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Abstract

World population reached the 7-billion mark in 2011. This event took place during the middle of a world wide economic crisis, combined with a record-year of damages due to natural disasters (Zeit, 2011). Therefore the question arises if and how these systems are related. This thesis provides an overview of several papers, which have treated this topic with various approaches. Afterwards we present an *Optimal Control Model* (OCM) model where fertility, capital and emissions are treated as endogenous controls, connecting the three systems of population, economy and environment. We extend the model by introducing more complex dynamics of two kinds. First a mortality function of either capital or environmental quality or both is introduced. Second advanced dynamics of the environment, called Shallow Lake Dynamics are discussed. This study shows the consistency of the analyzed model and provides results which are overlapping with real world developments.

Acknowledgment

Finishing this thesis marks a special point in my life, as it means finishing the studies of Technical Mathematics at the Technical University of Vienna. This instructive time would not have been possible without the all-embracing support of my family, first of all my dear mother. Thanks for taking off my mind from certain difficulties, showing a light at the end of the tunnel and creating an environment where I was able to focus on my studies, no matter where in the world you have been when I needed support. Thanks also to my brother who supported me with visions and ideas all the time, starting from sports to experiences abroad and many more. He and many friends made the last years unforgettable and enriched my life with many personal experiences, great moments and life-long lasting memories. I am also thankful to my teachers, especially Prof. Dr. Fürnkranz-Prskawetz for supervising this thesis, for making this time the interesting and exciting years, which they have become.

Contents

Chapter 1

Introduction

In October 2011 the number of humans on earth reached seven billion. The last billion of growth in population size occurred in only 12 years. The Economist (2011) illustrated this with ”a tale of three islands”. 60 years ago, when the world wide population was only 2.5 billion, all could have stood on a 381-square kilometer rock, the Isle of Weight. Reaching 3.5 billion mankind has moved to the Isle of Man, 572 square kilometers. Nowadays we already need an island as big as Zanzibar, 1,554 square kilometers. But are these islands always able to survive under this immense pressure of foots trampling over their surface? Can mankind survive on these islands?

Fortunately, as half the population lives in cities on four percent of earths surface, as can be seen in *National Geographics*, Kunzig (2012), we still have enough space to live. And our planet is a very generous keeper. It hosts us since hundreds and thousands of years. It provides us as keeper with everything we need, but we are not always modest. As we eat, consume and produce, we take a lot from the environment. The danger of overfishing certain species or overuse of resources such as wood, metal, fuels and water is evident. And because this host is so generous the human population is growing in a manner earth has never seen before. Not only through the things we take, but also through the things we leave behind, we destroy our home and complicate our keepers life. Side products of production like waste and emissions are threatening air and rivers with pollution, what not only has a bad influence on the health of people, but is going to change the climate as well. Already with this very general description, we shed light on feedback mechanisms of human actions.

Obviously, climate affects everybody. Economically speaking, it is a *non-rivalrous* good, as the use by one does not restrict its use for somebody else, like public television or a search engine online. Furthermore it is also a *non-exclusive* good, which means people can not be excluded from consumption like it is the case for public defense or street lights. So because of these two characteristics it is difficult to determine who may do what under which regulations or costs. Should the consumer - everyone, meaning taxpayers - pay for the well-being of the climate? Who cares about the environment? Firms and people often do not consider the damage done to the environment due to their actions, because it does not affect them directly.

Fortunately, the climate and many other ecological systems and resources have abilities to regenerate themselves. Forests and fish swarms can grow or pollution is to some extend naturally degraded. This is useful as we depend on the health of the environment and availability of resources. As we grow agriculture, build our houses with natural materials and dig in the ground for fuels, ecosystems must provide the basis. We have to consider the regeneration process in the way we act. The United Nations Framework Convention on Climate Change UNFCCC (2012) therefore formulates this problem as follows:

> [...] to allow ecosystems to adapt naturally to climate change, to ensure that food production is not threatened and to enable economic development to proceed in a sustainable manner.

However, the condition of the environment is manifold. Diverse wild-life is of importance as well as low CO2-emissions. Therefore, the *environmental performance index* (EPI, Yale (2012)) incorporates 16 indicators which are summarized to the areas environmental health, air quality, water, bio diversity, natural resources and energy. When comparing the EPI to life expectancy as shown in figure 1.1 we see a clear correlation between these two factors. But it is difficult to answer what is the reason for this correlation. It does not seem obvious that people first invest in environmental quality and then gain a longer life. In fact, technological advancement especially in medicine prolonged lifetime which was a driving factor for economic growth. Also it made it possible to reach the tipping point of the environmental Kuznets curve (EKC, Shafik (1994)). This generates investment into environmental quality instead of production or otherwise.

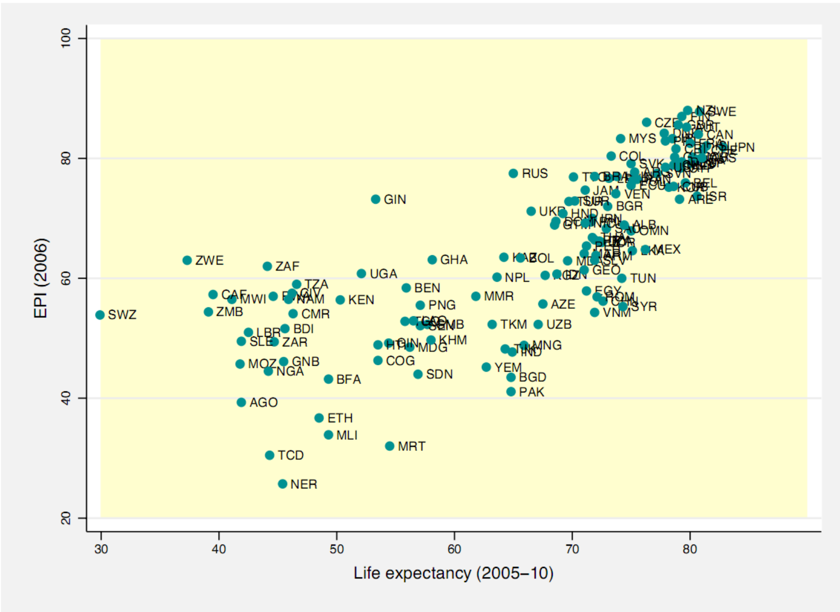


Figure 1.1: Correlation between EPI and life expectancy. Source: Mariani et al. (2009)

It is difficult to argue whether or not economic growth and wealth are a bless or a curse for the environment. On the one hand resources are obviously necessary for economic growth and the rising living standards generate more pollution. Many improvements are based on the intensive use of raw materials, energy and other resources. Together with recklessness to natural habitats concerning e.g. rain forests, these trends have had and will continue to have unforeseen effects on the environment. On the other hand poverty is dangerous as well. The poor and hungry have no other choice than to use what's closest to them in order to survive. Their livestock overgrazes grasslands, marginal land is overused and forests are cut down without foresight, as people have no other choices.

A very prestigious example for abuse of the environment due to economic prosperity, which has already taken place was the desiccation of the *Aral Sea*, see Glantz (1999) or Micklin et al. (2012). Water from many inflows of the Aral Sea has been used for various purposes, mainly irrigation. Huge fields of cotton and grain were grown thanks to water, which should have flown into the basin. In less than 30 years, the sea level fell by 12.9m, it contracted by 40% in area and 60% in volume. Obviously fishers found their boats grounded. A once fruitful area completely dried up and transformed into a salt desert, which it is still today and will continue in the future, since this process of desertification is almost irreversibel.

Figure 1.2: Grounded ship in the salt desert of ancient Aral sea, source: Kazakhstan (2012)

If this process continues, the Aral Sea may become a *Shallow Lake*. This is an example for a complex environmental system. As stated in Mäler et al. (2003) these lakes suffered as extensive disposal point for waste water of cities or industries. Up to a certain point plants and animals are able to clean the water and keep it clear (the self-regeneration rate). But as soon as a certain degree of impurity is reached, the water becomes turbid. In this state sunlight can't penetrate it anymore and the flora degrades. With it many species using the plants as nutrition disappear and therefore can't clean the water anymore. The shallow lake has lost its self-cleaning ability and will remain turbid. Once turbid, it is hard to restore the clean state, sometimes even impossible.

As pointed out, there are certain *thresholds*, like desertification or shallow lakes, where regeneration becomes impossible. A system crossing such a threshold is called *irreversible*. Others like the rise of the ocean can even endanger survival on earth. This is mainly caused by the accumulation of CO2 and other green house gases in the atmosphere, leading to global warming.

An economic crisis also poses a challenge to the environment as it limits abilities of all nations to react and adjust altruistic. The last points can be followed up in Brundtland (1990, part I).

All these aspects raise vital questions such as

1. Is economic growth or a growing population compatible with environmental protection?
2. Which are the main driving factors for environmental deterioration?
3. If we experience environmental deterioration, what are the consequences for human or animal life on earth? What consequences does it have on sustainable growth?
4. How can we anticipate it? What political instruments are available and which ones are effective?

and many more.

To sum up, the characteristics of and links between population growth, economy and the environment are very complex and manifold. Various approaches are necessary to capture as many details as possible and economic studies may help to explain certain relations and predict consequences of todays actions and decisions, as well as answer the above questions. Economists must work together with ecologists and other scientists of all kind in order to understand and incorporate environmental topics in their models. The literature about population and economy is vast, also between economy and environment. However it is important to keep in mind that *"little progress can be made without tackling both issues at the same time"* as stated in Peretto and Valente (2011, p.3) and we will shed light upon the existing literature in Chapter 2.

After having evaluated these contributions, we see that they consider different aspects of relations between population growth (or size), economy and the environment. The goal of this thesis is to investigate dependencies between population, economy and environmental dynamics in an optimal control model. From our point of view, an approach for industrialized countries should include the following characteristics:

- The number of children is a decision taken by the parents and it yields utility caused by pleasure.
- Physical capital is necessary for industrial production to create an unified consumption good. Emissions are a by-product of or input to production.
- Due to the flow of emissions a stock of pollution accumulates, affecting life quality and therefore utility.

The work of Quaas (2004) lives up to these characteristics and will therefore be used as base model. This model incorporates three systems namely demography, economy and the environment in an *Optimal Control Model* (OCM). Every system is characterized by one state and one control variable. The product of emissions per capita and population size represents the dirty flow, affecting the environment which has a certain self regeneration rate itself. Environmental deterioration and population growth is a consequence of inter temporal decisions taken by a representative household who aims to maximize utility drawn from consumption, fertility and environmental quality. Mortality is assumed to be given exogenous and households decide upon fertility. The economic control is consumption and the environmental control is the amount of emissions used for production. In the used framework of dynamic optimization, a steady state can be interpreted as state of *sustainable development*. Therefore the question of existence and the according properties are of interest for a sustainable development of population and economy. We will introduce this model in detail in chapter 3. In the analytical analysis we proof the existence of a balanced growth path. While population size increases, emissions decrease at the same rate. As the environmental impact is a product of population size and emissions, the value of this impact stays constant. After a transformation to a stationary state, we analyze it numerically in three parts. First transition dynamics are used with different projections on the control-state space as well as projections over time in order to understand the connections between the states and controls. Second the paths of system-characterizing functions (e.g. shadow prices, marginal productivity, overall emissions) are analyzed to provide deeper insight. Finally the stability of this model is tested with sensitivity and bifurcation analysis.

As pointed out above, the impacts of population and economy on the environment are not at all one-way and can become very complex. To pay tribute to these challenges, we will perform some adaptations on our model in chapter 4. In the base model, mortality has been exogenous. This is a very limiting assumption, since especially environmental developments like air pollution or extreme scenarios can severely raise mortality. Contrary economic welfare like medicine, good nutrition or clear water clearly reduces mortality. To pay tribute to these dynamics we will introduce an endogenous mortality function with three different specifications. One is a function of economic prosperity, the second a function of pollution and the third a function of prosperity and pollution combined. We will study the obtained steady states and transition dynamics.

In the base model, environmental dynamics are modeled quite simple. As pointed out above with the example of the Aral Sea, environmental dynamics can become very complex and even irreversible. Therefore shallow lake dynamics as motivated above are implemented into the base model, leading to multiple steady states. The possibility to perform such adaptations is very important to consider more complex dynamics and pay tribute to real world situations. Summary and conclusion will be shown in chapter 5.

Chapter 2

Literature review

The first economist to discuss the relations between environment, economy and fertility was Malthus (1798). He came to his essay after evaluating recent developments in England of the 18^{th} century. He argued that when the size of population is low in comparison to a certain resource, the economic situation is profitable. Therefore people will choose to have more children, resulting in a growing population size. A larger population puts pressure on the availability of resources (called demographic pressure) or per capita output and the economic situation worsens. Hence fertility will decline, until an equilibrium is reestablished with no population growth and low per-capita income, named subsistence level. This pessimistic view of the relationship between fertility and economic growth is called in honor of his founder *Malthusian trap*.

Especially the last century has proven that Malthus was wrong. Population growth as well as economic growth took place at the same time. Simultaneously we experience growing pollution, destruction of natural environments, over-fishing and further burdens on the environment. Therefore it becomes necessary to incorporate the environment in a new way. Even if the existing research on the relations between population growth and the environment is not large, there is some literature showing many different approaches. As mentioned in the introduction, it is important to treat all 3 systems (population, economy and environment) at the same time. We will provide an overview over some of the existing papers in this field. We investigate if the nature of the discussed problem is positive or normative. Question 2 of the introduction is positive while 1, 3 and 4 are normative. To answer these questions, one can apply different frameworks. *Simulations* can be used to reproduce historical data as seen in Fröling (2011). To test hypothesis on effects of population on environment, one may use *econometric analysis*, e.g. Van (2002) where the rate of deforestation is analyzed. In *overlapping generation* (OLG) models, the individual lives two to three periods and faces decisions in each period. Commonly decisions are taken in the second period of adulthood. Often it is uncertain if the third period is reached (longevity) and may for example depend on the quality of the environment. *Optimal control models* (OCM) assume that a central planner has certain controls at hand to influence the development of the systems. We will use this framework for our own analysis in the next chapter. The models also differ in scale starting from global (our planet) over local (e.g. nations) to regional (e.g. urban vs. rural). Models of different scale are obviously using different indicators of the environment. A global model may study pollution or energy use while a local model does take care of deforestation or water use see Quaas (2004); Fröling (2011); Van (2002); Pimentel et al. (1997) respectively. As the links between economy, population and the used indicator of environment vary significantly, different functions need to be used for modeling. Different forms of the economic-environmental relations are intensively studied in Xepapadeas (2005). For modeling environmental dynamics the regeneration rate is of special interest. In most papers, some kind of constant regeneration is assumed. Especially interesting are the mentioned shallow lake dynamics, an introduction can be found in Wagener (2009).

The way we use resources is determined by our knowledge. Knowledge accumulation is considered as main driving force of economic growth. *Technological change* (TC) concerning environmental topics can raise the efficiency of resource use on the one hand or can lower the cost of abatement policies on the other hand. Classic economists distinguish between Hicks-, Harrods- or Solow- technological progress, whether it augments total output, labor or capital respectively, see Prettner (2010). Loschel (2002) describes various ways to model induced or even endogenous TC and compares technological progress in the literature. Whether or not investment in technology pays off and to what extend is the question of many research papers, e.g. Romer (1990). TC generates positive spillovers, positive externalities or may even make backstop technologies possible. Backstop technologies can be introduced as *target time* in models for the switch of scarce resources (e.g. fossil fuels) to a substitute (e.g. solar energy), see for example Tahvonen and Salo (2001).

We previously mentioned that often decisions need to be taken. They are taken in order to optimize a certain function, commonly called welfare function. Its form may strongly influence the results, as explained in Canton and Meijdam (1997). They discuss 3 distinct sorts of utility function, represented by

$$U(0) = \int_0^\infty \frac{c(t)^{1-\theta}}{1-\theta} L(t)^\epsilon e^{-\rho t}$$

where utility U is drawn from consumption c. The Ramsey-model accounts for a fix number of dynasties in the population with two specifications: First the Ramsey-Benthamite variant, where utility is weighted by population size L ($\epsilon = 1$) or the Ramsey-Millian function ($\epsilon = 0$). ρ denotes the discount rate, $\rho > 0$ implies that future utility is less valued than todays utility. We see that in the Ramsey-model people behave altruistic due to the budget constraint

$$\frac{dk}{dt} = (r - \lambda n)k + w - c$$

where k represents assets, w wage income and $(r - \lambda n)$ the effective interest rate. If $\lambda = 1$, we have a Ramsey-model with altruism in the sense of Barro (1974), where transfers are given to new generations. On the contrary $\lambda = 0$ represents a model of *unloved children* where all generations are heterogeneous, called model of Weil (1989). In the next chapter we will use a Ramsey-Millian function as it fits our problem.

As we see, utility is mainly drawn from consumption with regard to various aspects like the shown altruism. This is extended by two-sided altruism, where the successors take care of their parents, e.g. Srinivasan (1988), Raut (1996). Another term in the welfare function is often some kind of environmental quality. If it is represented by pollution, it may influence utility negatively as we will it introduce later. Contrary, fertility gives pleasure to the parents and hence has a positive effect.

After having discussed the major parts of a model and their characteristics in general, we will summarize 8 existing models. These models have been chosen, as they incorporate different kinds of population and economic developments with various effects on the environment. The frameworks are used with very diverse settings and analyzed with different methods. This overview aims to display the range of possibilities in economic environmental and population modeling. After a separated discussion of each, they will be compared in table 2.1. Note that as we tried to exhibit the most important aspects, the summaries do not capture all details. The interested reader may be referred to the corresponding papers.

1. The work of Fröling (2011) is motivated by the energy and population development from 1800 to 1970. During the Industrial Revolution world economy and population size switched from the neither population nor economic growth Malthusian period to a post-Malthusian era with growth in both variables. The use of energy played an important role in this transition and is therefore incorporated in growth theory and modeled as input for production. It is separated into biomass (renewable) and coal (nonrenewable) energy. Energy choice is driven by accumulating technology for coal in order to produce the switch from biomass to coal around 1930. Fröling uses a dynamic equilibrium approach and is able to reproduce historical data broadly in terms of energy use, population size and GDP and points out the importance of energy for the industrial revolution.

2. Van (2002) uses deforestation as an indicator for environmental quality. First a cross-section regression over various regions is used to test the relations of deforestation to population growth and GDP. This is mainly interesting for developing countries, since wood is still a very important resource there. The results state that GDP per capita has a small negative but not significant effect on the rate of deforestation. Yet demographic pressure significantly drives the rate of deforestation up. Secondly Van supports his empirical results with an optimal control model, showing the existence of a balanced growth path (BGP) with consumption and income per capita growing at the same rate. The theoretical and empirical results in this paper are consistent, especially for developing countries.

