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DISSERTATION

Adapted Dependence with Applications to Financial and Actuarial Risk Management

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Kurzfassung der Dissertation

Diese Dissertation beschäftigt sich mit Finanz- und Versicherungsprodukten, bei denen die Auszahlung, welche durch einen stochastischen Prozess bestimmt wird, zu einem zufälligen Zeitpunkt stattfindet. Dieser Zeitpunkt wird durch eine Stoppzeit modelliert und soll innerhalb eines vorher bestimmten Zeitintervalls liegen. Das Besondere an dieser Stoppzeit ist, dass sie einer gegebenen Verteilung folgen soll und durchaus vom Prozess der Auszahlungen abhängig sein darf. Es wird dabei das Supremum über die zu erwartenden Auszahlungen betrachtet, um eine Worst-Case-Abschätzung zu erhalten. Interessant ist hierbei auch das Auffinden einer optimalen Stoppzeit, die diesen Höchstwert liefert. Eine Erweiterung des Problems ergibt sich durch die Verwendung von adaptierten zufälligen Wahrscheinlichkeitsmaßen anstelle der Stoppzeiten, die eine prozentuale Entnahme modellieren.

Damit das betrachtete Problem wohldefiniert ist, bedarf es einiger Annahmen an den stochastischen Prozess, der die Auszahlung modelliert. Die drei wichtigsten Annahmen dieser Arbeit sind, dass das Supremum der Beträge der Elemente des Prozesses fast sicher endlich ist, einen endlichen Erwartungswert hat und dass der Prozess gleichgradig integrierbar ist.

Für das beschriebene Problem gibt es zwei einfache Schranken. Einerseits erhält man eine untere Schranke, wenn man annimmt, dass der stochastische Prozess der Auszahlungen unabhängig von der Stoppzeit bzw. dem adaptierten zufälligen Wahrscheinlichkeitsmaß ist. Andererseits gibt es eine obere Schranke durch den Wert eines optimalen Stopp-Problems mit demselben zugrunde liegenden Auszahlungsprozess. Der Wert eines solchen Problems wird nämlich durch eine Stoppzeit bestimmt, für die allerdings keine Verteilungsannahme getroffen wird. Im Fall endlich vieler Perioden lässt sich der Wert eines solchen optimalen Stopp-Problems sowie die optimale Ausübungsstrategie leicht durch Verwendung der Snell-Einhüllenden finden.

Neben diesen beiden einfachen Schranken gibt es noch eine Vielzahl anderer Schranken, die in dieser Dissertation hergeleitet werden. Einige davon sind allgemein für das Problem gültig, während andere von der Struktur des zugrunde liegenden Prozesses abhängig sind.

Ein wichtiger Punkt bei der Betrachtung dieses Problems ist die Frage nach einer optimalen Strategie. In dieser Arbeit wird die Existenz einer optimalen Strategie in diskreter Zeit bewiesen. Diese Strategie ist nicht immer eindeutig, wie anhand von Beispielen gezeigt wird.

Für bestimmte Klassen von stochastischen Prozessen ist es möglich, eine optimale Strategie, sowie den daraus resultierenden Wert zu bestimmen. Dazu gehören unter anderem unabhängige Prozesse, Prozesse mit unkorrelierten Zuwächsen und Prozesse, bei denen der vorhersehbare Prozess in der Doob-Zerlegung unkorrelierte Zuwächse hat.

Um das Problem auch in stetiger Zeit betrachten zu können, müssen die adaptierten zufälligen Wahrscheinlichkeitsmaße durch einen stochastischen Übergangskern ersetzt werden. Bei der Betrachtung des Problems unter Verwendung der Stoppzeit bietet sich die

Dynkin-Formel für das Finden einer Lösung an. Weiters wird eine diskrete Approximation vorgestellt, mit deren Hilfe sich Ergebnisse, die in diskreter Zeit gefunden wurden, auch in stetige Zeit überführen lassen.

Der letzte Teil der Dissertation ist den verschiedenen Anwendungsgebieten dieser Problemstellung im Bereich der Finanz- und Versicherungsmathematik gewidmet. Ein interessantes Anwendungsgebiet sind fondsgebundene Lebensversicherungen. Für diese bietet sich durch die Modellierung des Vertrags ohne Annahme der Unabhängigkeit von biometrischen und Finanzmarktrisiken die Möglichkeit, eine Abschätzung für den Fall der ungünstigsten Abhängigkeit dieser Risiken zu erhalten. Diese könnte in die Berechnung des Solvenzkapitals einfließen. In den technischen Spezifikationen zu LTGA oder QIS5 für Solvency II finden sich bereits Hinweise auf die Berücksichtigung möglicher Abhängigkeiten bei der Berechnung des Solvenzkapitals. In diesen technischen Spezifikationen wird eine Korrelationsmatrix angegeben, die die lineare Korrelation zwischen verschiedenen Risikoarten beschreibt. Weitere Anwendungsmöglichkeiten sind unter anderem eine stochastische Modellierung in der Krankenversicherungsmathematik, die Liquidierung eines Investmentportfolios oder die Bewertung von Swing Optionen.

Abstract

This thesis is about financial and insurance products with pay-outs that are determined by the value of a stochastic process at a random point in time. This time point is modeled by a stopping time taking values in a predetermined time interval. What makes this stopping time special is the assumption that it follows a given distribution and that it may even depend on the stochastic process modeling the pay-out. The supremum over the expected pay-out is considered in order to obtain a worst-case estimate. It is also of special interest to find an optimal stopping time that yields this maximal value. An extension of the problem is given by the use of adapted random probability measures, which model a withdrawal measured in percentage, instead of the stopping times.

In order to have a well-defined problem, we need some assumptions on the stochastic process that models the pay-out. The three main assumptions in this thesis are that the supremum of the absolute values of the elements of the process is almost surely finite, that it has finite expectation, and that the process is uniformly integrable.

The presented problem has two simple bounds. On the one hand, a lower bound is found if one assumes that the stochastic process of pay-outs is independent of the stopping time or of the adapted random probability measure, respectively. On the other, there is an upper bound given by the value of an optimal stopping problem with the same underlying process. The value of such a problem is in fact determined by a stopping time, for which there is no assumption about its distribution. In the case of finitely many periods the value of such an optimal stopping problem and its optimal strategy can easily be found using the Snell envelope.

In addition to these two simple bounds, a number of other bounds exist that are derived in this dissertation. Some of them are valid for the problem in general, while others depend on the structure of the underlying process.

An important aspect in considering this problem is the question of an optimal strategy. In this thesis, the existence of an optimal strategy in a discrete-time setting is proven. This strategy is not unique in general as explained with the use of some examples.

It is possible to find both an optimal strategy and the resulting value for certain classes of stochastic processes. These include independent processes, processes with uncorrelated increments and processes for which the predictable process in the Doob decomposition has uncorrelated increments.

In order to consider the problem in continuous time, it is necessary to replace the adapted random probability measures with a stochastic transition kernel. When considering the problem using stopping times, the Dynkin formula offers a tool to find a solution. Furthermore, a discrete approximation is presented that can help transfer the results found in discrete time to some in continuous time.

The final part of the thesis deals with various applications of the problem in the areas of financial and actuarial mathematics. Unit-linked life insurances are an interesting area

of application. By modeling the contract without the assumption of independence between biometric and financial risks, it is possible to compute an estimate for the case of the most disadvantageous dependence between these risks. This could contribute to the computation of the solvency capital requirement. Signs for a possible dependence in computing the solvency capital can already be found in the technical specifications of LTGA or QIS5 for Solvency II. A correlation matrix is presented in these technical specifications that describes the linear correlation between different types of risk. Other possible applications include a stochastic model for health insurances, the liquidation of an investment portfolio or the valuation of swing options.

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Contents

Chapter 1

Introduction

There are many situations in financial and actuarial mathematics where independence is assumed for two stochastic components. It is questionable whether this assumption is always correct. This thesis presents a general framework to show how such situations can be handled without the assumption of independence.

For unit-linked life insurances it is normally assumed that financial and biometric risks are independent, see e.g. [28]. If surrender of the contract is allowed, this reason for dropping out, which also leads to a pay-out, should not be set independent of the financial market. It is possible that a downturn in the economy, which is often followed by high unemployment rates, leads to more lapses for an insurance company. It is equally conceivable that a flu epidemic could influence the financial markets. In the technical specifications of the long-term guarantees assessment (LTGA, [35]) or the fifth quantitative impact study (QIS5, [47]) for Solvency II there are assumptions about a positive correlation between financial and biometric risks used to compute the solvency capital requirement. Similar ideas are followed in current research. In [9] worst-case scenarios for pricing and reserving life insurance products are considered where a mutual dependence between interest rates and mortality is allowed. In [34] a valuation framework is presented with a given correlation between the dynamics of mortality and interest rates. Further upper and lower bounds for the value of a guaranteed annuity option are found using comonotonicity theory.

It is also possible to use this setting for health insurance contracts. These are often modeled in a similar way to life insurance contracts. The pay-outs for these contracts, called claims amount per risk in this setting, is normally a deterministic number, corresponding to the value the insurer expects to pay, and based on historical data. Using the setting of this thesis, such claims amount per risk can be modeled stochastically. This is more appropriate, since it is influenced by many factors, such as modern techniques in health care, the status of the corresponding country (social turmoil, peace or war, ...) and political decisions. These factors also influence the probability of occurrence of an insured event. Improvements in the medical system will guarantee that people are cured more rapidly and that the probability of a relapse declines.

The dependence between severe medical diseases and crises or catastrophes in the surroundings of the patients is a matter of paramount interest to medical research. One especially interesting work with regard to this thesis is about the impact of the socioeconomic crisis in Greece on acute myocardial infarction [36]. In [36] the authors find that the financial crisis may have led to a higher incidence of acute myocardial infarction in the population of Messinia and assert the need for an analysis of this phenomenon for the entire Greek population. In [29] and [42] the aftermath of the earthquake in Japan in March 2011 on coronary syndromes is analyzed. Both studies seem to demonstrate that disaster stress has increased the number of hospitalized patients. Similarly, an alteration in the pattern of acute myocardial infarction onset followed in the wake of hurricane Katrina in New Orleans. This is discussed in [45].

The setting presented in this thesis can also be used to model the liquidation of an investment portfolio. Here, our setting offers the possibility of liquidating the portfolio step by step throughout a given time interval. We want to maximize the expected amount gained by liquidating the portfolio. In this situation the amount liquidated depends on the current prices on the market. Further applications will be discussed at the end of this thesis, where some applications will also be illustrated by examples.

From a mathematical point of view, the problem considered is a special kind of an optimal stopping problem. The difference to standard optimal stopping problems is our assumption that we are given a probability distribution for the stopping times considered. We start by taking a look at the problem in discrete time. Thus we are interested in the value

$$\sup_{\substack{\tau\colon \Omega\to I \\ \tau \text{ stopping time} \\ \mathcal{L}(\tau)=\nu}} \mathbb{E}[Z_\tau] \, ,$$

where I is a predefined discrete time interval, Z is the underlying process modeling the pay-out and ν is the given distribution of the stopping times τ considered. The problem is extended further to the use of adapted random probability measures instead of stopping times. These are stochastic processes $\gamma = \{\gamma_t\}_{t\in I}$ with $\sum_{t\in I}\gamma_t \overset{\text{a.s.}}{=} 1$, $\gamma_t \geq 0$ a.s. and γ_t is $\mathcal{F}_t$-measurable for all $t \in I$. Similar to the stopping times we assume that we know something about the distribution ν of the process γ, which means that we assume $\mathbb{E}[\gamma_t] = \nu_t$ for all $t \in I$. We denote the set of all adapted random probability measures with these properties by $\mathcal{M}_I^\nu$. Using these adapted random probability measures we are interested in

$$\sup_{\gamma\in\mathcal{M}_I^\nu} \mathbb{E}\left[\sum_{t\in I} Z_t\gamma_t\right] .$$

In the study of these two problems we make no assumption about independence between the underlying process and the stopping time or the adapted random probability measure, respectively. We therefore have an adapted dependence between Z and τ or Z and γ.

We see that by considering stopping times we only allow one pay-out, which occurs at some random time defined by the stopping time. When using adapted random probability measures, several withdrawals within the time interval I are possible. We can interpret them as a stochastic component telling us the percentage of our portfolio that will have a pay-out.

One can easily find a lower bound for the problems by assuming independence between these stochastic components. In this case a lower bound is then given by

$$\sum_{t\in I} \mathbb{E}[Z_t]\,\nu_t \, ,$$

where $\nu_t = \mathbb{P}(\tau = t)$ or $\nu_t = \mathbb{E}[\gamma_t]$ for each $t \in I$. By dropping the assumption of a given distribution, an upper bound can be found. In the case of stopping times this upper bound can be computed using the Snell envelope, as the problem is then transformed to a standard optimal stopping problem with the same underlying stochastic process.

Thinking about these two problems raises a number of questions. Does an optimal strategy exist that yields the extremal value? If so, under which conditions is it unique? What does it look like for a given adapted stochastic process Z? If we are given an adapted process Z, for which we cannot find an optimal strategy and therefore do not know the exact value of the supremum, can we find bounds for the value other than the two presented above?

The answer to the first question is one of the main results of the thesis. It is proven that an optimal adapted random probability measure always exists if the underlying stochastic process satisfies the necessary assumptions. Unfortunately this strategy is not unique in general. There are types of processes, e.g. martingales or processes with uncorrelated increments, for which every stopping time or every adapted random probability measure yields the same value. For a martingale we can show that the extremal value in our problems equals the expected value of the martingale at the first time point of the interval I. For stopping times this is due to Doob's optional stopping theorem. For an adapted random probability measure this result can be shown using a theorem presented in this thesis, which links adapted random probability measures and stopping times.

As stated, we can compute the extremal value for martingales and processes with uncorrelated increments, as they have the same value for every stopping time or every adapted random probability measure such that the necessary assumptions are satisfied. But there are also other types of processes for which the extremal value can be computed; for example, independent processes or processes, for which the predictable process in the Doob decomposition has uncorrelated increments. We can also give a result about a sufficient condition for the optimal strategy for processes, which can be represented as the product of a deterministic function and a martingale. Moreover, it is possible to write down the optimal strategy for a binomial model if the distribution ν is given in a certain way. In the one-period model, the extremal value can even be computed for general processes Z. This is possible using the Doob decomposition.

For those processes, for which we cannot compute the extremal value or find an optimal strategy, we can take a look at bounds for the problems. There are some bounds that are valid for general processes Z and some that assume that the process Z is of special type. An easy upper bound is given by

$$\mathbb{E}\left[\sup_{t\in I} |Z_t|\right] ,$$

but there are also other upper bounds represented in this thesis that are valid for general processes Z. For special types of processes it is sometimes possible to find another bound, which is closer to the real value than the generally valid ones. For instance, this is possible for processes, which can be represented as the product of a deterministic function and a martingale. It is also possible to find special bounds for sub- and supermartingales. If for example $I = \{0, 1, \dots, T\}$ for some $T \in \mathbb{N}$, then these are given by $\mathbb{E}[Z_T]$ and $\mathbb{E}[Z_0]$, respectively.

Some modifications become necessary when the problem is considered in continuous time. The problem using stopping times can still be formulated as though we were using discrete time. The adapted random probability measures have to be replaced by a stochastic transition kernel in continuous time. Similar as in discrete time, $\mathcal{M}_I^\nu$ denotes the set of all adapted stochastic transition kernels Γ with marginal ν. The problem is then formulated as

$$\sup_{\Gamma \in \mathcal{M}_I^\nu} \mathbb{E}\left[\int_I Z_t \, \Gamma(dt)\right].$$

As in discrete time, every stopping time and every stochastic transition kernel yields the same value for martingales and processes with uncorrelated increments such that the necessary assumptions are satisfied. The result for processes with uncorrelated increments can be transferred from discrete time to continuous time by a discrete approximation, which is presented in this thesis.

Similar as in discrete time, we are also interested in computing the extremal value or at least some bounds. The main difference is that in continuous time we need to be more careful about the assumptions we make about the process Z. When using stopping times we can now try to use the Dynkin formula, in order to compute this extremal value for some special types of processes Z.

This thesis is structured in the following way.

Part I concentrates on the problem in discrete time. In Chapter 2 we introduce the notation and the problem. The necessary assumptions that have to be made to guarantee that the problem is well-posed are then discussed. Some first general results, useful for the further discussion of the problem, are also presented in this chapter. Chapter 3 answers the question of the existence of an optimal strategy. In proving the existence of an optimal strategy it is shown that the vector space of admissible stochastic processes Z is a Banach space. In Chapter 4 we concentrate on upper and lower bounds which can help us to estimate the value. The chapter starts by looking at the risk measures expected shortfall and conditional expected shortfall and their properties, which will be very useful in the other sections of this chapter and later on. We then show some general bounds valid for all types of processes. Later on we concentrate on special types of stochastic processes, e.g. sub- and supermartingales. Not all of the bounds presented take into account the given distribution of the stopping time or the adapted random probability measure. The effort in computing the bounds for a given underlying process also differs. Some of the bounds are therefore compared in examples at the end of the chapter. In Chapter 5 we take a look at some special classes of processes for which explicit solutions can be found, including an optimal strategy. These special classes include for example independent processes or processes with uncorrelated increments. Chapter 6 deals with the problem of pricing claims under an equivalent martingale measure and a recursive formula for deriving the extremal value. Risk neutral pricing is an important topic in mathematical finance and should therefore be mentioned in this thesis. Nevertheless, most of the problem of risk-neutral pricing is left for further research. We also present a way to compute the value by using a recursive formula although, unfortunately, this is only possible for independent processes. The idea of searching for a recursive formula comes from the Snell envelope.

Part II concentrates on the problem in continuous time and gives some first results for extending the problem to continuous time. Again we first introduce the notation and take a look at general results, which are useful for further computations, in Chapter 7. Chapter 8 is devoted to bounds for the problem and results for special types of processes. The use of the Dynkin formula for the stopping problem in continuous time is discussed in Section 8.1. There are some special types of process, for which, using the Dynkin formula, it can be proven that every strategy yields the same value. This is due to the form of the generator of these processes. In continuous time we also take a look at the use of utility functions within the given framework by means of an illustrative example. This is done in Section 8.2. For right-continuous processes a discrete approximation can be found, which is introduced in Section 8.3. This discrete approximation is useful for transforming results found in discrete time to results in continuous time. Again we take a look at special classes of processes for which explicit results can be found (see Section 8.4) and bounds for the problem (see Section 8.5).

In Part III we finally show different applications for the problem posed. Chapter 9 concentrates on applications in actuarial mathematics. As already mentioned, unit-linked life insurance products and health insurance contracts could be modeled using the presented model, allowing for a stochastic computation of the value of a contract without the need to assume independence between different stochastic components. This might not be necessary for the computation of the insurance premium but it might be worth considering when computing the solvency capital requirement for an insurance company. Chapter 10 discusses applications in risk management. It is shown that risk measures for stochastic processes are modeled in a way that is quite similar to our problem. Further, credit risk modeling is a field, where our results could be used. Finally in Chapter 11, applications in mathematical finance are introduced, which include the liquidation of an investment portfolio or the pricing of swing options that are often used in electricity markets.

Part I

Adapted Dependence in Discrete Time

Chapter 2

The Problem

Here we introduce the problem that is analyzed in this thesis. First the notation and the necessary assumptions are introduced in Section 2.1. We will see that the problem is a modified version of an optimal stopping problem. The difference to classical optimal stopping problems is our assumption that we have some information about the distribution of the stopping times considered. Furthermore, the problem is extended to the use of adapted random probability measures instead of stopping times. First results for the problem are shown in Section 2.2. These are general results, which give a better understanding of the problem and are useful, later on, in the computation of bounds and optimal strategies. In Section 2.3 we will take a look at some possible assumptions on the process Z which help to understand why stricter assumptions such as the ones noted in Assumption 2.2 are needed from time to time.

2.1 Notation

Let I be a countable, i.e. a finite or countably infinite, totally-ordered index set; for simplicity we assume that $I \subset \mathbb{R} \cup \{-\infty, \infty\}$. Typical examples are $I = \mathbb{N}_0$, $I = \mathbb{Z}$, $I = \mathbb{Q}$, $I = \mathbb{N}_0 \cup \{\infty\}$ or $I = \{0, 1, \ldots, T\}$ for some $T \in \mathbb{N}$. Let t be a time parameter in I.

Definition 2.1. (a) For $t \in I$ we define the set $I_{<t} := \{s \in I \,|\, s < t\}$ of all times before t, the set $I_{\leq t} := \{s \in I \,|\, s \leq t\}$ of all times up to t, the set $I_{\geq t} := \{s \in I \,|\, s \geq t\}$ of all times from t on, and the set $I_{>t} := \{s \in I \,|\, s > t\}$ of all times after t.

(b) For a probability measure $\nu = \{\nu_t\}_{t \in I}$ on I we define $\nu_{<t} := \sum_{s \in I_{<t}} \nu_s$ and $\nu_{>t} := \sum_{s \in I_{>t}} \nu_s$, as well as $\nu_{\leq t} := \nu_{<t} + \nu_t$ and $\nu_{\geq t} := \nu_{>t} + \nu_t$. Relations like $\nu_{\leq t} + \nu_{>t} = 1$ will be used without mentioning it.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in I}$. Let $Z = \{Z_t\}_{t \in I}$ be a real-valued process of discounted pay-offs, which is adapted to the filtration $\mathbb{F}$.

In this thesis we will sometimes decompose the process Z into its positive and negative part. We will write $Z_t^+ := \max\{Z_t, 0\}$ and $Z_t^- := \max\{-Z_t, 0\}$ for every $t \in I$. Then $Z_t = Z_t^+ - Z_t^-$ for every $t \in I$.

We will first make some assumptions about the process Z to ensure that the problem is well-posed. At the end of this chapter, we will take a look at some weaker assumptions and see why they cannot be used in this setting.

Assumption 2.2. *When we work on a finite-time setting it is enough to assume that the adapted process Z is in $L^1(\mathbb{P})$. Within this thesis we will therefore assume $Z_t \in L^1(\mathbb{P})$ for all $t \in I$, i.e. $\mathbb{E}[|Z_t|] < \infty$ for all $t \in I$, except stated otherwise, e.g. assuming $Z_t^+ \in L^1(\mathbb{P})$ or $Z_t^- \in L^1(\mathbb{P})$ for all $t \in I$. In order to be able to define all the given problems for a countably infinite index set I we also need one of the following assumptions*

(a) $\mathbb{P}(\sup_{t\in I} |Z_t| < \infty) = 1$,

(b) $\mathbb{E}[\sup_{t\in I} |Z_t|] < \infty$,

(c) the process Z is uniformly integrable.

Remarks 2.3. (i) It is clear that (b) implies (a).

(ii) It follows from [62, 13.3(b), p. 127/128] that (b) implies (c).

(iii) Next we want to show that if Z is a martingale, (c) implies (a). If Z is a uniformly integrable martingale, then it is bounded in $L^1(\mathbb{P})$ (see [62, 13.2]). This means that the limit $Z_{\sup(I)}$ exists and is finite a.s. (see [62, Theorem 11.5]). Therefore $\mathbb{P}(\sup_{t\in I} |Z_t| < \infty) = 1$.

(iv) (a) does not imply (b) or (c). Consider a process Z with $Z_t = X$ for all $t \in I$, where X is a non-integrable random variable. Take for example X satisfying $\mathbb{P}(X = 2^k) = 2^{-k}$ for all $k \in \mathbb{N}$. Then
$$\sup_{t\in I} |Z_t| = |X| \quad \text{and} \quad \mathbb{E}[|X|] = \infty \,.$$
In particular Z is not uniformly integrable.

(v) The following example will show that (c) does not imply (a) or (b). Assume $I = \mathbb{N}_0$ and consider the probability space $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}_{[0,1]}, \lambda)$, where λ is the Lebesgue–Borel measure of the Borel σ-algebra $\mathcal{B}_{[0,1]}$ on the unit interval. For $n \in \mathbb{N}_0$ we can find unique $k \in \mathbb{N}_0$ and $j \in \{0, \dots, 2^k - 1\}$ such that $n = 2^k + j$. Using this representation we define the process Z for every $\omega \in [0,1]$ by
$$Z_n(\omega) = \begin{cases} k1_{[\frac{j}{2^k}, \frac{j+1}{2^k}]}(\omega) & \text{if } n \in \mathbb{N}\,, \\ 0 & \text{if } n = 0\,. \end{cases}$$
Then $\mathbb{E}[Z_n] = k2^{-k} \to 0$ as $n \to \infty$. Given $\varepsilon > 0$ there exists an $M \in \mathbb{N}$ such that $M2^{-M} \le \varepsilon$. For every $n \in \mathbb{N}_0$ we have
$$\mathbb{E}\big[Z_n 1_{\{|Z_n| \ge M\}}\big] = \begin{cases} 0 & \text{if } n \le 2^M - 1\,, \\ k2^{-k} & \text{otherwise.} \end{cases}$$
As $k2^{-k} \le M2^{-M}$ for $n > 2^M - 1$, we have $\mathbb{E}\big[Z_n 1_{\{|Z_n| \ge M\}}\big] \le M2^{-M} \le \varepsilon$. This implies that the process Z is uniformly integrable. But $\sup_{n\in\mathbb{N}} Z_n = \infty$ on $[0,1]$.

Definition 2.4. In the following let $\mathcal{T}_I$ denote the set of all stopping times $\tau\colon \Omega \to I$. Further, for a given probability distribution ν on I, let $\mathcal{T}_I^\nu$ be the set of all I-valued stopping times with distribution ν, i.e. $\mathcal{L}(\tau) = \nu$.

We now assume that Assumption 2.2(b) is satisfied, i.e. we assume $\mathbb{E}[\sup_{t\in I}|Z_t|] < \infty$, unless otherwise stated. This will guarantee that the values we are interested in exist and are finite. If we just want to make sure that the values exist, we can also assume that we are given an adapted process Z in $L^1(\mathbb{P})$ with $\mathbb{E}\big[\sup_{t\in I} Z_t^-\big] < \infty$ or $\mathbb{E}\big[\sup_{t\in I} Z_t^+\big] < \infty$.

When we consider an optimal stopping problem, its value, which we will denote by V, is given by

$$V := \sup_{\tau\in\mathcal{T}_I} \mathbb{E}[Z_\tau] \ . \tag{2.5}$$

For a non-negative process Z, this value coincides with the value of a standard American option without any hedging possibilities.

Remark 2.6. The pricing of American options or optimal stopping problems are well known problems in the literature. See for example [57] for optimal stopping policies for Markov processes. In [31, Chapter 2] and [14, Chapter 6] the pricing of American options using the Snell envelope in a finite-time setting is shown. In [41, Chapter VI-1] the Snell envelope is discussed for $I = \mathbb{N}_0$.

If the stopping time in the optimal stopping problem follows a given probability distribution ν, then this value may change. Let $V(\nu)$ denote this new value. Then for $\mathcal{T}_I^\nu \neq \varnothing$

$$V(\nu) := \sup_{\tau\in\mathcal{T}_I^\nu} \mathbb{E}[Z_\tau] \ . \tag{2.7}$$

In case $\mathcal{T}_I^\nu = \varnothing$, we set $V(\nu) = -\infty$. If a stopping time $\tau \in \mathcal{T}_I^\nu \neq \varnothing$ is independent of the adapted process Z, then it is easy to calculate $\mathbb{E}[Z_\tau]$, since using Corollary 2.39 below we get

$$V^{\mathrm{ind}}(\nu) := \mathbb{E}[Z_\tau] = \sum_{t\in I} \mathbb{E}\big[Z_t 1_{\{\tau=t\}}\big] = \sum_{t\in I} \mathbb{E}[Z_t]\,\nu_t\,, \tag{2.8}$$

with $\nu_t := \mathbb{P}(\tau = t)$ for all $t \in I$. If such an independent stopping time $\tau \in \mathcal{T}_I^\nu$ does not exist, then we set $V^{\mathrm{ind}}(\nu) = -\infty$.

Looking at these three values, we see that $V^{\mathrm{ind}}(\nu) \le V(\nu) \le V$. If $I \subset \mathbb{N}_0$ is a discrete interval with $0 \in I$, if the process Z is a uniformly integrable martingale and if $\mathcal{T}_I^\nu \neq \varnothing$, then $V(\nu) = V$, because of Doob's optional stopping theorem (given below) that states that $\mathbb{E}[Z_\tau] = \mathbb{E}[Z_0]$ for all stopping times τ. If further there exists a stopping time $\tau \in \mathcal{T}_I^\nu \neq \varnothing$, which is independent of the uniformly integrable martingale Z, then this stopping time proves $V^{\mathrm{ind}}(\nu) = V(\nu) = V$. Of course these equalities are also true for all martingales Z and stopping times τ that satisfy the necessary conditions for using Doob's optional stopping theorem. The different conditions on the martingale and the stopping time in Doob's optional stopping theorem are noted, for example, in [62, Theorem 10.10], [25] or [10], which we will cite here in order to see the conditions for later use.

Theorem 2.9. *[Doob's optional stopping theorem]*

(a) *Given $I \subset \mathbb{N}_0$ with $0 \in I$. Let τ be a stopping time and let Z be a supermartingale. Then Z_τ is integrable and $\mathbb{E}[Z_\tau] \le \mathbb{E}[Z_0]$ in each of the following situations:*

   1. *τ is bounded a.s., i.e. for some $N \in \mathbb{N}$ we have $\mathbb{P}(\tau \le N) = 1$,*
   2. *τ is finite a.s. and Z is bounded a.s., i.e. for some $K > 0$ we have $\mathbb{P}(|Z_t| \le K) = 1$ for all $t \in I$,*
   3. *$\mathbb{E}[\tau] < \infty$ and for some $K > 0$ we have $\mathbb{P}(|Z_t - Z_{t-1}| \le K) = 1$ for all $t \in I \setminus \{0\}$,*

   4. The process Z is uniformly integrable. In this case Z is closable, which means that Z_∞ exists and is well-defined. Therefore Z_τ is also well-defined if $\tau = \infty$.

(b) If $I \subset \mathbb{N}_0$ with $0 \in I$, any of the conditions 1.–4. holds and Z is a martingale, then $\mathbb{E}[Z_\tau] = \mathbb{E}[Z_0]$.

(c) If $I \subset \mathbb{Z}$ is a countably infinite index set, Z is a martingale and τ is a bounded stopping time, then Z_τ is integrable and $\mathbb{E}[Z_\tau] = \mathbb{E}[Z_t]$ for all $t \in I$.

Proof. For the proof of (a1), (a2) and (a3) we refer to [62] and note that the proof can be adapted to the above statements. If condition (a4) is satisfied the result follows from [56, Corollary VII.2.2 and p. 486]. Further Z_∞ exists and is well-defined by [62, Theorem 11.5], as the uniform integrability implies that Z is bounded in $L^1(\mathbb{P})$. (b) is also proven in [62] and [56]. (c) follows from [25, Theorem 6.12]. □

Remark 2.10. All the results stated can also be used if one is interested in the infimum instead of the supremum, since

$$\inf_{\tau \in \mathcal{T}_I^\nu} \mathbb{E}[Z_\tau] = -\sup_{\tau \in \mathcal{T}_I^\nu} \mathbb{E}[-Z_\tau] \,.$$

Remark 2.11. If we consider the time interval $I = \{0, \dots, T\}$ for some $T \in \mathbb{N}$ and $\nu_T = 0$, we can reduce I to some time interval $\{0, \dots, S\}$ with $S < T$ such that $\nu_S > 0$ and $\sum_{t=0}^{S} \nu_t = 1$.

If we have $\nu_t = 0$ for some $t \in I$, we can neglect the value Z_t in our problem, because the sets $\{\tau = t\}$ and $\{\gamma_t > 0\}$ become null sets and $1_{\{\tau = t\}} \overset{\text{a.s.}}{=} 0$ as well as $\gamma_t \overset{\text{a.s.}}{=} 0$.

Unless notational problems arise, we can restrict our attention to the smaller index set $\{t \in I \,|\, \nu_t > 0\}$.

It is possible to extend the introduced problem to adapted random probability measures with marginal ν instead of stopping times. There is only one pay-out at a random time point when stopping times are used. Using adapted random probability measures, we can model several withdrawals measured in percentage during the predefined time interval I.

Definition 2.12. Given a probability measure ν on I, we say that a real-valued process $\gamma = \{\gamma_t\}_{t \in I}$ is in $\mathcal{M}_I^\nu$, if

(a) $\gamma_t \geq 0$ a.s. for all $t \in I$,

(b) $\sum_{t \in I} \gamma_t \overset{\text{a.s.}}{=} 1$,

(c) $\{\gamma_t\}_{t \in I}$ is adapted,

(d) $\mathbb{E}[\gamma_t] = \nu_t$ for all $t \in I$.

Definition 2.13. When we consider a finite time interval I and a $\gamma \in \mathcal{M}_I^\nu$, we simply define

$$Z_\gamma := \sum_{t \in I} Z_t \gamma_t \,. \tag{2.14}$$

For a countably infinite index set I, $\gamma \in \mathcal{M}_I^\nu$ and an adapted processes Z with

$$\mathbb{P}\left(\left\{ \sup_{t \in I} Z_t^+ < \infty \right\} \cup \left\{ \sup_{t \in I} Z_t^- < \infty \right\} \right) = 1 \,,$$

we define, for an increasing sequence of finite intervals $(I_k)_{k\in\mathbb{N}}$ such that $\bigcup_{k\in\mathbb{N}} I_k = I$,

$$Z_\gamma := \lim_{k\to\infty} \sum_{t\in I_k} Z_t\gamma_t \quad \text{a.s.} \tag{2.15}$$

For $\gamma \in \mathcal{M}_I^\nu$ and $t \in I$ the notation $\gamma_{<t}$, $\gamma_{\le t}$, $\gamma_{\ge t}$ and $\gamma_{>t}$ is used analogously to Definition 2.1. Relations like $\gamma_{\le t} + \gamma_{>t} = 1$ a.s. will be used without further mention.

Remark 2.16. What could also be of interest is to assume $\sum_{t\in I}\gamma_t = x$ a.s. or $\gamma_t \in [0, y]$ a.s. for $t \in I$, $x, y \in [0,\infty)$. Some of the results, which are shown later, can be adjusted to such a problem. We will nevertheless concentrate on $x = y = 1$ in the following.

The value we now want to compute, assuming $\mathbb{E}\big[\sup_{t\in I} Z_t^+\big] < \infty$ or $\mathbb{E}\big[\sup_{t\in I} Z_t^-\big] < \infty$, is

$$V^+(\nu) := \sup_{\gamma\in\mathcal{M}_I^\nu} \mathbb{E}[Z_\gamma] = \sup_{\gamma\in\mathcal{M}_I^\nu} \mathbb{E}\left[\sum_{t\in I} Z_t\gamma_t\right]. \tag{2.17}$$

Remark 2.18. We have $V^+(\nu) \ge V(\nu)$, because $V(\nu)$ is a special case of $V^+(\nu)$, where a stopping time τ corresponds to an adapted random probability measure γ, given by $\gamma_t(\omega) = 1_{\{t\}}(\tau(\omega))$ for all $t \in I$, $\omega \in \Omega$.

If a stopping time τ is used for modeling a claim, then an adapted random probability measure can be used to model a portfolio of such claims. If a portfolio consists of $N \in \mathbb{N}$ claims modeled by stopping times $\tau_1, \dots, \tau_N$, then the whole portfolio can be modeled using the adapted random probability measure $\gamma = \{\gamma_t\}_{t\in I}$ given by $\gamma_t(\omega) = \sum_{i=1}^N w_i 1_{\{t\}}(\tau_i(\omega))$ for $t \in I$, $\omega \in \Omega$ with non-negative weights $w_1 + \dots + w_N = 1$. Note that $\tau_1, \dots, \tau_N$ in $\mathcal{T}_I^\nu$ implies $\gamma \in \mathcal{M}_I^\nu$.

Remark 2.19. We already saw that $V(\nu) \le V$. This implies existence of the value $V(\nu)$ whenever the corresponding optimal stopping problem is well-defined. For further information about the value V we refer to the corresponding literature about optimal stopping problems. The existence of the value $V^+(\nu)$ is guaranteed by Assumption 2.2(b), as the assumptions stated there make sure that the problem is well-posed.

Remark 2.20. For the computation of the value $V(\nu)$ we assume that the filtration in our model is chosen appropriately. Otherwise it could happen that $\mathcal{T}_I^\nu = \varnothing$, as shown in Example 2.21, since a set might not exist in $\mathcal{F}_t$ with probability ν_t for some $t \in I$. This is not necessary for the computation of $V^+(\nu)$ since at least one adapted random probability measure exists in $\mathcal{M}_I^\nu$, namely the one defined by $\gamma_t = \nu_t$ for all $t \in I$.

Example 2.21. Given a one-period model with $I = \{0, 1\}$, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \{0, 1\}$, $\mathcal{F} = \{\varnothing, \{0\}, \{1\}, \Omega\}$ and $\mathbb{P}(\omega) = \frac{1}{2}$ for all $\omega \in \Omega$. Assume that the process Z is such that $Z_0 = 0$ and $Z_1(\omega) = \omega$. Let the filtration be given by $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_1 = \mathcal{F}$. If the distribution ν is given by $\nu_0 = \frac{1}{2}$ and $\nu_1 = \frac{1}{2}$, then it is impossible to find a random time τ such that the set $\{\tau = 0\}$ is in $\mathcal{F}_0$ and has probability $\frac{1}{2}$. Thus the problem cannot be solved.

We already noted that $V(\nu) \le V^+(\nu)$ and that the inequality can be strict if $\mathcal{T}_I^\nu = \varnothing$. As we will see in the following example it is possible that $V(\nu) < V^+(\nu)$ even in the case $\mathcal{T}_I^\nu \ne \varnothing$.

Example 2.22. We take a look at a one-period model with $I = \{0, 1\}$ and $\Omega = \{\omega_0, \omega_1\}$. We assume that we are given a probability distribution ν on I with $\nu_0 \ne \nu_1$ and $\nu_1, \nu_2 \in (0, 1)$. Let $\mathcal{F}_0 = \mathcal{F}_1 = \mathcal{P}(\Omega) = \{\varnothing, \{\omega_0\}, \{\omega_1\}, \Omega\}$. Furthermore we assume $\mathbb{P}(\{\omega_i\}) = \nu_i$ for

$i \in \{0,1\}$. Now we know that the only stopping time $\tau \in \mathcal{T}_I^\nu$ with the given distribution ν is given by

$$\{\tau = 0\} = \{\omega_0\}\,, \quad \{\tau = 1\} = \{\omega_1\}\,.$$

An adapted random probability measure $\gamma \in \mathcal{M}_I^\nu$ different from the stopping time τ is given by $\gamma_i = \nu_i$ for $i \in \{0,1\}$. If the process Z is given by $Z_0(\omega_0) = 0$, $Z_1(\omega_0) = 1$, $Z_0(\omega_1) = 1$ and $Z_1(\omega_1) = 0$, then we have $V(\nu) = 0$, whereas

$$V^+(\nu) \geq \mathbb{E}[Z_0\gamma_0] + \mathbb{E}[Z_1\gamma_1] = \nu_0\nu_1 + \nu_1\nu_0 > 0\,.$$

Therefore we have $V(\nu) < V^+(\nu)$.

Note that we fixed only the marginal distribution of the real-valued process γ in the definition of $\mathcal{M}_I^\nu$. As it might also be interesting to give a statement about the conditional distribution of the process, we will give a similar definition for a subset of $\mathcal{M}_I^\nu$, which takes these conditional distributions into account.

In the context of portfolio liquidation, the definition below means that for s, t in I with $s \leq t$ the expected part liquidated at time t given the information up to s depends linearly on the fraction $1 - \gamma_{\leq s}$ not liquidated up to s.

Definition 2.23. Given a probability measure ν on I, we say that a real-valued process $\gamma = \{\gamma_t\}_{t\in I}$ is in $\mathcal{N}_I^\nu$, if $\gamma \in \mathcal{M}_I^\nu$ and further for all $s < t$ in I

$$\mathbb{E}[\gamma_t \,|\, \mathcal{F}_s] \overset{\text{a.s.}}{=} \begin{cases} \frac{\nu_t}{1-\nu_{\leq s}}(1-\gamma_{\leq s}) & \text{if } \nu_{\leq s} < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.24}$$

The value we now want to compute, assuming $\mathbb{E}\big[\sup_{t\in I} Z_t^+\big] < \infty$ or $\mathbb{E}\big[\sup_{t\in I} Z_t^-\big] < \infty$, is

$$V'(\nu) := \sup_{\gamma\in\mathcal{N}_I^\nu} \mathbb{E}[Z_\gamma] = \sup_{\gamma\in\mathcal{N}_I^\nu} \mathbb{E}\Bigg[\sum_{t\in I} Z_t\gamma_t\Bigg]\,. \tag{2.25}$$

Remark 2.26. Obviously $\mathcal{N}_I^\nu \subset \mathcal{M}_I^\nu$, which implies $V'(\nu) \leq V^+(\nu)$. Previously, we also noted that $\mathcal{T}_I^\nu \subset \mathcal{M}_I^\nu$ and therefore $V(\nu) \leq V^+(\nu)$. In the following example we will show that we do not have $\mathcal{T}_I^\nu \subset \mathcal{N}_I^\nu$ in general.

Remark 2.27. Note that $\mathcal{N}_I^\nu \neq \varnothing$, as there exists at least one $\gamma \in \mathcal{N}_I^\nu$, namely the one defined by $\gamma_t = \nu_t$ for all $t \in I$.

Example 2.28. For a time interval $I = \{0,\dots,4\}$ let the process $Z = \{Z_t\}_{t\in I}$ be given by a simple random walk starting in 0, i.e. we assume there exists a process of symmetric independent random variables $X = \{X_t\}_{t\in I\setminus\{0\}}$, which are $\{-1,1\}$-valued, and that $Z_t = \sum_{s=1}^t X_s$ for $t \in I$. Let the filtration be given by $\mathcal{F}_t = \sigma(Z_0,\dots,Z_t)$ for $t \in I$. Consider a stopping time τ given by

$$\tau(\omega) := \min\{t \in I \,|\, Z_t(\omega) = 1\} \wedge 4\,, \quad \omega \in \Omega\,.$$

Then the distribution ν of the stopping time τ is given by

$$\nu_0 = 0\,, \quad \nu_1 = \frac{1}{2}\,, \quad \nu_2 = 0\,, \quad \nu_3 = \frac{1}{8}\,, \quad \nu_4 = \frac{3}{8}\,.$$

Then

$$\mathbb{P}(\tau = 3 \,|\, \mathcal{F}_2) = \frac{1}{2} 1_{\{Z_2=0\}} 1_{\{\tau>2\}}\,, \tag{2.29}$$

whereas the right side of (2.24) would be given by

$$\frac{\nu_3}{1-\nu_{\le 2}} 1_{\{\tau>2\}} = \frac{1}{4} 1_{\{\tau>2\}} \,. \tag{2.30}$$

This implies that (2.24) cannot be satisfied for the process $\{1_{\{t\}}(\tau)\}_{t\in I}$ as (2.29) and (2.30) differ on the event $\{Z_1=-1, Z_2=-2\}$, which has probability $\frac{1}{4}$.

Lemma 2.31. *Consider a discrete interval $I \subset \mathbb{Z}$ and $\gamma \in \mathcal{M}_I^\nu$. Then $\gamma \in \mathcal{N}_I^\nu$ if and only if for all $t \in I$ with $t+1 \in I$*

$$\mathbb{E}[\gamma_{t+1} \,|\, \mathcal{F}_t] \overset{a.s.}{=} \begin{cases} \frac{\nu_{t+1}}{1-\nu_{\le t}}(1-\gamma_{\le t}) & \text{if } \nu_{\le t} < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.32}$$

Proof. It is clear that (2.24) implies (2.32). By induction we can also prove the converse.

Fix some $s \in I$. By (2.32) it is clear that (2.24) is true for $s+1$. We assume that (2.24) is satisfied for all $s < t \in I$ with $t - s \le n$ for some $n \in \mathbb{N}$ and prove that it is also satisfied for $s < t \in I$ with $t - s \le n+1$.

First consider the case $\nu_{\le s+1} < 1$. By the tower property of the conditional expectation ([62, 9.7(i)]), (2.32) and as (2.24) is valid for all $s < t \in I$ with $t - s \le n$, we get

$$\begin{aligned} \mathbb{E}[\gamma_t \,|\, \mathcal{F}_s] &\overset{\text{a.s.}}{=} \mathbb{E}[\mathbb{E}[\gamma_t \,|\, \mathcal{F}_{s+1}] \,|\, \mathcal{F}_s] \overset{\text{a.s.}}{=} \mathbb{E}\left[\frac{\nu_t}{1-\nu_{\le s+1}}(1-\gamma_{\le s+1}) \,\middle|\, \mathcal{F}_s\right] \\ &\overset{\text{a.s.}}{=} \frac{\nu_t}{1-\nu_{\le s+1}}(1-\gamma_{\le s} - \mathbb{E}[\gamma_{s+1} \,|\, \mathcal{F}_s]) \overset{\text{a.s.}}{=} \frac{\nu_t}{1-\nu_{\le s+1}}\left(1-\gamma_{\le s} - \frac{\nu_{s+1}}{1-\nu_{\le s}}(1-\gamma_{\le s})\right) \\ &\overset{\text{a.s.}}{=} \frac{\nu_t}{1-\nu_{\le s}}(1-\gamma_{\le s}) \end{aligned}$$

If $\nu_{\le s} < 1$, but $\nu_{\le s+1} = 1$, we have to be more careful. Then $\nu_v = 0$ for all $v \in I$ with $v > s$, which implies $\gamma_v = 0$ a.s. for all $v \in I$ with $v > s$. This implies that the right-hand sides of (2.32) and (2.24) would equal 0 for t as $t > s$.

If $\nu_{\le s} = 1$, then $\nu_{\le v} = 1$ for all $v \in I$ with $v \ge s$. Therefore the right-hand sides of (2.32) and (2.24) would equal 0. ☐

If we also want to use adapted random probability measures for an American type problem, we have to drop condition (d) of Definition 2.12.

Definition 2.33. For a real-valued process $\gamma = \{\gamma_t\}_{t\in I}$, we say $\gamma \in \mathcal{M}_I$ if

(a) $\gamma_t \ge 0$ a.s. for all $t \in I$,

(b) $\sum_{t\in I} \gamma_t \overset{\text{a.s.}}{=} 1$,

(c) γ_t is $\mathcal{F}_t$-measurable for all $t \in I$.

For an adapted process Z with $\mathbb{E}\big[\sup_{t\in I} Z_t^-\big] < \infty$ or $\mathbb{E}\big[\sup_{t\in I} Z_t^+\big] < \infty$ and Z_γ defined as in Definition 2.13 (and replacing $\gamma \in \mathcal{M}_I^\nu$ by $\gamma \in \mathcal{M}_I$) we can take a look at the value

$$\tilde{V} := \sup_{\gamma\in\mathcal{M}_I} \mathbb{E}[Z_\gamma] \,. \tag{2.34}$$

We also have $V = \tilde{V}$, if $I \subset \mathbb{N}_0$ and the process Z is a closable martingale by Theorem 2.49(b) below. Further we have $V^+(\nu) \le \tilde{V}$ as $\mathcal{M}_I^\nu \subset \mathcal{M}_I$.

If we assume that Ω is finite and that $\mathcal{F}_t = \mathcal{P}(\Omega)$ for every $t \in I$, we can compute the value $\tilde{V}$ using linear optimization. For information about linear optimization see e.g. [4] or [20].

2.2 General Results

Lemma 2.35. *For a totally-ordered countable set I and a given process $Z = \{Z_t\}_{t\in I}$ with $\mathbb{E}\big[\sup_{t\in I} Z_t^-\big] < \infty$ or $\mathbb{E}\big[\sup_{t\in I} Z_t^+\big] < \infty$ we have for every $\gamma \in \mathcal{M}_I$ as in Definition 2.33 that*

$$\mathbb{E}[Z_\gamma] = \sum_{t\in I} \mathbb{E}[Z_t \gamma_t] \,. \tag{2.36}$$

Remark 2.37. As $\mathcal{M}_I^\nu \subset \mathcal{M}_I$ this lemma is especially applicable for $\gamma \in \mathcal{M}_I^\nu$ as in Definition 2.12.

Proof. (a) First consider a non-negative process Z. Then $Z_t^{(n)} := \min\{Z_t, n\} \nearrow Z_t$ as $n \to \infty$ for every $t \in I$. For every $n \in \mathbb{N}$ the process $Z^{(n)} = \{Z_t^{(n)}\}_{t\in I}$ is bounded. Using $\sum_{t\in I} \gamma_t \overset{\text{a.s.}}{=} 1$ as in Definition 2.33(b) we get that

$$Z_\gamma^{(n)} = \sum_{t\in I} Z_t^{(n)} \gamma_t \le \Big(\sup_{t\in I} Z_t^{(n)}\Big) \sum_{t\in I} \gamma_t \overset{\text{a.s.}}{=} n \,.$$

By dominated convergence (see [62, Theorem 5.9]) we see that we can exchange the expected value and the series, which proves (2.36) for $Z^{(n)}$. By monotone convergence (see [62, Theorem 5.3]) the process Z satisfies (2.36).

(b) Now we want to show the result for general processes Z with $\mathbb{E}\big[\sup_{t\in I} Z_t^-\big] < \infty$, which implies that Z_γ is well-defined as it implies $\mathbb{P}(\sup_{t\in I} Z_t^- < \infty) = 1$. For this we set $Z_\gamma^{(+)} := \sum_{t\in I} Z_t^+ \gamma_t$ and $Z_\gamma^{(-)} := \sum_{t\in I} Z_t^- \gamma_t$. Note that $\mathbb{E}\big[Z_\gamma^{(-)}\big] < \infty$, because using $\sum_{t\in I} \gamma_t \overset{\text{a.s.}}{=} 1$ by Definition 2.33(b) we get that

$$Z_\gamma^{(-)} = \sum_{t\in I} Z_t^- \gamma_t \le \Big(\sup_{t\in I} Z_t^-\Big) \sum_{t\in I} \gamma_t \overset{\text{a.s.}}{=} \sup_{t\in I} Z_t^- \,.$$

We have that

$$Z_\gamma = \sum_{t\in I} Z_t \gamma_t = \sum_{t\in I} (Z_t^+ - Z_t^-)\gamma_t = Z_\gamma^{(+)} - Z_\gamma^{(-)} \quad \text{a.s.}\,,$$

and hence $\mathbb{E}[Z_\gamma] = \mathbb{E}\big[Z_\gamma^{(+)}\big] - \mathbb{E}\big[Z_\gamma^{(-)}\big]$, because $\mathbb{E}\big[Z_\gamma^{(-)}\big] < \infty$. Also

$$\sum_{t\in I} \mathbb{E}[Z_t \gamma_t] = \sum_{t\in I} \big(\mathbb{E}\big[Z_t^+ \gamma_t\big] - \mathbb{E}\big[Z_t^- \gamma_t\big]\big) = \sum_{t\in I} \mathbb{E}\big[Z_t^+ \gamma_t\big] - \sum_{t\in I} \mathbb{E}\big[Z_t^- \gamma_t\big] \,,$$

because $\sum_{t\in I} \mathbb{E}\big[Z_t^- \gamma_t\big] < \infty$ by part (a). We know that $Z^+ = \{Z_t^+\}_{t\in I}$ and $Z^- = \{Z_t^-\}_{t\in I}$ are non-negative processes and further $\mathbb{E}\big[\sup_{t\in I} Z_t^-\big] < \infty$. Therefore, by part (a),

$$\mathbb{E}\big[Z_\gamma^{(+)}\big] = \sum_{t\in I} \mathbb{E}\big[Z_t^+ \gamma_t\big]$$

and

$$\mathbb{E}\big[Z_\gamma^{(-)}\big] = \sum_{t\in I} \mathbb{E}\big[Z_t^- \gamma_t\big] \,.$$

Altogether we see that (2.36) is true for a process Z with $\mathbb{E}\big[\sup_{t\in I} Z_t^-\big] < \infty$. The case $\mathbb{E}\big[\sup_{t\in I} Z_t^+\big] < \infty$ is similar. □

Alternative proof of Lemma 2.35. First assume that the process Z is non-negative. Let η be the counting measure on I. Then the product measure $\eta \otimes \mathbb{P}$ is defined on the product σ-algebra $\mathcal{P}(I) \otimes \mathcal{F}$. We can define another measure $\mathbb{P}_\gamma$ on $\mathcal{P}(I) \otimes \mathcal{F}$ by

$$\frac{d\mathbb{P}_\gamma}{d(\eta \otimes \mathbb{P})}(t, \omega) = \gamma_t(\omega)\,, \quad t \in I, \omega \in \Omega\,.$$

The measure $\mathbb{P}_\gamma$ is a probability measure, as $\mathbb{E}\big[\sum_{t\in I} \gamma_t\big] = 1$ by Definition 2.33(b). Using this new probability measure

$$\mathbb{E}[Z_\gamma] = \mathbb{E}_{\mathbb{P}_\gamma}[Z]\,.$$

Since Z is a non-negative process, η is a σ-finite measure as I is countable and $\mathbb{P}$ is a probability measure, (2.36) follows from the Tonelli version of the Fubini theorem, see [13, Theorem 14.2].

To prove the result for a process Z with $\mathbb{E}\big[\sup_{t\in I} Z_t^-\big] < \infty$ or $\mathbb{E}\big[\sup_{t\in I} Z_t^+\big] < \infty$, which is not necessarily non-negative, proceed similar to part (b) of the proof of Lemma 2.35 above. □

Remark 2.38. Note that such a representation of the product measure as in the alternative proof of Lemma 2.35 is used, for example, in [1].

Corollary 2.39. *Consider a totally-ordered countable set I, a process $Z = \{Z_t\}_{t\in I}$ with $\mathbb{E}\big[\sup_{t\in I} Z_t^-\big] < \infty$ or $\mathbb{E}\big[\sup_{t\in I} Z_t^+\big] < \infty$ and a $\gamma \in \mathcal{M}_I$ as in Definition 2.33. If Z and γ are independent, then*

$$\mathbb{E}[Z_\gamma] = \sum_{t\in I} \mathbb{E}[Z_t]\,\mathbb{E}[\gamma_t]\;.$$

Remark 2.40. If in Corollary 2.39 we further assume $\gamma \in \mathcal{M}_I^\nu$ given by Definition 2.12, then

$$\mathbb{E}[Z_\gamma] = \sum_{t\in I} \mathbb{E}[Z_t]\,\nu_t\,.$$

Proof. By Lemma 2.35 we can exchange the expected value and the sum. Due to the independence of Z and γ, we have $\mathbb{E}[Z_t\gamma_t] = \mathbb{E}[Z_t]\,\mathbb{E}[\gamma_t]$ for all $t \in I$. □

Theorem 2.41. *Given a totally-ordered countable set I. Consider an adapted random probability measure $\gamma \in \mathcal{M}_I$ as in Definition 2.33 w.r.t. the filtration $\{\mathcal{F}_t\}_{t\in I}$. By extending the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in I}, \mathbb{P})$ if necessary, we may assume w.l.o.g. that there exists a random variable U, uniformly distributed on $[0, 1]$ and independent of $\mathcal{F}_\infty := \sigma(\bigcup_{t\in I} \mathcal{F}_t)$. Define the enlarged filtration $\{\tilde{\mathcal{F}}_t\}_{t\in I}$ by $\tilde{\mathcal{F}}_t := \mathcal{F}_t \vee \sigma(U)$ for $t \in I$ and the random time τ by*

$$\{\tau = t\} = \{\gamma_{<t} < U \le \gamma_{\le t}\}\,, \quad t \in I\,. \tag{2.42}$$

Then the following holds:

(a) τ is a stopping time w.r.t. $\{\tilde{\mathcal{F}}_t\}_{t\in I}$ satisfying $\mathbb{P}(\tau = t\,|\,\mathcal{F}_\infty) \overset{a.s.}{=} \gamma_t$ for all $t \in I$.

