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The Optimal Trade-off between Green R&D, Abatement and Emission Permits in an Environmental Economic Growth Model

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Abstract

Die Umweltproblematik ist in den vergangenen Jahren stark in den Fokus der Öffentlichkeit gerückt: Sowohl die Gesellschaft als auch Politik und Wissenschaft setzen sich heute intensivst mit den Gefahren einer Ausbeutung der Umwelt auseinander, nicht zuletzt wegen den äußerst realen Bedrohungen durch die globale Klimaerwärmung oder den Rückgang der Ozonschicht. Das Bewusstsein um die Bedeutung des Umweltschutzes ist immer mehr in den Köpfen der Menschen verankert. Möglichkeiten, Umweltverschmutzung zu kontrollieren und einzudämmen, gibt es mehrere. Diese Diplomarbeit betrachtet eine Kombination aus einer vom Staat vorgegebenen Emissions-Vorschrift, an die sich Firmen im Zuge der Produktion halten müssen, sowie handelbaren Emissions-Zertifikaten, die gekauft, aber auch verkauft werden können. Die Arbeit basiert auf einem Papier von M. Rauscher [2009] und einer an der TU Wien verfassten Diplomarbeit von E. Moser [2010], entwickelt und analysiert aber ein eigenes Umweltmodell mit Wirtschaftswachstum. Mithilfe des Pontryagin´schen Maximumsprinzip aus der Optimalen Kontrolltheorie, werden die Auswirkungen von umwelttechnischen Kontrollinstrumenten auf das Verhalten von Firmen untersucht. Dabei wird besonders die Frage hervorgehoben, wie Firmen mit strengeren Umweltauflagen umgehen. Neben einem Produktionsrückgang stehen ihnen drei Alternativen zur Verfügung: Investition in grüne, umweltfreundlichere Technologien, Kompensation durch sog. Abatement-Maßnahmen am Ende des Produktionsprozesses und der Erwerb von Emissionszertifikaten.

Abstract

Environmental topics have become an important and highly discussed matter in society, politics and science due to changes occurring on a global scale including scarcity of resources, threat from the climate change and ozone depletion. More and more people become aware of the dangers resulting from the depletion of nature. A possibility of protecting earth from environmental pollution resulting from economic production is to set environmental standards which are not allowed to be exceeded by the firms of an economy. Another approach is to admit a certain amount of pollution allowed by each firm and to establish a market for tradeable pollution permits. This thesis considers a combination of both control instruments and analyzes the impacts on firms behavior. It is based on works of M. Rauscher [2009] and E. Moser [2010], but develops an own environmental model of economic growth. The analysis with Pontryagin´s Maximum Principle from Optimal Control Theory lays its focus on the question of firms reaction on stricter environmental constraints. As alternatives to a decrease in output, they have the possibility to invest in greener, more eco-friendly technologies, to invest in end-of-pipe abatement and to acquire permission permits to augment the tolerance.

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Contents

Chapter 1

Introduction

1.1 The Topic

The environment serves humankind as a source as well as a sink. As a source, it provides resources (such as land and water) indispensable to all life on earth and as a sink it absorbs waste (by-products of modern technology such as air and water pollutants, asbestos, pesticides and radioactive waste) also inseparably associated with human activity. In both roles, it is limited since we live in a finite world, although mankind has long behaved as it would not. Human activity has always had an impact on the environment, but on the one hand, accepting that the world imposes restrictions on us means taking responsibility for our actions, which normally makes life harder and more complicate. There would have to be a rethinking of the whole economic behavior which has worked well for the past. On the other hand there was no urgent need to pay attention. Humankind was simply not forced to deal with nature in a responsible way because the environment has not yet reached its limits and showed no warning signals. This has changed over the last decades and the protection of the environment has become an important matter and a highly discussed topic due to changes occurring on a global scale including scarcity of resources, threat from the climate change and ozone depletion. Scientific research clearly outlined the environmental dangers of climate change. More and more people are now aware of the dangers resulting from depletion of nature as a source as well as a sink[1]. The term "sustainability" has arisen. Sustainable development refers to a development "that meets the needs of the present without compromising the ability of future generations to meet

[1] In 2007, the EU's leaders endorsed an approach to climate and energy policy that aims to combat climate change and increase the EU's energy security. The goal was to transform Europe into a highly energy-efficient, low carbon economy. In 2008, the European Commission proposed binding legislation to implement the targets of 2007 (reducing greenhouse gas emissions by at least 20% below 1990 levels in 2020, increasing the share of renewable energy to 20% by 2020 and reducing primary energy use by 20%, to be achieved by improving energy efficiency). The "Climate and Energy Package" was agreed by the European Parliament and Council. It became law in 2009. http://ec.europa.eu/clima/policies/package/index_en.htm

their own needs"[2]. On the one hand overexploitation and abuse of the "waste container" nature have already undesirable effects, on the other hand humankind has the ability and at the same time the duty to think of the future and future generations which do also have the right to live in a world which gives them the chance to live. Whereas the earth can exist without humankind, this does not (yet) hold the other way round. As long as earth is the only planet in our solar system that provides the requirements for life, humankind should have a firm interest in conserving it. To destroy the environment means to destroy its own livelihood. It is not only self-evident but necessary to think about impacts of human activity on the welfare of nature since it is the source of life. It is a fact that human activity and especially economic production harm the environment which is and will not be without consequences. An irresponsible attitude concerning exploitation of nature means a lack of interest on the welfare of humankind.

In the economic system, environmental protection does not happen automatically because of market failure in the field of environment. Taking care of the environment is in most cases not rewarded with immediately monetary gains (but in contrary is expensive). An efficient economic system does not suffice to protect the environment because of externalities and environmental quality (e.g. a beautiful view) being a public good:

- The environment is subject to externalities of both consumption and production. In this thesis production and its negative externalities (resulting from emissions such as sulphur dioxide, carbon dioxide and other greenhouse gases) are considered. Negative externalities are costs which accrue during the production process, but are not reflected in the market price, because the producer does not have to bear the costs himself, but they are accrued to the society as a whole. Externalities arise when activities (production or consumption) of one entity has impacts on another entity, which are outside the market mechanism. The behavior of economic agents is therefore not efficient in a macroeconomic context, because the externalities are not considered. In the case of negative production externalities (for example, when there is air pollution during the production process), the firm produces too much to be macroeconomic optimal, because it does not consider the costs of pollution. Firms which pollute create a cost to society but not a cost to themselves. Because the firm does not have an accurate view of its costs of production, it does not set its production at the level that maximizes efficiency in the economy. Its environmentally unfriendly products are too cheap. In the long term external effects could become internal ones, but for now the government has to intervene in form of taxes or standards to force firms to internalize negative production externalities.

- Being a public good (like the national defense) means that there is no rivalry (two

[2] 1987 United Nations Commission on Environment and Development- Brundtland Commission

economic agents can consume/ use the good at the same time) as well as no possibility to exclude anyone (like excluding a class of population from having a car by making the price too high for this class to afford it). Especially with the environment goods clean air and clean water an exclusion is almost impossible: Acid Rain falls also on confined areas and running water as well as the air do not stop at a frontier. With non-exclusion there are no inconvenients due to irresponsible consumption. There is no market price and it is hard to define a price for a public good, because there is no incentive for consumers to pay for the good the value of it. Everybody can consume it regardless of whether he paid for the production or not. Firms will therefore not offer this good and it must be subsidized or provided by the government.

These are the reasons which require general regulations and public intervention.

Nature gives warning signals, but it cannot protect itself from human exploitation. It is therefore in the responsibility of humankind and especially of the world´s governments to pay attention and to think about future generations. Developing and analyzing models can be of help in this process, but unless political action is taken immediately, all scientific efforts will be of little use.

1.2 A Review of selected Environmental Models

There are many ways to model the environment as well as policies to protect it. There are also many ways to model the reaction possibilities of the private sector of an economy to deal with these policies. Much in this field has already been discussed in the literature (see for example Helfand [1991], Bréchet et al. [2010], Bretschger et al. [2007], Jørgensen et al. [2010], Smulders [1995] and Brock et al. [2004]). Three of the already existing works will be presented briefly in the following sections, because my work is based on them:

1. Green R&D versus End-of-Pipe Emission Abatement: A Model of Directed Technical Change (Rauscher [2009])

2. Optimal Controls in Models of Economic Growth and the Environment (Moser [2010])

3. Bankable Pollution Permits under Uncertainty and Optimal Risk Management Rules: Theory and Empirical Evidence (Chevallier et al. [2008])

Chapter 5 will give an overview of differences and similarities with the basic model presented in this thesis which is an extension of the first two models and includes an idea of the third work.

1.2.1 The Model of Rauscher: Green R&D versus End-of-Pipe Abatement

Rauscher [2009] addresses the question whether strict environmental regulation fosters innovation and economic growth with a dynamic model of directed technical change in an environmental-economics context. In this model, firms can decide between two types of capital for production which differ in their impact on the environment (in form of polluting emissions). Whereas one production process is more efficient, the other one is less environmentally harmful. Firms can abate emissions at the end of pipe, which means to repair some of the generated damage to the environment with a part of the output, to meet an environmental standard, set by the government. In the model of Rauscher, these standards are binding and require abatement costs which are proportional to the stock of conventional capital. The firm has therefore two possibilities to avoid pollution: producing with cleaner technology and end-of-pipe-abatement. It can choose between more expensive production with green capital (instead of conventional capital) and abatement costs at the end of the production process.

The two capital stocks are denoted as K for conventional or brown capital and G for green capital and form the state variables of the problem. They accumulate via the decision variables investment in Research&Development (R&D), R_K and R_G. Existing capital advances the accumulation of new capital, albeit at a decreasing rate. Additionally, positive knowledge spillovers in the R&D sector are assumed which yields the following processes of accumulation[3]:

$$\begin{aligned} \dot{K(t)} &= A(K(t), K^*(t), R_K(t)) \\ \dot{G(t)} &= B(G(t), G^*(t), R_G(t)) \end{aligned}$$

with $K^*(t)$ and $G^*(t)$ standing for the economy-wide stocks of conventional and green capital, respectively modeling knowledge spillovers. The stringency of environmental regulation is exogenously given by ϵ. Conventional capital pollutes the environment when used in production, whereas green capital does not. As a result, the abatement costs are modeled as

$$\chi(\epsilon)K(t).$$

Costs also occur by spending resources on investement, the so-called opportunity costs $w(R_K+R_G)$ as well as the expenses on consumption $C(t)$. Consumption in turn generates utility, just like the state of the environment, approximated by ϵ. The utility function

[3]Dots above variables denote derivatives with respect to time.

which has to be maximized is then given as

$$\log C(t) + u(\epsilon)$$

where $\log C(t)$ stands for the utility derived from consumption and $u(\epsilon)$ is the utility due to an intact environment. The produced output $F(K(t), G(t))$ has to be divided between consumption, opportunity costs of R&D investment in brown and green capital and abatement yielding the budget constraint

$$F(K(t), G(t)) - C(t) - w(R_K(t) + R_G(t)) - \chi(\epsilon)K(t) = 0.$$

The initial levels of the two capital stocks, $K(0)$ and $G(0)$, are given historically. Summing up, the whole optimization model with the discount rate being δ yields

$$\begin{aligned}
\max_{R_K(t), R_G(t)} \quad & \int_0^\infty (\log C(t) + u(\epsilon))e^{-\delta t}\,\mathrm{d}t \\
\text{s.t.} \quad & \dot{K(t)} = A(K(t), K^*(t), R_K(t)) \\
& \dot{G(t)} = B(G(t), G^*(t), R_G(t)) \\
& F(K(t), G(t)) - C(t) - w(R_K(t) + R_G(t)) - \chi(\epsilon)K(t) = 0
\end{aligned}$$

The core results from solving the model are the following statements about the impact of environmental regulation on the allocation of resources to conventional R&D, green R&D and end-of-pipe abatement.

- Stricter environmental standards induce declines in the steady-state rates of investment in both conventional and green capital (related to the corresponding capital stocks), but the share of green capital in conventional capital rises:

  $$\frac{\partial \frac{R_K}{K}}{\partial \epsilon} < 0$$
  $$\frac{\partial \frac{R_G}{G}}{\partial \epsilon} < 0$$
  $$\frac{\partial \frac{G}{K}}{\partial \epsilon} > 0.$$

  The steady-state growth rate of the economy is negatively affected by stricter environmental policy:
  $$\frac{\partial \frac{\dot{K}}{K}}{\partial \epsilon} = \frac{\partial \frac{\dot{G}}{G}}{\partial \epsilon} < 0.$$
  The economic explanation for these findings is straightforward. Tighter environmental standards raise the cost of using conventional capital which is therefore replaced

by green capital. The higher cost of using conventional capital reduces the incentive to accumulate this type of capital. Due to the shift from conventional to green capital, the marginal productivity of green capital is reduced and this reduces the incentive to invest in green capital as well.

- The effects of stricter environmental regulation on the R&D expenditures of K-type capital as a share of GDP is the following:

$$\frac{\partial \frac{wR_K}{F(K,G)}}{\partial \epsilon} < 0.$$

The share of GDP spent on conventional R&D will unambiguously decline. The effects on the R&D expenditure shares of G-type capital and on the share of end-of-pipe abatement cost in GDP are ambiguous and depend on the parameters of the model, in particular on the cost of using end-of-pipe abatement technologies and the weight of knowledge spillovers. A shift from end-of-pipe to process-integrated abatement is likely if the cost of end-of-pipe abatement measured as a share of GDP is high and the spillovers in green R&D are large compared to those in conventional R&D.

The model does not support the hypothesis, that stricter emission standards should be used to spur R&D and accelerate innovation and economic growth. Instead of supporting green R&D and long-term economic growth, tighter environmental standards rather retard them.

1.2.2 The Model of Moser: A quantitative Version of the Model of Rauscher

For her investigation, Moser [2010] uses also the endogenous growth model of Rauscher with the modification that positive knowledge spillovers in the R&D sector are neglected for the sake of simplicity. Whereas Rauscher considers the problem in a rather general formulation, Moser investigates various scenarios with different model functions and state dynamics.

The functional forms for the basic model of Moser, which will be subject to a comparison with the model formulated in this thesis, are given as

$$\begin{aligned}
&F(K,G) = bK^{\alpha_1}G^{\alpha_2}\\
&\chi(\epsilon) = a\epsilon^{\beta}\\
&u(\epsilon) = c\epsilon^{\gamma}\\
&A(K,R_K) = dK^{\delta_1}R_K^{\delta_2} - \phi K\\
&B(G,R_G) = eG^{\sigma_1}R_G^{\sigma_2} - \psi G.
\end{aligned}$$

The basic model of Moser is then

$$\begin{aligned}
&\max_{R_K,R_G} \int_0^{\infty} (\ln(\tau + bK^{\alpha_1}G^{\alpha_2} - w(R_K+R_G) - a\epsilon^{\beta}K) + c\epsilon^{\gamma})e^{-rt}\,\mathrm{d}t\\
&\text{s.t.}\quad \dot{K} = dK^{\delta_1}R_K^{\delta_2} - \phi K\\
&\qquad\ \ \dot{G} = eG^{\sigma_1}R_G^{\sigma_2} - \psi G\\
&\qquad\ \ 0 \le R_K\\
&\qquad\ \ 0 \le R_G\\
&\qquad\ \ 0 \le bK^{\alpha_1}G^{\alpha_2} - w(R_K+R_G) - a\epsilon^{\beta}K
\end{aligned}$$

where r denotes the discount rate and total utility, which consists of utility derived from consumption, $\ln(\tau + bK^{\alpha_1}G^{\alpha_2} - w(R_K+R_G) - a\epsilon^{\beta}K)$, and utility due to an intact environment, $c\epsilon^{\gamma}$, is maximized.

The key findings from the analysis are the following:

- A higher ϵ leads to a proportionally greater equilibrium accumulation of green capital, but in total, accumulated capital of both types as well as production output decline towards zero.
- Utility has its maximum at a very low level of ϵ.

1.2.3 The Model of Chevallier: Pollution Permits

Completely new in the present work is the introduction of pollution permits as an environmental regulation tool. Additionally to the trade-off between investment in clean capital and end-of-pipe-abatement, firms have the possibility to buy or sell tradeable permits as in the work of Chevallier et al. [2008].

Tradeable pollution permits represent the rights to emit or discharge a specific volume of actual or potential pollution e.g. 100 units of Carbone Dioxide per year. They can be sold and bought in artificially created markets[4]. Normally, the government of a country

[4]See http://stats.oecd.org/glossary/detail.asp?ID=2737

determines the total acceptable level of pollution over the area concerned and sets a limit on the amount of a pollutant that can be emitted. The amount of pollution permits is allocated (depending for example on the sector) or sold to firms in form of emissions permits. Firms are required to hold a number of permits equivalent to their emissions, but they have the possibility to sell the balance of their permits to other firms if they can reduce their emissions below the designated level or to buy some additionally from firms who require fewer permits. The total number of permits cannot exceed the limit set by the government, assuring that total emissions stay under that level[5]. In this thesis, firms have no initial set of permits and there is no limit, it is assumed that every firm can buy and sell as many permits as it wants. The government does not fix a certain amount of available pollution rights but an environmental standard which the firms have to meet. One goal of this thesis is to find out, what the optimal environmental regulation value is. Furthermore, the price of a tradeable permit is constant and does not change due to demand and supply at the market of permits.

With tradeable pollution permits the question arises, if companies really attempt to reduce pollution or if they will simply bear the cost of pollution. It is likely, that those firms who easily can avoid pollution, will sell their permits, so that in the long run, some firms will pollute heavily and others will not, to the end that some areas of the world or a country become highly polluted and other areas will be relatively clean. From a global point of view, countries who pollute more than their quotas allow can simply buy permits from other countries.

It also has to be said that there are difficulties to measure exactly how much a company is polluting. There is potential for hiding pollution emissions.

There are administration costs of implementing the scheme.

There is abuse as was in January 2011[6] when nearly half a million pollution permits were stolen from a Czech carbon bank.

Nevertheless, pollution permits are widely considered as efficient instruments for regulating the emissions of pollutants by firms. One advantage of market-based approaches such as emission permits that can be traded among firms is that it allows firms to reduce pollution at lowest cost (because only the firms which are able to reduce pollution efficiently will do so- the others will buy permits from these firms), unlike emission taxes or cumpolsory technologies which do not consider differences in firms production possi-

[5] See Stavins [2001]

[6] See http://www.newscientist.com/article/dn20012-black-market-steals-half-a-million-pollution-perm html

bilities[7]. Such regulatory limits would impose very different costs in different industries. Further, it is in the interests of firms to pollute as little as possible, because pollution permits work by obliging polluters to pay for their noxious emissions. If they pollute at a level higher than the government dictates, they have to buy additional permits. If they pollute less than they are allowed to, they can sell their permits. Every firm then has a clear incentive to make reductions and avoid the cost of buying the licenses. Those for whom it is easiest do most, while those who find it harder have to pay. Through a system of tradeable permits, firms that can easily and cheaply cut their emissions will do so, because they can sell their remaining permits to other firms whose emissions are harder to reduce. Accordingly, cleaner companies benefit, while polluters are forced to pay to acquire additional permits. This puts them under pressure to cut back on their emission levels in order to maintain their profitability and competitiveness. If the nature of the production process makes it hard or very expensive for them to reduce emissions, they still can trade with other firms that have already made cuts. So the environment gains, either way, but firms which cannot reduce emissions are not closed down by strict regulations and even light polluters have a financial incentive to reduce their emissions even further.

There are active trading programs in several air pollutants. For greenhouse gases the largest is the European Union Emission Trading Scheme[8]. In the United States there is a national market to reduce acid rain and several regional markets in nitrogen oxides[9]. Markets for other pollutants tend to be smaller and more localized. The Acid Rain Program of the United States launched in 1995 allowed companies to trade permits in sulphur dioxide, which is mainly produced by power generators burning high-sulphur coal. The results have been better than planned. So far the initiative is ahead of target with participating firms reducing compliance costs by up to 50 per cent[10].

From a regulator's point of view, tradeable pollution permits provide greater certainty about pollution levels, provided the enforcement regime is sufficiently robust. The system costs less to administer than traditional regulation, and provides a clear commercial incentive on businesses to reduce emissions to the maximum amount that can be justified in terms of commercial cost-benefit. In effect, the buyer is paying a charge for polluting, while the seller is being rewarded for having reduced emissions.

Chevallier et al. model the costs of tradeable permits such that firms can decide if they want to sell or buy, i.e. whether to stay below epsilon or above. For every time t, firms

[7] See Parry [2002]

[8] See http://www.decc.gov.uk/en/content/cms/emissions/eu_ets/eu_ets.aspx

[9] See http://www.epa.gov/airmarkets/

[10] See http://www.epa.gov/airmarkets/progsregs/arp/

choose how many pollution permits they want to use, denoted by P_t. $\bar{P}_t$ are the permits allocated to firms at time t and S_t the permits bank computed as the difference between the initial permits endowment and the number of permits used by the firm, $S_t = \bar{P}_t - P_t$. The gains or costs resulting from tradeable permits are described by the following term:

$$q_{t+1}(\bar{P}_{t+1} + S_t - P_{t+1})$$

where q_{t+1} is the pollution permits price at time $t + 1$.

Chevallier et al. analyze the behavior of firms facing uncertainty with respect to political decisions concerning the permits program (permit price, allocation rules). It can happen that firms do not participate in tradeable permits markets due to the risk of political decision changes. It seems as if the performance of pollution permits is critically linked to the clarity of political decisions. In this work, however, uncertainty is completely neglected to concentrate on other things.

1.3 The Goal of this Thesis

This thesis tries to demonstrate different connections concerning firms production behavior and the environment. It presents a way how reality can be formulated in theory. This was one main task: to create a mathematical model that extends an already existing and represents a firm´s profit maximization problem and its relationship with exogenous variables such as the environmental standard imposed by government. A challenge was to model the possibility to buy and sell tradeable permits. The newly created model has furthermore another tool of control. The next goal was to analyze the model to learn about the firm´s behavior and its dependence on the exogenous variables. Because the problem is too complex to be solved analytically, efficient numerical methods were applied to solve the optimality conditions resulting from Pontryagin´s Maximum Principle. For a chosen set of parameter values, this leads to an unique equilibrium point which was further analyzed in a bifurcation analysis. The bifurcation analysis helped to understand and to describe the underlying relations. The results have been explained and compared with already existing literature on the topic.

Note that this thesis follows a strictly theoretical approach and no exact data have been used to derive the parameter values and functional forms so that reasonable results are obtained but conclusions for actual quantitative policy cannot be drawn. Note also that an economic as well as a mathematical model is always just a simple and abstract representation of reality. For the modeling here, I tried to find a way between realistic but at the same time not too complex to stay understandable and solvable. This was achieved by concentrating on the parts of interest and limiting the influencing factors by eliminating the effects which are not important for the analysis of the concrete topic.

