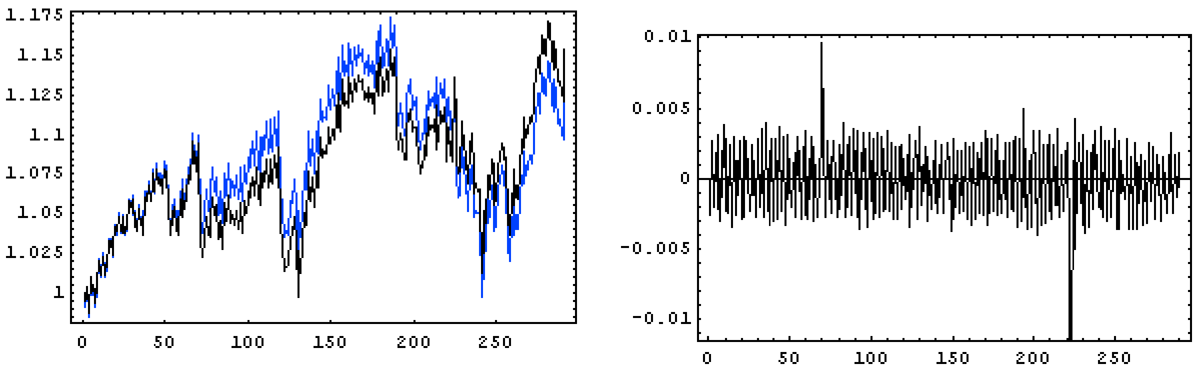


Figure 1.4: Chart and Rel. Returns of Entropy vs. Market.

# 1.3 Relative Arbitrage and Dominating Portfolios

## 1.3.1 Dominating Portfolios and Relative Arbitrage Opportunities

One of the most remarkable aspects of the model outlined in Sections 1.1 and 1.2 is the existence of arbitrage under certain circumstances. For a detailed discussion of the mathematical foundations and concepts of arbitrage we refer to the seminal works of Delbaen and Schachermayer, e.g. [28], [29], [30] or [31].

In our semimartingale model for the equity market it is possible to construct relative arbitrage opportunities in the sense of dominating portfolios. For the following considerations

[52] In this vein we particularly refer to [43], [44], [46], [47], [48], [49], [50].

it will be necessary to impose some restrictions on equity portfolios as defined in Definition 1.6 (see e.g. [44], [51]).

**Definition 1.14:** *A portfolio $\pi$ is called admissible*[53] *if:*

1. $\pi_i(t) \geq 0, \;\; t \in [0,T]$ *for* $\;\; i = 1, \ldots, n.$
2. $\exists \; c > 0$ *s.t.*
$$\hat{Z}_\pi(t)/\hat{Z}_\pi(0) \geq c\hat{Z}_\mu(t)/\hat{Z}_\mu(0), \;\; t \in [0,T] \;\; a.s.$$
3. $\exists \; M$ *s.t. for* $i = 1, \ldots, n$
$$\pi_i(t)/\mu_i(t) \leq M, \;\; t \in [0,T] \;\; a.s.$$

The three conditions stated in Definition 1.14 ensure that short sales are prohibited (1), furthermore that negative performance versus the market portfolio as numéraire is limited (2) and lastly that arbitrary overweighting of one single stock vs. the market is prohibited (3) (see [44]). In the next step we will define domination of one portfolio vs. another.

**Definition 1.15:** *Given two portfolios $\eta$ and $\pi$, we say that $\eta$ dominates*[54] *$\pi$ in the time interval $[0,T]$ if:*

$$\hat{Z}_\eta(T)/\hat{Z}_\eta(0) \geq \hat{Z}_\pi(T)/\hat{Z}_\pi(0) \;\; a.s. \tag{1.65}$$

*and*

$$P\left[\hat{Z}_\eta(T)/\hat{Z}_\eta(0) > \hat{Z}_\pi(T)/\hat{Z}_\pi(0)\right] > 0. \tag{1.66}$$

*If*

$$\hat{Z}_\eta(T)/\hat{Z}_\eta(0) > \hat{Z}_\pi(T)/\hat{Z}_\pi(0) \;\; a.s., \tag{1.67}$$

*then $\eta$ strictly dominates $\pi$.*

Alternatively, in the above setup one may call $\eta$ a relative arbitrage opportunity with respect to $\pi$ if the notions (1.65) and (1.66) prevail and a strict relative arbitrage opportunity in the case of (1.67).[55] Furthermore, in Fernholz and Karatzas [49] the notion of a portfolio

[53] See Fernholz [44], Definition 3.3.1.
[54] See Fernholz [44], Definition 3.3.2.
[55] See Fernholz and Karatzas [49], Section 6. or [46], Definition 2.1.

$\eta$ constituting a superior long-term growth opportunity relative to $\pi$ is introduced, namely that:[56]

$$\mathcal{L}_{\eta,\pi} := \liminf_{T\to\infty} \frac{1}{T} \log\left(\frac{\hat{Z}_\eta(T)}{\hat{Z}_\pi(T)}\right) > 0 \; a.s. \tag{1.68}$$

Now it is intuitively clear that the existence of two admissible portfolios $\eta$ and $\pi$, where $\eta$ strictly dominates $\pi$ yields an arbitrage opportunity [44]. We can start by investing 1 \$ in portfolio $\eta$ at time 0 and by short-selling 1 \$ in portfolio $\pi$. At time $T$ the value of our investment is as follows. The value of the long leg of this strategy is given by:

$$v_L = \hat{Z}_\eta(T)/\hat{Z}_\eta(0). \tag{1.69}$$

On the other hand, due to our short position in $\pi$ we owe

$$v_S = \hat{Z}_\pi(T)/\hat{Z}_\pi(0). \tag{1.70}$$

At time $T$, it holds with positive probability that $v_L > v_S$. Hence, starting with initial wealth 0 through this strategy one can obtain a terminal wealth $> 0$ with positive probability. Consequently the strategy sketched above constitutes an arbitrage opportunity (see [44]).

In a market $\mathfrak{M}$ which is defined as above and thereby permitting certain forms of arbitrage one may further obtain the following results. Let us therefore consider the market price of risk process $\theta : [0,\infty) \times \Omega \to \mathbb{R}^n$ which is given by[57]

$$\theta(t) = \xi^{-1}(t)\alpha(t), \; \forall\, 0 \leq t \leq T \; and \; \int_0^T \|\theta(t)\|^2 dt < \infty \; a.s. \tag{1.71}$$

By means of $\theta(\cdot)$ one may now define the following process $H(\cdot)$ as[58]

$$H(t) := \exp\left(-\int_0^t \theta^T(s) dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right), \; 0 \leq t < \infty, \tag{1.72}$$

[56] See Fernholz and Karatzas [49], Definition 6.1.

[57] See e.g. Karatzas and Shreve [67], Theorem 1.4.2. N.b. that in Equation 1.71 we only use the rate of return process $\alpha(\cdot)$, assuming zero dividend- and interest rate. The market price of risk in a general setup considering a non-zero dividend rate $\delta(\cdot)$ and risk-free interest rate $r(\cdot)$ is given by $\theta(t) = \xi^{-1}(t)(\alpha(t) + \delta(t) - r(t)\mathbf{1})$, where $\mathbf{1}$ is the vector with every entry equal to one.

[58] See e.g. Karatzas and Shreve [67], Definition 1.5.1.

as well as the shifted Brownian Motion $\tilde{W}(\cdot)$:[59]

$$\tilde{W}(t) := W(t) + \int_0^t \theta(s)ds,\ 0 \leq t < \infty. \tag{1.73}$$

$H(\cdot)$ is an exponential local martingale and a supermartingale, a martingale if and only if $E(H(t)) = 1$.[60] In the case that the market $\mathfrak{M}$ is of bounded variance and that there exists a relative arbitrage opportunity for two portfolios on a certain time horizon $[0,T]$, then the process $H(\cdot)$ is a strict local martingale, $E(H(T)) < 1$.[61] Furthermore, in this context, no equivalent martingale measure can exist in the market $\mathfrak{M}$ if we use the canonical filtration.[62]

By straightforward calculation, one obtains that

$$\begin{aligned} H(t)X_i(t) &= X_i(0)\exp\left(\int_0^t \underbrace{\gamma_i(s)}_{=\alpha_i(s)-\frac{1}{2}\sum_{\nu=1}^n \xi_{i\nu}^2(s)} ds + \int_0^t \sum_{\nu=1}^n \xi_{i\nu}(s)dW_\nu(s)\right) \\ &\quad \exp\left(-\int_0^t \theta^T(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right) \\ &= X_i(0)\exp\left(-\int_0^t \sum_{\nu=1}^n (\theta_\nu(s) - \xi_{i\nu}(s))dW_\nu(s) - \frac{1}{2}\int_0^t \sum_{\nu=1}^n (\theta_\nu(s) - \xi_{i\nu}(s))^2 ds\right) \\ &= X_i(0)\exp\left(-\int_0^t (\kappa_i(s))^T dW(s) - \frac{1}{2}\int_0^t \|\kappa_i(s)\|^2 ds\right), \end{aligned}$$

where the $\nu^{th}$ component of $\kappa_i(\cdot)$ is given by $\kappa_{i,\nu}(t) = \theta_\nu(t) - \xi_{i\nu}(t)$ for $\nu = 1,\ldots,n$ and $0 \leq t < \infty$. Each $H(\cdot)X_i(\cdot)$ is also a local martingale and a supermartingale.[63]

In Fernholz and Karatzas [46], the authors also retrieve sufficient conditions for relative arbitrage opportunities, which we will re-state in the following Propositions.

**Proposition 1.8:** [64]

*Let* $\Gamma : [0,\infty) \to [0,\infty)$ *be continuous and strictly increasing with* $\Gamma(0) = 0$ *and* $\Gamma(\infty) = \infty$. *Let furthermore the following condition with respect to the excess growth rate of the market* $\gamma_\mu^*(\cdot)$ *hold almost surely*

$$\Gamma(t) \leq \int_0^t \gamma_\mu^*(s)ds < \infty,\ \ \forall\, 0 \leq t < \infty. \tag{1.74}$$

[59] See e.g. Fernholz and Karatzas [49], Section 6.1.
[60] See e.g. Fernholz and Karatzas [46], Section 2.1. or [49], Section 6.1.
[61] See Fernholz and Karatzas [49], Proposition 6.1.
[62] See Fernholz and Karatzas [49], Proposition 6.2.
[63] See e.g. Fernholz and Karatzas [46], Section 2.1.
[64] See Fernholz and Karatzas [46], Proposition 3.1.

*Further, let* $S(x) = -\sum_{i=1}^{n} x_i \log x_i$ *be the entropy function as specified in Example 1.1. Then for any investment horizon* $[0,T]$ *satisfying:*

$$\Gamma^{-1}\left(S(\mu(0))\right) =: T^* < T < \infty, \tag{1.75}$$

*there exists a sufficiently large* $0 < c \in \mathbb{R}$*, such that the modified entropy-weighted portfolio given by the portfolio rule*

$$\pi_i(t) = \frac{c\mu_i(t) - \mu_i(t)\log\mu_i(t)}{c - \sum_{j=1}^{n}\mu_j(t)\log\mu_j(t)}, \quad i = 1, \ldots, n, \tag{1.76}$$

*constitutes an arbitrage opportunity relative to* $\mu$ *with:*

$$P[Z_\pi(T) > Z_\mu(T)] = 1. \tag{1.77}$$

*Proof:* For the proof we refer to Fernholz and Karatzas [46], proof of Proposition 3.1. □

Hence, by Proposition 1.8 one obtains a method to construct a relative arbitrage strategy with respect to the market portfolio for the standard market $\mathfrak{M}$. However, this almost sure outperformance comes at the price of a necessary investment horizon which is required to realize it. We will discuss this aspect further in the following Section 1.3.2, where we will see that the required time horizon may become fairly large. Furthermore, on may obtain the following result for the exponential local martingale $H(\cdot)$.

**Proposition 1.9:** [65] *Let condition (1.74) hold, together with the conditions on the market price of risk process given in (1.71). Let furthermore*

$$\int_0^t \sum_{i=1}^{n}\left(|\alpha_i(s)| + \sum_{\nu=1}^{n}\xi_{i\nu}^2(s)\right) ds < \infty, \ a.s., \ t \in (0,\infty). \tag{1.78}$$

*Suppose that in addition to that for every* $T > 0$ *there exists a real-valued constant* $K_T > 0$*, s.t.:*

$$\int_0^T \mu^T(s)\sigma(s)\mu(s)ds \equiv \int_0^T \sum_{\nu=1}^{n}\left(\sum_{i=1}^{n}\mu_i(s)\xi_{i\nu}(s)\right)^2 ds \le K_T, \ a.s. \tag{1.79}$$

*Then the exponential local martingale* $H(\cdot)$ *defined in Equation 1.72 is strict.*

*Proof:* For the proof we refer to Fernholz and Karatzas [46], proof of Proposition 3.4. □

[65] See Fernholz and Karatzas [46], Proposition 3.4.

It is worth noting, that in Fernholz and Karatzas [46] a concrete choice for the function $\Gamma(t)$ is given in the case that the market $\mathfrak{M}$ is non-degenerate and weakly diverse, which leads to $\Gamma(t) = \tilde{\gamma}t$ where $\tilde{\gamma} = (\epsilon\delta)/2$ [66], $\epsilon$ and $\delta$ stemming from the ellipticity and diversity assumptions respectively, which illustrates that the investment horizon needed for realizing the arbitrage opportunity will usually be rather large since it is specified by $\Gamma^{-1}(S(\mu(0))) =: T^* < T < \infty$.

## 1.3.2 Implementation of Relative Arbitrage Strategies

The construction of admissible, market-dominating portfolios is discussed in some detail in [44], [48], [49] and [51]. Consequently we will only outline some ideas in a very succinct way and refer to the cited papers for further details.

**Example 1.3:** [67]

*Let $\mathfrak{M}$ be a non-degenerate, weakly diverse market in $[0,T]$ without dividends and let $\boldsymbol{S}$ be a generating function.*

$$\boldsymbol{S}(x) = 1 - \frac{1}{2}\sum_{i=1}^{n} x_i^2.$$

*The portfolio generated through $\boldsymbol{S}$ possesses the following weights and drift process:*

$$\pi_i(t) = \left(\frac{2-\mu_i(t)}{\boldsymbol{S}(\mu(t))} - 1\right)\mu_i(t), \quad i = 1, \ldots, n.$$

$$d\Theta(t) = \frac{1}{2\boldsymbol{S}(\mu(t))}\sum_{i=1}^{n} \mu_i^2(t)\tau_{ii}(t)dt.$$

*It can be shown that $\pi$ is admissible and strictly dominates the market portfolio (see [44]) on a sufficiently long time horizon $T$. More precisely, it is shown, that:*

$$T > \frac{2n\log 2}{\epsilon\delta^2}. \tag{1.80}$$

[66] See Fernholz and Karatzas [46], Remark 3.2.
[67] See Fernholz [44], Example 3.3.3.

It is quite evident, that the constraint on the time horizon $T$ needed to realize the above arbitrage opportunity as it is given in Equation 1.80 will lead to fairly large values for $T$. For an exemplary market with $n = 4$ stocks, and the smallest eigenvalue of order $10^{-5}$ we obtain a similar chart as in Figure 1.1. The best values for $T$ as given in Inequality 1.80 which we obtain are in excess of 2200 years. N.b. that this is the result for a highly simplified example for a market which only consists of four stocks and will be correspondingly larger in a realistic market setup.

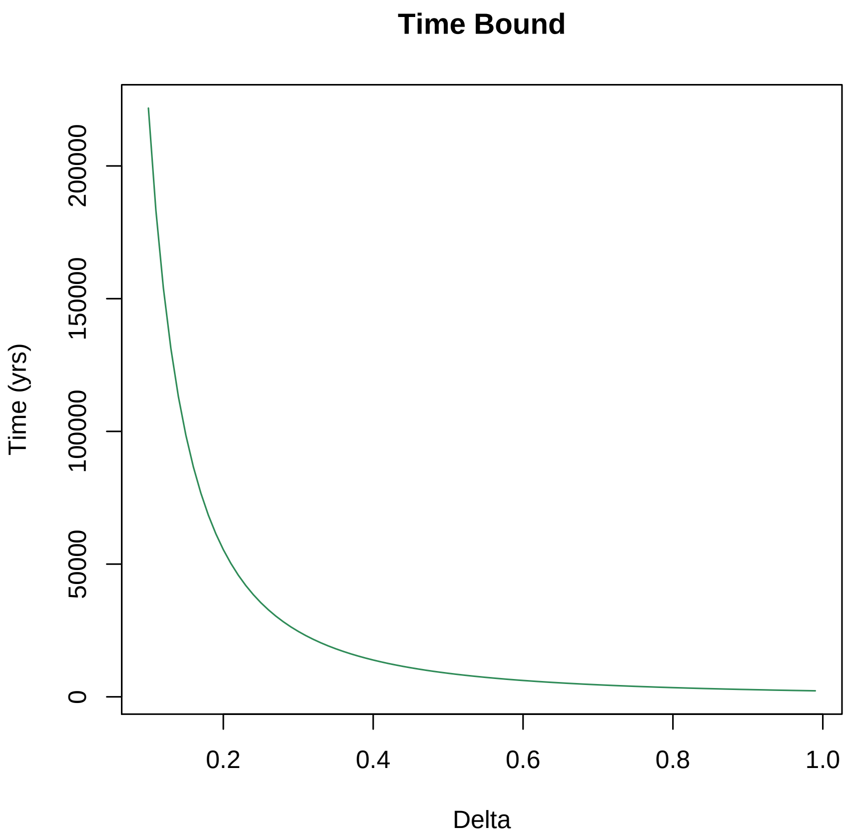


Figure 1.5: Timebound (1.80) dependant on $\delta$ (Yrs.).

Therefore it is fair to conclude that the arbitrage opportunity as constructed in Example 1.3 will not be available in everyday financial markets. Fernholz also concludes that *"...perhaps short-term arbitrage can be excluded even in a nondegenerate, weakly diverse market"*[68]. A similar result is given in [51] for the portfolio generated by $D_p(x) = \left(\sum_{i=1}^n x_i^p\right)^{1/p}$ . Here,

[68] See Fernholz [44], Chapter 3, Remark, p. 60.

the time horizon $T$ necessary in order to have $\pi$ dominating the market $\mu$ has to fulfill $T \geq \frac{2 \log n}{p\epsilon\delta}$, which again is a fairly large number for reasonably small $\delta$ and $\epsilon$[69].

Hence the crucial point is, whether relative arbitrage opportunities can be constructed on arbitrarily small time horizons. This question is addressed in some detail in Fernholz, Karatzas and Kardaras [51]. In markets where short-selling is permitted, i.e. where we deviate from the constraints imposed on a portfolio by the admissibility conditions as given in Definition 1.14, a somewhat reverse method to construct arbitrage opportunities is used [51]. This technique is introduced in [51] under the name of *Mirror Portfolios* and is also discussed in some detail in Fernholz and Karatzas [48] and [49].

**Remark 1.6:** It is worth noting that in the discussion of relative arbitrage in Fernholz, Karatzas and Kardaras [51] a distinction between **portfolios** and **extended portfolios** is being introduced. The difference between those two is that a portfolio $\pi$ in the sense of [51] consists only of nonnegative portfolio weights $\pi_i \geq 0$ for all $i = 1, \dots, n$, whereas an extended portfolio $\hat{\pi}$ only has to fulfill the condition $\sum_{i=1}^{n} \hat{\pi}_i = 1$ together with the conditions of uniform boundedness and measurability. This however corresponds to the more general characterization of a portfolio in the sense of Fernholz [44] which is also the one given in Definition 1.6. In order to avoid any confusion we will not use the notion of extended portfolios but rather stick to the general definition of a portfolio which allows short sales from the very beginning. In those cases, where nonnegativity of portfolio weights is needed, it will be noted explicitly.

Any portfolio as specified in Definition 1.6 renders it possible to sell one or more stocks short, but certainly not all of them. In the following considerations, we will use the market portfolio $\mu$ as numéraire.

**Definition 1.16:** *For a portfolio $\pi$ and $p \in \mathbb{R}\backslash\{0\}$, let us define the* ***p-mirror image***[70] *of $\pi$ w.r.t. $\mu$ as:*

$$\tilde{\pi}^{(p)}(t) := p\pi(t) + (1-p)\mu(t), \;\; t \in [0,T]. \tag{1.81}$$

If a portfolio $\pi$ has only nonnegative portfolio weights this will also be the case for the p-mirror image $\tilde{\pi}$, provided that $0 \leq p \leq 1$. Setting $p = -1$ one obtains $\pi^{(-1)}(t) = 2\mu(t) - \pi(t)$, the "mirror image" of $\pi$ with respect to the market. Recalling the covariance process $\tau^{\mu} = \{\tau_{ij}^{\mu}(t)\}_{1 \leq i,j \leq n}$ the relative covariance of $\pi$ w.r.t. $\mu$ is given by the following Equation[71], as already stated in Equation 1.34:

[69] See Fernholz, Karatzas and Kardaras [51], Equation 4.5. and Appendix A.

[70] See Fernholz, Karatzas and Kardaras [51], Equation 8.1.

[71] See Fernholz [44], Equation 1.2.8. and Fernholz, Karatzas and Kardaras [51], Equation 8.3.

$$\tau_{\pi\pi}^{\mu}(t) := (\pi(t) - \mu(t))^T \sigma(t)(\pi(t) - \mu(t)) \geq \epsilon \|\pi(t) - \mu(t)\|^2. \tag{1.82}$$

Through Equation 1.32 it is possible to derive that[72]:

$$\log\left(\frac{Z_{\tilde{\pi}^{(p)}(T)}}{Z_\mu(T)}\right) = p \cdot \log\left(\frac{Z_\pi(t)}{Z_\mu(T)}\right) + \frac{p(1-p)}{2} \int_0^T \tau_{\pi\pi}^{\mu}(t)dt. \tag{1.83}$$

N.b. that the last term in Equation 1.83 is non-negative due to 1.82. Furthermore one can state the following Lemma [51].

**Lemma 1.2:** *Suppose that the portfolio $\pi$ satisfies the following conditions*[73]*:*

$$P\left(\frac{Z_\pi(T)}{Z_\mu(T)} \geq \beta\right) = 1 \quad \text{or} \quad P\left(\frac{Z_\pi(T)}{Z_\mu(T)} \leq \frac{1}{\beta}\right) = 1, \tag{1.84}$$

*and*

$$P\left(\int_0^T \tau_{\pi\pi}^{\mu}(t)dt \geq \eta\right) = 1, \tag{1.85}$$

*for $\beta > 0$ and $\eta > 0$. There exists a portfolio $\hat{\pi}$ s.t.*

$$P\left(Z_{\hat{\pi}}(T) < Z_\mu(T)\right) = 1. \tag{1.86}$$

Note that conditions 1.84 and 1.85 mandate, that $\pi$ differs from the market portfolio, yet not too much. Basically, these conditions are satisfied if $\|\pi - \mu\|_{L^2([0,T])}$ is bounded away from zero a.s. (see [51]).

*Proof:* See [51]. Let $P\left(\frac{Z_\pi(T)}{Z_\mu(T)} \leq \frac{1}{\beta}\right) = 1$, then one can take $p > 1 + (2/\eta) \cdot \log(1/\beta)$. Then for $\hat{\pi} \equiv \tilde{\pi}^{(p)}$ it holds, that:

$$\log\left(\frac{Z_{\hat{\pi}(T)}}{Z_\mu(T)}\right) \leq p \cdot \left[\log\left(\frac{1}{\beta}\right) + \frac{\eta}{2}(1-p)\right] < 0 \ a.s.$$

[72] See Fernholz, Karatzas and Kardaras [51], Equation 8.7.
[73] See Fernholz, Karatzas and Kardaras [51], Lemma 8.1.

In the case, that $P\left(\frac{Z_\pi(T)}{Z_\mu(T)} \geq \beta\right) = 1$, one may take $p < \min(0, 1-(2/\eta)\cdot\log(1/\beta))$. Then from Equation 1.83 one can observe that $\hat{\pi} \equiv \tilde{\pi}^{(p)}$ satisfies:

$$\log\left(\frac{Z_{\hat{\pi}(T)}}{Z_\mu(T)}\right) \leq p \cdot \left[-\log\left(\frac{1}{\beta}\right) + \frac{\eta}{2}(1-p)\right] < 0 \ \ a.s.$$

□

Hence, Lemma 1.2 states that, under certain conditions, one may construct a portfolio in a weakly diverse market, which a.s. underperforms the market. The idea used in [51] is, that once one has such an underperforming portfolio, one may also construct a portfolio which almost surely outperforms the market. We will give one example which can - among others - be found in [51].

**Example 1.4:** [74]

*Let $\pi = e_1 = (1,0,\ldots,0)^T$, and let $\mu$ be the market portfolio. Furthermore we consider $p > 1$ where $p$ also satisfies $p > p(T) := 1 + \frac{2}{\epsilon\delta^2 T}\cdot\log\left(\frac{1}{\mu_1(0)}\right)$, $\epsilon, \delta > 0$ (see [51]). In this context we can define the portfolio $\hat{\pi}$ as follows:*

$$\hat{\pi}(t) := \tilde{\pi}^{(p)}(t) = pe_1 + (1-p)\mu(t), \ \ 0 \leq t < \infty. \tag{1.87}$$

*Let us consider a strategy $\eta$ which invests $p/(\mu_1(0))^p$ currency units in the market portfolio and $-1$ unit in $\hat{\pi}$ at time $t = 0$ and makes no change thereafter. The value $Z_\eta$ of this strategy is given by (see [51]):*

$$Z_\eta(t) = \frac{p}{(\mu_1(0))^p}\cdot Z_\mu(t) - Z_{\hat{\pi}}(t) \geq \frac{Z_\mu(t)}{(\mu_1(0))^p}[p - (\mu_1(t))^p] > 0, \ \ 0 \leq t < \infty.$$

*Hence we are dealing with a portfolio $\eta$ whose portfolio weights are given by:*

$$\eta_i(t) = \frac{1}{Z_\eta(t)}\left[\frac{p}{(\mu_1(0))^p}\cdot\mu_i(t)Z_\mu(t) - \hat{\pi}_i(t)\cdot Z_{\hat{\pi}}(t)\right], \ \ i = 1,\ldots,n. \tag{1.88}$$

*In fact, $\eta$ is an all-long portfolio [51]. In addition to that, $\eta$ outperforms the market portfolio a.s. at $t = T$ with the same initial investment of $\zeta := Z_\eta(0) = p/(\mu_1(0))^p - 1 > 0$. This is the case, because $\eta$ is long in the market portfolio and short in the portfolio $\hat{\pi}$ which underperforms the market at time $t = T$ a.s.:*

$$Z_\eta(T) = \frac{p}{(\mu_1(0))^p}Z_\mu(T) - Z_{\hat{\pi}}(T) > \zeta Z_\mu(T) \ \ a.s., from \ 1.86. \tag{1.89}$$

[74] See Fernholz, Karatzas and Kardaras [51], Example 8.1.

It is worth noting, that the result outlined in Example 1.4 depends on the ellipticity of the market $\mathfrak{M}$. We will revisit this Example in Section 2.2 and discuss some limitations of this approach. The fundamental question is, whether functionally generated arbitrage exists on arbitrarily small time horizons. In the general setup for the equity market $\mathfrak{M}$ which we have employed so far, we have encountered substantial challenges in this respect. However stronger results are possible, if the structure imposed on the market model is more specific. This leads to more complex market models which we will briefly discuss in the following Section 1.3.3.

### 1.3.3 Relative Arbitrage in Abstract Markets

The aim of this Section is to give a brief overview of the considerable efforts which have been undertaken in the course of the past few years to address the problem of the time horizon needed to realize arbitrage opportunities. The concept of stabilization by volatility will also be put into the context of our work in Chapter 3.

#### Arbitrage by Change of Measure

One attempt to tackle the problem of constructing arbitrage opportunities on arbitrarily short time horizons was undertaken by Osterrieder and Rheinländer [81]. In their work, the authors construct arbitrage opportunities by means of a non-equivalent change of measure. It is worth noting, that this approach somehow deviates from the paradigms of the Fernholz-Karatzas model which we have discussed so far, whose results do explicitly not depend on any special properties imposed on the probability measure or for example the usage of an equivalent martingale measure. The arbitrage opportunities considered by Osterrieder and Rheinländer follow the notion of arbitrage as introduced by Delbaen and Schachermayer in [29].

**Definition 1.17:** [75] *Let us consider a semimartingale $Y$ and a predictable process $\Phi$ which is integrable with respect to $Y$. $\Phi$ is admissible if the integral $\int \Phi dY$ is uniformly bounded from below. The semimartingale $Y$ is said to satisfy the no-arbitrage property for admissible integrands under a probability measure $P$, if $\Phi$ is admissible and:*

$$\int_0^T \Phi(s)dY(s) \geq 0 \;\; P-a.s. \Rightarrow \int_0^T \Phi(s)dY(s) = 0 \;\; P-a.s. \tag{1.90}$$

[75] See Osterrieder and Rheinländer [81], Definition 2.4.

Starting with the probability measure $P^0$, let $\mathcal{M}(Y)$ be the set of all equivalent probability measures such that the price process $Y$ is a local martingale. Then one may impose the following condition on the market:[76]

$$
\begin{aligned}
0 &< \inf_{P \in \mathcal{M}(Y)} P\left(\sup_{0 \le t \le T} \mu_{max}(t) \ge 1-\delta\right), && (1.91)\\
1 &> P^0\left(\sup_{0 \le t \le T} \mu_{max}(t) \ge 1-\delta\right), && (1.92)
\end{aligned}
$$

where $\delta \in (0,1)$ is the bound stemming from the well known diversity condition which we have introduced in Definition 1.10. In this setup one can now define a probability measure $Q$ which is absolutely continuous but not equivalent to $P^0$ via the Radon-Nikodym[77] density:

$$
\frac{dQ}{dP^0} = \begin{cases} 0, & \text{if } \mu_{max}(t) \ge 1-\delta \text{ for } t \in [0,T]; \\ c, & \text{else,} \end{cases} \tag{1.93}
$$

whereby $c \in \mathbb{R}$ is taken as a normalizing constant.[78] Then, it is possible to prove the following result.

**Proposition 1.10:** [79] *Let $Q$ be a probability measure which is absolutely continuous but not equivalent with respect to $P^0$. If the condition*

$$
\sup_{P \in \mathcal{M}(Y)} P\left(\frac{dQ}{dP^0} > 0\right) < 1 \tag{1.94}
$$

*holds, then there exists a $Q$-arbitrage opportunity which may be realized by means of an admissible strategy.*

*Proof:* For the proof we refer to Osterrieder and Rheinländer [81], proof of Proposition 2.8. □

This result, remarkable though it is, does not answer the questions raised in the context of relative arbitrage. In the Fernholz-Karatzas model one is looking for a portfolio which dominates the market or any other reasonable benchmark portfolio. This kind of relative arbitrage is not the one which is considered in the paper and the authors state that *"...although the sum of our arbitrage portfolio and the market portfolio dominates the market portfolio, it is not necessarily bounded from below relative to the market portfolio as numéraire"*[80].

[76] See Osterrieder and Rheinländer [81], Definition 2.3.
[77] See e.g. Rudin [90], Theorem 6.10.
[78] See Osterrieder and Rheinländer [81], Definition 2.5.
[79] See Osterrieder and Rheinländer [81], Proposition 2.8.
[80] See Osterrieder and Rheinländer [81], Remark 2.10., p.292.

### Strong Relative Arbitrage on Arbitrary Time Horizons

Building on the results obtained by Fernholz and Karatzas in the context of volatility stabilized markets (VSM)[81], Banner and D. Fernholz [14] constructed a setup in which relative arbitrage may be realized over arbitrary time horizons. We will briefly sketch their insights and conclude this topic by a brief discussion of the outlined results.

In their analysis, the authors investigate strong relative arbitrage opportunities of the following form. Let $\pi, \rho$ be portfolios and consider a time horizon $[t_0, T]$. Then $\pi$ constitutes a strong relative arbitrage opportunity relative to $\rho$, if there exists a constant $q = q(\pi, \rho, t_0, T) > 0$, such that:[82]

$$P\left(\frac{Z_\pi(t)}{Z_\rho(t)} \geq q\right) = 1 \; for\, all\, t_0 \leq t \leq T \; and \tag{1.95}$$

$$P\left(Z_\pi(T) > Z_\rho(T)\right) = 1. \tag{1.96}$$

Furthermore, one sets $Z_\pi(t_0) = Z_\rho(t_0) = z > 0$. In the VSM, which will be discussed in further detail in Chapter 4, relative arbitrage opportunities can be realized on fairly small time horizons as set forth in Inequality (4.103). The aim of the work by Banner and D. Fernholz is to strengthen this result and overcome the restrictions the VSM imposes on the dynamics of the stock market. We will restate the central result from [14] in the following Proposition 1.11.

**Proposition 1.11:** [83] *Let $\mathfrak{M}$ be a market where the dynamics of the individual stocks are modeled via the semimartingale model specified in Equation 1.1:*

$$d \log X_i(t) = \gamma_i(t)dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t) dW_\nu(t).$$

*Let further $\mathfrak{M}$ satisfy the following condition for the relative variance (w.r.t. $\mu(\cdot)$) of the smallest stock $\tau_{(n,n)}(\cdot)$:*

$$\tau_{(n,n)}(t) \geq \frac{k}{\mu_{(n)}(t)} \; \forall t \geq 0, \tag{1.97}$$

*where $\mu_{(n)}(t) = \min\{\mu_1(t), \ldots, \mu_n(t)\}$ is the market weight of the smallest stock and $k > 0$ constant. Then for any $T > 0$ there exists a strong relative arbitrage opportunity over the investment horizon $[0, T]$.*

*Proof:* We will sketch the central ideas used in the proof of Banner and D. Fernholz and refer to [14] for the detailed result.

[81] See Fernholz and Karatzas [46] and [49].
[82] See Banner and D. Fernholz [14], Section 2.
[83] See Banner and D. Fernholz [14], Proposition 1.

Taking $c > 1$ constant, the first step is to define a function $f$ on $[0, 1]$ by:

$$f(y) = \begin{cases} \Gamma(c+1, -\log y) = \int_{-\log y}^{\infty} e^{-r} r^c dr, & 0 < y \leq 1; \\ 0, & y = 0. \end{cases} \tag{1.98}$$

Here $\Gamma(\cdot, \cdot)$ denotes the incomplete Gamma function[84] and we note that $f$ is continuous. Its first and second derivatives on $(0, 1)$ are given by:

$$f'(y) = (-\log y)^c \;\; \textit{and} \;\; f''(y) = -\frac{c(-\log y)^{c-1}}{y}.$$

The idea used in the proof is to define a portfolio generating function

$$S(x_1, \ldots, x_n) = \sum_{i=1}^{n} f(x_i) \tag{1.99}$$

and apply Theorem 1.1 and the theory on portfolio generating functions as set out in Fernholz [44], especially the following almost sure notion:[85]

$$\log\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) = \log S(\mu(t)) - \log S(\mu(0)) + \int_0^t \frac{1}{2S(\mu(s))} \sum_{i=1}^{n} -f''(\mu_i(s))\tau_{ii}(s)\mu_i^2(s)ds. \tag{1.100}$$

It is shown in [14], that $S(x) \leq nf(1/n)$ and $S(x) \geq f(1)$. By applying these bounds to Equation 1.100, one obtains:

$$\log\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) \geq S_1(\mu_{(n)}(t)) - \log\left(nf\left(\frac{1}{n}\right)\right) + \int_0^t \Theta_1(\mu_{(n)}(s))ds, \tag{1.101}$$

where:

$$S_1(\mu_{(n)}(t)) = \log((n-1)f(\mu_{(n)}(t)) + f(1 - (n-1)\mu_{(n)}(t))), \tag{1.102}$$

$$\Theta_1(\mu_{(n)}(s)) = \frac{-c\mu_{(n)}(s)f''(\mu_{(n)}(s))}{2\left(f(\mu_{(n)}(s)) + (n-1)f\left(\frac{1-\mu_{(n)}(s)}{n-1}\right)\right)}. \tag{1.103}$$

By this it can be seen that Condition (1.95) is fulfilled with:

$$q = \frac{f(1)}{nf(1/n)}. \tag{1.104}$$

It is then shown in [14], that $\Theta_1(\cdot)$ is nonnegative and decreasing on $(0, 1/n)$. In the following steps of the proof, a stopping time $T_0$ is introduced together with a new portfolio

[84] See Abramowitz and Stegun [2], Section 6.5.

[85] See Fernholz [44], Definition 3.1.1. and Proof of Theorem 3.1.5.

$\tilde{\pi}(\cdot)$ which is equal to the originally generated portfolio $\pi(\cdot)$ up to the stopping time $T_0$ and equal to the market portfolio afterwards. It can then be shown that:

$$\log\left(\frac{Z_{\tilde{\pi}}(T)}{Z_\mu(T)}\right) = \log\left(\frac{Z_\pi(T_0)}{Z_\mu(T_0)}\right) \geq \cdots \geq \frac{T}{2}\Theta_1\left(\frac{1}{n}\right), \quad a.s. \tag{1.105}$$

By this, Condition 1.96 is established and the proof is completed. Once again we refer to Banner and D. Fernholz [14] for details and a closer discussion of the technical intricacies of this proof. □

At first sight it seems, that the arbitrage opportunity obtained in this Section prevails more or less in the general setup which we have outlined in Section 1.1. However, once again we obtain the necessary overall stability of the market by imposing restrictions on its (relative) variance structure. Condition 1.97 is also rather strong in the sense that it may mandate the smallest stock in the market to have an extremely high relative variance with respect to the market.

**Further Developments in Stochastic Portfolio Theory**

We will conclude Section 1.3 by briefly referring to further literature and ongoing work in the field of stochastic portfolio theory, in order to give a complete overview of the literature.

In the context of relative arbitrage, D. Fernholz and Karatzas [50] recently investigated the optimal arbitrage in a Markovian model for the financial market, where they deal with the question, what the smallest amount of initial capital is, starting with which one can match or exceed the performance of the market.

A further extension to the concept of functionally generated relative arbitrage is due to Strong [98] who recently investigated the performance of functionally generated portfolios versus the market if the number of stocks is stochastic. In the case that the number of stocks is growing due to breakups of companies and new entries to the market and that no stock ever loses all of its capital, Strong [98] shows that the above mentioned arbitrage opportunities as e.g. created by the entropy- or the $D_p$-weighted portfolios relative to the market are annihilated. In fact, it is even shown, that for the proposed generating functions an arbitrarily large underperformance of the market may occur with positive probability.[86]

Another area which has enjoyed substantial attention in the past few years is the set of questions linked to the dynamics of ranked market weights and the overall capital structure

[86] See Strong [98], Proposition 3.11. and Corollary 3.12.

of the market. The general results together with the available literature as well as our ideas will be discussed in Chapters 3 and 4.

This concludes our brief discussion of relative arbitrage in the Fernholz-Karatzas model. For further examples and results we refer to Banner and D. Fernholz [14], Fernholz [44], Fernholz and Karatzas [46], [49], Fernholz, Karatzas and Kardaras [51], Osterrieder and Rheinländer [81], Strong [98] and Delbaen and Schachermayer [28], [29], [30] and [31].

## 1.4 Some Optimality Results

Apart from the framework of functionally generated portfolios, the modeling approach outlined in the above sections may also be used to the ends of solving classical optimization problems. These issues are covered to some extent in Fernholz [44] and Fernholz and Karatzas [48]. For general results on the maximization of the logarithmic utility of the portfolio value we refer to Karatzas and Shreve [67]. Furthermore these issues are comprehensively treated in Platen [85] and in Platen and Heath [87]. At this point we will only give a brief introduction to this topic and mainly focus on the construction of the growth-optimal portfolio (see [48] or [87]). Further, we outline the result that the optimization of the portfolio growth rate and the logarithmic utility lead to the same result[87].

The growth optimal portfolio (GOP) was first discussed by Kelly in 1956 [71]. Since then the GOP has been applied in fields as diverse as gambling, optimization and derivatives pricing. The GOP may be characterized as maximizing the logarithmic utility of terminal wealth, i.e. the quantity $E(\log(Z_\pi(T)))$ for $T \in [0, \infty)$.

Let us consider the value process of a portfolio $\pi$ as it is given in Equation 1.15:

$$d \log Z_\pi(t) = \gamma_\pi(t)dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i,\nu}(t)dW_\nu(t),$$

for $t \in [0, \infty)$ a.s., where:

$$\gamma_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \underbrace{\frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{ij}(t)\right)}_{=:\gamma_\pi^*}.$$

[87] See Fernholz [44], Example 1.1.8.

Then we may define the growth optimal portfolio in a general version which holds for any continuous financial market model (see [85] or [87]):

**Definition 1.18:** *In a continuous financial market a strictly positive portfolio process* $Z_{\pi_*} = \{Z_{\pi_*}(t),\ t \in [0,\infty)\}$ *is called a* ***growth optimal portfolio (GOP)***[88]*, if for all strictly positive portfolios* $\eta \in \mathfrak{M}$ *the growth rates satisfy the inequality:*

$$\gamma_{\pi_*}(t) \geq \gamma_\eta(t),\ \ \forall\ t \in [0,\infty)\ \ a.s. \tag{1.106}$$

The optimization problem associated to the GOP can be formulated as follows[89]: one is looking for a portfolio $\pi_*(\cdot)$, such that for all $t \in [0,\infty)$ the vector $\pi_*(t)$ maximizes with probability 1 the expression

$$\sum_{i=1}^{n} x_i\gamma_i(t) + \frac{1}{2}\left(\sum_{i=1}^{n} x_i\sigma_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\sigma_{ij}(t)x_j\right) = x^T\alpha(t) - \frac{1}{2}x^T\sigma(t)x \tag{1.107}$$

over all vectors $(x_1,\ldots,x_n) \in \mathbb{R}^n$ with $\sum_{i=1}^n x_i = 1$. We recall that $\alpha(t) = (\alpha_1(t),\ldots,\alpha_n(t))$ is the vector of the stocks' rates of return which are defined as in Definition 1.3 by:

$$\alpha_i(t) = \gamma_i(t) + \frac{1}{2}\sum_{\nu=1}^{n}\xi_{i,\nu}^2(t).$$

Furthermore, the vector $\pi_*(t)$ has to satisfy the first order condition[90] of this optimization problem, i.e.:

$$(x - \pi_*(t))^T(\alpha(t) - \sigma(t)\pi_*(t)) \leq 0,\ \ \textit{for every vector}\ (x_1,\ldots,x_n) \in \mathbb{R}^n\ \textit{with}\ \sum_{i=1}^{n} x_i = 1. \tag{1.108}$$

Then, for any portfolio $\pi$ in a market $\mathfrak{M}$ it holds almost surely, that the growth rates satisfy the inequality[91]:

$$\gamma_\pi(t) \leq \gamma_{\pi_*}(t),\ \ \ \forall\ t \in [0,\infty). \tag{1.109}$$

[88]See Platen [85], Section 2.8. or Platen and Heath [87], Definition 10.2.1.
[89]See Fernholz and Karatzas [48], Problem 4.6.
[90]See Fernholz and Karatzas [48], Problem 4.6.
[91]See Fernholz and Karatzas [48], Equation 4.3.

If the property of the portfolio growth rate, which is formulated in Equation 1.24 in Remark 1.4 is satisfied, i.e. if Equation 1.25 holds, then

$$d\log\left(\frac{Z_\pi(t)}{Z_{\pi^*}(t)}\right) = (\gamma_\pi(t) - \gamma_{\pi^*}(t))\,dt + \sum_{\nu=1}^{n}(\sigma_{\pi\nu}(t) - \sigma_{\pi^*\nu}(t))\,dW_\nu(t), \tag{1.110}$$

where $\sigma_{\pi\nu}(t) := \sum_{i=1}^{n}\pi_i(t)\xi_{i\nu}(t)$ for $\nu = 1;\dots n$, leads to the growth-optimality property[92]:

$$\limsup_{T\to\infty}\frac{1}{T}\log\left(\frac{Z_\pi(t)}{Z_{\pi^*}(t)}\right) \leq 0 \;\; a.s. \tag{1.111}$$

for every portfolio $\pi$. If furthermore for some $\mathfrak{F}$-stopping time $T$ it holds that

$$E\left(\int_0^T \|\sigma(t)\|dt\right) < \infty,$$

then Equations 1.109 and 1.110 lead to the **log-optimality** property

$$E(\log Z_\pi(T)) \leq E(\log Z_{\pi^*}(T)), \tag{1.112}$$

for every portfolio $\pi$.

**Remark 1.7:** This result can be obtained directly by formulating the optimization problem for the logarithmic utility as well. The general problem is also formulated in Fernholz[93]. The aim is to find the portfolio $\pi$ which maximizes the objective function:

$$E\left(\log Z_\pi(t)\right),\;\; t \in [0,\infty). \tag{1.113}$$

Recalling the definition of the process $\{Z_\pi\}_{t\geq 0}$ as given in Equation 1.15, one obtains the following expression for the objective function 1.113:

$$E\left(\log Z_\pi(t)\right) = E\left(\log Z_\pi(0)\right) + \underbrace{E\left(\int_0^t \gamma_\pi(s)ds\right)}_{(\diamond)} + E\left(\int_0^t \sum_{i,\nu=1}^{n}\pi_i(s)\xi_{i,\nu}(s)dW_\nu(s)\right). \tag{1.114}$$

The first term on the right hand side of Equation 1.114 is constant and the last term on the right hand side vanishes as it is the expectation of an Itô integral by a Brownian

[92] See Fernholz and Karatzas [48], Equation 4.5.

[93] See Fernholz [44], Example 1.1.8.

motion. Hence the above problem reduces to maximizing the expected portfolio growth rate, thus illustrating that the questions of maximizing the expected growth rate and of maximizing the logarithmic utility lead to the same optimization problem. One can now insert the characterization of the portfolio growth rate which was outlined in Equation 1.16 in Proposition 1.1 into ($\diamondsuit$), thereby obtaining:

$$\begin{aligned}
(\diamondsuit) &= E\left[\int_0^t \left(\sum_{i=1}^n \pi_i(s)\gamma_i(s) + \frac{1}{2}\sum_{i=1}^n \pi_i(s)\sigma_{ii}(s) - \frac{1}{2}\sum_{i,j=1}^n \pi_i(s)\pi_j(s)\sigma_{ij}(s)\right) ds\right] \\
&= \int_0^t \underbrace{E\left[\sum_{i=1}^n \pi_i(s)\gamma_i(s) + \frac{1}{2}\sum_{i=1}^n \pi_i(s)\sigma_{ii}(s) - \frac{1}{2}\sum_{i,j=1}^n \pi_i(s)\pi_j(s)\sigma_{ij}(s)\right]}_{(\diamondsuit\diamondsuit)} ds,
\end{aligned}$$

where we can change the integrals by Fubini's Theorem[94]. Hence we seek to maximize the term ($\diamondsuit\diamondsuit$), which may be written in matrix / vector notation as follows:

$$(\diamondsuit\diamondsuit) = (\gamma + \frac{1}{2}\text{diag}\sigma)\pi^T - \frac{1}{2}\pi^T\sigma\pi = \alpha\pi^T - \frac{1}{2}\pi^T\sigma\pi. \tag{1.115}$$

In Equation 1.115 $\alpha$ is specified as in Definition 1.3. Taking the gradient of the right hand side in Equation 1.115 and setting the whole expression equal to zero yields:

$$\nabla\left(\alpha\pi^T - \frac{1}{2}\pi^T\sigma\pi\right) = \alpha - \sigma\pi \overset{!}{=} 0. \tag{1.116}$$

The matrix / vector notation used in Equation 1.116 may easily be verified by looking at the component-wise result:

$$\begin{aligned}
\frac{\partial}{\partial\pi_i}(\diamondsuit\diamondsuit) &= \alpha_i(t) - \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^n \pi_j(t)\sigma_{ij}(t) - \frac{1}{2}2\pi_i(t)\sigma_{ii}(t) \\
&= \alpha_i(t) - \sum_{j=1}^n \pi_j(t)\sigma_{ij}(t).
\end{aligned}$$

[94]See e.g. Rudin [90], Theorem 8.8.

Hence, if the inverse of $\sigma$, denoted by $\sigma^{-1}$ exists, then the log-optimal (and in fact also growth-optimal) portfolio weights are given by:

$$\pi^* = \sigma^{-1}\alpha. \tag{1.117}$$

Thus, for every stock $i$ its corresponding weight in the log-optimal portfolio is given by the fraction of the stock's drift (i.e. rate of return, $\alpha_i$) and of its variance (i.e. $\sigma$), which corresponds to the classical result for this problem[95].

In the last part of this section we will state the numéraire property of the GOP (i.e. of the log-optimal portfolio) as well as the formal proof concerning the growth optimality result stated in Equation 1.111. One may obtain the numéraire property of the GOP as follows[96]. Let $R^*_\pi(\cdot) := \frac{Z_\pi(\cdot)}{Z_{\pi^*}(\cdot)}$, then an application of Itô's rule yields:

$$\begin{aligned}\frac{dR^*_\pi(t)}{R^*_\pi(t)} &= \left[\gamma_\pi(t) - \gamma_{\pi^*}(t) + \frac{1}{2}\sum_{\nu=1}^{n}(\sigma_{\pi\nu}(t) - \sigma_{\pi^*\nu}(t))^2\right] dt + \sum_{\nu=1}^{n}(\sigma_{\pi\nu}(t) - \sigma_{\pi^*\nu}(t))dW_\nu(t) \\ &= (\pi(t) - \pi^*(t))^T \left[(\alpha(t) - \sigma(t)\pi^*(t))dt + \xi(t)dW(t)\right].\end{aligned}$$

Together with the first order condition 1.108, this semi-martingale decomposition shows, that $R^*_\pi(\cdot)$ is a local supermartingale. Since it is positive, an application of Fatou's Lemma (see e.g. Rudin [90]) yields, that it is in fact a supermartingale. Consequently one obtains the **numéraire property of the GOP** $\pi^*(\cdot)$:

$$R^*_\pi(\cdot) = \frac{Z_\pi(\cdot)}{Z_{\pi^*}(\cdot)} \ \textit{is a supermartingale for every portfolio}\ \pi(\cdot)^{97}. \tag{1.118}$$

The results sketched in this section are fairly remarkable, since they essentially state that the log-optimal portfolio $\pi^*$ will dominate any other portfolio as $T \to \infty$. We will conclude this section with the formal proof of the optimality of $\pi^*$.

**Theorem 1.2:** [98]

*Let $\theta(t) = \xi^{(-1)}(t)\alpha(t)$ denote the market price of risk. Let further the optimal*[99] *portfolio value be given by:*

[95]See e.g. Karatzas and Shreve [67], Section 3.10.
[96]See Fernholz and Karatzas [48], Equation 4.7.
[97]See Fernholz and Karatzas [48], Equation 4.7.
[98]See Karatzas and Shreve [67], Theorem 3.10.1.
[99]Here the term "optimal" refers to the log- and growth-optimality of the portfolio.

$$Z_{\pi^*}(t) = Z_{\pi^*}(0) \underbrace{\exp\left(\int_0^t \theta^T(s)dW(s) + \int_0^t \frac{1}{2}\|\theta(s)\|^2 ds\right)}_{=:\frac{1}{H_0(t)}}, \tag{1.119}$$

*where $H_0(t)$ is called the state price density process*[100]*. For the optimal portfolio weights $\pi^*$ we denote by $\tilde{\pi}^*(t) := \pi^*(t)Z_{\pi^*}(t)$ the vector of money-equivalent portfolio weights. Then for every initial endowment $x > 0$ and for any portfolio process $\pi$ it holds almost surely that:*

$$\limsup_{T\to\infty} \frac{1}{T} \log Z_\pi^x(T) \leq \limsup_{T\to\infty} \frac{1}{T} \log Z_{\pi^*}^x(T). \tag{1.120}$$

*Hence for $P$-almost every $\omega \in \Omega$, $\pi^*$ maximizes the growth of portfolio value (wealth) over all other portfolios $\pi$.*

*Proof:* We follow the proof outlined in Karatzas and Shreve [67].

Step 1: Define the ratio $R(t)$ as follows:

$$R(t) := \frac{Z_\pi^x(t)}{Z_{\pi^*}^x(t)} = \frac{1}{x} H_0(t) Z_\pi^x(t). \tag{1.121}$$

Step 2: One may now write $R(t)$ in differential form.

$$dR(t) := \frac{1}{x} H_0(t) \left[\xi^T(t)\pi(t) - Z_\pi^x(t)\theta(t)\right]^T dW(t). \tag{1.122}$$

This result can be derived from Remark 3.3. in Karatzas and Shreve [67] where it is stated that:

$$H_0(t)Z_\pi^x(t) = x + \underbrace{\int_0^t H_0(u)\left[\xi^T(u)\pi(u) - Z_\pi^x(u)\theta(u)\right] dW(u)}_{(\sharp)} \quad 0 \leq t \leq T. \tag{1.123}$$

Furthermore, via application of Fatou's Lemma[101] it can be shown [67], that ($\sharp$) is a supermartingale. Equation 1.123 implies that $E\left[H_0(T)Z_\pi^x(T)\right] \leq x$. Therefore the expected terminal wealth, discounted by the state price density process, cannot exceed the initial endowment.

Step 3: In this step it will be shown, that $R(t)$ satisfies the following inequality:

[100] See Karatzas and Shreve [67], Remark 1.5.8.
[101] See e.g. Rudin [90], Lemma 1.28.

$$e^{\delta n} P\left[\sup_{n\leq t<\infty} R(t) > e^{\delta n}\right] \leq E\left[R(n)\right] \leq R(0) = \frac{x}{x} = 1 \;\; \forall\, n \in \mathbb{N},\; 0 < \delta < 1. \tag{1.124}$$

This notion is based on two results in Karatzas and Shreve [66], namely Problem 1.3.16. and Theorem 1.3.8.(ii). We will state the respective propositions here and refer to [66] for details and proofs.

*Problem 1.3.16. [66]:* Let $\{X_t\}_{t\geq 0}$ be a right continuous, nonnegative supermartingale w.r.t. the filtration $\{\mathfrak{F}_t\}_{t\geq 0}$; then $X_\infty(\omega) = \lim_{t\to\infty} X(t,\omega)$ exists for $P$-almost every $\omega \in \Omega$ and $X(t)$ is a supermartingale.

*Theorem 1.3.8. (ii) [66]:* Let $\{X_t\}_{t\geq 0}$ be a right continuous submartingale w.r.t. the filtration $\{\mathfrak{F}_t\}_{t\geq 0}$; let further $[\kappa, \tau]$ be a subinterval of $[0,\infty)$ and let $0 < \lambda \in \mathbb{R}$. Then one can state the $2^{nd}$ submartingale ineqality:

$$\lambda P\left[\inf_{\kappa\leq t\leq\tau} X_t \leq -\lambda\right] \leq E[X_\tau^+] - E[X_\kappa]. \tag{1.125}$$

We have that $R(t)$ is a nonnegative, continuous supermartingale. Hence by Problem 1.3.16. in [66] $R_\infty(\omega) = \lim_{t\to\infty} R_t(\omega)$ exists for $P$-a.e. $\omega \in \Omega$.

Furthermore, through adapting Theorem 1.3.8.(ii) in [66] for supermartingales one obtains:

$$\lambda P\left[\sup_{n\leq t<\infty} R(t) > \lambda\right] \leq E[R(n)] - E[R_\infty]. \tag{1.126}$$

Hence, if we replace $\lambda$ by $e^{\delta n}$ and use the nonnegativity and the supermartingale property of $R(t)$, then we obtain Inequality 1.124. Shifting the multiplicative $e^{\delta n}$ to the right hand side and taking the probability of the logarithm, one obtains:

$$P\left[\sup_{n\leq t<\infty} \log R(t) > \delta n\right] \leq e^{-\delta n}. \tag{1.127}$$

One may now take the sum over all $n$ in Inequality 1.127 in order to obtain:

$$\sum_{n=1}^{\infty} P\left[\sup_{n\leq t<\infty} \log R(t) > \delta n\right] \leq \sum_{n=1}^{\infty} e^{-\delta n} < \infty. \tag{1.128}$$

STEP 4: Now an application of the Borel-Cantelli Lemma[102] yields:

$$P\left[\lim_{n\to\infty} \sup_{n\leq t<\infty} \log R(t) > \delta n\right] = 0. \tag{1.129}$$

[102] See e.g. Durett [34], Section 1.6., Lemma 1.6.1. The Borel-Cantelli Lemma states that if: $\sum_{n=1}^{\infty} P(A_n) < \infty$ for a sequence of events $A_n$ in the probability space, then $P(A_n\ \mathit{infinitely\ often}) = 0$, i.e. the probability that infinitely many $A_n$ occur is zero. This may also be denoted by $P[\limsup_{n\to\infty} A_n] = 0$.