(b) Let Z be an $\mathcal{F}_\infty$-measurable process such that $\mathbb{E}\big[\sup_{t\in I} Z_t^-\big] < \infty$ or $\mathbb{E}\big[\sup_{t\in I} Z_t^+\big] < \infty$. Then

$$\mathbb{E}[Z_\tau|\mathcal{F}_\infty] \overset{a.s.}{=} Z_\gamma \qquad \text{and} \qquad \mathbb{E}[Z_\tau] = \mathbb{E}[Z_\gamma]\,.$$

Remark 2.43. Note that if $\gamma \in \mathcal{M}_I^\nu$ as in Definition 2.12 for some distribution ν on I, then the stopping time τ has distribution ν, i.e. $\tau \in \mathcal{T}_I^\nu$. Further note that we use the notation $\mathcal{F}_t \vee \sigma(U) = \sigma(\mathcal{F}_t \cup \sigma(U))$.

Proof. We see that τ defined as in (2.42) is really a stopping time w.r.t. $\{\tilde{\mathcal{F}}_t\}_{t\in I}$ and as U is independent of $\mathcal{F}_\infty$ we also have $\mathbb{P}(\tau = t\,|\,\mathcal{F}_\infty) \overset{\text{a.s.}}{=} \gamma_t$. Then using Lemma 2.35

$$\mathbb{E}[Z_\tau] = \sum_{t\in I} \mathbb{E}[Z_t 1_{\{\tau=t\}}] = \sum_{t\in I} \mathbb{E}[Z_t\, \mathbb{P}(\tau = t\,|\,\mathcal{F}_\infty)] = \sum_{t\in I} \mathbb{E}[Z_t\gamma_t] = \mathbb{E}[Z_\gamma]\ .$$

$\square$

Remark 2.44. If it is necessary to enlarge the probability space, this can be done by setting $\tilde{\Omega} := [0,1]\times\Omega$, $\tilde{\mathcal{F}} := \mathcal{B}_{[0,1]}\otimes\mathcal{F}$ and $\tilde{\mathbb{P}} := \lambda\otimes\mathbb{P}$, where λ denotes the Lebesgue–Borel measure. $U\colon \tilde{\Omega} \to [0,1]$ is the projection onto the first component. For a filtration on the extended probability space it would be sufficient to consider a filtration given by $\hat{\mathcal{F}}_t := \mathcal{F}_t\otimes\{\varnothing, [0,1]\}$ for $t \in I$. The filtration needed for τ to be a stopping time is $\tilde{\mathcal{F}}_t = \hat{\mathcal{F}}_t \vee \sigma(U)$ for $t \in I$. Let $\pi\colon [0,1] \times \Omega \to \Omega$ be the projection on the second component. We then consider a process $\tilde{Z}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that $\tilde{Z}_t := Z_t \circ \pi$ for all $t \in I$. Similarly we consider processes $\tilde{\gamma}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with $\tilde{\gamma}_t := \gamma_t \circ \pi$ for all $t \in I$. Note that $\tilde{Z}$ and $\tilde{\gamma}$ are adapted to $\{\hat{\mathcal{F}}_t\}_{t\in I}$.

Remark 2.45. Due to the construction of the stopping time τ in Theorem 2.41 by using the given adapted random probability measure γ we have $\{\tau = t\} \subset \{\gamma_t > 0\}$ for all $t \in I$.

In the following example we will show that the enlargement of the filtered probability space can have effects on the value $V(\nu)$.

Example 2.46. We consider the situation of Example 2.22 and enlarge the probability space and the filtration according to Remark 2.44. $\mathcal{F}_0$ and $\mathcal{F}_1$ are enlarged to $\tilde{\mathcal{F}}_t = \hat{\mathcal{F}}_t\vee\sigma(U)$ for $t \in \{0,1\}$, where U is independent of $\tilde{Z}$ and uniformly distributed on $[0,1]$. Then we can find a stopping time $\tau^* \in \mathcal{T}_I^\nu$ given by

$$\{\tau^* = 0\} = \{U \le \nu_0\} \quad \text{and} \quad \{\tau^* = 1\} = \{U > \nu_0\}\,.$$

For this stopping time we get

$$\tilde{\mathbb{E}}[\tilde{Z}_{\tau^*}] = \nu_0\nu_1 + \nu_1\nu_0 > 0\,.$$

We see that by enlargement of the filtered probability space, the value $V(\nu)$ increased. Note that the stopping time τ^* corresponds to the stopping time found in Theorem 2.41 for γ given as in Example 2.22.

An enlargement of the filtration enlarges the set $\mathcal{M}_I^\nu$ and can increase the value $V^+(\nu)$. We will illustrate this effect in Example 2.47, where future information is added to the filtration. In Lemma 2.48 we will show that not every enlargement of a σ-algebra changes the conditional expectation of a random variable.

Example 2.47. We will start with the setting of Example 2.21. Consider $\Omega = \{0,1\}$ with $\mathbb{P}(\omega) = \frac{1}{2}$ for all $\omega \in \Omega$. Assume $Z_0 = 0$ and $Z_1(\omega) = \omega$ for all $\omega \in \Omega$. The filtration is given by $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_1 = \mathcal{P}(\Omega)$. We assume that the distribution ν is given by $\nu_0 = \nu_1 = \frac{1}{2}$. Then the only possible process $\gamma \in \mathcal{M}_I^\nu$ is given by $\gamma_0 = \gamma_1 = \frac{1}{2}$. Therefore we have

$$\mathbb{E}[Z_\gamma] = \mathbb{E}[Z_1\gamma_1] = \frac{1}{2}\,\mathbb{E}[Z_1] = \frac{1}{4}\,.$$

Now we enlarge the filtration and set $\mathcal{F}_0 = \mathcal{P}(\Omega)$. Then we can define an adapted random probability measure by using the stopping time τ defined by $\tau(\omega) = \omega$ for all $\omega \in \Omega$. Then we have

$$\mathbb{E}[Z_\tau] = \mathbb{E}[Z_0\,|\,\tau = 0]\,\mathbb{P}(\tau = 0) + \mathbb{E}[Z_1\,|\,\tau = 1]\,\mathbb{P}(\tau = 1) = 0\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = \frac{1}{2}\,.$$

Lemma 2.48. *Given σ-algebras $\mathcal{F} \subset \mathcal{G}$ and a σ-algebra $\mathcal{H}$ which is independent of $\mathcal{G}$, we have for every $\sigma(\mathcal{F}\cup\mathcal{H})$-measurable random variable X, which is integrable or non-negative,*

$$\mathbb{E}[X\,|\,\mathcal{G}] \overset{a.s.}{=} \mathbb{E}[X\,|\,\mathcal{F}]\ .$$

Proof. Define

$$\mathscr{H} = \{X\colon \Omega \to \mathbb{R}\,|\, X \text{ bounded, } \sigma(\mathcal{F}\cup\mathcal{H})\text{-measurable, } \mathbb{E}[X\,|\,\mathcal{G}] \overset{\text{a.s.}}{=} \mathbb{E}[X\,|\,\mathcal{F}]\}\,.$$

We will use the monotone-class theorem (see e.g. [62, Theorem 3.14]) in order to show that $\mathscr{H}$ contains all bounded $\sigma(\mathcal{F}\cup\mathcal{H})$-measurable X on Ω.

It can easily be checked that $\mathscr{H}$ is a vector space over $\mathbb{R}$ and that the constant function 1 is an element of $\mathscr{H}$.

Using monotone convergence for conditional expectations (see [62, 9.7(e)]) it can be shown that for a sequence $(X_n)_{n\in\mathbb{N}}$ of non-negative functions in $\mathscr{H}$ such that $X_n \nearrow X$, where X is a bounded function on Ω, we also have $X \in \mathscr{H}$.

The collection $\{F\cap H\,|\,F \in \mathcal{F}, H \in \mathcal{H}\}$ is stable under finite intersections and generates $\sigma(\mathcal{F}\cup\mathcal{H})$. We now have to show that $\mathscr{H}$ contains the indicator function of every set in $\{F\cap H\,|\,F \in \mathcal{F}, H \in \mathcal{H}\}$. Since $F \in \mathcal{F} \subset \mathcal{G}$

$$\mathbb{E}[1_F 1_H\,|\,\mathcal{G}] \overset{\text{a.s.}}{=} 1_F\,\mathbb{E}[1_H\,|\,\mathcal{G}]\ .$$

and

$$\mathbb{E}[1_F 1_H\,|\,\mathcal{F}]\,1_G \overset{\text{a.s.}}{=} 1_F\,\mathbb{E}[1_H\,|\,\mathcal{F}]\ .$$

Since $\mathcal{H}$ is independent of $\mathcal{G} \supset \mathcal{F}$, $\mathbb{E}[1_H\,|\,\mathcal{G}] = \mathbb{P}(H) = \mathbb{E}[1_H\,|\,\mathcal{F}]$ a.s. Therefore

$$\mathbb{E}[1_F 1_H\,|\,\mathcal{G}] \overset{\text{a.s.}}{=} \mathbb{E}[1_F 1_H\,|\,\mathcal{F}]\ .$$

The result for non-negative processes now follows from conditional monotone convergence. The result is also true for integrable processes, as they can be decomposed into their positive and negative part, which are both non-negative. The result then follows from linearity of conditional expectation. □

Theorem 2.49. *(a) Given $I \subset \mathbb{N}_0$ with $0 \in I$. Let Z be a uniformly integrable supermartingale. Then, for every $\gamma \in \mathcal{M}_I$, the random variable Z_γ is well-defined, integrable and satisfies $\mathbb{E}[Z_\gamma] \le \mathbb{E}[Z_0]$. If, further, Z is a martingale, then, for every $\gamma \in \mathcal{M}_I$, $\mathbb{E}[Z_\gamma] = \mathbb{E}[Z_0]$.*

(b) Given a totally ordered countable set I. Let Z be a closable martingale. Then, for every $\gamma \in \mathcal{M}_I$, the random variable Z_γ is well-defined, integrable and satisfies $\mathbb{E}[Z_\gamma] = \mathbb{E}[Z_t]$ for all $t \in I$.

(c) Given $I \subset \mathbb{N}_0$ with $0 \in I$, let ν be a probability distribution on I. Let Z be a supermartingale. Then, for every $\gamma \in \mathcal{M}_I^\nu$, the random variable Z_γ is well-defined, integrable and satisfies $\mathbb{E}[Z_\gamma] \le \mathbb{E}[Z_0]$ in each of the following situations:

1. *There exists a $t \in I$ with $\nu_{\le t} = 1$,*
2. *Z is bounded a.s.,*
3. *$\sum_{t\in I} t\nu_t < \infty$ and for some $K > 0$ we have $\mathbb{P}(|Z_t - Z_{t-1}| \le K) = 1$ for all $t \in I\backslash\{0\}$,*

(d) If $I \subset \mathbb{N}_0$ with $0 \in I$, any of the conditions (c1), (c2) or (c3) holds and Z is a martingale, then, for every $\gamma \in \mathcal{M}_I^\nu$, Z_γ is well-defined, integrable and $\mathbb{E}[Z_\gamma] = \mathbb{E}[Z_0]$.

(e) If $I \subset \mathbb{Z}$ is a countably infinite index set, ν is a probability distribution on I, Z is a martingale and there exists a $t \in I$ with $\nu_{\leq t} = 1$, then Z_γ is integrable and $\mathbb{E}[Z_\gamma] = \mathbb{E}[Z_t]$ for all $t \in I$.

Proof. First we will prove (b). Using monotone convergence and Jensen's inequality, we get by Definition 2.33(b)

$$\begin{aligned}\mathbb{E}[|Z_\gamma|] &\leq \mathbb{E}\left[\sum_{t\in I} |Z_t|\gamma_t\right] = \sum_{t\in I} \mathbb{E}[|Z_t|\gamma_t] \leq \sum_{t\in I} \mathbb{E}[\mathbb{E}[|Z_\infty|\,|\mathcal{F}_t]\,\gamma_t] = \mathbb{E}\left[|Z_\infty|\sum_{t\in I}\gamma_t\right] \\ &= \mathbb{E}[|Z_\infty|] < \infty\,.\end{aligned}$$

This implies that Z_γ is well-defined and integrable. Repeating this calculation without the absolute values, which is allowed due to the absolute convergence almost surely of Z_γ, the result follows.

The rest of the results of this theorem follows by Theorem 2.41 and Doob's optional stopping theorem, cited in Theorem 2.9. The conditions imposed on Z and ν in this theorem are equivalent to the conditions on the process and the stopping time in Theorem 2.9, where the stopping time is now found using Theorem 2.41.

If a $t \in I$ with $\nu_{\leq t} = 1$ exists, then the corresponding stopping time is bounded by t a.s., giving Condition (a1) of Theorem 2.9. Since we assume that ν is a probability distribution, we see that the corresponding stopping time is finite a.s., which implies that is sufficient to assume that Z is bounded a.s. to get Condition (a2) of Theorem 2.9. If $\sum_{t\in I} t\nu_t < \infty$, then we know that the corresponding stopping time is integrable and therefore Condition (c3) of Theorem 2.49 is equivalent to Condition (a3) of Theorem 2.9.

□

Remark 2.50. When using Condition (c3) of Theorem 2.49 note that

$$\sum_{t\in I} t\nu_t = \sum_{t\in I}(1-\nu_{\leq t})\,.$$

For an I-valued stopping time τ with probability distribution ν both these sums equal $\mathbb{E}[\tau]$.

We now want to take a closer look at the relation between $V^+(\nu)$ as in (2.17) and V as in (2.5). We know that

$$V^+(\nu), V \text{ in } [V(\nu), \tilde{V}]\,,$$

with $V(\nu)$ as in (2.7) and $\tilde{V}$ as in (2.34). In the following lemma we will prove, that $V^+(\nu) \leq V$ for every probability distribution on I.

Lemma 2.51. *Given a finite discrete time interval I and an adapted process $Z \in L^1(\mathbb{P})$. Let $U = \{U_t\}_{t\in I}$ be the Snell envelope of the process Z and let $U = M + A$ be the Doob decomposition of U. Then for every probability distribution ν on I, such that M and ν satisfy one of the conditions of Theorem 2.49, we have*

$$V^+(\nu) \leq V\,.$$

Proof. Fix $\gamma \in \mathcal{M}_I^\nu$. By [30, Theorem 1.2.1] U is the smallest supermartingale majorant of Z. Therefore

$$\mathbb{E}[Z_\gamma] \le \mathbb{E}[U_\gamma] = \mathbb{E}[M_\gamma] + \mathbb{E}[A_\gamma] \ .$$

As U is a supermartingale and $A_0 = 0$ by the Doob decomposition, we know that $A_t \le 0$ for all $t \in I$. This implies that

$$\mathbb{E}[U_\gamma] \le \mathbb{E}[M_\gamma] \ .$$

Since M and ν satisfy one of the conditions of Theorem 2.49, we have

$$\mathbb{E}[M_\gamma] = \mathbb{E}[M_{\tau^*}] \ ,$$

where τ^* is the optimal stopping time for the American option. The stopping time τ^* given by

$$\tau^* = \inf\{t \in I \,|\, U_t = Z_t\}\,, \tag{2.52}$$

is optimal by [30, Theorem 1.3.1], as $U_{\tau^*} = Z_{\tau^*}$ and the stopped process U^{τ^*} is a martingale. This implies that for this stopping time

$$\mathbb{E}[M_{\tau^*}] = \mathbb{E}[U_{\tau^*}] = \mathbb{E}[Z_{\tau^*}] = V \ .$$

Therefore $\mathbb{E}[Z_\gamma] \le V$ for every $\gamma \in \mathcal{M}_I^\nu$, which implies $V^+(\nu) \le V$. □

Remark 2.53. The result of Lemma 2.51 is also true on $I = \mathbb{N}$, if the stopping time τ^* defined in (2.52) is finite a.s.

Remark 2.54. When we also want to use utility functions we have to be careful in the extended problem. If we want to use a utility function u and we consider only stopping times, we have

$$\mathbb{E}[u(Z_\tau)] = \mathbb{E}\left[\sum_{t\in I} u(Z_t) 1_{\{\tau=t\}}\right],$$

where we can simply define the new process $\tilde{Z}_t = u(Z_t)$ for $t \in I$. When using adapted random probability measures, we cannot do this. Then we actually have to compute

$$\mathbb{E}[u(Z_\gamma)] = \mathbb{E}\left[u\left(\sum_{t\in I} Z_t\gamma_t\right)\right].$$

The problem can easily be solved if we assume that we know the expectation and the variance of Z_t and γ_t as well as their correlation for each $t \in I$. On page 121 of the technical specifications of LTGA ([35]) or on page 96 of the technical specifications of QIS5 ([47]) for Solvency II a fixed correlation is assumed between the risks inherent in the life insurance business and in the financial market. This is a motivation for considering this case in the following lemma.

Lemma 2.55. *Given a totally-ordered countable set I and a process Z with $\mathbb{E}\big[\sup_{t\in I} Z_t^-\big] < \infty$ or $\mathbb{E}\big[\sup_{t\in I} Z_t^+\big] < \infty$. Assume that for some $\gamma \in \mathcal{M}_I$ and for each $t \in I$ the covariance $\mathrm{Cov}(Z_t, \gamma_t)$ between Z_t and γ_t is known. Then for this $\gamma \in \mathcal{M}_I$*

$$\mathbb{E}[Z_\gamma] = \sum_{t\in I}\big(\mathrm{Cov}(Z_t,\gamma_t) + \mathbb{E}[Z_t]\,\mathbb{E}[\gamma_t]\big)\,.$$

Remark 2.56. This lemma is also applicable for $\gamma \in \mathcal{M}_I^\nu$. Then, by Definition 2.12(d) we have $\mathbb{E}[\gamma_t] = \nu_t$ for all $t \in I$.

Proof. Since

$$\operatorname{Cov}(Z_t, \gamma_t) = \mathbb{E}[Z_t \gamma_t] - \mathbb{E}[Z_t]\,\mathbb{E}[\gamma_t] ,$$

the result follows from Lemma 2.35. □

Remark 2.57. If we assume that the covariances needed in Lemma 2.55 are known and we consider a subset $M \subset \mathcal{M}_I^\nu$ such that the covariance is the same for all $\gamma \in M$, we can compute

$$\sup_{\gamma \in M} \mathbb{E}[Z_\gamma] = \sum_{t \in I} \bigl(\operatorname{Cov}(Z_t, \gamma_t) + \mathbb{E}[Z_t]\,\nu_t\bigr) .$$

Remark 2.58. If we assume that for all $t \in I$ the correlation $\rho(Z_t, \gamma_t) =: \rho \in [-1, 1]$ is constant and that it is the same for all $\gamma \in \mathcal{M}_I^\nu$, then we have

$$\sup_{\gamma \in \mathcal{M}_I^\nu} \mathbb{E}[Z_\gamma] = \sum_{t \in I} \mathbb{E}[Z_t]\,\nu_t + \rho \sup_{\gamma \in \mathcal{M}_I^\nu} \sum_{t \in I} \sqrt{\operatorname{Var}(Z_t)}\sqrt{\operatorname{Var}(\gamma_t)} .$$

2.3 Analysis of Assumptions for the Process Z

As we will show in Lemma 2.59, Lemma 2.60 and Remark 2.63, we have to be careful concerning the assumptions about the process Z if I is countably infinite, since there are cases, where the problem is not well-posed.

Lemma 2.59. *Let $I = \mathbb{N}_0$ and let X be a real-valued random variable. Then there exists a process $\{Z_t\}_{t \in I} \subset L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and a stopping time τ w.r.t. the filtration $\mathcal{F}_t = \sigma(Z_0, \ldots, Z_t)$, $t \in I$, such that $Z_\tau = X$. In particular, if $\mathbb{E}[|X|] = \infty$, then $\mathbb{E}[|Z_\tau|] = \infty$.*

Proof. For $\omega \in \Omega$ define the stopping time $\tau \colon \Omega \to I$ by

$$\tau(\omega) = \begin{cases} 0 & \text{if } X(\omega) = 0, \\ t & \text{if } t - 1 < |X(\omega)| \le t \text{ with } t \in I, \end{cases}$$

Consider a process Z defined by $Z_0(\omega) = 1_{\{\tau(\omega) \neq 0\}}$ and $Z_t(\omega) = X(\omega) 1_{\{\tau(\omega) = t\}}$ for all $t \in I$, $\omega \in \Omega$. Then Z_t is bounded by t for every $t \in I$ and $Z_\tau = X$. Note that $\{\tau = 0\} = \{Z_0 = 0\} \in \mathcal{F}_0$ and $\{\tau = t\} = \{Z_t \neq 0\} \in \mathcal{F}_t$ for all $t \in I$, hence τ is a stopping time. □

As we show in the following lemma, there can also be problems if the underlying process is uniformly integrable.

Lemma 2.60. *Set $I = \mathbb{N}_0$ and let ν be a probability measure on I with $\nu_t > 0$ for infinitely many $t \in I$. Then there exists a non-negative process $Z = \{Z_t\}_{t \in I} \subset L^\infty(\mathbb{N}_0, \mathcal{P}(\mathbb{N}_0), \nu)$ with $\mathbb{E}_\nu[Z_t] \to 0$ as $t \to \infty$ (hence Z is uniformly integrable) and a stopping time τ w.r.t. the filtration $\mathcal{F}_t = \sigma(Z_0, \ldots, Z_t)$, $t \in I$, such that $\mathcal{L}(\tau) = \nu$ and $\mathbb{E}_\nu[Z_\tau] = \infty$.*

Proof. Define $N_t = \sharp\{k \in \{0, 1, \ldots, t\} \,|\, \nu_k > 0\}$ and $M = \{t \in I \,|\, \nu_t > 0\}$. Furthermore, on $\Omega = \mathbb{N}_0$ define

$$Z_t(\omega) = \begin{cases} 1_{\{t\}}(\omega) & \text{if } t \in I \setminus M, \\ \frac{1}{N_t \nu_t} 1_{\{t\}}(\omega) & \text{if } t \in M, \end{cases}$$

and $\tau(\omega) = \omega$ for all $\omega \in \Omega$. Then $\{\tau = t\} = \{Z_t \neq 0\} \in \mathcal{F}_t$ for all $t \in I$, hence τ is a stopping time. Note that

$$\mathbb{E}_\nu[Z_t] = \begin{cases} 0 & \text{if } t \in I \setminus M, \\ \frac{1}{N_t} & \text{if } t \in M, \end{cases}$$

hence $\mathbb{E}_\nu[Z_t] \to 0$ as $t \to \infty$ and by monotone convergence, as $\mathbb{E}_\nu[Z_t 1_{\{\tau=t\}}] = \mathbb{E}_\nu[Z_t]$ for all $t \in I$ and by divergence of the harmonic series

$$\mathbb{E}_\nu[Z_\tau] = \sum_{t\in I} \mathbb{E}_\nu[Z_t 1_{\{\tau=t\}}] = \sum_{t\in M} \frac{1}{N_t} = \sum_{n\in\mathbb{N}} \frac{1}{n} = \infty.$$

□

Remark 2.61. We want to note, briefly, why $\mathbb{E}[|Z_t|] \to 0$ as $t \to \infty$ implies that Z is uniformly integrable. We need to show that for every $\varepsilon > 0$ there exists an $M > 0$ such that $\mathbb{E}\big[|Z_t| 1_{\{|Z_t|>M\}}\big] \le \varepsilon$ for all $t \in I$. For every $\varepsilon > 0$ there exists a $t_\varepsilon \in I$ such that $\mathbb{E}[|Z_t|] \le \varepsilon$ for all $t \ge t_\varepsilon$ in I. Now we only need to show that the condition is satisfied for $Z_0, \dots, Z_{t_\varepsilon}$. As this is a finite set of integrable random variables, it is uniformly integrable by [27, Theorem 6.18].

Remark 2.62. One simple condition that can guarantee that the suprema in the computation of $V(\nu)$ and $V^+(\nu)$ exist is to assume that the process Z is bounded.

Remark 2.63. If the distribution ν of the adapted random probability measure $\gamma \in \mathcal{M}_I^\nu$ satisfies some moment condition, then the problem is well-posed if the corresponding moments of the process Z are bounded. In the proof of Lemma 2.35 we saw that we can exchange the series and the expectation for non-negative processes. Therefore

$$\mathbb{E}[|Z_\gamma|] = \mathbb{E}\left[\left|\sum_{t\in I} Z_t \gamma_t\right|\right] \le \mathbb{E}\left[\sum_{t\in I} |Z_t|\gamma_t\right] = \sum_{t\in I} \mathbb{E}[|Z_t|\gamma_t] \; .$$

For $p, q \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, Hölder's inequality (see e.g. [27, Theorem 7.16]) implies

$$\sum_{t\in I} \mathbb{E}[|Z_t|\gamma_t] \le \sum_{t\in I} \sqrt[p]{\mathbb{E}[|Z_t|^p]}\, \sqrt[q]{\mathbb{E}[\gamma_t^q]}\,.$$

Since $\gamma_t \le 1$ a.s. for all $t \in I$, we have $\mathbb{E}[\gamma_t^q] \le \mathbb{E}[\gamma_t] = \nu_t$ for all $t \in I$. If $\sum_{t\in I} \sqrt[q]{\nu_t} < \infty$, then $\sup_{t\in I} \|Z_t\|_{L_p} < \infty$ implies $\mathbb{E}[|Z_\gamma|] < \infty$.

Chapter 3

Existence of an Optimal Strategy

After the introduction of the problem in the last chapter, an important question is whether an optimal strategy exists that yields the supremum we want to compute. Again we take a look at a discrete time interval I. As we will see in this chapter an optimal $\gamma \in \mathcal{M}_I^\nu$ as in Definition 2.12 and an optimal $\gamma \in \mathcal{N}_I^\nu$ as in Definition 2.23 always exist for a process Z satisfying $\mathbb{E}[\sup_{t\in I} |Z_t|] < \infty$.

In Theorem 2.49 we saw that for a martingale Z and a probability distribution ν satisfying one of the conditions of the theorem the value of $\mathbb{E}[Z_\gamma]$ is the same for all $\gamma \in \mathcal{M}_I^\nu$. Therefore, all strategies are optimal and we do not have uniqueness. In Chapter 5 we will see that there are also other types of stochastic processes, for which the optimal strategy is not unique, e.g. processes with uncorrelated increments.

Now we will show that the set $\mathcal{M}_I^\nu$ is convex and we also want to check whether $\mathcal{M}_I^\nu$ is compact. For this, we will use results from functional analysis. We therefore have to define the vector spaces we are working with. In the following we will write down the results in general, with conjugate Hölder exponents $p, p', q, q' \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{p'} + \frac{1}{q'} = 1$. The interesting case for our problem will then be $p = 1$ and $p' = \infty$.

Again, let I be a countable index set for which we assume that $I \subset \mathbb{R} \cup \{-\infty, \infty\}$; this could, for example, be $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$ or $\{0, 1, \ldots, T\}$ with $T \in \mathbb{N}_0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a collection of σ-algebras $\mathbb{F} = \{\mathcal{F}_t\}_{t\in I}$. For Theorem 3.2 and Lemma 3.4 they do not need to form a filtration.

Lemma 3.1. *Given γ and $\tilde{\gamma}$ in $\mathcal{M}_I^\nu$ and a $[0,1]$-valued random variable Λ independent of γ and $\tilde{\gamma}$, which is $\mathcal{F}_t$-measurable for all $t \in I$. Then also $\Lambda\gamma + (1-\Lambda)\tilde{\gamma} \in \mathcal{M}_I^\nu$. In particular the set $\mathcal{M}_I^\nu$ is convex. The same result holds for γ and $\tilde{\gamma}$ in $\mathcal{N}_I^\nu$.*

Proof. Using Definition 2.12 it is easy to check that for γ and $\tilde{\gamma}$ in $\mathcal{M}_I^\nu$ we have $\Lambda\gamma + (1-\Lambda)\tilde{\gamma} \in \mathcal{M}_I^\nu$. Further, if γ and $\tilde{\gamma}$ satisfy (2.24), then so does $\Lambda\gamma + (1-\Lambda)\tilde{\gamma}$, which implies $\Lambda\gamma + (1-\Lambda)\tilde{\gamma} \in \mathcal{N}_I^\nu$. □

For $p, p' \in [1,\infty]$ we define the vector space $X_{p,p'}$ of real-valued $\mathbb{F}$-adapted processes $Z = (Z_t)_{t\in I}$ by

$$X_{p,p'} := \left\{ (Z_t)_{t\in I} \in \prod_{t\in I} L^p(\Omega, \mathcal{F}_t, \mathbb{P}) \,\middle|\, \|Z\|_{X_{p,p'}} < \infty \right\},$$

where

$$\|Z\|_{X_{p,p'}} := \big\| \|Z\|_{l^{p'}} \big\|_{L^p}.$$

Here $\|\cdot\|_{l^{p'}}$ is the p'-norm on the sequence space $(I, \mathcal{P}(I), \eta)$ with power set $\mathcal{P}(I)$ and counting measure η, which is taken pathwise of the process Z. Hence $\|Z\|_{l^{p'}}$ is a $[0,\infty]$-valued random variable that is measurable with respect to $\mathcal{F}_\infty := \sigma(\bigcup_{t\in I}\mathcal{F}_t)$. Then $\|Z\|_{X_{p,p'}}$ denotes the usual p-norm of $\|Z\|_{l^{p'}}$ on the Lebesgue space $L^p(\Omega,\mathcal{F}_\infty,\mathbb{P})$. For example, if $p\in[1,\infty)$ and $p'=\infty$, this means

$$\|Z\|_{X_{p,p'}} = \left(\mathbb{E}\left[\sup_{t\in I}|Z_t|^p\right]\right)^{1/p}.$$

Note that for every $t\in I$ the space $(L^p(\Omega,\mathcal{F}_t,\mathbb{P}), \|\cdot\|_{L^p})$ is a Banach space, where random variables are identified if they are $\mathbb{P}$-a.s. equal. With componentwise addition and scalar multiplication, it follows using Minkowski's inequality (see e.g. [27, Theorem 7.17]) for the norms $\|\cdot\|_{l^{p'}}$ and $\|\cdot\|_{L^p}$, that $(X_{p,p'}, \|\cdot\|_{X_{p,p'}})$ is a normed vector space. The following lemma gives completeness.

Theorem 3.2. *For every $p,p'\in[1,\infty]$, the vector space $(X_{p,p'}, \|\cdot\|_{X_{p,p'}})$ is a Banach space.*

Proof. To prove completeness let $(Z^n)_{n\in\mathbb{N}}$ be a $\|\cdot\|_{X_{p,p'}}$-Cauchy sequence. Fix $t\in I$. Since $\|Z_t\|_{L^p}\le\|Z\|_{X_{p,p'}}$ for every $Z\in X_{p,p'}$, it follows that the t-components $(Z^n_t)_{n\in\mathbb{N}}$ form a $\|\cdot\|_{L^p}$-Cauchy sequence in $L^p(\Omega,\mathcal{F}_t,\mathbb{P})$. By completeness (see [51, Theorem 3.11]) there exists $Z_t\in L^p(\Omega,\mathcal{F}_t,\mathbb{P})$ such that $\|Z_t-Z^n_t\|_{L^p}\to 0$ as $n\to\infty$. Therefore, we have constructed an adapted process $Z=(Z_t)_{t\in I}\in\prod_{t\in I}L^p(\Omega,\mathcal{F}_t,\mathbb{P})$.

Next we will show that the sequence $(Z^n)_{n\in\mathbb{N}}$ converges to Z also with respect to $\|\cdot\|_{X_{p,p'}}$. Since I is countable, there exists an increasing sequence $(I_k)_{k\in\mathbb{N}}$ of finite index sets with $\bigcup_{k\in\mathbb{N}}I_k=I$. By monotone convergence ([62, Theorem 5.3]) we have that

$$\|Z-Z^n\|_{X_{p,p'}} = \lim_{k\to\infty}\Big\|\|(Z_t-Z^n_t)_{t\in I_k}\|_{l^{p'}(I_k)}\Big\|_{L^p},\quad n\in\mathbb{N}, \tag{3.3}$$

where $\|\cdot\|_{l^{p'}(I_k)}$ denotes the p'-norm in $\mathbb{R}^{I_k}$, taken pathwise of the process with $|I_k|$ components. Fix $\varepsilon>0$. Since $(Z^n)_{n\in\mathbb{N}}$ is a $\|\cdot\|_{X_{p,p'}}$-Cauchy sequence, there exists $N_\varepsilon\in\mathbb{N}$ such that $\|Z^m-Z^n\|_{X_{p,p'}}\le\varepsilon$ for all $m,n\in\mathbb{N}$ with $m,n\ge N_\varepsilon$. Fix $k,n\in\mathbb{N}$ with $n\ge N_\varepsilon$. Since $\|Z_t-Z^m_t\|_{L^p}\to 0$ as $m\to\infty$ for every t in the finite set I_k, we may iteratively find a subsequence $(m_l)_{l\in\mathbb{N}}$ with $m_l\ge N_\varepsilon$ for all $l\in\mathbb{N}$ such that $(Z^{m_l}_t)_{l\in\mathbb{N}}$ converges a.s. to Z_t for every $t\in I_k$. Then Fatou's lemma ([62, Section 5.4]) and $I_k\subset I$ imply that

$$\begin{aligned}\Big\|\|(Z_t-Z^n_t)_{t\in I_k}\|_{l^{p'}(I_k)}\Big\|_{L^p} &\le \liminf_{l\to\infty}\Big\|\|(Z^{m_l}_t-Z^n_t)_{t\in I_k}\|_{l^{p'}(I_k)}\Big\|_{L^p}\\ &\le \liminf_{l\to\infty}\|Z^{m_l}-Z^n\|_{X_{p,p'}}\le\varepsilon\,.\end{aligned}$$

Combination with (3.3) shows that $\|Z-Z^n\|_{X_{p,p'}}\le\varepsilon$ for all $n\ge N_\varepsilon$. We also have $Z\in X_{p,p'}$, because $Z=(Z-Z^n)+Z^n$, where $(Z-Z^n)$ and Z^n are both elements of the vector space $X_{p,p'}$ □

Denote the topological dual space of $(X_{p,p'}, \|\cdot\|_{X_{p,p'}})$ by $X^*_{p,p'}$ with operator norm $\|\phi\|_{X^*_{p,p'}} := \sup\{|\phi(Z)|\,|\,Z\in X_{p,p'}, \|Z\|_{X_{p,p'}}\le 1\}$ for $\phi\in X^*_{p,p'}$. $(X^*_{p,p'}, \|\cdot\|_{X^*_{p,p'}})$ is a Banach space by [50, Theorem 4.1].

Note that for $p\in[1,\infty)$ the space $(L^q(\Omega,\mathcal{F}_t,\mathbb{P}),\|\cdot\|_{L^q}) = (L^p(\Omega,\mathcal{F}_t,\mathbb{P}),\|\cdot\|_{L^p})^*$ is a Banach space for every $t\in I$, where random variables are identified if they are $\mathbb{P}$-a.s. equal. This is proven, for example, in [51, Theorem 6.16]. The dual space of $(L^p(\Omega,\mathcal{F}_t,\mathbb{P}),\|\cdot\|_{L^p})$ for the case $p=\infty$ is discussed in [52, 29.31(c)].

Lemma 3.4. *For $p, p' \in [1,\infty]$ let $q, q' \in [1,\infty]$ denote the conjugate Hölder exponents.*

(a) For every $\gamma = (\gamma_t)_{t\in I} \in X_{q,q'}$ the map $\Phi_\gamma \colon X_{p,p'} \to \mathbb{R}$ defined by

$$\Phi_\gamma(Z) := \mathbb{E}\Big[\sum_{t\in I} Z_t\gamma_t\Big], \quad Z = (Z_t)_{t\in I} \in X_{p,p'} \tag{3.5}$$

is a well-defined element of $(X^*_{p,p'}, \|\cdot\|_{X^*_{p,p'}})$ and satisfies $\|\Phi_\gamma\|_{X^*_{p,p'}} \le \|\gamma\|_{X_{q,q'}}$.

(b) Fix $p' = \infty$. For $q \in (1,\infty)$ consider the set

$$\hat{X}_{q,q'} := \big\{\gamma \in X_{q,q'} \,\big|\, \|\gamma\|_{l^{q'}} \text{ is } \mathcal{F}_t\text{-measurable for all } t \in I\big\}.$$

The map $T \colon \hat{X}_{q,q'} \to X^*_{p,p'}$ with $T(\gamma) := \Phi_\gamma$ for every $\gamma \in \hat{X}_{q,q'}$ is a homogeneous, isometric embedding of $(\hat{X}_{q,q'}, \|\cdot\|_{X_{q,q'}})$ into $(X^*_{p,p'}, \|\cdot\|_{X^*_{p,p'}})$.

(c) For $p = p'$ in $(1,\infty]$ the map $T \colon X_{q,q'} \to X^*_{p,p'}$ with $T(\gamma) := \Phi_\gamma$ for every $\gamma \in X_{q,q'}$ is a linear, isometric embedding of $(X_{q,q'}, \|\cdot\|_{X_{q,q'}})$ into $(X^*_{p,p'}, \|\cdot\|_{X^*_{p,p'}})$.

Remark 3.6. Note that $\mathcal{M}^\nu_I \subset \hat{X}_{q,1}$, as $\|\gamma\|_{l^1} = 1$ a.s. for all $\gamma \in \mathcal{M}^\nu_I$ by Definition 2.12(b).

Proof. (a) For every $\gamma \in X_{q,q'}$, using Jensen's (see e.g. [62, Theorem 6.6] or [27, Theorem 7.9]) and Hölder's inequality (see e.g. [27, Theorem 7.16]), we have for all $Z \in X_{p,p'}$

$$\left|\mathbb{E}\Big[\sum_{t\in I} Z_t\gamma_t\Big]\right| \le \mathbb{E}\Big[\sum_{t\in I} |Z_t\gamma_t|\Big] \le \mathbb{E}\big[\|Z\|_{l^{p'}}\|\gamma\|_{l^{q'}}\big] \le \|Z\|_{X_{p,p'}}\|\gamma\|_{X_{q,q'}} < \infty, \tag{3.7}$$

hence Φ_γ as defined in (3.5) is a bounded linear functional, which implies that it is continuous. Therefore, it is a well-defined element of $(X^*_{p,p'}, \|\cdot\|_{X^*_{p,p'}})$ for every $\gamma \in X_{q,q'}$. By (3.7) and the definition of the operator norm $\|\Phi_\gamma\|_{X^*_{p,p'}} \le \|\gamma\|_{X_{q,q'}}$.

(b) Due to (a) it remains to prove that $\|\Phi_\gamma\|_{X^*_{p,p'}} \ge \|\gamma\|_{X_{q,q'}}$ for all $\gamma \in \hat{X}_{q,q'}$.

As $p' = \infty$, we have $q' = 1$. Consider the process Z defined by $Z_t := \|\gamma\|^{q-1}_{l^1}\operatorname{sign}(\gamma_t)$ for $t \in I$. Then $Z \in \prod_{t\in I} L^p(\Omega, \mathcal{F}_t, \mathbb{P})$ as $\gamma \in \hat{X}_{q,q'}$. We get $\|Z\|_{l^\infty} = \|\gamma\|^{q-1}_{l^1}$. Then, as $(q-1)p = q$,

$$\|Z\|_{X_{p,p'}} = \big\|\|\gamma\|^{q-1}_{l^1}\big\|_{L^p} = \big\|\|\gamma\|^{(q-1)p/q}_{l^1}\big\|^{q/p}_{L^q} = \big\|\|\gamma\|_{l^1}\big\|^{q/p}_{L^q} = \|\gamma\|^{q/p}_{X_{q,q'}} < \infty.$$

Further

$$\Phi_\gamma(Z) = \mathbb{E}\Big[\|\gamma\|^{q-1}_{l^1}\sum_{t\in I}|\gamma_t|\Big] = \big\|\|\gamma\|^q_{l^1}\big\|_{L^1} = \big\|\|\gamma\|^{q/q}_{l^1}\big\|^q_{L^q} = \|\gamma\|^q_{X_{q,q'}},$$

while on the other hand

$$|\Phi_\gamma(Z)| \le \|\Phi_\gamma\|_{X^*_{p,p'}}\|Z\|_{X_{p,p'}}.$$

Altogether

$$\|\gamma\|^q_{X_{q,q'}} \le \|\Phi_\gamma\|_{X^*_{p,p'}}\|\gamma\|^{q/p}_{X_{q,q'}}.$$

Therefore $\|\gamma\|_{X_{q,q'}} \le \|\Phi_\gamma\|_{X^*_{p,p'}}$.

(c) Due to (a) it remains to prove that $\|\Phi_\gamma\|_{X^*_{p,p'}} \ge \|\gamma\|_{X_{q,q'}}$ for all $\gamma \in X_{q,q'}$.

We first assume $p = p'$ in $(1,\infty)$, which implies $q = q'$ in $(1,\infty)$. Consider the process Z defined by $Z_t := |\gamma_t|^{q-1}\operatorname{sign}(\gamma_t)$ for $t \in I$. Then, as $(q-1)p' = q$,

$$\|Z\|_{X_{p,p'}} = \left\| \left(\sum_{t\in I} |\gamma_t|^q \right)^{1/p'} \right\|_{L^p} = \big\| \|\gamma\|_{l^q}^{(q/p')(p/q)} \big\|_{L^q}^{q/p} = \big\| \|\gamma\|_{l^{q'}} \big\|_{L^q}^{q/p} = \|\gamma\|_{X_{q,q'}}^{q/p} < \infty\,.$$

Further

$$\Phi_\gamma(Z) = \mathbb{E}\bigg[\sum_{t\in I} |\gamma_t|^q \bigg] = \big\| \|\gamma\|_{l^{q'}}^q \big\|_{L^1} = \big\| \|\gamma\|_{l^{q'}}^{q/q} \big\|_{L^q}^q = \|\gamma\|_{X_{q,q'}}^q\,,$$

which implies, similarly to the proof of (b), $\|\gamma\|_{X_{q,q'}} \leq \|\Phi_\gamma\|_{X^*_{p,p'}}$.

Now assume $p = p' = \infty$ and consider the process Z defined by $Z_t = \operatorname{sign}(\gamma_t)$ for all $t \in I$. Then

$$\|Z\|_{X_{p,p'}} = \big\| \|Z\|_{l^\infty} \big\|_{L^\infty} \leq \|1\|_{L^\infty} \leq 1\,.$$

Further

$$\Phi_\gamma(Z) = \mathbb{E}\bigg[\sum_{t\in I} |\gamma_t| \bigg] = \big\| \|\gamma\|_{l^1} \big\|_{L^1} = \|\gamma\|_{X_{q,q'}}\,.$$

Therefore

$$\|\gamma\|_{X_{q,q'}} = |\Phi_\gamma(Z)| \leq \|\Phi_\gamma\|_{X^*_{p,p'}} \|Z\|_{X_{p,p'}} \leq \|\Phi_\gamma\|_{X^*_{p,p'}}\,.$$

□

Define

$$V_{p,p'} := \{ Z \in X_{p,p'} \,|\, \|Z\|_{X_{p,p'}} \leq 1 \}\,.$$

By the theorem of Banach–Alaoglu (see e.g. [50, Theorem 3.15]), we have that the polar set

$$K_{q,q'} := \big\{ \phi \in X^*_{p,p'} \,\big|\, |\phi(Z)| \leq 1 \text{ for all } Z \in V_{p,p'} \big\}$$

is weak-$*$-compact.

Lemma 3.8. *Consider $q, q' \in [1,\infty]$ with $q \neq 1$. The set $\{\Phi_\gamma\}_{\gamma\in\mathcal{M}_I^\nu}$ is contained in $K_{q,q'}$ and weak-$*$-compact.*

Proof. $\{\Phi_\gamma\}_{\gamma\in\mathcal{M}_I^\nu}$ is a subset of $K_{q,q'}$, because $\|\Phi_\gamma\|_{X^*_{p,p'}} \leq \|\gamma\|_{X_{q,q'}}$ by Lemma 3.4(a) and $\|\gamma\|_{X_{q,q'}} \leq 1$ for every $\gamma \in \mathcal{M}_I^\nu$, as for $q' \in [1,\infty)$

$$\|\gamma\|_{X_{q,q'}} = \big\| \|\gamma\|_{l^{q'}} \big\|_{L^q} = \left\| \left(\sum_{t\in I} \gamma_t^{q'} \right)^{1/q'} \right\|_{L^q} \leq \left\| \left(\sum_{t\in I} \gamma_t \right)^{1/q'} \right\|_{L^q} = 1\,.$$

and for $q' = \infty$

$$\|\gamma\|_{l^\infty} = \sup_{t\in I} |\gamma_t| \leq 1\,.$$

In several steps we will now show that it is also weak-$*$-closed, which implies that it is weak-$*$-compact by [48, Chapter IV.3, p. 99]. For this we consider $\Psi \in X^*_{p,p'}$, which is in the weak-$*$-closure of $\{\Phi_\gamma\}_{\gamma\in\mathcal{M}_I^\nu}$ and prove $\Psi \in \{\Phi_\gamma\}_{\gamma\in\mathcal{M}_I^\nu}$ by proving that there exists a $\tilde\gamma \in \mathcal{M}_I^\nu$ such that $\Psi = \Phi_{\tilde\gamma}$.

(a) **Existence of $\tilde{\gamma}$.** Fix $t \in I$ and define $\Psi_t\colon L^p(\Omega, \mathcal{F}_t, \mathbb{P}) \to \mathbb{R}$ by $\Psi_t(Z_t) = \Psi(Z^{(t)})$ for all $Z_t \in L^p(\Omega, \mathcal{F}_t, \mathbb{P})$, where $Z^{(t)} \in X_{p,p'}$ is given by

$$Z_s^{(t)} = \begin{cases} Z_t & \text{for } s = t, \\ 0 & \text{for } s \in I \setminus \{t\}. \end{cases} \tag{3.9}$$

The functional Ψ_t is linear and

$$|\Psi_t(Z_t)| = |\Psi(Z^{(t)})| \leq \|\Psi\|_{X^*_{p,p'}} \|Z^{(t)}\|_{X_{p,p'}} = \|\Psi\|_{X^*_{p,p'}} \|Z_t\|_{L^p} \,,$$

hence $\Psi_t \in (L^p(\Omega, \mathcal{F}_t, \mathbb{P}))^*$. By [51, Theorem 6.16] there exists a $\tilde{\gamma}_t \in L^q(\Omega, \mathcal{F}_t, \mathbb{P})$ with $\Psi_t(Z_t) = \mathbb{E}[\tilde{\gamma}_t Z_t]$ for all $Z_t \in L^p(\Omega, \mathcal{F}_t, \mathbb{P})$. Hence we have $\tilde{\gamma} = (\tilde{\gamma}_t)_{t\in I} \in \prod_{t\in I} L^q(\Omega, \mathcal{F}_t, \mathbb{P})$.

(b) **Distribution of $\tilde{\gamma}$.** For $t \in I$ define $Z_t = 1_\Omega$ and $Z^{(t)}$ via (3.9). Then $\Phi_\gamma(Z^{(t)}) = \mathbb{E}[\gamma_t] = \nu_t$ for every $\gamma \in \mathcal{M}_I^\nu$, hence $\mathbb{E}[\tilde{\gamma}_t] = \Psi(Z^{(t)}) = \nu_t$.

(c) **$\tilde{\gamma}$ is non-negative.** Fix $t \in I$. Define $A = \{\tilde{\gamma}_t < 0\}$ and note that $Z_t := 1_A \in L^p(\Omega, \mathcal{F}_t, \mathbb{P})$. Define $Z^{(t)}$ via (3.9). For every $\gamma \in \mathcal{M}_I^\nu$ we have $\Phi_\gamma(Z^{(t)}) = \mathbb{E}[\gamma_t 1_A] \geq 0$, hence $0 \leq \Psi(Z^{(t)}) = \Phi_{\tilde{\gamma}}(Z^{(t)}) = \mathbb{E}[\tilde{\gamma}_t 1_A]$. This implies $\mathbb{P}(A) = 0$.

(d) **$\tilde{\gamma}$ is a probability measure.** Consider an increasing sequence $(I_n)_{n\in\mathbb{N}}$ of finite index sets with $\bigcup_{n\in\mathbb{N}} I_n = I$. Define $Z^{\Omega,n} \in X_{p,p'}$ by $Z_s^{\Omega,n} = 1_\Omega$ for all $s \in I_n$ and $Z_s^{\Omega,n} = 0$ for all $s \in I \setminus I_n$. Since for all $\gamma \in \mathcal{M}_I^\nu$

$$\Phi_\gamma(Z^{\Omega,n}) = \mathbb{E}\Bigg[\sum_{t\in I_n} \gamma_t\Bigg] = \sum_{t\in I_n} \nu_t \leq 1 \,,$$

we have $\Psi(Z^{\Omega,n}) = \sum_{t\in I_n} \nu_t$, hence

$$\mathbb{E}\Bigg[\sum_{t\in I_n} \tilde{\gamma}_t\Bigg] = \Phi_{\tilde{\gamma}}(Z^{\Omega,n}) = \Psi(Z^{\Omega,n}) = \sum_{t\in I_n} \nu_t \nearrow \sum_{t\in I} \nu_t = 1 \quad \text{as } n \to \infty.$$

Further, by monotone convergence,

$$\mathbb{E}\Bigg[\sum_{t\in I_n} \tilde{\gamma}_t\Bigg] \nearrow \mathbb{E}\Bigg[\sum_{t\in I} \tilde{\gamma}_t\Bigg] \quad \text{as } n \to \infty.$$

Therefore $\mathbb{E}\big[\sum_{t\in I} \tilde{\gamma}_t\big] = 1$.

Define $A = \{\sum_{t\in I} \tilde{\gamma}_t > 1\} \in \sigma(\bigcup_{t\in I} \mathcal{F}_t)$. Further, set $A_n = \{\sum_{t\in I_n} \tilde{\gamma}_t > 1\} \in \sigma(\bigcup_{t\in I} \mathcal{F}_t)$. Then $A = \bigcup_{n\in\mathbb{N}} A_n$. For $n \in \mathbb{N}$ consider the processes Z^{A_n} with $Z_t^{A_n} = \mathbb{E}[1_{A_n} \,|\, \mathcal{F}_t]$ for all $t \in I_n$ and $Z_t^{A_n} = 0$ for all $t \in I \setminus I_n$. For every $\gamma \in \mathcal{M}_I^\nu$,

$$\Phi_\gamma(Z^{A_n}) = \sum_{t\in I_n} \mathbb{E}[\mathbb{E}[1_{A_n} \,|\, \mathcal{F}_t]\, \gamma_t] = \sum_{t\in I_n} \mathbb{E}[1_{A_n} \gamma_t] \leq \mathbb{E}\Bigg[1_{A_n} \sum_{t\in I} \gamma_t\Bigg] = \mathbb{P}(A_n) \,,$$

where the sum and the expected value can be exchanged due to monotone convergence. Therefore, $\Psi(Z^{A_n}) \leq \mathbb{P}(A_n)$. By the same calculation

$$\mathbb{E}\Bigg[1_{A_n} \sum_{t\in I_n} \tilde{\gamma}_t\Bigg] = \Phi_{\tilde{\gamma}}(Z^{A_n}) = \Psi(Z^{A_n}) \leq \mathbb{P}(A_n) \,,$$

which implies $\mathbb{P}(A_n) = 0$. Then A is a countable union of null sets, which implies $\mathbb{P}(A) = 0$. Hence

$$\sum_{t\in I} \tilde{\gamma}_t \overset{\text{a.s.}}{=} 1\,.$$

(e) $\boldsymbol{\Psi(Z) = \Phi_{\tilde{\gamma}}(Z)}$ **for all** $\boldsymbol{Z \in X_{p,p'}}$**.** Consider $Z \in X_{p,p'}$ and $\varepsilon > 0$. We have that $\Phi_{\tilde{\gamma}}(Z)$ is well-defined by Lemma 3.4(a), as the above steps show that $\tilde{\gamma} \in \mathcal{M}_I^\nu$. As $Z \in X_{p,p'}$ we have $Z^* := \sup_{t\in I} |Z_t| \in L^p(\Omega, \mathcal{F}, \mathbb{P}) \subset L^1(\Omega, \mathcal{F}, \mathbb{P})$. Therefore, by dominated convergence an $M > 0$ exists such that $\mathbb{E}\big[Z^* 1_{\{Z^*>M\}}\big] \le \varepsilon$. Further there exists a finite set $J \subset I$ such that $\sum_{t\in I\setminus J} \nu_t \le \frac{\varepsilon}{M}$. Define the process $Z^J \in X_{p,p'}$ by

$$Z_t^J = \begin{cases} 0 & \text{for } t \in J, \\ Z_t & \text{for } t \in I \setminus J. \end{cases}$$

Then for every $\gamma \in \mathcal{M}_I^\nu$

$$\begin{aligned} |\Phi_\gamma(Z^J)| &\le \mathbb{E}\Big[Z^* \sum_{t\in I\setminus J} \gamma_t\Big] \le \mathbb{E}\Big[(M 1_{\{Z^*\le M\}} + Z^* 1_{\{Z^*>M\}}) \sum_{t\in I\setminus J} \gamma_t\Big] \\ &\le M\,\mathbb{E}\Big[\sum_{t\in I\setminus J} \gamma_t\Big] + \mathbb{E}\big[Z^* 1_{\{Z^*>M\}}\big] \le 2\varepsilon\,. \end{aligned}$$

Then also $|\Psi(Z^J)| \le 2\varepsilon$. By the construction of $\tilde{\gamma}$ in (a) and linearity of Ψ

$$\Psi(Z - Z^J) = \Phi_{\tilde{\gamma}}(Z - Z^J)\,.$$

Then

$$|\Psi(Z) - \Phi_{\tilde{\gamma}}(Z)| \le |\Psi(Z - Z^J) - \Phi_{\tilde{\gamma}}(Z - Z^J)| + |\Psi(Z^J)| + |\Phi_{\tilde{\gamma}}(Z^J)| \le 4\varepsilon\,.$$

Therefore $\Psi(Z) = \Phi_{\tilde{\gamma}}(Z)$.

