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Real Rational Conchoids of Curves and Surfaces

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Zusammenfassung

Diese Arbeit beschäftigt sich mit der Konstruktion von Konchoiden und deren Eigenschaften. Ursprünglich handelt es sich dabei um eine Kurvenkonstruktion, die auf *Nikomedes* (280 v. Chr. – 210 v. Chr.) zurückgeführt wird. Er entwickelte die *Konchoide des Nikomedes*, das ist die Konchoide ausgehend von einer Geraden, unter anderem zur Dreiteilung des Winkels. Die Konstruktion kann relativ grob, wie folgt beschrieben werden: Von einer gegebenen Kurve wird eine Abstandskurve konstruiert, wobei der Abstand bezüglich eines Referenzpunkts gemessen wird.

Diese Konstruktion lässt sich auch direkt auf Flächen erweitern. Die Konchoide F_d zu einer gegebenen Fläche F im Abstand d in Bezug auf den Referenzpunkt O ist der Zariski-Abschluss der Punktmenge

$$F_d = \{Q \in OP \text{ mit } P \in F, \text{ und } \overline{QP} = d\}.$$

Konchoiden haben Anwendung zum Beispiel in folgenden Bereichen:

Medizin	der Kopf des Hüftgelenks (Menschik (1997))
Optik	als Kaustiken (Szmulowicz (1996))
Astronomie	Positionsbestimmung (Kerrick (1959))
Akustik	Richtcharackteristiken von Mikrofonen (Streicher and Dooley (2003))
Mechanik	Flüssigkeitsausbreitung (Sultan (2005))
Elektronik	elektromagnetische Felder (Lin et al. (2001))

Hierbei tritt in erster Linie die *Limacon des Pascal*, das sind Konchoiden eines Kreises durch den Referenzpunkt, auf.

Das Hauptaugenmerk der vorliegenden Arbeit liegt im Erstellen von rationalen Parametrisierungen von Konchoiden von Flächen. Dazu wird ein Modell vorgestellt bei dem die Flächen im dreidimensionalen Raum, auf Flächen auf einem Kegel in einem vierdimensionalen Raum abgebildet werden. Rationale Parametrisierungen haben vor allem durch den Einsatz von CAD-Systemen eine wichtige Bedeutung in der Geometrie.

Ziel ist es, Klassen von rationalen Flächen zu bestimmen, deren Konchoiden ebenfalls rationale Parametrisierungen besitzen. Das einfachste Beispiel für eine solche Fläche ist die Ebene. Diese besitzt rationale Parametrisierungen mit rationaler Distanz in den

Flächenparametern zum Referenzpunkt. Somit sind auch die Konchoiden der Ebene bezüglich des Referenzpunkts rational parametrisierbar.

Die Arbeit stützt sich im wesentlichen auf drei Publikationen die gemeinsam mit Martin Peternell und Juana Sendra verfasst wurden:

- Peternell, Gruber und Sendra (2011)
- Peternell, Gruber und Sendra (2013)
- Gruber und Peternell (2013)

Die Arbeit ist folgendermaßen gegliedert: Kapitel 1 beinhaltet einige Grundlagen die dem Verständnis der Arbeit dienen. Kapitel 2 widmet sich den Konchoiden der Kurven. Dabei wird das Kegel-Modell präsentiert und die Konchoiden einiger Kurven berechnet. Dieses Kapitel dient in erster Linie dem besseren Verständnis des Kegel-Modells.

Das dritte Kapitel behandelt Konchoiden von Flächen und deren rationale Parameterisierungen. Dazu wird das Kegel-Modell auf Flächen erweitert und daraus resultierende Parametrisierungsmöglichkeiten vorgestellt. Mit Hilfe des Kegel-Modells wird die rationale Parametrisierbarkeit von Konchoiden von rationalen Regelflächen und Quadriken gezeigt.

An dieser Stelle möchte ich mich ganz herzlich bei Martin Peternell für die gute Betreuung meiner Dissertation bedanken. Weiters möchte ich mich bei all meinem Kollegen und Kolleginnen für die vielen interessanten und wissenschaftlich fruchtbaren Gespräche bedanken.

Ebenso großer Dank gebührt meiner Freundin Lisa Bauer sowie meiner Familie und all meinen Freunden, für deren Unterstützung und Aufmunterung.

Abstract

This thesis deals with the construction of conchoids and their attributes. The conchoid construction dates back to the ancient Greeks. It was *Nicomedes* (280 bc – 210 bc) who discovered the *Conchoid of Nicomedes*, the conchoid of a line, and used it for example for angle trisection. The construction of the conchoid results, roughly speaking, in a distance curve to a given curve, where the distance is measured with respect to a given reference point.

This construction can be directly extended to surfaces. The conchoid F_d to a given surface F at distance d with respect to O is the Zariski closure of the set of points Q,

$$F_d = \{Q \in OP \text{ with } P \in F, \text{ and } \overline{QP} = d\}.$$

There are several applications of conchoids, for example in the following fields:

Medicine - hip joint (Menschik (1997))
Optics - caustics (Szmulowicz (1996))
Astronomy - astronomic positions (Kerrick (1959))
Acoustics - polar patterns of microphones (Streicher and Dooley (2003))
Mechanics - fluid processing (Sultan (2005))
Electronics - electromagnetic fields (Lin et al. (2001))

In most cases the *Limacon of Pascal*, a conchoid to a circle going through the reference point, appears.

The focus of this thesis is on rational parameterizations of conchoid surfaces. Mainly because of representations of geometric objects in CAD systems, such rational parameterizations are of scientific interest. A model to find such parameterizations is presented. The surfaces of the three-dimensional space are mapped to surfaces on a cone in a four-dimensional space.

The aim is to determine which rational surfaces have conchoids, admitting rational parameterizations. A simple example is the plane, it possesses a rational parameterization with rational distance in the surface parameters to the reference point. Hence the conchoid surfaces of the plane also admit rational parameterizations.

The structure of the thesis is the following: Chapter 1 gives an introduction to projective geometry and to the conchoid construction, for better understanding of the thesis. Chapter 2 introduces the cone model for the calculation of rational parameterizations of curves and their conchoids. This chapter primarily motivates the idea of the cone model for the surface case.

Chapter 3 covers the rational parameterizations of surfaces and their conchoids. The cone model of Chapter 2 is extended to the three-dimensional space and we prove that rational ruled surfaces and quadrics and their conchoids posses rational parameterizations.

The thesis is mainly based on three articles written together with the supervisor of this thesis, Martin Peternell and Juana Sendra:

- Peternell, Gruber and Sendra (2011)
- Peternell, Gruber and Sendra (2013)
- Gruber and Peternell (2013)

Here I want to thank Martin Peternell for the good supervision of my thesis. I would also like to thank all my colleagues at the institute for many interesting and scientifically productive discussions.

I also want to thank my girlfriend Lisa Bauer, my family and all my friends for their support and encouragement.

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Chapter 1

Fundamentals

In this chapter we give some necessary background information. In Section 1.1 we present basic ideas of *projective geometry* and in Section 1.2 of *algebraic geometry*. Section 1.3 contains the definition of *conchoids* and their attributes.

1.1 Projective Geometry

We start with some information on projective geometry, in particular we will deal with the projective 3−space which we denote as $\mathbb{P}^3$ and its automorphic mappings. For more information about projective geometry see for example Pottmann and Wallner (2001).

1.1.1 The Projective Space

Given a Euclidean space $\mathbb{R}^3$, we denote a point $X \in \mathbb{R}^3$ with the column vector $\mathbf{x} = (x, y, z)^T$ and the scalar product of two vectors as $\mathbf{x}^T \cdot \mathbf{y}$. The squared Euclidean norm of a vector is $\|\mathbf{x}\|^2 = \mathbf{x}^T \cdot \mathbf{x}$. The product of two matrices $\mathbf{A}, \mathbf{B}$ and matrix times vector are also denoted as $\mathbf{A} \cdot \mathbf{B}$ respectively $\mathbf{A} \cdot \mathbf{x}$.

We extend the Euclidean space $\mathbb{R}^3$ by points U at infinity, these can be defined as common points of parallel lines, and receive the projective space $\mathbb{P}^3 = \mathbb{R}^3 \cup \{U\}$. A point X in projective space $\mathbb{P}^3$ is denoted by the column vector $\mathbf{x}\mathbb{R} = (x_0, x_1, x_2, x_3)^T\mathbb{R}$. These are projective coordinates and they are only unique up to scalar multiplication. We denote planes $E \in \mathbb{P}^3$ with a vector $\mathbb{R}\mathbf{e} = \mathbb{R}(e_0, e_1, e_2, e_3)^T$, to distinguish between point and plane coordinates we place the $\mathbb{R}$ on the right or on the left side of the vector. The incidence relation between a point and a plane in $\mathbb{P}^3$ is given by $X \,\mathrm{I}\, E : \mathbf{x}^T \cdot \mathbf{e} = 0$.

Improper points are points at infinity, they are characterized by $x_0 = 0$. The plane containing all improper points is called the *ideal plane* $\mathbb{R}\omega = \mathbb{R}(1, 0, 0, 0)^T$. For proper points we can convert the projective coordinates to Euclidean coordinates as follows

$$x = \frac{x_1}{x_0},\ y = \frac{x_2}{x_0} \text{ and } z = \frac{x_3}{x_0}.$$

Inverting the roles of points and planes in the projective space $\mathbb{P}^3$ and exchanging connecting points, by intersecting planes, leads to the *dual* projective space $\mathbb{P}^{3\star}$. Every true statement in $\mathbb{P}^3$ for points, translated to the dual space, is a true statement for planes in $\mathbb{P}^{3\star}$. A statement typically differs from its dual statement. If this is not the case, it is called *self-dual*. A line g as a set of points in $\mathbb{P}^3$ is dual to a line $g^\star$ as a sheaf of planes in $\mathbb{P}^{3\star}$.

1.1.2 Projective Mappings

In this section we give a short introduction to linear automorphic transformations in $\mathbb{P}^3$ and to transformations of $\mathbb{P}^3$ to $\mathbb{P}^{3\star}$.

Definition 1.1 *A regular matrix* $\mathbf{A} \in \mathbb{R}^{4\times 4}$ *induces a projective mapping* $\varkappa$ *in* $\mathbb{P}^3$

$$\begin{aligned} \varkappa: \quad \mathbb{P}^3 &\rightarrow \mathbb{P}^3 \\ \mathbf{x}\mathbb{R} &\rightarrow \mathbf{x}'\mathbb{R} = (\mathbf{A}\cdot\mathbf{x})\mathbb{R}. \end{aligned}$$

We call $\varkappa$ *a* projective collineation.

A collineation is a bijective map in $\mathbb{P}^3$ and keeps collinearity of points. Planes are transformed by $\varkappa$ in the following way

$$\mathbb{R}\mathbf{e} \rightarrow \mathbb{R}\mathbf{e}' = \mathbb{R}(\mathbf{A}^{-T}\cdot\mathbf{e}),$$

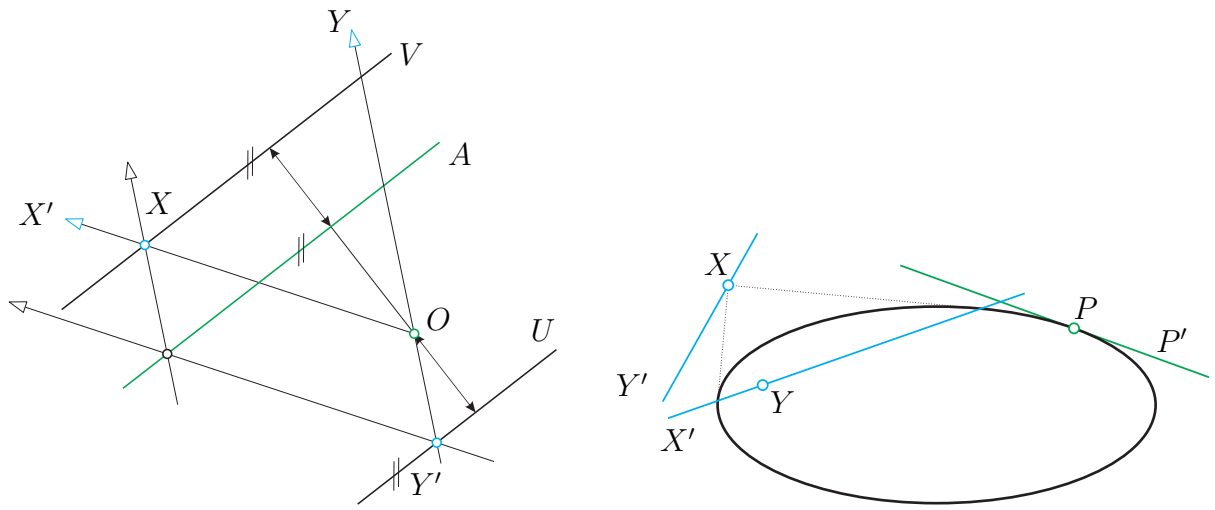
where $\mathbf{A}^{-T}$ is the transposed inverse of the defining matrix $\mathbf{A}$ of $\varkappa$.

We want to determine *fixed points* of a projective collineation $\varkappa$. Being a fixed point can be expressed by $\mathbf{A}\cdot\mathbf{x} = \lambda\mathbf{x}$ and therefore fixed points are determined as the eigenvectors of $\mathbf{A}$. Thus a projective collineation has typically four fixed points, correlated to the four distinct eigenvalues of $\mathbf{A}$.

If the multiplicity of one eigenvalue is three and the according eigenspace is three dimensional, all points of a plane are fixed, we call this plane the *axis plane* A of $\varkappa$. Furthermore $\varkappa$ has another fixed point belonging to the second distinct eigenvalue, the so called *center* of $\varkappa$. Such a projective collineation, with center and axis plane, is called a *perspective collineation*.

In the following we need the special perspective collineation, where the center is the origin O of the coordinate system. Every line g through the fixed point O intersects the axis plane A in another fixed point $P = g \cap A$. Since $\varkappa$ is a collineation the line g containing O is fixed, $g = g'$. Furthermore the image of an arbitrary point X lies on the line connecting X and O. Hence we are able to write the image of the point X under the perspective collineation with center in O as

$$\mathbf{x}'\mathbb{R} = \lambda\mathbf{x} + \mathbf{a}^T\cdot\mathbf{x}(1,0,0,0)^T.$$



(a) Perspective collineation $\varkappa$ with center in O and axis plane A.

(b) Polarity $\varkappa$ with two conjugate points X, Y and the self-conjugate point P.

Figure 1.1: Perspective collineation and polarity.

Points that fulfill $\mathbf{a}^T \cdot \mathbf{x} = 0$ are fixed, hence $\mathbb{R}\mathbf{a} = \mathbb{R}(a_0, \ldots, a_3)^T$ are the coordinates of the axis plane A of the collineation. Points with an improper image are given by the equation

$$\lambda x_0 + \mathbf{a}^T \cdot \mathbf{x} = \mathbf{v}^T \cdot \mathbf{x} = 0, \tag{1.1}$$

hence they are contained in the plane V with coordinates $\mathbb{R}\mathbf{v} = (\lambda + a_0, a_1, a_2, a_3)^T$. For $\lambda = 1$, equation 1.1 can be shortly written as

$$\begin{pmatrix} \mathbf{v}^T \cdot \mathbf{x} \\ \overline{\mathbf{x}} \end{pmatrix} \mathbb{R} = \begin{pmatrix} v_0 & \overline{\mathbf{v}} \\ 0 & \mathbf{I}^3 \end{pmatrix} \mathbf{x}\mathbb{R}, \tag{1.2}$$

with $\overline{\mathbf{v}} = (v_1, v_2, v_3)^T$, $\overline{\mathbf{x}} = (x_1, x_2, x_3)^T$ and $\mathbf{I}_3 = \mathrm{diag}(1, 1, 1)$. *Vanishing points*, these are image points $\mathbf{x}'$ with an improper pre-image $\mathbf{x} = (0, x_1, x_2, x_3)^T\mathbb{R}$, have the coordinates $(v_1x_1 + v_2x_2 + v_3x_3, x_1, x_2, x_3)^T\mathbb{R}$. They lie in the plane $U : \mathbb{R}\mathbf{u} = \mathbb{R}(-1, v_1, v_2, v_3)^T$, the so called *vanishing plane*. See Figure 1.1(a) for an illustration of a perspective collineation, with center O and axis plane A.

In Euclidean coordinates (1.2) reads

$$\mathbf{x}' = \frac{1}{v_0 + v_1x + v_2y + v_3z}\mathbf{x}.$$

1.1.3 Polarities and Quadrics

So far we have dealt with automorphic mappings in $\mathbb{P}^3$. Now we discuss mappings $\varkappa$ from $\mathbb{P}^3$ to its dual space $\mathbb{P}^{3\star}$.

Definition 1.2 *A regular matrix* $\mathbf{A} \in \mathbb{R}^{4\times 4}$ *induces an projective mapping* $\varkappa$ *from* $\mathbb{P}^3$ *to* $\mathbb{P}^{3\star}$

$$\begin{aligned} \varkappa: \quad \mathbb{P}^3 &\rightarrow \mathbb{P}^{3\star} \\ \mathbf{x}\mathbb{R} &\rightarrow \mathbb{R}\mathbf{e}' = \mathbb{R}(\mathbf{A}\cdot\mathbf{x}). \end{aligned}$$

Such mappings are called correlations.

A correlation is a bijective map from $\mathbb{P}^3$ to $\mathbb{P}^{3\star}$ and maps collinear points to a sheaf of planes.

We are interested in special correlations, namely if $\mathbf{A}$ is a symmetric matrix. We call such a correlation a *polarity*, see Figure 1.1(b). The image of a point X under a polarity $\varkappa$ is called the *polar plane* π of X. Two points X, Y are called *conjugate*, if X is contained in the image plane of Y and vice versa, hence

$$\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{y} = 0 \Leftrightarrow \mathbf{y}^T \cdot \mathbf{A} \cdot \mathbf{x} = 0,$$

the equivalence holds since $\mathbf{A}$ is a symmetric matrix.

Definition 1.3 *Given a polarity* $\varkappa : \mathbf{x}\mathbb{R} \rightarrow \mathbb{R}\mathbf{e} = \mathbb{R}(\mathbf{A}\cdot\mathbf{x})$ *from* $\mathbb{P}^3 \rightarrow \mathbb{P}^{3\star}$*, with* $\mathbf{A}$ *symmetric. The self-conjugate points of the induced conjugacy relation form a* quadric *in* $\mathbb{P}^3$*. The points of the quadric* F *are given as the zero-set of the polynomial*

$$F(x_0, \ldots, x_3) : \mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} = 0. \tag{1.3}$$

In the following we will denote both, the quadric and the defining polynomial with F, the meaning should be clear from the context. The polar plane T to a self-conjugate point X, hence $\mathbb{R}\mathbf{t} = \mathbb{R}(\mathbf{A}\cdot\mathbf{x})$ is the *tangent plane* to the quadric.

The pair (X, π) with $\mathbb{R}\pi = \mathbb{R}(\mathbf{A}\cdot\mathbf{x})$ is the polar plane to X, defines a perspective collineation $\varkappa$ with center X and axis plane π. If $\varkappa$ maps one point $Y \notin \pi$ on F to a point $Y' \in F$, every point of F is mapped to a point on F. Therefore such collineations keep the quadric F fixed. Given such a perspective collineation with improper center X and $\pi = \mathbf{A}\cdot\mathbf{x} = \lambda\mathbf{x}$, the plane π is called a *symmetry plane* of F.

In the dual projective space $\mathbb{P}^{3\star}$ the same quadric $F^\star$ to $F : \mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} = 0$, is given as the envelope of tangential planes, in plane coordinates $\mathbb{R}\mathbf{e} = \mathbb{R}(e_0, \ldots, e_3)^T$ it reads

$$F^\star : \mathbf{e}^T \cdot \mathbf{A}^{-1} \cdot \mathbf{e} = 0.$$

In the projective space we distinguish between oval, $\det(\mathbf{A}) < 0$, and ringlike, $\det(\mathbf{A}) > 0$, quadrics. The latter carry real lines. Depending on the intersection of F with the ideal plane we distinguish the affine types, see Table 1.1

Given a paraboloid the ideal plane is a tangent plane, hence paraboloids possess one real improper self-conjugate point. For the other affine types, the pre-image M of the ideal plane, hence $\mathbf{m} = \mathbf{A}^{-1}\omega$, is called the *mid-point* of the quadric. Quadrics that possess a mid-point are called *mid-point quadrics*, these are ellipsoids and hyperboloids.

Affine name of F	projective type	intersection with ω
Ellipsoid	oval	complex conic
Hyperboloid of two sheets	oval	real conic
Elliptic paraboloid	oval	conjugate complex lines
Hyperboloid of one sheet	ringlike	real conic
Hyperbolic paraboloid	ringlike	real lines
Complex quadric	oval	complex conic

Table 1.1: Quadrics.

Equation (1.3) is a quadratic homogeneous equation and it defines a quadratic surface $F \subset \mathbb{P}^3$ even if $\det(\mathbf{A}) = 0$. We call the quadric defined by (1.3) and a singular symmetric matrix $\mathbf{A}$ a *singular quadric*. We distinguish the following cases depending on the rank of $\mathbf{A}$:

- Let $\mathrm{rk}(\mathbf{A}) = 3$, then the surface defined by $F : \mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} = 0$ is a quadratic cone. The vertex of F is defined by the null space of $\mathbf{A}$.
- Let $\mathrm{rk}(\mathbf{A}) = 2$, then the polynomial $F(x_0, \ldots, x_3) = 0$ factorizes into two linear factors, hence F consists of two planes.
- Let $\mathrm{rk}(\mathbf{A}) = 1$, then the polynomial $F(x_0, \ldots, x_3) = 0$ is one squared linear term, hence F is a plane, double covered.

1.1.4 Pencil of Quadrics

Let $F : \mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} = 0$ and $G : \mathbf{x}^T \cdot \mathbf{B} \cdot \mathbf{x} = 0$ be two quadrics in $\mathbb{P}^3$. The linear combination

$$(\lambda F + \mu G) : \mathbf{x}^T \cdot (\lambda \mathbf{A} + \mu \mathbf{B}) \cdot \mathbf{x} = 0$$

with homogeneous parameters $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R} \setminus (0, 0)$, defines a family of quadrics, a so called *pencil of quadrics*.

We want to analyze the singular quadrics in a pencil of quadrics. Let $\det(\lambda \mathbf{A} + \mu \mathbf{B}) \not\equiv 0$, hence not all quadrics are singular. The polynomial $\det(\lambda \mathbf{A} + \mu \mathbf{B})$ is maximum of degree four in $(\lambda : \mu)$, therefore a pencil of quadrics includes at most four, not necessary distinct, singular quadrics. See Figure 3.17 for an illustration of a quadratic pencil with three different singular quadrics, in this example the cone K has to be counted twice.

1.1.5 Focal Conics

In Section 3.3, points with a rotational tangential cone to a quadric are of interest. Therefore we want to derive the locus of these points with methods of projective geometry. For a Euclidean construction see for example McCrea (1960).

The tangential cone Γ of a point P to a quadric F is a quadratic cone. Its vertex is P and a directrix is given by the intersection $\pi \cap F$ of the polar plane π to P, with respect to F, with the quadric F, hence a conic section.

Definition 1.4 *Given a quadric $F : \mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x} = 0 \subset \mathbb{P}^3$ and a point $P \in \mathbb{P}^3$. We call P a* focal point *of F iff the tangential cone Γ with vertex in P to F has rotational symmetry.*

We will construct the locus of focal points at first for mid-point quadrics, these are ellipsoids and hyperboloids. Since the construction for paraboloids follows analogous lines we only indicate the main results afterwards.

Let us assume we have given the mid-point quadric as a set of points in normal form, hence O is the mid- point and the coordinate axes coincide with the axes of the quadric,

$$F : x_0^2 + \frac{x_1^2}{a} + \frac{x_2^2}{b} + \frac{x_3^2}{c} = 0 \text{ with } a, b, c \in \mathbb{R}\backslash\{0\} \text{ and } a \leq b \leq c, . \tag{1.4}$$

Depending on the sign of a, b, c we get all affine mid-point quadric types, see Table 1.2.

$a \leq b \leq c < 0$	Ellipsoid
$a < 0 < b \leq c$	Hyperboloid of two sheets
$a \leq b < 0 < c$	Hyperboloid of one sheet
$0 < a \leq b \leq c$	Complex quadric

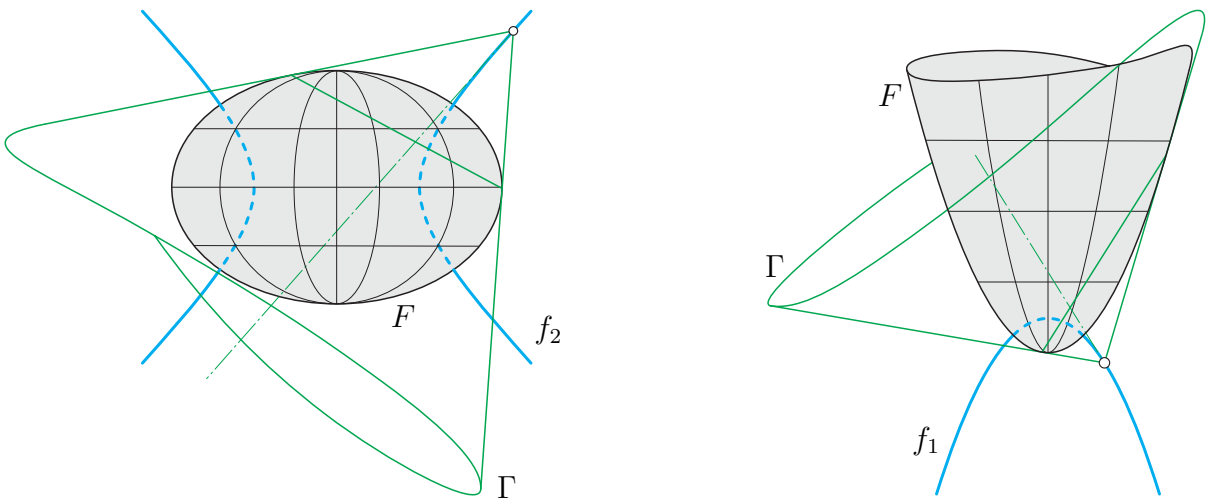
Table 1.2: Mid-point quadrics.

The calculation of the tangential cone simplifies, if we describe the quadric as envelope of planes. The dual representation of the quadric (1.4) reads

$$F^\star : e_0^2 + ae_1^2 + be_2^2 + ce_3^2 = 0, \text{ with } a \leq b \leq c. \tag{1.5}$$

The tangential cone through a point $P \notin F : \mathbf{p}\mathbb{R} = (p_0, \ldots, p_3)^T\mathbb{R}$ is the envelope of the set of tangential planes ε of F through P. We substitute the incidence relation $\mathbf{p}^T \cdot \mathbf{e} = 0$ in (1.5), and obtain the tangential cone Γ in the dual space

$$\Gamma^\star : \mathbf{e}^T \cdot \mathbf{B} \cdot \mathbf{e} = \mathbf{e}^T \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p_1^2 + p_0^2 a & p_1 p_2 & p_1 p_3 \\ 0 & p_1 p_2 & p_2^2 + p_0^2 b & p_2 p_3 \\ 0 & p_1 p_3 & p_2 p_3 & p_3^2 + p_0^2 c \end{pmatrix} \cdot \mathbf{e} = 0. \tag{1.6}$$



(a) Ellipsoid with focal hyperbola.

(b) Ell. paraboloid with focal parabola.

Figure 1.2: Focal conics of quadrics and a rotational tangential cone.

For $P \notin F$, $\mathrm{rk}(\mathbf{B}) = 3$ holds, and equation (1.6) defines a quadratic cone that can be written with $\overline{\mathbf{p}} = (0, p_1, p_2, p_3)^T \mathbb{R}$ as

$$\Gamma^\star : \mathbf{e}^T \cdot (\overline{\mathbf{p}} \cdot \overline{\mathbf{p}}^T + p_0^2 \,\mathrm{diag}(0, a, b, c)) \cdot \mathbf{e} = 0.$$

An arbitrary quadratic cone Γ has three symmetry planes $\tau_i,\ i = \{1, \ldots, 3\}$, given by the eigenvectors of B. Two symmetry planes contain the axis of the cone and one orthogonal to the axis through the vertex. The symmetry planes have poles $T_i,\ i = \{1, \ldots, 3\}$ on the normal to τ_i through the vertex P, hence $\mathbf{B}\tau_i = \lambda\tau_i,\ \lambda \neq 0$, holds and therefore the symmetry planes are given through the eigenvectors of $\mathbf{B}$.

The characteristic polynom $G(\lambda)$ of $\mathbf{B}$,

$$\begin{aligned} G(\lambda) = \ & \lambda((bp_0^2 - \lambda)(cp_0^2 - \lambda)p_1^2 + (ap_0^2 - \lambda)(cp_0^2 - \lambda)p_2^2 + \\ & (ap_0^2 - \lambda)(bp_0^2 - \lambda)p_3^2 + (ap_0^2 - \lambda)(bp_0^2 - \lambda)(cp_0^2 - \lambda)), \end{aligned} \tag{1.7}$$

is a polynomial of degree four in λ with three relevant zeros, since we have chosen $\lambda \neq 0$. Initially we assume that $p_i \neq 0,\ i = \{0, \ldots, 3\}$, and substitute λ as $p_0^2 a,\ p_0^2 b,\ p_0^2 c$ and ∞ in (1.7) which leads to alternating signs of G, therefore the polynomial has besides $\lambda = 0$ three different zeros. To achieve a rotational cone Γ, its defining matrix $\mathbf{B}$ has to have a two dimensional eigenspace hence a double eigenvalue.

Remain the cases where at least one coordinate $p_i,\ i = \{0, \ldots, 3\}$, equals zero. Substituting $p_0 = 0$ in (1.7) leads to

$$\lambda^3(p_1^2 + p_2^2 + p_3^2) = 0$$

and therefore only to the excluded zero $\lambda = 0$.