Thus there exists an $N_\delta$, s.t. $\forall n \geq N_\delta(\omega)$, $\forall t \geq n:\ \log R(t,\omega) \leq \delta n \leq \delta t \ \ for\ P-a.e.\ \omega \in \Omega$. In particular we have:

$$\limsup_{t\to\infty} \frac{1}{t} \log R(t,\omega) \leq \delta. \tag{1.130}$$

Hence by the definition of $R(t)$ in Equation 1.121 one obtains:

$$\limsup_{t\to\infty} \frac{1}{t} \log Z_\pi^x(t,\omega) - \limsup_{t\to\infty} \frac{1}{t} \log Z_{\pi^*}^x(t,\omega) \leq \limsup_{t\to\infty} \frac{1}{t} \log R(t,\omega) \leq \delta. \tag{1.131}$$

So, finally the result follows from 1.131 and the arbitrariness of $\delta$:

$$\limsup_{t\to\infty} \frac{1}{t} \log Z_\pi^x(t,\omega) \leq \limsup_{t\to\infty} \frac{1}{t} \log Z_{\pi^*}^x(t,\omega) + \delta. \tag{1.132}$$

This concludes the proof. $\square$

The short discussion of the growth-optimal portfolio concludes our brief overview of topics in stochastic portfolio theory.

## 1.5 Conclusion

In the first chapter we have outlined a semimartingale model for the stock price process in Definition 1.2. This representation of single stock price movements can be incorporated into the larger framework of a stock market $\mathfrak{M}$ which is a family of stocks. In Sections 1.1 and 1.2 some basic properties of stocks and markets as well as the concept of portfolios in a stock market are presented. In Section 1.2 we also dwell on the issue of portfolio-generation through so-called generating functions. Furthermore, in this Section some attention is paid to the behavior of functionally generated portfolios under real market conditions. The possibility to obtain portfolio weights and investment decisions based on clearly defined rules without having to undergo any optimization routines may be regarded as one of the most attractive features of stochastic portfolio theory.

A remarkable aspect of the model outlined is, that arbitrage may exist under certain circumstances. However, the time horizon necessary in order to realize these arbitrage opportunities is extremely large. This issue is succinctly treated in Sections 1.2.2 and 1.3. In Section 1.3, we also present some results as to how the investment horizon needed for the realization of relative arbitrage opportunities may shortened. It is discussed, that these improvements come at the cost of imposing a more restrictive structure on the dynamics of the individual stocks. In the last Section of the first Chapter, 1.4, we digress into the field of portfolio optimization and present some properties of the growth-optimal portfolio. With respect to all topics outlined in Chapter 1, we present the available literature and the results therein. At some points we provide further discussions and analyses as well as practical implementations in order to visualize some effects and assumptions prevalent in the Fernholz-Karatzas model.

# Chapter 2

# Market Ellipticity in Practice

In this chapter we will revisit the concept of non-degeneracy of the market $\mathfrak{M}$ which was specified in Chapter 1 in Definition 1.5. We will equivalently refer to the non-degeneracy of the market as ellipticity of the market. Let us recall that a model like the one outlined in the previous Chapter is called elliptic if the instantaneous covariance matrix $\sigma(t)$ is invertible.[1] Since for $\mathfrak{M}$ to be non-degenerate it has to hold that $x\sigma(t)x^T \geq \epsilon\|x\|^2$, i.e. that all eigenvalues of $\sigma(t)$ are positive and bounded away from zero by $\epsilon$, it clearly holds that the notions of ellipticity and non-degeneracy are equivalent in our setup.

## 2.1 An Empirical Analysis of Market Ellipticity

### 2.1.1 A Fourier Estimator for Instantaneous Covariances

In this section, we will analyze the extent to which traces of (uniform) non-degeneracy may be found in real-life data. In order to estimate the instantaneous covariance matrices a nonparametric estimation method based on Fourier transforms will be used. This approach was outlined in Malliavin and Mancino [75] and is also briefly discussed in Malliavin and Thalmaier [76]. For general results on Fourier analysis we refer to Grafakos [57] and Rudin [90]. There are several reasons for applying a Fourier based estimator for instantaneous covariance instead of classical local estimators which are comprehensively discussed in the recent work of Cuchiero and Teichmann [26]. In this paper the authors use a Fourier-Féjer estimator similar to the one presented in [75]. Unlike usual local estimators which use

[1] See e.g. Malliavin and Thalmaier [76], Theorem 2.6.

small sliding data windows for covariance estimation, the Fourier-based method utilizes the information of the whole time series for the estimation of instantaneous covariance.[2] Furthermore, the authors show, that in fact the variance of the Fourier-Féjer estimator is smaller than that of classical estimators.[3] In the case that the underlying process of the log-price is a drifted Brownian Motion, the Fourier-Féjer estimator is in fact $\frac{2}{3}$ of the classical one.[4] These properties should in fact be beneficial to the stability of the estimator.

The approach presented in [75] is based on a semimartingale model for the log-price of stocks as we have introduced it in Chapter 1. For the sake of completeness we will restate Equation 1.1 here:

$$d \log X(t) = \gamma(t)dt + \sum_{\nu=1}^{n} \xi_\nu(t) dW_\nu(t), \;\; t \in [0, \infty).$$

Further we recall that the instantaneous covariance matrix is given by $\sigma(t) = \xi(t)\xi(t)^T$. In practice, the covariance matrix is mainly computed over discrete time intervals using the quadratic variation formula. This leads to an estimator for the integrated covariance matrix over the considered time interval. Calculation of the instantaneous covariance matrix from these results would request the numerical calculation of the derivative which is potentially unstable.[5]

For the calculation of the Fourier transform we will rescale the finite time horizon $[0, T]$ to $[0, 2\pi]$, thus for every function $\phi$ on the circle $S^1$ and for its differential form $d\phi$ we can define the respective Fourier transforms as:[6]

$$\begin{aligned} \mathcal{F}(\phi)(k) &:= \frac{1}{2\pi}\int_0^{2\pi} \phi(\theta)\exp(-\mathbf{i}k\theta)d\theta \quad for\; k \in \mathbb{Z}, & (2.1)\\ \mathcal{F}(d\phi)(k) &:= \frac{1}{2\pi}\int_0^{2\pi} \exp(-\mathbf{i}k\theta)d\phi(\theta) \quad for\; k \in \mathbb{Z}. & (2.2) \end{aligned}$$

Let us further denote the Bohr convolution[7] by $*_B$ if for two functions $\Theta$ and $\Psi$ on $\mathbb{Z}$ the following limit exists $\forall k \in \mathbb{Z}$:

$$(\Theta *_B \Psi)(k) := \lim_{N\to\infty} \frac{1}{2N+1} \sum_{s=-N}^{N} \Theta(s)\Psi(k-s). \tag{2.3}$$

[2]See Cuchiero and Teichmann [26], Section 2.
[3]See Cuchiero and Teichmann [26], Theorem 6.6. and Remark 6.7.
[4]See Cuchiero and Teichmann [26], Section 2.
[5]See Malliavin and Mancino [75], Section 1.
[6]See e.g. Malliavin and Mancino [75], Section 1. or Rudin [90], Chapter 9.
[7]See Malliavin and Mancino [75], Section 1. or Malliavin and Thalmaier [76], Section A.1.

The central result presented in Malliavin and Mancino [75] provides us with a straightforward method to estimate the constituents of the instantaneous covariance matrix. We will restate this result in the following Theorem.

**Theorem 2.1:** [8] *Consider a market $\mathfrak{M}$ consisting of $i = 1, \ldots, n$ stocks whose log-prices follow semimartingale processes as specified in Equation 1.1, where the drift and local volatility terms of the $i^{th}$ stock satisfy for $\nu = 1, \ldots, n$:*

$$E\left[\int_0^T (\gamma_i(t))^2 dt\right] < \infty, \tag{2.4}$$

$$E\left[\int_0^T (\xi_{i,\nu}(t))^4 dt\right] < \infty. \tag{2.5}$$

*Then it holds for $i, j \in \mathfrak{M}$, that:*

$$\frac{1}{2\pi}\mathcal{F}(\sigma_{ij}) = \mathcal{F}(d\log X_i) *_B \mathcal{F}(d\log X_j). \tag{2.6}$$

*Proof:* For the proof we refer to Malliavin and Mancino [75], Section 7., Proof of Theorem 2.1. □

**Remark 2.1:** It should be noted that the regularity conditions specified in Equations 2.4 and 2.5 complement those conditions set forth in Definition 1.2. However they should not impose any unreasonable restrictions on our model. We furthermore remark that in [75] and [76] the instantaneous (co-)variances $\sigma_{ij}(t)$ are referred to as instantaneous (co-)volatilities. We are going to stick to the market convention and use the term "volatility" as a synonym for the standard deviation of instantaneous log-returns and not for the variance. Consequently we shall not use the term "volatility" in this context and digress thereby from the nomenclature used in Malliavin and Mancino [75].

The result stated in Equation 2.6 yields that the instantaneous covariances may be retrieved directly from the Bohr-convoluted Fourier transforms of the log-returns. Hence in order to estimate the instantaneous covariances we will have to calculate a numerical approximation of:

$$\sigma_{ij}(t) = \mathcal{F}^{-1}\left(2\pi\, \mathcal{F}(d\log X_i) *_B \mathcal{F}(d\log X_j)\right)(t). \tag{2.7}$$

To that end, one has to calculate the Fourier coefficients[9] of the instantaneous covariance function. Let us denote for all stocks $i = 1, \ldots, n$ the $k^{th}$ Fourier coefficient of the log-returns as $\phi_N^i(k) := \mathcal{F}(d\log X_i)(k)$. Applying the specification of the Bohr convolution

[8] See Malliavin and Mancino [75], Theorem 2.1.

[9] See e.g. Rudin [90], Section 4.26.

given in Equation 2.3, one may calculate the $k^{th}$ Fourier coefficient of the convoluted log-returns for the $i^{th}$ and $j^{th}$ stock as:[10]

$$\psi_N^{i,j}(k) := \frac{1}{2N+1} \sum_{s \in \mathbb{Z}} \phi_N^i(s)\phi_N^j(k-s), \; |k| \leq N. \tag{2.8}$$

It has to be remarked that Malliavin and Mancino [75] formulated Equation 2.8 for $j = 1,2$ whereas a more general formulation may be found in Clément and Gloter [24]. In general, for higher dimensional problems and especially with respect to their practical implementation some care has to be taken with the concrete choice of $N$. We refer to Clément and Gloter [24] for further details.

In the case that $\sigma_{ij}(t)$ is continuous, the Fourier-Fejér summation yields almost everywhere:[11]

$$\sigma_{ij}(t) = \lim_{N \to \infty} \sum_{|k|<N} \left(1 - \frac{|k|}{N}\right) \psi_N^{i,j}(k) e^{\mathbf{i}kt}, \;\; \forall t \in [0, 2\pi]. \tag{2.9}$$

For the practical implementation we consider a discrete time-grid for all $j = 1, \ldots, n$ stocks where we assume each time-grid to have $m$ observation points and furthermore $t_0^j = 0$ and $t_m^j = 2\pi$. Thus let the set of trading times for the $j^{th}$ stock be given by $\mathfrak{T}_m^j := \{t_l^j, l = 1, \ldots, m\}$. It is worth noting, that the time-grids need not be equally spaced and that the trading times of two assets need not necessarily coincide. In Malliavin and Mancino [75] special emphasis is put on the fact that the proposed estimator is especially suitable for calculations based on high frequency data over a short time horizon (e.g. intraday) with unevenly spaced time-grids.[12] For our ensuing calculations we will use end-of-day quotes over longer time horizons which result in evenly spaced identical time grids for all stocks. Denote further by $\rho^j(m) := \max_{0 \leq h \leq m-1} |t_{h+1}^j - t_h^j|$ the mesh of the time grid, which in our case will be the same for all stocks. We consider the time intervals $I_l^j := [t_l^j, t_{l+1}^j)$ for the $j^{th}$ stock and denote the log-return corresponding to $I_l^j$ by $\delta_{I_l^j}(X_j) = \log X_j(t_{l+1}^j) - \log X_j(t_l^j)$. Consequently for $k \in \mathbb{Z}$, $|k| < N$ the discrete approximation for the $k^{th}$ Fourier coefficient[13] of $d \log X_j$ is given by:

$$c^{(k,m)}(d \log X_j) := \frac{1}{2\pi} \sum_{l=0}^{m-1} \exp(-\mathbf{i}kt_l^j) \delta_{I_l^j}(X_j). \tag{2.10}$$

[10] See Clément and Gloter [24], Section 1 and Malliavin and Mancino [75], Section 2.
[11] See Malliavin and Mancino [75], Equation 8.
[12] See Malliavin and Mancino [75], Sections 1. and 2.
[13] See Malliavin and Mancino [75], Equation 9.

Moreover, denote the $k^{th}$ Fourier coefficient[14] for $\sigma_{ij}$ by:

$$c^{(k)}(\sigma_{ij}) := \frac{1}{2\pi}\int_0^{2\pi} \exp(-\mathbf{i}kt)\sigma_{ij}(t)dt. \tag{2.11}$$

Then we can accordingly define the discrete approximation of the $k^{th}$ Fourier coefficient of the Bohr-convoluted Fourier transforms of the log-returns of $X_i$ and $X_j$ as follows:[15]

$$\alpha^{(k)}(N, m, X_i, X_j) := \frac{2\pi}{2N+1}\sum_{|s|\leq N} c^{(s,m)}(d\log X_i)c^{(k-s,m)}(d\log X_j). \tag{2.12}$$

By Equation 2.9 one may finally calculate the following estimator for the instantaneous covariance of stocks $i$ and $j$ through the subsequent formula whereby the number of Fourier coefficients is given by $N$ :[16]

$$\hat{\sigma}_{ij}^{(m,N)}(t) := \sum_{|k|\leq N}\left(1 - \frac{|k|}{N}\right)\alpha^{(k)}(N, m, X_i, X_j)e^{\mathbf{i}kt}. \tag{2.13}$$

The convergence of the estimator $\hat{\sigma}_{ij}^{(m,N)}(t)$ towards $\sigma_{ij}(t)$ is comprehensively covered in Malliavin and Mancino [75]. Firstly it is shown that for $\sigma(t)$ continuous and $\rho(m)N \to 0$ as $m, N \to \infty$ the following limit converges in probability:[17]

$$\lim_{m,N\to\infty} \alpha^{(k)}(N, m, X_i, X_j) = c^{(k)}(\sigma_{ij}). \tag{2.14}$$

Secondly, for $\sigma(t)$ continuous and $\rho(m)N \to 0$, the following convergence in probability holds:[18]

$$\lim_{m,N\to\infty} \sup_{0\leq t\leq 2\pi}\left|\hat{\sigma}_{ij}^{(m,N)}(t) - \sigma_{ij}(t)\right| = 0. \tag{2.15}$$

From the proofs of the above limits and from the further convergence results in [75] one also gets that the number of Fourier coefficients is related to the mesh of the time-grid by $N = \rho(m)^{-2/3}$.[19] Further results on the optimal number of Fourier coefficients may be found in Clément and Gloter [24]. As already stated at the beginning of this section, a slightly modified Fourier-Féjer estimator has been recently discussed in Cuchiero and Teichmann [26]. In this work, a similar estimator albeit without the use of Bohr convolutions is applied

[14] See Malliavin and Mancino [75], Equation 10.
[15] See Malliavin and Mancino [75], Equation 11.
[16] See Malliavin and Mancino [75], Equation 12.
[17] See Malliavin and Mancino [75], Theorem 3.3.
[18] See Malliavin and Mancino [75], Theorem 3.4.
[19] See Malliavin and Mancino [75], Section 5.

to models incorporating jumps and the authors examine the convergence behavior of the estimator including the choice of the number of Fourier coefficients.

In the following Section 2.1.2 we will put the concept developed in Section 2.1.1 to work and evaluate the behavior of the instantaneous covariance matrix of the S&P 100 index.

### 2.1.2 Empirical Analysis of the S&P 100 Index

The Standard & Poors (S&P) 100 Index is a subset of the S&P 500 Index which comprises large cap blue chip stocks in the US market. The S&P 100 features 100 individual blue chips across a broad set of industry sectors. The index constituents are domiciled in the United States of America and represent some of the most important players in their respective industry sector globally. The current index constituents as well as further information on the index characteristics may be accessed through S&P's web portal [96]. The main reason for choosing the S&P 100 instead of the larger S&P 500 is that the number of corporations with similar characteristics is smaller in the S&P 100 than in the larger index. Thus interdependence between stocks can be expected to be more limited in the smaller index, implying that our chances of finding traces of ellipticity in the market data will be better with the S&P 100 than with the S&P 500.

For all ensuing calculations we use the index constituents as of April $14^{th}$, 2011. The time series of the respective stock prices were retrieved through the Bloomberg® system and comprise daily closing prices for a time period of up to five years. We will discuss some results which have been calculated for the whole index universe.

**Covariance Structure of the S&P 100**

In this section we will examine the patterns of ellipticity prevalent in the whole S&P 100 universe. Even though the estimation algorithm exhibits reasonable stability when examined for a low dimensional problem, one observes certain numerical instabilities which are commonplace when a problem of dimension 100 is being analyzed. In our case we observe that the resulting matrix is symmetric, yet some of its eigenvalues take extremely small negative values, a phenomenon which leads to a loss of positive semi-definiteness. Such effects occur frequently when estimating high dimensional covariance structures.

In the following, we have examined several simulation runs of the whole index universe with differing lengths of input data and different evaluation times. In the performed calculations, the negative eigenvalues are usually of order $10^{-17}$ or smaller. The following "heat map" as depicted in Figure 2.1 shows the spectra of 100 estimated covariance matrices based on 350 and 400 days of data history respectively. We observe, that the structure of the spectra stays essentially the same if the number of data points input into the algorithm is being varied from 350 to 400, however it has to be acknowledged that given the dimension of the problem, the number of data points used for this first set of calculations certainly needs to be further extended. Hence we conclude that some of the observed instabilities are also due to the fact that we do not have enough observation points.

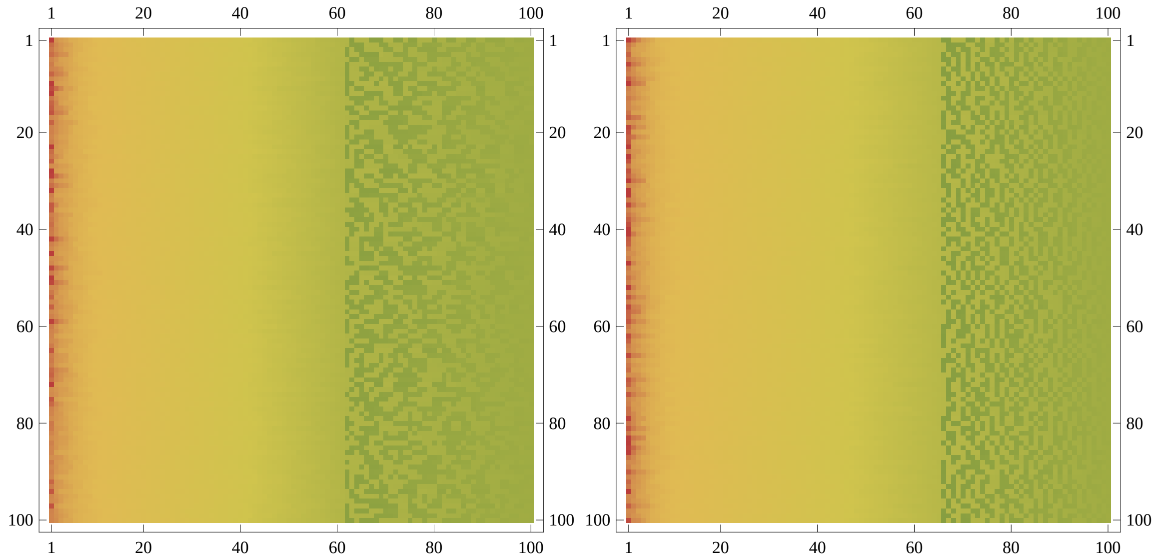


Figure 2.1: Heat-Maps for Estimators based on 350 (left) and 400 (right) Data Points.

In Figure 2.1 one clearly sees that all simulated instantaneous covariance matrices exhibit a similar pattern in their respective spectrum. To the left of the heat-map we have a small number of relatively large eigenvalues (i.e. of size $10^{-1}$). The further we go to the right the steeper we descend towards zero. Around the $60^{th}$ eigenvalue we generally reach a size of order $10^{-17}$ or smaller. From this point on one can observe a certain instability among the smallest eigenvalues which start to oscillate around zero, comprising extremely small positive and negative values. This pattern is also observable in the rightmost area of the plotted heat-map.

This generally observable structure of the spectra of estimated instantaneous covariance matrices may be further illustrated by conducting a principal component analysis (PCA)[20] of an exemplary covariance matrix. The scree-plot of the principal components depicted in Figure 2.2 illustrates the phenomenon that the contribution of the principal components to total variance declines rapidly. In fact, as can be seen in the following output of the

[20]See e.g. Joliffe [62].

statistical analysis conducted in R, the first 6 principal components account for approximately 99.5% of total variance and the first 10 principal components explain more than 99.9% of total variance.

Importance of components:

| | Comp.1 | Comp.2 | Comp.3 | Comp.4 | Comp.5 |
|---|---|---|---|---|---|
| Std. deviation | 0.02519988 | 0.01416013 | 0.01124600 | 0.006452271 | 0.004621237 |
| Prop. of Variance | 0.60566608 | 0.19123641 | 0.12062364 | 0.039706475 | 0.020368200 |
| Cum. Proportion | 0.60566608 | 0.79690249 | 0.91752612 | 0.957232600 | 0.977600800 |

| | Comp.6 | Comp.7 | Comp.8 | Comp.9 | Comp.10 |
|---|---|---|---|---|---|
| Std. deviation | 0.004256226 | 0.001615083 | 0.0008867617 | 0.0007392009 | 0.0006775836 |
| Prop. of Variance | 0.017277681 | 0.002487860 | 0.0007499806 | 0.0005211481 | 0.0004378869 |
| Cum. Proportion | 0.994878481 | 0.997366341 | 0.9981163215 | 0.9986374696 | 0.9990753565 |

Table 2.1: Summary of Principal Component Analysis.

Since the principal components of the instantaneous covariance matrix are in fact its eigenvectors, the result of the PCA highlights that the eigenspace of our covariance matrix is in fact spanned by only a handful of eigenvectors whereas the remaining eigenvectors add next to nothing to the spanned space. From a statistical point of view, it would be highly justifiable to reduce the dimensionality of the problem from 100 to probably around 10, as indicated by the PCA.

The instantaneous covariance matrices computed on basis of these time series indicate, that their smallest eigenvalues will in practice be extremely close to zero if we consider the usual rounding procedures applied by financial practitioners which may possibly have a precision goal somewhere around 12 after comma digits. The natural question arising in this context clearly is, how small the smallest (positive) eigenvalues in our simulations will get.

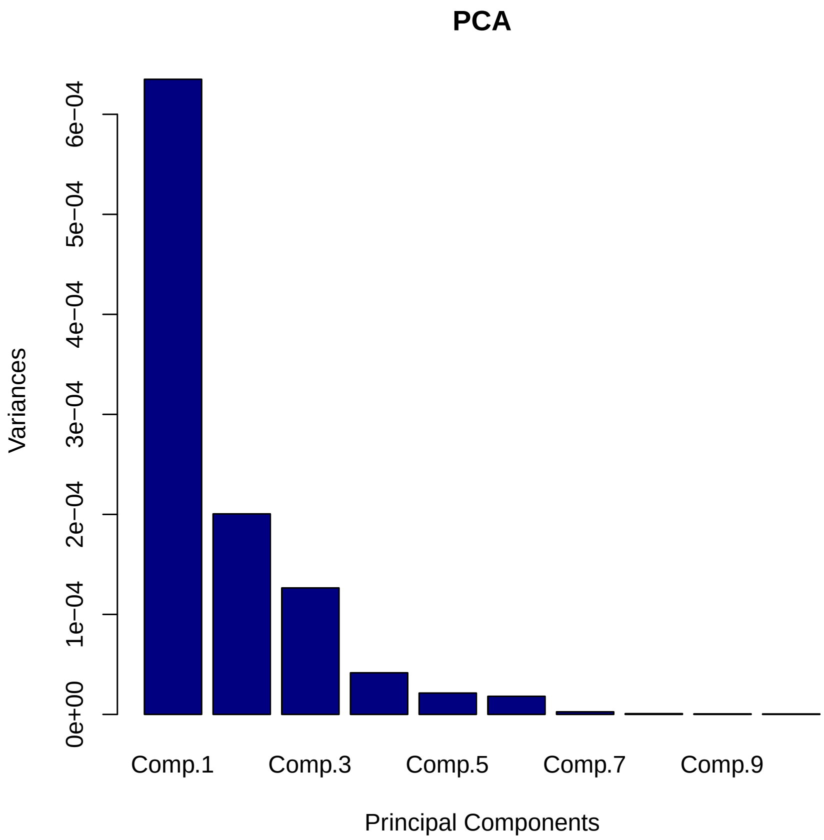


Figure 2.2: Scree-Plot of Principal Components of an Exemplary Covariance Matrix.

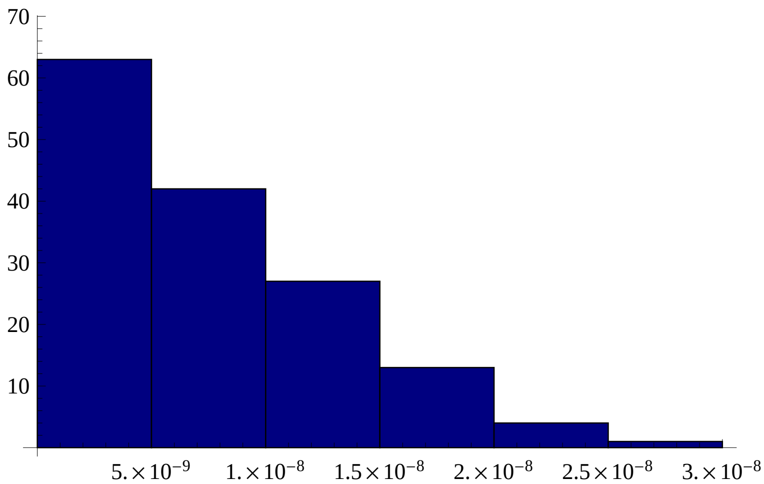


Figure 2.3: Smallest Eigenvalues for 150 Estimated Covariance Matrices multiplied by $10^{10}$.

In the next step we extend the number of utilized data points for the estimation of the instantaneous covariance matrix to 750 days. The overall pattern of the computed spectra of covariance matrices remains relatively stable, however we observe two important effects. Firstly, all calculated eigenvalues are strictly positive, i.e. we no longer observe the undesired effect of the smallest eigenvalues oscillating around zero as can be seen below in Figure 2.4.

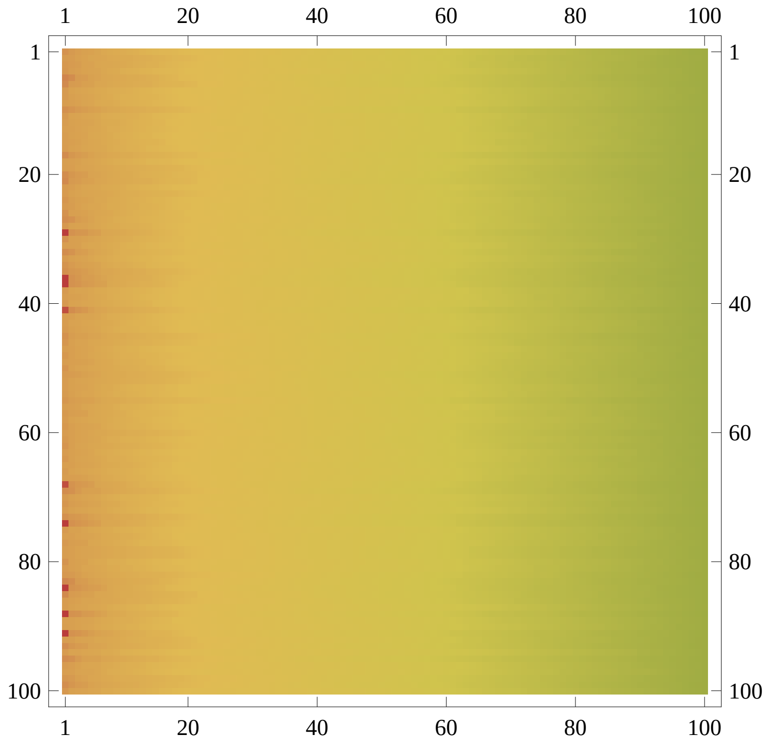


Figure 2.4: Heat-Maps for Estimators based on 750 Data Points.

Secondly, the size of the smallest eigenvalues has improved in terms of market ellipticity. For those estimators based on 750 data points, the size of the smallest eigenvalue is of order $10^{-8}$ which is a significant difference relative to the observations based on 350 or 400 data points. Furthermore the size of the smallest eigenvalues is quite stable over all 100 computed estimates as can be seen in Figure 2.5. Thus, if we estimate instantaneous covariance matrices based on longer time series we cannot rule out market ellipticity in a straightforward way. This is a valuable observation in the way that it helps to classify empirical market ellipticity as a long term effect rather than something which may be observable on arbitrarily short time horizons.

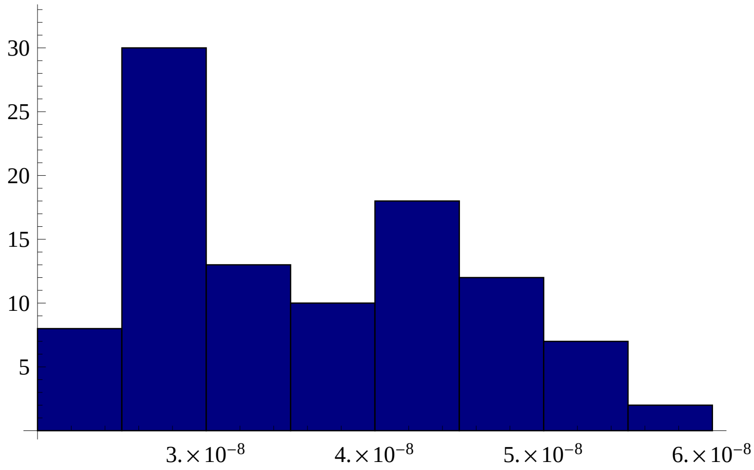


Figure 2.5: Smallest Eigenvalues for 100 Estimated Covariance Matrices (750 data points).

The empirical analysis conducted on the time series comprising 750 data points does not really rule out the concept of market ellipticity and unlike in the first set of analyses computed on the shorter time windows it is not straightforward to claim that the smallest eigenvalues are technically zero. However what may be done quite easily is to revisit for example the time bounds provided for relative arbitrage strategies as already presented in Sections 1.2 and 1.3.

We recall the outperformance time bound for the entropy portfolio as given in (1.54):

$$T > \frac{1}{\epsilon\delta^2}\log n(\log\log n - \log\delta_1),$$

whereby $\epsilon$ stems from the ellipticity assumption, $\delta$ is obtained from the condition of market diversity and we already set $\delta_1 = -\delta\log\delta$ in Section 1.2. If we plot this time bound as a function of $\delta$ as we have done before, we observe that the necessary time for outperforming the market will be in excess of eight million years for realistically chosen $\delta$ in the range of 0.7 to 0.8 and even larger for smaller values of $\delta$ as depicted in Figure 2.6. Hence even though in this case traces of market ellipticity are certainly observable, the resulting time bound for the entropy portfolio becomes fairly large.

A similar pattern may be observed in the case of the portfolio presented in Example 1.4[21], where we have a time bound for outperforming the market given by:

$$T > \frac{2n\log 2}{\epsilon\delta^2}.$$

[21]See Fernholz [44], Example 3.3.3.

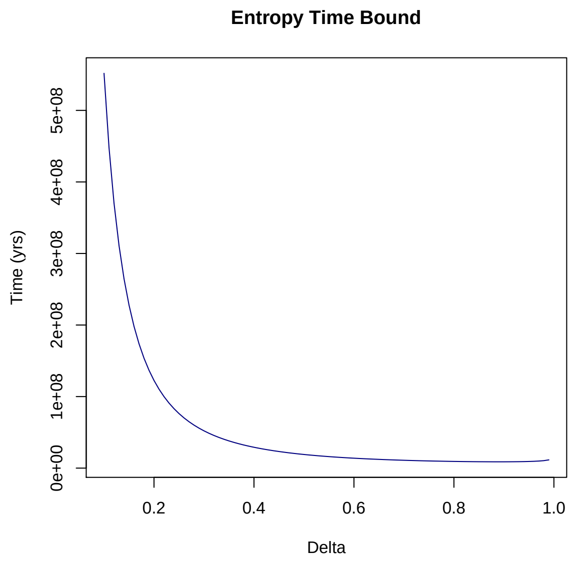


Figure 2.6: Simulated time bounds for the entropy portfolio.

The results for this simulation are depicted in Figure 2.7. For this portfolio we obtain results for the best time bounds lying in the range of 56 to 60 million years.

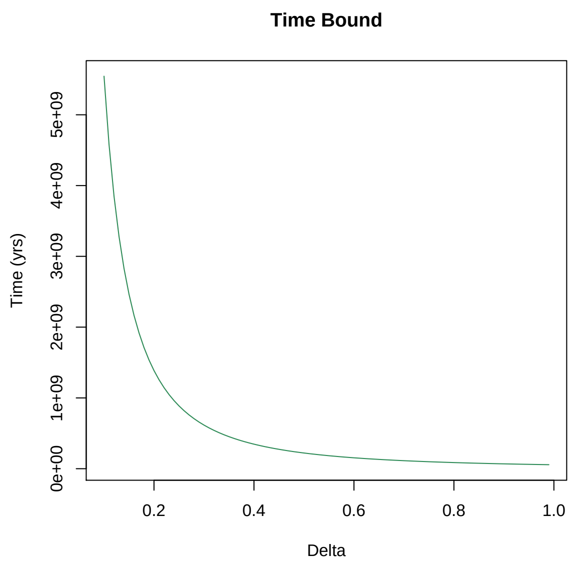


Figure 2.7: Simulated time bounds for the entropy portfolio.

It is of course also possible to select an elliptic investment universe if we arbitrarily define our "market" as a selection of a limited number of stocks with highly different characteristics. Even though such a voluntary restriction of the investment universe may seem plausible at first sight, it is in fact not an option for the vast majority of institutional investment managers who are obliged to manage the entrusted funds relative to a certain benchmark index. Since many institutional clients do not only prescribe the benchmark index but in addition to that specify limits on the tracking error[22] of their portfolio relative to the benchmark, the possibilities of investment managers to restrict themselves to a non-degenerate sub-market rather than the potentially degenerate investment universe specified by the benchmark index are limited.

## 2.2 Relaxation of the Ellipticity Assumption

Even though the empirical analysis in Section 2.1.2 does not totally discourage the notion of market non-degeneracy (ellipticity), it is certainly worth considering a relaxation of the uniform boundedness of the smallest eigenvalue by $\epsilon > 0$ as mandated by Definition 1.5. In the following sections we will introduce a weaker concept of ellipticity and outline what happens if the classical assumption of market-ellipticity is dropped.

We have introduced a market $\mathfrak{M}$ as a family of stocks $X_1, \ldots, X_n$ which are defined by the semimartingale model for the log-price given in Equation 1.11:

$$d \log X_i(t) = \gamma_i(t)dt + \sum_{\nu=1}^{n} \xi_{i,\nu}(t)dW_\nu(t), \;\; i = 1, \ldots, n.$$

Recall that the matrix of sensitivities $\xi(t) = (\xi_{i,\nu}(t))_{1 \leq i,\nu \leq n}$ directly leads to the definition of the instantaneous covariance matrix $\sigma(t) = \xi(t)\xi(t)^T$. We have introduced two fundamental properties of the market $\mathfrak{M}$ in Chapter 1 which were broadly used for proving the results outlined in the aforementioned chapter, namely non-degeneracy and bounded variance:

- $\mathfrak{M}$ is non-degenerate, if $\exists \epsilon > 0$, s.t. $x\sigma(t)x^T \geq \epsilon \|x\|^2$, for all $x \in \mathbb{R}^n$.
- $\mathfrak{M}$ is of bounded variance if $\exists M > 0$, s.t. $x\sigma(t)x^T \leq M\|x\|^2$, for all $x \in \mathbb{R}^n$.

Let us recall, that $\sigma(t)$ is positive semidefinite by construction and that the non-degeneracy (or ellipticity) assumption ensured that it would actually be positive definite. We will relax

[22] Here the term "tracking error" is used for the relative standard deviation of a portfolio $\pi$ relative to a benchmark portfolio $\eta$. Following the specification given in Equation 1.34, we use the term tracking error to describe $\sqrt{\tau^{\eta}_{\pi\pi}}$. See e.g. Jorion [63].

this assumption in the ensuing considerations. N.b. that we do not explicitly drop the assumption of bounded variance, i.e. of an upper bound for the largest eigenvalue of the instantaneous covariance matrix.

**Definition 2.1:** *Let $\mathfrak{M}$ be an equity market. Then we will call $\mathfrak{M}$ weakly non-degenerate or weakly elliptic, if there exists a random $\epsilon(t) \geq 0$ such that for all $t \in [0, \infty)$*

$$x\sigma(t)x^T \geq \epsilon(t)\|x\|^2, \; for\, all\, x \in \mathbb{R}^n. \tag{2.16}$$

By means of introducing the concept of weak ellipticity, we create a more general setup which should be able to accommodate those facets of ellipticity which are identifiable in market data. Since in this case, $\epsilon(\cdot)$ is a function of time $t \in [0, \infty)$ we allow the degree of non-degeneracy to vary, even admitting the case that $\epsilon(t) = 0$ for certain $t \in [0, \infty)$. In the sequel, we will revisit some properties and results presented in Chapter 1 in a weakly elliptic setup.

### 2.2.1 The Relative Covariance Matrix

Let us recall the definition of the relative covariance matrix $\tau(t)$ with respect to a portfolio $\eta$ which was given in Definition 1.7:

$$\tau^\eta_{ij}(t) = \sigma_{ij}(t) - \sigma_{i\eta}(t) - \sigma_{j\eta}(t) + \sigma_{\eta\eta}(t),$$

whereby $\sigma_{i\eta}(t) = \sum_{k=1}^n \eta_k(t)\sigma_{ik}(t)$. If we assume ellipticity, $\tau^\eta(t)$ is positive semidefinite with rank $n-1$ and the kernel (null space) $ker(\tau(t))$ is spanned by $\eta(t)$.[23] The proof of this result utilizes the positive definiteness of $\sigma(t)$[24], however we may attain this result if the ellipticity assumption is relaxed to weak ellipticity as well. Let us re-state the component-wise specification of $\tau(t)$ from Equation 1.29 in matrix terms:

$$\tau(t) = \sigma(t) - \underbrace{\begin{pmatrix} \sigma_{1\eta}(t) & \cdots & \sigma_{1\eta}(t) \\ \vdots & & \vdots \\ \sigma_{n\eta}(t) & \cdots & \sigma_{n\eta}(t) \end{pmatrix}}_{=:\sigma^\eta(t)} - \underbrace{\begin{pmatrix} \sigma_{1\eta}(t) & \cdots & \sigma_{n\eta}(t) \\ \vdots & & \vdots \\ \sigma_{1\eta}(t) & \cdots & \sigma_{n\eta}(t) \end{pmatrix}}_{=\sigma^\eta(t)^T} + \sigma_{\eta\eta}(t) \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$

If we calculate $x\tau(t)x^T$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we get:

[23]See Fernholz [44], Lemma 1.2.2.
[24]See Fernholz [44], Proof of Lemma 1.2.2.

$$
\begin{aligned}
x\tau(t)x^T &= x\sigma(t)x^T - x\sigma^\eta(t)x^T - x\sigma^\eta(t)^T x^T + \sigma_{\eta\eta}(t)x \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} x^T \qquad (2.17) \\
&= x\sigma(t)x^T - 2\underbrace{\sum_{i=1}^n x_i\sigma_{i\eta}}_{=x\sigma(t)\eta(t)^T}\sum_{i=1}^n x_i + \sigma_{\eta\eta}(t)\left(\sum_{i=1}^n x_i\right)^2. \qquad (2.18)
\end{aligned}
$$

Following the Proof as it is outlined in Fernholz [44], one may distinguish between two cases.

Firstly, let $\sum_{i=1}^n x_i = a \neq 0$. Then we may consider the re-scaled vector $y = \frac{1}{a}x$ which has $\sum_{i=1}^n y_i = 1$ and $x\tau(t)x^T = a^2 y\tau(t)y^T$. Hence, for $y$ we have:

$$
\begin{aligned}
y\tau(t)y^T &= y\sigma(t)y^T - 2y\sigma(t)\eta(t)^T + \eta(t)\sigma(t)\eta(t)^T \\
&= (y-\eta(t))\sigma(t)(y-\eta(t))^T \geq 0, \qquad (2.19)
\end{aligned}
$$

since $\sigma(t)$ is positive semidefinite in any case.

Secondly, if $a = 0$ then (2.18) reduces to $x\tau(t)x^T = x\sigma(t)x^T \geq 0$ again by the positive semidefiniteness of $\sigma(t)$. Hence the positive semidefiniteness of $\tau(t)$ still holds if the ellipticity assumption is dropped. From (2.19) one directly sees that we have equality to zero in the case that $y = \eta(t)$, i.e. $x = a\eta(t)$, hence the kernel $ker(\tau(t))$ is spanned by the vector $\eta(t)$ and therefore $\dim ker(\tau(t)) = 1$. According to the dimension formula for linear maps[25] we have that $\dim \mathbb{R}^n = n = \dim im(\tau(t)) + \dim ker(\tau(t))$. Since the dimension of the kernel of $\tau(t)$ is one, it follows that the dimension of the image, i.e. the rank of the matrix $\tau(t)$ is $n-1$.

### 2.2.2 Market Coherence and Diversity

In this Section we will revisit some of the results on market coherence and diversity which were outlined in Sections 1.2.1 and 1.2.2. Let us recall that a market $\mathfrak{M}$ is called coherent if the market weights for any stock do not decline exponentially or faster, i.e.:

$$\lim_{t\to\infty}\frac{1}{t}\log\mu_i(t) = 0 \ \ a.s.,\ i = 1,\ldots,n.$$

The most remarkable result concerning coherence was stated in Proposition 1.4, namely that in a coherent and elliptic market any constant-weighted portfolio with at least two

[25] See e.g. Fischer [53], Section 2.2.4.

positive weights outperforms the market. The proof of this Proposition strongly relies on ellipticity for which reason we will revisit the notions which were used to retrieve this result.

An essential pre-requisite for the proof of Proposition 1.4 is given in Fernholz[26] namely that for any portfolio $\pi$ in an elliptic market $\mathfrak{M}$ there exists an $\epsilon > 0$, such that:

$$\tau_{ii}^{\pi}(t) \geq \epsilon(1 - \pi_i(t))^2, \;\; a.s., \; t \in [0, \infty), \; i = 1, \ldots, n. \tag{2.20}$$

The proof of this result uses the non-degeneracy assumption of $\mathfrak{M}$ in the following way. Define $x(t) := (\pi_1(t), \ldots, \pi_i(t) - 1, \ldots, \pi_n(t))$, then by Definition 1.7 $\tau_{ii}^{\pi}$ is given by:

$$\begin{aligned} \tau_{ii}^{\pi}(t) &= \sigma_{ii}(t) - 2\sigma_{i\pi}(t) + \sigma_{\pi\pi}(t) & (2.21) \\ &= x(t)\sigma(t)x(t)^T \geq \epsilon \|x(t)\|^2, & (2.22) \end{aligned}$$

where clearly (2.22) only holds in the setup of an elliptic market. In a weakly-elliptic market, on may only state the weaker result, that:

$$\tau_{ii}^{\pi}(t) \geq \epsilon(t) \|x(t)\|^2 \geq 0 \;\; a.s., \; t \in [0, \infty). \tag{2.23}$$

Hence, although we are able to preserve the positive semidefiniteness of $\tau^{\pi}(t)$ in a weakly-elliptic setup, we lose the positivity result for its diagonal elements, i.e. the relative variances of the stocks w.r.t. the portfolio $\pi$.

This fact also has implications on the excess growth rate of portfolio $\pi$ as it was introduced in Equation 1.16 of Proposition 1.1:

$$\gamma_{\pi}^*(t) = \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sigma_{ij}(t)\right).$$

Due to the numéraire invariance of the excess growth rate[27], the excess growth rate $\gamma_{\pi}^*(t)$ may also be expressed in terms of the relative covariance matrix with respect to some arbitrary portfolio $\eta$ in $\mathfrak{M}$:

$$\gamma_{\pi}^*(t) = \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\eta}(t) - \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\tau_{ij}^{\eta}(t)\right). \tag{2.24}$$

Under the prevalence of ellipticity Equation 2.24 may be combined with Equation 2.22 to

[26] See Fernholz [44], Lemma 2.1.5.
[27] See Fernholz [44], Lemma 1.3.4.

obtain the following result (2.27) for $\gamma^*_\pi(t)$. Since for $\tau^\pi$

$$\begin{aligned}\sum_{i,j=1}^{n}\pi_i(t)\pi_j(t)\tau^\pi_{ij}(t) &= \sum_{i,j=1}^{n}\pi_i(t)\pi_j(t)\sigma_{ij}(t)-\sum_{i,j=1}^{n}\pi_i(t)\pi_j(t)\sigma_{i\pi}(t)- \\ &\quad \sum_{i,j=1}^{n}\pi_i(t)\pi_j(t)\sigma_{j\pi}(t)+\sigma_{\pi\pi}(t)\sum_{i,j=1}^{n}\pi_i(t)\pi_j(t) \qquad (2.25)\\ &= 2\sigma_{\pi\pi}(t)-2\sum_{i,j=1}^{n}\pi_i(t)\pi_j(t)\sigma_{ij}(t)=0, \qquad (2.26)\end{aligned}$$

one gets:

$$\gamma^*_\pi(t)=\frac{1}{2}\sum_{i=1}^{n}\pi_i(t)\tau^\pi_{ii}(t)\geq\frac{\epsilon}{2}(1-\pi_{max}(t))^2. \qquad (2.27)$$

In a weakly-elliptic setup one obtains instead

$$\gamma^*_\pi(t)\frac{1}{2}\sum_{i=1}^{n}\pi_i(t)\tau^\pi_{ii}(t)\geq\frac{\epsilon(t)}{2}(1-\pi_{max}(t))^2\geq 0\ for\, all\, t\in[0,\infty). \qquad (2.28)$$

Hence, assuming only weak ellipticity, the excess growth rate may become arbitrarily small or even zero. Furthermore, we recall that the excess growth rate corresponds to half of the diversification benefit in terms of risk (covariance) which one attains due to the covariance structure of the market. Thus, in the case of weak ellipticity of the market one may observe cases where such a diversification benefit vanishes. It is worth noting that such a loss of commonly observed diversification patterns within and across asset classes is precisely what could be observed during the financial crisis of 2008 and 2009. Furthermore, it should be mentioned that supervisory authorities explicitly mandate the effects of distortions in generally observed correlation structures to be considered in banks' stress testing scenarios.[28] Nonetheless, if we preserve the property of market coherence, one may modify Proposition 1.4 to show the following result.

**Proposition 2.1:** *Suppose that $\mathfrak{M}$ is coherent and weakly-elliptic and that $\pi$ is a constant-weighted portfolio with at least two positive weights, then:*

$$\liminf_{T\to\infty}\frac{1}{T}\log\left(\frac{Z_\pi(T)}{Z_\mu(T)}\right)\geq 0\ \ a.s. \qquad (2.29)$$

Thus, if we drop the uniform ellipticity assumption prevalent in Proposition 1.4 but maintain the assumption of market coherence it still holds, that any constant-weighted portfolio with at least two positive weights will at least not underperform the market. Even though

[28] See Basel Committee on Banking Supervision [10].

this result is clearly much weaker than the almost sure outperformance stated in Proposition 1.4, it is still quite a strong statement that any constant-weighted portfolio will at least not perform worse than the market with probability one. This effect makes it clear that the assumption of market coherence is fairly strong in its own right.

*Proof:* We follow the proof given in Fernholz [44] for Proposition 1.4[29] and digress from it at the points where the ellipticity property comes into play.

Let $\pi_i(t) = p_i$ be the constant portfolio weights, $t \in [0, \infty)$, $i = 1, \ldots, n$. In a weakly-elliptic market we have:

$$\frac{1}{T}\int_0^T \gamma_\pi^*(t)dt \geq 0, \;\; T \in [0, \infty) \; a.s. \tag{2.30}$$

Since by Proposition 1.2

$$d\log\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) = \sum_{i=1}^n p_i d\log\mu_i(t) + \gamma_\pi^*(t)dt \;\; a.s.$$

one obtains:

$$\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}\left(\log\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) - \int_0^T \gamma_\pi^*(t)dt\right) &= \lim_{T\to\infty}\frac{1}{T}\left(\sum_{i=1}^n p_i \log\mu_i(T)\right) \\
&= \sum_{i=1}^n p_i\left(\lim_{T\to\infty}\frac{1}{T}\log\mu_i(T)\right) \\
&= 0
\end{aligned}$$

by market coherence. Therefore and by (2.30) we have:

$$\liminf_{T\to\infty}\frac{1}{T}\log\left(\frac{Z_\pi(T)}{Z_\mu(T)}\right) \geq 0 \;\; a.s.$$

□

Analogously, dropping the uniform ellipticity assumption leads to problems with some results concerning market diversity which are given in Fernholz [44]. E.g. it can be shown[30] that in a non-degenerate and diverse market the excess growth rate of the market portfolio is strictly positive, i.e. $\gamma_\mu^*(t) \geq \delta > 0$ a.s. for $t \in [0, \infty)$ as we have already stated in Equation 1.47. This is proved using the fact that in an elliptic market $\gamma_\mu^*(t) \geq \epsilon(1 - \mu_{max}(t))^2$ as stated in Equation 2.27. Clearly enough, for the market portfolio as a special example it

[29] See Fernholz [44], Proof of Proposition 2.1.9.
[30] See Fernholz [44], Proposition 2.2.2.

also holds that in a weakly-elliptic setup we only have $\gamma_\mu^*(t) \geq \epsilon(t)(1 - \mu_{max}(t))^2 \geq 0$ for all $t \in [0, \infty)$.

This has a direct implication on the finite time bounds for outperforming the market portfolio which we have presented in Chapter 1. For the entropy-weighted portfolio $\pi$ we showed in Lemma 1.1 that $\frac{Z_\pi(T)}{Z_\pi(0)} > \frac{Z_\mu(T)}{Z_\mu(0)}$ a.s. if $T > \frac{1}{\delta_2} \log n(\log \log n - \log \delta_1)$, where $\delta_2$ is the lower bound for the excess growth rate of the market portfolio. As outlined above, in a weakly-elliptic setting $\delta_2$ may become zero and hence the time bound will become infinite. As a matter of fact, the practical analysis of outperformance time bounds as presented in Sections 1.2.2, 1.3.2 and 2.1.2 suggests that the time bounds presented in the literature may become extremely large. In a weakly-elliptic market $\epsilon(t) \geq 0$ for all $t \in [0, , \infty)$ may become arbitrarily small and even zero, therefore the concept of weak ellipticity allows us to accommodate the observable behavior.

### 2.2.3 Functionally Generated Portfolios and Relative Arbitrage

Relieving the ellipticity assumption does not prevent us from maintaining the central results for the functional generation of portfolios. Since ellipticity is not required in the proofs of Theorem 1.1 and Proposition 1.7, the approach for calculating the portfolio weights and the drift term of a functionally generated portfolio $\pi$ stays the same. The proof of Theorem 1.1 only uses the fact, that the null-space of the relative covariance matrix $\tau^\mu(t)$ is spanned by $\mu$ which still holds in a weakly-elliptic setting as outlined above.

The impact of relaxing the uniform ellipticity assumption becomes more severe if we revisit the results on relative arbitrage which we outlined in Section 1.3.1. In Example 1.3 we have seen that the portfolio generated by $S(x) = 1 - \frac{1}{2}\sum_{i=1}^n x_i^2$ outperforms the market portfolio almost surely if the investment horizon is larger than $T > \frac{2n \log 2}{\epsilon \delta^2}$. Here once again the $\epsilon$ comes from the non-degeneracy condition of $\mathfrak{M}$. Hence, clearly in a weakly-elliptic setup this time bound may become infinite since $\epsilon(t) \geq 0$ for all $t \in [0, \infty)$. N.b. that if $\epsilon(t) = 0$ for some $t \in [0, \infty)$ it holds for the drift of the functionally generated portfolio that $\Theta(T) \geq 0$. Hence, if the required investment horizon for outperforming the market becomes infinite, the drift of the functionally generated portfolio may vanish as well.

Similarly, one has to revisit the results concerning the construction of arbitrage opportunities based on mirror portfolios which are developed in Fernholz, Karatzas and Kardaras [51] and discussed in the ensuing works by Fernholz and Karatzas [48] and [49]. These results, some of which were outlined in Section 1.3, also rely on the ellipticity and bounded variance of the market $\mathfrak{M}$. The first important observation one makes concerns the relative variance of portfolios $\pi$ and $\eta$ for which we stated the following notion in Equation 1.82:

$$\tau^{\eta}_{\pi\pi}(t) = (\pi(t) - \eta(t))\, \sigma(t)\, (\pi(t) - \eta(t))^T \geq \epsilon \|\pi(t) - \eta(t)\|^2.$$

In the case of a weakly-elliptic market, this reduces to the statement

$$\tau^{\eta}_{\pi\pi}(t) = (\pi(t) - \eta(t))\, \sigma(t)\, (\pi(t) - \eta(t))^T \geq \epsilon(t) \|\pi(t) - \eta(t)\|^2 \geq 0, \;\; for\, all\, t \in [0, \infty), \tag{2.31}$$

since $\sigma(t)$ is only positive semidefinite. Hence, for the relative return of the p-mirror image $\tilde{\pi}^{(p)}$ and $\mu$ which is described by:

$$\log\left(\frac{Z^{\tilde{\pi}^{(p)}}(T)}{Z^{\mu}(T)}\right) = p \log\left(\frac{Z^{\pi}(T)}{Z^{\mu}(T)}\right) + \underbrace{\frac{p(1-p)}{2} \int_0^T \tau^{\mu}_{\pi\pi}(t)dt}_{(\maltese)},$$

as stated in Equation 1.83, we notice that we still have nonnegativity of ($\maltese$) by Inequality 2.31. However, ($\maltese$) need not be strictly positive as implied by $\tau^{\mu}_{\pi\pi}(t) \geq \epsilon(t)\|\pi(t) - \mu(t)\|^2$. This leads to problems in the proof of Lemma 1.2 [31] where the condition is used that $\int_0^T \tau^{\mu}_{\pi\pi}(t)dt \geq \kappa > 0$ a.s. Thus in a weakly-elliptic market it holds that

$$\int_0^T \tau^{\mu}_{\pi\pi}(t)dt \geq \int_0^T \epsilon(t)\|\pi(t) - \eta(t)\|^2 dt \geq 0.$$

**Example 2.1:** *Let us at this point revisit Example 1.4*[32] *and the results presented there. For a portfolio* $\pi = e_1 = (1, 0, \ldots, 0)$ *the first unit vector, one may define the portfolio:*

$$\hat{\pi}(t) := \tilde{\pi}^{(p)}(t) = pe_1 + (1-p)\mu(t), \;\; t \in [0, \infty),$$

*where the leverage parameter* $p = p(T)$ *will depend on the investment horizon* $T$*. Hence, one takes a long position in the first stock and shorts the market against it, whereby one observes that the strategy gets ever more aggressive, the higher the leverage factor* $p(T)$ *becomes. The relative log-returns of* $\hat{\pi}$ *versus* $\mu$ *can be written as:*[33]

$$\log\left(\frac{Z_{\hat{\pi}}(T)}{Z_{\mu}(T)}\right) = p \left[\underbrace{\log\left(\frac{\mu_1(T)}{\mu_1(0)}\right)}_{(*)} - \underbrace{\frac{p-1}{2} \int_0^T \tau^{\mu}_{11}(t)dt}_{(**)}\right].$$

*The first term* $(*)$ *indicates the performance of stock number one in terms of its market capitalization. The second term* $(**)$ *needs to be examined more closely. Setting* $\beta := \mu_1(0)$

[31] See also Fernholz, Karatzas and Kardaras [51], Lemma 8.1.
[32] See also Fernholz Karatzas and Kardaras [51], Example 8.1.
[33] See Fernholz Karatzas and Kardaras [51], Equation 8.12.