By (a) and (e) we see that an adapted process $\tilde{\gamma}$ exists with $\Psi(Z) = \Phi_{\tilde{\gamma}}(Z)$ for all $Z \in X_{p,p'}$, while by (b), (c) and (d) we see that $\tilde{\gamma} \in \mathcal{M}_I^\nu$. □

Theorem 3.10. *For $p, p' \in [1,\infty]$ with $p \neq \infty$ and $Z \in X_{p,p'}$ there always exists an optimal adapted random probability measure $\gamma^* \in \mathcal{M}_I^\nu$ solving*

$$\sup_{\gamma\in\mathcal{M}_I^\nu} \mathbb{E}[Z_\gamma] = \mathbb{E}[Z_{\gamma^*}]\,.$$

Remark 3.11. Note that for the unrestricted optimization problem over $\gamma \in \mathcal{M}_I$ or $\tau \in \mathcal{T}_I$, a corresponding optimal γ^* or τ^*, respectively, might not exist. Consider for example the deterministic process $Z = \{Z_n\}_{n\in\mathbb{N}}$ defined by $Z_n = 1 - \frac{1}{n}$ for $n \in \mathbb{N}$.

Proof. We can rewrite the problem for every $Z \in X_{p,p'}$ by

$$\sup_{\gamma\in\mathcal{M}_I^\nu} \mathbb{E}[Z_\gamma] = \sup_{\gamma\in\mathcal{M}_I^\nu} \Phi_\gamma(Z) = \sup_{\phi\in\{\Phi_\gamma\}_{\gamma\in\mathcal{M}_I^\nu}} \phi(Z)\,.$$

As $\{\Phi_\gamma\}_{\gamma\in\mathcal{M}_I^\nu}$ is weak-$*$-compact by Lemma 3.8 and $\mathcal{M}_I^\nu$ is not empty by Remark 2.20, there exists a $\gamma^*\in\mathcal{M}_I^\nu$ such that

$$\sup_{\phi\in\{\Phi_\gamma\}_{\gamma\in\mathcal{M}_I^\nu}} \phi(Z)=\Phi_{\gamma^*}(Z)\,,$$

as every continuous function on a non-empty compact set attains its supremum on this set (see e.g. [58, Theorem 4.5.3] for topological spaces or [48, Chapter IV.3, p. 99] for compact sets that are Hausdorff). □

Corollary 3.12. *For $p\in[1,\infty)$ and $Z\in X_{p,\infty}$ there always exists an optimal $\gamma^*\in\mathcal{N}_I^\nu$, with $\mathcal{N}_I^\nu$ as in Definition 2.23, solving*

$$\sup_{\gamma\in\mathcal{N}_I^\nu}\mathbb{E}[Z_\gamma]=\mathbb{E}[Z_{\gamma^*}]\,.$$

Proof. By Lemma 3.8 we know that $\{\Phi_\gamma\}_{\gamma\in\mathcal{M}_I^\nu}$ is weak-$*$-closed. By Remark 2.27 we know that $\mathcal{N}_I^\nu\neq\varnothing$. Similar to the proof of Lemma 3.8 we take a look at Ψ in the closure of $\{\Phi_\gamma\}_{\gamma\in\mathcal{N}_I^\nu}$ and repeat the steps of that proof. By Lemma 3.8 we know that a $\tilde\gamma\in\mathcal{M}_I^\nu$ exists such that $\Psi=\Phi_{\tilde\gamma}$. Last but not least we have to show that $\tilde\gamma$ satisfies condition (2.24) of Definition 2.23 of the set $\mathcal{N}_I^\nu$. For this we need to show that, for every $t\in I$ and $A\in\mathcal{F}_s$ with $s<t$ in I,

$$\mathbb{E}[\tilde\gamma_t 1_A]=\begin{cases}\frac{\nu_t}{1-\nu_{\le s}}\,\mathbb{E}[(1-\tilde\gamma_{\le s})1_A] & \text{if } \nu_{\le s}<1,\\ 0 & \text{otherwise.}\end{cases}$$

For $s<t$ in I with $\nu_{\le s}<1$ and $A\in\mathcal{F}_s$ define the process $Z^{A,s,t}$ by

$$Z_n^{A,s,t}=\begin{cases}1_A & \text{for } n=t,\\ \frac{\nu_t}{1-\nu_{\le s}}\,\mathbb{E}[1_A\,|\,\mathcal{F}_n] & \text{for } n\in I_{\le s},\\ 0 & \text{otherwise.}\end{cases}$$

For every $\gamma\in\mathcal{N}_I^\nu$ we have that

$$\Phi_\gamma(Z^{A,s,t})=\frac{\nu_t\,\mathbb{P}(A)}{1-\nu_{\le s}}\,,$$

which implies that

$$\Psi(Z^{A,s,t})=\frac{\nu_t\,\mathbb{P}(A)}{1-\nu_{\le s}}\,.$$

On the other hand,

$$\Psi(Z^{A,s,t})=\sum_{n\in I_{\le s}}\frac{\nu_t}{1-\nu_{\le s}}\,\mathbb{E}[\mathbb{E}[1_A\,|\,\mathcal{F}_n]\,\tilde\gamma_n]+\mathbb{E}[\tilde\gamma_t 1_A]=\sum_{n\in I_{\le s}}\frac{\nu_t}{1-\nu_{\le s}}\,\mathbb{E}[\tilde\gamma_n 1_A]+\mathbb{E}[\tilde\gamma_t 1_A]\,.$$

Therefore

$$\mathbb{E}[\tilde\gamma_t 1_A]=\frac{\nu_t\,\mathbb{P}(A)}{1-\nu_{\le s}}-\sum_{n\in I_{\le s}}\frac{\nu_t}{1-\nu_{\le s}}\,\mathbb{E}[1_A\tilde\gamma_n]=\frac{\nu_t}{1-\nu_{\le s}}\,\mathbb{E}[1_A(1-\tilde\gamma_{\le s})]\,.$$

For $s<t$ in I with $\nu_{\le s}=1$ and $A\in\mathcal{F}_s$ we set $Z_t:=1_A$ and define $Z^{(t)}$ via (3.9). Then $\Phi_\gamma(Z^{(t)})=0$ for all $\gamma\in\mathcal{N}_I^\nu$. This implies $\Psi(Z^{(t)})=\mathbb{E}[\tilde\gamma_t 1_A]=0$. □

Chapter 4

Some Bounds

In Chapter 2 we already derived some first bounds for our problem. In this chapter we will present more upper and lower bounds. These might be useful for finding optimal strategies or for estimating the values for those types of processes for which we cannot find an optimal stopping time or an optimal adapted random probability measure, respectively. The chapter is divided into four sections. First Section 4.1 gives general results about the expected shortfall and the conditional expected shortfall, which will be useful for finding upper bounds in the following sections. Section 4.2 focuses on bounds that are valid for general processes Z. Some of the bounds presented in that section take the distribution ν of the stopping time $\tau \in \mathcal{T}_I^\nu$ or the adapted random probability measure $\gamma \in \mathcal{M}_I^\nu$ into account, while others do not. In Section 4.3 bounds for special types of processes are derived. In Section 4.4 some examples of stochastic processes are considered for which different bounds are computed. This helps to compare the bounds. As shown in some examples, which of the bounds presented is best depends on the situation. Within this chapter we again consider discrete time intervals I, which are finite or countably infinite, and adapted stochastic processes $Z = \{Z_t\}_{t\in I}$ with $Z \in L^1(\mathbb{P})$.

We now state an obvious observation, which gives a sufficient criterion for the optimality of some $\gamma \in \mathcal{M}_I$.

Lemma 4.1. *Given a totally-ordered countable discrete time interval I, a probability distribution ν on I and an adapted process $Z = \{Z_t\}_{t\in I} \subset L^1(\mathbb{P})$. If a strategy $\gamma^* \in \mathcal{M}_I$ exists such that $\mathbb{E}[Z_{\gamma^*}]$ equals the value of an upper bound, then γ^* is optimal.*

Remark 4.2. Note that the optimal strategy γ^* yielding the value of an upper bound may not be unique and that the result also holds for stopping times $\tau \in \mathcal{T}_I^\nu \neq \varnothing$ or adapted random probability measures $\gamma \in \mathcal{M}_I^\nu$ or $\gamma \in \mathcal{N}_I^\nu$.

4.1 Expected Shortfall and Conditional Expected Shortfall

In this section we will make a short excursion into the realm of expected shortfall and conditional expected shortfall. Subsection 4.1.1 will concentrate on results about the expected shortfall, which may be quite technical, but useful for deriving upper bounds. Subsection 4.1.2 deals with the definition of the conditional expected shortfall and its properties. The results of Subsection 4.1.2 may not be particularly important for later chapters, but they are of mathematical interest and give a nice generalization of the results of the expected shortfall.

4.1.1 Notes on the Expected Shortfall

In this subsection we will state the definitions of quantiles and the expected shortfall, as they will be used later on. The result of Lemma 4.6 will be especially useful for deriving upper bounds.

Definition 4.3. Given a random variable $X\colon \Omega \to \mathbb{R}$ and $\delta \in [0,1]$.

(a) Define the δ-quantile of X by

$$q_\delta(X) := \inf\{x \in \mathbb{R} \,|\, \mathbb{P}(X \le x) \ge \delta\}\,.$$

Note that $q_0(X) = -\infty$ and if $\mathbb{P}(X \le x) < 1$ for all $x \in \mathbb{R}$, then $q_1(X) = \infty$.

(b) Define $f_{\delta,X}\colon \Omega \to [0,1]$ by

$$f_{\delta,X} := \begin{cases} 0 & \text{if } \delta = 1, \\ 1_{\{X > q_\delta(X)\}} + \beta_{\delta,X} 1_{\{X = q_\delta(X)\}} & \text{if } \delta \in [0,1), \end{cases}$$

where

$$\beta_{\delta,X} := \begin{cases} \frac{\mathbb{P}(X \le q_\delta(X)) - \delta}{\mathbb{P}(X = q_\delta(X))} & \text{if } \mathbb{P}\left(X = q_\delta(X)\right) > 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

(c) The expected shortfall of X at level δ is given by $(1-\delta)\,\mathrm{ES}_\delta(X) = \mathbb{E}[f_{\delta,X} X]$. Note that $\mathrm{ES}_0(X) = \mathbb{E}[X]$ and $\mathrm{ES}_1(X) = 0$ as $\beta_{1,X} = f_{1,X} = 0$.

Remarks 4.4. (a) Note that $\beta_{\delta,X} \in [0,1]$, because

$$\mathbb{P}(X < q_\delta(X)) \le \delta \le \mathbb{P}(X \le q_\delta(X))\,.$$

Therefore $f_{\delta,X}$ is $[0,1]$-valued.

(b) For $\delta \in [0,1]$ we have $\mathbb{E}[f_{\delta,X}] = 1 - \delta$. This is due to the fact that for $\delta \in [0,1)$

$$\mathbb{E}[f_{\delta,X}] = \mathbb{P}(X > q_\delta(X)) + \beta_{\delta,X} \mathbb{P}(X = q_\delta(X))\,.$$

Remark 4.5. Such a representation of the expected shortfall is for example used in [54]. The definition used in [3] amounts to a similar representation. Note that there are several representations for the expected shortfall and also different names for it, for example conditional value-at-risk. [38, Chapter 2.2.4] or [23, Chapter 5.6] discuss different representations and names for the expected shortfall.

Lemma 4.6. *Let X and Y be real-valued random variables, $\delta \in [0,1]$. Assume $X \ge 0$, $\mathbb{E}[X] < \infty$ and $\mathbb{E}[|Y|] < \infty$. Define*

$$\mathcal{F}^X_{\delta,Y} := \left\{ f\colon \Omega \to [0,1] \,\middle|\, f \text{ measurable}, \mathbb{E}[f\,X] = \mathbb{E}[f_{\delta,Y} X] \right\}.$$

Then the following holds:

(a) $\mathbb{E}[f_{\delta,Y} XY]$ *is well-defined and*

$$\sup_{f \in \mathcal{F}^X_{\delta,Y}} \mathbb{E}[f\,XY] = \mathbb{E}[f_{\delta,Y} XY]\ .$$

(b) If $f^* \in \mathcal{F}^X_{\delta,Y}$ satisfies $\mathbb{E}[f^*XY] = \mathbb{E}[f_{\delta,Y}XY] < \infty$, then $f^* \overset{a.s.}{=} f_{\delta,Y}$ on $\{X > 0, Y \neq q_\delta(Y)\}$.

(c) If X and $f_{\delta,Y}Y$ are uncorrelated, then

$$\mathbb{E}[f_{\delta,Y}XY] = \mathbb{E}[X]\,\mathbb{E}[f_{\delta,Y}Y] = (1-\delta)\,\mathbb{E}[X]\,\mathrm{ES}_\delta(Y)\,.$$

Remark 4.7. Note that X and $f_{\delta,Y}Y$ are uncorrelated, if X and Y are independent.

Proof. (a) $\mathbb{E}[f_{\delta,Y}XY]$ is well-defined, because $(f_{\delta,Y}XY)^- = X(f_{\delta,Y}Y)^- \leq X\min\{0, q_\delta(Y)\}$ as $X \geq 0$ and X is integrable. Assume $\delta \in (0,1)$. By Remark 4.4(a) $f_{\delta,Y}$ is $[0,1]$-valued. If $\mathbb{E}[f_{\delta,Y}XY] = \infty$, the result follows trivially, as $f_{\delta,Y} \in \mathcal{F}^X_{\delta,Y}$.

Assume $f \in \mathcal{F}^X_{\delta,Y}$ with $\mathbb{E}[fXY^-] < \infty$ and $\mathbb{E}[f_{\delta,Y}XY] < \infty$. As $\mathbb{E}[(f - f_{\delta,Y})Xq_\delta(Y)] = 0$, we get

$$\begin{aligned}
\mathbb{E}[fXY] - \mathbb{E}[f_{\delta,Y}XY] &= \mathbb{E}[(f - f_{\delta,Y})X(Y - q_\delta(Y))] \\
&= \mathbb{E}[\underbrace{(f - f_{\delta,Y})}_{\leq 0}\,X\,\underbrace{(Y - q_\delta(Y))}_{>0}\,1_{\{Y > q_\delta(Y)\}}] \\
&\quad + \mathbb{E}[\underbrace{(f - f_{\delta,Y})}_{\geq 0}\,X\,\underbrace{(Y - q_\delta(Y))}_{<0}\,1_{\{Y < q_\delta(Y)\}}] \leq 0\,.
\end{aligned}$$

If $\delta = 0$, we have $f_{\delta,Y} = 1$. Therefore $f = 1$ for all $f \in \mathcal{F}^X_{\delta,Y}$. If $\delta = 1$, then $f_{\delta,Y} = 0$, which implies $f = 0$ for all $f \in \mathcal{F}^X_{\delta,Y}$.

(b) Assume there exists a $f^* \in \mathcal{F}^X_{\delta,Y}$, such that $\mathbb{E}[f^*XY] = \mathbb{E}[f_{\delta,Y}XY] < \infty$. Then by the above calculations $\mathbb{P}(f^* < f_{\delta,Y}, X > 0, Y > q_\delta(Y)) = 0$ and $\mathbb{P}(f^* > f_{\delta,Y}, X > 0, Y < q_\delta(Y)) = 0$. Therefore $f^* \overset{\text{a.s.}}{=} f_{\delta,Y}$ on $\{X > 0, Y \neq q_\delta(Y)\}$.

(c) follows from the definition of correlation. □

4.1.2 Notes on the Conditional Expected Shortfall

In this subsection we introduce conditional quantiles and the conditional expected shortfall. We also discuss some properties of the conditional expected shortfall and show some results for it. For the definition of conditional quantiles and the conditional expected shortfall we will need to use the essential supremum and the essential infimum, for which we refer to [19, Chapter I.3] or [14, Chapter A.5]. In addition, we use a general version of the conditional expectation, where this conditional expectation is defined for random variables that are σ-integrable with respect to the σ-algebra on which the expected value is conditioned. For this definition of the conditional expected value we refer to [19, Chapter I.4].

Definition 4.8. Consider a random variable $X\colon \Omega \to \mathbb{R}$ and a sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$. Define $X^{\mathcal{G}}$ as the $\mathcal{G}$-measurable upper envelope of X, i.e. as the essential infimum of all $\mathcal{G}$-measurable random variables $Y\colon \Omega \to \mathbb{R} \cup \{\infty\}$ satisfying $\mathbb{P}(Y \geq X) = 1$.

The notion of a $\mathcal{G}$-measurable upper envelope is also used, for example, in [16].

Remarks 4.9. (a) We can show that $X^{\mathcal{G}}$ is $\mathcal{G}$-measurable and satisfies $X^{\mathcal{G}} \geq X$ a.s. For this let $\Phi_{\mathcal{G}}(X)$ be the set of all $\mathcal{G}$-measurable random variables, which are greater than or equal to X almost surely. For Y and $\tilde{Y}$ in $\Phi_{\mathcal{G}}(X)$, we have $\min\{Y, \tilde{Y}\} \in \Phi_{\mathcal{G}}(X)$. Therefore we can apply [14, Theorem A.33(b)]. Consider a sequence $Y_n \searrow Y$ for $n \to \infty$ with $Y_n \in \Phi_{\mathcal{G}}(X)$ for all $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ and some null set N_n we have

$Y_n(\omega) \geq X(\omega)$ for all $\omega \in \Omega \setminus N_n$. Set $N = \bigcup_{n\in\mathbb{N}} N_n$. Then N is a null set and for all $n \in \mathbb{N}$ we have $Y_n(\omega) \geq X(\omega)$ for all $\omega \in \Omega \setminus N$. Therefore $Y(\omega) \geq X(\omega)$ for all $\omega \in \Omega \setminus N$, which means $\mathbb{P}(Y \geq X) = 1$. This implies that $X^{\mathcal{G}} \in \Phi_{\mathcal{G}}(X)$, as we can represent the essential infimum as the limit of elements of the set by [14, Theorem A.33(b)].

(b) For $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ we have $X^{\mathcal{G}} \leq X^{\mathcal{H}}$ almost surely, because in the notation used above we have $\Phi_{\mathcal{H}}(X) \subset \Phi_{\mathcal{G}}(X)$.

(c) If X is $\mathcal{G}$-measurable, then $X^{\mathcal{G}} = X$ almost surely.

(d) For two random variables X and Y we have $(X+Y)^{\mathcal{G}} \leq X^{\mathcal{G}} + Y^{\mathcal{G}}$ a.s., because $X \leq X^{\mathcal{G}}$ a.s. and $Y \leq Y^{\mathcal{G}}$ a.s. imply $X + Y \leq X^{\mathcal{G}} + Y^{\mathcal{G}}$ a.s. Therefore $(X^{\mathcal{G}} + Y^{\mathcal{G}}) \in \Phi_{\mathcal{G}}(X+Y)$.

(e) For two random variables X and Y we have $(XY)^{\mathcal{G}} \leq X^{\mathcal{G}} Y^{\mathcal{G}}$ a.s., because $X \leq X^{\mathcal{G}}$ a.s. and $Y \leq Y^{\mathcal{G}}$ a.s. imply $XY \leq X^{\mathcal{G}} Y^{\mathcal{G}}$ a.s. Therefore $(X^{\mathcal{G}} Y^{\mathcal{G}}) \in \Phi_{\mathcal{G}}(XY)$.

(f) For two random variables X and Y, where Y is $\mathcal{G}$-measurable, we have $(X+Y)^{\mathcal{G}} = X^{\mathcal{G}} + Y$ a.s., because $(X+Y)^{\mathcal{G}} \leq X^{\mathcal{G}} + Y^{\mathcal{G}} = X^{\mathcal{G}} + Y$ a.s. by (d) and (c). On the other hand $((X+Y)^{\mathcal{G}} - Y) \in \Phi_{\mathcal{G}}(X)$, which implies $X^{\mathcal{G}} \leq (X+Y)^{\mathcal{G}} - Y$ a.s.

(g) For two random variables X and Y, where $Y > 0$ is $\mathcal{G}$-measurable, we have $(XY)^{\mathcal{G}} = X^{\mathcal{G}} Y$ a.s., because $(XY)^{\mathcal{G}} \leq X^{\mathcal{G}} Y^{\mathcal{G}} = X^{\mathcal{G}} Y$ a.s. by (e) and (c). On the other hand $((XY)^{\mathcal{G}}/Y) \in \Phi_{\mathcal{G}}(X)$, which implies $X^{\mathcal{G}} \leq (XY)^{\mathcal{G}}/Y$ a.s.

Definition 4.10. Consider a random variable $X\colon \Omega \to \mathbb{R}$, a sub-$\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$ and $\delta \in [0,1]$. Define the δ-quantile $q_{\mathcal{G},\delta}(X)$ of X given $\mathcal{G}$ as the essential infimum of all $\mathcal{G}$-measurable random variables $Y\colon \Omega \to \mathbb{R} \cup \{\infty\}$ satisfying $\mathbb{P}(Y \geq X \,|\, \mathcal{G}) \geq \delta$ almost surely.

Remarks 4.11. (a) In the definition of the conditional quantile in Definition 4.10, note that $q_{\mathcal{G},0}(X) = -\infty$ and that $q_{\mathcal{G},1}(X) \overset{\text{a.s.}}{=} X^{\mathcal{G}}$.

(b) The conditional δ-quantile $q_{\mathcal{G},\delta}(X)$ is $\mathcal{G}$-measurable and satisfies $\mathbb{P}(q_{\mathcal{G},\delta}(X) \geq X \,|\, \mathcal{G}) \geq \delta$ almost surely. To prove this, let $\Phi_{\mathcal{G},\delta}(X)$ be the set of all $\mathcal{G}$-measurable random variables Y satisfying $\mathbb{P}(Y \geq X \,|\, \mathcal{G}) \geq \delta$ almost surely. For a random variable $Y \in \Phi_{\mathcal{G},\delta}(X)$ we have for every $G \in \mathcal{G}$

$$\mathbb{E}\big[1_{\{Y \geq X\}} 1_G\big] \geq \delta\, \mathbb{E}[1_G] = \delta\, \mathbb{P}(G)\,.$$

Note that for $Y, \tilde{Y} \in \Phi_{\mathcal{G},\delta}$ we have $\{Y \geq \tilde{Y}\} \in \mathcal{G}$ and that

$$\{\min\{Y, \tilde{Y}\} \geq X\} = \{Y \geq X\} \cap \{\tilde{Y} \geq X\}\,.$$

Then

$$\begin{aligned}
\mathbb{E}\Big[1_{\{\min\{Y,\tilde{Y}\} \geq X\}} 1_G\Big] &= \mathbb{E}\Big[1_{\{Y \geq X\}} 1_{\{\tilde{Y} \geq X\}} 1_G\Big] \\
&= \mathbb{E}\Big[1_{\{Y \geq X\}} 1_{\{\tilde{Y} \geq X\}} 1_{\{Y \geq \tilde{Y}\}} 1_G\Big] + \mathbb{E}\Big[1_{\{Y \geq X\}} 1_{\{\tilde{Y} \geq X\}} 1_{\{Y < \tilde{Y}\}} 1_G\Big] \\
&\geq \delta\, \mathbb{P}(\{Y \geq \tilde{Y}\} \cap G) + \delta\, \mathbb{P}(\{Y < \tilde{Y}\} \cap G) = \delta\, \mathbb{P}(G)\,.
\end{aligned}$$

Therefore $\min\{Y, \tilde{Y}\} \in \Phi_{\mathcal{G},\delta}(X)$. Again we can apply [14, Theorem A.33(b)]. Consider a sequence $Y_n \searrow Y$ for $n \to \infty$ with $Y_n \in \Phi_{\mathcal{G},\delta}(X)$ for all $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$

$$\mathbb{E}\big[1_{\{Y_n \geq X\}} 1_G\big] \geq \delta\, \mathbb{P}(G)\,.$$

We have $1_{\{Y_n \geq X\}} \searrow 1_{\{Y \geq X\}}$ as $\{Y \geq X\} = \bigcap_{n\in\mathbb{N}}\{Y_n \geq X\}$. Therefore for $n \to \infty$

$$\mathbb{E}\big[1_{\{Y_n \geq X\}} 1_G\big] \searrow \mathbb{E}\big[1_{\{Y \geq X\}} 1_G\big] \geq \delta\, \mathbb{P}(G)\,.$$

(c) For $\delta \leq \delta'$ we have $q_{\mathcal{G},\delta}(X) \leq q_{\mathcal{G},\delta'}(X)$ almost surely.

Definition 4.12. Consider a random variable $X\colon \Omega \to \mathbb{R}$, a sub-$\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$ and $\delta \in [0,1]$. Define $f_{\mathcal{G},\delta,X}\colon \Omega \to [0,1]$ by

$$f_{\mathcal{G},\delta,X} := 1_{\{X > q_{\mathcal{G},\delta}(X)\}} + \beta_{\mathcal{G},\delta,X} 1_{\{X = q_{\mathcal{G},\delta}(X)\}},$$

where $\beta_{\mathcal{G},\delta,X}\colon \Omega \to [0,1]$ is a $\mathcal{G}$-measurable random variable satisfying a.s.

$$\beta_{\mathcal{G},\delta,X} = \begin{cases} \frac{\mathbb{P}(X \leq q_{\mathcal{G},\delta}(X) \,|\, \mathcal{G}) - \delta}{\mathbb{P}\left(X = q_{\mathcal{G},\delta}(X) \,|\, \mathcal{G}\right)} & \text{on the event } \left\{\mathbb{P}\big(X = q_{\mathcal{G},\delta}(X) \,\big|\, \mathcal{G}\big) > 0\right\}, \\ 0 & \text{otherwise}\,. \end{cases}$$

Remarks 4.13. (a) Note that $\beta_{\mathcal{G},\delta,X} \in [0,1]$ a.s., because

$$\mathbb{P}(X < q_{\mathcal{G},\delta}(X) \,|\, \mathcal{G}) \leq \delta \leq \mathbb{P}(X \leq q_{\mathcal{G},\delta}(X) \,|\, \mathcal{G}) \quad \text{a.s.}$$

Therefore $f_{\mathcal{G},\delta,X}$ is $[0,1]$-valued a.s.

(b) For $\delta \in [0,1]$ we have $\mathbb{E}[f_{\mathcal{G},\delta,X} \,|\, \mathcal{G}] = 1 - \delta$ almost surely. For $\delta \in [0,1)$ this is due to the fact that

$$\mathbb{E}[f_{\mathcal{G},\delta,X} \,|\, \mathcal{G}] \overset{\text{a.s.}}{=} \mathbb{P}(X > q_{\mathcal{G},\delta}(X) \,|\, \mathcal{G}) + \beta_{\mathcal{G},\delta,X}\mathbb{P}(X = q_{\mathcal{G},\delta}(X) \,|\, \mathcal{G})\,.$$

For $\delta = 1$ we have $\{X > q_{\mathcal{G},\delta}(X)\} = \varnothing$ a.s. and $\beta_{\mathcal{G},\delta,X} = 0$ a.s.

Definition 4.14. Consider a random variable $X\colon \Omega \to \mathbb{R}$, a sub-$\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$ and $\delta \in [0,1]$. The conditional expected shortfall of X at level δ given $\mathcal{G}$ is defined by

$$\mathrm{ES}_\delta(X \,|\, \mathcal{G}) = \begin{cases} X^{\mathcal{G}} & \text{if } \delta = 1, \\ \frac{1}{1-\delta}\, \mathbb{E}[f_{\mathcal{G},\delta,X} X \,|\, \mathcal{G}] & \text{if } \delta \in (0,1). \\ \operatorname{ess\,inf}_{\delta' \in (0,1)} \frac{1}{1-\delta'}\, \mathbb{E}\left[f_{\mathcal{G},\delta',X} X \,|\, \mathcal{G}\right] & \text{if } \delta = 0. \end{cases}$$

Remark 4.15. (a) If X is a $\mathcal{G}$-measurable random variable, then $\mathrm{ES}_\delta(X \,|\, \mathcal{G}) = X$ almost surely. For $\delta \in (0,1)$ this follows either from Lemma 4.22(b) or Definition 4.14, [19, Theorem 1.21] and Remark 4.13(b). For $\delta = 1$ this result follows from Remark 4.9(c) and for $\delta = 0$ this follows from the result for $\delta \in (0,1)$ and from the representation of $\mathrm{ES}_0(X \,|\, \mathcal{G})$ in Definition 4.14.

(b) In Definition 4.14, we have that $(f_{\mathcal{G},\delta,X} X)^-$ is σ-integrable with respect to $\mathcal{G}$, because $f_{\mathcal{G},\delta,X} X \geq \min\{0, q_{\mathcal{G},\delta}(X)\}$ a.s., where $\min\{0, q_{\mathcal{G},\delta}(X)\} =: \tilde{X}$ is $\mathcal{G}$-measurable. Therefore $\Omega_n = \{|\tilde{X}| \leq n\} \in \mathcal{G}$ for all $n \in \mathbb{N}$ and $\Omega_n \nearrow \Omega$ for $n \to \infty$. Then for every $n \in \mathbb{N}$ we have $\mathbb{E}[(f_{\mathcal{G},\delta,X} X)^- 1_{\Omega_n}] \leq \mathbb{E}[\tilde{X} 1_{\Omega_n}] \leq n$, which implies that $(f_{\mathcal{G},\delta,X} X)^-$ is σ-integrable with respect to $\mathcal{G}$. For the definition of σ-integrability with respect to $\mathcal{G}$ we refer to [19, Definition 1.15]. For a non-negative random variable $X \geq 0$ we can write $\mathbb{E}[X \,|\, \mathcal{G}] = \operatorname{ess\,sup}_{n\in\mathbb{N}} \mathbb{E}[\min\{X, n\} \,|\, \mathcal{G}]$ a.s., where for every $n \in \mathbb{N}$ the random variable $\min\{X, n\}$ is σ-integrable with respect to $\mathcal{G}$, as it is even integrable.

Remark 4.16. By setting $\mathcal{G}$ trivial, i.e. we assume $\mathbb{P}(G) \in \{0,1\}$ for all $G \in \mathcal{G}$, and $\delta \in (0,1)$ this amounts to the standard definitions of the δ-quantile and the expected shortfall, as in Subsection 4.1.1.

Remark 4.17. The conditional expected shortfall for random variables is used, for example, in [37], where for continuous distribution functions the representation there equals the representation of Definition 4.14. In [38] different methods for measuring market risk in both conditional and unconditional cases are discussed. [44] and [33] also discuss how a conditional expected shortfall can be estimated. In [11] conditional convex risk measures and their representation in terms of conditional expectation are discussed. In [2, Example 1.10] it is noted that the conditional expected shortfall is a coherent risk measure. This will also be shown in Lemma 4.22 using the representation of Definition 4.14.

Lemma 4.18. *Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$. The conditional quantile of Definition 4.10 at level $\delta \in [0,1]$ has, for all real-valued random variables X and all $\mathcal{G}$-measurable random variables Z, the following properties:*

(a) $q_{\mathcal{G},\delta}(X+Z) = q_{\mathcal{G},\delta}(X) + Z$ almost surely.

(b) If further $Z > 0$, then $q_{\mathcal{G},\delta}(XZ) = q_{\mathcal{G},\delta}(X)Z$ almost surely.

Proof. Using the notation of Remark 4.11(b) we have $(q_{\mathcal{G},\delta}(X) + Z) \in \Phi_{\mathcal{G},\delta}(X+Z)$, as $(q_{\mathcal{G},\delta}(X) + Z)$ is $\mathcal{G}$-measurable and $\mathbb{P}((q_{\mathcal{G},\delta}(X) + Z) \ge X + Z \,|\, \mathcal{G}) = \mathbb{P}(q_{\mathcal{G},\delta}(X) \ge X \,|\, \mathcal{G}) \ge \delta$. Therefore $q_{\mathcal{G},\delta}(X+Z) \le (q_{\mathcal{G},\delta}(X) + Z)$ a.s.

On the other hand $(q_{\mathcal{G},\delta}(X+Z) - Z) \in \Phi_{\mathcal{G},\delta}(X)$, as $(q_{\mathcal{G},\delta}(X+Z) - Z)$ is $\mathcal{G}$-measurable and $\mathbb{P}((q_{\mathcal{G},\delta}(X+Z) - Z) \ge X \,|\, \mathcal{G}) = \mathbb{P}(q_{\mathcal{G},\delta}(X+Z) \ge X + Z \,|\, \mathcal{G}) \ge \delta$. Therefore $q_{\mathcal{G},\delta}(X) \le (q_{\mathcal{G},\delta}(X+Z) - Z)$ a.s.

Altogether $q_{\mathcal{G},\delta}(X+Z) \overset{\text{a.s.}}{=} q_{\mathcal{G},\delta}(X) + Z$.

For $\delta = 0$ we have $q_{\mathcal{G},\delta}(Z+X) = q_{\mathcal{G},\delta}(X) = -\infty$. For $\delta = 1$ the result follows from Remark 4.9(f).

Similarly $q_{\mathcal{G},\delta}(ZX) = Zq_{\mathcal{G},\delta}(X)$ almost surely for $Z > 0$, using Remark 4.9(g) for $\delta = 1$. □

Lemma 4.19. *Let X and Y be real-valued random variables, $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra, and $\delta \in [0,1]$. Assume that $X \ge 0$ and that X and Y are σ-integrable with respect to $\mathcal{G}$. Define*

$$\mathcal{F}^X_{\mathcal{G},\delta,Y} := \left\{ f \colon \Omega \to [0,1] \;\middle|\; f \text{ is } \mathcal{F}\text{-measurable}, \mathbb{E}[fX \,|\, \mathcal{G}] \overset{a.s.}{=} \mathbb{E}[f_{\mathcal{G},\delta,Y} X \,|\, \mathcal{G}] \right\}.$$

Then the following holds:

(a) $\mathbb{E}[f_{\mathcal{G},\delta,Y} XY \,|\, \mathcal{G}]$ is a well-defined random variable with values in $\mathbb{R} \cup \{\infty\}$ (for $\delta = 0$ we further need to assume that XY is σ-integrable w.r.t. $\mathcal{G}$) and

$$\operatorname*{ess\,sup}_{f \in \mathcal{F}^X_{\mathcal{G},\delta,Y}} \mathbb{E}[fXY \,|\, \mathcal{G}] \overset{a.s.}{=} \mathbb{E}[f_{\mathcal{G},\delta,Y} XY \,|\, \mathcal{G}].$$

(b) If $f^* \in \mathcal{F}^X_{\mathcal{G},\delta,Y}$ satisfies $\mathbb{E}[f^* XY \,|\, \mathcal{G}] = \mathbb{E}[f_{\mathcal{G},\delta,Y} XY \,|\, \mathcal{G}] < \infty$ a.s., then $f^* \overset{a.s.}{=} f_{\mathcal{G},\delta,Y}$ on the event $\{X > 0, Y \ne q_{\mathcal{G},\delta}(Y)\}$.

(c) If $\delta \in (0,1)$ and X and $f_{\mathcal{G},\delta,Y} Y$ are conditionally uncorrelated given $\mathcal{G}$ (in particular if X and Y are conditionally independent given $\mathcal{G}$), then

$$\mathbb{E}[f_{\mathcal{G},\delta,Y} XY \,|\, \mathcal{G}] = \mathbb{E}[X \,|\, \mathcal{G}]\, \mathbb{E}[f_{\mathcal{G},\delta,Y} Y \,|\, \mathcal{G}] = (1-\delta)\, \mathbb{E}[X \,|\, \mathcal{G}]\, \mathrm{ES}_\delta(Y \,|\, \mathcal{G}).$$

Remark 4.20. Note that $f_{\mathcal{G},\delta,Y} \in \mathcal{F}^X_{\mathcal{G},\delta,Y}$.

Proof. (a) Assume $\delta \in (0,1)$. $\mathbb{E}[f_{\mathcal{G},\delta,Y}XY\,|\,\mathcal{G}]$ is a well-defined random variable with values in $\mathbb{R}\cup\{\infty\}$, because $(f_{\mathcal{G},\delta,Y}XY)^- = X(f_{\mathcal{G},\delta,Y}Y)^- \le X\min\{0,q_{\mathcal{G},\delta}(Y)\}$ a.s. by Remark 4.15(b) and as $X \ge 0$, where X is σ-integrable with respect to $\mathcal{G}$ and $\min\{0,q_{\mathcal{G},\delta}(Y)\}$ is $\mathcal{G}$-measurable.

In Remark 4.13(a) we noted that $f_{\mathcal{G},\delta,Y}$ is $[0,1]$-valued a.s. and by Remark 4.20 we have $f_{\mathcal{G},\delta,Y} \in \mathcal{F}^X_{\mathcal{G},\delta,Y}$. Consider $f \in \mathcal{F}^X_{\mathcal{G},\delta,Y}$ such that fXY^- is σ-integrable with respect to $\mathcal{G}$. If $\mathbb{E}[f_{\mathcal{G},\delta,Y}XY\,|\,\mathcal{G}] = \infty$ the result of the Lemma follows trivially. Assume $\mathbb{E}[f_{\mathcal{G},\delta,Y}XY\,|\,\mathcal{G}] < \infty$. Then for every $G \in \mathcal{G}$, as

$$\mathbb{E}[(f - f_{\mathcal{G},\delta,Y})Xq_{\mathcal{G},\delta}(Y)1_G\,|\,\mathcal{G}] \overset{\text{a.s.}}{=} 1_G\, q_{\mathcal{G},\delta}(Y)\,\mathbb{E}[(f - f_{\mathcal{G},\delta,Y})X\,|\,\mathcal{G}] \overset{\text{a.s.}}{=} 0\,,$$

we get

$$\begin{aligned}
\mathbb{E}[fXY1_G\,|\,\mathcal{G}] - \mathbb{E}[f_{\mathcal{G},\delta,Y}XY1_G\,|\,\mathcal{G}] &\overset{\text{a.s.}}{=} \mathbb{E}[(f - f_{\mathcal{G},\delta,Y})X(Y - q_{\mathcal{G},\delta}(Y))1_G\,|\,\mathcal{G}]\\
&\overset{\text{a.s.}}{=} \mathbb{E}[\underbrace{(f - f_{\mathcal{G},\delta,Y})}_{\le 0}\,X\,\underbrace{(Y - q_{\mathcal{G},\delta}(Y))}_{>0}\,1_{\{Y>q_\delta(Y)\}}1_G\,|\,\mathcal{G}]\\
&\quad + \mathbb{E}[\underbrace{(f - f_{\mathcal{G},\delta,Y})}_{\ge 0}\,X\,\underbrace{(Y - q_{\mathcal{G},\delta}(Y))}_{<0}\,1_{\{Y<q_\delta(Y)\}}1_G\,|\,\mathcal{G}]\\
&\overset{\text{a.s.}}{\le} 0\,.
\end{aligned}$$

If $\delta = 0$, we have $f_{\mathcal{G},\delta,Y} \overset{\text{a.s.}}{=} 1$. Therefore $f \overset{\text{a.s.}}{=} 1$ for all $f \in \mathcal{F}^X_{\mathcal{G},\delta,Y}$. If $\delta = 1$, then $\beta_{\mathcal{G},\delta,Y} \overset{\text{a.s.}}{=} 0$, which implies $f_{\mathcal{G},\delta,Y} \overset{\text{a.s.}}{=} 0$ and $f \overset{\text{a.s.}}{=} 0$ for all $f \in \mathcal{F}^X_{\mathcal{G},\delta,Y}$.

(b) Assume there exists a $f^* \in \mathcal{F}^X_{\mathcal{G},\delta,Y}$, such that $\mathbb{E}[f^*XY\,|\,\mathcal{G}] \overset{\text{a.s.}}{=} \mathbb{E}[f_{\mathcal{G},\delta,Y}XY\,|\,\mathcal{G}] < \infty$. Then by the above calculations $\mathbb{P}(f^* < f_{\mathcal{G},\delta,Y}, X > 0, Y > q_{\mathcal{G},\delta}(Y)\,|\,\mathcal{G}) = 0$ and $\mathbb{P}(f^* > f_{\mathcal{G},\delta,Y}, X > 0, Y < q_{\mathcal{G},\delta}(Y)\,|\,\mathcal{G}) = 0$. Therefore $f^* \overset{\text{a.s.}}{=} f_{\mathcal{G},\delta,Y}$ on $\{X > 0, Y \ne q_{\mathcal{G},\delta}(Y)\}$.

(c) This follows from the definition of correlation and the definition of the conditional expected shortfall. □

In [54, Lemma 3.32] different properties of the expected shortfall are noted. As we could not find a list of the different properties for the conditional expected shortfall in the representation we used, we will note them, similar to [54, Lemma 3.32], in the following lemma. Similar to [54], for some $\delta \in (0,1)$ we let $\mathcal{F}_{\mathcal{G},\delta}$ denote the set of all conditional probability densities given $\mathcal{G}$ on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ bounded by $\frac{1}{1-\delta}$, i.e.

$$\mathcal{F}_{\mathcal{G},\delta} := \Big\{ f\colon \Omega \to [0,1] \,\Big|\, \mathbb{E}[f\,|\,\mathcal{G}] \overset{\text{a.s.}}{=} 1,\ f \overset{\text{a.s.}}{\le} \frac{1}{1-\delta} \Big\}.$$

Further, for a real-valued random variable X we define

$$\mathcal{F}_{\mathcal{G},\delta,X} := \{ f \in \mathcal{F}_{\mathcal{G},\delta} \,|\, \mathbb{E}\big[X^+ f\,|\,\mathcal{G}\big] \overset{\text{a.s.}}{<} \infty \text{ or } \mathbb{E}\big[X^- f\,|\,\mathcal{G}\big] \overset{\text{a.s.}}{<} \infty \}\,.$$

Remark 4.21. Note that for $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ we have $\mathcal{F}_{\mathcal{G},\delta} \subset \mathcal{F}_{\mathcal{H},\delta}$. Further $\mathcal{F}_{\mathcal{G},\delta'} \subset \mathcal{F}_{\mathcal{G},\delta}$ for $\delta' < \delta$.

Lemma 4.22. *Given a probability space* $(\Omega,\mathcal{F},\mathbb{P})$ *and a sub-σ-algebra* $\mathcal{G} \subset \mathcal{F}$. *The conditional expected shortfall of Definition 4.14 at level* $\delta \in [0,1]$ *has, for all real-valued random variables* X *and* Y, *the following properties:*

(a) *Positive Homogeneity: If $Z > 0$ is $\mathcal{G}$-measurable, then*

$$\mathrm{ES}_\delta(ZX \,|\, \mathcal{G}) = Z\,\mathrm{ES}_\delta(X \,|\, \mathcal{G}) \text{ a.s.}$$

(b) *Translation (or cash) invariance: If Z is $\mathcal{G}$-measurable, then*

$$\mathrm{ES}_\delta(X + Z \,|\, \mathcal{G}) = \mathrm{ES}_\delta(X \,|\, \mathcal{G}) + Z \text{ a.s.}$$

(c) *Sub-additivity:*

$$\mathrm{ES}_\delta(X + Y \,|\, \mathcal{G}) \le \mathrm{ES}_\delta(X \,|\, \mathcal{G}) + \mathrm{ES}_\delta(Y \,|\, \mathcal{G}) \text{ a.s.}$$

(d) *Monotonicity: If $X \le Y$ a.s., then*

$$\mathrm{ES}_\delta(X \,|\, \mathcal{G}) \le \mathrm{ES}_\delta(Y \,|\, \mathcal{G}) \text{ a.s.}$$

(e) *Convexity: If $\alpha \in (0,1)$, then*

$$\mathrm{ES}_\delta(\alpha X + (1-\alpha)Y \,|\, \mathcal{G}) \le \alpha\,\mathrm{ES}_\delta(X \,|\, \mathcal{G}) + (1-\alpha)\,\mathrm{ES}_\delta(Y \,|\, \mathcal{G}) \text{ a.s.}$$

(f) *Bounds: By setting $\mathbb{E}[X^+ | \mathcal{G}]\,(1-\delta)^{-1} = \infty$ for $\delta = 1$, we get*

$$q_{\mathcal{G},\delta}(X) \overset{a.s.}{\le} \mathrm{ES}_\delta(X \,|\, \mathcal{G}) \overset{a.s.}{\le} \frac{\mathbb{E}[X^+ | \mathcal{G}]}{1-\delta}\,.$$

The conditional expected shortfall of Definition 4.14 at level $\delta \in (0,1)$ has, for all real-valued random variables X and Y, the following properties:

(g) *Scenario representation:*

   (i) $\mathrm{ES}_\delta(X \,|\, \mathcal{G}) = \frac{1}{1-\delta}\operatorname{ess\,sup}_{f \in \mathcal{F}^1_{\mathcal{G},\delta,X}} \mathbb{E}[f\,X | \mathcal{G}]$ *a.s.*

   (ii) $\mathrm{ES}_\delta(X \,|\, \mathcal{G}) = \operatorname{ess\,sup}_{f \in \mathcal{F}_{\mathcal{G},\delta,X}} \mathbb{E}[f\,X | \mathcal{G}]$ *a.s.*

   (iii) *If $\mathbb{E}[X^+ | \mathcal{G}] < \infty$ a.s., then* $\mathrm{ES}_\delta(X \,|\, \mathcal{G}) = \operatorname{ess\,sup}_{f \in \mathcal{F}_{\mathcal{G},\delta}} \mathbb{E}[f\,X | \mathcal{G}]$ *a.s.*

(h) *Let $\{X_n\}_{n\in\mathbb{N}}$ be bounded from below by some constant C. Then, if $X := \liminf_{n\to\infty} X_n$ is σ-integrable with respect to $\mathcal{G}$,*

$$\mathrm{ES}_\delta(X \,|\, \mathcal{G}) \le \liminf_{n\to\infty} \mathrm{ES}_\delta(X_n \,|\, \mathcal{G}) \text{ a.s.} \tag{4.23}$$

(i) *Let $\{X_n\}_{n\in\mathbb{N}}$ be bounded from below and converging in probability to a random variable X. Then (4.23) also holds.*

(j) *Let $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ be two σ-algebras. If $(X f_{\mathcal{G},\delta,X})^-$ is σ-integrable with respect to $\mathcal{H}$, then*

$$\mathrm{ES}_\delta(X \,|\, \mathcal{H}) \ge \mathbb{E}[\mathrm{ES}_\delta(X \,|\, \mathcal{G}) | \mathcal{H}] \text{ a.s.}$$

Remark 4.24. Note that (a), (b), (c) and (d) imply that the conditional expected shortfall is a coherent risk measure. This class of risk measure was introduced in [5].

Proof. (a) We start with $\delta \in (0,1)$. By Lemma 4.18(b), we have $q_{\mathcal{G},\delta}(ZX) = Zq_{\mathcal{G},\delta}(X)$ a.s. Therefore we know that $\beta_{\mathcal{G},\delta,ZX} = \beta_{\mathcal{G},\delta,X}$ a.s. and $f_{\mathcal{G},\delta,ZX} = f_{\mathcal{G},\delta,X}$ a.s. As Z is $\mathcal{G}$-measurable we get $\mathrm{ES}_\delta(ZX\,|\,\mathcal{G}) = Z\,\mathrm{ES}_\delta(X\,|\,\mathcal{G})$ a.s. using [19, Theorem 1.21]. For $\delta = 0$ the result follows from the representation of the conditional expected shortfall at level 0 using the essential infimum. For $\delta = 1$ the result follows from Remark 4.9(g), as $\mathrm{ES}_1(X\,|\,\mathcal{G}) = X^{\mathcal{G}}$.

(b) First consider $\delta \in (0,1)$. By Lemma 4.18(a) we have $q_{\mathcal{G},\delta}(X+Z) = q_{\mathcal{G},\delta}(X) + Z$ a.s. Again this implies $\beta_{\mathcal{G},\delta,X+Z} = \beta_{\mathcal{G},\delta,X}$ a.s. and $f_{\mathcal{G},\delta,X+Z} = f_{\mathcal{G},\delta,X}$ a.s. Using the linearity of the conditional expectation ([19, Theorem 1.18]) and as Z is $\mathcal{G}$-measurable ([19, Theorem 1.21]) we get $\mathrm{ES}_\delta(X+Z\,|\,\mathcal{G}) = \mathrm{ES}_\delta(X\,|\,\mathcal{G}) + Z$ a.s. For $\delta = 0$ the result follows from the representation of the conditional expected shortfall at level 0 using the essential infimum. For $\delta = 1$ the result follows from Remark 4.9(f), as $\mathrm{ES}_1(X\,|\,\mathcal{G}) = X^{\mathcal{G}}$.

(g) (gi) follows directly from Lemma 4.19. The proof of (gii) and (giii) works similar to the proof of Lemma 4.19. If $\frac{1}{1-\delta} f_{\mathcal{G},\delta,X} =: \tilde{f}_{\mathcal{G},\delta,X} \in \mathcal{F}_{\mathcal{G},\delta,X}$, the essential supremum is an upper bound for $\mathrm{ES}_\delta(X\,|\,\mathcal{G})$ and (gii) holds in the case $\mathrm{ES}_\delta(X\,|\,\mathcal{G}) = \infty$ a.s. If $\mathrm{ES}_\delta(X\,|\,\mathcal{G}) < \infty$ a.s., then necessarily $\mathbb{E}[X^+|\mathcal{G}] < \infty$ a.s., hence $\mathcal{F}_{\mathcal{G},\delta,X} = \mathcal{F}_{\mathcal{G},\delta}$ a.s. Consider $f \in \mathcal{F}_{\mathcal{G},\delta}$ with $\mathbb{E}[Xf\,|\,\mathcal{G}] > -\infty$ a.s. We have $\mathbb{E}\big[f - \tilde{f}_{\mathcal{G},\delta,X}\,\big|\,\mathcal{G}\big] = 0$ a.s., hence for every $G \in \mathcal{G}$

$$
\begin{aligned}
\mathbb{E}\big[fX1_G\,\big|\,\mathcal{G}\big] - \mathbb{E}\big[\tilde{f}_{\mathcal{G},\delta,X}X1_G\,\big|\,\mathcal{G}\big] &\overset{\text{a.s.}}{=} \mathbb{E}\big[(f - \tilde{f}_{\mathcal{G},\delta,X})(X - q_{\mathcal{G},\delta}(X))1_G\,\big|\,\mathcal{G}\big] \\
&\overset{\text{a.s.}}{=} \mathbb{E}\big[\underbrace{(f - \tilde{f}_{\mathcal{G},\delta,X})}_{\le 0}\underbrace{(X - q_{\mathcal{G},\delta}(X))}_{>0}1_{\{X>q_\delta(X)\}}1_G\,\big|\,\mathcal{G}\big] \\
&\quad + \mathbb{E}\big[\underbrace{(f - \tilde{f}_{\mathcal{G},\delta,X})}_{\ge 0}\underbrace{(X - q_{\mathcal{G},\delta}(X))}_{<0}1_{\{X<q_\delta(X)\}}1_G\,\big|\,\mathcal{G}\big] \overset{\text{a.s.}}{\le} 0\,,
\end{aligned}
$$

which means that the supremum is identical with $\mathbb{E}\big[\tilde{f}_{\mathcal{G},\delta,X}X\,\big|\,\mathcal{G}\big]$.

(c) Consider $\delta \in (0,1)$. As $\mathbb{E}[f_{\mathcal{G},\delta,X}\,|\,\mathcal{G}] = \mathbb{E}[f_{\mathcal{G},\delta,Y}\,|\,\mathcal{G}] = \mathbb{E}[f_{\mathcal{G},\delta,X+Y}\,|\,\mathcal{G}] = 1-\delta$ a.s., we note that $\mathcal{F}^1_{\mathcal{G},\delta,X} = \mathcal{F}^1_{\mathcal{G},\delta,Y} = \mathcal{F}^1_{\mathcal{G},\delta,X+Y}$ a.s. By (gi) and the linearity of the conditional expectation ([19, Theorem 1.18]) we get

$$
\begin{aligned}
\mathrm{ES}_\delta(X+Y\,|\,\mathcal{G}) &\overset{\text{a.s.}}{=} \frac{1}{1-\delta} \operatorname*{ess\,sup}_{f \in \mathcal{F}^1_{\mathcal{G},\delta,X+Y}} \mathbb{E}[f(X+Y)\,|\,\mathcal{G}] \\
&\overset{\text{a.s.}}{\le} \frac{1}{1-\delta}\Big(\operatorname*{ess\,sup}_{f \in \mathcal{F}^1_{\mathcal{G},\delta,X+Y}} \mathbb{E}[fX\,|\,\mathcal{G}] + \operatorname*{ess\,sup}_{f \in \mathcal{F}^1_{\mathcal{G},\delta,X+Y}} \mathbb{E}[fY\,|\,\mathcal{G}]\Big) \\
&\overset{\text{a.s.}}{=} \mathrm{ES}_\delta(X\,|\,\mathcal{G}) + \mathrm{ES}_\delta(Y\,|\,\mathcal{G})\,.
\end{aligned}
$$

For $\delta = 0$ the result follows from the representation of the conditional expected shortfall at level 0 using the essential infimum. For $\delta = 1$ the result follows from Remark 4.9(d).

(d) By (c) we get for $\delta \in [0,1]$

$$
\mathrm{ES}_\delta(X\,|\,\mathcal{G}) \overset{\text{a.s.}}{\le} \mathrm{ES}_\delta(X-Y\,|\,\mathcal{G}) + \mathrm{ES}_\delta(Y\,|\,\mathcal{G})\,.
$$

Consider $\delta \in (0,1)$. We know that $X - Y \le 0$ a.s. This implies $q_{\mathcal{G},\delta}(X-Y) \le 0$ a.s. and

$$
\mathbb{E}\Big[(X-Y)1_{\{(X-Y)>q_{\mathcal{G},\delta}(X-Y)\}}\Big|\,\mathcal{G}\Big] \overset{\text{a.s.}}{\le} 0\,.
$$

Therefore by Definition 4.14

$$
\mathrm{ES}_\delta(X-Y\,|\,\mathcal{G}) \overset{\text{a.s.}}{=} \frac{1}{1-\delta}\Big(\mathbb{E}\Big[(X-Y)1_{\{(X-Y)>q_{\mathcal{G},\delta}(X-Y)\}}\Big|\,\mathcal{G}\Big] + q_{\mathcal{G},\delta}(X-Y)\beta_{\mathcal{G},\delta,X-Y}\Big) \overset{\text{a.s.}}{\le} 0\,.
$$

For $\delta = 0$ the same result follows from the representation of the conditional expected shortfall at level 0 using the essential infimum. For $\delta = 1$ we have $\mathrm{ES}_1(X - Y \,|\, \mathcal{G}) = (X-Y)^{\mathcal{G}} \le 0$ a.s., because $X - Y \le 0$ a.s.

