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Operator extensions of Gödel logics

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Kurzfassung

Die Gödellogik erster Ordnung über $[0, 1]$ ist eine äußerst prominente Logik, da sie einerseits eine intermediäre Logik ist, d. h. sie ist stärker als die intuitionistische, aber schwächer als die klassische Logik, und andererseits ist sie einer der drei Hauptvertreter der t-Norm-basierten Fuzzylogiken (siehe Hájek [15]). Sie teilt mit der klassischen Logik viele Eigenschaften wie z. B. die Idempotenz der Junktoren $\wedge$, $\vee$, das Äquivalenzschema, die gleiche Komplexitätsklasse und den Abwärts-Löwenheim-Skolem-Satz, was sie zu einem idealen Ausgangspunkt für Verallgemeinerungen von gut verstandenen Erweiterungen der klassischen Logik macht, wie z. B. modale Operatoren [11]. Mit der intuitionistischen Logik teilt sie eine Kripke-Semantik; das Beweissystem von Horn [17] für die Gödellogik erster Ordnung umfasst die intuitionistische Logik erster Ordnung, das Axiom der Linearität und eine Quantorenvertauschungsregel. Die Semantik der Gödellogik gibt Anlass zu vielen Varianten [6], da sich Belegungen auf nichttriviale Teilmengen von $[0, 1]$ einschränken lassen und eine Reihe neuer Logiken erzeugen.

Die Hauptresultate dieser Dissertation sind einerseits der Nachweis der Axiomatisierbarkeit der gültigen Formeln des propositionellen Fragments einer Erweiterung der Gödellogik um einen addierenden Operator durch einen hilbertschen Kalkül mit endlich vielen Schemata und andererseits der Beweis der Vermutung, dass sich die gültigen prädikatenlogischen Formeln derselben Erweiterung nicht rekursiv aufzählen lassen und sie daher kein berechenbarer Kalkül charakterisieren kann. Diese Dissertation enthält auch Ergebnisse wie z. B. die Nichtkompaktheit der Folgerungsrelation im propositionellen Fragment sowie eine Variante des Hebelemmas und die Komplexitätsklasse der propositionellen Erfüllbarkeit. Die Motivation, diese Erweiterung der Gödellogik zu untersuchen, war folgende:

Gödellogik und Łukasiewiczlogik sind gemeinsam mit der Produktlogik die Hauptvertreter der t-Norm-basierten Fuzzylogiken. Die Gültigkeit von Formeln in den propositionellen Fragmenten dieser Logiken kann genau durch Kalküle beschrieben werden, von denen jeder durch die Hinzunahme nur eines Axioms zum hájekschen Kalkül der Basislogik entsteht. Für die prädikatenlogischen Fragmente tritt hingegen das Phänomen ein, dass zwar die Gödellogik dank ihrem ordnungstheoretischen Hintergrund eine Charakterisierung der gültigen Formeln durch einen Kalkül zulässt, dass aber ein solcher Kalkül nach einem Resultat von Scarpellini [20] für die

Łukasiewiczlogik unmöglich ist. Nach Ragaz [19] ist die Łukasiewiczlogik sogar Π_2-vollständig. Zwischen Gödellogik und Łukasiewiczlogik ergibt sich daher ein im weiermannschen [23, 24] Sinne aufgefasster Phasenübergang. Diese Dissertation trägt zum besseren Verständnis bei, was diesen Übergang ausmacht: Die Komplexitätsklasse der Gültigkeit von Formeln in der Gödellogik wird durch die Eigenschaft des betrachteten Operators, eine Distanz von Wahrheitswerten auszudrücken, gehoben.

Abstract

First-order Gödel logic on [0, 1] is a prominent logic as it is—on one hand—an intermediate logic, i. e. it is stronger than intuitionistic logic, but weaker than classical logic, and—on the other—it is one of the three main t-norm-based fuzzy logics (see [15]). With classical logic, it shares properties like idempotency of the connectives $\wedge$, $\vee$, the equivalence scheme, the same complexity class and the downward Löwenheim-Skolem theorem. It is therefore a good starting point for the generalisation of well-known extensions of classical logic like, e. g. modal operators [11]. With intuitionistic logic, it shares a Kripke semantics; the proof system of Horn [17] for first-order Gödel logic encompasses first-order intuitionistic logic, the axiom of linearity and a quantifier shift rule. The semantic of Gödel logic gives rise to many variants [6], as the possibility to restrict interpretations to non-trivial subsets of [0, 1] creates a number of new logics.

The main results of this thesis are (1) a proof that a Hilbert-style calculus with finitely many axioms axiomatises the valid formulae of the propositional fragment of an extension of Gödel logic by an adding operator and (2) the demonstration of the conjecture that the valid formulae of the first-order fragment of the considered logic cannot be recursively enumerable so that, in particular, no recursive calculus for them can exist. The thesis includes also smaller results like the non-compactness of entailment in the propositional fragment as well as a variant of the lifting lemma and the complexity class of propositional satisfiability. In the following, the motivation to investigate this extension of Gödel logic will be sketched:

Gödel logic and Łukasiewicz logic are, together with product logic, the three main t-norm-based fuzzy logics. The validity of formulae in the propositional fragments of these logics can be exactly characterised by calculi that result from Hájek's basic logic by adjunction of a single axiom scheme. For first-order logic, however, we see the phenomenon that Gödel logics admits such a characterisation of the valid formulae thanks to its order-theoretic nature but that Łukasiewicz logic cannot have such a calculus due to a result by Scarpellini [20]. In fact, Ragaz [19] proved Łukasiewicz logic to be Π_2-complete. In the sense of Weiermann [23, 24], this leads to a phase transition between Gödel logic and Łukasiewicz logic.

This thesis contributes to a better understanding what constitutes this transition: The ability of the considered operator to express distances of truth values is the feature that increases the complexity class of the valid formulae in Gödel logic.

Oliver Fasching

Operator extensions of Gödel logics

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Contents

Chapter 1

Preface

§1.1 Introduction

The semantics of Gödel logic gives rise to a number of variants and extensions of the logic presented by Gödel [14]. The original propositional variant, built on the truth value set $\mathbb{N}$, answers Hahn's question negatively whether propositional intuitionistic logic has a characteristic finite matrix. Dummett [12] gave a sound and complete proof system for propositional Gödel logic by adjunction of the *linearity* axiom

$$(A \supset B) \vee (B \supset A)$$

to the intuitionistic calculus. Therefore Gödel logic is often called Gödel-Dummett logic, or LC as Dummett calls it. The interpretation of the connectives allowed Gödel to consider the restriction of this logic to a sequence of finite sets, whose valid formulae LC_n form a strictly decreasing tower. Classical logic contains all these LC_n, and LC is the intersection of all LC_n. The fact that Gödel logic is in between intuitionistic logic and classical logic makes it a so-called *intermediate* logic. Thomas [21] and Kubin [18] give proof systems for these restricted logics LC_n by adding to LC an axiom that expresses that only n truth values are available.

In Gödel's and Dummett's original formulation, the logic appears to be 'discrete' in a certain sense because the truth value set is taken to be $\mathbb{N}$; it also somewhat peculiar that $\wedge$ must be defined as the maximum, and $\vee$

as minimum. A more natural presentation can be obtained by taking (a countable subset of) $[0, 1]$ as the truth value set; see §3.2.4 for the according definitions. For propositional Gödel logics, this difference in presentation is neglectable, but it creates a rich structure for first-order logic. This way, Gödel logic can be understood also as a t-norm based fuzzy logic.

Gödel's interpretation of the implication agrees with the definition of the implication in t-norm based fuzzy logics on $[0, 1]$ as residuation. While $\wedge$ and $\vee$ correspond to continuous functions on the truth value set, this amounts to a non-continuous function for implication. On one hand, this feature is sometimes avoided in applications because similar inputs should yield similar outputs, but on the other it enables to crisply distinguish truth values (except at value 1) from each other: The criterion here is the ordering of truth values alone, not the distance. In short, Gödel logics is the logics of comparison; this formulation is supported by completeness w. r. t. linearly ordered Heyting algebras [17].

Hájek [15] showed that his proof system Basic Logic, which is sound and complete for t-norm fuzzy logics, plus the axiom of idempotency yields another sound and complete proof system for propositional Gödel logic. Idempotency causes Gödel logics to appear rather classical in many respects, e. g.: Contraction of $\wedge$, $\vee$ allows us to bring a conjunctive normal form into a disjunctive normal form and vice versa; The equivalence schema

$$(a \leftrightarrow b) \supset (E[a] \leftrightarrow E[b])$$

also holds so that substitutions can be made 'in depth'. (For other t-norms, the antecedent in the implication would need to be raised to a higher power.) Indeed, the two mentioned properties are corner stones in the proof of Dummett. Further shared properties are the deduction theorem, and the downward Löwenheim-Skolem theorem in the first-order fragment. This almost classical behaviour makes Gödel logic an ideal starting point for trying generalisations of extensions out that are well-understood for classical logic, e. g., extension by modal operators; see, e. g., [10], or [11] and the references contained therein. For the link between Gödel logic and Kripke semantics, see e. g. [9]. For connections to relevance logic see [13]. Beside the proof systems of Dummett and Hájek, a third way to axiomatise validity in Gödel logics was given by Avron [1]. His hypersequent calculus shows that also the analytical proof system $\mathrm{LK}^{\mathrm{prop}}$ of classical logic has a counterpart for Gödel logics.