1.4 The Organization of this Work

The thesis is organized as follows.

In Chapter 2 the basic model is introduced. After the ideas are presented, the model is formulated first in a rather general form and then with specified model functions, which are explained in detail.

Chapter 3 explains the chosen approach, namely the approach after Pontryagin´s Maximum Principle which is used to solve the optimal intertemporal decision problem of the firms. The canonical system is derived for different cases. Obtaining steady states and checking the sufficiency condition is not possible analytically and therefore only illustrated numerically.

Chapter 4 deals with the parameter values used for the numerical analysis and the numerical results for the steady states and the bifurcation analysis obtained with Newton´s method. Furthermore, the stability of the steady states are studied.

In Chapter 5, the basic model and its results from the numerical analysis are compared to other models in literature.

Finally, Chapter 6 concludes the document and gives a brief summary and discussion of the numerical results.

Chapter 2

The Basic Model

The here introduced basic model will be analyzed in the following chapters.

2.1 The Framework

The model considered is a microeconomic one. Not the economy as a whole, but the behavior of a single firm is analyzed. The firms represent the supply side of the market and concentrate on their profit. The firm is characterized by a production function which describes the firm´s level of technology. The profit of a firm is given as the difference between revenue and cost. The terms of the objective function are all monetary values.

Consider a perfectly competitive market[1] economy consisting of identical profit-maximizing firms run by capital-owning entrepreneurs using identical technologies to produce a single homogeneous good at each instant of time.

The analysis is carried out in an economic growth framework, where economic growth is driven by the accumulation of capital.

A single pollutant discharged into the environment is a by-product of the output production process. The representative firm can accumulate and produce with two dif-

[1] The term Competitive Market refers to a theoretical construct in economics. It is a simplified model of the market which should help to investigate and understand complex relationships. The main properties of this fictitious market are the following:

1. Infinite number of buyers and sellers which are all price-takers (no participant is large enough to have the market power to set the price of a certain product)
2. Free entry into and exit from the market for both buyers and sellers (zero entry and exit barriers such as costs)
3. Perfect information concerning the conditions of transaction such as price and quality of the products (transparency)
4. Homogeneous products (all units of products supplied by the different sellers are identical)

ferent types of capital: conventional or "brown" capital and "green" capital, which is less emission-intensive, but at the same time less productive and harder to accumulate and therefore more expensive. Additionally, the firm can avoid emissions by end-of-pipe-abatement, which means to clean up wastes after they have been generated, for example with CCS (Carbon Dioxide Capture and Storage) technologies which trap and store emitted CO2 (a possible abatement effort would be the investment in the development of such technologies)[2]. The amount of pollution released into the atmosphere may therefore differ from the amount produced during the production process. To provide an incentive for pollution reduction (achieved by investement in green capital and abatement activities which both are associated with monetary costs and would therefore not take place voluntarily), the firms have to meet some environmental constraint, in the form of emission charges or emission limits, set by the government. Whereas in the model of Rauscher (see Section 1.2.1) these standards are binding, here, the firms have the possibility to exceed the standards and pay for the difference (in fact, they are forced to buy some tradeable pollution permits in order to be in compliance) or they remain under the threshold and increase their profit by selling allowances. The so-called end-of-pipe-abatement will either make it possible to keep emissions within the specified limits or will reduce the total amount paid for emissions. With the decision of the value of the abatement share the entrepreneur decides himself, how much of the produced emissions he wants to abate. He can clean up more than the government requires (in order to make profits with pollution permits) or less, the abatement effort is completely in the hands of the entrepreneur. How much of the two types of capital the firm wants to accumulate and use, it decides through the Research&Development (R&D) expenses for the two types. Consequently, the firms´s decision is the allocation of resources between conventional R&D, green R&D and end-of-pipe-abatement, therefore the decision whether to concentrate on a clean production process by adopting more expensive cleaner technologies (and to spend less on end-of-pipe-abatement and tradeable permits to meet the standard) or to spend money at the end of the process in the form of cleaning or abatement costs or to accept the exceeding of the standard and spend money by buying permits. Summing up, the choice of the firm follows two steps depending on each other:

1. First, the firm has to determine the extend of R&D investments that are made for the two types of capital and the associated inputs, as well as production and pollution level. In doing so, the firm has to decide if it prefers to invest more in conventional or green capital or rather, which alternative is the better one in view

[2] At this point, it has to be mentioned that due to several reasons (e.g. unproven nature of the technology concerning long term effects, limited number of suitable locations for storage) CCS is a controversial technology. Another reason which opposes the use of CCS is the amount of R&D that will be needed if these technologies are used more intensely in the near future. These R&D capacities could instead be used for further development of renewable energy sources (e.g. making them more cost-efficient) or to increase energy efficiency (e.g. in the industrial sector).

of profit optimization. Conventional capital leads to more output (benefit), but also to more pollution (costs). Therefore, the use of conventional capital requires more abatement activity or pollution permits. Green capital produces less output and less pollution. As one can see, there is a trade-off between more expensive production with clean technologies and environmental costs (in form of abatement or permits) at the end of the process due to the use of brown capital.

2. Second, the firm has to decide how to handle produced pollution. It has the choice of internal emissions abatement efforts and tradeable pollution permits.

The stringency of environmental standard is as in Rauscher´s model still exogenous, but there will be an analysis of the effects of environmental policy on the firm´s optimal choices of productive and abatement inputs.

2.2 Mathematical Formulation

This section expresses the above formulated ideas in mathematical terms.

The problem considered is of the kind of an “optimal control model”, which means that the decision-maker optimizes in a dynamic context- the decisions today (or at a certain moment) have an impact on the problem of tomorrow (and the whole future). The system, described by so-called “state variables” which are functions of time, is dynamical since it evolves over time (following the “state dynamics”) and the process of change often depends on the decision variables. The decision variables are the decision-makers instrument of controlling the problem. The goal is to optimize a certain objective function over time (for example the utility of the decision-maker or the profit in case of a firm) with the help of these variables, which can be chosen freely within a certain frame. Because the problem is not a static one, it does not suffice to consider it at a certain moment in time, but the whole period of time (for example a few years) has to be taken into account. Therefore, the decision-maker has to keep not only the current impact of his decision (increased value of the objective function) in mind, but also how the situation (the state variables which describe the status of the system) changes because of his decision. The state variables reflect the accumulation of all historical decisions and developments. They start at a certain level (initial state) and change according to current levels of the control variables and current levels of the state variables themselves. Capital, for example, grows with investment (decision variable) and becomes less because of depreciation (depending on the state variable capital level). The function describing this relationship can depend, as well as the objective function, itself explicitly on the time variable. If they do not (as in this thesis), the problem is called autonomous. The dynamics, which describe the evolution of the state variables to be controlled, are formulated as differential equations, because we face a problem in continuous time (in discrete time they would be difference equation). For the same reason, the objective function is modeled in form of an integral (and not as a sum), summing up continuously the current utility or profit function over the given time period (here: infinity). Because present value of the utility/ profit function is normally seen as more worth than the value of the function at some future point, a discount rate is included. This rate can be the rate of interest in the case of monetary profit. Anyway it should reflect the time preference of the decision maker.

In summary, the problem consists of (three) control or decision variables (which are chosen by the decision maker), (two) state variables (which describe the system), (two) constraints in form of differential equations for the state variables (one for each), the so-called state equations, an objective function (which should be maximized by the decision-maker) and (two) initial conditions for the state variables (which give the state at starting time). In the following, these variables and functions will be introduced and discussed.

2.2.1 State Variables

The system is described by two state variables: the amount of conventional capital at time t, $K(t)$, and the amount of green capital at time t, $G(t)$.

One may expect that it becomes easier to accumulate capital, the higher the already existing stock of capital is on the one hand, and the greater the investment effort of R&D for this type of capital is on the other hand. Both types are therefore influenced by themselves and by the decisions of the decision maker (for more details see Section 2.2.4). The use of capital is necessary to produce an output (benefit)[3], but also produces pollution, which comes at a cost due to the required environmental standard set by the government. Per definition there are differences between the two types concerning both the output and the emission production. Whereas conventional capital produces more output than the cleaner, green type (and is therefore cheaper in production), its damages to the environment are higher. Furthermore, conventional capital is easier to accumulate than green capital. This makes sense insofar that conventional capital is more likely to be established in the economy. These differences are shown graphically in Figures 2.1-2.3. The production function can change due to changes in K and G as shown in Figure 2.1, where the dashed curve shows the impact of changes in G to F with a constant level of K and the other curve vice versa. F grows more with an increase of K as with an increase of G. Figure 2.2 depicts the dependences of emissions E on K and G which are similar to the dependences of F. The dashed line in Figure 2.3 shows the growth path of green capital plot against the current level of green capital, the other line shows the growth

[3]It is a common assumption that without input no output can be produced.

path of conventional capital as a function of conventional capital.

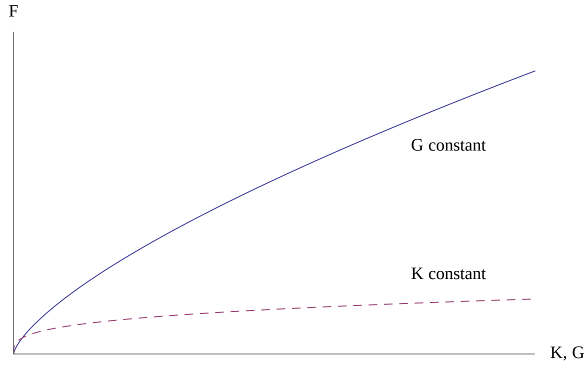


Figure 2.1: Dependence of the production function F on K and G.

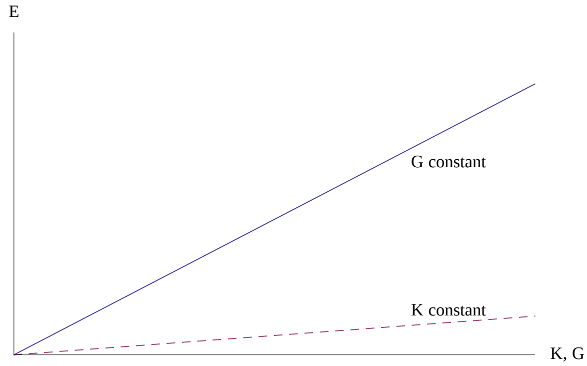


Figure 2.2: Dependence of the produced emissions E on K and G.

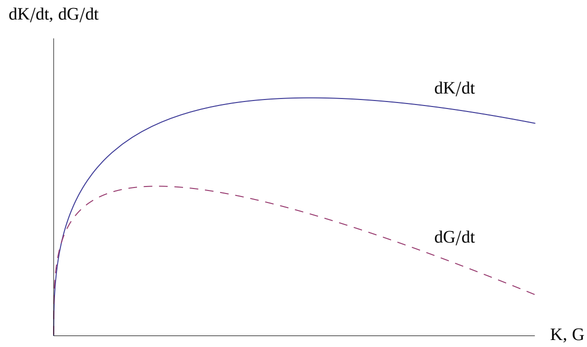


Figure 2.3: Growth paths of conventional and green capital with constant levels of investment.

2.2.2 Control Variables

The decision-maker has three variables to choose: $R_K(t)$, the investment for R&D to generate new capital of type K which can be positive or zero, $R_G(t)$, the investment in green capital which has to be positive or zero too, and $a(t)$, the share of abatement in the total amount of produced emissions $E(K(t), G(t))$: $0 \leq a \leq 1$. With the production input factors K and G, the firm not only generates output, but also pollution. To decrease the produced amount of emissions, investment in abatement is possible. How much the firm wants to reduce, it can decide by choosing a value of a. The abatement level a is a percentage and therefore takes values between 0 and 1 with 0 meaning that the firm does not abate anything and 1 that it abates all the pollution generated.

The implications of the taken decisions are modeled in the following subsections and are shortly explained here. Investments in the two types of capital increase the levels of K and G and therefore the production output (benefit) as well as the emission level (cost in form of pollution permits). Some further costs of investment which reduce the profit are opportunity and adjustment costs. Abatement a reduces the level of emissions released into the atmosphere (and therefore the amount of required pollution permits), but is not free of cost either. The abatement costs can be interpreted as some sort of opportunity costs too. With the resources spent on abatement, investment in capital could have been financed.

2.2.3 Terms of the Objective Function

The objective function, which has to be maximized by the decision-maker, is an integral over infinite time of the discounted total sum of the firm´s profit. A firm´s profit is calculated as the difference between revenues (here from production together with the potential benefit from selling tradeable permits, in case the firm has after the process of abatement a smaller level of emissions than allowed) and costs (here opportunity costs of research, adjustment costs of research, abatement costs and the costs of tradeable permits, if the firm has to buy some).

The discount rate r can take any value greater than zero and is exogenously given.

The production function F with its input factors $K(t)$ and $G(t)$ should be a well-behaved, neoclassical production function, satisfying the Inada conditions[4] to guarantee an interior equilibrium (in fact, they guarantee first, that capital is highly productive when scarce and barely productive when abundant and second, that both types of capital are necessary to produce an output)[5]. Per definition, conventional capital is more productive

[4]

$$\lim_{K\to 0} F_K(K,G) = \infty, \quad \lim_{K\to\infty} F_K(K,G) = 0, \quad F(0,G) = 0$$
$$\lim_{G\to 0} F_K(K,G) = \infty, \quad \lim_{G\to\infty} F_K(K,G) = 0, \quad F(K,0) = 0$$

[5]See Prettner [2010]

than green capital:

$$F = F(K(t), G(t)), \quad \frac{\partial F}{\partial K} > 0, \; \frac{\partial^2 F}{\partial K^2} < 0, \; \frac{\partial F}{\partial G} > 0, \; \frac{\partial^2 F}{\partial G^2} < 0, \; \frac{\partial F}{\partial K} > \frac{\partial F}{\partial G}.$$

Firms use therefore conventional and green capital to produce an output and they can decide how much of each type of capital they want to adopt.

The total amount of produced emissions E is modeled as a function of $K(t)$ and $G(t)$, which means that both types of capital produce emissions, whereas the production with the less pollution-intensive technology $G(t)$ leads per assumption to less emissions than the use of conventional technology $K(t)$:

$$E = E(K(t), G(t)), \quad \frac{\partial E}{\partial K} > \frac{\partial E}{\partial G} > 0.$$

The produced emissions $E(K(t), G(t))$ can be reduced by the abatement percentage rate $a(t)$. The amount of pollution abated is given by $a(t)E(K(t), G(t))$ and the total amount of "net emissions" (cf. Xepapadeas [1991]) is therefore

$$R(a(t), E(K(t), G(t))) = (1 - a(t))E(K(t), G(t))$$

with a positive first derivate with respect to E and a negative first derivative with respect to a

$$\frac{\partial R}{\partial E} = 1 - a > 0, \quad \frac{\partial R}{\partial a} = -E < 0.$$

The remaining emissions R (which are always positive) increase with an increasing E for constant a and decrease with an increasing a (see Figure 2.4(a) and Figure 2.4(b)). The level of the remaining emissions has to be compared with the given standard ϵ. The environmental quality ϵ determined by the government due to their required standards is considered as an exogenous given positive parameter of the total amount of allowed emissions, $\epsilon > 0$. $\epsilon = 0$ would mean the complete ban of pollution, which could lead to a high environmental quality, but makes production too costly for firms to be realistic. The emissions which are left after abatement activity, R, can be above or below ϵ. As the case may be, firms have to either pay for permission permits or they get a reward. Consider therefore the difference between ϵ and R

$$D(a, E) = \epsilon - R = \epsilon - (1 - a)E$$

with first partial derivatives

$$\frac{\partial D}{\partial E} = -(1-a) < 0, \quad \frac{\partial D}{\partial a} = E > 0.$$

Figure 2.4(a) and Figure 2.4(b) show the evolution of the remaining emissions in dependence of E, respectively a (the other variable is always set constant), as well as the constant parameter ϵ. The distance between these two curves gives the difference D for a certain level of E, respectively a. For a small E, the difference is positive (but for every level of E diminishing) until E reaches the level, where R equals ϵ (denoted with E'). For $E \geq E'$, the difference becomes negative- the firm produces more emissions then allowed. With a it is the other way round. A small a implies a negative difference, but the difference increases as a increases. For $a \geq a'$ (with a' denoting the level, where $R = \epsilon$), the difference is positive and still increasing. In Figures 2.5(a) and 2.5(b) the difference itself is plotted against E and a and the situation described above can be seen clearly. Given the definition of the difference D, one can formulate the "permits term" (with an exogenous positive scaling parameter p)

$$P(D) = pD^3 = p(\epsilon - (1-a)E)^3.$$

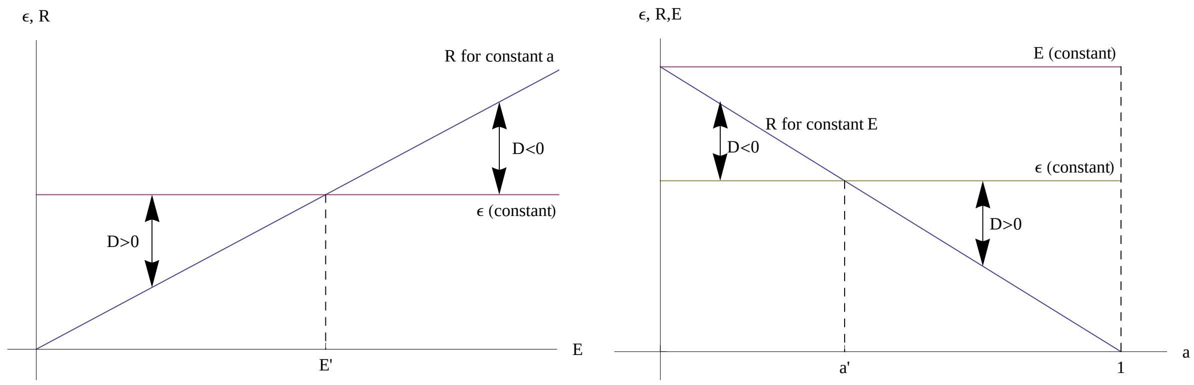
It gives either the penalty costs (one kind of costs in the objective function) which the firm has to pay if its remaining emissions are above ϵ (and the difference as well as the permits term is negative), or a monetary reward (which adds to the revenues of the profit) for a positive difference (in case the amount of remaining emissions is below the standard, which means that investments in green R&D and abatement are great enough to not only meet the norm but to even stay under the allowed pollution level). In this case, firms do not have to buy additional permits, but they can sell the ones they do not need and get a reward (the permits term is positive). The permits term has the following first partial derivatives

$$\frac{\partial P}{\partial D} = 3pD^2 > 0, \quad \frac{\partial P}{\partial E} = -3pD^2(1-a) < 0, \quad \frac{\partial P}{\partial a} = 3pD^2E > 0$$

which means that P increases with an increasing difference and becomes positive (and therefore a reward) for $D \geq 0$ (Figure 2.7). It decreases with an increasing E, changing sign for $E = E'$, and increases with an increasing a, changing from negative to positive for $a = a'$ (see Figures 2.6(a)-2.6(b)). The cubic form makes the reward (the costs) infinitely high with the difference approaching ∞ ($-\infty$): The second partial derivative of P with respect to the difference D is given by

$$\frac{\partial^2 P}{\partial D^2} = 6pD$$

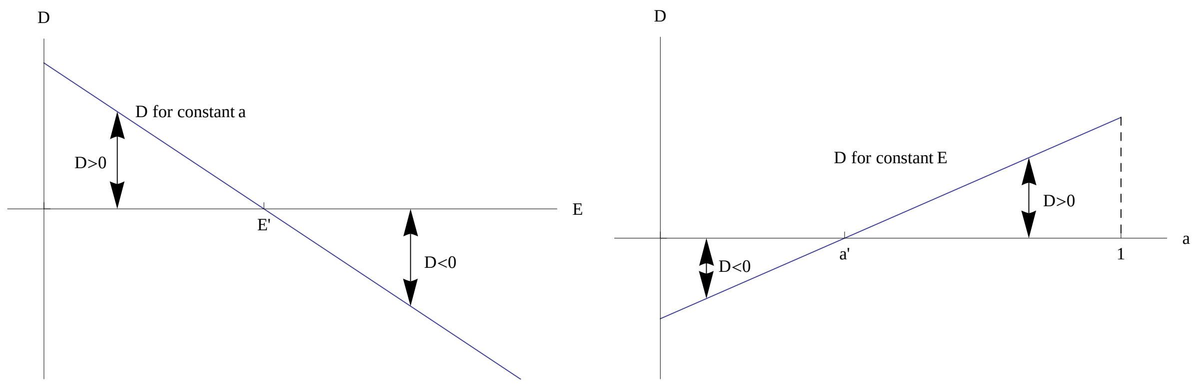
which is positive (and the function therefore convex) for a positive difference and vice versa for a negative difference D. For D between 0 and 1 (in case of a positive difference), the cubic term D^3 is less than D (in contrary to $D > 1$ where $D^3 > D$), which means that small deviations from the environmental standard are tolerated and only little penalized.



(a) Dependence of the Remaining Emissions R on E.

(b) Dependence of the Remaining Emissions R on a.

Figure 2.4: Dependence of the Remaining Emissions R on its input factors.



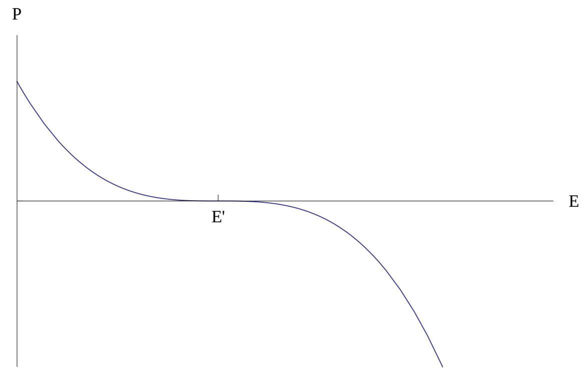
(a) Dependence of the Difference D on E.

(b) Dependence of the Difference D on a.

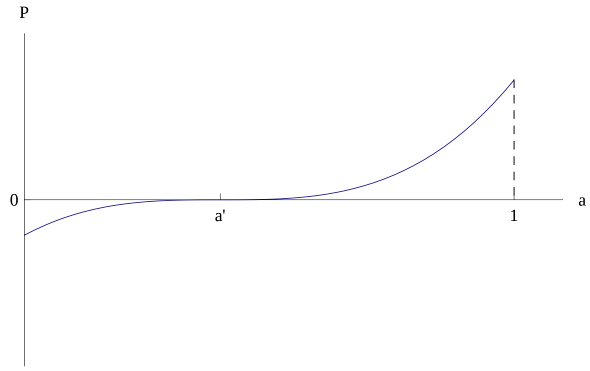
Figure 2.5: Dependence of the Difference D on its input factors.