*and thus obtaining* $\frac{\mu_1(T)}{\mu_1(0)} \leq \frac{1}{\beta}$ *and by using weak diversity one obtains in an elliptic market:*

$$\int_0^T \tau_{11}^{\mu}(t)dt \geq \epsilon \int_0^T (1 - \mu_{max}(t))^2 dt > \epsilon\delta^2 T =: \kappa,$$

*by the ellipticity result* $\tau_{ii}^{\mu}(t) \geq \epsilon(1 - \mu_i(t))^2$.[34]

*In order to apply the approach to construct relative arbitrage strategies w.r.t. the market as outlined in Section 1.3 one needs*

$$\log\left(\frac{\mu_1(T)}{\mu_1(0)}\right) - \frac{p-1}{2}\int_0^T \tau_{11}^{\mu}(t)dt \leq 0,$$

*from which we may conclude that by construction:*

$$\frac{p-1}{2} \geq \frac{1}{\epsilon\delta^2 T}\log\left(\frac{1}{\mu_1(0)}\right) \Longrightarrow p(T) \geq 1 + \frac{2}{\epsilon\delta^2 T}\log\left(\frac{1}{\mu_1(0)}\right). \tag{2.32}$$

*Thus, if the uniform ellipticity assumption is relaxed to weak ellipticity, our leverage parameter* $p(T)$ *in fact becomes* $p(t,T)$ *for* $t, T \in [0,\infty)$, $t \leq T$. *Yet,* $p(t,T)$ *may not be bounded from above for any pair of* $t, T \in [0,\infty)$ *since* $\epsilon(t)$ *may become arbitrarily small and even zero for any* $t \in [0,\infty)$.

Consequently, the investment approach suggested by Example 1.4 would potentially require an infinite degree of leverage in the shorted portfolio $\hat{\pi}$ in a weakly elliptic setup. The reason for this may be found in the way Lemma 1.2 is applied in this example. Even though ellipticity is not requested explicitly for Lemma 1.2, it is needed for the construction of $\kappa > 0$ in $\int_0^T \tau_{11}^{\mu}(t)dt > \kappa$, which is based on an ellipticity result in Fernholz.[35]

It is quite clear that extremely leveraged investment in a single stock vs. shorting the market is not a very sensible strategy to pursue and will in many cases end up in catastrophic losses. However, if the $1^{st}$ stock sufficiently outperforms the market (i.e. the short position leads to a loss) which in real life may happen with positive probability for any investment horizon, the relative arbitrage strategy outlined in Examples 1.4 and 2.1 will lead to the investor who applies it to incur substantial losses.

[34] See Fernholz, Karatzas and Kardaras [51], Remark 5.1.
[35] By Lemma 2.1.5. in Fernholz [44] one obtains $\tau_{ii}^{\pi}(t) \geq \epsilon(1 - \pi_i(t))^2$.

## 2.3 Conclusion

In this Chapter 2 we have started with a numerical investigation of market ellipticity, investigating the structure of spectra of instantaneous covariance matrices, estimated via a Fourier algorithm first presented by Malliavin and Mancino [75]. The result of our numerical investigation on the S&P 100 index universe shows relatively weak traces of market ellipticity in the short and medium run with smallest eigenvalues in the area of $10^{-19}$. However, it has to be acknowledged, that in the first calculations, the number of observation points was not sufficient and that some of the observed instabilities of the estimation procedure are also due to this aspect. The stability of the estimation substantially improves with longer time horizons which also lead to eigenvalues in the range of $10^{-8}$. In general it is fair to conclude that market ellipticity may not be ruled out in a well diversified market setup over a sufficiently long time horizon, yet observable ellipticity may be weak in the case of market universes with a higher amount of similar stocks and resulting collinearities.

In order to cover the effects prevalent in market data we propose a relaxation of the uniform ellipticity as it is given in Definition 1.5. To this end we introduce the notion of weak ellipticity in Section 2.2 and revisit some results presented in Chapter 1 in this weaker setup. In general it can be observed that some results can only be obtained in a weaker form, given a weakly elliptic setup, yet many general properties may be preserved.

# Chapter 3

# Capitalization Structure of the Equity Market

In this chapter, we will outline the dynamics of stocks ranked by their market capitalization. This concept provides the mathematical basis for formulating classical investment approaches based on the distinction between large- and small-cap stocks. Furthermore, a closer study of the structure of the market's capital distribution will lead to a broad set of problems which will be discussed in the ensuing considerations. In Section 3.1 some mathematical preliminaries are given and the behavior of ranked stocks and market weights is outlined. In Section 3.2, we outline some modeling approaches which permit conserving the capital distribution and give an overview of ongoing research in this field.

## 3.1 Interaction of Stocks through their Ranks

In order to characterize the dynamics of ranked stocks, we will first introduce the concept of local times for Brownian Motion and continuous semimartingales. For a comprehensive treatment of local times we refer to Kallenberg[1], Karatzas and Shreve[2] or Revuz and Yor[3]. We will first characterize local time of a Brownian motion and will then state the characterization for continuous semimartingales.

The elementary question underlying the concept of local time is, how one may quantify

[1]See Kallenberg [64], Chapter 22.
[2]See Karatzas and Shreve [66], Sections 3.6. and 3.7.
[3]See Revuz and Yor [89], Chapter VI.

the time a Brownian motion or more generally a continuous semimartingale spends at a certain level. Denoting by $W(\cdot)$ a standard Brownian motion and by $\lambda(\cdot)$ the Lebesgue measure, one may introduce the level set $\mathscr{Z}_\omega(\cdot)$ of $W$ as:[4]

$$\mathscr{Z}_\omega(x) = \{0 \leq t < \infty \mid W(t,\omega) = x\}. \tag{3.1}$$

Intuitively, one would calculate the Lebesgue measure of this set in order to gain some insight on the amount of time $W$ spends at level $x$, however, it turns out that $\lambda(\mathscr{Z}_\omega(x)) = 0$ $P-a.e.$ for $\omega \in \Omega$.[5] In order to overcome this shortfall, Paul Lévy introduced the so-called "mésure du voisinage" or local time as:[6]

$$L(t,x) = \lim_{\epsilon \to 0} \frac{1}{4\epsilon} \lambda \left( \{0 \leq s \leq t \mid |W(s) - x| \leq \epsilon\} \right); \quad t \in [0,\infty),\, x \in \mathbb{R}. \tag{3.2}$$

**Remark 3.1:** For clarity of notation we will denote Brownian local time by $L(\cdot,\cdot)$ and the local time of a semimartingale $X$ by $\Lambda_X(\cdot,\cdot)$ or simply $\Lambda(\cdot,\cdot)$ if it is clear which process we are considering. Further we write $L(t) := L(t,0)$ and $\Lambda_X(t) := \Lambda_X(t,0)$.

Let us now state the formal Definition of Brownian local time and of the occupation time of a certain Borel set.

**Definition 3.1:** *For a Borel set $B \in \mathcal{B}(\mathbb{R})$, its occupation time by a Brownian motion is defined as:*[7]

$$\Gamma(t,B) := \int_0^t \mathbb{1}_B(W(s))ds = \lambda \left( \{0 \leq s \leq t \mid W(s) \in B\} \right). \tag{3.3}$$

*Let further $L = \{L(t,x,\omega) \mid (t,x) \in [0,\infty) \times \mathbb{R},\, \omega \in \Omega\}$ denote the random field taking values in $[0,\infty)$, such that for each fixed tuple $(t,x)$ the random variable $L(t,x)$ is $\mathfrak{F}_t$-measurable. Denote by $P^z(A) = P^0(A - z)$ the probability measure corresponding to a Brownian motion with starting point $z$ and suppose there exists $\Omega^* \in \mathfrak{F}$ with $P^z(\Omega^*) = 1$ for all $z \in \mathbb{R}$ such that for all $\omega \in \Omega^*$ the function $(t,x) \mapsto L(t,x,\omega)$ is continuous and it holds that:*

$$\Gamma(t,B,\omega) = \int_B 2L(t,x,\omega)dx;\ 0 \leq t < \infty,\ B \in \mathcal{B}(\mathbb{R}). \tag{3.4}$$

*Then $L$ is called Brownian local time.*[8]

As stated in Equation 3.4, $2L(t,x)$ serves as density for the occupation time with respect to Lebesgue measure.[9] It is worth noting, that the definitions of local time in the literature

[4] See e.g. Karatzas and Shreve [66], Section 2.9.B.
[5] See Karatzas and Shreve [66], Theorem 2.9.6.
[6] See e.g. Karatzas and Shreve [66], Equation (3.6.2.)
[7] See Karatzas and Shreve [66], Example 3.6.2.
[8] See Karatzas and Shreve [66], Definition 3.6.3.
[9] See also Karatzas and Shreve [66], Section 3.6.A.

vary, especially concerning the fact whether $L(t,x)$ or $2L(t,x)$ is to be called local time. In the ensuing considerations we will follow the characterization as given in Karatzas and Shreve [66] which is also in line with the notion of local time used in Fernholz [44]. Usually, Brownian local time is expressed by means of Tanaka's formulae:[10]

$$L(t,a) = (W(t)-a)^+ - (W(0)-a)^+ - \int_0^t \mathbb{1}_{(a,\infty)}(W(s))dW(s), \tag{3.5}$$

$$L(t,a) = (W(t)-a)^- - (W(0)-a)^- + \int_0^t \mathbb{1}_{(-\infty,a]}(W(s))dW(s), \tag{3.6}$$

$$L(t,a) = \frac{1}{2}\left(|W(t)-a| - |W(0)-a| - \int_0^t \operatorname{sgn}(W(s)-a)dW(s)\right), \tag{3.7}$$

whereby the sign function is defined as:

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0; \\ -1, & x \leq 0. \end{cases} \tag{3.8}$$

The existence of Brownian local time was proved by Trotter.[11] Analogously to the characterization of local time for Brownian motion, one may obtain the corresponding results for continuous semimartingales.[12] Analogously to the Tanaka formulae 3.5 to 3.7, local time for a continuous semimartingale $X(\cdot)$ is given by the Tanaka-Meyer formulae:[13]

$$\Lambda_X(t,a) = (X(t)-a)^+ - (X(0)-a)^+ - \int_0^t \mathbb{1}_{(a,\infty)}(X(s))dX(s), \tag{3.9}$$

$$\Lambda_X(t,a) = (X(t)-a)^- - (X(0)-a)^- + \int_0^t \mathbb{1}_{(-\infty,a]}(X(s))dX(s), \tag{3.10}$$

$$\Lambda_X(t,a) = \frac{1}{2}\left(|X(t)-a| - |X(0)-a| - \int_0^t \operatorname{sgn}(X(s)-a)dX(s)\right). \tag{3.11}$$

By employing the concept of local time, one may now investigate the dynamics of ranked stocks and the behavior of portfolios based on ranked market weights. Let us consider the following ordering, following for the sake of consistency the notation introduced in Fernholz [44]. Let us define the $k^{th}$ ranked stock as:[14]

$$X_{(k)}(t) = \max_{1\leq i_1<\cdots<i_k\leq n} \min\left(X_{i_1},\ldots,X_{i_k}\right),\ t\in[0,T]. \tag{3.12}$$

We denote by $X_{(\cdot)}(t) = (X_{(1)}(t),\ldots,X_{(n)}(t))$ the vector of ranked stocks at time $t\in[0,T]$ and by $\mu_{(\cdot)}(t) = (\mu_{(1)}(t),\ldots,\mu_{(n)}(t))$ the vector of ranked market weights, ordered from the largest to the smallest.

[10] See e.g. Karatzas and Shreve [66], (3.6.11) - (3.6.13).
[11] See e.g. Karatzas and Shreve [66], Theorem 3.6.11.
[12] See e.g. Kallenberg [64], Chapter 22. or Karatzas and Shreve [66], Section 3.7.
[13] See Karatzas and Shreve [66], (3.7.7) to (3.7.9) or Fernholz [44], Section 4.1.
[14] See e.g. Fernholz [44], Definition 4.0.1.

**Definition 3.2:** [15] *Let us call the processes $X_1, \ldots, X_n$ pathwise mutually non-degenerate, if they satisfy:*

1. $\forall\, i \neq j:\ \lambda\left(\{t \mid X_i(t) = X_j(t)\}\right) = 0.$
2. $\forall\, i < j < k:\ \{t \mid X_i(t) = X_j(t) = X_k(t)\} = \varnothing$ *a.s.*

Hence, the pathwise mutual non-degeneracy of processes ensures, that the set of times, when two processes collide has zero Lebesgue measure and that three processes never collide. Furthermore, we shall denote by $p_t(k)$ the permutation which yields the name (index) of the $k^{th}$ ranked stock at time $t$, i.e. $X_{(k)}(t) = X_{p_t(k)}(t)$ and let us impose the ex-aequo condition $p_t(k) < p_t(k+1)$ if $X_{(k)}(t) = X_{(k+1)}(t)$ on our market $\mathfrak{M}$. Then, assuming now that our family of stocks $X_1, \ldots, X_n$ fulfills the assumption of pathwise mutual non-degeneracy, the dynamics of ranked stocks may be expressed as follows:[16]

$$dX_{(k)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{i\}}(p_t(k))dX_i(t) + \frac{1}{2}d\Lambda_{X_{(k)}-X_{(k+1)}}(t) - \frac{1}{2}d\Lambda_{X_{(k-1)}-X_{(k)}}(t). \tag{3.13}$$

For the largest and smallest stocks, Equation 3.13 changes to:

$$\begin{aligned} dX_{(1)}(t) &= \sum_{i=1}^{n} \mathbb{1}_{\{i\}}(p_t(1))dX_i(t) + \frac{1}{2}d\Lambda_{(X_{(1)}-X_{(2)})}(t), && (3.14)\\ dX_{(n)}(t) &= \sum_{i=1}^{n} \mathbb{1}_{\{i\}}(p_t(n))dX_i(t) - \frac{1}{2}d\Lambda_{(X_{(n-1)}-X_{(n)})}(t). && (3.15) \end{aligned}$$

Building on these dynamics, one may reformulate Theorem 1.1 as follows.

**Theorem 3.1:** [17] *Denote the relative rank covariance process by $\tau_{(ij)}(t) = \tau_{p_t(i)p_t(j)}(t)$ and consider a market $\mathfrak{M}$ consisting of $n$ stocks satisfying the pathwise mutual non-degeneracy condition. Let further $\mathcal{S}$ be a function defined on a neighborhood $U$ of the simplex $\Delta^n$ and let $S$ be a positive $\mathcal{C}^2$ function on $U$, such that for $(x_1, \ldots, x_n) \in U$:*

$$\mathcal{S}(x_1, \ldots, x_n) = S(x_{(1)}, \ldots, x_{(n)}), \tag{3.16}$$

*and suppose that $x_i \frac{\partial}{\partial x_i} \log S(x)$ is bounded for all $i = 1, \ldots, n$. Then $\mathcal{S}$ generates a portfolio $\pi$ with weights:*

$$\pi_{p_t(k)}(t) = \left(\frac{\partial}{\partial x_k} \log S(\mu_{(\cdot)}(t)) + 1 - \sum_{j=1}^{n} \mu_{(j)}(t) \frac{\partial}{\partial x_j} \log S(\mu_{(\cdot)}(t))\right) \mu_{(k)}(t), \tag{3.17}$$

[15] See Fernholz [44], Definition 4.1.2.
[16] See Fernholz [44], Proposition 4.1.11.
[17] See Fernholz [44], Theorem 4.2.1.

*for* $t \in [0,T]$ *a.s. with drift* $\Theta(\cdot)$ *satisfying:*

$$\begin{aligned} d\Theta(t) &= \frac{-1}{2\mathcal{S}(\mu(t))}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}S(\mu_{(\cdot)}(t))\mu_{(i)}(t)\mu_{(j)}(t)\tau_{(ij)}(t)dt + \\ & \frac{1}{2}\sum_{k=1}^{n-1}\left(\pi_{p_t(k+1)}(t) - \pi_{p_t(k)}(t)\right) d\Lambda_{\log\mu_{(k)}-\log\mu_{(k+1)}}(t), \end{aligned} \tag{3.18}$$

*for all* $t \in [0,T]$ *a.s.*

Recalling that a market $\mathfrak{M}$ is called coherent if $\lim_{t\to\infty}\frac{1}{t}\log\mu_i(t) = 0 \ \forall\, i = 1,\dots,n$ as set out in Definition 1.9, one may directly conclude that this also holds in the case of ranked market weights. This is evident since for any rank $k = 1,\dots,n$ the ranked market weight $\mu_{(k)}(t)$ equals the market weight corresponding to the name $i = p_t(k)$ for any time $t \in [0,T]$, whence:

$$\lim_{t\to\infty}\frac{1}{t}\log\mu_{(k)}(t) = 0 \ \forall k = 1,\dots,n. \tag{3.19}$$

Furthermore, the dynamics of the ranked market weights can be seen to follow:[18]

$$\begin{aligned} d\log\mu_{(k)}(t) &= \sum_{i=1}^{n}\mathbb{1}_{\{i\}}(p_t(k))d\log\mu_i(t) + \\ & \frac{1}{2}d\Lambda_{\log\mu_{(k)}-\log\mu_{(k+1)}}(t) - \frac{1}{2}d\Lambda_{\log\mu_{(k-1)}-\log\mu_{(k)}}(t) \ a.s.,\ t\in[0,T]. \end{aligned} \tag{3.20}$$

The central question which arises in this context is, whether the ranked dynamics of a certain market model exhibit a certain stability over time. This question was raised by Fernholz[19] and has played a central role in recent works by Chatterjee and Pal [22], Ichiba et al. [61], Pal and Pitman [82], Pal and Shkolnikov [83] and Shkolnikov [93], [94]. In order to study this question, Fernholz [44] introduced the notion of asymptotic stability.

**Definition 3.3:** [20] *Let* $\mathfrak{M}$ *be a coherent market. Then* $\mathfrak{M}$ *is called asymptotically stable if it holds almost surely that:*

1. *for* $k = 1,\dots,n-1$ *it holds that* $\lim_{t\to\infty}\frac{1}{t}\Lambda_{\log\mu_{(k)}-\log\mu_{(k+1)}}(t) = \lambda_{k,k+1}$*;*
2. *for* $k = 1,\dots,n-1$ *we have* $\lim_{t\to\infty}\frac{1}{t}\langle\log\mu_{(k)} - \log\mu_{(k+1)}\rangle_t = \sigma^2_{k,k+1}$*;*

[18]See Fernholz [44], Corollary 4.1.12.
[19]See Fernholz [44], Chapter 5.
[20]See Fernholz [44], Definition 5.3.1.

*whereby* $\lambda_{k,k+1}$ *and* $\sigma^2_{k,k+1}$ *are positive constants and* $\lambda_{0,1} = \lambda_{n,n+1} = 0$*. Furthermore, if we denote the centered growth rate of the* $k^{th}$ *ranked stock at time* $t$ *by* $g_k(t)$*:*

$$g_k(t) = \gamma_{p_t(k)}(t) - \gamma_\mu(t),\ t \in [0, \infty), \tag{3.21}$$

*then, in the case of asymptotic stability it holds that:*

$$g_k = \lim_{t\to\infty} \frac{1}{t} \int_0^t g_k(s)ds. \tag{3.22}$$

In the case of asymptotic stability, the following result by Fernholz [44] permits to actually calculate local times for the ranked market weights.

**Proposition 3.1:** [21] *If* $\mathfrak{M}$ *is asymptotically stable, then for* $k = 1, \ldots, n$*:*

$$g_k = \frac{1}{2}\lambda_{k-1,k} - \frac{1}{2}\lambda_{k,k+1}\ a.s. \tag{3.23}$$

*By this, one directly obtains:*

$$\lambda_{k,k+1} = -2\left(g_1 + \ldots + g_k\right)\ a.s. \tag{3.24}$$

The asymptotic stability of market models is directly linked to the preservation of the capital distribution curve (CDC). The CDC is the curve which one obtains by plotting the logarithm of the ordered ranks of stocks versus the logarithm of their respective market weights. It was one of the most crucial insights in Fernholz's inspiring work [44] that the CDC, if computed on the universe of stocks taken from the Center for Research in Securities Prices (CRSP)[22] which essentially comprises all relevant US stock markets, does not change its structure between 1929 and 1999.[23]

A similar pattern as observed by Fernholz [44] for the US market may analogously be observed if we calculate the CDC for the MSCI EMU universe which we have already used for our calculations in Section 1.2.3. Figure 3.1 shows the CDC for the MSCI EMU universe for selected dates between August $2^{nd}$, 2006 and October $30^{th}$, 2007. The CDCs for the MSCI EMU universe exhibit a similar stability as the data for the US markets. Considering that the time window we used is only slightly more than a year, the fluctuations in the number of stocks in the market are clearly less pronounced than in the case studied in [44], where a period of 70 years is covered.

[21] See Fernholz [44], Proposition 5.3.2.
[22] See CRSP [21].
[23] See Fernholz [44], Figure 5.1.

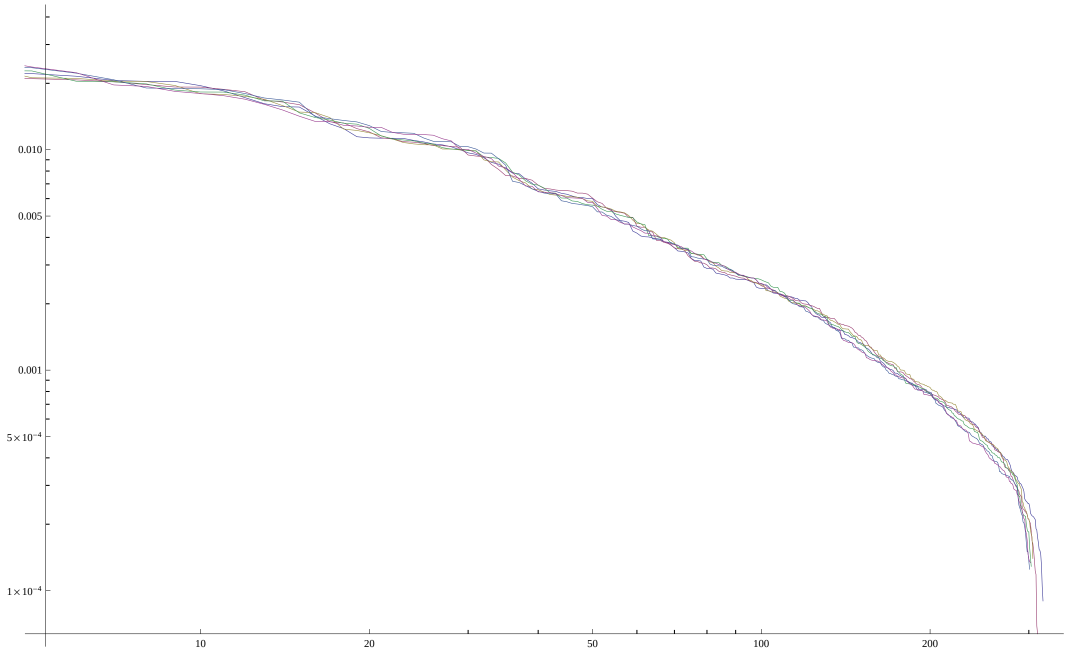


Figure 3.1: Capital distribution curves of the MSCI EMU universe.

The analysis of market data as e.g. performed above suggests that the distribution of capital is nearly invariant in time. This simple observation leads to fairly delicate modeling issues, since for example the classical log-normal model for stocks fails to reproduce the structures observable in market data.[24] In Section 3.2, we will present some approaches towards this problem which have been undertaken so far. In Chapter 4 we will develop our own approach for a market model which preserves the CDC.

## 3.2 Complex Market Models and Preservation of the Capital Curve

### 3.2.1 The Atlas Model

The Atlas model of an equity market is a special case of the general semimartingale model which has been introduced in Chapter 1. A detailed study of the Atlas model can be found in Banner et al. [13] and also in Fernholz [44]. The basic idea of the Atlas model is that

[24]See Stanley et al. [97].

only the smallest stock in a market gets a positive drift whereas all other stocks get zero drift. This setup will generally lead to the smallest stock increasing in value and finally changing positions with the formerly second smallest stock. For the two stocks which have changed places, the one which now occupies the smallest rank will get the positive drift, whereas the drift of the formerly smallest stock is set to zero. This setup leads to the situation that the stock currently in the smallest rank is literally bearing the growth of the entire market on its shoulders whence the name Atlas model in honor of the titan Atlas of Greek mythology, who supported the heavens with his shoulders.

In its most general formulation, the Atlas model may be written as follows:[25]

$$d\log X_i(t) = \gamma_i(t)dt + \sigma_i(t)dW_i(t), \tag{3.25}$$

whereby:

$$\begin{aligned} \gamma_i(t) &= g + \sum_{k=1}^{n} g_k \mathbb{1}_{\{X_i(t)=X_{p_t(k)}\}}(t) & (3.26)\\ \sigma_i(t) &= \sum_{k=1}^{n} \sigma_k \mathbb{1}_{\{X_i(t)=X_{p_t(k)}\}}(t), & (3.27) \end{aligned}$$

and $g > 0$, $g_k = -g$ for $k = 1, \ldots, n-1$ and $g_n = (n-1)g$ and $\sigma_k$ is a rank-dependant constant volatility for all ranks $k = 1, \ldots n$. The model given by Equation 3.25 to 3.27 may be formalized by means of the following polyhedral domains for $i = 1, \ldots, n$ and $k = 1, \ldots, n-2$:[26]

$$\begin{aligned} Q_1^{(i)} &= \{y \in \mathbb{R}^n \mid y_i \geq y_j \ \forall j \neq i\}\,; & (3.28)\\ Q_n^{(i)} &= \{y \in \mathbb{R}^n \mid y_i < y_j \ \forall j \neq i\}\,; & (3.29)\\ Q_{k+1}^{(i)} &= \left\{ y \in \mathbb{R}^n \mid y_i < \min_{1\leq r\leq k} y_{j_r} \ for\, j_1, \ldots j_k \, and \, y_i \geq y_l \ \forall\, l \notin \{j_1, \ldots, j_k\} \right\}. & (3.30) \end{aligned}$$

The families $\{Q_k^{(i)}\}_{1\leq i\leq n}$ and $\{Q_k^{(i)}\}_{1\leq k\leq n}$ are partitions of $\mathbb{R}^n$ for fixed $k$ and $i$ respectively.[27] Using these polyhedral domains one may reformulate Equation 3.25 as:[28]

$$d\log X_i(t) = \left( \sum_{k=1}^{n} g_k \mathbb{1}_{\{\log X(t) \in Q_k^{(i)}\}} + g \right) dt + \sum_{k=1}^{n} \sigma_k \mathbb{1}_{\{\log X(t) \in Q_k^{(i)}\}} dW_i(t), \tag{3.31}$$

[25] See Banner et al. [13], (1.1) to (1.6).
[26] See Banner et al. [13], Section 2.
[27] See Banner et al. [13], (2.2).
[28] See Banner et al. [13], (2.3).

whereby we denote by $\log X(t) = (\log X_1(t), \ldots, \log X_n(t))$ the $n$-dimensional vector of log-capitalizations of the stocks in our market $\mathfrak{M}$. The log-capitalization of the entire market can then be expressed as:[29]

$$\sum_{i=1}^{n} \log X_i(t) = \sum_{i=1}^{n} \log X_i(0) + ngt + \sum_{k=1}^{n} \sigma_k B_k(t), \tag{3.32}$$

with $B_k(t) = \sum_{i=1}^{n} \int_0^t \mathbb{1}_{\{\log X(s) \in Q_k^{(i)}\}} dW_i(s)$. By Lévy's Characterization Theorem[30] one directly obtains that the $B_k(\cdot)$ are Brownian Motions. The Atlas model satisfies several properties of markets which have been introduced so far. First and foremost it can be seen directly that the Atlas model is coherent.[31] For this it suffices to consider that

$$\lim_{T\to\infty} \frac{1}{T} \log X(T) = \max_{1\le i\le n} \left( \lim_{T\to\infty} \frac{1}{T} \log X_i(T) \right) = g, \tag{3.33}$$

from which it follows that $\lim_{T\to\infty} \frac{1}{T} \log \mu_i(T) = 0$, i.e. that $\mathfrak{M}$ is coherent.

Furthermore, the solution of the SDE 3.31 satisfies the ergodic relation:[32]

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{\log X(s) \in Q_k^{(i)}\}} ds = \frac{1}{n}, \tag{3.34}$$

thus asymptotically each stock spends the same amount of time in every rank, i.e. $T/n$ if we consider the finite time horizon $[0,T]$. This includes the smallest rank which means that each stock is acting as Atlas stock for the market one $n-th$ of the time. Considering now the ranked capitalization process, we shall denote the vector of ranked log-capitalizations as $Z(\cdot) = (Z_1(\cdot), \ldots, Z_n(\cdot))$:[33]

$$\begin{aligned} Z_k(t) &= \sum_{i=1}^{n} \mathbb{1}_{\{\log X(t) \in Q_k^{(i)}\}} \log X_i(t), && (3.35)\\ X_{(k)}(t) &= \exp\left(Z_k(t)\right). && (3.36) \end{aligned}$$

By this one directly obtains the ranked market weights $\mu_{(k)}(t) = \frac{X_{(k)}(t)}{X_1(t)+\ldots+X_n(t)}$. If we denote by $\Lambda_{k,k+1}(\cdot) := \Lambda_{Z_k - Z_{k+1}}(\cdot)$ the local time at the origin of the nonnegative semimartingale $Z_k - Z_{k+1}$ and using $\Lambda_{0,1}(\cdot) \equiv \Lambda_{n,n+1} \equiv 0$, then:

$$dZ_k(t) = \sum_{i=1}^{n} \mathbb{1}_{\{\log X(t) \in Q_k^{(i)}\}} d\log X_i(t) + \frac{1}{2}\left(d\Lambda_{k,k+1}(t) - d\Lambda_{k-1,k}(t)\right), \tag{3.37}$$

[29] See Banner et al. [13], (2.5).
[30] See e.g. Karatzas and Shreve [66], Theorem 3.3.16.
[31] See Banner et al. [13], Remark 2.1. or Fernholz [44], Example 5.3.3.
[32] See Banner et al. [13], Proposition 2.3.
[33] See Banner et al. [13], Section 3.

by which one obtains:

$$Z_k(t) = Z_k(0) + (g_k + g)t + \sigma_k B_k(t) + \frac{1}{2}\left(\Lambda_{k,k+1}(t) - \Lambda_{k-1,k}(t)\right). \tag{3.38}$$

N.b. that the above Equations 3.37 and 3.38 are merely the special case for the Atlas model of the general dynamics of ranked stocks as given in Equation 3.13. For the above notions it is needed, that the $\log X_1, \ldots, \log X_n$ are pathwise mutually non-degenerate as specified in Definition 3.2.[34] This however follows from an application of Girsanov's Theorem[35] whose application is justified by the non-degeneracy of the instantaneous covariance structure and the boundedness of the drift.[36]

The reason for discussing the Atlas model at this point is its asymptotic stability which is broadly discussed in Banner et al.[37] and Fernholz[38]. To this end, one studies the ergodic behavior of:[39]

$$\Xi_k(t) := \log\left(\frac{\mu_{(k)}(t)}{\mu_{(k+1)}(t)}\right) = Z_k(t) - Z_{k+1}(t) = \Xi_k(0) + \Theta_k(t) + \Lambda_{\Xi_k}(t), \tag{3.39}$$

whereby

$$\Theta_k(t) := (g_k - g_{k+1})t - \frac{1}{2}\left[\Lambda_{k-1,k}(t) + \Lambda_{k+1,k+2}(t)\right] + s_k \tilde{W}^{(k)}(t) \tag{3.40}$$

and

$$s_k = \sqrt{\sigma_k^2 + \sigma_{k+1}^2}, \tag{3.41}$$

$$\tilde{W}^{(k)}(t) = \frac{1}{s_k}\left(\sigma_k B_k(t) - \sigma_{k+1} B_{k+1}(t)\right). \tag{3.42}$$

Again, by Lévy's characterization[40] one obtains, that $\tilde{W}^{(k)}(\cdot)$ is a standard Brownian Motion. For $\Xi_k(\cdot)$ the following limit exists in distribution:[41]

$$\lim_{t\to\infty} \Xi_k(t) = \lim_{t\to\infty} \log\left(\frac{\mu_{(k)}(t)}{\mu_{(k+1)}(t)}\right) = \xi_k, \tag{3.43}$$

with $\xi_k \sim Exp(r_k)$ and $r_k := \frac{2\lambda_{k,k+1}}{s_k^2} = \frac{-4(g_1+\ldots+g_n)}{\sigma_k^2 - \sigma_{k+1}^2} > 0$.

[34] See Banner et al. [13], Section 3.
[35] See e.g. Kallenberg [64], Theorem 18.19. and Cor. 18.25. or Karatzas and Shreve [66], Section 3.5.
[36] See Banner et al. [13], Section 3.
[37] See Banner et al. [13], Sections 3. and 4.
[38] See Fernholz [44], Example 5.3.3.
[39] See Banner et al. [13], Equations (3.8) to (3.10) and (4.1).
[40] See e.g. Karatzas and Shreve [66], Theorem 3.3.16.
[41] See Banner et al. [13], Equation (4.2).

By this one obtains the asymptotic Pareto distribution for the ratios of successively ranked capitalizations:[42]

$$\lim_{t\to\infty} P\left[\frac{\mu_{(k)}(t)}{\mu_{(k+1)}(t)} > y\right] = y^{-r_k} P\left(\xi_k > \log y\right)\ \forall y \geq 1. \tag{3.44}$$

The results for the Atlas model can furthermore be put into the context of a portfolio $\pi$ in the Atlas market with $n$ stocks where we denote the asymptotics of the portfolio growth rate $\gamma_\pi(\cdot)$ and the excess growth rate $\gamma_\pi^*(\cdot)$ as follows:[43]

$$G_\pi(n) := \lim_{T\to\infty} \frac{1}{T}\int_0^T \gamma_\pi(t)dt, \tag{3.45}$$

$$G_\pi^*(n) := \lim_{T\to\infty} \frac{1}{T}\int_0^T \gamma_\pi^*(t)dt, \tag{3.46}$$

by which one obtains $G_\pi(n) = \lim_{T\to\infty} \frac{1}{T}\log Z_\pi(T)$. In this context it may be shown, that in an Atlas market the equally weighted portfolio outperforms the market if the variances are modeled in a way such that they are linearly growing by rank.[44]

Considering diversity it can be seen that the Atlas model admits a unique equivalent martingale measure on every finite time-horizon since it has a constant invertible instantaneous covariance matrix and bounded growth rates by which it follows, that the Atlas model can not be weakly diverse.[45] However, in fact the probability

$$P\left(\frac{1}{T}\int_0^T \mu_{(1)}(t)dt \leq 1-\delta\right) \tag{3.47}$$

is actually very close to one[46], which indicates that although we do not have weak diversity formally, the behavior of an Atlas market will actually be similar to a weakly diverse one.

### 3.2.2 Hybrid Models for the Equity Market

The Atlas model introduced in Section 3.2.1 may be further extended to cover both rank- and name-dependant characteristics. This Hybrid Atlas Model was extensively discussed in Ichiba et al. [61] and the study of such hybrid models is the subject of ongoing research.[47]

[42] See Banner et al. [13], Equation (4.4) or Fernholz [44], Example 5.3.3.
[43] See Banner et al. [13], Section 5.
[44] See Banner et al. [13], Example 5.2.
[45] See Banner et al. [13], Section 7.
[46] See Banner et al. [13], Section 7.
[47] See e.g. the lecture given by Fernholz [45].

Starting with the polyhedral domains $\left\{Q_k^{(i)}\right\}_{1\le i,k\le n} \in \mathbb{R}^n$ which were introduced in Section 3.2.1 we shall consider the symmetric group of permutations $\Sigma_n$ and let us introduce for each permutation $p \in \Sigma_n$ the polyhedral chamber $\mathcal{R}_p := \bigcap_{k=1}^n Q_k^{p(k)}$ which comprises all points $x \in \mathbb{R}^n$, such that $x_{p(k)}$ is ranked $k$-th among $x_1, \dots, x_n$. Since ties are resolved with favor to the lower index, there is a unique $p \in \Sigma_n$ for every $x \in \mathbb{R}^n$, such that $x \in \mathcal{R}_p$.[48] Hence for every $x \in \mathbb{R}^n$, one may define the associated indicator map $x \mapsto p^x \in \Sigma_n$, such that $x_{p^x(1)} \ge \dots \ge x_{p^x(n)}$ which assigns to each given rank $k$ the corresponding index (name) $p^x(k)$.

The aim of the Hybrid Atlas Model is to extend the modeling framework discussed in the above Sections in order to cover effects attributable to the name and rank of a certain stock. Considering as usual the log-capitalization of stocks, the Hybrid Atlas Model may be formulated as follows:[49]

$$\begin{aligned} d\log X_i(t) &= \left(\sum_{k=1}^n g_k \mathbb{1}_{\{\log X(t) \in Q_k^{(i)}\}} + \gamma_i + \gamma\right) dt \\ &\quad + \sum_{j=1}^n \rho_{ij} dW_j(t) \\ &\quad + \sum_{k=1}^n \sigma_k \mathbb{1}_{\{\log X(t) \in Q_k^{(i)}\}} dW_i(t), \end{aligned} \tag{3.48}$$

whereby $X_i(0) = x_i$, $0 \le t < \infty$, $g_k$ denotes the constant drift for rank $k$, $\gamma_i$ denotes the constant drift for name $i$, $\gamma$ denotes the common drift, $\sigma_k$ denotes the volatility for rank $k$ and $\rho_{ij}$ denotes the name-based correlation. The system specified by 3.48 behaves like a Brownian Motion with drift $(g_k + \gamma_i + \gamma)$ and variance $(\sigma_k + \rho_{ii})^2 + \sum_{i\ne j} \rho_{ij}^2$.[50] This setup provides a considerable amount of flexibility in incorporating effects stemming from different patterns observable in equity markets, furthermore it permits remarkable stability in the structure of the capital curve as will be discussed in the following considerations. These results however come at the price that the Hybrid Atlas Model has to be restricted to a setting fulfilling certain assumptions.

**Assumption 3.1:** *The following assumptions have to be made:*[51]

[48] See Ichiba et al. [61], Section 1.
[49] See Ichiba et al. [61], (2.1)
[50] See Ichiba et al. [61], Section 2.
[51] See Ichiba et al. [61], (2.2) and (2.3).

*Stability Condition:*

$$\sum_{k=1}^{n} g_k + \sum_{i=1}^{n} \gamma_i = 0. \tag{3.49}$$

*Ellipticity Assumption:*

$$\mathfrak{s}_p := diag\left(\sigma_{p^{-1}(1)}, \ldots, \sigma_{p^{-1}(n)}\right) + (\rho_{ij})_{1\le i,j\le n}\,. \tag{3.50}$$

*The $(n \times n)$ matrices $\mathfrak{s}_p$ are positive definite for $p \in \Sigma_n$ and $\sigma_k > 0\ \forall k$.*

Setting $\log X(t) =: Y(t)$, one may now reformulate Equation 3.48 in vector form as follows:[52]

$$dY(t) = G(Y(t))dt + S(Y(t))dW(t), \tag{3.51}$$

whereby $Y(0) = y \in \mathbb{R}^n$, $0 \le t < \infty$ and $G : \mathbb{R}^n \to \mathbb{R}^n,\ S : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ with:

$$\begin{aligned} G(y) &:= \sum_{p\in\Sigma_n} \mathbb{1}_{\{y\in\mathcal{R}_p\}} \left(g_{p^{-1}(1)} + \gamma_1 + \gamma, \ldots, g_{p^{-1}(n)} + \gamma_n + \gamma\right)^T, && (3.52)\\ S(y) &:= \sum_{p\in\Sigma_n} \mathbb{1}_{\{y\in\mathcal{R}_p\}} \mathfrak{s}_p,\ y \in \mathbb{R}^n. && (3.53)\end{aligned}$$

Provided that Assumption 3.1 holds, i.e. that the $\mathfrak{s}_p$ are positive definite, the system (3.48) resp. (3.51) has a weak solution $(Y, W)$ on a filtered probability space $(\Omega, \mathfrak{F}, \{\mathfrak{F}(t)\}_{t\ge0}, P)$ satisfying the usual[53] conditions.[54]

The ergodic behavior of the system 3.48 is comprehensively discussed in Ichiba et al. [61] and was also briefly treated in Fernholz's talk [45]. At this point, we will give an overview of the ergodicity results for the Hybrid Atlas model, followed by results on the stability of capital distribution and the discussion of remaining problems with this kind of second order stock market models. Let us therefore consider the average log-capitalization process $\bar{Y}(\cdot) = \frac{1}{n}\sum_{i=1}^{n} Y_i(\cdot)$ which has the dynamics:[55]

$$\bar{Y}(t) = \frac{1}{n}\sum_{i=1}^{n} y_i + \gamma t + \frac{1}{n}\sum_{k=1}^{n} \sigma_k B_k(t) + \frac{1}{n}\sum_{i,j=1}^{n} \rho_{ij} W_j(t), \tag{3.54}$$

where

$$B_k(t) := \sum_{i=1}^{n} \int_0^t \mathbb{1}_{\{Y(s)\in Q_k^{(i)}\}} dW_i(s),\ k = 1, \ldots, n; \tag{3.55}$$

[52]See Ichiba et al. [61], (2.4).
[53]See e.g. Karatzas and Shreve [66], Definition 1.2.25.
[54]See Ichiba et al. [61], Section 2.
[55]See Ichiba et al. [61], Section 3.

and $0 \leq t < \infty$, $\bigcup_{i=1}^{n} Q_k^{(i)} = \mathbb{R}^n$. $B_1(\cdot), \ldots, B_n(\cdot)$ are continuous local martingales with quadratic cross variations $\langle B_k, B_l \rangle(t) = \delta_{k,l} t$, hence by Knight's Theorem[56] one obtains that the $B_k,\ k = 1, \ldots, n$ are independent Brownian Motions.

It follows from Equation 3.54 that the average log-capitalization $\bar{Y}(\cdot)$ grows at a rate equal to the common drift $\gamma$, i.e.:

$$\lim_{T \to \infty} \frac{\bar{Y}(T)}{T} = \gamma \ a.s. \tag{3.56}$$

by the strong law of large numbers for Brownian Motions.[57]

Introducing the subspace $\Pi$ of $\mathbb{R}^n$, $\Pi = \{y \in \mathbb{R}^n | (1, \ldots, 1)^T y = 0\}$ and imposing in addition to Assumption 3.1 the stability condition $\sum_{k=1}^{l} (g_k + \gamma_{p(k)}) < 0$, it is shown in Ichiba et al. [61] that the vector of centered log-capitalizations $\tilde{Y}(t) = (Y_1(t) - \bar{Y}(t), \ldots, Y_n(t) - \bar{Y}(t))$ is stable in distribution,[58] i.e., there is a unique invariant (ergodic) probability measure $\mu$ for $\tilde{Y}(\cdot)$ and for any bounded, measurable function $g : \Pi \to \mathbb{R}$, the Strong Law of Large Numbers holds:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\tilde{Y}(t)) dt = \int_{\Pi} f(y) \mu(dy). \tag{3.57}$$

**Remark 3.2:** It is worth noting at this point, that although the Hybrid Atlas model permits non-zero drifts for all those stocks larger than the smallest, the stability condition $\sum_{k=1}^{l} (g_k + \gamma_{p(k)}) < 0$ mandates that cumulative rank- and name-dependant growth rates up to rank $n-1$ be negative. Thus, looking at the centralized vector of log-capitalizations $\tilde{Y}(\cdot)$ where the overall (average) growth rate of the market $\gamma$ is subtracted, we are again studying a system very much akin to the standard Atlas Model, where all stocks but the smallest have a cumulative negative growth contribution to the market.

An economic interpretation of such a system would be that the larger stocks - and the largest stock in particular - would switch to a regime of decline or at best stagnation in market capitalization growth once they have reached a dominating position in the market. This concept must be put into the context of the general framework of Stochastic Portfolio Theory which we have summarized in Chapter 1. Many results given in the monograph by Fernholz [44] are based on the assertion that large-cap stocks will tend to lag behind

[56]See e.g. Karatzas and Shreve [66], Theorem 3.4.13. The Knight Theorem states the following: letting $M = \{M_t = (M_t^{(1)}, \ldots, M_t^{(d)}),\ \mathfrak{F}_t,\ 0 \leq t < \infty\}$ be continuous local martingales with $P$-almost sure limit $\lim_{t \to \infty} \langle M_t^{(i)} \rangle = \infty$ and cross variations $\langle M^{(i)}, M^{(j)} \rangle_t = 0$ for $1 \leq i \neq j \leq d$ and $0 \leq t < \infty$. Defining $T_i(s) = \inf\{t \geq 0 : \langle M^{(i)} \rangle_t > s\}$, $0 \leq s < \infty$, $1 \leq i \leq d$, so that for each $i$ and $s$ the random time $T_i(s)$ is a stopping time for the right-continuous filtration $\{\mathfrak{F}_t\}$, then the processes $B_s^{(i)} := M_{T_i(s)}^{(i)}$, $0 \leq s < \infty$, $1 \leq i \leq d$ are independent, standard one-dimensional Brownian Motions.

[57]See e.g. Karatzas and Shreve [66], Problem 2.9.3.

[58]See Ichiba et al. [61], Proposition 1.

in terms of growth rate but will be superior in terms of dividend rate as opposed to small cap stocks.[59]

Although this pattern may certainly be observed to some extent in real-life equity markets, the aspect that the largest stock inevitably has to grow slower than the overall market on average is certainly questionable. It would be an interesting empirical question - one which is however beyond the scope of this work - whether the actually observable growth behavior and rank-occupation time of the largest stock in various markets strongly contradicts the pattern insinuated by the stability assumptions of the Hybrid Atlas Model.

Let us now consider the long-run average occupation time which a certain company $i$ spends in the $k^{th}$ rank which is given by:[60]

$$\theta_{k,i} := \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{X(t)\in Q_k^{(i)}\}} dt, \quad i,k = 1,\dots,n. \tag{3.58}$$

In this way, average occupation time is expressed as percentage of the total trading time considered and it can be shown that $\theta_{k,i}$ exists almost surely in $[0,1]$ for all ranks $k$ and all names $i$ under the regularity conditions imposed by Assumption 3.1 and the already mentioned stability condition $\sum_{k=1}^{l}(g_k + \gamma_{p(k)}) < 0$.[61] Furthermore, as can be expected by this construction, it holds that

$$\sum_{j=1}^{n} \theta_{k,j} = \sum_{l=1}^{n} \theta_{l,i} = 1, \tag{3.59}$$

hence summing the proportions of average occupation of the $k^{th}$ rank over all names or summing the proportions of occupation of all ranks by a certain stock yields the whole trading time, i.e. 100%. By this it clearly follows that the matrix $\vartheta := (\theta_{k,i})_{1\le k,i\le n}$ is doubly stochastic.[62] Similarly it is shown in [61] that the long term average occupation time of the market in a certain polyhedral chamber $\mathcal{R}_p$ given by:[63]

$$\theta_p := \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{X(t)\in\mathcal{R}_p\}} dt \tag{3.60}$$

exists almost surely in $[0,1]$ for all $p \in \Sigma_p$. Finally we have that:[64]

$$\theta_{k,i} = \sum_{\{p\in\Sigma_p | p(k)=i\}} \theta_p. \tag{3.61}$$

[59] See e.g. Fernholz [44], Theorem 2.3.4. and Corollaries 2.3.5. and 2.3.6.
[60] See Ichiba et al. [61], (3.7).
[61] See Ichiba et al. [61], Corollary 1.
[62] See also Ichiba et al. [61], Corollary 1.
[63] See Ichiba et al. [61], (3.8).
[64] See Ichiba et al. [61], Corollary 1.

By Theorem 1 in [61] it holds that:

$$\theta_{k,i} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{X(t)\in Q_k^{(i)}\}} dt = \lim_{t\to\infty} \frac{1}{T} \int_0^T \mathbb{1}_{\{\tilde{Y}(t)\in Q_k^{(i)}\cap\Pi\}} dt = \mu(Q_k^{(i)}), \tag{3.62}$$

where $\mu$ is the unique invariant (ergodic) probability measure for $\tilde{Y}(\cdot)$.

At this point we shall move on to discuss the dynamics of ranked stock prices under the Hybrid Atlas Model. To this end we will denote the $k$-th ranked log-price as $Z_k(t) := \log X_{(k)}(t) = \sum_{i=1}^n \mathbb{1}_{\{\log X(t)\in Q_k^{(i)}\}} \log X_i(t)$. By this construction we obtain $Z_1(t) \geq \ldots \geq Z_n(t)$ for all $t \in [0,T]$. Let us further denote the index process by $\mathfrak{p}_t := p^{X(t)} = p^{\log X(t)}$ which yields the name of the stock occupying the $k$-th rank at time $t$, $X_{\mathfrak{p}_t(1)}(t) \geq \ldots \geq X_{\mathfrak{p}_t(n)}(t)$, thus $Z_k(\cdot) = \log X_{\mathfrak{p}_\cdot(k)}(\cdot)$. Let us also recall the market weight of the $i$-th stock as introduced in Chapter 1, $\mu_i(\cdot) = \frac{X_i(\cdot)}{\sum_{j=1}^n X_j(\cdot)}$, and the ranked market weights $\mu_{(k)}(\cdot) = \frac{X_{(k)}(\cdot)}{\sum_{j=1}^n X_j(\cdot)}$ for $i,k = 1,\ldots,n$.

Under Assumption 3.1 and the additional stability condition $\sum_{k=1}^l (g_k + \gamma_{p(k)}) < 0$ for $l = 1,\ldots,n-1$ the process of ranked deviations $\tilde{Z}(\cdot) := (Z_1(\cdot)-\bar{Y}(\cdot),\ldots,Z_n(\cdot)-\bar{Y}(\cdot)$ from the average log-capitalization $\bar{Y}(\cdot) = \frac{1}{n}\sum_{i=1}^n \log X_i(\cdot)$ is also stable in distribution and so is the $(\mathbb{R}_t^{n-1}\times\Sigma_n)$-valued process $(\Xi(\cdot),\mathfrak{p}_\cdot)$, where $\Xi(\cdot) = (Z_1(\cdot)-Z_2(\cdot),\ldots,Z_{n-1}(\cdot)-Z_n(\cdot))$.[65]

Let us again denote by $\Lambda_{Z_k-Z_l}(\cdot)$ the local time for the $k$-th and $l$-th ranked stocks coinciding, then the dynamics of $Z_k(\cdot)$ are given by:[66]

$$dZ_k(t) = \sum_{i=1}^n \mathbb{1}_{\{\log X(t)\in Q_k^{(i)}\}} d\log X_i(t) + \frac{1}{N_k(t)}\left(\sum_{l=k+1}^n d\Lambda_{Z_k-Z_l}(t) - \sum_{l=1}^{k-1} d\Lambda_{Z_l-Z_k}(t)\right), \tag{3.63}$$

whereby $N_k(t) = |\{i| \log X_i(t) = Z_k(t)\}|$. It is worth noting, that under Assumption 3.1, the local times of collisions of three or more stocks are identically equal to zero.[67] Furthermore, under Assumption 3.1 and the stability condition $\sum_{k=1}^l (g_k + \gamma_{p(k)}) < 0$, $l = 1,\ldots,n-1$, one obtains the following limit properties for the Hybrid Atlas Model:[68]

[65] See Ichiba et al. [61], Corollary 2.
[66] See Ichiba et al. [61], (4.3).
[67] See Ichiba et al. [61], Lemma 1.
[68] See Ichiba et al. [61], (4.5) - (4.7).

$$
\begin{aligned}
\lim_{t\to\infty}\frac{1}{T}\Lambda_{Z_k-Z_l}(T) &= -2\sum_{l=1}^{k}\left(g_l+\sum_{i=1}^{n}\gamma_i\theta_{l,i}\right),\ a.s.\, k=1,\dots,n-1, && (3.64)\\
\lim_{T\to\infty}\frac{X_i(T)}{T} &= \lim_{T\to\infty}\frac{\log X_i(T)}{T}=\gamma,\ a.s., && (3.65)\\
\lim_{T\to\infty}\frac{\log X(T)}{T} &= \lim_{T\to\infty}\frac{1}{T}\log\left(\sum_{i=1}^{n}X_i(t)\right)=\gamma,\ a.s. && (3.66)
\end{aligned}
$$

From Equations 3.64 to 3.66 it directly follows, that a market $\mathfrak{M}$ in the Hybrid Atlas Model is coherent, i.e. that $\lim_{T\to\infty}\frac{1}{T}\log\mu_i(T)=0$ for all $i=1,\dots,n$. Furthermore it is shown in [61], that under the above assumptions the long term average occupation times $\theta_{k,i}$ satisfy the equilibrium identity[69]

$$
\sum_{k=1}^{n}\theta_{k,i}g_k+\gamma_i=0\ i=1,\dots,n. \tag{3.67}
$$

### Capital Distribution in the Hybrid Atlas Model

Akin to the standard Atlas Model which has been presented in Section 3.2.1, the Hybrid Atlas Model also permits to calculate an invariant distribution of the ranked market weights $\mu_{(k)}(\cdot)$, however only under certain restrictions to the setup specified in Equation 3.48.

**Assumption 3.2:** [70] *Let us assume that the conditions set forth in Assumption 3.1 hold together with the stability condition* $\sum_{k=1}^{l}(g_k+\gamma_{p(k)})<0$, $l=1,\dots,n-1$. *Then we further assume that the rank based variances grow linearly, i.e.*

$$
\sigma_2^2-\sigma_1^2=\sigma_3^2-\sigma_2^2=\dots=\sigma_n^2-\sigma_{n-1}^2, \tag{3.68}
$$

*and that*

$$
\rho_{i,j}=0,\ 1\le i,j,\le n,\ i\ne j. \tag{3.69}
$$

Under the conditions specified in Assumption 3.2, the ranked market weights $\mu_{(k)}(\cdot)$, $k=1,\dots,n$ have the invariant distribution with:[71]

$$
\mathfrak{f}(m_1,\dots,m_{n-1})=\sum_{p\in\Sigma_n}\left[\theta_p\prod_{k=1}^{n-1}\lambda_{p,k}\left(\prod_{j=1}^{n}m_j^{\lambda_{p,j}-\lambda_{p,j-1}+1}\right)^{-1}\right] \tag{3.70}
$$

[69] See Ichiba et al. [61], Corollary 3.
[70] See Ichiba et al. [61], Lemma 3 and Proposition 3.
[71] See Ichiba et al. [61], Corollary 5.

as density for $0 < m_n < \ldots < m_1 < 1$, $m_n = 1 - \sum_{j=1}^{n-1} m_j$ and

$$\lambda_{p,k} := \frac{-4\left(\sum_{l=1}^{k} g_l + \gamma_{p(l)}\right)}{\sigma_k^2 + \sigma_{k+1}^2},\ p \in \Sigma_n,\ 1 \leq k \leq n-1, \tag{3.71}$$

and $\lambda_{p,0} = \lambda_{p,n} = 0$. Similarly, the log-market weights $\mathfrak{m}_k(\cdot) := \log \mu_{(k)}(\cdot)$ have an invariant distribution with density:[72]

$$\mathfrak{g}(c_1, \ldots, c_{n-1}) = \sum_{p \in \Sigma_n} \left[\theta_p \prod_{j=1}^{n-1} \left(\lambda_{p,j} e^{-(\lambda_{p,j} - \lambda_{p,j+1})c_j}\right) e^{\lambda_{p,n-1} c_n}\right],\ -\infty < c_n \leq \ldots \leq c_1 < 0. \tag{3.72}$$

From these invariant distributions one obtains the piecewise linear Capital Distribution Curve with expected slope:[73]

$$E\left[\frac{\log \mu_{(k+1)} - \log \mu_{(k)}}{\log(k+1) - \log(k)}\right] = -\frac{E[\Xi_k]}{\log(1+k^{-1})} = -\frac{\sum_{p \in \Sigma_n} \theta_p \lambda_{p,k}^{-1}}{\log(1+k^{-1})}. \tag{3.73}$$

## 3.3 Conclusion

In this Chapter 3 we have presented some general aspects concerning the capital structure of equity markets and the analytic complexities arising once the dynamics of ranked stocks in a market $\mathfrak{M}$ are considered instead of the name-based dynamics. We have also given an overview of results available in the literature on special market models, namely the Atlas and the Hybrid Atlas models which permit to reproduce some observable patterns of equity markets, albeit at the cost of imposing stronger modeling assumptions.