3. Quaas (2004, Ch.7) also presents empirical and theoretical findings. First a field study of Bombay is carried out to motivate and support the following model. Special regard is paid to demographic changes (due to fertility as well as migration), emissions and their sources as well as economic aspects and poverty. He wants to study three groups of questions. First the impact of infrastructure on the environment, second the links of migration to economic and ecological surroundings. Third the reasons for poverty resulting in slums (nonofficial

settlement) and their environmental impact. The used framework is a spatial equilibrium economy. Emissions are generated as side effect of consumption (as movitated by the field study). As the individuals decide independently, external effects occur leading to a non-optimal *laissez-faire* equilibrium. However, these external effects can be incorporated by a Pigou-tax on consumption. Infrastructure is helpful to reduce the marginal damage of consumption as well. By studying the effects of slums, Quaas delivers a new input to economic literature. He shows that in the social optimum, there would be no slums. By comparing the laissez-faire scenario and an equilibrium with regulations on migration, he shows that regulations are not effective as long as they can be avoided, which is the assumption for the emergence of slums. This article is strong in incorporating empirical findings of the field study into a spatial economy.

4. Bretschger (2010) takes the challenge of modeling positive innovation and consumption growth with bounded resources and growing population. This paper is worth mentioning because of certain characteristics. First the dynamic law of population growth pays tribute to the Malthusian perspective and demographic transition at the same time. Second, not only the production sector, but also the innovation sector is in need of the non-renewable resource. Third, there is poor substitution between resources and labor or capital, which is often disregarded due to its complexity. Fourth, sectoral change is intensively discussed and modeled in terms of labor switch from employment between production and innovation. Last, physical capital does not have any impact on growth. Even under these conditions a path of sustainable growth emerges. By using an optimal control model to incorporate the mentioned characteristics, he studies the steady state as well as the transition phase. It is shown that population growth is important to foster innovation and therefore reduce the dependency of economic growth on resources.

5. Balestra and Dottori (2009) assume that longevity depends on the one hand on health care and on the other hand on environmental quality. The age of people is characterized by an OLG model, where young prefer investment in environmental protection as it takes longer to yield effects, but is more persistent. The older generation spends more on health care for their own good. This framework is adapted for a political economy (where agents vote for taxation, public expenditures on environmental maintenance and health care) as well as a social planner (maximizing of a social welfare function). Generally, health care is much more intensified in the political economy, but the social planner achieves a higher life expectancy due to the consequent investments in environmental quality. However, this model only captures longevity and omits population growth. It is argued that the demographic transition would strengthen the political economy, as it leads to a higher ratio of old to young people. An interesting topic is also the relation between education and life expectancy (which is also featured by the demographic transition, especially rise of life expectancy).

6. Mariani et al. (2009) is similar to the previous described paper in the sense, that environmental

quality affects life expectancy in an OLG model. But also higher life expectancy will motivate agents to invest more in environmental quality, as they expect to live longer from the beginning. Therefore we have a two-way dynamic in this model. Consequently the focus is on the existence of multiple steady states, caused by this two-way dynamic between environmental quality and life expectancy. One steady state is a *environmental poverty trap* of low environmental quality and low life expectancy. Therefore possible strategies to escape this trap are discussed as for example a permanent income expansion or an externally raised life expectancy (e.g. medical care).

7. Requate and Cronshaw (1997) consider an economy, where pollution occurs as side effect of production. It can be abated by restricting pollution or using labor. Population growth is given exogenous and used for comparative statics. A Malthusian perspective is proven, despite the absence of resource limits. At a high population size, consumers are worse off, but not necessarily because of increased pollution. Reduced consumption or increased labor supply could also be the reason for a lower level of utility. This effect is offset by the introduction of technological progress, which leads to a positive correlation of consumption and emissions to population.

8. Peretto and Valente (2011) aims to answer the question how to retain income growth in a finite habitat. This is done by studying the interactions of resource use and population (both endogenous). They generate a long-run equilibrium with growing income and constant population size. Innovations in this model can be horizontal (increase of total factor productivity) or vertical (design of new products or establishment of new firms). A so-called pseudo-Malthusian equilibrium with constant population and growing income per capita is identified. Especially interesting is the introduction of a preventive check (e.g. minimum resource requirement like nutrition or living space) which is identified as important function for constant population size.

	Fröling (2011)	Van (2002)	Quaas (2004, Ch.7)	Bretschger (2010)
problem nature	positive	normative	normative	normative
model	simulation using dynamic equilibrium	econometric analysis, OCM	spatial equilibrium	OCM
scale	global	local (developing countries)	regional (urban - rural)	global
production fct.	CD(A,E^{Coal},$E^{Biomass}$,L)	CD(A,k,n)	CES(L,E)	CES(resources, L)
utility fct.	u(c,n)	u(c,n,S)	u(c, l, T_f, S)	u(c)
techn. progress	Harrod accumulated by labor	none	none	Hicks, accumulated by labor
population growth	the individual splits time between fertility and working	controlled - utility from fertility	rural-urban migration	demographic transition
environmental indicator	energy renewable vs. nonrenewable	deforestation	pollution	scarce natural resource (nonrenewable)
impact on environment	none	demographic pressure	consumption	resource use
regeneration	none	implicitly	none	none
specials	labor shares for sectors	low use → forest growth	modeling of city development	intermediate goods sector
main results	·) broadly reproduce historical data (1800-1970) ·) technological progress necessary for economic development	·) demographic pressure worse than economic growth	·) modeling of Bombay problems ·) advices to the city management ·) income has positive effects on environment	·) labor and backstop are ultimate resources ·) Hartwick-Rule

Table 2.1: Overview of models 1-4

[0] Abbreviations used in the tables: OCM - Optimal Control Model, OLG - Overlapping Generations, CD - Cobb Douglas, CES - constant elasticity substitution, A - technology E - energy, L - labor, k - capital,l - space, n - fertility, N population size, S - environmental quality, T_f - spare time

	Balestra and Dottori (2009)	Mariani et al. (2009)	Requate and Cronshaw (1997)	Peretto and Valente (2011)
problem nature	normative	normative	normative	positive
model	general-equilibrium OLG	OLG (3^{th} period depends on environmental condition)	general-equilibrium and comparative static analysis	general equilibrium
scale	global	local	global	local
production fct.	only capital	none	only labor	none
utility fct.	u(c)	u(c, S)	u(c,L,S)	u(c,n,N)
techn. progress	none	none	resource augmenting constant	endogenous horizontal & vertical innovation
population growth	longevity dependent on health expenditures and environmental quality, but no population growth	longevity dependent on environmental quality	exogenous	endogenous by marginal value of *children value*
environmental indicator	pollution	local indicator (EPI)	pollution	scarce resource
impact on environment	side-output of production	consumption	side-output of production	resources are input to production
regeneration	endogenous maintenance	natural, maintenance by labor-abatement	restricting production, use of labor for abatement	none
specials	age structure, health expenditures	human capital	abatement activity	different forms of technological change
main results	·) social planner establishes higher environmental quality than political economy	·) poverty trap and ways to escape it ·) initial conditions matter	·) first-best allocation, ·)pigouvian tax vary with population size	·) growing income with constant population ·) different results if resource and labor are complementary/substitute products

Table 2.1 continued: Overview of models 5-8

To sum up, we have considered eight examples concerning the relations between population, economy and the environment. We evaluated papers starting from the Malthusian population pressure, tackling the Industrial Revolution, treating demographic transition, considering recent development of Bombay and taking care of deforestation in developing countries. Van (2002) and Peretto and Valente (2011) consider in particular developing countries, where children contribute to the output of a family. This is not the case for industrial countries. In this setting the environment is in general modeled in two ways. The one option is to model emissions as side-output of production as in Balestra and Dottori (2009); Requate and Cronshaw (1997) or the other option is to study resource use as done by Fröling (2011); Bretschger (2010); Peretto and Valente (2011) where they investigate the different cases of renewable versus nonrenewable or scarce resources.

The work of Mariani et al. (2009) distinguishes itself from other papers as it models not only the influence of environmental quality on life expectancy, but even considers a two-way causality. Environmental quality increases life expectancy and the decision of environmental investment depends on life expectancy. However, in their framework they do not consider population growth. Requate and Cronshaw (1997) also did not treat population growth as endogenous, but thanks to comparative statics, they were able to point out that not only the environmental quality is responsible for bad social welfare at a high population size.

The considered environmental indicator in the more abstract models is always some kind of ”pollution”, which is a side effect of production or consumption. A more precise definition is mainly given in empirical studies, like deforestation in Van (2002) or energy use in Fröling (2011).

Nowadays there are some models tackling the connections between all three systems. But none of these models considers endogenous fertility, capital accumulation and environmental deterioration at the same time and endogenously. This makes the approach of Quaas (2004) unique and we will now proceed to analyze this model.

Chapter 3

Optimal Control Model - Base scenario

When dealing with questions of economic growth, there are different frameworks. In this thesis an *Optimal Control Model (OCM)* will be used. In this model a central planner has various *controls* at hand to behave optimal. Optimality is usually defined over a *value function.* But todays choice does not only influence the value function, it is essential for the behavior of the *states.* Their development is given by the differential equations with respect to time, consisting of parameters, exogenous variables and the endogenous states and controls. As the name OCM points out, the controls are the central part of the model. Objects of inquiry can be split in two parts. First the question of existence, uniqueness and optimality of a *steady-state*[1]. Such a state or path can be economically interpreted as "sustainable development" where the system dynamics do not change anymore and the future therefore is given. The existence will be shown in the analytical section. Secondly its behavior dependent on different values of the parameters and starting points is of interest. As the world can not be supposed to start in the steady state, we need to find out under which circumstances an optimal path, leading to the steady state, is possible. The according transition paths and phase portraits may give important insights. These questions will be tackled in the numerical section. With answers to these questions, the central planner will know how to behave optimally in which situation. We investigate the relations between states and controls, how states and controls develop over time as well as the time evolution of production, emissions, etc... . We seek to find out if e.g. fertility rises with population size or if consumption is in correlation to pollution. This analysis may help a policy maker to behave better in the real world. Of course a precise control of fertility is not realistic, but people can be motivated to get children by e.g. financial support. In general it is easier to control production than fertility as laws can be incorporated easier in industrial policy than family policy.

To analyze the given questions about the development of and relations between population, economy and the environment, the optimal control model presented in *Quaas (2004) - Bevölkerung*

[1] all states grow at a constant rate (*Balance Growth Path* (BGP)) or are constant, meaning the growth rates equal 0 (stationary state)

und Umweltökonomie[2] will be discussed. This model is in line with our expectations pointed out in the introduction.

For the investigation, a global model characterized by the systems population, environment and economy is considered. Each one is represented by a state and control variable in a way, the systems are connected to each other. We assume that a central planner has absolute control over all three systems and aims to maximize welfare[3]. This assumption is commonly used in large scale economic models. In chapter 6 of Quaas (2004) decentralized decisions of independent households are considered as well. Nevertheless, the discussion with a central planner who maximizes social welfare provides many important insights and therefore this approach is used. As mentioned in the introduction the analysis is split into an analytical and numerical part. After a detailed description of the model dynamics and functions, we proof the existence of a BGP in the analytical section. Using a transformation, a unique stationary state can be derived and analyzed numerically. Transition dynamics exhibit the relations of variables to each other and over time and the sensitivity analysis shows the dependencies on certain parameters.

3.1 Welfare Function

To determine what is the optimal development of the system, we have to choose a value function, which is representative for the welfare of society. A priori it is not clear if the results will strongly depend on the form and variables choosen for the welfare function. As described in Chapter 2 we use a Ramsey-Millian utility function, as given by the following expression:

$$U = \int_0^\infty u(c(t), n(t), S(t)) \cdot exp(-\rho t) dt \tag{3.1}$$

To simplify notation the time variable (t) will be omitted in future equations. Welfare is considered over an infinite time horizon, while future utility is discounted by ρ. This means, that for $\rho > 0$ utility in the future is less important than utility today. This assumption is often criticized in environmental models, because in the context of sustainability, utility of future generations should not be discounted. Welfare depends on the utility drawn from the three different sectors. The considered variables are consumption c, fertility n and pollution S. Consumption is the main driving force of utility. We assume that getting children (fertility) conveys pleasure for the parents and therefore has a positive effect on utility. Other aspects of getting children (e.g. investment in pension) are neglected. Pollution reduces environmental quality and therefore influences utility negatively. Using the convention that derivatives are denoted by a subscript, we can write these assumptions as follows:

[2] Population and environmental economics
[3] the value function

$$u_c(c,n,S) > 0 \tag{3.2}$$

$$u_n(c,n,S) > 0 \tag{3.3}$$

$$u_S(c,n,S) < 0 \tag{3.4}$$

This is a general specification of the utility function. To derive concrete results in the numerical analysis, we have to exactly specify the form of the utility function. The following function is used, which satisfies the above assumptions:

$$u(c,n,S) = ln(c) + \nu\frac{n^{1-\epsilon}-1}{1-\epsilon} + \sigma ln(\bar{S}-S) \tag{3.5}$$

The parameters $\nu > 0$ and $\sigma > 0$ represent the relative gain in utility from getting children and environmental quality to consumption. $\bar{S} > 0$ can be seen as maximum pollution level, which is tolerated by the central planner. Above this level human life on earth is not acceptable any more. ϵ is the inverse intertemporal substitution elasticity of fertility.

3.2 System Dynamics

As mentioned above, the model consists of three systems. Their dynamics are described by the corresponding differential equations and their stock and control variables. Next an overview of the systems and their equations will be provided, with explanations about the given assumptions.

3.2.1 Economy

Our economy is characterized by the production of an homogeneous consumption good. The necessary inputs are labor, capital and emissions. With the assumption of a neoclassical production function, we have $F(K,N,E,t)$, where N is the size of population (everyone is working), K the absolute capital value of the economy and E the emissions used. The last can be used as a partly substitute for labor and capital, but it is not possible to neglect it completely ($F(K,N,E=0,t)=0$). For our analysis we will use values in efficiency units as in the function

$$f(k,e,t) = \frac{F(K,N,E,t)}{N} = F(k,1,e,t) \tag{3.6}$$

where k is per capita capital and e are the emissions per capita. Per capita production will be represented by the Cobb-Douglas function

$$f(k,e,t) = k^{\alpha}e^{\beta}exp(xt) \tag{3.7}$$

where x is an exogenous Hicks-neutral technical advancement in each period t. The parameters satisfy $\alpha > 0, \beta > 0, \alpha+\beta < 1$, implying decreasing returns to scale and $x > 0$.

The state variable describing the system of the economy is capital k. The output of the production $f(k, e, t)$ is used for consumption c and various child costs and capital accumulation. $\dot{k}$ denotes the investment in new capital and is described by

$$\dot{k} = f(k, e) - c - (n - d)k - bnk - b_0 n \tag{3.8}$$

where d represents exogenous given mortality and n fertility, which is a control variable. The size of costs for children depends on three factors. First a constant investment b_0. Secondly the factor bnk accounts for the fact, that child costs rise with the assets of their parents. Note that b_0 and b are exogenously given constants. The term $(n-d)k$ represents capital endowment used for a growing (or declining) population. This approach follows Barro and Sala-i Martin (2004).

An interesting approach is also to consider physical and human capital separately, as done in Jöst et al. (2006).

3.2.2 Population

The considered state of population is size N. Details like age structure will be neglected. The control variable is fertility n and therefore endogenous. Mortality d is treated as exogenous and constant. This assumption is acceptable as long as there are no considerable changes in the fields of medical treatments, nutrition or similar factors that strongly influences mortality. Another restriction caused by this assumption is that the environmental development does not influence mortality, for example because of dirty water. In the analysis, this aspect will be neglected, and fertility is the only endogenous treated decision in the population system. This yields the differential equation

$$\dot{N} = (n - d) \cdot N \tag{3.9}$$

where for given n, population grows with the rate $(n - d)$ (declines with $(d - n)$) if $n > d$ $(n < d)$. It is constant for $n = d$.
Since this is a global model, we do not consider migration in any way.

3.2.3 Environment

In the system of the environment, various approaches or indicators may be used to model the dynamics. The environment can be regarded as e.g. regenerative, accumulating, irreversible, etc... . States can be for example precise forms like deforestation (Van, 2002) or energy use (Fröling, 2011) or more general ones like pollution. Here we consider the stock of pollution S as our state variable. This is suitable for a global model. It is assumed to be accumulated by the size of population N and use of emissions e in production. Regeneration in the base scenario is constant with the regeneration rate δ. The relationship between accumulation and regeneration is linear and yields the differential equation

$$\dot{S} = Ne - \delta S \tag{3.10}$$

This equation is just a very simple approximation of the complex relations between production, environment and the regeneration process. Also irreversible damages are excluded. Nevertheless it is useful to understand the connections to the other systems.

3.2.4 Summary of the optimal control model

To sum up, the considered optimal control model is

$$\max_{c,n,e} \quad \int_0^\infty (ln(c) + \nu\frac{n^{1-\epsilon}-1}{1-\epsilon} + \sigma ln(\bar{S} - S)) \cdot exp(-\rho t)dt \tag{3.11}$$

$$\begin{aligned}
\text{subject to} \quad & \dot{N} = (n-d)\cdot N \\
& \dot{S} = Ne - \delta S \\
& \dot{k} = k^\alpha e^\beta exp(xt) - c - (n-d)k - bnk - b_0 n \\
& 0 < x, \rho, \nu, \sigma, \bar{S}, \delta, b_0, b \\
& 0 < \alpha, \beta \text{ and } \alpha + \beta < 1 \\
& 0 < d, \epsilon, \delta < 1.
\end{aligned}$$

3.3 Analytical solution of the base scenario

The aim of this section is to provide the analytical basis for the following investigation of the relations between the chosen systems population, economy and environment. We first investigate whether there exists an equilibrium in the long run, if it is unique, optimal and what are the properties of a path to such a state. To achieve a solution, we use Pontryagin's Maximum Principle as explained for example in Grass et al. (2008) or earlier Feichtinger and Hartl (1986).

3.3.1 Maximum Principle

First we need to set up the current-value Hamiltonian

$$H \quad = \quad \lambda^0 u(c,n,S) + \lambda^k \cdot \dot{k} + \lambda^N \cdot \dot{N} + \lambda^S \cdot \dot{S} \tag{3.12}$$

with the specified functions (3.8), (3.9) and (3.10).

$$\begin{aligned}
H = \lambda^0 (ln(c) + \nu\frac{n^{1-\epsilon}-1}{1-\epsilon} + \sigma ln(\bar{S} - S)) \\
+ \lambda^k \cdot (k^\alpha e^\beta exp(xt) - c - (n-d)k - bnk - b_0 n) \\
+ \lambda^N \cdot ((n-d)\cdot N) \\
+ \lambda^S \cdot (Ne - \delta S)
\end{aligned} \tag{3.13}$$

The idea is to combine the utility function u and the state dynamics $(\dot{N}, \dot{k}, \dot{S})$, weighted by the so called co-state variables λ. These co-state variables can be economically interpreted as shadow prices, meaning the increase (or decrease if $\lambda < 0$) in utility if the state is increased. This concept will be explained later in detail.