Substituting $p_1 = 0$ in (1.7) leads to

$$G(\lambda) = (ap_0^2 - \lambda)((cp_0^2 - \lambda)p_2^2 + (bp_0^2 - \lambda)p_3^2 + (bp_0^2 - \lambda)(cp_0^2 - \lambda)).$$

It has a multiple zero if $\lambda = ap_0^2$ is a zero of the second factor, hence

$$p_0^2((c-a)p_2^2 + (b-a)p_3^2 + (b-a)(c-a)p_0^2) = 0$$

which defines a complex conic section f_1, that is given in affine coordinates as

$$f_1 : x = 0 \cap \frac{y^2}{b-a} + \frac{z^2}{c-a} = -1.$$

In the same way, substituting $p_2 = 0$ and $p_3 = 0$ in (1.7) leads to a hyperbola f_2 and an ellipse f_3, in affine coordinates these are given as

$$f_2 : y = 0 \cap \tfrac{x^2}{b-a} - \tfrac{z^2}{c-b} = 1 \text{ and } f_3 : z = 0 \cap \tfrac{x^2}{c-a} + \tfrac{y^2}{c-b} = 1.$$

See Figure 1.2(a) for an ellipsoid with one focal conic f_2. If the quadric has rotation symmetry, for example $a = b$, the hyperbola f_2 degenerates to the $z-$axis, the axis of rotation, and f_3 is a circle.

For completeness we also present the solutions for paraboloids. A paraboloid in normal form is given as

$$F : x_0^2 + \frac{x_1^2}{a} + \frac{x_2^2}{b} + 2\frac{x_0 x_3}{c} = 0, \text{ with } a \leq b.$$

We distinct the following affine types. The quadric is a *hyperbolic paraboloid* if $a < 0 < b$, else an *elliptic paraboloid*. We again use the dual representation

$$F^\star : 2ce_0e_3 + ae_1^2 + be_2^2 - ce_3^2 = 0$$

to find rotational tangential cones. This leads to the focal parabolas

$$f_1 : x = 0 \cap \tfrac{y^2}{(a+c^2)(b-a)} + \tfrac{z}{2c(a+c^2)} = -\tfrac{1}{c^2} \text{ and } f_2 : y = 0 \cap \tfrac{x^2}{(b+c^2)(b-a)} - \tfrac{z}{2c(b+c^2)} = \tfrac{1}{c^2}.$$

Figure 1.2(b) shows an elliptic paraboloid with one focal parabola.

1.2 Algebraic Geometry

In this section we give a short introduction to algebraic geometry. Since we are interested in real rational parameterizations of real curves and real surfaces in $\mathbb{R}^n$ and $\mathbb{P}^n$ we restrict the definitions in this section to real affine and projective spaces. For more general definitions and detailed information about algebraic geometry we refer to Cox et al. (2010), Griffiths and Harris (1978) and Sendra et al. (2008).

1.2.1 Affine Variety

Given the affine space $\mathbb{R}^n$ and polynomials $F_i(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]$.

Definition 1.5 *The set of points*

$$\mathbf{V}(F_1, \ldots, F_s) = \{(x_1, \ldots, x_n)^T \in \mathbb{R}^n : F_i(x_1, \ldots, x_n) = 0, \forall i \in \{1, \ldots, s\}\}$$

is called an affine variety *in* $\mathbb{R}^n$.

In the following we will deal with hypervarieties, these are varieties defined by one polynomial, hence $s = 1$, in the Euclidean plane $\mathbb{R}^2$ and in the Euclidean space $\mathbb{R}^3$.

- A variety $\mathbf{V}(F) \subset \mathbb{R}^2$ defined by the zero-set of one polynomial $F(x, y) = 0$ is called a *curve*.
- A variety $\mathbf{V}(F) \subset \mathbb{R}^3$ defined by the zero-set of one polynomial $F(x, y, z) = 0$ is called a *surface*.

We extend the definition of a hypervariety to the projective space $\mathbb{P}^n$ and give

Definition 1.6 *The set of points*

$$\mathbf{V}(F) = \{(x_0, \ldots, x_n)^T \in \mathbb{P}^n : F(x_0, \ldots, x_n) = 0\}$$

is called a projective hypervariety *in* $\mathbb{P}^n$.

In the latter we denote both, the hypervariety and its defining polynomial with the same letter, e.g. F.

An arbitrary set of points is not necessarily a variety. Therefore we give

Definition 1.7 *Given a set of points* $\mathcal{P} \subset \mathbb{R}^n$*, the smallest variety* $\mathbf{V}(F)$ *containing* $\mathcal{P}$ *is called the* Zariski closure *of* $\mathcal{P}$.

1.2.2 Parameterization

Given a hypervariety $\mathbf{V}(F) \subset \mathbb{R}^n$, the map

$$\begin{aligned} \mathbf{f}: \quad \mathbb{R}^m &\to \mathbb{R}^n \\ (u_1, \ldots, u_m) &\to \mathbf{f}(u_1, \ldots, u_m) = (p_0(u_1, \ldots, u_m), \ldots, p_n(u_1, \ldots, u_m))^T \end{aligned} \tag{1.8}$$

with $F(p_i(u_j)) = 0$ is called a *parameterization* of $\mathbf{V}(F)$.

Definition 1.8 *Given a hypervariety* $\mathbf{V}(F)$ *parameterized by a rational map* $\mathbf{f}$ *of the form* (1.8). *We call* $\mathbf{V}(F)$ *a* uni-rational variety.

A *rational variety* is a variety, that admits a birational parameterization over an algebraic closed field.

Bijective parameterizations are called *proper*, for more information about proper parameterizations see for example Pérez-Díaz et al. (2002, 2006).

The real rational parameterizations presented in this thesis are typically not proper, arising from the conchoid construction, see Section 1.3. Such parameterizations are called *improper parametrizations*. An improper parameterization $\mathbf{f}$ maps two, or more points of the parameter domain to one point on the variety. Roughly speaking, the number of points of the parameter domain, that are mapped to the same point is called the *degree of improperness*.

Example 1.9 *Unit circle*

The unit circle $\mathbf{V}(F) \subset \mathbb{R}^2$ with $F(x, y) = x^2 + y^2 = 1$, possess the rational proper parameterization

$$\mathbf{f}: t \in \mathbb{R} \to \mathbf{f}(t) = \left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2} \right)^T \subset \mathbb{R}^2.$$

Therefore the unit circle is a rational hypervariety. Note, that according to Lüroth's theorem every uni-rational curve is rational, see Sendra et al. (2008).

Another rational parameterization of the unit circle is

$$\mathbf{f}': t \in \mathbb{R} \to \mathbf{f}(t) = \left(\frac{1 - t^4}{1 + t^4}, \frac{2t^2}{1 + t^4} \right)^T \subset \mathbb{R}^2.$$

This parameterization is improper since $\mathbf{f}(t) = \mathbf{f}(-t)$, and the degree of improperness is two. Furthermore it parameterizes only one half of the real part of the unit circle. $\star$

1.2.3 Resultant

Given polynomials $F(x),\ G(x) \in \mathbb{R}[x]$ of degree m and n,

$$F(x) = \sum_{i=0}^{m} a_i x^i,\ G(x) = \sum_{i=0}^{n} b_i x^i.$$

The determinant of the matrix $\mathbf{M}_{FGx} \in \mathbb{R}^{(m+n)\times(m+n)}$,

$$R_{FGx} = \det(\mathbf{M}_{FGx}) \text{ with } \mathbf{M}_{FGx} = \begin{pmatrix} a_0 & \dots & a_m & & \mathbf{0} \\ & \ddots & & \ddots & \\ \mathbf{0} & & a_0 & \dots & a_m \\ b_0 & \dots & b_n & & \mathbf{0} \\ & \ddots & & \ddots & \\ \mathbf{0} & & b_0 & \dots & b_n \end{pmatrix}$$

is called *resultant* of $F(x)$ and $G(x)$ with respect to x. If $F(x)$ and $G(x)$ are two relatively prime polynomials $R_{FGx} \neq 0$. Else there exist unique polynomials $\overline{F}(x), \overline{G}(x) \in \mathbb{R}[x]$ with degrees $\deg \overline{F} < \deg F,\ \deg \overline{G} < \deg G$ and

$$R_{FGx} = F(x)\overline{G}(x) + G(x)\overline{F}(x) = 0.$$

Since R_{FGx} is independent of x one may use this method to eliminate the unknown x from the given polynomials $F(x),\ G(x)$.

Example 1.10 *Eliminate an unknown*

The unit circle C possess the rational parameterization

$$\mathbf{c}(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)^T,$$

according to Example 1.9. Splitting this vector in x and y coordinates leads to two polynomials

$$\begin{aligned} F(t) &= (1+t^2)x - (1-t^2) = (x-1) + (x+1)t^2 \text{ and} \\ G(t) &= (1+t^2)y - 2t = y - 2t + yt^2. \end{aligned}$$

The points on the circle C are the common zeros of $F(t)$ and $G(t)$, hence $R_{FGt} = 0$, with

$$R_{FGt} = \det \begin{pmatrix} x-1 & 0 & x+1 & 0 \\ 0 & x-1 & 0 & x+1 \\ y & -2 & y & 0 \\ 0 & y & -2 & y \end{pmatrix} = 4(x^2+y^2-1) = 0. \tag{1.9}$$

Equation (1.9) is the well known implicit equation of the unit circle. $\star$

1.3 The Conchoid Construction

In this section we present the classical construction of a conchoid to a given curve and the extension for surfaces. Furthermore we give definitions and attributes of *rational conchoid curves* and *rational conchoid surfaces*. For more geometric details on conchoids of curves see for example Wieleitner (1908); Lawrence (1972); Kunz (2000) and for more information about algebraic attributes see for example Sendra and Sendra (2008, 2010) and Albano and Roggero (2010).

1.3.1 Definition

Definition 1.11 *Given a curve $C \subset \mathbb{R}^2$, a distance $d \in \mathbb{R}$ and the fixed* reference point *$O \in \mathbb{R}^2$. The conchoid curve C_d to C at distance d with respect to O is the Zariski closure of the set of points P_d on the line OP, at distance d to the moving point $P \in C$,*

$$C_d = \{P_d \in \mathbb{R}^2 : P_d \in OP, P \in C \text{ and } \|PP_d\| = d\}^*,$$

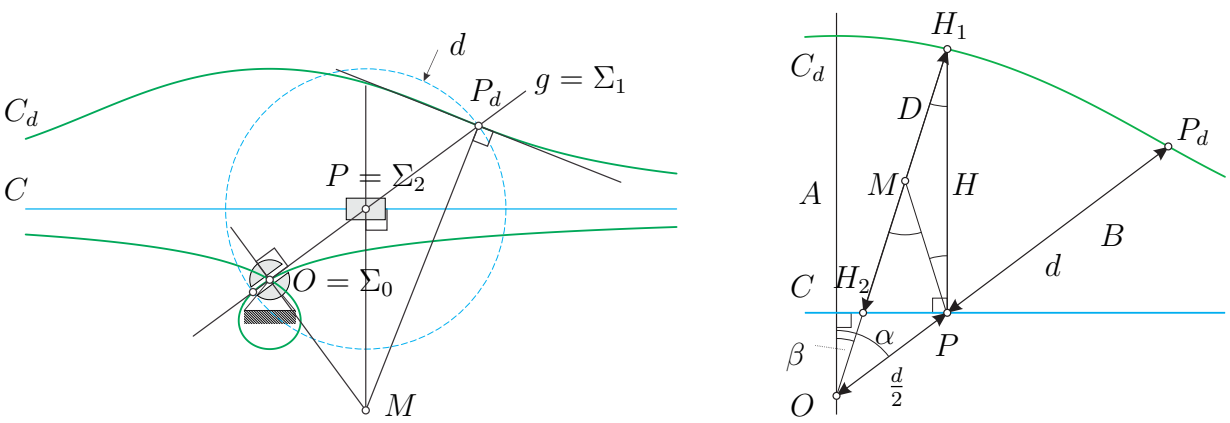
where the asterisk denotes the Zariski closure, see Figure 1.3(a) for an example.

Roughly speaking the conchoid to a given curve is a curve at distance d measured with respect to O. The conchoid construction dates back to the ancient Greeks, see Wieleitner (1908). Nicomedes (*280 b.c.; †210 b.c.) discovered it while studying the problem of angle trisection. A solution of this problem uses the conchoid of a line, its shell shape caused the name *conchoid*, it originates in the ancient Greek word concha - Greek: $\varkappa\acute{o}\gamma\chi\eta$, Lat. concha: muscle, shell; from Liddell and Scott (1883). Figure 1.3(b) shows the conchoid C_d of a line C.

We want to show the construction for angle trisection with the help of the conchoid of Nicomedes. Given an angle α with the sides A, B and the vertex O, we construct an angle β and proof that $\alpha = 3\beta$ holds, see Figure 1.3(b).

Construction of angle trisection We choose the line C perpendicular to A, such that $O \notin C$. The intersection $P = C \cap B$ defines the distance $d = 2\|OP\|$ for the conchoid construction. We construct the conchoid C_d to C with respect to O at distance d, and intersect a perpendicular line H through P to C with the conchoid C_d. The intersection point $H_1 = H \cap C_d$ and the point O define a line D. The angle β with vertex in O and sides A and D is one third of α.
Proof: First we observe that the angle $\angle OH_1P = \beta$. We choose M as the midpoint of H_1H_2, which is also the midpoint of the circumcircle of the right-angled triangle H_1H_2P, hence $\angle H_1MP = 2\beta$. The triangles H_1PM, H_2PM and OPM are equal-sided triangles and we can deduce $\angle POM = 2\beta$. And therefore $\alpha = 3\beta$ holds. □

Definition 1.11 can be directly extended to surfaces.



(a) Kinematic construction.

(b) Angular trisection with $\alpha = 3\beta$.

Figure 1.3: Conchoid of Nicomedes.

Definition 1.12 *Given a surface $F \subset \mathbb{R}^3$, a distance $d \in \mathbb{R}$ and the fixed* reference point *$O \in \mathbb{R}^2$. The conchoid surface F_d to F at distance d with respect to O is the Zariski closure of all points P_d on the line OP, at distance d to the moving point $P \in F$. As a set of points it reads*

$$F_d = \{P_d \in \mathbb{R}^3 : P_d \in OP, P \in F, \|PP_d\| = d\}^*,$$

where the asterisk denotes the Zariski closure, see Figure 1.5(a) for an example.

This construction allows a kinematic realization which we will discuss in the next section.

1.3.2 Kinematic Construction

In this section we present the kinematic realization of the conchoid construction and show that it is a constrained motion. For more information about kinematics we refer to Husty et al. (1997).

The kinematic chain consists of two links. First, the fixed one Σ_0 containing O and the curve C. Second, the moving link Σ_1, containing the line g. The links are connected via two sliding joints. One in O, where g slides through O and rotates about O. Another in the point $P = g \cap C$, where P is a fixed point on g and slides along C, furthermore g rotates about P. In each joint the movement has two free parameters, the sliding length and the rotation angle. Figure 1.3(a) shows a conchoid of a line and its kinematic realization.

We want to calculate the degree of freedom of the kinematic chain. Therefore we use the degrees of freedom of each joint and the formula of Gruebler. The degree of freedom of the kinematic chain according to Gruebler is

$$F_G = 3(n-1) - 2f_1 - f_2,$$

with n is the number of links and f_1, f_2 the numbers of joints with one free parameter respectively two free parameters. The kinematic realization of the conchoid construction, with $n = 2, f_1 = 0$ and $f_2 = 2$ has the degree of freedom according to Gruebler, $F_G = 1$, hence it is a *constrained motion*.

Performing only an infinitesimal motion of the line g, it momentarily rotates about a point M, the so called instantaneous center of rotation. For the kinematic realization of the conchoid construction, M is the intersection of the normal to C in P and the normal to g in O. This holds, since the point $P \in g$ moves along C and O, considered as a point on g, slides in direction of g during the infinitesimal motion.

Obviously every point P_d on g momentarily rotates about the same point M. Hence the the tangents in the points $P_d \in g$ to the conchoid C_d, for a fixed position of the kinematic chain and varying d, envelope a parabola with focal point M.

1.3.3 Polar Representation

To calculate parameterizations of conchoids C_d starting with a parameterization of C we use polar representations of C. We choose without loss of generality the reference point O as the origin of the coordinate system, this choice is made throughout the whole thesis if not stated otherwise. A curve C has the *polar representation*

$$\mathbf{c}(t) = \varrho(t)\mathbf{k}(t), \tag{1.10}$$

with $\varrho(t) = \|\mathbf{c}(t)\|$ is the distance to the origin and $\mathbf{k}(t) = \frac{1}{\|\mathbf{c}(t)\|}\mathbf{c}(t)$ is a parameterization of the unit circle, hence $\|\mathbf{k}(t)\| = 1$, see Figure 1.4(a). We call $\varrho(t)$ the *radius* of $\mathbf{c}(t)$ and $\mathbf{k}(t)$ the *spherical part* of the parameterization. It is obvious that the conchoid C_d of C at distance d with respect to the origin O has the parameterization

$$\mathbf{c}_d(t) = (\varrho(t) \pm d)\mathbf{k}(t).$$

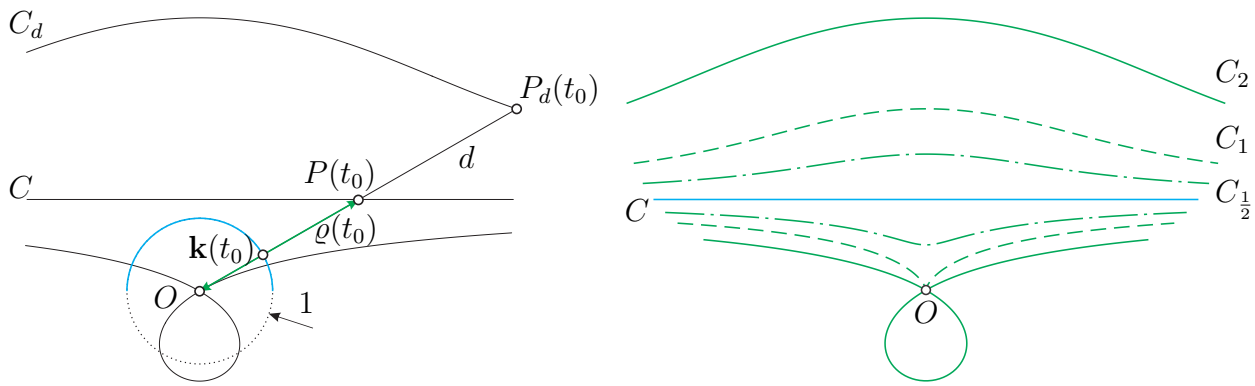
Definition 1.13 *A curve C is called a* rational conchoid curve *with respect to the origin O, if it admits a rational polar representation of the form* (1.10)*, with a rational radius function $\varrho(t)$ and a rational parameterization of the unit circle $\mathbf{k}(t)$.*

In an analogous way we can define *rational conchoid surfaces*.

Definition 1.14 *A surface F is called a* rational conchoid surface *with respect to the origin O, if it admits a rational polar representation of the form*

$$\mathbf{f}(u, v) = \varrho(u, v)\mathbf{k}(u, v), \tag{1.11}$$

with a rational radius function $\varrho(u, v) = \|\mathbf{f}(u, v)\|$ and a rational parameterization $\mathbf{k}(u, v)$ of the unit sphere S^2.



(a) Radius ϱ and the unit circle.

(b) Conchoids of the line $y = 1$ for $d \in \{\frac{1}{2}, 1, 2\}$.

Figure 1.4: Conchoid of Nicomedes.

Conchoids F_d, of surface F with a rational parameterization according to (1.11), have the rational parameterization

$$\mathbf{f}_d(u, v) = (\varrho(u, v) \pm d)\mathbf{k}(u, v).$$

Typically the parameterizations $\mathbf{f}_{+d} = (\varrho(u, v) + d)\mathbf{k}(u, v)$ and $\mathbf{f}_{-d} = (\varrho(u, v) - d)\mathbf{k}(u, v)$ correspond to two distinct real parts of the surface F_d.

Theoretically it might be possible that a surface F and its conchoid surface F_d admit real rational polar representations with distinct spherical parts $\mathbf{k}$. Up to now we do not know an example for this case.

To conclude the chapter on fundamentals we give two examples for rational polar representations of a line and a plane. In Chapters 2 and 3 we present a method to find rational polar representations for certain curves and surface classes.

Example 1.15 *Conchoid of Nicomedes*

The conchoid construction is obviously invariant with respect to rotations about the reference point O and central similarities with center in O.

We study admissible mappings in Section 2.1.1 in more detail. Therefore we can assume without loss of generality, that the line C is given by the equation $y = 1$. This line has the polar trigonometric representation

$$\mathbf{c}(t) = \begin{pmatrix} \frac{\sin(t)}{\cos(t)} \\ 1 \end{pmatrix}, \tag{1.12}$$

with $\varrho = \frac{1}{\cos(t)}$ and $\mathbf{k}(t) = (\sin(t), \cos(t))^T$. Therefore the conchoid of Nicomedes at distance d has the trigonometric parameterization

$$\mathbf{c}_d(t) = \frac{1 \pm d\cos(t)}{\cos(t)} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}, \tag{1.13}$$

see Figure 1.4(b) for and illustration of Conchoids C_d of a line C with different $d-$values.

Using the Weierstrass substitutions

$$\cos(t) = \frac{1-u^2}{1+u^2} \text{ and } \sin(t) = \frac{2u}{1+u^2}, \tag{1.14}$$

in (1.12) and (1.13) leads to rational polar representations of C and C_d. Therefore the line C is a rational conchoid curve, according to Definition 1.13. In Example 1.10 we illustrated a method to gain the implicit equation from a rational parameterization using the resultant R_{FGt} of

$$\begin{aligned} F(u) &= -x + 2(1+d)u + 2(1-d)u^3 + xu^4 \\ G(u) &= (1-y+d) + (1-y-d)u^2. \end{aligned} \tag{1.15}$$

Calculating the resultant R_{FGu} leads to the implicit equation of the Zariski closure of the set of points $\mathbf{c}_d(u)$

$$C_d : R_{FGt} = (x^2+y^2)(y-1)^2 - d^2y^2 = 0. \tag{1.16}$$

This is a polynomial of degree four, hence the algebraic degree of the curve is four. There are no linear terms in x and y and one can easily see that the reference point $O = (0,0)$ is a double point of the conchoid independent on d. Changing to homogeneous coordinates, equation (1.16) reads

$$C_d : (x_1^2 + x_2^2)(x_2 - x_0)^2 - d^2x_0^2x_2^2 = 0.$$

Intersecting the conchoid with the line at infinity, $x_0 = 0$, shows that the conchoid contains the points $(0,1,i)\mathbb{R}$, $(0,1,-i)\mathbb{R}$ and $(0,1,0)\mathbb{R}$. A curve passing through the first two of these points is called circular, again this is independent on d. $\star$

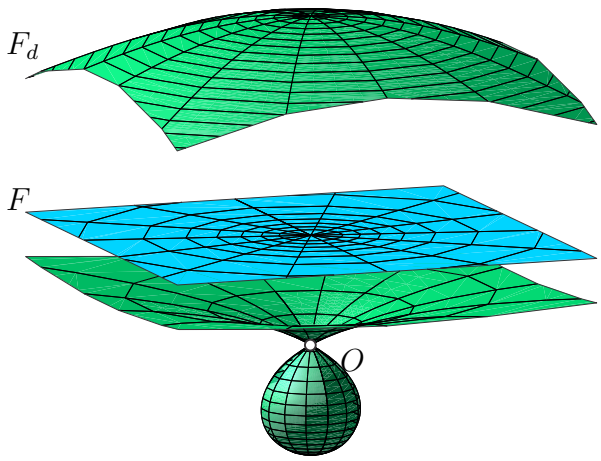
Example 1.16 *Conchoids of a plane*

Analogue to the conchoid of Nicomedes in $\mathbb{R}^2$ we can parameterize the conchoid of a plane in $\mathbb{R}^3$. Starting with the trigonometric polar representation of the plane $F : z = 1$

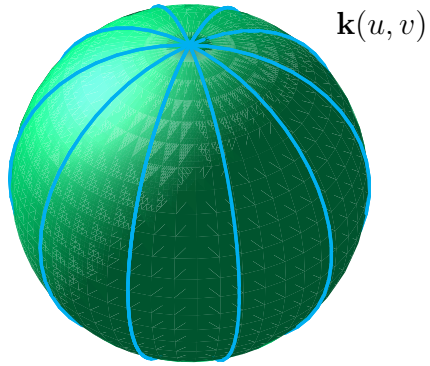
$$\mathbf{f}(u,v) = \frac{1}{\sin(u)} \begin{pmatrix} \cos(u)\cos(v) \\ \cos(u)\sin(v) \\ \sin(u) \end{pmatrix},$$

with $\varrho(u) = \frac{1}{\sin(u)}$ and $\mathbf{k}(u,v) = (\cos(u)\cos(v), \cos(u)\sin(v), \sin(u))^T$, the conchoid F_d of F at distance d has the trigonometric parameterization

$$\mathbf{f}_d(u,v) = \frac{1 \pm d\sin(u)}{\sin(u)} \begin{pmatrix} \cos(u)\cos(v) \\ \cos(u)\sin(v) \\ \sin(u) \end{pmatrix}. \tag{1.17}$$



(a) Conchoid of a plane.



(b) Spherical part: Great circles on S^2.

Figure 1.5: Conchoid of a plane and its spherical part.

See Figure 1.5(a) for the conchoid to the plane $F : z = 1$ at distance $d = 2$ and Figure 1.5(b) illustrates one parameter family of circles of $\mathbf{k}(u, v)$. The conchoid F_d is a surface of algebraic order four with the equation

$$F_d(x, y, z) = (x^2 + y^2 + z^2)(z - 1)^2 - z^2 d^2 = 0. \tag{1.18}$$

Similar to the conchoid of Nicomedes, the origin $O = (0, 0, 0)$ is a singular point of F_d and the intersection with the plane at infinity consists of the point $(0, 0, 0, 1)\mathbb{R}$ and the ideal conic $x_1^2 + x_2^2 + x_3^2 = 0 \cap x_0 = 0$, independent on the choice of d. Hence the conchoid F_d is a surface of rotation about the $z-$axis, the meridian curve is obviously a conchoid of Nicomedes.

$\star$

Chapter 2

Rational Conchoids of Curves

The main contribution of the thesis is to find rational parameterizations of conchoid curves and surfaces. Therefore we introduce a cone model. This was at first presented by Peternell, Gruber and Sendra (2013) and it is similar to the cylinder model used for the construction of rational offset curves and offset surfaces, see for example Krasauskas and Peternell (2009).

2.1 The Cone Model

Let us assume that C is a rational conchoid curve according to Definition 1.13 with

$$\mathbf{c}(t) = (c_1, c_2)^T(t), \ \|\mathbf{c}(t)\| = \varrho(t)$$

and rational functions $c_1(t), c_2(t), \varrho(t)$. We denote the distance of a point $X \in \mathbb{R}^2$ to the origin as $\|\mathbf{x}\| = w$, hence the hence the coordinates $c_1(t), c_2(t)$ of C satisfy the equation

$$D : x^2 + y^2 - w^2 = 0. \tag{2.1}$$

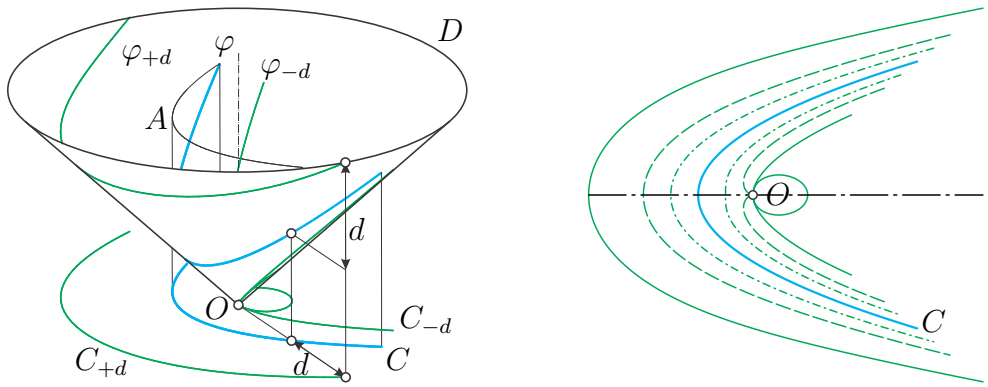
Interpreting w as a third coordinate of an extended space $\mathbb{R}^3$ of $\mathbb{R}^2$, equation (2.1) defines a cone D with vertex in the origin and opening angle $\pi/2$. The rational conchoid curve C corresponds to two rational curves φ on the cone D with

$$\varphi(t) = (c_1, c_2, \pm\varrho)^T(t),$$

Where $\varphi(t)^+ = (c_1, c_2, +\varrho)^T(t)$ and $\varphi(t)^- = (c_1, c_2, -\varrho)^T(t)$ are at least locally different curves. On the other hand, a curve $\varphi \subset D$ with rational parametrization $\varphi(t) = (\varphi_1, \varphi_2, \varphi_3)^T(t)$ corresponds to a rational conchoid curve C in $\mathbb{R}^2$ with

$$\mathbf{c}(t) = (\varphi_1, \varphi_2)^T(t) \text{ and } \|\mathbf{c}(t)\| = |\varphi_3(t)|.$$

So we can state the following



(a) Cone model.

(b) C is a parabola with O as its focal point.

Figure 2.1: Conchoid of a parabola.

Theorem 2.1 *Rational curves $\varphi \subset D \subset \mathbb{R}^3$ correspond to rational conchoid curves $C \subset \mathbb{R}^2$ and vice versa, where the rational parameterizations of C and φ are connected in the following way*

$$\mathbf{c}(t) = (\varphi_1, \varphi_2)^T(t) \text{ and } \|\mathbf{c}(t)\| = |\varphi_3(t)|.$$

We denote this correspondence as $C = \pi(\varphi)$.

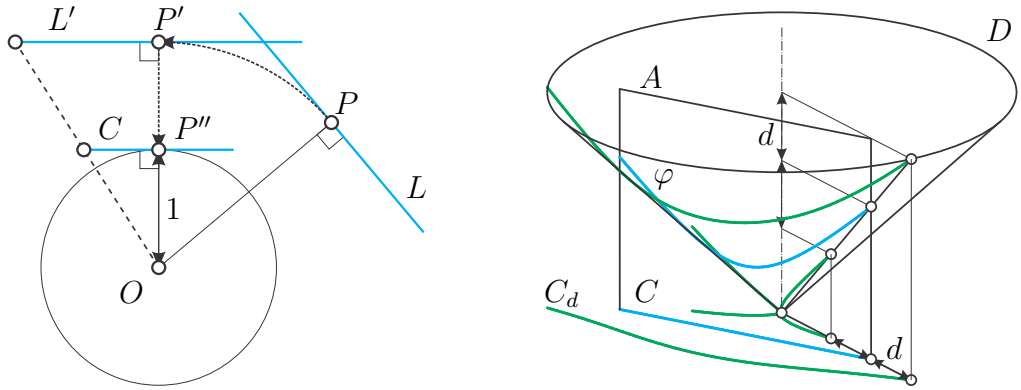
We embed the Euclidean plane $[x, y]$ in a Euclidean space $[x, y, w]$ as the plane $w = 0$. Then the corresponding curves C and φ span a cylinder A with generating lines parallel to the $w-$axis. Since $C = A \cap (w = 0)$, the cylinder A has the same implicit equation as the curve $C \subset \mathbb{R}^2$. The corresponding curve on D to C is the intersection $A \cap D$. Figure 2.1(a) illustrates the situation in $\mathbb{R}^3$ for a parabola C.