[72] See Ichiba et al. [61], (5.19).
[73] See Ichiba et al. [61], (5.20).

# Chapter 4

# A Market Model Preserving the Capital Curve

The preceding considerations in Chapter 3 demonstrated the interesting empirical fact that the Capital Distribution Curve (CDC) possesses a characteristic shape which remains similar both over time and across markets.[1] Furthermore it ought to be noted, that the usual log-normal modeling approach for equity markets fails to reproduce the desired shape of the CDC.[2] In this Chapter we will present the central contribution of this thesis, namely the construction of a model for the stock market, based on the idea of modeling stocks according to the dynamics of a Squared Brownian Motion (SqBM). We will provide a detailed motivation for our model in Section 4.1, also highlighting similarities to the dynamics of volatility stabilized markets which have been studied in the context of stochastic portfolio theory for several years. In Section 4.2 we will equip our market with a correlation structure and present the results of our model implementation. It ought to be underlined that the correlated model which we develop in Section 4.2.1 has been developed with special attention to applications in risk management where the incorporation of observed dependence structures is of paramount importance. While the independent SqBM model which is introduced in Section 4.1.2 possesses some similarities to volatility stabilized market models, the enhanced correlated model outlined in Section 4.2.1 has - to the best of our knowledge - not been applied so far in order to address the problem of replicating the general structure of stock markets' capital distribution and general market dynamics.

[1]See Figure 3.1 and Fernholz [44].
[2]See Stanley et al. [97].

# 4.1 Motivation

## 4.1.1 Analysis of Market Weights for Positive Affine Processes

### General Properties of Affine Processes

In the ensuing considerations of Section 4.1, we will motivate our approach of utilizing a squared Brownian Motion as building block of our market model. In the first step we will state some general results on (positive) affine processes and discuss the distributional behavior of market weights. In Section 4.1.2 we will further elaborate on the special case of squared Brownian Motion.

Affine processes have been introduced and discussed with a view to applications in finance by Duffie, Filipović and Schachermayer [33]. Further research in this class of processes has been conducted for instance in the areas of affine LIBOR models[3], yield curve shapes[4] or affine stochastic volatility models[5]. For a general overview of affine processes we refer e.g. to Keller-Ressel [68] and to Keller-Ressel, Papapantoleon and Teichmann [70], the one-dimensional special case is discussed in Keller-Ressel and Steiner [69].

In the first step we will introduce affine processes defined on the general state space $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ with total dimension $\mathrm{Dim}(D) = d = m + n$ and we shall then move on to the special case of $\tilde{D} = \mathbb{R}_{\geq 0}$. With respect to the notation we will follow the standards and conventions as set out in Keller-Ressel [68]. Let therefore $M = I \cup J = \{1, \ldots, d\}$ be the index vector, whereby the subset $I = \{1, \ldots, m\}$ corresponds to the $\mathbb{R}_{\geq 0}$-valued components and $J = \{m + 1, \ldots, m + n\}$ is the index set of the $\mathbb{R}$-valued components. Furthermore, for a $d$-dimensional vector $x$ let $x_I = (x_i)_{i \in I}$ denote its projection on the components with index $i \in I$ and let us denote the scalar product $\langle x, y \rangle = \sum_{k=1}^d x_k y_k$ for $x, y$ in $\mathbb{R}^d$ or $\mathbb{C}^d$.

We shall furthermore define:[6]

$$\mathcal{U} = \left\{ u \in \mathbb{C}^d | \mathrm{Re}\, u_I \leq 0,\ \mathrm{Re}\, u_J = 0 \right\}, \tag{4.1}$$

$$\mathcal{U}^\circ = \left\{ u \in \mathbb{C}^d | \mathrm{Re}\, u_I < 0,\ \mathrm{Re}\, u_J = 0 \right\}. \tag{4.2}$$

[3] See Keller-Ressel, Papapantoleon and Teichmann [70].
[4] See Keller-Ressel and Steiner [69].
[5] See Keller-Ressel [68].
[6] See e.g. Keller-Ressel [68], Equations 1.1. and 1.2.

**Definition 4.1:** [7] *Let $(X(t))_{t\geq 0}$ be a stochastically continuous*[8]*, time-homogenous Markov*[9] *process on the state space $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ with starting point $X(0) = x$ and law $P_x$. Then $(X(t))_{t\geq 0}$ is called affine if its characteristic function is an exponentially-affine function of the state vector. Thus, on $i\mathbb{R}^d$ there exist functions $\phi : \mathbb{R}_{\geq 0} \times i\mathbb{R}^d \to \mathbb{C}$ and $\psi : \mathbb{R}_{\geq 0} \times i\mathbb{R}^d \to \mathbb{C}^d$ such that:*

$$E_x\left[e^{\langle X(t),u\rangle}\right] = \exp\left(\phi(t,u) + \langle x, \psi(t,u)\rangle\right), \tag{4.3}$$

*for $x \in D$ and $(t,u) \in \mathbb{R}_{\geq 0} \times i\mathbb{R}^d$.*

Let furthermore $(P_t)_{t\geq 0}$ be the semi-group of operators associated[10] to the Markov process $(X(t))_{t\geq 0}$, acting on the bounded Borel functions $\mathcal{B}_b(D)$ by:

$$P_t[f(x)] = E_x\left[f(X(t))\right], \text{ for all } x \in D,\ t \geq 0,\ f \in \mathcal{B}_b(D), \tag{4.4}$$

and let us define the set $\mathcal{O}$

$$\mathcal{O} = \left\{(t,u) \in \mathbb{R}_{\geq 0} \times \mathcal{U} | P_s[\exp(\langle 0,u\rangle)] \neq 0 \,\forall\, s \in [0,t]\right\}. \tag{4.5}$$

Then we can state the following properties of the functions $\phi$ and $\psi$:[11]

1. $\phi$ maps $\mathcal{O}$ to $\mathbb{C}_-$, whereby $\mathbb{C}_- := \{u \in \mathbb{C} | \operatorname{Re} u \leq 0\}$.
2. $\psi$ maps $\mathcal{O}$ to $\mathcal{U}$.
3. $\phi(0,u) = 0$ and $\psi(0,u) = u$ for all $u \in \mathcal{U}$.
4. $\phi$ and $\psi$ possess the *semi-flow property*:

$$\begin{aligned} \phi(t+s,u) &= \phi(t,u) + \phi(s,\psi(t,u)), \\ \psi(t+s,u) &= \psi(s,\psi(t,u)), \end{aligned} \tag{4.6}$$

   for all $t, s \geq 0$ with $(t+s,u) \in \mathcal{O}$.

[7] See e.g. Keller-Ressel [68], Definition 1.1. Nota bene that we stick to the definition as given in [68] and include stochastic continuity whereas this property is introduced in Duffie, Filipovic and Schachermayer [33] for *regular* affine processes.

[8] See e.g. Keller-Ressel [68]. A process $X(\cdot)$ is called stochastically continuous if for any $t_n \to t$ in $\mathbb{R}_{\geq 0}$ it holds that $X(t_n) \overset{P_x}{\to} X(t)$.

[9] See e.g. Karatzas and Shreve [66], Section 2.5.B.

[10] See e.g. Keller-Ressel, Section 1. For the general background we refer to Kallenberg [64].

[11] See Keller-Ressel [68], Proposition 1.3.

5. $\phi$ and $\psi$ are jointly continuous on $\mathcal{O}$.

6. With the remaining arguments fixed, $u_I \mapsto \phi(t,u)$ and $u_I \mapsto \psi(t,u)$ are analytic functions in $\{u_I|\,\mathrm{Re}\,u_I < 0,\,(t,u) \in \mathcal{O}\}$.

7. Let $(t,u)$, $(t,w) \in \mathcal{O}$ with $\mathrm{Re}\,u \leq \mathrm{Re}\,w$. Then:

$$\begin{aligned}\mathrm{Re}\,\phi(t,u) &\leq \phi(t,\mathrm{Re}\,w),\\ \mathrm{Re}\,\psi(t,u) &\leq \psi(t,\mathrm{Re}\,w).\end{aligned} \tag{4.7}$$

Due to the above properties, $\phi(t,\cdot)$ and $\psi(t,\cdot)$ map real numbers to real numbers.[12] A more detailed discussion of the semi-flow property can be found in Section 1.2 of Keller-Ressel [68] together with the reference to more general results on semi-flows as e.g. given in Filipović and Teichmann [52].

**Definition 4.2:** [13] *An affine process $X(\cdot)$ is called regular, if the right-sided derivatives:*

$$F(u) := \frac{\partial\phi}{\partial t}(t,u)\bigg|_{t=0+} \quad , \quad R(u) := \frac{\partial\psi}{\partial t}(t,u)\bigg|_{t=0+} \tag{4.8}$$

*exist for all $u \in \mathcal{U}$, and if they are continuous at $u = 0$. $F(\cdot)$ and $R(\cdot)$ are called functional characteristics of the affine process $X(\cdot)$.*[14]

For a regular affine process, the semi-flow equations given in (4.6) may be differentiated w.r.t. $s$ and for $s = 0$ one obtains the following ordinary differential equations for $\phi$ and $\psi$ and $(t,u) \in \mathcal{O}$, which are called generalized Riccati equations with the variable $u$ entering as an initial condition:[15]

$$\frac{\partial}{\partial t}\phi(t,u) = F(\psi(t,u)), \;\; \phi(0,u) = 0, \tag{4.9}$$

$$\frac{\partial}{\partial t}\psi(t,u) = R(\psi(t,u)), \;\; \psi(0,u) = u. \tag{4.10}$$

It can furthermore be shown, that $F(\cdot)$ and $R(\cdot)$ are in fact log-characteristic functions of sub-stochastic infinitely divisible measures subject to a certain set of admissibility conditions.[16] This establishes a connection to the theory of Lévy processes, wherefrom we

[12]See Keller-Ressel [68], Proposition 1.3.
[13]See e.g. Duffie, Filipović and Schachermayer [33], Definition 2.5., or Keller-Ressel [68], Definition 2.1.
[14]See Keller-Ressel [68], Remark 2.2.
[15]See Duffie, Filipović and Schachermayer [33], Section 6.
[16]See Duffie, Filipović and Schachermayer [33], Section 2. and Keller-Ressel [68], Chapter 2.

have that the characteristic function of an infinitely divisible probability measure can be described by the Lévy-triplet $(a, b, m)$, whereby $a$ denotes the diffusion matrix, $b$ denotes the drift vector and $m$ denotes the Lévy measure.[17] In the case of a sub-stochastic infinitely divisible measure one obtains as a fourth parameter the defect of the measure $c = -\log m(D)$.[18] For the scalar function $F(\cdot)$ and the $d$-dimensional, vector-valued function $R(\cdot)$ one ends up with a Lévy-quadruplet $(a, b, c, m)$ describing $F$ and with Lévy-quadruplets $(\alpha_i, \beta_i, \gamma_i, \mu_i)$ for $i = 1, \ldots, d$ describing $R$.

**Definition 4.3:** [19]

*The parameter set for an affine process $X(\cdot)$ is given by positive semi-definite real $d \times d$ matrices $a, \alpha^1, \ldots, \alpha^d$, by $\mathbb{R}^d$-valued vectors $b, \beta^1, \ldots, \beta^d$ and by non-negative numbers $c, \gamma^1, \ldots, \gamma^d$ and by the Lévy measures $m, \mu^1, \ldots, \mu^d$ on $\mathbb{R}^d$. This parameter set is called admissible for an affine process with state space $D$, if the following conditions hold:*

$$a_{kl} = 0 \text{ if } k \in I \text{ or } l \in I, \tag{4.11}$$
$$\alpha^j = 0 \text{ for all } j \in J, \tag{4.12}$$
$$\alpha^i_{kl} = 0 \text{ if } k \in I\backslash\{i\} \text{ or } l \in I\backslash\{i\}, \tag{4.13}$$
$$b \in D, \tag{4.14}$$
$$\beta^i_k \geq 0 \text{ for all } i \in I \text{ and } k \in I\backslash\{i\}, \tag{4.15}$$
$$\beta^j_k = 0 \text{ for all } j \in J \text{ and } k \in I, \tag{4.16}$$
$$\gamma^j = 0 \text{ for all } j \in J, \tag{4.17}$$
$$\operatorname{supp} m \subseteq D \text{ and } \int_{D\backslash\{0\}} \left\{ \left(|x_I| + |x_J|^2\right) \wedge 1 \right\} m(dx) < \infty, \tag{4.18}$$
$$\mu^j = 0 \text{ for all } j \in J, \tag{4.19}$$
$$\operatorname{supp} \mu^i \subseteq D \text{ for all } i \in I, \text{ and} \tag{4.20}$$
$$\int_{D\backslash\{0\}} \left\{ \left(|x_{I\backslash\{i\}}| + |x_{J\cup\{i\}}|^2\right) \wedge 1 \right\} \mu_i(dx) < \infty \text{ for all } i \in I. \tag{4.21}$$

The conditions outlined in Definition 4.3 look cumbersome, yet they are far less complex in the one-dimensional case which will be outlined below. A concise visualization of the structure of $a, \alpha^1, \ldots, \alpha^d$, and $b, \beta^1, \ldots, \beta^d$ can be found in Section 2 of Keller-Ressel [68]. We will conclude this general overview of affine processes by stating a central result of Duffie, Filipović and Schachermayer [33] on the generator of an affine process.

[17] See e.g. Applebaum [9], Section 1.2.4. or Schoutens [91].
[18] See Keller-Ressel [68], Chapter 2.
[19] See Duffie, Filipović and Schachermayer [33], Definition 2.6. or Keller-Ressel [68], Definition 2.3.

**Definition 4.4:** [20] *Let us define the following truncation functions* $h, \chi^1, \ldots, \chi^m$ *from* $\mathbb{R}^d \to [-1,1]^d$ *coordinate-wise by:*

$$h_k(\xi) := \begin{cases} 0, & k \in I;\, \forall \xi \in \mathbb{R}^d; \\ \frac{\xi_k}{1+\xi_k^2}, & k \in J;\, \forall \xi \in \mathbb{R}^d; \end{cases} \tag{4.22}$$

*and*

$$\chi_k^i(\xi) := \begin{cases} 0, & k \in I\backslash\{i\};\, \forall \xi \in \mathbb{R}^d,\, i \in I; \\ \frac{\xi_k}{1+\xi_k^2}, & k \in J \cup \{i\};\, \forall \xi \in \mathbb{R}^d,\, i \in I. \end{cases} \tag{4.23}$$

**Theorem 4.1:** [21]

*Let* $(X(t))_{t\geq 0}$ *be a regular affine process and* $D$ *its state space. Then there exist some admissible parameters* $(a, \alpha^i, b, \beta^i, c, \gamma^i, m, \mu^i)_{i\in\{1,\ldots,d\}}$ *such that:*

1. *the functions* $F$ *and* $R$ *given in Definition 4.2 are of the following Lévy-Khintchine form:*

$$\begin{aligned} F(u) &= \langle u, au\rangle + \langle b, u\rangle - c + \int_{\mathbb{R}^d\backslash\{0\}} \left(\exp\left(\langle \xi, u\rangle\right) - 1 - \langle h(\xi), u\rangle\right) m(d\xi), \\ R_i(u) &= \langle u, \alpha^i u\rangle + \langle \beta^i, u\rangle - \gamma^i + \int_{\mathbb{R}^d\backslash\{0\}} \left(\exp\left(\langle \xi, u\rangle\right) - 1 - \langle \chi^i(\xi), u\rangle\right) \mu^i(d\xi); \end{aligned} \tag{4.24}$$

   *and*

2. *the generator*[22] $\mathcal{A}$ *of* $(X(t))_{t\geq 0}$ *is given by:*

$$\begin{aligned} \mathcal{A}f(x) = &\sum_{k,l=1}^{d} \left(a_{kl} + \sum_{i=1}^{m} \alpha_{kl}^i x_i\right) \frac{\partial^2 f(x)}{\partial x_k \partial x_l} \\ &+ \left\langle b + \sum_{i=1}^{d} \beta^i x_i, \nabla f(x)\right\rangle - \left(c + \sum_{i=1}^{m} \gamma^i x_i\right) f(x) \\ &+ \int_{D\backslash\{0\}} \left(f(x+\xi) - f(x) - \langle h(\xi), \nabla f(x)\rangle\right) m(d\xi) \\ &+ \sum_{i=1}^{m} \int_{D\backslash\{0\}} \left(f(x+\xi) - f(x) - \langle \chi^i(\xi), \nabla f(x)\rangle\right) x_i \mu^i(d\xi), \end{aligned} \tag{4.25}$$

   *for all* $f \in C_0^2(D)$ *and* $x \in D$.

[20] See e.g. Keller-Ressel [68], Definition 2.4.

[21] See Duffie, Filipović and Schachermayer [33], Theorm 2.7.

[22] See e.g. Øksendal [80], Definition 7.3.1. and Theorem 7.3.3. The generator of an Itô diffusion is defined as $\mathcal{A}f(x) = \lim_{t\downarrow 0} \frac{E_x[f(X(t))]-f(x)}{t}$ for $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$.

*Proof:* For a proof of this theorem we refer to Duffie, Filipović and Schachermayer [33], Proof of Theorem 2.7. or Keller-Ressel [68], Proof of Theorem 2.6. for an alternative approach. □

The general results and simplifications in the one-dimensional case have been outlined in Keller-Ressel and Steiner [69] and we refer to their paper for further details. Specifically we shall concentrate on the domain $D = \mathbb{R}_{\geq 0}$, since we are interested in stock prices which should not become negative. In this setup, we have:[23]

$$\mathcal{U} = \{u \in \mathbb{C} | \mathrm{Re}\, u \leq 0\}. \tag{4.26}$$

Again in this context we may simplify Definition 4.1, thus calling a Markov process $(X(t))_{t\geq 0}$ and its semi-group $(P_t)_{t\geq 0}$ affine if the characteristic function is exponentially affine, i.e. there exist $\mathbb{C}$-valued functions $\phi(t,u)$ and $\psi(t,u)$ on $\mathbb{R}_{\geq 0} \times \mathcal{U}$, such that:

$$E_x \left[ e^{X(t)u} \right] = \exp\left(\phi(t,u) + x\psi(t,u)\right). \tag{4.27}$$

Furthermore, let us outline how the set of admissible parameters from Definition 4.3 simplifies in the one-dimensional case. The parameters $(a, \alpha, b, \beta, c, \gamma, m, \mu)$ are called admissible for a process with state space $D = \mathbb{R}_{\geq 0}$, if:[24]

$$a = 0, \tag{4.28}$$
$$\alpha,\, b,\, c,\, \gamma \in \mathbb{R}_{\geq 0}, \tag{4.29}$$
$$\beta \in \mathbb{R}_{\geq 0}, \tag{4.30}$$
$$\int_{(0,\infty)} (\xi \wedge 1) m(d\xi) < \infty, \tag{4.31}$$

and $m, \mu$ are Lévy measures on $(0, \infty)$. Similarly the truncation functions $h$ and $\chi$ given in Definition 4.4 simplify to:

$$h(\xi) := \begin{cases} 0, & \text{if } D = \mathbb{R}_{\geq 0} \\ \frac{\xi}{1+\xi^2}, & \text{if } D = \mathbb{R}; \end{cases} \tag{4.32}$$

[23] See Keller-Ressel and Steiner [69], Definition 2.1.
[24] See Keller-Ressel and Steiner [69], Definition 2.3.

and

$$\chi_k(\xi) := \begin{cases} \frac{\xi}{1+\xi^2}, & \text{if } D = \mathbb{R}_{\geq 0}, \\ 0, & \text{if } D = \mathbb{R}. \end{cases} \tag{4.33}$$

Since we are merely focusing on $D = \mathbb{R}_{\geq 0}$, the truncation functions simplify considerably and the functions $F(\cdot)$ and $R(\cdot)$ as stated in Equation 4.24 are given by:

$$\begin{aligned} F(u) &= au^2 + bu - c + \int_{D\setminus\{0\}} \left( \exp(\xi u) - 1 - \underbrace{h(\xi)u}_{=0} \right) m(d\xi), \\ R(u) &= \alpha u^2 + \beta u - \gamma + \int_{D\setminus\{0\}} \left( \exp(\xi u) - 1 - \underbrace{\chi(\xi)u}_{=\frac{\xi}{1+\xi^2}u} \right) \mu(d\xi); \end{aligned} \tag{4.34}$$

Similarly the second result from Theorem 4.1 may be simplified, so that the infinitesimal generator is given by:[25]

$$\begin{aligned} \mathcal{A}f(x) &= (a + \alpha x)f''(x) + (b + \beta x)f'(x) - (c + \gamma x)f(x) \\ &\quad + \int_{D\setminus\{0\}} \left(f(x+\xi) - f(x) - f'(x)h(\xi)\right) m(d\xi) \\ &\quad + x \int_{D\setminus\{0\}} \left(f(x+\xi) - f(x) - f'(x)\chi(\xi)\right) \mu(d\xi). \end{aligned} \tag{4.35}$$

This concludes our brief discussion of general properties of affine processes and we will now move on to the next step of investigating the structure of market weights in an affine model.

### Behavior of Market Weights

Let us now assume that our market $\mathfrak{M}$ consists of $n$ independent instances of a positive affine process $X(\cdot)$ with $D = \mathbb{R}_{\geq 0}$, i.e. $\mathfrak{M} = \{X_1(\cdot), \ldots, X_n(\cdot)\}$, whereby all $X_i(\cdot)$ have the same distributional behavior $X_i(\cdot) \sim G_i = G \sim X(\cdot)$ for all $i = 1, \ldots, n$. We shall

[25] See Keller-Ressel and Steiner [69], Theorem 2.4.

furthermore assume that all initial values $X_i(0) = x_i = x$ are equal for $i = 1, \ldots, n$. Hence for all particles in our market, the characteristic function is of the same exponential affine form as given in Equation 4.27. Since we assume independence between all particles in $\mathfrak{M}$, the distribution of the market capitalization $S(t) = \sum_{i=1}^{n} X_i(t)$ may be defined in a straightforward way by means of multiplying the individual characteristic functions.[26]

Since $X_i$, $i = 1, \ldots, n$ are independent, we have

$$E\left[\sum_{j=1}^{n} X_j(t)\right] = nE[X_i(t)] \text{ and } Var\left[\sum_{j=1}^{n} X_j(t)\right] = nVar[X_i(t)]. \tag{4.36}$$

Summarizing, for the process $S(t) = \sum_{j=1}^{n} X_j(t)$ it holds that:

$$E\left[\exp(uS(t))\right] = \exp\left(n\phi(t,u) + nx\psi(t,u)\right), \tag{4.37}$$

whereby $u \in \mathcal{U}$ and $X_i(0) = x \ \ for\, all\, i \in \{1, \ldots, n\}$. Since $n$ is deterministic, this expectation can be written with respect to the starting point $x$ of individual particles and may be expressed as:

$$E_x\left[\exp\left(\frac{uS(t)}{n}\right)\right] = \exp\left(\phi^{(n)}\left(t, \frac{u}{n}\right) + x\psi^{(n)}\left(t, \frac{u}{n}\right)\right). \tag{4.38}$$

Thereby, $S(\cdot)$ is a positive affine process as well, with

$$\phi^{(n)}(t,u) = n\phi\left(t, \frac{u}{n}\right), \tag{4.39}$$

$$\psi^{(n)}(t,u) = n\psi\left(t, \frac{u}{n}\right), \tag{4.40}$$

for $n \in \mathbb{N}$ and $(t,u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$. Furthermore, we obtain

$$\left.\frac{\partial}{\partial t}\phi^{(n)}(t,u)\right|_{t=0+} = \left.n\frac{\partial}{\partial t}\phi(t,u)\right|_{t=0+} = nF\left(\frac{u}{n}\right) =: F^{(n)}(u), \tag{4.41}$$

$$\left.\frac{\partial}{\partial t}\psi^{(n)}(t,u)\right|_{t=0+} = \left.n\frac{\partial}{\partial t}\psi(t,u)\right|_{t=0+} = nR\left(\frac{u}{n}\right) =: R^{(n)}(u). \tag{4.42}$$

Now, that we have analyzed the general behavior of $\mathfrak{M} = \{X_1(\cdot), \ldots, X_n(\cdot)\}$ and of $S(\cdot)$, we want to retrieve an approximation for the ordered market weights $\mu_{(k)}(t) = \frac{X_{(k)}(t)}{\sum_{j=1}^{n} X_j(t)}$. Recall, that we have assumed all particles in the market $\mathfrak{M}$ to follow the distribution function $G$. Let us coherently define the quantile function as the (generalized) inverse $G^{-1}(p)$

[26]See e.g. Kallenberg [64], Chapter 5.

for $p \in [0,1]$[27]. If we want to estimate the market weight of the $k$-th largest ordered particle, we can do so by calculating the quantile corresponding to the probability $\frac{n-k}{n}$ for $k \in \{1,\dots,n\}$. The approximation for the market capitalization may be obtained by straightforward calculation of $E[S(t)]$. We shall therefore utilize the following approximation for $\mu_{(k)}(t),\ t \geq 0$

$$\hat{\mu}_{(k)}(t) = \frac{G^{-1}\left(\frac{n-k}{n}\right)}{E\left[\sum_{j=1}^{n} X_j(t)\right]} = \frac{G^{-1}\left(\frac{n-k}{n}\right)\frac{1}{n}}{E\left[\frac{1}{n}\sum_{j=1}^{n} X_j(t)\right]}. \tag{4.43}$$

The rationale behind (4.43) is to approximate the $k^{th}$ largest stock $X_{(k)}(\cdot)$ by the corresponding quantile of its probability distribution $G(\cdot)$. Let us write $p = G(x)$, then for all $f : [0,1] \to \mathbb{R}$ we have

$$\begin{aligned} \int_0^\infty f(G(x))x dG(x) &= \int_0^1 f(p)G^{-1}(p)dp \\ &= \frac{1}{n}\sum_{i=1}^{n} f(G(X_i(t)))X_i(t) \\ &= \frac{1}{n}\sum_{k=1}^{n} f(G(X_{(k)}(t)))X_{(k)}(t), \end{aligned} \tag{4.44}$$

which holds due to the Law of Large Numbers (LLN)[28]. Hence, again by virtue of the Law of Large Numbers we have

$$G(X_{(k)}(t)) \stackrel{LLN}{=} \frac{n-k}{n}.$$

Consequently we obtain

$$\frac{1}{\frac{1}{N}\sum_{k=1}^{n} X_{(k)}(t)} G^{-1}\left(\frac{n-k}{n}\right)\frac{1}{n} \stackrel{LLN}{=} \frac{\frac{1}{n}X_{(k)}(t)}{\frac{1}{n}\sum_{k=1}^{n} X_{(k)}(t)} = \mu_{(k)}(t). \tag{4.45}$$

Therefore in the limit we can use the approximation

$$\hat{\mu}_{(k)}(t) = \frac{\frac{1}{n}G^{-1}\left(\frac{n-k}{n}\right)}{E[X_1(t)]},$$

which is exactly the term which we have defined in Equation 4.43.

For $u \in \mathcal{U}$ it holds that

$$\varphi_t^x(u) = E_x[\exp(uX(t))] = \left(\mathcal{L}g_t^x\right)(u), \tag{4.46}$$

[27] See also Definition 5.6.

[28] See e.g. Kallenberg [64], Theorem 4.23.

whereby $\mathcal{L}$ denotes the Laplace transform[29] and $g_t^x$ denotes the density corresponding to $G$ (if it exists). Now let us assume, that $g_t^x$ exists, then one may recover the distribution function of $X_i(t)$ for arbitrary $i \in \{1, \dots, n\}$ by calculating the Fourier-Mellin integral[30], whereby $\gamma = \mathrm{Re}(u)$:

$$\begin{aligned} G_{X(t)}(\xi) &= \mathcal{L}^{-1}\left[\frac{(\mathcal{L}g_t^x)(u)}{u}\right] && (4.47) \\ &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} \exp(u\xi)\frac{(\mathcal{L}g_t^x)(u)}{u}du \\ &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} \exp(u\xi)\frac{1}{u}\exp(\phi(t,u)+x\psi(t,u))du \\ &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} \frac{1}{u}\exp\left(u\xi+\phi(t,u)+x\psi(t,u)\right)du. \end{aligned}$$

Provided the above limit exists, one has retrieved the distribution function $G_{X(t)}(\xi)$, $\xi \in \mathbb{R}_{\geq 0}$ and consequently, the approximation for ordered market weights may be obtained by means of the (generalized) inverse $G_{X(t)}^{-1}(\cdot)$[31].

## 4.1.2 The Special Case: Squared Brownian Motion

### General Characteristics of the Squared Brownian Motion

Let $B(\cdot)$ be a standard Brownian Motion and let further $X(\cdot) = (B(\cdot))^2$ be the squared process which will be used as basis for the modeling of single particles in our stock market. Let $X(0) = x$, unless stated otherwise we will mostly use $x = 1$ or $x = 0$. It obviously holds that $X(\cdot)$ takes paths in $\mathbb{R}_{\geq 0}$. Furthermore, we can deduct the distributional properties of $X(t)$ at any time $t \in \mathbb{R}_{\geq 0}$ from the distributional properties of the standard Brownian Motion $B(t)$. Hence, $X(t) = \left(\sqrt{t}Z\right)^2$, whereby $Z \sim N(0,1)$. Since the sum of $m$ squared standard Gaussian random variables follows a $\chi^2(m)$ distribution[32], it holds that

$$X(t) = t\zeta \quad \zeta \sim \chi^2(1). \tag{4.48}$$

The Squared Brownian Motion process together with the $\chi^2(1)$ distribution can easily be

[29] See e.g. Kallenberg [64], Chapter 5.

[30] See e.g. Abate et al. [1], Section 1.1 and Abramowitz and Stegun [2], Section 29.

[31] See also Definition 5.6.

[32] See e.g. Abramowitz and Stegun [2], Section 26.4.

seen to be the one-dimensional special case of a matrix valued Wishart process whose distribution corresponds to the non-central Wishart distribution.[33] The characteristic function of Wishart processes together with the specification of $\phi(t,u)$ and $\psi(t,u)$ is discussed in some detail in Cuchiero and Teichmann [27], its one-dimensional special case being given by

$$\varphi_{X(t)}(u) = \exp\left(-\frac{1}{2}\log(1-2tu) + x\frac{u}{1-2tu}\right), \tag{4.49}$$

whence we directly observe that it is of the form given in Equation 4.3 with:

$$\phi(t,u) = -\frac{1}{2}\log(1-2tu), \tag{4.50}$$

$$\psi(t,u) = \frac{u}{1-2tu}. \tag{4.51}$$

Hence, $X(\cdot)$ is a positive affine process, whose dynamics are yielded by a straightforward application of Itô's formula[34] for $X(t) = (B(t))^2$.

$$\begin{aligned} dX(t) &= 2B(t)dB(t) + \frac{1}{2}2dt \\ &= dt + 2\sqrt{X(t)}dW(t), \end{aligned} \tag{4.52}$$

where the process $W(t) = \int_0^t \frac{B(s)}{\sqrt{X(s)}}dB(s)$ is a continuous local martingale with quadratic variation $\langle W\rangle_t = t$. Hence by Lévy's characterization[35], $W(\cdot)$ is a Brownian Motion. Thus for a stock market $\mathfrak{M} = \{X_1,\dots,X_n\}$ consisting of independent particles governed by above dynamics, each single stock is described by the SDE:

$$dX_i(t) = dt + 2\sqrt{X_i(t)}dW_i(t), \tag{4.53}$$

whereby $(W_1(\cdot),\dots,W_n(\cdot))$ are independent standard Brownian Motions. A further application of Itô's rule to $X_i(\cdot)$ for $f(x) = \log(x)$ yields the dynamics of the instantaneous log-returns of each stock.

[33] See Cuchiero and Teichmann [27], Section 1.
[34] See e.g. Karatzas and Shreve [66] or Øksendal [80].
[35] See e.g. Karatzas and Shreve [66], Theorem 3.3.16.

$$\begin{aligned} d \log X_i(t) &= \frac{1}{X_i(t)} dX_i(t) + \frac{1}{2}\left(-\frac{1}{X_i(t)^2}\right) d\langle X_i, X_i\rangle_t \\ &= \frac{1}{X_i(t)} dt + \frac{2}{\sqrt{X_i(t)}} dW_i(t) - \frac{4}{2X_i(t)^2} X_i(t) dt \\ &= -\frac{1}{X_i(t)} dt + \frac{2}{\sqrt{X_i(t)}} dW_i(t). \end{aligned} \tag{4.54}$$

Thus, following the notation in Fernholz [44], one has a time-dependant growth rate $\gamma_i(t) = \frac{-1}{X_i(t)}$ and stochastic volatility directions $\xi_{ii}(t) = \frac{2}{\sqrt{X_i(t)}}$. Furthermore, again using Fernholz's notation, the instantaneous (co-)variance matrix $\sigma(t) = \xi(t)^T \xi(t)$ has the following diagonal matrix structure:

$$\sigma(t) = \begin{pmatrix} \frac{4}{X_1(t)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{4}{X_n(t)} \end{pmatrix}. \tag{4.55}$$

If all stocks stay away from zero, the whole system behaves well and the above instantaneous covariance matrix of log-returns is invertible with eigenvalues equal to its diagonal elements $\mathfrak{S}(\sigma(t)) = \left\{\frac{4}{X_1(t)}, \ldots, \frac{4}{X_n(t)}\right\}$. This is indeed the case since the local time of the process $X(\cdot)$ at 0 can be seen to be zero. Using the Tanaka-Meyer formula 3.11 we can calculate the local time $\Lambda_X(t, 0)$:

$$\begin{aligned} \Lambda_X(t,0) &= \frac{1}{2}\left(|X(t) - 0| - |X(0) - 0| - \int_0^t \operatorname{sgn}(X(s) - 0) dX(s)\right) \\ &= \frac{1}{2}\left(X(t) - 1 - \int_0^t 1 dX(s)\right) \\ &= \frac{1}{2}\left(X(t) - 1 - X(t) + 1\right) \\ &= 0. \end{aligned}$$

Therefore, for all $t \in \mathfrak{T}^\circ = \{0 \le s < \infty | X_i(t) > 0\}$, $i = 1, \ldots, n$ the market $\mathfrak{M}$ is elliptic and for all $t \in \mathfrak{T} = \{0 \le s < \infty | X_i(t) \ge 0\}$, $i = 1, \ldots, n$ the market $\mathfrak{M}$ is weakly elliptic. At any time $t$ the ellipticity- (non-degeneracy)- bound $\epsilon(t)$ can be set as:

$$\epsilon(t) = \min_{i \in \{1,\ldots,n\},\ 0 \le s \le t} \left\{\frac{4}{X_i(s)} | i = 1, \ldots, n\right\}. \tag{4.56}$$

At time $t$ one obtains that $\sigma(t)$ is positive definite and $x^T\sigma(t)x \geq \epsilon(t)\|x\|^2,\ x \in \mathbb{R}$. This approach may similarly be applied to the upper bound $M(t)$, setting it to the be the maximum in Equation 4.56, yet it is clear, that this bound will become extremely large in the case that a single stock takes values close to zero.

In the next step we will briefly assess the quantile function of the SqBM process. Recalling our setting of $X_i(\cdot) = W_i(\cdot)^2$, where $W_i(\cdot)$ are independent standard Brownian Motions for $i = 1, \ldots, n$, we can now investigate the general distributional properties of $X_i(\cdot)$ which can be directly derived from the properties of standard Brownian Motion.[36]

Since $W_i(t) \sim N(0,t)$ one directly obtains that $\frac{W_i(t)}{\sqrt{t}} \sim N(0,1)$. Defining

$$Y_i(t) := \frac{X_i(t)}{t} = \frac{W_i(t)^2}{t} = \left(\frac{W_i(t)}{\sqrt{t}}\right)^2, \tag{4.57}$$

it clearly follows, that $Y_i(t) \sim \chi^2(1)$ as square of a standard normally distributed random variable.[37] For the sake of simplicity we will in the first step discuss the behavior of the time-scaled particles $Y_i(t),\ i = 1, \ldots, n$ and we will show in the second step, that the approximation of market weights is in fact invariant with respect to any time scaling. Hence, $Y_i(\cdot)$ has the density

$$f(y,1) = \begin{cases} \frac{1}{\sqrt{2}\Gamma(\frac{1}{2})} y^{\frac{1}{2}-1} \exp(-\frac{y}{2}), & x \geq 0; \\ 0, & \text{else}; \end{cases} \tag{4.58}$$

and cumulative distribution function

$$P(Y \leq y) = F(y,1) = \frac{\gamma(\frac{1}{2}, \frac{y}{2})}{\Gamma(\frac{1}{2})} = \frac{\int_0^{y/2} \frac{1}{\sqrt{(t)}} \exp(-t)dt}{\int_0^{\infty} \frac{1}{\sqrt{(t)}} \exp(-t)dt}, \tag{4.59}$$

whereby $\gamma(s,y) = \int_0^y t^{s-1} \exp(-t)dt$ is the lower incomplete Gamma-function[38] and $\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t)dt$ denotes the Gamma-function.[39] Since $\Gamma(1/2) = \sqrt{\pi}$ [40], the cumulative distribution function (4.59) simplifies to:

$$P(Y \leq y) = F(y,1) = \frac{1}{\sqrt{\pi}} \int_0^{y/2} \frac{1}{\sqrt{t}} \exp(-t)dt = \frac{1}{\sqrt{\pi}} \gamma\left(\frac{1}{2}, \frac{y}{2}\right). \tag{4.60}$$

$\gamma\left(\frac{1}{2}, \frac{y}{2}\right)$ can further be expressed in terms of the error function $\text{Erf}(\cdot)$[41]:

[36] For the general properties of Brownian Motion we refer e.g. to Karatzas and Shreve [66].
[37] See e.g. Abramowitz and Stegun [2], Section 26.4.
[38] See Abramowitz and Stegun [2], Section 6.3.
[39] See Abramowitz and Stegun [2], Section 6.1.
[40] See Abramowitz and Stegun [2], 6.1.8.
[41] See Abramowitz and Stegun [2], Section 7.1.

$$\mathrm{Erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y \exp(-t^2)dt. \tag{4.61}$$

Thus, for $\gamma(\cdot,\cdot)$ one obtains

$$\begin{aligned}
\gamma\left(\frac{1}{2}, \frac{y}{2}\right) &= \int_0^{y/2} \frac{1}{\sqrt{t}} \exp(-t)dt \\
&= \sqrt{\pi}\left(1 - \mathrm{Erf}(\sqrt{y/2})\right) \\
&= \sqrt{\pi} - 2\int_0^{\sqrt{y/2}} \exp(-t^2)dt.
\end{aligned} \tag{4.62}$$

Furthermore, the cumulative distribution function $F(y,1)$ may also be expressed in terms of the regularized Gamma function $Q(a,z) = \frac{\gamma(a,z)}{\Gamma(a)}$. We have already seen above in Equation 4.59, that $F(y,1) = Q(\frac{1}{2}, \frac{y}{2})$. The inverse of the regularized Gamma Function may be calculated numerically which provides us with an approach to calculate the quantile function of the $\chi^2(1)$ distribution. An alternative approach for the calculation of the quantile function would be to use the error function, which will also be outlined below. The numerical results of both approaches are of course the same.

Using the correspondence

$$Q\left(\frac{1}{2}, \frac{y}{2}\right) = p \leftrightarrow Q^{-1}\left(\frac{1}{2}, p\right) = \frac{y}{2}, \tag{4.63}$$

one can directly calculate the quantile corresponding to the probability $p$ as:

$$y = 2Q^{-1}\left(\frac{1}{2}, p\right). \tag{4.64}$$

Alternatively, one may use the relationship with the error function stated in Equation 4.62, to calculate the quantile function. By Equation 4.61 one obtains the relationship to the cumulative distribution function of the standard normal distribution $\Phi(\cdot)$ in a straightforward way as:

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp(-t^2/2)dt = \frac{1}{2}\left[1 + \mathrm{Erf}\left(\frac{y}{\sqrt{2}}\right)\right]. \tag{4.65}$$

Using Equation 4.65, we may now restate the relationship outlined in Equation 4.60.

$$\begin{aligned}
F(y,1) &= \frac{1}{\sqrt{\pi}}\gamma\left(\frac{1}{2},\frac{y}{2}\right) = \frac{1}{\sqrt{\pi}}\left(\sqrt{\pi}\left(1-\mathrm{Erf}\left(\frac{\sqrt{y}}{\sqrt{2}}\right)\right)\right) \\
&= 1-(2\Phi(\sqrt{y})-1) \\
&= 2\left(1-\Phi(\sqrt{y})\right) = 2\Phi(-\sqrt{y}). \qquad (4.66)
\end{aligned}$$

Using (4.66), we can alternatively calculate the quantile for probability $p$ as:

$$y = \left(-\Phi^{-1}(p/2)\right)^2. \qquad (4.67)$$

After this little digression on the calculation of quantiles for our stock price process $X(\cdot)$ we can now approximate the ordered market weights $\mu_{(i)}(\cdot)$ for $i = 1,\ldots,n$ via the following ratio:

$$\hat{\mu}_{(i)}(t) = \frac{2Q^{-1}_{Y(t)}\left(\frac{1}{2},\frac{n-i}{n}\right)}{\underbrace{E\left[\sum_{i=1}^{n} Y_i(t)\right]}_{=n}}. \qquad (4.68)$$

We can directly compute the characteristic function of $\tilde{S}(t) = \sum_{i=1}^{n} Y_i(t)$ as product of the individual characteristic functions of the $Y_i(t),\ i = 1,\ldots,n$.

$$\varphi_{\tilde{S}(t)}(u) = \prod_{i=1}^{n} \varphi_{Y_i(t)}(u) = \exp\left(-\frac{n}{2}\log(1-2u)\right). \qquad (4.69)$$

Due to Equation 4.69, one may directly conclude that $\tilde{S}(t) \sim \chi^2(n)$[42] and therefore it holds that $E[\tilde{S}(t)] = n$.

The ratio given in 4.68 remains unchanged if we re-scale $Y_i(t) \sim \chi^2(1),\ i = 1,\ldots,n$ by a factor $t$ in order to regain the characteristics of $X_i(t),\ i = 1,\ldots,n$. In this case we can use the fact that for $Y \sim \chi^2(1)$ a random variable $\Xi = tY$ follows a $\Gamma(1/2, 2t)$ distribution. In this case, $E(\Xi) = t$ and $Var(\Xi) = t^2$. For the calculation of the quantile function we can once again employ the fact that $P(X \leq x) = Q(\frac{1}{2}, \frac{x}{2t})$ which permits the direct calculation of the $p$-quantile as $z = 2tQ^{-1}\left(\frac{1}{2}, p\right)$. Thus in case of re-scaling we obtain the following approximation for the market weights:

$$\tilde{\mu}_{(i)}(t) = \frac{2tQ^{-1}_{X(t)}\left(\frac{1}{2},\frac{n-i}{n}\right)}{nt} = \frac{2Q^{-1}_{X(t)}\left(\frac{1}{2},\frac{n-i}{n}\right)}{n}, \qquad (4.70)$$

[42] See e.g. Abramowitz and Stegun [2], Section 26.4.

which is exactly the same as in Equation 4.68, i.e. this estimator for the ordered market weights is time invariant. The ordered market weights for four markets consisting of 100, 500, 1000 and 10000 stocks respectively which have been obtained by this approximation are plotted in Figure 4.1 on a $\log - \log$ scale.

The log-log plot of the ordered market weights reproduces the desired shape of the Capital Distribution Curve as depicted in Figure 3.1 and as discussed in Chapter 5 of Fernholz [44] very well. Therefore, it is fair to conclude, that the approach of modeling individual stocks in a market $\mathfrak{M}$ through independent particles of a Squared Brownian Motion is successful in reproducing the observable distributional structure of market capitalizations and is therefore a well founded basis for further analysis. Following our general discussion of market properties of this type of market model, we will further enhance the model in Section 4.2 by endowing it with the correct correlation structure.

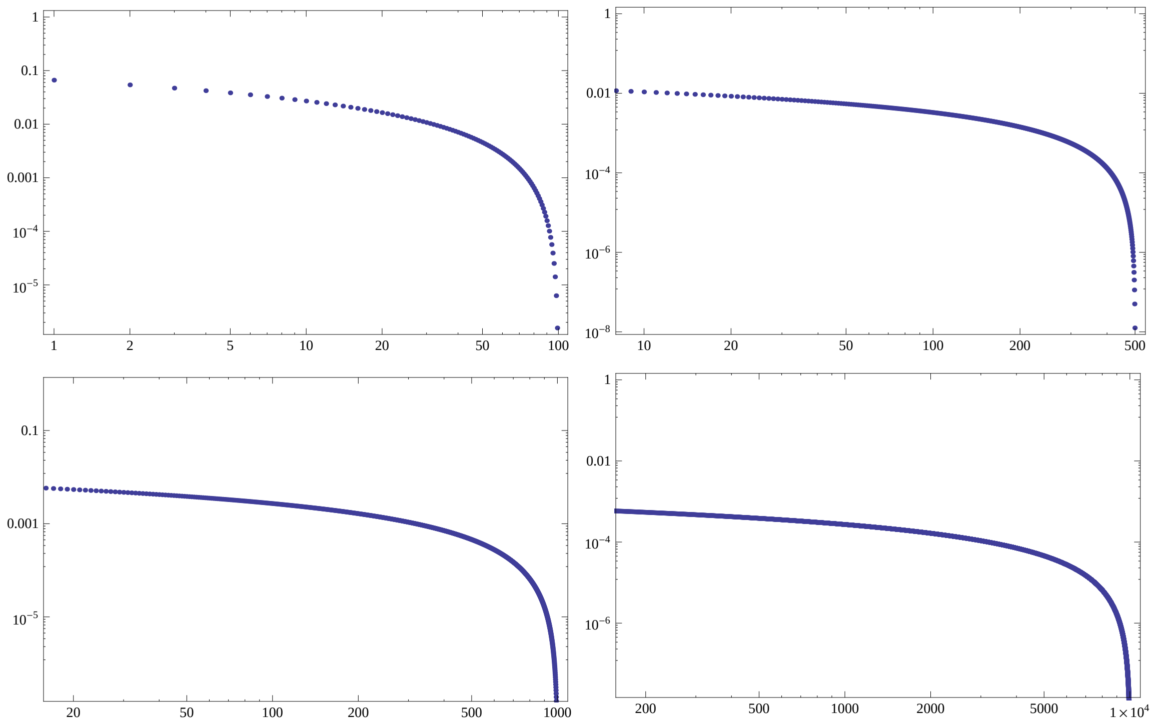


Figure 4.1: Log-log-plot of estimated market weights for Squared Brownian Motion markets consisting of 100 (top left), 500 (top right), 1000 (bottom left) and 10000 (bottom right) particles.

**Portfolio Dynamics for $(W(t))^2$**

Let us now consider a portfolio $\pi = (\pi_i(\cdot))$ in the market $\mathfrak{M}$ which fulfills the self-financing condition

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \sum_{i=1}^n \pi_i(t)\frac{dX_i(t)}{X_i(t)}, \tag{4.71}$$

then we can calculate the dynamics of the portfolio and of the portfolio log-returns in a straightforward way as follows:[43]

$$dZ_\pi(t) = Z_\pi(t)\sum_{i=1}^n \frac{\pi_i(t)}{X_i(t)}\left(dt + 2\sqrt{X_i(t)}dW_i(t)\right); \tag{4.72}$$

and

$$\begin{aligned}
d\log Z_\pi(t) &= \frac{1}{Z_\pi(t)}dZ_\pi(t) + \frac{1}{2}\frac{-1}{Z_\pi(t)^2}d\langle Z_\pi, Z_\pi\rangle_t \\
&= \sum_{i=1}^n \frac{\pi_i(t)}{X_i(t)}\left(dt + 2\sqrt{X_i(t)}dW_i(t)\right) - 2\sum_{i=1}^n \pi_i(t)^2\frac{1}{X_i(t)}dt \\
&= \sum_{i=1}^n \pi_i(t)\underbrace{\frac{2}{\sqrt{X_i(t)}}}_{=\xi_i(t)}dW_i(t) + \underbrace{\frac{1}{2}\sum_{i=1}^n \pi_i(t)\frac{2}{X_i(t)}dt}_{=\frac{1}{2}\sum_{i=1}^n \pi_i(t)\frac{4}{X_i(t)}dt - \frac{1}{2}\sum_{i=1}^n \pi_i(t)\frac{2}{X_i(t)}dt} \\
&\quad -\frac{1}{2}\sum_{i=1}^n \pi_i(t)^2\frac{2}{X_i(t)}dt - \frac{1}{2}\sum_{i=1}^n \pi_i(t)^2\frac{2}{X_i(t)}dt \qquad (4.73)\\
&= \sum_{i=1}^n \pi_i(t)\frac{2}{\sqrt{X_i(t)}}dW_i(t) - \sum_{i=1}^n \pi_i(t)\frac{1}{X_i(t)}dt \\
&\quad +\frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\frac{4}{X_i(t)} - \sum_{i=1}^n \pi_i(t)^2\frac{4}{X_i(t)}\right)dt; \qquad (4.74)
\end{aligned}$$

with $\xi_i(t) = \frac{2}{\sqrt{X_i(t)}}$, $\gamma_i(t) = -\frac{1}{X_i(t)}$ and $\sigma_{ii}(t) = \frac{4}{X_i(t)}$ we obtain the dynamics corresponding to Proposition 1.1 with the following portfolio growth rate $\gamma_\pi(\cdot)$ and excess growth rate $\gamma_\pi^*(\cdot)$:

[43] See also Chapter 1.

$$d\log Z_\pi(t) = \underbrace{\left[-\sum_{i=1}^n \pi_i(t)\frac{1}{X_i(t)} + \frac{1}{2}\underbrace{\left(\sum_{i=1}^n \pi_i(t)\frac{4}{X_i(t)} - \sum_{i=1}^n \pi_i(t)^2\frac{4}{X_i(t)}\right)}_{=\gamma^*_\pi(t)}\right]}_{=\gamma_\pi(t)} dt$$
$$+\sum_{i=1}^n \pi_i(t)\frac{2}{\sqrt{X_i(t)}}dW_i(t). \tag{4.75}$$

As a special case, we may now compute the dynamics of the market portfolio, setting $\mu_i(t) = \frac{X_i(t)}{\sum_{j=1}^n X_j(t)}$, we obtain the following dynamics out of (4.75).

$$\begin{aligned}
d\log Z_\mu(t) &= \left[-\sum_{i=1}^n \frac{X_i(t)}{X_i(t)\sum_{j=1}^n X_j(t)}\right.\\
&\quad \left.+\frac{1}{2}\left(\sum_{i=1}^n \frac{X_i(t)}{\sum_{j=1}^n X_j(t)}\frac{4}{X_i(t)} - \sum_{i=1}^n \frac{X_i(t)^2}{\left(\sum_{j=1}^n X_j(t)\right)^2}\frac{4}{X_i(t)}\right)\right] dt\\
&\quad +\sum_{i=1}^n \frac{X_i(t)}{\sum_{j=1}^n X_j(t)}\frac{2}{\sqrt{X_i(t)}}dW_i(t)\\
&= \left[\frac{n}{\sum_{j=1}^n X_j(t)} - 2\frac{1}{\sum_{j=1}^n X_j(t)}\sum_{i=1}^n \underbrace{\frac{X_i(t)}{\sum_{j=1}^n X_j(t)}}_{=\mu_i(t)}\right] dt\\
&\quad +2\sum_{i=1}^n \frac{\sqrt{X_i(t)}}{\sum_{j=1}^n X_j(t)}dW_i(t)\\
&= \frac{n-2}{\sum_{j=1}^n X_j(t)}dt + 2\sum_{i=1}^n \frac{\sqrt{X_i(t)}}{\sum_{j=1}^n X_j(t)}dW_i(t). \qquad (4.76)
\end{aligned}$$

Hence, by (4.76), $Z_\mu(t)$ takes the form

$$Z_\mu(t) = Z_\mu(0)\exp\left(\frac{1}{\sum_{j=1}^n X_j(t)}\left[(n-2)t + 2\int_0^t \sum_{i=1}^n \sqrt{X_i(s)}dW_i(s)\right]\right). \tag{4.77}$$

Since we can set $Z_\mu(t) = \sum_{j=1}^n X_j(t)$ [44], we get the dynamics of $Z_\mu(\cdot)$ by application of

[44] See Fernholz [44], (1.2.10).

Itô's formula to $f(\cdot) = \exp(\cdot)$ to (4.76) as[45]

$$dZ_\mu(t) = ndt + \sum_{i=1}^{n} 2\sqrt{X_i(t)}dW_i(t) = \sum_{i=1}^{n} dX_i(t). \tag{4.78}$$

Furthermore, in order to render our model more amenable to comparison with volatility stabilized markets one may introduce the process

$$\tilde{W}(t) = \int_0^t \sum_{i=1}^{n} \sqrt{\frac{X_i(t)}{\sum_{j=1}^{n} X_j(t)}} dW_i(t),$$

which is a local martingale and since we are actually taking the sum over the square roots of all market weights, it clearly holds that

$$\langle \tilde{W} \rangle_t = \int_0^t \sum_{i=1}^{n} \mu_i(s)ds = t,$$

and therefore that $\tilde{W}(\cdot)$ is a Brownian Motion by Lévy's characterization theorem.[46] Thus, Equations 4.76, 4.77 and 4.78 may also be expressed in terms of $\tilde{W}(\cdot)$ as:

$$d\log Z_\mu(t) = \frac{n-2}{Z_\mu(t)}dt + \frac{2}{\sqrt{Z_\mu(t)}}d\tilde{W}(t), \tag{4.79}$$

$$Z_\mu(t) = Z_\mu(0)\exp\left(\int_0^t \frac{n-2}{Z_\mu(s)}ds + \int_0^t \frac{2}{\sqrt{Z_\mu(s)}}d\tilde{W}(s)\right), \tag{4.80}$$

$$dZ_\mu(t) = Z_\mu(t)\left(\frac{n}{Z_\mu(t)}dt + \frac{2}{\sqrt{Z_\mu(t)}}d\tilde{W}(t)\right). \tag{4.81}$$

### Covariance Matrix and Relative Covariance Matrix

Starting from the instantaneous covariance matrix $\sigma(t)$ which was given in Equation 4.55, we can calculate the covariance of a certain stock $i$ relative to a portfolio $\eta$ and the relative covariance matrix $\tau(\cdot)$ as introduced in Chapter 1. Following Definition 1.7, one obtains:

$$\sigma_{i\eta}(t) = \sum_{j=1}^{n} \eta_j(t)\sigma_{ij}(t) = \eta_i(t)\frac{4}{X_i(t)}, \tag{4.82}$$

[45] See also Fernholz [44]

[46] See e.g. Karatzas and Shreve [66], Theorem 3.3.16.

and

$$\tau_{ij}^{\eta}(t) = \sigma_{ij}(t) - \sigma_{i\eta}(t) - \sigma_{j\eta}(t) + \sigma_{\eta\eta}(t). \tag{4.83}$$

Thus, for $i \neq j$ we have:

$$\begin{aligned} \tau_{ij}^{\eta}(t) &= 0 - \eta_i(t)\frac{4}{X_i(t)} - \eta_j(t)\frac{4}{X_j(t)} + \sigma_{\eta\eta}(t) \\ &= -\eta_i(t)\frac{4}{X_i(t)} - \eta_j(t)\frac{4}{X_j(t)} + \sum_{l=1}^{n}\eta_l^2(t)\frac{4}{X_l(t)}, \end{aligned} \tag{4.84}$$

and for $i = j$ one gets:

$$\tau_{ii}^{\eta}(t) = \frac{4}{X_i(t)} - \eta_i(t)\frac{8}{X_i(t)} + \sum_{l=1}^{n}\eta_l^2(t)\frac{4}{X_l(t)}. \tag{4.85}$$

Furthermore, we can now restate Lemma 1.2.2 from Fernholz [44] for this special case.