Altogether, for $\delta \in [0,1]$ we get

$$\mathrm{ES}_\delta(X \,|\, \mathcal{G}) \overset{\text{a.s.}}{\le} \mathrm{ES}_\delta(X - Y \,|\, \mathcal{G}) + \mathrm{ES}_\delta(Y \,|\, \mathcal{G}) \overset{\text{a.s.}}{\le} \mathrm{ES}_\delta(Y \,|\, \mathcal{G}) \,.$$

(e) follows from (a) and (c).

(f) First consider $\delta \in (0,1)$ and note that $X \le X^+$ a.s., which implies $\mathrm{ES}_\delta(X \,|\, \mathcal{G}) \le \mathrm{ES}_\delta(X^+ \,|\, \mathcal{G})$ a.s. by (d). Further $f_{\mathcal{G},\delta,X} \le 1$ a.s. Using these observations for the upper bound, both bounds now follow directly from Definition 4.14. For $\delta = 0$ we have $q_{\mathcal{G},0}(X) = -\infty$ and as $f_{\mathcal{G},\delta,X} \le 1$ a.s. for all $\delta \in (0,1)$ and $X \le X^+$ a.s., we also see that the upper bound is true. For $\delta = 1$ we have $\mathbb{E}[X^+ \,|\, \mathcal{G}]\,(1-\delta)^{-1} = \infty$ and $q_{\mathcal{G},1}(X) = \mathrm{ES}_1(X \,|\, \mathcal{G}) = X^{\mathcal{G}}$.

(h) By translation invariance from (b), we may assume without loss of generality that X_n is non-negative for every $n \in \mathbb{N}$. Similar to the proof of (c) we can show that $\mathcal{F}^1_{\mathcal{G},\delta,X} = \mathcal{F}^1_{\mathcal{G},\delta,X_n}$ a.s. for every $n \in \mathbb{N}$. By Definition 4.14 we can write

$$(1-\delta)\,\mathrm{ES}_\delta(X \,|\, \mathcal{G}) = \mathbb{E}[f_{\mathcal{G},\delta,X} X \,|\, \mathcal{G}] \;.$$

Using Fatou's Lemma for conditional expectations ([19, Theorem 1.19(2)]) for non-negative processes, we get

$$\mathbb{E}[f_{\mathcal{G},\delta,X}\, X \,|\, \mathcal{G}] \overset{\text{a.s.}}{\le} \liminf_{n\to\infty} \mathbb{E}[f_{\mathcal{G},\delta,X}\, X_n \,|\, \mathcal{G}] \;.$$

Further by Lemma 4.19 for every $n \in \mathbb{N}$

$$\mathbb{E}[f_{\mathcal{G},\delta,X}\, X_n \,|\, \mathcal{G}] \overset{\text{a.s.}}{\le} \operatorname*{ess\,sup}_{f \in \mathcal{F}^1_{\mathcal{G},\delta,X_n}} \mathbb{E}[f\, X_n \,|\, \mathcal{G}] \overset{\text{a.s.}}{=} (1-\delta)\,\mathrm{ES}_\delta(X_n \,|\, \mathcal{G}) \,.$$

Dividing by $1-\delta$ proves the result.

(i) By passing to a subsequence if necessary, we may assume that the sequence $\{\mathrm{ES}_\delta(X_n \,|\, \mathcal{G})\}_{n\in\mathbb{N}}$ converges to the limit inferior on the right-hand side of (4.23). By passing to a further subsequence if necessary, we may assume that $\{X_n\}_{n\in\mathbb{N}}$ converges almost surely to X. Then (4.23) follows from (h).

(j) By (giii) and the tower property of conditional expectation ([19, Theorem 1.22])

$$\mathrm{ES}_\delta(X \,|\, \mathcal{H}) \overset{\text{a.s.}}{=} \operatorname*{ess\,sup}_{f \in \mathcal{F}_{\mathcal{H},\delta}} \mathbb{E}[f\, X \,|\, \mathcal{H}] \overset{\text{a.s.}}{=} \operatorname*{ess\,sup}_{f \in \mathcal{F}_{\mathcal{H},\delta}} \mathbb{E}[\mathbb{E}[f\, X \,|\, \mathcal{G}] \,|\, \mathcal{H}] \;.$$

By Remark 4.21 and defining $\frac{1}{1-\delta} f_{\mathcal{G},\delta,X} =: \tilde{f}_{\mathcal{G},\delta,X} \in \mathcal{F}_{\mathcal{G},\delta}$ with $f_{\mathcal{G},\delta,X}$ as in Definition 4.12

$$\begin{aligned} \operatorname*{ess\,sup}_{f \in \mathcal{F}_{\mathcal{H},\delta}} \mathbb{E}[\mathbb{E}[f\, X \,|\, \mathcal{G}] \,|\, \mathcal{H}] &\overset{\text{a.s.}}{\ge} \operatorname*{ess\,sup}_{f \in \mathcal{F}_{\mathcal{G},\delta}} \mathbb{E}[\mathbb{E}[f\, X \,|\, \mathcal{G}] \,|\, \mathcal{H}] \overset{\text{a.s.}}{\ge} \mathbb{E}[\mathbb{E}[\tilde{f}_{\mathcal{G},\delta,X} X \,|\, \mathcal{G}] \,|\, \mathcal{H}] \\ &\overset{\text{a.s.}}{=} \mathbb{E}[\mathrm{ES}_\delta(X \,|\, \mathcal{G}) \,|\, \mathcal{H}] \;, \end{aligned}$$

which proves the result.

□

Corollary 4.25. *Assume X^+ is integrable. Given $\delta \in (0,1)$, a totally-ordered countable discrete time interval I and a filtration $\{\mathcal{F}_t\}_{t\in I}$, the process $\{\mathrm{ES}_\delta(X \,|\, \mathcal{F}_t)\}_{t\in I}$ is a supermartingale.*

Proof. The supermartingale property follows from Lemma 4.22(j). Further it is clear that the process $\{\mathrm{ES}_\delta(X\,|\,\mathcal{F}_t)\}_{t\in I}$ is adapted. $\mathrm{ES}_\delta(X\,|\,\mathcal{F}_t)$ is integrable for every $t \in I$, since the positive part of $Xf_{\mathcal{F}_t,\delta,X}$ is integrable by assumption that X^+ is integrable and the negative part of $Xf_{\mathcal{F}_t,\delta,X}$ is integrable, because it is bounded from below as stated in Remark 4.15(b). □

Remark 4.26. Note that the result of Corollary 4.25 would also be true for $I = [0,\infty)$, i.e. for a continuous time interval. If in that case the filtration $\{\mathcal{F}_t\}_{t\in I}$ is right-continuous, $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets and the mapping $t \to \mathbb{E}[\mathrm{ES}_\delta(X\,|\,\mathcal{F}_t)]$ is right-continuous, then there exists a càdlàg modification of $\{\mathrm{ES}_\delta(X\,|\,\mathcal{F}_t)\}_{t\in I}$ by [27, Theorem 21.24].

In the following lemma we will take a look at the conditional expected shortfall of a conditional expectation.

Lemma 4.27. *Given a probability space* $(\Omega,\mathcal{F},\mathbb{P})$ *and two sub-*σ*-algebras* $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$*. Assume* $X \in L^1(\Omega,\mathcal{F},\mathbb{P})$*. Then*

$$\mathrm{ES}_\delta(\mathbb{E}[X\,|\,\mathcal{G}]\,|\,\mathcal{H}) \overset{a.s.}{\leq} \mathrm{ES}_\delta(X\,|\,\mathcal{H}) \quad \forall\, \delta \in [0,1]\,.$$

Proof. First consider $\delta \in (0,1)$. Let $\mathcal{F}^1_{\mathcal{H},\delta,X}$ be defined as in Lemma 4.19 and define $\mathcal{F}^1_{\mathcal{H},\delta,X}(\mathcal{G})$ as the set of all $f \in \mathcal{F}^1_{\mathcal{H},\delta,X}$, which are $\mathcal{G}$-measurable. Then, as for every $\mathcal{F}$-measurable f

$$\mathbb{E}[\mathbb{E}[X\,|\,\mathcal{G}]\,f\,|\,\mathcal{H}] = \mathbb{E}[\mathbb{E}[X\,|\,\mathcal{G}]\,\mathbb{E}[f\,|\,\mathcal{G}]\,|\,\mathcal{H}]\ ,$$

and using Lemma 4.19 we have

$$\begin{aligned}
\mathrm{ES}_\delta(\mathbb{E}[X\,|\,\mathcal{G}]\,|\,\mathcal{H}) &\overset{\text{a.s.}}{=} \frac{1}{1-\delta}\operatorname*{ess\,sup}_{f\in\mathcal{F}^1_{\mathcal{H},\delta,X}} \mathbb{E}[\mathbb{E}[X\,|\,\mathcal{G}]\,f\,|\,\mathcal{H}] \overset{\text{a.s.}}{=} \frac{1}{1-\delta}\operatorname*{ess\,sup}_{f\in\mathcal{F}^1_{\mathcal{H},\delta,X}(\mathcal{G})} \mathbb{E}[\mathbb{E}[X\,|\,\mathcal{G}]\,f\,|\,\mathcal{H}] \\
&\overset{\text{a.s.}}{=} \frac{1}{1-\delta}\operatorname*{ess\,sup}_{f\in\mathcal{F}^1_{\mathcal{H},\delta,X}(\mathcal{G})} \mathbb{E}[Xf\,|\,\mathcal{H}] \overset{\text{a.s.}}{\leq} \frac{1}{1-\delta}\sup_{f\in\mathcal{F}^1_{\mathcal{H},\delta,X}} \mathbb{E}[Xf\,|\,\mathcal{H}] = \mathrm{ES}_\delta(X\,|\,\mathcal{H})\,.
\end{aligned}$$

Now set $\delta = 1$. Then

$$\mathrm{ES}_\delta(\mathbb{E}[X\,|\,\mathcal{G}]\,|\,\mathcal{H}) \overset{\text{a.s.}}{=} \mathbb{E}[X\,|\,\mathcal{G}]^{\mathcal{H}} \quad \text{and} \quad \mathrm{ES}_\delta(X\,|\,\mathcal{H}) \overset{\text{a.s.}}{=} X^{\mathcal{H}}\,.$$

Let Y be a $\mathcal{H}$-measurable random variable with $Y \geq X$ almost surely. Then $Y \geq X^{\mathcal{H}}$ almost surely, be the definition of $X^{\mathcal{H}}$ in Definition 4.8. Then

$$Y \overset{\text{a.s.}}{=} \mathbb{E}[Y\,|\,\mathcal{G}] \overset{\text{a.s.}}{\geq} \mathbb{E}[X\,|\,\mathcal{G}]\ .$$

This implies $\{Y\colon \Omega \to \mathbb{R}\cup\{\infty\}\,|\,Y\,\mathcal{H}\text{-measurable}, Y \overset{\text{a.s.}}{\geq} X\} \subset \{Y\colon \Omega \to \mathbb{R}\cup\{\infty\}\,|\, Y\,\mathcal{H}\text{-measurable}, Y \overset{\text{a.s.}}{\geq} \mathbb{E}[X\,|\,\mathcal{G}]\}$. Therefore $\mathbb{E}[X\,|\,\mathcal{G}]^{\mathcal{H}} \leq X^{\mathcal{H}}$ almost surely.

Finally we consider the case $\delta = 0$. For every $\delta' \in (0,1)$ we have

$$\mathrm{ES}_{\delta'}(\mathbb{E}[X\,|\,\mathcal{G}]\,|\,\mathcal{H}) \overset{\text{a.s.}}{\leq} \mathrm{ES}_{\delta'}(X\,|\,\mathcal{H})$$

Therefore

$$\mathrm{ES}_\delta(\mathbb{E}[X\,|\,\mathcal{G}]\,|\,\mathcal{H}) = \operatorname*{ess\,inf}_{\delta'\in(0,1)} \mathrm{ES}_{\delta'}(\mathbb{E}[X\,|\,\mathcal{G}]\,|\,\mathcal{H}) \overset{\text{a.s.}}{\leq} \operatorname*{ess\,inf}_{\delta'\in(0,1)} \mathrm{ES}_{\delta'}(X\,|\,\mathcal{H}) = \mathrm{ES}_\delta(X\,|\,\mathcal{H})\,.$$

□

Remark 4.28. For a given discrete time interval I let $\{M_t\}_{t\in I}$ be a martingale. Then for every $\delta \in [0,1]$ and a filtration $\{\mathcal{F}_t\}_{t\in I}$ by Lemma 4.27 for all $s \leq t \leq u$ in I

$$\mathrm{ES}_\delta(M_t\,|\,\mathcal{F}_s) \overset{\text{a.s.}}{\leq} \mathrm{ES}_\delta(M_u\,|\,\mathcal{F}_s)\,.$$

4.2 General Bounds

In Chapter 2 we already saw some first bounds for the problem. In particular we noted that we can find a lower bound by assuming that the process Z and the stopping time or the adapted random probability measure are independent and that an upper bound is given by the value of an optimal stopping problem with the same underlying process Z. In our notation this means

$$V^{\mathrm{ind}}(\nu) \leq V(\nu) \leq V^{+}(\nu) \leq V \leq \tilde{V} \quad \text{and} \quad V^{\mathrm{ind}}(\nu) \leq V'(\nu) \leq V^{+}(\nu) \leq V \leq \tilde{V}\,,$$

with $V^{\mathrm{ind}}(\nu)$ as in (2.8), $V(\nu)$ as in (2.7), $V^{+}(\nu)$ as in (2.17), V as in (2.5), $\tilde{V}$ as in (2.34) and $V'(\nu)$ as in (2.25). Due to these results most upper bounds will be derived for $V^{+}(\nu)$.

We will start with very general upper bounds for a totally-ordered countable discrete time interval I. We assume that the stochastic process $Z = \{Z_t\}_{t\in I}$ satisfies $\mathbb{E}\big[\sup_{t\in I} Z_t^{-}\big] < \infty$ or $\mathbb{E}\big[\sup_{t\in I} Z_t^{+}\big] < \infty$. We know that for every $\gamma \in \mathcal{M}_I$ we have $\gamma_t \in [0,1]$ a.s. for all $t \in I$ by Definition 2.33. This implies that for every $\gamma \in \mathcal{M}_I$

$$\mathbb{E}\Bigg[\sum_{t\in I} Z_t\gamma_t\Bigg] \leq \mathbb{E}\Bigg[\sum_{t\in I} |Z_t|\gamma_t\Bigg] \leq \mathbb{E}\Bigg[\sum_{t\in I} |Z_t|\Bigg] = \sum_{t\in I} \mathbb{E}[|Z_t|]\ .$$

Therefore we can easily find that for a non-negative process $Z = \{Z_t\}_{t\in I}$

$$\tilde{V} \leq \sum_{t\in I} \mathbb{E}[Z_t]\ .$$

For a general process Z we get

$$\tilde{V} \leq \sum_{t\in I} \mathbb{E}[|Z_t|]\ .$$

As we assume $\sum_{t\in I} \gamma_t \overset{\text{a.s.}}{=} 1$ by Definition 2.33(b) for all $\gamma \in \mathcal{M}_I$, we also have

$$V^{+}(\nu) \leq \mathbb{E}\Big[\sup_{t\in I} Z_t\Big]\ .$$

These upper bounds are quite general and do not incorporate the distribution ν. In many situations the values found by these upper bounds will be quite high. We will therefore derive other bounds for our problem.

Proposition 4.29. *Given a totally-ordered countable discrete time interval I and a probability distribution ν on I. Assume we are given a process $Z = \{Z_t\}_{t\in I}$ which satisfies $\mathbb{E}\big[\sup_{t\in I} Z_t^{-}\big] < \infty$ or $\mathbb{E}\big[\sup_{t\in I} Z_t^{+}\big] < \infty$. Then*

$$V^{+}(\nu) \leq \sum_{t\in I} \nu_t\, \mathrm{ES}_{1-\nu_t}(Z_t)\,.$$

Proof. By definition of $V^{+}(\nu)$ and using Lemma 2.35 we get

$$V^{+}(\nu) \leq \sum_{t\in I} \sup_{\gamma\in\mathcal{M}_I^{\nu}} \mathbb{E}[Z_t\gamma_t]\ .$$

Using Lemma 4.6 and by noticing that $\gamma_t \in \mathcal{F}^1_{1-\nu_t,Z_t}$ for all $t \in I$, we have

$$\sup_{\gamma\in\mathcal{M}_I^{\nu}} \mathbb{E}[Z_t\gamma_t] = \nu_t\, \mathrm{ES}_{1-\nu_t}(Z_t)\,.$$

□

Remark 4.30. We have $\gamma_t \in \mathcal{F}^1_{1-\nu_t, Z_t}$, because

$$\mathbb{E}[f_{1-\nu_t, Z_t} \cdot 1] = \mathbb{P}(Z_t > q_{1-\nu_t}(Z_t)) + \mathbb{P}(Z_t \leq q_{1-\nu_t}(Z_t)) - (1-\nu_t) = \nu_t = \mathbb{E}[\gamma_t \cdot 1] \,.$$

Proposition 4.31. *Given a discrete time interval $I \subset \mathbb{N}_0$ with $0 \in I$ and a probability distribution ν on I. Using the Doob decomposition $Z = M + A$ for an adapted process Z in $L^1(\mathbb{P})$ (see e.g. [62, Theorem 12.11]) and assuming that the martingale part M together with the distribution ν satisfies one of the conditions of Theorem 2.49 and $\mathbb{E}\big[\sup_{t\in I} A_t^-\big] < \infty$ or $\mathbb{E}\big[\sup_{t\in I} A_t^+\big] < \infty$ for the predictable process A, we can show that*

$$V^+(\nu) \leq \mathbb{E}[M_0] + \sum_{t\in I} \nu_t \operatorname{ES}_{1-\nu_t}(A_t) \,.$$

Proof. After using the Doob decomposition it is clear that $\mathbb{E}[M_\tau] = \mathbb{E}[M_\gamma] = \mathbb{E}[M_0]$ for $\tau \in \mathcal{T}_I^\nu$ and $\gamma \in \mathcal{M}_I^\nu$ by Theorem 2.49 and Doob's optional stopping theorem. Proceed with the process A according to Proposition 4.29. □

Remark 4.32. This second upper bound computes the exact value for the martingale part in the Doob decomposition. For a part of the problem the exact value is therefore computed, which might be an advantage of this bound. On the other hand, computing this bound is more demanding, since one needs to start by determining the two processes of the Doob decomposition. As we will see later in Section 4.4 we cannot state generally whether the bound of Proposition 4.29 or that Proposition 4.31 works better.

Note that it might transpire that just one of the upper bounds of Proposition 4.29 or Proposition 4.31 is applicable for the given process. This will be illustrated in the following example.

Example 4.33. For $I = \mathbb{N}$ consider a process $Z = \{Z_t\}_{t\in I}$ of independent, uniformly distributed, symmetric, $\{-1,1\}$-valued random variables. Assume that $\nu_t = 2^{-t}$ for $t \in I$. As $\mathbb{E}[\sup_{t\in I} |Z_t|] = 1$, we can apply Proposition 4.29, which yields

$$V^+(\nu) \leq \sum_{t\in I} \frac{1}{2^t} = 1 \,.$$

The Doob decomposition of Z is given by $Z = M + A$ with $M_t = \sum_{s=0}^t Z_s$ for $t \in I$, $A_0 = 0$ and $A_t = -M_{t-1}$ for $t \in I \setminus \{0\}$. We cannot apply Proposition 4.31 for these processes M and A. On a finite time interval we could apply Proposition 4.31. If now $T \in \mathbb{N}$ and we consider $I_{\leq T}$, then we would have

$$V^+(\nu) \leq \sum_{I_{\leq T}} \frac{1}{2^t} t \nearrow 1 \,\text{for}\; T \to \infty.$$

Proposition 4.34. *Given a discrete time interval $I \subset \mathbb{N}_0$ with $0 \in I$ and a probability distribution ν on I. Using again the Doob decomposition $Z = M + A$ for an adapted process Z in $L^1(\mathbb{P})$, where we assume that the martingale part M together with the distribution ν satisfies one of the conditions of Theorem 2.49 and $\mathbb{E}\big[\sup_{t\in I} A_t^-\big] < \infty$ or $\mathbb{E}\big[\sup_{t\in I} A_t^+\big] < \infty$ for the predictable process A, and if we define $\Delta A_t := A_t - A_{t-1}$ for $t \in I \setminus \{0\}$ and $\Delta A_0 := 0$, we have for all $\gamma \in \mathcal{M}_I^\nu$*

$$\mathbb{E}[Z_\gamma] \leq \mathbb{E}[M_0] + \sum_{t\in I} \nu_{\geq t} \operatorname{ES}_{\nu_{<t}}(\Delta A_t) \,.$$

Proof. For the predictable process A of the Doob decomposition we can write, as it will be proven in the proof of Theorem 5.25,

$$A_\gamma = \sum_{t\in I} \Delta A_t \gamma_{\geq t} \,.$$

The rest follows similar to the proofs of Proposition 4.29 and Proposition 4.31. □

4.3 Bounds for Special Classes of Processes

Within this section we will now take a look at certain types of processes, for which we can derive bounds that may be closer to the values we are interested in than the bounds presented in the last section.

Lemma 4.35. *Assume $I = \mathbb{N}_0$ and that the process Z is a supermartingale.*

(a) If $\mathbb{E}[\sup_{t\in I} |Z_t|] < \infty$, then for every $\gamma \in \mathcal{M}_I$

$$\mathbb{E}[Z_0] \geq \mathbb{E}[Z_\gamma] \geq \mathbb{E}[Z_\infty] \,.$$

(b) If we consider the Doob decomposition $M + A$ of the supermartingale Z and assume M (or in case $\gamma \in \mathcal{M}_I^\nu$ consider M and ν) satisfies one of the conditions of Theorem 2.49, we get

$$\mathbb{E}[M_0] \geq \mathbb{E}[Z_\gamma] \geq \mathbb{E}[M_0] + \mathbb{E}[A_\infty] \,.$$

Proof. (a) By Doob's convergence theorem (see [62, Theorem 11.5.]) we know that the limit $Z_\infty = \lim_{t\to\infty} Z_t$ exists a.s., because Z is bounded in L^1 (see Remark 4.36).

Using Lemma 2.35, $Z_t \geq \mathbb{E}[Z_\infty | \mathcal{F}_t]$ for all $t \in I$ and that by Definition 2.33 the process γ is adapted and satisfies $\sum_{t\in I} \gamma_t = 1$ a.s., we have

$$\mathbb{E}[Z_\gamma] = \mathbb{E}\left[\sum_{t\in I} Z_t\gamma_t\right] \geq \mathbb{E}\left[Z_\infty \sum_{t\in I} \gamma_t\right] = \mathbb{E}[Z_\infty] \,.$$

On the other hand $\mathbb{E}[Z_\gamma] \leq \mathbb{E}[Z_0]$ by Theorem 2.49, as $\mathbb{E}[\sup_{t\in I} |Z_t|] < \infty$ implies that Z is uniformly integrable.

(b) The Doob decomposition of the process Z is given by a martingale M and a non-increasing, predictable process A with $A_0 = 0$. Using this decomposition and Theorem 2.49, we get

$$\mathbb{E}[Z_\gamma] = \mathbb{E}\left[\sum_{t\in I} Z_t\gamma_t\right] = \mathbb{E}[M_0] + \mathbb{E}\left[\sum_{t\in I} A_t\gamma_t\right].$$

As $A_t \geq A_\infty$ for all $t \in I$ and $\sum_{t\in I} \gamma_t \overset{\text{a.s.}}{=} 1$ by Definition 2.33(b), we have

$$\mathbb{E}\left[\sum_{t\in I} A_t\gamma_t\right] \geq \mathbb{E}\left[A_\infty \sum_{t\in I} \gamma_t\right] = \mathbb{E}[A_\infty] \,.$$

Therefore

$$\mathbb{E}[Z_\gamma] \geq \mathbb{E}[M_0] + \mathbb{E}[A_\infty] \,.$$

Since A is non-increasing, we have $A_t \leq 0$ a.s. for all $t \in I$. This implies

$$\mathbb{E}[Z_\gamma] = \mathbb{E}[M_0] + \mathbb{E}\left[\sum_{t\in I} A_t\gamma_t\right] \leq \mathbb{E}[M_0] = \mathbb{E}[Z_0] \,.$$

□

Remark 4.36. For every $t \in I$ we have

$$\mathbb{E}[|Z_t|] \leq \mathbb{E}\left[\sup_{t\in I} |Z_t|\right] < \infty\,.$$

Since the supremum is defined to be the smallest upper bound of the sequence, we have

$$\sup_{t\in I} \mathbb{E}[|Z_t|] \leq \mathbb{E}\left[\sup_{t\in I} |Z_t|\right] < \infty\,.$$

Therefore the process Z is bounded in L^1.

Lemma 4.37. *Assume $I = \mathbb{N}_0$ and that the process Z is a submartingale.*

(a) If $\mathbb{E}[\sup_{t\in I} |Z_t|] < \infty$, then for every $\gamma \in \mathcal{M}_I$

$$\mathbb{E}[Z_0] \leq \mathbb{E}[Z_\gamma] \leq \mathbb{E}[Z_\infty]\ .$$

(b) Using the Doob decomposition $M + A$ of the submartingale Z and assuming that M (or in case $\gamma \in \mathcal{M}_I^\nu$ we consider M and ν) satisfies one of the conditions of Theorem 2.49, we find

$$\mathbb{E}[M_0] \leq \mathbb{E}[Z_\gamma] \leq \mathbb{E}[M_0] + \mathbb{E}[A_\infty]\ .$$

Proof. The proof works similar to that of Lemma 4.35. The process A in the Doob decomposition is now a non-decreasing process. □

Remark 4.38. In Lemma 4.35 and in Lemma 4.37 we showed the result for $I = \mathbb{N}_0$. If $I = \{0, \dots, T\}$ we simply have to replace Z_∞ and A_∞ by Z_T and A_T, respectively.

Lemma 4.39. *Let $I = \{0, \dots, T\}$ and assume that $Z_0 \geq Z_1 \geq \dots \geq Z_T$ a.s. Consider some weights $\gamma_0, \dots, \gamma_T$ with $\sum_{t\in I} \gamma_t = 1$ a.s.*

(a) For $\gamma_0 \geq \gamma_1 \geq \dots \geq \gamma_T$ a.s. we have

$$\mathbb{E}\left[\sum_{t\in I} Z_t\gamma_t\right] \geq \frac{1}{T+1}\sum_{t\in I} \mathbb{E}[Z_t]\ . \tag{4.40}$$

(b) For $\gamma_0 \leq \gamma_1 \leq \dots \leq \gamma_T$ a.s. we get

$$\mathbb{E}\left[\sum_{t\in I} Z_t\gamma_t\right] \leq \frac{1}{T+1}\sum_{t\in I} \mathbb{E}[Z_t]\ . \tag{4.41}$$

Proof. (a) By Chebyshev's sum inequality (see e.g. [18, Section 2.17]) for $\gamma_0 \geq \gamma_1 \geq \dots \geq \gamma_T$ a.s. we get

$$\frac{1}{T+1}\sum_{t\in I} Z_t\gamma_t \geq \left(\frac{1}{T+1}\sum_{t\in I} Z_t\right)\left(\frac{1}{T+1}\sum_{t\in I} \gamma_t\right) \text{ a.s.}$$

Therefore, since $\sum_{t\in I} \gamma_t = 1$ a.s.,

$$\sum_{t\in I} Z_t\gamma_t \geq \frac{1}{T+1}\sum_{t\in I} Z_t \text{ a.s.}$$

Taking expectations gives (4.40).
(b) If $\gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_T$ a.s. Chebyshev's sum inequality states that

$$\frac{1}{T+1}\sum_{t\in I} Z_t\gamma_t \leq \left(\frac{1}{T+1}\sum_{t\in I} Z_t\right)\left(\frac{1}{T+1}\sum_{t\in I}\gamma_t\right) \ a.s.$$

Similarly we get (4.41). □

Remarks 4.42.

1. Lemma 4.39 is especially applicable for $\gamma \in \mathcal{M}_I^\nu$.
2. Similar results can be found for a process Z with $Z_0 \leq Z_1 \leq \cdots \leq Z_T$ a.s.
3. What we need to find is a permutation σ such that $Z_{\sigma(0)} \leq Z_{\sigma(1)} \leq \cdots \leq Z_{\sigma(T)}$ and $\gamma_{\sigma(0)} \leq \gamma_{\sigma(1)} \leq \cdots \leq \gamma_{\sigma(T)}$ or $\gamma_{\sigma(0)} \geq \gamma_{\sigma(1)} \geq \cdots \geq \gamma_{\sigma(T)}$, respectively. This permutation does not need to be measurable.

In the following proposition we will try to find an upper bound for a process which is a product of a deterministic function and a martingale. Note that the strategy found is not adapted.

Proposition 4.43. *Assume $I = \{0,\dots,T\}$ and that the process Z is given by $Z_t = f(t)M_t$ for $t \in I$, where the deterministic function f is increasing. Further define $\nu_{\leq -1} := 0$. Then*

$$V^+(\nu) \leq \sum_{t\in I} f(t)\,\mathbb{E}\Big[M_T 1_{\{q_{\nu_{\leq t-1}}(M_T) < M_T \leq q_{\nu_{\leq t}}(M_T)\}}\Big]\,.$$

Proof. Using the martingale property and that γ_t is $\mathcal{F}_t$-measurable for all $t \in I$ for every $\gamma \in \mathcal{M}_I^\nu$ by Definition 2.12(c), we get

$$V^+(\nu) = \sup_{\gamma\in\mathcal{M}_I^\nu} \mathbb{E}\Big[\sum_{t\in I} Z_t\gamma_t\Big] = \sup_{\gamma\in\mathcal{M}_I^\nu}\sum_{t\in I} f(t)\,\mathbb{E}[M_T\gamma_t]\,.$$

Following the result of the rearrangement inequality ([18, Chapter X, Theorem 368]) big values of $f(t)$ should be multiplied with big values of $\mathbb{E}[M_T\gamma_t]$. As $f(t)$ is increasing in t, we should therefore find a strategy γ such that $\mathbb{E}[M_T\gamma_t]$ is non-decreasing in t. We will use a strategy given by

$$\gamma_t := 1_{\{q_{\nu_{\leq t-1}}(M_T) < M_T \leq q_{\nu_{\leq t}}(M_T)\}}\,,$$

for $t \in I$ with $\nu_{\leq -1} := 0$, which guarantees that $\mathbb{E}[M_T\gamma_s] \leq \mathbb{E}[M_T\gamma_t]$ for $s < t$ in I. Unfortunately this strategy is not adapted. Finally we get the inequality

$$V^+(\nu) \leq \sum_{t\in I} f(t)\,\mathbb{E}\Big[M_T 1_{\{q_{\nu_{\leq t-1}}(M_T) < M_T \leq q_{\nu_{\leq t}}(M_T)\}}\Big]\,.$$

□

Remark 4.44. If we assume that M_T has a continuous distribution, we can rewrite this upper bound by

$$V^+(\nu) \leq \sum_{t\in I} f(t)((1-\nu_{\leq t-1})\,\mathrm{ES}_{\nu_{\leq t-1}}(M_T) - (1-\nu_{\leq t})\,\mathrm{ES}_{\nu_{\leq t}}(M_T))\,.$$

If the deterministic function is non-decreasing (hence Z is a submartingale), we can find a similar upper bound as shown in the next proposition.

Proposition 4.45. *Given a discrete time interval $I \subset \mathbb{N}_0$ with $0 \in I$ and a probability distribution ν on I, assume that $Z_t = g(t)M_t$ for all $t \in I$, with g being a non-decreasing function and M a martingale. Further assume that Z satisfies $\mathbb{E}[\sup_{t\in I}|Z_t|] < \infty$. Then we can find an upper bound given by*

$$V^+(\nu) \leq g(0)\,\mathbb{E}[M_0] + \sum_{\substack{t\in I\\ t+1\in I}} (g(t+1)-g(t))(1-\nu_{\leq t})\,\mathrm{ES}_{\nu_{\leq t}}(M_t)\,.$$

Using the densities f of Definition 4.3(b), we can write

$$\gamma_0 = 1 - f_{\nu_0,M_0}$$

and

$$\gamma_t = f_{\nu_{\leq t-1},M_{t-1}} - f_{\nu_{\leq t},M_t} \quad \text{for } t \in I \setminus \{0\}\,.$$

If $\gamma_t \geq 0$ a.s. for all $t \in I$, then $\gamma = \{\gamma_t\}_{t\in I} \in \mathcal{M}_I^\nu$. Further this strategy γ yields the value of the upper bound, which implies that it is an optimal strategy if $\gamma \in \mathcal{M}_I^\nu$.

Proof. As $\sum_{t\in I}\gamma_t = 1$ a.s. by Definition 2.12(b), we can rewrite Z_γ as

$$Z_\gamma = \sum_{t\in I} g(t)M_t\gamma_t \overset{\text{a.s.}}{=} \sum_{t\in I} g(t)M_t(\gamma_{\geq t}-\gamma_{>t}) \overset{\text{a.s.}}{=} g(0)M_0 + \sum_{\substack{t\in I\\ t+1\in I}} (g(t+1)M_{t+1}-g(t)M_t)\gamma_{>t}\,.$$

Using the martingale property of M and that $\gamma \in \mathcal{M}_I^\nu$ is adapted by Definition 2.12(c), we get for all $t \in I$ with $t+1 \in I$

$$\mathbb{E}[g(t+1)M_{t+1}\gamma_{>t}] = g(t+1)\,\mathbb{E}[\mathbb{E}[M_{t+1}|\mathcal{F}_t]\,(1-\gamma_{\leq t})] = \mathbb{E}[g(t+1)M_t\gamma_{>t}]\ .$$

Therefore

$$\begin{aligned}\mathbb{E}[Z_\gamma] &= g(0)\,\mathbb{E}[M_0] + \sum_{\substack{t\in I\\ t+1\in I}} \mathbb{E}[(g(t+1)M_{t+1}-g(t)M_t)\cdot\gamma_{>t}]\\ &= g(0)\,\mathbb{E}[M_0] + \sum_{\substack{t\in I\\ t+1\in I}} (g(t+1)-g(t))\,\mathbb{E}\Big[M_t\gamma_{>t}\Big]\,.\end{aligned}$$

Now we can try to find an upper bound by maximizing each summand. We have

$$\sup_{\gamma\in\mathcal{M}_I^\nu}\mathbb{E}\bigg[\sum_{t\in I} Z_t\gamma_t\bigg] \leq g(0)\,\mathbb{E}[M_0] + \sum_{\substack{t\in I\\ t+1\in I}} \sup_{\gamma\in\mathcal{M}_I^\nu}(g(t+1)-g(t))\,\mathbb{E}[M_t\gamma_{>t}]\ .$$

For every $t \in I$ we want to maximize

$$\mathbb{E}[M_t\gamma_{>t}]\ ,$$

because we know that $g(t+1)-g(t) \geq 0$ for all $t \in I$ with $t+1 \in I$. For all $t \in I$ with $t+1 \in I$ we have

$$\gamma_{>t} \in \mathcal{F}^1_{\nu_{\leq t},M_t}\,,$$

where $\mathcal{F}^1_{\nu_{\leq t},M_t}$ is defined as in Lemma 4.6. Therefore the strategy for the upper bound is given by

$$\gamma_{>t} = f_{\nu_{\leq t},M_t}\,.$$

Altogether this yields

$$V^+(\nu) \le g(0)\,\mathbb{E}[M_0] + \sum_{\substack{t\in I\\ t+1\in I}} (g(t+1)-g(t))(1-\nu_{\le t})\,\mathrm{ES}_{\nu_{\le t}}(M_t)\,.$$

The process $\gamma = \{\gamma_t\}_{t\in I}$ yielding the upper bound is given by

$$\gamma_t = \gamma_{>t-1} - \gamma_{>t} = f_{\nu_{\le t-1},M_{t-1}} - f_{\nu_{\le t},M_t} \quad \text{for } t \in I\setminus\{0\}$$

and

$$\gamma_0 = 1-\gamma_{>0} = 1 - f_{\nu_0,M_0}\,.$$

□

Next we will use comonotonicity for finding an upper bound. We will use that for two real-valued random variables X and Y we have

$$\mathbb{E}[XY] \le \mathbb{E}[X^*Y^*]\,, \tag{4.46}$$

where X^* and Y^* are two comonotonic random variables with $\mathcal{L}(X) = \mathcal{L}(X^*)$ and $\mathcal{L}(Y) = \mathcal{L}(Y^*)$. This result is due to [38, Theorem 5.25].

Lemma 4.47. *Given a totally-ordered countable discrete time interval I. If the process Z is given in the form $Z_t = f(t)M_t$ for $t\in I$, with a deterministic function f and a closable martingale M, and satisfies $\mathbb{E}\big[\sup_{t\in I} Z_t^-\big] < \infty$ or $\mathbb{E}\big[\sup_{t\in I} Z_t^+\big] < \infty$, then we have*

$$\mathbb{E}[Z_\gamma] \le \sum_{t\in I} f(t)\int_{l_t}^{u_t} F^{\leftharpoonup}(s)\,ds\,, \quad \gamma\in\mathcal{M}_I^\nu\,,$$

where F is the cumulative distribution function of M_∞ and $F^{\leftharpoonup}$ its lower inverse. Further

$$l_t := \sum_{\substack{s\in I\\ f(s)<f(t)}} \nu_s + \sum_{\substack{s\in I_{<t}\\ f(s)=f(t)}} \nu_s\,, \quad t\in I\,,$$

and

$$u_t := l_t + \nu_t\,, \quad t\in I\,.$$

Proof. By extending the probability space if necessary for using Theorem 2.41 and further using Lemma 2.35 and the martingale property we get

$$\mathbb{E}[Z_\gamma] = \mathbb{E}\Big[M_\infty \sum_{t\in I} f(t)1_{\{\tau=t\}}\Big]$$

for some stopping time τ. Let F be the cumulative distribution function of M_∞, then $M_\infty \overset{(d)}{=} F^{\leftharpoonup}(U)$ for a random variable $U\sim U(0,1)$. Now we need to find a random variable X such that X and $F^{\leftharpoonup}(U)$ are comonotonic and $X \overset{(d)}{=} \sum_{t\in I} f(t)1_{\{\tau=t\}}$. $F^{\leftharpoonup}(U)$ is an increasing function in U, therefore X should also be increasing in U. Using a stochastic time θ defined by

$$\{\theta = t\} = \{l_t < U \le u_t\}\,, \quad t\in I\,,$$

with

$$l_t := \sum_{\substack{s\in I\\ f(s)<f(t)}} \nu_s + \sum_{\substack{s\in I_{<t}\\ f(s)=f(t)}} \nu_s\,, \quad t\in I\,,$$

and

$$u_t := l_t + \nu_t\,, \quad t\in I\,,$$

we can set

$$X = \sum_{t\in I} f(t)1_{\{\theta=t\}}\,,$$

where equality in distribution $X \overset{(d)}{=} \sum_{t\in I} f(t)1_{\{\tau=t\}}$ is due to the fact that for every $t\in I$

$$\mathbb{P}(\tau=t) = \mathbb{P}(\theta=t)\,.$$

Using (4.46) and Lemma 2.35 we get for $\gamma\in\mathcal{M}_I^\nu$

$$\begin{aligned}\mathbb{E}[Z_\gamma] &\le \mathbb{E}[F^{\leftharpoonup}(U)X] = \mathbb{E}\Big[F^{\leftharpoonup}(U)\sum_{t\in I} f(t)1_{\{\theta=t\}}\Big]\\ &= \sum_{t\in I} f(t)\,\mathbb{E}\big[F^{\leftharpoonup}(U)1_{\{l_t<U\le u_t\}}\big] = \sum_{t\in I} f(t)\int_{l_t}^{u_t} F^{\leftharpoonup}(s)\,ds\,.\end{aligned}$$

□

In Lemma 2.55 we gave a short note about using correlation between the processes Z and γ. Using this result we can find another upper bound.

Lemma 4.48. *For a given totally-ordered countable discrete time interval I fix $\gamma\in\mathcal{M}_I$ and assume that expectation and variance of Z_t and γ_t are known for all $t\in I$. Then we have*

$$\mathbb{E}[Z_\gamma] \le \sum_{t\in I}\Big(\mathbb{E}[Z_t]\,\mathbb{E}[\gamma_t] + \sqrt{\mathrm{Var}(Z_t)}\sqrt{\mathrm{Var}(\gamma_t)}\Big)\,.$$

Proof. In Remark 2.58 we saw that for $\gamma\in\mathcal{M}_I$

$$\mathbb{E}[Z_\gamma] = \sum_{t\in I}\mathbb{E}[Z_t]\,\mathbb{E}[\gamma_t] + \rho\sum_{t\in I}\sqrt{\mathrm{Var}(Z_t)}\sqrt{\mathrm{Var}(\gamma_t)}\,.$$

The upper bound can now easily be found by recognizing that $\rho\in[-1,1]$ and noticing that the standard deviation is non-negative. □

Remark 4.49. The result of Lemma 4.48 is especially interesting if, given a probability distribution ν on I, the set $\mathcal{M}_I^\nu$ is reduced to only those processes γ that all have the same variance.

4.4 Comparison of Different Bounds

So far we have seen different upper bounds for different types of processes and also some that are valid in general. In the following we want to take a closer look at some of the bounds in illustrative examples and compare them.

As it is shown in the following example, it can happen that the upper bound found in Proposition 4.31 is closer to $V^+(\nu)$ than the upper bound presented in Proposition 4.29 or the upper bound using the American option.

Example 4.50. For $I = \{0, \dots, T\}$ with $T = 2$ we are given a process Z defined by $Z_0 = U, Z_1 = 0, Z_2 = 2U$, where U is $\mathcal{F}_0$-measurable and $U \sim U(0,1)$. The distribution ν is given by $\nu_0 = \frac{2}{6}, \nu_1 = \frac{3}{6}, \nu_2 = \frac{1}{6}$.

If we use the result from Proposition 4.29, we have

$$\begin{aligned} V^+(\nu) &\leq \nu_0 \,\mathbb{E}[U \,|\, U > 1-\nu_0] + 2\nu_2 \,\mathbb{E}[U \,|\, U > 1-\nu_2] \\ &= \frac{2}{6} \cdot \frac{10}{12} + \frac{2}{6} \cdot \frac{11}{12} = \frac{42}{72} \,. \end{aligned}$$

The Doob decomposition of the process Z is given by $M_0 = M_1 = M_2 = U$ and $A_0 = 0, A_1 = -U, A_2 = U$. By using the upper bound given in Proposition 4.31, we get

$$V^+(\nu) \leq \frac{1}{2} + \frac{3}{6} \operatorname{ES}_{\frac{3}{6}}(-U) + \frac{1}{6} \operatorname{ES}_{\frac{5}{6}}(U) = \frac{1}{2} - \frac{3}{6} \cdot \frac{3}{12} + \frac{1}{6} \cdot \frac{11}{12} = \frac{38}{72} \,.$$

Now we try to compute the value of the corresponding American option. For this we use the Snell envelope, which, as U is $\mathcal{F}_0$-measurable, is given for this example by

$$E_2 = Z_2 = 2U \,,$$

$$E_1 = \max\{Z_1 = 0, \mathbb{E}[E_2 \,|\, \mathcal{F}_1] = 2U\} = 2U \,,$$

$$E_0 = \max\{Z_0 = U, \mathbb{E}[E_1 \,|\, \mathcal{F}_0] = 2U\} = 2U \,.$$

This means that for $t \in \{0, \dots, T\}$ the optimal stopping time is given by

$$\tau_t = \min\{s \in \{t, \dots, T\} | E_s = Z_s\} = T = 2 \,.$$

So we have that the value of the American option equals

$$V = \mathbb{E}[Z_{\tau_0}] = \mathbb{E}[Z_2] = 2\,\mathbb{E}[U] = 1 \,.$$

If we also want to take a look at the value given if we assume that Z is independent of γ or τ, which gives a lower bound, we get

$$V^{\text{ind}}(\nu) = \nu_0 \,\mathbb{E}[Z_0] + \nu_1 \,\mathbb{E}[Z_1] + \nu_2 \,\mathbb{E}[Z_2] = \frac{2}{6} \cdot \frac{1}{2} + 0 + \frac{1}{6} \cdot 1 = \frac{1}{3} = \frac{24}{72} \,.$$

In Example 5.45 we will show that for this process Z and this distribution ν we have $V(\nu) = V^+(\nu) = \frac{38}{72}$.

On the other hand we can find an example for which the bound of Proposition 4.29 works better than the one of Proposition 4.31.

Example 4.51. Set $I = \{0, \dots, 4\}$ and let $X_1, \dots, X_4$ be independent, symmetric, $\{-1, 1\}$-valued random variables. Set $Z_0 = 0$ and $Z_t = X_t$ for $t \in \{1, \dots, 4\}$. Let the distribution ν be given by $\nu_0 = \nu_1 = \nu_2 = 0$ and $\nu_3 = \nu_4 = \frac{1}{2}$. Then by Proposition 4.29

$$V^+(\nu) \leq \nu_3 \operatorname{ES}_{1-\nu_3}(Z_3) + \nu_4 \operatorname{ES}_{1-\nu_4}(Z_4) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1 \,.$$

The Doob decomposition of the process Z is given by $Z = M + A$ with $M_t = \sum_{s=1}^{t} X_s$ for $t \in \{0, \dots, 4\}$, $A_0 = 0$ and $A_t = -M_{t-1}$ for $t \in \{1, \dots, 4\}$. Then by Proposition 4.31

$$V^+(\nu) \leq \mathbb{E}[M_0] + \nu_3 \operatorname{ES}_{1-\nu_3}(A_3) + \nu_4 \operatorname{ES}_{1-\nu_4}(A_4) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{3}{2} = \frac{5}{4} \,.$$

As we will show in the following example, there are also cases in which the American option works as well as an upper bound as the one found in Proposition 4.31 and both yield a better result than the upper bound of Proposition 4.29.

Example 4.52. Assume $I = \{0, 1\}$ and Z_0 and Z_1 are independent random variables with values in $\{0, 2\}$, which are uniformly distributed. Let the filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\in\{0,1\}}$ be the one generated by the process Z. Further we assume that $\nu_0 = \nu_1 = \frac{1}{2}$.

By using the result from Proposition 4.29 we get

$$V^+(\nu) \le \nu_0 \operatorname{ES}_{1-\nu_0}(Z_0) + \nu_1 \operatorname{ES}_{1-\nu_1}(Z_1) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2 = 2\,.$$

The Doob decomposition of the process Z is given by $M_0 = Z_0$, $M_1 = Z_1 - 1 + Z_0$ and $A_0 = 0$, $A_1 = 1 - Z_0$. By Proposition 4.31

$$V^+(\nu) \le 1 + \frac{1}{2} \operatorname{ES}_{\frac{1}{2}}(1 - Z_0) = 1 + \frac{1}{2} = \frac{3}{2}.$$

Taking a look at the Snell envelope, which is given by

$$\begin{aligned} E_1 &= Z_1 \\ E_0 &= \max\{Z_0, \mathbb{E}[E_1\,|\,\mathcal{F}_0]\} \end{aligned}$$

where due to the independence of the process Z we have $\mathbb{E}[E_1\,|\,\mathcal{F}_0] = 1$. We see that the stopping time used to compute the value of the American option has to be $\tau = 1_{\{Z_0=0\}}$, i.e. $\{\tau = 0\} = \{Z_0 = 2\}$ and $\{\tau = 1\} = \{Z_0 = 0\}$. Now we have

$$V^+(\nu) \le \mathbb{E}[Z_\tau] = \mathbb{E}\big[Z_0 1_{\{Z_0=2\}} + Z_1 1_{\{Z_0=0\}}\big] = 2 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{3}{2}\,.$$

As this stopping time also follows the given distribution ν, i.e. $\tau \in \mathcal{T}_I^\nu$, it is also an optimal stopping time for computing the value $V(\nu)$.

Remark 4.53. Note that the upper bound using the Doob decomposition in Example 4.52 equals the optimal value found later for the one-period model in Theorem 5.4.

We will now take a look at an example which considers a time-discrete exponential Brownian motion. This type of process is interesting, since it is a time-discrete version of the exponential Brownian motion of the Black–Scholes model, which is often used in financial mathematics.

Example 4.54. Assume that the process Z is given in the form $Z_t = z_0 \exp(\mu t + \sigma W_t)$ for $t \in I$, where $\{W_t\}_{t\in I}$ is a discrete version of a standard Brownian motion with $W_0 = 0$, $\sigma > 0$, $\mu \in \mathbb{R}$ and $z_0 \in \mathbb{R}$. Since we know that $W_t \sim N(0, t)$, we have that $\mu t + \sigma W_t \sim N(\mu t, \sigma^2 t)$ for all $t \in I$. Therefore Z_t has a lognormal distribution with parameters μt and $\sigma\sqrt{t}$ for $t \in I$.

For a random variable $Y \sim LN(\mu, \sigma^2)$, which has a lognormal distribution with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$, the expected shortfall at level $\alpha \in (0, 1)$ can be computed as

$$\operatorname{ES}_\alpha(Y) = \frac{1}{1-\alpha} z_0 \exp\left(\mu + \frac{\sigma^2}{2}\right)\left(1 - \Phi(\Phi^{-1}(\alpha) - \sigma)\right),$$

where Φ is the cumulative distribution function of the standard normal distribution. We now want to take a look at some of the bounds already presented for this type of process.

To keep this example simple, we will assume that $I = \{0, 1, 2\}$ and that the distribution ν is given by $\nu_0 = 0$ and $\nu_1 = \nu_2 = \frac{1}{2}$.

Using Proposition 4.29, we get

$$\begin{aligned} V^+(\nu) &\leq \frac{1}{2}(\mathrm{ES}_{\frac{1}{2}}(Z_1) + \mathrm{ES}_{\frac{1}{2}}(Z_2)) \\ &= z_0 \sum_{t \in \{1,2\}} \left(1 - \Phi\left(\Phi^{-1}\left(\frac{1}{2}\right) - \sigma\sqrt{t}\right)\right) \exp\left(\left(\mu + \frac{\sigma^2}{2}\right)t\right). \end{aligned}$$

Next we turn our attention to the upper bound presented in Proposition 4.43. In order to be able to use this proposition, we have to represent Z as a product of a deterministic function and a martingale. For this, for every $t \in I$ we set

$$g(t) := z_0 \exp\left(\left(\mu + \frac{\sigma^2}{2}\right)t\right)$$

and

$$M_t := \exp\left(\sigma W_t - \frac{\sigma^2}{2}t\right).$$

Then $Z_t = g(t)M_t$ for every $t \in I$, where g is a deterministic function and M is a martingale with expectation $\mathbb{E}[M_t] = 1$ for all $t \in I$. The upper bound of Proposition 4.43 is now given by

$$\begin{aligned} V^+(\nu) &\leq g(1)(\mathbb{E}[M_2] - \frac{1}{2}\mathrm{ES}_{\frac{1}{2}}(M_2)) + g(2)(\frac{1}{2}\mathrm{ES}_{\frac{1}{2}}(M_2) - 0) \\ &= g(1) + \frac{1}{2}\mathrm{ES}_{\frac{1}{2}}(M_2)(g(2) - g(1)) \\ &= z_0 \exp\left(\mu + \frac{\sigma^2}{2}\right) + z_0\left(1 - \Phi\left(\Phi^{-1}\left(\frac{1}{2}\right) - \sqrt{2}\sigma\right)\right) \\ &\quad \cdot \left(\exp\left(\left(\mu + \frac{\sigma^2}{2}\right)2\right) - \exp\left(\mu + \frac{\sigma^2}{2}\right)\right). \end{aligned}$$

The lower bound using the assumption of independence is given by

$$\begin{aligned} V^{\mathrm{ind}}(\nu) &= \sum_{t \in I} \mathbb{E}[Z_t]\,\nu_t = \frac{1}{2}(g(1) + g(2))\,\mathbb{E}[M_1] \\ &= \frac{1}{2}\left(\exp\left(\mu + \frac{\sigma^2}{2}\right) + \exp\left(\left(\mu + \frac{\sigma^2}{2}\right)2\right)\right). \end{aligned}$$

We will now show some graphs that illustrate how these bounds evolve depending on μ and σ. For this we set $z_0 = 1$. First we have a look at the lower bound found by assuming independence between Z and τ or γ. This bound is illustrated in Figure 4.1. Next we take a look at the upper bound of Proposition 4.29 in Figure 4.2, which is valid for general processes Z satisfying the necessary assumptions. Last but not least we look at the upper bound given by Proposition 4.43 in Figure 4.3, which in the following will be called the special bound, as it is only valid for processes which can be represented as the product of a deterministic function and a martingale. In Figure 4.1, Figure 4.2 and Figure 4.3 we see that for all three bounds the values of the bound becomes larger as μ and σ grow.

We now look at the three bounds simultaneously in order to compare them. In Figure 4.4 the values of the three bounds are shown for $\sigma \in [0, 2]$. In Figure 4.4 we see that the values of the two upper bounds seem to be quite close to the value of the lower bound for smaller

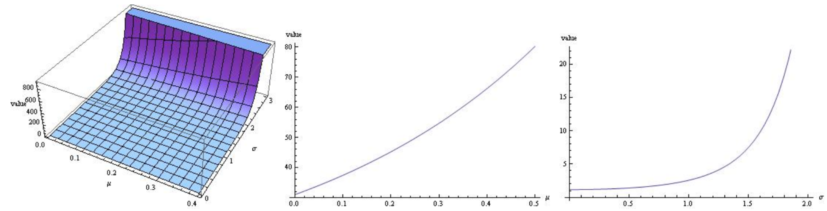

Figure 4.1: The evolution of the lower bound given by $V^{\text{ind}}(\nu)$ for different values of μ and σ. For the second graph we fixed $\sigma = 2$ and for the third one $\mu = 0,1$.

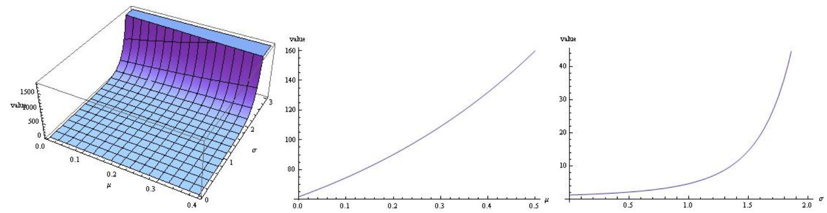

Figure 4.2: The evolution of the upper bound of Proposition 4.29 for different values of μ and σ. For the second graph we fixed $\sigma = 2$ and for the third one $\mu = 0,1$.