Other costs are the opportunity costs of research which reflects the fact, that resources spend on investment in R&D could be used profitably otherwise. They are modeled as a fraction of the expenditures for R&D, $w(R_K(t) + R_G(t))$, with w being an exogenous given parameter between 0 and 1, $w \in (0, 1)$.

Furthermore, there are so-called adjustment costs (costs of making changes in the control variables $R_K(t)$ and $R_G(t)$) c_K and c_G, which are increasing functions of $R_K(t)$



(a) Dependence of the Permits Term P on E.



(b) Dependence of the Permits Term P on a.

Figure 2.6: Dependence of the Permits Term P on its input factors.

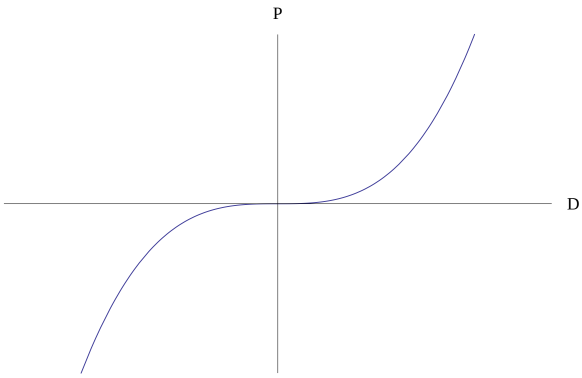


Figure 2.7: Dependence of the Permits Term P on D.

and $R_G(t)$:

$$c_K = c_K(R_K), \quad \frac{\partial c_K}{\partial R_K} > 0;$$
$$c_G = c_G(R_G), \quad \frac{\partial c_G}{\partial R_G} > 0.$$

The total abatement costs are given as χ, an increasing function of the firms´s effort of abatement $a(t)$ and produced emissions $E(K(t), G(t))$ (abatement becomes more expensive not only with increasing abatement effort, but also with a higher level of produced emissions which means it is more costly to abate the half of a big amount of emissions than the half of a small amount), which should be convex in a, so that the costs approach ∞ as the abatement effort a gets near 1:

$$\chi = \chi(a, E), \quad \frac{\partial \chi}{\partial E} > 0, \ \frac{\partial \chi}{\partial a} > 0, \ \frac{\partial^2 \chi}{\partial a^2} > 0.$$

The first unit abated is the cheapest one and every additional unit of abatement effort is less effective which is a quite realistic assumption. For a graphical demonstration, see

Figure 2.8 which shows the dependencies of χ on a and E.

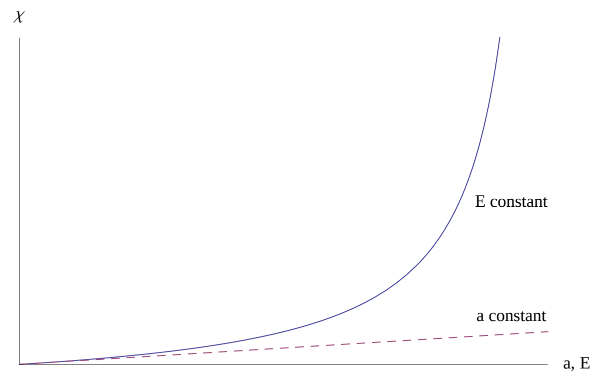


Figure 2.8: Dependence of abatement costs χ on a and E.

Consequently, the profit is given as

$$\begin{aligned}&F(K(t), G(t)) - w(R_K(t) + R_G(t)) - c_K(R_K(t)) - c_G(R_G(t)) - \\ &\chi(a(t), E(K(t), G(t))) + p(\epsilon - (1 - a(t))E(K(t), G(t)))^3\end{aligned}$$

and since the objective is the maximization of the discounted profit over time, the objective function is

$$\begin{aligned}\int_0^\infty e^{-rt}(&F(K(t), G(t)) - w(R_K(t) + R_G(t)) - c_K(R_K(t)) - c_G(R_G(t)) - \\ &\chi(a(t), E(K(t), G(t))) + p(\epsilon - (1 - a(t))E(K(t), G(t)))^3)\,\mathrm{d}t\end{aligned}$$

with r being the discount rate.

2.2.4 Terms of the Constraints

The process of capital accumulation is modeled such, that existing capital levels and investment in Research&Development (R&D) have a positive feedback on the accumulation of new capital and are necessary to accumulate new capital. Investment has an even greater positive feedback than capital which has additionally a negative feedback too in form of depreciation. Figure 2.9 shows the evolution of conventional capital due to a change in investment with constant level of capital and vice versa. Naturally, the growth path of green capital is similar, only with a flatter slope due to smaller partial elasticities. Moreover, brown capital is easier to accumulate than green capital. The model dynamics are described by the following differential equations[6]:

$$\dot{K}(t) = A(K(t), R_K(t))$$

[6] $\dot{K}(t)$ stands for the derivative of $K(t)$ with respect to time t: $\dot{K}(t) = \frac{\partial K(t)}{\partial t}$

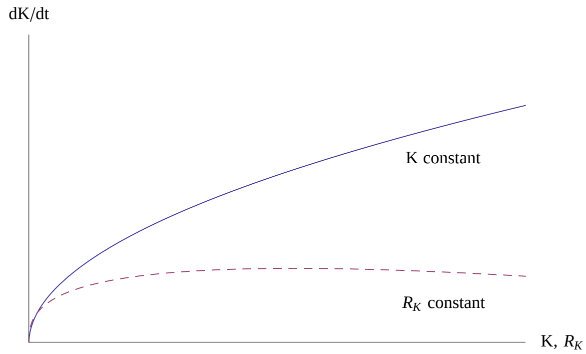


Figure 2.9: Growth path of conventional capital.

$$\dot{G}(t) = B(G(t), R_G(t))$$

with $A(.,.)$ and $B(.,.)$ being again well-behaved, neoclassical production functions, satisfying the Inada conditions[7]. Because the production function of capital accumulation is assumed to depend only on two input factors (capital and technological process) and the third one (labor) is neglected in this approach assuming that it is exogenous or predetermined, decreasing instead of constant returns to scale are required.

2.2.5 The General Model

Summing up, an infinite-horizon model of a profit-maximizing firm facing emission limits and tradeable permits is formulated. The representative capitalist-entrepreneur maximizes the present value of future profit

$$\int_0^\infty e^{-rt}(F(K(t),G(t)) - w(R_K(t) + R_G(t)) - c_K(R_K(t)) - c_G(R_G(t)) - \\ \chi(a(t), E(K(t),G(t))) + p(\epsilon - (1-a(t))E(K(t),G(t)))^3)\,\mathrm{d}t$$

where $F(K(t),G(t))$ denotes the production function, $w(R_K(t) + R_G(t))$ the opportunity costs, $c_K(R_K(t))$ and $c_G(R_G(t))$ the adjustment costs, $\chi(a(t), E(K(t),G(t)))$ the abatement costs and $p(\epsilon - (1-a(t))E(K(t),G(t)))^3$ the permits term.

The optimization takes place subject to the constraints concerning the development of the state variables

[7]

$$\lim_{K\to 0} A_K(K,R_K) = \infty, \quad \lim_{K\to\infty} A_K(K,R_K) = 0$$
$$\lim_{R_K\to 0} A_{R_K}(K,R_K) = \infty, \quad \lim_{R_K\to\infty} A_{R_K}(K,R_K) = 0$$
$$\lim_{G\to 0} B_G(G,R_G) = \infty, \quad \lim_{G\to\infty} B_G(G,R_G) = 0$$
$$\lim_{R_G\to 0} B_{R_G}(G,R_G) = \infty, \quad \lim_{R_G\to\infty} B_{R_G}(G,R_G) = 0$$

$$\dot{K(t)} = A(K(t), R_K(t)),$$

$$\dot{G(t)} = B(G(t), R_G(t)),$$

the non-negativity conditions for the controls

$$R_K(t) \geq 0,$$

$$R_G(t) \geq 0,$$

$$a(t) \geq 0$$

and the considered initial values K_0 and G_0 of state variables

$$K(0) = K_0,$$

$$G(0) = G_0.$$

The optimal control model then is given as follows[8]:

$$\max_{R_K, R_G, a} \int_0^\infty e^{-rt}(F(K,G) - w(R_K + R_G) - c_K(R_K) - c_G(R_G) - \chi(a, E(K,G)) + p(\epsilon - (1-a)E(K,G))^3) \,\mathrm{d}t \tag{2.1}$$

s.t.:

$$\dot{K} = A(K, R_K) \tag{2.1a}$$

$$\dot{G} = B(G, R_G) \tag{2.1b}$$

$$R_K \geq 0 \tag{2.1c}$$

$$R_G \geq 0 \tag{2.1d}$$

$$a \geq 0 \tag{2.1e}$$

$$K(0) = K_0 \tag{2.1f}$$

$$G(0) = G_0 \tag{2.1g}$$

All definitions of introduced variables can be found in Table 2.1.

[8] Note that as of here, the time argument t is often omitted (if there is no ambiguity) for the ease of notation.

Variable	Name	Description	Constraints
K	conventional capital	function of time t	
R_K	investment in conventional capital	function of time t	$R_K \geq 0$
G	green capital	function of time t	
R_G	investment in green capital	function of time t	$R_G \geq 0$
a	percentage rate of abatement	function of time t	$0 \leq a \leq 1$
λ_1	costate associated with K	function of time t	
λ_2	costate associated with G	function of time t	
F	production function	function of K and G	
w	opportunity cost of R&D	exogenous parameter	$0 < w < 1$
c_K	adjustment costs of R_K	function of R_K	
c_G	adjustment costs of R_G	function of R_G	
χ	abatement costs	function of a and $E(K, G)$	
p	scaling parameter of the permits term	exogenous parameter	$p > 0$
ϵ	total amount of "allowed" emissions	exogenous parameter	$\epsilon > 0$
E	emission function	function of K and G	
r	discount rate	exogenous parameter	$r > 0$
A	accumulation function of K	function of K and R_K	
B	accumulation function of G	function of G and R_G	

Table 2.1: Overview of introduced variables.

2.3 Functional Forms

In this section, the functions of the basic model formulated in the previous section will be specified satisfying the necessary properties from above. Additionally, the properties, the parameters of the functions have to fulfill, will be established.

Table 2.2 shows all the introduced functions and their specified forms. In the following, there is an explanation of how the functional forms are chosen.

Function	Form	Name
$F(K, G)$	$fK^{\alpha_1}G^{\alpha_2}$	production function
$\chi(a, E)$	$c\frac{a}{1-a}E(K, G)$	abatement costs
$A(K, R_K)$	$dK^{\delta_1}R_K^{\delta_2} - \phi K$	accumulation function of K
$B(G, R_G)$	$bG^{\sigma_1}R_G^{\sigma_2} - \psi G$	accumulation function of G
$E(K, G)$	$\kappa K + \gamma G$	emission function
$c_K(R_K)$	kR_K^2	adjustment costs of R_K
$c_G(R_G)$	gR_G^2	adjustment costs of R_G

Table 2.2: Functional forms in the basic model.

2.3.1 Objective Function

For the production function F a strictly concave[9] (the second unit of input should be less productive than the first) Cobb-Douglas function is taken, because it satisfies all the requirements. Inputs are the two types of capital. Each of them is therefore essential to produce a positive output and has positive, but diminishing returns.

$$F(K,G) = fK^{\alpha_1}G^{\alpha_2} \text{ with } f > 0, 0 < \alpha_2 \leq \alpha_1 < 1, \alpha_1 + \alpha_2 \leq 1$$

α_1 and α_2 specify the partial elasticities of production[10]. Conventional capital is considered to be more productive than green capital (reflected by the assumption $\alpha_2 \leq \alpha_1$), see Figure 2.1. This means that the output increases more or the same, if the input of conventional capital increases by one percent, as if the input of green capital is raised by one percent. Returns to scale[11] can be both constant or decreasing. The term Returns to Scale describes how an output changes if all inputs increase by a constant factor. Constant returns to scale means that output increases by the same proportional change, decreasing returns to scale describe an output which increases by less than that proportional change. Returns to scale faced by a firm are purely technologically imposed and not influenced by economic decisions, therefore exogenously given in this model. The constant f is only a scale parameter[12] and has to be greater than 0, $f > 0$, so that output cannot become negative or zero, without one of the input variables being zero.

The total amount of produced emissions is modeled linear as

$$E(K,G) = \kappa K + \gamma G \text{ with } 0 < \gamma < \kappa < 1$$

The emission intensities κ and γ should both be positive, because both types of capital produce emissions during the production process. However, brown capital is considered to be more pollutive ($\kappa > \gamma$), see Figure 2.2. This is the main difference between the two types of capital- they are differently pollutive. Furthermore, they are differently productive and differently easy to accumulate.

For the adjustment costs a quadratic function is used, as usual, with positive scaling

9

$$\begin{aligned} F_{KK} &= f\alpha_1(\alpha_1-1)K^{\alpha_1-2}G^{\alpha_2} < 0, \quad \text{because} \quad \alpha_1 < 1 \\ F_{GG} &= f\alpha_2(\alpha_2-1)K^{\alpha_1}G^{\alpha_2-2} < 0, \quad \text{because} \quad \alpha_2 < 1 \end{aligned}$$

10 The Elasticity of a function $f(z)$ is defined as $\frac{\partial f(z)}{\partial z}\frac{z}{f(z)}$ and indicates how many percents the output changes, if the input increases by one percent.

11 $F(\zeta K, \zeta G) = f\zeta^{\alpha_1+\alpha_2}K^{\alpha_1}G^{\alpha_2} \leq \zeta F(K,G) = f\zeta K^{\alpha_1}G^{\alpha_2}$, because $\alpha_1 + \alpha_2 \leq 1$

12 Scaling parameters only determine the absolute level of a function and have no qualitative influence.

parameters k respectively g. Note that the controls are integrated in the objective function in a nonlinear way. Otherwise, the model (2.1) would be a singular control problem and a so-called bang-bang solution, where the controls jump from one boundary to another, would be optimal.

$$c_K(R_K) = kR_K^2 \text{ with } k > 0$$

$$c_G(R_G) = gR_G^2 \text{ with } g > 0$$

The abatement costs, χ, depending on a and $E(K,G)$, are set as

$$\chi(a, E(K,G)) = c\frac{a}{1-a}E(K,G) \text{ with } c > 0 \tag{2.2}$$

c is again a scaling parameter, which has to be positive. Note that, as required, χ is increasing in a and $E(K,G)$ and convex in a[13]. Furthermore, the abatement costs are linear in E[14], so that the costs increase proportional with the level of produced emissions, and the cross-derivation is positive[15]. Note, that the function $\chi(a,E)$ is not defined for $a \geq 1$. For $a > 1$, the costs would be negative which makes no sense and for $a = 1$, the denominator would be zero. Thus, there are no boundary arc solutions with $a = 1$ and the search for solutions can be limited to the interior.

13

$$\begin{aligned}
\chi_a(a, E(K,G)) &= c\frac{1}{(1-a)^2}E(K,G) > 0 \\
\chi_E(a, E(K,G)) &= c\frac{a}{1-a} > 0 \\
\chi_{aa}(a, E(K,G)) &= 2c\frac{1}{(1-a)^3}E(K,G) > 0
\end{aligned}$$

14

$$\chi_{EE}(a, E(K,G)) = 0$$

15

$$\chi_{Ea}(a, E(K,G)) = \chi_{aE}(a, E(K,G)) = c\frac{1}{(1-a)^2} > 0$$

which means that the marginal cost of E increases with an increase of a and vice versa.

2.3.2 Constraints

To model the R&D output I assume again a strictly concave[16] Cobb Douglas production function with decreasing returns to scale ($\delta_1+\delta_2<1$, $\sigma_1+\sigma_2<1$) for the same reasons as above for the production function F. Here capital stock and investment are the production factors (and both necessary to produce capital[17]). The controls R_K and R_G are assumed to be positive, so that no disinvestment in capital is possible. Whereas the already existing amount of conventional respectively green capital and the expenditure for R&D increase the capital stocks (whereas investments in R&D increase the capital more than the existing stock of capital do, taken into account by the assumptions $\delta_1 \le \delta_2$ and $\sigma_1 \le \sigma_2$ and shown graphically in Figure 2.9), at the same time they are reduced by the constant and positive depreciation rates ϕ and ψ. Depreciation of capital is a popular and very realistic assumption for all kinds of capital (human capital, monetary capital, technical capital etc.). It yields the fact that without investment the capital stock decreases[18]. To summarize, capital accumulates via investments and depreciates at rate ϕ, respectively ψ. The result of all the described assumptions are the following differential equations for the state dynamics.

$$\dot{K} = A(K,R_K) = dK^{\delta_1}R_K^{\delta_2} - \phi K \text{ with } d>0, 0<\phi<1, 0<\delta_1\le\delta_2<1, \delta_1+\delta_2<1$$

$$\dot{G} = B(G,R_G) = bG^{\sigma_1}R_G^{\sigma_2} - \psi G \text{ with } b>0, 0<\psi<1, 0<\sigma_1\le\sigma_2<1, \sigma_1+\sigma_2<1$$

Per definition of the differences between the two types of capital, brown capital K is easier to accumulate than green capital. The requirements $\delta_1 \ge \sigma_1$ and $\delta_2 \ge \sigma_2$ take this into account.

16

$$\begin{aligned}
A_{KK} &= d\delta_1(\delta_1-1)K^{\delta_1-2}R_K^{\delta_2} < 0, \quad \text{because} \quad \delta_1<1\\
A_{R_KR_K} &= d\delta_2(\delta_2-1)K^{\delta_1}R_K^{\delta_2-2} < 0, \quad \text{because} \quad \delta_2<1\\
B_{GG} &= b\sigma_1(\sigma_1-1)G^{\sigma_1-2}R_G^{\sigma_2} < 0, \quad \text{because} \quad \sigma_1<1\\
B_{R_GR_G} &= b\sigma_2(\sigma_2-1)G^{\sigma_1}R_G^{\sigma_2-2} < 0, \quad \text{because} \quad \sigma_2<1
\end{aligned}$$

17

$$A(0,R_K)=0,\ A(K,0)<0,\ B(0,R_G)=0,\ B(G,0)<0$$

18

$$A(K,0)<0,\ B(G,0)<0$$

2.3.3 The General Model with Specified Functions

The optimal control model- including the specified functional forms- is given as follows:

$$\max_{R_K,R_G,a} \int_0^\infty e^{-rt}(fK^{\alpha_1}G^{\alpha_2} - w(R_K + R_G) - kR_K^2 - gR_G^2 - c\frac{a}{1-a}(\kappa K + \gamma G) + p(\epsilon - (1-a)(\kappa K + \gamma G))^3)\,\mathrm{d}t \tag{2.3}$$

s.t.:

$$\dot{K} = dK^{\delta_1}R_K^{\delta_2} - \phi K \tag{2.3a}$$

$$\dot{G} = bG^{\sigma_1}R_G^{\sigma_2} - \psi G \tag{2.3b}$$

$$R_K \geq 0 \tag{2.3c}$$

$$R_G \geq 0 \tag{2.3d}$$

$$a \geq 0 \tag{2.3e}$$

$$K(0) = K_0 \tag{2.3f}$$

$$G(0) = G_0 \tag{2.3g}$$

Chapter 3

Analysis of the Basic Model

Solving the problem means to find an admissible control (R_K, R_G, a) maximizing the objective function (2.3) subject to the state dynamics (2.3a)-(2.3b), the non-negativity constraints (2.3c)-(2.3e) and the initial conditions (2.3f)-(2.3g). The discounted autonomous model with infinite planning horizon will be solved with Pontryagin´s Maximum Principle which can be applied to dynamic, time continuously, nonlinear optimization problems as this one. The qualitative analysis includes the computation of steady states as well as the determination of stability. It provides a valuable structural insight into the shape of the optimal paths.