According to Kubota (1972) and Farouki and Sakkalis (1990) all real polynomial curves φ on the cone D are of the form

$$\varphi(t) = \begin{pmatrix} c(t)(a(t)^2 - b(t)^2) \\ 2a(t)b(t)c(t) \\ c(t)(a(t)^2 + b(t)^2) \end{pmatrix}$$

with polynomials $a(t), b(t), c(t) \in \mathbb{R}[t]$. The polynomials $a(t), b(t)$ can be chosen relatively prime, else we put the factor $\gcd(a(t), b(t))$ to $c(t)$. Furthermore we exclude the trivial cases $c(t) = 0$ and $a(t) = b(t) = 0$. We give some examples for the corresponding curves C of φ for choosing $a(t), b(t), c(t)$ with low degree.

- Let $a(t) = a_1 t + a_0$ be linear and the others constant $b(t) = b_0$, $c(t) = c_0$. The corresponding curve C has the parameterization $\mathbf{c}(t) = (c_0((a_1 t + a_0)^2 - b_0^2), 2(a_1 t +$



(a) Transformation $L \to C : y = 1$.

(b) Cone model.

Figure 2.2: Conchoid of Nicomedes.

$a_0)b_0c_0)^T$. For $b_0 \neq 0$ it defines the parabola

$$C : x - \frac{1}{4c_0b_0^2}y^2 + c_0b_0^2 = 0.$$

The origin is the focal point of C, see Figure 2.1(b). See Section 2.3 for more information about the rationality of conchoids of conic sections.

- Let $c(t)$ be linear and the others constant. The corresponding curve C is obviously a line $\mathbf{c}(t) = ((c_1t + c_0)(a_0^2 - b_0^2), 2a_0b_0(c_1t + c_0))^T$ through the origin for $t = -\frac{c_1}{c_0}$, with the defining polynomial

$$C : 2a_0b_0x - (a_0^2 - b_0^2)y = 0.$$

2.1.1 Admissible Mappings

We study mappings that keep the rationality of the polar representation of a curve. Consider the mapping $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$\sigma(\mathbf{x}) = \mathbf{x}' = \frac{r(\mathbf{x})}{s(\mathbf{x})}\mathbf{R} \cdot \mathbf{x}, \text{ with } \mathbf{R} \in \mathbb{R}^{2\times 2}, \text{ and } \mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}_2 = \mathrm{diag}(1, 1), \tag{2.2}$$

and relatively prime polynomials $r(\mathbf{x})$ and $s(\mathbf{x})$. Consequently the norm of $\mathbf{x}'$ is

$$\|\mathbf{x}'\| = \sqrt{\mathbf{x}' \cdot \mathbf{x}'} = \frac{r(\mathbf{x})}{s(\mathbf{x})}\|\mathbf{x}\|.$$

Thus the rational map (2.2) preserves rational polar representations. It can be decomposed into a rotation $\mathbf{x} \mapsto \mathbf{R} \cdot \mathbf{x}$ about O and a 'scaling' $\mathbf{x} \mapsto f(\mathbf{x})\,\mathbf{x}$ with a rational function $f(\mathbf{x})$, fixing all lines through O.

If one chooses $s(\mathbf{x}) = s_0 + s_1 x + s_2 y$ as a linear polynomial, $r \in \mathbb{R}$ and $\mathbf{R} = \mathbf{I}_2$, then the rational map (2.2) becomes a *perspective collineation*. This is a projective linear map with fixed point O, keeping the axis line $r - s(\mathbf{x}) = 0$ point-wise fixed. The line $s(\mathbf{x}) = 0$ contains points with an improper image, the line $-r + s_1 x + s_2 y = 0$ is called vanishing line and consists of points with improper pre-images, see Section 1.1.2.

Corollary 2.2 *Any rational map of the form* (2.2) *preserves rational polar representations with respect to the reference point* $O = (0, 0)$. *Choosing* $r \in \mathbb{R}$ *and a linear polynomial* $s(\mathbf{x})$ *these mappings are rotations about* O *combined with perspective collineations with center* O.

In the following sections we discuss two well known examples for rational conchoid curves. We use the cone model to proof rationality of the distance to the reference point.

2.2 Conchoids of Nicomedes

We already presented a trigonometric polar representation of the conchoid of a line in Example 1.15. We investigate the corresponding curve in the cone model and compare the results to the previous ones. Given a line L in $w = 0 \subset \mathbb{R}^3$, the cylinder A through C is a plane parallel to the $w-$axis. The intersection $A \cap D$ is a hyperbola with main axis direction parallel to w. A hyperbola in $\mathbb{R}^3$ allows a rational parametrization.

Instead of analyzing an arbitrary line L in the plane we use mappings of the form (2.2) and map L to the line $C : y = 1$. We rotate L about O such that the foot-point P on L with respect to O is a point P' on the $y-$axis after the rotation. A perspective collineation with center O and axis line at infinity maps P' to P'', with $\|OP''\| = 1$. Figure 2.2(a) illustrates the mapping $L \to C$.

The line C defines the plane $A : y = 1$ in the cone model, which intersects the cone D in a hyperbola $\varphi : w^2 - x^2 = 1 \cap y = 1$, see Figure 2.2(b). This hyperbola has the parameterization

$$\varphi(t) = \begin{pmatrix} \sinh(t) \\ 1 \\ \cosh(t) \end{pmatrix}.$$

The corresponding curve $\pi(\varphi)$ is the line C with the polar representation

$$\mathbf{c}(t) = \begin{pmatrix} \sinh(t) \\ 1 \end{pmatrix} \text{ with } \varrho(t) = \cosh(t)$$

and its conchoids read

$$\mathbf{c}_d(t) = (\cosh(t) \pm d) \begin{pmatrix} \frac{\sinh(t)}{\cosh(t)} \\ \frac{1}{\cosh(t)} \end{pmatrix}. \tag{2.3}$$

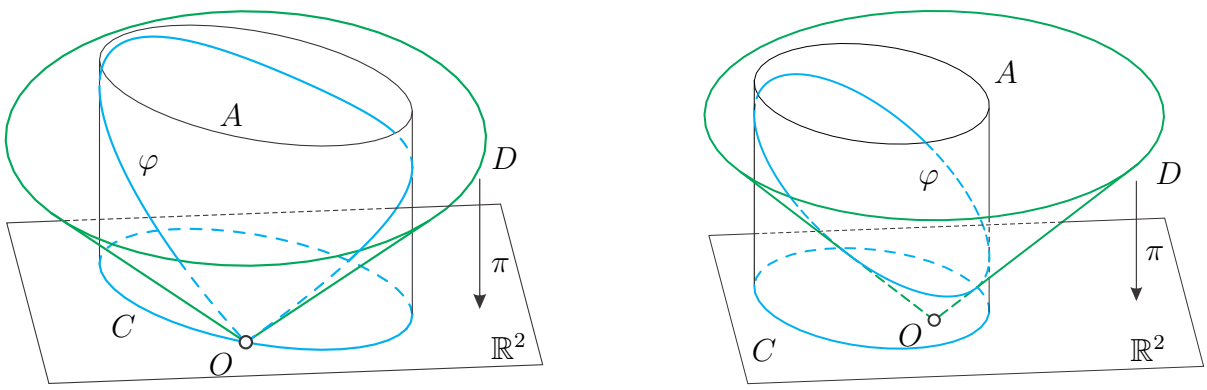
Substituting the hyperbolic functions with rational ones,

$$\cosh(t) = \frac{1+u^2}{1-u^2} \text{ and } \sinh(t) = \frac{2u}{1-u^2},$$

in (2.3) leads to the rational polar representation of the conchoids of Nicomedes

$$\mathbf{c}_d(u) = \frac{(1+u^2) \pm d(1-u^2)}{1-u^4} \begin{pmatrix} 2u \\ 1-u^2 \end{pmatrix}. \tag{2.4}$$

Using the Weierstrass substitutions (1.14) in equation (1.13) of Example 1.15, shows that the parameterizations (1.13) and (2.4) are equal.



(a) C is a conic with $O \in C$.

(b) C is a conic with O is a focal point.

Figure 2.3: Cylinder A through φ and C and projection π.

2.3 Conchoids of Conic Sections

Conic sections are well studied according to their rational polar representations. It is known that for an arbitrary reference point in the plane, the conchoid curves of conic sections are not rational. Except if the reference point O lies on C and if O is a focal point of C, for an elementary proof see Wieleitner (1908) and for an algebraic proof see Sendra and Sendra (2010). We want to show this result by analyzing the curve φ in the cone model.

The cylinder A through a conic section is a quadratic cylinder, hence $\varphi = A \cap D$ is a curve of degree four, see Figure 2.3. Quartic curves in $\mathbb{R}^3$ are well studied and they are rational, if φ has a singularity or if it is reducible, see Telling (1936). We want to examine these two cases.

First, let the intersection of the cone D and the cylinder A have a singularity. The singularity has to be in the vertex of D, hence $O \in A$ is a necessary condition, and so $O \in C$ follows. If C is a circle and $O \in C$, the conchoids are so called *Limacons of Pascal*, see Figure 2.3(a) and Example 2.3.

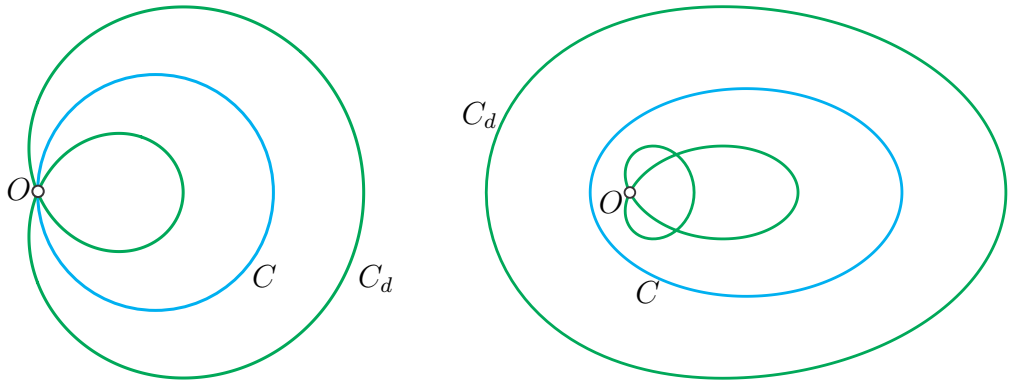
Second, Let us assume φ is reducible, see Figure 2.3(b). Since the generators of A are parallel to the axis of D, φ can only consist of two conic sections. Furthermore D is symmetric to the plane $w = 0$, which implies that φ decomposes into two conic sections symmetric to the plane $w = 0$. The corresponding curve $C = \pi(\varphi)$ is a conic section with one focal point in O, see Wunderlich (1966). We discuss a conic with focal point in O in Example 2.4 in more detail.

Example 2.3 *Limacon of Pascal*

Given an arbitrary circle C with radius r through the origin O, we rotate C about O, such that its mid-point lies on the $x-$axis. This circle is defined by

$$C : x^2 - 2rx + y^2 = 0, \tag{2.5}$$

see Figure 2.4(a). The cylinder A through C in the cone model has rotational symmetry



(a) *Limacon of Pascal* (b) Ellipse with O is a focal point.

Figure 2.4: Rational conchoid of conic sections.

and the same implicit equation as C, for an illustration see Figure 2.5(a). We parameterize the cone D through the circle in the plane $w = 1$,

$$\mathbf{d}(t, s) = \begin{pmatrix} s\cos(t) \\ s\sin(t) \\ s \end{pmatrix}, \tag{2.6}$$

and compute the intersection with A by inserting (2.6) in (2.5),

$$s^2 - 2rs\cos(t) = 0.$$

There are two solutions for s, first $s = 0$ which yields the origin and second $s = 2r\cos(t)$ which yields to a trigonometric parameterization of the quartic φ with a double point in O, see Figure 2.3(a). It reads

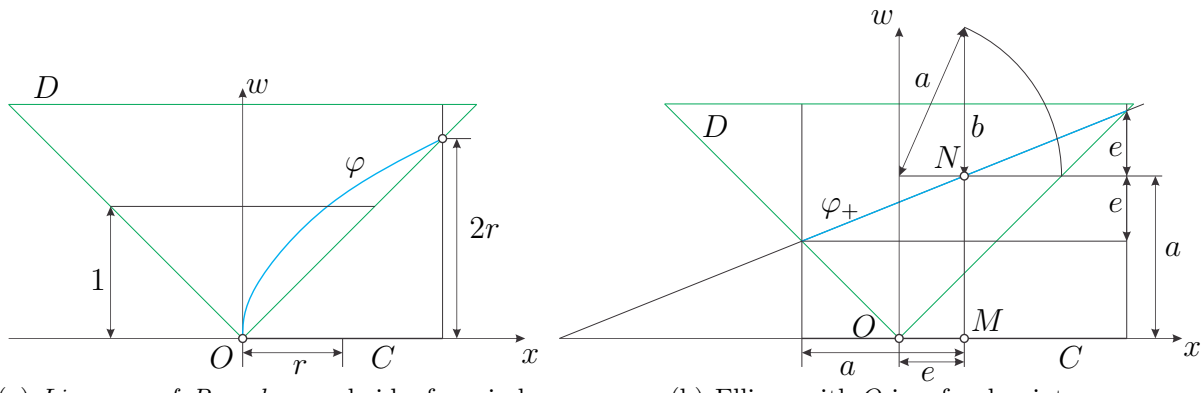
$$\varphi(t) = 2r\cos(t) \begin{pmatrix} \cos(t) \\ \sin(t) \\ 1 \end{pmatrix}, \tag{2.7}$$

this defines the well known *Viviani curve*. Equation (2.7) leads to a trigonometric parameterization of the corresponding circle $C = \pi(\varphi)$ and its conchoids C_d,

$$\mathbf{c}_d(t) = (2r\cos(t) \pm d) \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}. \tag{2.8}$$

In this example the parameterizations $\mathbf{c}_{+d}$ and $\mathbf{c}_{-d}$ parameterize the same curve since $\mathbf{c}_{+d}(t) = \mathbf{c}_{-d}(t + \pi)$. Using the Weierstrass substitutions (1.14) in (2.8) leads to a rational parameterization of the Limacons of Pascal, its equation is

$$C_d : d^2(x^2 + y^2) - (x^2 - 2xr + y^2)^2 = 0.$$



(a) *Limacon of Pascal:* conchoid of a circle with $O \in C$.

(b) Ellipse with O is a focal point.

Figure 2.5: Notation for the parameterization in the cone model.

Obviously $O \in C_d$ holds and $d = 0$ leads to the circle (2.5). $\star$

Example 2.4 *Ellipse with the focal point O*

We want to give a rational polar representation of an ellipse C with one focal point in O, see Figures 2.4(b) and 2.5(b). With possible admissible mappings, C is given through

$$C : \frac{(x-e)^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

with $e^2 + b^2 = a^2$. Since φ decomposes into two ellipses φ_+, φ_-, symmetric to the $w = 0$ plane, we have to parameterize only one of them, for instance φ_+ with $w > 0$. The midpoint N of φ_+ has the coordinates $N = (e, 0, a)^T$ and the plane carrying φ_+ is spanned by the vectors $\mathbf{e}_1 = (a, 0, e)^T$ and $\mathbf{e}_2 = (0, 1, 0)^T$. The semi-major axis is $\sqrt{a^2 + e^2}$ and the semi-minor axis is b, hence the parameterization of φ_+ is

$$\varphi_+(t) = N + \cos(t)\mathbf{e}_1 + b\sin(t)\mathbf{e}_2 = \begin{pmatrix} e + a\cos(t) \\ b\sin(t) \\ a + e\cos(t) \end{pmatrix}.$$

The corresponding curve $\pi(\varphi_+)$ yields a trigonometric parameterization of the ellipse C

$$\mathbf{c}(t) = \begin{pmatrix} e + a\cos(t) \\ b\sin(t) \end{pmatrix} \text{ with } \varrho(t) = a + e\cos(t). \tag{2.9}$$

The conchoid C_d has the trigonometric parameterization

$$\mathbf{c}_d(t) = \frac{a + e\cos(t) \pm d}{a + e\cos(t)} \begin{pmatrix} e + a\cos(t) \\ b\sin(t) \end{pmatrix}. \tag{2.10}$$

The Weierstrass substitutions (1.14) applied in (2.9) and (2.10) leads to real rational polar representations of the ellipse C and its conchoids C_d. The conchoids are algebraic curves of degree eight that factorize into two algebraic curves C_{+d} and C_{-d}, both of degree four,

$$C_{\pm d} : d^2(b^2x^2 + a^2y^2) \pm 2dab^2(x^2 + y^2) - a^2b^2(x^2 + y^2)\left(\frac{(x-e)^2}{a^2} + \frac{y^2}{b^2} - 1\right) = 0.$$

$\star$

Chapter 3

Rational Conchoids of Surfaces

This chapter contains the main contribution of the thesis. It is a cumulation of the articles Peternell, Gruber and Sendra (2011), Peternell, Gruber and Sendra (2013) and Gruber and Peternell (2013). The content of the first one is adapted to the notation and concept of the cone model. The other two have been adapted to fit the notation and the reading flow of the thesis and some detailed information was added.

We expand the cone model presented in Section 2.1 to Euclidean three-space $\mathbb{R}^3$ and give classes of rational conchoid surfaces together with algorithms to calculate the necessary rational polar representations.

Note that this is a similar matter to Pythagorean-Normal surfaces, denoting surfaces with rational Gaussian image, hence they possess rational normal vectors with rational length. For more information about PN surfaces see for example Peternell and Pottmann (1998) and Lávička and Bastl (2008).

3.1 The Cone Model

Lets assume F is a rational conchoid surface according to Definition 1.14 with

$$\mathbf{f}(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v))^T \text{ and } \|\mathbf{f}(u,v)\| = \varrho(u,v)$$

and rational functions $f_1(u,v), f_2(u,v), f_3(u,v)$ and $\varrho(u,v)$. We denote the distance of a point $X \in \mathbb{R}^3$ to the origin as $\|\mathbf{x}\| = w$, hence the coordinates $f_i(u,v), i = \{1,2,3\}$ of $\mathbf{f}(u,v)$ satisfy the equation

$$D : x^2 + y^2 + z^2 - w^2 = 0. \tag{3.1}$$

Interpreting w as a fourth coordinate of an extended space $\mathbb{R}^4$ of $\mathbb{R}^3$, equation (3.1) defines a cone $D \subset \mathbb{R}^4$ with vertex in the origin and opening angle $\pi/2$. The rational conchoid surface F corresponds to a rational surface Φ on the cone D with

$$\varphi(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v), \pm\varrho(u,v))^T.$$

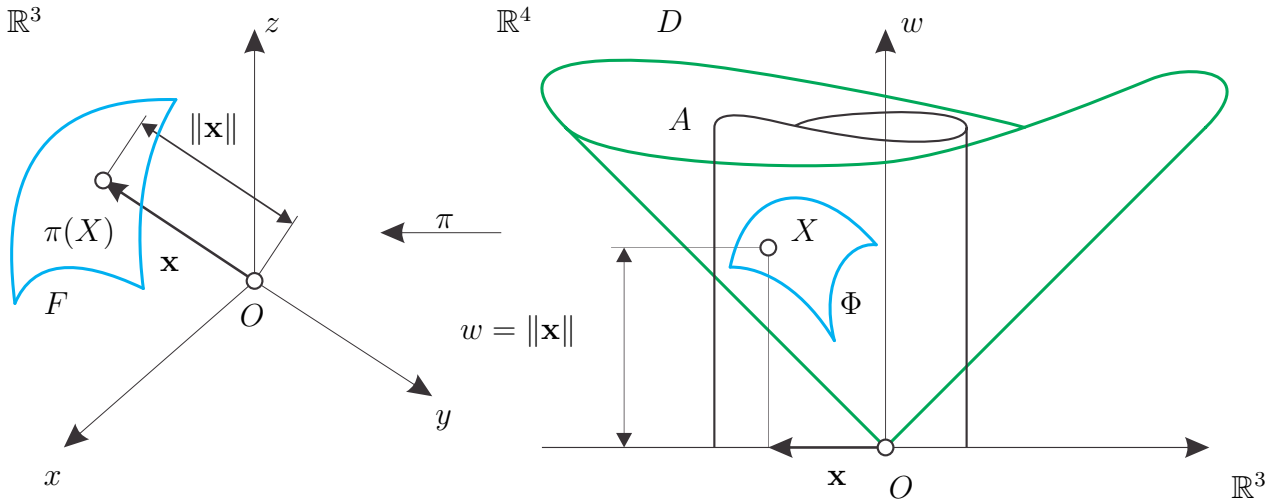


Figure 3.1: The cone model.

On the other hand, a rational surface $\varphi(u,v) \subset D$ corresponds to a rational conchoid surface F in $\mathbb{R}^3$ with

$$\mathbf{f}(u,v) = (\varphi_1(u,v), \varphi_2(u,v), \varphi_2(u,v))^T \text{ and } \|\mathbf{f}(u,v)\| = |\varphi_4(u,v)|.$$

So we can state the following

Theorem 3.1 *Rational surfaces $\Phi \subset D \subset \mathbb{R}^4$ correspond to rational conchoid surfaces $F \subset \mathbb{R}^3$ and vice versa. Where F and Φ are connected in the following way*

$$\mathbf{f}(u,v) = (\varphi_1(u,v), \varphi_2(u,v), \varphi_3(u,v))^T \text{ and } \|\mathbf{f}(u,v)\| = |\varphi_4(u,v)|.$$

We denote this correspondence as $F = \pi(\Phi)$. For an illustration see Figure 3.1.

We embed the Euclidean space $\mathbb{R}^3$ with coordinates $[x,y,z]$ in an Euclidean space $\mathbb{R}^4$ with coordinates $[x,y,z,w]$ as the hyperplane $w = 0$. Then the corresponding surfaces $F \subset \mathbb{R}^3 \subset \mathbb{R}^4$ and $\Phi \subset \mathbb{R}^4$ span a cylinder A with generating lines parallel to the $w-$axis. Since $F = A \cap (w = 0)$, the cylinder A has the same implicit equation as the surface $F \subset \mathbb{R}^3$. The corresponding surface Φ on D to F is the intersection $A \cap D$.

According to Dietz et al. (1993) all polynomial surfaces F on the cone D can be parameterized as $F : \mathbf{f}(u,v) = (x(u,v), y(u,v), z(u,v), w(u,v))^T$ with

$$\begin{aligned} x(u,v) &= 2\sigma(u,v)(a(u,v)b(u,v) - c(u,v)d(u,v)), \\ y(u,v) &= 2\sigma(u,v)(b(u,v)d(u,v) + a(u,v)c(u,v)), \\ z(u,v) &= \sigma(u,v)(-a(u,v)^2 + b(u,v)^2 + c(u,v)^2 - d(u,v)^2), \\ w(u,v) &= \sigma(u,v)(a(u,v)^2 + b(u,v)^2 + c(u,v)^2 + d(u,v)^2) \end{aligned} \tag{3.2}$$

and polynomials $a(u,v), b(u,v), c(u,v), d(u,v), \sigma(u,v) \in \mathbb{R}[u,v]$.

Example 3.2 *Linear polynomials*

We choose $a(u,v),\ldots,\sigma(u,v)$ of (3.2) as linear polynomials, hence the parameterization $\mathbf{f}(u,v)$ is of maximum bi-degree $(2,2)$. For more details on quadratically parameterizable surfaces see Coffman et al. (1996). For example the choice

$$\sigma(u,v)=1,\ a(u,v)=u,\ b(u,v)=1,\ c(u,v)=1 \text{ and } d(u,v)=v$$

leads to the two dimensional surface

$$\varphi(u,v)=(2(u-v),2(u+v),-(u^2+v^2)+2,u^2+v^2+2)^T$$

contained in the three space $z+w=4\subset\mathbb{R}^4$. Inserting $w=4-z$ in (3.1) leads to the cylinder A with generators parallel to the $w-$axes containing Φ, hence to the corresponding surface $F=\pi(\Phi)$,

$$F: x^2+y^2+8z=16.$$

This is a rotational paraboloid with O as focal point, see Figure 3.2(a).

We want to derive another example by choosing the linear polynomials

$$\sigma(u,v)=1,\ a(u,v)=u,\ b(u,v)=u,\ c(u,v)=1 \text{ and } d(u,v)=v.$$

The surface in $\mathbb{R}^4$ is given with the parameterization

$$\varphi(u,v)=(2(u^2-v),2u(v+1),1-v^2,2u^2+v^2+1)^T.$$

To gain an implicit equation of the corresponding surface $F=\pi(\Phi)$, we eliminate the parameter u,v from the polynomials

$$x-2(u^2-v)=0,\ y-2u(1+v)=0 \text{ and } z+v^2-1=0.$$

This yields to the quartic algebraic surface F containing O,

$$F: 4xy^2z+4x^2z^2+y^4-16y^2+16y^2z-8y^2x+16z^3-16z^2=0.$$

See Figure 3.2(b) for an illustration of this surfaces. $\star$

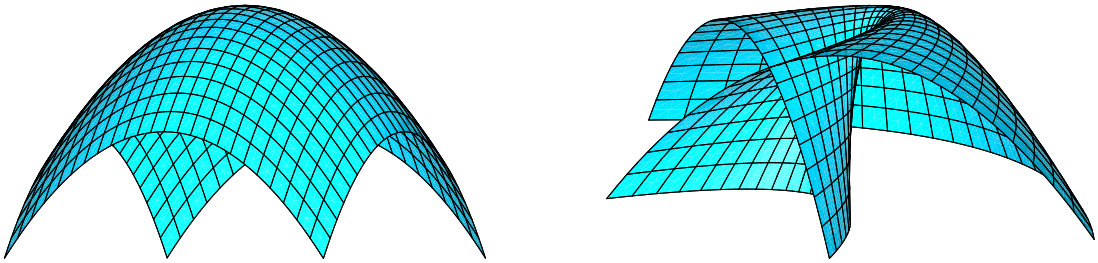
3.1.1 Admissible Rational Mappings

We study mappings that keep the rationality of the polar representation of a surface. Consider the mapping $\sigma:\mathbb{R}^3\to\mathbb{R}^3$ with

$$\sigma(\mathbf{x})=\mathbf{x}'=\frac{r(\mathbf{x})}{s(\mathbf{x})}\mathbf{R}\cdot\mathbf{x},\ \text{with } \mathbf{R}\in\mathbb{R}^{3\times3},\ \text{and } \mathbf{R}^T\cdot\mathbf{R}=\mathbf{I}_3=\operatorname{diag}(1,1,1), \tag{3.3}$$

and relatively prime polynomials $r(\mathbf{x})$ and $s(\mathbf{x})$. Consequently the norm of $\mathbf{x}'$ is

$$\|\mathbf{x}'\|=\sqrt{\mathbf{x}'\cdot\mathbf{x}'}=\frac{r(\mathbf{x})}{s(\mathbf{x})}\|\mathbf{x}\|.$$



(a) Rotational paraboloid. (b) Quartic surface.

Figure 3.2: Surfaces arising from linear polynomials.

Thus the rational map (3.3) preserves rational polar representations. It can be decomposed into a rotation $\mathbf{x} \mapsto \mathbf{R}\cdot\mathbf{x}$ around a line through O and a 'scaling' $\mathbf{x} \mapsto f(\mathbf{x})\,\mathbf{x}$ with a rational function $f(\mathbf{x})$, fixing all lines through O.

If one chooses $s(\mathbf{x}) = s_0 + s_1x + s_2y + s_3z$ as a linear polynomial, $r \in \mathbb{R}$ and $\mathbf{R} = \mathbf{I}_3$, then the rational map (3.3) becomes a *perspective collineation*. This is a projective linear map with fixed point O keeping the axis plane $r - s(\mathbf{x}) = 0$ point-wise fixed. The plane $s(\mathbf{x}) = 0$ contains points with improper image points, the plane $-r + s_1x + s_2y + s_3z = 0$ is called vanishing plane and consists of points with improper pre-images, see Section 1.1.2. In Section 3.3.2 these perspective collineations together with rotations around lines through O are used to transform a quadric to a particular normal form.

Corollary 3.3 *Any rational map of the form* (3.3) *preserves rational polar representations with respect to the reference point* $O = (0, 0, 0)$. *Choosing* $r \in \mathbb{R}$ *and a linear polynomial* $s(\mathbf{x})$ *these mappings are rotations about lines through* O *combined with perspective collineations with center* O.

3.2 Conchoids of Rational Ruled Surfaces

In this section we prove, that *real rational ruled surfaces* are real rational conchoid surfaces. Furthermore we provide an algorithm to compute a rational polar representation of a given ruled surface and some examples. This section is the main contribution of Peternell, Gruber and Sendra (2011).

3.2.1 Ruled Surfaces

A ruled surface F carries a one parameter family of lines, called generating lines, and therefore it can be parameterized as follows

$$\mathbf{f}(u,v) = \mathbf{c}(u) + v\mathbf{e}(u),$$

where $\mathbf{c}(u)$ is a curve on F, called *directrix*, and $\mathbf{e}(u)$ is the *direction vector field* of the generating lines. A ruled surface is rational, if its directrix and the direction vector field are rational. In the following we want to choose the *foot-point curve* $\mathbf{p}(u)$, with respect to the reference point O, as directrix curve. If $\mathbf{c}(u), \mathbf{e}(u)$ are real and rational, then the foot-point curve is also a real curve with rational parameterization

$$\mathbf{p}(u) = \mathbf{c}(u) - \frac{\mathbf{c}(u)^T \cdot \mathbf{e}(u)}{\mathbf{e}(u)^T \cdot \mathbf{e}(u)}\mathbf{e}(u).$$

This foot-point curves always exists as long as $\|\mathbf{e}\| \neq 0,\ \forall u$, which would not define a ruled surface. If F is a cone with vertex in O, the curve $\mathbf{p}(u)$ degenerates to O and $\|\mathbf{p}(u)\| = 0,\ \forall u$.