The main motivation for the research in this thesis has been the question whether it is possible to transfer results on complexity of the validity problem between first-order Gödel logic and Łukasiewicz logic. First-order

Łukasiewicz logic is one of the oldest logics based on an infinite truth value set: In older literature, it is called just the infinite predicate calculus. Together with Gödel logic and product logic, it is one of the three major t-norm based fuzzy logics and it features connectives that induce continuous functions on the truth value set. Hájek's proof system Basic Logic constitutes a framework to uniformly handle the propositional fragments of Łukasiewicz logic, Gödel logic and product logic because the valid formulae in these fragments can be characterised by proof systems that arise from Basic Logic by adjunction of a single axiom scheme for the respective logic [15]. This gives a certain insight into the relation between them. However, for the first-order fragments, the valid formulae of Gödel logic are recursively enumerable, i. e. Σ_1, while for Łukasiewicz logics they are not [20], in fact they are Π_2-complete [19]. Due to their common background as t-norms, a phase transition in the sense of Weiermann [23, 24] can be observed and the question arises which (semantical) property is decisive for that phenomenon to appear. This thesis shows that the key to this phase transition is the ability to express an absolute distance. The adjunction of an adding operator to Gödel logic proved to provide the required semantical strength, but it can also be understood as an example of Gödel logic extended by a modal operator $\bigcirc$. We show that validity of the propositional fragment of Gödel logic extended by such an adding operator can be characterised by a sound and complete proof system, and that the valid formulae of the corresponding first-order fragment are not recursively enumerable. For the proof of the latter, we will use a translation between classical logic and the extended Gödel logic. Unfortunately, this translation involves a binary predicate symbol so that the complexity of the monadic fragment of the extended Gödel logic remains an open question. It is therefore unclear if there is a connection between this monadic fragment and the one of Łukasiewicz logic, where the problem on the decidability status of validity is also open.

In [16], Hájek investigates a very general setting and he considers a subdiagonal interpretation of the ring operator, i. e. $\mathrm{I}(\bigcirc A) \leq \mathrm{I}(A)$. Hájek's starting point is the system of Basic Logic together with the axioms $\bigcirc A \supset A$, $\bigcirc(A \supset B) \supset (\bigcirc A \supset \bigcirc B)$, $\bigcirc(A \vee B) \supset (\bigcirc A \vee \bigcirc B)$ and the rule $\frac{A}{\bigcirc A}$. The system we will use for propositional logic is of similar size but does not involve a rule beside modus ponens. We will consider the converse equality $\mathrm{I}(A) \leq \mathrm{I}(\bigcirc A)$ but a generalisation of the method presented in this thesis to almost order-preserving interpretations of the $\bigcirc$-operator seems to be possible for the propositional fragment, i. e., both $\mathrm{I}(\bigcirc A) < \mathrm{I}(A)$ and

		without $\triangle$			with $\triangle$	
	number of predicate symbols	1	2	3	1	2
VAL	finite V	✓	✓	✓	✓	✓
	$\mathsf{VAL}(V_\uparrow)$	✓	?	?	✓	✗
	infinite V except $\mathsf{VAL}(V_\uparrow)$	?	✗	✗	?	✗
	infinite V witnessed	?	✗	✗	?	✗
SAT	finite V	✓	✓	✓	✓	✓
	witnessed	✓	✓	✓	?	✗
	prenex	✓	✓	✓	?	✗
	0 isolated in V	✓	✓	✓	?	✗
	$\mathsf{SAT}(V_\downarrow)$	✓	?	?	✓	✗
	all other cases	?	?	✗	?	✗

Legend: ✓… decidable, ✗… undecidable, ?… unknown

Figure 1.1: Decidability status

$\mathrm{I}(B) < \mathrm{I}(\bigcirc B)$ may occur for propositional variables A and B, for the same interpretation I; cf. §4.2.

In contrast to the other two main t-norm fuzzy logics, Łukasiewicz logic and product logic, the semantic of Gödel logics allows us to consider nontrivial restrictions of the truth value set: For a detailed analysis on validity and entailment, see, e. g., [6] and [5]. Table §1.1 is meant to give an impression of the diversity of open questions left; there, VAL means validity, SAT satisfiability, $V_\uparrow := \{1\} \cup \{1 - \frac{1}{n};\, n \geq 1\}$, and $V_\downarrow := \{0\} \cup \{\frac{1}{n};\, n \geq 1\}$. This topic goes beyond this thesis, but preliminary results of the adding operator on a finite set have been obtained and further research on it is planned. Table §1.1 can also serve as a starting point for many other 'combinatorial' questions, i. e., one can ask for the complexity class of validity or satisfiability, given a restricted truth value set and an operator that does not add (which would yield values that are not contained in the truth value set) but skips a given number of truth values etc.

§1.2 Technical remarks

Although I have written this thesis alone, I shall adopt the usual academic habit of using the majestic plural.

We will work in ZFC exclusively. The set $\mathbb{N}$ of natural numbers includes

0. If $\overline{x}$ is a list $x_0, \ldots, x_{n-1}$ of bound variables, let $\forall\overline{x}$ and $\exists\overline{x}$ abbreviate $\forall x_0 \ldots \forall x_{n-1}$ and $\exists x_0 \ldots \exists x_{n-1}$. Propositional variables are denoted by Var; Free variables by FrVar, and bounded ones by BdVar. However, we often will not strictly distinguish free and bounded ones.

§1.2.1 Convention. We will tacitly apply the standard practice to re-use a meta variable ranging in the domain of a first-order interpretation as a constant, which is to be interpreted by itself; e. g., in an expression like $\inf_{x \in |\mathcal{I}|} A(x)$, where A is a context with a term gap. We will proceed similarly with lists or vectors of meta variables. We will only occasionally write $A(\underline{x})$ to emphasise that we deal with a constant. ☡

For philosophical reasons, we will avoid the term "tautology" and instead speak of *valid* formulae, of course, this depends on a given semantics.

§1.3 Overview on the contents

In §2, we will present Scarpellini's theorem in a way that allows us to generalise his theorem for our purposes. In §3, we will introduce an extension of Gödel logic by an operator that carries out a limited form of addition (§3.2.4), discuss entailment relations (§3.2.9), show that a theorem analogue to the one of Scarpellini holds for the first-order prenex fragment of this logic, and prove the soundness and completeness of propositional logic (§3.4.19) w. r. t. a proof system that extends Dummett's original proof system for Gödel logic. Likewise, Hájek's Basic Logic can be used. We generalise this result to the fragment that contains the Baaz-Takeuti-Titani $\triangle$-operator (§3.5.4). We briefly discuss satisfiability in the propositional fragment (§3.4.23).

Chapter 2

Scarpellini's theorem revisited

§2.1 Introduction

Although it is an established result of Scarpellini that the first-order formulae valid w. r. t. Łukasiewicz semantics are not recursively enumerable, his method of proof is not widely known, apparently because his article [20] was published in German. Since we will transfer this method to Gödel logic with ring, we will present it here in a way that fits our needs and we will repeat all major steps. Note carefully that the content of §2.2.1–§2.2.9 is contained in [20], sometimes only implicitly and with a deviating notation.

Scarpellini's idea rests on a reduction of validity in Łukasiewicz semantics to classical validity in finite models by specifying an effective translation of formulae. As §2.2.4 will show, the classical values 0 and 1 correspond to closed intervals S_0 and S_1 in the truth value set $[0, 1]$ of Łukasiewicz logics. As Trakhtenbrot's theorem [22] says that classical validity in finite models is not recursively enumerable, we conclude that validity in Łukasiewicz semantics is it also.

Scarpellini's idea does not give an answer whether the monadic fragment of Łukasiewicz logic is decidable because the full fragment of classical logic is embedded to Łukasiewicz logic. This problem is still open.

Many aspects of Łukasiewicz logic are covered in [15]. We give here only the definition of the syntax and the semantics of Łukasiewicz logic.

§2.1.1 Definition (Łukasiewicz logic). Let $\mathscr{L}'_{Ł}$ be the first-order language with connectives $\supset/2$ and $\perp/0$, with a quantifier $\forall$, without equality, without function symbols, without constants, but with predicate symbols of any arity, including nullary predicates. $\mathsf{Fm}'_{Ł}$ denotes the set of formulae in $\mathscr{L}'_{Ł}$. Other connectives are introduced as abbreviations: $\neg A := A \supset \perp$, $\top := \perp \supset \perp$, $A \wedge B := \neg(A \supset \neg B)$, $A \vee B := \neg A \supset B$, $A \dot{-} B := A \wedge \neg B$, $A \uparrow B := (A \supset B) \supset B$, $A \downarrow B := A \wedge (A \supset B)$, $\mathrm{dist}(A,B) := (A \dot{-} B) \uparrow (B \dot{-} A)$, $\Downarrow_{i=1}^{n} A_i := A_1 \downarrow (A_2 \downarrow (\ldots \downarrow A_n))$, and $\exists x A := \neg \forall x \neg A$.