The *first order conditions (FOC)* are:

$$
\begin{aligned}
H_c =& 0 & \lambda^0 u_c - \lambda^k = 0 \quad (3.14)\\
H_n =& 0 & \lambda^0 u_n - \lambda^k[k + bk + b_0] + \lambda^N N = 0 \quad (3.15)\\
H_e =& 0 & \lambda^k f_e + \lambda^S N = 0 \quad (3.16)\\
\rho\lambda^k - H_k =& \dot{\lambda}^k & \lambda^k[\rho - f_k + (n - d) + bn] = \dot{\lambda}^k \quad (3.17)\\
\rho\lambda^N - H_N =& \dot{\lambda}^N & \lambda^N[\rho - (n - d)] - \lambda^S e = \dot{\lambda}^N \quad (3.18)\\
\rho\lambda^S - H_S =& \dot{\lambda}^S & u_s + \lambda^S(\rho + \delta) = \dot{\lambda}^S \quad (3.19)
\end{aligned}
$$

Proposition 3.3.1. *We can assume $\lambda^0 = 1$ without loss of generality.*

Proof. If $\lambda^0 \neq 0$ we can divide H by λ^0 and the assumption is proven. Therefore we just need to prove that $\lambda^0 = 0$ is not valid. From (3.14) it follows that, in this case, $\lambda^k = 0$. From (3.15) and (3.16) we obtain $\lambda^N N = 0$ and $\lambda^S N = 0$ (due to $\lambda^k = 0$). We have to distinguish between the cases $N = 0$ and $N \neq 0$. If $N \neq 0$, this implies $\lambda^N = 0$ and $\lambda^S = 0$. This would imply $(\lambda_0, \lambda^k, \lambda^N, \lambda^S) = 0$ which would violate the Maximum Principle, namely that there exists a vector $(\lambda_0, \lambda) \neq 0$ that solves $H(x^*, u^*, \lambda, \lambda_0, t) = \max_u (H(x^*, u, \lambda, \lambda_0, t))$, see Grass et al. (2008, p.156).
If $N = 0$, we would have no population at all. We will exclude this boundary case and conclude $\lambda^0 \neq 0$ for our further analysis. □

The 6-dimensional canonical system of the differential equations (3.8),(3.9), (3.10) and (3.17)-(3.19) are the necessary conditions for an optimal path. To assure sufficiency, we use the *"Arrow Sufficiency Conditions for Infinite Time Horizon"* as can be found in Grass et al. (2008, p.159). (x^*, u^*)[4] is an optimal solution, if it satisfies Pontryagin Maximum Principle together with λ, the transversality condition $\lim_{t\to\infty} \exp(-\rho t)H^0 = 0$ holds and the *maximized Hamiltonian* H^0 is concave in x for all t and λ. To obtain H^0 we replace the control variables by the values obtained from the FOC (3.14)-(3.16)

$$
c = \frac{1}{\lambda^k} \tag{3.20}
$$

$$
n = \nu^{\frac{1}{\epsilon}} \left[\lambda^k[(1 + b)k + b_0\right] - \lambda^N N]^{-\frac{1}{\epsilon}} \tag{3.21}
$$

$$
e = \left(-\frac{N\lambda^S}{\lambda^k k^\alpha \beta}\right)^{\frac{1}{\beta-1}} \tag{3.22}
$$

and obtain

[4]A star on top of a variable denotes an optimal solution

$$
\begin{aligned}
H^0 =& \frac{\nu-(1-\epsilon)}{1-\epsilon}\nu^{-\frac{1}{\epsilon}}\left[\lambda^k[(1+b)k+b_0]-\lambda^N N\right]^{-\frac{1-\epsilon}{\epsilon}} \\
&+\frac{1-\beta}{\beta}(\beta\lambda^k)^{\frac{1}{1-\beta}}k^{\frac{\alpha}{1-\beta}}(-\lambda^S N)^{-\frac{\beta}{1-\beta}}-\ln\lambda^k+\sigma(\bar{S}-S)-1+\lambda^k dk-\lambda^N dN-\lambda^S\delta S.
\end{aligned}
$$

A function is concave if the Hessian matrix is *negative definite*. We use the *criteria of principal minors*, which states: If the determinants of the principal minors are alternating, starting with negative, the function is negative definite (Havlicek, 2006, p.277). In our case, the Hessian matrix is

$$
\mathcal{H}=\begin{pmatrix} \frac{d^2H^0}{dN^2} & \frac{d^2H^0}{dNdk} & 0 \\ \frac{d^2H^0}{dkdN} & \frac{d^2H^0}{dk^2} & 0 \\ 0 & 0 & \frac{d^2H^0}{dS^2} \end{pmatrix} \tag{3.23}
$$

where we have already taken into account that the mixed derivates with S disappear. Because H^0 is twice differentiable and because of the *Lemma of Schwarz* (Kaltenbäck, 2008) the terms $\frac{d^2H^0}{dNdk}$ and $\frac{d^2H^0}{dkdN}$ must be equal. The terms of the Hessian matrix are

$$
\begin{aligned}
\frac{d^2H^0}{dN^2} =& (\nu-(1-\epsilon))\frac{((1+b)k+b_0-vcn^{-\epsilon})^2}{v\epsilon^2c^2N^2n^{-1-\epsilon}}+\frac{\beta}{1-\beta}\frac{k^\alpha e^\beta}{cN^2} && (3.24)\\
\frac{d^2H^0}{dNdK} =& (\nu-(1-\epsilon))\frac{((1+b)k+b_0-vcn^{-\epsilon})(1+b)}{v\epsilon^2c^2Nn^{-1-\epsilon}}+\frac{\alpha\beta}{1-\beta}\frac{k^{\alpha-1}e^\beta}{cN} && (3.25)\\
\frac{d^2H^0}{dk^2} =& (\nu-(1-\epsilon))\frac{(1+b)^2}{v\epsilon^2c^2n^{-1-\epsilon}}-\frac{\alpha(1-\alpha-\beta)}{1-\beta}\frac{k^{\alpha-2}e^\beta}{c} && (3.26)\\
\frac{d^2H^0}{dS^2} =& -\frac{\sigma}{(\bar{S}-S)^2} && (3.27)
\end{aligned}
$$

We need $\frac{d^2H^0}{dN^2}\leq 0$, $\frac{d^2H^0}{dk^2}\leq 0, \frac{d^2H^0}{dS^2}\leq 0$ and $\frac{d^2H^0}{dN^2}\frac{d^2H^0}{dk^2}-\left(\frac{d^2H^0}{dkdN}\right)^2\geq 0$. In general, the condition for $\frac{d^2H^0}{dS^2}$ is fulfilled, since $S<\bar{S}$. To assure $\frac{d^2H^0}{dN^2}\leq 0$, we need $\nu<1-\epsilon$, since all other terms are positive for all feasible values. Nevertheless, we can not derive any proposition to assure sufficency of the necessary conditions and therefore we always have to check if our solution satisfies the two inequalities

$$
\begin{aligned}
\frac{d^2H^0}{dk^2} &\leq 0 && (3.28)\\
\frac{d^2H^0}{dN^2}\frac{d^2H^0}{dk^2}-\left(\frac{d^2H^0}{dkdN}\right)^2 &\geq 0. && (3.29)
\end{aligned}
$$

3.3.2 Growth Rates of the Controls

We have now derived constraints under which a solution path is optimal. If these constraints are satisfied the necessary conditions are sufficient. To find this path, we are interested in the growth rates of the controls.

Using equations (3.14)-(3.16) to get the values of the co-state variables yield

$$\lambda^k = \frac{1}{c} \tag{3.30}$$

$$\lambda^N = \left(\frac{k + bk + b_0}{c} - \nu n^{-\epsilon}\right)\frac{1}{N} \tag{3.31}$$

$$\lambda^S = -\frac{f_e}{cN}. \tag{3.32}$$

As mentioned before, these values can be economically interpreted as shadow prices. If consumption is low, an additional increase in capital will be valued highly. Therefore λ^k is inverse proportional to c. For λ^N we see a positive dependency on fertility costs in relation to the consumption level, which may be lowered because of an already high level of fertility. This will be scaled by the size of population. Therefore it may be that an increase in population size affects the welfare negative as well as positive. This interpretation also makes economically sense, since we consider fertility as utile and not the size of population. Also the interpretation for (3.32) is straight forward. It is equal to the negative marginal utility of emissions used in production in relation to the product of consumption and population size. Therefore the direct effect of the increase of pollution on utility is always negative, as we wanted it to be because of our model specification.

Taking the time derivatives of the above functions (3.30)-(3.32), eliminating the co-state variables and using the utility function (3.5) as well as the productivity function (3.7) yields the growth rates of the control variables:

$$g_c = \frac{\dot{c}}{c} = \alpha k^{\alpha-1} e^{\beta} \exp(xt) - bn - \rho - (n - d) \tag{3.33}$$

$$g_n = \frac{\dot{n}}{n} = -\frac{\rho}{\epsilon} + \left[\beta k^{\alpha} e^{\beta} \exp(xt) + [(1+b)k + b_0]\left[\frac{\dot{c}}{c} + \rho\right] - (1+b)\dot{k}\right]\frac{n^{\epsilon}}{\nu \epsilon c} \tag{3.34}$$

$$g_e = \frac{\dot{e}}{e} = -\frac{1}{1-\beta}\left[\rho + \delta + (n-d) + \frac{\dot{c}}{c} - \alpha\frac{\dot{k}}{k} - \frac{\sigma N c}{(\bar{S} - S)\beta k^{\alpha} e^{\beta-1} \exp(xt)}\right] \tag{3.35}$$

3.3.3 Steady State Analysis

We have derived a 6-dimensional differential system $(\dot{N}, \dot{k}, \dot{S}, \dot{c}, \dot{n}, \dot{e})$, which can be used to find an optimal steady state, as it is sufficient for optimality under the mentioned constraints. The next step is to search for a BGP, where all growth rates $g_N, g_S, g_k, g_c, g_n, g_e$[5] are constant. Afterwards we will transform the system in a way, that makes all growth rates equal to zero.

[5] where g_i denotes $\frac{\dot{i}}{i}$

This means to transform it into a stationary state for the numerical investigation of transition dynamics as well as sensitivity and bifurcation analysis. If the resulting values are positive, unique and fulfill the transversality conditions, we have an optimal steady state of our model.

Proposition 3.3.2. *Under the assumption of constant growth rates of control and state variables, we obtain*

$$g_k = g_c = g_S = g_n = 0 \text{ and } g_N = -g_e = \frac{x}{\beta}.$$

Proof. See Appendix (A.1) □

In this BGP, all states and controls are constant except population size and emissions. Their growth rates depend on the relation of technological advancement x to the output-elasticity of emissions β. The growth rates of population and emissions are exactly negative correlated, meaning that the overall value of pollution impact is constant:

$$\frac{d}{dt}(Ne) = \dot{N}e + N\dot{e} = Ne\left(\frac{\dot{N}}{N} + \frac{\dot{e}}{e}\right) = 0 \tag{3.36}$$

The precise value depends on the productivity of emissions β and technical advancement x. This means technical advancement is only used for improving productivity of emissions, while productivity of capital stays constant. In the long run, utility will also be constant, as it is just dependent on c, n and S, which are all constant in the steady state. Without technological advancement (i.e. $x = 0$), the steady state would even be a stationary state, where all growth rates are zero.

To obtain a stationary state we will transform the size of population N and the per-capita emissions e to obtain a stationary state. This is done by using $\hat{N} = N \cdot \exp(-\frac{x}{\beta}t)$ and $\hat{e} = e \cdot \exp(\frac{x}{\beta}t)$. The growth rates therefore become

$$\begin{aligned}
\dot{\hat{N}} &= (n - d - \frac{x}{\beta}) \cdot \hat{N} && (3.37)\\
\dot{k} &= k^{\alpha}\hat{e}^{\beta} - c - (n-d)k - bnk - b_0 n && (3.38)\\
\dot{S} &= \hat{N}\hat{e} - \delta S && (3.39)\\
\frac{\dot{c}}{c} &= \alpha k^{\alpha-1}\hat{e}^{\beta} - bn - \rho - (n-d) && (3.40)\\
\frac{\dot{n}}{n} &= -\frac{\rho}{\epsilon} + \left[\beta k^{\alpha}\hat{e^*}\beta + [(1+b)k + b_0]\left[\frac{\dot{c}}{c} + \rho\right] - (1+b)\dot{k}\right]\frac{n^{1+\epsilon}}{\nu\epsilon c} && (3.41)\\
\frac{\dot{\hat{e}}}{\hat{e}} &= -\frac{\hat{e}}{1-\beta}\left[\rho + \delta + (n-d) + \frac{\dot{c}}{c} - \alpha\frac{\dot{k}}{k} - \frac{\sigma N c}{(\bar{S}-S)\beta k^{\alpha}\hat{e}^{\beta-1}}\right] + \frac{x}{\beta}\hat{e} && (3.42)
\end{aligned}$$

Now we can derive the steady-state values of all state and control variables.

Proposition 3.3.3. *The steady state of the transformed system $(\hat{N}, S, k, c, n, \hat{e})$ is unique, feasible and a stationary solution under the restriction*[6]

[6] to assure $k, c > 0$

$$\nu\rho\frac{(1-\alpha)(1+b)\frac{x}{\beta}+(1-\alpha)bd+\rho}{\beta bd+(1+b)x+(\beta+(1+b)\alpha)\rho} \quad > \quad \left(d+\frac{x}{\beta}\right)^{\epsilon} \tag{3.43}$$

Proof. See Appendix (A.2) □

3.3.4 Stability in the Steady State

The existence, uniqueness and optimality is an important result. But it can't be assumed that the system starts in the steady state, and therefore rests at this point. For all other values of state variables, the system is dynamically changing. The central planer is expected to choose his controls optimal. This poses the question for which starting values an optimal development is possible. We can answer this by finding a stable manifold, where for given starting values (N_0, S_0, k_0) and the corresponding optimal controls (n_0, e_0, c_0) the system converges to the steady state $(N^*, S^*, k^*, n^*, e^*, c^*)$ as has been found before. This stable manifold is given by the eigenvectors according to the negative eigenvalues of the Jacobi-Matrix J as proven in Feichtinger and Hartl (1986, p.133).

$$\mathcal{J}=\begin{pmatrix}
\frac{\partial \dot{N}}{\partial N} & \frac{\partial \dot{N}}{\partial k} & \frac{\partial \dot{N}}{\partial S} & \frac{\partial \dot{N}}{\partial n} & \frac{\partial \dot{N}}{\partial e} & \frac{\partial \dot{N}}{\partial c}\\
\frac{\partial \dot{k}}{\partial N} & \frac{\partial \dot{k}}{\partial k} & \frac{\partial \dot{k}}{\partial S} & \frac{\partial \dot{k}}{\partial n} & \frac{\partial \dot{k}}{\partial e} & \frac{\partial \dot{k}}{\partial c}\\
\frac{\partial \dot{S}}{\partial N} & \frac{\partial \dot{S}}{\partial k} & \frac{\partial \dot{S}}{\partial S} & \frac{\partial \dot{S}}{\partial n} & \frac{\partial \dot{S}}{\partial e} & \frac{\partial \dot{S}}{\partial c}\\
\frac{\partial \dot{n}}{\partial N} & \frac{\partial \dot{n}}{\partial k} & \frac{\partial \dot{n}}{\partial S} & \frac{\partial \dot{n}}{\partial n} & \frac{\partial \dot{n}}{\partial e} & \frac{\partial \dot{n}}{\partial c}\\
\frac{\partial \dot{c}}{\partial N} & \frac{\partial \dot{c}}{\partial k} & \frac{\partial \dot{c}}{\partial S} & \frac{\partial \dot{c}}{\partial n} & \frac{\partial \dot{c}}{\partial e} & \frac{\partial \dot{c}}{\partial c}\\
\frac{\partial \dot{e}}{\partial N} & \frac{\partial \dot{e}}{\partial k} & \frac{\partial \dot{e}}{\partial S} & \frac{\partial \dot{e}}{\partial n} & \frac{\partial \dot{e}}{\partial e} & \frac{\partial \dot{e}}{\partial c}
\end{pmatrix} \tag{3.44}$$

We will analyze this behavior in the numerical section.

3.4 Numerical Analysis

The numerical analysis serves to understand the behavior of the model, mainly in a close environment of the steady state. We use the transformed system and will from now on omit the $\hat{}$ sign for N and e. The calculations and plots are done in *Mathematica 8.0*[7].

3.4.1 Parameters

For the numerical analysis we have to fix a set of parameters. Table 3.1 gives an overview over the parameters of the base model. The choice is justified in Quaas (2004, p.89) for a given time period of 10 years.

Parameter	Description	Value
ϵ	inverse inter temporal substitution elasticity of fertility	0.01
ν	utility-weight of fertility	0.9
σ	utility-weight of environmental quality	0.33
$\bar{S}$	maximum pollution level	1
d	mortality rate	0.1
δ	degradation rate of pollution	0.1
x	technical advancement	0.1
ρ	discount rate	0.1
α	production elasticity of capital k	0.2
β	production elasticity of emissions e	0.04
b	capital-dependent child-raising cost	0.1
b_0	fixed fertility cost	0.035

Table 3.1: Model Parametrization

This choice satisfies the given constraint (3.43) $\nu < 1 - \epsilon$ and the optimality conditions (3.28) as well as (3.29), evaluated at the steady state.

3.4.2 Steady State

Using the given parameters from Table (3.1), we can compute steady state values using equations (A.38)-(A.43). Note that the absolute values do not always allow immediate interpretation. We have given them as an overview and will from now on normalize all results by these values. The first three are the state variables, followed by the control variables and the shadow prices. Next we see interesting values of the model, namely utility, production and the marginal productivity of the inputs as well as overall emissions. Last are the eigenvalues of the Jacobian presented to analyze the stability afterwards.

In this base scenario there are a few notable points about the steady state values. First the high level of fertility which is a result of the transformation, see (3.37). The value of fertility $n = d + x/\beta$ is that high caused by technological progress $x > 0$. If $x = 0$, we would have $n = d$. In the steady state the shadow price λ^N is negative, in absolute terms even greater

[7] `http://www.wolfram.com/`

Variable/Term	Description	Value
N	population size	0.1529
k	capital	0.0335
S	pollution	0.0368
n	fertility	2.6
c	consumption	0.2954
e	emissions	0.2409
λ^k	shadow price of capital	3.384
λ^N	shadow price of population	-13.7127
λ^S	shadow price of pollution	-1.7594
u	utility	0.0597
f	production	0.4789
f_e	marginal productivity of emissions	0.0794
f_k	marginal productivity of capital	2.86
$N \cdot e$	overall emissions	0.0364
τ_1	first eigenvalue of $\mathcal{J}$	6.43
τ_2	second eigenvalue of $\mathcal{J}$	-6.33
τ_3	third eigenvalue of $\mathcal{J}$	0.6014
τ_4	fourth eigenvalue of $\mathcal{J}$	-0.4995
τ_5	fifth eigenvalue of $\mathcal{J}$	0.3369
τ_6	sixth eigenvalue of $\mathcal{J}$	-0.1388

Table 3.2: Steady State Values

than the shadow price of higher pollution. This means higher population would decrease utility. This happens because of two reasons. First fertility is considered as utile and not population size itself. As can be seen in the transitional dynamics later, fertility dynamics behave inverse (consequently also utility) to the dynamics of population size, see figure 3.4(a) and 3.2(a). Second a higher population implies higher pollution because of (3.39), which obviously affects utility negatively. Pollution overall is very low, as it takes up only 3.6% of the allowed capacity $\bar{S}$. Therefore the world in our steady state is environmental-friendly. Furthermore, an additional increase of emissions used in production is far less more productive than an increase in capital ($f_e < f_k$). This is partly a consequence of $\beta < \alpha$, yet the steady state could have been very different. For the following stability analysis we note that all eigenvalues are real, unequal to 0 and three of them negative.