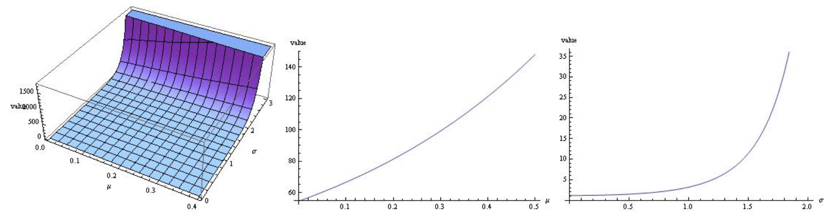

Figure 4.3: The evolution of the upper bound of Proposition 4.43 for different values of μ and σ. For the second graph we fixed $\sigma = 2$ and for the third one $\mu = 0,1$.

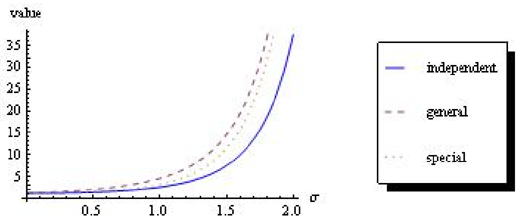

Figure 4.4: The evolution of the bounds for $\mu = 0,1$.

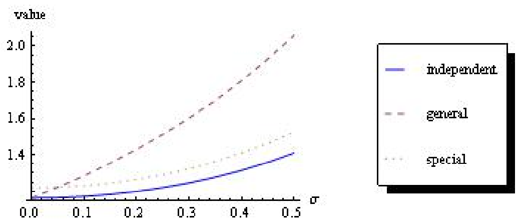

Figure 4.5: The evolution of the bounds for $\mu = 0,1$.

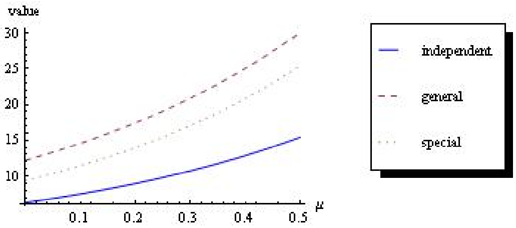

Figure 4.6: The evolution of the bounds for $\sigma = 1,5$.

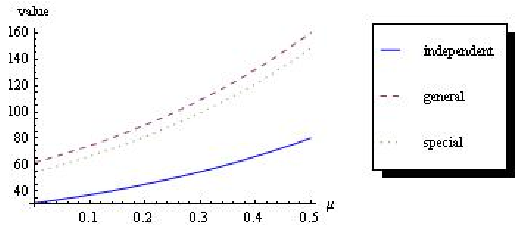

Figure 4.7: The evolution of the bounds for $\sigma = 2$.

values of σ. We therefore take a closer look at the values of the bounds for $\sigma \in [0, 0.5]$ in Figure 4.5. From Figure 4.4 and Figure 4.5 we see that the value of the upper bound of Proposition 4.43 is closer to the value of the lower bound than the upper bound of Proposition 4.29 for $\sigma \geq 0.05$.

Figure 4.6 and 4.7 show the values of the bounds for different values of μ. In Figure 4.6 we fix $\sigma = 1,5$ and in Figure 4.7 we fix $\sigma = 2$. In Figure 4.6 and Figure 4.7 we see that the value of the upper bound of Proposition 4.43 is closer to the value of the lower bound than the upper bound of Proposition 4.29.

Chapter 5

Results for Special Cases

In this chapter some special classes of processes are considered, for which it is possible to find an optimal strategy and the extremal value resulting from it. First we take a short look at deterministic processes in Section 5.1, as this will be useful later on. Then the use of the Doob decomposition is discussed in Section 5.2. Using this decomposition the extremal value can be computed for general processes Z in $L^1(\mathbb{P})$ in the one-period model. In Section 5.3 we concentrate on processes which have uncorrelated increments either themselves or in the predictable process of the Doob decomposition. In Section 5.4 the greedy strategy is introduced. We will see that this is an optimal strategy for independent processes. Section 5.5 deals with $\mathcal{F}_0$-measurable processes. An important class of processes are those stochastic processes that can be represented as the product of a deterministic function and a martingale. These are the main topic of Section 5.6, in which we present a representation for an optimal strategy for this type of process. Using this representation we can find results for processes which can be represented by convex functions and martingales in Section 5.7 and the binomial model in Section 5.8. Again we consider totally-ordered finite or countably infinite discrete time intervals I and adapted stochastic processes $Z = \{Z_t\}_{t\in I}$ with $Z \in L^1(\mathbb{P})$ in this chapter.

5.1 Deterministic Processes

In this section we will briefly discuss deterministic processes, as they will appear from time to time.

Lemma 5.1. *Let I be a totally-ordered countable discrete time interval. Assume that the process $Z = \{Z_t\}_{t\in I}$ is deterministic. Further assume that for $\gamma \in \mathcal{M}_I$ given by Definition 2.33 one of the following conditions holds:*

(a) $\mathbb{E}\big[\sup_{t\in I} Z_t^+\big] < \infty$,

(b) $\mathbb{E}\big[\sup_{t\in I} Z_t^-\big] < \infty$,

(c) $\sum_{t\in I} |Z_t|\,\mathbb{E}[\gamma_t] < \infty$.

Then for this $\gamma \in \mathcal{M}_I$ the random variable Z_γ is well-defined and

$$\mathbb{E}[Z_\gamma] = \sum_{t\in I} Z_t\,\mathbb{E}[\gamma_t]\ . \tag{5.2}$$

If condition (c) is satisfied, then the random variable Z_γ is integrable.

Remark 5.3. Note that Z_γ would also be integrable for $\gamma \in \mathcal{M}_I$, if $\mathbb{E}[\sup_{t\in I}|Z_t|] < \infty$. This is satisfied if both condition (a) and (b) hold. If we want to consider $\gamma \in \mathcal{M}_I^\nu$, then (c) is replaced by $\sum_{t\in I} Z_t\nu_t$. This condition will then imply that (5.2) is true for all $\gamma \in \mathcal{M}_I^\nu$.

Proof. If either (a) or (b) is satisfied, the result follows from Corollary 2.39.

Now assume that (c) is satisfied and consider $\gamma \in \mathcal{M}_I$. We will first prove that Z_γ is well-defined. We have

$$\mathbb{E}[|Z_\gamma|] \le \mathbb{E}\left[\sum_{t\in I}|Z_t|\gamma_t\right] .$$

By monotone convergence

$$\mathbb{E}\left[\sum_{t\in I}|Z_t|\gamma_t\right] = \sum_{t\in I}\mathbb{E}[|Z_t|\gamma_t] .$$

As Z is deterministic, we have $\mathbb{E}[|Z_t|\gamma_t] = |Z_t|\,\mathbb{E}[\gamma_t]$ for all $t \in I$. Therefore $\sum_{t\in I} Z_t\gamma_t$ converges absolutely almost surely if (c) is satisfied, which implies that Z_γ is well-defined. Further we see that Z_γ is integrable.

We now want to compute $\mathbb{E}[Z_\gamma]$. Due to the almost sure absolute convergence, we can exchange the series and the expected value and obtain

$$\mathbb{E}[Z_\gamma] = \mathbb{E}\left[\sum_{t\in I} Z_t\gamma_t\right] = \sum_{t\in I}\mathbb{E}[Z_t\gamma_t] = \sum_{t\in I} Z_t\,\mathbb{E}[\gamma_t] .$$

□

5.2 Results Using the Doob Decomposition

Within this section we assume $I \subset \mathbb{N}_0$ with $0 \in I$. Using the Doob decomposition the adapted process $Z = \{Z_t\}_{t\in I}$ can be decomposed into a martingale $M = \{M_t\}_{t\in I}$ and a predictable process $A = \{A_t\}_{t\in I}$ with $A_0 = 0$, i.e. $Z_t = M_t + A_t$ for all $t \in I$, see for example [62, Theorem 12.11] or [25, Lemma 6.10]. Using this decomposition we can try to compute the values $V(\nu)$ from (2.7), $V'(\nu)$ from (2.25) and $V^+(\nu)$ from (2.17).

Theorem 5.4 (One-period model)**.** *Assume $I = \{0, 1\}$. Given a probability distribution ν on I and an adapted stochastic process $Z \in L^1(\mathbb{P})$, where $Z = M + A$ denotes the Doob decomposition of Z, we get*

$$V^+(\nu) = \mathbb{E}[M_0] + \nu_1\,\mathrm{ES}_{1-\nu_1}(A_1) .$$

An optimal adapted random probability measure is therefore given by $\gamma_1^* = f_{1-\nu_1,A_1}$, where $f_{1-\nu_1,A_1}$ is defined as in Definition 4.3(b) and $\gamma_0^* = 1 - \gamma_1^*$. The optimal strategy γ^* is a.s. unique on $\{A_1 \neq q_{1-\nu_1}(A_1)\}$.

Remark 5.5. Note that the strategy γ^* in Theorem 5.4 is adapted, because A is predictable. We can apply Theorem 2.49 for M, because we work on a finite-time setting. Further note that $\gamma^* \in \mathcal{N}_I^\nu$, because $\mathbb{E}[\gamma_1^*|\mathcal{F}_0] = 1 - \gamma_0^*$, where $\frac{\nu_1}{1-\nu_0} = 1$.

Proof. Given $\nu_0 \in [0, 1]$ and $\nu_1 := 1 - \nu_0$, the value $V^+(\nu)$ can be computed by

$$\begin{aligned} V^+(\nu) &= \sup_{\gamma\in\mathcal{M}_I^\nu} \mathbb{E}[Z_0\gamma_0 + Z_1\gamma_1] \\ &= \sup_{\gamma\in\mathcal{M}_I^\nu} \left(\mathbb{E}[M_0] + \mathbb{E}[A_1\gamma_1]\right) , \end{aligned}$$

where the Doob decomposition has been used in order to get a better representation for this equation. Looking at this equation for $V^+(\nu)$, one discovers that the supremum is only dependent on the values of A_1 and we have $\gamma_1 \in \mathcal{F}^1_{1-\nu_1,A_1}$, where $\mathcal{F}^1_{1-\nu_1,A_1}$ is defined in Lemma 4.6. As we already saw in Lemma 4.6 the optimal strategy is therefore given by $\gamma_1 = f_{1-\nu_1,A_1}$ and we get

$$V^+(\nu) = \mathbb{E}[M_0] + \mathbb{E}[A_1 f_{1-\nu_1,A_1}] = \mathbb{E}[M_0] + \nu_1 \, \mathrm{ES}_{1-\nu_1}(A_1)\,.$$

The a.s. uniqueness of γ^* on $\{A_1 \neq q_{1-\nu_1}(A_1)\}$ follows from Lemma 4.6. □

Remark 5.6. In Theorem 2.41 we showed that for a given $\gamma \in \mathcal{M}^\nu_I$ we can find a stopping time defined on an enlarged probability space, which yields the same value. We will now use this lemma to search for a stopping time that corresponds to the optimal adapted random probability measure presented in Theorem 5.4. We will show how this is done for the case $\nu_0, \nu_1 \in (0,1)$ as the case $\nu_0 = 1$ or $\nu_1 = 1$ is trivial. In Theorem 2.41 we saw that we have to define the stopping time τ by

$$\{\tau = 0\} = \{0 < U \leq \gamma_0\} \quad \text{and} \quad \{\tau = 1\} = \{\gamma_0 < U \leq \gamma_0 + \gamma_1\}\,.$$

We found that the optimal strategy is given by

$$\gamma_1 = 1_{\{A_1 > q_{1-\nu_1}(A_1)\}} + \frac{\mathbb{P}\left(A_1 \leq q_{1-\nu_1}(A_1)\right) - 1 + \nu_1}{\mathbb{P}\left(A_1 = q_{1-\nu_1}(A_1)\right)} 1_{\{A_1 = q_{1-\nu_1}(A_1)\}}$$

and $\gamma_0 = 1 - \gamma_1$. We set

$$\beta_{1-\nu_1,A_1} = \frac{\mathbb{P}\left(A_1 \leq q_{1-\nu_1}(A_1)\right) - 1 + \nu_1}{\mathbb{P}\left(A_1 = q_{1-\nu_1}(A_1)\right)}$$

This implies

$$\begin{aligned}\{\tau = 1\} &= \{1 - 1_{\{A_1 > q_{1-\nu_1}(A_1)\}} - \beta_{1-\nu_1,A_1} 1_{\{A_1 = q_{1-\nu_1}(A_1)\}} < U \leq 1\} \\ &= \{A_1 > q_{1-\nu_1}(A_1)\} \cup \{A_1 = q_{1-\nu_1}(A_1), 1 - \beta_{1-\nu_1,A_1} < U \leq 1\}\end{aligned}$$

Remark 5.7. In the one-period model a choice must be made whether to exercise the option immediately or not, where we have given probabilities for these decisions. Such a setting is also used in randomized tests, which are explained, for example, in [32].

In the multi-period model this computation is rather demanding. When we look at $\mathbb{E}[Z_\gamma]$, we see that

$$\mathbb{E}[Z_\gamma] = \mathbb{E}[M_\gamma] + \mathbb{E}[A_\gamma]\ ,$$

because of the Doob decomposition of the adapted process Z. If M and ν satisfy one of the conditions of Theorem 2.49, then $\mathbb{E}[M_\gamma] = \mathbb{E}[M_0]$. So the only interesting value is $\mathbb{E}[A_\gamma]$, where we still have $A_0 = 0$.

Remark 5.8. If we assume, that for $I \subset \mathbb{N}_0$ with $0 \in I$ the process A of the Doob decomposition is deterministic and satisfies one of the conditions of Lemma 5.1, then

$$\mathbb{E}[A_\gamma] = \sum_{t \in I} \mathbb{E}[A_t \gamma_t] = \sum_{t \in I} A_t \nu_t\,.$$

Examples are processes Z with independent increments, where $A_t = \mathbb{E}[Z_t] - \mathbb{E}[Z_0]$ for $t \in I$. This includes integrable Lévy processes (like Brownian motion with drift and the Poisson process), considered at times $t \in I$. For processes with independent increments the result is also proven using different techniques in Lemma 5.10.

If the process A has uncorrelated increments, then the computation can be done as explained in Theorem 5.25.

Remark 5.9. To use the Doob decomposition several times by always defining a new adapted process each time with the help of the predictable process of the decomposition does not help to solve the problem. This is due to the fact that after the first decomposition we can neglect the set $\{\tau = 0\}$. If we now define a new adapted process P by $P_t = A_{t+1}$ for $t \in I \setminus \{0\}$, we now do not look at $\mathbb{E}[P_\tau]$ as the set $\{\tau = 0\}$ is already neglected and can no longer use Doob's optional stopping theorem for the martingale part of the decomposition.

5.3 Results for Processes with Uncorrelated Increments

5.3.1 Processes with Uncorrelated Increments

Lemma 5.10. *Given a discrete time interval $I \subset \mathbb{N}_0$ with $0 \in I$ and a probability distribution ν on I. For a given adapted stochastic process Z we define the increments of Z by $\Delta Z_0 := Z_0$ and $\Delta Z_t := Z_t - Z_{t-1}$ for all $t \in I \setminus \{0\}$. Assume the increments are integrable and there exists a sequence $\{c_t\}_{t\in I} \subset [1, \infty)$ such that they satisfy*

$$\mathbb{E}[\Delta Z_t \,|\, \mathcal{F}_{t-1}] \overset{a.s.}{=} \mathbb{E}[\Delta Z_t] \tag{5.11}$$

and

$$\mathbb{E}\big[|\Delta Z_t| \,\big|\, \mathcal{F}_{t-1}\big] \overset{a.s.}{\leq} c_t\, \mathbb{E}[|\Delta Z_t|] \tag{5.12}$$

for all $t \in I \setminus \{0\}$ with $\nu_{\leq t-1} < 1$, as well as

$$\sum_{t\in I} c_t\, \mathbb{E}[|\Delta Z_t|]\, (1 - \nu_{\leq t-1}) < \infty\,, \tag{5.13}$$

with the understanding that $1 - \nu_{\leq t-1} = 1$ for $t = 0$. Then, for all $\gamma \in \mathcal{M}_I^\nu$, Z_γ is well-defined, integrable and

$$\mathbb{E}[Z_\gamma] = \sum_{t\in I} \mathbb{E}[Z_t]\, \nu_t\,.$$

Remark 5.14. If the increment ΔZ_t is independent of $\mathcal{F}_{t-1}$ for every $t \in I \setminus \{0\}$, then (5.11) and (5.12) are satisfied with equality for $c_t = 1$ for all $t \in I \setminus \{0\}$. However (5.11) and (5.12) can also be satisfied in other cases, which will be demonstrated in Example 5.17. Note that (5.11) implies that the process Z has uncorrelated increments. Further note that it would be possible to replace condition (5.13) by $\mathbb{E}[\sup_{t\in I} |Z_t|] < \infty$.

Proof. Consider $\gamma \in \mathcal{M}_I^\nu$. First we will show that Z_γ is well-defined and integrable. Using that all the summands are non-negative, we have a.s.

$$\sum_{t\in I} |Z_t| \gamma_t = \sum_{t\in I} \gamma_t \Bigg| \sum_{s\in I_{\leq t}} \Delta Z_s \Bigg| \leq \sum_{t\in I} \gamma_t \sum_{s\in I_{\leq t}} |\Delta Z_s| = \sum_{s\in I} |\Delta Z_s| \gamma_{\geq s}\,.$$

By monotone convergence (see [62, Theorem 5.3]) and as $\sum_{t\in I} \gamma_t \overset{\text{a.s.}}{=} 1$ by Definition 2.12(b), we have

$$\mathbb{E}\Bigg[\sum_{s\in I} |\Delta Z_s| \gamma_{\geq s}\Bigg] = \sum_{s\in I} \mathbb{E}[|\Delta Z_s|(1 - \gamma_{<s})]\,.$$

By (5.12) and by understanding, that $1-\gamma_{<s}=1$ for $s=0$, we get

$$\begin{aligned}\sum_{s\in I}\mathbb{E}[|\Delta Z_s|(1-\gamma_{\le s-1})] &= \sum_{s\in I}\mathbb{E}\big[\mathbb{E}\left[|\Delta Z_s|\,\middle|\,\mathcal{F}_{s-1}\right](1-\gamma_{\le s-1})\big]\\ &\le \sum_{s\in I} c_s\,\mathbb{E}[|\Delta Z_s|]\,(1-\nu_{\le s-1})<\infty\,.\end{aligned}$$

This implies that $\sum_{t\in I} Z_t\gamma_t$ converges absolutely almost surely. Therefore Z_γ is well-defined. Since $|Z_\gamma|\le\sum_{t\in I}|Z_t|\gamma_t$, the calculations above show that Z_γ is integrable.

Repeating the above calculations without the absolute value, and using the dominated convergence theorem [62, Theorem 5.9] as well as (5.11) and the a.s. absolute convergence for exchanging the sums, we get

$$\mathbb{E}[Z_\gamma]=\sum_{t\in I}\mathbb{E}[\Delta Z_t]\,\nu_{\ge t}=\sum_{s\in I}\nu_s\sum_{t\in I_{\le s}}\mathbb{E}[\Delta Z_t]=\sum_{s\in I}\nu_s\,\mathbb{E}[Z_s]\;.$$

□

Remark 5.15. If we further assume in Lemma 5.10 that $\mathbb{E}[|Z_t|]\le C$ for all $t\in I$ and some $C>0$ and that $\sum_{t\in I}\nu_{>t}<\infty$, we can prove the result of the lemma using martingales.

The assumptions above imply that Z and ν satisfy Condition (c2) of Theorem 2.49. Define $\Delta Z_0:=Z_0$ and $\Delta Z_t:=Z_t-Z_{t-1}$ for all $t\in\mathbb{N}$. Then the process $M=\{M_t\}_{t\in I}$ defined by $M_t:=Z_t-\mathbb{E}[Z_t]=\sum_{s=0}^{t}(\Delta Z_s-\mathbb{E}[\Delta Z_s])$ for every $t\in I$ is a martingale, because of the uncorrelated increments of Z. Also M and ν satisfy Condition (c2) of Theorem 2.49. Using Theorem 2.49 we get for $\gamma\in\mathcal{M}_I^\nu$

$$\mathbb{E}[M_\gamma]=\mathbb{E}[M_0]=0\,.$$

On the other hand

$$\mathbb{E}[M_\gamma]=\mathbb{E}\left[\sum_{t\in I}(Z_t-\mathbb{E}[Z_t])\gamma_t\right]=\mathbb{E}\left[\sum_{t\in I}Z_t\gamma_t\right]-\mathbb{E}\left[\sum_{t\in I}\mathbb{E}[Z_t]\,\gamma_t\right],$$

where the two series are almost surely absolutely convergent and integrable due to our assumptions, as they also guarantee that one of the conditions of Lemma 5.1 is satisfied for the deterministic process $\{\mathbb{E}[Z_t]\}_{t\in I}$. This implies by Lemma 2.35

$$\mathbb{E}[Z_\gamma]=\mathbb{E}\left[\sum_{t\in I}Z_t\gamma_t\right]=\mathbb{E}\left[\sum_{t\in I}\mathbb{E}[Z_t]\,\gamma_t\right]=\sum_{t\in I}\mathbb{E}[Z_t]\,\nu_t\,,\quad \gamma\in\mathcal{M}_I^\nu\,.$$

Remark 5.16. The result found in Lemma 5.10 is the same as the one for the value $V^{\mathrm{ind}}(\nu)$. This means that the result is the same as if we would assume that the process Z and the process γ were independent, which is actually a lower bound for our problem. This value also coincides with the value found in Remark 5.8, where we stated that for $I\subset\mathbb{N}_0$ with $0\in I$, if M satisfies one of the conditions of Theorem 2.49 and A satisfies one of the conditions of Lemma 5.1,

$$V^+(\nu)=\mathbb{E}[M_0]+\sum_{t\in I}A_t\nu_t=\sum_{t\in I}(\mathbb{E}[M_t]\,\nu_t+\mathbb{E}[A_t]\,\nu_t)=\sum_{t\in I}\mathbb{E}[Z_t]\,\nu_t\,.$$

We will now give an example of a process, which does not have independent increments, but for which Lemma 5.10 applies, in particular an example that satisfies (5.11) and (5.12).

Example 5.17. Consider $I = \mathbb{N}_0$ and let $\{X_t\}_{t\in I}$ be independent, symmetric, $\{-1,1\}$-valued random variables. Define $\mathcal{F}_t = \sigma(X_0,\dots,X_t)$ for all $t \in I$. For a given deterministic sequence $\{a_t\}_{t\in I}$ with $a_t > 0$ for all $t \in I\setminus\{0\}$, set $Z_0 = a_0X_0$ and define the increments of the process Z by

$$\Delta Z_t = a_t X_t 1_{\{X_0=X_1=\cdots=X_{t-1}=-1\}} \quad \text{for all } t \in I\setminus\{0\}\,.$$

For every $t \in I\setminus\{0\}$ the events $\{\Delta Z_t = 0\} = \bigcup_{n=0}^{t-1}\{X_n = 1\} \in \mathcal{F}_{t-1}$ and $\{X_0 = X_1 = \cdots = X_{t-1} = -1\} \in \mathcal{F}_{t-1}$ have strictly positive probabilities, but $\{\Delta Z_t = 0, X_0 = X_1 = \cdots = X_{t-1} = -1\} = \varnothing$. Hence ΔZ_t is not independent of $\mathcal{F}_{t-1}$ for $t \in I\setminus\{0\}$. But

$$\mathbb{E}\left[|\Delta Z_t|\,\middle|\,\mathcal{F}_{t-1}\right] = a_t 1_{\{X_0=X_1=\cdots=X_{t-1}=-1\}}$$

and

$$\mathbb{E}[|\Delta Z_t|] = \frac{a_t}{2^t}\,,$$

which implies that (5.12) is satisfied for $c_t = 2^t$ for all $t \in I$. Also

$$\mathbb{E}[\Delta Z_t\,|\,\mathcal{F}_{t-1}] = 0 = \mathbb{E}[\Delta Z_t]\ ,$$

which implies that (5.11) is satisfied.

We can apply Lemma 5.10 to every distribution ν, satisfying (5.13), which is the case if

$$\sum_{t\in I} a_t(1-\nu_0-\cdots-\nu_{t-1}) = \sum_{t\in I} a_t \sum_{\substack{s\in I\\ s\geq t}} \nu_s = \sum_{s\in I}\nu_s\sum_{t=0}^{s} a_t\,.$$

By choosing $a_t = t^r$ for $t \in I$ and some $r > 0$, then

$$\sum_{s\in I}\nu_s\sum_{t=0}^{s} a_t \leq \sum_{s\in I}\nu_s s^{r+1}\,.$$

This is a convergent sum if the corresponding moment of the distribution ν exists.

We will now give an example, which shows that there are martingales with independent increments, for which Lemma 5.10 can be used, but not Theorem 2.49.

Example 5.18. Let $I = \mathbb{N}_0$ and let $\{X_t\}_{t\in I}$ be independent, $\{-1,1\}$-valued random variables with $\mathbb{P}(X_t = 1) = p$ and $\mathbb{P}(X_t = -1) = 1-p =: q$ for all $t \in I$ and $p \in [0,1]$. Define $\mathcal{F}_t = \sigma(X_0,\dots,X_t)$ for all $t \in I$. Then $M = \{M_t\}_{t\in I}$ defined by

$$M_t := \sum_{s=0}^{t} 2^s X_s\,,\quad t \in I\,,$$

is a martingale with independent increments, hence it satisfies (5.11) and (5.12) for $c_t = 1$ for all $t \in I$. Let $\tau := \inf\{t \in I\,|\,X_t = 1\}$. Then $\{\tau \geq t\} = \{X_0 = X_1 = \cdots = X_{t-1} = -1\}$ for all $t \in I$, which implies that τ is a stopping time. The distribution of this stopping time is given by $\mathbb{P}(\tau \geq t) = q^t$ for all $t \in I$. Actually τ follows a geometric distribution with $\mathbb{P}(\tau = t) = pq^t$ for all $t \in I$. Further the stopped martingale $M^\tau = \{M_t^\tau\}_{t\in I}$ is also a martingale. The increments of the stopped martingale M^τ are given by $\Delta M_0^\tau = M_0^\tau$ and

$$\Delta M_t^\tau = M_t^\tau - M_{t-1}^\tau = 2^t X_t 1_{\{\tau\geq t\}}\,,\quad t \in I\setminus\{0\}\,.$$

This implies

$$\mathbb{E}[|\Delta M_t^\tau|] = 2^t\,\mathbb{E}[|X_t|]\,\mathbb{P}(\tau \geq t) = (2q)^t\,.$$

Then

$$\sum_{t\in I} \mathbb{E}[|\Delta M_t^\tau|]\,\mathbb{P}(\tau \geq t) = \sum_{t\in I}(2q^2)^t\,.$$

This last geometric series converges, if $2q^2 < 1$, i.e. if $q < \frac{\sqrt{2}}{2} \approx 0.707107$. We can therefore apply Lemma 5.10 to the stochastic process M^τ by setting $p = 0.3$.

On the other hand we have $M_\tau^\tau \overset{\text{a.s.}}{=} 1$, which implies $\mathbb{E}[M_\tau^\tau] = 1 \neq 0 = \mathbb{E}[M_0^\tau]$, so that Doob's optional stopping theorem is not valid.

Remark 5.19. As we already noted in Chapter 3, we see that for martingales satisfying one of the conditions of Theorem 2.49 and for processes with uncorrelated increments satisfying the conditions of Lemma 5.10 we have that the expected value of $\sum_{t\in I} Z_t\gamma_t$ is the same for all $\gamma \in \mathcal{M}_I^\nu$. Therefore the optimal strategy cannot be unique in these cases.

We can now use Lemma 5.10 to modify Wald's equation, which states that for a random sum

$$S = \sum_{i=1}^N X_i,$$

where N is a non-negative integer-valued random variable with finite expectation, $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of real valued random variables with the same expectation, which satisfy $\mathbb{E}\big[X_n 1_{\{N\geq n\}}\big] = \mathbb{E}[X_n]\,\mathbb{P}(N \geq n)$ and $\sum_{n\geq 1}\mathbb{E}\big[|X_n| 1_{\{N\geq n\}}\big] < \infty$, the expectation of S can be computed by

$$\mathbb{E}[S] = \mathbb{E}[N]\,\mathbb{E}[X_1]\ .$$

More detailed information about Wald's equation can be found in [60] and [61].

Remark 5.20. By dropping the assumptions that the expectation of N is finite and that the $\{X_n\}_{n\in\mathbb{N}}$ all have the same expectation, and while still assuming that they are integrable and keeping the other assumptions as stated above, one can show that

$$S_N := \sum_{i=1}^N X_i \quad \text{and} \quad T_N := \sum_{i=1}^N \mathbb{E}[X_i]$$

are integrable and that they have the same expectation, i.e. $\mathbb{E}[S_N] = \mathbb{E}[T_N]$.

Lemma 5.21 (Modification of Wald's equation)**.** *Set $I = \mathbb{N}$. Given a stopping time τ and an independent sequence of integrable random variables $\{X_t\}_{t\in I}$. Assume*

$$\sum_{t\in I}\mathbb{E}\left[\sum_{i=1}^t |X_i| 1_{\{\tau=t\}}\right] < \infty \tag{5.22}$$

and

$$\sum_{t\in I}\mathbb{E}\left[\sum_{i=1}^t \mathbb{E}[|X_i|]\, 1_{\{\tau=t\}}\right] < \infty\,. \tag{5.23}$$

Then, for $S_\tau := \sum_{i=1}^\tau X_i$ and $T_\tau := \sum_{i=1}^\tau \mathbb{E}[X_i]$

$$\mathbb{E}[S_\tau] = \mathbb{E}[T_\tau]\ .$$

If further the random variables $\{X_t\}_{t\in I}$ all have the same expectation and τ has finite expectation, then

$$\mathbb{E}[S_\tau] = \mathbb{E}[\tau]\,\mathbb{E}[X_1]\;.$$

Proof. Define a process $\{Z_t\}_{t\in I}$ by $Z_t := X_1 + \cdots + X_t$ for $t \in I$. Since $\{X_t\}_{t\in I}$ is an i.i.d. sequence, the process Z has independent increments. We can now set $\gamma_t = 1_{\{\tau=t\}}$, where the distribution of the process γ is given by $\nu_t = \mathbb{P}(\tau = t)$ for $t \in I$. Then $S_\tau = Z_\tau$, where the process Z satisfies the conditions (5.11) and (5.12) of Lemma 5.10, because $\{X_t\}_{t\in I}$ is a sequence of independent random variables. We have that S_τ is well-defined and integrable, because

$$\mathbb{E}[|S_\tau|] = \mathbb{E}\Bigg[\Bigg|\sum_{i=1}^{\tau} X_i\Bigg|\Bigg] \le \mathbb{E}\Bigg[\sum_{i=1}^{\tau} |X_i|\Bigg] = \mathbb{E}\Bigg[\sum_{t\in I}\sum_{i=1}^{t} |X_i| 1_{\{\tau=t\}}\Bigg]\,.$$

By monotone convergence we can exchange the series and the expectation and can thus prove by (5.22) absolute convergence almost surely of the random series S_τ, which implies that it is well-defined and integrable.

Now we want to prove that also T_τ is well-defined and integrable.

$$\mathbb{E}[|T_\tau|] = \mathbb{E}\Bigg[\Bigg|\sum_{i=1}^{\tau} \mathbb{E}[X_i]\Bigg|\Bigg] \le \mathbb{E}\Bigg[\sum_{i=1}^{\tau} |\,\mathbb{E}[X_i]\,|\Bigg] = \mathbb{E}\Bigg[\sum_{t\in I}\sum_{i=1}^{t} |\,\mathbb{E}[X_i]\,| 1_{\{\tau=t\}}\Bigg]\,.$$

By monotone convergence we get

$$\mathbb{E}\Bigg[\sum_{t\in I}\sum_{i=1}^{t} |\,\mathbb{E}[X_i]\,| 1_{\{\tau=t\}}\Bigg] = \sum_{t\in I}\mathbb{E}\Bigg[\sum_{i=1}^{t} |\,\mathbb{E}[X_i]\,| 1_{\{\tau=t\}}\Bigg]\,.$$

By Jensen's inequality (see e.g. [62, Theorem 6.6] or [27, Theorem 7.9]), we have $|\,\mathbb{E}[X_t]\,| \le \mathbb{E}[|X_t|]$ for every $t \in I$. Therefore

$$\mathbb{E}\Bigg[\sum_{i=1}^{t} |\,\mathbb{E}[X_i]\,| 1_{\{\tau=t\}}\Bigg] \le \mathbb{E}\Bigg[\sum_{i=1}^{t} \mathbb{E}[|X_i|]\, 1_{\{\tau=t\}}\Bigg] = \mathbb{E}\Bigg[\mathbb{E}\Bigg[\sum_{i=1}^{t} |X_i|\Bigg] 1_{\{\tau=t\}}\Bigg]\,.$$

This implies that by (5.23) the stochastic series T_τ converges absolutely almost surely, which implies that T_τ is well-defined and integrable. By the absolute convergence of the stochastic series T_τ we can exchange the series and the expected value in oder to get

$$\mathbb{E}[T_\tau] = \mathbb{E}\Bigg[\sum_{t\in I}\mathbb{E}[Z_t]\,\gamma_t\Bigg] = \sum_{t\in I}\mathbb{E}[\mathbb{E}[Z_t]\,\gamma_t] = \sum_{t\in I}\mathbb{E}[Z_t]\,\nu_t\,,$$

with $\mathbb{E}[Z_t] = \sum_{i=1}^{t}\mathbb{E}[X_i]$ by linearity of the expected value.

By Lemma 5.10

$$\mathbb{E}[S_\tau] = \mathbb{E}\Bigg[\sum_{t\in I} Z_t\gamma_t\Bigg] = \sum_{t\in I}\mathbb{E}[Z_t]\,\nu_t\,.$$

If, further, all the random variables $\{X_t\}_{t\in I}$ have the same expectation and τ has finite expectation, then

$$\mathbb{E}[Z_t] = t\,\mathbb{E}[X_1] \quad \text{for all } t \in I\,.$$

This implies

$$\mathbb{E}[T_\tau] = \mathbb{E}[S_\tau] = \sum_{t\in I} t\,\mathbb{E}[X_1]\,\mathbb{P}(\tau = t) = \mathbb{E}[X_1]\,\mathbb{E}[\tau]\;.$$

□

Remark 5.24. In Lemma 5.21 we do not need any assumption about independence between τ and $\{X_t\}_{t\in I}$.

5.3.2 Predictable Processes with Uncorrelated Increments

Theorem 5.25. *Given a discrete time interval $I \subset \mathbb{N}_0$ with $0 \in I$ and a probability distribution ν on I, assume that the adapted process $Z = \{Z_t\}_{t\in I}$ can be decomposed into $Z_t = M_t + N_t + A_t$ for $t \in I$, where M is a martingale such that M and ν satisfy one of the conditions of Theorem 2.49, N is a process such that N and ν satisfy the conditions of Lemma 5.10 and A is a predictable process, with $A_0 = 0$. Denote the increments of the process A by $\Delta A_0 = A_0 = 0$ and $\Delta A_t := A_t - A_{t-1}$ for $t \in I \setminus \{0\}$. Assume that for the density of Definition 4.3(b) we have for every $t \in I \setminus \{0\}$ with $t+1 \in I$*

$$\mathbb{E}\big[f_{\delta_t,\Delta A_{t+1}}\,|\,\mathcal{F}_{t-1}\big] \overset{a.s.}{=} 1-\delta_t \tag{5.26}$$

and that for some sequence $\{c_t\}_{t\in I}$ for every $t \in I \setminus \{0\}$

$$\mathbb{E}\big[|\Delta A_t|\,\big|\,\mathcal{F}_{t-1}\big] \overset{a.s.}{\le} c_t\,\mathbb{E}[|\Delta A_t|]\;. \tag{5.27}$$

and that for each $t \in I \setminus \{0,1\}$ we have that $f_{\delta_{t-1},\Delta A_t}\Delta A_t$ and $(1-\gamma_0-\cdots-\gamma_{t-2})$ are uncorrelated. Further assume that the process A satisfies either

$$\sum_{t\in I\setminus\{0\}} c_t\,\mathbb{E}[|\Delta A_t|]\,\nu_{\ge t} < \infty \tag{5.28}$$

or

$$\mathbb{E}\Big[\sup_{t\in I}|A_t|\Big] < \infty\,. \tag{5.29}$$

With these assumptions we have that Z_γ is well-defined and integrable. Then an optimal adapted random probability measure γ^* is given by

$$\gamma_t^* = \begin{cases} (1-\gamma^*_{\le t-1})(1-f_{\delta_t,\Delta A_{t+1}}) & \text{if} \quad t+1 \in I\,,\\ 1-\gamma^*_{\le t-1} & \text{if} \quad t+1 \notin I\,, \end{cases} \tag{5.30}$$

where $f_{\delta_t,\Delta A_{t+1}}$ is defined as in Definition 4.3(b) and

$$\delta_t = \begin{cases} 1-\frac{1-\nu_{\le t}}{1-\nu_{\le t-1}} = \frac{\nu_t}{1-\nu_{\le t-1}} & \text{if} \quad \nu_{\le t-1} < 1\,,\\ 0 & \text{if} \quad \nu_{\le t-1} = 1\,. \end{cases}$$

Using this strategy we have

$$V^+(\nu) = \mathbb{E}[M_0] + \sum_{t\in I}\mathbb{E}[N_t]\,\nu_t + \sum_{t\in I\setminus\{0\}}(1-\nu_{\le t-1})\,\mathrm{ES}_{\delta_{t-1}}(\Delta A_t)\,.$$

If $\mathbb{P}(\Delta A_{t+1} = q_{\delta_t}(\Delta A_{t+1})) = 0$ for all $t \in I$ with $t+1 \in I$, then the optimal strategy γ^* is a.s. unique.

Remark 5.31. By admitting $N = 0$, the Doob decomposition can be used to find a decomposition of the adapted process Z. In Example 5.36 we will see, that there exist processes, for which the decomposition $Z = M + N + A$ works better than the standard Doob decomposition.

Remark 5.32. The theorem is especially valid for predictable processes A with independent increments, that satisfy (5.29).

Remark 5.33. For the strategy γ^* presented in Theorem 5.25 we even have $\gamma^* \in \mathcal{N}_I^\nu$, because for all $t \in I \setminus \{0\}$

$$\mathbb{E}[\gamma_t^* \,|\, \mathcal{F}_{t-1}] = (1-\gamma_{\le t-1}^*)\big(1 - \mathbb{E}\big[f_{\delta_t,\Delta A_{t+1}} \,|\, \mathcal{F}_{t-1}\big]\big)\,,$$

where $\mathbb{E}\big[f_{\delta_t,\Delta A_{t+1}} \,|\, \mathcal{F}_{t-1}\big] \overset{\text{a.s.}}{=} 1 - \delta_t$ by (5.26). By Lemma 2.31 we see that $\gamma^* \in \mathcal{N}_I^\nu$.

Proof. In Theorem 2.49 and Lemma 5.10 we showed that M_γ and N_γ are well-defined and integrable for $\gamma \in \mathcal{M}_I^\nu$. Next we will show that A_γ is well defined and integrable. For $\gamma \in \mathcal{M}_I^\nu$ we have a.s., using that all the summands are non-negative,

$$\sum_{t\in I\setminus\{0\}} |A_t|\gamma_t = \sum_{t\in I\setminus\{0\}} \gamma_t \Bigg| \sum_{\substack{s\in I\setminus\{0\}\\ s\le t}} \Delta A_s \Bigg| \le \sum_{t\in I\setminus\{0\}} \gamma_t \sum_{\substack{s\in I\setminus\{0\}\\ s\le t}} |\Delta A_s| = \sum_{s\in I\setminus\{0\}} |\Delta A_s| \sum_{\substack{t\in I\setminus\{0\}\\ t\ge s}} \gamma_t\,.$$

By monotone convergence (see [62, Theorem 5.3]) and by Definition 2.12(b) we have

$$\mathbb{E}\Bigg[\sum_{s\in I\setminus\{0\}} |\Delta A_s| \sum_{\substack{t\in I\setminus\{0\}\\ t\ge s}} \gamma_t\Bigg] = \sum_{s\in I\setminus\{0\}} \mathbb{E}[|\Delta A_s|(1-\gamma_{\le s-1})]\,.$$

Further by (5.27) and (5.28)

$$\begin{aligned}\sum_{s\in I\setminus\{0\}} \mathbb{E}[|\Delta A_s|(1-\gamma_{\le s-1})] &= \sum_{s\in I\setminus\{0\}} \mathbb{E}\big[\mathbb{E}\big[|\Delta A_s|\,\big|\,\mathcal{F}_{s-1}\big](1-\gamma_{\le s-1})\big]\\ &\le \sum_{s\in I\setminus\{0\}} c_s\,\mathbb{E}[|\Delta A_s|]\,(1-\nu_{\le s-1}) < \infty\,.\end{aligned}$$

If (5.29) is satisfied, then simply by Definition 2.12(b)

$$\mathbb{E}[|A_\gamma|] \le \mathbb{E}\Big[\sup_{t\in I}|A_t| \sum_{t\in I}\gamma_t\Big] = \mathbb{E}\Big[\sup_{t\in I}|A_t|\Big] < \infty.$$

This implies that A_γ is well-defined and integrable. It also implies that Z_γ is well-defined and integrable for $\gamma \in \mathcal{M}_I^\nu$.

Next we will show that we really have $\gamma^* \in \mathcal{M}_I^\nu$ and afterwards we will prove optimality. It can easily be seen that γ^* is adapted and that $\sum_{t\in I}\gamma_t^* \overset{\text{a.s.}}{=} 1$ by the definition of γ^* in (5.30). Now we will show that $\mathbb{E}[\gamma_t^*] = \nu_t$ and that $\gamma_t^* \ge 0$ a.s. for all $t \in I$. This is done by induction. We know that γ_0^* is $[0,1]$-valued a.s., as $f_{\delta_0,\Delta A_1}$ is $[0,1]$-valued a.s. by Remark 4.4(a), due to the definition of the process γ^*. Using Remark 4.4(b) we can easily compute

$$\mathbb{E}[\gamma_0^*] = 1 - \mathbb{E}[f_{\nu_0,\Delta A_1}] = 1 - (1-\nu_0) = \nu_0\,.$$

Fix $t \in I$ with $t+1 \in I$. We assume $\gamma_s^* \ge 0$ a.s. for $s \in \{1,\dots,t-1\}$, $\gamma_{\le t-1}^* \le 1$ a.s. and $\mathbb{E}\big[\gamma_{\le t-1}^*\big] = \nu_{\le t-1}$. We have that γ_t^* is $[0,1]$-valued a.s., because as $\gamma_{\le t-1}^* \le 1$ a.s. and by Remark 4.4(a)

$$1-\gamma_{\le t-1}^* \in [0,1] \text{ a.s.} \quad \text{and} \quad 1 - f_{\delta_t,\Delta A_{t+1}} \in [0,1] \text{ a.s.}$$

We know that

$$\mathbb{E}\big[1-\gamma^*_{\le t-1}\big] = 1-\nu_{\le t-1}\,.$$

For $t\in I$ with $t+1\in I$ we have, using (5.26),

$$\begin{aligned}\mathbb{E}[\gamma^*_t] &= \mathbb{E}\big[(1-\gamma^*_{\le t-1})\big(1-\mathbb{E}\big[f_{\delta_t,\Delta A_{t+1}}|\mathcal{F}_{t-1}\big]\big)\big] = \mathbb{E}\big[(1-\gamma^*_{\le t-1})\delta_t\big]\\ &= (1-\nu_{\le t-1})\delta_t = \nu_t\,.\end{aligned}$$

Further $\gamma^*_{\le t}\le 1$ a.s., because $\gamma^*_t\le 1-\gamma^*_{\le t-1}$ by Definition of γ^* in (5.30).

For $t\in I$ with $t+1\notin I$, we have $\gamma^*_t = 1-\gamma^*_{\le t-1}$. As $\gamma^*_{\le t-1}\in[0,1]$ a.s., we get $\gamma^*_t\ge 0$ a.s. and $\mathbb{E}[\gamma^*_t] = 1-\nu_{\le t-1} = \nu_t$, because ν is a probability distribution. Altogether we have that $\gamma^*\in\mathcal{M}^\nu_I$ as in Definition 2.12.

We still have to show that γ^* is optimal. For $\gamma\in\mathcal{M}^\nu_I$ we have, using Theorem 2.49 and Lemma 5.10,

$$\mathbb{E}[Z_\gamma] = \mathbb{E}[M_\gamma]+\mathbb{E}[N_\gamma]+\mathbb{E}[A_\gamma] = \mathbb{E}[M_0]+\sum_{t\in I}\mathbb{E}[N_t]\,\nu_t+\mathbb{E}[A_\gamma]\;.$$

Therefore

$$V^+(\nu) = \mathbb{E}[M_0]+\sum_{t\in I}\mathbb{E}[N_t]\,\nu_t+\sup_{\gamma\in\mathcal{M}^\nu_I}\mathbb{E}[A_\gamma]\;.$$

Repeating the above calculations without the absolute value, and using the dominated convergence theorem [62, Theorem 5.9] and the a.s. absolute convergence for exchanging the sums, we get, using Definition 2.12(b)

$$\mathbb{E}[A_\gamma] = \mathbb{E}\Bigg[\sum_{t\in I\setminus\{0\}}\Delta A_t\sum_{\substack{s\in I\setminus\{0\}\\ s\ge t}}\gamma_s\Bigg] = \mathbb{E}\Bigg[\sum_{t\in I\setminus\{0\}}\Delta A_t(1-\gamma_{\le t-1})\Bigg]\,.$$

We define

$$B_t := \{1-\gamma_{\le t}>0\}\,.$$

By dominated convergence we can exchange the series and the expected value and get

$$\begin{aligned}\mathbb{E}[A_\gamma] &= \sum_{t\in I\setminus\{0\}}\mathbb{E}\big[1_{B_{t-2}}(1-\gamma_{\le t-1})\Delta A_t\big]\\ &= \sum_{t\in I\setminus\{0\}}\mathbb{E}\left[(1-\gamma_{\le t-2})\left(1-\frac{\gamma_{t-1}}{1-\gamma_{\le t-2}}1_{B_{t-2}}\right)\Delta A_t\right]\,.\end{aligned}$$

Define $X := 1-\gamma_{\le t-2}$, which is $\mathcal{F}_{t-2}$-measurable. Then

$$\left(1-\frac{\gamma_{t-1}}{1-\gamma_{\le t-2}}1_{B_{t-2}}\right)\in\mathcal{F}^X_{\delta_{t-1},\Delta A_t}\,.$$

The value we are interested in is $\sup_{\gamma\in\mathcal{M}^\nu_I}\mathbb{E}[A_\gamma]$. For this we have

$$\begin{aligned}\sup_{\gamma\in\mathcal{M}^\nu_I}\mathbb{E}[A_\gamma] &\le \sum_{t\in I\setminus\{0\}}\sup_{\gamma\in\mathcal{M}^\nu_I}\mathbb{E}[(1-\gamma_{\le t-1})\Delta A_t]\\ &= \sum_{t\in I\setminus\{0\}}\sup_{\gamma_0,\dots,\gamma_{t-2}\in\mathcal{M}^\nu_I}\;\sup_{\gamma_{t-1}\in\mathcal{M}^\nu_I}\mathbb{E}[(1-\gamma_{\le t-1})\Delta A_t]\;.\end{aligned}$$

The last supremum with respect to $\gamma_{t-1} \in \mathcal{M}_I^\nu$ can be computed with the help of Lemma 4.6. We get

$$\begin{aligned}
&\sup_{\gamma_{t-1}\in\mathcal{M}_I^\nu} \mathbb{E}\left[(1-\gamma_{\le t-2})\left(1-\frac{\gamma_{t-1}}{1-\gamma_{\le t-2}}1_{B_{t-2}}\right)\Delta A_t\right] \\
&\le \sup_{f\in\mathcal{F}^X_{\delta_{t-1},\Delta A_t}} \mathbb{E}[(1-\gamma_{\le t-2})f\Delta A_t] \\
&= \mathbb{E}\big[(1-\gamma_{\le t-2})f_{\delta_{t-1},\Delta A_t}\Delta A_t\big]
\end{aligned}$$

As $f_{\delta_{t-1},\Delta A_t}\Delta A_t$ and $(1-\gamma_{\le t-2})$ are uncorrelated for each $t \in I\setminus\{0\}$, we get

$$\begin{aligned}
&\mathbb{E}\big[(1-\gamma_{\le t-2})f_{\delta_{t-1},\Delta A_t}\Delta A_t\big] \\
&= \mathbb{E}[(1-\gamma_{\le t-2})]\,(1-\delta_{t-1})\,\mathrm{ES}_{\delta_{t-1}}(\Delta A_t) \\
&= (1-\nu_{\le t-1})\,\mathrm{ES}_{\delta_{t-1}}(\Delta A_t)\,.
\end{aligned}$$

The upper bound computed here is exactly attained for γ^*, since $1-\frac{\gamma^*_{t-1}}{1-\gamma^*_{\le t-2}} = f_{\delta_{t-1},\Delta A_t}$.

The a.s. uniqueness of the optimal strategy if $\mathbb{P}(\Delta A_{t+1} = q_{\delta_t}(\Delta A_{t+1})) = 0$ for all $t \in I$ with $t+1 \in I$ follows from Lemma 4.6. $\square$

Remark 5.34. If we consider a finite discrete time interval $\{0,\dots,T\}$ we clearly have $\delta_T = 1$. Theorem 2.41 states that there exists a stopping time on an enlarged probability space, which yields the same value. In this case an optimal stopping time would be given by

$$\begin{aligned}
\{\tau^* = t\} &= \{\tau^* \ge t\} \\
&\quad \cap \left(\big\{\Delta A_t > q_{\delta_{t-1}}(\Delta A_t)\big\} \cup \big\{\Delta A_t = q_{\delta_{t-1}}(\Delta A_t), 1-\beta_{\delta_{t-1},\Delta A_t} < U\big\}\right),
\end{aligned}$$

for $t \in I$, with δ_{t-1} defined as in Theorem 5.25, $\beta_{\delta_{t-1},\Delta A_t}$ as in Definition 4.3(b) and $U \sim U(0,1)$ independent of A.

We know that for every process with independent increments, the predictable process in the Doob decomposition is deterministic and therefore also has independent increments. We will now give an example of an adapted process Z, for which the process A of the Doob decomposition has independent increments, but the process Z itself does not.

Example 5.35. Consider $I \subset \mathbb{N}_0$ with $0 \in I$ and an adapted process $Z = \{Z_t\}_{t\in I}$, where the random variables Z_t are independent for $t \in I$. Then Z does not necessarily have independent increments. The process $A = \{A_t\}_{t\in I}$ of the Doob decomposition of the process Z is given by

$$A_t = \sum_{k=1}^{t} \mathbb{E}[Z_k - Z_{k-1}\,|\,\mathcal{F}_{k-1}] = \sum_{k=1}^{t} (\mathbb{E}[Z_k] - Z_{k-1})\,.$$

Therefore

$$\Delta A_t = A_t - A_{t-1} = \mathbb{E}[Z_t] - Z_{t-1}\,.$$

We see that the increments of the predictable process in the Doob decomposition of Z are independent.

We will now give an example of a process that admits the use of Theorem 5.25 for the appropriate choice of ν and I by setting $Z = M + N + A$, but not if we set $Z = M + A$.

Example 5.36. Given a discrete time interval $I \subset \mathbb{N}_0$ with $0 \in I$ and a probability distribution ν on I. Let $X = \{X_t\}_{t\in I}$ be an i.i.d. process, for which X_t has a standard normal distribution for each $t \in I$. Define a process $N = \{N_t\}_{t\in I}$ with independent increments by $N_t := \sum_{s=0}^{t} X_s$ for each $t \in I$ and assume that N and ν satisfy the conditions of Lemma 5.10. Let $M = \{M_t\}_{t\in I}$ be a martingale such that M and ν satisfy one of the conditions of Theorem 2.49. Further let $A = \{A_t\}_{t\in I}$ be a predictable process satisfying the conditions of Theorem 5.25. For an appropriate choice of ν we can obviously use Theorem 5.25 for the computation of $V^+(\nu)$ for an adapted process $Z = \{Z_t\}_{t\in I}$ defined by $Z_t := M_t + N_t + A_t$ for all $t \in I$ with M, N and A defined as above.

We will now look at the Doob decomposition of the process Z. For the martingale part, which we will now denote by $\tilde{M} = \{\tilde{M}_t\}_{t\in I}$ we have

$$\tilde{M}_t = Z_0 + \sum_{k=1}^{t} (Z_k - \mathbb{E}[Z_k \,|\, \mathcal{F}_{k-1}]) = M_t + N_t \,.$$

As $X_t \sim N(0,1)$ for all $t \in I$ and therefore $N_t \sim N(0,t)$ for all $t \in I$ we have that neither the process $\tilde{M}$ nor its increments are bounded. Further $\tilde{M}$ is not uniformly integrable and it is not necessarily closable as well as it does not necessarily satisfy $\mathbb{E}[\sup_{t\in I} \tilde{M}_t^+] < \infty$ or $\mathbb{E}[\sup_{t\in I} \tilde{M}_t^-] < \infty$. This means that we cannot use Theorem 2.49 if we are given a probability distribution ν that does not satisfy $\nu_0 + \cdots + \nu_t = 1$ for some $t \in I$.

5.4 Greedy Strategy

To follow a greedy strategy means to exercise the option only if the value of the underlying process is as big as possible under the given circumstances. This strategy is optimal if the underlying process is independent, which will be shown in the following lemma. In Theorem 5.25 the optimal strategy in the case of random variables with uncorrelated increments in the predictable process of the Doob decomposition is to exercise the option if the following increment is small and to wait if it is going to be big – hence a somewhat greedy strategy. In Example 5.45 we will look at the situation, where the increments of the process are dependent.

We will start by presenting a lemma for stopping times, which will later on be generalized to adapted random probability measures.

Lemma 5.37. *Given a discrete time interval $I \subset \mathbb{N}_0$ with $0 \in I$ and a probability distribution ν on I. Assume that the adapted process $Z = \{Z_t\}_{t\in I}$ is a process of independent random variables such that $\mathbb{E}[\sup_{t\in I} |Z_t|] < \infty$. Further let $U = \{U_t\}_{t\in I}$ be an adapted process of independent random variables uniformly distributed on $[0,1]$, such that Z and U are independent. For $t \in I$ set*

$$\delta_t = 1 - \frac{\nu_t}{1 - \nu_{<t}}$$

and

$$E_t := \{Z_t > q_{\delta_t}(Z_t)\} \cup \{Z_t = q_{\delta_t}(Z_t), 1 - \beta_{\delta_t, Z_t} < U_t \le 1\}\,, \tag{5.38}$$

with β_{δ_t, Z_t} as in Definition 4.3(b). Then an optimal stopping time τ^* solving

$$\sup_{\tau \in \mathcal{T}_I^\nu} \mathbb{E}[Z_\tau] = \mathbb{E}[Z_{\tau^*}]$$

is given by

$$\{\tau^* = t\} = \begin{cases} \left(\bigcap_{I_{<t}} E_s^c\right) \cap E_t & \text{if } \nu_{\leq t} < 1\,, \\ \bigcap_{I_{<t}} E_s^c & \text{if } \nu_{\leq t} = 1\,. \end{cases}$$

$\mathbb{E}[Z_{\tau^*}]$ *can be computed as*

$$\mathbb{E}[Z_{\tau^*}] = \sum_{t \in I} \mathbb{E}\big[Z_t 1_{\{\tau^*=t\}}\big] = \sum_{t \in I} \nu_t \, \mathrm{ES}_{1-\frac{\nu_t}{1-\nu_{<t}}}(Z_t)\,.$$

Proof. This proof will be done by induction. The parameter that will change from one induction step to the next will be the end point $T \in \mathbb{N}_0$ of the time interval $I = \{0, \dots, T\}$ of the interval of the first T time points in I, which means that we will show that this strategy is optimal for every finite time interval. We will therefore start with $T = 0$.