A Łukasiewicz interpretation I consists (1) of a non-empty domain $|\mathrm{I}|$, (2) of an interpretation of predicate symbols $\mathrm{I}(P)\colon\ |\mathrm{I}|^n \to [0,1]$ for each n-ary predicate symbol P, (3) of an interpretation $\mathrm{I}\colon\ \mathsf{FrVar} \to |\mathrm{I}|$, (4) of an interpretation $\mathrm{I}\colon\ \mathsf{Fm}'_{Ł} \to [0,1]$ such that $\mathrm{I}(\perp) = 0$, $\mathrm{I}(A \supset B) = \min\{1, 1 - \mathrm{I}(A) + \mathrm{I}(B)\}$, $\mathrm{I}(\forall x A(x)) = \inf_{x \in |\mathrm{I}|} \mathrm{I}(A(\underline{x}))$, cf. §1.2.1. It is immediately clear that the interpretation $\mathrm{I}(A)$ of any formula A without free variables does not depend on (3). ☡

§2.1.2 Proposition. With the notation from above, the following hold:

$\mathrm{I}(\neg A) = 1 - \mathrm{I}(A)$,
$\mathrm{I}(\top) = 1$,
$\mathrm{I}(A \wedge B) = \max\{0, \mathrm{I}(A) + \mathrm{I}(B) - 1\}$,
$\mathrm{I}(A \vee B) = \min\{1, \mathrm{I}(A) + \mathrm{I}(B)\}$,
$\mathrm{I}(A \dot{-} B) = \max\{0, \mathrm{I}(A) - \mathrm{I}(B)\}$,
$\mathrm{I}(A \uparrow B) = \max\{\mathrm{I}(A), \mathrm{I}(B)\}$,
$\mathrm{I}(A \downarrow B) = \min\{\mathrm{I}(A), \mathrm{I}(B)\}$,
$\mathrm{I}(\mathrm{dist}(A,B)) = |\mathrm{I}(A) - \mathrm{I}(B)|$,
$\mathrm{I}(\Downarrow_{i=1}^{n} A_i) = \min_{i=1}^{n} \mathrm{I}(A_i)$,
$\mathrm{I}(\exists x P(x)) = \sup_{x \in |\mathrm{I}|} \mathrm{I}(A(\underline{x}))$. ☡

We omit the elementary proof. The above proposition justifies that our approach of defining $\wedge$ and $\vee$ as abbreviations is semantically equivalent to the introduction of $\wedge$ and $\vee$ as connectives in their own right and with their own semantics. We neglect the fact that this approach changes proof-theoretic properties like the length of proofs because we are not interested in these in the following. We will save parentheses in expressions like $A_1 \wedge A_2 \wedge A_3$, $A_1 \vee A_2 \vee A_3$, $A_1 \uparrow A_2 \uparrow A_3$, and $A_1 \downarrow A_2 \downarrow A_3$ since §2.1.2 shows immediately that parenthesisation is irrelevant.

§2.2 Scarpellini's proof

§2.2.1 Definition. It is assumed that Q will always denote a nullary predicate symbol in the remainder. For $eA := A \vee A$, the parentheses are omitted for the sake of brevity. For every n-ary predicate symbol P, define

$$\Gamma(Q;P) := \forall\overline{x}(P(\overline{x}) \downarrow (eQ \dot{-} P(\overline{x})) \downarrow \mathrm{dist}(P(\overline{x}), Q))$$

for a fresh list $\overline{x}$ of n bound variables. For all predicates $P_1, \ldots, P_m$, let

$$\Lambda(Q;P_1,\ldots,P_m) := Q \downarrow \neg eQ \downarrow \Gamma(Q;P_1) \downarrow \ldots \downarrow \Gamma(Q;P_m).$$ ☡

§2.2.2 Proposition. Suppose I is a Łukasiewicz interpretation, $P_1, \ldots, P_m$ are predicate symbols and Q is a nullary predicate symbol such that

$$0 < \delta \leq \mathrm{I}(\Lambda(Q;P_1,\ldots,P_m)).$$

If P_i, $i < m$, is an n-ary predicate symbol, then

$$\delta \leq \mathrm{I}(Q) - \delta < \mathrm{I}(Q) + \delta \leq 2\cdot\mathrm{I}(Q) - \delta < 2\cdot\mathrm{I}(Q) + \delta \leq 1$$

and

$$\forall x \in |\mathrm{I}|^n.\ \mathrm{I}(P_i)(x) \in [\delta, \mathrm{I}(Q) - \delta] \cup [\mathrm{I}(Q) + \delta, 2\cdot\mathrm{I}(Q) - \delta].$$ ☡

Proof. Since

$$\begin{aligned}0 < \delta &\leq \mathrm{I}(\Lambda(Q;P_1,\ldots,P_m)) = \\ &= \mathrm{I}(Q \downarrow \neg eQ \downarrow \Gamma(Q;P_1) \downarrow \ldots \downarrow \Gamma(Q;P_m)) = \\ &= \min\{\mathrm{I}(Q),\ 1 - \min\{1, 2\cdot\mathrm{I}(Q)\},\ \mathrm{I}(\Gamma(Q;P_1)),\ldots,\mathrm{I}(\Gamma(Q;P_m))\},\end{aligned}$$

we have $\delta \leq \mathrm{I}(Q)$, $0 < \delta \leq \max\{0, 1 - 2\cdot\mathrm{I}(Q)\}$ and $\delta \leq \mathrm{I}(P_i(x))$, $0 < \delta \leq \max\{0, \min\{1, 2\cdot\mathrm{I}(Q)\} - \mathrm{I}(P_i(x))\}$, $\delta \leq |\mathrm{I}(P_i(x)) - \mathrm{I}(Q)|$ for all $x \in |\mathrm{I}|^n$. We conclude the following for every $x \in |\mathrm{I}|^n$: $\delta \leq \mathrm{I}(Q)$, $\delta \leq 1 - 2\cdot\mathrm{I}(Q)$, $\delta \leq \mathrm{I}(P_i(x))$, $\delta \leq \min\{1, 2\cdot\mathrm{I}(Q)\} - \mathrm{I}(P_i(x))$, and moreover, (1) $\delta \leq \mathrm{I}(Q) - \mathrm{I}(P_i(x))$ or (2) $\delta \leq \mathrm{I}(P_i(x)) - \mathrm{I}(Q)$. We clearly have $2\cdot\mathrm{I}(Q) - \delta < 2\cdot\mathrm{I}(Q) + \delta \leq 1$. In case (1), we find $\delta \leq \mathrm{I}(P_i(x)) \leq \mathrm{I}(Q) - \delta$ so that then $\delta \leq \mathrm{I}(Q) - \delta$ and $\mathrm{I}(Q) + \delta \leq 2\cdot\mathrm{I}(Q) - \delta$. In case (2), we see $\mathrm{I}(Q) + \delta \leq \mathrm{I}(P_i(x)) \leq \min\{1, 2\cdot\mathrm{I}(Q)\} - \delta \leq 2\cdot\mathrm{I}(Q) - \delta$ so that also now we obtain $\mathrm{I}(Q) + \delta \leq 2\cdot\mathrm{I}(Q) - \delta$ and thus $\delta \leq \mathrm{I}(Q) - \delta$. This proves all the claimed inequalities since $|\mathrm{I}| \neq \emptyset$. ♢

§2.2.3 Definition. The variant $\mathscr{L}'_{\mathrm{CL}}$ of the language of classical first-order logic is generated by $\bot/0$, $\neg/1$, $\wedge/2$, $\forall$. The set of formulae is denoted by $\mathsf{Fm}'_{\mathrm{CL}}$ and the set of semi formulae by $\mathsf{SFm}'_{\mathrm{CL}}$. It is needless to introduce the semantics of $\mathscr{L}'_{\mathrm{CL}}$. We use the abbreviations $A \vee B := \neg(\neg A \wedge \neg B)$, $A \supset B := \neg(A \wedge \neg B)$, $A \leftrightarrow B := \neg(\neg(A \wedge B) \wedge \neg(\neg A \wedge \neg B))$, and $\top := \neg\bot$.

Define a transformation α_Q: $\mathsf{SFm}'_{\mathrm{CL}} \to \mathsf{SFm}'_{\text{Ł}}$ recursively as follows: $\alpha_Q(P) := P$ for any atom P, $\alpha_Q(\neg A) := eQ \dot{-} \alpha_Q(A)$, $\alpha_Q(A \wedge B) := \alpha_Q(A) \downarrow \alpha_Q(B)$, $\alpha_Q(\forall x A) := \forall x \alpha_Q(A)$. Since $\wedge$ in $\mathscr{L}'_{\mathrm{CL}}$ and $\downarrow$ in $\mathscr{L}'_{\text{Ł}}$ are both associative w. r. t. semantics, we will neglectfully write $\alpha_Q(A \wedge B \wedge C) = \alpha_Q(A) \downarrow \alpha_Q(B) \downarrow \alpha_Q(C)$ as we are not interested in proof-theoretic properties. ☡

Next, we will prove that $\alpha_Q(A)$ embeds classical logic into Łukasiewicz logic.