3.4.3 Stability and transitional Dynamics

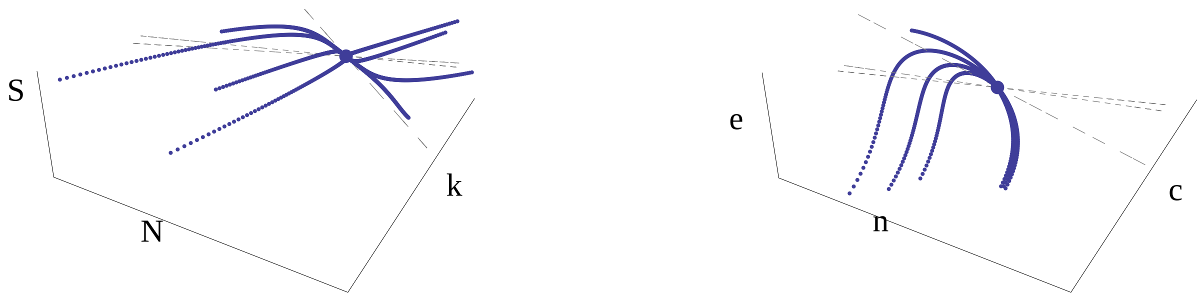
As mentioned in (3.3.4) we want to analyze the behavior of the system to answer the question, under which circumstances it will converge to the steady state. For this purpose we need the Jacobian with the according eigenvalues and eigenvectors. With the given parametrization and the calculated steady state values listed in table (3.2), there are three negative eigenvalues τ_2, τ_5, τ_6 with τ_2 clearly dominating. The corresponding eigenvectors are spanning the stable manifold. Every time we start in this stable manifold, the system will converge to the optimal steady state thanks to the optimal control.

To find optimal trajectories, one has to choose a starting point in the stable manifold, near to the steady state. This is done by using

$$X = S + \vec{e}_1 * a_1 + \vec{e}_2 * a_2 + \vec{e}_3 * a_3 \tag{3.45}$$

where X is the calculated starting point, S is the steady state, the vectors e_i are the normalized eigenvectors and a_i are freely choosen weights. Starting from this point, the differential equations describing the growth rates will be solved backwards in time. With this method, we can analyze the system with starting points in all directions. The objective of this analysis is to find out if the optimal paths have special characteristics (e.g. monotonous), in which relations are certain variables to each other (e.g. proportional) and so on. The optimal result would be an explicit conclusion to give policy advice how to set the controls.

As a matter of fact, we have a 6-dimensional system consisting of 3 state and 3 control variables. Appropriate projections need to be made to prepare the results for interpretations. We will present 3 different ways to shed light on the transition dynamics. First we investigate 3D Plots, one for all 3 states and another one for all 3 controls. Second we are interested in the way the states and controls of one system relate to each other. Therefore we will analyze the state-control spaces (N, n), (k, c) and (S, e). Third the behavior over time is of interest, so all variables will be plotted over time. Because not only the main variables are of interest, but also various other terms (like per capita production $f(k, e)$ or overall emissions $N \cdot e$), we will also take a look at these terms. All values are normalized with division by their steady-state values, so the steady state value itself is always 1.



(a) state variables (b) control variables

Figure 3.1: Behavior close to the steady state in the stable manifold. Blue lines are transition paths, black lines eigenvectors corresponding to the negative eigenvalues.

Because we have real valued eigenvalues with the same number of positives and negatives, the steady state is an attractive saddle point in the 3-dimensional manifold, which can be nicely seen in figure 3.1. However, it is difficult to interpret the behavior of the variables in these plots. Therefore we will choose two transition paths with significant different characteristics by two different sets of values for (a_1, a_2, a_3) (3.45) and proceed to the 2-dimensional phase portraits.

The shown optimal path in figures 3.2-3.3 is characterized by a starting point with high population size, low capital and low pollution in comparison to the steady state, while the other one seen in figures 3.4-3.5 is characterized by a starting point with low population size, low capital and low pollution stock.

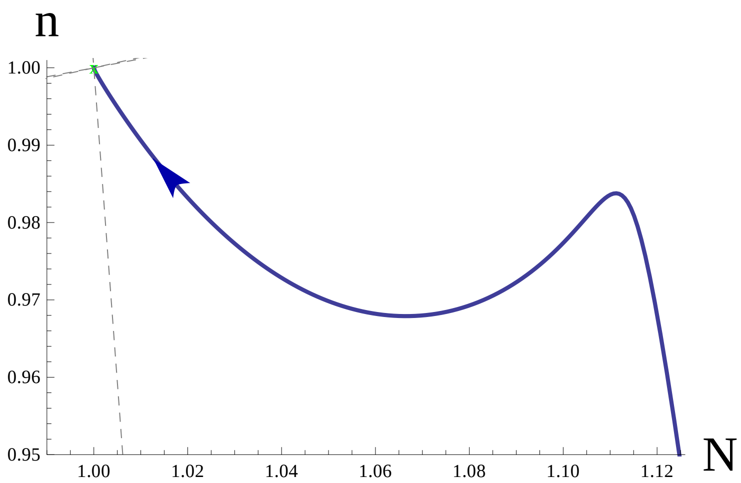
We first note correlations and differences between the starting points and the steady state values. When initial population size is lower/higher, initial fertility is higher/lower than in the steady state. As we start with low capital in both cases, independent of the states population size and pollution, we see that both transitions experience a rise in capital. Even though we have low pollution in both cases, emissions can be greater or lower than in the steady state (because of the different population size levels), resulting obviously in different correlations of emissions to pollution. Economically speaking, the amount of initial pollution in figure 3.4 is driven by emissions while the same amount of initial pollution in 3.2 is driven by high population size.

The next step is to analyze the transition to the steady state. In transition path 3.2, we see declining population size. The relation of fertility to population is changing. As it first rises, it declines for a short period to rise again with declining population. The fertility peak is associated with growing capital, consumption and pollution. So as population declines, we have an increase in production to afford high consumption and fertility at the same time. This increase (even though it is financed by capital, not emissions which are low at that time) results in an extensive accumulation of pollution. After the fertility peak is passed, it declines together with population size, leading to an increased use of emissions as overall emissions $N \cdot e$ are less challenged due to low population size. Nevertheless, the constant regeneration rate of the environment lowers pollution in the following phase faster than it accumulates even due to increased use of emissions. With the increased production due to higher emissions, see figure 3.7, fertility rises again during the period of population decline resulting in lower consumption. This path leads to the steady state.

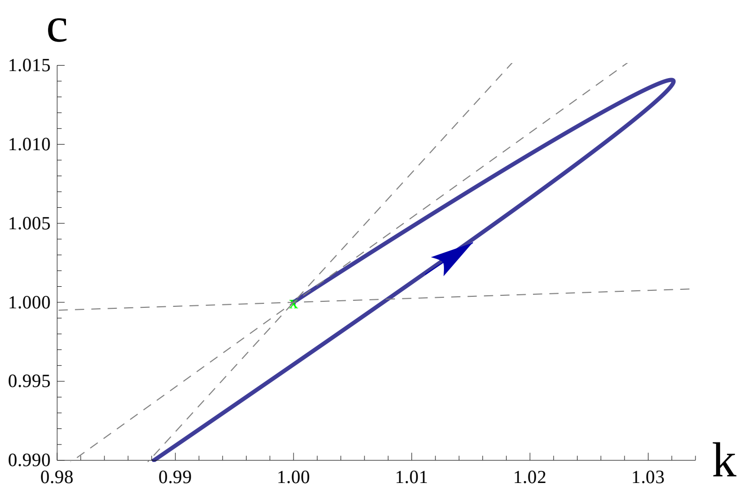
Investigating figure 3.4, we see fertility declining with rising population size. This is in line with the general inverse correlation seen in figure 3.2, but because of the different situation, we have other economic effects. Growing population has to be supplied with additional production, mainly from an increase in capital (not due to shortened consumption). As the initial level of emissions is very high, pollution accumulates to a high level before the constant regeneration rate exceeds the accumulation.

The main take aways from these phase portraits are:

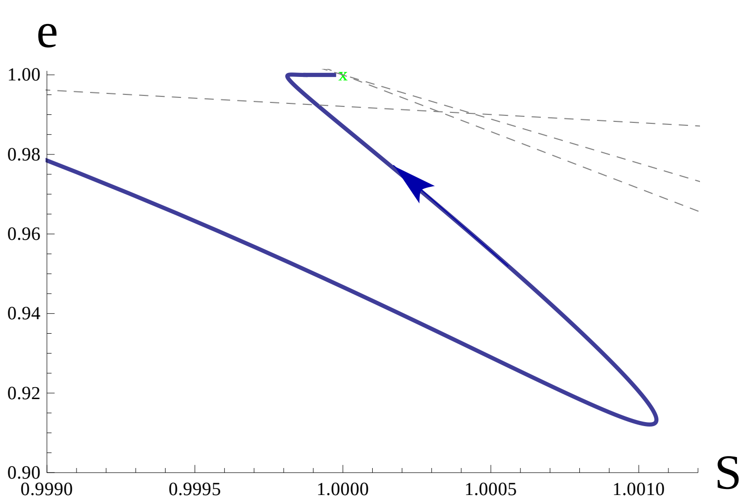
- In general, the development of fertility and population size are inversely correlated.
- The pollution peak does not need to coincide with an emissions peak, as it is accumulated by population as well. The constant regeneration rate becomes effective in a period associated with low overall emissions (low emissions and low population size), meaning it is more effective to reduce overall emissions than to rely on the regeneration rate.



(a) Population

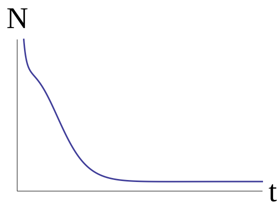


(b) Economy

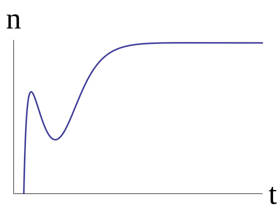


(c) Environment

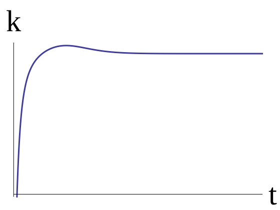
Figure 3.2: Projections on the systems, control over state variables. Dashed lines are eigenvectors



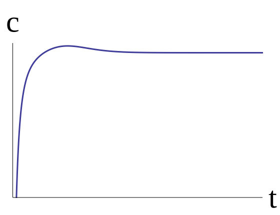
(a) Population size



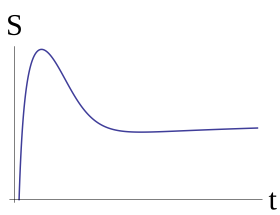
(b) Fertility



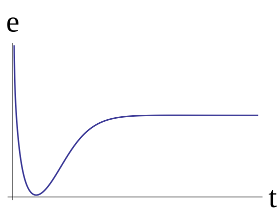
(c) Capital



(d) Consumption



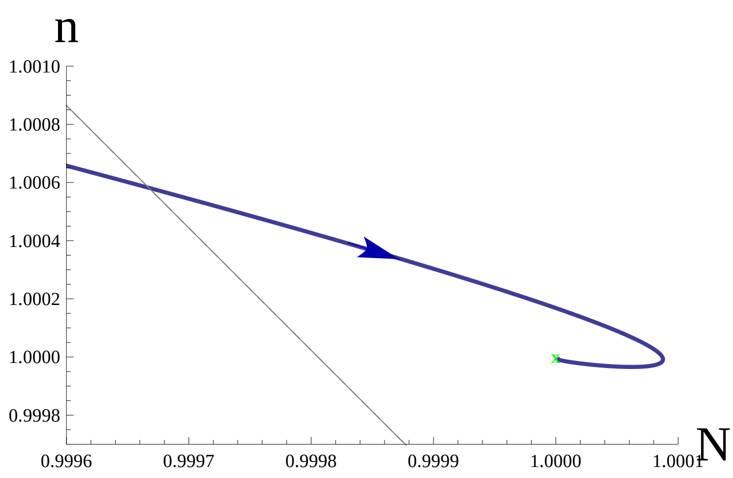
(e) Pollution



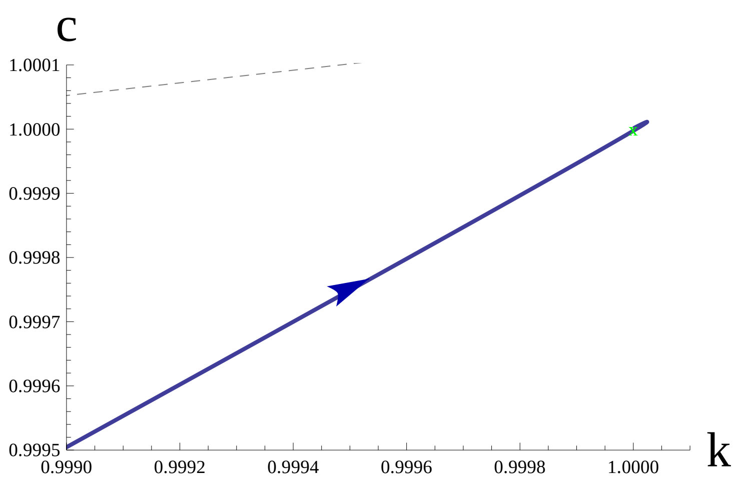
(f) Emissions

Figure 3.3: Projections on the time scales

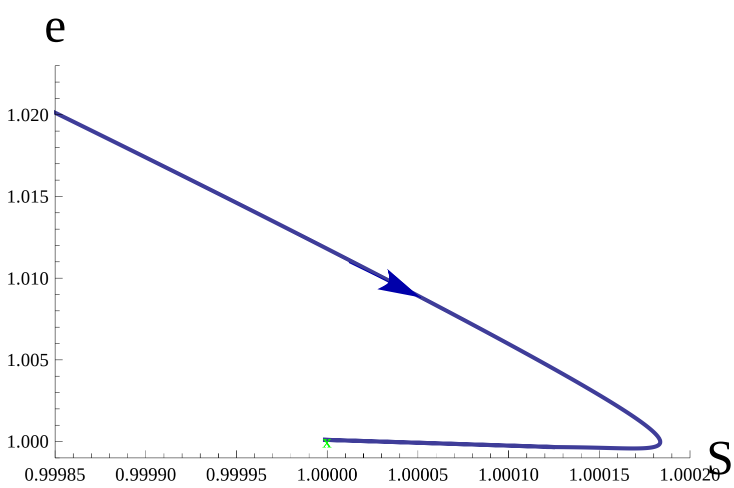
Model starting with high population size, low capital and low pollution stock



(a) Population

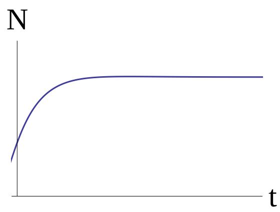


(b) Economy

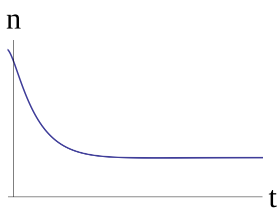


(c) Environment

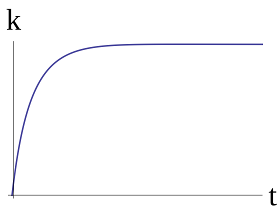
Figure 3.4: Projections on the systems, control over state variables. Dashed lines are eigenvectors



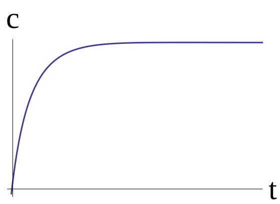
(a) Population size



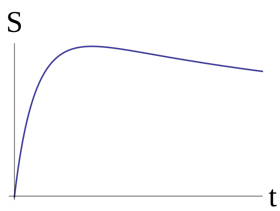
(b) Fertility



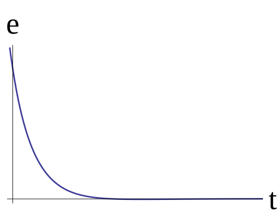
(c) Capital



(d) Consumption



(e) Pollution



(f) Emissions

Figure 3.5: Projections on the time scales

Model starting with low capital, low population size and low pollution stock

After having discussed the transitional dynamics of states and controls, we will proceed to the analysis of other aspects for deeper insight. The following investigation will be done only for the path shown in 3.2. The discussion for the other path can be derived in a similiar way.

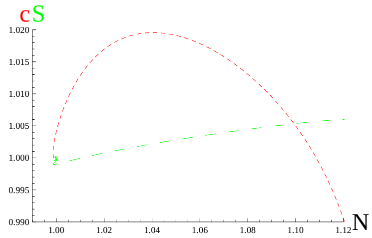
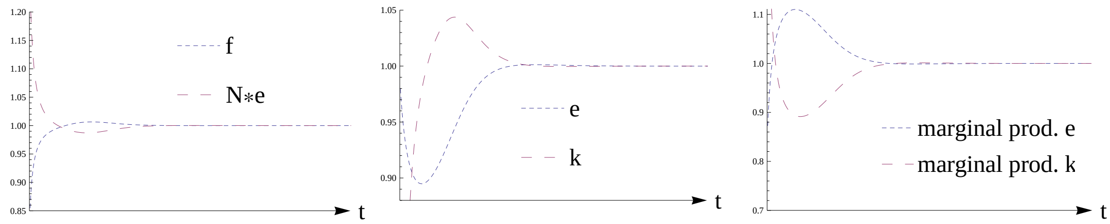


Figure 3.6: Consumption and pollution over population size

If one poses the question, if the effects of population on consumption and on pollution somehow correlates, we may consider figure 3.6. On the one hand decreasing population implies also decreasing pollution. The path of consumption on the other hand describes a parable. This does indicate a changing correlation. To understand this parable we also have to consider figures 3.3 and 3.7(b). First capital rises strongly together with consumption and fertility. Fertility has a peak before any other dynamic changes qualitatively. During the period of population decline, it becomes sustainable to raise emissions as well, which is possible without increasing pollution. After the valley of emissions is passed, also fertility rises again due to increased production. As production shares have to be used for child costs, consumption declines (it is more useful in this moment to invest in fertility than consumption). Therefore we have a parable.



(a) Production and overall emissions (b) Emissions and capital per capita (c) Marginal productivity

Figure 3.7: Time evolution and properties of the production function

An important aspect of the economic system is the production itself and the marginal productivity of the inputs used. Per capita production f has a maximum before converging to the steady state. Surprisingly it is inverse proportional to overall emissions $N \cdot e$, even though the rising level of emissions. We see that the maximum/minimum of capital/emissions does not occur at the same time. The evolution of marginal productivity of the inputs is inverse correlated as can be seen in figure 3.7(c).