3.1 Canonical System

Because the optimal control problem does not consist only of the objective function (2.3) and the state dynamics (2.3a)- (2.3b) but there are also inequality constraints in form of non-negativity conditions (2.3c)-(2.3e), the Lagrangian $\mathcal{L}$ instead of the Hamiltonian $\mathcal{H}$ has to be considered in a first step:

$$\mathcal{L} = \mathcal{H} + \mu\mathcal{C}$$

where $\mathcal{H}$ denotes the current value[1] Hamiltonian, μ the Lagrangian multiplicator and $\mathcal{C}$ the non-negativity constraint. The Hamiltonian summarizes all effects incurred by a change of the control. These are the direct effect resulting from a change in the objective function (current benefit) and the indirect one given by the state constraint (benefit in the future). The second effect is weighted by the so-called costate (also called adjoint variable and shadow price) which also changes over time t and measures the value of a marginal increment in the associated state at time t when moving along the optimal trajectory. It therefore expresses the highest (hypothetical) price that the decision-maker is willing

[1] The present value Hamiltonian evaluates the optimal behavior at the time the optimization is actually done. All future effects are therefore discounted contrary to the current value Hamiltonian.

to pay in the optimum for an additional unit of the state variable at time t (see Grass et al. [2008], p.118). The current value Hamiltonian $\mathcal{H}$ with costate variables (λ_1, λ_2) is formulated as

$$\begin{aligned}\mathcal{H}(K,G,R_K,R_G,a,\lambda_1,\lambda_2) &= F(K,G) - w(R_K+R_G) - c_K(R_K) - c_G(R_G) - \chi(a,E(K,G)) + \\ &\quad p(\epsilon-(1-a)E(K,G))^3 + \lambda_1 A(K,R_K) + \lambda_2 B(G,R_G) \\ &= fK^{\alpha_1}G^{\alpha_2} - w(R_K+R_G) - kR_K^2 - gR_G^2 - \\ &\quad c\frac{a}{1-a}(\kappa K+\gamma G) + p(\epsilon-(1-a)(G\gamma+K\kappa))^3 + \\ &\quad \lambda_1\left(dK^{\delta_1}R_K^{\delta_2} - \phi K\right) + \lambda_2\left(bG^{\sigma_1}R_G^{\sigma_2} - \psi G\right)\end{aligned}$$

which yields the Lagrangian $\mathcal{L}$ (in current-value notation)

$$\begin{aligned}\mathcal{L}(K,G,R_K,R_G,a,\lambda_1,\lambda_2,\mu_1,\mu_2,\mu_3) &= F(K,G) - w(R_K+R_G) - c_K(R_K) - c_G(R_G) - \\ &\quad \chi(a,E(K,G)) + p(\epsilon-(1-a)E(K,G))^3 + \\ &\quad \lambda_1 A(K,R_K) + \lambda_2 B(G,R_G) + \\ &\quad \mu_1 R_K + \mu_2 R_G + \mu_3 a \\ &= fK^{\alpha_1}G^{\alpha_2} - w(R_K+R_G) - kR_K^2 - gR_G^2 - \\ &\quad c\frac{a}{1-a}(\kappa K+\gamma G) + p(\epsilon-(1-a)(\kappa K+\gamma G))^3 + \\ &\quad \lambda_1\left(dK^{\delta_1}R_K^{\delta_2} - \phi K\right) + \lambda_2\left(bG^{\sigma_1}R_G^{\sigma_2} - \psi G\right) + \\ &\quad \mu_1 R_K + \mu_2 R_G + \mu_3 a.\end{aligned}$$

The necessary First Order Conditions[2] can be stated as

$$\mathcal{L}_{R_K}(K,R_K,\lambda_1,\mu_1) = -w - 2kR_K + \lambda_1\delta_2 dK^{\delta_1}R_K^{\delta_2-1} + \mu_1 = 0 \tag{3.1a}$$

$$\mathcal{L}_{R_G}(G,R_G,\lambda_2,\mu_2) = -w - 2gR_G + \lambda_2\sigma_2 bG^{\sigma_1}R_G^{\sigma_2-1} + \mu_2 = 0 \tag{3.1b}$$

$$\mathcal{L}_a(K,G,a,\mu_3) = -c\frac{1}{(1-a)^2}E(K,G) + 3pE(K,G)(\epsilon-(1-a)E(K,G))^2 + \mu_3 = 0 \tag{3.1c}$$

$$\begin{aligned}\dot{\lambda_1}(K,G,R_K,a,\lambda_1) &= r\lambda_1 - \mathcal{L}_K(K,G,R_K,a,\lambda_1) = r\lambda_1 - (f\alpha_1 K^{\alpha_1-1}G^{\alpha_2} - c\frac{a}{1-a}\kappa - \\ &\quad 3p(\epsilon-(1-a)E(K,G))^2(1-a)\kappa + \lambda_1(d\delta_1 K^{\delta_1-1}R_K^{\delta_2} - \phi))\end{aligned} \tag{3.1d}$$

$$\begin{aligned}\dot{\lambda_2}(K,G,R_G,a,\lambda_2) &= r\lambda_2 - \mathcal{L}_G(K,G,R_G,a,\lambda_2) = r\lambda_2 - (f\alpha_2 K^{\alpha_1}G^{\alpha_2-1} - c\frac{a}{1-a}\gamma - \\ &\quad 3p(\epsilon-(1-a)E(K,G))^2(1-a)\gamma + \lambda_2(b\sigma_1 G^{\sigma_1-1}R_G^{\sigma_2} - \psi))\end{aligned} \tag{3.1e}$$

[2]The necessary conditions only imply that the resulting control is an extremum. That it is optimal indeed the Sufficiency Condition has to hold as well, see Section 3.4.

where subscripts denote partial derivatives of multivariate functions and dots the derivative with respect to time. The Complementary Slackness Conditions are

$$\mu_1 \geq 0, \quad \mu_1 R_K = 0 \tag{3.2a}$$

$$\mu_2 \geq 0, \quad \mu_2 R_G = 0 \tag{3.2b}$$

$$\mu_3 \geq 0, \quad \mu_3 R_a = 0 \tag{3.2c}$$

To derive the canonical system it has to be distinguished between the case of an interior arc and the case of a boundary arc. The first case describes a solution in which all of the non-negativity constraints are inactive, whereas in the second case at least one of the constraints is active which means that either R_K or R_G or a or two or all three of them are 0. Since it is easier to understand, I start with the interior arc first.

3.1.1 Interior Arc

If the controls R_K, R_G and a are all strictly greater than zero, the complementary slackness conditions (3.2a)-(3.2c) imply that the Lagrangian multiplicators μ_1, μ_2 and μ_3 have to be zero, $\mu_1 = \mu_2 = \mu_3 = 0$, and the Lagrangian $\mathcal{L}$ reduces to the Hamiltonian $\mathcal{H}$. Therefore, the Hamiltonian will be maximized

$$(R_K^*, R_G^*, a^*) = arg \underset{R_K, R_G, a}{max} \mathcal{H}$$

$$\mathcal{H}_{R_K}(K, R_K, \lambda_1) = -w - 2kR_K + d\lambda_1\delta_2 K^{\delta_1} R_K^{\delta_2 - 1} = 0 \tag{3.3a}$$

$$\mathcal{H}_{R_G}(G, R_G, \lambda_1) = -w - 2gR_G + b\lambda_2\sigma_2 G^{\sigma_1} R_G^{\sigma_2 - 1} = 0 \tag{3.3b}$$

$$\mathcal{H}_a(K, G, a) = -c\frac{1}{(1-a)^2}(\kappa K + \gamma G) + 3p(\kappa K + \gamma G)(\epsilon - (1-a)(\kappa K + \gamma G))^2 = 0 \tag{3.3c}$$

$$\begin{aligned}\dot{\lambda}_1(K, G, R_K, a, \lambda_1) = r\lambda_1 - \mathcal{H}_G(K, G, R_K, a, \lambda_1) = r\lambda_1 - (f\alpha_1 K^{\alpha_1 - 1} G^{\alpha_2} - c\frac{a}{1-a}\kappa - \\ 3p(\epsilon - (1-a)E(K,G))^2(1-a)\kappa + \lambda_1(d\delta_1 K^{\delta_1 - 1} R_K^{\delta_2} - \phi))\end{aligned} \tag{3.3d}$$

$$\begin{aligned}\dot{\lambda}_2(K, G, R_G, a, \lambda_2) = r\lambda_2 - \mathcal{H}_G(K, G, R_G, a, \lambda_2) = r\lambda_2 - (f\alpha_2 K^{\alpha_1} G^{\alpha_2 - 1} - c\frac{a}{1-a}\gamma - \\ 3p(\epsilon - (1-a)E(K,G))^2(1-a)\gamma + \lambda_2(b\sigma_1 G^{\sigma_1 - 1} R_G^{\sigma_2} - \psi))\end{aligned} \tag{3.3e}$$

Because a maximum is required, the Legendre-Clebsch Condition[3] must hold, which says that the Hessian matrix

$$D^2\mathcal{H}(K^*, G^*, R_K^*, R_G^*, a^*, \lambda_1^*, \lambda_2^*) = \begin{pmatrix} \mathcal{H}_{R_K R_K} & \mathcal{H}_{R_K R_G} & \mathcal{H}_{R_K a} \\ \mathcal{H}_{R_G R_K} & \mathcal{H}_{R_G R_G} & \mathcal{H}_{R_G a} \\ \mathcal{H}_{a R_K} & \mathcal{H}_{a R_G} & \mathcal{H}_{aa} \end{pmatrix} = \begin{pmatrix} \mathcal{H}_{R_K R_K} & 0 & 0 \\ 0 & \mathcal{H}_{R_G R_G} & 0 \\ 0 & 0 & \mathcal{H}_{aa} \end{pmatrix}$$

has to be negative semidefinite. Sylvester's criterion gives necessary and sufficient conditions for a matrix being negative-definite. It says that a Hermitian matrix H is negative-definite if and only if the leading principal minors are alternately negative and positive. The leading principal minors are the determinants of the following sub-matrices:

$$\begin{aligned} S_1 &= \begin{pmatrix} \mathcal{H}_{R_K R_K} \end{pmatrix} \\ S_2 &= \begin{pmatrix} \mathcal{H}_{R_K R_K} & 0 \\ 0 & \mathcal{H}_{R_G R_G} \end{pmatrix} \\ S_3 &= \begin{pmatrix} \mathcal{H}_{R_K R_K} & 0 & 0 \\ 0 & \mathcal{H}_{R_G R_G} & 0 \\ 0 & 0 & \mathcal{H}_{aa} \end{pmatrix}. \end{aligned}$$

For the determinants, respectively the minors, should hold:

$$M_1 = \mathcal{H}_{R_K R_K} < 0 \tag{3.4a}$$
$$M_2 = \mathcal{H}_{R_K R_K}\mathcal{H}_{R_G R_G} > 0 \tag{3.4b}$$
$$M_3 = \mathcal{H}_{R_K R_K}\mathcal{H}_{R_G R_G}\mathcal{H}_{aa} < 0. \tag{3.4c}$$

From Equation (3.4a) it follows that $\mathcal{H}_{R_G R_G} < 0$ must hold to fulfill Equation (3.4b). With $\mathcal{H}_{R_K R_K} < 0$ and $\mathcal{H}_{R_G R_G} < 0$, $\mathcal{H}_{aa} < 0$ must hold too for assuring Equation (3.4c). A maximum is therefore assured by the following equations

$$\mathcal{H}_{R_K R_K} = -2k + \lambda_1^*\delta_2(\delta_2 - 1)d(K^*)^{\delta_1}(R_K^*)^{\delta_2 - 2} < 0 \tag{3.5a}$$
$$\mathcal{H}_{R_G R_G} = -2g + \lambda_2^*\sigma_2(\sigma_2 - 1)b(G^*)^{\sigma_1}(R_G^*)^{\sigma_2 - 2} < 0 \tag{3.5b}$$
$$\mathcal{H}aa = -2c\frac{1}{(1-a^*)^3}(\kappa K^* + \gamma G^*) + 6p(\kappa K^* + \gamma G^*)^2(\epsilon - (1-a^*)(\kappa K^* + \gamma G^*)) < 0. \tag{3.5c}$$

From Equation (3.3a) $\lambda_1^*(K, R_K)$ can be obtained,

$$\lambda_1^*(K, R_K) = \frac{K^{\delta_1}R_K^{1-\delta_2}(2kR_K + w)}{d\delta_2},$$

[3] See Grass et al. [2008], p. 113

which is positive for every admissible parameter set ($k > 0$, $0 < w < 1$, $d > 0$, $0 < \delta_2 < 1$). Since $\delta_2 < 1$, Equation (3.5a) is satisfied for all admissible parameter sets. Solving Equation (3.3b) for $\lambda_2^*(G, R_G)$ yields

$$\lambda_2^*(G, R_G) = \frac{G^{\sigma_1} R_G^{1-\sigma_2}(2gR_G + w)}{b\sigma_2},$$

which is also positive. It is assumed that $\sigma_2 < 1$ too and therefore Equation (3.5b) is satisfied for all admissible parameter sets. Equation (3.3c) is of fourth degree in a, whereas two solutions can be excluded (see Appendix). The remaining two are:

$$a_2^*(K, G) = 1 - \frac{\epsilon - \sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}}{2(\kappa K + \gamma G)},$$
$$a_4^*(K, G) = 1 - \frac{\epsilon + \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}}{2(\kappa K + \gamma G)}.$$

Inserting $a^*(K, G)$ into Equation (3.5c) yields

$$\mathcal{H}_{aa}(a_2^*) = -\frac{8(\kappa K + \gamma G)^3}{\left(\epsilon - \sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}\right)^3} \left(4cE - \sqrt{3cp}\epsilon \left(\epsilon - \sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}\right)\right) < 0 \tag{3.6a}$$

$$\mathcal{H}_{aa}(a_4^*) = -\frac{8(\kappa K + \gamma G)^3}{\left(\epsilon + \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}\right)^3} \left(4cE + \sqrt{3cp}\epsilon \left(\epsilon + \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}\right)\right) < 0. \tag{3.6b}$$

Whereas Equation (3.6b) is fulfilled for every parameter set, Equation (3.6a) has to be checked for every solution obtained. However, note that (3.5c) can be satisfied only for $a < 1$ (which is always true for a_4^*), since E and ϵ are positive. This makes sense in so far, as the abatement costs (2.2) are not defined for $a \geq 1$.

Normally, the canonical system is derived in the state-costate-space. The approach would be the following:

1. Solving the first order conditions (3.3a)-(3.3c) for the controls R_K, R_G and a. The controls are then given as functions of the states (K, G) and costates (λ_1, λ_2):

$$H_{R_K}(K, R_K, \lambda_1) = 0 \Rightarrow R_K(K, \lambda_1)$$
$$H_{R_G}(G, R_G, \lambda_2) = 0 \Rightarrow R_G(G, \lambda_2)$$
$$H_a(K, G, a) = 0 \Rightarrow a(K, G)$$

2. Inserting these functions into the state (Equations (2.3a) and (2.3b)) and costate dynamics (Equations (3.3d) and (3.3e)) yields the four-dimensional canonical system in the state-costate-space

$$\begin{aligned}\dot{K}(K, R_K(K, \lambda_1)) &= \dot{K}(K, \lambda_1)\\ \dot{G}(G, R_G(G, \lambda_2)) &= \dot{G}(G, \lambda_2)\\ \dot{\lambda}_1(K, G, R_K(K, \lambda_1), a(K, G), \lambda_1) &= \dot{\lambda}_1(K, G, \lambda_1)\\ \dot{\lambda}_2(K, G, R_G(G, \lambda_2), a(K, G), \lambda_2) &= \dot{\lambda}_2(K, G, \lambda_2)\end{aligned}$$

Because the Equations (3.3a)-(3.3b) appear to involve the control variables R_K and R_G in an essentially non-algebraic way, it is not possible to eliminate them explicitly and to generally formulate the canonical system in the state-costate-space. Since the number of controls (three) is not the same as the number of states (two) and therefore as the number of costates, it is neither possible to derive the canonical system in the state-control-space by replacing the costate dynamics with control dynamics. Instead, the canonical system has to be considered in the state-costate-control-space with four differential equations (for the state and costates) and three algebraic equations.

State-Costate-Control-Space

The canonical system consists of seven differential algebraic equations: the two state dynamics (2.3a)-(2.3b), the two costate dynamics (3.3d)-(3.3e) and the three algebraic Equations (3.3a)-(3.3c). The seven-dimensional canonical system is given as

$$\dot{K}(K, R_K) = dK^{\delta_1}R_K^{\delta_2} - K\phi \tag{3.7a}$$

$$\dot{G}(G, R_G) = bG^{\sigma_1}R_G^{\sigma_2} - G\psi \tag{3.7b}$$

$$\begin{aligned}\dot{\lambda}_1(K, G, R_K, a, \lambda_1) = r\lambda_1 - (f\alpha_1 K^{\alpha_1 - 1}G^{\alpha_2} - c\frac{a}{1-a}\kappa-\\ 3p(\epsilon - (1-a)E(K,G))^2(1-a)\kappa + \lambda_1(d\delta_1 K^{\delta_1-1}R_K^{\delta_2} - \phi))\end{aligned} \tag{3.7c}$$

$$\begin{aligned}\dot{\lambda}_2(K, G, R_G, a, \lambda_2) = r\lambda_2 - (f\alpha_2 K^{\alpha_1}G^{\alpha_2 - 1} - c\frac{a}{1-a}\gamma-\\ 3p(\epsilon - (1-a)E(K,G))^2(1-a)\gamma + \lambda_2(b\sigma_1 G^{\sigma_1-1}R_G^{\sigma_2} - \psi))\end{aligned} \tag{3.7d}$$

$$\mathcal{H}_{R_K}(K, R_K, \lambda_1) = -w - 2kR_K + d\lambda_1\delta_2 K^{\delta_1}R_K^{\delta_2 - 1} = 0 \tag{3.7e}$$

$$\mathcal{H}_{R_G}(G, R_G, \lambda_1) = -w - 2gR_G + b\lambda_2\sigma_2 G^{\sigma_1}R_G^{\sigma_2 - 1} = 0 \tag{3.7f}$$

$$\mathcal{H}_a(K, G, a) = -c\frac{1}{(1-a)^2}(\kappa K + \gamma G) + 3p(\kappa K + \gamma G)(\epsilon - (1-a)(\kappa K + \gamma G))^2 = 0 \tag{3.7g}$$

The canonical system together with the non-negativity conditions

$$R_K \geq 0$$
$$R_G \geq 0$$
$$a \geq 0$$

describe the economic system.

State-Costate-Space: Special Case $\delta_2 = \sigma_2 = \frac{1}{2}$

For the special case of $\delta_2 = \sigma_2 = \frac{1}{2}$ in $\dot{K} = dK^{\delta_1} R_K^{\delta_2} - \phi K$ and $\dot{G} = bG^{\sigma_1} R_G^{\sigma_2} - \psi G$ which means that R_G has the same production elasticity in $\dot{G}$ as R_K in $\dot{K}$, the controls can be eliminated from (3.3a)-(3.3c) and the canonical system can be formulated in the state-costate-space. The procedure is the following: Express R_K and R_G in terms of the states and costates λ_1 and K, respectively λ_2 and G by solving (3.3a) and (3.3b) for R_K and R_G. Three solutions are obtained, but two of them are complex, so the admissible controls as function of states and co-states are well-defined as:

$$R_K^*(K, \lambda_1) = \frac{1.67989 w^2}{k \sqrt[3]{6912 d^2 k \lambda_1^2 K^{2\delta_1} + 3762.42 \sqrt{k} K^{\delta_1} \lambda_1 d \sqrt{3.375 d^2 k \lambda_1^2 K^{2\delta_1} + w^3} + 1024 w^3}} + \frac{0.0165354 \sqrt[3]{6912 d^2 k \lambda_1^2 K^{2\delta_1} + 3762.42 \sqrt{k} K^{\delta_1} \lambda_1 d \sqrt{3.375 d^2 k \lambda_1^2 K^{2\delta_1} + w^3} + 1024 w^3}}{k} - \frac{0.333333 w}{k} \tag{3.8}$$

$$R_G^*(G, \lambda_2) = \frac{1.67989 w^2}{g \sqrt[3]{6912 b^2 g \lambda_2^2 G^{2\sigma_1} + 3762.42 \sqrt{g} G^{\sigma_1} \lambda_2 b \sqrt{3.375 b^2 g \lambda_2^2 G^{2\sigma_1} + w^3} + 1024 w^3}} + \frac{0.0165354 \sqrt[3]{6912 b^2 g \lambda_2^2 G^{2\sigma_1} + 3762.42 \sqrt{g} G^{\sigma_1} \lambda_2 b \sqrt{3.375 b^2 g \lambda_2^2 G^{2\sigma_1} + w^3} + 1024 w^3}}{g} - \frac{0.333333 w}{g}. \tag{3.9}$$

Solving Equation (3.3c) for a yields four solutions:

$$a_1^*(K,G) = 1 - \frac{\epsilon + \sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}}{2(\kappa K + \gamma G)}, \tag{3.10a}$$

$$a_2^*(K,G) = 1 - \frac{\epsilon - \sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}}{2(\kappa K + \gamma G)}, \tag{3.10b}$$

$$a_3^*(K,G) = 1 - \frac{\epsilon - \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}}{2(\kappa K + \gamma G)}, \tag{3.10c}$$

$$a_4^*(K,G) = 1 - \frac{\epsilon + \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}}{2(\kappa K + \gamma G)}. \tag{3.10d}$$

Solution (3.10a) and (3.10c) are not admissible, because the first one cannot be real and a maximum at the same time and the third is not a maximum if it is less than 1, as required (see Appendix).

The optimal intertemporal evolution of the system is given by the dynamic system formed by the four differential Equations (2.3a), (2.3b), (3.3d), (3.3e), taking into account the temporary equilibrium conditions (3.8), (3.9), (3.10b) or (3.10d),

$$\begin{aligned}
\dot{K}(K, R_K^*(K,\lambda_1)) &= dK^{\delta_1} R_K^*(K,\lambda_1)^{\delta_2} - K\phi \\
\dot{G}(G, R_G^*(G,\lambda_2)) &= bG^{\sigma_1} R_G^*(G,\lambda_2)^{\sigma_2} - G\psi \\
\dot{\lambda}_1(K, G, R_K^*(K,\lambda_1), a^*(K,G), \lambda_1) &= c\frac{a^*(K,G)}{1-a^*(K,G)}\kappa - f\alpha_1 G^{\alpha_2} K^{\alpha_1 - 1} + r\lambda_1 + \\
&\quad 3(1-a^*(K,G))p\kappa(\epsilon - (1-a^*(K,G))(\kappa K + \gamma G))^2 - \\
&\quad \lambda_1 \left(d\delta_1 K^{\delta_1 - 1} R_K^*(K,\lambda_1)^{\delta_2} - \phi\right) \\
\dot{\lambda}_2(K, G, R_G^*(G,\lambda_2), a^*(K,G), \lambda_2) &= c\frac{a^*(K,G)}{1-a^*(K,G)}\gamma - f\alpha_2 G^{\alpha_2 - 1} K^{\alpha_1} + r\lambda_2 + \\
&\quad 3(1-a^*(K,G))p\gamma(\epsilon - (1-a^*(K,G))(\kappa K + \gamma G))^2 - \\
&\quad \lambda_2 \left(b\sigma_1 G^{\sigma_1 - 1} R_G^*(G,\lambda_2)^{\sigma_2} - \psi\right)
\end{aligned}$$

The two canonical systems with inserted $a(K,G)$ (two possibilities), $R_K(K,\lambda_1)$ and $R_G(G,\lambda_2)$ are shown in Appendix.

3.1.2 Boundary Arc

If one of the multiplicators gets greater than zero, the corresponding optimal control has to take its boundary value and does not necessarily maximize the Hamiltonian, but the Lagrangian. To demonstrate the derivation of the canonical system in the boundary arc case, assume that all multiplicators can be greater than zero and there is zero investment in conventional capital, $R_K = 0$, zero investment in green capital, $R_G = 0$, and zero investment in abatement, $a = 0$.