When $T = 0$, we have to stop at time 0, since this is the only possible moment. So we will have $\nu_0 = 1$ and $\tau^* = 0$.

We now assume that for some $t \in I \setminus \{0\}$ the formula presented in the lemma is true for $T = t - 1$ and show that it also has to be true for $T = t$.

We have

$$\begin{aligned} V(\nu) &= \sup_{\tau \in \mathcal{T}^{\nu}_{\{0,\dots,t\}}} \mathbb{E}[Z_\tau] = \sup_{\tau \in \mathcal{T}^{\nu}_{\{0,\dots,t\}}} \mathbb{E}\big[Z_0 1_{\{\tau=0\}} + Z_\tau 1_{\{\tau \geq 1\}}\big] \\ &\leq \underbrace{\sup_{\tau \in \mathcal{T}^{\nu}_{\{0,\dots,t\}}} \mathbb{E}\big[Z_0 1_{\{\tau=0\}}\big]}_{(*)} + \underbrace{\sup_{\tau \in \mathcal{T}^{\nu}_{I}} \mathbb{E}\big[\big(Z_1 1_{\{\tau=1\}} + \dots + Z_t 1_{\{\tau=t\}}\big) 1_{\{\tau \geq 1\}}\big]}_{(**)}\,. \end{aligned}$$

$(*)$ and $(**)$ can now be computed separately.

When we look at $(*)$ we see, that Z_0 should be made as big as possible. This implies that the optimal stopping time at time 0 should satisfy $\{\tau^* = 0\} = E_0$.

For computing $(**)$ a change of measure can be done. Then the problem is reduced to a problem with time horizon $t - 1$, for which we already know the result due to our assumption. This means

$$(**) = \sup_{\tau \in \mathcal{T}^{\mu,\mathbb{Q}}_{\{1,\dots,t\}}} \mathbb{E}_{\mathbb{Q}}\left[Z_\tau\right]\,,$$

where $\mathbb{Q}$ is an equivalent probability measure and μ is the distribution of the new stopping time τ under $\mathbb{Q}$ with $\mu_i = \frac{\nu_i}{1-\nu_0}$ for $i = 1, \dots, t$. By $\mathcal{T}^{\mu,\mathbb{Q}}_{\{1,\dots,t\}}$ we denote the set of all stopping times with values in $\{1, \dots, t\}$ which have distribution μ under $\mathbb{Q}$. Now we use the formula stated in the lemma in order to compute the optimal stopping time τ^*, but we have to intersect with the set $E_0^c = \{\tau^* \geq 1\}$, since we do our computation on this set. Then for every $s \in \{1, \dots, t\}$ we have

$$\{\tau^* = s\} = \left(\bigcap_{i=1}^{s-1} E_i^c\right) \cap E_0^c \cap E_s = \left(\bigcap_{i=0}^{s-1} E_i^c\right) \cap E_s\,.$$

Actually the stopping time τ^* found is just an upper bound, but it can easily be checked that $\mathbb{P}(\tau^* = t) = \nu_t$ for all $t \in I$ and that τ^* is really a stopping time. $\square$

Remark 5.39. Note that in the proof of Lemma 5.37 the stopping time τ^* is for $s \in \{1, \dots, t\}$ found as

$$\{\tau^* = s\} = \left(\bigcap_{i=1}^{s-1} E_i^c\right) \cap E_0^c \cap E_s\,,$$

where by

$$\{\hat{\tau}^* = s\} = \left(\bigcap_{i=1}^{s-1} E_i^c\right) \cap E_s$$

the optimal stopping time for $(**)$ is found, i.e. $\hat{\tau}^* \in \mathcal{T}^{\mu,\mathbb{Q}}_{\{1,\dots,t\}}$. We can use the sets E_t of (5.38) for $t \in I \setminus \{0\}$ also for $\hat{\tau}^*$, because

$$1 - \frac{\mu_t}{1-\mu_{\leq t-1}} = 1 - \frac{\nu_t}{1-\nu_{\leq t-1}}\,.$$

Remark 5.40. If we look at such a set $\{\tau^* = t\}$, then we see that it is only dependent on Z_s with $s \in I_{\leq t}$. Since all the random variables Z_t with $t \in I$ are independent, this set $\{Z_s \,|\, s \in I_{<t}\}$ is independent of $\{Z_s \,|\, s \in I_{>t}\}$. Even the set $\{\tau^* \leq t\}$ is independent of $\{Z_s \,|\, s \in I_{>t}\}$. So whatever happens up to time t, does not effect the future. Therefore, it is best to act in an optimal way in each moment of time. Since we have to stop with a given probability, this means that we only take values as big as possible each moment in time.

We can extend Lemma 5.37 to the use of adapted random probability measures.

Theorem 5.41. *Given a totally-ordered countable discrete time interval I. Assume we are given an adapted process $Z = \{Z_t\}_{t\in I}$ of independent random variables such that $\mathbb{E}[\sup_{t\in I}|Z_t|] < \infty$. Then there exists an optimal strategy γ^* solving*

$$\sup_{\gamma\in\mathcal{M}_I^\nu} \mathbb{E}[Z_\gamma] = \mathbb{E}[Z_{\gamma^*}]\,,$$

where γ^* is given by

$$\gamma_t^* = \begin{cases} (1-\gamma_{<t}^*) f_{1-\delta_t, Z_t} & \text{if } \nu_{\leq t} < 1\,,\\ 1-\gamma_{<t}^* & \text{if } \nu_{\leq t} \geq 1\,,\end{cases}$$

with $f_{1-\delta_t,Z_t}$ as in Definition 4.3(b) and

$$\delta_t = \begin{cases} 1 - \frac{1-\nu_{\leq t}}{1-\nu_{<t}} = \frac{\nu_t}{1-\nu_{<t}} & \text{if } \nu_{<t} < 1\,,\\ 0 & \text{if } \nu_{<t} = 1\,.\end{cases}$$

Then

$$\mathbb{E}[Z_{\gamma^*}] = \sum_{t\in I} \nu_t \,\mathrm{ES}_{1-\frac{\nu_t}{1-\nu_{<t}}}(Z_t)\,.$$

Remark 5.42. Note that for $I \subset \mathbb{N}_0$ with $0 \in I$ the strategy γ^* presented in Theorem 5.41 is an element of $\mathcal{N}_I^\nu$, because for all $t \in I \setminus \{0\}$

$$\mathbb{E}[\gamma_t^* \,|\, \mathcal{F}_{t-1}] = (1-\gamma_{\leq t-1}^*)\big(\mathbb{E}[f_{1-\delta_t,Z_t} \,|\, \mathcal{F}_{t-1}]\big)\,,$$

where, due to the assumption of independence of the random variables Z_t for all $t \in I$, we get $\mathbb{E}[f_{1-\delta_t,Z_t} \,|\, \mathcal{F}_{t-1}] = \mathbb{E}[f_{1-\delta_t,Z_t}] = \delta_t$ by Remark 4.4(b). By Lemma 2.31 we see that $\gamma^* \in \mathcal{N}_I^\nu$.

Proof. This result follows from the proof of Theorem 5.25. We define

$$B_{<t} := \{1-\gamma_{<t} > 0\}\,.$$

We have by Lemma 2.35

$$\sup_{\gamma \in \mathcal{M}_I^\nu} \mathbb{E}[Z_\gamma] = \sup_{\gamma \in \mathcal{M}_I^\nu} \sum_{t\in I} \mathbb{E}[Z_t \gamma_t] \leq \sum_{t\in I} \sup_{\gamma \in \mathcal{M}_I^\nu} \mathbb{E}[Z_t\gamma_t] = \sum_{t\in I} \sup_{\gamma\in\mathcal{M}_I^\nu} \underbrace{\sup_{\gamma_t\in\mathcal{M}_I^\nu} \mathbb{E}[Z_t\gamma_t]}_{=(*)} .$$

$(*)$ can be computed with the help of Lemma 4.6, when setting $X := 1 - \gamma_{<t}$ and noticing $\frac{\gamma_t}{1-\gamma_{<t}} 1_{B_{<t}} \in \mathcal{F}^X_{1-\delta_t, Z_t}$. We have

$$\begin{aligned}(*) &= \sup_{\gamma_t\in\mathcal{M}_I^\nu} \mathbb{E}\left[(1-\gamma_{<t})\left(\frac{\gamma_t}{1-\gamma_{<t}} 1_{B_{<t}}\right) Z_t\right] \\ &\leq \sup_{f\in\mathcal{F}^X_{1-\delta_t,Z_t}} \mathbb{E}[(1-\gamma_{<t}) f Z_t] \\ &= \mathbb{E}[(1-\gamma_{<t}) f_{1-\delta_t,Z_t} Z_t]\end{aligned}$$

As Z_t and X are independent, because Z is an independent process and both processes are adapted, we have

$$\begin{aligned}\mathbb{E}[(1-\gamma_{<t}) f_{1-\delta_t,Z_t} Z_t] &= \mathbb{E}[(1-\gamma_{<t})]\, \delta_t \,\mathrm{ES}_{1-\delta_t}(Z_t) \\ &= \nu_t \,\mathrm{ES}_{1-\delta_t}(Z_t) .\end{aligned}$$

The upper bound computed here is exactly attained for γ^*, since $\frac{\gamma_t^*}{1-\gamma^*_{<t}} = f_{1-\delta_t,Z_t}$. The process γ^* is adapted and it is easy to prove $\mathbb{E}[\gamma_t^*] = \nu_t$. Assume T is the last element of I, for which $\nu_T > 0$, i.e. $\nu_{\leq T} = 1$ and $\nu_{<T} < 1$. Then $\delta_T = 1$ and $\gamma_T^* = 1 - \gamma^*_{<T}$, which implies that $\sum_{t\in I} \gamma_t^* \overset{\text{a.s.}}{=} 1$. Similar as in the proof of Theorem 5.25 it can be shown, that $\gamma_t^* \geq 0$ a.s. for all $t \in I$. Therefore $\gamma^* \in \mathcal{M}_I^\nu$ as in Definition 2.12. □

Example 5.43 (Greedy strategy - independent process)**.** Assume $I = \{0,1,2\}$ and let Z be a process with $Z_t \sim U(0, t+1)$ and Z_t independent for $t \in I$. Let $\{\mathcal{F}_t\}_{t\in I}$ be the filtration generated by the process Z and $\nu_0 = \nu_2 = \frac{1}{4}$ and $\nu_1 = \frac{1}{2}$.

The Doob decomposition of the process Z is given by

$$M_0 = Z_0 , \quad M_1 = Z_1 - \mathbb{E}[Z_1] + Z_0 , \quad M_2 = M_1 + Z_2 - \mathbb{E}[Z_2]$$

and

$$A_0 = 0 , \quad A_1 = \mathbb{E}[Z_1] - Z_0 , \quad A_2 = A_1 + \mathbb{E}[Z_2] - Z_1 .$$

Due to the independence of the process Z we have that the increments of the process A, given by

$$\Delta A_1 = A_1 - A_0 = \mathbb{E}[Z_1] - Z_0 , \quad \Delta A_2 = A_2 - A_1 = \mathbb{E}[Z_2] - Z_1 ,$$

are independent. Using the result of Theorem 5.25 we get

$$\begin{aligned}V^+(\nu) &= \mathbb{E}[M_0] + \sum_{t\in I\setminus\{0\}} (1 - \nu_0 - \cdots - \nu_{t-1}) \,\mathrm{ES}_{\delta_{t-1}}(\Delta A_t) \\ &= \frac{1}{2} + (1-\nu_0)\,\mathrm{ES}_{\nu_0}(\Delta A_1) + (1-\nu_0-\nu_1)\,\mathrm{ES}_{\frac{\nu_1}{1-\nu_0}}(\Delta A_2) \\ &= \frac{1}{2} + \frac{3}{4}\,\mathrm{ES}_{\frac{1}{4}}(\underbrace{1-Z_0}_{\sim U(0,1)}) + \frac{1}{4}\,\mathrm{ES}_{\frac{2}{3}}\Bigg(\underbrace{\frac{3}{2} - Z_1}_{\sim U(-\frac{1}{2},\frac{3}{2})}\Bigg) \\ &= \frac{1}{2} + \frac{3}{4}\cdot\frac{1}{2}\left(\frac{1}{4}+1\right) + \frac{1}{4}\cdot\frac{1}{2}\left(\frac{5}{6}+\frac{3}{2}\right) = \frac{121}{96} \approx 1.26042 .\end{aligned}$$

A greedy strategy (as found in Lemma 5.37) is modeled by a stopping time given by

$$\begin{aligned}
\{\tau^* = 0\} &= \{Z_0 > q_{1-\nu_0}(Z_0)\}\,,\\
\{\tau^* = 1\} &= \{Z_0 \le q_{1-\nu_0}(Z_0)\} \cap \{Z_1 > q_{1-\frac{\nu_1}{1-\nu_0}}(Z_1)\}\,,\\
\{\tau^* = 2\} &= \{Z_0 \le q_{1-\nu_0}(Z_0)\} \cap \{Z_1 \le q_{1-\frac{\nu_1}{1-\nu_0}}(Z_1)\}\\
&\qquad \cap \{Z_2 > q_{1-\frac{\nu_2}{1-\nu_0-\nu_1}}(Z_2) = q_0(Z_2)\}\,,
\end{aligned}$$

This yields

$$\begin{aligned}
\mathbb{E}\Big[\sum_{t\in I} Z_t 1_{\{\tau=t\}}\Big] &= \frac{1}{4}\,\mathbb{E}\Big[Z_0\,\Big|\,Z_0 > q_{\frac{3}{4}}(Z_0)\Big] + \frac{1}{2}\,\mathbb{E}\Big[Z_1\,\Big|\,Z_1 > q_{\frac{1}{3}}(Z_1)\Big]\\
&\quad + \frac{1}{4}\,\mathbb{E}[Z_2\,|\,Z_2 > q_0(Z_2)]\\
&= \frac{1}{4}\,\mathbb{E}\Big[Z_0\,\Big|\,Z_0 > \frac{3}{4}\Big] + \frac{1}{2}\,\mathbb{E}\Big[Z_1\,\Big|\,Z_1 > \frac{2}{3}\Big] + \frac{1}{4}\,\mathbb{E}[Z_2\,|\,Z_2 > 0]\\
&= \frac{1}{4}\cdot\frac{1}{2}\Big(\frac{3}{4}+1\Big) + \frac{1}{2}\cdot\frac{1}{2}\Big(\frac{2}{3}+2\Big) + \frac{1}{4}\cdot\frac{3}{2} = \frac{121}{96} \approx 1.26042\,.
\end{aligned}$$

Using the upper bound found in Proposition 4.29, we get

$$\begin{aligned}
V^+(\nu) &\le \sum_{t\in I} \nu_t\,\mathrm{ES}_{1-\nu_t}(Z_t)\\
&= \frac{1}{4}\,\mathbb{E}\Big[Z_0\,\Big|\,Z_0 > \frac{3}{4}\Big] + \frac{1}{2}\,\mathbb{E}[Z_1\,|\,Z_1 > 1] + \frac{1}{4}\,\mathbb{E}\Big[Z_2\,\Big|\,Z_2 > \frac{9}{4}\Big]\\
&= \frac{1}{4}\cdot\frac{1}{2}\Big(\frac{3}{4}+1\Big) + \frac{1}{2}\cdot\frac{3}{2} + \frac{1}{4}\cdot\frac{1}{2}\Big(\frac{9}{4}+3\Big) = \frac{1}{4}\Big(\frac{7}{8}+3+\frac{21}{8}\Big) = \frac{13}{8} = 1.625\,.
\end{aligned}$$

Using the upper bound of Proposition 4.31 and the sub-additivity of the expected shortfall, we find

$$\begin{aligned}
V^+(\nu) &\le \mathbb{E}[M_0] + \sum_{t\in I} \nu_t\,\mathrm{ES}_{1-\nu_t}(A_t)\\
&\le \mathbb{E}[M_0] + \nu_1\,\mathrm{ES}_{1-\nu_1}(A_1) + \nu_2(\mathrm{ES}_{1-\nu_2}(\Delta A_1) + \mathrm{ES}_{1-\nu_2}(\Delta A_2))\\
&= \frac{1}{2} + \frac{1}{2}\cdot\frac{3}{4} + \frac{1}{4}(\frac{7}{8}+\frac{5}{4}) = \frac{45}{21} \approx 1.40625\,,
\end{aligned}$$

which gives a smaller value than the upper bound of Proposition 4.29.

Under the assumption of independence, we get

$$V^{\mathrm{ind}}(\nu) = \sum_{t\in I} \mathbb{E}[Z_t]\,\nu_t = \frac{1}{2}\cdot\frac{1}{4} + 1\cdot\frac{1}{2} + \frac{3}{2}\cdot\frac{1}{4} = 1\,.$$

Remark 5.44. In Example 5.43 we saw that for the considered process the values using the strategies of Theorem 5.25 and of Lemma 5.37 are equal. On the other hand, we noted that the strategy of Theorem 5.25 is unique for processes with continuous distribution functions. Therefore, we will now show that these two strategies really coincide, using the greedy strategy for the process after applying the Doob decomposition.

$$\begin{aligned}
\mathbb{E}[Z_{\tau^*}] &= \mathbb{E}\left[M_2 \sum_{t\in I} 1_{\{\tau^*=t\}}\right] + \sum_{t\in I} \mathbb{E}\big[A_t 1_{\{\tau^*=t\}}\big] = \mathbb{E}[M_2] + \mathbb{E}\big[A_1 1_{\{\tau^*=1\}}\big] + \mathbb{E}\big[A_2 1_{\{\tau^*=2\}}\big] \\
&= \mathbb{E}[M_2] + \mathbb{E}\left[\Delta A_1 1_{\{\Delta A_1 > q_{\nu_0}(\Delta A_1)\}} 1_{\{Z_1 > q_{1-\frac{\nu_1}{1-\nu_0}}(Z_1)\}}\right] \\
&\quad + \mathbb{E}\left[(\Delta A_1 + \Delta A_2) 1_{\{Z_0 \le q_{1-\nu_0}(Z_0)\}} 1_{\{Z_1 \le q_{1-\frac{\nu_1}{1-\nu_0}}(Z_1)\}} 1_{\{Z_2 > q_0(Z_2)\}}\right] \\
&= \mathbb{E}[M_0] + \mathbb{E}\Big[A_1 1_{\{\Delta A_1 > q_{\nu_0}(\Delta A_1)\}}\Big] \frac{\nu_1}{1-\nu_0} \\
&\quad + \mathbb{E}\Big[\Delta A_1 1_{\{\Delta A_1 > q_{\nu_0}(\Delta A_1)\}}\Big] \Big(1 - \frac{\nu_1}{1-\nu_0}\Big) + \mathbb{E}\left[\Delta A_2 1_{\{\Delta A_2 > q_{\frac{\nu_1}{1-\nu_0}}(\Delta A_2)\}}\right](1-\nu_0) \\
&= \mathbb{E}[M_0] + (1-\nu_0)\,\mathrm{ES}_{\nu_0}(\Delta A_1) \frac{\nu_1}{1-\nu_0} + (1-\nu_0)\,\mathrm{ES}_{\nu_0}(\Delta A_1)\Big(1 - \frac{\nu_1}{1-\nu_0}\Big) \\
&\quad + \Big(1 - \frac{\nu_1}{1-\nu_0}\Big)\,\mathrm{ES}_{\frac{\nu_1}{1-\nu_0}}(\Delta A_2)(1-\nu_0) \\
&= \mathbb{E}[M_0] + (1-\nu_0)\,\mathrm{ES}_{\nu_0}(\Delta A_1) + (1-\nu_0-\nu_1)\,\mathrm{ES}_{\frac{\nu_1}{1-\nu_0}}(\Delta A_2)\,,
\end{aligned}$$

because

$$\{Z_0 \le q_{1-\nu_0}(Z_0)\} = \{\Delta A_1 > q_{\nu_0}(\Delta A_1)\}$$

and

$$\{Z_1 \le q_{1-\frac{\nu_1}{1-\nu_0}}(Z_1)\} = \{\Delta A_2 > q_{\frac{\nu_1}{1-\nu_0}}(\Delta A_2)\}$$

We saw that we can use a greedy strategy for independent processes. In the following example we will look at dependent processes and notice that the optimal strategy is not of greedy type.

Example 5.45. Assume $I = \{0, \dots, T\}$ with $T = 2$ and $U \sim U(0,1)$. Further assume that the process Z is given by $Z_0 = U, Z_1 = 0, Z_2 = 2U$. Using the Doob decomposition for this process, we get $M_0 = M_1 = M_2 = U$ and $A_0 = 0, A_1 = -U, A_2 = U$. The increments $A_1 - A_0 = -U$ and $A_2 - A_1 = 2U$ are clearly dependent. Assume for example that the distribution for the stopping time is given by $\nu_0 = \frac{2}{6}, \nu_1 = \frac{3}{6}, \nu_2 = \frac{1}{6}$. An optimal stopping time one can find for this special problem is given by

$$\begin{aligned}
\{\tau^* = 0\} &= \{\nu_1 < U \le 1 - \nu_2\}\,, \\
\{\tau^* = 1\} &= \{U \le \nu_1\}\,, \\
\{\tau^* = 2\} &= \{U > 1 - \nu_2\}\,,
\end{aligned}$$

because it yields

$$\begin{aligned}
\mathbb{E}[Z_{\tau^*}] &= \nu_0\,\mathbb{E}[U\,|\,\tau^* = 0] + \nu_1\,\mathbb{E}[0\,|\,\tau^* = 1] + \nu_2\,\mathbb{E}[2U\,|\,\tau^* = 2] \\
&= \nu_0 \frac{\nu_1 + (1-\nu_2)}{2} + 2\nu_2\Big(1 - \frac{\nu_2}{2}\Big) \\
&= \frac{2}{6}\left(\frac{3}{12} + \frac{5}{12}\right) + 2 \cdot \frac{1}{6} \cdot \frac{11}{12} = \frac{38}{72}\,.
\end{aligned}$$

In Example 4.50 we saw that this value equals the value of the upper bound found in Proposition 4.31. Therefore this strategy is optimal.

When taking a closer look at the upper bound of Proposition 4.31, we can find another strategy that yields this optimal value. This is given by

$$\gamma_0 = 1 - \gamma_1 - \gamma_2 \,,$$
$$\gamma_1 = f_{1-\nu_1,-U} \,,$$
$$\gamma_2 = f_{1-\nu_2,U} \,,$$

where γ_1 and γ_2 are defined as in Definition 4.3(b). Actually this strategy equals the strategy used for finding the upper bound in Proposition 4.31, because

$$\begin{aligned}\mathbb{E}[Z_\gamma] &= \mathbb{E}[U(1-\gamma_1-\gamma_2) + 0\cdot\gamma_1 + 2U\gamma_2] \\ &= \mathbb{E}[U] + \mathbb{E}[-Uf_{1-\nu_1,-U}] + \mathbb{E}[Uf_{1-\nu_2,U}] \\ &= \frac{1}{2} + \frac{3}{6}\,\mathrm{ES}_{\frac{3}{6}}(-U) + \frac{1}{6}\,\mathrm{ES}_{\frac{5}{6}}(U) = \frac{38}{72}\,.\end{aligned}$$

We have to check whether we have $\gamma \in \mathcal{M}_I^\nu$. The process γ is adapted, because the whole process is $\mathcal{F}_0$-measurable. Obviously $\sum_{t\in I}\gamma_t \overset{\text{a.s.}}{=} 1$. Also we can easily check that $\mathbb{E}[\gamma_t] = \nu_t$ for all $t \in I$. We see that $\gamma_1 \geq 0$ a.s. and $\gamma_2 \geq 0$ a.s. Now we have to check whether $\gamma_0 \geq 0$ a.s. We have

$$\gamma_1 = 1_{\{-U > q_{1-\nu_1}(-U)\}} = 1_{\{-U > -\nu_1\}} = 1_{\{U < \nu_1\}}$$

and

$$\gamma_2 = 1_{\{U > q_{1-\nu_2}(U)\}} = 1_{\{U > 1-\nu_2\}}\,.$$

We see that $\gamma_0 \geq 0$ a.s. because $\{U < \nu_1\} \cap \{U > 1-\nu_2\} = \varnothing$.

Based on the results of this example we can try to find optimal strategies for $\mathcal{F}_0$-measurable processes, which will be the subject of the following section.

5.5 $\mathcal{F}_0$-Measurable Processes

In this section we will consider discrete time intervals $I \subset \mathbb{N}_0$ with $0 \in I$. Assuming that the process Z is $\mathcal{F}_0$-measurable, we know at time 0 which values the process Z will take. In Example 5.45 we assumed we are given a process Z which is $\mathcal{F}_0$-measurable and we also found the optimal stopping time for this process. In this example we considered a process Z for which there exists a $\mathcal{F}_0$-measurable random variable U with $U \sim U(0,1)$, such that we can represent $Z_t = f_t(U)$ for every $t \in I$, with some deterministic function f_t. Intuitively it would make sense to optimize with respect to the deterministic function determining Z_t for each $t \in I$ and then to optimize with respect to ν. Conversely, the optimization could be done the other way around by starting with ν. In the following definition we explain what we mean by optimizing with respect to the deterministic functions or ν.

Definition 5.46. Again we only consider stopping times which follow the given distribution ν. For a process Z satisfying $Z_t = f_t(U)$ for every $t \in I$, with some deterministic function f_t and a $\mathcal{F}_0$-measurable random variable U with $U \sim U(0,1)$, we say that a stopping time τ^1 optimizes with respect to the deterministic function, if the stopping time τ^1 is defined by

$$\{\tau^1 = t\} = \{b_t < U \leq b_t + \nu_t\} \quad \forall\, t \in I\,,$$

with

$$b_t := \sum_{\substack{s\in I\\ f_s(U)<f_t(U)}} \nu_s + \sum_{\substack{s\in I\\ f_s(U)=f_t(U)\\ \nu_s<\nu_t}} \nu_s \,.$$

We say that a stopping time τ^2 optimizes with respect to ν, if the stopping time τ^2 is defined by

$$\{\tau^2 = t\} = \{c_t < U \le c_t + \nu_t\} \quad \forall\, t \in I\,,$$

with

$$c_t := \sum_{\substack{s\in I\\ \nu_s<\nu_t}} \nu_s + \sum_{\substack{s\in I\\ \nu_s=\nu_t\\ f_s(U)<f_t(U)}} \nu_s \,.$$

In the following examples we will take a closer look at these two strategies and see which one gives the higher value in different situations. This will show that we cannot give a general statement about which of these two strategies works better.

Example 5.47. For $I = \{0, \dots, 5\}$ consider a process Z given by $Z_0 = U$, $Z_1 = 3U$, $Z_2 = 4U$, $Z_3 = 2U$, $Z_4 = 3U$ and $Z_5 = 2U$, where U is $\mathcal{F}_0$-measurable and $U \sim U(0,1)$. We assume that the distribution of the stopping times considered is given by $\nu_0 = \frac{1}{6}$, $\nu_1 = \frac{1}{8}$, $\nu_2 = \frac{1}{4}$, $\nu_3 = \frac{1}{4}$, $\nu_4 = \frac{1}{12}$ and $\nu_5 = \frac{1}{8}$. For this distribution $\nu_4 < \nu_1 = \nu_5 < \nu_0 < \nu_2 = \nu_3$. If we write $Z_t = f_t(U)$ for $t = 0, \dots, 5$, then $f_0(x) = x < f_3(x) = f_5(x) = 2x < f_1(x) = f_4(x) = 3x < f_2(x) = 4x$ for all $x > 0$. If $x = 0$ we have $f_t(x) = 0$ for $t = 0, \dots, 5$.
If we want to optimize with respect to the deterministic function, we will choose a stopping time defined by

$$\{\tau^1 = 0\} = \{0 < U \le \nu_0\} = \left\{0 < U \le \frac{1}{6}\right\}\,,$$

$$\{\tau^1 = 1\} = \{\nu_0 + \nu_3 + \nu_4 + \nu_5 < U \le \nu_0 + \nu_3 + \nu_4 + \nu_5 + \nu_1\} = \left\{\frac{5}{8} < U \le \frac{3}{4}\right\}\,,$$

$$\{\tau^1 = 2\} = \{\nu_0 + \nu_1 + \nu_3 + \nu_4 + \nu_5 < U \le 1\} = \left\{\frac{3}{4} < U \le 1\right\}\,,$$

$$\{\tau^1 = 3\} = \{\nu_0 + \nu_5 < U \le \nu_0 + \nu_5 + \nu_3\} = \left\{\frac{7}{24} < U \le \frac{13}{24}\right\}\,,$$

$$\{\tau^1 = 4\} = \{\nu_0 + \nu_3 + \nu_5 < U \le \nu_0 + \nu_3 + \nu_5 + \nu_4\} = \left\{\frac{13}{24} < U \le \frac{5}{8}\right\}$$

and

$$\{\tau^1 = 5\} = \{\nu_0 < U \le \nu_0 + \nu_5\} = \left\{\frac{1}{6} < U \le \frac{7}{24}\right\}\,.$$

Using this stopping time we get

$$\begin{aligned}
\mathbb{E}[Z_{\tau^1}] &= \sum_{t\in I} \nu_t\, \mathbb{E}\big[Z_t \big| \tau^1 = t\big] \\
&= \frac{1}{6}\cdot\frac{1}{12} + \frac{1}{8}\cdot 3\cdot\frac{11}{16} + \frac{1}{4}\cdot 4\cdot\frac{7}{8} + \frac{1}{4}\cdot 2\cdot\frac{10}{24} + \frac{1}{12}\cdot 3\cdot\frac{7}{12} + \frac{1}{8}\cdot 2\cdot\frac{11}{48} \\
&= \frac{1795}{1152} \approx 1.55816\,.
\end{aligned}$$

A stopping time τ^2 optimizing with respect to ν would be given by

$$\{\tau^2 = 0\} = \{\nu_1 + \nu_4 + \nu_5 < U \leq \nu_1 + \nu_4 + \nu_5 + \nu_0\} = \left\{\frac{1}{3} < U \leq \frac{1}{2}\right\},$$

$$\{\tau^2 = 1\} = \{\nu_4 + \nu_5 < U \leq \nu_4 + \nu_5 + \nu_1\} = \left\{\frac{5}{24} < U \leq \frac{1}{3}\right\},$$

$$\{\tau^2 = 2\} = \{\nu_0 + \nu_1 + \nu_3 + \nu_4 + \nu_5 < U \leq 1\} = \left\{\frac{3}{4} < U \leq 1\right\},$$

$$\{\tau^2 = 3\} = \{\nu_0 + \nu_1 + \nu_4 + \nu_5 < U \leq \nu_0 + \nu_1 + \nu_4 + \nu_5 + \nu_3\} = \left\{\frac{1}{2} < U \leq \frac{3}{4}\right\},$$

$$\{\tau^2 = 4\} = \{0 < U \leq \nu_4\} = \left\{0 < U \leq \frac{1}{12}\right\}$$

and

$$\{\tau^2 = 5\} = \{\nu_4 < U \leq \nu_4 + \nu_5\} = \left\{\frac{1}{12} < U \leq \frac{5}{24}\right\}.$$

This stopping time yields

$$\begin{aligned}
\mathbb{E}[Z_{\tau^2}] &= \sum_{t\in I} \nu_t \, \mathbb{E}\left[Z_t \middle| \tau^2 = t\right] \\
&= \frac{1}{6} \cdot \frac{5}{12} + \frac{1}{8} \cdot 3 \cdot \frac{13}{48} + \frac{1}{4} \cdot 4 \cdot \frac{7}{8} + \frac{1}{4} \cdot 2 \cdot \frac{5}{8} + \frac{1}{12} \cdot 3 \cdot \frac{1}{24} + \frac{1}{8} \cdot 2 \cdot \frac{7}{48} \\
&= \frac{1619}{1152} \approx 1.40538\,.
\end{aligned}$$

Example 5.48. Assume $I = \{0, 1, 2\}$ and $Z_0 = U$, $Z_1 = U^2$ and $Z_2 = U$, where U is a $\mathcal{F}_0$-measurable random variable with $U \sim U(0,1)$. Set $\nu_0 = \frac{1}{2}$, $\nu_1 = \frac{1}{6}$ and $\nu_2 = \frac{1}{3}$. Then the two stopping times coincide, i.e. $\tau^1 = \tau^2$ and we have

$$\{\tau^1 = 0\} = \left\{\frac{1}{2} < U \leq 1\right\}, \quad \{\tau^1 = 1\} = \left\{0 < U \leq \frac{1}{6}\right\}, \quad \{\tau^1 = 2\} = \left\{\frac{1}{6} < U \leq \frac{1}{2}\right\}.$$

We get

$$\mathbb{E}[Z_{\tau^1}] = \mathbb{E}[Z_{\tau^2}] = \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{6} \int_{\frac{5}{6}}^{1} x^2 \, dx + \frac{1}{3} \cdot \frac{1}{3} = \frac{1891}{3888} \approx 0.486368\,.$$

Example 5.49. Consider $I = \{0, 1\}$, $Z_0 = U + \frac{1}{3}$ and $Z_1 = U + \frac{1}{4}$, where U is a $\mathcal{F}_0$-measurable random variable with $U \sim U(0,1)$. Further assume $\nu_0 = \frac{1}{4}$ and $\nu_1 = \frac{3}{4}$. Then

$$\{\tau^1 = 0\} = \{U > \nu_0\} \quad \text{and} \quad \{\tau^1 = 1\} = \{U \leq \nu_0\}$$

yields

$$\mathbb{E}[Z_{\tau^1}] = \frac{1}{4}\left(\mathbb{E}\left[U \middle| U > \frac{1}{4}\right] + \frac{1}{3}\right) + \frac{3}{4}\left(\mathbb{E}\left[U \middle| U \leq \frac{1}{4}\right] + \frac{1}{4}\right) = \frac{25}{48} \approx 0.520833\,,$$

while on the other hand

$$\{\tau^2 = 0\} = \{U \leq \nu_0\} \quad \text{and} \quad \{\tau^2 = 1\} = \{U > \nu_0\}$$

yields

$$\mathbb{E}[Z_{\tau^2}] = \frac{1}{4}\left(\mathbb{E}\left[U \middle| U \leq \frac{1}{4}\right] + \frac{1}{3}\right) + \frac{3}{4}\left(\mathbb{E}\left[U \middle| U > \frac{1}{4}\right] + \frac{1}{4}\right) = \frac{37}{48} \approx 0.770833\,.$$

The following proposition will deal with a type of process for which the stopping time τ^1 of Definition 5.46 is optimal.

Proposition 5.50. *Given $I = \{0, \dots, T\}$, assume Z_t is $\mathcal{F}_0$-measurable for all $t \in I$ and that the distribution ν of the stopping times considered is a uniform distribution, i.e. $\nu_t = \frac{1}{T+1}$ for all $t \in I$. If there exists a $\mathcal{F}_0$-measurable random variable U with $U \sim U(0,1)$, such that we can represent $Z_t = f_t(U)$ for every $t \in I$, with some function $f_t(x) = d_t \cdot x$ for $x \in \mathbb{R}$ and a fixed $d_t \in \mathbb{R}$, then the strategy defined by the stopping time τ^1 of Definition 5.46 given by*

$$\{\tau^1 = t\} = \{b_t < U \le b_t + \nu_t\} \quad \forall\, t \in I\,,$$

with

$$b_t := \sum_{\substack{s \in I \\ d_s < d_t}} \nu_s + \sum_{\substack{s \in I \\ d_s = d_t \\ s < t}} \nu_s$$

is optimal.

Proof. We have

$$\mathbb{E}[Z_\tau] = \sum_{t \in I} \nu_t\, \mathbb{E}[Z_t \,|\, \tau = t] = \frac{1}{T+1} \sum_{t \in I} d_t\, \mathbb{E}[U \,|\, \tau = t]\ .$$

We know that the sum becomes as large as possible if the d_t and the conditional expectations are arranged in the same order for $t \in I$. This result is due to the rearrangement inequality, see e.g. [18, Chapter X, Theorem 368]. We see that the strategy defined by τ^1 satisfies this. □

Remark 5.51. Note that the optimal strategy in Proposition 5.50 is not necessarily unique. Further we replaced the condition $\nu_s < \nu_t$ in the second sum of the definition of b_t by $s < t$ for s, t in I as we assumed that the stopping time would follow a uniform distribution. Further we replaced the conditions $f_s(U) < f_t(U)$ and $f_s(U) = f_t(U)$ by $d_s < d_t$ and $d_s = d_t$, respectively, due to the assumptions made for the deterministic function f.

Lemma 5.52. *Given a discrete time interval $I \subset \mathbb{N}_0$ with $0 \in I$ and a distribution ν on I, assume Z_t is $\mathcal{F}_0$-measurable for all $t \in I$ and $\mathbb{E}\big[\sup_{t \in I} Z_t^-\big] < \infty$ or $\mathbb{E}\big[\sup_{t \in I} Z_t^+\big] < \infty$. Set*

$$X := Z_0 - \sum_{t \in I \setminus \{0\}} \frac{\nu_t}{1 - \nu_0} Z_t\,.$$

Then the strategy $\gamma^* \in \mathcal{N}_I^\nu$ defined by

$$\gamma_0^* := f_{1-\nu_0, X} \quad \text{and} \quad \gamma_t^* := \frac{\nu_t}{1-\nu_0}(1 - f_{1-\nu_0,X})\,, \quad t \in I \setminus \{0\}\,,$$

with $f_{1-\nu_0,X}$ as in Definition 4.3(b), is optimal and we get

$$V'(\nu) = \sum_{t \in I \setminus \{0\}} \frac{\nu_t}{1-\nu_0}\, \mathbb{E}[Z_t] + \nu_0\, \mathrm{ES}_{1-\nu_0}(X)\,.$$

Remark 5.53. Note that by positive homogeneity and sub-additivity of expected shortfall (see e.g. Lemma 4.22 for the conditional case or [54, Lemma 3.32]), we have

$$\nu_0\, \mathrm{ES}_{1-\nu_0}(X) \le \nu_0\, \mathrm{ES}_{1-\nu_0}(Z_0) + \frac{\nu_0}{1-\nu_0} \sum_{t \in I \setminus \{0\}} \nu_t\, \mathrm{ES}_{1-\nu_0}(Z_t)\,.$$

Proof. Using Lemma 2.35 we get for $\gamma \in \mathcal{N}_I^\nu$

$$\mathbb{E}[Z_\gamma] = \sum_{t\in I} \mathbb{E}[Z_t\, \mathbb{E}[\gamma_t\,|\,\mathcal{F}_0]] = \mathbb{E}[Z_0\gamma_0] + \sum_{t\in I\setminus\{0\}} \frac{\nu_t}{1-\nu_0}\, \mathbb{E}[Z_t(1-\gamma_0)]\ .$$

Therefore

$$V'(\nu) = \sum_{t\in I\setminus\{0\}} \frac{\nu_t}{1-\nu_0}\, \mathbb{E}[Z_t] + \sup_{\gamma\in\mathcal{N}_I^\nu} \mathbb{E}\left[\gamma_0\left(Z_0 - \sum_{t\in I\setminus\{0\}} \frac{\nu_t}{1-\nu_0} Z_t\right)\right],$$

where by Lemma 4.6 we get

$$\sup_{\gamma\in\mathcal{N}_I^\nu} \mathbb{E}[\gamma_0 X] \le \nu_0\, \mathrm{ES}_{1-\nu_0}(X)\, .$$

Using the strategy γ^* as defined above and Lemma 2.35 we have

$$\begin{aligned}\mathbb{E}\left[\sum_{t\in I} Z_t\gamma_t\right] &= \mathbb{E}\left[Z_0 f_{1-\nu_0,X} + \sum_{t\in I\setminus\{0\}} Z_t \frac{\nu_t}{1-\nu_0}(1-f_{1-\nu_0,X})\right]\\ &= \sum_{t\in I\setminus\{0\}} \frac{\nu_t}{1-\nu_0}\, \mathbb{E}[Z_t] + \mathbb{E}[f_{1-\nu_0,X}X]\ ,\end{aligned}$$

which yields the upper bound found above and therefore γ^* is an optimal strategy by Lemma 4.1. We really have $\gamma^* \in \mathcal{N}_I^\nu$ as γ^* is adapted, $\sum_{t\in I}\gamma_t^* = 1$ almost surely and for all $t \in I$ we have $\gamma_t^* \ge 0$ almost surely by Remark 4.4(a) and $\mathbb{E}[\gamma_t^*] = \nu_t$ by Remark 4.4(b). Further note that for all $t \in I \setminus \{0\}$

$$\gamma_t^* = \frac{\nu_t}{1-\nu_{\le t-1}}(1-\gamma_0^*)\left(1 - \frac{\nu_1}{1-\nu_0} - \dots \frac{\nu_{t-1}}{1-\nu_0}\right) = \frac{\nu_t}{1-\nu_{\le t-1}}(1-\gamma_{\le t-1}^*)$$

and $\mathbb{E}[\gamma_t^*\,|\,\mathcal{F}_{t-1}] = \gamma_t^*$. □

5.6 The Product of a Martingale and a Deterministic Function

In the following example we will show that we have to be cautious in finding the optimal strategy when the adapted process Z is given in the form $Z_t = f(t)M_t$ for $t \in I$, where f is a deterministic function and M is a martingale. In this situation the gain of information with time also has to be considered, which is why the most intuitive strategy might not be the optimal one.

Example 5.54. Set $I = \{0, 1, 2\}$ and $\nu_t = \frac{1}{3}$ for all $t \in I$. Assume $\Omega = \{\omega_1, \dots, \omega_6\}$ with $\mathbb{P}(\omega_i) = \frac{1}{6}$ for $i = 1, \dots, 6$. Further let $\mathcal{F}_0$ contain the atoms $\{\omega_1\}, \{\omega_2\}, \{\omega_3, \omega_4\}$ and $\{\omega_5, \omega_6\}$ and set $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{P}(\Omega)$.

We now consider a process Z defined by $Z_t = f(t)M_t$ for $t \in I$, with $f(0) = f(1) = 1$ and $f(2) = 0$. The martingale M is given by

$$\begin{aligned} M_t(\omega_i) &= 1 \quad \forall\, t \in I, i \in \{1,2\}\, ,\\ M_0(\omega_i) &= 2 \quad \forall\, i \in \{3,\dots,6\}\, ,\\ M_1(\omega_i) &= 4 \quad \forall\, i \in \{3,5\}\, ,\\ M_1(\omega_i) &= 0 \quad \forall\, i \in \{4,6\}\, ,\\ M_2 &= M_1\, . \end{aligned}$$

Intuitively it would be optimal to follow a greedy strategy, because the deterministic function is non-increasing. The strategy would be given by

$$\{\tau = 0\} = \{\omega_3, \omega_4\}\,, \quad \{\tau = 1\} = \{\omega_1, \omega_5\}\,, \quad \{\tau = 2\} = \{\omega_2, \omega_6\}\,.$$

Using this strategy we get

$$\mathbb{E}[Z_\tau] = \sum_{t=0}^{2} \mathbb{E}\big[Z_t 1_{\{\tau=t\}}\big] = 2 \cdot \frac{1}{3} + 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} = \frac{9}{6} = \frac{3}{2}\,.$$

An optimal strategy in this example is given by

$$\{\tau = 0\} = \{\omega_1, \omega_2\}\,, \quad \{\tau = 1\} = \{\omega_3, \omega_5\}\,, \quad \{\tau = 2\} = \{\omega_4, \omega_6\}\,,$$

because it yields

$$\mathbb{E}[Z_\tau] = \sum_{t=0}^{2} \mathbb{E}\big[Z_t 1_{\{\tau=t\}}\big] = 1 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} = \frac{5}{3} > \frac{3}{2}\,.$$

This value exactly equals the value found by the upper bound in Lemma 4.35, which can be used, because Z is a supermartingale. Then

$$V^+(\nu) \leq \mathbb{E}[M_0] = \mathbb{E}[Z_0] = 1 \cdot \frac{1}{3} + 2 \cdot \frac{2}{3} = \frac{5}{3}\,.$$

This shows that the gain of information is important in this setting.

If we had $\mathcal{F}_0 = \mathcal{P}(\Omega)$, then the greedy strategy would be given by

$$\{\tau = 0\} = \{\omega_4, \omega_6\}\,, \quad \{\tau = 1\} = \{\omega_3, \omega_5\}\,, \quad \{\tau = 2\} = \{\omega_1, \omega_2\}\,.$$

Using this strategy we would get

$$\mathbb{E}[Z_\tau] = \sum_{t=0}^{2} \mathbb{E}\big[Z_t 1_{\{\tau=t\}}\big] = 2 \cdot \frac{1}{3} + 4 \cdot \frac{1}{3} = \frac{6}{3} = 2\,,$$

which is the highest possible value. Note that in this case the process Z would no longer be a supermartingale. Due to the structure of the filtration this strategy cannot be used.

The gain of information is not only important in the case of a non-increasing function, but also in that of a non-decreasing function. Let us consider the same setting as before, but now assume that we consider an increasing function $\tilde{f}$ with $\tilde{f}(0) = 1$, $\tilde{f}(1) = 2$ and $\tilde{f}(2) = 3$. Using a greedy strategy would result in stopping according to

$$\{\tau = 0\} = \{\omega_3, \omega_4\}\,, \quad \{\tau = 1\} = \{\omega_1, \omega_5\}\,, \quad \{\tau = 2\} = \{\omega_2, \omega_6\}\,,$$

which yields

$$\mathbb{E}[Z_\tau] = \sum_{t=0}^{2} \mathbb{E}\big[Z_t 1_{\{\tau=t\}}\big] = 2 \cdot \frac{1}{3} + 8 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} = \frac{17}{6}\,.$$

Also in this case we can get better results, if we follow a different strategy, e.g.

$$\{\tau = 0\} = \{\omega_1, \omega_2\}\,, \quad \{\tau = 1\} = \{\omega_4, \omega_6\}\,, \quad \{\tau = 2\} = \{\omega_3, \omega_5\}\,.$$

Using this strategy we get

$$\mathbb{E}[Z_\tau] = \sum_{t=0}^{2} \mathbb{E}\big[Z_t 1_{\{\tau=t\}}\big] = 1 \cdot \frac{1}{3} + 12 \cdot \frac{1}{3} = \frac{13}{3}\,.$$

The next lemma will give a characterization of an optimal strategy.

Lemma 5.55. *Given a totally-ordered countable discrete time interval I and a probability distribution ν on I. Let $\{M_t\}_{t\in I}$ be a martingale bounded from below by some constant C and $f\colon I \to \mathbb{R}$ a non-decreasing deterministic function. Assume that M and the distribution ν satisfy one of the conditions of Theorem 2.49. Define $Z_t = f(t)M_t$ for all $t \in I$ and assume $\mathbb{E}\big[\sup_{t\in I} Z_t^+\big] < \infty$ or $\mathbb{E}\big[\sup_{t\in I} Z_t^-\big] < \infty$. If $\gamma^* \in \mathcal{M}_I^\nu$ satisfies*

$$\mathbb{E}\Big[\sum_{t\in I_{>s}} M_t\gamma_t^*\Big] \geq \mathbb{E}\Big[\sum_{t\in I_{>s}} M_t\gamma_t\Big] \tag{5.56}$$

for all $s \in I$ and $\gamma \in \mathcal{M}_I^\nu$, then γ^* is optimal for $\{Z_t\}_{t\in I}$.

Proof. In the following we will assume that $\mathbb{E}[M_0 - C] > 0$, since the case $\mathbb{E}[M_0] = C$ is trivial. For every $t \in I$ we set

$$\mu_t^* := \frac{1}{\mathbb{E}[M_0 - C]}\,\mathbb{E}[(M_t - C)\gamma_t^*]$$

and for $\gamma \in \mathcal{M}_I^\nu$

$$\mu_t := \frac{1}{\mathbb{E}[M_0 - C]}\,\mathbb{E}[(M_t - C)\gamma_t]\ .$$

Due to the result of Theorem 2.49, we have that μ^* and μ are probability distributions. Due to (5.56) we have that μ^* dominates μ in stochastic order. Now we need to prove that this implies

$$\mathbb{E}[Z_{\gamma^*}] \geq \mathbb{E}[Z_\gamma]\ , \quad \forall\, \gamma \in \mathcal{M}_I^\nu\,.$$

Now, let X^* and X be discrete random variables taking values in I such that $\mathbb{P}(X^* = t) = \mu_t^*$ and $\mathbb{P}(X = t) = \mu_t$ for every $t \in I$. Then X^* dominates X in stochastic order. This implies that $\mathbb{E}[g(X^*)] \geq \mathbb{E}[g(X)]$ for every non-decreasing function g, as stated for example in [55, (1.A.7)]. Therefore using Lemma 2.35

$$\begin{aligned}\mathbb{E}[Z_{\gamma^*}] &= \sum_{t\in I} f(t)\,\mathbb{E}[(M_t - C)\gamma_t^*] + C\sum_{t\in I} f(t)\nu_t = \mathbb{E}[M_0 - C]\sum_{t\in I} f(t)\mu_t^* + C\sum_{t\in I} f(t)\nu_t\\ &= \mathbb{E}[M_0 - C]\,\mathbb{E}[f(X^*)] + C\sum_{t\in I} f(t)\nu_t\,.\end{aligned}$$

Since $\mathbb{E}[M_0 - C] > 0$, we get

$$\mathbb{E}[M_0 - C]\,\mathbb{E}[f(X^*)] + C\sum_{t\in I} f(t)\nu_t \geq \mathbb{E}[M_0 - C]\,\mathbb{E}[f(X)] + C\sum_{t\in I} f(t)\nu_t\,,$$

which implies $\mathbb{E}[Z_{\gamma^*}] \geq \mathbb{E}[Z_\gamma]$ as $\mathbb{E}[Z_\gamma]$ can be represented using $\mathbb{E}[f(X)]$ similar as $\mathbb{E}[Z_{\gamma^*}]$. □

Corollary 5.57. *Given a totally-ordered countable discrete time interval I and a probability distribution ν on I. Consider a process Z defined by $Z_t = f(t)M_t$ for all $t \in I$, where $\{M_t\}_{t\in I}$ is a martingale bounded from below by some constant C and $f\colon I \to \mathbb{R}$ a non-decreasing deterministic function. Assume that M and the distribution ν satisfy one of the conditions of Theorem 2.49. An adapted random probability measure $\gamma^* \in \mathcal{M}_I^\nu$ is optimal, if for all $s \in I$*

$$\mathbb{E}\Big[\sum_{t\in I_{>s}} M_t\gamma_t^*\Big] = (1 - \nu_{\leq s})\,\mathrm{ES}_{\nu_{\leq s}}(M_s)\,.$$

Proof. Since M and the distribution ν satisfy one of the conditions of Theorem 2.49, we know that for $s \in I$

$$\mathbb{E}\Big[\sum_{t\in I} M_t\gamma_t\Big] = \mathbb{E}[M_s]\ .$$

Therefore

$$\mathbb{E}\Big[\sum_{t\in I_{>s}} M_t\gamma_t\Big] = \mathbb{E}[M_s] - \mathbb{E}\Big[\sum_{t\in I_{\le s}} \mathbb{E}[M_s\,|\,\mathcal{F}_t]\,\gamma_t\Big] = \mathbb{E}[M_s] - \sum_{t\in I_{\le s}} \mathbb{E}[M_s\gamma_t] = \mathbb{E}[M_s(1-\gamma_{\le s})]\ .$$

This last expected value is maximal, if

$$\mathbb{E}[M_s(1-\gamma_{\le s})] = (1-\nu_{\le s})\,\mathrm{ES}_{\nu_{\le s}}(M_s)\,.$$

This follows from Lemma 4.6, as $1-\gamma_{\le s} \in \mathcal{F}^1_{\nu_{\le s},M_s}$. □

Remark 5.58. Equivalently a stopping time $\tau^* \in \mathcal{T}_I^\nu$ is optimal, if for all $t \in I$

$$\mathbb{E}\big[M_{\tau^*}1_{\{\tau^*>t\}}\big] = (1-\mathbb{P}(\tau^*\le t))\,\mathrm{ES}_{\mathbb{P}(\tau^*\le t)}(M_t)\,.$$

Remark 5.59. In the setting of Corollary 5.57 a stopping time $\tau^* \in \mathcal{T}_I^\nu$ is optimal, if for all $t \in I$

$$\mathbb{E}\big[M_{\tau^*}1_{\{\tau^*>t\}}\big] = \mathbb{E}\big[M_t1_{\{\tau^*>t\}}\big] = \|M_t\|_\infty\mathbb{P}(\tau^*>t)\,.$$

This implies that up to a null set $\{\tau^*>t\}$ is contained in $\{M_t = \|M_t\|_\infty\}$ for all $t\in I$, i.e. $\mathbb{P}(\{\tau^*>t\}\setminus\{M_t=\|M_t\|_\infty\}) = 0$ for all $t \in I$. This representation will be useful in the following sections.

5.7 Convex Functions

In the following we will take a look at a special case, where the process Z can be represented using a convex function.

Proposition 5.60. *Given $I = \{0,\dots,T\}$ assume $Z_t = \exp(M_t)$ for all $t\in I$, where M is a simple random walk, i.e. $M_0 = 0$ and $M_t = \sum_{s=1}^t X_s$, where X_t are independent with $\mathbb{P}(X_t = 1) = \mathbb{P}(X_t=-1) = \frac{1}{2}$ for all $t \in I$. Let the distribution ν be given by*

$$\nu_t = \begin{cases} 0 & \text{if } t = 0,\\ \frac{1}{2^t} & \text{if } t \in \{1,\dots,T-1\},\\ \frac{1}{2^{T-1}} & \text{if } t = T.\end{cases}$$

Then the optimal stopping time for this problem is given by

$$\tau^* = T \wedge \min\{t\in\{1,\dots,T\}\,|\,X_t=-1\}\,.$$

Using this optimal strategy we get

$$V(\nu) = \sum_{t\in I\setminus\{0,T\}} \exp(t-1)\nu_t + \frac{1}{2}(\exp(T-1)+\exp(T))\nu_T\,.$$

Proof. The result follows from Corollary 5.57. We can write $Z_t := f(t)\tilde{M}_t$ for $t \in I$, where f is a deterministic function and $\tilde{M} = \{\tilde{M}_t\}_{t\in I}$ a martingale, by setting $f(0) := 1$ and $\tilde{M}_0 := Z_0$ as well as for $t \in I \setminus \{0\}$

$$f(t) := \prod_{s=1}^{t} \mathbb{E}[\exp(X_s)]$$

and

$$\tilde{M}_t := \frac{Z_t}{\prod_{s=1}^{t} \mathbb{E}[\exp(X_s)]} .$$

We have that $\tilde{M} = \{\tilde{M}_t\}_{t\in I}$ is a non-negative martingale, because $X = \{X_t\}_{t\in I}$ is an independent process.