Economically speaking, production does not need to be the main driving factor for the accumulation of pollution. The production can be raised sustainable even with the use of emissions when population size is low at the same time. However the use of emissions pays off just until a certain point in time.

As we have seen, the analysis of the transitional dynamics yield various important qualitative results. To test if this results are robust, we will perform now a sensitivity analysis.

3.4.4 Sensitivity analysis

At the beginning of the numerical analysis we have chosen a parameter set. Even though it has been carefully chosen, different parameterizations need to be considered. Also the parameters may be different for various economic regions and can vary over time. Therefore it is crucial to find out whether or not the model is stable for varying parameters or if the behavior becomes qualitatively different. Therefore the investigation of different parameter choices is very important and will be performed in this section.

The investigation is as follows:

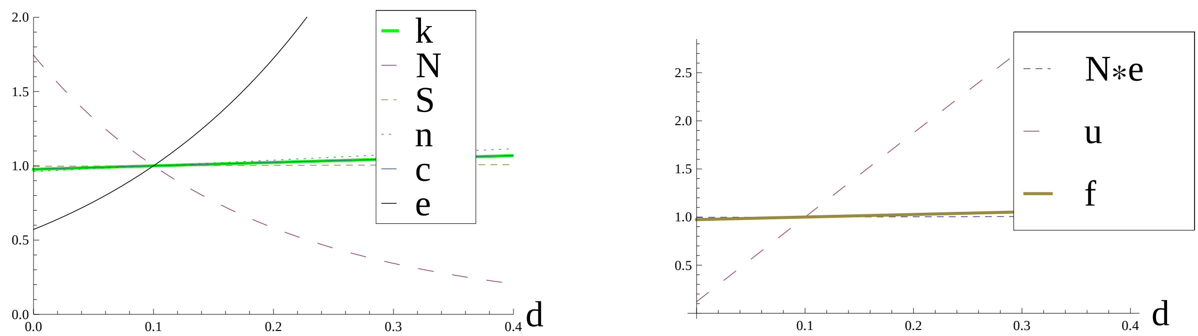
1. Evaluating steady state values for varying parameters.

2. Evaluating overall emissions, production and utility as well as shadow prices in order to obtain deeper insight.

3. Evaluate the stability of the eigenvalues, especially if sign or head[8] is changing.

Variation of mortality rate d

The dynamics of population size are driven by fertility and mortality. As mortality has been chosen as exogenous, it is not varying during the transition. Especially when considering the evolution over the long term, mortality has obviously changed. Mortality decline even marked the initialization of the demographic transition. Evidently mortality changes are really happening and so we will perform a sensitivity analysis with respect to the exogenous given mortality rate d.



(a) States and controls

(b) Overall emissions, utility, production

Figure 3.8: Steady-State Values with respect to variations in the mortality rate d

We note that the parameter condition (3.43) is fulfilled for all positive values of d with the rest of the parameters kept as before. In figure 3.8(a) we see the steady state values of states

[8] real or complex

and controls over morality rate d in the interval $[0, 0.4]$. This range is chosen in order to cover the range of the demographic transition from 0.4 over the actual mortality rate (used before, 0.1) in industrialized countries to no mortality at all, meaning $d = 0$. The direct effect of d on population dynamics $\dot{N} = (n - d)N$ yields a declining population with rising mortality. In the steady state, fertility equals mortality and therefore it is directly correlated. Emissions rises heavily as population decline allows for additional pollution generated in production, keeping in mind that overall emissions stay constant. These results coincide with the previous analyzed transitional dynamics and implies that pollution does not depend on mortality. We see that mortality does not change the agents environmental attitude as pollution stays constant. Increasing emissions allow higher production leading to an increase of capital and consumption.

As fertility and consumption increases while pollution stays constant with rising mortality, the level of utility also rises.

The results coincide with a demographic transition as we have an higher population size and lower fertility at a lower level of mortality rate d.

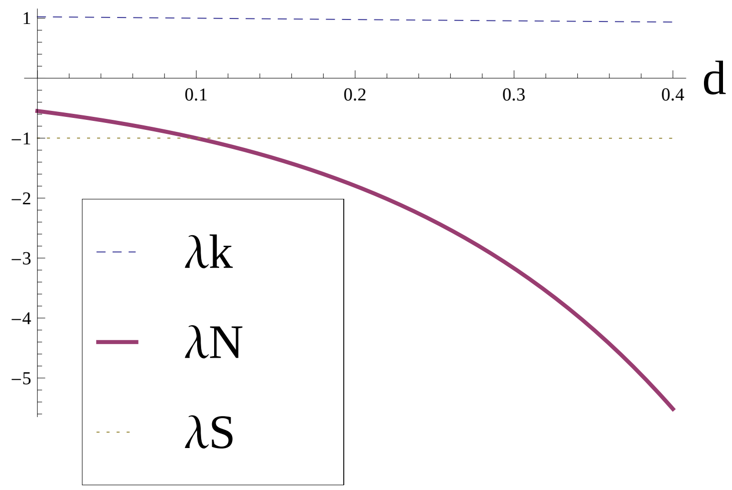


Figure 3.9: Shadow Prices with respect to variations in the mortality rate d

As mortality rate rises, the value of population growth obviously diminishes. A shorter and more risky life has less value and therefore the shadow price of population size is declining as can be seen in figure 3.9. The other shadow prices are independent from d. Additional capital and therefore production will add value to consumption, regardless of mortality (only $n - d$ is of importance and this equals 0 in the steady state). Also the shadow price of pollution is independent from mortality, as the effect of a declining population is already considered by λ^N.

Concerning the stability of the systems, the eigenvalues of the Jacobian are plotted in figure 3.10. We see a stable development without any bifurcation. Therefore variations in the mortality rate does not change the stability of the system as d enters only linearly.

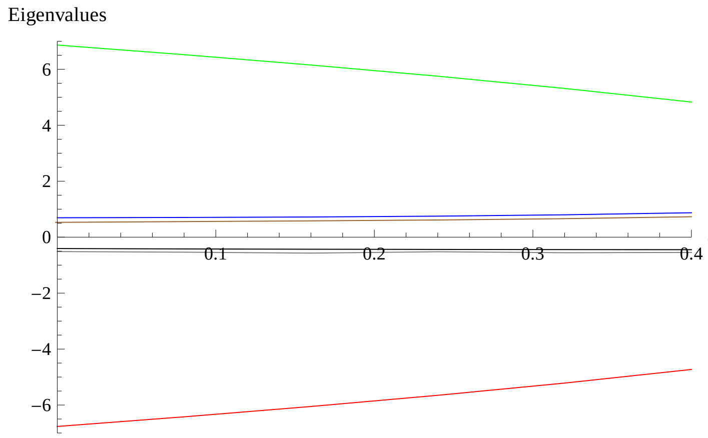
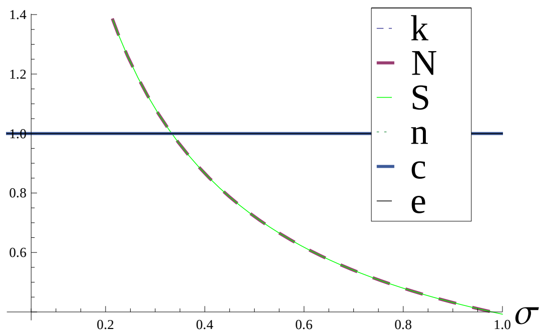


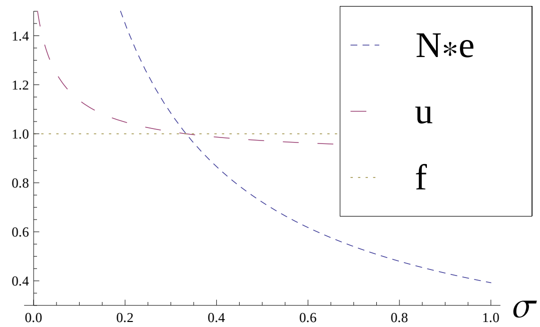
Figure 3.10: Eigenvalues with respect to variations in the mortality rate d

Variation of utility-weight of environmental quality σ

The next parameter to be investigated is the utility-weight of environmental quality. Around the globe we see very different ways to deal with the environment. Obviously *Greenpace* values a forest very different than a sawmill. Also nations have different attitudes toward pollution and preservation of environmental quality. This can be seen for example in the actual debate about nuclear energy.



(a) States and controls



(b) Overall emissions, utility, production

Figure 3.11: Steady-State Values with respect to variations in the utility-weight of environmental quality σ

Note that σ does not affect the parameter constraint (3.43) at all and therefore we don't have any boundaries for varying σ. However, we assume environmental quality can be maximal as important as consumption and therefore $\sigma = 1$ is the upper bound of the investigated interval. Changes in the utility-weight of environmental quality does affect only population size and the stock of pollution. In fact just the relative importance of environmental quality changes. The reduction of pollution could be done via two channels. Reduction of emissions or decline of population size. In this case, the central planer chooses to reduce population size in order to keep production (emissions and consequently capital and consumption) at a constant level. As the steady state value of fertility is determined by $n^* = d + x/\beta$ (see A.2), it is independent from

σ in the stationary state. This is the utility-maximizing strategy even though utility declines with a rising importance of environmental quality.

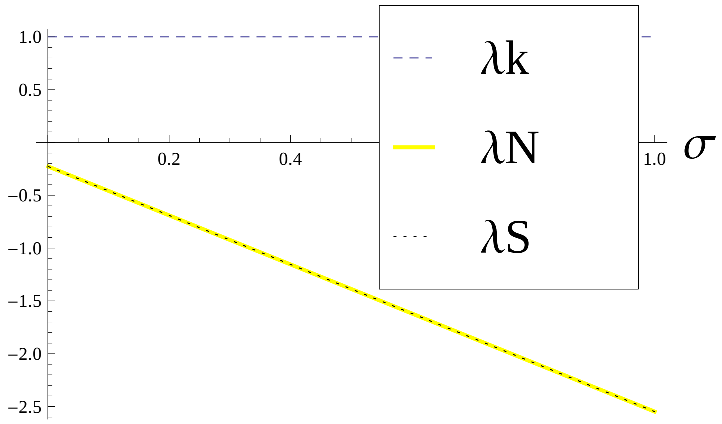


Figure 3.12: Shadow Prices with respect to variations in the utility-weight of environmental quality σ

As mentioned above population size and pollution is decreasing with rising utility-weight of environmental quality. Consequently also the shadow prices are declining as can be seen in figure 3.12. Greater importance of environmental quality affects utility as the shadow prices of pollution and the pollution generating population size worsen.

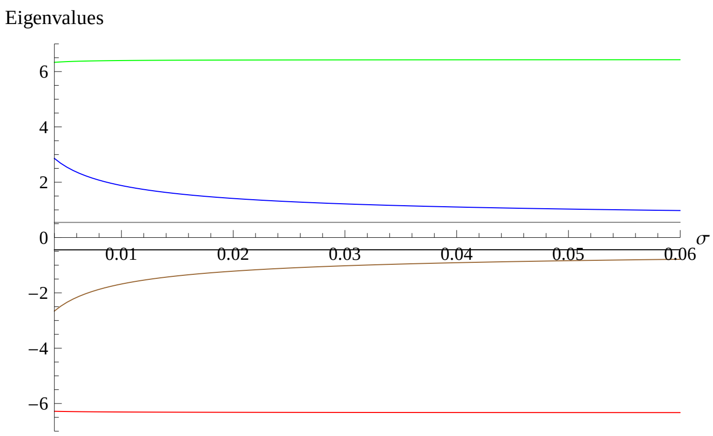


Figure 3.13: Eigenvalues with respect to variations in the utility-weight of environmental quality σ

Concerning steady state stability, we see in figure 3.13 that again no eigenvalue changes its sign or head. Therefore the dynamical system is stable with respect to variations of σ.

With this sensitivity analysis we close the investigation of the base model and go on by performing some adaptations to the base model.

Chapter 4

Adaptations of the base model

In chapter 3 we have analyzed an optimal control model consisting of the systems population, economy and environment, represented by the corresponding state dynamics. Every system was characterized by one state and one control variable. Of course, all controls also had effects on other states, e.g. fertility causes capital endowment. The system dynamics were chosen quite simple in order to assure solvability of the model. But as mentioned in the introduction, relations between demography, economy and the environment can become very complex. Therefore we will perform some adaptations on the base model. The following investigation is split into two steps:

1. Endogenizing mortality with a function depending on either capital, environmental quality or both.
2. Implementing *Shallow Lake Dynamics* (SLD) for the environment into the base model.

4.1 Introducing a mortality function

In the base model, we treated mortality as constant and exogenous. With this assumption, we have been able to analyze the systems of population, economy and the environment at the same time and were able to provide several insights. When introducing endogenous mortality into the optimal control model, we have to reduce the complexity of our model in order to keep it solvable. The first step is to omit the explicit treatment of population dynamics, meaning we will not further regard $\dot{N}$. However, it becomes necessary to change the dynamics of the environment $\dot{S}$ as well because it depended on N. Without explicit population size in our model, we will denote pollution in efficiency units as $s = S/N$, accumulated as side product of per capita production ϕf. Pollution in efficiency units has also been considered for example in a Solow Model (Xepapadeas, 2005, p.1227) or in the OLG model of Mariani et al. (2009) who assumed that longevity depends on environmental quality (and therefore changes mortality rate). As pollution is a side product of production, it can include aspects like waste, deforestation, factories destroying the view on a landscape and many more alternative aspects. Note that s stands for "inverse" environmental quality, as an higher value implies worse conditions. For better understanding, the word pollution will be used from now on. Note that as mentioned above *emissions* are not an input to production any more as in the base model, here the burden on the environment is a side product. The production function consequently changes to $f(k) = k^\alpha$ where we also omit technological progress. Production will also be needed in this model to "pay" for the environment, e.g. waste disposal. Therefore costs as a function of pollution $cost(s)$ will be added to capital dynamics $\dot{k}$. For the following investigation we choose $cost(s) = s^2$. As it is not economically reasonable to consider pollution in efficiency units in the utility function, we will omit the utility term.

The dynamics of the environment become

$$\dot{s} = \left(\frac{\dot{S}}{N}\right) = \frac{\dot{S}N - S\dot{N}}{N^2} = \phi f(k) - \delta s - s(n-d) \tag{4.1}$$

The term $-s(n-d)$ can be interpreted as follows: In a growing population $n > d$, additional resources can be devoted to "clean" the environment, resulting in a lower per capita pollution. We will call this effect *fragmentation* of pollution.

As pointed out in the introduction, it is plausible to assume on the one hand that pollution raises mortality because of worse living conditions (scarce resources, undrinkable water,...). On the other hand welfare decreases mortality as we profit from medicine, better nutrition and so on. Because of this relationship, a mortality function with the following specifications would satisfy these assumptions:

$$0 < d[k, s] < 1 \text{ with } d_k < 0 \text{ and } d_s > 0 \tag{4.2}$$

This general specification begs the question, in which relation capital and environmental quality should be placed with respect to each other concerning their influence on mortality. It is not clear whether the effects can be separated or if they need to be considered at the same time. For deeper insight, we will split this analysis in three parts. First we will analyze mortality as a function of capital k and second as a function of environmental quality s. Finally it will be a function of both simultaneously.

We proceed by introducing the different functions, followed by the analytical solution, a comparison of the different steady states and investigation of the transitional paths.

4.1.1 Mortality as a function of economic prosperity

As mentioned it is plausible that economic prosperity reduces mortality. We will use the function

$$d[k] = d_0 + \frac{\overline{d}}{1 + exp(k_s(k - k_p))} \tag{4.3}$$

following Dockner and Feichtinger (2000). The constant d_0 is a lower bound for mortality, representing a natural death rate (as in the base model), while $\overline{d}$ denotes the range of mortality due to changes in capital. The maximum death rate is $(d_0 + \overline{d})$ for $exp(k_s(k - k_p)) \to 0$ and the minimum is d_0 for $k \to \infty$. The exponent of the exponential function includes the constant

k_s, determining the (negative) slope of the function, and k_p, determining the turning point of the logistic function, yielding the development seen in figure 4.1 with the parametrization $k_s = 10, k_p = 1/2, d_0 = 0.1, \overline{d} = 0.3$ in order to cover a reasonable range of the mortality function, namely $d \in [0.1, 0.4]$.

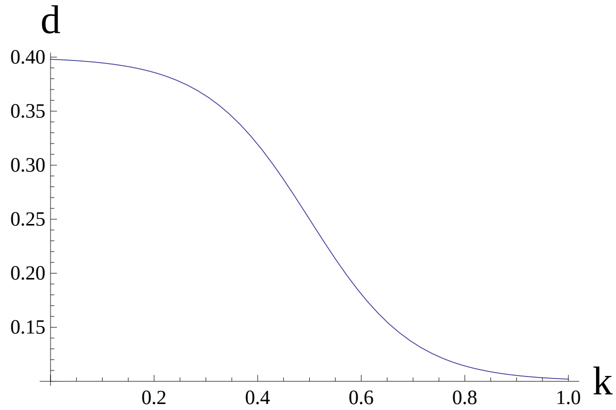


Figure 4.1: Mortality function (4.3)

4.1.2 Mortality as a function of environmental quality

After having introduced a mortality function dependent on capital k, we proceed by introducing mortality as a function of pollution s. Similar to (4.3) we will specify the mortality function as

$$d[s] = d_0 - \frac{\overline{d}}{1 + exp(s_s(s - s_p))} \tag{4.4}$$

where d_0 now is an upper bound and reduced by the weighted value $\overline{d}$. Note that now $d_0 - \overline{d}$ needs to be greater than the lower bound $\underline{d} = 0.1$ in order to guarantee a reasonable mortality base level. Mortality is a function of per capita pollution as the direct environment is of importance for the living conditions of an individual. The interpretations of s_s and s_p are as before slope and turning point. The dependency of d on s can be seen in figure 4.2

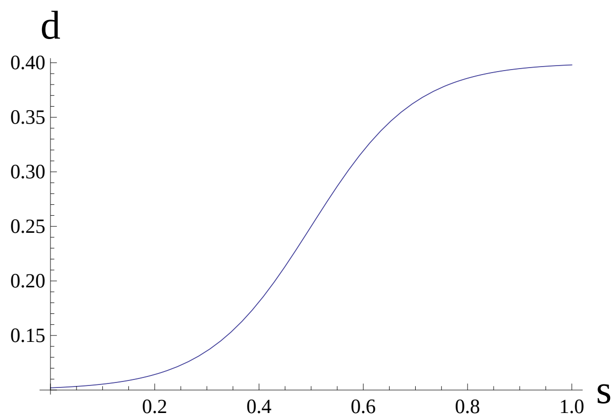


Figure 4.2: Mortality function (4.4)

The used parameters for the mortality function can be seen in table 4.1.