**Lemma 4.1:** *For any given portfolio $\eta(\cdot)$ in the market $\mathfrak{M}$, the instantaneous relative covariance matrix with respect to this portfolio , $\tau^{\eta}(\cdot)$ is positive semi-definite with kernel $ker(\tau^{\eta}(\cdot)) = \{\eta(\cdot)\}$.*

*Proof:* Let $x \in \mathbb{R}^n$, then:

$$\begin{aligned} x\tau^{\eta}(t)x^T &= x\sigma(t)x^T - x\begin{pmatrix} \sigma_{1\eta}(t) & \cdots & \sigma_{1\eta}(t) \\ \vdots & & \vdots \\ \sigma_{n\eta}(t) & \cdots & \sigma_{n\eta}(t) \end{pmatrix} x^T \\ &\quad - x\begin{pmatrix} \sigma_{1\eta}(t) & \cdots & \sigma_{1\eta}(t) \\ \vdots & & \vdots \\ \sigma_{n\eta}(t) & \cdots & \sigma_{n\eta}(t) \end{pmatrix} x^T + \sigma_{\eta\eta}(t)x\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \\ &= x\sigma(t)x^T - 2\underbrace{\sum_{i=1}^{n} x_i\sigma_{i\eta}(t)}_{x\sigma(t)\eta(t)^T}\sum_{i=1}^{n} x_i + \sigma_{\eta\eta}(t)\sum_{i=1}^{n} x_i^2. \end{aligned} \tag{4.86}$$

We will now distinguish between two cases:

$1^{st}$ **Case:** $\sum_{i=1}^{n} x_i = a \neq 0$

In this case we perform the following re-scaling $y := \frac{1}{a}x$. Hence we have the correspondence $x\tau^{\eta}(t)x^T = a^2 y\tau^{\eta}(t)y^T$.

$$\begin{aligned} y\tau^{\eta}(t)y^T &= y\sigma(t)y^T - 2y\sigma(t)\eta(t) + \eta(t)\sigma(t)\eta(t)^T \\ &= (y-\eta(t))\sigma(t)(y-\eta(t))^T \geq 0. \end{aligned} \tag{4.87}$$

(4.87) holds due to the fact that in our setup $\sigma(\cdot)$ is positive definite, given that no particle $X_i(\cdot)$ should become zero.

$2^{nd}$ **Case:** $\sum_{i=1}^{n} x_i = 0$

Here, Expression 4.86 simplifies to:

$$x\tau^{\eta}(t)x^T = x\sigma(t)x^T \geq 0.$$

From (4.87) we also directly get, that $ker(\tau^{\eta}(\cdot)) = \{\eta(\cdot)\}$.

□

**A Remark on Arbitrage in the SqBM Model**

A natural question arising at this point is whether a market $\mathfrak{M}$ whose components are governed by the dynamics of Equation 4.53 permits the construction of an equivalent martingale measure and is therefore arbitrage-free in the sense of Delbaen and Schachermayer [30], Theorem 1.1. Since we are considering independent particles at this point, it suffices to study the construction of an equivalent martingale measure for one coordinate. Let $i \in \{1, \ldots, n\}$ denote an arbitrary particle in the market whose dynamics are - as discussed previously - given by

$$dX_i(t) = dt + 2\sqrt{X_i(t)}dW_i(t).$$

We want to apply Girsanov's theorem[47], to which end we re-state the market price of risk process

$$\theta_i(t) = \frac{1}{2\sqrt{X_i(t)}}. \tag{4.88}$$

Then it holds that $2\sqrt{X_i(t)}\theta_i(t) = 1 - \beta(t)$, whereby we choose $\beta(t) = 0$.[48] Let us now put

$$M(t) = \exp\left(-\int_0^t \frac{1}{2\sqrt{X_i(t)}}dW_i(s) - \frac{1}{2}\int_0^t \frac{1}{4X_i(s)}ds\right),\ t \leq T.$$

[47] See e.g. Øksendal [80], Theorem 8.6.6.

[48] See Øksendal [80], Theorem 8.6.6. and following Remarks.

If $M(\cdot)$ is a martingale, then the equivalent martingale measure $Q(\omega)$ is given by

$$dQ(\omega) = M(T,\omega)dP(\omega) \ on \ \mathfrak{F}_T, \tag{4.89}$$

and the process $\hat{W}(\cdot)$

$$\hat{W}_i(t) := \int_0^t \frac{1}{2\sqrt{X_i(s)}} ds + W(t) \tag{4.90}$$

is a $Q$-Brownian Motion. Thereby $X_i(\cdot)$ would have the following dynamics under $Q$:

$$dX_i(t) = 2\sqrt{X_i(t)} d\hat{W}_i(t). \tag{4.91}$$

The $P$-dynamics of the process $X_i(\cdot)$ and its previously discussed distributional properties imply that $P(X_i(t) = 0) = 0$. On the other hand, under $Q$ the process $X_i(\cdot)$ as given in Equation 4.91 is a squared Bessel process of dimension $\delta = 0$ ($BESQ^\delta$).[49] It is evident from the dynamics given in Equation 4.91 that once the process has reached zero, it will stay there. Since the ($BESQ^\delta$) process is recurrent for $\delta \leq 2$ and the set $\{0\}$ is an absorbing point of the process for $\delta = 0$, one obtains that $Q(X_i(t) = 0) > 0$.[50] This however means that $Q \nsim P$, i.e. we do not have equivalence of the measures $P$ and $Q$.

Nonetheless, one may construct an equivalent martingale measure if we consider the following stopping time

$$T_i^{\mathfrak{M}} := \min\left\{0 \leq t \leq \infty | X_i(t) \leq X_{min}^{\mathfrak{M}}\right\}, \ i = 1, \ldots, n, \tag{4.92}$$

whereby $X_{min}^{\mathfrak{M}} > 0$ is the minimum admissible market capitalization. One may interpret $X_{min}^{\mathfrak{M}} > 0$ as a restructuring threshold which once it is passed leads to the company being subject to a bankruptcy action akin to Chapter 11 in the United States or, thinking of a benchmark index, which leads to the elimination of a certain stock from the index universe. Altogether, imposing such a minimum boundary does not seem unreasonable and does not really infringe the flexibility of our model. Now one may define the stopped processes as

$$\hat{X}_i(t) = X_i(t \wedge T_i^{\mathfrak{M}}), \ i = 1, \ldots, n. \tag{4.93}$$

Then the Novikov[51] condition reads

$$\begin{aligned} E\left[\exp\left(\frac{1}{2}\int_0^T \frac{1}{2\sqrt{\hat{X}_i(s)}} ds\right)\right] &\leq E\left[\exp\left(\frac{1}{2}\int_0^T \frac{1}{2\sqrt{\hat{X}_{min}^{\mathfrak{M}}}} ds\right)\right] \\ &= E\left[\exp\left(\frac{T}{4\sqrt{X_{min}^{\mathfrak{M}}}}\right)\right] \\ &= \exp\left(\frac{T}{4\sqrt{X_{min}^{\mathfrak{M}}}}\right) < \infty. \end{aligned} \tag{4.94}$$

[49] See Revuz and Yor [89], Chapter XI, Definition 1.1.
[50] See Revuz and Yor [89], Chapter XI, §1 and Proposition 1.5.
[51] See e.g. Øksendal [80], (8.6.22).

Thus the market $\hat{\mathfrak{M}}$ consisting of $i = 1, \ldots, n$ independent particles following stopped Squared Brownian Motions $\hat{\mathfrak{M}} = \{\hat{X}_1(\cdot), \ldots, \hat{X}_n(\cdot)\}$ admits the construction of an equivalent martingale measure and is therefore arbitrage free in the sense of "No Free Lunch with Vanishing Risk (NFLVR)" according to the Fundamental Theorem of Asset Pricing.[52] Furthermore, the existence of an equivalent martingale measure implies that $\hat{\mathfrak{M}}$ is not weakly diverse.[53]

### Visualization of a SqBM-Market

In Figure 4.2 we provide a first visualization of a market consisting of 100 stocks, modeled by independent SqBM-processes. The Capital Distribution Curves (CDCs) depicted in Figure 4.2 are obtained by simulation of 100 independent SqBM-particles for 100 time-steps. The simulated CDCs correspond reasonably well to the observable pattern which we have depicted in Figure 3.1.

[52] See Delbaen and Schachermayer [30], Theorem 1.1.

[53] See Fernholz [44], Problem 5.3.6. together with Example 3.3.3.

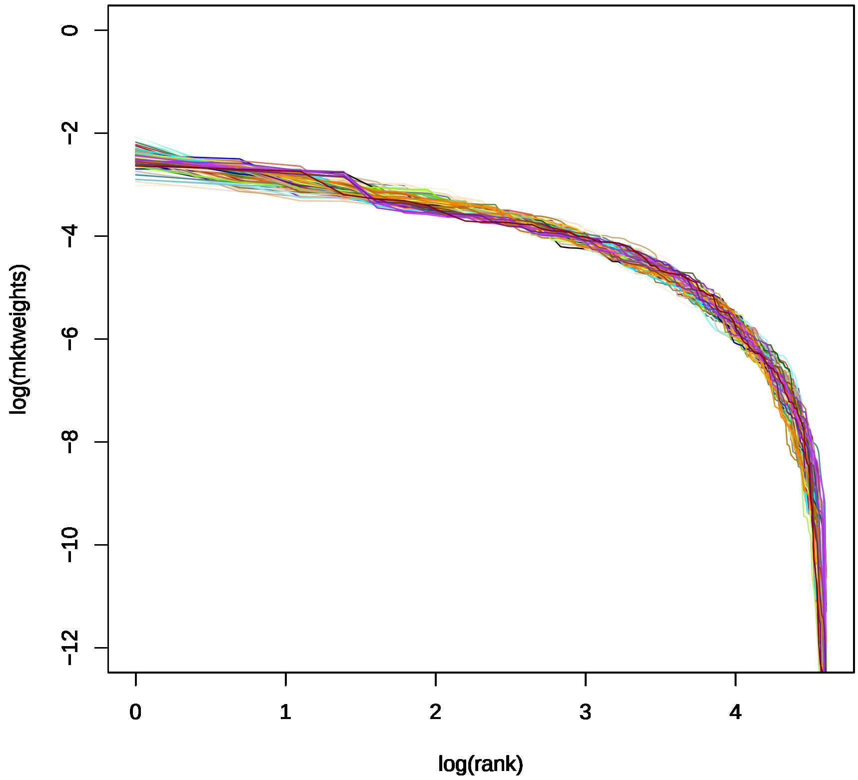


Figure 4.2: Capital Distribution Curves for 100 simulated time-steps in a SqBM-Market.

**Connection to Volatility Stabilized Markets**

The approach to utilize a Squared Brownian Motion as model for stock price dynamics may seem rater far fetched at first, however we will see in Section 4.2 that this model may be equipped with some minor modifications which lead to reasonable patterns for the depiction of stock price movements. Further, at this point a connection ought to be made to the concept of Volatility Stabilized Markets which have been studied predominantly from the perspective of relative arbitrage so far. Furthermore similar dynamics for asset prices are incorporated in the SABR model for stochastic volatility[54] and the Constant

[54]See Hagan et al. [58].

Elasticity of Variance (CEV) model.[55]

For the general concept of Volatility Stabilized Markets (VSM), we refer to the relevant literature, above all Fernholz and Karatzas [46], [48] and [49], Pal [84] and Shkolnikov [95]. The concept of a volatility stabilized model (VSM) for equity markets was introduced by Fernholz and Karatzas in [46] and is further discussed in [48] and [49]. The more abstract modeling framework of these markets permits to obtain some rather remarkable results on relative arbitrage and other aspects of stochastic portfolio theory. The VSM is also a key feature in ongoing research. So for instance in his recent work, Pal [84] derived the joint density of market weights in volatility stabilized models and Shkolnikov [95] investigated the behavior of large volatility stabilized markets if the number of diffusions tends to infinity. These strong results, however, come at the cost of less analytic tractability and interpretability of the behavior of such models.

In its general form, the VSM may be specified as follows:[56]

$$d\log X_i(t) = \frac{a}{2\mu_i(t)}dt + \frac{1}{\sqrt{\mu_i(t)}}dW_i(t), \tag{4.95}$$

for stocks $X_i(\cdot)$, $i = 1, \ldots, n$ and $a \geq 0$ constant. By application of Itô's formula[57] for $f(x) = \exp(x)$ one obtains:

$$\begin{aligned} dX_i(t) &= X_i(t)\left(\frac{a}{2\mu_i(t)}dt + \frac{1}{\sqrt{\mu_i(t)}}dW_i(t)\right) + \frac{1}{2}X_i(t)\frac{1}{\mu_i(t)}dt \\ &= X_i(t)\left(\frac{a+1}{2\mu_i(t)}dt + \frac{1}{\sqrt{\mu_i(t)}}dW_i(t)\right). \end{aligned}$$

By substituting $\mu_i(t) = \frac{X_i(t)}{\sum_{j=1}^n X_j(t)}$ we get:[58]

$$dX_i(t) = \frac{1+a}{2}\left(\sum_{j=1}^n X_j(t)\right)dt + \sqrt{X_i(t)\left(\sum_{j=1}^n X_j(t)\right)}dW_i(t). \tag{4.96}$$

In this setup, one has a rate of return process $\alpha_i(\cdot)$ and sensitivities $\xi_{i\nu}(\cdot)$ given by:

$$\alpha_i(t) = \frac{(1+a)}{2\mu_i(t)} \;\; and \;\; \xi_{i\nu}(t) = \frac{\delta_{i\nu}}{\sqrt{\mu_i(t)}}. \tag{4.97}$$

[55] See e.g. Carr and Linetsky [20].

[56] See e.g. Fernholz and Karatzas [46], Section 6 or [49], Chapter IV, Section 12.

[57] See e.g. Øksendal [80], Theorem 4.1.2.

[58] See also Fernholz and Karatzas [46], Section 6 or [49], Chapter IV, Section 12.

With these return rates and volatilities and interest rate $r(\cdot)$ set to zero, the VSM fulfills the regularity condition:[59]

$$\sum_{j=1}^{n} \int_0^T \left( |\alpha_j(t)| + \sum_{\nu=1}^{n} (\xi_{j\nu}(t))^2 \right) dt < \infty. \tag{4.98}$$

Furthermore, the market price of risk in this setup is given by $\theta_\nu(\cdot) = \frac{(1+a)}{2\sqrt{\mu_i(\cdot)}}$ which also fulfills the regularity conditions given in Equation 1.71.[60]

Hence, in a VSM we are looking at uncorrelated stocks whose drift and volatility are largest for small stocks. This setup will lead to large fluctuations for single stocks but to a relatively stable behavior of the overall market.[61] We note, that by setting $a = 0$ as it is done in the basic setup in [46], Section 4., one obtains a model for stocks whose log-returns solely depend on the fluctuations of the respective Brownian Motion, amplified by by the inverse of the square-root of its market weight. Furthermore, it should be noted that the VSM is not diverse.[62]

Since in this model, the instantaneous covariance matrix of the log-returns is the diagonal matrix with entries $\sigma_{ii} = 1/\mu_i(t)$, one directly obtains that the portfolio variance of the market portfolio is $\sigma_{\mu\mu}(t) = \mu(t)\sigma\mu(t)^T = 1$ for all $t \in [0, \infty)$. By using the definition of the portfolio growth rate given in Proposition 1.1, we can directly calculate the growth rate and excess growth rate for the market portfolio:

$$\begin{aligned} \gamma_\mu(t) &= \sum_{i=1}^{n} \mu_i(t) \frac{a}{2\mu_i(t)} + \frac{1}{2} \left( \sum_{i=1}^{n} \mu_i(t) \frac{1}{\mu_i(t)} - \sum_{i=1}^{n} \mu_i^2(t) \frac{1}{\mu_i(t)} \right) \\ &= \frac{n(a+1)-1}{2}. \end{aligned}$$

By this calculation, one directly sees that the excess growth rate of the market portfolio is given by $\gamma_\mu^*(t) = (n-1)/2$. Hence both the growth rate and excess growth rate of the market portfolio are constant and independent of $t$.[63] By this and Proposition 1.1 one directly obtains the following characterization of the log-return process of the market portfolio:[64]

$$d \log Z_\mu(t) = \frac{n(a+1)-1}{2} dt + \sum_{\nu=1}^{n} \sqrt{\mu_\nu(t)} dW_\nu(t). \tag{4.99}$$

[59] See Fernholz and Karatzas [49], Chapter IV, Section 12.
[60] See Fernholz and Karatzas [49], Chapter IV, Section 12.
[61] See Fernholz and Karatzas [46], Section 4.
[62] See Fernholz and Karatzas [46], Section 4. or [49], Chapter IV, Section 12.
[63] See also Fernholz and Karatzas [46] and [49].
[64] See also Fernholz and Karatzas [46], Section 6.

As already outlined before, the process $\tilde{W}(t) = \int_0^t \sum_{\nu=1}^n \sqrt{\mu_\nu(s)} dW_\nu(s)$ is a local martingale and $\langle \tilde{W} \rangle_t = \int_0^t \sum_{\nu=1}^n \mu_\nu(s) ds = t$. By Lévy's characterization Theorem[65] $\tilde{W}(\cdot)$ is a Brownian motion.[66] Hence, the portfolio value process for the market portfolio takes the form of a geometric Brownian motion with drift:

$$Z_\mu(t) = Z_\mu(0) \exp\left( \frac{n(a+1)-1}{2} t + \tilde{W}(t) \right), \tag{4.100}$$

whose dynamics are given by

$$dZ_\mu(t) = Z_\mu(t) \frac{n(a+1)}{2} dt + Z_\mu(t) d\tilde{W}(t). \tag{4.101}$$

By this it clearly follows, that the constant growth rate of the market capitalization is at the same time its long-term limit. It can further be shown, that the long term growth of the market and the individual stocks $X_i(\cdot)$, $i = 1, \dots, n$ are actually the same.[67]

$$\lim_{T\to\infty} \frac{1}{T} \log Z_\mu(T) = \lim_{T\to\infty} \frac{1}{T} \log X_i(T) = \frac{n(1+a)-1}{2}. \tag{4.102}$$

The fact that asymptotically, all stocks have the same growth rate, signifies that a market governed by VSM dynamics is coherent.[68] Even though the VSM fails to be diverse as stated before, the fact that the excess growth rate of the market portfolio is constant in time permits the application of the sufficient conditions given in Proposition 1.8 for $\Gamma(t) = t\frac{n-1}{2}$. With this we obtain the necessary investment horizon to realize arbitrage opportunities by $T > T^* = \Gamma^{-1}(S(\mu(0)))$.[69] Hence:

$$\frac{2S(\mu(0))}{n-1} \leq \frac{2 \log n}{n-1} = T^*. \tag{4.103}$$

Equation 4.103 leads to fairly small bounds on the requested investment horizon for a realistic number of assets in the market, which means that in VSM arbitrage opportunities may be realized on realistically small time horizons. This result however comes at the cost of imposing a more complex structure on the dynamics of the equity market.

However with respect to the dynamics and characteristics of the market portfolio as described in Equations 4.99, 4.100 and 4.101 it has to be remarked that the large positive drift produced by the model will dominate the evolution of market capitalization an will

[65] See e.g. Karatzas and Shreve [66], Theorem 3.3.16.
[66] See also Fernholz and Karatzas [46], Section 4. and Section 6.
[67] See Fernholz and Karatzas [46], Equation 6.9. and Proposition 6.1. This result is based on the insight, that the stock price process in the VSM may be re-written as a time-changed Bessel process.
[68] See Fernholz [44], Proposition 2.1.2.
[69] See Fernholz and Karatzas [49], Equation 12.6.

lead to an overly optimistic growth of total capitalization which may be problematic for risk management applications.

Comparing VSM to the dynamics of the Squared Brownian Motion and its respective log-returns as given in Equations 4.53 and 4.54 one directly observes the similarity of the terms $X(t)$ and $\sqrt{X_i(t)}$ in the denominator for the drift and volatility terms of the log-returns. In the case of VSM, the large impact on the fluctuations of the stock in case of very small stock prices is amplified by the multiplication with the total market capitalization, which leads to the effect, that stocks in a VSM fluctuate *"all over the place"*[70]. Still, the feature that volatility substantially increases when stocks decline to small values close to zero is a feature of both models. As a matter of fact, this property may also be observed in real-life markets, where the volatility structure of so-called "Penny-Stocks" may diverge substantially from normal stocks.

With respect to the effect of incorporating correlations versus uncorrelated models we also refer to Section 5.3.3, where we have visualized the behavior of the uncorrelated SqBM which possesses certain similarities to the VSM together with the same dynamics for the correlated model. Even though the positive drift in the SqBM model is less pronounced than in the VSM, one still observes an overly optimistic monotonous growth behavior of the market capitalization, albeit not as pronounced as for the VSM. Therefore it is fair to conclude that the inclusion of correlation structures is essential for risk management purposes.

### 4.1.3 Conclusion

The above considerations lead to the conclusion that a stock market where individual particles follow independent squared Brownian Motions will eventually generate a Capital Distribution Curve very much akin to the structures observable from market data. Indicators for this assertion are the shape of Capital Distribution Curves resulting from simulating independent SqBM particles as depicted in Figure 4.2 as well as the time invariant approximation of the Capital Distribution Curve obtained through the quantile function of the $\chi^2(1)$ distribution as shown in Figure 4.1.

Modeling the dynamics of a stock price by means of a Squared Brownian Motion may at first sight seem unconventional however there exist similarities to the SABR[71] and the CEV[72] models. Moreover, the comparison to Volatility Stabilized Market models - which

[70] See Fernholz and Karatzas [46], Section 1, p. 150

[71] See Hagan et al. [58].

[72] See e.g. Carr and Linetsky [20].

have been studied for quite some time now - shows certain similarities in the way stock price dynamics are being constructed, however, the SqBM approach should lead to a smaller propensity for abnormally large fluctuations of single stocks relative to VSM models. We will show in the following Section 4.2 that through application of some minor modifications to the SqBM model one may obtain a thoroughly reasonable modeling framework for equity markets. At this point we also refer to the work of Teichmann and Wunsch [99] in which the same set of questions as above is addressed with alternative modeling approaches based on the particle approximation of the Wasserstein diffusion[73] and on the eigenvalue processes of general Wishart matrices.[74]

The biggest drawback of our current modeling framework is the absence of correlations between stocks, a problem which we will address in the following Section 4.2 as well. Altogether, the approach of a stock market model based on squared Brownian Motions seems to merit closer scrutiny and it appears promising with respect to the problem of replicating the observable structure of the Capital Distribution Curve in stock markets.

## 4.2 Construction of Correlated Stocks in the SqBM-Model

In this section we will take the stock market model based on squared Brownian Motion as basis and implement slight adaptations in order to ensure a better alignment with the dynamics observable in stock markets. Furthermore, we will equip our stock market with a time-varying covariance structure which is one of the essential prerequisites for a mathematical model to claim credibility in the market and which is of paramount importance for any risk management application.

Hand-in-hand with the general procedure outlined in this Section goes the practical implementation of this model. We have chosen the S&P 100 universe as basis for our model and we utilize time series going back for five years. In the case of stocks with shorter available price time series, the remainder of the time series was filled backward with the last available quote. When working on a large-cap blue chip index as the S&P 100, the issue of incomplete or insufficiently long time series may be less pressing, yet for smaller or more exotic markets this can be a major problem. In such cases it is recommendable to apply a $\beta$-mapping [75] onto a benchmark index or another suitable time series.

[73]See e.g. Andres and von Renesse [6].

[74]See e.g. Teichmann and Wunsch [99], Section 4.

[75]See e.g. Jorion [63]. In order to perform a $\beta$-mapping of time series $Y$ onto $Z$, a $\beta$ factor has to be

### 4.2.1 Algorithm for Constructing Correlated Stocks

As first step, let us recall the general dynamics for the process $X(\cdot) = (W(\cdot))^2$ as specified in Equation 4.53:

$$dX_i(t) = dt + 2\sqrt{X_i(t)}dW_i(t).$$

Now let us adjust the above SDE in order to permit for more modeling flexibility. We shall equip the volatility term in Equation 4.53 with an additional sensitivity $\xi_i(\cdot)$ and we will introduce an additional drift $b_i(t) = \alpha_i(t)\xi_i(t)$. Hence the new SDE under scrutiny is of the form:

$$dX_i(t) = b_i(t)dt + \xi_i(t)\sqrt{X_i(t)}dW_i(t). \tag{4.104}$$

Applying Itô's formula[76] to $f(x) = \log x$, one directly obtains the dynamics of the instantaneous log-returns as follows:

$$\begin{aligned} d\log X_i(t) &= \frac{1}{X_i(t)}dX_i(t) - \frac{1}{2}\frac{1}{X_i(t)^2}d\langle X_i(t), X_i(t)\rangle \\ &= \frac{1}{X_i(t)}\left(b_i(t)dt + \xi_i(t)\sqrt{X_i(t)}dW_i(t)\right) - \frac{1}{2X_i(t)^2}\xi_i(t)^2 X_i(t)dt \\ &= \frac{1}{X_i(t)}\left(b_i(t) - \frac{\xi_i(t)^2}{2}\right)dt + \frac{\xi_i(t)}{\sqrt{X_i(t)}}dW_i(t). \end{aligned} \tag{4.105}$$

Nota bene that we have stuck to the notation of Fernholz [44] with $\xi_i(\cdot)$ denoting the sensitivity w.r.t. the $i$-th Brownian Motion. At a later point, when we have equipped the whole model with a correlation structure and a resulting instantaneous covariance matrix we shall again use the notation $\sigma(t) = \xi(t)\xi(t)^T$ for the instantaneous covariance matrix as in Chapter 1. As long as only independent particles are involved, we shall denote by $\xi_i(t)^2$ the instantaneous variance.

For the ensuing considerations and the practical implementation we will discretize our market setup and move to an evenly spaced time grid with mesh $\tau$, whereby the continuous time interval $[0, T]$ is translated into the discrete time set $\mathfrak{T} = \{t_0, \dots, t_N\}$. The dynamics

calculated as $\beta_Y = \frac{Cov(Y,Z)}{Var(Z)}$, based on the available time series. The missing log-returns of $Y$ may now be substituted with the $\beta$-weighted log-returns of $Z$.

[76] See e.g. Karatzas and Shreve [66] or Øksendal [80].

for the instantaneous log-returns for any stock $i \in \mathfrak{M}$ as given in Equation 4.105 now translate into:

$$\log\left(\frac{X_i(t_{j+1})}{X_i(t_j)}\right)\sqrt{X_i(t_j)} = \frac{1}{\sqrt{X_i(t_j)}}\left(\alpha_i(t_j)\xi_i(t_j) - \frac{1}{2}\xi_i(t_j)^2\right)\Delta t_j + \xi_i(t_j)\Delta W_i(t_j). \tag{4.106}$$

Acting on a discrete time grid now makes our model amenable to calibration on real market data. As already mentioned, we will calibrate our market $\mathfrak{M}$ on the S&P 100 index universe. For fitting our model we utilize daily log-returns from the time period January 3rd, 2006 to April 14th, 2011, resulting in 1330 data points.[77] We shall denote the market data time series by $Y_i(\cdot)$ for $i = 1,\dots 100$. In the first step, we will implement a GARCH(1,1)[78] estimator for stochastic volatility. To this end we will assume that our time series mimic the dynamics given in Equation 4.106 with $\alpha_i(t_j) = \frac{1}{2}\xi_i(t_j)$, resulting in zero drift. We note that in order for the algorithm to converge, one needs $b_i(t_j) = \alpha_i(t_j)\xi_i(t_j) \geq \frac{1}{2}\xi_i(t_j)^2$. Then we can calculate the scaled log-returns

$$r_i(t_j) = \log\left(\frac{Y_i(t_{j+1})}{Y_i(t_j)}\right)\sqrt{Y_i(t_j)},\;\; i = 1,\dots,100,\;\; j = 1,\dots,N-1. \tag{4.107}$$

Based on $r_i(t_j)$ we can now calculate the GARCH(1,1) volatility $\hat{\xi}_i(t_j)$ as follows.

$$\hat{\xi}_i(t_j) = \sqrt{\theta_0 + \theta_1 r_i(t_{j-1})^2 + \beta\hat{\xi}_i(t_{j-1})^2}, \tag{4.108}$$

whereby we use the following standard parametrization for the GARCH(1,1) model:

| Parameter | Value |
|---|---|
| $h_i$ | unconditional variance of $Y_i$ |
| $\theta_1$ | 0.1 |
| $\beta$ | 0.85 |
| Persistence | $\theta_1 + \beta = 0.95$ |
| $\theta_0$ | $h_i(1-\theta_1-\beta)$ |

[77] The constituent time series of the S&P 100 index were obtained from Bloomberg® utilizing the index ticker OEX and the time series history wizard for each featured stock.

[78] See e.g. McNeal, Frey and Embrechts [78], Section 4.3.

According to our model in Equation 4.106, the scaled log-returns $r_i(t_j)$ follow a conditionally normal distribution $r_i(t_j) \sim N(0, \hat{\xi}_i(t_j)^2 \tau)$. Since we are working on a daily data base, we will utilize a daily time scale with daily time steps, resulting in a mesh of $\tau = 1$ day. Hence normalizing $r_i(t_j)$ with the respective stochastic volatility yields:

$$\tilde{r}_i(t_j) = \log\left(\frac{Y_i(t_{j+1})}{Y_i(t_j)}\right)\sqrt{Y_i(t_j)}\frac{1}{\hat{\xi}_i(t_j)}, \;\; i = 1, \ldots, 100, \;\; j = 1, \ldots, N-1. \tag{4.109}$$

By construction, the normalized log-returns $\tilde{r}_i(\cdot)$ are i.i.d standard normally distributed for $i = 1, \ldots, 100$. Hence each time series of log-returns is component-wise i.i.d., however, across stocks these time series are still correlated, a fact which we will exploit in the next step.

**Remark 4.1:** At this point a remark on the normality assumption of log-returns is due. Ever since the seminal works of Black and Scholes [17], this assumption has been at the center of scientific discussions. It is quite clear, that modeling log-returns via a normal distribution may lead to a serious underestimation of risk in the case that the true distribution of the returns is leptokurtic. The treatment of market data time series outlined in this section is similar to the filtered historical simulation proposed by Barone-Adesi et al. [15], the main difference being that we will utilize the results of (4.109) - as will be outlined in the following paragraphs - as input for a Monte-Carlo inspired simulation algorithm. Thus our algorithm for the construction and simulation of correlated particles may be seen as a hybrid approach, incorporating facets of filtered historical simulation and Monte Carlo simulation. The Gaussian modeling of log-returns may certainly be regarded as a simplification and it comes with well-known drawbacks (among others no heavy tails, no jumps), but still this class of models remains overwhelmingly popular in practice and also in research.

With respect to the effectiveness of the normalization procedure outlined above we observe that the results for mean and variance / standard deviation of the normalized time series are quite satisfactory as can be seen in Figure 4.3. One ought to observe that we have larger deviations from one for the standard deviations, however the fact that the standard deviations of these time series are smaller than one indicate that we are actually overestimating the volatility of the stock, which is the comforting case from the point of view of risk management. With respect to the tail behavior of normalized log-returns it is certainly observable that deviations to the normal distribution are prevalent in market data time series as can be seen from the sampled normal quantile-quantile (QQ) plots in Figure 4.4 which can be regarded as a usual problem when working with a Gaussian model.

Finally we also provide exemplary sample plots of time series of normalized log-returns in

Figure 4.5 which do not exhibit any striking autocorrelation in the data. Therefore our claim that the time series of normalized log-returns are component-wise i.i.d. standard normally distributed as insinuated by our model in Equation 4.109 certainly appears defendable with the most important shortcoming certainly being the imperfect coverage of tail events which constitues a challenge for future research.

One of the major advantages of models such as the one given in Equation 4.105 is its relatively easy implementation and calibration which lead to considerable stability in everyday application. On the other hand, the increased flexibility of more delicate models involving Lévy processes[79] often comes at the cost of more challenging calibration procedures and less numerical stability. An enhancement of the given model (4.104 and 4.105) towards Lévy processes certainly seems a promising area for future research, yet it is well beyond the scope of this work. For the time being, we shall therefore continue our work with the normality assumption, always bearing in mind its evident shortcomings in case of tail events.

We remark that a commonly used approach in order to improve the tail behavior of models - especially in the area of risk management - would be to switch from Gaussian increments to $t$-distributed increments[80] whereby heavier tails are obtained in simulation algorithms. At this stage, we only point out at this possible extension of our proposed model, a detailed discussion of the market characteristics based on $t$-distributed increments is beyond the scope of this work.

In the next step we can calculate the correlation matrix $\tilde{\rho}(\mathfrak{M}, N)$, depending on our input market data with $\|\mathfrak{M}\| = n$ stocks and the available time series. In the straightforward implementation, one obtains $\tilde{\rho}$ by applying the correlation estimator of the utilized software - `R` in our case - on the time series of normalized log-returns, thus obtaining the matrix of correlation coefficients[81]:

$$\tilde{\rho}(\mathfrak{M}, N) = \begin{pmatrix} 1 & \tilde{\rho}_{1,2} & \cdots & \tilde{\rho}_{1,n} \\ \tilde{\rho}_{2,1} & \ddots & & \tilde{\rho}_{2,n} \\ \vdots & & \ddots & \vdots \\ \tilde{\rho}_{n,1} & \cdots & \tilde{\rho}_{n,n-1} & 1 \end{pmatrix}. \tag{4.110}$$

[79] See e.g. Applebaum [9] or Schoutens [91].
[80] See e.g. McNeil et al. [78], Example 2.14.
[81] See e.g. Casella and Berger [19].

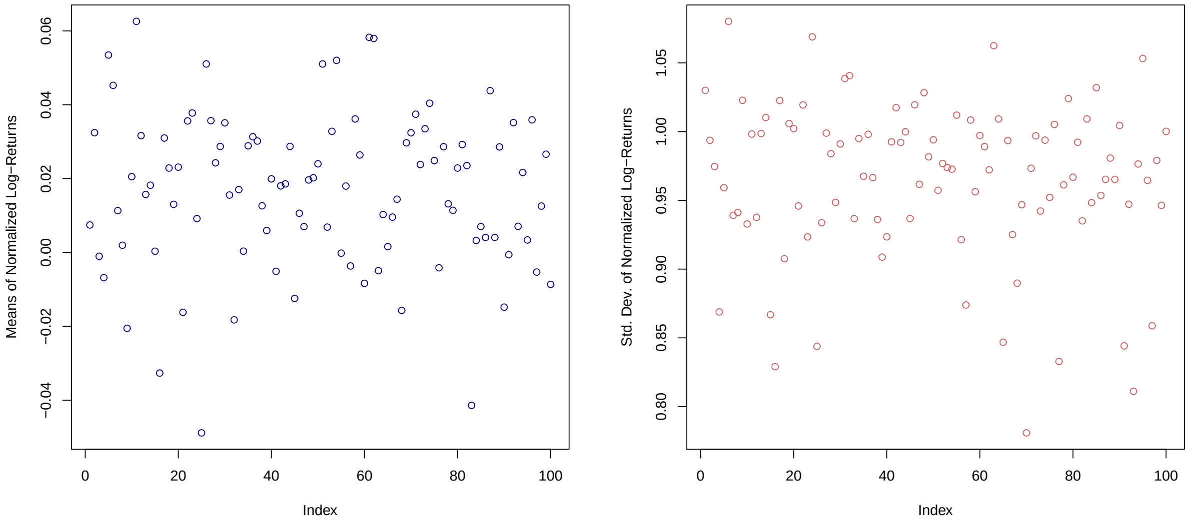


Figure 4.3: Means and standard deviations of normalized time series.

Utilizing the above matrix $\tilde{\rho}(\mathfrak{M}, N)$ one may now construct a stochastic covariance matrix for every time step of our simulation. In order to calculate the first simulated return we first compute

$$\Xi(t_N) = \hat{\xi}(t_N)^T \hat{\xi}(t_N) = \left(\hat{\xi}_l(t_N)\hat{\xi}_k(t_N)\right)_{l,k=1,\ldots,n}. \tag{4.111}$$

It is worth noting, that the vector $\hat{\xi}(t_N)$ contains the GARCH(1,1) volatilities for all stocks calculated according to Equation 4.108 based on the log-return between $t_{N-1}$ and $t_N$ and the GARCH(1,1) volatility $\hat{\xi}(t_{N-1})$. Denoting by $*$ the component-wise product of two matrices we can calculate the estimator for the instantaneous covariance matrix as:

$$\tilde{\sigma}(t_N) = \Xi(t_N) * \tilde{\rho}(\mathfrak{M}, N). \tag{4.112}$$

The instantaneous covariance matrix $\tilde{\sigma}(t_N)$ shall now be utilized to calculate our first simulated return. Therefore, we compute the Cholesky[82] decomposition of $\tilde{\sigma}(t_N)$:

[82]See e.g. Platen and Heath [87], Section 1.4.

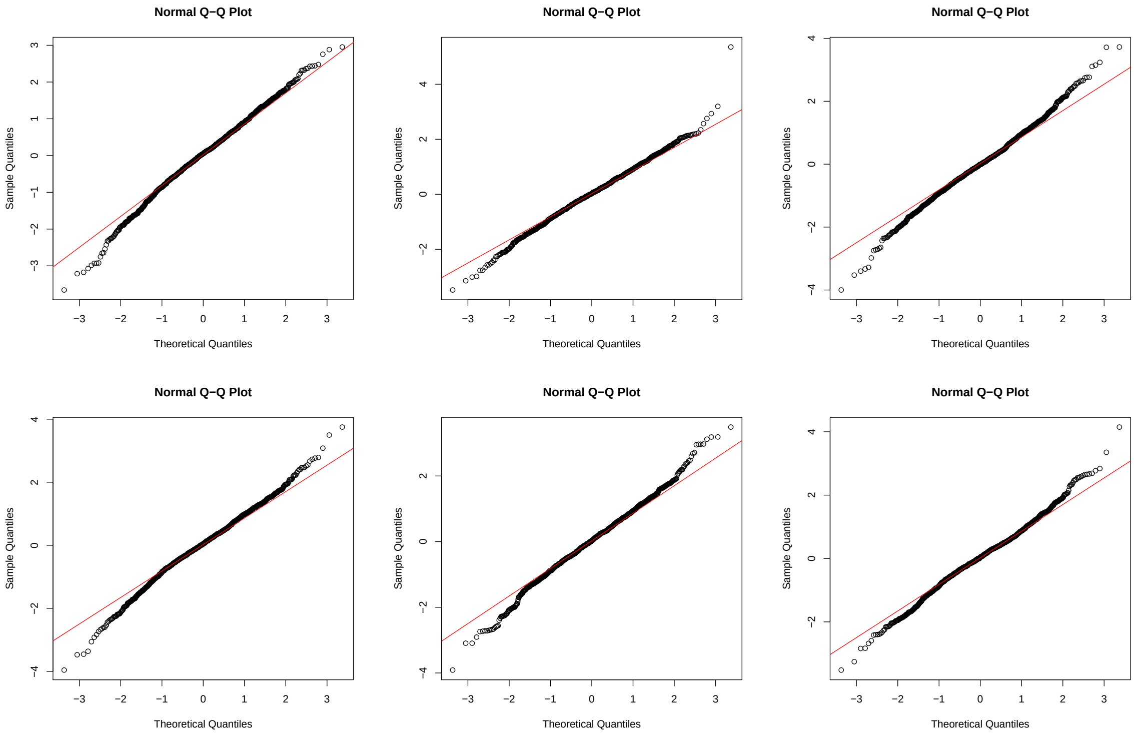


Figure 4.4: QQ-plots of exemplary normalized log-return time series.

$$\tilde{\sigma}(t_N) = \tilde{\xi}(t_N)^T \tilde{\xi}(t_N). \tag{4.113}$$

**Remark 4.2:** At this point it is worth noting that the outlined approach avoids any explicit ellipticity assumptions as discussed in Chapter 2. In our model, we do not postulate any specific structure of the correlation matrix $\tilde{\rho}(\mathfrak{M}, N)$ or of the resulting instantaneous covariance matrix $\tilde{\sigma}(t_N)$ but we merely extract the utilized market structure out of the available data. Thus, our model and the quality of its results do not hinge on the concept of non-degeneracy (ellipticity) of $\mathfrak{M}$.

We can now revisit the assumed stock price dynamics of Equation 4.105 and endow this structure with the instantaneous covariance structure we have just constructed. To this end, we reformulate Equation 4.105 as:

$$d \log X_i(t) = \frac{1}{X_i(t)} \left( b_i(t) - \frac{\tilde{\xi}_{ii}(t)^2}{2} \right) dt + \frac{1}{\sqrt{X_i(t)}} \sum_{\nu=1}^{n} \tilde{\xi}_{i\nu}(t) dW_\nu(t), \tag{4.114}$$

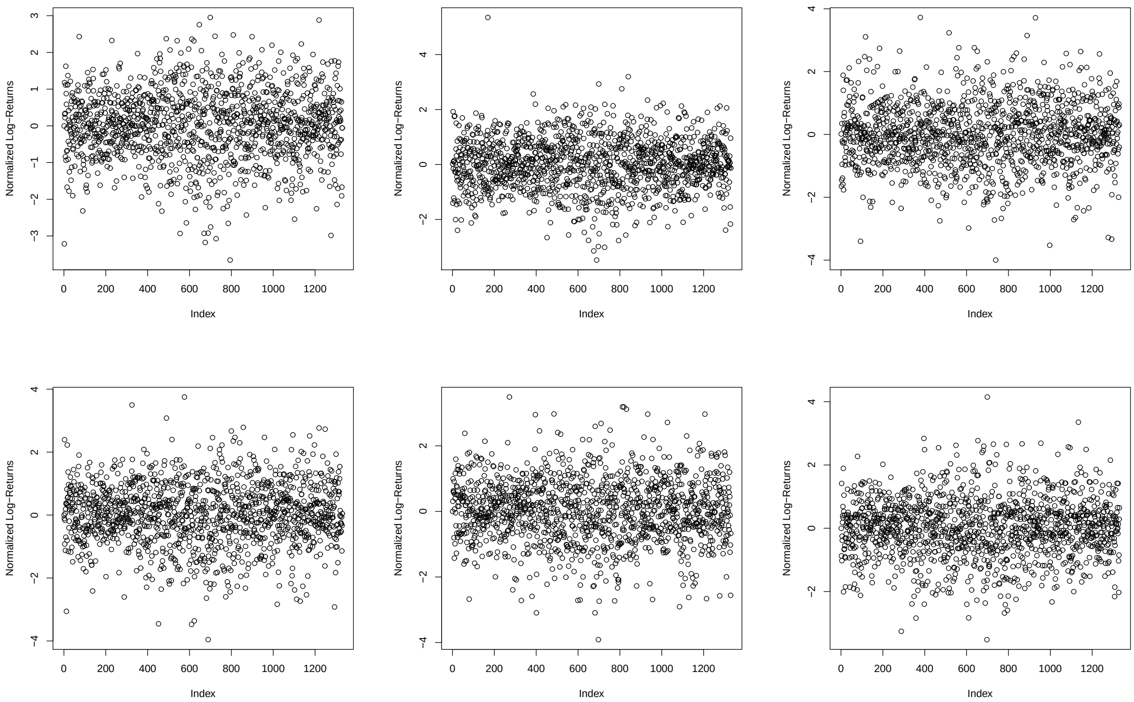


Figure 4.5: Sample plots of exemplary normalized log-return time series.

which gives us the continuous setup of our final model of correlated particles. Hereby $W(\cdot)$ denotes an $n$-dimensional Brownian Motion and $\tilde{\xi}_{ii}(\cdot)^2$ denotes the instantaneous variance of the $i$-th stock. The dynamics of $X_i(\cdot)$ may be derived directly from (4.114) by application of Itô's formula to $f(x) = \exp(x)$.

$$\begin{aligned} dX_i(t) &= X_i(t) d\log X_i(t) + \frac{1}{2}\tilde{\xi}_{ii}(t)^2 dt \\ &= \left(b_i(t) - \frac{\tilde{\xi}_{ii}(t)^2}{2}\right) dt + \sqrt{X_i(t)} \sum_{\nu=1}^{n} \tilde{\xi}_{i\nu}(t) dW_\nu(t) + \frac{1}{2}\tilde{\xi}_{ii}(t)^2 dt \\ &= b_i(t)dt + \sqrt{X_i(t)} \sum_{\nu=1}^{n} \tilde{\xi}_{i\nu}(t) dW_\nu(t). \end{aligned} \tag{4.115}$$

This continuous setup may be discretized in a straightforward way so as to obtain the first simulated return. We recall that we have extracted the correlation structure of log-returns from our historical time series in the form of the correlation matrix $\tilde{\rho}(\mathfrak{M}, N)$ and the vector of GARCH(1,1) volatilities $\hat{\xi}(t_N)$ which we have utilized to construct the instantaneous covariance matrix $\tilde{\sigma}(t_N)$. We shall denote the simulated stock prices by $\tilde{X}_i(t_j)$ for $j \in \mathbb{N}$

and $j \geq N+1$. For the first simulation, we are utilizing the last available stock price $X_i(t_N)$ for every stock $i = 1, \ldots, n$, which - without loss of generality - we re-scale to 1 for each stock. Thereby one obtains the following discrete calculation scheme for the log-return $\bar{r}(t_N, t_{N+1})$ which constitutes the logarithmic return of the $i$-th stock realized between time $t_N$ - i.e. the final point of our time series - and time $t_{N+1}$ which is the first simulated time step.

$$\begin{aligned} \bar{r}_i(t_N, t_{N+1}) \quad = \quad & \frac{1}{X_i(t_N)} \left( \alpha_i(t_N) \tilde{\xi}_{ii}(t_N) - \frac{1}{2} \tilde{\xi}_{ii}(t_N)^2 \right) \Delta t_N + \\ & \frac{1}{\sqrt{X_i(t_N)}} \sum_{\nu=1}^{n} \tilde{\xi}_{i\nu}(t_N) \Delta W_\nu(t_N). \end{aligned} \tag{4.116}$$

We will pursue the simulation approach of computing a long-term projection of 5000 time steps. Assuming 250 business days per year this corresponds to a 20 year horizon. It is worth noting that for classical risk management applications one would often compute several runs of short period returns rather than long term projections. Since we are investigating the capital distribution of the market at this point, we will stick to the long-term projections. The classical applications in risk management will be discussed in Chapter 5.

With this set of simulated log-returns $\bar{r}_i(t_N, t_{N+1})$ for stocks $i = 1, \ldots, n$ we can now calculate the next GARCH(1,1) volatility $\tilde{\xi}_i(t_{N+1})$ for each stock as:

$$\tilde{\xi}_i(t_{N+1}) = \sqrt{\theta_0 + \theta_1 \bar{r}_i(t_N, t_{N+1})^2 + \beta \tilde{\xi}_i(t_N)^2}. \tag{4.117}$$

Furthermore, we obtain the simulated stock price for $t_{N+1}$:

$$\tilde{X}_i(t_{N+1}) = X_i(t_N) \exp\left(\bar{r}_i(t_N, t_{N+1})\right). \tag{4.118}$$

This permits us to return to the step described in Equation 4.111 and perform the computation described in the above steps for the next time step from $t_{N+1}$ to $t_{N+2}$ and so on.

The simulation algorithm which we have just outlined in the steps between Equations 4.111 and 4.118 will be the backbone of all our further calculations. The outlined constructive approach provides us with a model which incorporates the correlation information which may be extracted from market data and which at the same time yields a reasonably shaped

Capital Distribution Curve. In the following charts, we will visualize our correlated SqBM-model.

### 4.2.2 Visualization of Results

In the first step we shall depict the heat-map of the correlation matrix $\tilde{\rho}(\mathfrak{M}, N)$ which has been utilized for the ensuing simulations. It should be borne in mind that the data base for our calculations is the S&P 100 index which should provide us per definition with a quite well-diversified investment universe.

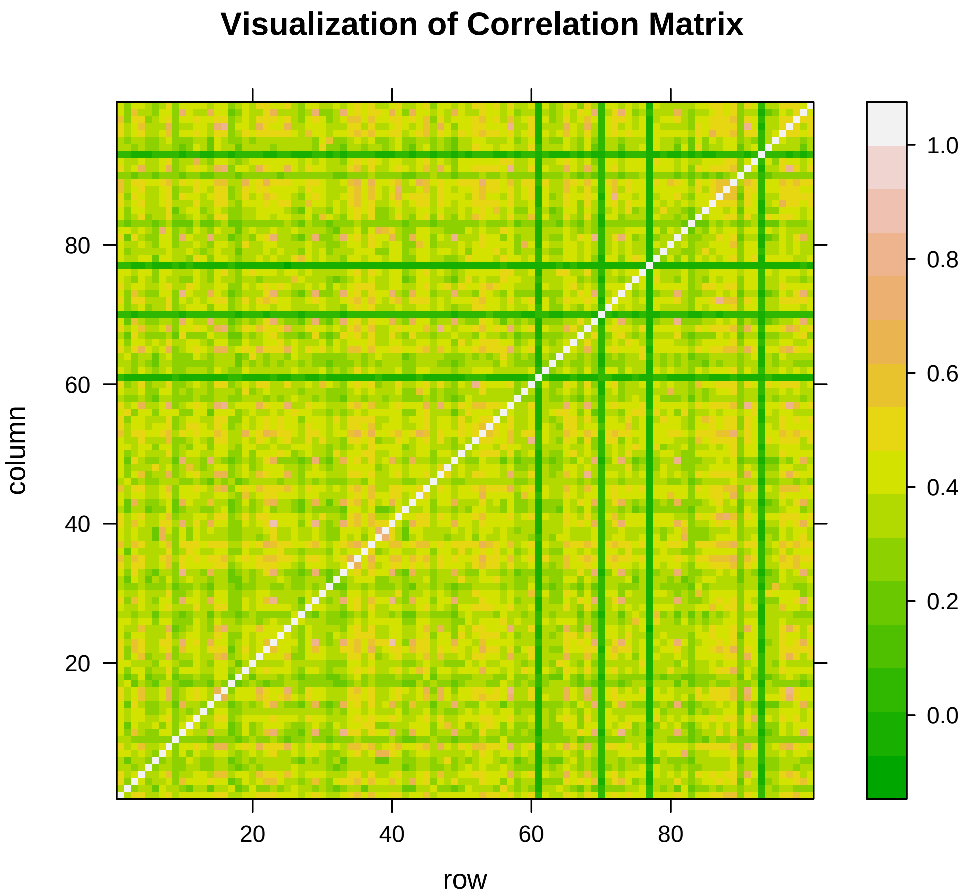


Figure 4.6: Heat-map of the correlation matrix $\tilde{\rho}(\mathfrak{M}, N)$.

It can be observed in Figure 4.6 that the majority of inter-particle correlations are hovering between zero and 0.4. Low negative correlation between particles is observable in some cases as well as a few larger positive correlations in the area of $\geq 0.6$. It is worth noting that different index structures can lead to a substantially different correlation pattern. In indices with larger constituent bases, collinearities between similar stocks are usually more pronounced than in condensed benchmark indices like the S&P 100. Specialized theme or industry indices are - by definition - depicting a special universe consisting of strongly dependant particles.

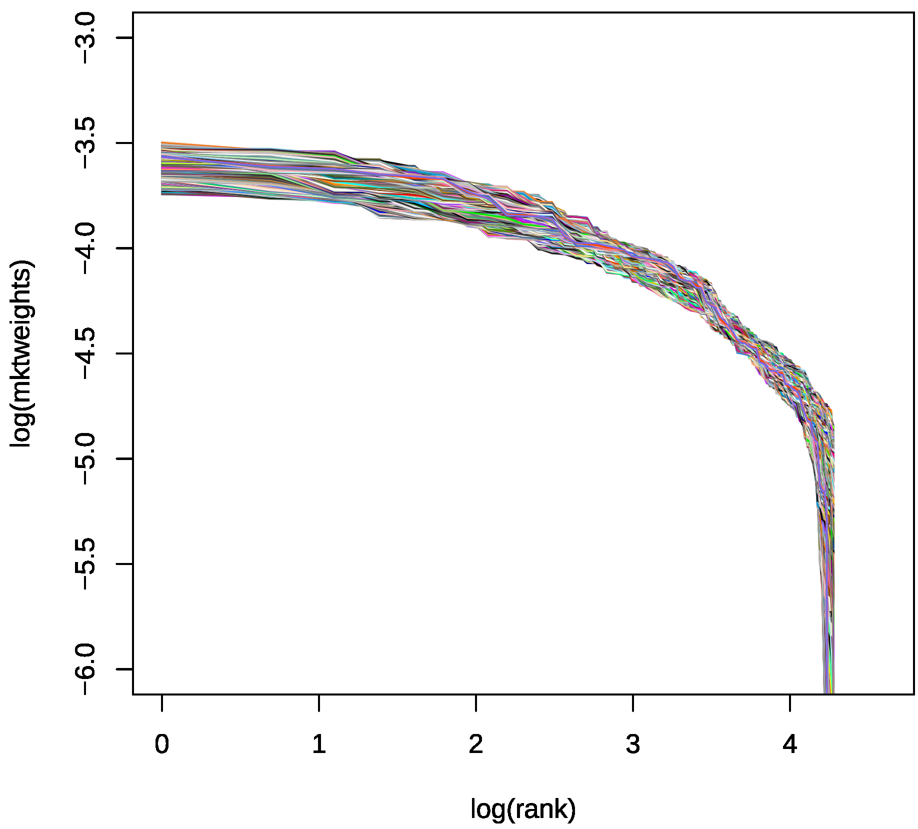


Figure 4.7: Capital Distribution Curves for the correlated SqBM model for 2000 time steps.

In Figure 4.7 we have visualized 2000 CDCs which were obtained from the simulated market structures in time steps 3000 to 5000. The pattern which we have reproduced through the model outlined in Section 4.2.1 does a reasonable job in reproducing the structure observable in market data as depicted in Figure 3.1. Furthermore, the fact that we have started at simulation time 1 with a market with evenly spread market weights permits us to visualize the convergence of the CDC from the artificially created straight line to the characteristically sloped structure which has already been presented various times (see Figure 4.8).