The necessary conditions (3.1a)-(3.1e) become then

$$\frac{\partial \mathcal{L}}{\partial R_K} = -w + \mu_1 = 0 \tag{3.11a}$$

$$\frac{\partial \mathcal{L}}{\partial R_G} = -w + \mu_2 = 0 \tag{3.11b}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial a} &= -\chi_a(0, E(K,G)) + 3pE(K,G)(\epsilon - E(K,G))^2 + \mu_3 = \\ &= -cE + 3pE(\epsilon - E)^2 + \mu_3 = 0\end{aligned} \tag{3.11c}$$

$$\begin{aligned}\dot{\lambda}_1 &= r\lambda_1 - \mathcal{L}_K = r\lambda_1 - (F_K(K,G) - \chi_E(0, E(K,G))E_K(K,G) - \\ &\quad 3p(\epsilon - E(K,G))^2 E_K(K,G) + \lambda_1 A_K(K,0)) = \\ &= r\lambda_1 - F_K(K,G) + 3p(\epsilon - E)^2\kappa + \lambda_1\phi\end{aligned} \tag{3.11d}$$

$$\begin{aligned}\dot{\lambda}_2 &= r\lambda_2 - \mathcal{L}_G = r\lambda_2 - (F_G(K,G) - \chi_E(0, E(K,G))E_G(K,G) - \\ &\quad 3p(\epsilon - E(K,G))^2 E_G(K,G) + \lambda_2 B_G(G,0)) = \\ &= r\lambda_2 - F_G(K,G) + 3p(\epsilon - E)^2\gamma + \lambda_2\psi.\end{aligned} \tag{3.11e}$$

The canonical system reduces to

$$\dot{K}(K,0) = -\phi K, \tag{3.12a}$$

$$\dot{G}(G,0) = -\psi G, \tag{3.12b}$$

$$\dot{\lambda}_1 = r\lambda_1 - F_K(K,G) + 3p(\epsilon - E)^2\kappa + \lambda_1\phi, \tag{3.12c}$$

$$\dot{\lambda}_2 = r\lambda_2 - F_G(K,G) + 3p(\epsilon - E)^2\gamma + \lambda_2\psi. \tag{3.12d}$$

3.2 Steady States

Steady states denote equilibrium solutions (also called fixed points, stationary points or critical values)[4] of the canonical system, which means that the controls and states do not change (any more), but stay constant forever. If the system starts exactly at one of these special points, it will remain there. If it starts elsewhere, it will converge into a steady state and do not change any more from the moment on, the system reaches the steady state. The term steady state refers therefore to the long-run (asymptotic) behavior of the dynamical system. The stationary solution(s) $(\hat{K}, \hat{G}, \hat{R_K}, \hat{R_G}, \hat{a})$ of the system (2.3), if existing, are obtained by setting all the equations of the canonical system zero and

[4]See Grass et al. [2008], p.12

solving them simultaneously for states and controls.

In the inner case this would mean to solve the seven-dimensional system of equations

$$
\begin{aligned}
0 &= dK^{\delta_1} R_K^{\delta_2} - K\phi \\
0 &= bG^{\sigma_1} R_G^{\sigma_2} - G\psi \\
0 &= r\lambda_1 - (f\alpha_1 K^{\alpha_1 - 1} G^{\alpha_2} - c\frac{a}{1-a}\kappa - 3p(\epsilon - (1-a)E(K,G))^2(1-a)\kappa + \\
&\quad \lambda_1(d\delta_1 K^{\delta_1 - 1} R_K^{\delta_2} - \phi)) \\
0 &= r\lambda_2 - (f\alpha_2 K^{\alpha_1} G^{\alpha_2 - 1} - c\frac{a}{1-a}\gamma - 3p(\epsilon - (1-a)E(K,G))^2(1-a)\gamma + \\
&\quad \lambda_2(b\sigma_1 G^{\sigma_1 - 1} R_G^{\sigma_2} - \psi)) \\
0 &= -w - 2kR_K + d\lambda_1\delta_2 K^{\delta_1} R_K^{\delta_2 - 1} \\
0 &= -w - 2gR_G + b\lambda_2\sigma_2 G^{\sigma_1} R_G^{\sigma_2 - 1} \\
0 &= -c\frac{1}{(1-a)^2}E(K,G) + 3pE(K,G)(\epsilon - (1-a)E(K,G))^2
\end{aligned}
$$

for the controls R_K, R_G and a, the states K and G and the costates λ_1 and λ_2, respectively the four-dimensional system of equations

$$
\begin{aligned}
0 &= dK^{\delta_1} R_K(K,\lambda_1)^{\delta_2} - K\phi \\
0 &= bG^{\sigma_1} R_G(G,\lambda_2)^{\sigma_2} - G\psi \\
0 &= c\frac{a(K,G)}{1-a(K,G)}\kappa - f\alpha_1 G^{\alpha_2} K^{\alpha_1 - 1} + r\lambda_1 + \\
&\quad 3(1-a(K,G))p\kappa(\epsilon - (1-a(K,G))(\kappa K + \gamma G))^2 - \\
&\quad \lambda_1\left(d\delta_1 K^{\delta_1 - 1} R_K(K,\lambda_1)^{\delta_2} - \phi\right) \\
0 &= c\frac{a(K,G)}{1-a(K,G)}\gamma - f\alpha_2 G^{\alpha_2 - 1} K^{\alpha_1} + r\lambda_2 + \\
&\quad 3(1-a(K,G))p\gamma(\epsilon - (1-a(K,G))(\kappa K + \gamma G))^2 - \\
&\quad \lambda_2\left(b\sigma_1 G^{\sigma_1 - 1} R_G(G,\lambda_2)^{\sigma_2} - \psi\right)
\end{aligned}
$$

for the special case of $\delta_2 = \sigma_2 = \frac{1}{2}$. Both systems cannot be solved analytically and have therefore be considered with concrete parameter values to get a numeric solution (see Chapter 4).

In the boundary arc case, steady states can be found easily by setting the state dynamics and the adjoint equations zero and solving for K, G, λ_1 and λ_2 simultaneously. From the state dynamics (Equation (3.12a) and (3.12b)), it follows that

$$\hat{K} = \hat{G} = 0$$

in the steady state. The adjoint Equations (Equation (3.12c) and (3.12d)) yield

$$\hat{\lambda_1} = -\frac{3p\epsilon^2\kappa}{\phi + r}$$

$$\hat{\lambda_2} = -\frac{3p\epsilon^2\gamma}{\psi + r}.$$

From the First Order Conditions (3.11a)-(3.11c), the steady state values of the Lagrange multiplicators can be derived: $\hat{\mu_1} = w$, $\hat{\mu_2} = w$, $\hat{\mu_3} = 0$. Note that all of them are greater or equal 0 as required.

3.3 Stability

The local stability of an equilibrium point refers to its qualitative behavior and can be read off the eigenvalues of the Jacobian evaluated at the equilibrium. In the case of the canonical system in the state-costate-space, the Jacobian has the following form:

$$J = \begin{pmatrix} \dot{K}_K & \dot{K}_G & \dot{K}_{\lambda_1} & \dot{K}_{\lambda_2} \\ \dot{G}_K & \dot{G}_G & \dot{G}_{\lambda_1} & \dot{G}_{\lambda_2} \\ \dot{\lambda}_{1K} & \dot{\lambda}_{1G} & \dot{\lambda}_{1\lambda_1} & \dot{\lambda}_{1\lambda_2} \\ \dot{\lambda}_{2K} & \dot{\lambda}_{2G} & \dot{\lambda}_{2\lambda_1} & \dot{\lambda}_{2\lambda_2} \end{pmatrix} = \begin{pmatrix} \dot{K}_K & 0 & \dot{K}_{\lambda_1} & 0 \\ 0 & \dot{G}_G & 0 & \dot{G}_{\lambda_2} \\ \dot{\lambda}_{1K} & \dot{\lambda}_{1G} & \dot{\lambda}_{1\lambda_1} & 0 \\ \dot{\lambda}_{2K} & \dot{\lambda}_{2G} & 0 & \dot{\lambda}_{2\lambda_2} \end{pmatrix}$$

The Eigenvalues are determined by solving the characteristic polynomial

$$D(J - \zeta E_4) = 0$$

for ζ, where E_4 denotes the four-dimensional identity matrix and D the determinant of the matrix

$$J - \zeta E_4 = \begin{pmatrix} \dot{K}_K & 0 & \dot{K}_{\lambda_1} & 0 \\ 0 & \dot{G}_G & 0 & \dot{G}_{\lambda_2} \\ \dot{\lambda}_{1K} & \dot{\lambda}_{1G} & \dot{\lambda}_{1\lambda_1} & 0 \\ \dot{\lambda}_{2K} & \dot{\lambda}_{2G} & 0 & \dot{\lambda}_{2\lambda_2} \end{pmatrix} - \zeta \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \dot{K}_K - \zeta & 0 & \dot{K}_{\lambda_1} & 0 \\ 0 & \dot{G}_G - \zeta & 0 & \dot{G}_{\lambda_2} \\ \dot{\lambda}_{1K} & \dot{\lambda}_{1G} & \dot{\lambda}_{1\lambda_1} - \zeta & 0 \\ \dot{\lambda}_{2K} & \dot{\lambda}_{2G} & 0 & \dot{\lambda}_{2\lambda_2} - \zeta \end{pmatrix}.$$

For the boundary arc case, the following eigenvalues are obtained:

$$
\begin{aligned}
J^* - \zeta E_4 &= \begin{pmatrix} -\phi-\zeta & 0 & 0 & 0 \\ 0 & -\psi-\zeta & 0 & 0 \\ -6p\epsilon\kappa^2 & -6p\epsilon\kappa\gamma & r+\phi-\zeta & 0 \\ -6p\epsilon\kappa\gamma & -6p\epsilon\gamma^2 & 0 & r+\psi-\zeta \end{pmatrix} \\
D(J^* - \zeta E_4) &= (-\phi-\zeta)(-\psi-\zeta)(r+\phi-\zeta)(r+\psi-\zeta) \\
\zeta_1 &= -\phi \\
\zeta_2 &= -\psi \\
\zeta_3 &= r+\phi \\
\zeta_4 &= r+\psi
\end{aligned}
$$

with J^* being the Jacobian valuated at the considered steady state. The unique equilibrium where all three controls are at their boundaries is therefore a saddle point with a two-dimensional stable manifold, since the number of positive eigenvalues as well as the number of negative eigenvalues are greater than zero[5] and equal.

Since for the inner case, the equilibrium cannot be computed analytically, the stability will only be determined numerically in the next chapter.

3.4 Sufficiency Conditions

The First Order Conditions only yield candidates for an optimal solution. To be sure, that a candidate is in fact an optimal solution, one has to check the Sufficiency Condition[6], which requires that the maximized Hamiltonian $\mathcal{H}^*$, i.e. the Hamiltonian with the maximizing control, is concave in the state variable(s). In the multidimensional case as here, this means that the Hessian matrix of $\mathcal{H}^*$

$$D^2\mathcal{H}^* = \begin{pmatrix} \mathcal{H}^*_{KK} & \mathcal{H}^*_{KG} \\ \mathcal{H}^*_{GK} & \mathcal{H}^*_{GG} \end{pmatrix}$$

[5] see Grass et al. [2008], p.45

[6] In fact, it is called "The Arrow Sufficiency Conditions for Infinite Time Horizon", see Grass et al. [2008], p. 159.

has to be negative semidefinite. Applying again Sylvester's criterion, this is assured by the following conditions:

$$\mathcal{H}^*_{KK} = 6(1-a^*)^2 p\kappa^2(\epsilon - (1-a^*)E) + d\,(\delta_1 - 1)\,\delta_1\lambda_1 K^{\delta_1 - 2} {R^*_K}^{\delta_2} + f\,(\alpha_1 - 1)\,\alpha_1 G^{\alpha_2} K^{\alpha_1 - 2} < 0,$$

$$\mathcal{H}^*_{GG} = 6(1-a^*)^2 p\gamma^2(\epsilon - (1-a^*)E) + d\,(\sigma_1 - 1)\,\sigma_1\lambda_2 G^{\sigma_1 - 2} {R^*_G}^{\sigma_2} + f\,(\alpha_2 - 1)\,\alpha_2 G^{\alpha_2 - 2} K^{\alpha_1} < 0,$$

$$\mathcal{H}^*_{KK}\mathcal{H}^*_{GG} > (\mathcal{H}^*_{KG})^2 \Leftrightarrow$$

$$\Big(6(a^*-1)^2 p\kappa^2((a^*-1)E + \epsilon) + d\lambda_1\,(\delta_1 - 1)\,\delta_1 K^{\delta_1 - 2} {R^*_K}^{\delta_2} + f\,(\alpha_1 - 1)\,\alpha_1 G^{\alpha_2} K^{\alpha_1 - 2}\Big)$$
$$\big(6(a^*-1)^2 p\gamma^2((a^*-1)E + \epsilon) + b\lambda_2\,(\sigma_1 - 1)\,\sigma_1 G^{\sigma_1 - 2} {R^*_G}^{\sigma_2} + f\,(\alpha_2 - 1)\,\alpha_2 G^{\alpha_2 - 2} K^{\alpha_1}\big) >$$
$$\big(6(a^*-1)^2 p\gamma\kappa((a^*-1)E + \epsilon) + f\alpha_1\alpha_2 G^{\alpha_2 - 1} K^{\alpha_1 - 1}\big)^2$$

To summarize, analytically only in the boundary arc case an equilibrium could be obtained for which the stability was determined. For the inner case, the analysis follows numerically in the next chapter.

Chapter 4

Numerical Results

Because the canonical system (both the seven-dimensional as well as the four-dimensional) cannot be solved analytically, I searched for solutions numerically with Newton´s method. Newton´s method is a method for finding approximations to the roots of a real-valued function.

4.1 Parameter Values

To investigate the behavior of the model (steady states, local stability) and to carry out bifurcation analysis, the values for the parameters have to be specified. The parameter values used in a first analysis are listed in Table 4.1. They are chosen in view of the values used in the thesis of Moser [2010] in order to allow a comparison (see Section 5.2). Note that all of the chosen parameter values satisfy the necessary requirements, formulated in the previous subsection and listed also in Table 4.1.

4.2 Steady States

The steady states of the problem are determined by solving

$$\dot{K}(K, R_K) = 0 \tag{4.1a}$$

$$\dot{G}(G, R_G) = 0 \tag{4.1b}$$

$$\dot{\lambda}_1(K, G, R_K, a, \lambda_1) = 0 \tag{4.1c}$$

$$\dot{\lambda}_2(K, G, R_G, a, \lambda_2) = 0 \tag{4.1d}$$

$$\frac{\partial \mathcal{H}}{\partial R_K}(K, R_K, \lambda_1) = 0 \tag{4.1e}$$

$$\frac{\partial \mathcal{H}}{\partial R_G}(G, R_G, \lambda_2) = 0 \tag{4.1f}$$

$$\frac{\partial \mathcal{H}}{\partial a}(K, G, a) = 0 \tag{4.1g}$$

Parameter	Value	Description	Constraints
b	1	scale parameter of B	$b > 0$
c	1	scale parameter of χ	$c > 0$
d	1	scale parameter of A	$d > 0$
f	1	scale parameter of F	$f > 0$
k	1	scale parameter of c_K	$k > 0$
g	1	scale parameter of c_G	$g > 0$
p	1	scale parameter of the permits term	$p > 0$
r	0.05	discount rate	$r > 0$
w	0.1	opportunity cost of research	$0 < w < 1$
α_1	0.7	production elasticity of K in F	$0 < \alpha_2 \leq \alpha_1 < 1, \alpha_1 + \alpha_2 \leq 1$
α_2	0.3	production elasticity of G in F	$0 < \alpha_2 \leq \alpha_1 < 1, \alpha_1 + \alpha_2 \leq 1$
δ_1	0.3	production elasticity of K in $\dot{K}$	$0 < \delta_1 \leq \delta_2 < 1, \delta_1 + \delta_2 < 1, \sigma_1 \leq \delta_1$
δ_2	0.5	production elasticity of R_K in $\dot{K}$	$0 < \delta_1 \leq \delta_2 < 1, \delta_1 + \delta_2 < 1, \sigma_2 \leq \delta_2$
σ_1	0.3	production elasticity of G in $\dot{G}$	$0 < \sigma_1 \leq \sigma_2 < 1, \sigma_1 + \sigma_2 < 1, \sigma_1 \leq \delta_1$
σ_2	0.4	production elasticity of R_G in $\dot{G}$	$0 < \sigma_1 \leq \sigma_2 < 1, \sigma_1 + \sigma_2 < 1, \sigma_2 \leq \delta_2$
ϕ	0.05	depreciation rate of $\dot{K}$	$0 < \phi < 1$
ψ	0.05	depreciation rate of $\dot{G}$	$0 < \psi < 1$
κ	0.7	emission intensity of K	$0 < \gamma < \kappa < 1$
γ	0.1	emission intensity of G	$0 < \gamma < \kappa < 1$
ϵ	10	total amount of "allowed" emissions	$\epsilon > 0$

Table 4.1: Parameter values in the basic model.

simultaneously. The functional forms are shown in Equations (3.7a)-(3.7g). From Equation (4.1a) R_K can be expressed as a function of K, from Equation (4.1b) R_G as a function of G:

$$dK^{\delta_1} R_K^{\delta_2} - \phi K = 0 \Rightarrow R_K(K) = \left(\frac{\phi K^{1-\delta_1}}{d} \right)^{\frac{1}{\delta_2}} \tag{4.2}$$

$$bG^{\sigma_1} R_G^{\sigma_2} - \psi G = 0 \Rightarrow R_G(G) = \left(\frac{\psi G^{1-\sigma_1}}{b} \right)^{\frac{1}{\sigma_2}} \tag{4.3}$$

From $\mathcal{H}_{R_K} = 0$ and $\mathcal{H}_{R_G} = 0$, $\lambda_1(K, R_K)$ and $\lambda_2(G, R_G)$ are obtained. Inserting $R_K(K)$ and $R_G(G)$ yields the following functions of K and G:

$$\lambda_1(K) = \frac{K^{\frac{1-\delta_1}{\delta_2}} \left(\frac{\phi}{d}\right)^{\frac{1-\delta_2}{\delta_2}} \left(2k \left(\frac{\phi K^{1-\delta_1}}{d}\right)^{\frac{1}{\delta_2}} + w\right)}{d\delta_2} \tag{4.4}$$

$$\lambda_2(G) = \frac{G^{\frac{1-\sigma_1}{\sigma_2}} \left(\frac{\psi}{b}\right)^{\frac{1-\sigma_2}{\sigma_2}} \left(2g \left(\frac{\psi G^{1-\sigma_1}}{b}\right)^{\frac{1}{\sigma_2}} + w\right)}{b\sigma_2} \tag{4.5}$$

Solving $H_a = 0$ for $a(K,G)$ yields four solutions for a. Two of them are not admissible (see Appendix). The other two are:

$$a_2^*(K,G) = 1 - \frac{\epsilon - \sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}}{2(\kappa K + \gamma G)},$$
$$a_4^*(K,G) = 1 - \frac{\epsilon + \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}}{2(\kappa K + \gamma G)}.$$

The remaining equations from the canonical system are Equation (4.1c) and (4.1d). Combining these two with Equations (4.2)-(4.5) yields two systems (one for each $a(K,G)$), each with two equations depending on the state variables K and G. The first system is given as:

$$\dot{\lambda}_1(K,G) = c\kappa\left(\frac{2E}{\epsilon - \sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}E}} - 1\right) + \frac{\kappa\sqrt{3cp}}{2}\left(\sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}E} + \epsilon\right) -$$
$$f\alpha_1 K^{\alpha_1 - 1}G^{\alpha_2} + \frac{K^{\frac{1-\delta_1}{\delta_2}}\left(\frac{\phi}{d}\right)^{\frac{1-\delta_2}{\delta_2}}\left(2k\left(\frac{\phi K^{1-\delta_1}}{d}\right)^{\frac{1}{\delta_2}} + w\right)}{d\delta_2}(r + \phi - \delta_1\phi) = 0$$
$$\dot{\lambda}_2(K,G) = c\gamma\left(\frac{2E}{\epsilon - \sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}E}} - 1\right) + \frac{\gamma\sqrt{3cp}}{2}\left(\sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}E} + \epsilon\right) -$$
$$f\alpha_2 K^{\alpha_1 -}G^{\alpha_2 - 1} + \frac{G^{\frac{1-\sigma_1}{\sigma_2}}\left(\frac{\psi}{b}\right)^{\frac{1-\sigma_2}{\sigma_2}}\left(2g\left(\frac{\psi G^{1-\sigma_1}}{b}\right)^{\frac{1}{\sigma_2}} + w\right)}{b\sigma_2}(r + \psi - \sigma_1\psi) = 0$$

The second system (with $a_4(K,G)$) is

$$\dot{\lambda}_1(K,G) = c\kappa\left(\frac{2E}{\epsilon + \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}E}} - 1\right) + \frac{\kappa\sqrt{3cp}}{2}\left(\sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}E} - \epsilon\right) -$$
$$f\alpha_1 K^{\alpha_1 - 1}G^{\alpha_2} + \frac{K^{\frac{1-\delta_1}{\delta_2}}\left(\frac{\phi}{d}\right)^{\frac{1-\delta_2}{\delta_2}}\left(2k\left(\frac{\phi K^{1-\delta_1}}{d}\right)^{\frac{1}{\delta_2}} + w\right)}{d\delta_2}(r + \phi - \delta_1\phi) = 0$$
$$\dot{\lambda}_2(K,G) = c\gamma\left(\frac{2E}{\epsilon + \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}E}} - 1\right) + \frac{\gamma\sqrt{3cp}}{2}\left(\sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}E} - \epsilon\right) -$$
$$f\alpha_2 K^{\alpha_1 -}G^{\alpha_2 - 1} + \frac{G^{\frac{1-\sigma_1}{\sigma_2}}\left(\frac{\psi}{b}\right)^{\frac{1-\sigma_2}{\sigma_2}}\left(2g\left(\frac{\psi G^{1-\sigma_1}}{b}\right)^{\frac{1}{\sigma_2}} + w\right)}{b\sigma_2}(r + \psi - \sigma_1\psi) = 0$$

Figure 4.1 shows the second systems graphically for the parameters in Table 4.1. The first system has no admissible solution. For the parameters in Table 4.1, only

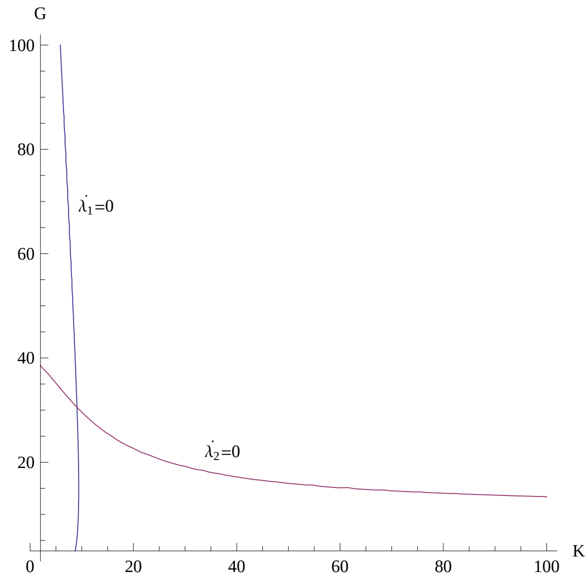


Figure 4.1: Contour Plot of $\dot{\lambda_1}(K,G)=0$ and $\dot{\lambda_2}(K,G)=0$ for the second system.

one steady state can therefore be found which lies at $\hat{K} = 12.9027$, $\hat{G} = 33.6878$, $\hat{a} = 0.139488$, $\hat{R_K} = 0.089721$, $\hat{R_G} = 0.263331$, $\hat{\lambda_1} = 0.0777256$, $\hat{\lambda_2} = 0.244925$. The terms of the objective function take the following values: $F(\hat{K},\hat{G}) = 17.2075$, $w(\hat{R_K}+\hat{R_G}) = 0.0353052$, $c_K(\hat{R_K}) = 0.00804985$, $c_G(\hat{R_G}) = 0.0693431$, $\chi(\hat{a}, E(\hat{K},\hat{G})) = 2.01014$ and $p(\epsilon - (1-\hat{a})E(\hat{K},\hat{G})^3 = -0.302028$. The profit is then $F(\hat{K},\hat{G}) - w(\hat{R_K}+\hat{R_G}) - c_K(\hat{R_K}) - c_G(\hat{R_G}) - \chi(\hat{a}, E(\hat{K},\hat{G})) + p(\epsilon - (1-\hat{a})E(\hat{K},\hat{G})^3 = 14.7827$. In the steady state, the produced emissions are $E(\hat{K},\hat{G}) = 12.4007$ and the emissions released into the atmosphere (net or remaining emissions) are $(1-\hat{a})E(\hat{K},\hat{G}) = 10.6709$. The eigenvalues of the Jacobian are given as: $\zeta_1 = 0.647362$, $\zeta_2 = -0.604724$, $\zeta_3 = 0.167452$ and $\zeta_4 = -0.0941876$. Since the number of eigenvalues with a positive real part as well as the number of eigenvalues with a negative real part are both positive and equal (two), the equilibrium is a saddle point with a two-dimensional stable manifold. For a better overview, all results are listed in Table 4.2.