§2.2.4 Proposition. Suppose I is a Łukasiewicz interpretation, Q is a nullary predicate symbol, $P_1, \dots, P_m$ are predicate symbols such that

$$0 < \delta \leq \mathrm{I}(\Lambda(Q; P_0, \dots, P_m))$$

for some δ. Define $S_0 := [\delta, \mathrm{I}(Q) - \delta]$, $S_1 := [\mathrm{I}(Q) + \delta, 2 \cdot \mathrm{I}(Q) - \delta]$, and $S := S_0 \cup S_1$. Let $A, B \in \mathsf{Fm}'_{\text{Ł}}(P_1, \dots, P_m)$, and suppose $C(\cdot)$ is a context in $\mathsf{Fm}'_{\text{Ł}}(P_1, \dots, P_m)$ with a gap to be filled by an unbound variable or constant. Then the following hold:

(a) If $\mathrm{I}(\alpha_Q(A)) \in S_i$, $i \in \{0, 1\}$, then $\mathrm{I}(\alpha_Q(\neg A)) \in S_{1-i}$.
(b) If $\mathrm{I}(\alpha_Q(A)) \in S_i$ and $\mathrm{I}(\alpha_Q(B)) \in S_j$, then $\mathrm{I}(\alpha_Q(A \wedge B)) \in S_{\min\{i,j\}}$.
(c) If $\mathrm{I}(\alpha_Q(A)) \in S_i$ and $\mathrm{I}(\alpha_Q(B)) \in S_j$, then $\mathrm{I}(\alpha_Q(A \vee B)) \in S_{\max\{i,j\}}$.
(d) If $\mathrm{I}(\alpha_Q(C(\underline{x}))) \in S_{i(x)}$ for all $x \in |\mathrm{I}|$, then $\mathrm{I}(\alpha_Q(\forall x C(x))) \in S_{\min\{i(x);\, x \in |\mathrm{I}|\}}$.
(e) $\mathrm{I}(\alpha_Q(A)) \in S$.
(f) Moreover, if I' is a classical interpretation such that $|\mathrm{I}| = |\mathrm{I}'|$, $\mathrm{I}'(a) = \mathrm{I}(a)$ for all free variables a, and

$$\mathrm{I}'(P_i)(d_1, \dots, d_n) = \begin{cases} 1 & \text{if } \mathrm{I}(P_i)(d_1, \dots, d_n) \in S_1 \\ 0 & \text{if } \mathrm{I}(P_i)(d_1, \dots, d_n) \in S_0, \end{cases}$$

then $\mathrm{I}(\alpha_Q(A)) \in S_{\mathrm{I}'(A)}$. ∂

Proof. By §2.2.2, we have $\delta \leq \mathrm{I}(Q) - \delta < \mathrm{I}(Q) + \delta \leq 2 \cdot \mathrm{I}(Q) - \delta < 2 \cdot \mathrm{I}(Q) < 2 \cdot \mathrm{I}(Q) + \delta \leq 1$ and $\forall x \in |\mathrm{I}|^{n_i}.\ \mathrm{I}(P_i)(x) \in S$ for all $i < m$; here n_i is the arity of P_i. In particular, $S_1 \cap S_2 = \emptyset$.

For item (a), observe $\mathrm{I}(eQ \dot{-} F) = \max\{0, \min\{1, 2 \cdot \mathrm{I}(Q)\} - \mathrm{I}(F)\} = \max\{0, 2 \cdot \mathrm{I}(Q) - \mathrm{I}(F)\}$. If $\mathrm{I}(F) \in S_0$, i. e. $\delta \leq \mathrm{I}(F) \leq \mathrm{I}(Q) - \delta$, it follows $\mathrm{I}(Q) + \delta = 2 \cdot \mathrm{I}(Q) - (\mathrm{I}(Q) - \delta) \leq 2 \cdot \mathrm{I}(Q) - \mathrm{I}(F) \leq 2 \cdot \mathrm{I}(Q) - \delta$, i. e. $\mathrm{I}(eQ \dot{-} F) \in S_1$. If $\mathrm{I}(F) \in S_1$, i. e. $\mathrm{I}(Q) + \delta \leq \mathrm{I}(F) \leq 2 \cdot \mathrm{I}(Q) - \delta$, it follows $\delta = 2{\cdot}\mathrm{I}(Q) - (2{\cdot}\mathrm{I}(Q) - \delta) \leq 2{\cdot}\mathrm{I}(Q) - \mathrm{I}(F) \leq 2{\cdot}\mathrm{I}(Q) - (\mathrm{I}(Q) + \delta) = \mathrm{I}(Q) - \delta$, i. e. $\mathrm{I}(eQ \dot{-} F) \in S_0$.

Item (b) follows immediately from $\mathrm{I}(F \downarrow G) = \min\{\mathrm{I}(F), \mathrm{I}(G)\}$. Item (c) follows from (a) and (b). Item (d) follows from $\mathrm{I}(\alpha_Q(\forall x C(x))) = \mathrm{I}(\forall x \alpha_Q(C(x))) = \inf_{x \in |\mathrm{I}|} \mathrm{I}(\alpha_Q(C(x)))$ and from the fact that both S_0 and S_1 are closed and disjoint. Item (e) follows by induction on the complexity of (semi) formulae: The case of atoms was dealt with in §2.2.2, and the induction steps are explained in (a), (b), (c). Observe that the case of atoms is trivial for (e); the result follows then by induction and (a), (b), (c). ◇

The formula $\Phi_{\mathrm{CL}}(P, a, b)$ in the following definition will help us to factorise an infinite model to a finite one by identifying elements in the model that behave in the same way for all predicate symbols.

§2.2.5 Definition. For any $(n+1)$-ary predicate symbol P and variables a, b, we define the following formula in $\mathscr{L}'_{\mathrm{CL}}$

$$\begin{aligned}\Phi_{\mathrm{CL}}(P,a,b) := \forall x_1 \dots \forall x_n((P(a,x_1,\dots,x_n) \leftrightarrow P(b,x_1,\dots,x_n)) \wedge \\ \wedge (P(x_1,a,x_2,\dots,x_n) \leftrightarrow P(x_1,b,x_2,\dots,x_n)) \wedge \\ \wedge \dots \wedge (P(x_1,x_2,\dots,x_n,a) \leftrightarrow P(x_1,x_2,\dots,x_n,b))).\end{aligned}$$

For any nullary predicate symbol P, define $\Phi_{\mathrm{CL}}(P,a,b) := \top$. ☡

The easy proof of the following proposition is omitted.

§2.2.6 Proposition. Let P be a predicate symbol and I' be a classical interpretation. Then Φ_{CL} induces an equivalence relation on $|\mathrm{I}'|$ in the following sense: $\mathrm{I}'(\Phi_{\mathrm{CL}}(P,a,a)) = 1$; $\mathrm{I}'(\Phi_{\mathrm{CL}}(P,a,b)) = \mathrm{I}'(\Phi_{\mathrm{CL}}(P,b,a))$; moreover, $\mathrm{I}'(\Phi_{\mathrm{CL}}(P,a,b)) = \mathrm{I}'(\Phi_{\mathrm{CL}}(P,b,c)) = 1$ implies $\mathrm{I}'(\Phi_{\mathrm{CL}}(P,a,c)) = 1$. ☡

§2.2.7 Definition. For any predicate symbol P and semiterms r, s, let $\Phi(Q;P,r,s)$ denote the $\mathscr{L}'_{Ł}$-formula $\alpha_Q(\Phi_{\mathrm{CL}}(P,r,s))$. For all predicate symbols $P_1, \dots, P_M$, and every unary predicate symbol G, define the $\mathscr{L}'_{Ł}$-formula

$$\begin{aligned}\Psi(Q;G,P_1,\dots,P_M) &:= \forall x \forall y \\ &(\mathrm{dist}(G(x),G(y)) \uparrow ((\Downarrow_{m\le M} \Phi(Q;P_m,x,y)) \dot{-} Q)).\end{aligned}$$

In $\mathscr{L}'_{Ł}$, let

$$\begin{aligned}\Xi(Q;A,B) &:= \\ &eQ \dot{-} (eQ \dot{-} (A \downarrow B) \downarrow eQ \dot{-} ((eQ \dot{-} A) \downarrow (eQ \dot{-} B))).\end{aligned}$$ ☡

It is easy to see that then $\Phi(Q;P,r,s)$ is

$$\begin{aligned}\forall x_1 \dots \forall x_n(\Xi(Q;P(a,x_1,\dots,x_n),P(b,x_1,\dots,x_n)) \downarrow \\ \downarrow \Xi(Q;P(x_1,a,x_2,\dots,x_n),P(x_1,b,x_2,\dots,x_n)) \downarrow \\ \downarrow \dots \downarrow \Xi(Q;P(x_1,x_2,\dots,x_n,a),P(x_1,x_2,\dots,x_n,b))).\end{aligned}$$

§2.2.8 Theorem. Let $P_1, \dots, P_M$ be predicate symbols, and suppose we have $A \in \mathsf{Fm}'_{\mathrm{CL}}(P_1, \dots, P_M)$, let Q be a fresh nullary predicate symbol, and G a fresh unary predicate symbol. Then the following statements are equivalent:
(1) There is a classical interpretation with finite domain that satisfies A.
(2) There is a Łukasiewicz interpretation I such that

$$0 < \mathrm{I}((\alpha_Q(A) \dot{-} Q) \downarrow \Lambda(Q;P_1,\dots,P_M) \downarrow \Psi(Q;G,P_1,\dots,P_M)).$$ ☡

Proof. (1)→(2): Let I' be a classical interpretation with finite domain $B := \{b_1,\dots,b_N\}$ such that $\mathrm{I}'(A) = 1$. We define a Łukasiewicz interpretation I as follows: Let I have the domain B; interpret free variables as in I'; take

$\mathrm{I}(G(\underline{b_i})) := \frac{i}{5N}$, $\mathrm{I}(Q) := \frac{2}{5}$, and take

$$\mathrm{I}(P_m(d_1,\dots,d_n)) := \tfrac{3}{5} \text{ if } \mathrm{I}'(P_m(d_1,\dots,d_n)) = 1$$

and

$$\mathrm{I}(P_m(d_1,\dots,d_n)) := \tfrac{1}{5} \text{ if } \mathrm{I}'(P_m(d_1,\dots,d_n)) = 0$$

for all $d_j \in B$ and $m \in \{1, \dots, M\}$.