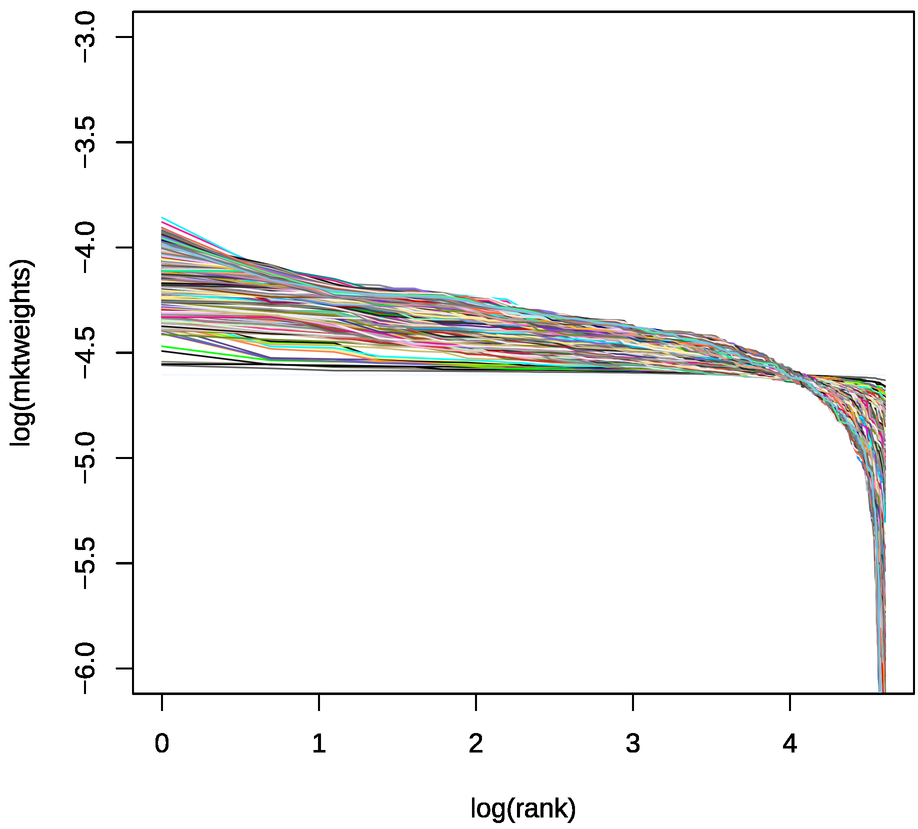


Figure 4.8: Capital Distribution Curves for the correlated SqBM model starting with time 1 (straight line) up to time step 300.

Furthermore, we can also visualize the market entropy in our correlated SqBM model (Figure 4.9), utilizing the results from Section 1.2.2. In this plot we visualize the simulated market entropy for our correlated model (in blue) together with the entropy in the case of uncorrelated particles (in red) and in the case of randomly correlated particles (in green). It is worth noting that in our simulation example we have a fairly high value of normalized market entropy in the area of 0.92 which is due to the fact that we started our simulation with a perfectly diverse market.

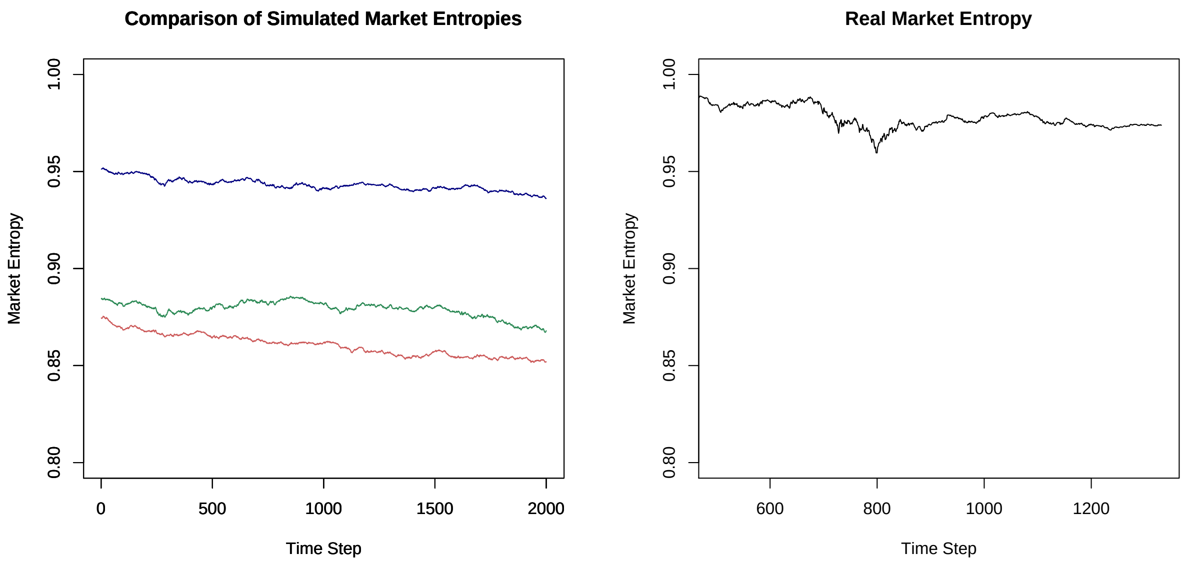


Figure 4.9: Left chart: simulated market entropy for the correlated model (blue), the uncorrelated model (red) and randomly correlated particles (green); right chart: market entropy calculated from realized fluctuations of market data time series.

Altogether the market entropy which is obtained from our simulation exhibits a reasonable degree of stability and the mean-reverting behavior which may also be observed in market data.[83] If we further compare our simulated entropy with the values observable from our market data as depicted in Figure 4.9 in the chart to the right, it may be observed

[83] See Fernholz [44], Section 5.5, Figure 5.7.

that the results of our correlated model as outlined in Section 4.2.1 fit the observable market structure reasonably well. Furthermore it is apparent from the charts in Figure 4.9, that the inclusion of inter-stock correlations improves the dynamics and the overall level of market entropy when compared to market data. Our model also attains the goal of reproducing the volatility pattern of market entropy as observable from data. In order to render the market data based entropy calculation and the simulation results comparable we have also re-scaled the initial "market prices" of all stocks to one, the different lengths of the simulated charts and the data based charts in Figure 4.9 stem from the length of our market data time series.

Finally in Figure 4.10 we visualize the Capital Distribution Curves for our correlated model (in blue) and for uncorrelated particles (in red). As compared to some CDCs calculated directly from our market data it is apparent that endowing our baseline SqBM model with the above described correlation structure substantially improves the replication of observable CDC patterns. The correlated model also leads to a slower decline of market capitalizations relative to the uncorrelated version where the structural break between larger stocks and small penny stocks is more pronounced. We remark that both the correlated and uncorrelated model contain the same number of particles. In the left chart in Figure 4.10 the tail of the uncorrelated CDCs is cut off by the plot range.

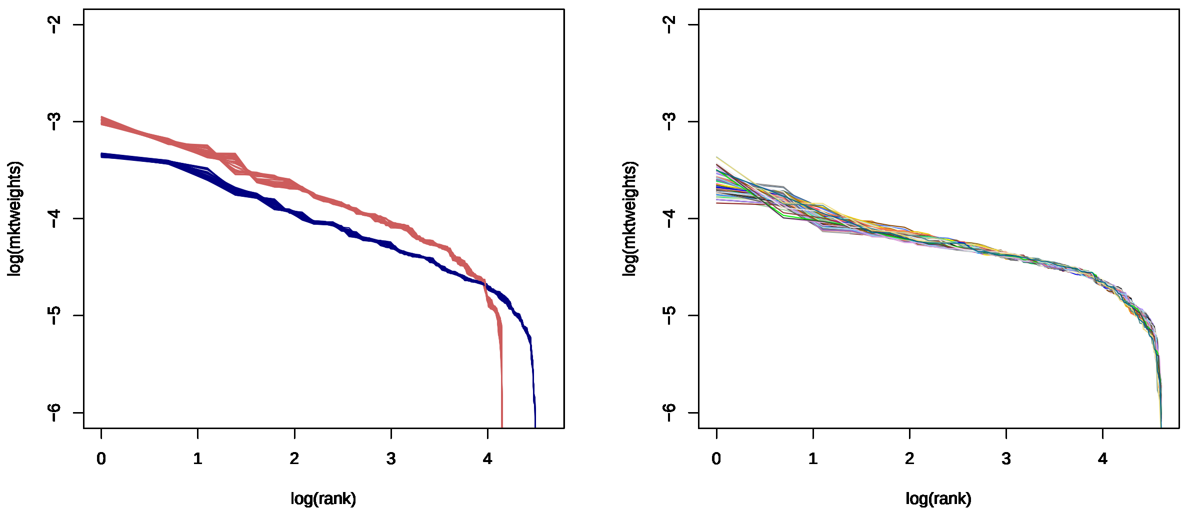


Figure 4.10: CDC for simulated markets with and without correlation (blue resp. red charts on the left) and for real market data (right).

Based on these results we can justifiably claim that our correlated SqBM model reasonably captures both the diversity and the capital structure of the equity market if we compare it to our market data. The endowment of our model with the real life correlation structure which is utilized as base for the stochastic evolution of instantaneous covariance matrices clearly leads to an improved imaging of real market conditions. The algorithm outlined in Section 4.2.1 can therefore be seen as a substantial step forward towards the end of establishing a model which solidly replicates market dynamics.

### 4.2.3 Alternative Approaches for Incorporating the Correlation Structure

At this point it should be remarked, that the approach of endowing our market model with a correlation structure which was presented in Equations 4.110 to 4.112 is by no means the only approach one may think of.

One alternative approach for deriving an estimator for the instantaneous covariance matrix would be to apply the Fourier Estimator we have introduced in Section 2.1.1.[84] In this vein, one could consider applying the Fourier Estimator given in Equation 2.9 to the normalized time series of log-returns $\tilde{r}(\cdot)$. This would yield an alternative estimator $\tilde{\sigma}_F(\cdot)$ which would then be a function of time and could be evaluated at every step of the simulation. Furthermore, every simulated log-return could be used to update the time series for the Fourier estimator and thus to provide an updated $\tilde{\sigma}_F(\cdot)$ at every step of the simulation.

Another alternative approach would be to utilize the results of Ahdida and Alfonsi [4] on stochastic differential equations on correlation matrices. In this work the authors propose a matrix valued mean-reverting SDE model for the evolution of stochastic correlation matrices $C(\cdot)$. Such a framework of stochastic correlation matrices may be incorporated in our model and calibrated to our normalized time series of log-returns $\tilde{r}(\cdot)$.

Both alternative approaches possess substantial mathematical beauty, yet they have one central drawback in common: they would dramatically increase model complexity and would pose a bigger challenge to numerical stability. The approach we have outlined in Section 4.2.1 possesses the great advantage of relative ease in implementation and good performance in practical calculations. Therefore we will not further elaborate these alternative approaches at this point and leave these issues open for future research.

[84] See Malliavin and Mancino [75].

## 4.3 Further Ramifications of the SqBM-Model

### 4.3.1 Particles as Sums of Two Squared Brownian Motions

A natural extension of the model for independent stocks as outlined in Section 4.1.2 would be to model individual stocks as sums of two independent Squared Brownian Motions. This approach is of course arbitrarily extendable. In this case any stock $X_i(\cdot)$ for $i \in \{1, \ldots, n\}$ may be written as

$$X_i(t) = \operatorname{tr}\left[\begin{pmatrix} W_{i_1}(t) & 0 \\ 0 & W_{i_2}(t) \end{pmatrix}^2\right], \tag{4.119}$$

whereby the $W_{i_l}(\cdot)$, $i = 1, \ldots, n$ and $l = 1, 2$ are independent standard Brownian Motions. Therefore we have the following dynamics for $X_i(\cdot)$, $i = 1, \ldots, n$

$$dX_i(t) = 2dt + 2W_{i_1}(t)dW_{i_1}(t) + 2W_{i_2}(t)dW_{i_2}(t). \tag{4.120}$$

With respect to the distributional properties of the resulting process, we have only minor changes compared to the results described in Section 4.1.2, since for $i = 1, \ldots, n$

$$\frac{X_i(t)}{t} = \left(\frac{W_{i_1}(t)}{\sqrt{t}}\right) + \left(\frac{W_{i_2}(t)}{\sqrt{t}}\right) \sim \chi^2(2). \tag{4.121}$$

Hence in this case, the scaled process is the sum of two squared standard normally distributed random variables and therefore $\chi^2(2)$ distributed.[85] Furthermore, if one has $\zeta \sim \chi^2(2)$, then it holds for $\Xi = t\zeta$ that $\Xi \sim \Gamma[1, 2t]$. Thus in analogy to the estimated capital curves in Figure 4.1 we can display the same type of charts for a market where each stock is constructed as sum of two Squared Brownian Motions.

Furthermore, akin to the capital distribution curves depicted in Figure 4.2, one may also simulate the capital distribution structure for this extended model, the result for 300 simulation runs may be found in Figure 4.12. The results are very similar, the important aspect being that overall structure of the CDC remains as desired in the extended SqBM model as well. The only difference which may be observed based on our simulations is the somehow dampened decline into nothingness happening at the right side of the chart. Thus if we model individual stocks as sums of two Squared Brownian Motions, then the market weights of the smallest stocks decline slower than in the standard case.

[85] See e.g. Abramowitz and Stegun [2], Section 26.4.

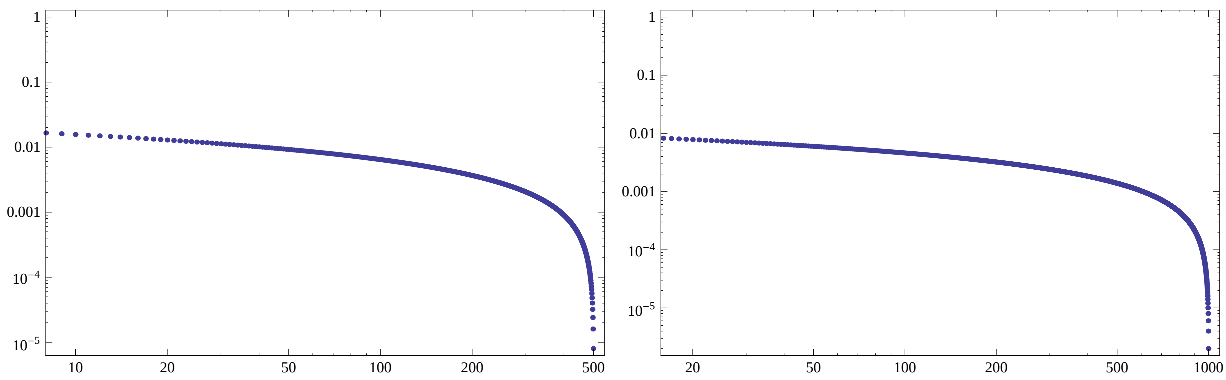


Figure 4.11: Log-log plots for estimated market weights based on sums of two Squared Brownian Motions for 500 (left) and 1000 (right) particles.

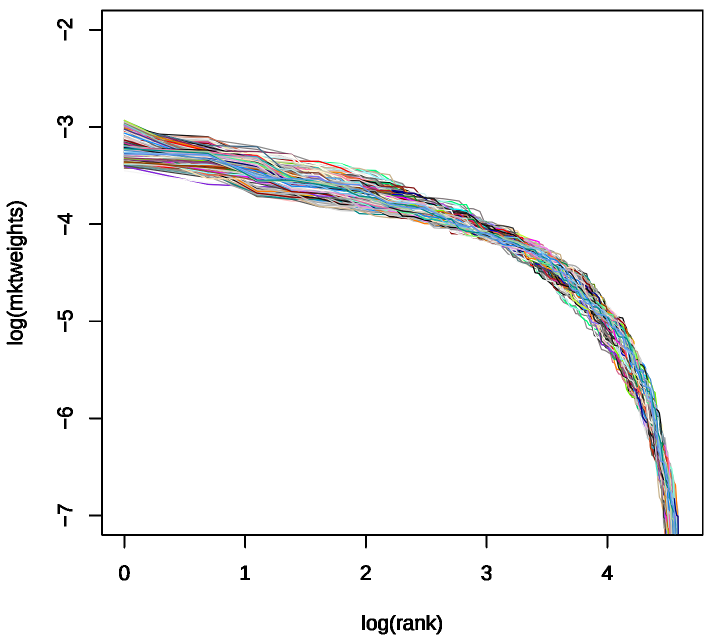


Figure 4.12: Capital Distribution Curves for 300 simulated time-steps in a market consisting of 100 particles as sums of two SqBMs.

### 4.3.2 Generalization to Wishart Processes

In this section we will briefly outline the next obvious extension of the SqBM model described in Section 4.1.2, namely the generalization to Wishart processes. We will briefly outline the dynamics of Wishart processes and we will provide the results of the simulation of a Wishart model calibrated to the S&P 100 market data which we have also used for the other simulations in this chapter. The family of Wishart processes and more generally that of affine processes on positive semi-definite matrices provide us with a very flexible modeling framework, however a detailed discussion of this class of processes is well beyond the scope of this work. A comprehensive introduction to the field of Wishart processes may be found in Bru [18], the general topic of affine processes on positive semi-definite matrices is extensively discussed in Cuchiero et el. [25].

Let us consider the Wishart process $\{X(t)\}_{t\geq 0}$ living on the positive semi-definite $d \times d$ matrices $\mathcal{S}_d^+$ whose dynamics are given by[86]

$$\begin{aligned} dX(t) \;&=\; \left(b + HX(t) + X(t)H^T\right) dt \\ &\quad+ \sqrt{X(t)}dW(t)\Sigma \\ &\quad+ \Sigma^T dW(t)\sqrt{X(t)}. \end{aligned} \tag{4.122}$$

We will model our stock market $\mathfrak{M}$ as the diagonal elements of the matrix-valued process $X(\cdot)$. At this point it is furthermore worth mentioning, that the definition given in Equation 4.122 may be further generalized by incorporating a jump component as it is done in Cuchiero et al.[87] The individual components of the dynamics for $X(\cdot)$ may be characterized as follows:

- $b$ is a $d \times d$ matrix for which it has to hold that $b - (d-1)\Sigma^T\Sigma \in \mathcal{S}_d^+$.[88] Due to this and the discussion in Bru[89] we set $b = \delta\Sigma^T\Sigma,\ \delta = d + 1$.
- We choose $\Sigma$ such that $\Sigma^T\Sigma = A$ whereby $A$ is a $d \times d$ covariance matrix. If the eigendecomposition of $A$ is given by $A = O\Lambda O^T$ where $\Lambda$ is the diagonal matrix comprising the eigenvalues of $A$, then we define $\Sigma$ as the square root of $A$, i.e. $\Sigma = O\sqrt{\Lambda}O^T$.
- $H \in \mathbb{R}^{d\times d}$, and furthermore $-H \in \mathcal{S}_d^+$.[90] Therefore we shall choose $H$ to be a

[86] See Bru [18], Section 5.2.
[87] See Cuchiero et al. [25], Equation (1.2).
[88] See Cuchiero et al. Section 1.
[89] See Bru [18], Section 5.2.
[90] See Cuchiero et al, Section 1.

negative definite matrix which also serves the goal that the drift term in (4.122) does not become too large.

- $W(\cdot)$ denotes a $d \times d$ matrix of Brownian Motions.

We will accordingly call $X(\cdot)$ a $\text{Wis}_d(x, \delta, H, \Sigma; t)$ process. Its characteristic function is given as follows. Let $v = iv_I,\ v_I \in \mathbb{R}^{d \times d}$, then the characteristic function of a $\text{Wis}_d(x, \delta, H, \Sigma; t)$ process is given by[91]

$$E_x\left[\exp\left(\text{tr}\left(vX(t)\right)\right)\right] = \frac{\exp\left(\text{tr}\left[v(I_d - 2q_t v)^{-1} m_t x m_t^T\right]\right)}{\det(I_d - 2q_t v)^{\delta/2}}, \tag{4.123}$$

where $I_d$ denotes the $d$-dimensional identity matrix and $q_t$ and $m_t$ are given by

$$q_t = \int_0^t \exp(sH)\Sigma^T \Sigma \exp(sH^T) ds, \tag{4.124}$$

$$m_t = \exp(tH). \tag{4.125}$$

Hence it obviously holds that the characteristic function of a $\text{Wis}_d(x, \delta, H, \Sigma; t)$ process is of exponentially-affine form with

$$\phi(v, t) = -\frac{\delta}{2} \log(\det(I_d - 2q_t v)), \text{ and} \tag{4.126}$$

$$\psi(v, t) = \exp\left(\text{tr}\left[v(I_d - 2q_t v)^{-1} m_t x m_t^T\right]\right). \tag{4.127}$$

If $X(\cdot)$ is of the form as given in Equation 4.122, then it holds that $X(t) = Y(t)^T Y(t)$ where $Y(\cdot)$ follows the dynamics[92]

$$dY(t) = dB(t)\Sigma + Y(t)H dt, \tag{4.128}$$

whereby $Y(\cdot)$ is a $(d+1) \times d$ matrix and $B(\cdot)$ is a $(d+1) \times d$ Brownian matrix. For the initial value it holds that $X(0) = x = y^T y$. Based on the dynamics of $Y(\cdot)$ we will implement a simulation scheme for $X(\cdot)$. The implemented steps of the simulation scheme are outlined as follows:

**1. Ad-Hoc-Estimation of $\Sigma$**

We will resort to some of the results which we have retrieved for the simulation of correlated particles in the SqBM model in Section 4.2. Our aim is to construct a covariance matrix $A$ which is calibrated to market data. Since we are looking at a time homogeneous model in (4.122) we have to implement a slight adaptation to the construction of

[91] See Ahdida and Alfonsi [3], Proposition 5.

[92] See Bru [18], (5.6) and (5.7)

the instantaneous covariance matrices in Section 4.2. As basis we will utilize the estimated correlation matrix $\tilde{\rho}(\mathfrak{M}, N)$ which has been defined in (4.110). Based on the GARCH(1,1) volatilities $\hat{\xi}_i(t_j),\ i = 1, \ldots, n;\ j = 1, \ldots, N$ we will calculate the vector comprising the mean volatilities for each stock $\bar{\xi}_i,\ i = 1, \ldots, n$. We recall that we named the number of stocks in the market $\|\mathfrak{M}\| = n$ in Section 4.2. In order to be in line with the dimensionality parameter $d$ in this section, we will set $d = n$. Akin to the approach outlined in Equation 4.111 we will set

$$\bar{\Xi} = \bar{\xi}^T \bar{\xi} = \left(\bar{\xi}_i \bar{\xi}_j\right)_{i,j=1,\ldots,d}.$$

Then, again denoting by $*$ component-wise multiplication, we obtain $A$ as

$$A = \bar{\Xi} * \tilde{\rho}(\mathfrak{M}, N). \tag{4.129}$$

Denoting again by $A = O \Lambda O^T$ the eigendecomposition of $A$ we obtain the $d \times d$ matrix $\Sigma$ as:

$$\Sigma = O \sqrt{\Lambda} O^T. \tag{4.130}$$

This choice for $\Sigma$ as square root of $A$ ensures that we have $\Sigma^T \Sigma = A$ and thereby the observed covariance structure is duly accounted for and we remark that this construction is in line with the structure of the admissible parameter set admitting the representation (4.122).[93]

**2. Ad-Hoc-Estimation of $H$**

As outlined above, we want the $d \times d$ matrix $H$ to be negative definite. In order not to complicate things, we will set

$$H = \kappa \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix},\ \kappa \in \mathbb{R}^-. \tag{4.131}$$

Hereby the interesting question is, how to choose $\kappa$. Our aim is to keep the drift term in Equation 4.122 under control, in order to prevent our matrix valued process $X(\cdot)$ from "exploding". To this end we apply the following one-dimensional heuristic. We are translating the drift term into the scalar form

$$\delta \sigma^2 + 2\kappa x \stackrel{!}{=} 0,$$

whereby we obtain

$$\kappa = -\frac{\delta}{2} \sigma^2$$

[93] See Cuchiero et al. [25], Theorem 2.6.

for some suitably chosen variance term $\sigma^2$. Utilizing the mean of the diagonal elements of $A$ as $\sigma^2$ yields

$$\kappa = -0.02466859$$

which works quite well for our simulation.

**3. Obtaining the starting point of $Y(\cdot)$**
As initial value for the $d \times d$ matrix valued process $X(\cdot)$ we will utilize $I_d$, an approach which is well aligned with our usual procedure of setting the initial stock prices to one. Since the initial value $Y(0) = y$ is of dimension $(d+1) \times d$, we will set $y_{ii}^2 = x_{ii},\ i = 1, \dots, d$ and $y_{i,j} = 0,\ i \neq j$.

**4. Simulation step**
The central simulation step reads as follows

$$\Delta Y(t) = \Delta B(t)\Sigma + Y(t_-)H\Delta t.$$

Based on the thus simulated value of $Y(t)$ we obtain

$$X(t) = Y(t)^T Y(t)$$

and consequently our vector of stock prices at time $t$ is given by

$$Z(t) = \text{diag}(X(t)).$$

This procedure is then repeated according to the defined simulation horizon.

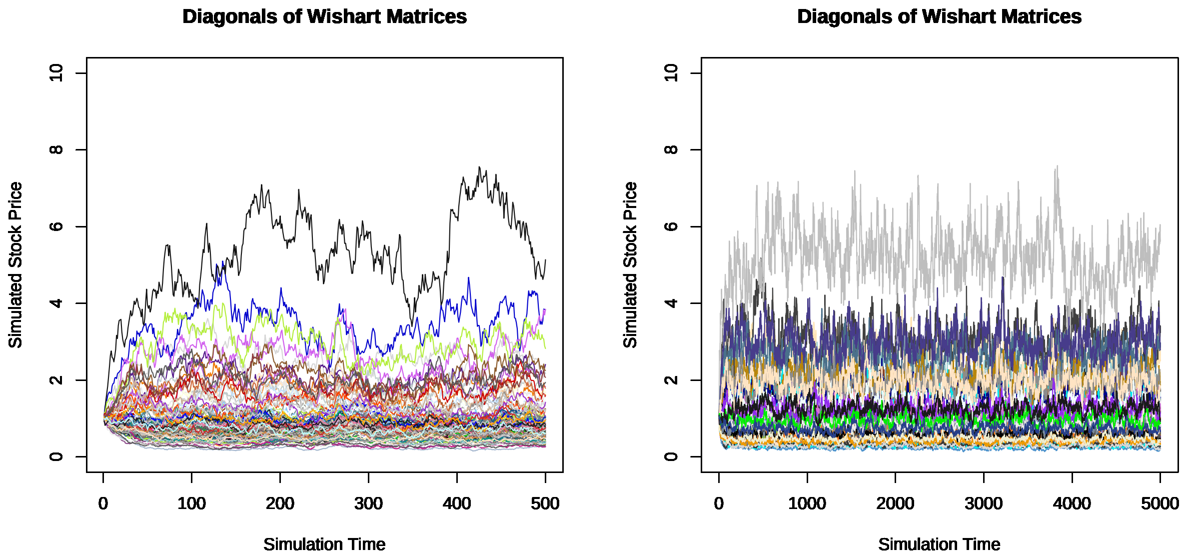


Figure 4.13: Evolution of stock prices in the Wishart model for 500 (left) and 5000 (right) time steps.

The results of this simulation scheme are remarkably stable and certainly look promising with respect to future research. In Figure 4.13 we display the evolution of simulated stock prices over the time horizons of 500 and 5000 days. Even though a handful of stocks exhibit a somehow quite extreme behavior, the vast majority of simulated particles possesses a very reasonable amplitude of dynamics and the overall market behavior appears quite stable. In the next step, we have produced the simulated Capital Distribution Curves for 3000 days (simulation runs) for our market consisting of 100 stocks. The resulting curves are displayed in Figure 4.14. The overall shape and structure of the capital distribution is quite similar to the ones obtained in the standard SqBM model and in the correlated market model, however two major differences are observable. Firstly, the descent among the largest stocks is steeper than in the other cases and secondly the decline of the smallest stock visible at the rightmost part of the chart is less pronounced than in the previously discussed cases.

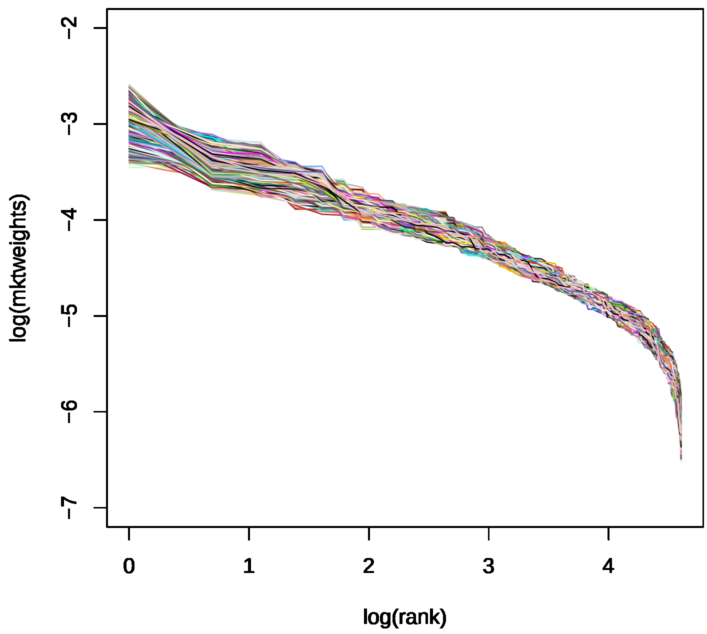


Figure 4.14: Capital Distribution Curves for 3000 simulated days in the Wishart model.

Finally we have simulated the market entropy in our Wishart model which may be seen in Figure 4.15 together with the entropy calculated based on real market fluctuations.

The evolution of market entropy in the Wishart model shows the desired mean-reverting behavior and the overall level of market entropy is comparable to the correlated model in Section 4.2. The major difference to both the model in Section 4.2 and the market data based fluctuations of the entropy is the larger amplitude of fluctuations in the Wishart model. Nonetheless, the Wishart model yields realistic stock price dynamics together with a reasonable reproduction of the capital structure and with the desired dynamics of the market entropy. Therefore the modeling approach based on Wishart processes can certainly be considered a promising area for future research.

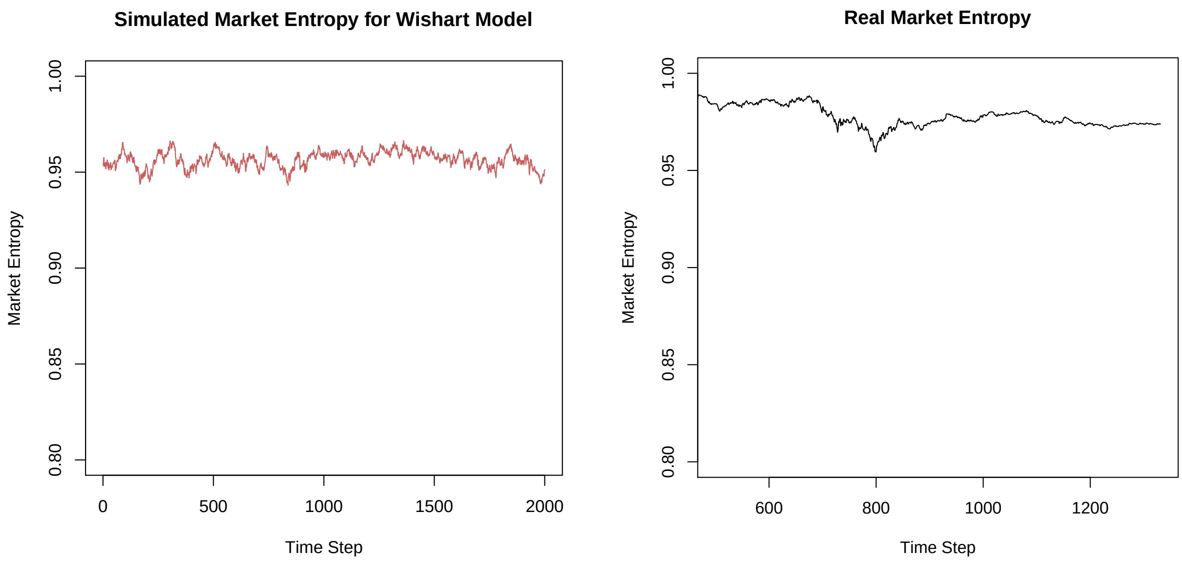


Figure 4.15: Simulated market entropy in the Wishart model and data-based market entropy.

## 4.4 Conclusion

In Chapter 4 we have introduced a model for equity markets based on squared Brownian Motions (SqBM). Slight adaptations to the dynamics of SqBM processes permitted us to endow our model with a sufficient degree of flexibility in order to reproduce a realistic pattern of stock price dynamics. In the central Section 4.2.1 we have equipped our SqBM model with a stochastic covariance structure, thus obtaining a model which describes the simultaneous evolution of correlated particles. Furthermore the successful reproduction of market characteristics yielded by our model does not depend on any ellipticity assumptions, thus eliminating a possible source of problems.

This model can be calibrated in a way which does not only reproduce volatility patterns of single stocks but also performs a good job at reproducing the structure of the Capital Distribution Curve and the behavior of market entropy which are observable in real markets. In our final Chapter 5 we will utilize this model to solve classical problems in the field of risk management.

Furthermore we investigate possible ramifications of the SqBM approach, namely the case when single stock dynamics are modeled by the sum of two Squared Brownian Motions and the more general case of $d \times d$ matrix valued Wishart processes. Whilst the differences between the first ramification and the standard SqBM model are naturally small, the more general Wishart model produces steeper Capital Distribution Curves. Yet altogether, the Wishart approach still attains the goal of producing reasonable dynamics for single particles, the distribution of capital and the market entropy, a fact which renders this class of processes a promising field for future research.

# Chapter 5

# Application to Risk Management

In the following Chapter 5 we will discuss approaches to quantify the downside risk of a certain financial asset, like e.g. a stock $X_i \in \mathfrak{M}$. For all general results on risk management techniques and on the properties of commonly used risk measures we refer to the comprehensive works of Föllmer and Schied [54] and McNeil et al. [78]. Grasping the downside risk or loss potential of a certain financial position is the central challenge in everyday risk management in financial institutions. In this vein, classical measures of dispersion like variance / standard deviation (or volatility as it is commonly referred to in financial markets) will only provide a limited picture and need to be complemented by further measures which are amenable to quantifying a position's loss potential. Further discussions of this challenge together with a comprehensive introduction to risk measures and their applications can be found in Föllmer and Schied [54], a book which we refer to with regard to all general results stated in the ensuing introduction.

From the perspective of an investor, a risk measure may indicate potential loss levels linked to a certain financial position. From the point of view of a supervising agency however, a risk measure represents an indication for the amount of capital which a credit institution ought to hold in order to cover the risks linked to a certain position. The current regulations for credit institutions in the European Union as mandated by the directives of the European Communities 2006/48/EC [38] and 2006/49/EC [39], the Capital Requirements Directive and Capital Adequacy Directive as amended by directive 2010/76/EU [40], in accordance with the guidelines set out by the Basel Committee on Banking Supervision (BCBS) [11] put this principle to action. Under the so-called "Internal Model-Based Approach"[1], banks have to calculate a Value-at-Risk figure for the positions held in their trading book which is then used to derive the regulatory capital requirement for market risk in the trading

[1]See e.g. the national implementation in Austria as effected in Art. 22p Austrian Banking Act [16].

book.

In the following sections we will formally define risk measures and discuss their general characteristics. Furthermore we will introduce some broadly used risk measures and outline their properties, underlining strengths and weaknesses and we will establish a connection to those risk measures currently employed for the calculation of regulatory capital requirements and provide an update on regulatory novelties in this field. The general discussion of important risk measures will finally lead us to utilizing the correlated SqBM model which we have constructed in Section 4.2 as a base for measuring portfolio risk.

For general mathematical properties and numerous results on risk measures we refer e.g. to Artzner et. al. [8], Föllmer and Schied [54], Jorion [63] or McNeil et al. [78]. With regard to the regulations discussed we refer to the binding texts published by the European Union [38], [39] and [40] as well as to the guidelines issued by BCBS [11].

## 5.1 Risk Measures - Definitions and Properties

In the ensuing considerations we will slightly generalize the setup introduced in Chapter 1, in order to introduce risk measures in a general way. We will follow the classical results on risk measures introduced by Artzner et al. [8] and also refer to Föllmer and Schied [54] for a comprehensive treatment of the topic.

Let us consider a finite investment horizon $[0, T]$ and let us suppose that the cardinality $\|\Omega\| = N < \infty$. Hence one is interested in the terminal value of a general financial position $Y(T, \omega)$ and in our case specifically in the terminal value of the stocks $X_i(T, \omega), \ldots i \in \mathfrak{M}$. Let furthermore $\mathfrak{X}$ be the set of all risks, i.e. the set of all bounded functions $Y : \Omega \to \mathbb{R}$, clearly, $\mathfrak{M} \subset \mathfrak{X}$.[2] Moreover we denote the discounted positions as $\tilde{Y}(t) = \frac{Y(t)}{B(t)}$, where $B(\cdot)$ denotes the value of the bank account, hence $\tilde{Y}(T, \omega)$, $\omega \in \Omega$ denotes the discounted terminal value of a financial position $Y$. Let further[3]:

$$\begin{aligned}
\mathfrak{L}_+ &= \{Y \in \mathfrak{X} | Y(T, \omega) \geq 0, \, \forall \, \omega \in \Omega\}, && (5.1) \\
\mathfrak{L}_- &= \{Y \in \mathfrak{X} | Y(T, \omega) \leq 0, \, \forall \, \omega \in \Omega\}, && (5.2) \\
\mathfrak{L}_{--} &= \{Y \in \mathfrak{X} | Y(T, \omega) < 0, \, \forall \, \omega \in \Omega\}. && (5.3)
\end{aligned}$$

**Definition 5.1:** *We shall denote by $\mathfrak{A}$ the set of all terminal values of financial positions which are accepted by the regulatory authority and accordingly $\mathfrak{A}$ will be called **acceptance***

[2] See Artzner et al. [8], Section 2.2. and Föllmer and Schied [54], Section 4.1.
[3] See Artzner et al. [8], Section 2.2.

***set*** [4].

The concept of acceptance sets can be generalized to comprise different currencies and different regulatory bodies[5]. This generalization permits for situations where a certain terminal value of a position might be acceptable to a certain regulator in one currency but not in another and it may also be unacceptable to other regulators. However in all further considerations we will restrict ourselves to the elementary case of just one currency and one authority involved.

Let us now set forth the necessary axioms on the acceptance set as specified in Artzner et al.[6]:

**Axiom 5.1:**

$$\mathfrak{L}_+ \subseteq \mathfrak{A}. \tag{5.4}$$

**Axiom 5.2:**

$$\mathfrak{A} \cap \mathfrak{L}_{--} = \{\}. \tag{5.5}$$

*This Axiom may also be strengthened to:*

$$\mathfrak{A} \cap \mathfrak{L}_{-} = \{0\}. \tag{5.6}$$

**Axiom 5.3:**

$$\textit{The acceptance set } \mathfrak{A} \textit{ is convex.} \tag{5.7}$$

**Axiom 5.4:**

$$\textit{The acceptance set } \mathfrak{A} \textit{ is a positively homogeneous cone.} \tag{5.8}$$

**Definition 5.2:** *A **measure of risk**[7] is defined as a mapping $\rho : \mathfrak{X} \to \mathbb{R}$.*

For those $Y \in \mathfrak{X}$ for which $\rho(Y(T)) \geq 0$ one may interpret the amount $\rho(Y(T))$ as the cash or equity needed in order to make the combined position acceptable for the supervisor. Conversely negative values of $\rho(Y(T))$ imply that the position is already acceptable and may still be acceptable, if a negative cash position up to $\rho(Y(T))$ is included.[8] Risk measures and acceptance sets may be linked to each other in a straightforward way.

[4] See Artzner et al. [8], 2.2.(c).
[5] See Artzner et al. [8], 2.2.(c) and (d).
[6] See Artzner et al. [8], Axioms 2.1. - 2.4.
[7] See Artzner et al. [8], Definition 2.1.
[8] See Artzner et al. [8], Definition 2.1.

**Definition 5.3:** *The acceptance set $\mathfrak{A}_\rho$ which is associated to a risk measure $\rho$ is defined as:*[9]

$$\mathfrak{A}_\rho = \{Y(T) \in \mathfrak{X} | \rho(Y(T)) \leq 0\}. \tag{5.9}$$

*Conversely, one may define the risk measure $\rho_{\mathfrak{A}}$ which is induced by an acceptance set $\mathfrak{A}$ as:*[10]

$$\rho_{\mathfrak{A}} = \inf\{m \in \mathbb{R} | mB(T) + X(T) \in \mathfrak{A}\}. \tag{5.10}$$

*Equation 5.10 may also be stated in discounted terms as* $\rho_{\mathfrak{A}} = \inf\{m \in \mathbb{R} | m + \tilde{X}(T) \in \mathfrak{A}\}$.

Risk measures are generally classified according to their properties. In the following Definition 5.4 we will introduce the properties commonly deemed desirable in risk measures. Nonetheless, not all risk measures which are commonly used in practice fulfill all desirable properties which leads potential problems and caveats in their implementation.

**Definition 5.4:** *One can specify the following desired properties of risk measures*[11] *for all* $X, Y \in \mathfrak{X}$*:*

1. *Monotonicity: if* $X(T) \leq Y(T)$*, then* $\rho(X(T)) \geq \rho(Y(T))$.
2. *Translation Invariance (cash invariance): let* $\alpha \in \mathbb{R}$*, then* $\rho(X(T) + \alpha B(T)) = \rho(X(T)) - \alpha$.
3. *Subadditivity: for all* $X, Y \in \mathfrak{X}: \ \ \rho(X(T) + Y(T)) \leq \rho(X(T)) + \rho(Y(T))$.
4. *Positive Homogeneity: for all* $\lambda \geq 0$ *and all* $X \in \mathfrak{X}: \ \ \rho(\lambda X(T)) = \lambda\rho(X(T))$.
5. *Convexity:* $\rho : \mathfrak{X} \to \mathbb{R}$ *is called a convex measure of risk if* $\rho(\lambda X(T) + (1-\lambda)Y(T)) \leq \lambda\rho(X(T)) + (1-\lambda)\rho(Y(T)), \ \ for\ 0 \leq \lambda \leq 1$.
6. *Relevance: for all* $X \in \mathfrak{X}$ *with* $X \leq 0,\ X \neq 0$ *it holds that* $\rho(X(T)) > 0$.

**Remark 5.1:** [12]

The property of Monotonicity may be interpreted in the sense that a position with a higher payoff profile will have less downside risk as compared to a position with a poorer payoff profile.

The property of Translation Invariance (or Cash Invariance) represents the fact that the riskiness of a position can be reduced by simply adding cash. Therefore, from a regulatory

[9] See Artzner et al. [8], Definition 2.3. and Föllmer and Schied [54], Section 4.1.
[10] See Artzner et al. [8], Definition 2.2. and Föllmer and Schied [54], Proposition 4.6.
[11] See Artzner et al. [8], Section 2.2. and Föllmer and Schied [54], Section 4.1.
[12] See Föllmer and Schied [54], Section 4.1.

point of view $\rho(X)$ is the amount of cash needed, in order to compensate the risk inherent in a certain position $X$ and consequently to render it acceptable to the regulator, i.e.:

$$\rho(X(T) + \rho(X(T))B(T)) = 0. \tag{5.11}$$

The Subadditivity property formalizes the fact that a combination of two risky positions should not bear more risk than the sum of the individual positions. This also formalizes the effect of portfolio diversification. Conversely, application of a risk measure which is not subadditive might sometimes lead to the undesirable effect of punishing portfolio diversification. In this case it might be possible to reduce the overall level of risk of the positions held - and therefore also the amount of equity requested from a regulatory perspective - if the positions were simply split up and distributed to various independent trading desks or legal entities[13].

A positively homogenous risk measure is also normalized, i.e. $\rho(0) = 0$. Furthermore, under the assumption of Positive Homogeneity, convexity is equivalent to Subadditivity.[14] Positive Homogeneity of a risk measure depicts the effect when no netting or diversification occurs.[15]

Convexity of a risk measure also generally formalizes the effect of portfolio diversification, namely that the risk profile of a portfolio investing the proportion $\lambda$ in one asset and $1-\lambda$ in another should have a lower risk profile than the weighted sum of the individual risk profiles.[16]

A risk measure satisfying the properties of Monotonicity and Translation Invariance is sometimes called a monetary measure of risk.[17]

**Definition 5.5:** *A risk measure $\rho$ is called a **coherent**[18] measure of risk if it satisfies the properties of:*

- *Monotonicity,*
- *Translation Invariance,*
- *Subadditivity and*

[13] See also Artzner et al. [8], Axiom S.
[14] See Föllmer and Schied [54], Definition 4.5.
[15] See Artzner et al. [8], Remark 2.9.
[16] See Föllmer and Schied [54], Definition 4.4.
[17] See Föllmer and Schied [54], Definition 4.1.
[18] See e.g. Artzner et al. [8], Definition 2.4. or Jorion [63].

- *Positive Homogeneity.*

It should be noted that Positive Homogeneity and Subadditivity of a risk measure ensure convexity of $\rho$.[19] In particular, $\rho$ is coherent if and only if $\mathfrak{A}_\rho$ is a convex cone.[20] Moreover, any risk measure satisfying Translation Invariance and Monotonicity can be seen to be Lipschitz continuous with respect to the supremum norm[21]:

$$|\rho(X(T)) - \rho(Y(T))| \leq \|X(T) - Y(T)\|. \tag{5.12}$$

In the following Section 5.2 we will introduce some commonly used risk measures and discuss their merits and drawbacks.

## 5.2 Important Measures of Risk

### 5.2.1 Value-at-Risk (VaR)

Value-at-Risk (VaR) is probably the most commonly used risk measure in the finance industry. This popularity is not only due to its relative ease of computation but also due to the fact that VaR has become the risk measure of choice in financial regulation.[22] Formally, VaR is defined as quantile of the P&L distribution for a given confidence level and a given holding period.

**Definition 5.6:** [23] *Let $\lambda \in (0,1)$ be a given confidence level and let further $X(\cdot)$ be a given financial position and let $\tau = T - t_0$ be the given holding period. Let us furthermore assume without loss of generality that all financial positions are treated in a normalized way, i.e. $X(t_0) = 1$ and also for the bank account $B(t_0) = 1$. Thus, the log-return of the financial position over holding period $\tau$ is given by $\log X(T)$ and $\log B(T) = r\tau$.*

*In general, for a random variable $Y$, a $\lambda$-quantile is any $y \in \mathbb{R}$ with:*

$$P(Y \leq y) \geq \lambda \text{ and } P(Y < y) \leq \lambda. \tag{5.13}$$

*The formal definition of quantiles and thus of VaR may be formulated by means of the generalized inverse:*

[19] See Artzner et al. [8], Proof of Proposition 2.2.

[20] See Föllmer and Schied [54], Proposition 4.6.(d).

[21] See Föllmer and Schied [54], Lemma 4.3.

[22] See e.g. the relevant EU directives 2006/48/EC [38] and 2006/49/EC [39] as well as the national implementation in Austria in Article 22p Austrian Banking Act [16]. For the general methodological outline we refer to the policy papers of the Basel Committee on Banking Supervision, above all [11].

[23] See e.g. Föllmer and Schied [54], Definition 4.40. or McNeil et al. [78], Definition 2.10.

- *Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing function, then the generalized inverse of $F$ which we will denote by $F^{(-1)}$ is defined as $F^{(-1)}(u) := \inf\{x \in \mathbb{R} | F(x) \geq u\}$. By convention, the* inf *of the empty set is defined as $-\infty$.*[24]

- *In the case that $F : \mathbb{R} \to \mathbb{R}$ is a distribution function, its generalized inverse is its quantile function, whereby for $\lambda \in (0,1)$ the $\lambda$-quantile is given by:*

$$q_\lambda(F) := F^{(-1)}(\lambda) = inf\{x \in \mathbb{R} | F(x) \geq \lambda\}.$$

*By means of the quantile function and assuming $\log X \sim F$, $VaR(\lambda, \tau, X)$ can be defined as the negative of the $1-\lambda$ quantile:*

$$VaR(\lambda, \tau, X) := -q_{1-\lambda}(\log X) = -\inf\{x \in \mathbb{R} | P(\log X(T) \leq x) \geq 1-\lambda\}. \tag{5.14}$$

**Remark 5.2:** At this stage, several points ought to be noted. Firstly, we digress from the notational setup in Föllmer and Schied [54] by defining $VaR(\lambda, \tau, X)$ as the $(1-\lambda)$-quantile of the P&L distribution. This is due to the notational practice in financial markets, where a confidence level of $\lambda = 99\%$, say, is often utilized synonymously for the calculation of the 1% quantile. Our notation is similar to the one utilized in McNeil et al. [78] with the sole difference, that there the authors consider the *loss*-distribution in a way that losses have a positive sign, whereas we consider the P&L-distribution where losses have negative signs and gains have positive signs. Furthermore, in order to be consistent with the notation introduced in Section 5.1, we add a negative sign to the definition of $VaR(\lambda, \tau, X)$ in Equation 5.14. The pure quantile of the P&L distribution for common confidence levels will usually yield negative values, hence we have to add the negative sign in order to be in line with the utilized convention of a risk measure $\rho(Y(t)) \geq 0$ representing the amount of cash which has to be added in order to render the risky position $Y(t)$ acceptable. This clearly also holds when our quantile is positive in which case the risk measure should be smaller than zero, corresponding to the amount of cash that might be shorted while still keeping the risky position acceptable.

Secondly, as already mentioned we have always described VaR in terms of the P&L distribution, whereas in Föllmer and Schied [54], VaR is usually expressed in terms of the discounted net terminal value of the financial position. These two formulations can easily be seen to be equivalent. In Equation 5.14, $VaR(\lambda, \tau, X)$ is expressed via the (usually negative) log-return (loss) which the financial position $X$ will not surpass over the defined holding period $\tau$ at a certain confidence level $\lambda$. Conversely in Section 4.4 of Föllmer and Schied [54], VaR is defined as the amount of cash which has to be added to the financial

[24] See e.g. McNeil et al. [78], Definition 2.12.

position $X(\cdot)$ in order to ensure that the discounted net terminal value of the combined position over holding period $\tau$ will be negative only with probability $1-\lambda$.

These minor technical deviations in formulating VaR should however not blur the overall picture and the central economic message that VaR describes a loss level which should not be surpassed at a given confidence level over the specified holding period.

**Remark 5.3:** A second remark is due on Value-at-Risk with respect to the axioms of coherent risk measures which we have stated in Definition 5.5. It has been discussed extensively in the literature that VaR is not a coherent measure of risk. This observation is due to Artzner et al. [7] and [8] and is also extensively discussed e.g. in Föllmer and Schied [54] as well as in McNeil et al. [78], where a particularly useful discussion of the topic is given in Section 6.1.

Nonetheless, due to its broad utilization both in regulation of financial markets as well as in internal risk management of financial institutions, the VaR-concept deserves considerable attention, notwithstanding its theoretical shortcomings. Despite its theoretical deficiency of not being coherent, VaR is a risk measure which is easily and robustly computable and facile to communicate. It ought to be underlined at this point that any risk measure requires a certain "street credibility" on the trading / market side as well, in order to be an efficient tool for risk steering and limitation. Its tangibility is therefore a major asset of the VaR-concept, even though the fact, that VaR does indeed not provide any information on the size of losses if they surpass the VaR-level, sometimes leads to misunderstandings and also to wrong management decisions.

We have already mentioned that VaR has become the risk measure of choice in financial regulation. Hence at this point we will state, how the minimum capital requirements for market risks in the Trading Book of a credit institution are calculated using the VaR-concept. According to the legal framework in the European Union[25], credit institutions may choose between different quantification schemes for the capital requirements connected to their trading activities (i.e. their regulatory Trading Book).

In the "Internal Model-Based Approach", credit institutions have to implement a quantitative risk model which satisfies a pre-defined set of qualitative and quantitative criteria. In the "Standardized Approach" on the other hand, capital charges are computed as percentage rates defined by the regulator for different types of assets and risk categories. In its fully fledged form, an internal VaR-model covers the general and specific interest rate risk as well as the general and specific equity risk and the risks stemming from foreign exchange (FX) and commodity positions. However, credit institutions are free to apply for

[25] See the respective guidelines [38] and [39].

model admission only in some of these risk categories and calculate the remaining capital charges via the Standardized Approach (so-called "partial use"). Therefore the capital requirement for market risks stemming from the risk position in the Trading Book $Z$ at time $t$ can be formulated as:[26]

$$\begin{aligned} CC_{MR}(t) &= \max\left\{VaR_t(99\%, 10\,d, \hat{Z}(t)), k\frac{1}{60}\sum_{i=1}^{60} VaR_{t-i+1}(99\%, 10\,d, \hat{Z}(t-i+1))\right\} + \\ &\quad \max\left\{sVaR_t(99\%, 10\,d, \hat{Z}(t)), k\frac{1}{60}\sum_{i=1}^{60} sVaR_{t-i+1}(99\%, 10\,d, \hat{Z}(t-i+1))\right\} + \\ &\quad IRC_t(\hat{Z}_{spec.\ IR}(t)) + APR_t(\hat{Z}_{Cor.}(t)) + STA_t(\tilde{Z}(t)), \end{aligned} \tag{5.15}$$

whereby $Z(t) = \hat{Z}(t) \uplus \tilde{Z}(t)$ is decomposed into those risky positions covered by the internal model $(\hat{Z}(\cdot))$ and those covered by the standardized approach $(\tilde{Z}(\cdot))$. In the case that the credit institution does not possess an internal model, $\hat{Z}(\cdot) \equiv 0$. The factor $k$ denotes the regulatory multiplier which is at least 3.0[27] and which may be increased according to observed backtesting violations of the model and based on regulatory model assessment.

The risky position covered by the internal model $\hat{Z}(\cdot)$ is not just subject to risk quantification by means of VaR. Since implementation of the CRD III [40], credit institutions are obliged to calculate a so-called "stressed Value-at-Risk" (sVaR) measure for all positions in the application range of the internal model. This sVaR measure is methodologically equivalent to VaR-calculation, however guideline 2010/76/EU [40] mandates that the internal model be calibrated to a continuous 12 month period of market stress instead of the "normal" calibration for usual VaR calculation. The capital charge resulting from sVaR calculation is computed analogously to the one based on VaR, the corresponding sVaR is denoted by $sVaR_{\cdot}(99\%, 10\,d, \hat{Z}(\cdot))$ in Equation 5.15.

In the case that the application range of a credit institution's internal model also comprises specific interest rate risk, the capital charge also has to comprise a so-called "Incremental Risk Charge" (IRC).[28] The IRC is designed to cover migration and default risks resulting from positions bearing specific interest rate risk. To this end, credit institutions have to model and calculate the losses stemming from credit migrations and defaults in their portfolio over a capital horizon of one year at 99.9% confidence level.

Since positions are subject to maturing or turnover during one year, a minimum liquidity horizon of three months has to be assumed and banks have to model portfolio changes

[26] See e.g. Art. 22p ABA [16] or McNeil et al. [78], Equation 2.21.
[27] See e.g. Art 22p Austrian Banking Act [16] and Art. 229 Solvability Bylaw.
[28] See e.g. Art. 22p para. 2 no. 4 Austrian Banking Act [16].

according to the "constant level of risk" premise thus assuming that positions which mature or which are eliminated from the portfolio are replaced by positions of the same risk level. For details on the intricacies of sVaR and IRC models we refer to the guidelines issued by the European Banking Authority on these topics (see [36] and [37]). In Equation 5.15 we denote the resulting capital charge by $IRC.(\hat{Z}_{spec.\ IR}(\cdot))$.

Finally the term $APR.(\hat{Z}_{Cor.}(\cdot))$ in Equation 5.15 denotes the measure of "All Price Risks" (APR) to be computed for the positions of the correlation trading portfolio. As its name alludes to, the APR measure ought by definition to cover all price risks which are connected to the specific features of n-th-to-default instruments and other positions belonging to a bank's correlation trading portfolio.[29]

### 5.2.2 Expected Shortfall (ES)

The concept of Expected Shortfall (ES) is closely connected to VaR since it is defined as the conditional expectation of losses exceeding VaR. For a general discussion of Expected Shortfall we refer to Föllmer and Schied [54] or McNeil et al. [78]. Economically, ES can be seen as a risk measure complementing VaR since it provides information on the tail behavior of the loss distribution.[30] Furthermore, ES is a coherent measure of risk[31] which constitutes a substantial advantage over VaR. Due to these reasons, ES is increasingly being used for internal risk management purposes in credit institutions, a development which is also observable in international regulation. In this vein, the Basel Committee on Banking Supervision is proposing the utilization of ES or similar coherent risk measures for calculation of capital requirements in the Trading Book.[32]

**Definition 5.7:** [33]

*Formally ES can be defined as follows. Let $VaR(\lambda,\tau,X)$ be the Value-at-Risk as given in Definition 5.6 for the financial position $X(\cdot)$ at confidence level $\lambda$ and with holding period $\tau$. Then the corresponding Expected Shortfall $ES(\lambda,\tau,X)$ is defined as:*

$$ES(\lambda,\tau,X) = \frac{1}{1-\lambda}\int_0^{1-\lambda} VaR(u,\tau,X)du = -E(\log X(T)|\log X(T) \leq -VaR(\lambda,\tau,X)). \tag{5.16}$$

[29]See Art. 22p para. 2 no. 3 Austrian Banking Act [16]
[30]See e.g. McNeil et al. [78], Definition 2.15.
[31]See e.g. McNeil et al [78], Proposition 6.9.
[32]See the Basel Committee's Fundamental review of the trading book [12].
[33]See e.g. McNeil et al. [78], Definition 2.15 and Lemma 2.16.