Parameter	Description	Value
d_0	upper bound of mortality	0.4
$\bar{d}$	maximal decline of mortality	0.3
s_s	slope	10
s_p	turning point	0.5

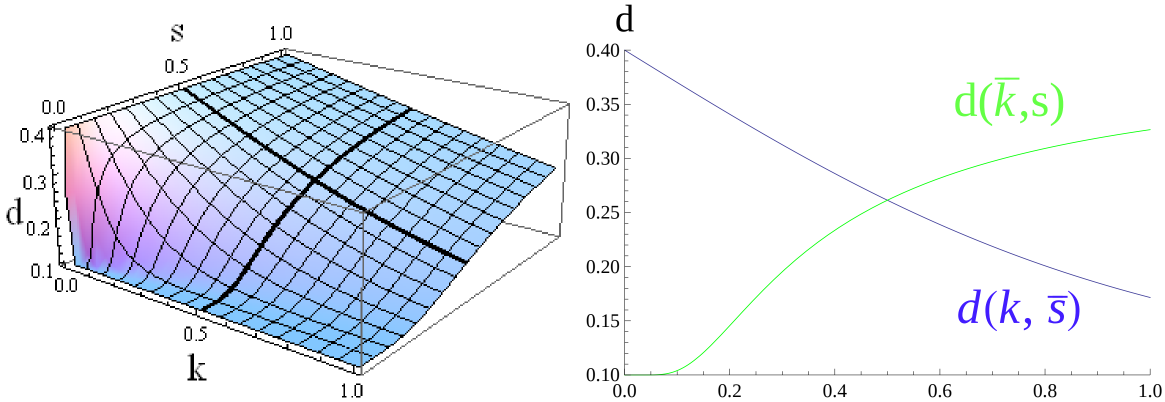
Table 4.1: Parameters for endogenous mortality dependent on pollution

4.1.3 Mortality as a function of pollution and capital

The last - and most complex - function to be introduced is a function of capital and pollution. The function

$$d[k,s] = d_0 + \frac{\bar{d}}{1 + exp(k/s)} \tag{4.5}$$

exhibits the desired behavior for the square $(k,S) \in \Big((0\text{x}1),(0\text{x}1)\Big)$ as can be seen in figure 4.3. However, capital k and pollution s could theoretically grow infinitely, resulting in $d[k,s] \to d_0$ for $k \to \infty$ or $d[k,s] \to d_0 + \bar{d}$ for $s \to \infty$, which is consistent with our assumption. d_0 is a lower bound and $\bar{d}$ is a scale parameter for the impact of pollution or welfare. In the left panel of figure 4.3 mortality is plotted for varying k and s. In the right panel of figure 4.3 we plot mortality as a function of only one variable by fixing the other variable at the value 0.5.



(a) Mortality function (4.5). The thick lines show the sections plotted in panel (b)

(b) Mortality d as a function of k or s with the other state fixed to 0.5

Figure 4.3: Properties of mortality function (4.5)

Note at this point, mortality does not affect utility directly. We keep considering fertility as utile and therefore it is up to the complex dynamics of the OCM if it is of interest to lower mortality due to high economic prosperity/low pollution or allow high mortality by accepting a high level of pollution.

4.1.4 Summary of the Optimal Control Model with the different mortality functions and analytical analysis

We will analyze the following optimal control model

$$\max_{c,n} \quad \int_0^\infty (ln(c) + \nu ln(n)) \cdot exp(-\rho t)dt \tag{4.6}$$

$$\begin{aligned}
\text{subject to} \quad & \dot{s} = \phi k^\kappa - \delta s - s(n - d_i) \\
& \dot{k} = k^\alpha - c - (n - d_i)k - bnk - b_0 n - s^2 \\
& 0 < \rho, \nu, \sigma, b_0, b \\
& 0 < \alpha, \kappa \text{ and } \alpha, \kappa < 1 \\
& 0 < d_0, \overline{d} < 1, \text{ and } 0 < d_0 + \overline{d} < 1 \\
& 0 < \delta, \phi < 1.
\end{aligned}$$

with the three different mortality functions:

$$\begin{aligned}
d_1 := \quad d[k] \quad &= d_0 + \frac{\overline{d}}{1 + exp(k_s(k - k_p))} \\
d_2 := \quad d[s] \quad &= d_0 - \frac{\overline{d}}{1 + exp(s_s(s - s_p))} \\
d_3 := \quad d[k, s] \quad &= d_0 + \frac{\bar{d}}{1 + exp(k/s)}.
\end{aligned}$$

We use the Pontryagin's Maximums Principle similar as in the base model and derive the first order conditions.

$$H = u(c, n, s) + \lambda^k \cdot \dot{k} + \lambda^s \cdot \dot{s} \tag{4.7}$$

$$\begin{aligned}
\Leftrightarrow \qquad H = \; & ln(c) + \nu \cdot ln(n) \\
& + \lambda^k \cdot \left[k^\alpha - c - \left(n - d^i\right) k - bnk - b_0 n - s^2\right] \\
& + \lambda^s \cdot \left[\phi k^\kappa - \delta s - s(n - d^i)\right]
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
\text{FOC:} \quad & H_n = 0 & u_n - \lambda^k(k + bk + b_0) - s\lambda^s = 0 \quad (4.9) \\
& H_c = 0 & u_c - \lambda^k = 0 \quad (4.10) \\
& \rho\lambda^k - H_k = \dot{\lambda}^k & \lambda^k\left(\rho - d^i + d^i_k - f_k + n + bn\right) + \lambda^s(d^i_k - \phi\kappa k^{\kappa-1}) = \dot{\lambda}^k \quad (4.11) \\
& \rho\lambda^s - H_s = \dot{\lambda}^s & \lambda^s(\rho - d^i + n + sd^i_s + \delta) + 2\lambda^k s = \dot{\lambda}^s \quad (4.12)
\end{aligned}$$

We proceed by calculating the steady state of the 4-dimensional differential system $\dot{s}, \dot{k}, \dot{\lambda^k}$

and $\dot{\lambda^s}$. First the controls can be replaced by the optimal values taken from (4.9) and (4.10). Next analytical expressions for λ^k and λ^s can be derived by solving (4.11) $\dot{\lambda^k} = 0$ for λ^k and (4.12) $\dot{\lambda^s} = 0$ for λ^s for all three mortality functions. However because of the complexity (multiple non-linearities and even exponential functions) of the model, it is not possible to obtain analytical solutions for capital k and pollution s. We can derive steady states numerically in each case. They are always feasible and unique. All parameters are taken as in the base model, where it was possible. Parameters of the mortality functions are described in the corresponding subsections, waste elasticity is choosen as $\kappa = 0.4$ and the waste coefficient is $\phi = 0.04$, meaning environmental impact equals $0.04 \cdot k^{0.4}$. This value is choosen as it equals output elasticity of emissions in the base model. The calculated steady states values can be seen in table 4.2.

Variable/Term	Description	I d(k)	II d(s)	III d(k,s)
k	capital	0.7960	4.0024	2.4273
s	pollution in efficiency units	0.1166	0.6625	0.2901
n	fertility	0.3277	0.3557	0.1966
c	consumption	0.6920	1.1272	1.0525
d^i	mortality (various functions)	0.1147	0.3506	0.1000
λ^k	shadow price of capital	1.4448	0.8872	0.9500
λ^s	shadow price of pollution	-0.8163	-4.2457	-1.8468
$f(k)$	production	0.9127	1.7415	1.4257
f_k	marginal productivity of capital	0.4586	0.1740	0.2349
τ_1	first eigenvalue of $\mathcal{J}$	$0.05 + 0.77i$	$0.42 + 0.21i$	$0.26 + 0.11i$
τ_2	second eigenvalue of $\mathcal{J}$	$0.05 - 0.77i$	$0.42 - 0.21i$	$0.26 - 0.11i$
τ_3	third eigenvalue of $\mathcal{J}$	0.4237	$-0.32 + 0.21i$	$-0.16 + 0.11i$
τ_4	fourth eigenvalue of $\mathcal{J}$	-0.3237	$-0.32 - 0.21i$	$-0.16 - 0.11i$

Table 4.2: Steady State Values with endogenous mortality

It is evident that the use of the three different functions yields very different results. In the case of pollution-dependent mortality, pollution achieves the highest value and an high level of mortality is accepted. In the two cases where mortality depends on capital, mortality is almost touching the lower bound. Next we consider the relation of capital to pollution as it gives a value how "clean" capital is, regardless of the absolute production. As can be seen in table 4.3, capital is cleanest when it really needs to be as k/s is also the mortality reducing term in case III. Therefore when considering this broadest scenario of the three, we receive the cleanest

		$\frac{k}{s}$	$\frac{\lambda^k}{\lvert\lambda^s\rvert}$	$\frac{c}{f(k)}$	$\frac{s^2}{f(k)}$	$\frac{\text{Child costs}}{f(k)}$	Pop. Growth
I	d(k)	6.83	1.77	75,8%	1.5%	22.7%	+
II	d(s)	6.04	0.20	64.7%	25.2%	10.1%	=
III	d(k,s)	8.37	0.52	73.8%	5.9%	20.3%	+

Table 4.3: Comparison of the relation of capital to pollution, shadow prices, relations of consumption, environmental costs and child costs to production and population growth in the three mortality models

production. This is a very optimistic view regarding the relation capital to pollution. Even if pollution rises a little, it won't really affect mortality (as k/s stays high). Regarding the relation between the shadow prices especially case I steps apart from the others. Here higher capital does not only yield the advantage of additional consumption (and fertility), it also reduces mortality, resulting in a personal cleaner environment (remember pollution fragmentation $\dot{s} = ... - s(n-d)$), which lowers environmental costs and liberates production shares for both consumption and fertility. This effect is smaller in case III, but still evident, as mortality not only depends on capital, but also on pollution. The shares of production for case I and III in the steady state are quite similar. Case II steps apart with high environmental costs. Due to constant population there is no capital dilution. The absence of capital endowment is important as child costs are only 10% and a major share of production can still be used for consumption. In cases I and III we see population growth and one fifth of production is used for child costs. This yields the result that when capital is more important (i.e. mortality-reducing), fertility is also valued higher. This happens because of the following feedback-effect: Population growth (possible initiated by higher capital) reduces personal pollution (fragmentation effect) resulting in lower environmental costs allowing additional production shares to be used for consumption and fertility or capital accumulation. Therefore this feedback-effect leads to a capital multiplier. After this detailed analysis of the steady state, we will proceed to the transitional dynamics.

4.1.5 Transitional paths

In the last 4 lines of table 4.2 the eigenvalues of the Jacobian can be seen. As described in the base model, the negative real parts of the eigenvalues represent the stable manifold in the steady state. When the system starts in this stable manifold, it will converge to the steady state. In case I $d(k)$ there is only one negative real valued eigenvalue, indicating a 1-dimensional stable node. The transitions are therefore monotonous to the steady state. If rising or falling only depends on the choice of the starting value. As this does not yield any insights, the analysis of these transitional dynamics will be omitted. In case II and III the steady state is a 2-dimensional stable spiral. As we have complex eigenvalues and consequently complex eigenvectors $\vec{e} = \vec{a} \pm \vec{b}i$, the starting points are chosen by using

$$X = S + exp[\vec{a}] \cdot cos(\vec{b}) \cdot cos(h \cdot \pi) + exp[\vec{a}] \cdot sin(\vec{b}) \cdot sin(h \cdot \pi) \tag{4.13}$$

where X is the resulting starting point, S the steady state, a the real part and b the imaginary part of the eigenvector. The starting points are determined with $h \in [0, 2)$ in an ellipse around the steady state. The resulting transitional dynamics can be seen in figures 4.4-4.9, where case II $d(s)$ is on the left side and case III $d(k, s)$ on the right side. In figure 4.4 pollution is plotted over capital and the spiral can clearly be seen. In case III it is much more cyclical, because of the higher imaginary parts in relation to the real parts of the eigenvectors as can be seen in table 4.4. This more cyclical behavior yields a faster evolution to the steady states (compare the time paths figure 4.7, 4.8 and 4.9). It also implies smaller changes in all variables, meaning the evolution will proceed slower as when mortality only depends on pollution. This is economically meaningful as environmental changes have faster effects on mortality than capital changes. Research in medicine and assuring healthy life (meaning economic prosperity) takes longer as the impureness of water or any kind of environmental destruction.

$d(s)$	$\{0.976,$	$0.078 - 0.031i,$	$-0.165 + 0.002i,$	$-0.104 + 0.014i\}$
$d(k, s)$	$\{0.922,$	$0.081 - 0.088i,$	$-0.226 - 0.003i,$	$-0.172 + 0.230i\}$

Table 4.4: Eigenvectors according to the third eigenvalue in case II and III

Note that the values c and n are normalized by their steady state values in figures 4.5 and 4.6. The steady state itself is marked with the black "x". In both cases, consumption and fertility rises with growing capital and pollution until it reaches a peak at the same time (as they are related to the same state value) in both variables. The peak is characterized by high consumption, which is not sustainable due to high pollution, consequently high environmental costs which takes a large share of production. This share could be used for financing higher fertility or consumption, both yielding utility.

Concerning population growth[1], case II experiences different stages from a decline of -0.04 to a growth period of $+0.02$, resulting in a very stable population at the end (see figure 4.7). The changes happen due to the non-cyclical transition, especially the one of pollution. Declining pollution leads to declining mortality leading to a delayed decline in fertility, as the agent tries to keep constant child costs in favor of high consumption. The production shares can be seen in the area figures 4.8. On the bottom of the figures, the value of consumption is plotted. Above the share of environmental costs is added and last child costs. As capital accumulation equals zero in the steady state, it does not claim any production shares. However the share of environmental costs is very high ($s^2/f(k) \approx 25\%$) even though it was low in the beginning ($s^2/f(k) < 10\%$).

It is of crucial importance that the evolution of fertility and mortality is significant for the evolution of shadow prices. The valley of mortality and co-state of capital occur at the same time. As mortality rises, additional capital needs to be raised to afford future fertility. Obviously, the larger output is also used for consumption. Pollution is thanks to capital control not allowed to accumulate until the steady state of consumption is reached and the negative effect of pollution is considerable low ($\lambda^s \approx -1$). This means that the agent keeps the level of pollution down until the sustainable level of consumption is achieved. Only then pollution accumulates as the resulting high level of capital exceeds the environmental regeneration rate. This evolution is only possible thanks to the fragmentation effect, as it supports regeneration of the environment.

Case III to the right shows a much more stable behavior concerning population growth and production shares. As we have mentioned above, the dependency of mortality on economic prosperity and pollution at the same time yields smoother developments. So when the relation between welfare and pollution does not change significantly (as it is in the case due to the smooth spiral shown in figure 4.4) the according system is also stable, characterized by constant

[1]Population growth can be calculated with (n-d) from figure 4.7.

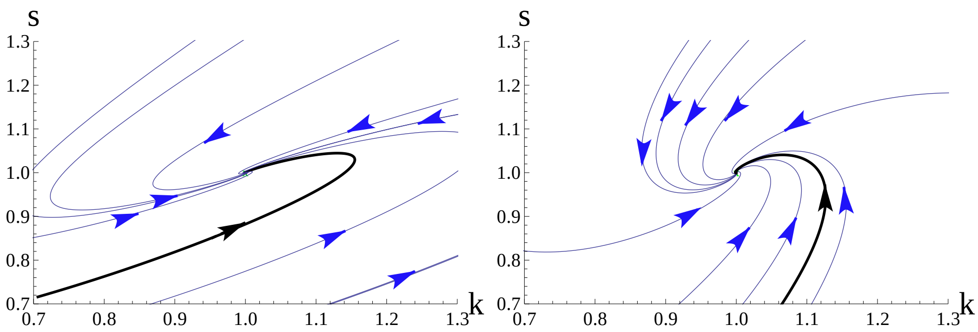


Figure 4.4: State space

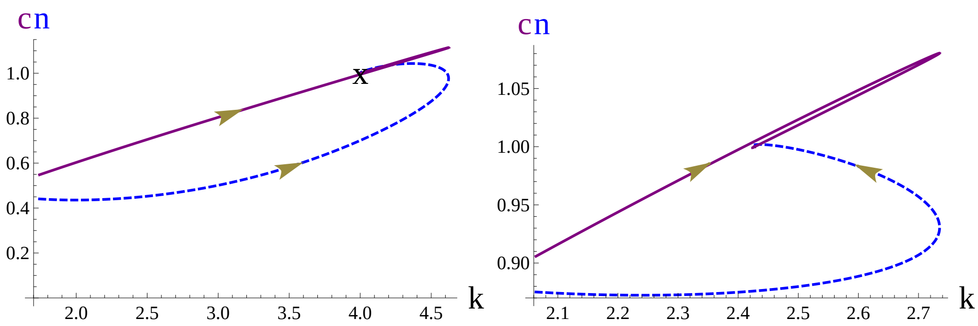


Figure 4.5: Controls over capital

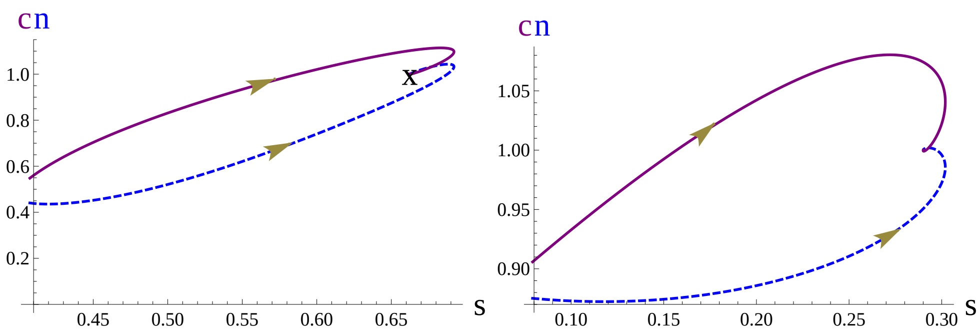


Figure 4.6: Controls over pollution

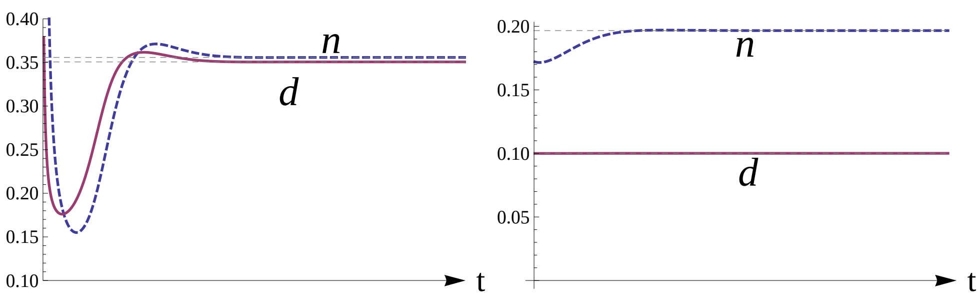


Figure 4.7: Fertility and mortality

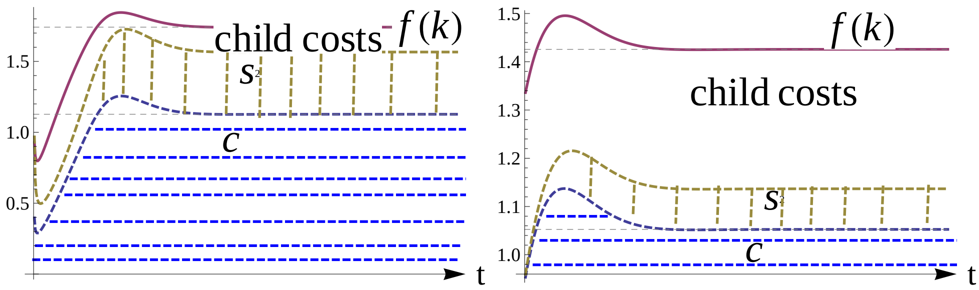


Figure 4.8: Production split up into the three shares: Consumption, environmental costs and child costs

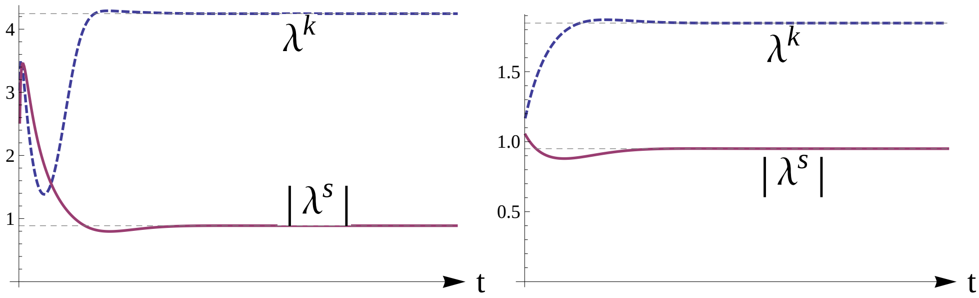


Figure 4.9: Shadow prices

population growth at the same rate and constant production shares.