The tangent planes of a ruled surface $\mathbf{f}(u,v) = \mathbf{p}(u) + v\mathbf{e}(u)$, are defined through the normal vectors

$$\mathbf{n}(u,v) = \mathbf{f}_u(u,v) \times \mathbf{f}_v(u,v) = (\mathbf{p}_u(u) + v\mathbf{e}_u(u)) \times \mathbf{e}(u) = \mathbf{n}_1(u) + v\mathbf{n}_2(u). \tag{3.4}$$

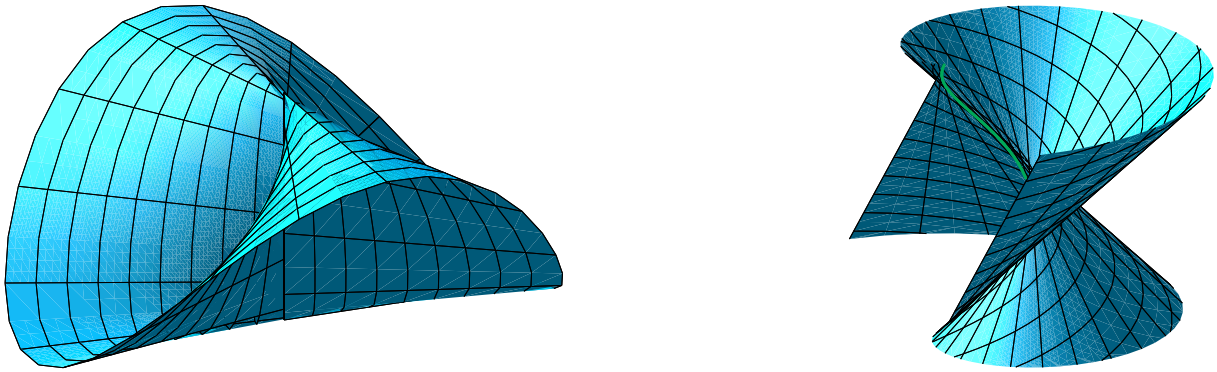
If the normals along a generating line are linearly dependent, the line is called *torsal*. A ruled surface that has only torsal generating lines is called a *torsal ruled surface* or *developable ruled surface*. If it has a finite set of torsal generating lines it is called *skew ruled surface*, see Figure 3.3(a) for an example. Torsal ruled surfaces are *cylinders, cones* and *tangent surfaces to spatial curves* or combinations of those. Given a tangent surface to a spacial curve, the curve itself is a locus of singular points of the surface, see Figure 3.3(b). A tangent surface is given by

$$\mathbf{f}(u,v) = \mathbf{c}(u) + v\dot{\mathbf{c}}(u)$$

where $\dot{\mathbf{c}}(u)$ denotes the derivative with respect to u. The normal vectors of F according to equation (3.4) then read

$$\mathbf{n}(u,v) = v(\dot{\mathbf{c}}(u) \times \ddot{\mathbf{c}}(u)).$$

If $\mathbf{c}(u)$ is a rational, $\dot{\mathbf{c}}(u)$ and $\ddot{\mathbf{c}}(u)$ s also rational and therefore the tangent surfaces to a rational spacial curve, is also rational.



(a) Plücker conoid (skew).

(b) Tangent surface of a helix (torsal).

Figure 3.3: Ruled surfaces.

3.2.2 Rational Conchoids of Rational Ruled Surfaces

Theorem 3.4 *Rational ruled surfaces F possess rational polar representations, hence they are rational conchoid surfaces. The real rational parameterizations can be constructed explicitly.*

Proof: We give a constructive proof of the theorem. Lets assume the rational ruled surface is given by a rational foot-point curve $\mathbf{p}(u)$ with respect to O and a rational direction vector field $\mathbf{e}(u)$,

$$\mathbf{f}(u,v) = \mathbf{p}(u) + v\mathbf{e}(u).$$

The corresponding cylinder A in the cone model, see Section 3.1, has the parameterization

$$\mathbf{a}(u,v,r) = \overline{\mathbf{p}}(u) + v\overline{\mathbf{e}}(u) + r\mathbf{w},$$

with $\overline{\mathbf{p}}(u) = (\mathbf{p}(u),0)^T$, $\overline{\mathbf{e}}(u) = (\mathbf{e}(u),0)^T$ and $\mathbf{w} = (0,0,0,1)^T$. Obviously this cylinder carries a one parameter family of planes $\varepsilon(u)$ spanned by $\overline{\mathbf{e}}(u)$ and $\mathbf{w}$. The intersection $\Phi = A \cap D$ is a surface that carries a one parameter family of conic sections $c(u) = \varepsilon(u) \cap D$. Since $\mathbf{p}(u)^T \cdot \mathbf{e}(u) = 0$, $\forall u$, substituting $\mathbf{a}(u,v,r)$ in $D : x^2 + y^2 + z^2 - w^2 = 0$ leads to a quadratic equation in v and r,

$$c(u) : \mathbf{p}(u)^2 + v^2\mathbf{e}(u)^2 - r^2 = 0.$$

Introducing homogeneous coordinates $v = x_2/x_1$ and $r = x_0/x_1$, the previous equation reads

$$c(u) : -x_0^2 + \mathbf{p}(u)^2 x_1^2 + \mathbf{e}(u)^2 x_2^2 = 0. \tag{3.5}$$

This defines a rational one parameter family of conic sections $c(u)$ in the projective plane $\mathbb{P}^2$ with coordinates x_0, x_1, x_2. A rational parameterization $\mathbf{z}(u,t)$ of $c(u)$ would permit a rational parameterization of Φ,

$$\Phi = \overline{\mathbf{p}}(u) + \frac{z_2(u,t)}{z_1(u,t)}\overline{\mathbf{e}}(u) + \frac{z_0(u,t)}{z_1(u,t)}\mathbf{w}.$$

To find a rational parameterization $\mathbf{z}(u,t)$ satisfying (3.5),

$$-z_0(u,t)^2 + \mathbf{p}(u)^2 z_1(u,t)^2 + \mathbf{e}(u)^2 z_2(u,t)^2 = 0, \tag{3.6}$$

we need the following

Lemma 3.5 *Let*

$$c(u) : (x_0, x_1, x_2) \cdot \mathbf{A}(u) \cdot (x_0, x_1, x_2)^T = 0, \tag{3.7}$$

be a one parameter family of conics in $\mathbb{P}^2$, with a symmetric matrix $\mathbf{A}(u) \in \mathbb{R}^{3\times 3}$ with real rational entries ($a_{ij}(u) \in \mathbb{R}(u)$), and $(x_0, x_1, x_2)\mathbb{R}$ are homogeneous coordinates in $\mathbb{P}^2$. If for all but finitely many $u \in \mathbb{R}$ the quadratic curve $c(u)$ contains more than one real point, then there exists real rational functions $y_0(u), y_1(u)$ and $y_2(u)$ which satisfy (3.7) *identically.*

Proof: For a full constructive proof see Peternell (1997) or Schicho (1997). We give an outline of the construction and assume that $\mathbf{A}(u)$ is a diagonal matrix. Let $\mathbf{A}(u) = \mathrm{diag}(-a_0(u)^2, a_1(u)^2, a_2(u)^2)$ with polynomials $a_0(u), a_1(u), a_2(u) \in \mathbb{R}(u)$ of degree l, m, n respectively. The family of conics (3.7) simplifies to

$$c(u) : -a_0(u)^2 x_0^2 + a_1(u)^2 x_1^2 + a_2(u)^2 x_2^2 = 0. \tag{3.8}$$

We assume that the polynomials $a_0(u)^2, a_1(u)^2, a_2(u)^2$ do not have common or multiple zeros and denote the conjugate complex zeros of $a_i(u)^2$ with $\alpha_{i,j}, \overline{\alpha}_{i,j},\ i \in \{0,1,2\}$. Substituting the zeros in 3.8 leads to a system of quadratic equations

$$\begin{aligned} +a_1(\alpha_{0,j})^2 x_1^2 + a_2(\alpha_{0,j})^2 x_2^2 &= 0, j \in \{0,\ldots,l\}, \\ -a_0(\alpha_{1,j})^2 x_0^2 + a_2(\alpha_{1,j})^2 x_2^2 &= 0, j \in \{0,\ldots,m\}, \\ -a_0(\alpha_{2,j})^2 x_0^2 + a_1(\alpha_{2,j})^2 x_1^2 &= 0, j \in \{0,\ldots,n\}. \end{aligned}$$

Factorizing these equations leads to at most $2(l+m+n)$ relevant linear equations. We make the general ansatz

$$y_0 = \sum_{i=0}^{m+n} y_{0i} u^i,\ y_1 = \sum_{j=0}^{l+n} y_{1j} u^j,\ y_2 = \sum_{k=0}^{l+m} y_{2k} u^k, \tag{3.9}$$

where we have $2(l+m+n)+3$ unknown. At least three degrees of freedom are left, and we choose them such that $-a_0(u)^2 y_0^2 + a_1(u)^2 y_1^2 + a_2(u)^2 y_2^2 = 0,\ \forall u \in \mathbb{R}$. The curve $\mathbf{y}(u)$

tracing the conic family has the degree $\max(m+n, l+n, l+m)$ $\square$

According to this Lemma, there exists a rational trajectory $\mathbf{y}(u)$ through a real rational family of conics where the conics possess more than one real point, except a finite set. The conics of the family (3.5) fulfill this condition for real rational $\mathbf{p}(u)$ and $\mathbf{e}(u)$, for an example of such a family of conics see Figure 3.6(a).

Stereographic projections with centers in $\mathbf{y}(u)$ of a line $\mathbf{g}(t)$ to the conics $c(u)$ provide a real rational parameterization $\mathbf{z}(u,t)$ according to (3.6). For example if we choose $\mathbf{g}(t)$ as the line at infinity $\mathbf{g}(t) = (0,1,t)^T\mathbb{R}$, the parameterization of the conics is given by

$$\mathbf{z}(u,t) = \mathbf{y}(u) - 2\frac{\mathbf{p}(u)^2 y_1(u) + \mathbf{e}(u)^2 y_2(u) t}{\mathbf{p}(u)^2 + \mathbf{e}(u)^2 t^2}\mathbf{g}(t). \tag{3.10}$$

The real rational polar representation of the ruled surface F follows

$$\mathbf{f}(u,t) = \varrho(u,t)\mathbf{k}(u,t) = \frac{1}{z_1(u,t)}(z_1(u,t)\mathbf{p}(u) + z_2(u,t)\mathbf{e}(u)), \tag{3.11}$$

with

$$\varrho(u,t) = \frac{z_0(u,t)}{z_1(u,t)} \text{ and } \mathbf{k}(u,t) = \frac{1}{z_0(u,t)}(z_1(u,t)\mathbf{p}(u) + z_2(u,t)\mathbf{e}(u)),$$

and the conchoids are given by

$$\mathbf{f}_d(u,v) = (\varrho(u,t) \pm d)\mathbf{k}(u,t) = \frac{z_0(u,t) \pm d z_1(u,t)}{z_0(u,t) z_1(u,t)}(z_1(u,t)\mathbf{p}(u) + z_2(u,t)\mathbf{e}(u)). \tag{3.12}$$

$\square$

The most difficult part in the algorithm is the calculation of the rational parameterization $\mathbf{z}(u,t)$ of the conics $c(u)$ in equation (3.5). More precisely the computation of the rational trajectory $\mathbf{y}(u)$. We show a numerical example in Appendix A to demonstrate the construction of $\mathbf{y}(u)$.

There are some special geometrically distinguished cases where $\mathbf{y}(u)$ can be computed in a simple way or it even degenerates to a point. We discuss some of this simple cases in the next sections.

3.2.3 Ruled Surfaces with Rational Length of Direction Vector Field

Let F be a rational ruled surface, with foot-point curve $\mathbf{p}(u)$ with respect to the origin and directions vectors $\mathbf{e}(u)$ with rational length. Since the direction vector field is unique up to scaling we choose $\frac{\mathbf{e}(u)}{\|\mathbf{e}(u)\|}$ as direction vector field of the ruled surface and denote it again

as $\mathbf{e}(u)$, therefore $\|\mathbf{e}(u)\| = 1$ holds. Consequently the corresponding family of conics (3.5) reads

$$c(u) : \mathbf{p}(u)^2 x_1^2 + x_2^2 = x_0^2. \tag{3.13}$$

Obviously these conics share the common points $(1,0,1)^T\mathbb{R}$ and $(1,0,-1)^T\mathbb{R}$. Applying the stereographic projection according to (3.10) to the family of conics $c(u)$ of (3.13), leads to the rational parameterization

$$\mathbf{z}(u,t) = (1 + \mathbf{p}(u)^2 t^2, 2t, 1 - \mathbf{p}(u)^2 t^2).$$

Furthermore the parameterization (3.11) of the ruled surface F simplifies to

$$\mathbf{f}(u,t) = \mathbf{p}(u) + \frac{1 - \mathbf{p}(u)^2 t^2}{2t}\mathbf{e}, \text{ with } \|\mathbf{f}(u,t)\| = \frac{1 + \mathbf{p}(u)^2 t^2}{2t}. \tag{3.14}$$

According to (3.12) one obtains a rational polar representation of F and its conchoid surfaces F_d,

$$\mathbf{f}_d(u,t) = \frac{1 + \mathbf{p}(u)^2 t^2 \pm 2td}{2t(1 + \mathbf{p}(u)^2 t^2)}(2t\mathbf{p}(u) + (1 - \mathbf{p}(u)^2 t^2)\mathbf{e}(u)). \tag{3.15}$$

Let us assume that the degree of the foot-point curve $\mathbf{p}(u)$ is n, and that the degree of the direction vector field is m. Then the rational bi-degree of the parameterization of the conchoids (3.15) is $(4n + m, 4)$. We want to give some examples with rational direction vector field $\mathbf{e}(u)$.

Example 3.6 *F is a rational cylinder*

We choose the coordinate system such that direction vector of F is $\mathbf{e} = (0,0,1)^T$. Hence the cross section curve $\mathbf{p}(u) = (p_1, p_2, 0)^T(u)$ with the $xy-$plane is the foot-point curve of F with respect to O. Therefore equations (3.14) and (3.15) read

$$\mathbf{f}(u,t) = \frac{1}{2t}\begin{pmatrix} 2tp_1(u) \\ 2tp_2(u) \\ 1 - \mathbf{p}(u)^2 t^2 \end{pmatrix} \text{ and } \mathbf{f}_d(u,t) = \frac{1 + \mathbf{p}(u)^2 t^2 \pm 2td}{2t(1 + \mathbf{p}(u)^2 t^2)}\begin{pmatrix} 2tp_1(u) \\ 2tp_2(u) \\ 1 - \mathbf{p}(u)^2 t^2 \end{pmatrix}.$$

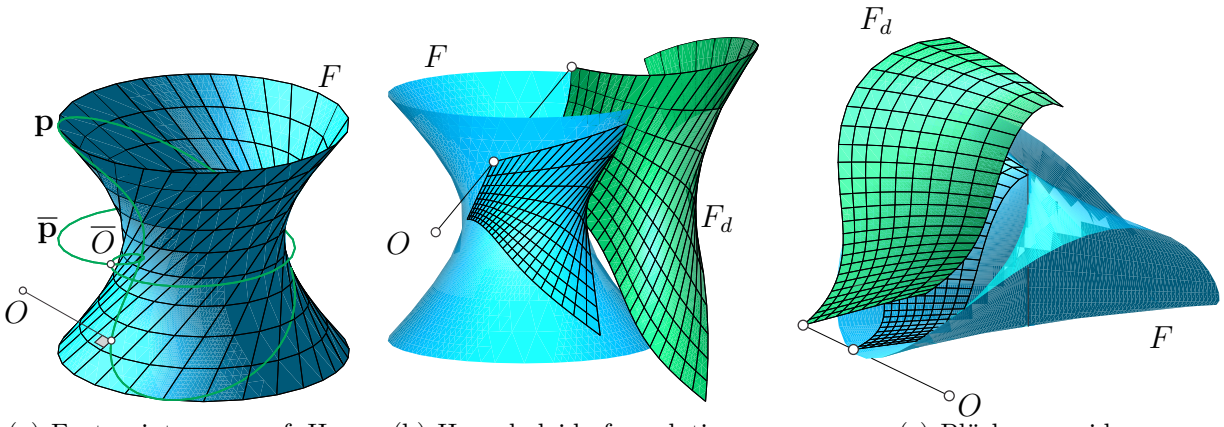
This is a rational parameterization of bi-degree $(4n, 4)$, with rational degree n of $\mathbf{p}(u)$. $\star$

Example 3.7 *F is a rotational ruled surface*

We may assume that the rotational axis of F is parallel to the z-axis of the coordinate system and that its direction vector is

$$\mathbf{e}(u) = (\cos(\alpha)\cos(u), \cos(\alpha)\sin(u), \sin(\alpha))^T,$$

with $\alpha = \text{const}$. Besides the trivial cases $\alpha = 0$ where F is a plane, and $\alpha = \pi/2$ where F is a rotational cylinder, F is a rotational hyperboloid of one sheet. An example is illustrated in Fig. 3.4(b).



(a) Foot-point curve of Hyperboloid.

(b) Hyperboloid of revolution.

(c) Plücker conoid.

Figure 3.4: Conchoids F_d of different ruled surfaces at distance d.

We want to discuss a rotational hyperboloid of one sheet F with the $xy-$plane as symmetry plane. The circle in the $xy-$plane has radius r and mid-point $(m, 0, 0)^T$. The surface has the parameterization

$$\mathbf{f}(u, v) = \begin{pmatrix} r\sin(u) + m \\ -r\cos(u) \\ 0 \end{pmatrix} + v \begin{pmatrix} \cos(\alpha)\cos(u) \\ \cos(\alpha)\sin(u) \\ \sin(\alpha) \end{pmatrix}.$$

The foot-point curve $\mathbf{p}$ with respect to O, see Figure 3.4(a), has the parameterization

$$\mathbf{p}(u) = \begin{pmatrix} r\sin(u) - m\cos(\alpha)^2\cos(u)^2 + m \\ -\cos(u)(r + m\cos(\alpha)^2\sin(u)) \\ -m\cos(\alpha)\sin(\alpha)\cos(u) \end{pmatrix}. \tag{3.16}$$

Shape of the foot-point curve It turns out that the normal projection $\overline{\mathbf{p}}(u)$ of $\mathbf{p}(u)$ to the $xy-$plane is a conchoid of the circle with mid-point $M = (m(1+\sin(\alpha)^2)/2, 0, 0)^T$ and Radius $R = m\cos(\alpha)^2/2$ at distance r with respect to the point $\overline{O} = (m\sin(\alpha)^2, 0, 0)^T$, see Figure 3.4(a).

Substituting (3.16) in (3.14) and (3.15) leads to trigonometric polar representations of F and F_d.

$\star$

Example 3.8 *Ruled surface by rational motion to a line*

More general examples are obtained by applying a rational motion to a line. Therefore the direction vector $\mathbf{e}(u)$ has constant length and defines a rational curve in the unit sphere. We want to discuss this special case exemplarily for the *Plücker conoid*.

The Plücker conoid F is an algebraic ruled surface of order three, also called cylindroid, it is projectively equivalent to the Whitney umbrella. A trigonometric parameterization with the double line as z-axis reads $(0, 0, \sin 2u)^T + v(\cos u, \sin u, 0)^T$. It can be generated in the following way. Rotate the x-axis around z and superimpose this rotation by the translation $(0, 0, \sin 2u)^T$ in z-direction. An implicit equation of F is $z(x^2 + y^2) = 2xy$.

Since the z-axis is a double line of F, the origin is a double point and the computation of the conchoid with respect to O is trivial. Thus we apply a translation by $(0, 1, 2)^T$. A rational parameterization of the translated surface, which is again denoted by F is given by

$$\mathbf{f}(u, v) = \frac{1}{(u^2 + 1)^2} \begin{pmatrix} (1 - u^4)v \\ (u^2 + 2vu + 1)(u^2 + 1) \\ 2(u^4 - 2u^3 + 2u^2 + 2u + 1) \end{pmatrix}. \tag{3.17}$$

The squared length of $\mathbf{f}(u, v)$ corresponds to the family of conics

$$c(u) : -(u^2 + 1)^4 x_0^2 + \alpha(u) x_1^2 + 4u(u^2 + 1)^3 x_1 x_2 + (u^2 + 1)^4 x_2^2 = 0, \tag{3.18}$$

with $\alpha(u) = 5(u^2 + 1)^4 + 16u(u^2 - 1)((u^2 + 1)^2 + u(u^2 - 1))$. It is obvious that these conics share the vertices $(\pm 1, 0, 1)^T \mathbb{R}$.

A rational solution $\mathbf{z}(u, t) = (z_0, z_1, z_2)^T(u, t)$ of (3.18) is computed by stereographic projection of the line $(0, 1, t)^T \mathbb{R}$ onto the conics from (3.18) with center $(1, 0, 1)^T \mathbb{R}$. This leads to the reparameterization along the generating lines of F,

$$v(u, t) = \frac{z_2}{z_1} = -\frac{\alpha(u)t^2 + (u^2 + 1)^2(4tu - 1)}{2(u^2 + 1)^3 t}. \tag{3.19}$$

The surface F has the rational radius function

$$\|\mathbf{f}(u, t)\| = \frac{z_0(u, t)}{z_1(u, t)} = -\frac{\alpha(u)t^2 + (u^2 + 1)^2}{2(u^2 + 1)^3 t}.$$

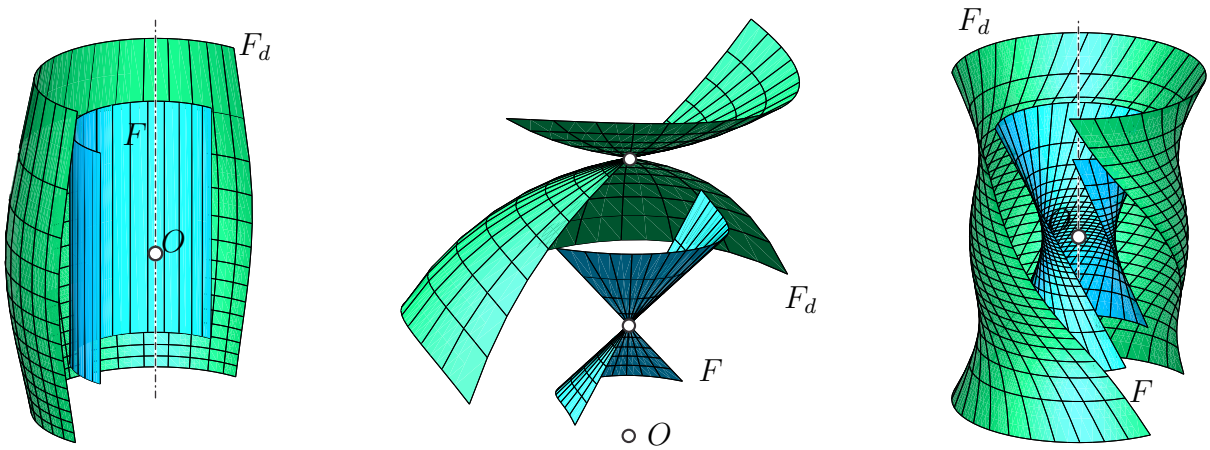
Substituting (3.19) in (3.17) leads to a rational parameterization of $\mathbf{f}(u, t)$ and a rational parameterization of the conchoid F_d with respect to O and distance $d = 1$

$$\mathbf{f}_d(u, t) = \begin{pmatrix} \beta(u, t)(u^2 - 1)(\alpha(u)t^2 + (u^2 + 1)^2(4tu - 1)) \\ \beta(u, t)(-\alpha(u)t^2 u + (u^4 - 1)^2 t + (u^2 + 1)u) \\ 4t\beta(u, t)(u^2 + 1)^2((u^2 + 1)^2 - 2(u^2 + 2)u) \end{pmatrix}$$

with

$$\beta(u, t) = \frac{\alpha(u)t^2 + 2(u^2 + 1)^3 t + (u^2 + 1)^2}{2t(u^2 + 1)^2(\alpha(u)(u^2 + 1)^2 t^2 + (u^2 + 1)^4)}.$$

$\star$



(a) Rot. cylinder. (b) Rational cone. (c) Rot. hyperboloid.

Figure 3.5: Ruled surfaces and their conchoids.

Other special cases of rational ruled surfaces F occur if the norms $\|\mathbf{p}(u)\|$ and $\|\mathbf{e}(u)\|$ in equation (3.5) are both rational. We may assume $\|\mathbf{e}\| = 1$ and we denote $\|\mathbf{p}(u)\| = \alpha(u)$. Thus there exists a spherical rational curve $\mathbf{a}(u)$ with $\|\mathbf{a}(u)\| = 1$ and $\mathbf{p}(u) = \alpha(u)\mathbf{a}(u)$. The spherical part $\sigma(F)$ consists of great circles being contained in planes spanned by the *rational orthogonal unit vectors* $\mathbf{a}(u)$ and $\mathbf{e}(u)$ and $\sigma(F)$ admits the parameterization

$$\mathbf{k}(u,t) = \mathbf{a}(u)\cos t + \mathbf{e}(u)\sin t.$$

To determine the radius function $\varrho(u,t)$ of $\mathbf{f}(u,t) = \varrho(u,t)\mathbf{k}(u,t)$, the parameterization $\mathbf{z}(u,t) = (\alpha(u), \cos t, \alpha(u)\sin t)$ of the conics $c(u)$ from (3.5) leads to $\varrho(u,t) = z_0/z_1$.

This case occurs when computing conchoid surfaces F_d of rotational ruled surfaces F with respect to a point O on the rotational axis. Examples are illustrated in Figure 3.5. It is evident that the conchoid surface F_d of F is a rotational surface. The generating curve of the conchoid F_d is the conchoid curve with respect to O of a generating line of F.

3.2.4 Conchoid Surfaces of Rational Cones

Let F be a rational cone with vertex $\mathbf{v} = (0,0,1)^T$ and directrix $\mathbf{c}(u) = (c_1(u), c_2(u), 0)^T$. For dealing with the general case we assume that $O \notin F$. Then F is parameterized by

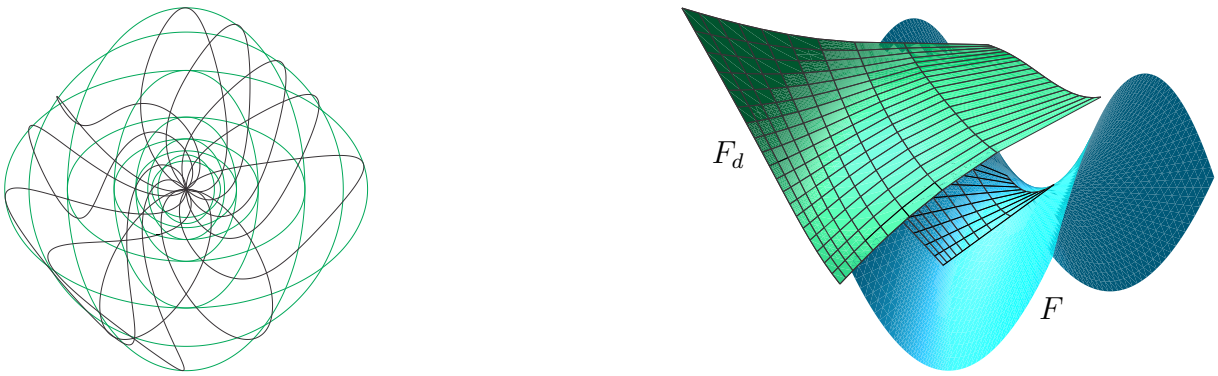
$$\mathbf{f}(u,v) = \mathbf{v} + v(\mathbf{c}(u) - \mathbf{v}) = \mathbf{v} + v\mathbf{e}(u),$$

with $\mathbf{e}(u) = (c_1(u), c_2(u), -1)$. With respect to these choices the squared length of $\mathbf{f}(u,v)$ is $\|\mathbf{f}(u,v)\|^2 = 1 - 2v + \mathbf{e}(u)^2 v^2$. The family of conics reads

$$c(u) : x_1^2 - 2x_1x_2 + x_2^2\mathbf{e}(u)^2 = x_0^2.$$

The conics $c(u)$ share two common points $(1,-1,0)^T\mathbb{R}$ and $(1,1,0)^T\mathbb{R}$, and a stereographic projection results in their rational parameterization

$$\mathbf{z}(u,t) = (1 - 2t + \mathbf{e}(u)^2t^2, 1 - \mathbf{e}(u)^2t^2, 2t(1-t))^T\mathbb{R}.$$



(a) Family of conics (green) with rational trajectories.

(b) Hyperbolic Paraboloid.

Figure 3.6: Conchoid of HP-Surface.

Substituting $v = z_2/z_1$ in $\mathbf{f}(u,v)$ gives the rational parameterization

$$\mathbf{f}(u,t) = \mathbf{v} + \frac{2t(1-t)}{1-\mathbf{e}(u)^2t^2}\mathbf{e}(u), \text{ with } \|\mathbf{f}(u,t)\| = \frac{z_0(u,t)}{z_1(u,t)} = \frac{1-2t+\mathbf{e}(u)^2t^2}{1-\mathbf{e}(u)^2t^2}.$$

Rational polar representations of F and its conchoid surfaces F_d are obtained with (3.11) and (3.12).

3.2.5 Conchoid Surfaces of Quadratic Ruled Surfaces

Let F be a real quadratic ruled surface, hence a hyperboloid of one sheet or a hyperbolic paraboloid, see Table 1.1. Depending on the incidence of O and F we can transform F, according to Sections 3.3.2 and 3.3.5, either to a hyperboloid of one sheet, in normal form $F' : -1 - a^2x^2 + b^2y^2 + c^2z^2 = 0$, or to the hyperbolic paraboloid $F' : z = -ax^2 + by^2$.

Instead of the method mentioned in the previous sections we use a different approach, that leads to lower bi-degrees for the polar representation of quadratic ruled surfaces. This approach can be used for every real rational ruled surface

Given the ruled surface F by

$$\mathbf{f}(u,v) = \mathbf{c}(u) + v\mathbf{e}(u).$$

We want to parameterize the spherical part $\mathbf{k}(u,v)$ of the polar parameterization $\mathbf{f}(u,v) = \varrho(u,v)\mathbf{k}(u,v)$ and then deduce the radius function $\varrho(u,v)$.