In order to establish $\mathrm{I}(\alpha_Q(A)) = \frac{3}{5}$, we prove $\mathrm{I}(\alpha_Q(C)) = \frac{3}{5}$ whenever $\mathrm{I}'(C) = 1$, and $\mathrm{I}(\alpha_Q(C)) = \frac{1}{5}$ whenever $\mathrm{I}'(C) = 0$ by induction on the complexity of the formula C in $\mathscr{L}'_{\mathrm{CL}}$. The claim is true for atomic formulae by definition. Suppose the claim already holds for formulae E and F. If $\mathrm{I}(E) = \frac{1}{5}$, then $\mathrm{I}(eQ \dot{-} E) = \max\{0, \min\{1, \frac{2}{5} + \frac{2}{5}\} - \mathrm{I}(C)\} = \frac{3}{5}$; if $\mathrm{I}(E) = \frac{3}{5}$, we similarly find $\mathrm{I}(eQ \dot{-} E) = \frac{1}{5}$. Thus the claim is true also for $\neg E$. Clearly, we have $\mathrm{I}(\alpha_Q(E \vee F)) = \mathrm{I}(\alpha_Q(E) \uparrow \alpha_Q(E)) = \max\{\mathrm{I}(\alpha_Q(E)), \mathrm{I}(\alpha_Q(F))\}$ so that the claim is true for $E \vee F$. Suppose the claim holds for all instances $E(\underline{c})$, $c \in |I|$, of a formula $E(\cdot)$ with a term gap. In particular, we have $\mathrm{I}(\alpha_Q(E(\underline{c}))) \in \{\frac{1}{5}, \frac{3}{5}\}$. Now $\mathrm{I}(\alpha_Q(\forall x E(x))) = \mathrm{I}(\forall x.\alpha_Q(E(x))) = \inf_{x\in B} \mathrm{I}(\alpha_Q(E(\underline{x}))) = \min_{x\in B} \mathrm{I}(\alpha_Q(E(\underline{x})))$ establishes the claim also for the quantifier.

Since $\mathrm{I}(\alpha_Q(A)) = \frac{3}{5}$, we find $\mathrm{I}(\alpha_Q(A) \dot{-} Q) = \max\{0, \frac{3}{5} - \frac{2}{5}\} = \frac{1}{5}$ and $\mathrm{I}(\neg eQ) = \max\{0, 1 - \min\{1, 2 \cdot \mathrm{I}(Q)\}\} = \frac{1}{5}$. If P_m has arity d, we have

$$\mathrm{I}(\Gamma(Q;P_m)) = \min_{\overline{x}\in B^d} \min \{\mathrm{I}(P_m)(\overline{x}), \max\{0, \min\{1, 2\cdot\tfrac{2}{5}\} - \mathrm{I}(P_m)(\overline{x})\}, |\mathrm{I}(P_m)(\overline{x}) - \tfrac{2}{5}|\}$$

so that $\mathrm{I}(\Gamma(Q;P_m)) \geq \min\{\frac{1}{5}, \frac{4}{5} - \frac{3}{5}, \frac{1}{5}\} \geq \frac{1}{5}$. Thus $\mathrm{I}(\Lambda(Q;P_1,\dots,P_M)) = \frac{1}{5}$.

Observe that $1 \leq i < j \leq N$ implies $\mathrm{I}(\mathrm{dist}(G(\underline{b_i}), G(\underline{b_j}))) = |\mathrm{I}(G(\underline{b_i})) - \mathrm{I}(G(\underline{b_j}))| = |\frac{i}{5N} - \frac{j}{5N}| \geq \frac{1}{5N}$. If F is a formula such that $\mathrm{I}(F) \in \{\frac{1}{5}, \frac{3}{5}\}$, then it is easy to verify that $\mathrm{I}(\Xi(Q;F,F)) = \max\{\mathrm{I}(eQ \dot{-} F), \mathrm{I}(eQ \dot{-} (eQ \dot{-} F))\} = \max\{\mathrm{I}(F), 0, \frac{4}{5} - \mathrm{I}(F)\} = \frac{3}{5}$; in particular, if F is $P_m(x_1, \dots, x_{k-1}, \underline{a}, x_k, \dots, x_n)$, where $a \in B$, $\overline{x} \in B^n$, $m \in \{1, \dots, M\}$, and $1 \leq k \leq n$. Therefore $\mathrm{I}((\Downarrow_{m\leq M} \Phi(Q;P_m,\underline{a},\underline{a})) \dot{-} Q) = \max\{0, (\min_{m\leq M} \mathrm{I}(\Phi(Q;P_m,\underline{a},\underline{a}))) - \mathrm{I}(Q)\} = \max\{0, \frac{3}{5} - \frac{2}{5}\} = \frac{1}{5}$ for all $a \in B$. Thus $\mathrm{I}(\Psi(Q;G,P_1,\dots,P_M)) \geq \frac{1}{5N} > 0$. This proves the claim.

(2)→(1): Let I be a Łukasiewicz interpretation and $\delta \in \mathbb{R}$ such that

$$0 < \delta < \mathrm{I}((\alpha_Q(A) \dot{-} Q) \downarrow \Lambda(Q;P_1,\dots,P_M) \downarrow \Psi(Q;G,P_1,\dots,P_M)).$$

From $\delta < \mathrm{I}(\Lambda(Q;P_1,\dots,P_M))$ and §2.2.4, we obtain $\delta \leq \mathrm{I}(Q) - \delta < \mathrm{I}(Q) + \delta \leq 2 \cdot \mathrm{I}(Q) - \delta < 2 \cdot \mathrm{I}(Q) + \delta \leq 1$ and $\mathrm{I}(\alpha_Q(A)) \in S := S_0 \cup S_1$, where $S_0 := [\delta, \mathrm{I}(Q) - \delta]$ and $S_1 := [\mathrm{I}(Q) + \delta, 2 \cdot \mathrm{I}(Q) - \delta]$. Since $0 < \delta < \mathrm{I}(\alpha_Q(A) \dot{-} Q) = \max\{0, \mathrm{I}(\alpha_Q(A)) - \mathrm{I}(Q)\}$, we find $\delta < \mathrm{I}(\alpha_Q(A)) - \mathrm{I}(Q)$ so that $\mathrm{I}(\alpha_{\mathrm{Q}}(A)) \in S_1$. Since $\delta < \mathrm{I}(\Psi(Q;G,P_1,\ldots,P_M))$, we find for each pair $p,\, q \in |\mathrm{I}|$ that

$$\delta \leq \max\{|\mathrm{I}(G(p)) - \mathrm{I}(G(q))|, \max\{0, (\min_{m\leq M} \mathrm{I}(\Phi(Q;P_m,p,q))) - \mathrm{I}(Q)\}\},$$

so that

$$\delta \leq |\mathrm{I}(G(p)) - \mathrm{I}(G(q))| \vee \delta + \mathrm{I}(Q) \leq \min_{m\leq M} \mathrm{I}(\Phi(Q;P_m,p,q)).$$

The relation $\sim$ on $|\mathrm{I}|$ defined by

$$p \sim q \;:\Leftrightarrow\; \mathrm{I}(\alpha_{\mathrm{Q}}(\textstyle\bigwedge_{m\leq M} \Phi_{\mathrm{CL}}(P_m,p,q))) \in S_1\,.$$

is an equivalence relation as can easily be seen by §2.2.6 and, §2.2.4(f).

By §2.2.4(c), we have $p \not\sim q$ if and only if $\mathrm{I}(\alpha_{\mathrm{Q}}(\Phi_{\mathrm{CL}}(P_m,p,q))) \in S_0$ for some $m \leq M$; in particular, $p \not\sim q$ implies $\mathrm{I}(\Phi(Q;P_m,p,q)) \leq \mathrm{I}(Q) - \delta$ for some $m \leq M$, and thus $\delta \leq |\mathrm{I}(G(p)) - \mathrm{I}(G(q))|$ by the above. It follows immediately that we have only a finite number of equivalence classes for otherwise $\{\mathrm{I}(G(p_i))\}_i \subseteq [0,1]$ were infinite and thus had an accumulation point, which is absurd.