Note, that all required conditions are satisfied:

- The found equilibrium is admissible, since the control variables are all positive and a is additionally less than 1.
- The complementary slackness conditions (3.2a)-(3.2c) are fulfilled, since all Lagrange

Variable/ Term	Name	Value
$\hat{K}$	conventional capital	12.9027
$\hat{G}$	green capital	33.6878
$\hat{a}$	percentage rate of abatement	0.139488
$\hat{R_K}$	investment in conventional capital	0.089721
$\hat{R_G}$	investment in green capital	0.263331
$\hat{\lambda_1}$	costate associated with K	0.0777256
$\hat{\lambda_2}$	costate associated with G	0.244925
$F(\hat{K},\hat{G})$	production function	17.2075
$w(\hat{R_K}+\hat{R_G})$	opportunity cost of R&D	0.0353052
$c_K(\hat{R_K})$	adjustment costs of brown R&D	0.00804985
$c_G(\hat{R_G})$	adjustment costs of green R&D	0.0693431
$\chi(\hat{a},E(\hat{K},\hat{G}))$	abatement costs	2.01014
$p(\epsilon-(1-\hat{a})E(\hat{K},\hat{G})^3$	permits term	-0.302028
profit term	profit	14.7827
$E(\hat{K},\hat{G})$	produced emissions	12.4007
$(1-\hat{a})E(\hat{K},\hat{G})$	net emissions	10.6709
ζ_1	first eigenvalue	0.647362
ζ_2	second eigenvalue	-0.604724
ζ_3	third eigenvalue	0.167452
ζ_4	fourth eigenvalue	-0.0941876

Table 4.2: Steady State Values of the Standard Parameter Set.

multipliers are zero. They can be computed from Equations (3.1a)-(3.1c):

$$\begin{aligned}
\mathcal{L}_{R_K}(\hat{K},\hat{R_K},\hat{\lambda_1},\hat{\mu_1}) &= -w-2k\hat{R_K}+\hat{\lambda_1}\delta_2 d\hat{K}^{\delta_1}\hat{R_K}^{\delta_2-1}+\hat{\mu_1}=0 \Rightarrow\\
\hat{\mu_1} &= w+2k\hat{R_K}-\hat{\lambda_1}\delta_2 d\hat{K}^{\delta_1}\hat{R_K}^{\delta_2-1}=0\\
\mathcal{L}_{R_G}(\hat{G},\hat{R_G},\hat{\lambda_2},\hat{\mu_2}) &= -w-2g\hat{R_G}+\hat{\lambda_2}\sigma_2 b\hat{G}^{\sigma_1}\hat{R_G}^{\sigma_2-1}+\hat{\mu_2}=0 \Rightarrow\\
\hat{\mu_2} &= w+2g\hat{R_G}-\hat{\lambda_2}\sigma_2 b\hat{G}^{\sigma_1}\hat{R_G}^{\sigma_2-1}=0\\
\mathcal{L}_a(\hat{K},\hat{G},\hat{a},\hat{\mu_3}) &= -c\frac{1}{(1-\hat{a})^2}E(\hat{K},\hat{G})+3pE(\hat{K},\hat{G})(\epsilon-(1-\hat{a})E(\hat{K},\hat{G}))^2+\hat{\mu_3}=0 \Rightarrow\\
\hat{\mu_3} &= \frac{c\hat{E}}{(1-\hat{a})^2}-3p\hat{E}(\epsilon-(1-\hat{a})\hat{E})^2=0
\end{aligned}$$

- Because the equilibrium is obtained with the solution $a_4(K,G)$, the Legendre-Clebsch Condition is fulfilled and the equilibrium is actually a maximum.

- The Hessian matrix of $\mathcal{H}$

$$D^2\mathcal{H}=\begin{pmatrix}\mathcal{H}_{KK} & \mathcal{H}_{KG}\\ \mathcal{H}_{GK} & \mathcal{H}_{GG}\end{pmatrix}=\begin{pmatrix}-1.48241 & -0.20035\\ -0.20035 & -0.0330695\end{pmatrix}$$

is negative semidefinite and the Sufficiency Condition is fulfilled too.

4.3 Bifurcation Analysis

For a deeper insight into the subject, bifurcation analysis is used. Bifurcation theory serves to analyze the possible changes in the dynamical behavior of the model due to some variation of the parameters. It measures therefore the sensitivity of the system with respect to changes in the parameters. The unique steady state, found in the previous section, is considered for different parameter values. The changes concerning equilibrium values of the states, costates, controls and function values as well as changes in the stability are presented in a table as well as in several figures. Because the main focus of this thesis is the investigation of the implications of climate policy on the firm´s investment decisions and on economic growth, I start with ϵ being the subject of variation.

Note that 0 and 1 are model inherited boundaries for a; thus the solutions are considered only within this range.

4.3.1 Variation of Environmental Standard Parameter ϵ

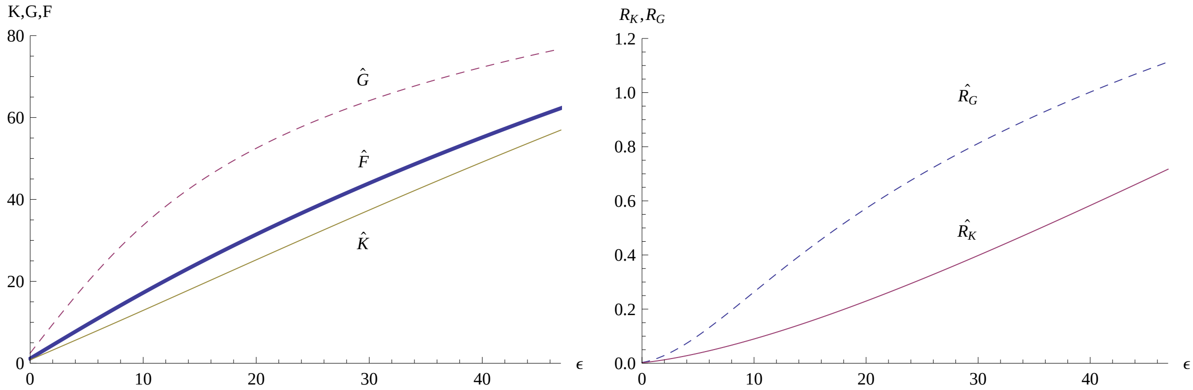
A higher ϵ means a less strict environmental policy, because the government allows more pollution. $\epsilon = 0$ reflects the strictest standard possible, no emissions at all are allowed, which is economical not reasonable and therefore excluded.

Figures 4.2(a) - 4.5(a) illustrate the impact of a variation of ϵ on the steady state values of K, G, F, R_K, R_G and a. The influence of an increasing ϵ on the accumulation of capital and hence on production (Figure 4.2(a)) is not surprising. As the environmental standard gets laxer, capital and therefore production too grows due to falling abatement and permit costs. That means that environmental standards are a drawback to economic growth. Whereas K grows nearly constantly, $G(\epsilon)$ is concave and dominant in production- it lies above K for the whole range. The greatest difference between green and brown capital is at approximately $\epsilon = 23.5$, as can be seen in Figure 4.3(a). Increasing ϵ also results in an augmentation of investment in capital of both types (Figure 4.2(b)). The difference between the two types of investment are shown in Figure 4.3(b) and is positive for all parameter values. The maximum lies at approximately $\epsilon = 35.7$, i.e. later than for the capital stocks. The share of green capital in the total amount of capital decreases with increasing ϵ (Figure 4.4(a)) which is obvious since brown capital is more productive. Nevertheless, the share is always greater than 0.5 which means that green capital staying dominant is optimal for all considered values of ϵ. The share of investment in green capital first increases, takes its maximum at $\epsilon = 9.9$ and then decreases (Figure 4.4(b)). The influence of the required environmental standards on the abatement share a (Figure 4.5(a)) is as expected. The greater ϵ is, which means the more emissions are allowed, the lesser are the abatement efforts. Figure 4.5(b) depicts the change of the steady state values of emissions E (dashed line), the emissions that remain after the abatement-process $(1-a)E$ (thick line) and the difference between the compulsory environmental standard ϵ and the remaining emissions: $\epsilon - (1-a)E$. Emissions grow of course since both "emissions producer" K and G grow. The remaining emissions are nearly the same as the emissions without abatement since the abatement effort a is rather low, but always chosen in a way that the remaining emissions are slightly above the required standard and the difference is constant over ϵ. The permits term is therefore negative and the firm is punished in form of costs for exceeding the standards. The evolution of all cost terms can be seen in Figure 4.6. The thick line depicts the abatement costs χ, the dashed line the permits term which forms costs resulting from an exceeding of the environmental standard and the other ones are the adjustment and opportunity costs of investment. For a very high ϵ, the abatement costs become zero, but before, they are rather large compared to other costs and therefore the main source of costs. The permits costs as well as the opportunity costs stay small, whereas the adjustment costs rise, which is clear since they are quadratic in investments and investments grow with increasing ϵ. The abatement costs maximum lies at $\epsilon = 19.7$. The profit also increases with ϵ (Figure 4.7), which affirms the assumption that a lower ϵ is worse for the firms. Note that both costates grow too (Figure 4.8), whereas the costate referring to the state G (dashed line) grows faster than that referring to K. The eigenvalues are all real (Figure 4.9) and for all values of ϵ the number of positive eigenvalues as well as the number of negative eigenvalues are greater than zero and equal. The stability of the steady state does not change therefore and the equilibrium point stays a saddle point. The numerical results of state variables, control variables, costate variables, the terms of the objective function and eigenvalues for selected values of ϵ are listed in Table 4.3.

In summary, in this model, decreasing environmental standards (a stricter policy) result in diminishing production inputs as well as outputs and an increasing share of green capital in total capital. Nevertheless, produced emissions are always greater than ϵ and even the emissions left after abatement are greater which yields to penalty costs.

Note, that all required conditions are satisfied:

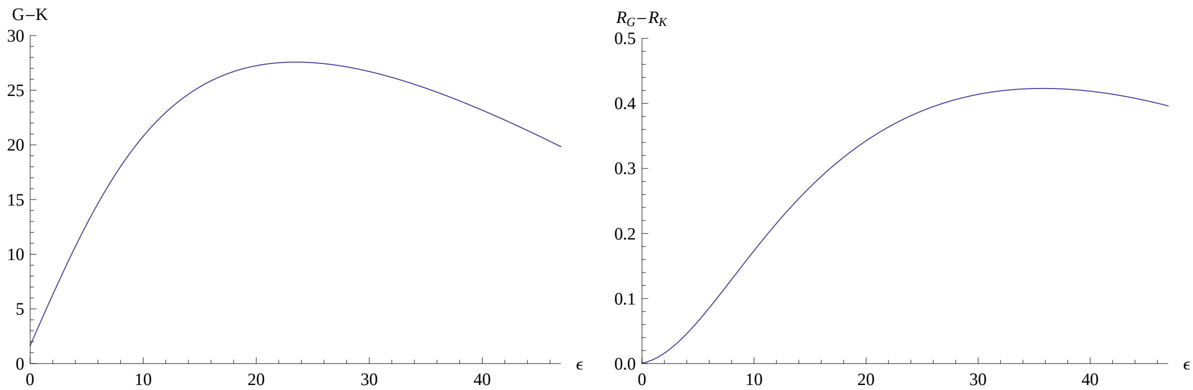
- The found equilibrium is admissible, since the control variables are all positive and a is additionally less than 1 (Figures 4.2(b) and 4.5(a)).
- The Complementary Slackness Conditions (3.2a)-(3.2c) are fulfilled, since all Lagrange multipliers are zero, even for the boundary arc $a = 0$ (for $\epsilon = 46.957$).



(a) Steady state levels of K, G and F.

(b) Steady state levels of R_K an R_G.

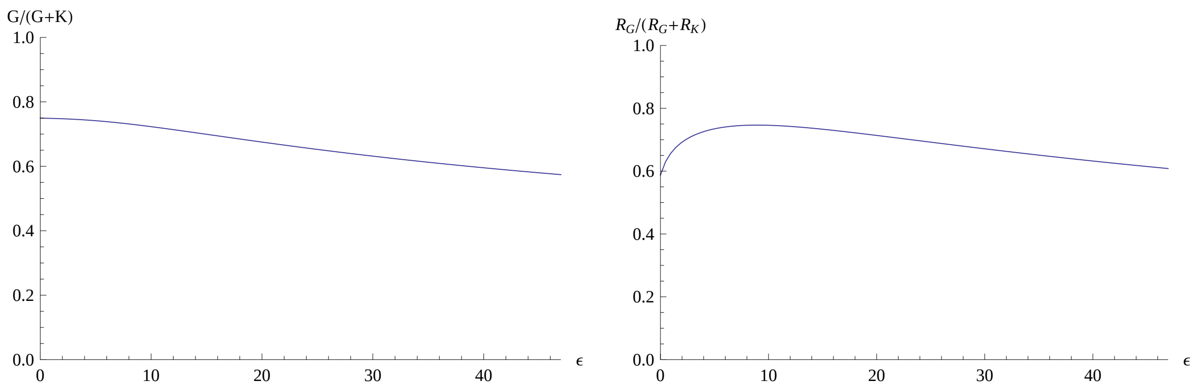
Figure 4.2: Bifurcation diagrams for steady state levels of K, G, F and R_K and R_G with respect to ϵ.



(a) Steady state levels of $G - K$.

(b) Steady state levels of $R_G - R_K$.

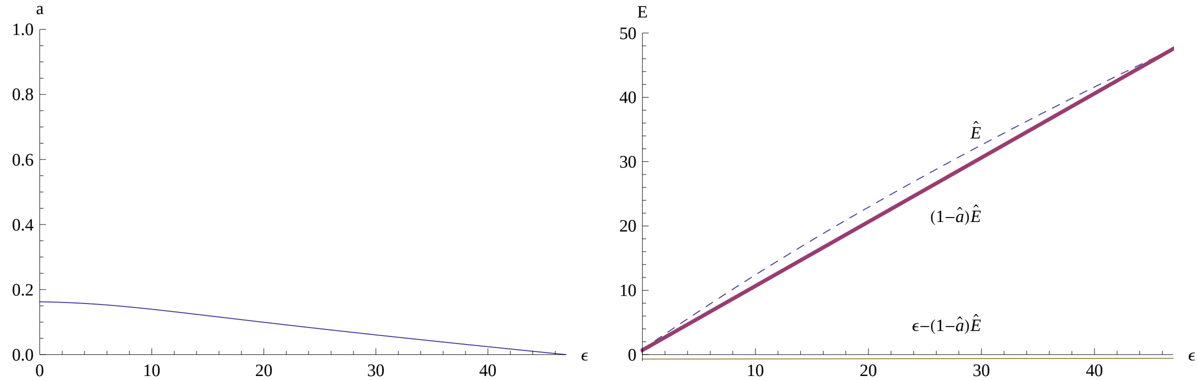
Figure 4.3: Bifurcation diagrams for steady state levels of $G - K$ and $R_G - R_K$ with respect to ϵ.



(a) Steady state levels of $G/(G + K)$.

(b) Steady state levels of $R_G/(R_G + R_K)$.

Figure 4.4: Bifurcation diagrams for steady state levels of $G/(G + K)$ and $R_G/(R_G + R_K)$ with respect to ϵ.



(a) Steady state levels of a.

(b) Steady state levels of E, $(1-a)E$ and $\epsilon-(1-a)E$.

Figure 4.5: Bifurcation diagrams for steady state levels of a and E, $(1-a)E$ and $\epsilon-(1-a)E$ with respect to ϵ.

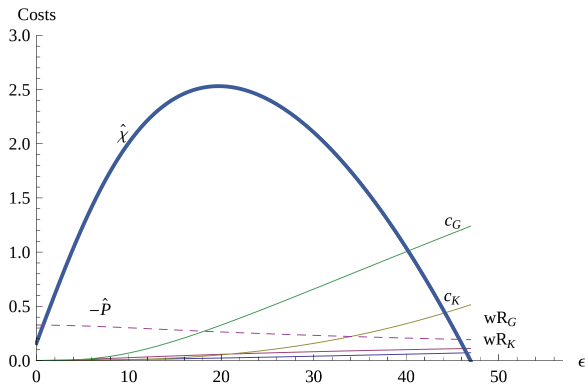


Figure 4.6: Bifurcation diagram for steady state levels of the cost components with respect to ϵ.

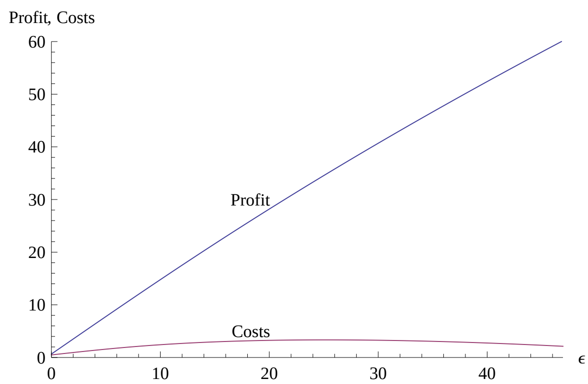


Figure 4.7: Bifurcation diagram for steady state level of the profit and overall costs with respect to ϵ.

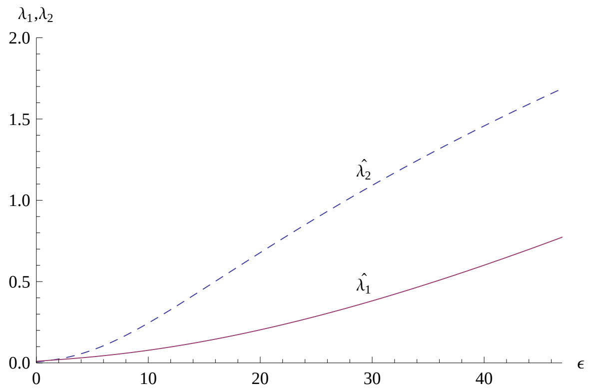


Figure 4.8: Bifurcation diagram for steady state level of λ_1 and λ_2 with respect to ϵ.

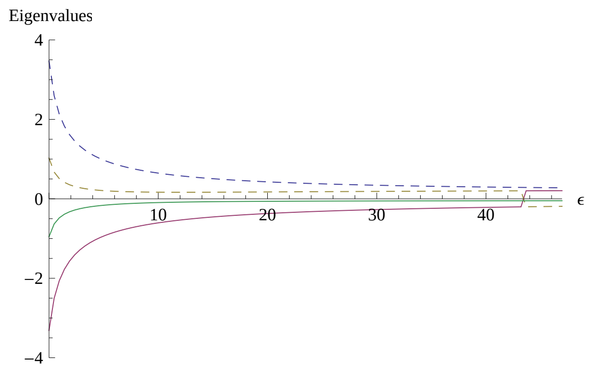


Figure 4.9: Bifurcation diagram for steady state levels of the eigenvalues with respect to ϵ.

- Because the equilibrium is obtained with the solution $a_4(K, G)$, the Legendre-Clebsch Condition is fulfilled and the equilibrium is actually a maximum.
- The Hessian matrix of $\mathcal{H}$ (shown for selected values of ϵ in Table 4.4) is always negative definite and the Sufficiency Condition is fulfilled too.

4.3.2 Variation of Scale Parameter p

Next to varying the environmental standards parameter ϵ, a governmental institution could make the reward or penalty of selling, respectively buying, permission permits higher. In the following, the parameter p, the scaling parameter of the permits term, is varied. The change of the equilibrium values are considered in Figures 4.10-4.16. Since $(1-a)E > \epsilon$ (see Figure 4.12(b)), the permits term represents costs to pay and a greater scaling parameter means more expensive costs.

Figure 4.10(a) shows the impact of a change in the scaling parameter on the steady state values of K, G and F: As exceeding the environmental standard ϵ becomes more costly, less input is used to produce less output. The same happens with investment R_K and R_G (Figure 4.10(b)). Figures 4.11(a) and 4.11(b) depict the change of the ratios $G/(G+K)$ and $R_G/(R_G+R_K)$. They are constant except for small p, since G and K as

Variable/ Term	$\epsilon = 0.001$	$\epsilon = 16$	$\epsilon = 31.5$	$\epsilon = 46.957$
$\hat{K}$	0.825	20.265	39.144	56.939
$\hat{G}$	2.468	46.105	65.472	76.777
$\hat{a}$	0.163	0.116	0.055	0
$\hat{R_K}$	0.002	0.169	0.424	0.717
$\hat{R_G}$	0.003	0.456	0.842	1.113
$\hat{\lambda_1}$	0.010	0.146	0.411	0.773
$\hat{\lambda_2}$	0.006	0.501	1.148	0.687
$F(\hat{K}, \hat{G})$	1.146	25.980	31.462	62.281
$w(\hat{R_K} + \hat{R_G})$	0	0.063	0.127	0.183
$c_K(\hat{R_K})$	0	0.029	0.180	0.514
$c_G(\hat{R_G})$	0	0.209	0.710	1.239
$\chi(\hat{a}, E(\hat{K}, \hat{G}))$	0.160	2.463	1.984	0
$p(\epsilon - (1 - \hat{a})E(\hat{K}, \hat{G})^3$	-0.328	-0.278	-0.228	-0.192
profit term	0.658	22.938	42.446	60.152
$E(\hat{K}, \hat{G})$	0.824	18.831	33.948	47.535
$(1 - \hat{a})E(\hat{K}, \hat{G})$	0.690	16.653	32.072	47.535
ζ_1	3.470	0.488	0.334	0.277
ζ_2	-3.314	-0.436	-0.261	0.205
ζ_3	1.025	0.168	0.189	-0.189
ζ_4	-0.955	-0.070	-0.053	-0.048

Table 4.3: Steady State values for selected values of ϵ.