The spherical part $\mathbf{k}(u,v)$ of a ruled surface is a family of great circles contained in planes $\alpha(u)$ on the unit sphere S^2, with carrier planes $\alpha(u) = \mathbb{R}(0,\mathbf{a}(u))^T = \mathbb{R}(0,\mathbf{c}(u) \times \mathbf{e}(u))^T$. The stereographic projection with center in $(0,0,1)^T$ maps this family of circles on the sphere, to a family of circles $C(u)$ in the plane $z = 0$,

$$C(u) : \left(x + \frac{a_1(u)}{a_3(u)}\right)^2 + \left(y + \frac{a_2(u)}{a_3(u)}\right)^2 - \frac{\|\mathbf{a}(u)\|^2}{a_3(u)^2} = 0.$$

Decomposing $\|\mathbf{a}(u)\|^2$ into the sum of two squares $h_1(u), h_2(u)$ leads to a rational trajectory Q of the circles C,

$$\mathbf{q}(u) = \frac{1}{a_3(u)} \begin{pmatrix} h_1(u) - a_1(u) \\ h_2(u) - a_2(u) \end{pmatrix}. \tag{3.20}$$

The decomposition is possible, if $\|\mathbf{a}(u)\|^2$ is a non-negative polynomial, this is fulfilled since $\mathbf{a}(u)$ defines a real curve on S^2.

With the trajectory $\mathbf{q}(u)$ of (3.20) we construct a rational parameterization $\mathbf{d}(u,t)$ of the circles $C(u)$. The inverse stereographic projection leads to a rational parameterization $\mathbf{k}(u,t)$ of S^2. See Figure 3.6(a) for the circles $C(u)$ of the hyperbolic paraboloid F shown in Figure 3.6(b).

In the next step we calculate the radius function $\varrho(u,t)$. Consider the planes $\nu(u)$ spanned by the generating lines $\mathbf{c}(u) + v\mathbf{e}(u)$ and the vector $\mathbf{a}(u)$. The intersection of the line bundle $\varrho(u,t)\mathbf{k}(u,t)$ with $\nu(u) : (\mathbf{x} - \mathbf{c}(u))^T \cdot \mathbf{n}(u) = 0$, with $\mathbf{n}(u) = \mathbf{e}(u) \times \mathbf{a}(u)$, leads to the radius function

$$\varrho(u,t) = \frac{\mathbf{c}(u) \cdot \mathbf{n}(u)}{\mathbf{k}(u,t) \cdot \mathbf{n}(u)} = \frac{\|\mathbf{c}(u) \times \mathbf{e}(u)\|}{\mathbf{k}(u,t) \cdot \mathbf{n}(u)} = \frac{\|\mathbf{a}(u)\|^2}{\mathbf{k}(u) \cdot \mathbf{n}(u)}. \tag{3.21}$$

Example 3.9 *Hyperboloid of one sheet*

Let the hyperboloid of one sheet $F : -1 - a^2x^2 + b^2y^2 + c^2z^2 = 0$ be parameterized by

$$\mathbf{f}(u,v) = \mathbf{c}(u) + v\mathbf{e}(u) = \begin{pmatrix} \frac{-u}{a} \\ \frac{-u}{b} \\ \frac{1}{c} \end{pmatrix} + v \begin{pmatrix} \frac{-u^2-1}{au} \\ \frac{u^2-1}{bu} \\ \frac{2}{c} \end{pmatrix}.$$

The family of circles $C(u)$ in the plane $z = 0$ read

$$C(u) : \left(x + \frac{a(1+u^2)}{2cu}\right)^2 + \left(y + \frac{b(1-u^2)}{2cu}\right)^2 - \frac{(\gamma(1-u^2))^2 + (2\beta u)^2}{4c^2u^2} = 0,$$

with $\beta = \sqrt{a^2+c^2}$ and $\gamma = \sqrt{a^2+b^2}$. Hence the quadratic rational trajectory Q is given by

$$\mathbf{q}(u) = -\frac{1}{2cu} \begin{pmatrix} \gamma(1-u^2) + a(1+u^2) \\ 2\beta u + b(1-u^2) \end{pmatrix}.$$

Parameterizing the family of circles $C(u)$ with Q and inverse stereographic projection leads to the rational spherical part $\mathbf{k}(u,t)$ of F of bi-degree $(4,2)$,

$$\mathbf{k}(u,t) = \frac{1}{\mu(u,t) + 2c^2(1+t^2)u^2} \begin{pmatrix} 4cu(\gamma(1-u^2)(1-t^2) - a(1+u^2)(1+t^2) + 4\beta ut) \\ 4cu(-2\beta u(1-t^2) + (2t\gamma - b(1+t^2))(1-u^2)) \\ \mu(u,t) \end{pmatrix}.$$

with

$$\begin{aligned}\mu(u,t) = \;& 2\gamma((u^4+1)(-2bt+\gamma(1+t^2))+(1-u^4)a(t^2-1))+\\ & (4(2bt\gamma+(a^2-b^2)(1+t^2)))u^2+4\beta u(-2at(u^2+1)+(1-u^2)b(1-t^2)).\end{aligned}$$

Equation (3.21) leads to the rational radius function $\varrho(u,t)$ of F of bi-degree $(4,2)$,

$$\varrho(u,t) = \frac{\mu(u,t)+2c^2(1+t^2)u^2}{\nu(u,t)},$$

with denominator

$$\begin{aligned}\nu(u,t) = \;& 2c(\gamma((-2bt+\gamma(1+t^2))(1-u^4)-a(1-t^2)(u^4+1))\\ & +2u\beta(b(1-t^2)(u^2+1)-2at(1-u^2))+2\gamma a(1-t^2)u^2).\end{aligned}$$

The hyperboloid of one sheet has a rational polar representation $\mathbf{f}(u,t) = \varrho(u,t)\mathbf{k}(u,t)$ of bi-degree $(4,2)$, since the denominator of $\mathbf{k}(u,t)$ equals the numerator of $\varrho(u,t)$. Obviously the bi-degree of the rational polar representation $\mathbf{f}_d(u,t) = (\varrho(u,t)+d)\mathbf{k}(u,t)$ of the conchoids F_d of F raises and it is $(8,4)$.

$\star$

3.3 Conchoids of Quadrics

In this section we prove that real *quadrics* are real rational conchoid surfaces. Furthermore we provide an algorithm to compute a rational polar representation of a given quadric and its conchoid surfaces and give some examples. This section is the main contribution of Gruber and Peternell (2013).

3.3.1 Conchoids of Quadrics – Theory

A quadric $F \subset \mathbb{R}^3$ is the zero set of a quadratic equation in x, y and z, see Section 1.1.3. In the following F denotes both, the quadric as well as its defining polynomial $F(x, y, z) = 0$, since it should be clear from the context whether F denotes a surface or a polynomial. We assume that the polynomial F has real coefficients and that the quadric F has more than one real point.

Quadrics $F \subset \mathbb{R}^3$ and conics $c \subset \mathbb{R}^2$ admit real rational parameterizations, see for example Lü (1996). The conchoid curves c_d of conics $c \subset \mathbb{R}^2$ with respect to an arbitrary reference point O are typically non-rational curves, see Section 2.3. Thus it is quite surprising that conchoid surfaces F_d of quadrics $F \subset \mathbb{R}^3$ admit real rational parameterizations, with respect to any reference point $O \in \mathbb{R}^3$.

The proof of this statement is constructive and is performed in several steps. We provide a symbolic computation of real rational polar representations of quadrics. An outline of the construction reads as follows:

- Apply admissible transformations to represent a quadric F by a normal form, see Section 3.1.1 for information about these transformations.
- Compute the associated pencil of quadrics in $\mathbb{R}^4$. Its base locus Φ carries a rational one-parameter family of real conics $L(u)$.

Theoretically these steps already proof the existence of a real rational parameterization of the quadric. To obtain a parameterization of low bi-degree we perform the following steps in addition.

- Conics $L(u) \subset \Phi$ are transformed to circles $\overline{C}(u)$ in S^2. We provide explicit parameterizations of $\overline{C}(u) \subset S^2$ which imply rational parameterizations of Φ.
- Rational parameterizations of Φ correspond to rational polar representations of F.

3.3.2 Transformation to Normal Form

A quadric F in $\mathbb{P}^3$ is given by the equation

$$F(x, y, z) = \mathbf{x}^T \cdot \mathbf{M} \cdot \mathbf{x} = 0, \text{ with } \mathbf{M}^T = \mathbf{M} \in \mathbb{R}^{4\times 4} \text{ and } \mathbf{x} = (1, x, y, z)^T,$$

where $\mathbf{M}$ is a symmetric 4×4 matrix with real entries. We assume that the matrix $\mathbf{M}$ has rank four, and that $O \notin F$. The case of $\mathrm{rk}(M) < 4$ and other special cases are discussed

in detail in Sections 3.3.4, 3.3.5 and 3.3.6. Due to some nice geometric properties, the case that F is a sphere with mid point not in O, will be discussed separately in Section 3.4.

We apply admissible transformations according to Section 3.1.1 and coordinate transformations, such that the image quadric is represented by a diagonal matrix. For the excluded quadrics of the previous paragraph, this transformation is either not possible or not necessary, hence we discuss them in distinct sections.

We perform this transformation in two steps. First we apply a perspective collineation $\varkappa$ with center O according to Corollary 3.3. Assume that $\mathbf{M}$ has entries $m_{ij} \in \mathbb{R}$, with $i, j = 1, \ldots, 4$, then this transformation reads

$$\varkappa : \overline{\mathbf{x}}' = \frac{1}{s(\mathbf{x})}\overline{\mathbf{x}}, \text{ with } s(\mathbf{x}) = m_{11} + m_{12}x + m_{13}y + m_{14}z \text{ and } \overline{\mathbf{x}} = (x, y, z^T). \tag{3.22}$$

The polar plane δ of O with respect to F is given by $s(\mathbf{x}) = 0$. Thus $\varkappa$ maps δ to the ideal plane $\omega = \mathbb{P}^3 \setminus \mathbb{R}^3$ of the projective space $\mathbb{P}^3$ extending $\mathbb{R}^3$. The perspective collineation $\varkappa$ maps F to the quadric

$$F' : \frac{1}{m_{11}} + \overline{\mathbf{x}}'^T \cdot \mathbf{M}' \cdot \overline{\mathbf{x}}' = 0,$$

with a symmetric 3×3 matrix $\mathbf{M}'$. The equation of F' does no longer contain linear terms in x, y and z. Since $O \notin F$ and thus $O \notin \delta$, the origin O becomes the center of the transformed quadric F'. Further we may assume that $m_{11} = \pm 1$. Depending on the position of O and δ with respect to F, there exist different affine types of F'. These types are determined by the intersection $F' \cap \omega$, the ideal conic of F'. It is the image of the conic $F \cap \delta$ with respect to the map $\varkappa$.

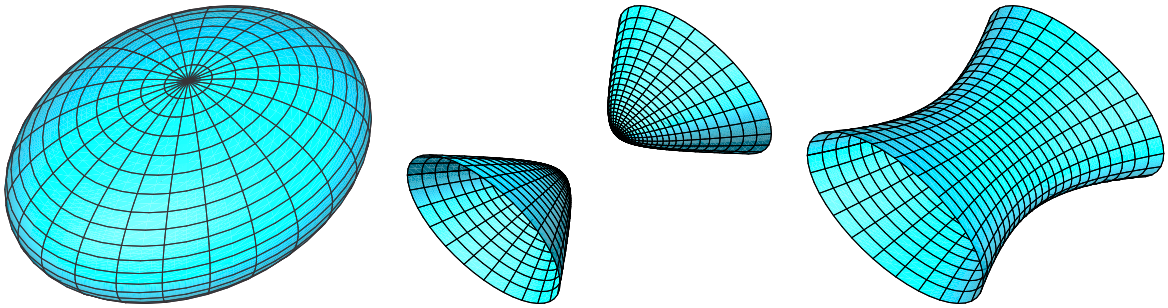
- If O is inside of F, the intersection $F \cap \delta$ does not contain real points. Since $\varkappa : \delta \mapsto \omega$, the quadric F' is an *ellipsoid*.
- Otherwise if O is outside F, the intersection $F \cap \delta$ is a conic containing real points. Same arguments imply that F' is a *hyperboloid*, either *of one sheet* or *of two sheets*.
- Since O is the center of F', it is never a paraboloid.

In a second step we apply a coordinate transformation where the new coordinate axes are chosen as eigenvectors of $\mathbf{M}'$. This can be considered as rotation fixing O. Thus F' is represented by a diagonal matrix, and reads

$$F' : \mathbf{x}^T \cdot \operatorname{diag}(\pm 1, \pm a^2, \pm b^2, \pm c^2) \cdot \mathbf{x} = 0. \tag{3.23}$$

If all signs in (3.23) are positive, F' is a quadric without real points. Otherwise in case of strict inequalities between a, b and c and with a proper re-ordering of the coordinate axes the different combinations of signs imply the normal forms of Table 3.1.

In case that there are two coincident eigenvalues, say $b = c$, F' is a rotational quadric with x as axis. In case that all three eigenvalues coincide, F' is a sphere, centered at O. These particular cases where the computation simplifies significantly, are postponed to Section 3.3.6.



(a) Ellipsoid. (b) Hyperboloid of two sheets. (c) Hyperboloid of one sheet.

Figure 3.7: Quadrics in normal form.

Ellipsoid	$F' : -1 + a^2x^2 + b^2y^2 + c^2z^2 = 0,$
Hyperboloid of two sheets	$F' : 1 - a^2x^2 + b^2y^2 + c^2z^2 = 0,$
Hyperboloid of one sheet	$F' : -1 - a^2x^2 + b^2y^2 + c^2z^2 = 0.$

Table 3.1: Normal forms of quadrics, illustrated in Figure 3.7.

Quadric Pencil and Base Locus

Consider a quadric represented by one of the normal forms, listed in Table 3.1. For simplicity it shall be denoted again by F instead of F'. We show that F contains a one-parameter family of conics admitting a real rational polar representation. Let $A \subset \mathbb{R}^4$ be the three-dimensional quadratic cylinder through F, whose generating lines are parallel to the w-axis. Thus A is represented by

$$A(x, y, z, w) = F(x, y, z) = 0.$$

Additionally consider the quadratic cone $D : x^2 + y^2 + z^2 - w^2 = 0$ from equation (3.1). Let $\mathcal{B}$ be the pencil of quadrics in $\mathbb{R}^4$ spanned by A and D, see Figure 3.8(a),

$$\mathcal{B}(\alpha, \beta) = \alpha A + \beta D \subset \mathbb{R}^4, \text{ with } (\alpha, \beta) \in \mathbb{R}^2 \setminus (0, 0).$$

A pencil of quadrics in $\mathbb{R}^4$ contains up to five singular quadrics. If one of these singular quadrics, say B, is a cone over a real ruled quadric, the cone B contains two one-parameter families of real planes $\psi(u)$, $u \in \mathbb{R}$, corresponding to the generating lines of the ruled quadric.

Consider the intersection surface $\Phi = A \cap D = \pi^{-1}(F)$, the base locus of the pencil of quadrics $\mathcal{B}$. The two-dimensional surface Φ is a so called *del Pezzo surface* of degree four, and it is known that it admits real rational parameterizations, even a proper one

over an algebraically closed field. For detailed information about del Pezzo surfaces see for example Griffiths and Harris (1978); Manin (1974); Schicho (1998).

Assume that the singular quadric B contains real planes $\psi(u)$. Then there exists a subset $\psi(s)$, $s \in I \subset \mathbb{R}$ of planes intersecting D in real conics $L(s) = D \cap \psi(s)$, compare equation (3.31) in Section 3.3.3. Thus Φ contains the real rational family of conics $L(s)$. It has been proved in Peternell (1997) and Schicho (1998) that such a family of conics always admits a real rational parameterization.

We have a look at the three different normal forms of F, and show that the pencil of quadrics $\mathcal{B}$ contains a cone over a ruled quadric, which implies that $\Phi = A \cap D$ carries a rational one-parameter family of real conics $L(s)$.

- Given an ellipsoid $F : -1 + a^2x^2 + b^2y^2 + c^2z^2 = 0$ in $\mathbb{R}^3$, with $0 < a^2 < b^2 < c^2$. The respective pencil of quadrics $\mathcal{B}$ contains the cylinder

$$B : -1 - (b^2 - a^2)x^2 + (c^2 - b^2)z^2 + b^2w^2 = 0, \tag{3.24}$$

  which is a cylinder over the ruled two-dimensional quadric $B \cap (y = 0)$.

- Given a hyperboloid of two sheets $F : 1 - a^2x^2 + b^2y^2 + c^2z^2 = 0$ in $\mathbb{R}^3$, with $0 < b^2 < c^2$. The respective pencil of quadrics contains the cylinder

$$B : 1 - (c^2 + a^2)x^2 - (c^2 - b^2)y^2 + c^2w^2 = 0, \tag{3.25}$$

  which is a cylinder over the ruled two-dimensional quadric $B \cap (z = 0)$.

- Given the hyperboloid of one sheet $F : -1 - a^2x^2 + b^2y^2 + c^2z^2 = 0$ in $\mathbb{R}^3$, with $0 < b^2 < c^2$. The corresponding cylinder A is already a cylinder over the ruled quadric F. The respective pencil of quadrics contains two further cylinders

$$\begin{aligned} B_1 &: -1 + (a^2 + b^2)y^2 + (a^2 + c^2)z^2 - a^2w^2 = 0, \text{ and} \\ B_2 &: -1 - (a^2 + b^2)x^2 + (c^2 - b^2)z^2 + b^2w^2 = 0, \end{aligned} \tag{3.26}$$

  over ruled quadrics $B_1 \cap x = 0$ and $B_2 \cap y = 0$, respectively.

Theorem 3.10 *A quadric $F \subset \mathbb{R}^3$ is a rational conchoid surface independent of the position of the reference point O and the distance d.*

Of course it remains to find an explicit rational parameterization of Φ which implies a rational polar representation of F. All details on this construction are found in Section 3.3.3.

The problem of computing a real rational parameterization of the intersection Φ of two quadrics in $\mathbb{R}^4$, has already been studied in Aigner et al. (2009). This particular method needs an educated guess two times. First, one has to find a point $P \in \Phi$. The projection of Φ from P to three-dimensional space is a cubic surface, say Ψ. Second, one has to find a line g on the cubic surface Ψ. Consider the one parameter family of planes $\varepsilon(t)$ through

g. The intersection $\varepsilon(t) \cap \Psi$ consists of g and a curve $k(t)$ of degree two, typically a conic. A real rational parameterization of this family of curves $k(t)$ on Ψ is lifted back to a real rational parameterization of Φ. The bi-degree of the parameterization of Ψ is $(7, 2)$. An explicit example in that contribution shows that one can expect $(12, 4)$ for the bi-degree of the final parameterization of Φ. We improve this result and provide an algorithm in Section 3.3.3 resulting in a real rational parameterization of Φ with bi-degree $(6, 2)$.

3.3.3 Parameterization of the Family of Conics

The details of the construction of a real rational polar representation of a quadric $F \subset \mathbb{R}^3$ are performed exemplarily, if the normal form is an ellipsoid. The construction is analogous in the case that F is a hyperboloid, for the formulae see Appendix B.

In case that F is an ellipsoid, the singular quadric B of the respective pencil of quadrics $\mathcal{B} \subset \mathbb{R}^4$ is given by (3.24). A rational parameterization of B reads

$$\mathbf{b}(u, v_1, v_2) = \mathbf{e}_0(u) + v_1\mathbf{e}_1(u) + v_2\mathbf{e}_2, \tag{3.27}$$

with

$$\mathbf{e}_0(u) = \left(-\frac{u}{\gamma}, 0, -\frac{u}{\alpha}, \frac{1}{b}\right)^T, \ \mathbf{e}_1(u) = \left(-\frac{u^2+1}{u\gamma}, 0, -\frac{u^2-1}{u\alpha}, \frac{2}{b}\right)^T, \text{ and } \mathbf{e}_2 = (0, 1, 0, 0)^T,$$

where we use the abbreviations $\gamma = \sqrt{b^2 - a^2}$ and $\alpha = \sqrt{c^2 - b^2}$. The generating planes $\psi(u)$ of B are spanned by $\mathbf{e}_1(u)$ and $\mathbf{e}_2$. In order to obtain a quadratic equation (3.28) without linear terms in v_1, the directrix curve of B in (3.27) is chosen as $\mathbf{e}_0 + \lambda(u)\mathbf{e}_1$, with

$$\lambda(u) = \frac{(a^2b^2(u^2+1) - c^2b^2(u^2-1) - 2a^2c^2)u^2}{a^2b^2(u^2+1)^2 - c^2b^2(u^2-1)^2 - 4a^2c^2u^2}.$$

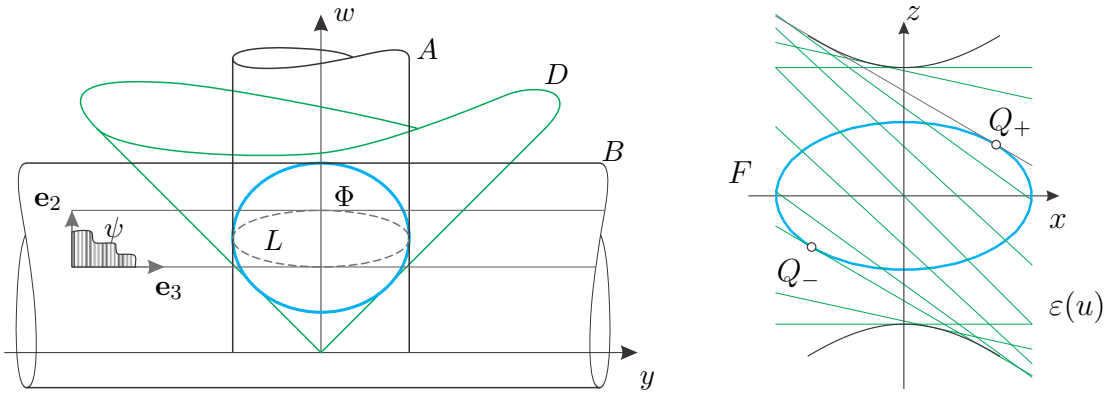
The family of conics $L(u) \subset \Phi$, where $\Phi = A \cap D$ is obtained by inserting (3.27) into the implicit representation of D, is given by

$$L(u) : l_0(u) + l_1(u)v_1^2 + l_2(u)v_2^2 = 0, \tag{3.28}$$

whose coefficients are the polynomials

$$\begin{aligned} l_0(u) &= b^2\alpha^2\gamma^2u^2(c^2(u^2-1)^2 - a^2(u^2+1)^2), \\ l_1(u) &= (b^2(c^2(u^2-1)^2 - a^2(u^2+1)^2) + 4a^2c^2u^2)^2, \\ l_2(u) &= -b^2\alpha^2\gamma^2u^2(b^2(c^2(u^2-1)^2 - a^2(u^2+1)^2) + 4a^2c^2u^2). \end{aligned}$$

The final aim is to find real rational functions $(v_1(u,t), v_2(u,t))$ satisfying equation (3.28) identically. First of all $L(u)$ has to contain real points for all $u \in \mathbb{R}$. If this is not the case it is necessary to substitute $u = (u_0s^2 + u_1)/(s^2+1)$ such that $L(s)$ satisfies this requirement for all $s \in \mathbb{R}$. In the next step one computes the zeros of the polynomials



(a) Pencil of quadrics in $\mathbb{R}^4$.

(b) Envelope of $\varepsilon(u)$.

Figure 3.8: Illustration of the situation in $\mathbb{R}^4$ and planes $\varepsilon(u)$.

$l_i(s)$. Real zeros appear with even multiplicities. In case that two of the polynomials $l_i(s)$ have common zeros, equation (3.28) can be simplified. Finally we end up with an equation $L(s)$ of the form (3.28) where no two polynomials have common zeros. In the present case these polynomials are of degrees ≤ 8. To construct real rational functions $(v_1(s,t), v_2(s,t))$ satisfying $L(s)$ identically, a linear system combined with a quadratic equation has to be solved. To our knowledge it is not possible to compute a symbolic solution for $v_1(s,t)$ and $v_2(s,t)$, but only numeric solutions are available. In addition, the degrees of the final parameterization of Φ are unnecessarily high.

Since this direct method does not result in a symbolic parameterization of Φ, further geometric properties of the family of conics $L(u)$ have to be investigated. All proposed computational steps can be carried out symbolically with help of a computer-algebra-system.

- The rational family of conics $L(u) \subset \Phi \subset \mathbb{R}^4$ is transformed to a rational family of circles $\overline{C}(u) \subset S^2 \subset \mathbb{R}^3$.
- A real rational parameterization of $\overline{C}(u)$ is constructed explicitly.
- A real rational parameterization of Φ corresponds to a real rational polar representation of the quadric F.

Cones of Revolution

Consider the top view projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ with $\pi(x,y,z,w) = (x,y,z)$. We intend to prove that the top-view projections $C(u) = \pi(L(u))$ are a family of conics which are contained in cones of revolution $\Gamma(u)$, with common vertex at the origin O. To achieve this we investigate at first the intersection of the cone $D \subset \mathbb{R}^4$ with a generic three-dimensional subspace E.

Lemma 3.11 *Consider the cone $D : x^2 + y^2 + z^2 - w^2 = 0$ and a hyperplane $E : a_1x + a_2y + a_3z - a_4w = 0$. If the intersection $K = D \cap E$ is a real cone $\subset \mathbb{R}^4$, its top view projection $\pi(K) = \Gamma$ is a cone of revolution with $\mathbf{a} = (a_1, a_2, a_3)$ as rotational axis and its half opening angle τ is determined by $\|\mathbf{a}\| \cos(\tau) = a_4$.*

Proof: The intersection $K = D \cap E$ is a quadratic cone with vertex $O \in \mathbb{R}^4$. Its projection $\Gamma = \pi(K)$ is a cone with vertex at O, given by

$$\Gamma : (a_1^2 - a_4^2)x^2 + (a_2^2 - a_4^2)y^2 + (a_3^2 - a_4^2)z^2 + 2(a_1a_2xy + a_2a_3yz + a_3a_1zx) = 0. \quad (3.29)$$

Since the origin in $\mathbb{R}^4$ coincides with the origin in $\mathbb{R}^3$, we use the same symbol O. Introducing the vector $\mathbf{x} = (x, y, z)$, we may write $D : \mathbf{x}^T \cdot \mathbf{x} - w^2 = 0$ and $E : \mathbf{a}^T \cdot \mathbf{x} - a_4w = 0$. Eliminating w from these two equations yields $\Gamma : \mathbf{x}^T \cdot \mathbf{M} \cdot \mathbf{x} = 0$, with $\mathbf{M} = \mathbf{a} \cdot \mathbf{a}^T - a_4^2\mathbf{I}_3$, and $\mathbf{I}_3 = \mathrm{diag}(1, 1, 1)$, which is just equation (3.29) in vector notation.

If $a_4 = 0$, it follows that $\mathrm{rk}(\mathbf{M}) = 1$, and Γ is the double plane $(\mathbf{x}^T \cdot \mathbf{a})^2 = 0$. If $a_4^2 = a_1^2 + a_2^2 + a_3^2$, E is tangent to D, and $\mathrm{rk}(\mathbf{M}) = 2$. The projection $\Gamma = \pi(D \cap E)$ consists of a real line carrying two conjugate complex planes.

Otherwise, $\mathrm{rk}(\mathbf{M}) = 3$ and its eigenvectors define the axes of symmetry of Γ. The eigenvalues and corresponding eigenvectors (eigenspaces) of $\mathbf{M}$ are

$$\begin{aligned} t_1 &= a_1^2 + a_2^2 + a_3^2 - a_4^2 &&\rightarrow \quad \mathbf{a} = (a_1, a_2, a_3), \\ t_2 &= t_3 = -a_4^2 &&\rightarrow \quad \lambda\mathbf{v}_1 + \mu\mathbf{v}_2, \text{ with } \mathbf{v}_1, \mathbf{v}_2 \perp \mathbf{a}. \end{aligned}$$

The eigenvalue t_1 corresponds to the axis $\mathbf{a}$ of Γ. The twofold eigenvalue $t_2 = t_3$ corresponds to a two-dimensional eigenspace spanned by two linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2$, both orthogonal to $\mathbf{a}$. Any plane passing through the axis with direction vector $\mathbf{a}$ is a plane of symmetry of Γ, and thus Γ is a cone of rotation. Intersecting $\Gamma : \mathbf{x}^T \cdot \mathbf{M} \cdot \mathbf{x} = 0$ with the unit sphere $\mathbf{x}^T \cdot \mathbf{I}_3 \cdot \mathbf{x} = 1$ shows that the half opening angle τ of Γ satisfies $\|\mathbf{a}\| \cos(\tau) = a_4$. □

Lemma 3.12 *Consider the cone $D : x^2 + y^2 + z^2 - w^2 = 0$. Let $\psi \subset \mathbb{R}^4$ be a plane with $O \notin \psi$ and assume that $L = D \cap \psi$ contains real points. Then the projection $\pi(L) = C$ is either a segment of a line or a conic contained in a rotational cone $\Gamma \subset \mathbb{R}^3$.*

Proof: The intersection $L = D \cap \psi$ is a conic in $\mathbb{R}^4$. Assume that its carrier plane ψ is not parallel to the w-axis, then the projection $C = \pi(L)$ is a conic as well. Consider the hyperplane E joining $O = (0, 0, 0, 0)$ and ψ. Lemma 3.11 says that the projection $\pi(D \cap E) = \Gamma$ is a cone of rotation. Since $\psi \subset E$, the projection $C = \pi(L)$ is a conic in the rotational cone Γ.