Let I' be the classical interpretation from §2.2.4(f) so that we obtain $\mathrm{I}'(A) = 1$ from $\mathrm{I}(\alpha_{\mathrm{Q}}(A)) \in S_1$, $\mathrm{I}(\alpha_{\mathrm{Q}}(A)) \in S_{\mathrm{I}'(A)}$ and $S_0 \cap S_1 = \emptyset$. Let I'_0 be the classical interpretation with $|\mathrm{I}''| := |\mathrm{I}'|/\sim$ and $\mathrm{I}''(P_i)([d_1],\ldots,[d_n]) := \mathrm{I}'(P_i)(d_1,\ldots,d_n)$, where $[d]$ denotes the equivalence class of d w. r. t. $\sim$; this is well-defined since $p_1 \sim q_1,\, \ldots,\, p_n \sim q_n$ implies $\mathrm{I}(\alpha_{\mathrm{Q}}(\Phi_{\mathrm{CL}}(P_i,p_k,q_k))) \in S_1$ and thus $\mathrm{I}'(\Phi_{\mathrm{CL}}(P_i,p_k,q_k)) = 1$ for all k by §2.2.4(f), which implies $\mathrm{I}'(P_i)(p_1,\ldots,p_n) = \mathrm{I}'(P_i)(q_1,\ldots,q_n)$ in turn. By induction on the complexity of a formula, it is easily seen that $\mathrm{I}''(B) = \mathrm{I}'(B)$ holds for all formulae B, in particular, we find $\mathrm{I}''(A) = \mathrm{I}'(A) = 1$. This completes the proof since I'' has a finite domain. ◇

We recall that a formula is *closed* if it does not contain free variables.

§2.2.9 Theorem. The set of all closed valid first-order formulae in Łukasiewicz logics is not recursively enumerable. ☡

Proof. Choosing a fixed ordering among all predicate symbols in $\mathsf{Fm}'_{\mathrm{CL}}$, one obtains a computable function $\beta\colon\ \mathsf{Fm}'_{\mathrm{CL}} \to \mathsf{Fm}'_{\text{Ł}}$, $A \mapsto (\alpha_Q(A) \mathbin{\dot-} Q) \downarrow \Lambda(Q;P_1,\ldots,P_M) \downarrow \Psi(Q;G,P_1,\ldots,P_M)$, where $P_1,\, \ldots,\, P_M$ are all the predicate symbols appearing in A, Q is a fresh nullary predicate symbol, and G is a fresh unary predicate symbol. For any closed $A \in \mathsf{Fm}'_{\mathrm{CL}}$, we find that the following statements are equivalent:

$\forall\mathrm{I}$, classical interpretation with finite domain: $\mathrm{I}(A) = 1$

$\neg\exists\mathrm{I}$, classical interpretation with finite domain: $\mathrm{I}(A) = 0$

$\neg\exists\mathrm{I}$, classical interpretation with finite domain: $\mathrm{I}(\neg A) = 1$

$\neg\exists$I, Łukasiewicz interpretation: $\mathrm{I}(\beta(\neg A)) > 0$; by §2.2.8
$\forall$I, Łukasiewicz interpretation: $\mathrm{I}(\beta(\neg A)) = 0$
$\forall$I, Łukasiewicz interpretation: $\mathrm{I}(\neg\beta(\neg A)) = 1$; by §2.1.2

Thus, if the set of all closed valid first-order formulae in Łukasiewicz logics was recursively enumerable, then also the classical formulae valid in all finite domains would be, contradicting Trakhtenbrot's theorem. ♢

§2.2.10 Remark. Let V be the set of formulae that are valid in Łukasiewicz logics. The result §2.2.9 just says that V is not in Σ_1: V contains a Π_1-set (via the embedding, this is the set of formulae that are valid in all finite classical models). Ragaz [19] proved that V is actually Π_2-complete; see [15, p. 162ff, 6.3.4–6.3.18] for further complexity results on Gödel, Łukasiewicz and product logic and other classes. ≀

Chapter 3

Gödel logics with ring

§3.1 Introduction

In §3.2, we will introduce all underlying arithmetics of Gödel logic (§3.2.2), the languages we will use, and the semantics of Gödel logic. We will define validity and we will also show that the entailment relation is not compact, even for the propositional fragment. Already in §3.3, we will start with predicate logic and transfer Scarpellini's result to an operator extension of Gödel logic. We postpone propositional logic to §3.4 because the definitions of the proof systems we will work with consume a large space and the proof of the main result is rather long, though not difficult. We will conclude with some results on satisfiability.

§3.2 Syntax and semantics of Gödel logics

§3.2.1 Definition. It will be convenient to make the following definitions for all $x, y \in \mathbb{R}$:

$$x \oplus y := \min\{1, x + y\},$$

$$x \trianglelefteq y := \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$$

$$x \lhd y := \begin{cases} 1 & \text{if } x < y \\ y & \text{if } x \geq y \end{cases}$$

$$x \bowtie y := \begin{cases} 1 & \text{if } x = y \\ \min\{x,y\} & \text{if } x \neq y \end{cases}$$

$$\pi_0(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

$$\pi_1(x) := \begin{cases} 1 & \text{if } x \geq 1 \\ 0 & \text{if } x < 1 \end{cases}$$

It is immediately clear that $\oplus$, $\unlhd$, $\lhd$, $\bowtie$ induce functions $[0,1] \times [0,1] \to [0,1]$, and π_0 and π_1 induce functions $[0,1] \to [0,1]$. ☡

§3.2.2 Proposition. Let $k \in \mathbb{N} \setminus \{0\}$ and x, y, z, r_1, …, $r_k \in [0,1]$. Then the following holds:

(A1) $1 \oplus x = 1$
(A2) $0 \oplus x = x$
(A3) $x \leq r \oplus x \leq 1$
(A4) $x \oplus y = y \oplus x$
(A5) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$
(A6) $r_1 \oplus \ldots \oplus r_k \oplus x = (r_1 + \cdots + r_k) \oplus x$
(A7) $(r \oplus x) \unlhd (r \oplus y) = r \oplus (x \unlhd y)$
(A8) $(r \oplus x) \lhd (r \oplus y) = r \oplus (x \lhd y)$
(A9) $\max\{(r \oplus x) \lhd y, (r \oplus y) \lhd x\} = \begin{cases} 1 & |x-y| \geq r \\ \max\{x,y\} & |x-y| < r \end{cases}$
(A10) We have $y \leq x \unlhd y$
(A11) We have $x \lhd y < 1$ if and only if $y \leq x$ and $y < 1$
(A12) We have $x \lhd y = 1$ if and only if $x < y$ or $x = y = 1$
(A13) We have $y < x \unlhd y$ if and only if $x \leq y < 1$
(A14) We have $y < x \lhd y$ if and only if $x < y < 1$
(A15) We have $x < x \lhd y$ if and only if $x < y$
(A16) We have $1 = x \unlhd y$ if and only if $x \leq y$
(A17) We have $1 = x \bowtie y$ if and only if $x = y$
(A18) $1 - \pi_0(1 - \pi_0(x)) = \pi_0(x)$ ☡

The proof is routine and can be done by case distinctions. For later use, observe that (A9) is an expression that allows us to measure a distance (when $x < 1$ and $y < 1$ therein).

$$\text{(MP)} \quad \frac{A \quad A \supset B}{B}$$

(IPL1) $\bot \supset A$
(IPL2) $(A \wedge B) \supset A$
(IPL3) $(A \wedge B) \supset B$
(IPL4) $A \supset B \supset (A \wedge B)$
(IPL5) $A \supset (A \vee B)$
(IPL6) $B \supset (A \vee B)$
(IPL7) $(A \supset C) \supset (B \supset C) \supset (A \vee B) \supset C$
(IPL8) $A \supset B \supset A$
(IPL9) $(A \supset B \supset C) \supset (A \supset B) \supset (A \supset C)$

Figure 3.1: Proof system IPL

§3.2.3 Definition (Languages). We will indicate by R/n that a logical connective, a predicate symbol or a predicate R has arity $n \in \mathbb{N}$.

The logics we want consider here are based on the language $\mathscr{L}^{\mathsf{p}}$ of propositional formulae generated by a countably infinite set Var of propositional variables and the connectives $\bot/0$, $\supset/2$, $\wedge/2$, $\vee/2$; $\supset$ is understood to be written right-associatively. We understand $\neg A$ as an abbreviation for $A \supset \bot$, $A \leftrightarrow B$ for $(A \supset B) \wedge (B \supset A)$, $\top$ for $\bot \supset \bot$, and $A \prec B$ for $(B \supset A) \supset B$.

The language $\mathscr{L}$ of first-order formulae contains quantifiers $\forall$ and $\exists$ and is constructed in the usual way from $\mathscr{L}^{\mathsf{p}}$ with the following features: For each arity n, the set of function symbols and the set of predicate symbols has to be denumerable. Functions of arity 0 will be called *constants.* We will distinguish free and bound variables, and we assume that both form denumerable sets. In proofs, we will occasionally drop the formal distinction between the two sorts of variables. A first-order formula is *closed* if no free variable occurs in it.

The Hilbert-style proof systems IPL and IL of intuitionistic propositional and predicate logic are given in Figures 3.1 and 3.2.