*Again, the negative signs in Equation 5.16 stem from the need for consistency with the conventions set forth in Section 5.1.*

**Remark 5.4:** [34]

It is worth noting that different names exist for the risk measure defined in the above Equation 5.16. Apart from Expected Shortfall, the names "conditional VaR" (cVaR), "Tail Conditional Expectation" (TCE) or "Worst conditional Expectation" (WCE) are also being used for variants of the ES-concept.

## 5.3 Application of the SqBM Model

In this section we will apply the correlated market model which we have constructed in Section 4.2.1 for classical applications in portfolio risk management. To this end we will explain, in what way the model may be applied in practice in order to address classical issues in risk modeling. As stated above, the current regulatory setup mandates measuring risk by means of VaR with a confidence level of $\lambda = 99\%$ and a holding period of $\tau = 10$ days in the context of calculating regulatory capital requirements in the Trading Book. We will stick to the regulatory holding period and simulate 10 day log-returns for selected portfolios. Our SqBM model can be seen as hybrid approach, blending aspects of historical simulation and data analysis with facets of Monte-Carlo simulation.

### 5.3.1 Regulatory Value-at-Risk Calculation

As described in detail in Section 4.2.1, the core of our model is based on historical data like the correlation matrix $\tilde{\rho}(\mathfrak{M}, N)$ and the GARCH volatilities which are calculated from time series data in a straightforward way. As stated in Equation 4.116 these historical, market data-based elements are then input into our model and thereby serve as a base for a broad set of possible simulations. In order to compute risk figures over a 10 day holding period, we utilize the SDE 4.116 for the simulation of 10 day time windows. The result of each simulation step is a real 10 day log-return, a calculation which is then repeated for a sufficiently large number of times, in order to ensure robust quantile computation. In this illustrative example, we utilize $M = 10000$ simulated 10 day log-returns, resulting in 100000 simulated one day log-returns.

In this example, we have calculated risk figures for four arbitrarily chosen portfolios: a

[34] See e.g. McNeil et al. [78], Section 2.2.

diversity-weighted portfolio with generating function $D_p(\cdot)$ as introduced in Section 1.2.3, an entropy-weighted portfolio (see Definition 1.11), the market portfolio (see Definition 1.8) and an equally weighted portfolio. Recalling that we have started our model simulation in Section 4.2.1 with an equally weighted market universe, we have arbitrarily picked the market weights resulting from simulation run No. 300 in Section 4.2.2 as initial weighting for the market. This initial weighting and the short time window of 10 days explain, why the portfolio weights and thus the resulting P&L distribution for the different portfolios are relatively similar. Above all, it should be noted that we do not use the real world constituent weights from the S&P 100 index, thus a comparison between the results of this simulation and the real world index volatility is not really adequate.

As can be seen from the histograms presented in Figure 5.1, the initial setup of our market model leads to very homogenous results in terms of simulated portfolio profit and loss.

| | Mean | Vol. | Vol. ann. | VaR 99% | VaR 95% | VaR 90% | ES 99% |
|---|---|---|---|---|---|---|---|
| Diversity | -0,05% | 3,34% | 16,71% | 8,14% | 5,65% | 4,27% | 9,37% |
| Entropy | -0,05% | 3,34% | 16,70% | 8,14% | 5,64% | 4,27% | 9,37% |
| Market | -0,05% | 3,33% | 16,67% | 8,17% | 5,63% | 4,25% | 9,36% |
| Equal | -0,05% | 3,37% | 16,87% | 8,15% | 5,72% | 4,32% | 9,46% |

The above table completes the picture suggested by the histograms in Figure 5.1 and illustrates the similar development of the four portfolios. Once again we have to underline that this result is not surprising since we have started with a fairly homogenous market and the different weighting approaches lead to fairly similar portfolio weights in this situation. This effect is further amplified by the very short time window of simulating 10 day returns as requested for regulatory applications. Numerical comparisons with different (randomly selected) starting points for the initial price vector show that the effect of the initial weighting is observable albeit small, which is not surprising considering the relatively short regulatory time window of 10 days over which the evolution of particles is calculated. This can also be seen from the figures in the table below where we have computed the same risk figures based on a simulation utilizing a vector of 100 $\chi^2(1)$ distributed random numbers as starting point. Thus, in Section 5.3.2 we will provide an analysis of risk figures over a longer simulation horizon.

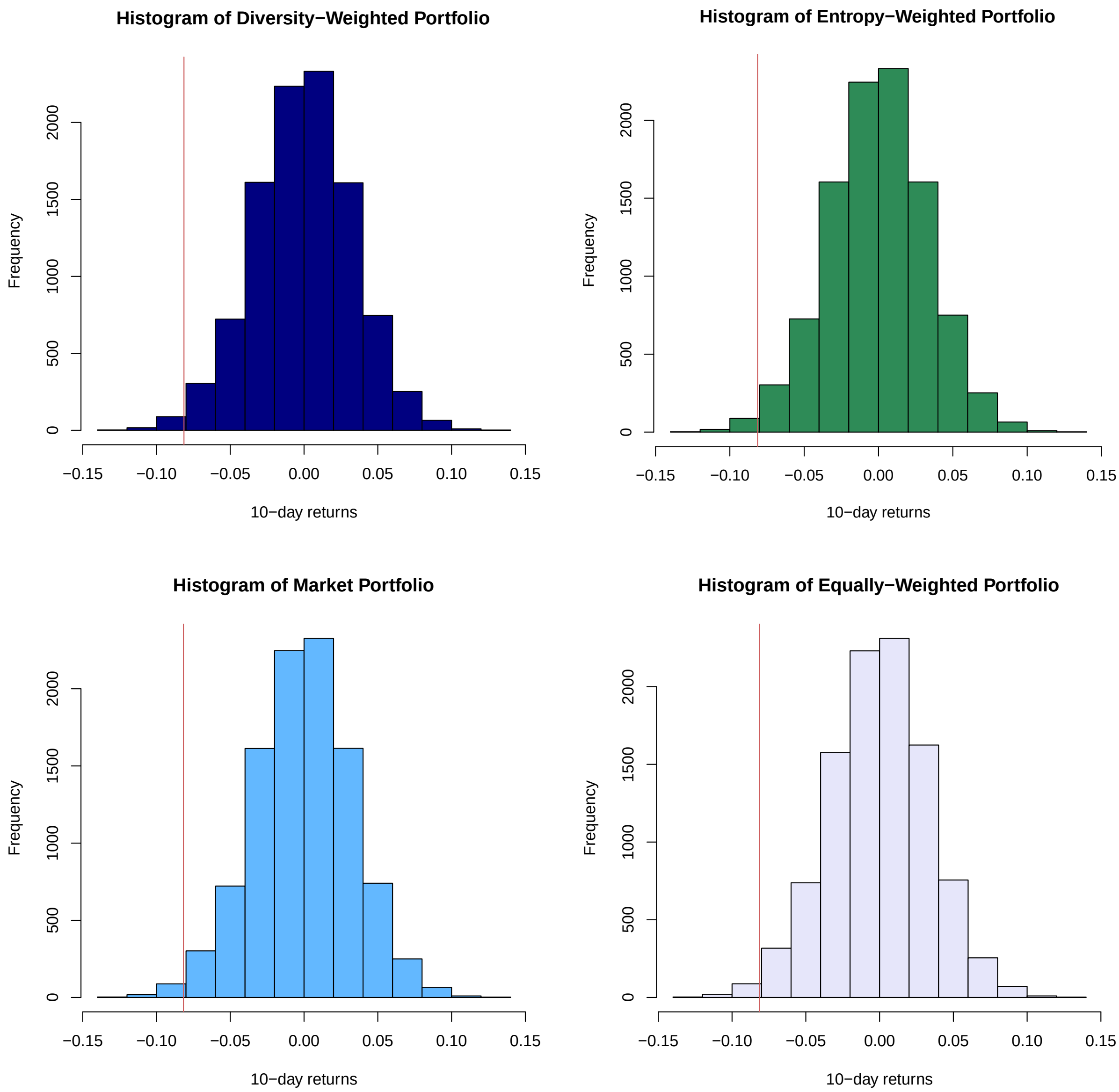


Figure 5.1: Histogram of simulated portfolio P&Ls and their 1% quantile (VaR).

| | Mean | Vol. | Vol. ann. | VaR 99% | VaR 95% | VaR 90% | ES 99% |
|---|---|---|---|---|---|---|---|
| Diversity | -0,04% | 3,45% | 17,25% | 8,50% | 5,74% | 4,39% | 9,87% |
| Entropy | -0,04% | 3,46% | 17,31% | 8,56% | 5,76% | 4,40% | 9,91% |
| Market | -0,05% | 3,52% | 17,60% | 8,69% | 5,82% | 4,47% | 10,08% |
| Equal | -0,03% | 3,36% | 16,82% | 8,34% | 5,49% | 4,30% | 9,59% |

It is quite evident that the simulation based on randomly selected initial market weights leads to more observable deviation between the different portfolio types. In this case, all portfolios exhibit more favorable characteristics than the market portfolio with the equally weighted portfolio suggesting the best risk / return relation.

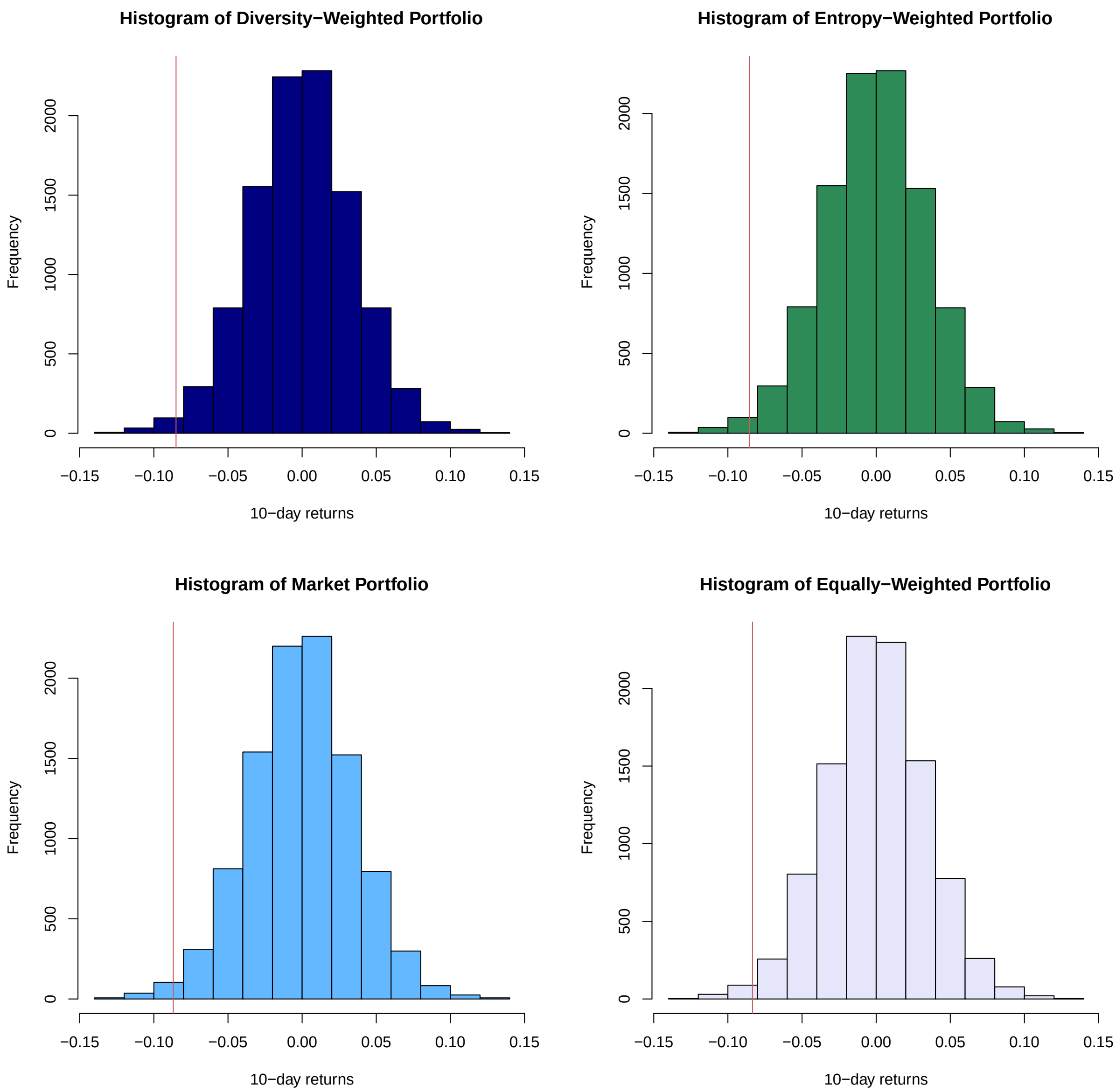


Figure 5.2: Histogram of simulated portfolio P&Ls and their 1% quantile (VaR) with random initial market capitalizations.

The histograms of the four portfolios are given in Figure 5.2. These plots also illustrate more distinguishable characteristics of the portfolio P&L distributions if compared to our first simulation and the histograms provided in Figure 5.1.

Apart from classical VaR-calculation our model may also be applied in a straightforward way for the calculation of stressed-Value-at-Risk (sVaR). In this vein, it would be sufficient to restrict the calibration procedure outlined in Section 4.2.1 to a continuous twelve months period of market stress. This would provide us with a correlation matrix $\tilde{\rho}_S(\mathfrak{M}, N_S)$ resulting from the time series in the chosen stress period. In order to ensure a conservative stress modeling, one could furthermore choose the stochastic volatility matrix which is utilized to retrieve the first simulated return as described in Equation 4.116 as $\tilde{\xi}(t_{N_{max}}) := \max_{t_i \in \{t_1,\ldots,t_{N_S}\}} \|\tilde{\xi}(t_i)\|$, where $\|\cdot\|$ denotes some suitable matrix norm, rather than $\tilde{\xi}(t_{N_S})$.

A similar approach may be utilized in order to explicitly incorporate event risk for risk management purposes as mandated by European regulation.[35] To this end one may apply a jump detection algorithm as e.g. proposed by Lee and Mykland [73] in order to identify name- and market-specific extreme events. Once such extreme events are being identified, one may utilize this data set to either re-calibrate the model in Equation 4.116 to some extreme volatility level or to enrich the current Brownian model with a jump component, thereby moving to a more general Lévy-based model.[36] At this point we shall only point out the different possible refinements of our modeling approach, a detailed elaboration of these topics is however beyond the scope of this work.

### 5.3.2 Long-Term Benchmark Calculation in the Correlated SqBM Model

In this section we shall analyze the results obtained for a long term simulation of the portfolios already utilized in Section 5.3.1 relative to the canonical benchmark - the market portfolio $\mu(\cdot)$. We start again with a market setup where all stocks have initial value 1. Due to this, at time $t = 0$ all four portfolios under scrutiny, the diversity-weighted, entropy-weighted, equally-weighted and the market portfolio are identical to the equally-weighted portfolio and their respective portfolio value is 1. This approach helps to get a feeling of

[35] For the national implementation in Austria we refer to Art. 232 Solvability Bylaw of the Austrian Banking Act [16].

[36] See e.g. Applebaum [9] or Schoutens [91].

the speed at which the different portfolio-generating functions diverge from the market. We set a simulation time-frame of $t_N = 5000$ business days, i.e. a 20 year time horizon.

The aim of this visualization is on the one hand to illustrate that our proposed model as constructed in Section 4.2 is also well suited for depicting stock price dynamics over a longer time horizon and on the other hand that it may be applied in a straightforward way for classical relative risk analysis versus a given benchmark. Here the market portfolio is being utilized in its role as canonical benchmark relative to which the other portfolios are assessed. It is worth noting at this point that there exists a major difference to the regulatory approach which we illustrated in Section 5.3.1. For the regulatory VaR-calculation, the portfolio weights have to be held constant for the simulation, since its aim is to quantify the risk level of the existing position without any further rebalancings. For the long-term benchmark analysis on the other hand it is clearly necessary to adapt the market weights in every simulation step in order to account for the concept of dynamic portfolio generation by means of generating functions as outlined in Section 1.2.

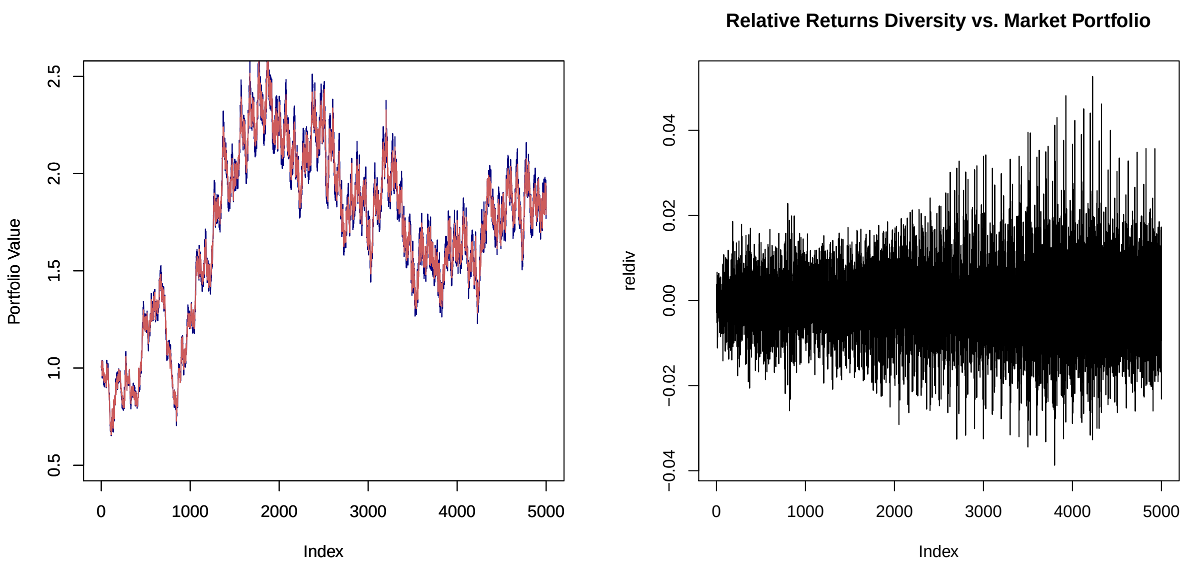


Figure 5.3: Diversity-weighted portfolio (red) and market portfolio (blue) and their relative returns.

In Figure 5.3 we depict the behavior of the diversity-weighted portfolio relative to the market. The left-hand plot illustrates the overall evolution of these portfolios, showing that our model reproduces bull- and bear-markets with a reasonable amplitude of market fluctuations over a long time horizon without leading to "explosions" or "erosions" of the aggregate market capitalization. The right chart illustrates the relative returns of the diversity-weighted portfolio relative to the market which range between $-4\%$ and $4\%$. Since we have started in an equally-weighted market universe it should be expected that the relative deviations between the two portfolios increase as time passes, which is indeed the behavior observable in the right chart. In Figure 5.4 we depict the histogram of the day-to-day log-returns of the diversity-weighted portfolio and of the market portfolio. These histograms suggest a more favorable behavior of the diversity-weighted portfolio from a risk management perspective since it exhibits less dispersion in its log-returns and less down-side potential with at the same time similar up-side versus the market.

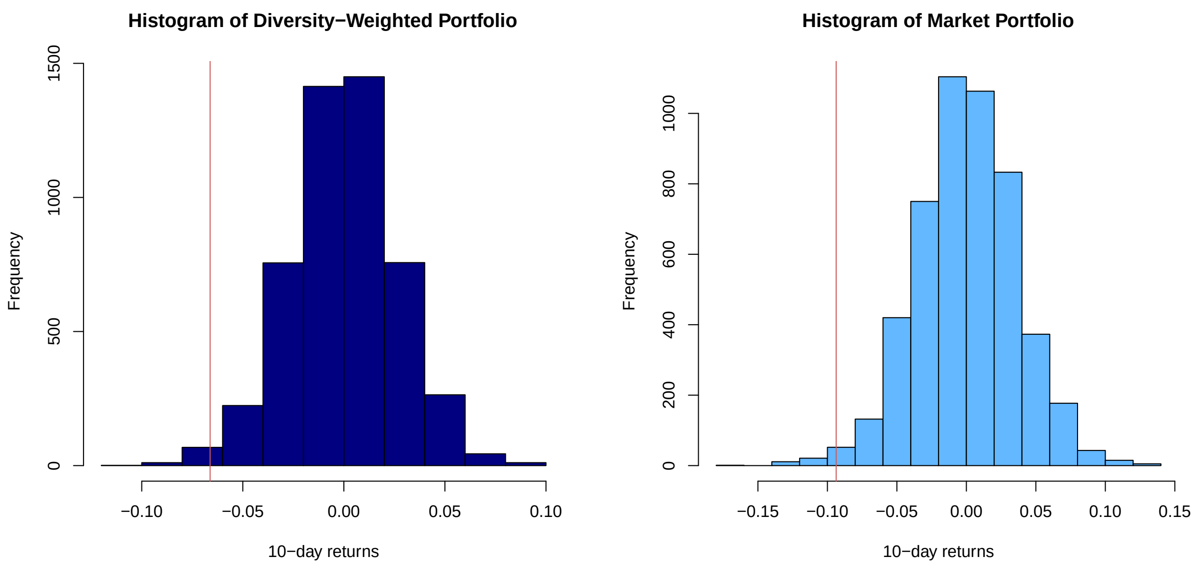


Figure 5.4: Histogram of day-to-day log-returns of the diversity-weighted and of the market portfolio.

Analogously to the above charts for the diversity-weighted portfolio, the portfolio evolution and relative returns of the entropy-weighted portfolio relative to the market are depicted in Figure 5.5. The behavior of the entropy-weighted portfolio is rather similar to the diversity-weighted portfolio as it was already suggested by our analysis of real market data in Section 1.2.3. It may be noted however, that the dispersion of relative returns is smaller than in the latter case.

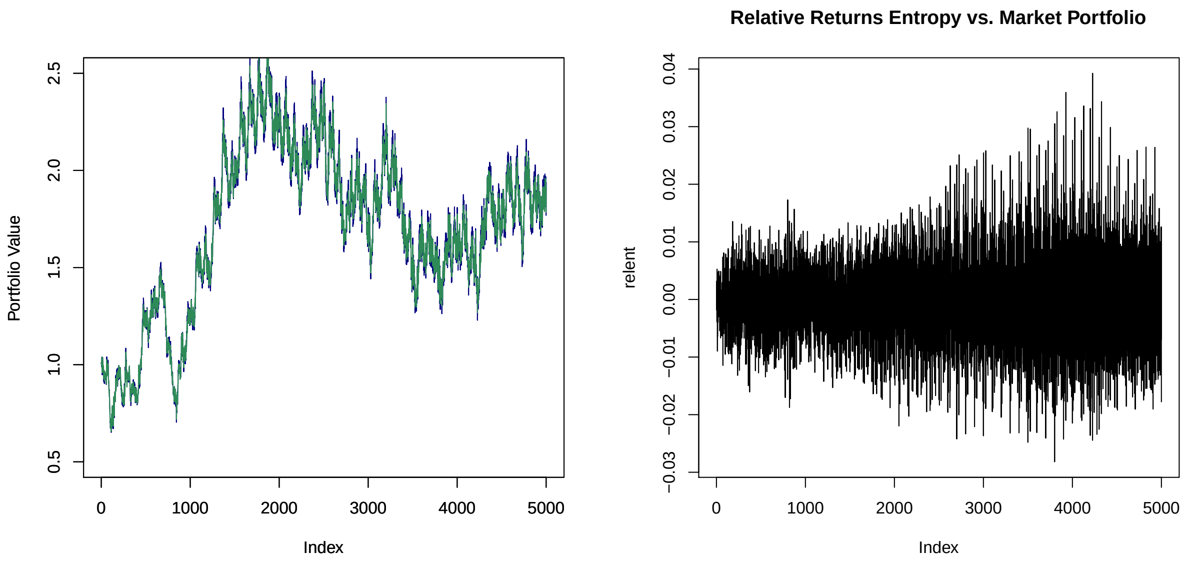


Figure 5.5: Entropy-weighted portfolio (green) and market portfolio (blue) and their relative returns.

The structure of the entropy portfolio's day-to-day log-returns bears more resemblance to those of the market portfolio than in the case of the diversity-weighted portfolio (see Figure 5.6). Finally the same charts as above for the equally-weighted portfolio are provided below. Not surprisingly, the deviations between the equally-weighted portfolio and the market are largest both in terms of the relative returns as provided in Figure 5.7 and in terms of the distributional structure of day-to-day log-returns as given in Figure 5.8. An interesting facet of the simulation for the equally weighted portfolio however is, that it reveals remarkable risk-return characteristics with little down-side and very generous up-side potential being suggested by its histogram. Hence one lesson to be taken from this little analysis is that even extremely simple portfolio management techniques which produce a high degree of diversification - in our case by simply taking the same proportion of the portfolio in every stock in the market - may lead to very attractive results.

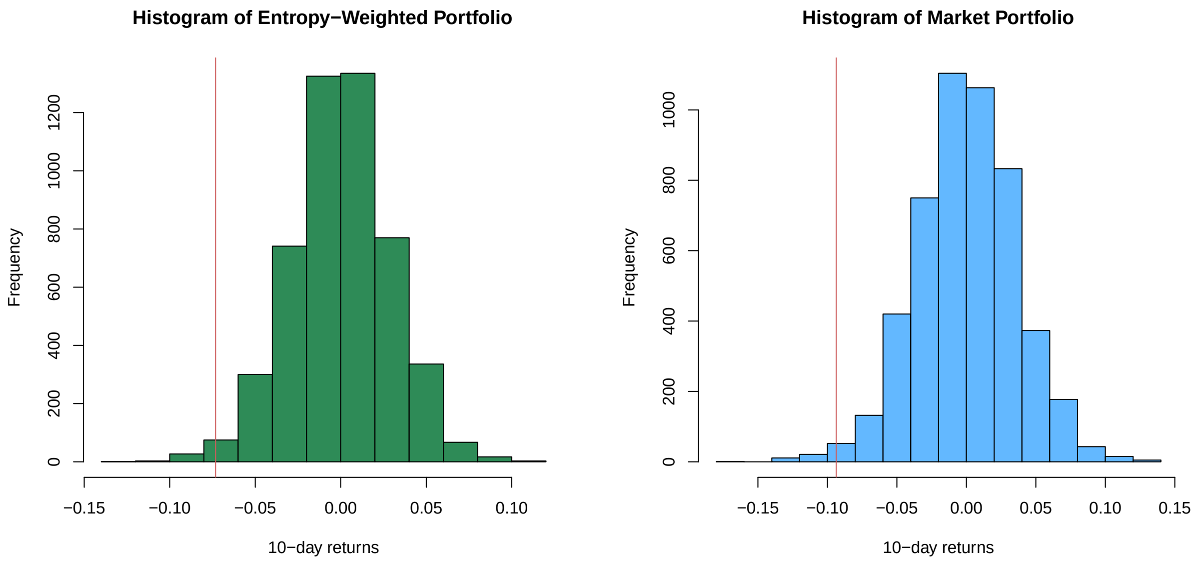


Figure 5.6: Histogram of day-to-day log-returns of the entropy-weighted and of the market portfolio.

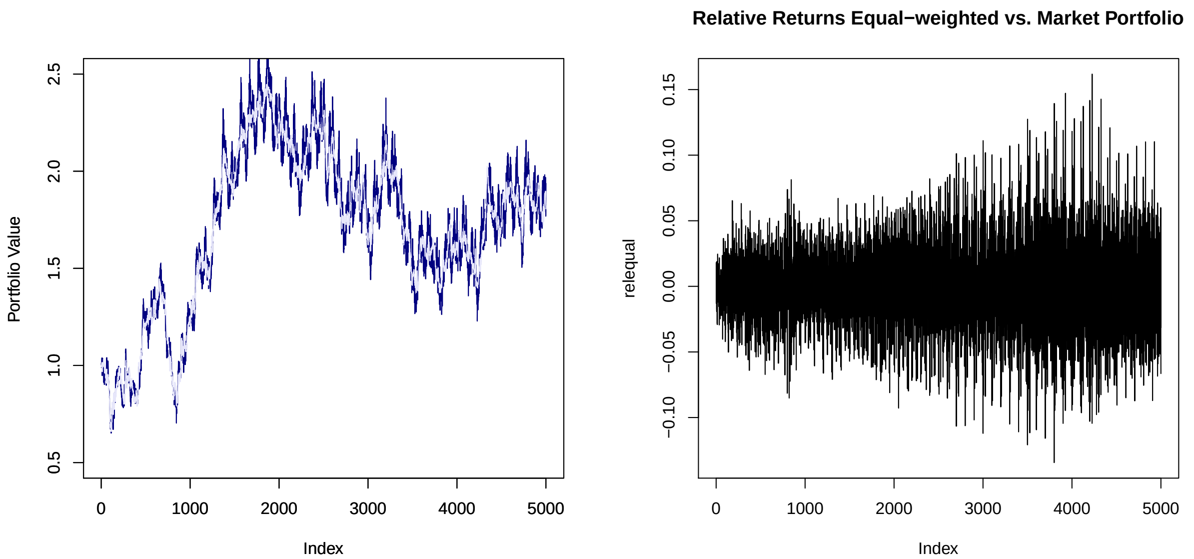


Figure 5.7: Equally-weighted portfolio (light-blue) and market portfolio (blue) and their relative returns.

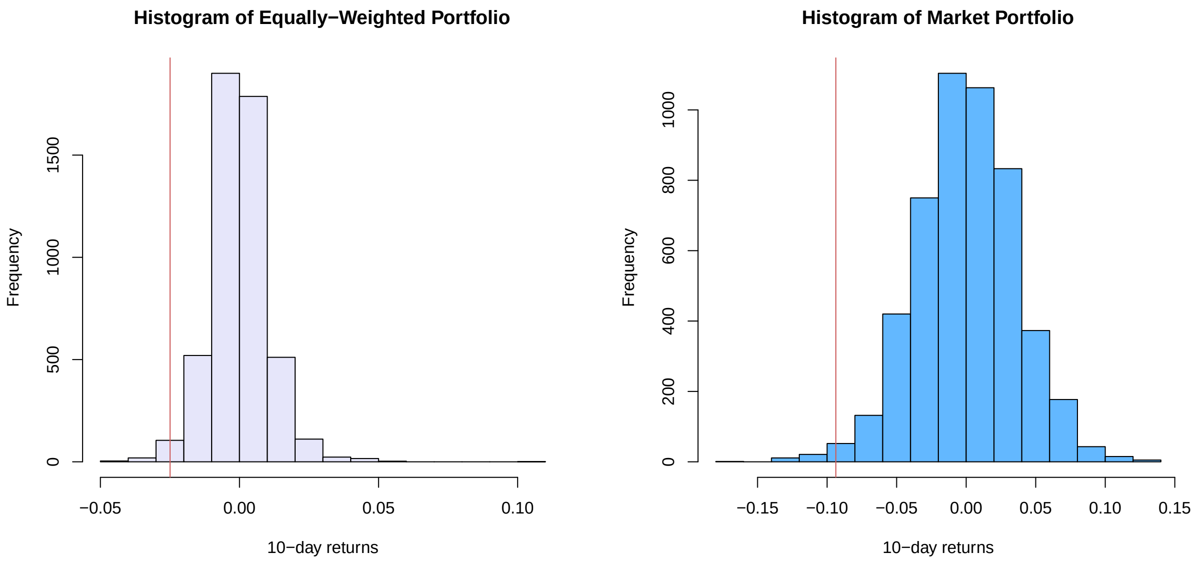


Figure 5.8: Histogram of day-to-day log-returns of the entropy-weighted and of the market portfolio.

### 5.3.3 On the Effect of Correlations

The final simulation study which we will describe in this chapter aims to assess the effect of capturing the market data inflicted correlation structure in our model. To this end we repeat the simulation performed in Section 5.3.2 for 100 times with 500 time steps each based on the correlated model and simultaneously we repeat this simulation for a simplified model where we replace the correlation matrix $\tilde{\rho}(\mathfrak{M}, N)$ by the identity matrix, keeping the GARCH volatilities, the drift and the utilized random increments unchanged. The result is a model where stocks possess individual drifts and volatility directions relative to their respective Brownian Motion but they are uncorrelated to each other. Such a cancelation of inter-stock correlations is for example used in parts of the works by Ichiba et al.[37], furthermore the volatility stabilized market model described in Section 4.1.2 is also based on uncorrelated particles.

In Figure 5.9 we present the sample trajectories of the market portfolio, the entropy-

[37] See Ichiba et al. [61], Examples 2, 3, 5, 7, 8.

weighted portfolio, the diversity-weighted portfolio and the equally-weighted in the correlated case. The displayed charts illustrate that the correlated model outlined in Section 4.2.1 reproduces a realistic structure of market fluctuations.

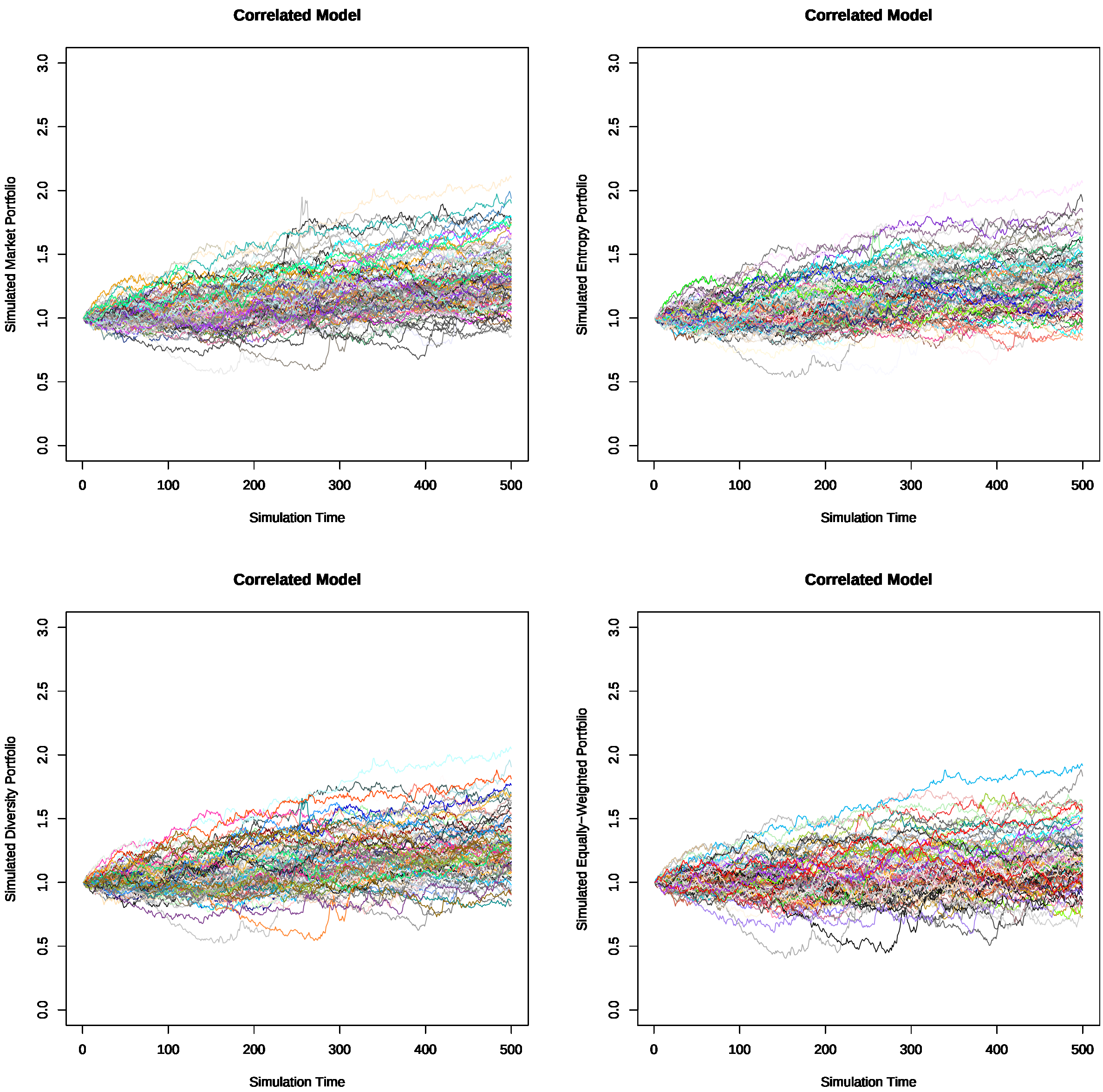


Figure 5.9: Sample paths of portfolios for the correlated model. Top row: market portfolio (left) and entropy portfolio (right); bottom row: diversity portfolio (left) and equally-weighted portfolio (right).

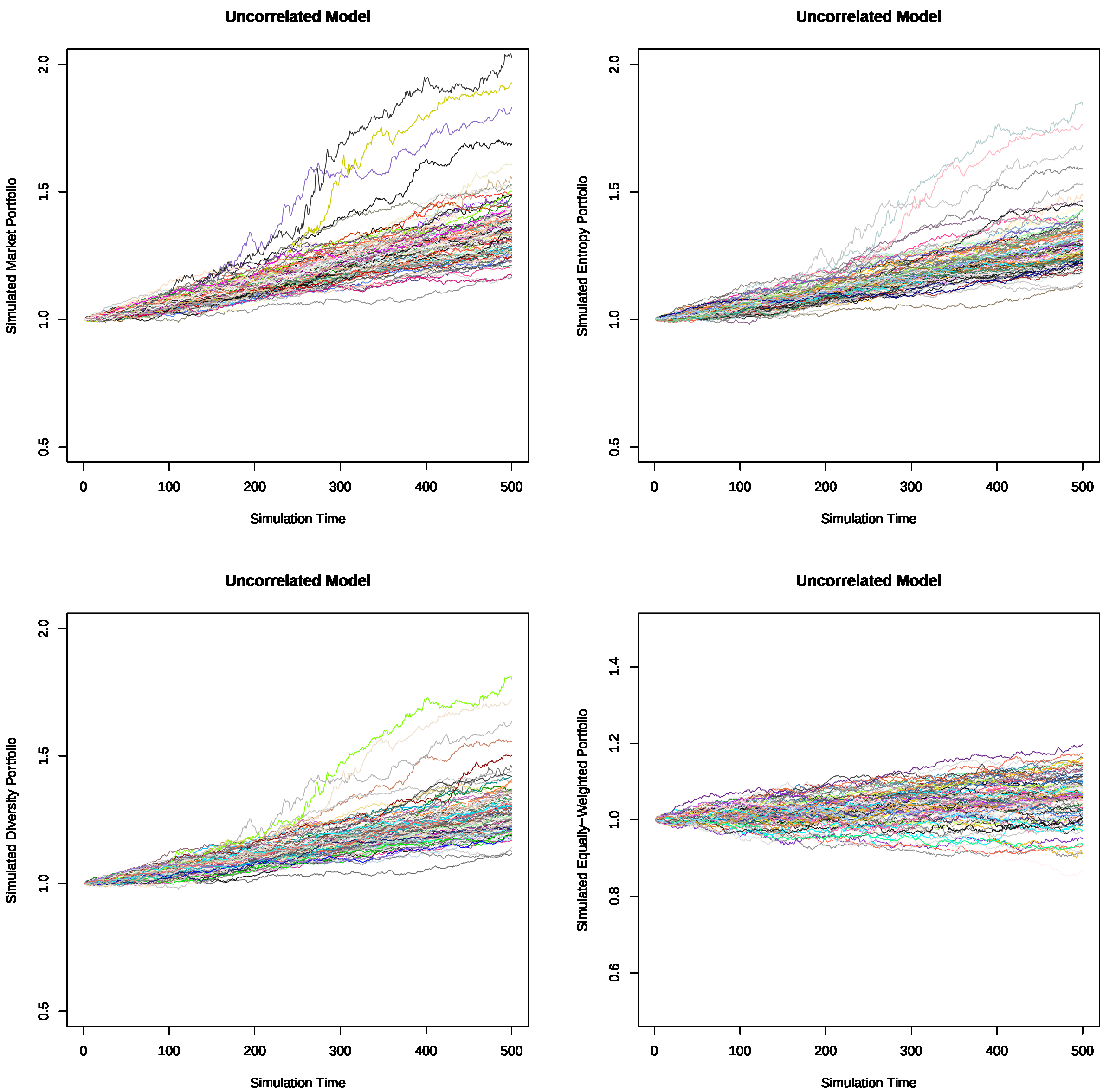


Figure 5.10: Sample paths of portfolios for the uncorrelated model. Top row: market portfolio (left) and entropy portfolio (right); bottom row: diversity portfolio (left) and equally-weighted portfolio (right).

In Figure 5.10 the sample paths of the same portfolio value processes as in Figure 5.9 are depicted for the uncorrelated case. It is striking to observe, that canceling the inter-stock correlations leads to an unrealistic structure of market fluctuations. The trajectories of portfolio values resemble monotonically increasing functions with some noise created by the market weight-based functional portfolio generation rules (market portfolio, diversity- and entropy-weighted portfolio). In the case of the equally-weighted portfolio the growth behavior of the market is essentially flattened out, the resulting charts correspond to the development of the average capitalization of individual stocks.

From these charts it is evident that the incorporation of the data-based correlation structure is crucial for the reproduction of realistic stock price and portfolio dynamics. The effect due to simply canceling out inter-stock correlations in the diffusion term of our model is massive and certainly leads to overly optimistic market evolutions. We note that this behavior is due to the large drift component of the market portfolio in the uncorrelated case as discussed in Section 4.1.2, as may directly be seen from the dynamics of the market portfolio in the pure SqBM model with respect to the Brownian Motion $\tilde{W}(\cdot)$

$$dZ_\mu(t) = Z_\mu(t)\left(\frac{n}{Z_\mu(t)}dt + \frac{2}{\sqrt{Z_\mu(t)}}d\tilde{W}(t)\right).$$

Due to this feature, the incorporation of market correlations is of paramount importance from the point of view of risk management applications. Furthermore, we note that this kind of behavior which is observable in the case of the uncorrelated model is even more pronounced in the VSM model which we have also discussed in Section 4.1.2, since in this case, the dynamics of the market portfolio are given as specified in Equation 4.101 by

$$dZ_\mu(t) = Z_\mu(t)\left(\frac{n(a+1)}{2}dt + d\tilde{W}(t)\right).$$

Hence in the VSM case the overly positive development of the market portfolio is even amplified by multiplication with the entire market capitalization. In the VSM the market portfolio takes the dynamics of an geometric Brownian Motion with an enormously large drift, whereas in our modeling approach based on Squared Brownian the process for the market portfolio again possesses dynamics akin to those of individual stocks as specified in Equation 4.52, more precisely we have that $dZ_\mu(t) = \sum_{i=1}^n dX_i(t)$ as outlined in Equation 4.78.

Another set of questions which we want to address is the risk and performance structure of functionally generated portfolios in the correlated and uncorrelated in order to assess the effect of incorporating correlations from this perspective as well.

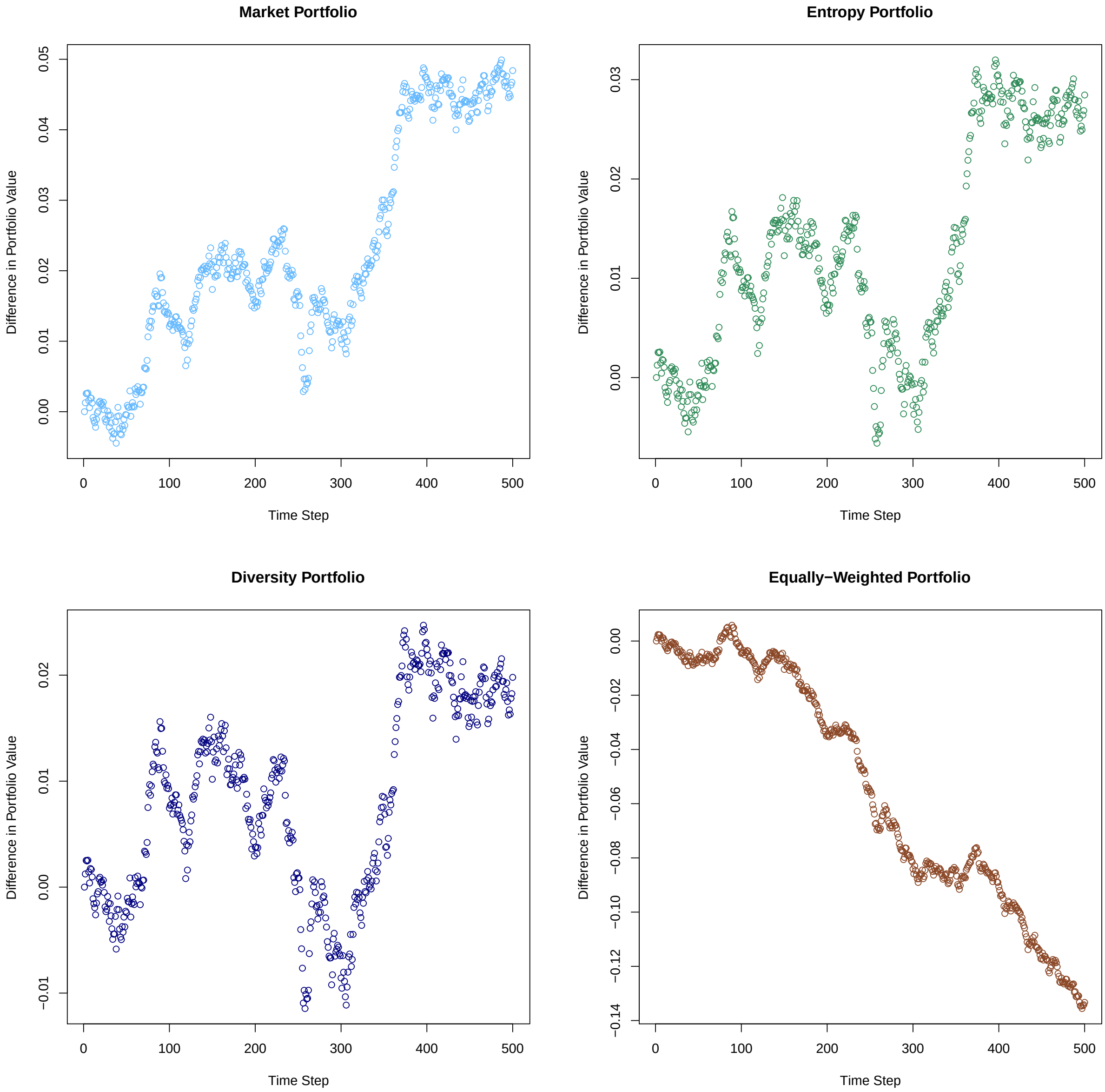


Figure 5.11: Difference in average aggregate portfolio value. Top row: market portfolio (left) and entropy portfolio (right); bottom row: diversity portfolio (left) and equally-weighted portfolio (right).

The first facet which will be outlined in Figure 5.11 is the performance of certain portfolios in the uncorrelated versus the correlated case. We recall that the simulated paths in the correlated and uncorrelated case are perfectly comparable since we have only altered the input correlation matrix in the calculation and kept all other terms including the utilized random increments equal. The charts in Figure 5.11 contain the differences of the average aggregate portfolio value based on the 100 paths containing 500 time steps which we have already utilized above. We always calculate the difference between the uncorrelated and the correlated case, thus positive values mean that the average aggregate portfolio value is larger in the uncorrelated case than in the correlated case.

The striking feature in Figure 5.11 is that the three market weight based portfolio generation rules (market portfolio, entropy- and diversity-weighted portfolio) on average lead to better results in the uncorrelated case, resulting in aggregate value differences between around 2% and 5% over the simulated time horizon of 500 days (i.e. two business years). Interestingly enough, the effect is exactly the opposite in the case of the equally-weighted portfolio. Here the average aggregate value is lagging behind in the uncorrelated case by some 14% over the simulated time horizon. We can easily obtain the portfolio dynamics in the uncorrelated case by applying Equation 4.72 or Equation 1.1.12 in Fernholz [44]. Let $\eta(t) = (\eta_1(t), \ldots, \eta_n(t))$ denote the equally weighted portfolio where for all $i \in \{1, \ldots, n\}$ and for all $t \geq 0$ $\eta_i(t) = \frac{1}{n}$. Then the dynamics of the portfolio value are given by

$$dZ_\eta(t) = \frac{Z_\eta(t)}{n} \left[ \left( \sum_{i=1}^{n} \frac{1}{X_i(t)} \right) dt + \sum_{i=1}^{n} \frac{1}{\sqrt{X_i(t)}} dW_i(t) \right].$$

Since for this simulation we have actually used the correlated model given by Equation 4.115 and then canceled out the inter-stock correlations, we are actually considering the dynamics

$$dZ_\eta(t) = \frac{Z_\eta(t)}{n} \left[ \left( \sum_{i=1}^{n} \frac{b_i(t)}{X_i(t)} \right) dt + \sum_{i=1}^{n} \frac{\tilde{\xi}_{ii}(t)}{\sqrt{X_i(t)}} dW_i(t) \right].$$

Similarly we have the dynamics in the correlated case given by

$$dZ_\eta(t) = \frac{Z_\eta(t)}{n} \left[ \left( \sum_{i=1}^{n} \frac{b_i(t)}{X_i(t)} \right) dt + \sum_{i=1}^{n} \frac{1}{\sqrt{X_i(t)}} \sum_{\nu=1}^{n} \tilde{\xi}_{i\nu}(t) dW_\nu(t) \right].$$

It is obvious that the drift terms will perfectly cancel out when assessing the differences between uncorrelated and correlated case and therefore the strong underperformance visible in Figure 5.11 for the equally-weighted portfolio is uniquely due to the effect of inter-stock correlations and their effect on the volatility term and the respective Brownian Motions.

Finally, if we take a closer look at the summary statistics of the distribution of daily log-returns and the daily VaRs for various portfolios which are summarized in the table below, we observe that the incorporation of correlations is essential for risk measurement purposes. One observes that all portfolios exhibit annualized volatilities between 3.4% and 3.7% in the uncorrelated case which is not really a plausible range for equity portfolios.

| | Mean | Vol. | Vol. ann. | VaR 99% | VaR 95% | VaR 90% |
|---|---|---|---|---|---|---|
| Market Cor. | 0.052% | 1.20% | 18.96% | 2.75% | 1.86% | 1.42% |
| Market Uncor. | 0.052% | 0.24% | 3.74% | 0.55% | 0.33% | 0.25% |
| Diversity Cor. | 0.046% | 1.25% | 19.69% | 2.87% | 1.94% | 1.48% |
| Diversity Uncor. | 0.044% | 0.22% | 3.52% | 0.54% | 0.31% | 0.24% |
| Entropy Cor. | 0.047% | 1.23% | 19.45% | 2.84% | 1.92% | 1.46% |
| Entropy Uncor. | 0.046% | 0.23% | 3.56% | 0.54% | 0.31% | 0.25% |
| Equally-Wgt. Cor. | 0.031% | 1.43% | 22.61% | 3.34% | 2.23% | 1.70% |
| Equally-Wgt. Uncor. | 0.015% | 0.22% | 3.41% | 0.53% | 0.34% | 0.25% |

In the correlated case on the other hand, all annualized portfolio volatilities range between 18.9% and 22.6% which is precisely the magnitude one would expect for well-diversified portfolios consisting of liquid large cap stocks. This aspect further demonstrated that our correlated market model is well suited for risk management applications since we do not only reproduce stylized facts like the capital distribution structure or the behavior of market entropy but also a realistic degree of fluctuations both in individual stocks and in compound portfolios.

## 5.4 Conclusion

In this chapter we have illustrated how the equity market model which we have constructed in Chapter 4 may be applied to classical risk management tasks. We have introduced two of the most commonly used risk measures, Value-at-Risk and Expected Shortfall, and given a brief introduction on the general properties of risk measures. Furthermore we have connected the mathematical properties of our model with the international regulatory setup in the area of market risk in the Trading Book. The exemplary applications in Section 5.3 illustrate that our proposed market model does not only do a good job in reproducing the structure of the Capital Distribution Curve and the dynamics of market entropy but it also provides us with a very flexible and numerically stable simulation environment for risk management applications with short-, medium- and long-term time horizons. Furthermore the simulation results presented in Section 5.3.2 demonstrate the ability of our proposed model to reproduce trustworthy long-term dynamics of market evolutions.

The simulation study in Section 5.3.3 further provides some valuable results on the importance of incorporating realistic inter-stock correlations in a market model focusing on risk management applications. We illustrate that an uncorrelated SqBM model leads to unrealistic market behavior when compared to the correlated model. Without due consideration of the effect of correlations, the market capitalization exhibits extremely low volatility and an upward trending behavior which renders the uncorrelated setup less suitable for risk management applications.