ϵ	0.001	16	31.5	46.957
$D^2\mathcal{H}$	$\begin{pmatrix} -1.78 & -0.09 \\ -0.09 & -0.07 \end{pmatrix}$	$\begin{pmatrix} -1.51 & -0.21 \\ -0.21 & -0.03 \end{pmatrix}$	$\begin{pmatrix} -1.61 & -0.23 \\ -0.23 & -0.04 \end{pmatrix}$	$\begin{pmatrix} -1.70 & -0.24 \\ -0.24 & -0.04 \end{pmatrix}$

Table 4.4: Hessian Matrix for selected values of ϵ.

well as R_G and R_G change with the same rate. As p grows, abatement grows too, which can be intuitively explained: the more expensive $\epsilon - (1-a)E$ becomes, the greater a will be in order to diminish the difference(depicted in Figure 4.12(a)). As already noticed, the difference is diminishing too in K and G and therefore E (Figure 4.12(b)). The difference stays negative nevertheless. The costates are also diminishing (Figure 4.13) as well as the permits term $-P$ (Figure 4.14), although p is increasing. Figure 4.14 also shows, that the steady state abatement costs χ are nearly constant and the other costs (adjustment and opportunity costs) are negligible small. An increasing p leads to a decreasing profit, as can be seen in Figure 4.15. Because the permits term are always costs, this is quite clear. The eigenvalues behave smoothly (Figure 4.16) and assure a saddle point for all values of p. The numerical results for steady state values of the variables of the model for different values of p are listed in Table 4.5.

Variable/ Term	$p = 0.001$	$p = 0.103$	$p = 0.201$	$p = 0.3$
$\hat{K}$	36.088	14.642	13.912	13.582
$\hat{G}$	63.048	37.027	35.661	35.026
$\hat{a}$	0.065	0.134	0.136	0.137
$\hat{R_K}$	0.379	0.107	0.100	0.096
$\hat{R_G}$	0.789	0.312	0.291	0.282
$\hat{\lambda}_1$	0.360	0.092	0.086	0.083
$\hat{\lambda}_2$	1.049	0.303	0.278	0.267
$F(\hat{K}, \hat{G})$	42.663	19.342	18.452	18.046
$w(\hat{R_K} + \hat{R_G})$	0.117	0.042	0.039	0.038
$c_K(\hat{R_K})$	0.143	0.011	0.010	0.009
$c_G(\hat{R_G})$	0.622	0.097	0.085	0.079
$\chi(\hat{a}, E(\hat{K}, \hat{G}))$	2.186	2.161	2.101	2.073
$p(\epsilon - (1 - \hat{a})E(\hat{K}, \hat{G})^3$	-7.440	-0.925	-0.666	-0.547
Profit Term	32.155	16.105	15.551	15.300
$E(\hat{K}, \hat{G})$	31.566	13.952	13.305	13.010
$(1 - \hat{a})E(\hat{K}, \hat{G})$	29.522	12.081	11.490	11.222
ζ_1	0.297	0.574	0.602	0.616
ζ_2	-0.227	-0.529	-0.588	-0.572
ζ_3	0.183	0.165	0.166	0.166
ζ_4	-0.052	-0.085	-0.088	-0.090

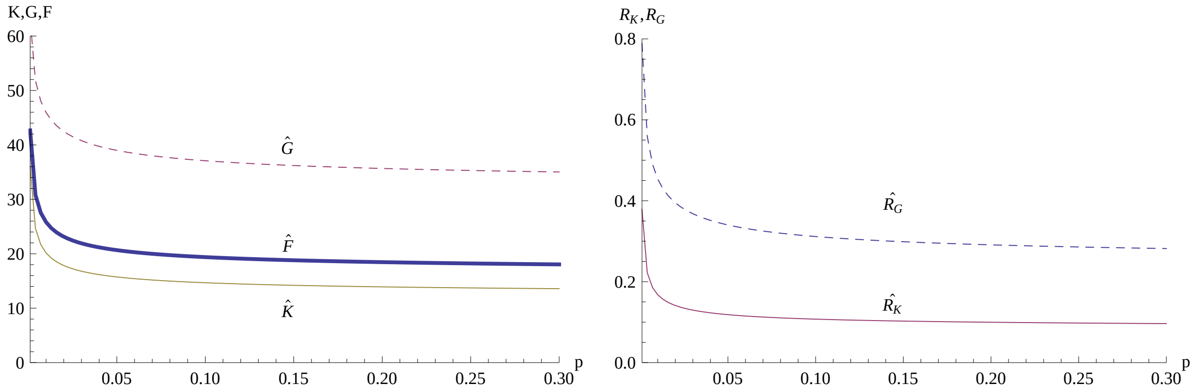
Table 4.5: Steady State Values for selected values of p.

Note, that also for varying p all required conditions are satisfied:

- The found equilibrium is admissible, since the control variables are all positive and a is additionally less than 1 (Figures 4.10(b) and 4.12(a)).
- The Complementary Slackness Conditions (3.2a)-(3.2c) are fulfilled, since all Lagrange multipliers are zero.
- Because the equilibrium is obtained with the solution $a_4(K, G)$, the Legendre-Clebsch Condition is fulfilled and the equilibrium is actually a maximum.
- The Hessian matrix of $\mathcal{H}$ (shown for selected values of p in Table 4.6) is always negative definite and the Sufficiency Condition is fulfilled too.

p	0.001	0.103	0.201	0.3
$D^2\mathcal{H}$	$\begin{pmatrix} -0.06 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -0.49 & -0.06 \\ -0.06 & -0.01 \end{pmatrix}$	$\begin{pmatrix} -0.68 & -0.09 \\ -0.09 & -0.02 \end{pmatrix}$	$\begin{pmatrix} -0.82 & -0.11 \\ -0.11 & -0.02 \end{pmatrix}$

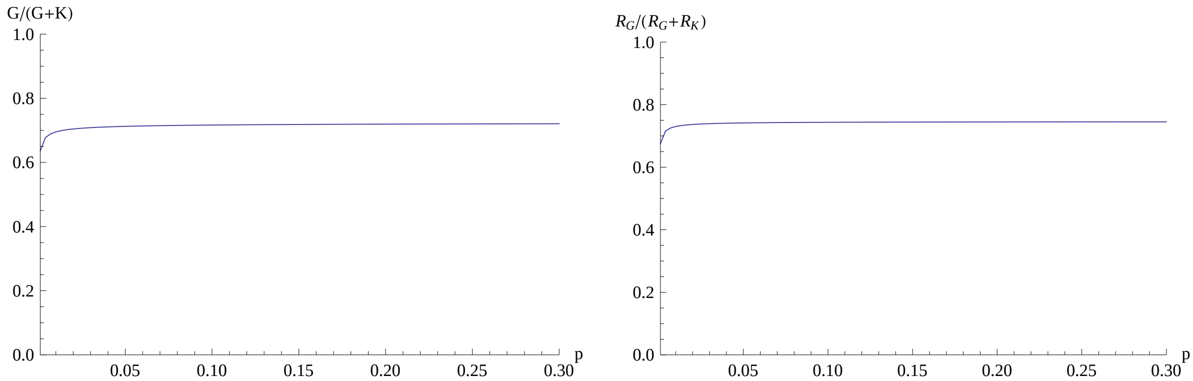
Table 4.6: Hessian Matrix for selected values of p.



(a) Steady state levels of K, G and F.

(b) Steady state levels of R_K and R_G.

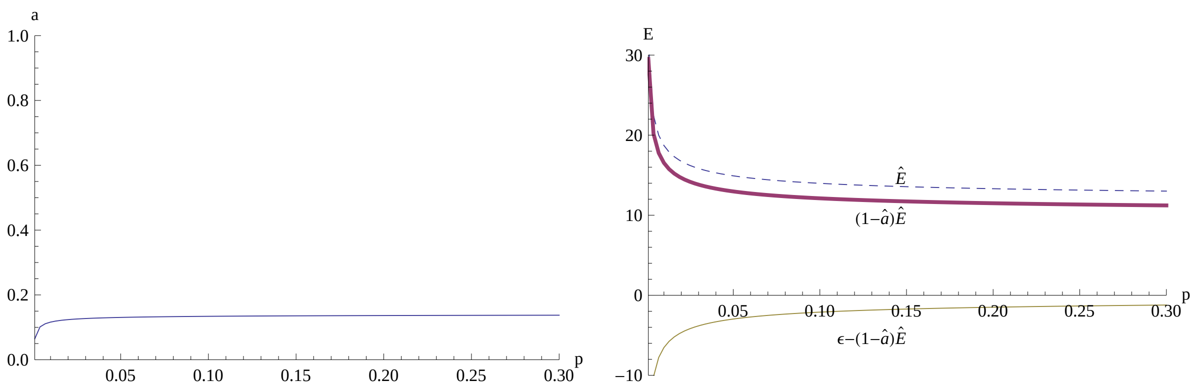
Figure 4.10: Bifurcation diagrams for steady state levels of K, G, F, R_K and R_G with respect to p.



(a) Steady state levels of $G/(G+K)$.

(b) Steady state levels of $R_G/(R_G+R_K)$.

Figure 4.11: Bifurcation diagrams for steady state levels of $G/(G+K)$ and $R_G/(R_G+R_K)$ with respect to p.



(a) Steady state levels of a.

(b) Steady state levels of E, $(1-a)E$ and $\epsilon-(1-a)E$.

Figure 4.12: Bifurcation diagrams for steady state levels of a and E, $(1-a)E$ and $\epsilon-(1-a)E$ with respect to p.

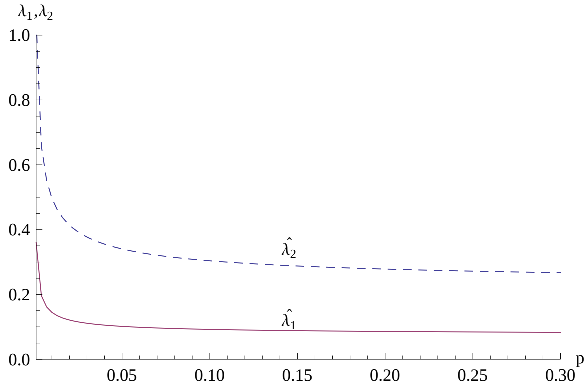


Figure 4.13: Bifurcation diagram for steady state level of λ_1 and λ_2 with respect to p.

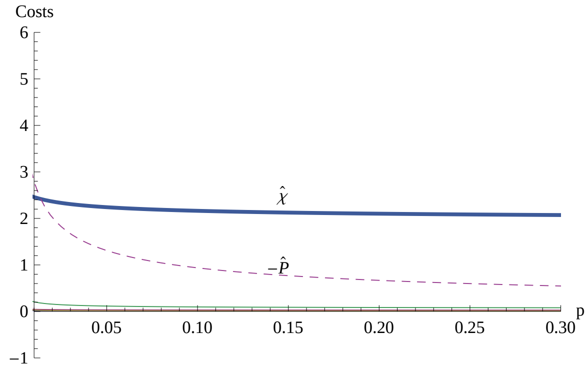


Figure 4.14: Bifurcation diagram for steady state levels of the cost components with respect to p.

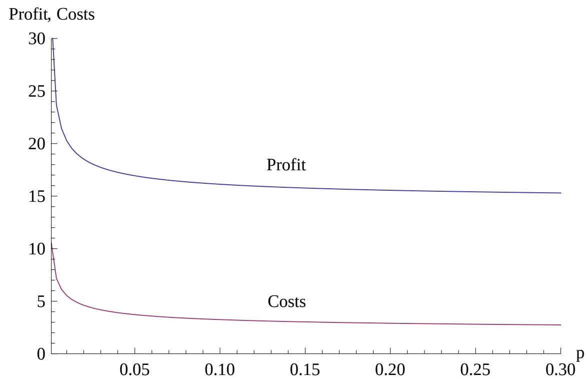


Figure 4.15: Bifurcation diagram for steady state level of the profit and overall costs with respect to p.

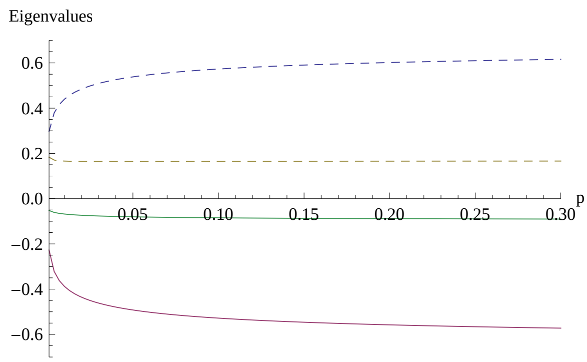


Figure 4.16: Bifurcation diagram for steady state levels of the eigenvalues with respect to p.

4.3.3 Variation of Emission Intensity Ratio $\frac{\kappa}{\gamma}$

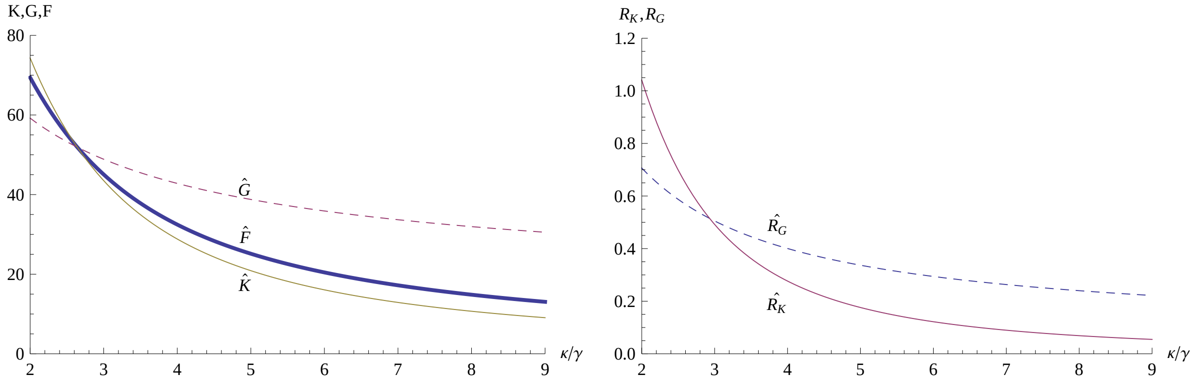
The ratio $\frac{\kappa}{\gamma}$ denotes how much capital of type K pollutes in proportion to capital G. A small ratio implies that K is not much more pollutive than G. With increasing $\frac{\kappa}{\gamma}$, K becomes more and more pollutive. Since $1 < \kappa < \gamma < 0$, the domain is given as $\frac{\kappa}{\gamma} \in (0, 10)$.

Figure 4.17(a) depicts the variation of K, G and F, if the ratio $\frac{\kappa}{\gamma}$ changes. All three of them are decreasing. For $\frac{\kappa}{\gamma} = 2$, more brown capital than green capital is accumulated in the steady state, but it decreases faster than G, so that for $\frac{\kappa}{\gamma} > 3$ K is lower than G. If the pollution intensity of K is high enough compared to that of G, a movement of production factors from brown capital to green capital takes place. The same happens with investment in K and G (Figure 4.17(b)): For $\frac{\kappa}{\gamma} = 3$ R_K is dominated by R_G. The share of G on total capital (Figure 4.18(a)) as well as the share of R_G on total investment (Figure 4.18(b)) increases. a (Figure 4.19(a)) becomes smaller since less emission is produced (Figure 4.19(b)). The remaining emissions as well as the difference to ϵ stay constant for all values of the ratio. The costates decreases too (Figure 4.20) and also the abatement costs χ are diminishing (Figure 4.21), since E diminishes. Because the difference $\epsilon - (1 - a)E$ stays constant, the permits term does too and the other costs are again negligible small. The profit decreases (Figure 4.22), because the output decreases. The number of positive eigenvalues as well as the number of negative eigenvalues is always positive and equal (Figure 4.23) and the equilibrium point stays a saddle point. The numerical results for steady state values of the variables of the model for different values of $\frac{\kappa}{\gamma}$ are listed in Table 4.7.

Summarizing, a more pollutive K, respectively a less pollutive G, leads to greener production.

Note, that also for varying $\frac{\kappa}{\gamma}$ all required conditions are satisfied:

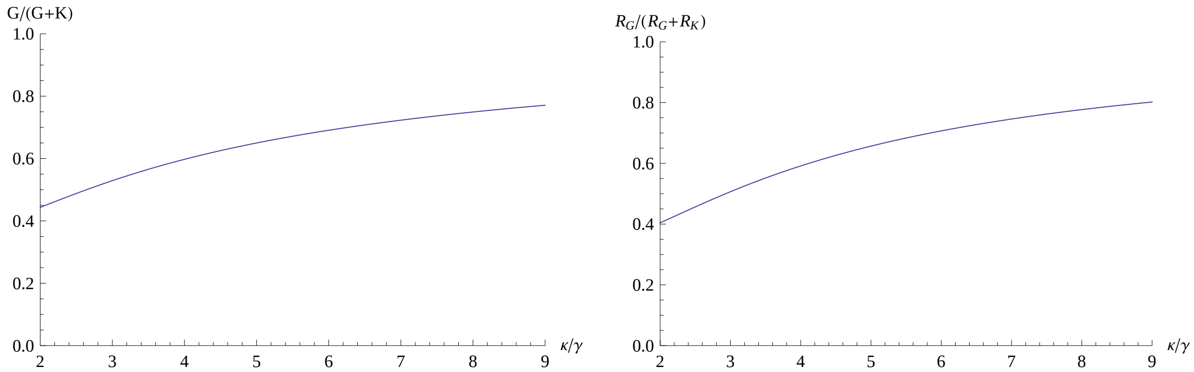
- The found equilibrium is admissible, since the control variables are all positive and a is additionally less than 1 (Figures 4.17(b) and 4.19(a)).
- The Complementary Slackness Conditions (3.2a)-(3.2c) are fulfilled, since all Lagrange multipliers are zero.
- Because the equilibrium is obtained with the solution $a_4(K, G)$, the Legendre-Clebsch Condition is fulfilled and the equilibrium is actually a maximum.
- The Hessian matrix of $\mathcal{H}$ (shown for selected values of $\frac{\kappa}{\gamma}$ in Table 4.8) is always negative definite and the Sufficiency Condition is fulfilled too.



(a) Steady state levels of K, G and F.

(b) Steady state levels of R_K and R_G.

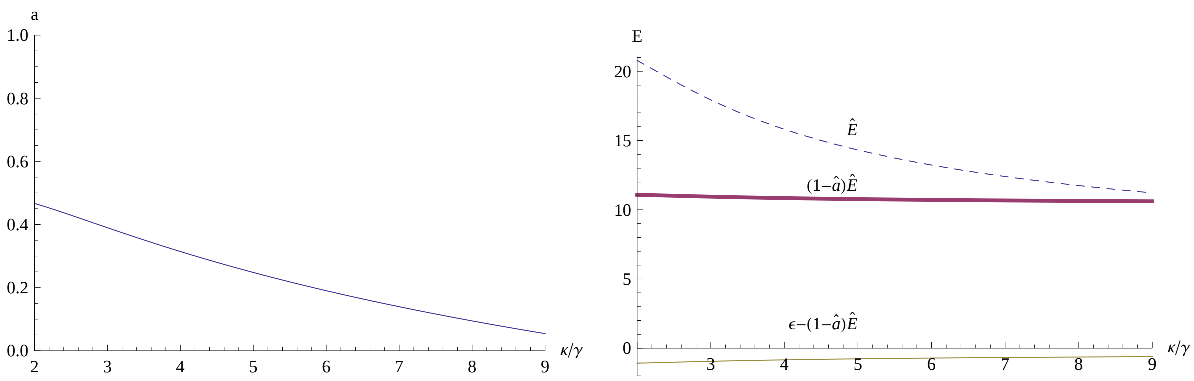
Figure 4.17: Bifurcation diagrams for steady state levels of K, G, F, R_K and R_G with respect to κ/γ.



(a) Steady state levels of $G/(G+K)$.

(b) Steady state levels of $R_G/(R_G+R_K)$.

Figure 4.18: Bifurcation diagrams for steady state levels of $G/(G+K)$ and $R_G/(R_G+R_K)$ with respect to κ/γ.



(a) Steady state levels of a.

(b) Steady state levels of E, $(1-a)E$ and $\epsilon-(1-a)E$.

Figure 4.19: Bifurcation diagrams for steady state levels of a and E, $(1-a)E$ and $\epsilon-(1-a)E$ with respect to κ/γ.

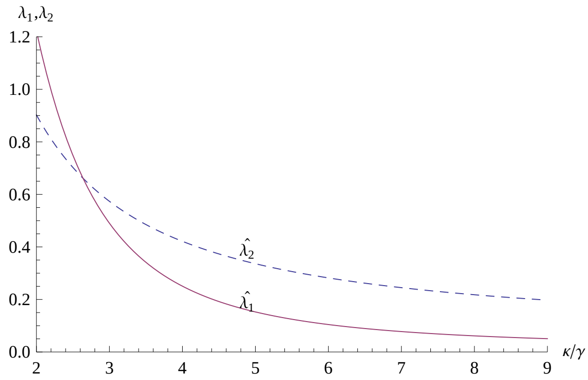


Figure 4.20: Bifurcation diagram for steady state level of λ_1 and λ_2 with respect to κ/γ.

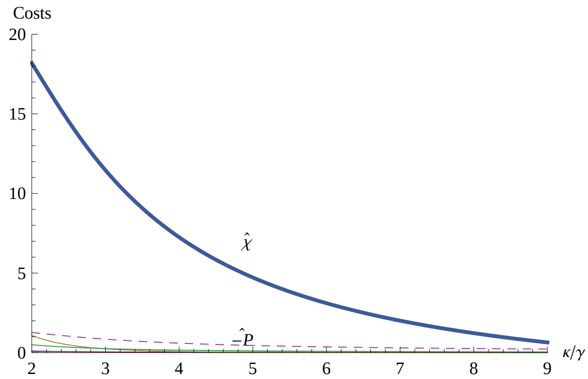


Figure 4.21: Bifurcation diagram for steady state levels of the cost components with respect to κ/γ.