Due to the structure of the stopping time, for all $t \in I$ we have

$$\{\tau^* > t\} = \{\tilde{M}_t = \|\tilde{M}_t\|_\infty\} = \{Z_t = \|Z_t\|_\infty\} ,$$

which implies optimality of τ^* by Remark 5.59. □

5.8 The Binomial Model

For $I \subset \mathbb{N}_0$ with $0 \in I$ let $X = \{X_t\}_{t\in I\setminus\{0\}}$ be an independent process of identically distributed $\{0, 1\}$-valued random variables and let the filtration be given by $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_t = \sigma(X_1, \dots, X_t)$ for $t \in I \setminus \{0\}$. We set $p := \mathbb{P}(X_t = 1)$ for all $t \in I \setminus \{0\}$. In the following we will consider a process Z modeled by $Z_0 > 0$ and $Z_t = Z_0 u^{N_t} d^{t-N_t}$ for $t \in I \setminus \{0\}$ with $u > d > 0$ and $N_t = \sum_{s=1}^{t} X_t$.

In this model the increments of the process Z are given by

$$\Delta Z_t := Z_t - Z_{t-1} = Z_{t-1}(u^{X_t} d^{1-X_t} - 1) , \quad t \in I \setminus \{0\} .$$

Therefore for $t \in I \setminus \{0\}$

$$\mathbb{E}[|\Delta Z_t|] = \mathbb{E}[Z_{t-1}]\, \mathbb{E}\big[|u^{X_t} d^{1-X_t} - 1|\big] = \mathbb{E}[Z_{t-1}]\,(p|u-1| + (1-p)|d-1|)$$

and

$$\mathbb{E}\big[|\Delta Z_t| \,\big|\, \mathcal{F}_{t-1}\big] = Z_{t-1}(p|u-1| + (1-p)|d-1|) .$$

We see that the conditions of Lemma 5.10 need not be satisfied. Also the increments of the predictable process A of the Doob decomposition of Z, which are for $t \in I \setminus \{0\}$ given by

$$\Delta A_t = \mathbb{E}[Z_t|\mathcal{F}_{t-1}] - Z_{t-1} = Z_{t-1}(\mathbb{E}\big[u^{X_t} d^{1-X_t} |\mathcal{F}_{t-1}\big] - 1) = Z_{t-1}(pu + (1-p)d - 1) ,$$

do not necessarily satisfy the conditions of Theorem 5.25.

In the following proposition we will see a special case where it is possible to find an optimal stopping time.

Proposition 5.61. *In the setting for the binomial model stated above we now assume $I = \{0, \dots, T\}$. Further we have to assume that the distribution ν is given by*

$$\nu_t = \begin{cases} 0 & \textit{if } t = 0, \\ \frac{1}{2^t} & \textit{if } t \in \{1, \dots, T-1\}, \\ \frac{1}{2^{T-1}} & \textit{if } t = T. \end{cases}$$

Then the optimal stopping time is given by

$$\tau^* = T \wedge \min\{t \in I \,|\, Z_t < Z_{t-1}\}\,.$$

Proof. The result of this proposition follows from Corollary 5.57. For this we need to represent Z as $Z_t = f(t) M_t$ for $t \in I$, setting $f(0) := 1$ and $M_0 := Z_0$ as well as for each $t \in I \setminus \{0\}$

$$f(t) := \prod_{s=1}^{t} \mathbb{E}\big[u^{X_s} d^{1-X_s}\big]$$

and

$$M_t := Z_0 \prod_{s=1}^{t} \frac{u^{X_s} d^{1-X_s}}{\mathbb{E}[u^{X_s} d^{1-X_s}]}\,.$$

We have that $M = \{M_t\}_{t \in I}$ is a non-negative martingale.

Due to the structure of the stopping time, for all $t \in I$ we have

$$\{\tau^* > t\} = \{\tilde{M}_t = \|\tilde{M}_t\|_\infty\} = \{Z_t = \|Z_t\|_\infty\}\,,$$

which implies optimality of τ^* by Remark 5.59. □

Using the result of Proposition 5.61, in the following example we show the difference in the value for dependence and independence.

Example 5.62. We will now consider a binomial model with $Z_0 = 1$, $u = 2$, $d = \frac{1}{2}$ and $p = \frac{1}{2}$. Let $I = \{0, \ldots, 5\}$. Further we assume that the distribution ν is given as in Proposition 5.61.

If we assume that the process Z and the stopping time τ are independent, we get

$$V^{\mathrm{ind}}(\nu) = \sum_{t \in I} \mathbb{E}[Z_t]\, \nu_t = \frac{5}{4} \cdot \frac{1}{2} + \frac{25}{16} \cdot \frac{1}{4} + \frac{125}{64} \cdot \frac{1}{8} + \frac{625}{256} \cdot \frac{1}{16} + \frac{3125}{1024} \cdot \frac{1}{16} = \frac{26265}{16384} \approx 1.60309\,.$$

Using the optimal stopping time of Proposition 5.61, we get

$$V(\nu) = \sum_{t \in I} \mathbb{E}\big[Z_t 1_{\{\tau^* = t\}}\big] = \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \frac{1}{2}(8 + 32)\frac{1}{16} = \frac{9}{4} = 2.25\,.$$

In this case the process Z is a submartingale. Therefore we know that we can compute an upper bound by Lemma 4.37

$$V(\nu) \le \mathbb{E}[Z_5] = \frac{1}{32}\left(32 + 40 + 20 + 5 + \frac{5}{8} + \frac{1}{32}\right) = \frac{3125}{1024} \approx 3.05176\,.$$

Chapter 6

Risk Neutral Pricing and a Recursive Formula

In this chapter we briefly discuss two topics of interest to mathematical finance. As they are not a crucial part of this thesis, we content ourselves with a short note on some preliminary results.

6.1 Risk Neutral Pricing

In mathematical finance an arbitrage-free price of a derivative is found by computing the expected value of the discounted pay-off under some equivalent martingale measure. As risk neutral pricing is an important topic in mathematical finance, we want to look at it briefly. In this section only a few results for pricing under an equivalent martingale measure are shown, which might lead to further research in this area. We consider discrete intervals $I \subset \mathbb{N}_0$ with $0 \in I$ and adapted stochastic processes $Z = \{Z_t\}_{t \in I}$ with $Z \in L^1(\mathbb{P})$.

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we want to compute the values $V^+(\nu)$, $V'(\nu)$ and $V(\nu)$. The distribution of the adapted random probability measure γ or the stopping time τ under the probability measure $\mathbb{P}$ are known. We now want to compute the price P for the problem using stopping times and P' and P^+ for the problems using adapted random probability measures under an equivalent martingale measure $\mathbb{Q}$. We assume that the process Z is already the discounted price process of the underlying asset and that Z and the distribution of τ or γ under $\mathbb{Q}$ satisfy one of the conditions of Theorem 2.49. Then using that Z is a martingale under $\mathbb{Q}$, we have for $s \in I$,

$$P^+ = \sup_{\gamma \in \mathcal{M}_I^\nu} \mathbb{E}_\mathbb{Q}\left[\sum_{t \in I} Z_t \gamma_t\right] = \mathbb{E}_\mathbb{Q}[Z_s] = P\,.$$

This result is due to Theorem 2.49, because the equivalence of $\mathbb{P}$ and $\mathbb{Q}$ implies $\sum_{t \in I} \gamma_t = 1$ $\mathbb{Q}$-a.s. and Lemma 2.35 is also valid under $\mathbb{Q}$.

In the applications that require risk neutral pricing, it is especially interesting to include guarantees. This means, that the pay-out would not be modeled by Z itself, but by some other process $\tilde{Z} = \max\{Z, G\}$, where $G > 0$ is the guaranteed value. We know that Z is a martingale under $\mathbb{Q}$. Therefore it is especially a submartingale under $\mathbb{Q}$. Using [27, Theorem 9.32(iv)], which states that the minimum of two supermartingales is a supermartingale, we have that $\tilde{Z}$ is a submartingale under $\mathbb{Q}$. We can therefore use our results about submartingales to have bounds for the risk neutral prices of products including guarantees.

In [31, Section 2.5.2] and [14, Section 6.2] American and European options are discussed and compared. In that section the authors showed that, under certain circumstances, it may happen that the value of an American call option coincides with the value of a European call option with the same underlying process modeling the pay-out. We will now take a closer look at these results.

In [31, Proposition 2.5.1] or [14, Example 6.24] it is shown that the value of the American call option and the value of the European call option coincide if the price process $\{c_t\}_{t\in I}$ of the European call option satisfies $c_t \geq Z_t$ for any $t \in I$, where $\{Z_t\}_{t\in I}$ models the pay-out of the American call option for each $t \in I$. In this situation the risk-neutral price for such a European call option is also an upper bound for the risk-neutral value P for our problem using stopping times.

Assume $I = \{0, \dots, T\}$ and let Z be a process modeling the discounted pay-off of an option. Then the risk-neutral value of the corresponding European option would be given by $\mathbb{E}_{\mathbb{Q}}[Z_T]$. This implies that the risk-neutral value of the corresponding European option is an upper bound for the risk-neutral value of our claim if the process Z is a submartingale under $\mathbb{Q}$. This was shown in Proposition 4.37. By Proposition 4.35 the corresponding European option is a lower bound, if Z is a supermartingale. In this case the risk-neutral value of our claim lies between the value of the corresponding European and American options. Note that we showed in Lemma 2.51 that the value of an optimal stopping problem, or for non-negative processes the value of an American option, is also an upper bound for $V'(\nu)$ and $V^+(\nu)$.

Remark 6.1. If $I \subset \mathbb{N}_0$ with $0 \in I$ and we neglect discounting, we have that the pay-out process of a call or put option is a supermartingale, by Jensen's inequality for conditional expectation ([62, 9.7(h)]). Let $S^0 = \{S^0_t\}_{t\in I}$ denote the process of the riskless asset used for discounting. Then the discounted value of a call option is a supermartingale if $\mathbb{E}_{\mathbb{Q}}\left[\frac{S^0_s}{S^0_t}|\mathcal{F}_s\right] \leq 1$ for all $s < t$ in I. For a put option, we need to assume $\mathbb{E}_{\mathbb{Q}}\left[\frac{S^0_s}{S^0_t}|\mathcal{F}_s\right] \geq 1$ for all $s < t$ in I.

6.2 A Recursive Formula

When contemplating optimal stopping problems or standard American options, one thing that crosses one's mind is the Snell envelope (see [14] or [31]). This envelope can be used to compute the value V mentioned before and it can even be used to solve the optimal stopping problem. In this chapter we will assume $I = \{0, \dots, T\}$ and that we are given an adapted process $Z = \{Z_t\}_{t\in I}$ with $Z \in L^1(\mathbb{P})$. We show how to compute the value $V(\nu)$ for an independent process Z using a recursive formula. As in Definition 2.4 we will let $\mathcal{T}_I^\nu$ denote the set of I-valued stopping times with distribution ν.

Lemma 6.2. *If $I = \{0, \dots, T\}$ and the random variables Z_t for $t \in I$ are independent, the value $V(\nu)$ can be computed by the recursion*

$$V_T = Z_T \,,$$

$$V_{T-1}(\mu_{T-1}) = \sup_{\tau_{T-1}\in\mathcal{T}^{\mu_{T-1}}_{\{T-1,T\}}} \mathbb{E}_{\mathbb{Q}_{T-1}}\left[Z_{T-1}1_{\{\tau_{T-1}=T-1\}} + V_T 1_{\{\tau_{T-1}=T\}}\right] \,,$$

and for $t \in \{0, \dots, T-2\}$

$$V_t(\mu_t) = \mathbb{Q}_t(\tau_t = t) \sup_{\tau_t\in\mathcal{T}^{\mu_t}_{\{t,\dots,T\}}} \mathbb{E}_{\mathbb{Q}_t}\left[Z_t|\tau_t = t\right] + \mathbb{Q}_t(\tau_t \geq t+1)V_{t+1}(\mu_{t+1}) \,,$$

where for $t \in \{0, \ldots, T-1\}$ we have that $\tau_t := \max\{\tau, t\}\colon \Omega \to \{t, \ldots, T\}$ is a stopping time with distribution μ_t under a new probability measure $\mathbb{Q}_t$ with $\mu_t(s) = \mathbb{Q}_t(\tau_t = s) = \frac{\mathbb{P}(\tau=s)}{\mathbb{P}(\tau \geq t)}$ for $s \in \{t, \ldots, T\}$.

Proof. We have

$$\begin{aligned} V(\nu) &= \sup_{\tau \in \mathcal{T}_I^\nu} \mathbb{E}[Z_\tau] \\ &= \sup_{\tau \in \mathcal{T}_I^\nu} \mathbb{E}\big[Z_0 1_{\{\tau=0\}} + Z_\tau 1_{\{\tau \geq 1\}}\big] \\ &\leq \mathbb{P}(\tau = 0) \sup_{\tau \in \mathcal{T}_I^\nu} \mathbb{E}[Z_0 | \tau = 0] + \mathbb{P}(\tau \geq 1) \underbrace{\sup_{\tau \in \mathcal{T}_I^\nu} \mathbb{E}[Z_\tau | \tau \geq 1]}_{=V_1(\mu_1)}, \end{aligned}$$

where $V_1(\mu_1)$ denotes the value of our claim at time 1. In the following each conditional expectation will be maximized for each time step. As we already saw in Lemma 5.37 this gives a stopping time that gives an upper bound and which is optimal as it follows the given distribution.

The value $V_1(\mu_1)$ can also be computed, by taking a look at a stopping time $\tau_1 := \max\{\tau, 1\}\colon \Omega \to \{1, 2, \ldots, T\}$, which has distribution μ_1 under a new probability measure $\mathbb{Q}_1$, which is given by $\mu_1(j) = \mathbb{Q}_1(\tau_1 = j) = \frac{\mathbb{P}(\tau=j)}{\mathbb{P}(\tau \geq 1)}$ for $j = 1, \ldots, T$.

$$\begin{aligned} V_1(\mu_1) &= \sup_{\tau \in \mathcal{T}_I^\nu} \mathbb{E}[Z_\tau | \tau \geq 1] = \sup_{\tau_1 \in \mathcal{T}^{\mu_1}_{\{1,\ldots,T\}}} \mathbb{E}_{\mathbb{Q}_1}[Z_{\tau_1}] \\ &= \sup_{\tau_1 \in \mathcal{T}^{\mu_1}_{\{1,\ldots,T\}}} \mathbb{E}_{\mathbb{Q}_1}\big[Z_1 1_{\{\tau_1=1\}} + Z_{\tau_1} 1_{\{\tau_1 \geq 2\}}\big] \\ &\leq \mathbb{P}(\tau_1 = 1) \sup_{\tau_1 \in \mathcal{T}^{\mu_1}_{\{1,\ldots,T\}}} \mathbb{E}[Z_1 | \tau_1 = 1] + \mathbb{P}(\tau_1 \geq 2) \underbrace{\sup_{\tau_1 \in \mathcal{T}^{\mu_1}_{\{1,\ldots,T\}}} \mathbb{E}[Z_{\tau_1} | \tau_1 \geq 2]}_{=V_2(\mu_2)}, \end{aligned}$$

where again $V_2(\mu_2)$ is the value of the claim at time 2. Now we can again find a new stopping time $\tau_2 := \max\{\tau, 2\}\colon \Omega \to \{2, \ldots, T\}$, which has distribution μ_2 under a new probability measure $\mathbb{Q}_2$, which is given by $\mu_2(j) = \mathbb{Q}_2(\tau_2 = j) = \frac{\mathbb{P}(\tau=j)}{\mathbb{P}(\tau \geq 2)}$ for $j = 2, \ldots, T$. We can continue with this procedure until time $T - 1$. It is clear that at time T the value of the option is equal to Z_T, i.e. $V_T = Z_T$. $\square$

Remark 6.3. The good thing about this recursive formula is that it gives us the value of the considered problem for every $t \in \{0, \ldots, T\}$.

Part II

Adapted Dependence in Continuous Time

Chapter 7

The Problem

In this chapter we introduce the problem in continuous time. Using stopping times does not significantly change the formulation of the problem. To formulate the problem of several withdrawals within the predefined time interval we must exchange the adapted random probability measures by stochastic transition kernels. Similar to discrete time, the notation and the necessary assumptions are first introduced in Section 7.1. Afterwards some general results are presented in Section 7.2. Again these results are useful later to find bounds and optimal strategies.

7.1 Notation

Let $I \subset [0, \infty)$ be a continuous time interval with $0 \in I$. We consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in I}, \mathbb{P})$. Again, we will need to make some assumptions about the adapted process $Z = \{Z_t\}_{t\in I}$ with $Z \in L^1(\mathbb{P})$, i.e. $\mathbb{E}[|Z_t|] < \infty$ for all $t \in I$, which will be needed from time to time. In continuous time we will further often need to assume that the process Z is càdlàg or that it is at least right-continuous.

Assumption 7.1. *In order to be able to define all the given problems, we can use some of the following assumptions for an adapted process $Z = \{Z_t\}_{t\in I}$ with $Z \in L^1(\mathbb{P})$:*

(a) $\mathbb{P}(\sup_{t\in I} |Z_t| < \infty) = 1$,

(b) $\mathbb{E}[\sup_{t\in I} |Z_t|] < \infty$,

(c) the process is uniformly integrable,

(d) $\sup_{\tau\in\mathcal{T}_I} \mathbb{E}[|Z_\tau|] < \infty$*, i.e.,* $\{Z_\tau\}_{\tau\in\mathcal{T}_I}$ *bounded in* $L^1(\mathbb{P})$*,*

(e) Z is of class D, i.e. $\{Z_\tau\}_{\tau\in\mathcal{T}_I}$ is uniformly integrable.

Remarks 7.2. (i) Again it is clear that (b) implies (a).

(ii) Also (b) implies (e). According to [46, Chapter I, Theorem 11] the process Z is uniformly integrable if $\sup_{t\in I} \mathbb{E}[|Z_t|] < \infty$ and if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $t \in I$ and $A \in \mathcal{F}$, $\mathbb{P}(A) \leq \delta$ implies $\mathbb{E}[|Z_t 1_A|] < \varepsilon$. Since $\{Z_\tau\}_{\tau\in\mathcal{T}_I}$ is bounded from above by $\sup_{t\in I} |Z_t|$, which is in $L^1(\mathbb{P})$, these two conditions are satisfied by (b) and [62, Lemma 13.1(a)].

(iii) We have that (e) implies (c). This is due to the fact that $\{Z_t\}_{t\in I} \subset \{Z_\tau\}_{\tau\in\mathcal{T}_I}$.

(iv) Further (e) implies (d) by definition of uniform integrability.

(v) Next we want to show that if Z is a martingale, (c) implies (a). By [46, Chapter I, Theorem 12] the limit Z_∞ exists and is integrable. Therefore $\mathbb{P}(\sup_{t\in I}|Z_t| < \infty) = 1$.

(vi) Also (c) implies (e), if Z is a martingale. This result is shown in [49, Chapter VI, Lemma 29.6].

(vii) (a) does not imply (b), (c), (d) or (e). The first three can be shown similar to discrete time in Remark 2.3(iv). As the process is not uniformly integrable, it cannot be of class D.

(viii) We have that (c) does not imply (a). This can be seen using a process as in discrete time in Remark 2.3(v) and setting $Z_t = \tilde{Z}_{\lfloor t \rfloor}$ and $\mathcal{F}_t = \tilde{\mathcal{F}}_{\lfloor t \rfloor}$ for all $t \in I$.

(ix) We have that (c) does not imply (d) and therefore nor does it imply (b). Using the discretization presented above we will consider a discrete time process on a probability space $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}_{[0,1]}, \lambda)$, where λ is the Lebesgue–Borel measure of the Borel σ-algebra $\mathcal{B}_{[0,1]}$ on the unit interval. For $n \in \mathbb{N}_0$ we can find unique $k \in \mathbb{N}_0$ and $j \in \{0, \dots, 2^k - 1\}$ such that $n = 2^k + j$. Similar to Remark 2.3(v) consider the process Z defined by

$$Z_n(\omega) = k 1_{[\frac{j}{2^k}, \frac{j+1}{2^k}]}(\omega), \quad \omega \in [0,1] .$$

We already saw that it is uniformly integrable and that it does not satisfy (b). Using the stopping times defined by $\tau_k := \inf\{n \geq 2^k \,|\, Z_n \geq k\}$, we get $\mathbb{E}[Z_{\tau_k}] = k$, as $Z_{\tau_k} = k$, and therefore assumption (d) is not satisfied.

(x) Also (d) and (c) do not imply (e). Consider a process Z defined by $Z_t := \frac{1}{\|W_t\|}$ for $t \in I$, where $\{W_t\}_{t\in I}$ is a three-dimensional Brownian motion starting in $x_0 = (1,0,0)$ and $\|\cdot\|$ being the distance to $(0,0,0)$. In [22] it is shown that this process is uniformly integrable and a supermartingale, which, by Doob's optional stopping theorem implies (d), but it is not of class D.

(xi) In [17, Chapter 12.3] two martingales (thus satisfying (d) by Doob's optional stopping theorem) are shown, which are not uniformly integrable. Therefore (d) does not imply (c).

(xii) (d) does not imply (b). This can be shown by considering the process $Z_t = \exp(W_t - \frac{1}{2}t)$ for $t \in I$, where $\{W_t\}_{t\in I}$ is a standard Brownian motion. Clearly this process is a martingale and therefore it satisfies (d), which can by shown using Doob's optional stopping theorem for bounded stopping times and Fatou's lemma. We now want to show that $Z_t \to Z_\infty = 0$ as $t \to \infty$ pointwise on Ω. We will therefore concentrate on the behavior of $\frac{W_t}{t}$ as $t \to \infty$. We can write $Z_t = \exp(t(\frac{W_t}{t} - \frac{1}{2}))$. Using a continuous time version of the ergodic theorem (see e.g. [25, Theorem 9.8]) we have that $\frac{W_t}{t} \to 0$ as $t \to \infty$. Therefore the martingale Z is not closable and it cannot therefore satisfy (b).

Remark 7.3 (Measurability of the Supremum)*.* In Assumption 7.1(b) we consider the expectation of $\sup_{t\in I}|Z_t|$. We therefore need to make sure that $\sup_{t\in I}|Z_t|$ is measurable. An easy way to guarantee this is to assume that the process Z is separable.

On the other hand, we are considering càdlàg processes. For these processes we have $\sup_{t\in I_k}|Z_t| \nearrow \sup_{t\in I}|Z_t|$ for $k\to\infty$ for a sequence $(I_k)_{k\in\mathbb{N}}$ of finite subsets of I with $I_k \subset I_{k+1}$, such that the end point of each I_k converges to $\sup(I)$ for $k\to\infty$ and $\bigcup_{k\in\mathbb{N}} I_k$ dense in I. By monotone convergence we also have $\mathbb{E}\big[\sup_{t\in I_k}|Z_t|\big] \nearrow \mathbb{E}[\sup_{t\in I}|Z_t|]$. Therefore we know that $\sup_{t\in I}|Z_t|$ is measurable.

It would also be possible to replace Assumption 7.1(b), by considering processes Z for which there exists a random variable $Z^* \in L^1(\Omega,\mathcal{F},\mathbb{P})$ and a null set $N\in\mathcal{F}$ with

$$\bigcup_{t\in I}\{|Z_t|\le Z^*\} \subset \Omega\setminus N\,.$$

Then the problem of measurability of $\sup_{t\in I}|Z_t|$ does not occur and the statements of Remarks 7.2 remain valid.

Again we let $\mathcal{T}_I$ denote the set of all stopping times $\tau\colon\Omega\to I$. For a given probability distribution ν on I, let $\mathcal{T}_I^\nu$ be the set of all I-valued stopping times with distribution ν, i.e. $\mathcal{L}(\tau)=\nu$.

For an adapted process Z with $\mathbb{E}[Z_\tau^-]<\infty$ or $\mathbb{E}[Z_\tau^+]<\infty$ for all $\tau\in\mathcal{T}_I^\nu$ we are interested in the value

$$V(\nu) := \sup_{\tau\in\mathcal{T}_I^\nu}\mathbb{E}[Z_\tau]\;.$$

If $\mathcal{T}_I^\nu=\varnothing$, we set $V(\nu)=-\infty$.

Remark 7.4. Since the problems are similar, we will use the same notation as in discrete time.

Remark 7.5. Similar to discrete time the value of an optimal stopping problem with the same underlying process Z, which we will again denote by V, is an upper bound for $V(\nu)$ and also for $V^+(\nu)$, which is proven in Lemma 8.17. American options in a continuous-time framework are discussed, for example, in [40, Chapter 5] or in [15, Chapter 8]. In [57] optimal stopping problems are also studied for continuous-time Markov processes.

Definition 7.6. For a fixed probability measure ν on $\mathcal{B}_I$ we say that a stochastic transition kernel $\Gamma\colon\Omega\times\mathcal{B}_I\to[0,1]$ is in $\mathcal{M}_I^\nu$ if for all $t\in I$

(a) $\Omega\ni\omega\mapsto\Gamma(\omega,[0,t])$ is $\mathcal{F}_t$-measurable,

(b) $\mathbb{E}[\Gamma(\cdot,[0,t])]=\nu([0,t])$.

For a $(\mathcal{F}\otimes\mathcal{B}_I)$-measurable process $Z\colon\Omega\times I\to\mathbb{R}$ and $\Gamma\in\mathcal{M}_I^\nu$ with

$$\mathbb{P}\bigg(\bigg\{\int_I Z_t^-\,\Gamma(dt)<\infty\bigg\}\cup\bigg\{\int_I Z_t^+\,\Gamma(dt)<\infty\bigg\}\bigg)=1$$

we define

$$Z_\Gamma := \int_I Z_t\,\Gamma(dt).$$

For an adapted process Z with $\mathbb{E}\big[\int_I Z_t^-\,\Gamma(dt)\big]<\infty$ or $\mathbb{E}\big[\int_I Z_t^+\,\Gamma(dt)\big]<\infty$ for all $\Gamma\in\mathcal{M}_I^\nu$, we are now interested in the value

$$V^+(\nu) := \sup_{\Gamma\in\mathcal{M}_I^\nu}\mathbb{E}[Z_\Gamma]\;.$$

Remark 7.7. For the definition of Z_Γ it is sufficient to assume that Z is a $(\mathcal{F}\otimes\mathcal{B}_I)$-measurable process. For finding a reasonable optimal stochastic transition kernel Γ^* satisfying

$$\mathbb{E}[Z_{\Gamma^*}] = \sup_{\Gamma\in\mathcal{M}_I^\nu} \mathbb{E}[Z_\Gamma]$$

it is necessary to assume that the process Z is adapted. Then there exists a modification of Z, which is progressively measurable (see [39, Theorem T46]).

7.2 General Results

Lemma 7.8. *Let $Z\colon \Omega\times I \to \mathbb{R}$ be an $(\mathcal{F}\otimes\mathcal{B}_I)$-measurable process. Then, for every adapted stochastic transition kernel $\Gamma\in\mathcal{M}_I^\nu$ which is independent of Z and satisfies $\mathbb{E}\big[\int_I Z_t^-\,\Gamma(dt)\big]<\infty$ or $\mathbb{E}\big[\int_I Z_t^+\,\Gamma(dt)\big]<\infty$,*

$$\mathbb{E}\left[\int_I Z_t\,\Gamma(dt)\right] = \int_I \mathbb{E}[Z_t]\;\nu(dt)\,. \tag{7.9}$$

Proof. We will use the monotone-class theorem to prove that the set of all processes satisfying (7.9) contains all bounded $(\mathcal{F}\otimes\mathcal{B}_I)$-measurable processes. We set

$$\begin{aligned}\mathscr{H} := \Bigg\{ & Z=\{Z_t\}_{t\in I}\,|\, Z \text{ bounded}, (\mathcal{F}\otimes\mathcal{B}_I)\text{-measurable},\\ & \mathbb{E}\left[\int_I Z_t\,\Gamma(dt)\right] = \int_I \mathbb{E}[Z_t]\;\nu(dt) \quad \forall\,\Gamma\in\mathcal{M}_I^\nu(Z) \text{ independent of } Z\\ & \text{with } \mathbb{E}\left[\int_I Z_t^-\,\Gamma(dt)\right]<\infty\Bigg\}\,.\end{aligned}$$

Due to the linearity of the expectation we have that the set $\mathscr{H}$ defines a vector space. The constant element 1 is in $\mathscr{H}$, because

$$1 = 1\cdot\mathbb{E}\left[\int_I \Gamma(dt)\right] = \int_I 1\,\nu(dt) = 1\,.$$

If we take a look at a non-negative sequence $Z^n \nearrow Z$ for $n\to\infty$, where Z is bounded, then we can show that $Z\in\mathscr{H}$. To prove this we can use monotone convergence which gives for $n\to\infty$

$$\mathbb{E}\left[\int_I Z_t^n\,\Gamma(dt)\right] \nearrow \mathbb{E}\left[\int_I Z_t\,\Gamma(dt)\right]$$

and

$$\int_I \mathbb{E}[Z_t^n]\;\nu(dt) \nearrow \int_I \mathbb{E}[Z_t]\;\nu(dt)\,.$$

Last but not least we need to prove that $\mathscr{H}$ contains the indicator function of every set in the π-system $\{F\cap J\,|\,F\in\mathcal{F}, J\in\mathcal{B}_I\}$. Due to the independence of the stochastic transition kernel Γ and the process Z we have

$$\mathbb{E}\left[\int_I 1_F 1_J\,\Gamma(dt)\right] = \mathbb{E}\left[1_F\int_I 1_J\,\Gamma(dt)\right] = \mathbb{E}[1_F\Gamma(\cdot,J)] = \mathbb{P}(F)\nu(J)$$

and

$$\int_I \mathbb{E}[1_F 1_J]\;\nu(dt) = \int_I 1_J\,\mathbb{E}[1_F]\;\nu(dt) = \mathbb{P}(F)\int_I 1_J\nu(dt) = \mathbb{P}(F)\nu(J)\,.$$

By the monotone-class theorem we know that $\mathscr{H}$ contains all bounded, $\mathcal{F}\otimes\mathcal{B}_I$-measurable processes.

The result so far is only valid for bounded processes Z. Assume we are given a non-negative process Z. Then there exists a sequence of bounded processes Z^n with $Z^n \nearrow Z$ for $n \to \infty$, such that the result follows by monotone convergence. Since we can decompose every process into its positive and negative part, which are both non-negative, we get the result for general processes Z.

Similarly the result can be found if $\mathbb{E}\big[\int_I Z_t^+\,\Gamma(dt)\big] < \infty$. ☐

Remark 7.10. The lemma applies to all deterministic processes Z with $\int_I Z_t^-\,\nu(dt) < \infty$ or $\int_I Z_t^+\,\nu(dt) < \infty$.

Remark 7.11. Similar to discrete time discussed in Part I we have that an assumed independence between Z and $\tau \in \mathcal{T}_I^\nu \neq \varnothing$ or $\Gamma \in \mathcal{M}_I^\nu$ gives a lower bound to our problem, i.e.

$$\int_I \mathbb{E}[Z_t]\,\nu(dt) \le V(\nu) \le V^+(\nu)\,.$$

Theorem 7.12. *Let Γ be an adapted stochastic transition kernel with respect to the filtration $\{\mathcal{F}_t\}_{t\in I}$. By extending the probability space if necessary, we may assume w.l.o.g. that there exists a random variable U, uniformly distributed on $[0,1]$ and independent of $\mathcal{F}_\infty := \sigma(\bigcup_{t\in I}\mathcal{F}_t)$. Then*

$$\tau(\omega) := \inf\{t \in I \,|\, U(\omega) \le \Gamma(\omega,[0,t])\}, \qquad \omega\in\Omega,$$

satisfies $\{\tau \le t\} = \{U \le \Gamma(\cdot,[0,t])\}$ for every $t\in I$, hence τ is a stopping time with respect to the filtration $\tilde{\mathcal{F}} = \{\tilde{\mathcal{F}}_t\}_{t\in I}$ defined by $\tilde{\mathcal{F}}_t := \mathcal{F}_t \vee \sigma(U)$ for $t\in I$ and satisfies $\mathbb{P}(\tau\le t|\mathcal{F}_t) \overset{a.s.}{=} \Gamma(\cdot,[0,t])$ for all $t\in I$. Let $Z\colon \Omega\times I\to\mathbb{R}$ be an $(\mathcal{F}_\infty\otimes\mathcal{B}_I)$-measurable process such that $\mathbb{E}[Z_\tau^-]<\infty$. Then

$$\mathbb{E}[Z_\tau\,|\,\mathcal{F}_\infty] \overset{a.s.}{=} Z_\Gamma \qquad \textit{and} \qquad \mathbb{E}[Z_\tau] = \mathbb{E}[Z_\Gamma]\,.$$

Remark 7.13. If $\Gamma\in\mathcal{M}_I^\nu$, then $\tau\in\mathcal{T}_I^\nu$ for the stopping time found in Theorem 7.12.

Proof. For a fixed $\omega\in\Omega$ the mapping $\mathcal{F}\ni A\mapsto \Gamma(\omega,A)$ is a probability measure. Therefore Γ is right-continuous in the second component, because for every sequence $t_n \searrow t$ for $n\to\infty$ we have $\Gamma(\omega,[0,t_n]) \searrow \Gamma(\omega,[0,t])$ for every $\omega\in\Omega$, $n\to\infty$. This is due to the fact that $\bigcap_{n\in\mathbb{N}}(t,t_n] = \varnothing$ and therefore for every $\omega\in\Omega$, as $\Gamma(\omega,(t,t_n])\to 0$ for $n\to\infty$,

$$\Gamma(\omega,[0,t_n]) = \Gamma(\omega,[0,t]) + \Gamma(\omega,(t,t_n]) \to \Gamma(\omega,[0,t]) \text{ as } n\to\infty\,.$$

Due to this right-continuity in the second component, we have the representation

$$\{\tau\le t\} = \{U \le \Gamma(\cdot,[0,t])\}\,,$$

where this set is $\tilde{\mathcal{F}}_t$-measurable for every $t\in I$. Using [40, Lemma A.0.1(v)] with $h(x,y) = 1_{\{x\le y\}}$ and $H(y)=y$, we get

$$\mathbb{P}(\tau\le t|\mathcal{F}_t) \overset{\text{a.s.}}{=} \Gamma(\cdot,[0,t])\,, \quad \forall\, t\in I\,.$$

Finally, we must prove $\mathbb{E}[Z_\tau\,|\,\mathcal{F}_\infty] \overset{\text{a.s.}}{=} \int_I Z_t\,\Gamma(dt)$. In a first step we will use the monotone-class theorem (see [62, Theorem 3.14]) to prove the result for bounded processes. We set

$$\begin{aligned}\mathscr{H} := \{&Z = \{Z_t\}_{t\in I}\,|\, Z \text{ bounded}, (\mathcal{F}_\infty\otimes\mathcal{B}_I)\text{-measurable}, \mathbb{E}\big[Z_\tau^-\big]<\infty,\\ &\mathbb{E}[Z_\tau\,|\,\mathcal{F}_\infty] \overset{\text{a.s.}}{=} Z_\Gamma\}\,.\end{aligned}$$

Due to the linearity of the expectation and the integral we have that the set $\mathscr{H}$ defines a vector space. Obviously the constant element 1 is in $\mathscr{H}$.

If we consider a non-negative sequence $Z^n \nearrow Z$ for $n \to \infty$, where Z is bounded, then we can show that $Z \in \mathscr{H}$. If $Z^n \nearrow Z$ for $n \to \infty$ then also $Z^n_\tau \nearrow Z_\tau$ and $Z^n_\Gamma \nearrow Z_\Gamma$ for $n \to \infty$. By monotone convergence we have for $n \to \infty$

$$\mathbb{E}[Z^n_\tau \,|\, \mathcal{F}_\infty] \nearrow \mathbb{E}[Z_\tau \,|\, \mathcal{F}_\infty] \; .$$

Last but not least we need to prove that $\mathscr{H}$ contains the indicator function of every set in the π-system $\{A \cap B \,|\, A \in \mathcal{F}_\infty, B \in \mathcal{B}_I\}$. W.l.o.g. we will set $B = (b, c] \subset I$ and $Z(t, \omega) = 1_A(\omega) 1_B(t)$. We have

$$\begin{aligned}
\mathbb{E}[Z_\tau \,|\, \mathcal{F}_\infty] &= \mathbb{E}[1_A(\omega)1_B(\tau) \,|\, \mathcal{F}_\infty] = 1_A(\omega)\, \mathbb{E}[1_B(\tau) \,|\, \mathcal{F}_\infty] = 1_A(\omega)\, \mathbb{P}(b < \tau \le c \,|\, \mathcal{F}_\infty) \\
&= 1_A(\omega)\, \mathbb{P}(\Gamma(\omega, [0, b]) < U \le \Gamma(\omega, [0, c]) \,|\, \mathcal{F}_\infty) = 1_A(\omega)\, \Gamma(\omega, (b, c]) \\
&= \int_I 1_A(\omega) 1_B(t)\, \Gamma(\omega, dt) \, .
\end{aligned}$$

By the monotone-class theorem we know that $\mathscr{H}$ contains all bounded, $\mathcal{F}_\infty \otimes \mathcal{B}_I$-measurable processes.

The result so far is only valid for bounded processes Z. Assume we are given a non-negative process Z. Then there exists a sequence of bounded processes Z^n with $Z^n \nearrow Z$ for $n \to \infty$, such that the result follows by monotone convergence. Since we can decompose every process into its positive and negative part, which are both non-negative, we get the result for general processes Z. □

Remark 7.14. For the existence of the random variable U in the proof of Theorem 7.12 it would be sufficient to enlarge the probability space to $\bar{\Omega} = \Omega \times [0, 1]$, $\bar{\mathcal{F}} = \mathcal{F} \otimes \mathcal{B}_{[0,1]}$, $\bar{\mathbb{P}} = \mathbb{P} \otimes \lambda$, where λ denotes the Lebesgue measure, and a filtration defined by $\bar{\mathcal{F}}_t := \mathcal{F}_t \otimes \{\varnothing, [0, 1]\}$ for $t \in I$. In the proof we need to enlarge the filtration we use to $\tilde{\mathcal{F}}_t = \mathcal{F}_t \vee \sigma(U)$ for $t \in I$ to ensure that τ really is a stopping time.

Lemma 7.15. *If the process Z is a closable right-continuous martingale, we have $V^+(\nu) = \mathbb{E}[Z_0]$.*

Proof. To prove the result use Theorem 7.12 and Doob's optional stopping theorem (see e.g. [46, Chapter I.2, Theorem 16] or [7, Theorem 3.11] for different versions in continuous time). □

Chapter 8

Results for Special Cases and Bounds

This chapter is devoted to the computation of optimal strategies or bounds for the problem in continuous time. If one is interested in finding optimal stopping times, one can try to use the Dynkin formula for computing the extremal value for some special types of processes. In Section 8.1 this is done by considering Itō diffusions and applying some functions on them. Due to the special form of the generators of these processes, results can be found using the Dynkin formula. In Section 8.2 a general result concerned with the use of utility functions is briefly discussed and an example of Section 8.1 is revisited. In Section 8.3 we introduce a discrete approximation for right-continuous processes. This is a useful tool for transferring results found in discrete time to continuous time. Using this discrete approximation we can find results for special classes of processes in Section 8.4 and some bounds in Section 8.5. Section 8.5 also shows some bounds that can be found right away, but that do not take care of the distribution ν of the stopping time τ or the stochastic transition kernel Γ. In this chapter we consider a time interval $I \subset [0, \infty)$ with $0 \in I$ and an adapted stochastic process $Z = \{Z_t\}_{t \in I}$ with $Z \in L^1(\mathbb{P})$. Once more, we then let $\mathcal{T}_I$ denote the set of all I-valued stopping times.

8.1 First Results Using the Dynkin Formula

In this section we will use the Dynkin formula, which is a result following from the Itō formula. We will therefore present two different versions of the Itō formula, which can be found in standard text books.

We will start with the version found in [43], for which we assume that the process Z is an Itō process. This means that X is the solution of a SDE given by

$$dX_t = \mu \, dt + \sigma \, dW_t \,, \quad X_0 = x \,, \quad t \in I \,,$$

where $\{W_t\}_{t \in I}$ is a Brownian motion with respect to our filtration, $x > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$. Assume further that we are given a function $f(t, Z)$ that is twice continuously differentiable in the second component and continuously differentiable in the first component. Itō's formula states

$$f(t, Z_t) = f(0, Z_0) + \int_0^t \frac{\partial}{\partial s} f(s, Z_s) \, ds + \int_0^t \frac{\partial}{\partial Z} f(s, Z_s) \, dZ_s + \frac{1}{2} \int_0^t \frac{\partial^2}{\partial Z^2} f(s, Z_s) (dZ_s)^2 \,.$$

If the time point t is now replaced by a stopping time τ with $\mathbb{E}[\tau] < \infty$ and we take a look at the expectations we get the Dynkin formula, which states

$$\mathbb{E}[f(\tau, Z_\tau)] = f(0, Z_0) + \mathbb{E}\left[\int_0^\tau \Big(\frac{\partial}{\partial s} f(s, Z_s) + \mu \frac{\partial}{\partial Z} f(s, Z_s) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial Z^2} f(s, Z_s)\Big)\, ds\right] ,$$

because

$$\mathbb{E}\left[\int_0^t \sigma \frac{\partial}{\partial Z} f(s, Z_s)\, dW_s\right] = 0 .$$

The expectation exists if the support of f is compact, because the integrand will then be bounded on the support and the stopping time has finite expectation.

In [46] we can find a more general version of the Itō formula if the process Z is a semimartingale. Consider a C^2 function f. Then, with the understanding that $Z_s^-(\omega) = \lim_{t \to s, t<s} Z_t(\omega)$,

$$\begin{aligned} f(Z_t) = f(Z_0) &+ \int_{0+}^t f'(Z_{s-})\, dZ_s + \frac{1}{2} \int_{0+}^t f''(Z_{s-}) d[Z, Z]_s^c \\ &+ \sum_{0<s\le t} (f(Z_s) - f(Z_{s-}) - f'(Z_{s-}) \Delta Z_s) , \end{aligned}$$

where $[Z, Z]^c$ denotes the path-by-path continuous part of $[Z, Z]$, which is the quadratic variation process of Z and the last sum, which is a convergent series, denotes the jump part of the stochastic integral $\int_{0+}^t f''(Z_{s-}) d[Z, Z]_s$.

8.1.1 Itō Diffusions

Assume that the process Z is an Itô diffusion solving

$$dZ_t = Z_t(\mu\, dt + \sigma\, dW_t) , \quad Z_0 = x , \quad t \in I , \tag{8.1}$$

where $\{W_t\}_{t\in I}$ is a Brownian motion with respect to our filtration, $x > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$. We consider stopping times τ with distribution ν for which we have $\mathbb{E}_x[\tau] < \infty$. Let $f \in C_0^2(\mathbb{R})$, which means that f is twice continuously differentiable and has compact support, i.e. the support $\operatorname{supp}(f) = \overline{\{x \in \mathbb{R} \,|\, f(x) \neq 0\}}$ is compact. Under these assumptions we can try to use the Dynkin formula to solve our problem. The generator of the process Z defined in (8.1) and the function f is given by

$$Af(x) = \mu x \frac{\partial}{\partial x} f(x) + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2}{\partial x^2} f(x) .$$

The Dynkin formula states

$$\mathbb{E}_x[f(Z_\tau)] = f(x) + \mathbb{E}_x\left[\int_0^\tau Af(Z_s) ds\right] .$$

Lemma 8.2. *For a process Z satisfying (8.1) we have for every $\tau \in \mathcal{T}_I$*

$$\mathbb{E}_x[\ln(Z_\tau)] = \ln x + \left(\mu - \frac{\sigma^2}{2}\right) \mathbb{E}[\tau] .$$

Proof. We will use the Dynkin formula for this proof. The generator of the process Z for the given function $f(x) = \ln x$ is given by

$$Af(x) = \mu x \frac{1}{x} + \frac{1}{2}\sigma^2 x^2 \frac{(-1)}{x^2} = \mu - \frac{\sigma^2}{2}\,.$$

Therefore using the Dynkin formula

$$\begin{aligned}\mathbb{E}_x[\ln(Z_\tau)] &= \ln x + \mathbb{E}_x\left[\int_0^\tau \left(\mu - \frac{\sigma^2}{2}\right) ds\right] = \ln x + \mathbb{E}_x\left[\left(\mu - \frac{\sigma^2}{2}\right)\tau\right] \\ &= \ln x + \left(\mu - \frac{\sigma^2}{2}\right)\mathbb{E}_x[\tau]\ .\end{aligned}$$

□

Remark 8.3. For the function $f(x) = \ln x$ in Lemma 8.2 we do not need to be concerned about the point $x = 0$, since $Z_t > 0$ a.s. for all $t > 0$. This means that the process Z takes values on $(0, \infty)$ and we can therefore use the Itō formula only on this set, thus avoiding problems with the continuously differentiability of f.

Example 8.4. This example is based on [43, Exercise 7.9]. Consider an I-valued stopping time τ. We will consider a process Z satisfying (8.1) and compute

$$\mathbb{E}_x[U(Z_\tau)]\ ,$$

where for $\sigma \neq \sqrt{2\mu}$ and $\mu > 0$ we assume that U is the utility function given by

$$U(x) = \frac{x^\alpha}{\alpha} \quad \text{with} \quad \alpha = 1 - \frac{2\mu}{\sigma^2}\,,\, x \in [0,\infty)\,.$$

A function U of this form is known as power utility. If we now use the given utility function U, we have

$$\begin{aligned}AU(x) &= \left(1 - \frac{2\mu}{\sigma^2}\right)^{-1} \cdot \left(\mu x \left(1 - \frac{2\mu}{\sigma^2}\right) x^{-\frac{2\mu}{\sigma^2}} + \frac{1}{2}\sigma^2 x^2 \left(1 - \frac{2\mu}{\sigma^2}\right)\left(-\frac{2\mu}{\sigma^2}\right) x^{-1-\frac{2\mu}{\sigma^2}}\right) \\ &= \left(1 - \frac{2\mu}{\sigma^2}\right)^{-1} \cdot \left(x^{1-\frac{2\mu}{\sigma^2}}\left(\mu\left(1 - \frac{2\mu}{\sigma^2}\right) - \mu\left(1 - \frac{2\mu}{\sigma^2}\right)\right)\right) = 0\,.\end{aligned}$$

Altogether we have

$$\mathbb{E}_x[U(Z_\tau)] = U(x) = \frac{1}{1 - \frac{2\mu}{\sigma^2}} x^{1-\frac{2\mu}{\sigma^2}}\,.$$

If we now switch to a stochastic process Z modeled by the stochastic differential equation

$$dZ_t = \mu\, dt + \sigma\, dW_t\,, \quad Z_0 = x\,, \quad t \in I\,, \tag{8.5}$$

where $\{W_t\}_{t\in I}$ is a Brownian motion with respect to our filtration, $x > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$, we can also find a result for a function $f(x) = x$.

Lemma 8.6. *For a stochastic process Z satisfying the stochastic differential equation (8.5) we have for every $\tau \in \mathcal{T}_I$*

$$\mathbb{E}_x[Z_\tau] = x + \mu\,\mathbb{E}_x[\tau]\ .$$

Proof. Using the Dynkin formula for the function $f(x) = x$, we get

$$\mathbb{E}_x[f(Z_\tau)] = \mathbb{E}_x[Z_\tau] = x + \mathbb{E}_x\left[\int_0^\tau \mu \cdot 1\, ds\right] = x + \mu\, \mathbb{E}_x[\tau]\ ,$$

as the generator is now given by $Af(x) = \mu$, due to the form of the process defined by the SDE in (8.5). □

Remark 8.7. This result can also be found by noticing that $Z_t = x + \mu t + \sigma W_t$ for all $t \in I$.

8.2 Utility Functions

Consider a utility function u and a stochastic process Z with $\mathbb{E}\big[\sup_{t\in I} Z_t^-\big] < \infty$ or $\mathbb{E}\big[\sup_{t\in I} Z_t^+\big] < \infty$. Define $\tilde{Z} = \{\tilde{Z}_t\}_{t\in I}$ by $\tilde{Z}_t := u(Z_t)$ for $t \in I$. Similarly as in Remark 2.54 for the discrete time setting, for a stopping time $\tau \in \mathcal{T}_I^\nu$ we have

$$\mathbb{E}[u(Z_\tau)] = \mathbb{E}\big[\tilde{Z}_\tau\big]\,. \tag{8.8}$$

For $\Gamma \in \mathcal{M}_I^\nu$ we have

$$\mathbb{E}[u(Z_\Gamma)] = \mathbb{E}\left[u\left(\int_I Z_t\, \Gamma(dt)\right)\right].$$

In some cases a good choice of utility function facilitates the solving of the problem for stopping times. As we will see in the following example, we could have found the result of Lemma 8.2 by simply using (8.8).

Example 8.9. We will now assume $I = [0, T]$ in order to avoid problems with the use of Doob's optional stopping theorem. In Lemma 8.2 we considered a process Z modeled in a similar way to the risky asset in a Black–Scholes model, i.e. we assumed that the process Z follows the stochastic differential equation

$$dZ_t = Z_t(\mu\, dt + \sigma\, dW_t)\,, \quad Z_0 = x\,, \quad t \in I\,,$$

where $\{W_t\}_{t\in I}$ is a Brownian motion with respect to our filtration, $x > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$. We know that the process Z is given in the form

$$Z_t = Z_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)\,, \quad Z_0 = x\,, \quad t \in I.$$

Using the utility function $u(x) = \ln(x)$, we get for $t \in I$

$$\tilde{Z}_t := u(Z_t) = \ln(Z_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\,.$$

By (8.8) we have

$$\mathbb{E}[u(Z_\tau)] = \ln(Z_0) + \mathbb{E}\left[\left(\mu - \frac{1}{2}\sigma^2\right)\tau\right] + \mathbb{E}[\sigma W_\tau] = \ln(Z_0) + \left(\mu - \frac{1}{2}\sigma^2\right)\mathbb{E}[\tau]\ ,$$

because $\mathbb{E}[W_\tau] = \mathbb{E}[W_0] = 0$, since $\{W_t\}_{t\in I}$ is a martingale and we can use Doob's optional stopping theorem.

8.3 Discrete Approximation

In this section we will look at a discrete approximation, which will allow us to transfer results from discrete time to continuous time.

Proposition 8.10. *Given a continuous time interval $I \subset [0,\infty)$ with $0 \in I$. Let $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{m_n}^{(n)}$ be a partition of the time interval I, such that the length of the corresponding subintervals tends to zero as $n \to \infty$ and that $t_{m_n}^{(n)} \to \sup(I)$ for $n \to \infty$ in case $\sup(I) \in I$ or $t_{m_n}^{(n)} \to \infty$ for $n \to \infty$ in case $I = [0,\infty)$. Given a stochastic transition kernel Γ, for a fixed $n \in \mathbb{N}$ define a discrete adapted random probability measure γ^n by*

$$\gamma^n_{t_k^{(n)}} = \begin{cases} \Gamma(\cdot,\{0\}) & \text{if } k = 0, \\ \Gamma(\cdot,[0,t_k^{(n)}]) - \Gamma(\cdot,[0,t_{k-1}^{(n)}]) & \text{if } k = 1,\dots,m_n - 1, \\ \Gamma(\cdot,I) - \Gamma(\cdot,[0,t_{m_n-1}^{(n)}]) & \text{if } k = m_n. \end{cases}$$

Define a sequence of stochastic transition kernels $(\Gamma^n)_{n\in\mathbb{N}}$ by

$$\Gamma^n = \sum_{k=0}^{m_n} \gamma^n_{t_k^{(n)}} \delta_{t_k^{(n)}} ,$$

where $\delta_{t_k^{(n)}}$ denotes the Dirac measure. Then for every right-continuous process $Z = \{Z_t\}_{t\in I}$ with $\mathbb{E}[\sup_{t\in I}|Z_t|] < \infty$

$$\lim_{n\to\infty} \int_I Z_t \, \Gamma^n(dt) = \int_I Z_t \, \Gamma(dt) \quad \textit{pointwise on } \Omega \textit{ and in } L^1.$$

Proof. Take $\omega \in \Omega$, $m_n \in \mathbb{N}$. We have

$$\begin{aligned} \int_{I\setminus\{0\}} Z_t(\omega)\,\Gamma^n(\omega,dt) &= \sum_{k=1}^{m_n-1} Z_{t_k^{(n)}}(\omega)\gamma^n_{t_k^{(n)}}(\omega) + Z_{t_{m_n}^{(n)}}(\omega)\Big(1 - \sum_{k=1}^{m_n-1} \gamma_{t_k^{(n)}}(\omega)\Big) \\ &= \sum_{k=1}^{m_n-1} Z_{t_k^{(n)}}(\omega)\,\Gamma(\omega,(t_{k-1}^{(n)},t_k^{(n)}]) + Z_{t_{m_n}^{(n)}}(\omega)\,\Gamma(\omega,(t_{m_n-1}^{(n)},\infty)\cap I) \\ &= \int_{I\setminus\{0\}} \sum_{k=1}^{m_n} Z_{t_k^{(n)}}(\omega) 1_{(t_{k-1}^{(n)},t_k^{(n)}]}(t \wedge t_k^{(n)})\,\Gamma(\omega,dt) . \end{aligned}$$

Therefore

$$\begin{aligned} &\left| \int_{I\setminus\{0\}} Z_t(\omega)\,\Gamma^n(\omega,dt) - \int_{I\setminus\{0\}} Z_t(\omega)\,\Gamma(\omega,dt) \right| \\ &= \left| \int_{I\setminus\{0\}} \underbrace{\left(\sum_{k=1}^{m_n} Z_{t_k^{(n)}}(\omega) 1_{(t_{k-1}^{(n)},t_k^{(n)}]}(t \wedge t_k^{(n)}) - Z_t(\omega) \right)}_{\overset{n\to\infty}{\to} 0} \Gamma(\omega,dt) \right| . \end{aligned}$$

As the integrand converges to 0 for $n \to \infty$ and the integral is bounded by $2\sup_{t\in I}|Z_t|$, which is a non-negative integrable random variable, we get convergence of the discrete approximation. $\square$

Remark 8.11. The result of Proposition 8.10 should also hold for processes whose paths are continuous from the left. In that case one would have to choose the decomposition of the interval I in such a way that the end points of the intervals of the decomposition coincide with the points of discontinuity.