Let $\bigcirc$ and $\triangle$ be two new unary connectives, which we will call operators for simplicity. In this chapter, we will consider only the language extensions $\mathscr{L}^{\mathsf{p}}_{\bigcirc}$, $\mathscr{L}^{\mathsf{p}}_{\triangle}$, $\mathscr{L}^{\mathsf{p}}_{\bigcirc,\triangle}$, $\mathscr{L}_{\bigcirc}$, $\mathscr{L}_{\triangle}$ and $\mathscr{L}_{\bigcirc,\triangle}$ that arise from adding $\bigcirc$ and $\triangle$. For $n \in \mathbb{N}$, we define $\bigcirc^0 A := A$ and $\bigcirc^{n+1} A := \bigcirc(\bigcirc^n A)$. ☡

§3.2.4 Definition (Semantics of Gödel logics). A *Gödel interpretation* of $\mathscr{L}^{\mathsf{p}}$ is a mapping $\mathrm{I}\colon \mathsf{Var} \to [0,1]$ that is extended to all formulae by (1) $\mathrm{I}(\bot) := 0$, (2) $\mathrm{I}(A \wedge B) := \min\{\mathrm{I}(A), \mathrm{I}(B)\}$, (3) $\mathrm{I}(A \vee B) := \max\{\mathrm{I}(A), \mathrm{I}(B)\}$ and (4) $\mathrm{I}(A \supset B) := \mathrm{I}(A) \trianglelefteq \mathrm{I}(B)$. In an extension of $\mathscr{L}^{\mathsf{p}}$ where $\triangle$ is present, (MP), (IPL1)–(IPL9) and (IL1)–(IL4), where

(IL1) $\dfrac{B \supset A(a)}{B \supset \forall x A(x)}$, where a is not free in B

(IL2) $\dfrac{A(a) \supset B}{\exists x A(x) \supset B}$, where a is not free in B

(IL3) $\forall x A(x) \supset A(t)$

(IL4) $A(t) \supset \exists x A(x)$

Figure 3.2: Proof system IL

we put (5) $\mathrm{I}(\triangle(A)) := \pi_1(\mathrm{I}(A))$. In presence of $\bigcirc$, we require that there exists some $r_\mathrm{I} \in [0,1]$ such that $\mathrm{I}(\bigcirc A) = r_\mathrm{I} \oplus \mathrm{I}(A)$ for every formula A; in this case, we speak also of a Gödel r_I-interpretation. It is immediately clear that all formulae are assigned a value in $[0,1]$.

A *Gödel interpretation* I of (an extension of) $\mathscr{L}$ consists of (1) a nonempty set $|\mathrm{I}|$, the *domain* of I, (2) a function $P^\mathrm{I}\colon |\mathrm{I}|^k \to [0,1]$ for each k-ary predicate symbol P, (3) a function $f^\mathrm{I}\colon |\mathrm{I}|^k \to |\mathrm{I}|$ for each k-ary predicate symbol f, (4) a value $v^\mathrm{I} \in |\mathrm{I}|$ for each free variable v, and (5) a mapping I from the set of formulae to $[0,1]$ such that (5) $\mathrm{I}(\forall x A(x)) = \inf\{\mathrm{I}(A(u))\colon u \in |\mathrm{I}|\}$, (6) $\mathrm{I}(\exists x A(x)) = \sup\{\mathrm{I}(A(u))\colon u \in |\mathrm{I}|\}$ and (7) $\bot$, $\supset$, $\wedge$, $\vee$ (and $\triangle$, $\bigcirc$ if present) are interpreted like in $\mathscr{L}^\mathrm{p}$. Throughout this paper we will, as usual, apply the notational convention that an occurrence of a domain element $u \in |\mathrm{I}|$ in a formula stands for a fresh constant $\underline{u}$ that is to be interpreted as u. This means that we will also speak of r-interpretations, for $r \in [0,1]$, when the languages under consideration are $\mathscr{L}_\bigcirc$ and $\mathscr{L}_{\bigcirc,\triangle}$. A *classical* interpretation of a language without ring is a Gödel interpretation that takes only values 0 and 1; the connectives and quantifiers clearly agree with the usual notion if 0 is understood as false and 1 as true.

§3.2.5 Proposition. If I is a Gödel interpretation, where $\bigcirc$ does not have to be in the language, and A and B are formulae, we have:

(B1) $\mathrm{I}(\top) = 1$,
(B2) $\mathrm{I}(\neg A) = 1 - \pi_0(\mathrm{I}(A))$,
(B3) $\mathrm{I}(\neg\neg A) = \pi_0(\mathrm{I}(A))$,
(B4) $\mathrm{I}(A \prec B) = \mathrm{I}(A) \lhd \mathrm{I}(B)$,
(B5) $\mathrm{I}(A \leftrightarrow B) = \mathrm{I}(A) \bowtie \mathrm{I}(B)$

If I is a Gödel r-interpretation where $\bigcirc$ is in the language, A and B are formulae, and $k \in \mathbb{N}$, we have:

(B6) $\mathrm{I}(A \supset \bigcirc A) = 1$,
(B7) $\mathrm{I}(\bigcirc^k A) = (k \cdot r) \oplus \mathrm{I}(A)$,
(B8) $\mathrm{I}(\bigcirc A \supset \bigcirc B) = r \oplus (\mathrm{I}(A) \trianglelefteq \mathrm{I}(B)) = \mathrm{I}(\bigcirc(A \supset B))$,
(B9) $\mathrm{I}(\bigcirc A \prec \bigcirc B) = r \oplus (\mathrm{I}(A) \triangleleft \mathrm{I}(B)) = \mathrm{I}(\bigcirc(A \prec B))$
(B10) $\mathrm{I}((\bigcirc A \supset B) \vee (\bigcirc B \supset A)) = \begin{cases} 1 & \text{if } |\mathrm{I}(A) - \mathrm{I}(B)| \geq r \\ \max\{\mathrm{I}(A), \mathrm{I}(B)\} & \text{if } |\mathrm{I}(A) - \mathrm{I}(B)| < r. \end{cases}$

Proof. For (B6), use (A3) and (A16) to obtain $\mathrm{I}(A \supset \bigcirc A) = \mathrm{I}(A) \trianglelefteq (r \oplus \mathrm{I}(A)) = 1$. The other cases follow easily from §3.2.2. ◇

§3.2.6 Proposition. (a) There is no formula $F(A)$ in $\mathscr{L}^{\mathrm{p}}_{\triangle}$ such that $\mathrm{I}(F(A)) = \mathrm{I}(\bigcirc A)$ for all Gödel interpretations I in $\mathscr{L}^{\mathrm{p}}_{\triangle}$. (b) There is no formula $F(A)$ in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$ such that $\mathrm{I}(F(A)) = \mathrm{I}(\triangle A)$. for all Gödel interpretations I in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$.

Proof. (a) Suppose $F(A)$ were a formula in $\mathscr{L}^{\mathrm{p}}_{\triangle}$ such that $\mathrm{I}(F(A)) = \mathrm{I}(\bigcirc A)$ for every Gödel interpretation I in $\mathscr{L}^{\mathrm{p}}_{\triangle}$, in particular, for every Gödel interpretation I such that $\mathrm{I}(A) = \mathrm{I}(B)$ for every propositional variable B. (Clearly, such interpretations exist.) We may thus assume that A is the only propositional variable that occurs in $F(A)$. Observe that $\{0,1\}$ is closed under any application of the functions π_1, $(x,y) \mapsto \min\{x,y\}$, $(x,y) \mapsto \max\{x,y\}$, $(x,y) \mapsto x \trianglelefteq y$, $x \mapsto 1$ and $x \mapsto 0$. Thus, we have $\mathrm{I}(F(A)) \in \{0,1\}$ for the interpretation I with $\mathrm{I}(A) := 0$ and $r_{\mathrm{I}} := \frac{1}{2}$. This contradicts $\mathrm{I}(F(A)) = \mathrm{I}(\bigcirc A) = \frac{1}{2} \oplus \mathrm{I}(A) = \frac{1}{2}$.

(b) Suppose $F(A)$ were a formula in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$ such that $\mathrm{I}(F(A)) = \mathrm{I}(\triangle A)$ for every Gödel interpretation I in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$, in particular, for every Gödel interpretation I such that $\mathrm{I}(A) = \mathrm{I}(B)$ for every propositional variable B and $r_{\mathrm{I}} = 0$. Thus there is some formula $G(A)$ in $\mathscr{L}^{\mathrm{p}}$ such that A is the only propositional variable in G and such that $\mathrm{I}(G(A)) = \mathrm{I}(\triangle A)$ for every Gödel interpretation I in $\mathscr{L}^{\mathrm{p}}$. Moreover, suppose w. l. o. g. that $G(A)$ is the shortest formula with this property.

We write $C \sim D$ in this proof whenever $\mathrm{I}(C) = \mathrm{I}(D)$ for every Gödel interpretation I in $\mathscr{L}^{\mathrm{p}}$.

If A is the only propositional variable of a formula H with three connectives, it is easy to show by a lengthy case distinction that there is a formula K such that $K \sim H$, K has ≤ 2 connectives and A is the only propositional variable in K. Thus G has at most 2 connectives. Again, by case distinction, one sees that we must have $H \sim \bot$, or $H \sim (\bot \supset \bot)$, or $H \sim A$, or $H \sim (\bot \supset A)$, or $H \sim (\bot \supset (\bot \supset A))$, or $H \sim A \vee (\bot \supset A)$. Then it is easy to find interpretations I that evaluate $\triangle A$ and the above formulae to different values. ♢

§3.2.7 Definition (Validity). If $\mathrm{I}(A) = 1$ holds for all Gödel interpretations in a language, we write $\models_{\mathrm{G}} A$ and say that the formula A is *valid* w. r. t. Gödel semantics in that language.