In case where the plane $\psi \subset \mathbb{R}^4$ is parallel to the w-axis, its projection $\pi(\psi) \subset \mathbb{R}^3$ is a line. Consequently the projection of the conic $L = D \cap \psi$ is a segment of that line. □

In the remainder of the section we give the explicit representations of the conics $L(u) \subset \Phi$ and their projections $C(u) = \pi(L(u))$ being contained in cones of revolution $\Gamma(u)$ with

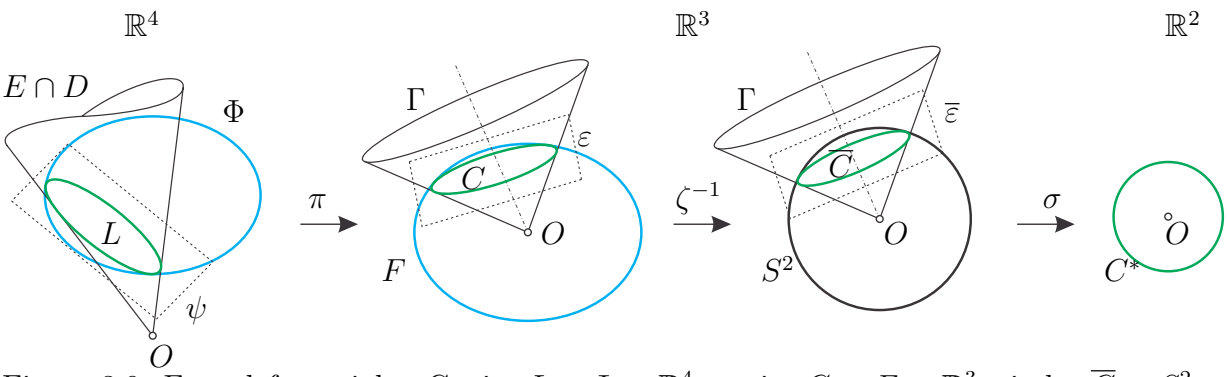


Figure 3.9: From left to right: Conics $L \subset \Phi \subset \mathbb{R}^4$, conics $C \subset F \subset \mathbb{R}^3$, circles $\overline{C} \subset S^2 \subset \mathbb{R}^3$, circles $C^* \subset \mathbb{R}^2$.

common vertex O. The intersection of a cone of revolution with vertex at O and the unit sphere S^2 consists of two circles. It is possible to define a rational map $C(u) \mapsto \overline{C}(u)$ between the family of conics $C(u) \subset F$ and a family of circles $\overline{C}(u) \subset S^2$. An explicit representation of this map is finally given by the radius function $\rho(s,t)$ in equation (3.34). The motivation to proceed in that way is that the practical parameterization of a one-parameter family of circles on S^2 is easier than parameterizing a general one-parameter family of conics $C(u)$ in space. Moreover it turns out that the map $C(u) \mapsto \overline{C}(u)$ and its inverse do not raise the degree of the final parameterization.

The family of conics $L(u) = \psi(u) \cap D$ on the surface $\Phi \subset \mathbb{R}^4$ are represented by

$$\left.\begin{aligned} \psi(u) &= \mathbf{e}_0(u) + v_1\mathbf{e}_1(u) + v_2\mathbf{e}_2 \\ D &: x^2 + y^2 + z^2 - w^2 = 0 \end{aligned}\right\} L(u),$$

where $\psi(u)$ is one family of generating planes of the cylinder B from (3.24). The top view projection $C(u) = \pi(L(u))$ is a family of conics $C = F \cap \varepsilon$. Thus their representation reads

$$\left.\begin{aligned} \varepsilon(u) = \pi(\psi(u)) &: -2u + \gamma(u^2-1)x - \alpha(u^2+1)z = 0 \\ F = \pi(\Phi) &: -1 + a^2x^2 + b^2y^2 + c^2z^2 = 0 \end{aligned}\right\} C(u). \tag{3.30}$$

Consider the hyperplanes $E(u)$ connecting planes $\psi(u)$ and the origin O in $\mathbb{R}^4$. According to Lemma 3.11, the conics $C(u)$ are the intersections of planes $\varepsilon = \pi(\psi(u))$ and rotational cones $\Gamma(u) = \pi(E(u) \cap D)$. An illustration is given in Figure 3.9.

The cones $\Gamma(u)$ with common vertex O are defined through a direction vector of their rotational axes $(\gamma(u^2+1), 0, -\alpha(u^2-1))$ and the half opening angles $\tau(u)$. Since an expression for the latter is rather lengthy, it is omitted here. Additionally we note that the planes $\varepsilon(u)$ envelope the hyperbolic cylinder $-(b^2-a^2)x^2 + (c^2-b^2)z^2 - 1 = 0$, see Figure 3.8(b). It is obtained as intersection $B \cap \mathbb{R}^3 : w = 0$.

Since $\Gamma(u)$ is a family of rotational cones in $\mathbb{R}^3$, their intersections $\overline{C}(u) = \Gamma(u) \cap S^2$ are a family of circles in S^2. The carrier planes $\overline{\varepsilon}(u)$ of the circles $\overline{C}(u)$ envelope the hyperbolic

cylinder $-(b^2-a^2)x^2+(c^2-b^2)z^2+b^2=0$. The family of circles $\overline{C}(u)$ is given by

$$\left.\begin{aligned}\overline{\varepsilon}(u): -2ub+\gamma(u^2+1)x-\alpha(u^2-1)z=0\\ S^2: x^2+y^2+z^2-1=0\end{aligned}\right\}\overline{C}(u).$$

Rational Parameterization of a Family of Circles

In Section 3.3.3 the family of conics $L(u)\subset\Phi\subset\mathbb{R}^4$ has been transformed to a family of circles $\overline{C}(u)\subset S^2$. Not all planes $\overline{\varepsilon}(u)$ intersect S^2 in a circle containing real points. To construct a real rational parameterization of the circles $\overline{C}(u)\subset S^2$ we have to restrict the parameter $u\in\mathbb{R}$ to a proper interval. The necessary re-parameterization reads

$$u(s)=\frac{u_0s^2+u_1}{s^2+1},\ \text{with } u_0=\frac{c-a}{\beta},\ u_1=-\frac{c-a}{\beta},\ \text{and } \beta=\sqrt{c^2-a^2}. \tag{3.31}$$

The curves $\overline{C}(u_0),\overline{C}(u_1)\subset S^2$ degenerate to the points $P_\pm=\pm\frac{1}{b\beta}(c\gamma,0,a\alpha)$. Let τ be the symmetry plane of P_+ and P_-. An illustration is given in Figure 3.10(a).

To gain a rational parameterization of the circles $\overline{C}(s)$, a stereographic projection $\sigma: S^2\to\tau$ with projection center P_+ is performed. Since σ is a conformal map, it transfers circles $\overline{C}\subset S^2$ to circles $C^\star\subset\tau$. An implicit representation for $C^\star(s)$ is obtained by choosing a Cartesian coordinate system $\{O,\xi,\eta\}$ in τ, with $\eta=y$ and $\xi=\eta\times\overrightarrow{OP_+}$. This gives

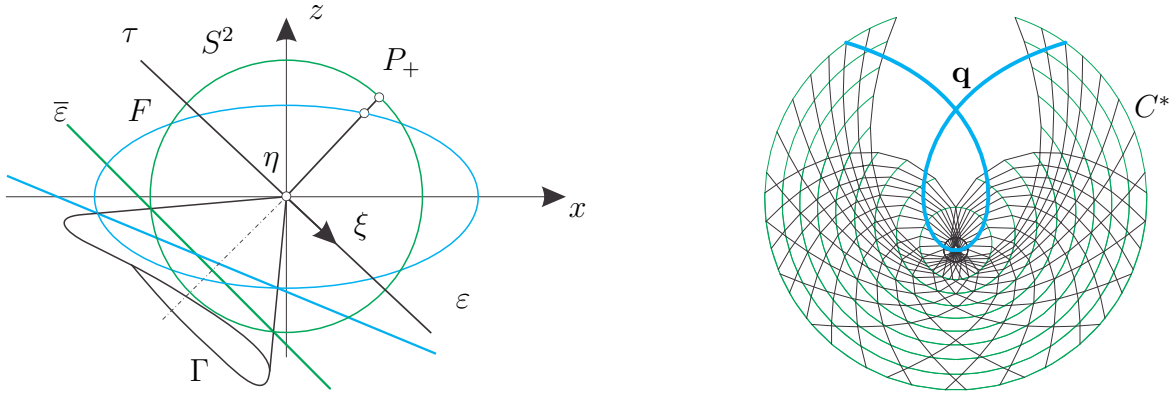
$$\begin{aligned}&C^*(s):(\xi-m(s))^2+\eta^2-r(s)^2=0,\ \text{with}\\ &m(s)=\frac{s^2\alpha\gamma}{b^2+s^2ac},\ \text{and } r(s)^2=\frac{b^2s^2(s^2a+c)(a+cs^2)}{(b^2+s^2ac)^2}.\end{aligned}$$

The denominator of $r(s)^2$ is a square of a polynomial. Its numerator is a non-negative polynomial and therefore it is the sum of the two squares $h_1(s)^2=(s^2b(a+c))^2$ and $h_2(s)^2=(sb\sqrt{ac}(s^2-1))^2$. The terms $h_1(s)$ and $h_2(s)$ together with $m(s)$ define a rational cubic trajectory $\mathbf{q}(s)$ of the family of circles $C^*(s)$, with the property that $\mathbf{q}(s)\in C^\star(s)$ for all $s\in\mathbb{R}$, see Figure 3.10(b). A parameterization reads

$$\mathbf{q}(s)=\frac{1}{b+s^2ac}\begin{pmatrix}s^2\alpha\gamma+h_1(s)\\ h_2(s)\end{pmatrix}=\frac{1}{b+s^2ac}\begin{pmatrix}s^2(b(c+a)+\alpha\gamma)\\ sb(s^2-1)\sqrt{ac}\end{pmatrix}$$

We construct a parameterization of that part of the plane τ being covered by the circles $C^\star(s)$ of bi-degree $(3,2)$, with the property that $\mathbf{c}^\star(s_0,t)$ represents the fixed circle $C^\star(s_0)$, with $s_0=\text{const}$. Using the abbreviation $\mu(s,t)=2\sqrt{ac}t(s^2-1)-(c+a)s(t^2-1)$, this parameterization reads

$$\mathbf{c}^*(s,t)=\frac{1}{(b^2+s^2ac)(1+t^2)}\begin{pmatrix}s(-b\mu(s,t)+s\alpha\gamma(t^2+1))\\ -sb(\sqrt{ac}(s^2-1)(t^2-1)+2ts(c+a))\end{pmatrix}.$$



(a) Relation between F and S^2.

(b) Circles $C^\star \subset \tau$, with cubic trajectory $\mathbf{q}$.

Figure 3.10: Correspondence between the sphere and the quadric and the circles in the plane τ.

The inverse stereographic projection $\sigma^{-1} : \tau \to S^2$ maps $\mathbf{c}^\star(s,t)$ to a rational parameterization $\overline{\mathbf{c}}(s,t)$ of S^2 of bi-degree $(6,2)$,

$$\overline{\mathbf{c}}(s,t) = (\overline{c}_1(s,t), \overline{c}_2(s,t), \overline{c}_3(s,t)) = \frac{1}{n(s,t)}\overline{\mathbf{g}}(s,t), \tag{3.32}$$

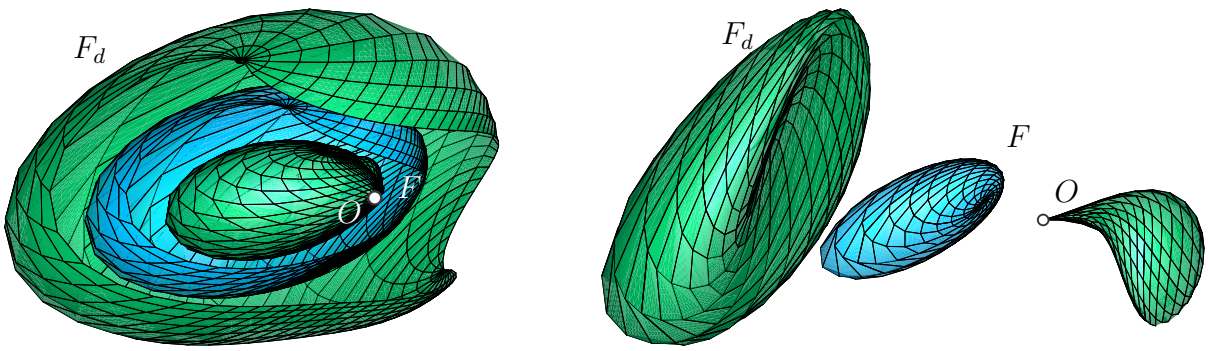
whose numerator $\overline{\mathbf{g}}(s,t) = (\overline{g}_1, \overline{g}_2, \overline{g}_3)(s,t)$ and denominator $n(s,t)$ are the polynomials

$$\begin{aligned}
\overline{g}_1(s,t) &= -2b\alpha s(a+cs^2)\mu(s,t) - \gamma(1+t^2)(c(s^4+1)(b^2-s^2ac) + 2s^2(b^2a - c^3s^2)) \\
\overline{g}_2(s,t) &= -2\beta s(b^2+s^2ac)(\sqrt{ac}(1-t^2)(1-s^2) + 2st(a+c)) \\
\overline{g}_3(s,t) &= 2b\gamma s(as^2+c)\mu(s,t) - \alpha(1+t^2)(a(s^4+1)(b^2-s^2ac) + 2s^2(b^2c - a^3s^2)) \\
n(s,t) &= \beta(-2\gamma\alpha s^3\mu(s,t) + b(1+t^2)((1-s^4)(b^2-acs^2) + 2s^2(s^2(a^2+c^2)+2ac))).
\end{aligned} \tag{3.33}$$

Rational Polar Representation of F and its Conchoid Surfaces

A polar representation $\mathbf{f}(s,t) = \rho(s,t)\mathbf{k}(s,t)$ of a surface F consists of a radius function $\rho(s,t)$ and a parameterization $\mathbf{k}(s,t)$ of S^2. The parameterization $\overline{\mathbf{c}}(s,t)$ from equation (3.32) is already the spherical part of the polar representation of the ellipsoid F. To determine the radius function $\rho(s,t)$, we have to determine the conics $C(s) \subset \varepsilon(s)$, compare equation (3.30). Using the substitution (3.31), the coefficients e_i and $\overline{e}_i$ of the planes $\varepsilon : e_0 + e_1x + e_2y + e_3z = 0$ and $\overline{\varepsilon} : \overline{e}_0 + \overline{e}_1x + \overline{e}_2y + \overline{e}_3z = 0$ read

$$\begin{aligned}
&e_0 = \beta(s^4-1), \quad e_1 = \gamma(a(s^4+1)+2cs^2), \quad e_2 = 0, \quad e_3 = \alpha(c(s^4+1)+2as^2), \text{ and} \\
&\overline{e}_0 = be_0, \qquad \overline{e}_1 = -\tfrac{\gamma}{\gamma}e_3, \qquad \overline{e}_2 = 0, \quad \overline{e}_3 = -\tfrac{\alpha}{\beta}e_1.
\end{aligned}$$



(a) O inside F. (b) O outside F.

Figure 3.11: Conchoids of ellipsoids.

The conics $C(s) \subset F$ are computed as intersection curves $C = \Gamma \cap \varepsilon$. Thus we have

$$\mathbf{f}(s,t) = \mathbf{c}(s,t) = \rho(s,t)\overline{\mathbf{c}}(s,t), \text{ with}$$
$$\rho(s,t) = \frac{-e_0}{e_1\overline{c}_1 + e_2\overline{c}_2 + e_3\overline{c}_3} = \frac{-e_0 n}{e_1\overline{g}_1 + e_2\overline{g}_2 + e_3\overline{g}_3}. \tag{3.34}$$

We note that $\overline{\mathbf{c}} \subset \overline{\varepsilon}$ for all $s \in \mathbb{R}$. In case that $\varepsilon(s_0) = \overline{\varepsilon}(s_0)$, it follows that $\overline{\mathbf{c}} \subset \varepsilon(s_0)$, and the denominator and numerator of (3.34) have the common factor $(s - s_0)$. The condition $\varepsilon(s_0) = \overline{\varepsilon}(s_0)$ holds for the zeros of s^2-1, corresponding to $u = 0$, and for the zeros of s^2+1, corresponding to $u = \infty$. This implies that the polynomial $\varepsilon : e_0 n + e_1\overline{g}_1 + e_2\overline{g}_2 + e_3\overline{g}_3 = 0$ is divisible by $s^4 - 1$. Since $e_0 = \alpha(s^4 - 1)$, also the denominator $e_1\overline{g}_1 + e_2\overline{g}_2 + e_3\overline{g}_3$ is divisible by $s^4 - 1$. Thus the radius function (3.34) is represented by

$$\begin{aligned} \rho(s,t) &= \frac{n(s,t)}{p(s,t)} \text{ with } n(s,t) \text{ from (3.33) and} \qquad (3.35) \\ p(s,t) &= b(-2\beta\gamma bs\mu(s,t) + (1+t^2)(ac(1-s^4)(b^2 - s^2ac) + 2s^2b^2(a^2 + c^2 + 2s^2ac))). \end{aligned}$$

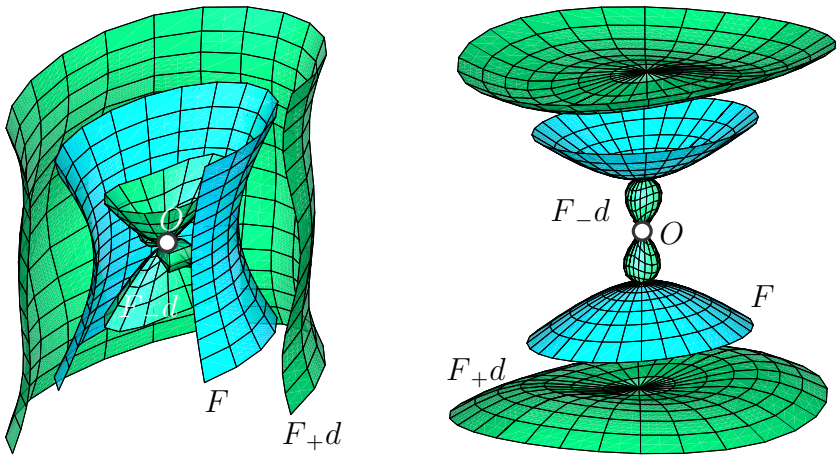
Combining equations (3.32), (3.34) and (3.35) leads to real rational polar representations of the ellipsoid F and its conchoid surfaces F_d at distance $d \in \mathbb{R}$,

$$F: \quad \mathbf{f}(s,t) = \frac{1}{p(s,t)}\overline{\mathbf{g}}(s,t) = \rho(s,t)\overline{\mathbf{c}}(s,t), \text{ with } \rho(s,t) = \frac{n(s,t)}{p(s,t)}, \tag{3.36}$$

$$F_d: \quad \mathbf{f}_d(s,t) = \frac{n(s,t) + d\,p(s,t)}{p(s,t)}\overline{\mathbf{c}}(s,t) = \frac{n(s,t) + d\,p(s,t)}{n(s,t)\,p(s,t)}\overline{\mathbf{g}}(s,t).$$

The parameterization $\mathbf{f}(s,t)$ of F is of bi-degree $(6,2)$, whereas the parameterization $\mathbf{f}_d(s,t)$ of F_d is typically of bi-degree $(12,4)$. Numerical examples show that the degree of improperness of the parameterization (3.36) is four. A rational parameterization of the del Pezzo surface $\Phi = D \cap A$ is of bi-degree $(6,2)$ and reads

$$\phi(s,t) = \frac{1}{p(s,t)}(\overline{g}_1(s,t), \overline{g}_2(s,t), \overline{g}_3(s,t), \pm n(s,t)). \tag{3.37}$$



(a) Hyperboloid of one sheet. (b) Hyperboloid of two sheets.

Figure 3.12: Conchoids of hyperboloids in normal form.

Theorem 3.13 *A quadric $F \subset \mathbb{R}^3$ admits a rational polar representation $\mathbf{f}(s,t)$ of bi-degree at most $(6,2)$ with respect to an arbitrarily chosen reference point O. Its conchoid surfaces F_d with respect to d and O admit rational polar representations $\mathbf{f}_d(s,t)$ of bi-degree at most $(12,4)$.*

See Figures 3.11 and 3.12 for illustrations of the conchoids of ellipsoids and hyperboloids. Appendix B.3 shows a numerical example for this algorithm.

In Sections 3.3.1 and 3.3.3 we have given a detailed investigation of real rational polar representations of regular quadrics F with respect to a reference point O in general position. What remains is a brief discussion of all excluded cases where the parameterizations are typically of lower degrees. These are

- F is a singular quadric,
- the reference point O lies on F,
- the reference point O lies on a focal conic of F or coincides with a focal point of a rotational quadric F.

3.3.4 Singular Quadrics

The quadric $F : \mathbf{x}^T \cdot \mathbf{M} \cdot \mathbf{x} = 0$, with $\mathbf{x} = (1, x, y, z)^T \in \mathbb{R}^4$, $\mathbf{M} = \mathbf{M}^T \in \mathbb{R}^{4\times 4}$, is called *singular* if $\det(\mathbf{M}) = 0$. Singular quadrics are cones and cylinders in case that $\mathrm{rk}(\mathbf{M}) = 3$ or pairs of planes if $\mathrm{rk}(\mathbf{M}) = 2$, or a double plane if $\mathrm{rk}(\mathbf{M}) = 1$. We again assume that F contains more than one real point. Rational polar representations of planes, cylinders and cones are discussed already in Section 1.3.3 and Section 3.2.

3.3.5 Reference Point lies on the Quadric

Consider the quadric $F : \mathbf{x}^T \cdot \mathbf{M} \cdot \mathbf{x} = 0$, with $\mathbf{x} = (1, x, y, z)^T$. The reference point $O = (0, 0, 0)^T$ is contained in F, if and only if the constant term of $F(x, y, z) = 0$ vanishes. We have a look at two methods to find a real rational polar representation of F.

On the one hand, consider a parameterization $\mathbf{f}(u, v) = \rho(u, v)\mathbf{k}(u, v)$ with an arbitrary rational parameterization $\mathbf{k}(u, v)$ of S^2. To determine the unknown radius function $\rho(u, v)$, one inserts $\mathbf{f}(u, v)$ into $F(x, y, z) = 0$. This gives the trivial solution $\rho(u, v) = 0$, and besides this a rational function $\rho(u, v)$, expressed by the coordinates of $\mathbf{k}(u, v)$.

On the other hand, a quadric F is mapped by a perspective collineation $\varkappa$ of the form (3.22) to a quadric F'. By choosing the denominator of $\varkappa$ as tangent plane of F we can assume that F' becomes a paraboloid. By an admissible rotation we can achieve $F' : z = ax^2 + by^2$, with $a, b \in \mathbb{R} \setminus 0$. Thus F' is either an elliptic or a hyperbolic paraboloid, depending whether $ab > 0$ or $ab < 0$.

We consider a one-parameter family of cones of rotation $\Gamma(v)$ with vertex O and axis z. An implicit equation of these cones is $\Gamma(v) : \sinh^2(v)(x^2 + y^2) - z^2 = 0$, and a possible parameterization reads

$$\mathbf{g}(u, t, v) = u \left(\frac{2t}{\sinh(v)}, \frac{1 - t^2}{\sinh(v)}, 1 + t^2 \right)^T .$$

Intersecting $\Gamma(v)$ with F' determines the function $u(v, t) = \sinh^2(v)(1 + t^2)/(4at^2 + b(1 - t^2)^2)$. The t-lines of the final parameterization $\mathbf{f}(t, v) = \mathbf{g}(u(t, v), t, v)$ are rational quartic curves, the intersection curves $\Gamma(v) \cap F'$. This polar representation of F' reads, see Figure 3.13(a),

$$\mathbf{f}(v, t) = \frac{(1 + t^2)\sinh(v)}{4at^2 + b(1 - t^2)^2} \begin{pmatrix} 2t \\ (1 - t^2) \\ (1 + t^2)\sinh(v) \end{pmatrix}, \text{ with } \|\mathbf{f}(v, t)\| = \frac{(1 + t)^2 \sinh(v)\cosh(v)}{4at^2 + b(1 - t^2)^2}.$$

Performing the rational substitutions $\cosh(v) = (1 + s^2)/(1 - s^2)$ and $\sinh(v) = 2t/(1 - s^2)$ yields a rational polar representation of F' of bi-degree $(4, 2)$. In case that $F' : z = a(x^2 + y^2)$ is a paraboloid of rotation, the parameterization simplifies and is of bi-degree $(2, 2)$, and its norm is independent on t,

$$\mathbf{f}(v, t) = \frac{\sinh(v)}{a(1 + t^2)} \left(2t, 1 - t^2, (1 + t^2)\sinh(v)\right)^T, \text{ with } \|\mathbf{f}(v, t)\| = \frac{\sinh(v)\cosh(v)}{a}.$$

3.3.6 Reference Point lies on a Focal Conic of the Quadric

Given a quadric $F : \mathbf{x}^T \cdot \mathbf{M} \cdot \mathbf{x} = 0$, the perspective collineation $\varkappa$ from equation (3.22) maps F to the quadric $F' : \pm 1 + \mathbf{x}'^T \cdot \mathbf{M}' \cdot \mathbf{x}' = 0$, whose center is the origin $O = (0, 0, 0)$. The tangential cone Δ of F' with vertex at O is fixed with respect to $\varkappa$. Thus it is the tangential cone of F and F', and reads

$$\Delta : \mathbf{x}'^T \cdot \mathbf{M}' \cdot \mathbf{x}' = 0.$$

The eigenvectors and eigenvalues of $\mathbf{M}'$ define the coordinate transformation to achieve the normal form (3.23). The case of pairwise distinct eigenvalues is already discussed, and the cases of coinciding eigenvalues remain. There is the case $b = c$ for all affine types of Table 3.1 and additionally $a = b$ and $a = b = c$ in case that F' is an ellipsoid. We discuss these particular cases exemplarily for an ellipsoid F'.

Example 3.14 *F' is a rotational quadric*

Consider the rotational ellipsoid $F' : a^2x^2 + b^2(y^2 + z^2) = 1$ with axis in direction of the $x-$axes. We substitute $b = c$ in the parameterization (3.33) and in (3.35). The parameterization of the spherical part $\overline{\mathbf{c}}(s,t)$ and the rational polar representation $\mathbf{f}(s,t)$ of F' are

$$\overline{\mathbf{c}}(s,t) = \frac{1}{n(s,t)}\overline{\mathbf{g}}(s,t), \text{ and } \mathbf{f}(s,t) = \frac{1}{p(s,t)}\overline{\mathbf{g}}(s,t) = \rho(s,t)\overline{\mathbf{c}}(s,t).$$

with coordinate functions $\overline{g}_i(s,t)$ of $\overline{\mathbf{g}}(s,t)$ and polynomials $n(s,t)$ and $p(s,t)$,

$$\begin{aligned}
\overline{g_1}(s,t) &= b(1+t^2)(s^4-1)\\
\overline{g_2}(s,t) &= -2s(\sqrt{ab}(1-t^2)(1-s^2) + 2st(a+b))\\
\overline{g_3}(s,t) &= 2s(2t\sqrt{ab}(s^2-1) - (a+b)s(t^2-1)) = 2s\mu(s,t)\\
p(s,t) &= b(1+t^2)(2bs^2 + a(1+s^4))\\
n(s,t) &= (1+t^2)(2as^2 + b(1+s^4)).
\end{aligned}$$

The t-lines of $\mathbf{f}(s,t)$ are parallel circles, the s-lines are rational curves of degree four. Figure 3.13(b) shows an illustration of an ellipsoid with one highlighted s-line. The parameterization $\mathbf{f}(s,t)$ is of bi-degree $(4,2)$ and has rational length $\rho(s)$ independent on t,

$$\rho(s) = \frac{n(s,t)}{p(s,t)} = \frac{2as^2 + b(1+s^4)}{b(2bs^2 + a(1+s^4))}.$$

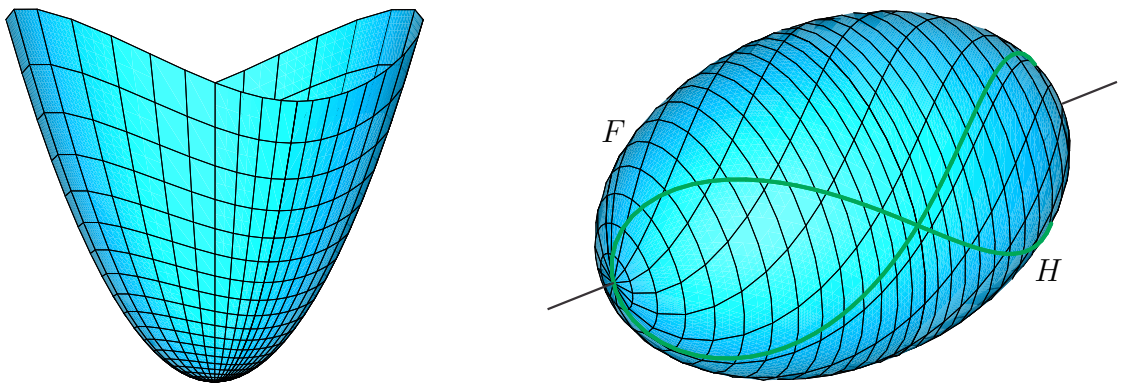
Alternatively, the parameterization $\mathbf{f}(s,t)$ could be derived using the cone model, see Section 3.3.3. The corresponding pencil of quadrics $\mathcal{B}$ contains a cylinder over the conic $-(b^2-a^2)x^2 + b^2w^2 = 1$, and its generating planes are parallel to the $yz-$plane.

Two or more similar eigenvalues of M' imply, that the tangential cone Δ to F and $F' = \varkappa(F)$, with vertex O, is a *rotational cone*. Hence quadrics F in $\mathbb{R}^3$ that have a rotational quadric F' as normal form are characterized as those which have a tangential rotational cone Δ with vertex at O. The possible positions of the vertices of these cones lie on the so called *focal conics* of the quadric F, see Section 1.1.5. Hence, if O lies on a focal conic of the given quadric F its normal form F' is a rotational quadric, see Figure 1.2.

$\star$

Example 3.15 *F' is a sphere*

Consider a sphere $F' : x^2 + y^2 + z^2 = 1$ of radius 1, centered at O. Any rational parameterization $\mathbf{f}(u,v) = \mathbf{k}(u,v)$, with $\|\mathbf{k}\| = 1$, of F' has rational length trivially. The



(a) Paraboloid. (b) Rotational ellipsoid.

Figure 3.13: Special cases.

pre-image F with respect to $\varkappa$ is an oval rotational quadric with O as focal point. Since F' admits a proper rational polar representation of bi-degree $(2,2)$, the same holds for the pre-image F. The conchoid surfaces of F with respect to O are reducible and each component admits proper rational polar representations.