For the decomposition of the time interval one can also choose a random partition which tends to the identity, as it is defined in [46, page 64].

Due to the result of Proposition 8.10, we can now use a discrete approximation for computations in continuous time if we are given a right-continuous process.

Remark 8.12. For the discrete approximation of Proposition 8.10, for each $n \in \mathbb{N}$ we define a probability measure ν^n on the discrete time interval $I' = \{t_0^{(n)}, \dots, t_{m_n}^{(n)}\}$ such that $\Gamma^n \in \mathcal{M}_{I'}^{\nu^n}$, with $\mathcal{M}_{I'}^{\nu^n}$ as in Definition 2.12. The probability measure ν^n is defined by

$$\nu^n_{t_0^{(n)}} := \mathbb{E}[\Gamma(\cdot, \{0\})] \,,$$

for $k = 1, \dots, m_n - 1$

$$\begin{aligned}\nu^n_{t_k^{(n)}} &:= \mathbb{E}\big[\gamma^n_{t_k^{(n)}}\big] = \mathbb{E}\big[\Gamma(\cdot, [0, t_k^{(n)}])\big] - \mathbb{E}\big[\Gamma(\cdot, [0, t_{k-1}^{(n)}])\big] \\ &= \nu([0, t_k^{(n)}]) - \nu([0, t_{k-1}^{(n)}]) = \nu((t_{k-1}^{(n)}, t_k^{(n)}])\,.\end{aligned}$$

and

$$\nu^n_{t_{m_n}^{(n)}} := \mathbb{E}\big[\Gamma(\cdot, I)\big] - \mathbb{E}\big[\Gamma(\cdot, [0, t_{m_n-1}^{(n)}])\big] = \nu((t_{m_n-1}^{(n)}, \infty) \cap I)$$

Then for $k = 0, \dots, m_n$ we have

$$\gamma_{t_k^{(n)}} \geq 0 \quad \text{a.s.},$$

and

$$\mathbb{E}\big[\gamma_{t_k^{(n)}}\big] = \nu^n_{t_k^{(n)}} \,.$$

Further for $k = 0, \dots, m_n - 1$

$$\gamma_{t_k^{(n)}} = \underbrace{\Gamma(\cdot, [0, t_k^{(n)}])}_{\mathcal{F}_{t_k^{(n)}}\text{-mb.}} - \underbrace{\Gamma(\cdot, [0, t_{k-1}^{(n)}])}_{\mathcal{F}_{t_{k-1}^{(n)}}\text{-mb.}} \quad \text{is } \mathcal{F}_{t_k^{(n)}}\text{-measurable}\,,$$

and $\gamma_{t_{m_n}^{(n)}}$ is $\mathcal{F}_{t_{m_n-1}^{(n)}}$-measurable. Last but not least

$$\sum_{k=1}^{m_n} \gamma_{t_k^{(n)}} = \Gamma(\cdot, I) = 1 \quad \text{a.s.}$$

Lemma 8.13. *For a right-continuous process Z with $\mathbb{E}[\sup_{t\in I} |Z_t|] < \infty$ and a discrete approximation as defined in Proposition 8.10 we have*

$$\mathbb{E}\bigg[\int_I Z_t\, \Gamma(dt)\bigg] = \mathbb{E}\bigg[\lim_{n\to\infty} \int_I Z_t\, \Gamma^n(dt)\bigg] = \lim_{n\to\infty} \mathbb{E}\bigg[\int_I Z_t\, \Gamma^n(dt)\bigg] \,.$$

Proof. We can exchange the expectation and the limit due to dominated convergence, because for all $n \in \mathbb{N}$

$$\int_I Z_t\, \Gamma^n(dt) \leq \sup_{t\in I} |Z_t| \,,$$

where $\sup_{t\in I} |Z_t|$ is a non-negative element of $L^1(\Omega, \mathcal{F}, \mathbb{P})$. $\square$

Using the discrete approximation of Proposition 8.10 and assuming that the process Z satisfies all necessary assumptions, we can find alternative proofs for Lemma 7.8 and Lemma 7.15.

Alternative proof for Lemma 7.8. Let Z be a right-continuous, $(\mathcal{F}\otimes\mathcal{B}_I)$-measurable process with $\mathbb{E}\big[\int_I Z_t^- \,\Gamma(dt)\big] < \infty$ or $\mathbb{E}\big[\int_I Z_t^+ \,\Gamma(dt)\big] < \infty$ and assume we are given a stochastic transition kernel $\Gamma \in \mathcal{M}_I^\nu$ independent of Z. Using the discrete approximation and Lemma 8.13 we get

$$\begin{aligned}\mathbb{E}[Z_\Gamma] &= \mathbb{E}\left[\int_I Z_t\,\Gamma(dt)\right] = \mathbb{E}\left[\lim_{n\to\infty}\sum_{k=0}^{m_n}\gamma^n_{t_k^{(n)}} Z_{t_k^{(n)}}\right] = \lim_{n\to\infty}\sum_{k=0}^{m_n}\mathbb{E}\big[\gamma^n_{t_k^{(n)}} Z_{t_k^{(n)}}\big]\\ &= \lim_{n\to\infty}\sum_{k=0}^{m_n}\mathbb{E}\big[Z_{t_k^{(n)}}\big]\,\mathbb{E}\big[\gamma^n_{t_k^{(n)}}\big] = \lim_{n\to\infty}\sum_{k=0}^{m_n}\mathbb{E}\big[Z_{t_k^{(n)}}\big]\,\nu^n_{t_k^{(n)}} = \int_I \mathbb{E}[Z_t]\,\nu(dt)\,.\end{aligned}$$

□

Alternative proof for Lemma 7.15. If the process Z is a closable right-continuous martingale, we have for all $\Gamma \in \mathcal{M}_I^\nu$ using Lemma 8.13

$$\begin{aligned}\mathbb{E}[Z_\Gamma] &= \mathbb{E}\left[\int_I Z_t\,\Gamma(dt)\right] = \mathbb{E}\left[\lim_{n\to\infty}\sum_{k=0}^{m_n}\gamma^n_{t_k^{(n)}} Z_{t_k^{(n)}}\right] = \lim_{n\to\infty}\sum_{k=0}^{m_n}\mathbb{E}\big[\gamma^n_{t_k^{(n)}} Z_{t_k^{(n)}}\big]\\ &= \lim_{n\to\infty}\sum_{k=0}^{m_n}\mathbb{E}\big[\gamma^n_{t_k^{(n)}}\,\mathbb{E}\big[Z_\infty\,\big|\,\mathcal{F}_{t_k^{(n)}}\big]\big] = \lim_{n\to\infty}\mathbb{E}\Bigg[\underbrace{\sum_{k=0}^{m_n}\gamma^n_{t_k^{(n)}}}_{\overset{\text{a.s.}}{=}1} Z_\infty\Bigg] = \mathbb{E}[Z_\infty] = \mathbb{E}[Z_0]\;.\end{aligned}$$

□

8.4 Results for Special Classes of Processes

In this section we will look at special classes of processes, for which we can derive explicit results. For this, the discrete approximation of Section 8.3 will also be useful.

In discrete time we have seen that by using the Doob decomposition the adapted process Z can be decomposed into a martingale M and a predictable process A. For a deterministic process A it was possible to solve the problem using Theorem 2.49 and Lemma 5.1. In continuous time we have to consider the Doob–Meyer decomposition of the adapted process Z. This decomposition is discussed, for example, in [21, Theorem 1.2.1.6], [26, Chapter 1, Theorem 4.10], [46, Chapter III.3] or [25, Chapter 22].

Lemma 8.14. *Assume that the filtration is right-continuous and let the adapted process Z be a right-continuous sub- or supermartingale of class D. Then there exists a unique decomposition $Z = M + A$, where $M = \{M_t\}_{t\in I}$ is a uniformly integrable right-continuous martingale and $A = \{A_t\}_{t\in I}$ a predictable process with $A_0 = 0$ and $\mathbb{E}[|A_\infty|] < \infty$. Then*

$$V^+(\nu) = \mathbb{E}[M_0] + \sup_{\Gamma\in\mathcal{M}_I^\nu}\mathbb{E}[A_\Gamma]\;.$$

If further the process A is deterministic, then

$$V^+(\nu) = \mathbb{E}[M_0] + \int_I A_t\,\nu(dt)\,.$$

Proof. The existence of the decomposition follows from the Doob–Meyer decomposition theorem (see [26, Chapter 1, Theorem 4.10] or others cited above). Lemma 7.15 gives the first result of the lemma, which can be applied as uniform integrability implies that M is closable. If now A is deterministic, Lemma 7.8 implies

$$\mathbb{E}[A_\Gamma] = \int_I A_t\,\nu(dt)\,.$$

$\square$

Corollary 8.15. *For an adapted, right-continuous process $Z = \{Z_t\}_{t\in I}$ with independent increments and $\mathbb{E}[\sup_{t\in I}|Z_t|] < \infty$ (which also implies that Z is of class D by Remark 7.2(ii)) we have*

$$\sup_{\Gamma\in\mathcal{M}_I^\nu} \mathbb{E}\left[\int_I Z_t\,\Gamma(dt)\right] = \int_I \mathbb{E}[Z_t]\;\nu(dt)\,.$$

Proof. For a process with independent increments the predictable process of the Doob–Meyer decomposition is given by $A_t = \mathbb{E}[Z_t] - \mathbb{E}[Z_0]$ for all $t \in I$. The result then follows from Lemma 8.14 and the fact that for every $\Gamma \in \mathcal{M}_I^\nu$

$$\mathbb{E}\left[\int_I Z_t\,\Gamma(dt)\right] = \mathbb{E}[M_0] + \int_I A_t\,\nu(dt) = \int_I \mathbb{E}[M_t]\;\nu(dt) + \int_I \mathbb{E}[A_t]\;\nu(dt) = \int_I \mathbb{E}[Z_t]\;\nu(dt)\,.$$

$\square$

The result of Corollary 8.15 also follows from the following lemma.

Lemma 8.16. *Given a continuous time interval I and a probability distribution ν on I. For a given adapted stochastic process $Z = \{Z_t\}_{t\in I}$ we define the increments of Z by $\Delta Z_0 := Z_0$ and $\Delta Z_t := Z_t - Z_{t-1}$ for all $t \in I\setminus\{0\}$. If the increments are integrable and there exists a sequence $\{c_t\}_{t\in I} \subset [1,\infty)$ such that they satisfy*

$$\mathbb{E}[\Delta Z_t\,|\,\mathcal{F}_{t-1}] \overset{a.s.}{=} \mathbb{E}[\Delta Z_t]$$

and

$$\mathbb{E}\left[|\Delta Z_t|\,\middle|\,\mathcal{F}_{t-1}\right] \overset{a.s.}{\leq} c_t\,\mathbb{E}[|\Delta Z_t|]$$

for all $t \in I$ with $\nu([0,t)) < 1$, as well as

$$\mathbb{E}\left[\sup_{t\in I}|Z_t|\right] < \infty\,.$$

Then, for all $\Gamma \in \mathcal{M}_I^\nu$, Z_Γ is well-defined, integrable and

$$\mathbb{E}[Z_\Gamma] = \int_I \mathbb{E}[Z_t]\;\nu(dt)\,.$$

Proof. By Proposition 8.10 and Lemma 8.13

$$\mathbb{E}\left[\int_I Z_t\,\Gamma(dt)\right] = \mathbb{E}\left[\lim_{n\to\infty}\sum_{k=1}^{m_n} Z_{t_k^{(n)}}\gamma^n_{t_k^{(n)}}\right] = \lim_{n\to\infty}\mathbb{E}\left[\sum_{k=1}^{m_n} Z_{t_k^{(n)}}\gamma^n_{t_k^{(n)}}\right].$$

By Lemma 5.10 we have

$$\mathbb{E}\left[\sum_{k=1}^{m_n} Z_{t_k^{(n)}}\gamma^n_{t_k^{(n)}}\right] = \sum_{k=1}^{m_n}\mathbb{E}\big[Z_{t_k^{(n)}}\big]\nu_n(t_k^{(n)})\,,$$

where we again define

$$\nu_n(t_k^{(n)}) := \mathbb{E}\big[\gamma^n_{t_k^{(n)}}\big] = \mathbb{E}\big[\Gamma(\cdot,[0,t_k^{(n)}])\big] - \mathbb{E}\big[\Gamma(\cdot,[0,t_{k-1}^{(n)}])\big] = \nu([0,t_k^{(n)}]) - \nu([0,t_{k-1}^{(n)}])\,.$$

We have

$$\lim_{n\to\infty}\sum_{k=1}^{m_n}\mathbb{E}\big[Z_{t_k^{(n)}}\big]\nu_n(t_k^{(n)}) = \int_I \mathbb{E}[Z_t]\;\nu(dt)\,.$$

□

8.5 Some Bounds

In this section we look at upper and lower bounds for our problem. Again, some are valid for all types of processes, while others are only valid for certain classes of processes. Moreover, some take the distribution of the stopping time or the stochastic transition kernel into account and some do not.

Similar to that in discrete time, an upper bound for the problem when considering stopping times is given by a general optimal stopping problem. A general lower bound is given by assuming that the process Z and the stopping time τ or the stochastic transition kernel Γ, respectively, are independent. This lower bound can be found using Lemma 7.8.

For an adapted càdlàg process Z with $\mathbb{E}\big[\int_I Z_t^-\,\Gamma(dt)\big] < \infty$ or $\mathbb{E}\big[\int_I Z_t^+\,\Gamma(dt)\big] < \infty$ for all $\Gamma \in \mathcal{M}_I^\nu$, we have

$$V(\nu) \le V^+(\nu) \le \mathbb{E}\left[\sup_{t\in I} Z_t\right]\,.$$

For the measurability of $\sup_{t\in I} Z_t$ we refer to Remark 7.3. Again this upper bound is quite general and does not incorporate the distribution ν. In many situations the values found by this upper bound will be quite high.

For deriving more bounds we can try to use the discrete approximation presented in Section 8.3.

Lemma 8.17. *Given a continuous time interval I and an adapted right-continuous process $Z \in L^1(\mathbb{P})$ with $\mathbb{E}[\sup_{t\in I}|Z_t|] < \infty$. Let $U = \{U_t\}_{t\in I}$ be the Snell envelope of the process Z and let $U = M + A$ be the Doob–Meyer decomposition of U and assume that M is uniformly integrable. Then we have*

$$V^+(\nu) \le V\,.$$

Proof. For every $\Gamma \in \mathcal{M}_I^\nu$ we have by Proposition 8.10 and Lemma 8.13

$$\mathbb{E}[Z_\Gamma] = \mathbb{E}\left[\int_I Z_t\,\Gamma(dt)\right] = \mathbb{E}\left[\lim_{n\to\infty}\int_I Z_t\,\Gamma^n(dt)\right] = \lim_{n\to\infty}\mathbb{E}\left[\int_I Z_t\,\Gamma^n(dt)\right],$$

where for each $n \in \mathbb{N}$ by Lemma 2.51, as the value of an optimal stopping problem on a continuous time interval is higher than for one with exercise possibilities in any discrete time interval,

$$\mathbb{E}\left[\int_I Z_t\,\Gamma^n(dt)\right] \le V\,.$$

Therefore, for every $\Gamma \in \mathcal{M}_I^\nu$,

$$\mathbb{E}\left[\int_I Z_t\,\Gamma(dt)\right] = \lim_{n\to\infty}\mathbb{E}\left[\int_I Z_t\,\Gamma^n(dt)\right] \le V$$

and

$$V^+(\nu) \le V\,.$$

□

Lemma 8.18. *Let $Z = \{Z_t\}_{t\in I}$ be a right-continuous supermartingale with $\mathbb{E}[\sup_{t\in I} |Z_t|] < \infty$. Then for every $\Gamma \in \mathcal{M}_I^\nu$*

$$\mathbb{E}[Z_0] \ge \mathbb{E}[Z_\Gamma] \ge \mathbb{E}[Z_\infty]\ .$$

If further Z is of class D, then

$$\mathbb{E}[M_0] \ge \mathbb{E}[Z_\Gamma] \ge \mathbb{E}[M_0] + \mathbb{E}[A_\infty]\ ,$$

where $M = \{M_t\}_{t\in I}$ is the uniformly integrable martingale and $A = \{A_t\}_{t\in I}$ the predictable non-increasing process with $A_0 = 0$ and $\mathbb{E}[A_\infty] < \infty$ of the Doob–Meyer decomposition.

Proof. For every $\Gamma \in \mathcal{M}_I^\nu$ we have by Proposition 8.10 and Lemma 8.13

$$\mathbb{E}[Z_\Gamma] = \mathbb{E}\left[\int_I Z_t\,\Gamma(dt)\right] = \mathbb{E}\left[\lim_{n\to\infty}\int_I Z_t\,\Gamma^n(dt)\right] = \lim_{n\to\infty}\mathbb{E}\left[\int_I Z_t\,\Gamma^n(dt)\right].$$

As $Z_t \ge \mathbb{E}[Z_\infty\,|\,\mathcal{F}_t]$ for all $t \in I$, we get using again Lemma 8.13

$$\begin{aligned}\mathbb{E}[Z_\Gamma] &\ge \lim_{n\to\infty}\mathbb{E}\left[\int_I \mathbb{E}[Z_\infty\,|\,\mathcal{F}_t]\ \Gamma^n(dt)\right] = \lim_{n\to\infty}\mathbb{E}\left[\int_I Z_\infty\,\Gamma^n(dt)\right] \\ &= \mathbb{E}\left[\lim_{n\to\infty}\int_I Z_\infty\,\Gamma^n(dt)\right] = \mathbb{E}\left[Z_\infty\int_I \Gamma(dt)\right] = \mathbb{E}[Z_\infty]\ .\end{aligned}$$

Note that the limit Z_∞ exists, due to Doob's convergence theorem (see [62, Theorem 11.5]). If now Z is of class D, we can use the Doob–Meyer decomposition, which states that $Z_t = M_t + A_t$ for $t \in I$. Then

$$\mathbb{E}[Z_\Gamma] = \mathbb{E}\left[\int_I (M_t + A_t)\,\Gamma(dt)\right] = \mathbb{E}\left[\int_I M_t\,\Gamma(dt)\right] + \mathbb{E}\left[\int_I A_t\,\Gamma(dt)\right]$$

By Lemma 7.15 we know

$$\mathbb{E}\left[\int_I M_t\,\Gamma(dt)\right] = \mathbb{E}[M_0]\ .$$

As Z is a supermartingale we know that A is non-increasing and therefore $A_t \ge A_\infty$ a.s. for all $t \in I$. Knowing that $\int_I \Gamma(dt) \overset{\text{a.s.}}{=} 1$, this implies

$$\mathbb{E}\left[\int_I A_t\,\Gamma(dt)\right] \ge \mathbb{E}[A_\infty]\ .$$

Altogether

$$\mathbb{E}[Z_\Gamma] \ge \mathbb{E}[M_0] + \mathbb{E}[A_\infty]\ .$$

Since A is a non-increasing process

$$\mathbb{E}[Z_\Gamma] \le \mathbb{E}[M_0] = \mathbb{E}[Z_0]\ .$$

□

Lemma 8.19. *Let $Z = \{Z_t\}_{t \in I}$ be a right-continuous submartingale with $\mathbb{E}[\sup_{t \in I} |Z_t|] < \infty$. Then for every $\Gamma \in \mathcal{M}_I^\nu$*

$$\mathbb{E}[Z_0] \leq \mathbb{E}[Z_\Gamma] \leq \mathbb{E}[Z_\infty] \; .$$

If further Z is of class D, we get

$$\mathbb{E}[M_0] \leq \mathbb{E}[Z_\Gamma] \leq \mathbb{E}[M_0] + \mathbb{E}[A_\infty] \; ,$$

where we assume that the uniformly integrable martingale $M = \{M_t\}_{t \in I}$ and the predictable non-decreasing process $A = \{A_t\}_{t \in I}$ with $A_0 = 0$ and $\mathbb{E}[A_\infty] < \infty$ are the processes of the Doob–Meyer decomposition.

Proof. The proof works in a similar way to that of Lemma 8.18, but now A is a non-decreasing predictable process. $\square$

Part III

Applications

Chapter 9

Applications in Actuarial Mathematics

This chapter briefly discusses applications of the problem in actuarial mathematics. The time interval will model the time points on which pay-outs occur. First the application for unit-linked life insurance products is discussed in Section 9.1. In this section we also explain how to determine the distribution ν of the stopping time $\tau \in \mathcal{T}_I^\nu$ or the adapted random probability measure $\gamma \in \mathcal{M}_I^\nu$ for a given discrete time interval I. Section 9.2 gives a short explanation of how the problem can be used in health insurance mathematics. Normally an insurance contract will only be considered on a finite-time setting. This is why we now assume $I = \{0, \dots, T\}$ for some $T \in \mathbb{N}$. We further assume that an adapted process $Z = \{Z_t\}_{t \in I} \in L^1(\mathbb{P})$ is given.

9.1 Unit-Linked Life Insurances

One interesting area where the results of the previous parts can be used are unit-linked life insurance contracts, where the pay-out is determined by the price of some underlying financial asset, index or investment fund at a random time point. This random time point might be dependent on the moment of death of the insured person. When considering these insurance contracts, we assume that the adapted process $Z = \{Z_t\}_{t \in I} \in L^1(\mathbb{P})$ now models the pay-out for each moment in time and that the stopping time τ models the time of a pay-out, for example following the demise of the insured, with a distribution given by a life table. Using adapted random probability measures, an entire portfolio of unit-linked life insurance contracts can be modeled. For each $t \in I$ the value γ_t then corresponds to the percentage of contracts that have a pay-out at time t. For these types of insurance contracts we drop the usual assumption that biometric risks are independent of those that exist in financial markets. With this assumption of an adapted dependence between the underlying fund and the stopping time or the adapted random probability measure, respectively, the value we obtain for an insurance contract is greater than the one computed by standard actuarial methods assuming independence (see e.g. [28]). In this way our value could be used to model a worst-case scenario for the insurer, which might be of interest for assessing the risk of the insurance contract.

Note that there are several papers in medical research which deal with the impact of crisis or catastrophes in the surroundings of the patients on severe medical diseases. For the application in unit-linked life insurances [36] is especially interesting as the authors find that the financial crisis, which also had impacts on the financial market and therefore on the underlying of unit-linked life insurance products, may have lead to a higher incidence of acute myocardial infarction in the population of Messinia and claim the need for an analysis of this phenomenon for the entire Greek population. Others analyze the effect of the earthquake in Japan in March 2011 on coronary syndromes ([29] and [42]) or the alteration in the pattern of acute myocardial infarction onset after hurricane Katrina in New Orleans ([45]).

In order to determine the distribution ν of the stopping time τ or the adapted random probability measure γ we need to use a life table. We denote by ${}_np_x$ the probability that a x-year old person will survive the next n years. Conversely, we denote by ${}_nq_x = 1 -_n p_x$ the probability that a x-year old person will die within the next n years. For $n = 1$ we write p_x and q_x. First of all we set $\nu_0 = 0$. This is a reasonable assumption, since there will not be a pay-out at the initiation the contract. For modeling one single unit-linked life insurance contract with pay-out at the end of the year of death of the insured or at the end of the contract, we set ν_t equal to the probability that a x-year old person survives $t-1$ years and dies within the t-th year for $t \in \{1, \dots, T-1\}$, i.e. $\nu_t = {}_{t-1}p_x \cdot q_{x+t-1}$ for $t \in \{1, \dots, T-1\}$. Importantly, we have to choose $\nu_T = {}_{T-1}p_x$. Then we have $\sum_{t=0}^{T} \nu_t = 1$. If one prefers to model a whole portfolio of $N \in \mathbb{N}$ homogeneous contracts using adapted random probability measures, one can choose $\mathbb{E}[\gamma_t] = \nu_t$ with ν_t defined as before for $t \in I$. Assume the process S models the evolution of the underlying fund for one single contract. Then $Z = N \cdot S$ models the evolution of the entire portfolio.

Remark 9.1. In continuous time the distribution ν would be found similar to the explanations above using a given force of mortality.

We will illustrate the computation of the value $V(\nu)$ for a unit-linked life insurance and a portfolio of unit-linked life insurances in the following examples using Lemma 5.37 and Theorem 5.25. In order to derive the values for the distribution ν we will use the values q_x given in the Austrian annuity table 2005, which was presented in [24].

Example 9.2. In order to be able to use Lemma 5.37 we need to assume that the process Z consists of independent random variables. We will assume that Z is an i.i.d. process with $Z_t \sim U(0, 2)$ for all $t \in I$. Let the distribution ν be given as explained above.

First we will compute the quantiles and expected shortfalls needed in general, which allows an easy implementation. Assume $X \sim U(a, b)$, then the δ-quantile of X is given by

$$q_\delta(X) = \delta(b-a) + a\,.$$

The expected shortfall of X for a given $\delta \in (0, 1)$ is given by

$$\begin{aligned} \mathrm{ES}_\delta(X) &= \frac{1}{\mathbb{P}(X > q_\delta(X))}\,\mathbb{E}\big[X 1_{\{X > q_\delta(X)\}}\big] = \frac{1}{1-\delta}\int_{q_\delta(X)}^{b} \frac{x}{b-a}\,dx \\ &= \frac{1}{2(b-a)(1-\delta)}\big(b^2 - q_\delta(X)^2\big)\,. \end{aligned} \tag{9.3}$$

Note that $\mathrm{ES}_\delta(X) = 0$ for $\delta = 1$ and $\mathrm{ES}_\delta(X) = \mathbb{E}[X]$ for $\delta = 0$.
Under the assumption of independence between the process Z and the stopping time τ we have $V^{\mathrm{ind}}(\nu) = 1$ for all maturities $T \in \mathbb{N}$. Assume that we want to compute the price for a unit-linked life insurance contract for a 20-year old male, with a maturity of $T = 10$. In order to be able to use Lemma 5.37 we have to use $\delta_t = 1 - \frac{\nu_t}{1-\nu_0-\dots-\nu_{t-1}}$ for each $t \in I$ for

the computation of the expected shortfall. This yields

$$V(\nu) = \sum_{t=0}^{T} \mathrm{ES}_{1-\frac{\nu_t}{1-\nu_0-\cdots-\nu_{t-1}}}(Z_t) \cdot \nu_t \approx 1.0071286\,.$$

Figure 9.1 shows the evolution of the values V, $V(\nu)$ and $V^{\mathrm{ind}}(\nu)$ for different maturities. We see that the difference between $V(\nu)$ and $V^{\mathrm{ind}}(\nu)$ becomes higher for larger maturities, while the difference between V and $V(\nu)$ becomes smaller for larger maturities.

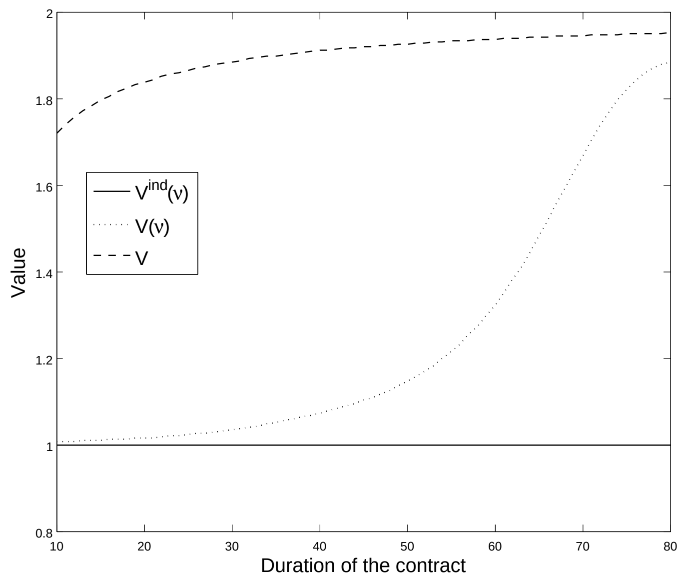

Figure 9.1: The values V, $V(\nu)$ and $V^{\mathrm{ind}}(\nu)$ for a unit-linked life insurance contract for a 20-year old male for different maturities.

They pay-out of the last example is not much realistic for an insurance contract. If the insured person would just receive the pay-out of the underlying fund, the contract is perfectly hedged. Therefore, we are also interested in insurance contracts including a guarantee. For such a contract hedging becomes more challenging. To model a unit-linked life insurance contract including a guarantee, let the process $S = \{S_t\}_{t\in I}$ model the underlying fund and let $G = \{G_t\}_{t\in I}$ model the guaranteed value. Then the pay-out of the insurance contract, which we will denote by $Z = \{Z_t\}_{t\in I}$, at each time point $t \in I$ is given by $Z_t = \max\{S_t, G_t\}$. This pay-out may then be represented as the sum of the fund and the value of a put option by writing $Z_t = S_t + \max\{0, G_t - S_t\}$.

In the following example we will now extend Example 9.2 by a guarantee. The pay-out is still modeled very simple by an independent process. This allows us to use Lemma 5.37.

Example 9.4. Let $S = \{S_t\}_{t\in I}$ be an i.i.d. process with $S_t \sim U(0,2)$ for all $t \in I$ and let $G = \{G_t\}_{t\in I}$ be a deterministic process with $G_t \in [0,2]$ for all $t \in I$. This means that G will model the guaranteed value. Let the process $Z = \{Z_t\}_{t\in I}$ be given by $Z_t = \max\{S_t, G_t\} =$

$G_t + \max\{S_t - G_t, 0\}$ for all $t \in I$. For each $t \in I$ the distribution of Z_t is given by

$$\mathbb{P}(Z_t \le x) = \begin{cases} 0 & \text{if } x < G_t, \\ \mathbb{P}(S_t \le G_t) & \text{if } x = G_t, \\ \mathbb{P}(S_t \le G_t) + \mathbb{P}(G_t < S_t \le x) & \text{if } G_t < x < 2, \\ 1 & \text{if } x \ge 2, \end{cases}$$

where $\mathbb{P}(S_t \le G_t) = \frac{G_t}{2}$ and $\mathbb{P}(G_t < S_t \le x) = \frac{x}{2} - \frac{G_t}{2} = \frac{x-G_t}{2}$. The δ-quantile of Z_t for $t \in I$ is therefore given by

$$q_\delta(Z_t) = \begin{cases} -\infty & \text{for } \delta = 0, \\ G_t & \text{for } 0 < \delta \le \mathbb{P}(S_t \le G_t), \\ 2\delta & \text{for } \mathbb{P}(S_t \le G_t) < \delta \le 1. \end{cases}$$

The expected shortfall is then computed as in (9.3).

For simplicity we will now assume $G_t = 1$ for all $t \in I$. Under the assumption of independence between the process Z and the stopping time τ we have $V^{\mathrm{ind}}(\nu) = 1.25$ for all maturities $T \in \mathbb{N}$. Assume that we again want to compute the price for a unit-linked life insurance contract for a 20-year old male, with a maturity of $T = 10$. In order to be able to use Lemma 5.37 we have to use $\delta_t = 1 - \frac{\nu_t}{1-\nu_0-\dots-\nu_{t-1}}$ for each $t \in I$ for the computation of the expected shortfall. This yields

$$V(\nu) = \sum_{t=0}^{T} \mathrm{ES}_{1-\frac{\nu_t}{1-\nu_0-\dots-\nu_{t-1}}}(Z_t) \cdot \nu_t \approx 1.5036\,.$$

Figure 9.2 shows the evolution of the values V, $V(\nu)$ and $V^{\mathrm{ind}}(\nu)$ for different maturities. We see that again the difference between $V(\nu)$ and $V^{\mathrm{ind}}(\nu)$ becomes higher for larger maturities, while the difference between V and $V(\nu)$ becomes smaller for larger maturities. Note that including the guarantee increases the difference between $V(\nu)$ and $V^{\mathrm{ind}}(\nu)$ from the beginning, compared to Figure 9.1.

Example 9.5. In this example we want to consider an entire portfolio of 100 homogeneous contracts. We will use Theorem 5.25 for the computation. We assume that the process Z is given in such a way that the processes M and A of the Doob decomposition satisfy the necessary conditions. We assume that the process A is one whose increments are i.i.d. with $\Delta A_t \sim U(0,1)$ and that the process M is given by a binomial model with parameters chosen in such a way that it is a martingale. We will assume that $M_0 = 1$ and that $M_t = M_{t-1}(1-X)$ with $\mathbb{P}(X = \frac{1}{2}) = \mathbb{P}(X = -\frac{1}{2}) = \frac{1}{2}$. This time the value of $V^{\mathrm{ind}}(\nu)$ is increasing with an increasing maturity, starting with $V^{\mathrm{ind}} = 598.1888$ for $T = 10$. Let

$$\delta_t = \begin{cases} 1 - \frac{1-\nu_{\le t}}{1-\nu_{\le t-1}} = \frac{\nu_t}{1-\nu_{\le t-1}} & \text{if} \quad \nu_{\le t-1} < 1\,, \\ 0 & \text{if} \quad \nu_{\le t-1} = 1\,. \end{cases}$$

For a portfolio of 20 year old males and a maturity of $T = 10$, we get

$$V^+(\nu) = \mathbb{E}[M_0] + \sum_{t \in I\setminus\{0\}} (1 - \nu_0 - \dots - \nu_{t-1})\,\mathrm{ES}_{\delta_{t-1}}(\Delta A_t) \approx 598.5452\,.$$

Figure 9.3 shows the evolution of the values $V^+(\nu)$ and $V^{\mathrm{ind}}(\nu)$ for different maturities. Again we see that the difference becomes larger for a longer duration of the contract.

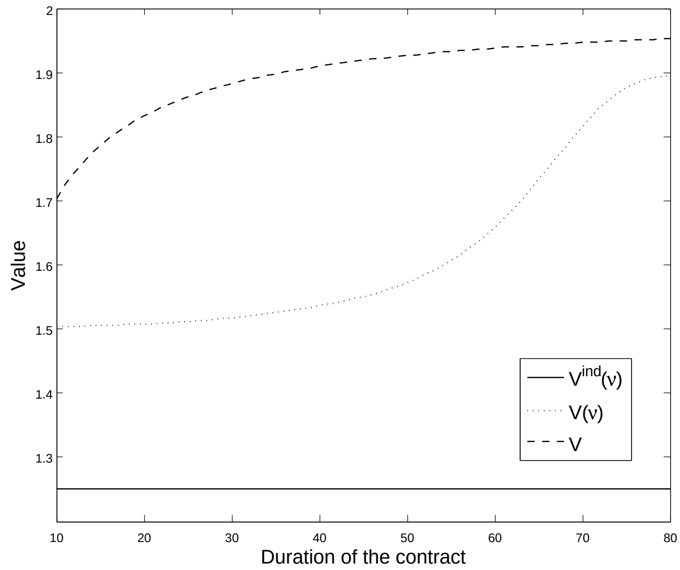

Figure 9.2: The values V, $V(\nu)$ and $V^{\text{ind}}(\nu)$ for a unit-linked life insurance contract for a 20-year old male for different maturities.

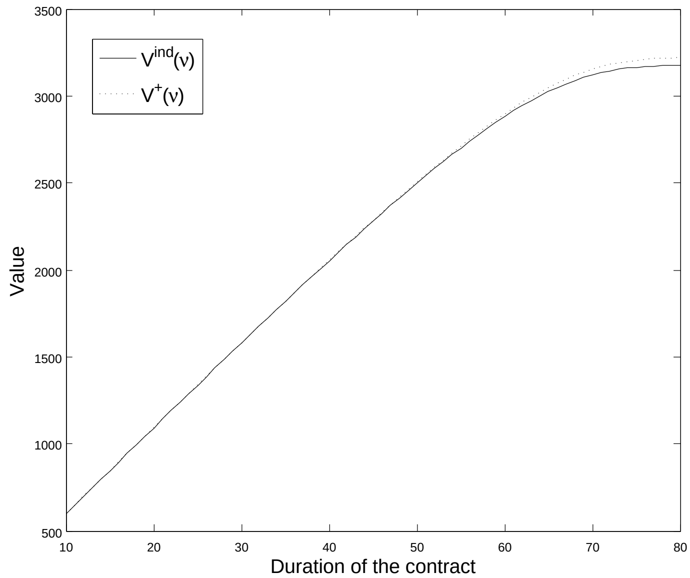

Figure 9.3: The values $V^{+}(\nu)$ and $V^{\text{ind}}(\nu)$ for a portfolio of unit-linked life insurance contracts for 20-year old males for different maturities.

Remark 9.6. So far we only considered endowment insurances, which trigger a pay-out whether the insured person dies or survives. For pure endowment insurances that have a pay-out if the insured person survives a predefined time period, say $I = \{0, \dots, T\}$, Lemma 5.55 is applicable for pay-outs modeled by a martingale or the product of a deterministic function and a martingale, as the pay-out process can be multiplied by a deterministic function given by $f(t) = 0$ for $t \in \{0, \dots, T-1\}$ and $f(T) = 1$. Further any strategy $\gamma \in \mathcal{M}_I^\nu$ satisfying $\mathbb{E}[Z_T \gamma_T] = \nu_T \operatorname{ES}_{1-\nu_T}(Z_T)$ is optimal by Lemma 4.1.

9.2 Health Insurances

If we consider a health insurance contract we can also use the setting presented before. A health insurance contract can be modeled in a similar way to a life insurance contract. If we transform this to our setting, we have that the process γ models the state of the insured person (or the whole portfolio), which means that it determines whether something has to be paid or not. The process γ could therefore be of the form $\gamma_t = 1_{S_t}$, where we define $S_t := \{$the insured person asserts a claim in the time interval $(t, t+1]\}$. The adapted process Z models what is normally called the claims amount per risk. This means that we are interested in the value

$$\sup_{\gamma \in \mathcal{M}_I^\nu} \mathbb{E}[Z_\gamma] = \sup_{\gamma \in \mathcal{M}_I^\nu} \mathbb{E}\left[\sum_{t \in I} Z_t \gamma_t\right].$$

Analogue to the standard calculations, the distribution of γ needs to be found from the data of the insurance company. Also the process Z is modeled based on the data of previous years. Not only the expected values of Z and γ are necessary, but the empirical distributions, which can be found using the data.

Chapter 10

Applications in Risk Management

This chapter concentrates on applications of the problem in risk management, which we briefly discuss. First we take a look at risk measures for stochastic processes. These types of risk measures are discussed in some papers which show that they have a representation, similar to the problem using adapted random probability measures. This representation is shown in Section 10.1. In Section 10.2 we give a short note on how the problem could be used in credit risk modeling.

10.1 Risk Measures for Stochastic Processes

If one is interested in the risk measure of a process one could instead look at the risk measure of a random variable on an appropriate product space. This is shown in [6] and this representation is used, for example, in [1]. Given a time interval I and a process $\{X_t\}_{t\in I}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ one can instead consider a random variable X on a product space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, which is defined by $\bar{\Omega} = \Omega \times I$, $\bar{\mathcal{F}} = \sigma(\{A_t \times \{t\} | A_t \in \mathcal{F}_t, t \in I\})$, $\bar{\mathbb{P}} = \mathbb{P} \otimes \gamma$, where $\gamma = \{\gamma_t\}_{t\in I}$ is some adapted reference process, with $\gamma_t > 0$ for all $t \in I$ and $\sum_{t\in I} \gamma_t = 1$. The correspondence between the random variable and the process is then given by

$$\mathbb{E}_{\bar{\mathbb{P}}}[X] := \mathbb{E}_{\mathbb{P}}\left[\sum_{t\in I} X_t \gamma_t\right].$$

There is a little difference to our setting, since, strictly speaking, the value of each γ_t has to be positive and there is no assumption about the distribution of the process γ. Even if this process γ is not unique it would nevertheless be interesting (if some assumption about the distribution is made) to take a look at the supremum over the various reference processes of this expectation.

In [1] a robust representation of a continuous conditional convex risk measure ρ_t for $t \in I$ for stochastic processes is given by

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t^{loc}} \operatorname*{ess\,sup}_{\gamma\in\Gamma_t(Q)} \left(\mathbb{E}_Q\left[-\sum_{s\in I_{\geq t}} \gamma_s X_s | \mathcal{F}_t\right] - \alpha_t(Q\otimes\gamma)\right), \quad X \in \mathcal{R}_t^\infty,$$

where α_t is the minimal penalty function of ρ_t and $I_{\geq t} := \{s \in I \,|\, s \geq t\}$. Further $\mathcal{Q}_t^{loc} := \{Q \in \mathcal{M}_{loc}(\mathbb{P}) \,|\, Q = \mathbb{P} \,\text{on}\, \mathcal{F}_t\}$ and $\Gamma_t(Q) := \{\gamma \in \Gamma(Q) \,|\, \gamma_s = 0 \,\forall s < t\}$, where $\mathcal{M}_{loc}(\mathbb{P})$ denotes the set of all probability measures Q on $(\Omega, \mathcal{F})$ which are locally absolutely continuous with respect to $\mathbb{P}$ (i.e. $Q \ll \mathbb{P} \,\text{on}\, \mathcal{F}_t$ for each $t \in I \cap \mathbb{N}_0$) and where $\Gamma(Q)$

is the set of all optional random measures $\gamma = \{\gamma_t\}_{t\in I}$ on I, which are normalized with respect to Q. $\mathcal{R}^\infty$ is the space of all adapted stochastic processes X with $\|X\|_\infty < \infty$ and $\mathcal{R}_t^\infty := \pi_{t,T}(\mathcal{R}^\infty)$, where for $0 \le t \le T$, with T being the end point of the time interval I, the projection $\pi_{t,T}\colon \mathcal{R}^\infty \to \mathcal{R}^\infty$ is defined by $\pi_{t,T}(X)_r = 1_{\{t\le r\}}X_{r\wedge T}$, $r \in I$.

10.2 Credit Risk

In credit risk modeling, an important factor is the expected loss at default. Using the above results it is possible to find an upper bound for this expectation in the case of dependence between the time of default and the loss given default. For a given time interval I the stochastic process would then model the loss given default for each time point $t \in I$ and the stopping time models the time of default, for which the probability of default is known. It is generally assumed that the probability of default is only known for one time period. One could therefore try to use the result of Theorem 5.4 for each time period or try to set up a model for the probabilities of default for a multi-period setting. Again adapted random probability measures $\gamma \in \mathcal{M}_I^\nu$, where I is given by the duration of the contracts and ν is given by the probability of default, can be used to model a whole portfolio of homogeneous credit contracts. There are different approaches for modeling the transition and default probabilities of the obligor. These different approaches are discussed in [53, Chapter 2.3].

The holder of a credit normally needs to have a credit life insurance. This insurance contract guarantees repayment of the credit should the insured person die. The insured sum is therefore non-increasing, as the insured person pays back parts of the credit at the different time points. The insured sum is usually modeled deterministically. If the insurance contract is set up in one currency but is entered on to the balance sheet in another, the insured sum becomes stochastic due to the stochastic exchange rate. One might wish to model the insured sum stochastically in any case. The pricing of such a credit life insurance can then be done as explained in Section 9.

Chapter 11

Applications in Mathematical Finance

This chapter will briefly consider applications of the introduced problem in mathematical finance. First we will take a look at American options in Section 11.1. As we already saw in Chapter 2 these are an upper bound. By assuming we know that the holder of the option will react irrationally, and where we account for this by choosing an appropriate distribution, we can turn our attention to the value of the option for this person. In Section 11.2 the liquidation of an investment portfolio is discussed. In this situation we try to earn as much as possible from liquidating the portfolio step by step using adapted random probability measures, where the distribution can be used to model preferences about how much should be liquidated at what point in time. Last but not least we take a look at swing options in Section 11.3. These are often used in electricity markets and have a representation similar to the problem that arises when using adapted random probability measures.

11.1 American Options

Already in the introduction of the problem we saw that an upper bound is given by the value of an optimal stopping problem, or for non-negative process by the value of the corresponding American option. For the price of the American option it is assumed that the buyer of the option acts rationally and uses an optimal strategy. If one has any information that the buyer of the option will not act rationally and that he or she will follow a strategy with a given distribution, the setting of the problem introduced could be used for pricing the option. This is illustrated in the following example.

Example 11.1. We will consider an American option with maturity $T = 4$, where the pay-out at time $t \in \{0, \dots, 4\}$ is given by a random variable $Z_t \sim U(0, t+1)$. We assume that the adapted process Z is an independent process. First we will take a look at the Snell envelope $U = \{U_t\}_{t\in\{0,\dots,4\}}$ to find the value and the optimal strategy for the American option. We have

$$U_4 = Z_4\,,$$

$$U_3 = \max\{Z_3, \mathbb{E}[U_4\,|\,\mathcal{F}_3]\} = \max\{Z_3, \mathbb{E}[Z_4]\} = \max\left\{Z_3, \frac{5}{2}\right\},$$

$$U_2 = \max\{Z_2, \mathbb{E}[U_3\,|\,\mathcal{F}_2]\} = \max\left\{Z_2, \mathbb{E}\left[\max\left\{Z_3, \frac{5}{2}\right\}\right]\right\} = \max\left\{Z_2, \frac{89}{32}\right\},$$

$$U_1 = \max\{Z_1, \mathbb{E}[U_2 \,|\, \mathcal{F}_1]\} = \max\left\{Z_1, \mathbb{E}\left[\max\left\{Z_2, \frac{89}{32}\right\}\right]\right\} = \max\left\{Z_1, \frac{17137}{6144}\right\} = \frac{17137}{6144}$$

and

$$U_0 = \max\{Z_0, \mathbb{E}[U_1 \,|\, \mathcal{F}_0]\} = \frac{17137}{6144}\,.$$

It is well known that the optimal stopping time is given by

$$\tau = \min\{s \in \{0, \dots, 4\} \,|\, U_s = Z_s\}\,.$$

Therefore

$$\{\tau = 0\} = \{\tau = 1\} = \varnothing\,,$$
$$\{\tau = 2\} = \left\{Z_2 \geq \frac{89}{32}\right\},$$
$$\{\tau = 3\} = \left\{Z_2 < \frac{89}{32}\right\} \cap \left\{Z_3 \geq \frac{5}{2}\right\}$$

and

$$\{\tau = 4\} = \left\{Z_2 < \frac{89}{32}\right\} \cap \left\{Z_3 < \frac{5}{2}\right\}.$$

The probability for exercising the option when following the optimal strategy is given by

$$\mathbb{P}(\tau = 0) = \mathbb{P}(\tau = 1) = 0\,, \quad \mathbb{P}(\tau = 2) = \frac{7}{96}\,, \quad \mathbb{P}(\tau = 3) = \frac{89}{576}\,, \quad \mathbb{P}(\tau = 4) = \frac{445}{576}\,.$$

Then, the value of the American option is given by

$$\mathbb{E}[Z_\tau] = \sum_{t=2}^{4} \mathbb{E}[Z_t \,|\, \tau = t]\,\mathbb{P}(\tau = t) = \frac{7}{96} \cdot \frac{185}{64} + \frac{89}{576} \cdot \frac{13}{4} + \frac{445}{576} \cdot \frac{5}{2} = \frac{16247}{6144} \approx 2.64437\,.$$

Now we assume that we want to consider an irrational buyer of the option. We assume that for this person the probabilities for exercising the option are given by

$$\nu_0 = 0\,, \quad \nu_1 = \frac{1}{12}\,, \quad \nu_2 = \frac{1}{6}\,, \quad \nu_3 = \frac{1}{3}\,, \quad \nu_4 = \frac{5}{12}\,.$$

Then we can compute the value $V(\nu)$ using the results of Lemma 5.37. We have

$$\begin{aligned}
V(\nu) &= \sum_{t \in I} \nu_t \,\mathrm{ES}_{1 - \frac{\nu_t}{1 - \nu_0 - \cdots - \nu_{t-1}}}(Z_t) \\
&= \frac{1}{12}\,\mathbb{E}\left[Z_1 \,\middle|\, Z_1 > \frac{11}{6}\right] + \frac{1}{6}\,\mathbb{E}\left[Z_2 \,\middle|\, Z_2 > -\frac{27}{11}\right] + \frac{1}{3}\,\mathbb{E}\left[Z_3 \,\middle|\, Z_3 > \frac{20}{9}\right] + \frac{5}{12}\,\mathbb{E}[Z_4] \\
&= \frac{1}{12} \cdot \frac{23}{12} + \frac{1}{6} \cdot \frac{71}{44} + \frac{1}{3} \cdot \frac{56}{18} + \frac{5}{12} \cdot \frac{5}{2} = \frac{11915}{4752} \approx 2.50737\,.
\end{aligned}$$

American options are normally taken to be financial options but they could also be real options, i.e. options on business initiatives within a company. These can be modeled in a similar way to financial options, which would allow the use some of the results of this thesis for financial American options. For more information about real options see, for example, [59].

11.2 The Liquidation of an Investment Portfolio

The results could also be helpful to model the liquidation of an investment portfolio, if we assume that the liquidation of the portfolio is done over some time interval I. Again adapted random probability measures are used. The adapted process $\gamma = \{\gamma_t\}_{t\in I}$ models the fraction of the portfolio that is liquidated. The adapted process $Z = \{Z_t\}_{t\in I} \in L^1(\mathbb{P})$ models the value of the investment portfolio. This means that the value $Z_t\gamma_t$ gives the amount we receive for the liquidation of a part of the portfolio. Naturally, we want to earn as much as possible from this liquidation so we are interested in an optimal liquidation strategy γ^* satisfying

$$\sup_{\gamma\in\mathcal{M}_I^\nu} \mathbb{E}[Z_\gamma] = \mathbb{E}[Z_{\gamma^*}] \; .$$

The distribution of the process γ would be given by the preferences of the person liquidating the investment portfolio. If a lot of money is needed right at the beginning, the expectation of γ_0 will be high. By an appropriate choice of the expected values of γ_t for all $t \in I$, the person liquidating the portfolio can decide at which point in time within the predefined time interval I what proportion of the portfolio to sell on the market.

If only one asset is considered, the problem formulation using stopping times could be of interest. It could be used to find bounds for the expected price of the asset, which has to be traded at a random time, modeled by the stopping time with distribution ν, due to some circumstances.

11.3 Swing Options

Another area for possible application is that of electricity markets, where swing options are traded. These options enable investors to buy a certain amount of the commodity at a fixed price at certain dates in the future, which can be nominated throughout the whole delivery period. In electricity markets these options are used to ensure the availability of additional power in periods of high demand.

In [8] the value of a swing option is computed in the following way:

$$V_{Swing} = \max_{\phi} \left\{ \sum_{i=1}^{N} \mathbb{E}^* \left[e^{-r(t_i-t_0)} \phi(t_i)(S_{t_i} - K) \right] \right\} , \tag{11.2}$$

with

$$\sum_{i=1}^{N} \phi(t_i) = A \, , \quad 0 \le \phi(t_i) \le C \quad \forall i \in \{1, \ldots, N\} \, ,$$

where S_{t_i} is the spot market price at time t_i, K is the strike price of the option, $\mathbb{E}^*$ denotes the expectation under a pricing measure $\mathbb{P}^*$, r is the interest rate, C is the capacity limit, A is the fixed energy amount which is allowed to be spread over the contract period and $\phi(t)$ is the exercising strategy. $t_1, \ldots, t_N$ are the moments in time when the option can be exercised, i.e. when additional power can be bought. This looks quite similar to the computation of the value $V^+(\nu)$, where the adapted process Z has to be modeled appropriately. What is different, however, is the assumption $\sum_{i=1}^N \phi(t_i) = A$ instead of $\sum_{t=0}^T \gamma_t = 1$ and $0 \le \phi(t_i) \le C$. If $C < A$ we can only use adapted random probability measures for modeling this problem.

In [8] the value of the swing option is computed by numerical methods and Monte Carlo simulation since it is difficult to find a realistic model for an electricity market admitting explicit results.

Lemma 11.3. *In the setting used in (11.2) assume that the interest rate r is deterministic and assume that $\mathbb{E}^*[\phi(t_i)] := \nu_{t_i}$ is known for $i \in \{1, \ldots, N\}$. Then we have*

$$V_{Swing} = A\mathbb{E}^*\left[S_{t_1}\right] - \sum_{i=1}^{N} K e^{-r(t_i - t_0)} \nu_{t_i} \,.$$

Proof. The discounted process $\{e^{-r(t_i-t_0)} S_{t_i}\}_{i \in \{1,\ldots,N\}}$ is a martingale under $\mathbb{P}^*$. The result then follows by linearity of the expectation and the result about martingales in Section 6.1. $\square$

Remark 11.4. If discounting is neglected, i.e. we assume $r = 0$ in the setting presented for the Swing option, the price introduced in (11.2) can be computed by similar considerations as in Section 6.1. By defining $Z_{t_i} := S_{t_i} - K$ for $i \in \{1, \ldots, N\}$, we have that Z is a martingale under $\mathbb{P}^*$, if the measure $\mathbb{P}^*$ is chosen such that S is a martingale under $\mathbb{P}^*$. Since we only consider finitely many time points, for every admissible ϕ we have

$$\begin{aligned}\sum_{i=1}^{N} \mathbb{E}^*\left[\phi(t_i)(S_{t_i} - K)\right] &= \sum_{i=1}^{N} \mathbb{E}^*\left[\phi(t_i)(S_{t_N} - K)\right] = A\mathbb{E}^*\left[(S_{t_N} - K)\right] \\ &= A(\mathbb{E}^*\left[S_{t_1}\right] - K)\,.\end{aligned}$$

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Professional Experience

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Teaching Experience

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Talks

June 28^{th}, 2012 **Adapted Dependence for Pricing Unit-linked Life Insurances**, IME 2012 – 16^{th} International Congress on Insurance: Mathematics and Economics, June 28–30, 2012, Hong Kong

June 20^{th}, 2012 **Adapted Dependence in Discrete Time**, BFS 2012 – 7^{th} World Congress of the Bachelier Finance Society, June 19–22, 2012, Sydney, Australia

Oct. 14^{th}, 2011 **Adapted Dependence**, PRisMa Day – Workshop on Portfolio Risk Management, TU Vienna, Austria

Oct. 8^{th}, 2010 **Adapted Dependence**, Evaluation of PRisMa Lab, TU Vienna, Austria

Jan. 22^{nd}, 2010 **Variable Annuities**, Internal Workshop with the Industry Partners of PRisMa Lab, TU Vienna, Austria

July 3^{rd}, 2009 **Adapted Dependence**, Internal Workshop with the Industry Partners of PRisMa Lab, TU Vienna, Austria

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Preprints

submitted S. Altay, S. Gerhold, R. Haidinger and K. Hirhager, *Digital double barrier options: several barrier periods and structure floors*

in preparation K. Hirhager and C. Weber, *On GMWBs for Life*

in preparation K. Hirhager, J. Hirz and U. Schmock, *Conditional quantiles, conditional weighted expected shortfall and application to risk capital allocation*