# Appendices

# Appendix A

# Software Code of Presented Models

## A.1 Fourier Estimator for Covariance Matrices

### A.1.1 R Implementation

```
# Implementation of Fourier estimator for instantaneous covariance
# matrices and its application to empirically assessing
# market ellipticity.
# Author: Florian Leisch
# Created: 04.05.2011
# Last Update: 31.05.2011



data <- read.csv("C:/A/Dissertation/Ellipticity_Analysis/
SP100_constituents_R_Test_7.csv",sep=";",dec=",", skip=0)


logreturns<-data

for(k in 1:length(data)) {
 logreturns[1,k]<-0
}
```

```
for(j in 1:length(data)) {
 for(i in 2:length(data[,1])) {
  logreturns[i,j]<-log(data[i,j]/data[i-1,j])
 }
}

write.table(logreturns,file="C:/A/Dissertation/Ellipticity_Analysis/
logret_test_7.csv",sep=";",dec=",")

cfun<-function(k,N,times,logret){
t1<-1:length(logret)
for(j in 1:length(logret)){
 t1[j]<-exp(-1i*k*times[j])
}
t2<-t1*logret
t3<-1/(2*pi)*sum(t2)
if (abs(k)<=2*N) t4<-t3 else t4<-0
return(t4)
}

afun<-function(k,N,logret1,logret2,times1,times2) {
tnum<-2*N
temp1<-(-tnum):tnum
temp2<-temp1
for(count in 1:(4*N+1)){
 temp2[count]<-cfun(temp1[count],N,times1,logret1)*
cfun(k-temp1[count],N,times2,logret2)
}
temp3<-(2*pi/(2*2*N+1))*sum(temp2)
return(temp3)
}

sfun<-function(t,N,logret1,logret2,times1,times2){
temp4<-(-N):N
temp5<-temp4
for(cc in 1:(2*N+1)){
 temp5[cc]<-(1-abs(temp4[cc])/N)*
afun(temp4[cc],N,logret1,logret2,times1,times2)*exp(1i*temp4[cc]*t)
}
```

```
temp6<-sum(temp5)
return(temp6)
}


mytime<-pi/2
rho<-2*pi/length(logreturns[,1])
mynum<-ceiling(rho^(-2/3))
timegrid<-0:(length(logreturns[,1]))
timegrid<-2*pi*timegrid/length(logreturns[,1])
covmat<-matrix(data=NA,nrow=length(data),ncol=length(data))

for(i in 1:length(data)){
 for(j in 1:length(data)){
  covmat[i,j]<-
Re(sfun(mytime,mynum,logreturns[,i],logreturns[,j],timegrid, timegrid))
  }
}



for(i in 95:length(data)){
 for(j in 1:length(data)){
  covmat[i,j]<-
Re(sfun(mytime,mynum,logreturns[,i],logreturns[,j],timegrid, timegrid))
  }
}
write.table(covmat,file="C:/A/Dissertation/Ellipticity_Analysis/
covmat_test_7.csv",sep=";",dec=",")

evs<-eigen(covmat,only.values=TRUE)
write.table(evs[1],file="C:/A/Dissertation/Ellipticity_Analysis/
evs_test_7.csv",sep=";",dec=",")


require(graphics)
pco<-princomp(covmat)
screeplot(pco,xlab="Principal Components",col="navyblue")
pcsum<-summary(pco)
```

```
write.table(pcsum,file="C:/A/Dissertation/Ellipticity_Analysis/
Princomp_Covm_test_7.csv",sep=";",dec=",")
```

### A.1.2 Mathematica Implementation

```
(* Data Import *)

TSClean=
Import["/u/leischf/FourierEstimator/
SP100_constituents_Math_Test_5.csv","CSV"];
myNames=TSClean[[1]];
NumStocks=Length[myNames]
100
Quotes=Take[Drop[TSClean,1],750];
temp1=Quotes;
For[k=0, k<Length[myNames] , k++;
temp1[[1,k]]=0;
]

For[j = 0, j<Length[myNames], j++;
For[i =1, i<Length[Quotes],i++;
temp1[[i,j]]=Log[Quotes[[i,j]]/Quotes[[i-1,j]]];
]
]

myDataHighDim=temp1;
Length[myDataHighDim[[All,1]]]
750
TimeFrame=2*N[Pi];
TimePoints=Table[i,{i,0,Length[myDataHighDim[[All,1]]]}];
TimeGrid=2*N[Pi]/(Length[TimePoints]-1)*TimePoints;
TimeGrid2=Take[TimeGrid,{1,Length[TimeGrid]-1}];
rho=2*N[Pi]/(Length[TimePoints]-1);
Num=Ceiling[rho^(-2/3)]
25
```

```
Length[TimePoints]
751


(* Functions *)

DiscreteFourier[k_,logret_,time_,Number_]:=(
temp=1/(2*N[Pi])*Sum[Exp[-Sqrt[-1]*k*time[[i]]]
*logret[[i]],{i,1,Length[time]-1}];

temp2=If[Abs[k]<= 2*Number,temp,0];
temp2
)

Alpha[k2_,Number_,logret1_, logret2_,time1_,time2_]:= (
t1=2*N[Pi]/(2*2*Number +1) *
Sum[DiscreteFourier[i,logret1,time1,Number]*
DiscreteFourier[(k2-i),logret2,time2,Number],{i,-2*Number,2*Number}];
t1
)


RescaledDirichletKernel[N_,t_]:=(
t1=If[t==0,1,1/(2*N+1)*(Sin[(2*N+1)/2*t])/Sin[t/2]];
t1
)


IntegratedVolatility[logret1_,logret2_,time1_,time2_,Num1_]:=(
temp=Sum[Sum[Sum[Exp[Sqrt[-1]*s*(time1[[i]]-time2[[j]])]*
logret1[[i]]*logret2[[j]],{j,1,Length[Logret2]}],
{i,1,Length[Logret1]}],{s,-Num1,Num1}];
temp
)


Kernelestimator[x_,y_,h_]:=(
t1=1/(Length[x]*h)*
Sum[If[-1<=(x[[i]]-y)/h<=1,1/h 3/4 (1-((x[[i]]-y)/h)^2),0],{i,Length[x]}];
```

```
t1
)


(* Calculation of Volatilities (higher dim) *)


VolaTable2=IdentityMatrix[NumStocks];
For[i=0,i<NumStocks,i++;
For[j=0,j<i,j++;
ts1=myDataHighDim[[All,i]];
ts2=myDataHighDim[[All,j]];
VolaTable2[[i,j]]=
Sum[(1-Abs[l]/Num)*Exp[l*Sqrt[-1]*t]*
Alpha[l,Num,ts1,ts2,TimeGrid, TimeGrid],{l,-Num,Num}];
VolaTable2[[j,i]]=VolaTable2[[i,j]];
]
]

Export["/u/leischf/FourierEstimator/covm_Math_11.csv",VolaTable2,"CSV"]

RandomTimes=2*Pi*Table[RandomReal[],{i,100}];
VmTable=Table[0,{i,Length[RandomTimes]}];
EvsTable=VmTable;
EvsNewTable=EvsTable;
For[count=0, count<Length[RandomTimes], count++;
vm=Re/@(VolaTable2/.t->RandomTimes[[count]]);
vm2=vm;
VmTable[[count]]=vm;

myevs=Eigenvalues[vm2];

EvsTable[[count]]=myevs;

]

Export["/u/leischf/FourierEstimator/EVS_Math_11.csv",EvsTable,"CSV"]

Export["/u/leischf/FourierEstimator/Covmat_11.csv",VmTable,"CSV"]
```

## A.2 Squared Brownian Motion Model

```
# Simulation of capital distribution curves for squared Brownian Motions
# Author: Florian Leisch
# Created: 23.04.2012
# Last Update: 17.07.2012
# Higher dimensional simulation of correlated stocks based on
# low-dim approach (n=4) in file SqBM_Simulation.R
# Simulation based on latest discussion on 05.04.2012

data <- read.csv("C:/A/Dissertation/Quad_BM_Modell/
logret_SnP_100.csv",sep=";",dec=",",skip=0)
library(Matrix)
library(PASWR)
library(bdsmatrix)
library(lattice)

# Some functions for price aggregation and BM simulation

AggregatePrice<-function(retvec){
timestep<-length(retvec)
pricevec<-1:(timestep+1)
pricevec[1]<-1
for(j in 1:timestep){
 pricevec[j+1]<-pricevec[j]*exp(retvec[j])
}
return(pricevec)
}

CalculateLogRet<-function(prices){
timestep<-length(prices)
returns<-1:(timestep-1)

for(j in 1:length(returns)){
returns[j]<-log(prices[j+1]/prices[j])
}

return(returns)
```

```
}

BMSimulation<-function(timestep){
deltatime<-1/timestep
noisevec<-rnorm(timestep,0,sqrt(deltatime))
bmvec1<-cumsum(noisevec)
bmvec2<-append(bmvec1,0,after=0)
return(bmvec2)

}

SqBMSimulation<-function(timestep){
deltatime<-1/timestep
noisevec<-rnorm(timestep,0,sqrt(deltatime))
bmvec1<-cumsum(noisevec)
bmvec2<-append(bmvec1,0,after=0)
bmvec3<-bmvec2^2
return(bmvec3)
}


# Function for calculation and plotting of
# capital distribution curves (CDC).

cdc<-function(data,day){
mktcaps<-sort(data[,day],decreasing=TRUE)
mktweights<-mktcaps/sum(mktcaps)
rank<-1:length(mktweights)
temp<-colors()
index<-sample.int(length(temp), size = 1, replace = FALSE, prob = NULL)

plot(log(rank),log(mktweights),ylim=c(-6,-2),col=temp[index],"l")
}

cdc2<-function(data,day,mycol){
mktcaps<-sort(data[,day],decreasing=TRUE)
mktweights<-mktcaps/sum(mktcaps)
rank<-1:length(mktweights)
temp<-colors()
```

```
index<-mycol

plot(log(rank),log(mktweights),ylim=c(-6,-3),col=temp[index],"l")
}

# Function for calculating the entropy of a given market vector

mktentropy<-function(data,day){
basecaps<-sort(data[,1],decreasing=TRUE)
baseweights<-basecaps/sum(basecaps)
logbaseweights<-log(baseweights)
baseval<-(-1)*(t(logbaseweights)%*%baseweights)

mktcaps<-sort(data[,day],decreasing=TRUE)
mktweights<-mktcaps/sum(mktcaps)
logmktweights<-log(mktweights)
eval<-((-1)*(t(logmktweights)%*%mktweights))/baseval
return(eval)
}

#Function for calculating market portfolio weights

calcmktweights<-function(data,day){
mktcaps<-data[,day]
mktweights<-mktcaps/sum(mktcaps)
return(mktweights)
}

# Function for calculating entropy weighted portfolios

calcentropyweights<-function(data,day){
mktcaps<-data[,day]
mktweights<-mktcaps/sum(mktcaps)
logmktweights<-log(mktweights)
eval<-((-1)*(t(logmktweights)%*%mktweights))
evec<-(-1)*mktweights*logmktweights

entwght<-(1/eval)*evec
```

```
return(entwght)

}

# Functions for calculating diversity weighted portfolios

diversityweights<-function(data,day){
p<-0.7
mktcaps<-data[,day]
mktweights<-mktcaps/sum(mktcaps)
pmktweights<-mktweights^p
psum<-sum(pmktweights)
divweights<-(1/psum)*pmktweights

return(divweights)

}




#########################################################################
# GARCH(1,1) estimator for local volatility of a stock

mystock1<-data[1]
mystock<-mystock1[,1]
a1<-0.1
beta<-0.85

stdvola<-sqrt(var(mystock))
h<-stdvola^2
pers<-a1+beta
a0<-h*(1-pers)

condvola<-1:length(mystock)
condvola[1]<-0

# Calculating GARCH(1,1) volatilities
```

```
for (j in 2:length(mystock)){
condvola[j]<-sqrt(a0+a1*mystock[j-1]^2+beta*condvola[j-1]^2)
}

condvola

stdvola

########################################################################



# GARCH vola calculation for all historical time series


GARCHestimate2<-function(tsdata){

mystock1<-tsdata[1]
mystock<-mystock1[,1]
pricevec<-1:(length(mystock)+1)

for (k in 2:length(pricevec)){
pricevec[k]<-pricevec[k-1]*exp(mystock[k-1])
}

mystocknormalized<-mystock

for(j in 1:length(mystock)){
mystocknormalized[j]<-mystock[j]*sqrt(pricevec[j])
}

a1<-0.1
beta<-0.85

stdvola<-sqrt(var(mystocknormalized))
h<-stdvola^2
pers<-a1+beta
a0<-h*(1-pers)
```

```
condvola<-1:length(mystocknormalized)
condvola[1]<-0

# Calculating GARCH(1,1) volatilities
for (j in 2:length(mystocknormalized)){
condvola[j]<-sqrt(a0+a1*mystocknormalized[j-1]^2+beta*condvola[j-1]^2)
}

outputmat<-condvola

for (k in 2:length(tsdata)){
mystock2<-tsdata[k]
mystock3<-mystock2[,1]

pricevec2<-1:(length(mystock3)+1)

for (l in 2:length(pricevec2)){
pricevec2[l]<-pricevec2[l-1]*exp(mystock3[l-1])
}

mystocknormalized2<-mystock3

for(j in 1:length(mystock)){
mystocknormalized2[j]<-mystock3[j]*sqrt(pricevec2[j])
}


stdvola<-sqrt(var(mystocknormalized2))
h<-stdvola^2
a0<-h*(1-pers)

condvola<-1:length(mystocknormalized2)
condvola[1]<-0

for (j in 2:length(mystock3)){
condvola[j]<-sqrt(a0+a1*mystocknormalized2[j-1]^2+beta*condvola[j-1]^2)
}
```

```
outputmat<-append(outputmat,condvola,after=length(outputmat))

}

outputmat2<-
matrix(data=outputmat,nrow=length(tsdata),ncol=length(mystock),byrow=TRUE)

return(outputmat2)
}

########################################################################

GARCHts<-GARCHestimate2(data)
numstocks<-length(GARCHts[,1])
numtimes<-length(GARCHts[1,])

stdvoltable<-1:numstocks
for (i in 1:numstocks){
ms1<-data[i]
ms2<-ms1[,1]

pricevec2<-1:(length(ms2)+1)

for (l in 2:length(pricevec2)){
pricevec2[l]<-pricevec2[l-1]*exp(ms2[l-1])
}

mystocknormalized2<-ms2

for(j in 1:length(ms2)){
mystocknormalized2[j]<-ms2[j]*sqrt(pricevec2[j])
}
stdvoltable[i]<-sqrt(var(mystocknormalized2))
}


# Normalization of historical log-returns with GARCH-volas
# s.t. every vector is i.i.d.
# Initial run of normalization to equip output matrix with a first row.
```

```
temp3<-GARCHts[1,]

temp4<-AggregatePrice(data[,1])

temp5<-temp6<-data[,1]

for (l in 2:numtimes){
temp6[l]<-temp5[l]*sqrt(temp4[l])/temp3[l]
}

output1<-temp6

# Loop to calculate remaining vectors.

for (k in 2:numstocks){

temp1<-GARCHts[k,]
temp2<-AggregatePrice(data[,k])
temp7<-temp8<-data[,k]

for (j in 2:numtimes){
temp8[j]<-temp7[j]*sqrt(temp2[j])/temp1[j]

}

output1<-append(output1,temp8,after=length(output1))

}

output2<-matrix(data=output1,nrow=numstocks,ncol=numtimes,byrow=TRUE)



#########################################################################

# Calculate correlation matrix obtained by cor(x) from
# normalized log-returns.
```

```
# Correlation Matrix obtained from real data
cormat<-cor(t(output2))

levelplot(cormat,col.regions=terrain.colors(100),
main="Visualization of Correlation Matrix")



# Correlation Matrix for independent particles
matdat<-1:10000
tmat<-matrix(data=matdat,ncol=numstocks,byrow=TRUE)
for(i in 1:100){
for(j in 1:100)
tmat[i,j]<-0
}
for(i in 1:100){
tmat[i,i]<-1
}

cormat2<-tmat

# Random correlation matrix

randdat<-rnorm(10000)
randmat<-matrix(data=randdat,nrow=100,ncol=100,byrow=TRUE)
randsym<-t(randmat)%*%randmat

randcor<-randsym

for (i in 1:100){
for (j in 1:100){

randcor[i,j]<-randsym[i,j]/(sqrt(randsym[i,i])*sqrt(randsym[j,j]))

}
}

cormat3<-randcor
```

```
# Simulation Setup

simtime<-5000
# Time expressed in days. Assuming 250 business days per year
# this corresponds to 20 years simulation horizon

mesh<-1


# Simulation of random increments ~ N(0,1) for d=100 stocks
simulationdata2<-rnorm(numstocks*simtime)
randommat2<-
  matrix(data=simulationdata2,nrow=numstocks,ncol=simtime,byrow=TRUE)

pricedat2<-1:(numstocks*simtime)
pricemat2<-matrix(data=pricedat2,nrow=numstocks,ncol=simtime,byrow=TRUE)

returnmat2<-pricemat2

for(i in 1:numstocks){
pricemat2[i,1]<-1
returnmat2[i,1]<-0
}

# Initial setup of volatility vectors for simulation.
# First volatility applied is the newest (last) GARCH estimator.

voladat<-1:(numstocks*simtime)
volamat<-matrix(data=voladat,nrow=numstocks,ncol=simtime, byrow=TRUE)

for(i in 1:numstocks){
volamat[i,1]<-GARCHts[i,length(GARCHts[1,])]
}

for (j in 1:(simtime-1)){

# Setup of stochastic volatility matrix for simulation.
tm1<-volamat[,j]%*%t(volamat[,j])
```

```
tm2<-cormat*tm1
xi<-t(chol(tm2))

alpha<-median(volamat[,j])


for (i in 1:numstocks){

tempret<-(1/pricemat2[i,j])*(alpha*volamat[i,j]-volamat[i,j]^2)*
(1/mesh)+(1/sqrt(pricemat2[i,j]))*(xi[i,]%*%randommat2[,j])

returnmat2[i,j+1]<-sign(tempret)*min(0.1,abs(tempret))
# setting an absolute ceiling of 10% for daily logreturns

pricemat2[i,j+1]<-pricemat2[i,j]*exp(returnmat2[i,j+1])

a1<-0.1
beta<-0.85

stdvola<-stdvoltable[i]
h<-stdvola^2
pers<-a1+beta
a0<-h*(1-pers)

# Calculating the next GARCH(1,1) volatility
# based on the simulated log-return.

volamat[i,j+1]<-sqrt(a0+a1*returnmat2[i,j]^2+beta*volamat[i,j]^2)

}

}

# write.csv(t(pricemat2),file="prices4.csv")

########################################################################
### Simulation of uncorrelated particles for comparison
########################################################################
```

```
volamat2<-volamat
returnmat3<-returnmat2
pricemat3<-pricemat2

for (j in 1:(simtime-1)){

# Setup of stochastic volatility matrix for simulation.
tm1<-volamat2[,j]%*%t(volamat2[,j])
tm2<-cormat2*tm1
xi<-t(chol(tm2))

alpha<-median(volamat2[,j])


for (i in 1:numstocks){

tempret<-(1/pricemat3[i,j])*(alpha*volamat2[i,j]-volamat2[i,j]^2)*
(1/mesh)+(1/sqrt(pricemat3[i,j]))*(xi[i,]%*%randommat2[,j])

returnmat3[i,j+1]<-sign(tempret)*min(0.1,abs(tempret))
# setting an absolute ceiling of 10% for daily logreturns

pricemat3[i,j+1]<-pricemat3[i,j]*exp(returnmat3[i,j+1])

a1<-0.1
beta<-0.85

stdvola<-stdvoltable[i]
h<-stdvola^2
pers<-a1+beta
a0<-h*(1-pers)

# Calculating the next GARCH(1,1) volatility
# based on the simulated log-return.

volamat2[i,j+1]<-sqrt(a0+a1*returnmat3[i,j]^2+beta*volamat2[i,j]^2)

}
```

```
}



###########################################################################
### Simulation of randomly correlated particles for comparison
###########################################################################

volamat3<-volamat
returnmat4<-returnmat2
pricemat4<-pricemat2

for (j in 1:(simtime-1)){

# Setup of stochastic volatility matrix for simulation.
tm1<-volamat3[,j]%*%t(volamat3[,j])
tm2<-cormat3*tm1
xi<-t(chol(tm2))

alpha<-median(volamat2[,j])


for (i in 1:numstocks){

tempret<-(1/pricemat4[i,j])*(alpha*volamat3[i,j]-volamat3[i,j]^2)*
(1/mesh)+(1/sqrt(pricemat4[i,j]))*(xi[i,]%*%randommat2[,j])

returnmat4[i,j+1]<-sign(tempret)*min(0.1,abs(tempret))
# setting an absolute ceiling of 10% for daily logreturns

pricemat4[i,j+1]<-pricemat4[i,j]*exp(returnmat4[i,j+1])

a1<-0.1
beta<-0.85

stdvola<-stdvoltable[i]
h<-stdvola^2
pers<-a1+beta
a0<-h*(1-pers)
```

```
# Calculating the next GARCH(1,1) volatility
# based on the simulated log-return.

volamat3[i,j+1]<-sqrt(a0+a1*returnmat4[i,j]^2+beta*volamat3[i,j]^2)

}

}


########################################################################

########################################################################

# Visualization

y1<-pricemat2[1,]
y2<-pricemat2[2,]
y3<-pricemat2[3,]
y4<-pricemat2[4,]
x1<-1:length(y1)


y5<-returnmat2[1,]
y6<-returnmat2[2,]
y7<-returnmat2[3,]
y8<-returnmat2[4,]
x2<-1:length(y5)


par(new=F)
plot(x1,y1, "l", ylab ="Stock Prices",ylim=c(0,3.5), col="navyblue")
par(new=T)
plot(x1, y2, "l", ylab =" ",ylim=c(0,3.5), col="indianred")
par(new=T)
plot(x1, y3, "l", ylab =" ",ylim=c(0,3.5), col="seagreen")
par(new=T)
plot(x1, y4, "l", ylab =" ",ylim=c(0,3.5), col="moccasin")
```

```
par(new=F)
plot(x2,y5, "l", ylab ="Log-Returns",ylim=c(-0.2,0.2), col="navyblue")
par(new=T)
plot(x2, y6, "l", ylab =" ",ylim=c(-0.2,0.2), col="indianred")
par(new=T)
plot(x2, y7, "l", ylab =" ",ylim=c(-0.2,0.2), col="seagreen")
par(new=T)
plot(x2, y8, "l", ylab =" ",ylim=c(-0.2,0.2), col="moccasin")


# Calculation of CDC

for(j in 1:300){
par(new=T)
cdc(pricemat2,j)
}


for(j in 1:20){
par(new=T)
cdc2(pricemat2,(3000+j),491) # correlated particles (blue)
#par(new=T)
#cdc2(pricemat3,(3000+j),372) # uncorrelated particles (red)
#par(new=T)
#cdc2(pricemat4,(3000+j),574) # randomly correlated particles (green)
}




# Calculating Market Entropy


t1<-3000
t2<-5000
entropyvec<-1:(t2-t1)
x3<-entropyvec
entropyvec2<-entropyvec
```

```
entropyvec3<-entropyvec

for (j in 1:length(entropyvec)){
entropyvec[j]<-mktentropy(pricemat2,(t1+j))
# correlated particles
entropyvec2[j]<-mktentropy(pricemat3,(t1+j))
# uncorrelated particles
entropyvec3[j]<-mktentropy(pricemat4,(t1+j))
# randomly correlated particles
}

plot(x3,entropyvec, "l",xlab="Time Step", ylab="Market Entropy",
  main="Comparison of Simulated Market Entropies",
ylim=c(0.8,1), col="navyblue")
  # correlated
par(new=T)

plot(x3,entropyvec2, "l",xlab="Time Step", ylab="Market Entropy",
  main="Comparison of Simulated Market Entropies",
ylim=c(0.8,1), col="indianred")
  # uncorrelated
par(new=T)

plot(x3,entropyvec3, "l",xlab="Time Step", ylab="Market Entropy",
  main="Comparison of Simulated Market Entropies",
ylim=c(0.8,1), col="seagreen")
  # randomly correlated
par(new=F)


mktentropy(pricemat2,50)


#########################################################################
### Visualization of real market entropy
#########################################################################

mktprices<-AggregatePrice(data[,1])
mylength<-length(mktprices)
```

```
for(j in 2:numstocks){
mktprices<-
append(mktprices,AggregatePrice(data[,j]),after=length(mktprices))
}

tdat<-1:length(mktprices)
for(i in 1:length(mktprices)){
tdat[i]<-mktprices[i]
}
tdat2<-c(tdat)

mktpricemat<-
matrix(data=mktprices,nrow=numstocks,ncol=mylength,byrow=TRUE)

entropyvec3<-1:length(mktpricemat[1,])
x4<-entropyvec3

for (j in 1:length(entropyvec3)){
entropyvec3[j]<-mktentropy(mktpricemat,j)
}

plot(x4,entropyvec3, "l",xlab="Time Step", ylab="Market Entropy",
xlim=c(500,length(entropyvec3)),ylim=c(0.8,1), col="seagreen")
# real data

for(j in 600:700){
par(new=T)
cdc(mktpricemat,j)

}

########################################################################
# Calculating the Market Portfolio

t1<-3000
t2<-5000
x3<-1:(t2-t1+1)
```

```
weightmat<-calcmktweights(pricemat2,t1)
pfval<-t(weightmat)%*%pricemat2[,t1]

for(j in 3001:5000){
temp1<-calcmktweights(pricemat2,j)
weightmat<-append(weightmat,temp1, after=length(weightmat))

temp2<-t(temp1)%*%pricemat2[,j]
pfval<-append(pfval,temp2, after=length(pfval))
}

weightmat2<-matrix(data=weightmat,nrow=numstocks,ncol=2001,byrow=TRUE)

plot(x3,pfval, "l")

mktreturns<-CalculateLogRet(pfval)
x4<-1:length(mktreturns)

plot(x4,mktreturns,"l")

######################################################################
# Calculating the Entropy Portfolio

t1<-3000
t2<-5000
x3<-1:(t2-t1+1)

entweightmat<-calcentropyweights(pricemat2,3000)
entval<-t(entweightmat)%*%pricemat2[,3000]

for(j in 3001:5000){
temp1<-calcentropyweights(pricemat2,j)
entweightmat<-append(entweightmat,temp1, after=length(entweightmat))

temp2<-t(temp1)%*%pricemat2[,j]
entval<-append(entval,temp2,after=length(entval))
}
entweightmat2<-
matrix(data=entweightmat, nrow=numstocks,ncol=2001,byrow=TRUE)
```

```
plot(x3,entval, "l")

entreturns<-CalculateLogRet(entval)

plot(x4,entreturns,"l")

##########################################################################
# Calculating the Diversity Weighted Portfolio

t1<-3000
t2<-5000
x3<-1:(t2-t1+1)

divweightmat<-diversityweights(pricemat2,3000)
divval<-t(divweightmat)%*%pricemat2[,3000]

for(j in 3001:5000){
t3<-diversityweights(pricemat2,j)
divweightmat<-
append(divweightmat,t3,after=length(divweightmat))

t4<-t(t3)%*%pricemat2[,j]
divval<-append(divval,t4,after=length(divval))

}

divweightmat2<-
matrix(data=entweightmat, nrow=numstocks,ncol=2001,byrow=TRUE)

plot(x3,divval, "l")

divreturns<-CalculateLogRet(divval)

plot(x4,divreturns, "l")
```

```
#########################################################################
###     Application of Simulation Algorithm for VaR Calculation      ###
#########################################################################


MCtime<-10000 # Number of simulation runs
simtime<-11
# Holding period for VaR calculation expressed in days +1
# (i.e. 10 day holding period).
mesh<-1
aggregatedata<-1:(MCtime*numstocks)
aggregatematrix<-
matrix(data=aggregatedata,nrow=numstocks,ncol=MCtime,byrow=TRUE)


for (k in 1:MCtime){

# Simulation of random increments ~ N(0,1) for d=100 stocks
simulationdata3<-rnorm(numstocks*simtime)
randommat3<-
matrix(data=simulationdata3,nrow=numstocks,ncol=simtime,byrow=TRUE)

pricedat3<-1:(numstocks*simtime)
pricemat3<-
matrix(data=pricedat3,nrow=numstocks,ncol=simtime,byrow=TRUE)

returnmat3<-pricemat3

for(i in 1:numstocks){
pricemat3[i,1]<-1
returnmat3[i,1]<-0
}

# Initial setup of volatility vectors for simulation.
# First volatility applied is the newest (last) GARCH estimator.

voladat<-1:(numstocks*simtime)
```

```
volamat<-matrix(data=voladat,nrow=numstocks,ncol=simtime, byrow=TRUE)

for(i in 1:numstocks){
volamat[i,1]<-GARCHts[i,length(GARCHts[1,])]
}

for (j in 1:(simtime-1)){

# Setup of stochastic volatility matrix for simulation.
tm1<-volamat[,j]%*%t(volamat[,j])
tm2<-cormat*tm1
xi<-t(chol(tm2))

alpha<-median(volamat[,j])


for (i in 1:numstocks){

tempret<-(1/pricemat3[i,j])*(alpha*volamat[i,j]-volamat[i,j]^2)*
(1/mesh)+(1/sqrt(pricemat3[i,j]))*(xi[i,]%*%randommat3[,j])

returnmat3[i,j+1]<-sign(tempret)*min(0.4,abs(tempret))
# setting an absolute ceiling of 50% for daily logreturns

pricemat3[i,j+1]<-pricemat3[i,j]*exp(returnmat3[i,j+1])

a1<-0.1
beta<-0.85

stdvola<-stdvoltable[i]
h<-stdvola^2
pers<-a1+beta
a0<-h*(1-pers)

# Calculating the next GARCH(1,1) volatility
# based on the simulated log-return.

volamat[i,j+1]<-sqrt(a0+a1*returnmat3[i,j]^2+beta*volamat[i,j]^2)
```

```
}

}

for(j in 1:numstocks){
aggregatematrix[j,k]<-sum(returnmat3[j,])
}

}

# Application of simulated 10 day returns to different portfolios

initialprices<-pricemat2[,300]
# Setting the starting point of the market to some
# arbitrarily chosen initial weightin from the above run (t=300).

initialdivweight<-diversityweights(pricemat2,300)
divreturns<-1:length(aggregatematrix[1,])

initialentweight<-calcentropyweights(pricemat2,300)
entreturns<-1:length(aggregatematrix[1,])

initialmktweight<-calcmktweights(pricemat2,300)
mktreturns<-1:length(aggregatematrix[1,])

equalweights<-1:numstocks
for (i in 1:numstocks){
equalweights[i]<-1/numstocks
}
equalreturns<-1:length(aggregatematrix[1,])

for (i in 1:MCtime){
divreturns[i]<-t(initialdivweight)%*%aggregatematrix[,i]
entreturns[i]<-t(initialentweight)%*%aggregatematrix[,i]
mktreturns[i]<-t(initialmktweight)%*%aggregatematrix[,i]
equalreturns[i]<-t(equalweights)%*%aggregatematrix[,i]
}
```

```
divresults<-
c(mean(divreturns),sqrt(var(divreturns)),
quantile(divreturns,0.01),quantile(divreturns,0.05),
quantile(divreturns,0.1))
entresults<-
c(mean(entreturns),sqrt(var(entreturns)),
quantile(entreturns,0.01),quantile(entreturns,0.05),
quantile(entreturns,0.1))
mktresults<-
c(mean(mktreturns),sqrt(var(mktreturns)),
quantile(mktreturns,0.01),quantile(mktreturns,0.05),
quantile(mktreturns,0.1))
equalresults<-
c(mean(equalreturns),sqrt(var(equalreturns)),
quantile(equalreturns,0.01),quantile(equalreturns,0.05),
quantile(equalreturns,0.1))



hist(divreturns, col="navyblue", xlab="10-day returns",
main="Histogram of Diversity-Weighted Portfolio")
par(new=T)
abline(v=quantile(divreturns,0.01), col="indianred")
par(new=F)

hist(entreturns, col="seagreen", xlab="10-day returns",
main="Histogram of Entropy-Weighted Portfolio")
par(new=T)
abline(v=quantile(entreturns,0.01), col="indianred")
par(new=F)

hist(mktreturns, col="steelblue1", xlab="10-day returns",
main="Histogram of Market Portfolio")
par(new=T)
abline(v=quantile(mktreturns,0.01), col="indianred")
par(new=F)

hist(equalreturns, col="lavender", xlab="10-day returns",
```

```
main="Histogram of Equally-Weighted Portfolio")
par(new=T)
abline(v=quantile(equalreturns,0.01), col="indianred")
par(new=F)


#######################################################################
###    Calculation of Expected Shortfall for Simulated Portfolios    ###
#######################################################################

divorder<-sort(divreturns)
entorder<-sort(entreturns)
mktorder<-sort(mktreturns)
equalorder<-sort(equalreturns)
conflevel<-0.01
confindex<-conflevel*MCtime

ESdiv<-ESent<-ESmkt<-ESequal<-1:confindex

for (j in 1:confindex){
ESdiv[j]<-divorder[j]
ESent[j]<-entorder[j]
ESmkt[j]<-mktorder[j]
ESequal[j]<-equalorder[j]
}

ESvalues<-c(mean(ESdiv),mean(ESent),mean(ESmkt),mean(ESequal))

divresults<-append(divresults,ESvalues[1],after=length(divresults))
entresults<-append(entresults,ESvalues[2],after=length(entresults))
mktresults<-append(mktresults,ESvalues[3],after=length(mktresults))
equalresults<-append(equalresults,ESvalues[4],after=length(equalresults))
title<-c("Mean","Volatility","VaR 99%","VaR 95%",
"VaR 90%","Expected Shortfall 99%")

outputdata<-c(title,divresults,entresults,mktresults,equalresults)

outputmatrix<-matrix(data=outputdata,ncol=length(title),byrow=TRUE)
```

```
write.csv(outputmatrix,file="RMoutput.csv")

########################################################################

########################################################################

# Simulation of independent stock paths
# Calculation of model log-returns: d log X_i(t) =
# = 1/X_i(t) (alpha*sigma_i(t) - sigma_i(t)^2/2) dt
# + sigma_i(t) / sqrt(X_i(t))d B_i(t)


pricedat<-1:(numstocks*numtimes)
pricemat<-matrix(data=pricedat,nrow=numstocks,ncol=numtimes,byrow=TRUE)
returnmat<-pricemat
for(i in 1:numstocks){
pricemat[i,1]<-1
returnmat[i,1]<-0
}

alpha<-((1/4)+0.0001)

for (i in 1:numstocks){

for (j in 2:numtimes){

returnmat[i,j]<-
(1/pricemat[i,j-1])*(alpha*GARCHts[i,j-1]-GARCHts[i,j-1]^2)*
(1/mesh)+(GARCHts[i,j-1]/sqrt(pricemat[i,j-1]))*randommat[i,j]

pricemat[i,j]<-pricemat[i,j-1]*exp(returnmat[i,j])


}
}

y1<-pricemat[1,]
y2<-pricemat[2,]
y3<-pricemat[3,]
```

```
y4<-pricemat[4,]
x1<-1:length(y1)


y5<-returnmat[1,]
y6<-returnmat[2,]
y7<-returnmat[3,]
y8<-returnmat[4,]
x2<-1:length(y5)



par(new=F)
plot(x1,y1, "l", ylab ="Stock Prices",ylim=c(0,2.5), col="navyblue")
par(new=T)
plot(x1, y2, "l", ylab =" ",ylim=c(0,2.5), col="indianred")
par(new=T)
plot(x1, y3, "l", ylab =" ",ylim=c(0,2.5), col="seagreen")
par(new=T)
plot(x1, y4, "l", ylab =" ",ylim=c(0,2.5), col="moccasin")

par(new=F)
plot(x2,y5, "l", ylab ="Log-Returns",ylim=c(-0.2,0.2), col="navyblue")
par(new=T)
plot(x2, y6, "l", ylab =" ",ylim=c(-0.2,0.2), col="indianred")
par(new=T)
plot(x2, y7, "l", ylab =" ",ylim=c(-0.2,0.2), col="seagreen")
par(new=T)
plot(x2, y8, "l", ylab =" ",ylim=c(-0.2,0.2), col="moccasin")



########################################################################
###     Application of Simulation Algorithm for VaR Calculation      ###
###      Long-Term Simulation                         ###
########################################################################


MCtime<-10000 # Number of simulation runs
```

```
simtime<-11 # Holding period for VaR calculation expressed in days +1
# (i.e. 250 day holding period).
mesh<-1
aggregatedata<-1:(MCtime*numstocks)
aggregatematrix<-matrix(data=aggregatedata,
nrow=numstocks,ncol=MCtime,byrow=TRUE)


for (k in 1:MCtime){

# Simulation of random increments ~ N(0,1) for d=100 stocks
simulationdata3<-rnorm(numstocks*simtime)
randommat3<-matrix(data=simulationdata3,
nrow=numstocks,ncol=simtime,byrow=TRUE)

pricedat3<-1:(numstocks*simtime)
pricemat3<-matrix(data=pricedat3,
nrow=numstocks,ncol=simtime,byrow=TRUE)

returnmat3<-pricemat3

for(i in 1:numstocks){
pricemat3[i,1]<-1
returnmat3[i,1]<-0
}

# Initial setup of volatility vectors for simulation.
# First volatility applied is the newest (last) GARCH estimator.

voladat<-1:(numstocks*simtime)
volamat<-matrix(data=voladat,nrow=numstocks,ncol=simtime, byrow=TRUE)

for(i in 1:numstocks){
volamat[i,1]<-GARCHts[i,length(GARCHts[1,])]
}

for (j in 1:(simtime-1)){

# Setup of stochastic volatility matrix for simulation.
```

```
tm1<-volamat[,j]%*%t(volamat[,j])
tm2<-cormat*tm1
xi<-t(chol(tm2))

alpha<-median(volamat[,j])


for (i in 1:numstocks){

tempret<-(1/pricemat3[i,j])*(alpha*volamat[i,j]-volamat[i,j]^2)
*(1/mesh)+(1/sqrt(pricemat3[i,j]))*(xi[i,]%*%randommat3[,j])
returnmat3[i,j+1]<-sign(tempret)*min(0.2,abs(tempret))
# setting an absolute ceiling of 20% for daily logreturns
pricemat3[i,j+1]<-pricemat3[i,j]*exp(returnmat3[i,j+1])

a1<-0.1
beta<-0.85

stdvola<-stdvoltable[i]
h<-stdvola^2
pers<-a1+beta
a0<-h*(1-pers)

# Calculating the next GARCH(1,1) volatility
# based on the simulated log-return

volamat[i,j+1]<-sqrt(a0+a1*returnmat3[i,j]^2+beta*volamat[i,j]^2)

}

}

for(j in 1:numstocks){
aggregatematrix[j,k]<-sum(returnmat3[j,])
}

}

# Application of simulated 10 day returns to different portfolios
```

```
initialprices<-rchisq(100,1)

ipm<-t(matrix(data=initialprices,nrow=1,byrow=TRUE))


initialdivweight<-diversityweights(ipm,1)
divreturns<-1:length(aggregatematrix[1,])

initialentweight<-calcentropyweights(ipm,1)
entreturns<-1:length(aggregatematrix[1,])

initialmktweight<-calcmktweights(ipm,1)
mktreturns<-1:length(aggregatematrix[1,])

equalweights<-1:numstocks
for (i in 1:numstocks){
equalweights[i]<-1/numstocks
}
equalreturns<-1:length(aggregatematrix[1,])

for (i in 1:MCtime){
divreturns[i]<-t(initialdivweight)%*%aggregatematrix[,i]
entreturns[i]<-t(initialentweight)%*%aggregatematrix[,i]
mktreturns[i]<-t(initialmktweight)%*%aggregatematrix[,i]
equalreturns[i]<-t(equalweights)%*%aggregatematrix[,i]
}



divresults<-c(mean(divreturns),sqrt(var(divreturns)),
quantile(divreturns,0.01),quantile(divreturns,0.05),
quantile(divreturns,0.1))

entresults<-c(mean(entreturns),sqrt(var(entreturns)),
quantile(entreturns,0.01),quantile(entreturns,0.05),
quantile(entreturns,0.1))

mktresults<-c(mean(mktreturns),sqrt(var(mktreturns)),
```

```
quantile(mktreturns,0.01),quantile(mktreturns,0.05),
quantile(mktreturns,0.1))

equalresults<-c(mean(equalreturns),sqrt(var(equalreturns)),
quantile(equalreturns,0.01),quantile(equalreturns,0.05),
quantile(equalreturns,0.1))



hist(divreturns, col="navyblue", xlab="10-day returns",
main="Histogram of Diversity-Weighted Portfolio") #, nclass=20)
par(new=T)
abline(v=quantile(divreturns,0.01), col="indianred")
par(new=F)

hist(entreturns, col="seagreen", xlab="10-day returns",
main="Histogram of Entropy-Weighted Portfolio") #, nclass=20)
par(new=T)
abline(v=quantile(entreturns,0.01), col="indianred")
par(new=F)

hist(mktreturns, col="steelblue1", xlab="10-day returns", #
main="Histogram of Market Portfolio") # , nclass=20)
par(new=T)
abline(v=quantile(mktreturns,0.01), col="indianred")
par(new=F)

hist(equalreturns, col="lavender", xlab="10-day returns",
main="Histogram of Equally-Weighted Portfolio") #, nclass=20)
par(new=T)
abline(v=quantile(equalreturns,0.01), col="indianred")
par(new=F)


########################################################################
###     Application of Simulation Algorithm for VaR Calculation
###      Correlated / Uncorrelated Simulation
########################################################################
```

```
# For initial run evaluate body of masterloop
# individually to initiate output data set.

masternum<-100

for (mastercount in 1:masternum){

MCtime<-1 # Number of simulation runs
simtime<-500 # Holding period for benchmark calculation +1.
mesh<-1
aggregatedata<-1:(MCtime*numstocks)
aggregatematrix<-matrix(data=aggregatedata,
nrow=numstocks,ncol=MCtime,byrow=TRUE)
aggregatematrixuncor<-aggregatematrix

for (k in 1:MCtime){

# Simulation of random increments ~ N(0,1) for d=100 stocks
simulationdata3<-rnorm(numstocks*simtime)
randommat3<-matrix(data=simulationdata3,
nrow=numstocks,ncol=simtime,byrow=TRUE)

pricedat3<-1:(numstocks*simtime)
pricemat3<-matrix(data=pricedat3,
nrow=numstocks,ncol=simtime,byrow=TRUE)

returnmat3<-pricemat3
returnmatuncor<-pricemat3
pricematuncor<-pricemat3


for(i in 1:numstocks){
pricemat3[i,1]<-1
returnmat3[i,1]<-0
pricematuncor[i,1]<-1
returnmatuncor[i,1]<-0
}

# Initial setup of volatility vectors for simulation.
```

```
# First volatility applied is the newest (last) GARCH estimator.

voladat<-1:(numstocks*simtime)
volamat<-matrix(data=voladat,nrow=numstocks,ncol=simtime, byrow=TRUE)

for(i in 1:numstocks){
volamat[i,1]<-GARCHts[i,length(GARCHts[1,])]
}

for (j in 1:(simtime-1)){

tm1<-volamat[,j]%*%t(volamat[,j])
tm2<-cormat*tm1
tm3<-cormat2*tm1
xi<-t(chol(tm2))
xi2<-t(chol(tm3))

alpha<-median(volamat[,j])

# alpha<-max(median(volamat[,j]),mean(volamat[,j]))


for (i in 1:numstocks){

tempret<-(1/pricemat3[i,j])*(alpha*volamat[i,j]-volamat[i,j]^2)
*(1/mesh)+(1/sqrt(pricemat3[i,j]))*(xi[i,]%*%randommat3[,j])
returnmat3[i,j+1]<-max(-0.1,tempret)
# setting an absolute floor of -10% for daily logreturns
pricemat3[i,j+1]<-pricemat3[i,j]*exp(returnmat3[i,j+1])

tempret2<-(1/pricematuncor[i,j])*(alpha*volamat[i,j]-volamat[i,j]^2)*
(1/mesh)+(1/sqrt(pricematuncor[i,j]))*(xi2[i,]%*%randommat3[,j])
returnmatuncor[i,j+1]<-sign(tempret2)*min(0.1,abs(tempret2))
# setting an absolute ceiling of 10% for daily logreturns
pricematuncor[i,j+1]<-pricematuncor[i,j]*exp(returnmatuncor[i,j+1])

a1<-0.1
beta<-0.85
```

```
stdvola<-stdvoltable[i]
h<-stdvola^2
pers<-a1+beta
a0<-h*(1-pers)

# Calculating the next GARCH(1,1) volatility
# based on the simulated log-return

volamat[i,j+1]<-sqrt(a0+a1*returnmat3[i,j]^2+beta*volamat[i,j]^2)

}

}

for(j in 1:numstocks){
aggregatematrix[j,k]<-sum(returnmat3[j,])
aggregatematrixuncor[j,k]<-sum(returnmatuncor[j,])
}

}



#########################################################################
# For the long-term benchmark simulation we
# calculate the initial pf-weights (correlated and uncorrelated)


initialmktweight<-calcmktweights(pricemat3,1)
mktreturns<-1:length(aggregatematrix[1,])
mktweightdat<-initialmktweight
for (j in 2:length(pricemat3[1,])){
temp<-calcmktweights(pricemat3,j)
mktweightdat<-append(mktweightdat,temp,after=length(mktweightdat))
}

mktweightmat<-t(matrix(data=mktweightdat,
 nrow=length(pricemat3[1,]),ncol=numstocks,byrow=TRUE))
```

```
mktpf<-1:length(pricemat3[1,])
for(j in 1:length(pricemat3[1,])){
mktpf[j]<-mktweightmat[,j]%*%pricemat3[,j]
}

initialmktweightuncor<-calcmktweights(pricematuncor,1)
mktreturnsuncor<-1:length(aggregatematrix[1,])
mktweightdatuncor<-initialmktweightuncor
for (j in 2:length(pricematuncor[1,])){
temp<-calcmktweights(pricematuncor,j)
mktweightdatuncor<-append(mktweightdatuncor,
 temp,after=length(mktweightdatuncor))
}

mktweightmatuncor<-t(matrix(data=mktweightdatuncor,
 nrow=length(pricematuncor[1,]),ncol=numstocks,byrow=TRUE))

mktpfuncor<-1:length(pricematuncor[1,])
for(j in 1:length(pricematuncor[1,])){
mktpfuncor[j]<-mktweightmatuncor[,j]%*%pricematuncor[,j]
}



initialdivweight<-diversityweights(pricemat3,1)
divreturns<-1:length(aggregatematrix[1,])
divweightdat<-initialdivweight
for (j in 2:length(pricemat3[1,])){
temp<-diversityweights(pricemat3,j)
divweightdat<-append(divweightdat,temp,after=length(divweightdat))
}
divweightmat<-t(matrix(data=divweightdat,
 nrow=length(pricemat3[1,]),ncol=numstocks,byrow=TRUE))
divpf<-1:length(pricemat3[1,])
for(j in 1:length(pricemat3[1,])){
divpf[j]<-divweightmat[,j]%*%pricemat3[,j]
}

initialdivweightuncor<-diversityweights(pricematuncor,1)
```

```
divreturnsuncor<-1:length(aggregatematrix[1,])
divweightdatuncor<-initialdivweightuncor
for (j in 2:length(pricematuncor[1,])){
temp<-diversityweights(pricematuncor,j)
divweightdatuncor<-append(divweightdatuncor,
 temp,after=length(divweightdatuncor))
}
divweightmatuncor<-t(matrix(data=divweightdatuncor,
 nrow=length(pricematuncor[1,]),ncol=numstocks,byrow=TRUE))
divpfuncor<-1:length(pricematuncor[1,])
for(j in 1:length(pricematuncor[1,])){
divpfuncor[j]<-divweightmatuncor[,j]%*%pricematuncor[,j]
}

initialentweight<-calcentropyweights(pricemat3,1)
entreturns<-1:length(aggregatematrix[1,])
entweightdat<-initialentweight
for (j in 2:length(pricemat3[1,])){
temp<-calcentropyweights(pricemat3,j)
entweightdat<-append(entweightdat,temp,after=length(entweightdat))
}
entweightmat<-t(matrix(data=entweightdat,
 nrow=length(pricemat3[1,]),ncol=numstocks,byrow=TRUE))
entpf<-1:length(pricemat3[1,])
for(j in 1:length(pricemat3[1,])){
entpf[j]<-entweightmat[,j]%*%pricemat3[,j]
}

initialentweightuncor<-calcentropyweights(pricematuncor,1)
entreturnsuncor<-1:length(aggregatematrix[1,])
entweightdatuncor<-initialentweightuncor
for (j in 2:length(pricematuncor[1,])){
temp<-calcentropyweights(pricematuncor,j)
entweightdatuncor<-append(entweightdatuncor,
 temp,after=length(entweightdatuncor))
}
entweightmatuncor<-t(matrix(data=entweightdatuncor,
 nrow=length(pricematuncor[1,]),ncol=numstocks,byrow=TRUE))
entpfuncor<-1:length(pricematuncor[1,])
```

```
for(j in 1:length(pricematuncor[1,])){
entpfuncor[j]<-entweightmatuncor[,j]%*%pricematuncor[,j]
}



equalweights<-1:numstocks
for (i in 1:numstocks){
equalweights[i]<-1/numstocks
}
equalpf<-1:length(pricemat3[1,])
for(j in 1:length(pricemat3[1,])){
equalpf[j]<-equalweights%*%pricemat3[,j]
}

equalweightsuncor<-1:numstocks
for (i in 1:numstocks){
equalweightsuncor[i]<-1/numstocks
}
equalpfuncor<-1:length(pricematuncor[1,])
for(j in 1:length(pricematuncor[1,])){
equalpfuncor[j]<-equalweightsuncor%*%pricematuncor[,j]
}

mktdatcor<-append(mktdatcor,mktpf,after=length(mktdatcor))
divdatcor<-append(divdatcor,divpf,after=length(divdatcor))
entdatcor<-append(entdatcor,entpf,after=length(entdatcor))
equdatcor<-append(equdatcor,equalpf,after=length(equdatcor))

mktdatuncor<-append(mktdatuncor,
 mktpfuncor,after=length(mktdatuncor))
divdatuncor<-append(divdatuncor,
 divpfuncor,after=length(divdatuncor))
entdatuncor<-append(entdatuncor,
 entpfuncor,after=length(entdatuncor))
equdatuncor<-append(equdatuncor,
 equalpfuncor,after=length(equdatuncor))

}
```

```
##############################################################
### Comparison between correlated and uncorrelated model  ###

mktdatcor<-mktpf
# Utilize the time series of one isolated run of correlated calc.
divdatcor<-divpf
entdatcor<-entpf
equdatcor<-equalpf


mktdatuncor<-mktpfuncor
# Utilize the time series of one isolated run of uncorrelated calc.
divdatuncor<-divpfuncor
entdatuncor<-entpfuncor
equdatuncor<-equalpfuncor

# Visualization of Results

#Market portfolio
mktmatcor<-matrix(data=mktdatcor,
 nrow<-masternum,ncol<-simtime,byrow=TRUE)
mktmatuncor<-matrix(data=mktdatuncor,
 nrow<-masternum,ncol<-simtime,byrow=TRUE)

xvec<-1:simtime
temp<-colors()

par(new=F)
for (j in 1:masternum){
index<-sample.int(length(temp),
 size = 1, replace = FALSE, prob = NULL)
par(new=T)
plot(xvec,mktmatcor[j,],ylim=c(0,3),
 xlab="Simulation Time",ylab="Simulated Market Portfolio",
 main="Correlated Model",col=temp[index],"l")
}

par(new=F)
```

```
for (j in 1:masternum){
index<-sample.int(length(temp),
 size = 1, replace = FALSE, prob = NULL)
par(new=T)
plot(xvec,mktmatuncor[j,],ylim=c(0.5,2),
 xlab="Simulation Time",ylab="Simulated Market Portfolio",
 main="Uncorrelated Model",col=temp[index],"l")
}

# Diversity-weighted portfolio
divmatcor<-matrix(data=divdatcor,
 nrow<-masternum,ncol<-simtime,byrow=TRUE)
divmatuncor<-matrix(data=divdatuncor,
 nrow<-masternum,ncol<-simtime,byrow=TRUE)


par(new=F)
for (j in 1:masternum){
index<-sample.int(length(temp),
 size = 1, replace = FALSE, prob = NULL)
par(new=T)
plot(xvec,divmatcor[j,],ylim=c(0,3),
 xlab="Simulation Time",ylab="Simulated Diversity Portfolio",
 main="Correlated Model",col=temp[index],"l")
}

par(new=F)
for (j in 1:masternum){
index<-sample.int(length(temp),
 size = 1, replace = FALSE, prob = NULL)
par(new=T)
plot(xvec,divmatuncor[j,],ylim=c(0.5,2),
 xlab="Simulation Time",ylab="Simulated Diversity Portfolio",
 main="Uncorrelated Model",col=temp[index],"l")
}

# Entropy-weighted portfolio
entmatcor<-matrix(data=entdatcor,nrow<-masternum,ncol<-simtime,byrow=TRUE)
entmatuncor<-matrix(data=entdatuncor,
```

```
 nrow<-masternum,ncol<-simtime,byrow=TRUE)


par(new=F)
for (j in 1:masternum){
index<-sample.int(length(temp), size = 1,
 replace = FALSE, prob = NULL)
par(new=T)
plot(xvec,entmatcor[j,],ylim=c(0,3),xlab="Simulation Time",
 ylab="Simulated Entropy Portfolio",
 main="Correlated Model",col=temp[index],"l")
}

par(new=F)
for (j in 1:masternum){
index<-sample.int(length(temp),
 size = 1, replace = FALSE, prob = NULL)
par(new=T)
plot(xvec,entmatuncor[j,],ylim=c(0.5,2),
 xlab="Simulation Time",ylab="Simulated Entropy Portfolio",
 main="Uncorrelated Model",col=temp[index],"l")
}

# Equally-weighted portfolio
equmatcor<-matrix(data=equdatcor,
 nrow<-masternum,ncol<-simtime,byrow=TRUE)
equmatuncor<-matrix(data=equdatuncor,
 nrow<-masternum,ncol<-simtime,byrow=TRUE)


par(new=F)
for (j in 1:masternum){
index<-sample.int(length(temp), size = 1,
 replace = FALSE, prob = NULL)
par(new=T)
plot(xvec,equmatcor[j,],ylim=c(0,3),
 xlab="Simulation Time",ylab="Simulated Equally-Weighted Portfolio",
 main="Correlated Model",col=temp[index],"l")
}
```

```
par(new=F)
for (j in 1:masternum){
index<-sample.int(length(temp),
 size = 1, replace = FALSE, prob = NULL)
par(new=T)
plot(xvec,equmatuncor[j,],ylim=c(0.5,1.5),
 xlab="Simulation Time",ylab="Simulated Equally-Weighted Portfolio",
 main="Uncorrelated Model",col=temp[index],"l")
}
```

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## Curriculum Vitae

### Personal Data

Name: Florian Clemens Leisch
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### Education

- 1992 - 2000: Akademisches Gymnasium Graz, high school education.
- 06.06.2000: Matura (A-Levels) with Honours at the Akademisches Gymnasium Graz.
- 02.10.2000 - 01.10.2001: Military Service at the Austrian Army.
- 01.10.2001 - 19.06.2006: University studies in Technical Mathematics (specialization on financial and actuary mathematics) at Graz University of Technology (TU Graz). Awarded academic degree: "Diplom-Ingenieur (DI)". Third diploma examination passed with Honors on 19.06.2006. The diploma thesis "Valuation of Mortgage-Backed Securities" was supervised by o.Univ.-Prof. Dr. Robert F. Tichy and Univ.-Prof. Dr. Hansjörg Albrecher (now HEC Lausanne).
- Since 01.10.2006: PhD-studies in mathematics (avocational) at Vienna University of Technology (TU Wien). PhD supervisor: Prof. Dr. Josef Teichmann (Eidgenössische Technische Hochschule (ETH) Zürich), Co-supervisor: o.Univ.-Prof. Dr. Walter Schachermayer (University of Vienna).
- 01.02.2011 - 31.07.2011: Research visit at the Department of Mathematics, ETH Zürich).

### Professional Career

- Since 01.01.2009 employee at Oesterreichische Nationalbank (OeNB), On-Site Banking Inspections Division - Large Banks.
  From 01.01.2009 - 13.10.2010 as Banking Examiner, since 01.11.2010 as Senior Examiner.
- From 01.09.2006 to 31.12.2008: Risk Analyst at Raiffeisen Capital Management (RCM), Vienna, Department of Risk & Performance Analysis.

- 2003 - 2005: Undergraduate research assistant at the Institute of Analysis and Computational Number Theory and at the Institute of Mathematical Structure Theory, Graz University of Technology.
- Internships: Bankhaus Krentschker & Co. (1998, 2000), Roche Diagnostics Ltd. (2002), Raiffeisen Capital Management (2004, 2005).

**Lecturing Activities**

- Lecture on liquidity risk from a regulator's point of view, held at the conference "Auswirkungen von CRR I & CRD IV Regelungen auf die Liquiditätssteuerung", hosted by the Institute for International Research, Vienna, 14.11.2012.
- Lecture on liquidity risk in on-site supervision, 3rd Liquidity Workshop, hosted by OeNB and the Financial Market Authority, Vienna, 07.08.2012.
- Lecture on banking supervision in Austria, Joint Vienna Institute, 11.07.2012.
- Lecture on banking supervision and related quantitive issues, Vienna University of Technology, 01.03.2010 and 01.03.2012.
- Lecturer of the exercise classes for the lecture "Advances in Stochastic Portfolio Theory", held by Prof. Dr. Josef Teichmann and Prof. Dr. Johannes Muhle-Karbe, ETH Zürich, Spring Semester 2011.
- Lecture on "Stress Testing - Supervisory Expectations"; held at the seminar "Stress Testing and Risk Management Techniques" hosted by the Financial Stability Institute and the Slovenian Central Bank, Ljubljana, 02.06.2010.

**Further Qualifications:**

- Financial Risk Manager (FRM), certified by the Global Association of Risk Professionals on 15.04.2010.
- Author of "Mortgage-Backed Securities aus bewertungsmethodischer Sicht" in: Funk M., Bienert S., Hrsg.: Immobilienbewertung Österreich; ÖVI Verlag Wien, 2007, 2nd edition published in 2009.
- Honorary award of the Austrian Association of Actuaries (AVÖ) 2006 for diploma thesis "Valuation of Mortgage-Backed Securities". The diploma thesis was partly published in the journal of the Austrian Association of Actuaries (2008).

Vienna, $11^{th}$ of March, 2013 Florian Clemens Leisch