We want to look at a rotational ellipsoid F with O as focal point in more detail. Let us assume, that the rotation axis of F is the $x-$axes of the coordinate system and F has the implicit equation

$$F : a^2(x-e)^2 + b^2(y^2+z^2) = 1 \text{ with } e^2 = \frac{1}{a^2} - \frac{1}{b^2}. \tag{3.38}$$

The perspective collineation

$$\varkappa : \mathbf{x}' = \frac{b^2}{a(1+b^2ex)}\mathbf{x}$$

maps the ellipsoid F to the sphere F'. This can be easily seen by substituting the inverse map

$$\varkappa^{-1} : \mathbf{x} = \frac{a}{b^2(1-aex')}\mathbf{x}'$$

in equation (3.38). If we choose a parameterization of the sphere F', for example the trigonometric one

$$\mathbf{f}'(u,v) = (\cos(u)\cos(v), \cos(u)\sin(v), \sin(u))^T,$$

we receive the polar representations of F and its conchoids F_d with

$$\mathbf{f}(u,v) = \frac{a}{b^2(1-ae\cos(u)\cos(v))}\mathbf{f}'(u,v) \text{ and}$$
$$\mathbf{f}_d(u,v) = \frac{a \pm db^2(1-ae\cos(u)\cos(v))}{b^2(1-ae\cos(u)\cos(v))}\mathbf{f}'(u,v).$$

$\star$

3.4 Conchoids of the Sphere

Although the sphere is a special quadric and therefore the algorithm provided in Section 3.3 leads to a rational polar representation of the sphere, we want to discuss this special surface in an own section. Mainly because there are nice geometric properties of the rational polar representation. This section is the main contribution of Peternell, Gruber and Sendra (2013).

3.4.1 Conchoids of Spheres

Given a sphere F in $\mathbb{R}^3$ and an arbitrary focus point O, according to Theorem 3.10 there exists a rational representation $\mathbf{f}(u,v)$ of F with the property that $\|\mathbf{f}(u,v)\|$ is a rational function of the parameters u and v. We describe the situation in the cone model in Section 3.4.2, later on in Section 3.4.6 we study a different method working in $\mathbb{R}^3$ directly. There are several relations between these methods which will be discussed along their derivation.

Let F be the sphere with center M and radius r. Without loss of generality, we can choose the coordinate system such that, the center is given by $\mathbf{m} = (m,0,0)^T$ and the reference point $O = (0,0,0)^T$ is the center of the coordinate system. Thus F is given by

$$F : (x-m)^2 + y^2 + z^2 - r^2 = 0. \tag{3.39}$$

If $m = 0$, the center of F coincides with O. In this trivial situation the conchoid surface of F is reducible and consists of two spheres, where one might degenerate to F's center if $d = r$. If $m^2 - r^2 = 0$, the focal point O is contained in F. To construct a rational polar representation, we make the ansatz $\mathbf{f}(u,v) = \rho(u,v)\mathbf{k}(u,v)$ with $\mathbf{k}(u,v) = (k_1,k_2,k_3)(u,v)$ and $\|\mathbf{k}(u,v)\| = 1$ and an unknown radius function $\rho(u,v)$. Plugging this into (3.39), we obtain a rational polar representation with rational radius function $\rho(u,v) = 2mk_1(u,v)$. Note that in this case the conchoid is irreducible and rational.

3.4.2 Pencil of Quadrics in $\mathbb{R}^4$

Consider the Euclidean space $\mathbb{R}^4$ with coordinate axes x, y, z and w and let $\mathbb{R}^3$ be embedded as the hyperplane $w = 0$. Let a sphere $F \subset \mathbb{R}^3$ be defined by (3.39) and $O = (0,0,0)^T$. To study the general case, we assume $m \neq 0$ and $m^2 \neq r^2$. The equation of the cylinder $A \subset \mathbb{R}^4$ through F agrees with the equation of F in $\mathbb{R}^3$,

$$A : (x-m)^2 + y^2 + z^2 - r^2 = 0.$$

Consider the pencil $\mathcal{B}(t) = A + tD$ of quadrics in $\mathbb{R}^4$, spanned by A and the quadratic cone $D : x^2 + y^2 + z^2 = w^2$ from Section 3.1. We study the geometric properties of the del Pezzo surface $\Phi = A \cap D$ of degree four, the base locus of the pencil of quadrics $\mathcal{B}(t)$. According to Section 3.3, the sphere F is a rational conchoid surface.

Besides A and D there exist two further singular quadrics in $\mathcal{B}(t)$. These quadrics are obtained for the zeros $t_1 = -1$ and $t_2 = r^2/\gamma^2$ of the characteristic polynomial

$$\det(\mathbf{A} + t\mathbf{D}) = -(1+t)^2 t(\gamma^2 t - r^2), \text{ with } \gamma^2 = m^2 - r^2 \neq 0,$$

with $\mathbf{A}, \mathbf{D}$ are the defining matrices of A respectively D. The quadric corresponding to the twofold zero $t_1 = -1$ is a cylinder

$$R : w^2 - 2mx + m^2 - r^2 = 0. \tag{3.40}$$

Its directrix is a parabola in the xw-plane and its two-dimensional generators are parallel to the yz-plane. The singular quadric S corresponding to $t_2 = r^2/\gamma^2$ is a quadratic cone and reads

$$S : \left(x - \frac{m^2 - r^2}{m}\right)^2 + y^2 + z^2 = \frac{r^2}{m^2} w^2.$$

Its vertex is the point $O' = (\frac{m^2-r^2}{m}, 0, 0, 0)^T$. The intersections of S with three-spaces $w = c$ are spheres $\sigma(c)$, whose top view projections in $w = 0$ are centered at O' and their radii are rc/m. The intersections of D with three-spaces $w = c$ are spheres $d(c)$ whose top view projections in $w = 0$ are centered at O with radii c. The intersections $k(c) = s(c) \cap d(c)$ of these spheres ($w = c$) are circles in planes $x = (c^2 + m^2 - r^2)/(2m)$. Thus Φ contains a family of conics, whose top view projections are the circles $k(c)$. The conics in Φ are contained in the planes

$$\varepsilon(c) : x = \frac{c^2 + m^2 - r^2}{2m}, w = c.$$

The half opening angle δ of D with respect to the w-axis is $\pi/4$, thus $\tan\delta = 1$. The half opening angle σ of S is given by $\tan\sigma = r/m$, see Figure 3.14(a). Applying the scaling

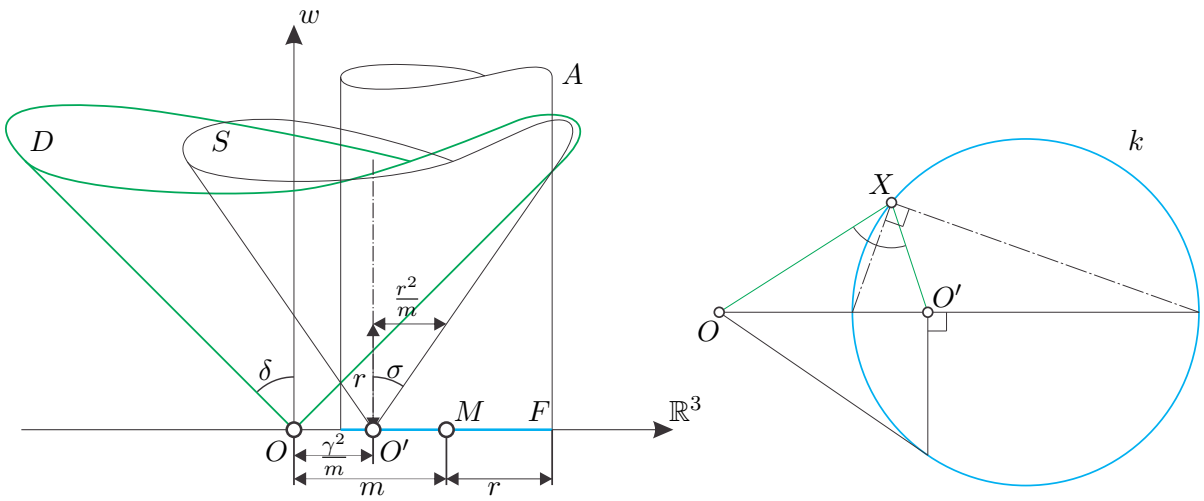
$$(x', y', z', w')^T = (fx, fy, fz, w)^T, \text{ with } f = \frac{r}{m}$$

in $\mathbb{R}^4$ maps D to a congruent copy of S. Consider a point $(x, y, z, w)^T$ in $\Phi = A \cap D$ and its projection $X = (x, y, z)^T$ in F. The distance $\mathrm{dist}(X, O)$ of X to O in $\mathbb{R}^3$ is w. For the distance $\mathrm{dist}(X, O')$ between X and O' we consequently obtain

$$\mathrm{dist}(X, O') = \frac{r}{m}\,\mathrm{dist}(X, O), \text{ for all } X \in F. \tag{3.41}$$

Remark on the circle of Apollonius Note that O' is the inverse point of O with respect to the sphere F. It is an old result by Apollonius Pergaeus (262–190 b.c.), that the set of points X in the plane having constant ratio of distances $f = d/d'$, with $d = \mathrm{dist}(O, X)$ and $d' = \mathrm{dist}(O', X)$, from two given fixed points O and O', respectively, is a circle k, see Figure 3.14(b). Rotating k around the line OO' gives the sphere F and O and O' are inverse points with respect to F (and the circle k).

If we consider a varying constant ratio f, one obtains a family of spheres $F(f)$ with inverse points O and O' which form an elliptic pencil of spheres. Their centers are on the line OO'. Ratio 1 ($d = d'$) corresponds to the bisector plane of O and O'.



(a) Sketch of the quadric pencil in $\mathbb{R}^4$.

(b) Apollonius circle.

Figure 3.14: Pencil of quadrics in $\mathbb{R}^4$ and Apollonius circle.

3.4.3 A Rational Quartic on the Sphere

The pencil of quadrics $\mathcal{B}(t)$ in $\mathbb{R}^4$ spanned by the cylinder A and the cone D contains the cylinder R. Expressing the variable x from (3.40) one gets

$$x = \frac{w^2 + m^2 - r^2}{2m}, \tag{3.42}$$

and inserting this into D results in the polynomial

$$\alpha(w) : 4m^2(y^2 + z^2) + p(w) = 0, \text{ with } p(w) = w^4 - 2w^2(m^2 + r^2) + (m^2 - r^2)^2. \tag{3.43}$$

Considering y and z as variables, $\alpha(w)$ is a one-parameter family of circles in the yz-plane, depending *rationally* on the parameter w. The circles $\alpha(w)$ do not possess real points for all w, but there exist intervals determining families of real circles $\alpha(w)$. To obtain real circles one has to perform a re-parameterization $w(u)$ within an appropriate interval. The factorization of $p(w)$ reads

$$p(w) = (w + a)(w - a)(w + b)(w - b), \text{ with } a = m + r, \text{ and } b = m - r.$$

If O is outside of F, thus $m > r$, the polynomial $-p(w)$ is positive in the interval $[m - r, m + r]$. Thus a possible re-parameterization is

$$w(u) = \frac{au^2 + b}{1 + u^2} = \frac{u^2(m + r) + m - r}{1 + u^2}. \tag{3.44}$$

Otherwise we could re-parameterize over another appropriate interval. Additionally we note that if O is inside of F, the inverse point O' is outside of F. Since equation (3.41) holds for the distances of a point $X \in F$ to O and O', we can exchange roles and perform the computation for the point O'.

We return to the family of conics $\alpha(w)$. Substituting (3.44) into (3.43) leads to a family of real conics

$$\alpha(u) : y^2 + z^2 = \frac{4r^2u^2}{m^2(1+u^2)^4}(au^2+m)(mu^2+b). \tag{3.45}$$

We are looking for rational functions $y(u)$ and $z(u)$ satisfying (3.45) identically. Therefore we introduce auxiliary variables $\tilde{y}$ and $\tilde{z}$ by the relations $y = 2\tilde{y}ru/(m(1+u^2)^2)$ and $z = 2\tilde{z}ru/(m(1+u^2)^2)$. We obtain $\tilde{y}^2 + \tilde{z}^2 = (au^2+m)(mu^2+b)$. Factorizing left and right hand side of this equation results in a linear system to determine $\tilde{y}$ and $\tilde{z}$,

$$\begin{aligned} \tilde{y} + \mathrm{i}\,\tilde{z} &= (\sqrt{a}u + \mathrm{i}\sqrt{m})(\sqrt{m}u + \mathrm{i}\sqrt{b}), \\ \tilde{y} - \mathrm{i}\,\tilde{z} &= (\sqrt{a}u - \mathrm{i}\sqrt{m})(\sqrt{m}u - \mathrm{i}\sqrt{b}). \end{aligned}$$

The solution $\tilde{y} = \sqrt{m}(\sqrt{a}u^2 - \sqrt{b})$, $\tilde{z} = u(m + \sqrt{ab})$ finally leads to

$$y(u) = \frac{2r\sqrt{m}u}{m(1+u^2)^2}\left(\sqrt{a}u^2 - \sqrt{b}\right), \text{ and } z(u) = \frac{2ru^2}{m(1+u^2)^2}\left(m + \sqrt{ab}\right), \tag{3.46}$$

which is a rational parameterization of a curve in the yz-plane, following the family of conics $\alpha(w)$.

The solution (3.46) together with (3.42) determines a curve $C \subset F$ which possesses the rational distance function

$$\|\mathbf{c}(u)\| = w(u) = \frac{u^2(m+r) + (m-r)}{1+u^2} \tag{3.47}$$

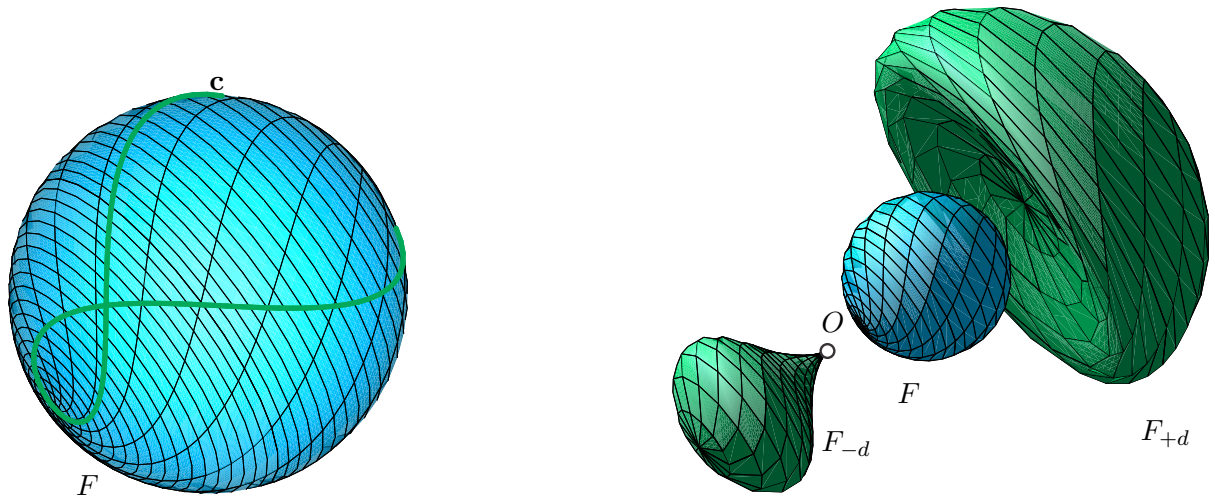
with respect to O. Its parameterization is

$$\mathbf{c}(u) = \frac{1}{m(1+u^2)^2}\begin{pmatrix} u^4m(m+r) + 2u^2(m^2-r^2) + m(m-r) \\ 2r\sqrt{m}u(u^2\sqrt{m+r} - \sqrt{m-r}) \\ 2ru^2(m + \sqrt{m^2-r^2}) \end{pmatrix}. \tag{3.48}$$

Theorem 3.16 *Let F be a sphere and let O be an arbitrary point in $\mathbb{R}^3$. Then there exists a rational quartic curve $C \subset F$ and a rational parameterization $\mathbf{c}(u)$ of C such that the distance of C to O is a rational function in the curve parameter u.*

Rotating C around the x-axis leads to a rational polar representation $r(u,v)\mathbf{k}(u,v)$ of F with rational distance function $\varrho(u,v) = w(u)$ from O. The quartic curve C together with this parameterization is illustrated in Figure 3.15(a). Figure 3.15(b) displays a sphere F together with both conchoid surfaces F_{+d} and F_{-d} for distances d and $-d$ with respect to O. We summarize the presented construction.

Theorem 3.17 *Spheres in $\mathbb{R}^3$ admit rational polar representations of bi-degree $(4,2)$ with respect to any focus point O. This implies that the conchoid surfaces of spheres admit rational parameterizations of bi-degree $(8,2)$. The construction is based on rational quartic curves on F with rational distance from O.*



(a) Base curve of the pencil in $\mathbb{R}^3$.

(b) Sphere F and both conchoid surfaces F_{+d} and F_{-d} w.r.t. O.

Figure 3.15: Rational polar representation of a sphere and its conchoid surfaces.

Example 3.18 *Implicit equation of the conchoids of a sphere*

The sphere F with center $\mathbf{m} = (3/2, 0, 0)$ and radius $r = 1$, and compute its conchoid F_d for variable distance d. We obtain parameterizations $\mathbf{f}_{+d}(u, v)$ and $\mathbf{f}_{-d}(u, v)$ from equation (3.48) for the real uni-rational varieties F_{+d} and F_{-d}. The algebraic variety $F_d = F_{+d} \cup F_{-d}$ is given by the equation

$$\begin{aligned} F_d: \quad & (x^2 + y^2 + z^2)(4(x^2 + y^2 + z^2) - 12x + 5)^2 \\ & + d^2(40(x^2 + y^2 + z^2) - 144x^2 + 96x(x^2 + y^2 + z^2) - 32(x^2 + y^2 + z^2)^2) \\ & + 16d^4(x^2 + y^2 + z^2) = 0. \end{aligned}$$

$\star$

Remarks on the parameterization The rational quartic C on F is of course not unique but depends on the re-parameterization (3.44). An admissible rational re-parameterization of a real interval is of even degree. Let us consider a quadratic re-parameterization. Since α is of degree four in w, the re-parameterized family is typically of degree ≤ 8 in u. This implies that the solutions $y(u)$ and $z(u)$ are of degree ≤ 4, which holds also for $x(u)$ because of (3.42). The coefficient functions $\mathbf{c}(u) = (x, y, z)^T(u)$ determine a rational quartic C on F, with rational norm $\|\mathbf{c}\| = w(u)$.

Different choices of the interval and a quadratic re-parameterization will typically result in different quartic curves on F. In (3.44) we have chosen the largest possible interval and a rational function satisfying $w(-u) = w(u)$ and obtained the curve C through antipodal points of F. By rotating we obtain the full sphere, doubly covered.

For any quadratic re-parameterization, the quartic C is the base locus of a pencil of quadrics $\mathcal{A}(t) = F + tK$, spanned by the sphere F and, for instance, the quadratic projection cone K with vertex at C's double point.

The particular choice (3.44) implies that the quartic C is symmetric with respect to the xz-plane. This holds since u appears only with even powers in x and z, thus we have $x(-u) = x(u)$ and $z(-u) = z(u)$. The orthogonal projection of C to the xz-plane is doubly covered, thus a conic. In this case $(x, z)^T(u)$ parameterizes a parabola, because of the factor $(1 + u^2)^2$ in $\mathbf{c}(u)$'s denominator. This implies that the pencil $\mathcal{A}(t)$ can also be spanned by the sphere F and the parabolic cylinder P passing through C, whose generating lines are parallel to the $y-$axis. It can be proved that all quadrics $\mathcal{A}(t)$ except P are rotational quadrics with parallel axes. This implies that K is a rotational cone, and the remaining singular quadric L is a rotational cone, too. For the particular choice (3.44) and for the generalized construction performed in Section 3.4.6, the rotational cone L has the vertex O. We note that for any admissible re-parameterization L's vertex is typically different from O.

3.4.4 Pencil of Quadrics in $\mathbb{R}^3$

The quartic curve C from (3.48) on the sphere F is the base locus of a pencil of quadrics $F + \lambda K$ in $\mathbb{R}^3$, spanned by F and the projection cone K of C from its double point $\mathbf{s}$, see Figure 3.16. The double point $\mathbf{s}$ is located in the symmetry plane of C and in the polar plane of the origin O with respect to F. Its coordinates are

$$\mathbf{s} = \frac{1}{m}(\gamma^2, 0, r\gamma)^T \text{ with } \gamma^2 = m^2 - r^2. \tag{3.49}$$

The pencil $F + \lambda K$ contains two further singular quadrics which are obtained for the zeros $\lambda_1 = 1/m$ and $\lambda_2 = -1/\gamma$ of the characteristic polynomial

$$\det(\mathbf{F} + \lambda\mathbf{K}) = r^2(m\lambda - 1)(\gamma\lambda + 1).$$

Corresponding to λ_1 there is a parabolic cylinder P with y-parallel generating lines passing through C. Corresponding to λ_2 we find the rotational cone L through C with vertex O.

To give explicit representations for the quadrics we use homogeneous coordinates $\mathbf{x}\mathbb{R} = (1, x, y, z)^T\mathbb{R}$. The coefficient matrices of F and K read

$$\mathbf{F} = \begin{pmatrix} m^2 - r^2 & -m & 0 & 0 \\ -m & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \mathbf{K} = \begin{pmatrix} \gamma^3 & -\gamma m & 0 & 0 \\ -\gamma m & \gamma & 0 & r \\ 0 & 0 & -m & 0 \\ 0 & r & 0 & -\gamma \end{pmatrix}.$$

An elementary computation shows that K is a cone of revolution with opening angle $\pi/2$ and $\mathbf{a} = (m + \gamma, 0, r)^T$ denotes a direction vector of its axis.

The cone L through C with vertex at O is again a cone of revolution, whose axis is parallel to $\mathbf{a}$. The parabolic cylinder P through the quartic C has generating lines parallel to the $y-$axes. The axis of the cross section parabola in the xz-plane is orthogonal to $\mathbf{a}$, see Figure 3.16(a). The coefficient matrices L and P are

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r \\ 0 & 0 & m+\gamma & 0 \\ 0 & -r & 0 & 2\gamma \end{pmatrix}, \ \mathbf{P} = \begin{pmatrix} \gamma^2(m+\gamma) & -m(m+\gamma) & 0 & 0 \\ -m(m+\gamma) & m+\gamma & 0 & r \\ 0 & 0 & 0 & 0 \\ 0 & r & 0 & m-\gamma \end{pmatrix}. \quad (3.50)$$

A trigonometric parameterization of the quartic C is obtained by intersecting the cone K with one quadric of the pencil $F+\lambda K$, for instance F. Let $\mathbf{a}$ be a unit vector in direction of K's axis, and $\mathbf{b}$ and $\mathbf{c}$ complete it to an orthonormal basis in $\mathbb{R}^3$. A trigonometric parameterization of K is given by

$$\mathbf{k}(t,v) := \mathbf{s} + v(\mathbf{a} + (\mathbf{b}\cos(t) + \mathbf{c}\sin(t))), \text{ with}$$
$$\mathbf{a} = \tfrac{1}{\sqrt{2m(m+\gamma)}}(m+\gamma, 0, r)^T, \ \mathbf{b} = (0,-1,0)^T, \text{ and } \mathbf{c} = \tfrac{1}{\sqrt{2m(m+\gamma)}}(r, 0, -(m+\gamma))^T.$$

Thus K admits the explicit parameterization

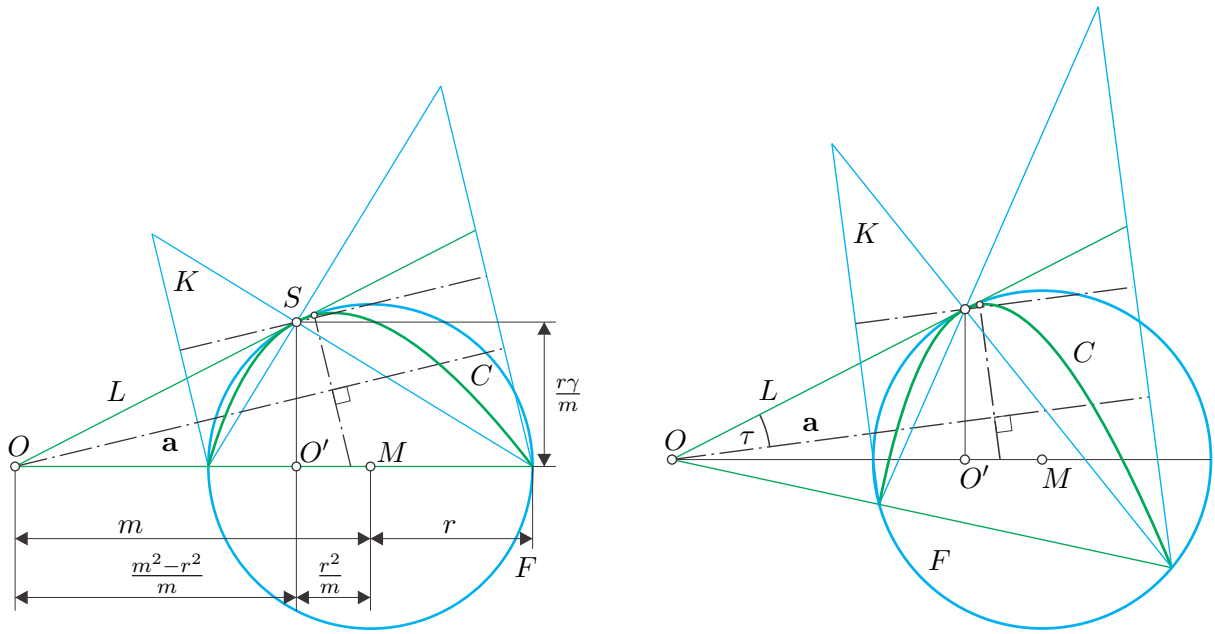
$$\mathbf{k}(t,v) = \frac{1}{2m\sqrt{m(m+\gamma)}} \begin{pmatrix} 2\gamma^2\sqrt{m(m+\gamma)} + v\sqrt{2}m(m+\gamma+r\sin(t)) \\ -2vm\sqrt{m(m+\gamma)}\cos(t) \\ 2r\gamma\sqrt{m(m+\gamma)} + v\sqrt{2}m(r-(m+\gamma)\sin(t)) \end{pmatrix}.$$

Finally, a trigonometric parameterization of the quartic C follows by

$$\mathbf{c}(t) = \frac{1}{2m} \begin{pmatrix} (m+r\sin(t))^2 + \gamma^2 \\ \sqrt{2}\sqrt{m(m+\gamma)}\cos(t)(\gamma - m - r\sin(t)) \\ r(m+\gamma)\cos^2(t) \end{pmatrix}, \quad (3.51)$$

with $\|\mathbf{c}(t)\| = m + r\sin(t)$. The correspondence of the trigonometric parameterization and its norm with the expressions (3.48) and (3.47) in terms of rational functions is realized by the Weierstrass substitutions and some rearrangement of the equations. Section 3.4.5 discusses relations to Viviani's curve (or Viviani's window). This particular quartic has a similar shape and its pencil of quadrics has similar properties. Viviani's curve has an additional symmetry.

Remark The inversion with center O at the sphere which intersects the given sphere F perpendicularly, maps the sphere F onto itself. Analogously this inversion fixes the rotational cone L. Thus the quartic intersection curve $C = F \cap L$ remains fixed as a whole, but of course not point-wise. The product of the distances $\mathrm{dist}(O,P)$ and $\mathrm{dist}(O,P')$ of two inverse points $P \in F$ and $P' \in F$ equals $\sqrt{m^2 - r^2}$. This property follows from the elementary tangent-secant-theorem of a circle.



(a) Symmetric case.

(b) General case.

Figure 3.16: Geometric properties of the conchoid construction.

3.4.5 Relations to Viviani's Curve

The quartic curve C, the base locus of the pencil of quadrics $F + tK$, can be considered as generalization of Viviani's curve V. This particularly well known curve V is the base locus of a pencil of quadrics, spanned by a sphere F and a cylinder of revolution L touching F and passing through the center of F. The pencil of quadrics of Viviani's curve also contains a right circular cone K with vertex in V's double point and opening angle $\pi/2$, and further a parabolic cylinder P. Viviani's curve V is obtained from C by letting $O \to \infty$. Consequently, the inverse point O' becomes the center of the sphere F.

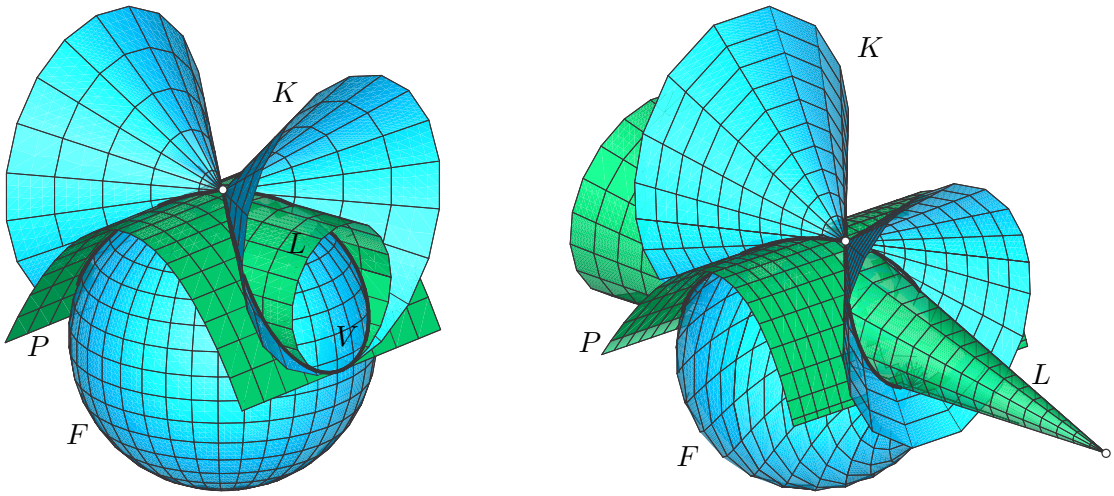
Choosing the inverse point $O' = (\frac{m^2-r^2}{m}, 0, 0)^T$ as origin, the parameterization (3.51) of C becomes

$$\mathbf{c}(t) = \frac{1}{2m} \begin{pmatrix} r^2(1+\sin^2(t)) + 2mr\sin(t) \\ \sqrt{2}\sqrt{m(m+\gamma)}\cos(t)(\gamma - m - r\sin(t)) \\ r(m+\gamma)\cos^2(t) \end{pmatrix}.$$

By letting $m \to \infty$ one obtains V as limit curve

$$\mathbf{v}(t) = \left(r\sin(t), -r\sin(t)\cos(t), r\cos^2(t)\right)^T.$$

Figure 3.17(a) illustrates Viviani's curve V, together with the sphere and the singular quadrics belonging to the pencil. The generalized Viviani curve C being the base locus



(a) Quadric pencil of Viviani's curve.