To sum up, we started the analysis by introducing three kinds of mortality rates as a function of

- economic prosperity
- environmental quality
- both at the same time

and were able to point out significantly different results in these cases. Therefore it is important to consider which functions fits best when applying an endogenous mortality function to a model studying relations of population, economy and environmental relations.

4.2 Shallow Lake Dynamics

The next adaptation is aiming towards more complex environmental dynamics. Until now, we assumed a constant regeneration rate of the environment δ. This assumption is convenient as it implies a unique steady state in environmental dynamics. But in reality dynamics of the environment can become very complex. In the introduction we already mentioned processes like desertification or shallow lakes. Especially the second has motivated economists to adapt the modeling of environmental dynamics. Usually it was a convex function, consisting of an impact term (depending on other states or agents actions) and a regeneration term. As long as both are kept simple, there is a unique steady-state to which the pollution stock can converge.

However, as pointed out in Scheffer et al. (2001) even though nature usually responds smoothly to gradual changes, there may be critical triggers and thresholds. After having crossed them the environment will converge to a new equilibrium, often connected with worse conditions. In the example of shallow lakes, there is a pristine state with rich flora and fauna, keeping the lake clean even when it is used as inflow for waste water. The animals live from the fauna, clean the water because of their natural habits and assure clear water. But if the water gets too turbid, the vegetations disappears due to missing sunlight and the animals loose their nutrition and die as well. This leads to the establishment of an alternative equilibrium. Mathematically speaking, we obtain a problem with multiple stable equilibria. We have to consider the aspects of irreversibility and possible hysteresis of environmental systems.

This kind of development can be modeled with a non-convex function, changing the qualitative outcome considerable. The shallow lake dynamics (SLD) are usually represented by

$$\dot{S} = D - \delta S + \frac{S^2}{1 + S^2} \tag{4.14}$$

with D being the impact on the environment (e.g. pollution, inflow of waste water,...), S the stock of environmental quality and δ remains the constant regeneration rate. The role of δ is now very important for the qualitative behavior of the environmental system. Let us assume that D is exogenous given and we will analyze the system for a varying regeneration rate δ. There are three cases to be distinguished.

The first case $\delta \geq 0.65$ is shown in figure 4.10. The path is monotonous even under the assumed SLD. We receive no special dynamics and the behavior is similar to a convex function, resulting in a unique steady state.

In the case of $0.5 < \delta < 0.65$, it becomes interesting as can be seen in figure 4.11. The black thick arrowheads mark the general behavior of the system for given values of environmental quality S and dirt flow D, meaning it will converge to the upper or lower branch, represented by the solid lines. The lower one corresponds to a clean lake and the upper one to a turbid lake with bad environmental quality. The dashed line between S_0 and S_1 represents the indifference barrier. Imagine, the lake is in its natural state. The locus would be somewhere on the lower branch, in its original steady state. As the dirt flow increases, we move along the lower branch to the right. As long as $\overline{D}$ is not reached, the self-cleaning dynamics are strong enough to keep

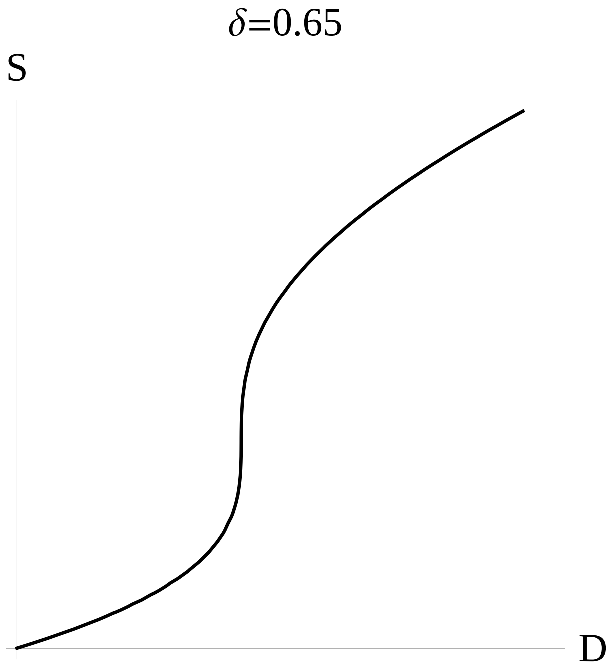


Figure 4.10: Monotonous path

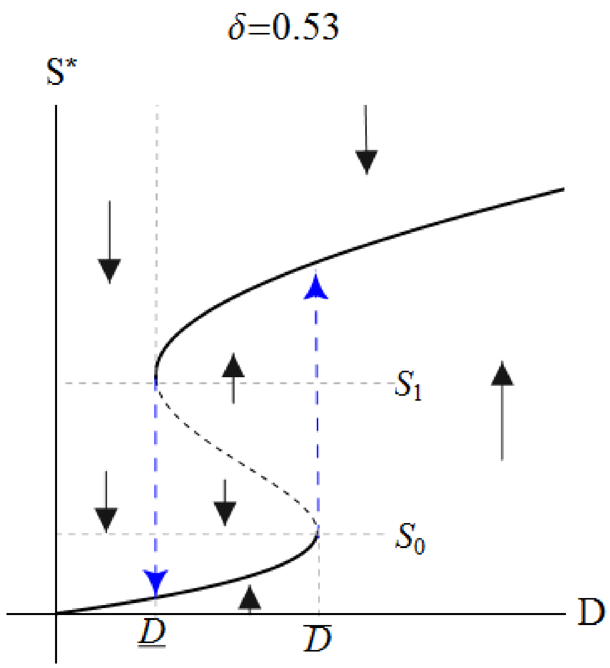


Figure 4.11: Two equilibria with *reversible* pollution stock

environmental quality on the lower branch. But as soon as $\overline{D}$ is surpassed, the system will converge to the upper branch and stabilize in an alternate equilibrium. In order to restore a clean lake, one would need to take the effort to establish $D < \underline{D}$, so the system may again converge to the lower branch. So all damage done to the environment is in theory *reversible*. If the environmental quality is between S_0 and S_1, it depends on the actual dirt flow if the system converges to the lower or upper branch. Here it is important to enact the right control at the right time in order to save a lot of efforts and inconveniences.

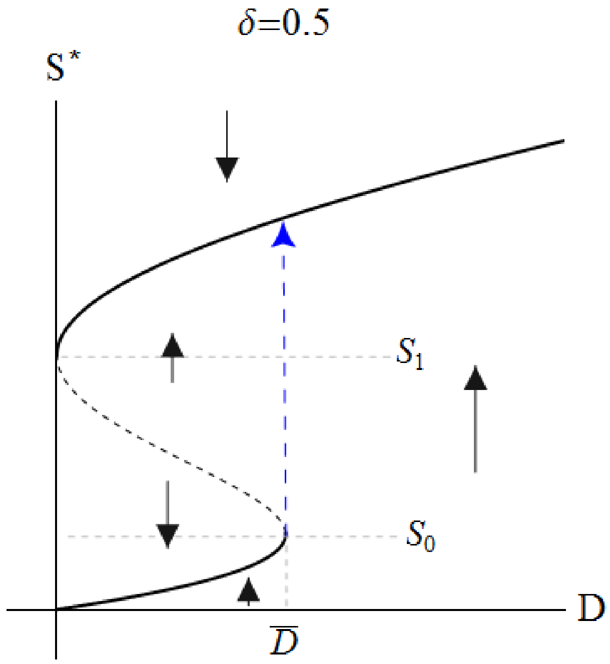


Figure 4.12: Two equilibria with *irreversible* pollution stock

Last but not least we consider the case $\delta \leq 0.5$ as shown in figure 4.12. It is similar to the previous discussed case, in the sense that we have a threshold $\overline{D}$ which is the critical point. But this time, once damage is done and the system is on the upper branch, there is no way to reestablish the clean lake (lower branch) as there is no "regeneration threshold" $\underline{D}$. It can not be brought back to the steady state even with $D = 0$, the system is *irreversible*.

4.2.1 SLD with population dynamics

After having discussed the main characteristics of SLD, we will implement this function into our base model 3.12 with the three states population size N, capital k and pollution S. The dirt flow D will now be represented by $N \cdot k$ and the production function is defined as $f(k) = k^{\alpha}$. Emissions are not considered as control anymore, as we assume environmental impact to be a side product of production. Before emissions have been an input to production. This change better fits the modeling of SLD and is therefore plausible. To summarize, the considered SLD model with population dynamics is

$$\max_{c,n} \quad \int_0^{\infty} (ln(c) + \nu ln(n) + \sigma ln(\bar{S} - S)) \cdot exp(-\rho t)dt \tag{4.15}$$

$$\begin{aligned}
\text{subject to} \quad & \dot{N} = (n - d) \cdot N \\
& \dot{S} = Nk - \delta S + \frac{S^2}{1 + S^2} \\
& \dot{k} = k^{\alpha} - c - (n - d)k - bnk - b_0 n \\
& 0 < \rho, \nu, \sigma, \bar{S}, \delta, b_0, b \\
& 0 < \alpha, \text{ and } \alpha < 1 \\
& 0 < d, \epsilon, \delta < 1.
\end{aligned}$$

The problem is solved similar to the base model with Pontryagin's Maximums Principle. First the Hamiltonian is set up followed by the derivation of the FOC. The interpretation of the co-states is analogous to the base model, as H_n and H_c have not changed. However as we don't use emissions as control anymore and λ^S does not appear in (4.19) or (4.18), a direct interpretation is not possible. Considering (4.22) we see that growth or decline of the shadow price of pollution depends on the marginal utility of environmental quality and the actual state of the environment. Capital or population dynamics do not have a direct influence.

$$H = u(c, n, S) + \lambda^k \cdot \dot{k} + \lambda^N \cdot \dot{N} + \lambda^S \cdot \dot{S} \tag{4.16}$$

$$\begin{aligned}
\Leftrightarrow \quad H = & \, (ln(c) + \nu ln(n) + \sigma ln(\bar{S} - S)) \\
& + \lambda^k \cdot (k^{\alpha} - c - (n - d)k - bnk - b_0 n) \\
& + \lambda^N \cdot ((n - d) \cdot N) \\
& + \lambda^S \cdot (Nk - \delta S + \frac{S^2}{1 + S^2})
\end{aligned} \tag{4.17}$$

$$
\begin{aligned}
H_c =& 0 & u_c - \lambda^k = 0 \quad (4.18)\\
H_n =& 0 & u_n - \lambda^k[k + bk + b_0] + \lambda^N N = 0 \quad (4.19)\\
\rho\lambda^k - H_k =& \dot{\lambda}^k & \lambda^k[\rho - f_k + (n-d) + bn] = \dot{\lambda}^k \quad (4.20)\\
\rho\lambda^N - H_N =& \dot{\lambda}^N & \lambda^N[\rho - (n-d)] - \lambda^S k = \dot{\lambda}^N \quad (4.21)\\
\rho\lambda^S - H_S =& \dot{\lambda}^S & u_s + \lambda^S\left(\rho + \delta - \frac{2S}{(1+S^2)^2}\right) = \dot{\lambda}^S \quad (4.22)
\end{aligned}
$$

To solve the optimized control, n^* and c^* are taken from $H_n = 0$ and $H_c = 0$. In order to assure solvability, we set output elasticity $\alpha = 1/2$. Inserting these and solving the resulting 6-dimensional system in the steady state (all dynamics equal 0) for the states and co-states yields

$$
\begin{aligned}
\dot{N} = 0 \quad &\Rightarrow \quad \lambda_k^* = \frac{\nu + dN\lambda_N}{d(b_0 + k + bk)}\\
\dot{\lambda_N} = 0 \quad &\Rightarrow \quad \lambda_S^* = -\frac{\lambda_N \rho}{k}\\
\dot{\lambda_k} = 0 \quad &\Rightarrow \quad N^* = -\frac{\sqrt{k}\nu - 2kv(bd+\rho)}{d\lambda_N(\sqrt{k} + 2b_0\rho + 2bk(\rho - d))}\\
\dot{\lambda_S} = 0 \quad &\Rightarrow \quad \lambda_N^* = \frac{k(1+S^2)^2\sigma}{(S - \bar{S})\rho(-2S + \delta + \rho + 2S^2(\delta+\rho) + S^4(\delta+\rho))}\\
\dot{k} = 0 \quad &\Rightarrow \quad k_1^* = \frac{1}{8b^2d^2(d - (1+\nu)\rho)^2}\Big(8bb_0d^3(1+\nu)\rho - d^2(-1 + 8bb_0(1+\nu)^2\rho^2)\\
&\qquad + 2\nu\rho(2\nu\rho + \sqrt{D_1}) - d(4\nu\rho + \sqrt{D_1})\Big)\\
\text{or} \quad &\qquad k_2^* = \frac{1}{8b^2d^2(d - (1+\nu)\rho)^2}\Big(8bb_0d^3(1+\nu)\rho - d^2(-1 + 8bb_0(1+\nu)^2\rho^2)\\
&\qquad + 2\nu\rho(2\nu\rho + \sqrt{D_1}) + d(-4\nu\rho + \sqrt{D_1})\Big)\\
\text{with} \quad &\qquad D_1 = -4dv\rho + 16bb_0d^3(1+\nu)\rho + 4\nu^2\rho^2 - d^2(-1 + 16bb_0(1+\nu)^2\rho^2).
\end{aligned}
$$

As we see, due to the specification $\alpha = 1/2$ we receive two possible steady-state values for capital k. Note that neither of them is dependent on the variable S, which can not be derived analytically. Therefore we need to calculate it numerically. When using k_1 we obtain attracting minimal steady states and will therefore omit this case.

We proceed to k_2, whose behavior can be seen in figure 4.13. As mentioned above, the system critically depends on the value of δ and so we will perform a bifurcation analysis with respect to the regeneration rate δ. Many steady-states here need to be excluded because population size $N < 0$, marked by the brown solid thin lines. When checking the *Arrow-Sufficiency Conditions* (see Appendix C) for optimality, we receive that the upper and the lower branch represents maxima. At $\delta = 0.52$ we see a blue sky bifurcation, as the equilibria ”appear out of nowhere”. For the small area $0.48 < \delta < 0.52$ we see the existence of an optimal steady state with high pollution. The optimal steady state with low pollution exists for almost all δ. Just for very low δ it becomes infeasible due to $N < 0$. This is not the same area as described before due to the influence of the other systems. For $\delta > 0.52$, we have the case of a monotonous path. This means economically that only in a short parameter range, the system may converge into the dirty equilibrium. At a higher regeneration rate δ, the lake may always converge to the clean steady state.

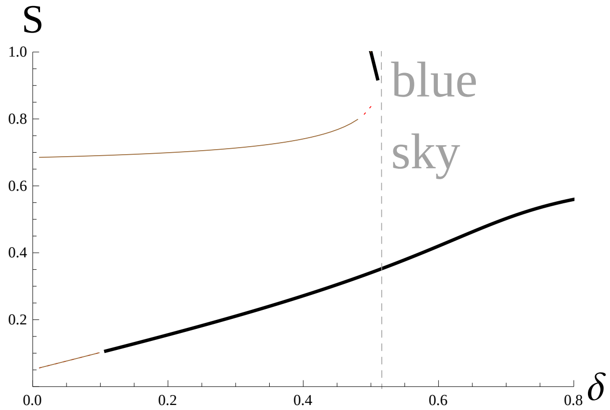


Figure 4.13: Steady-State values for varying δ, thick lines mark maxima, brown solid lines mark infeasible points ($N < 0$) and dotted lines mark indefinite points

Chapter 5

Result summary and conclusion

The subject of this thesis is to investigate the relations between the systems population, economy and the environment. Therefore an optimal control model inspired by Quaas (2004) has been analyzed in three stages.

1. Base model with simple dynamics. Emissions are an input to production.
2. Introduction of an endogenous mortality function dependent on either capital, environmental quality or both, to pay tribute to more complex relations of economy and environment to population dynamics.
3. Base model with shallow lake dynamics, implying multiple equilibria.

The analysis of the base model is split into an analytical and a numerical section. In the analytical section we have proven that a balanced growth path (BGP) exists. The BGP is characterized by increasing population size N and declining emissions used in production e at the same rate. Together these BGP assures that overall emissions $N \cdot e$ stay constant. After a transformation of the BGP to a stationary state, the optimal transition paths leading to the stable saddle point have bee analyzed in the numerical section. The dependence of the steady states on the parameters has been investigated by a bifurcation analysis. We will sum up the main results:

- An unique optimal steady state with positive values for state and control variables exists if the constraint (3.43) is fulfilled for the given parameters.
- Initial conditions do matter, as we have shown with two very different transition paths.
- The transition is non-monotonous. Especially during the period of population decline, fertility first rises, falls and rises again to a stable population level.
- The constant regeneration rate of the environment can't offset overall emissions until they fall by their own. Consequently environmental impact needs to be controlled active by the agent due decrease of emissions or fertility, as the environment can't cope with the impact of emissions and population.

- Nevertheless emissions can be sustainable increased due to declining population size.
- Even though it is theoretically possible that the shadow price of population size can be smaller or greater zero ($\lambda_N < 0$ or $\lambda_N > 0$), we do not obtain positive values neither in the steady state, nor in the transition path (when starting with a higher population size). Even while conducting sensitivity analysis, we do not obtain any case with $\lambda_N > 0$. This implies that growing population affects utility negatively.
- Consumption and pollution do have a changing correlation. Therefore the argument that higher consumption implies higher pollution can not be supported in general.
- The marginal productivity of emissions e and capital k behave inverse to each other in the transition to the steady state. The same is true for per capita production f and overall emissions $N \cdot e$.
- Pollution does not depend on mortality d.
- Changes in the utility-weight of environmental quality only affects the environmental sector and the according population size. Emissions are not reduced as the environment becomes more important, only population shrinks.