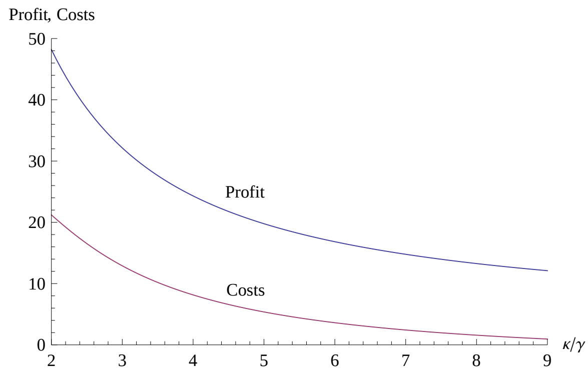


Figure 4.22: Bifurcation diagram for steady state level of the profit and overall costs with respect to κ/γ.

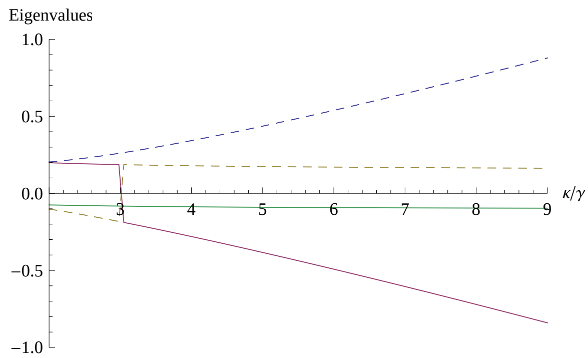


Figure 4.23: Bifurcation diagram for steady state levels of the eigenvalues with respect to κ/γ.

Variable/ Term	$\kappa/\gamma = 2$	$\kappa/\gamma = 4.3$	$\kappa/\gamma = 6.62$	$\kappa/\gamma = 9$
$\hat{K}$	74.299	25.883	13.966	9.072
$\hat{G}$	59.201	41.393	34.458	30.543
$\hat{a}$	0.467	0.293	0.158	0.054
$\hat{R_K}$	1.041	0.238	0.100	0.055
$\hat{R_G}$	0.706	0.378	0.274	0.222
$\hat{\lambda_1}$	1.222	0.211	0.086	0.051
$\hat{\lambda_2}$	0.902	0.390	0.258	0.197
$F(\hat{K},\hat{G})$	69.404	29.798	18.312	13.058
$w(\hat{R_K}+\hat{R_G})$	0.175	0.062	0.037	0.028
$c_K(\hat{R_K})$	1.083	0.057	0.010	0.003
$c_G(\hat{R_G})$	0.500	0.143	0.075	0.049
$\chi(\hat{a},E(\hat{K},\hat{G}))$	18.183	6.333	2.382	0.643
$p(\epsilon-(1-\hat{a})E(\hat{K},\hat{G})^3$	-1.269	-0.544	-0.322	-0.228
profit term	48.197	22.661	15.485	12.107
$E(\hat{K},\hat{G})$	20.780	15.295	12.691	11.219
$(1-\hat{a})E(\hat{K},\hat{G})$	11.083	10.816	10.686	10.611
ζ_1	0.203	0.370	0.605	0.879
ζ_2	0.199	-0.312	-0.561	-0.840
ζ_3	-0.102	0.177	0.168	0.163
ζ_4	-0.075	-0.088	-0.094	-0.096

Table 4.7: Steady State Values for selected values of κ/γ.

p	2	4.3	6.62	9
$D^2\mathcal{H}$	$\begin{pmatrix} -0.08 & -0.03 \\ -0.03 & -0.02 \end{pmatrix}$	$\begin{pmatrix} -0.47 & -0.10 \\ -0.10 & -0.03 \end{pmatrix}$	$\begin{pmatrix} -1.30 & -0.19 \\ -0.19 & -0.03 \end{pmatrix}$	$\begin{pmatrix} -2.69 & -0.29 \\ -0.29 & -0.04 \end{pmatrix}$

Table 4.8: Hessian Matrix for selected values of κ/γ.

Chapter 5

Comparison of the Models

In this chapter, connections and differences of the basic model with the models presented in Section 1.2.1 and 1.2.2 are summarized.

5.1 Rauscher

The model presented in this thesis is an extension of the model of Rauscher [2009]. The main modifications of Rauscher´s model are the following.

- In Rauscher´s work, the environmental standards are binding. In my work, the firms have the possibility to exceed the standards and to be punished (in fact, they are forced to pay for the difference) or they remain under the threshold and as a reward, increase their profit by selling the difference. An additional term, called the "permits term", is introduced which stands for either costs or gains. Thus, the firm does not have to abate enough emissions to satisfy the environmental regulation- it can choose freely how much it wants to abate. With the activity of cleaning up, firms can reduce their total amount of emissions. This benefit, though, comes at a cost (so-called abatement costs), which reduces the total profit.

- There is, next to the decision of how great the investments in brown and green capital are, a third decision variable, namely the share in produced emissions which is abated, a. The abatement costs do no longer depend on ϵ, but amongst other things on a.

- Contrary to Rauscher, not only brown capital pollutes but green capital as well (even though it pollutes less than conventional capital). The actual amount of produced emissions, E, is explicitly modeled and next to the abatement share a, the other factor determining the abatement costs. Whereas in Rauscher´s model end-of-pipe-abatement depends on the exogenous given stringency of environmental standard and the amount of brown capital and therefore exclusively on the decision of allocation between brown and green capital, here it depends on the share in produced emissions which the entrepreneur decides to abate, a, and the total amount of emissions produced during the production process.

- The present value of future profit instead of utility is maximized. The problem is considered at firm-level.

- The environmental standard ϵ gives the negative environmental quality. The smaller ϵ, the cleaner the environment will be and the higher the environmental quality will be. Rauscher assumes that next to the burden, an environmental standard imposes, the decision maker does also benefit from it. He considers utility derived from ϵ. Here, this approach is totally neglected. Firms do only have costs associated with ϵ.

- Adjustment costs are considered, which makes investment a bit more expensive. Rauscher models only opportunity costs.

- Positive knowledge spillovers in the R&D sector as considered in Rauscher´s model are neglected in this analysis for the sake of simplicity.

Summing up, the main difference is, that firms do no longer have just two, but three possibilities to avoid, respectively handle pollution:

1. Producing with cleaner, but more expensive (in accumulation and production) technology G (process-integrated abatement),

2. investing in end-of-pipe abatement after the production process,

3. accepting a punishment in form of permits costs.

Although the model in this thesis is much more complex than Rauscher´s model and considers additional effects, it supports the results of Rauscher: Stricter environmental standards (a smaller ϵ) reduce capital investment in both types of capital (Figure 4.2(b)), whereas the share of green capital in the total amount of capital rises (Figure 4.4(a)).

5.2 Moser

The model of Moser only differs from Rauscher´s model in the way of modeling the accumulation processes. As in my work, knowledge spillovers in the R&D sector taken into account by Rauscher are neglected. Therefore, Moser´s model is closer to the one in this work, but still differs in several ways: Similar to Rauscher, she assumes two decision variables and the allocation of resources to conventional R&D, green R&D and end-pf-pipe-abatement. The environmental standards are still binding and require abatement costs proportional to the polluting type of capital K, but influences the objective function also positive. There are no adjustment costs and additionally, the objective function still denotes the utility of the firm instead of the profit as in my work.

Nevertheless, because the functional forms and parameter values in this work are chosen similar to Moser´s assumption whenever possible, the results can be compared meaningfully. Note that in my model, the term environmental quality refers to $-\epsilon$!

- K and G: In both models, the steady state values of K as well as G decline with increasing environmental quality, but whereas in my model G is dominant in production over the whole period (Figure 4.2(a)), in the model of Moser, brown capital is dominant for small ϵ (whereas ϵ refers to environmental quality in contrary to this work, where ϵ is the amount of pollution allowed) but declines more than G with increasing ϵ until green capital gets dominant over brown capital.

- R_K and R_G: The steady states values of investment behave just the same as K and G in both models (Figure 4.2(b)).

- F: In both models, the output F is decreasing with a better environmental quality (Figure 4.2(a)).

- Objective function: Whereas in my model the profit declines constantly (Figure 4.7), the steady state utility of Moser´s model first rises up to a peak before it decreases due to the trade-off between consumption and environmental quality.

- Share of G: In both models, the share of G in total capital, $G/(G + K)$, augments with increasing environmental quality. In my model, this increase is constant, whereas in Moser´s model the ratio follows a convex-concave shape.

- Share of R_G: Also this ratio increases in the models. In Moser´s model the development of $R_G/(R_G + R_K)$ is similar to the development of $G/(G + K)$, whereas in my model it first increases and then decreases.

Chapter 6

Summary

In order to investigate the impact of different policy options, namely environmental standards and permission permits, an optimal control model was formulated in Chapter 2. Its analysis (analytically in Chapter 3 and numerically in Chapter 4) showed how firms react optimally on exogenous given standards. To test the sensitivity of the system, bifurcation analysis was applied. The results are compared to two other models of the literature in Chapter 5.

The investigation of the environmental economic-growth model has produced the following insights:

- For all considered variations of ϵ, p and $\frac{\kappa}{\gamma}$, the steady state remaining emissions $(1-a)E$ are always (slightly) above the environmental standard ϵ. As a result, firms pay a (marginal) punishment for the exceeding.
- The cleanest production (where $\frac{G}{G+K}$ has its maximum) is found at the lowest ϵ, the highest p and the highest ratio $\frac{\kappa}{\gamma}$.
- For a stricter environmental policy (decreasing ϵ and increasing p), production output as well as profit and investment decline. Economic growth is therefore rather repressed, but both policies have a positive impact on the accumulation of green capital and green R&D, so that they are an adequate tool to turn production greener.

Appendix
Canonical System in the State-Costate-Space

Solving $\mathcal{H}_a = 0$ for a yields four solutions for a, since $\mathcal{H}_a$ is an equation of fourth degree in a. These are the following:

$$a_1^*(K,G) = 1 - \frac{\epsilon + \sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}}{2(\kappa K + \gamma G)},$$
$$a_2^*(K,G) = 1 - \frac{\epsilon - \sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}}{2(\kappa K + \gamma G)},$$
$$a_3^*(K,G) = 1 - \frac{\epsilon - \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}}{2(\kappa K + \gamma G)},$$
$$a_4^*(K,G) = 1 - \frac{\epsilon + \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}}{2(\kappa K + \gamma G)}.$$

Solution a_1 and a_3 are not admissible, because the first one cannot be real and a maximum at the same time and the third is not a maximum if it is less than 1 as required:

- $a_1(K,G)$: For a_1 being real, the discriminant $\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)$ has to be positive, which means $\epsilon^2 > \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)$. For a_1 being a maximum, the second derivate

$\mathcal{H}_{aa}(K, G, (a_1(K, G))$ has to be negative:

$$2E\left(-\frac{c}{(1-a_1(K,G))^3}+3pE(\epsilon-(1-a_1(K,G))E)\right)<0$$

$$-\frac{c}{\left(\frac{\epsilon+\sqrt{\epsilon^2-\frac{4\sqrt{c}}{\sqrt{3p}}E}}{2E}\right)^3}+3pE\left(\epsilon-\frac{\epsilon+\sqrt{\epsilon^2-\frac{4\sqrt{c}}{\sqrt{3p}}E}}{2E}E\right)<0$$

$$-\frac{8E^3c}{\left(\epsilon+\sqrt{\epsilon^2-\frac{4\sqrt{c}}{\sqrt{3p}}E}\right)^3}+3pE\left(\frac{\epsilon}{2}-\frac{\sqrt{\epsilon^2-\frac{4\sqrt{c}}{\sqrt{3p}}E}}{2}\right)<0$$

$$-16E^2c+3p\left(\epsilon-\sqrt{\epsilon^2-\frac{4\sqrt{c}}{\sqrt{3p}}E}\right)\left(\epsilon+\sqrt{\epsilon^2-\frac{4\sqrt{c}}{\sqrt{3p}}E}\right)^3<0$$

$$-16E^2c+3p\left(\epsilon^2-\epsilon^2+\frac{4\sqrt{c}}{\sqrt{3p}}E\right)\left(\epsilon+\sqrt{\epsilon^2-\frac{4\sqrt{c}}{\sqrt{3p}}E}\right)^2<0$$

$$-16E^2c+3p\left(\epsilon^2-\epsilon^2+\frac{4\sqrt{c}}{\sqrt{3p}}E\right)\left(\epsilon^2+2\epsilon\sqrt{\epsilon^2-\frac{4\sqrt{c}}{\sqrt{3p}}E}+\epsilon^2-\frac{4\sqrt{c}}{\sqrt{3p}}E\right)<0$$

$$-16E^2c+6p\frac{4\sqrt{c}}{\sqrt{3p}}E\left(\epsilon^2+\epsilon\sqrt{\epsilon^2-\frac{4\sqrt{c}}{\sqrt{3p}}E}-\frac{2\sqrt{c}}{\sqrt{3p}}E\right)<0$$

$$-8E^2c-8E^2c+4\sqrt{3cp}E\left(\epsilon^2+\epsilon\sqrt{\epsilon^2-\frac{4\sqrt{c}}{\sqrt{3p}}E}\right)<0$$

$$\sqrt{3cp}\left(\epsilon^2+\epsilon\sqrt{\epsilon^2-\frac{4\sqrt{c}}{\sqrt{3p}}E}\right)<4Ec$$

$$\epsilon^2+\epsilon\sqrt{\epsilon^2-\frac{4\sqrt{c}}{\sqrt{3p}}E}<\frac{4\sqrt{c}}{\sqrt{3p}}E$$

which cannot be, if a_1 is real: $\epsilon^2 > \frac{4\sqrt{c}}{\sqrt{3p}}E$.

- $a_3(K,G)$: $a_3 < 1$ yields

$$\frac{\epsilon - \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}}{2(\kappa K + \gamma G)} > 0$$

$$\epsilon - \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}$$

$$\epsilon > \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}$$

$$\epsilon\sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)} > \epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)$$

$\mathcal{H}_{aa}(K, G, (a_3(K,G)) < 0$ yields

$$2E\left(-\frac{c}{(1-a_3(K,G))^3} + 3pE(\epsilon - (1-a_3(K,G))E)\right) < 0$$

$$-\frac{c}{\left(\frac{\epsilon - \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}E}}{2E}\right)^3} + 3pE\left(\epsilon - \frac{\epsilon - \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}E}}{2E}E\right) < 0$$

$$-16E^2c + 3p\left(\epsilon + \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}E}\right)\left(\epsilon - \sqrt{\epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}E}\right)^3 < 0$$

$$-16E^2c + 3p\left(\epsilon^2 - \epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}E\right)\left(\epsilon^2 - 2\epsilon\sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}E} + \epsilon^2 + \frac{4\sqrt{c}}{\sqrt{3p}}E\right) < 0$$

$$-8E^2c - 8E^2c - 4\sqrt{3cp}E\left(\epsilon^2 - \epsilon\sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}E}\right) < 0$$

$$-\epsilon^2 + \epsilon\sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}E} < \frac{4\sqrt{c}}{\sqrt{3p}}E$$

$$\epsilon\sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}E} < \frac{4\sqrt{c}}{\sqrt{3p}}E + \epsilon^2$$

The two resulting canonical systems (one for each remaining solution of a) are given by

$$\dot{K}(K, R_K(K,\lambda_1)) = \left(\frac{1.67989w^2}{k\sqrt[3]{6912d^2k\lambda_1^2K^{2\delta_1} + 3762.42\sqrt{k}d\lambda_1K^{\delta_1}\sqrt{3.375d^2k\lambda_1^2K^{2\delta_1} + w^3} + 1024w^3}} + \frac{0.0165354\sqrt[3]{6912d^2k\lambda_1^2K^{2\delta_1} + 3762.42\sqrt{k}d\lambda_1K^{\delta_1}\sqrt{3.375d^2k\lambda_1^2K^{2\delta_1} + w^3} + 1024w^3}}{k} - \frac{0.333333w}{k}\right)^{\delta_2} dK^{\delta_1} - K\phi$$

$$\dot{G}(G, R_G(G, \lambda_2)) = \Bigg(\frac{1.67989w^2}{g\sqrt[3]{6912b^2g\lambda_2^2G^{2\sigma_1} + 3762.42\sqrt{g}b\lambda_2G^{\sigma_1}\sqrt{3.375b^2g\lambda_2^2G^{2\sigma_1} + w^3} + 1024w^3}} + \frac{0.0165354\sqrt[3]{6912b^2g\lambda_2^2G^{2\sigma_1} + 3762.42\sqrt{g}b\lambda_2G^{\sigma_1}\sqrt{3.375b^2g\lambda_2^2G^{2\sigma_1} + w^3} + 1024w^3}}{g} - \frac{0.333333w}{g}\Bigg)^{\sigma_2} bG^{\sigma_1} - G\psi$$

$$\dot{\lambda}_1(K, G, R_K(K, \lambda_1), a_2(K, G), \lambda_1) = \frac{4(\kappa K + \gamma G)c\kappa}{\epsilon - \sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}} - c\kappa - d\delta_1\lambda_1K^{\delta_1 - 1}\Bigg(\frac{1.67989w^2}{k\sqrt[3]{6912d^2k\lambda_1^2K^{2\delta_1} + 3762.42\sqrt{k}d\lambda_1K^{\delta_1}\sqrt{3.375d^2k\lambda_1^2K^{2\delta_1} + w^3} + 1024w^3}} + \frac{0.0165354\sqrt[3]{6912d^2k\lambda_1^2K^{2\delta_1} + 3762.42\sqrt{k}d\lambda_1K^{\delta_1}\sqrt{3.375d^2k\lambda_1^2K^{2\delta_1} + w^3} + 1024w^3}}{k} - \frac{0.333333w}{k}\Bigg)^{\delta_2} - f\alpha_1G^{\alpha_2}K^{\alpha_1 - 1} + \lambda_1(r + \phi)$$

$$\dot{\lambda}_2(K, G, R_G(G, \lambda_2), a_2(K, G), \lambda_2) = \frac{4(\kappa K + \gamma G)c\gamma}{\epsilon - \sqrt{\epsilon^2 - \frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K + \gamma G)}} - c\gamma - b\sigma_1\lambda_2G^{\sigma_1 - 1}\Bigg(\frac{1.67989w^2}{g\sqrt[3]{6912b^2g\lambda_2^2G^{2\sigma_1} + 3762.42\sqrt{g}b\lambda_2G^{\sigma_1}\sqrt{3.375b^2g\lambda_2^2G^{2\sigma_1} + w^3} + 1024w^3}} + \frac{0.0165354\sqrt[3]{6912b^2g\lambda_2^2G^{2\sigma_1} + 3762.42\sqrt{g}b\lambda_2G^{\sigma_1}\sqrt{3.375b^2g\lambda_2^2G^{2\sigma_1} + w^3} + 1024w^3}}{g} - \frac{0.333333w}{g}\Bigg)^{\sigma_2} - f\alpha_2G^{\alpha_2 - 1}K^{\alpha_1} + \lambda_2(r + \psi)$$

and

$$\dot{K}(K, R_K(K, \lambda_1)) = \Bigg(\frac{1.67989w^2}{k\sqrt[3]{6912d^2k\lambda_1^2K^{2\delta_1} + 3762.42\sqrt{k}d\lambda_1K^{\delta_1}\sqrt{3.375d^2k\lambda_1^2K^{2\delta_1} + w^3} + 1024w^3}} + \frac{0.0165354\sqrt[3]{6912d^2k\lambda_1^2K^{2\delta_1} + 3762.42\sqrt{k}d\lambda_1K^{\delta_1}\sqrt{3.375d^2k\lambda_1^2K^{2\delta_2} + w^3} + 1024w^3}}{k} - \frac{0.333333w}{k}\Bigg)^{\delta_2} dK^{\delta_1} - K\phi$$

$$\dot{G}(G, R_G(G, \lambda_2)) = \Bigg(\frac{1.67989w^2}{g\sqrt[3]{6912b^2g\lambda_2^2G^{2\sigma_1} + 3762.42\sqrt{g}b\lambda_2G^{\sigma_1}\sqrt{3.375b^2g\lambda_2^2G^{2\sigma_1} + w^3} + 1024w^3}} + \frac{0.0165354\sqrt[3]{6912b^2g\lambda_2^2G^{2\sigma_1} + 3762.42\sqrt{g}b\lambda_2G^{\sigma_1}\sqrt{3.375b^2g\lambda_2^2G^{2\sigma_1} + w^3} + 1024w^3}}{g} - \frac{0.333333w}{g}\Bigg)^{\sigma_2} bG^{\sigma_1} - G\psi$$

$$\dot{\lambda}_1(K,G,R_K(K,\lambda_1),a_4(K,G),\lambda_1) = \frac{4(\kappa K+\gamma G)c\kappa}{\epsilon+\sqrt{\epsilon^2+\frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K+\gamma G)}} - c\kappa -$$

$$d\delta_1\lambda_1 K^{\delta_1-1}\Bigg(\frac{1.67989w^2}{k\sqrt[3]{6912d^2k\lambda_1^2K^{2\delta_1}+3762.42\sqrt{k}d\lambda_1K^{\delta_1}\sqrt{3.375d^2k\lambda_1^2K^{2\delta_1}+w^3}+1024w^3}}+$$

$$\frac{0.0165354\sqrt[3]{6912d^2k\lambda_1^2K^{2\delta_1}+3762.42\sqrt{k}d\lambda_1K^{\delta_1}\sqrt{3.375d^2k\lambda_1^2K^{2\delta_1}+w^3}+1024w^3}}{k}-$$

$$\frac{0.333333w}{k}\Bigg)^{\delta_2} - f\alpha_1 G^{\alpha_2}K^{\alpha_1-1} + \lambda_1(r+\phi)$$

$$\dot{\lambda}_2(K,G,R_G(G,\lambda_2),a_4(K,G),\lambda_2) = \frac{4(\kappa K+\gamma G)c\gamma}{\epsilon+\sqrt{\epsilon^2+\frac{4\sqrt{c}}{\sqrt{3p}}(\kappa K+\gamma G)}} - c\gamma -$$

$$b\sigma_1\lambda_2 G^{\sigma_1-1}\Bigg(\frac{1.67989w^2}{g\sqrt[3]{6912b^2g\lambda_2^2G^{2\sigma_1}+3762.42\sqrt{g}b\lambda_2G^{\sigma_1}\sqrt{3.375b^2g\lambda_2^2G^{2\sigma_1}+w^3}+1024w^3}}+$$

$$\frac{0.0165354\sqrt[3]{6912b^2g\lambda_2^2G^{2\sigma_1}+3762.42\sqrt{g}b\lambda_2G^{\sigma_1}\sqrt{3.375b^2g\lambda_2^2G^{2\sigma_1}+w^3}+1024w^3}}{g}-$$

$$\frac{0.333333w}{g}\Bigg)^{\sigma_2} - f\alpha_2 G^{\alpha_2-1}K^{\alpha_1} + \lambda_2(r+\psi)$$

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