(b) Quadric pencil of the quartic C.

Figure 3.17: Quadric pencils of Viviani's curve and its generalization.

of the pencil appearing in the conchoid construction of the sphere is illustrated in Figure 3.17(b). In contrast to the classical Viviani curve V whose single parameter r is the radius of the sphere F, the quartic curve C has two parameters r and m.

3.4.6 Rotational Quadrics with Parallel Axes

We consider the mentioned pencil of quadrics $\mathcal{B}(t) = A + tD$ from Section 3.4.2, and a hyperplane $E : ax + by + cz - dw = 0$ passing through $O = (0, 0, 0, 0)^T$. The intersection $D \cap E$ is a quadratic cone, whose projection onto $\mathbb{R}^3$ is a cone of revolution L with axis in direction of $\mathbf{a} = (a, b, c)^T$. Assuming $\|\mathbf{a}\| = 1$, the opening angle 2τ of L is determined by $d = \cos(\tau)$.

Consider the quartic intersection curve $C = F \cap L$ of a sphere F and the cone of revolution L. It is rational exactly if the cone L is touching F at a single point. Since this touching point has to be contained in the polar plane of $O = (0, 0, 0)^T$ with respect to F, we choose $\mathbf{s} = (\gamma^2/m, 0, r\gamma/m)^T$ (compare (3.49)) and prescribe an arbitrary opening angle 2τ for L. Thus the unit direction vector of L's axis is

$$\mathbf{a} = \frac{1}{m}(\gamma\cos(\tau) - r\sin(\tau), 0, \gamma\sin(\tau) + r\cos(\tau))^T = (a, b, c)^T.$$

The quartic C is real if the axis is contained in the wedge formed by $\mathbf{s}$ and the x-axis, see Figure 3.16(b). Thus $-r/\gamma \leq \tan\tau \leq 0$, because the rotation from $\mathbf{s}$ to $\mathbf{a}$ by $\tau \leq 0$ is counterclockwise. In the following we use the abbreviations $ct := \cos(\tau)$ and $st := \sin(\tau)$. The quadrics of the pencil with base locus C are denoted similarly to Section 3.4.4. The

coefficient matrix of the projection cone L reads

$$\mathbf{L}(\tau) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & r^2(ct^2 - st^2) + 2\gamma rstct & 0 & -\gamma r(ct^2 - st^2) + (r^2 - \gamma^2)stct \\ 0 & 0 & m^2ct^2 & 0 \\ 0 & -\gamma r(ct^2 - st^2) + (r^2 - \gamma^2)stct & 0 & \gamma^2(ct^2 - st^2) - 2\gamma rstct \end{pmatrix}.$$

Rewriting $\mathbf{L}(\tau)$ in terms of the double angle 2τ and substituting

$$\cos(2\tau) = \gamma/m, \text{ and } \sin(2\tau) = -r/m$$

we obtain L from equation (3.50). This holds for all equations and parameterizations in this section in an analogous way.

The pencil of quadrics $F + tL(\tau)$ contains two further singular quadrics. The first is a parabolic cylinder $P(\tau)$ passing through C. It corresponds to the eigenvalue $\frac{-1}{m^2ct^2}$ and its generating lines are parallel to the y-axis. Its coefficient matrix of cylinder reads

$$\mathbf{P}(\tau) = \begin{pmatrix} \gamma^2m^2ct^2 & -m^3ct^2 & 0 & 0 \\ -m^3ct^2 & \gamma^2(ct^2 - st^2) + m^2st^2 - 2r\gamma stct & 0 & (\gamma^2 - r^2)stct + r\gamma(ct^2 - st^2) \\ 0 & 0 & 0 & 0 \\ 0 & (\gamma^2 - r^2)stct + r\gamma(ct^2 - st^2) & 0 & m^2ct^2 - \gamma^2(ct^2 - st^2) + 2r\gamma stct \end{pmatrix}.$$

Our goal is not only to characterize the pencil of quadrics but to provide an explicit parameterization of the quartic curve C on F whose distance from O is rational. This is performed by using a parameterization of the second singular quadric K which corresponds to the zero $\frac{r}{\gamma m^2ctst}$ of the characteristic polynomial $\det(\mathbf{F}+t\mathbf{L}(\tau))$. K is a cone of revolution with axis parallel to $\mathbf{a}$, and its coefficient matrix reads

$$\mathbf{K}(\tau) = \begin{pmatrix} \gamma^2 & -m & 0 & 0 \\ -m & \frac{\gamma(m^2+2r^2)stct+r^3(ct^2-st^2)}{\gamma m^2stct} & 0 & \frac{-r((\gamma^2-r^2)stct+\gamma r(ct^2-st^2))}{\gamma m^2stct} \\ 0 & 0 & \frac{\gamma st+rct}{\gamma st} & 0 \\ 0 & \frac{-r((\gamma^2-r^2)stct+\gamma r(ct^2-st^2))}{\gamma m^2stct} & 0 & \frac{\gamma(\gamma^2-r^2)stct+r\gamma^2(ct^2-st^2)}{\gamma m^2stct} \end{pmatrix}.$$

A parameterization of the cone of revolution K with respect to its vertex $\mathbf{s}$ is

$$\mathbf{k}(u, v) = \mathbf{s} + v(\mathbf{a} + R(\mathbf{b}\cos(u) + \mathbf{c}\sin(u))),$$

where $\mathbf{a}$ is a unit vector in direction of its axis, and $\mathbf{b}$ and $\mathbf{c}$ complete $\mathbf{a}$ to an orthonormal basis in $\mathbb{R}^3$, and R denotes the radius of the cross section circle at distance 1 from $\mathbf{s}$ which has still to be determined. In detail this reads

$$\mathbf{k}(u, v) = \begin{pmatrix} \frac{\gamma^2}{m} + v(\frac{\gamma ct - rst}{m} + R\frac{\sin(u)(\gamma st+rct)}{m}) \\ -vR\cos(u) \\ \frac{\gamma r}{m} + v(\frac{\gamma st+rct}{m} + R\frac{\sin(u)(-\gamma ct+rst)}{m}) \end{pmatrix}.$$

Inserting $\mathbf{k}(u,v)$ into the equation $\mathbf{x}^T \cdot \mathbf{K}(\tau) \cdot \mathbf{x} = 0$ defines the radius

$$R = \frac{\sqrt{-ctst(\gamma st + rct)(\gamma ct - rst)}}{ct(\gamma st + rct)} = \sqrt{\frac{-st(\gamma ct - rst)}{ct(\gamma st + rct)}}.$$

The final parameterization of the quartic curve C is obtained for $v = \frac{2r(R\sin(u)ct - st)}{1+R^2}$ and is a bit lengthy. It reads

$$\mathbf{c}(u) = \begin{pmatrix} \frac{(4Rr\sin(u)ct(\gamma ct - rst) + 2rct(\gamma st + rct)(R^2\sin^2(u) - 1) + m^2 + r^2 + R^2(m^2 - r^2) - 2Rr\gamma\sin(u))}{m(1+R^2)} \\ \frac{-2Rr\cos(u)(Rct\sin(u) - st)}{1+R^2} \\ \frac{r(2R^2ct\sin^2(u)(rst - \gamma ct) + 4R\gamma\sin(u)ctst - \gamma(1-R^2) - 2rctst + 2\gamma ct^2 + 2Rr\sin(u)(ct^2 - st^2))}{m(1+R^2)} \end{pmatrix},$$

and its norm is

$$\|\mathbf{c}(u)\| = \frac{\gamma ct(1+R^2) - 2rst + 2rRct\sin(u)}{ct(1+R^2)}.$$

This is proved by using the incidence $\mathbf{c} \subset E$, thus $a\mathbf{c}_1 + b\mathbf{c}_2 + c\mathbf{c}_3 = ctw$, with $w = \|\mathbf{c}\|$. Note that R is *not rational* in any rational substitution for the trigonometric functions $\cos(\tau)$ and $\sin(\tau)$. Rotating C around the x-axis gives a rational polar representation $\mathbf{f}(u,v)$ of the sphere F. The resulting parameterization $\mathbf{f}$ of F is not proper, but almost all points of F are traced twice, therefore belonging to two parameter values (u_1, v) and (u_2, v). We summarize the construction.

Corollary 3.19 *There exists a one-parameter family of quartic curves $C(\tau) \subset F$ with double point at $\mathbf{s}$ and symmetry plane $y = 0$. The corresponding pencils of quadrics $\mathcal{A}(t) = F + \lambda L(\tau)$ contain rotational cones $K(\tau)$ and $L(\tau)$, where the vertex of the latter is at O, and a parabolic cylinder $P(\tau)$. Besides $P(\tau)$ all quadrics have rotational symmetry with parallel axes $\mathbf{a}(\tau)$. The distance function* $\mathrm{dist}(OC) = \|\mathbf{c}(u)\|$ *is rational in the curve parameter, but not rational in the angle-parameter τ.*

Conclusion

We presented a model to calculate real rational polar representations of curves and surfaces, to gain real rational parameterizations of their conchoids with respect to a reference point. Besides general information about the cone model and its attributes we dealt with admissible mappings, these are mappings that keep rationality of the polar representation. Furthermore we investigated special curves and surfaces and presented symbolic real rational polar representations.

We used this model to show the rationality of conchoids of lines, independent on the position of the reference point. The conchoids of conic sections possess rational polar representations, if the reference point lies on the conic or coincides with a focal point. Furthermore explicit symbolic real rational representations are given for some examples.

We proved that, real rational ruled surfaces and real quadrics are real rational conchoid surfaces, hence they possess a real rational polar representation with respect to the reference point. Additionally we gave explicit symbolic real rational polar representations for special ruled surfaces and quadrics. The latter ones are directly connected to rational parameterizations of del Pezzo surfaces of degree four.

Appendix A

Rational Polar Representation of Ruled Surfaces

A.1 Numerical Example: Hyperbolic paraboloid

The hyperbolic paraboloid is a quadratic ruled surface, hence we can use the algorithms presented in Sections 3.2.2, 3.2.5 or 3.3. We have already seen examples of the latter ones and therefore use the first approach. To recall, the tricky part was the construction of the rational trajectory of the family of conics 3.5.

The necessary computational steps are outlined. Given a hyperbolic paraboloid $F : xy - z = 1$ with the parameterization

$$\mathbf{f}(u, v) = (u, v, uv + 1)^T.$$

To obtain a diagonal normal form of the family of conics determined by the squared distance $\|\mathbf{f}(u, v)\|^2$, we might use the method presented in Section 3.2.1. This example shows another method. The conics $c(u)$ are represented in a coordinate system which is based on the vertices of a polar triangle of the conics $c(u)$ as base points. The squared length of $\mathbf{f}(u, v)$ reads $\|\mathbf{f}(u, v)\|^2 = (u^2 + 1)v^2 + 2uv + (u^2 + 1)$. For the corresponding one parameter family of conics $c(u)$ we get

$$c(u) : -x_0^2 + (u^2 + 1)x_1^2 + 2ux_1x_2 + (u^2 + 1)x_2^2 = 0.$$

By the rational transformation $x_0 = \overline{x}_0$, $x_1 = \overline{x}_1 + \overline{x}_2$, $x_2 = \overline{x}_1 - \overline{x}_2$, the conics are transformed into the normal form, where we use again x_i instead of $\overline{x}_i$,

$$c(u) = -x_0^2 + 2(u^2 + u + 1)x_1^2 + 2(u^2 - u + 1)x_2^2 = 0. \tag{A.1}$$

According to Lemma 3.5 there exist polynomials which satisfy (A.1) identically. Denote the coefficients of x_0, x_1 and x_2 in (A.1) by $a_0(u) = -1$, $a_1(u) = u^2+u+1$ and $a_2(u) = u^2-u+1$, and their degrees are $2l = 0$, $2m = 2$ $2n = 2$. We make the ansatz (3.9) and evaluate (A.1)

at the zeros $\alpha_{0,j}$, $\overline{\alpha}_{0,j}$, $\alpha_{1,j}$, $\overline{\alpha}_{1,j}$ and $\alpha_{2,j}, \overline{\alpha}_{2,j}$ of a_0, a_1 and a_2, respectively. The solution of these four equations in the seven unknowns y_{0i}, y_{1j}, y_{2k} reads

$$\begin{aligned}
&y_{00} = -t_1 - 2\sqrt{3}t_2 + 2\sqrt{3}t_3, && y_{01} = 3t_1 + 2\sqrt{3}t_2 - 4\sqrt{3}t_3, \quad y_{02} = t_1,\\
&y_{10} = t_2, && y_{11} = -\sqrt{3}t_1 - 2t_2 + 3t_3,\\
&y_{20} = \sqrt{3}t_1 + 3t_2 - 4t_3, && y_{21} = t_3.
\end{aligned}$$

We choose $t_2 = 1$ and $t_3 = 0$ and let t_1 such that $c(u)$ has an additional zero at $u = 0$, thus $t_1 = (2\sqrt{2} - 4\sqrt{3})/5$. The curve $\mathbf{y}(u) = (y_0, y_1, y_2)^T(u)$ following the conics $c(u)$ has the parameterization

$$\mathbf{y}(u) = \frac{1}{5}\begin{pmatrix} (2\sqrt{2} - 4\sqrt{3})u^2 + (6\sqrt{2} - 2\sqrt{3})u - 2\sqrt{2} - 6\sqrt{3} \\ (2\sqrt{6} + 2)u + 5 \\ 2\sqrt{6} + 3 \end{pmatrix}.$$

Stereographic projection applied to each conic $c(u)$ finally leads to a reparameterization of F. The center of the stereographic projection is $\mathbf{y}(u)$ and the line which is projected to $c(u)$ is chosen by $\mathbf{q}(t) = (0, 1, t)^T\mathbb{R}$. This leads to the rational polar representation of the hyperbolic paraboloid F,

$$\mathbf{f}(u, t) = (u, v(u, t), uv(u, t) + 1),$$

with

$$v(u, t) = \frac{b - 2t + bt^2 + u(-c + 2at - at^2) + u^2(c - 4t - at^2) + u^3(-1 - 2t + t^2)}{-1 - 2bt + t^2 + 2ut(2 + t) + 2u^2t(-a + t) + u^3(-1 + 2t + t^2)},$$

and the constant factors $a = 1 - \sqrt{6}$, $b = 2 - \sqrt{6}$ and $c = 3 - \sqrt{6}$. For the rational radius function of $\mathbf{f}(u, t)$ we obtain

$$\|\mathbf{f}(u, t)\| = \frac{\sqrt{2}(1 + t^2 + u(-1 + t^2) + u^2(1 + t^2))(4 - \sqrt{6} + \sqrt{6}u + 2u^2)}{2(-1 - 2bt + t^2 + 2ut(2 + t) + 2u^2t(-a + t) + u^3(-1 + 2t + t^2))}.$$

Similar to the approaches of Sections 3.2.2, 3.2.5 or 3.3, we end up with a rational parameterization of bi-degree $(4, 2)$.

Appendix B

Rational Polar Representation of Quadrics

B.1 Hyperboloid of Two Sheets

Formulae of Section 3.3.3 for a hyperboloid of two sheets. Let

$$F : 1 - a^2x^2 + b^2y^2 + c^2z^2 = 0,$$

then the parameterization of the cylinder B of (3.25) reads

$$B : \mathbf{b}(u, v_1, v_1) = \mathbf{e}_0(u) + v_1\mathbf{e}_1(u) + v_2\mathbf{e}_2$$

with

$$\mathbf{e}_0(u) = \left(-\frac{u}{\beta}, \frac{1}{\alpha}, 0, -\frac{u}{c}\right)^T, \ \mathbf{e}_1(u) = \left(\frac{1-u^2}{u\beta}, \frac{2}{\alpha}, 0, -\frac{1+u^2}{uc}\right)^T, \ \mathbf{e}_2 = (0,0,1,0)^T$$

and

$$\alpha = \sqrt{c^2 - b^2}, \ \beta = \sqrt{a^2 + c^2}, \ \gamma = \sqrt{a^2 + b^2}.$$

The conics $L(u), C(u), \overline{C}(u)$, see Figure 3.9, are given as

$$\left.\begin{aligned} \psi(u) = \mathbf{e}_0(u) + v_1\mathbf{e}_1(u) + v_2\mathbf{e}_2 \\ D : x^2 + y^2 + z^2 - w^2 = 0 \end{aligned}\right\} L(u),$$

$$\left.\begin{aligned} \varepsilon(u) = \pi(\psi(u)) : 1 + u^2 + 2\beta ux - \alpha(1-u^2)y = 0 \\ F = \pi(\Phi) : 1 - a^2x^2 + b^2y^2 + c^2z^2 = 0 \end{aligned}\right\} C(u),$$

$$\left.\begin{aligned} \overline{\varepsilon}(u) : c(1+u^2) + \beta(1-u^2)x + 2\alpha uy = 0 \\ S^2 : x^2 + y^2 + z^2 - 1 = 0 \end{aligned}\right\} \overline{C}(u).$$

The conics $\overline{C}$ degenerate to points $P_\pm$ for $u = u_0$ and $u = u_1$

$$P_\pm = (c\gamma, -b\beta, \pm a\alpha, 0).$$

To gain a real parameterization we have to reparameterize the parameter u as

$$u(s) = \frac{u_0 s^2 + u_1}{s^2+1}, \text{ with } u_0 = \frac{-b+\gamma}{a}, \ u_1 = -\frac{-b+\gamma}{a}.$$

Stereographic projection of $\overline{C}(s)$ from P_+ to the plane through O perpendicular to OP_+ leads to the family of circles

$$\begin{aligned} &C^*(s) : (\xi - m(s))^2 + \eta^2 - r(s)^2 = 0, \text{ with} \\ &m(s) = \frac{\alpha\beta(\gamma + bs^2)}{a\mu(s)}, r(s)^2 = \frac{c^2\gamma^2(\gamma s^2 + b)(\gamma + bs^2)}{a^2 s^2 \mu(s)^2}, \text{ and } \mu(s) = b\gamma - s^2\alpha^2. \end{aligned}$$

The squared radius is the sum of the squares

$$h_1(s)^2 = \left(\frac{c\gamma(b+\gamma)}{a\mu(s)}\right)^2, \ h_2(s)^2 = \left(\frac{c\gamma\sqrt{b\gamma}(s^2-1)}{as\mu(s)}\right)^2,$$

and therefore we find the cubic trajectory $\mathbf{q}(s)$ of the family of conics $C^*(s)$

$$\mathbf{q}(s) = \frac{1}{as\mu(s)} \begin{pmatrix} s(c\gamma(b+\gamma) + \alpha\beta(bs^2+\gamma)) \\ \sqrt{b\gamma}c\gamma(s^2-1) \end{pmatrix}.$$

A parameterization of the conics in the plane of bi-degree $(3,2)$ follows

$$\mathbf{c}^*(s,t) = \frac{1}{as\mu(s)(1+t^2)} \begin{pmatrix} \alpha\beta s(s^2 b + \gamma)(1+t^2) - c\gamma((b+\gamma)s(1-t^2) + 2\sqrt{b\gamma}t(s^2-1)) \\ c\gamma(\sqrt{b\gamma}(s^2(1+t^2) + 1 - t^2) - 2ts(b+\gamma)). \end{pmatrix}.$$

The parameterization of the family of conics $\overline{C}(s)$ on the unit sphere reads

$$\begin{aligned} \overline{g}_1(s,t) &= \beta(1+t^2)((s^4\alpha^2 + b^2)(s^2 b + \gamma) + c^2 s^2(s^2\gamma + b)) + 2c\alpha s(s^2\gamma + b)\nu(s) \\ \overline{g}_2(s,t) &= a(-(1+t^2)\alpha((3s^2 b + \gamma)(s^2\gamma + b) + a^2 s^2 + s^2\alpha^2(-s^4+1)) - 2c\beta s\nu(s)) \\ \overline{g}_3(s,t) &= 2as(s^2\alpha^2 - b\gamma)(\sqrt{b\gamma}(1-t^2)(-s^2+1) + 2ts(\gamma+b)) \\ n(s,t) &= c(1+t^2)(\alpha^2 s^2((s^4+1)\gamma + 2s^2 b) + \gamma^2(2s^2\gamma + (s^4+1)b)) + 2s\alpha\beta(s^2 b + \gamma)\nu(s) \end{aligned}$$

with

$$\nu(s) = (\gamma + b)s(t^2-1) - 2\sqrt{b\gamma}t(s^2-1).$$

Again the radius function $\rho(s,t)$ does not increase the degree of the parameterization since $e_1(u)\overline{g}_1(u,t) + e_3(u)\overline{g}_3(u,t) = e_0 p(u,t)$ holds, with

$$\begin{aligned} p(s,t) = \ & \tfrac{-a}{\gamma-b}(2\alpha\beta c s^3(2t(s^2-1)\sqrt{b\gamma}(\gamma-b) - a^2 s(t^2-1)) \\ & +(\gamma-b)(1+t^2)(b\alpha^2 s^2(-\gamma(s^4+3) - 2s^2 bg) - \gamma^2((2c^2 s^4 - b^2(s^4+1))))). \end{aligned}$$

Which yields a rational parameterization of the del Pezzo surface Φ and a rational polar representation of the corresponding hyperboloid of two sheets of bi-degree $(6,2)$.

B.2 Hyperboloid of One Sheet

Formulae of Section 3.3.3 for a hyperboloid of one sheet. Let

$$F : -1 - a^2x^2 + b^2y^2 + c^2z^2 = 0,$$

then the parameterization of one of the cylinders B of (3.26) reads

$$B : \mathbf{b}(u, v_1, v_1) = \mathbf{e}_0(u) + v_1\mathbf{e}_1(u) + v_2\mathbf{e}_2$$

with

$$\mathbf{e}_0(u) = \left(\frac{u}{\gamma}, 0, \frac{u}{\alpha}, -\frac{1}{b}\right)^T, \ \mathbf{e}_1(u) = \left(\frac{1}{u\gamma}, 0, \frac{1-u^2}{\alpha(1+u^2)}, \frac{2u}{(1+u^2)b}\right)^T, \ \mathbf{e}_2 = (0,1,0,0)^T$$

and

$$\alpha = \sqrt{c^2 - b^2}, \ \beta = \sqrt{a^2 + c^2}, \ \gamma = \sqrt{a^2 + b^2}.$$

The conics $L(u), C(u), \overline{C}(u)$, see Figure 3.9, are given as

$$\left.\begin{aligned} \psi(u) = \mathbf{e}_0(u) + v_1\mathbf{e}_1(u) + v_2\mathbf{e}_2 \\ D : x^2 + y^2 + z^2 - w^2 = 0 \end{aligned}\right\} L(u),$$

$$\left.\begin{aligned} \varepsilon(u) = \pi(\psi(u)) : 2u - \gamma(1-u^2)x + \alpha(1+u^2)z = 0 \\ F = \pi(\Phi) : -1 - a^2x^2 + b^2y^2 + c^2z^2 = 0 \end{aligned}\right\} C(u),$$

$$\left.\begin{aligned} \overline{\varepsilon}(u) : 2bu - \gamma(1+u^2)x + \alpha(1-u^2)z = 0 \\ S^2 : x^2 + y^2 + z^2 - 1 = 0 \end{aligned}\right\} \overline{C}(u).$$

The conics $\overline{C}(u)$ are real $\forall u \in \mathbb{R}$ hence no reparameterization of the parameter u is necessary. Stereographic projection of $\overline{C}(s)$ from $(0,0,1)^T$ to the plane $z = 0$ leads to the family of circles

$$\begin{aligned} &C^*(s) : (\xi - m(s))^2 + \eta^2 - r(s)^2 = 0, \text{ with} \\ &m(s) = -\frac{\gamma(1+u^2))}{2bu + \alpha(u^2-1)}, \ r(s)^2 = \frac{a^2(1+u^2)^2 + c^2(1-u^2)^2}{((2bu + \alpha(u^2-1))^2} \\ &\text{and } \mu(s) = (b\beta^2s^2 + c^2\gamma)(bs^2 + \gamma). \end{aligned}$$

The numer of the squared radius is the sum of the squares

$$h_1(s)^2 = (a(1+u^2))^2, \ h_2(s)^2 = (c(1-u^2))^2,$$

and therefore we find the quadratic trajectory $\mathbf{q}(s)$ of the family of conics $C^*(s)$

$$\mathbf{q}(s) = \frac{1}{2bu + \alpha(u^2-1)} \begin{pmatrix} (a-\gamma)(1+u^2) \\ c(u^2-1) \end{pmatrix}.$$

A parameterization of the conics in the plane of bi-degree $(2,2)$ follows

$$\mathbf{c}^*(s,t) = \frac{1}{(2bu + \alpha(u^2-1))(1+t^2)} \begin{pmatrix} -(1+u^2)(\gamma(1+t^2) + a(1-t^2)) + 2ct(1-u^2) \\ c(u^2-1)(t^2-1) - 2at(1+u^2) \end{pmatrix}$$

The parameterization of the family of conics $\overline{C}(s)$ on the unit sphere reads

$$\begin{aligned}
\overline{g}_1(u,t) &= -(2(2bu + \alpha(u^2-1)))((1+u^2)(\gamma(1+t^2) + a(1-t^2)) + 2ct(1-u^2)) \\
\overline{g}_2(u,t) &= (2(2bu + \alpha(u^2-1)))(c(u^2-1)(t^2-1) - 2at(1+u^2)) \\
\overline{g}_3(u,t) &= 2(1+t^2)(-2bu\alpha(u^2-1) + (u^2+1)^2\gamma^2 - 4u^2b^2) - 2(2ct(u^4-1) + a(1+u^2)^2(t^2-1))\gamma \\
n(s,t) &= 2(1+t^2)(2bu\alpha(u^2-1) + (u^4+1)\beta^2 + 2u^2(\gamma-\alpha)) - 2\gamma(2ct(u^4-1) + a(1+u^2)(t^2-1)).
\end{aligned}$$

Again the radius function $\rho(s,t)$ does not increase the degree of the parameterization since $e_1(u)\overline{g}_1(u,t) + e_3(u)\overline{g}_3(u,t) = e_0 p(u,t)$ holds, with

$$\begin{aligned}
p(u,t) = \; & 4u((b(1-u^2) + 2\alpha u)(-u^2a + 2u^2ct + u^2at^2 - a - 2ct + at^2) - \\
& (u^2+1)(t^2+1)(-u^2bc^2 - u^2ba^2 + 2\alpha ua^2 + bc^2 + ba^2)).
\end{aligned}$$

Which yields a rational parameterization of the del Pezzo surface Φ and a rational polar representation of the corresponding hyperboloid of one sheet of bi-degree $(4,2)$.

B.3 Numerical Example: Ellipsoid

We used the following example to draw the Figures 3.8(b), 3.10(b) and 3.11. Let an ellipsoid be given by

$$F : -2 + 2x^2 + 4x - 2xy + 2y^2 + z^2 = 0.$$

The polar plane $\delta : x - 1 = 0$ of the origin intersects F in a complex conic. The transformations to achieve a normal form are the following:

- Perspective collineation:

$$\varkappa : \mathbf{x}' = \frac{1}{x-1}\mathbf{x}$$

- Rotation about O with $\omega = \frac{3}{8}\pi$ combined with a re-ordering of the coordinate axes:

$$\widetilde{\mathbf{x}} = R \cdot \varkappa(\mathbf{x}) = \begin{pmatrix} 0 & 0 & 1 \\ \cos(\omega) & \sin(\omega) & 0 \\ -\sin(\omega) & \cos(\omega) & 0 \end{pmatrix} \cdot \varkappa(\mathbf{x}).$$

Applying these two transformations to F leads to the normal form, again

$$F' : x^2 + (3 - \sin 2\omega + \cos 2\omega)y^2 + (3 + \sin 2\omega - \cos 2\omega)z^2 - 2.$$

Since the coefficients of F' are trigonometric functions of the rotation angle ω, and because of the fact that the final parameterization $\mathbf{f}(s,t)$ contains square roots of these coefficients, e.g. $\alpha = \sqrt{c^2 - b^2}$, we use floating point numbers as approximations. Inserting the coefficients $a = 1/2, b = (3 - \sin 2\omega + \cos 2\omega)/2$ and $c = (3 + \sin 2\omega - \cos 2\omega)/2$ into the solution (3.37) and inverting the transformations to get the following rational parameterization of the quadric F,

$$\begin{aligned}
\mathbf{f}(s,t) = {} & \tfrac{1}{p(s,t)}(x(s,t), y(s,t), z(s,t)), \text{ with} \\
x(s,t) = {} & (-1.07s^6 + 0.80)(t^2+1) - 1.41(t^2 + 1.20t - 1)s^5 + (1.59t^2 - 2.01 - 6.02t)s^4 \\
& +0.34(t^2 - 5.47t - 1)s^3 + (6.11t^2 - 4.54t - 1.47)s^2 + 1.06(t^2 + 3.34t - 1)s \\
y(s,t) = {} & (0.44s^6 - 0.33)(t^2+1) - 3.40(t^2 - 0.20t - 1)s^5 + (-0.66t^2 - 14.53t + 0.83)s^4 \\
& +0.83(t^2 + .93t - 1)s^3 + (-2.53t^2 - 10.97t + 0.61)s^2 + 2.56(t^2 - 0.57t - 1)s \\
z(s,t) = {} & (1.10s^6 - 0.83)(t^2+1)s^6 - 8.43ts^5 + (12.82t^2 - 5.21)s^4 + 4.42ts^3 \\
& +(4.60t^2 - 3.98)s^2 + 4.01ts, \\
p(s,t) = {} & (0.82s^6 + 2.23)(t^2+1) - 1.41(t^2 + 1.20t - 1)s^5 + (5.86t^2 - 6.02t + 2.25)s^4 \\
& +0.34(t^2 - 17.26t - 1)s^3 + (15.84t^2 - 4.54t - 0.32)s^2 + 1.06(t^2 + 7.12t - 1)s.
\end{aligned}$$

The norm of $\mathbf{f}(s,t)$ reads

$$\|\mathbf{f}(s,t)\| = \tfrac{1}{p(s,t)}((1.60s^6 + 4.79s^2 + 1.21)(t^2+1) - 4.50s^3t(s^2-1) + (11.84t^2 + 2.21)s^4).$$

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Scientific Publications

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Scientific Talks

03. July 2012, *Rational Conchoid Surfaces.* 8. International Conference on Mathematical Methods for Curves and Surfaces, Oslo, Norway.
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