Even though the base model exhibits a lot of important results, it was extended by the introduction of mortality functions in three ways. Mortality dependent on economic prosperity, environmental quality or both, showing very different results due to the different specifications. The main results are:

- When mortality depends on economic welfare, the steady state is stable only in one dimension. The agent chooses his controls to enact low mortality accompanied by low pollution.
- When mortality depends negatively on capital higher fertility results due to the feedback of environmental quality. Population growth reduces personal pollution (fragmentation effect), resulting in lower environmental costs liberating production shares for consumption and fertility implying population growth or capital accumulation. This exhibits a multiplier effect.
- When mortality depends on environmental quality, namely case II, mortality and the co-state of capital are directly correlated to capital accumulation. This allows a delayed fertility adjustment to a higher mortality level.
- In the transition phase different stages of population growth and decline occur, resulting in difficult policy-making especially concerning fertility.
- Overall transitions in the third case $d(k, s)$ are smoother and exhibit smaller changes.

As we can see, the results of the three specifications are very different. Therefore when using an endogenous mortality function in an economic population and environment model, it must be carefully evaluated which specification captures the desired effects.

Implementing Shallow Lake Dynamics produces multiple equilibria. After an intensive discussion of SLD dynamics, showing that the constant regeneration rate is very important for the qualitative behavior, the base model has been tested with SLD. As expected, we see multiple optimal equilibria occurring. Many steady states need to be excluded because of infeasibility. The implementation of boundary conditions ($N > 0$) may yield further insights. The occurrence of multiple steady states indicate that after the passing of certain thresholds, the environmental system may stabilize in a dirty environment, where it becomes difficult to reestablish a clean one.

To sum up, even this model with the two adaptations already yields interesting results. It is difficult to capture the complex dynamics and relations between the three systems and the use of different functions can yield different results. However, as pointed out in the literature review chapter 2, there does already exist some research relating population, economy and the environment. This thesis concentrates on mathematical modeling aspects and the study of endogenous mortality functions and shallow lake dynamics.

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List of Figures

List of Tables

Appendix A

Base scenario - proofs

A.1 Proof of Proposition 3.3.2

For the proof we use the more general utility function

$$\tilde{u} = \frac{c^{1-\kappa} - 1}{1-\kappa} + \nu \frac{n^{1-\epsilon} - 1}{1-\epsilon} + \sigma ln(\bar{S} - S) \tag{A.1}$$

where the original function 3.5 is a special case for $\kappa = 1$. We note $\tilde{u}_c = c^{-\kappa}$ and

$$-\frac{\dot{\tilde{u}}_c}{\tilde{u}_c} = \kappa \frac{\dot{c}}{c}. \tag{A.2}$$

The growth rates with the new utility function become

$$\begin{aligned}
\frac{\dot{c}}{c} &= \frac{1}{\kappa}\left[\alpha k^{\alpha-1} e^{\beta} \exp(xt) - bn - \rho - (n-d)\right] && \text{(A.3)}\\
\frac{\dot{n}}{n} &= -\frac{\rho}{\epsilon} + \left[\beta k^{\alpha} e^{\beta} \exp(xt) + [(1+b)k + b_0]\left[\kappa \frac{\dot{c}}{c} + \rho\right] - (1+b)\dot{k}\right] \frac{n^{\epsilon} c^{-\kappa}}{\nu \epsilon} && \text{(A.4)}\\
\frac{\dot{e}}{e} &= -\frac{1}{1-\beta}\left[\rho + \delta + (n-d) + \kappa \frac{\dot{c}}{c} - \alpha \frac{\dot{k}}{k} - \frac{\sigma N c^{\kappa}}{(\bar{S} - S)\beta k^{\alpha} e^{\beta-1} \exp(xt)}\right] && \text{(A.5)}
\end{aligned}$$

Starting with $\frac{\dot{N}}{N}$ constant, we conclude

$$\frac{\dot{n}}{n} = 0 \tag{A.6}$$

meaning the change in fertility is zero. The derivative of 3.10 with respect to time yields

$$\frac{d}{dt}\left(\frac{\dot{S}}{S}\right) = (\dot{N}e + N\dot{e} - \delta\dot{S})/S \overset{!}{=} 0 \tag{A.7}$$

$$\dot{N}e + N\dot{e} = \delta\dot{S} \tag{A.8}$$

$$\frac{\dot{N}}{N}(Ne) + (Ne)\frac{\dot{e}}{e} = \delta S\frac{\dot{S}}{S} \tag{A.9}$$

$$\frac{\dot{N}}{N} + \frac{\dot{e}}{e} = \frac{\delta S}{Ne} \cdot \frac{\dot{S}}{S} \tag{A.10}$$

stating that the growth rate of weighted pollution is equal to the sum of the growth rates of population size and emissions. The derivative of A.3 with respect to time (and keeping A.6 in mind) yields

$$0 = \frac{d}{dt}f_k = \frac{d}{dt}\alpha k^{\alpha-1}e^{\beta}\exp(xt) \tag{A.11}$$

$$\Leftrightarrow \frac{\dot{k}}{k} = \alpha\frac{\dot{k}}{k} + \beta\frac{\dot{e}}{e} + x \tag{A.12}$$

Using this and the derivative of (3.8) with respect to time yields

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k}\left(1 + \frac{b_0 n}{c}\right) \tag{A.13}$$

and again using (A.12) and the derivative of (A.5), inserting the previous result (A.13) as well as (A.10)

$$0 = \frac{\dot{N}}{N} + \kappa\frac{\dot{c}}{c} + \frac{\dot{S}}{S}\frac{\dot{S}}{\bar{S} - S} - \alpha\frac{\dot{k}}{k} + (1-\beta)\frac{\dot{e}}{e} - x = \left[\kappa - 1 + \kappa\frac{b_0 n}{c}\right]\frac{\dot{k}}{k} + \frac{\dot{S}}{S}\left(\frac{\dot{S}}{\bar{S} - S} + \frac{\delta S}{Ne}\right) \tag{A.14}$$

Using the growth rate for (A.4) with $\dot{n} = 0$, we note

$$\xi \equiv -\beta f - [(1+b)k + b_0]\left[\kappa\frac{\dot{c}}{c} + \rho\right] + (1+b)k\frac{\dot{k}}{k} = -\frac{\rho}{\epsilon}\frac{u_n}{u_c} \tag{A.15}$$

and because of derivative with respect to time

$$
\begin{aligned}
0 &= \xi\frac{u_c}{u_n}\underbrace{\left[\frac{\dot{u}_c}{u_c}-\overbrace{\frac{\dot{u}_n}{u_n}}^{=0}\right]}_{=-\kappa\dot{c}/c,\text{ see (A.12)}} \underbrace{-\left[\beta f\left[\alpha\frac{\dot{k}}{k}+\beta\frac{\dot{e}}{e}+x\right]+(1+b)k\left[\frac{\dot{k}}{k}\left[\kappa\frac{\dot{c}}{c}+\rho\right]-\left[\frac{\dot{k}}{k}\right]^2\right]\right]}_{=\frac{\dot{k}}{k}[\xi+b_0[\kappa\frac{\dot{c}}{c}+\rho]]}\frac{u_c}{u_n} \\
\Leftrightarrow 0 &= -\xi\frac{u_c}{u_n}\kappa\frac{\dot{c}}{c}+\frac{\dot{k}}{k}\left[\xi+b_0\left[\kappa\frac{\dot{c}}{c}+\rho\right]\right]\frac{u_c}{u_n} \\
\overset{(A.15)}{\Leftrightarrow} 0 &= \kappa\frac{\dot{c}}{c}\frac{\rho}{\epsilon}+\frac{\dot{k}}{k}\left[-\frac{\rho}{\epsilon}+b_0[\kappa\frac{\dot{c}}{c}+\rho]\frac{n^\epsilon}{\nu\epsilon c^\kappa}\right] \\
\overset{(A.13)}{\Leftrightarrow} 0 &= \left[\left[\kappa\left[1+\frac{b_0 n}{c}\right]-1\right]\frac{\rho}{\epsilon}+b_0\left[\kappa\left[1+\frac{b_0 n}{c}\right]\frac{\dot{k}}{k}+\rho\right]\frac{n^\epsilon}{\nu\epsilon c^\kappa}\right]\frac{\dot{k}}{k}
\end{aligned}
$$

This equation is fulfilled for given values of n and c and $\dot{k}=0$ or

$$
\frac{\dot{k}}{k}=\left[b_0\kappa\left[1+\frac{b_0 n}{c}\right]\frac{n^\epsilon}{\nu\epsilon c^\kappa}\right]^{-1}\left[\kappa\left[1+\frac{b_0 n}{c}\right]-1+b_0\frac{n^\epsilon}{\nu c^\kappa}\right]\frac{\rho}{\epsilon} \tag{A.16}
$$

As the right side is just dependent on consumption c (n is constant), it is obvious that for an increasing (or declining) c the growth rate g_k is not constant. This is consistent with the assumption of a BGP and therefore $g_k=0$. With (A.13) we conclude $\dot{c}=0$ and furthermore with (A.14) we obtain $\dot{S}=0$.
For the growth rates g_N and g_e we use (A.10) and (A.12) to get

$$
\frac{\dot{N}}{N}=-\frac{\dot{e}}{e}=\frac{x}{\beta} \tag{A.17}
$$

and obtain a BGP because of the growing size of population and declining rate of emissions.

A.2 Proof of Proposition 3.3.3

We show that under the assumption of a stationary state (where all growth rates are 0), the transformed system has an unique, feasible and optimal solution. For feasibility all variables need to be positive. Using the given growth rates (3.37)-(3.42) we obtain

$$
\begin{aligned}
n^* &= d+\frac{x}{\beta} && \text{(A.18)}\\
f(k^*,\hat{e}^*) &= c^*+(n^*-d)k^*+bn^*k^*+b_0n^* && \text{(A.19)}\\
\hat{N}^*\hat{e}^* &= \delta S^* && \text{(A.20)}\\
\alpha(k^*)^{-1}f(k^*,e^*) &= bn^*+(n^*-d)+\rho && \text{(A.21)}\\
(n^*)^{-\epsilon}\nu c^* &= \frac{\beta}{\rho}f(k^*,\hat{e}^*)+(1+b)k^*+b_0 && \text{(A.22)}\\
\frac{\hat{N}^*\hat{e}^*\sigma}{\bar{S}-S^*} &= \beta\frac{f(k^*,\hat{e}^*)}{c^*}(\rho+\delta+x) && \text{(A.23)}
\end{aligned}
$$

From the first equation (A.18), we see that n^* is uniquely determined and positive, as all parameters are positive. With (A.21) we obtain

$$f(k^*, \hat{e}^*) = \frac{bn^* + (n^* - d) + \rho}{\alpha} k^* \equiv \zeta k^*,\ \zeta > 0 \tag{A.24}$$

Using this and (A.19) yields

$$c^* = (\zeta - (n^* - d) - bn^*)k^* - b_0 n^*. \tag{A.25}$$

Inserting this in (A.22) and solving for k^* yields the stationary-state value of capital

$$k^* = \frac{b_0(\nu n^* + (n^*)^\epsilon)}{\nu(\zeta - (n^* - d) - bn^*) - \frac{\beta}{\rho}(n^*)^\epsilon \zeta - (1+b)(n^*)^\epsilon} \tag{A.26}$$

$$= \frac{\alpha\rho b_0(\nu n^* + (n^*)^\epsilon)}{\nu\rho\Xi - \beta(n^*)^\epsilon(bn^* + (n^* - d) + \rho) - (1+b)(n^*)^\epsilon \alpha\rho} \tag{A.27}$$

where $\Xi = (1-\alpha)(n^* - d) + (1-\alpha)bn^* + \rho = \alpha(\zeta - (n^* - d) - bn^*)$. Thanks to the unique value of n^*, given in (A.18), k^* is also unique. Inserting k^* in (A.25) yields an unique value for c^* as follows:

$$c^* = \frac{\Xi}{\alpha} k^* - b_0 n^* \tag{A.28}$$

$$= b_0(n^*)^\epsilon \frac{\rho\Xi + \beta n^*(bn^* + (n^* - d) + \rho) + (1+b)n^*\alpha\rho}{\nu\rho\Xi - \beta(n^*)^\epsilon(bn^* + (n^* - d) + \rho) - (1+b)(n^*)^\epsilon \alpha\rho} \tag{A.29}$$

.

To validate the feasibility of c^* and k^*, we have to determine the parameter range where the denominator is positive, meaning

$$\nu\rho\Xi - \beta(n^*)^\epsilon(bn^* + (n^* - d) + \rho) - (1+b)(n^*)^\epsilon \alpha\rho > 0 \tag{A.30}$$

$$\Leftrightarrow \quad \nu\rho \frac{(1-\alpha)(1+b)\frac{x}{\beta} + (1-\alpha)bd + \rho}{\beta bd + (1+b)x + (\beta + (1+b)\alpha)\rho} > \left(d + \frac{x}{\beta}\right)^\epsilon \tag{A.31}$$

We can now use the resting equations (A.23), (A.21) and (A.20) to obtain values for S^*, e^* and N^*, respectively.

$$S^* = \bar{S} \frac{\beta\zeta(\rho + \delta + x)k^*}{\delta\sigma c^* + \beta\zeta(\rho + \delta + x)k^*} \tag{A.32}$$

$$\hat{e}^* = e^* \exp(\frac{x}{\beta} t) = \zeta^{\frac{1}{\beta}} (k^*)^{\frac{1-\alpha}{\beta}} \tag{A.33}$$

$$\hat{N}^* = N^* \exp(-\frac{x}{\beta} t) = \frac{\delta S^*}{\hat{e}^*} \tag{A.34}$$

Last, we need to check if the transversality condition holds

$$\lim_{t\to\infty}\exp(-\rho t)H^0 \overbrace{=}^{\dot{k}=\dot{S}=0} \lim_{t\to\infty}\exp(-\rho t)\left[u(c,n,S)+\frac{x}{\beta}\lambda^N N\right] \tag{A.35}$$

$$= \lim_{t\to\infty}\exp(-\rho t)\left[u(c,n,S)+\frac{x}{\beta}(u_c(k+bk+b_0)-u_n)\right] \tag{A.36}$$

$$= 0 \tag{A.37}$$

which is true, since all values in the brackets are constant.

To summarize, we obtained an optimal stationary state satisfying the transversality condition with the following values:

$$n^* = d+\frac{x}{\beta} \tag{A.38}$$

$$k^* = \frac{\alpha\rho b_0(\nu n^*+(n^*)^\epsilon)}{\nu\rho\Xi-\beta(n^*)^\epsilon(bn^*+(n^*-d)+\rho)-(1+b)(n^*)^\epsilon\alpha\rho} \tag{A.39}$$

$$c^* = \frac{\Xi}{\alpha}k^*-b_0 n^* \tag{A.40}$$

$$S^* = \bar{S}\frac{\beta\zeta(\rho+\delta+x)k^*}{\delta\sigma c^*+\beta\zeta(\rho+\delta+x)k^*} \tag{A.41}$$

$$\hat{e}^* = e^*\exp(\frac{x}{\beta}t)=\zeta^{\frac{1}{\beta}}(k^*)^{\frac{1-\alpha}{\beta}} \tag{A.42}$$

$$\hat{N}^* = N^*\exp(-\frac{x}{\beta}t)=\frac{\delta S^*}{\hat{e}^*} \tag{A.43}$$

Appendix B

Mortality model - optimality conditions

We need to proof that the Hamiltonian is concave to assure sufficiency of the necessary conditions. This implies that the obtained steady state is a maximum.

The optimized Hamiltonian H^0 is

$$\begin{aligned} H^0 =& \log(c^*) + \nu \log(n^*) && \text{(B.1)} \\ &+ \lambda^k \left(k^\alpha - c^* + k\left(-(b+1)n^* - d_i + 1 +\right) - b_0 n^*\right) && \text{(B.2)} \\ &+ \lambda^s \left(\phi k^\kappa - \delta s - s(n^* - d_i)\right) && \text{(B.3)} \\ \text{with } n^* =& \frac{\nu}{\lambda^k(bk + b_0 + k) + \lambda^s s} && \text{(B.4)} \\ c^* =& \frac{1}{\lambda^k} && \text{(B.5)} \end{aligned}$$

and taking the derivative with respect to the states twice yields

$$\mathcal{H}_{mort} = \begin{pmatrix} \frac{d^2 H^0}{dk^2} & \frac{d^2 H^0}{dkds} \\ \frac{d^2 H^0}{dsdk} & \frac{d^2 H^0}{ds^2} \end{pmatrix} \tag{B.6}$$

where for the given values of the steady state and the parameter set, we obtain optimality for all three steady states with the different mortality specifications. The calculation of the Hessian and of the principal minors has been done numerically and can be seen in table B.1.

mortality function	$det(\mathcal{H}_1)$	$det(\mathcal{H}_2)$
$d(k)$	-1.001	0.419
$d(s)$	-1.349	0.282
$d(k, s)$	-0.121	0.055

Table B.1: Principal minors of the Hessian $\mathcal{H}_{mort}$

Appendix C

SLD - optimality conditions

The optimized Hamiltonian in the three state-model is

$$
\begin{aligned}
H^0 =& \lambda^k \left(-\left(k(bn^* - d + n^*) + b_0 n^* + c - \sqrt{k}\right)\right) + \ln(c^*) + \lambda^N N(n^* - d) \\
&+ \lambda^S \left(kN + S\left(\frac{S}{S^2+1} - \delta\right)\right) + v\ln(n^*) + \sigma \ln(\bar{S} - S) \\
\text{with } n^* =& \frac{v}{\lambda^k(bk + b_0 + k) - \lambda^N N} \\
c^* =& \frac{1}{\lambda^k}
\end{aligned}
\tag{C.1}
$$

which needs to be concave in the steady state to make the necessary conditions (dynamics and FOCs) sufficient. In the case of convexity the steady state is a minimum. We use the Hessian Matrix

$$
\mathcal{H}_{SLD} = \begin{pmatrix} \frac{d^2H^0}{dN^2} & \frac{d^2H^0}{dNdk} & 0 \\ \frac{d^2H^0}{dkdN} & \frac{d^2H^0}{dk^2} & 0 \\ 0 & 0 & \frac{d^2H^0}{dS^2} \end{pmatrix}
\tag{C.2}
$$

as in the base model. If the Hamiltonian is concave, convex or indefinite in the points of interest can be seen in the corresponding plot 4.13.

The values for $\delta = 0.51$ are shown in table C.1 as example.

Branch	S^*	$det(\mathcal{H}_1)$	$det(\mathcal{H}_2)$	$det(\mathcal{H}_3)$
lower	0.347	$-7.01 \cdot 10^{-6}$	111.174	-1000.1
middle	0.901	0.002	-4102.6	31143.4
upper	0.912	-0.001	4420.04	71179

Table C.1: Steady State values and the principal minors of the Hessian $\mathcal{H}_{SLD}$