Although, strictly speaking, this amounts to different definitions depending on the presence of $\triangle$ or $\bigcirc$, it is immediately clear that a formula in the intersection of two of the considered languages is valid w. r. t. to the first if and only if it is valid w. r. t. to the second definition. For the sake for clarity, we remark that the above validity definition of a formula A in $\mathscr{L}_{\bigcirc}$ exactly says that $\mathrm{I}(A) = 1$ must hold for all r-interpretations I, for all $r \in [0,1]$. ☡

§3.2.8 Definition (Entailment). Let A be a formula and Γ be a set of formulae from the same language L, and let I be a Gödel interpretation w. r. t. L; in case L is first-order, we suppose that no formula has a free variable.

If $\inf\{\mathrm{I}(B) \colon B \in \Gamma\} \leq \mathrm{I}(A)$, we write $\Gamma \Vdash^{\mathrm{I}}_{\mathrm{G}} A$ and say Γ entails A under I (w. r. t. Gödel semantics in L). We write $\Gamma \Vdash_{\mathrm{G}} A$ if $\Gamma \Vdash^{\mathrm{I}}_{\mathrm{G}} A$ holds for every Gödel interpretation I. If $\mathrm{I}(A) = 1$ holds for every interpretation I such that $\mathrm{I}(B) = 1$ for all $B \in \Gamma$, we write $\Gamma \Vdash^{1}_{\mathrm{G}} A$ and say Γ one-entails A (w. r. t. Gödel semantics in L).

Let R be relation between a set of formulae and a formula, e. g. $\Vdash_{\mathrm{G}}$ or $\Vdash^{1}_{\mathrm{G}}$. We say R is *compact* if $\Gamma R A$ implies that there is a finite $\Gamma' \subseteq \Gamma$ with $\Gamma' R A$. ☡

While entailment for propositional Gödel logics (without restrictions on the truth value set) is compact, see Theorem 3.6 in [8], the semantics of the $\bigcirc$-operator destroys this property:

§3.2.9 Proposition. None of the entailment relations $\Vdash$ and $\Vdash^{1}$ between propositional formulae is compact for Gödel semantics if $\bigcirc$ is present. ☡

Proof. The following construction will show that it suffices to consider the language $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$. We have to show that there are Γ and A such that $\Gamma \Vdash A$ but there is no finite subset $\Pi \subseteq \Gamma$ such that $\Pi \Vdash A$; likewise for $\Vdash^{1}$. We will prove that $\Gamma := \{\bigcirc^{k} x \supset y;\ k \in \mathbb{N}\}$ and $A := y \vee \neg\bigcirc\bot$ have the required properties.

Suppose $\Gamma \Vdash A$ would not hold so that there is an r-interpretation I such that $\mathrm{I}(A) < \inf\{\mathrm{I}(B) \colon B \in \Gamma\}$. From $1 - \pi_0(r) \leq \max\{\mathrm{I}(y), 1 - \pi_0(r)\} < ((rk) \oplus \mathrm{I}(x)) \trianglelefteq \mathrm{I}(y) \leq 1$ for all $k \in \mathbb{N}$, we obtain $r > 0$ and $1 = \pi_0(r)$. By (A13), the resulting $\mathrm{I}(y) < ((rk) \oplus \mathrm{I}(x)) \trianglelefteq \mathrm{I}(y)$ yields $(rk) \oplus \mathrm{I}(x) \leq \mathrm{I}(y) < 1$ for all $k \in \mathbb{N}$. This contradicts the fact that for all integer $k > \frac{1}{r}$, we have $(rk) \oplus \mathrm{I}(x) = 1$. Hence we must have $\Gamma \Vdash A$.

Suppose $\Gamma \Vdash^1 A$ would not hold so that there is an r-interpretation I such that $\mathrm{I}(\bigcirc^k x \supset y) = 1$ for all $k \in \mathbb{N}$ but $\max\{\mathrm{I}(y), \mathrm{I}(\neg\bigcirc\bot)\} < 1$. Since $1 > \mathrm{I}(\neg\bigcirc\bot) = 1 - \pi_0(\bot)$, we have $r > 0$. By (A16), $\mathrm{I}(\bigcirc^k x \supset y) = 1$ yields $\mathrm{I}(\bigcirc^k x) \leq I(y)$ so that $(rk) \oplus \mathrm{I}(x) = \mathrm{I}(\bigcirc^k x) \leq I(y) < 1$ for all $k \in \mathbb{N}$, which is absurd because $rk \oplus \mathrm{I}(x) = 1$ for all integer $k > \frac{1}{r}$. Hence we must have $\Gamma \Vdash^1 A$.

Now, consider a finite $\Pi \subseteq \Gamma$ so that there is $K \in \mathbb{N}$ such that $\Pi \subseteq \{\bigcirc^k x \supset y; k \leq K\}$. If one takes $r := \frac{1}{K+2}$ and defines an r-interpretation by $\mathrm{I}(x) := 0$, $\mathrm{I}(y) := \frac{K+2}{K+3}$, we obtain $\mathrm{I}(y \vee \neg\bigcirc\bot) = \frac{K+2}{K+3} < 1$ and $\inf\{\mathrm{I}(B) \colon B \in \Pi\} \geq \min\{\mathrm{I}(\bigcirc^k x \supset y) \colon k \leq K\} = \min\{(k\frac{1}{K+2} \oplus 0) \trianglelefteq \frac{K+2}{K+3}; k \leq K\} = 1$ since $\frac{0}{K+2} < \frac{1}{K+2} < \frac{2}{K+2} < \ldots < \frac{K}{K+2} < \frac{K+2}{K+3}$. Thus neither $\Pi \Vdash A$ nor $\Pi \Vdash^1 A$, as required. ♢

§3.2.10 Remark. An important consequence of §3.2.9 is that the usual 'algebraic-abstract' way of proving completeness of some axiom system for the propositional fragment cannot succeed because the associated algebra of formulae would have to be compact, which it is not. ≀

We will come back to the propositional fragment in §3.4.

§3.3 Predicate logic

By a result of Horn [17], see also §3.4.2, it is well-known that the valid formulae in $\mathscr{L}_\bigcirc$ w. r. t. Gödel semantics are recursively enumerable. We will prove that this is not the case for the valid prenex formulae in $\mathscr{L}_\bigcirc$ w. r. t. Gödel semantics. The method of this proof will follow Scarpellini's ideas he used in [20] and that we have presented in §2. Like Scarpellini, we define a translation of formulae from classical predicate logic to $\mathscr{L}_\bigcirc$; this will be done in §3.3.3. Another important tool in the paper is Trakhtenbrot's theorem [22].

We start with the preparation of some propositions we will use later.

§3.3.1 Definition. Let A^* denote the result from replacing every atom $P(\bar{t})$, except $\bot$, by $\neg\neg P(\bar{t})$ in a first-order formula A. ≀

§3.3.2 Proposition. Let I be a Gödel interpretation for $\mathscr{L}$, and form I′ from I by changing the evaluation of every atom $P(\overline{x})$ to $\mathrm{I}'(P)(\overline{x}) := \pi_0(\mathrm{I}(P)(\overline{x}))$. Then I′ is a classical interpretation, and $\mathrm{I}'(A) = \mathrm{I}(A^*)$ holds for all formulae in $\mathscr{L}$. In particular, if A and B are classically equivalent formulae in $\mathscr{L}$, then A^* and B^* are equivalent w. r. t. Gödel semantics. ☡

Proof. The interpretation of $\bot$ is retained since $\mathrm{I}'(\bot) = \pi_0(\mathrm{I}(P)) = \pi_0(0) = 0$. Since $\pi_0(x) \in \{0, 1\}$ for all $x \in \mathbb{R}$, induction on the formula complexity yields that all formulae evaluate to 0 or to 1 under I′; thus I′ is classical.

By (B3), we have $\mathrm{I}(P^*) = \mathrm{I}(\neg\neg P) = \pi_0(\mathrm{I}(P)) = \mathrm{I}'(P) \in \{0, 1\}$ for all atoms P. Observe for the following that every classical interpretation is also a Gödel interpretation. Suppose we have formulae A and B such that $\mathrm{I}'(A) = \mathrm{I}(A^*)$ and $\mathrm{I}'(B) = \mathrm{I}(B^*)$. Then $\mathrm{I}'(A \vee B) = \max\{\mathrm{I}'(A), \mathrm{I}'(B)\} = \max\{\mathrm{I}(A^*), \mathrm{I}'(B^*)\} = \mathrm{I}(A^* \vee B^*) = \mathrm{I}((A \vee B)^*)$, $\mathrm{I}'(A \wedge B) = \min\{\mathrm{I}'(A), \mathrm{I}'(B)\} = \min\{\mathrm{I}(A^*), \mathrm{I}'(B^*)\} = \mathrm{I}(A^* \wedge B^*) = \mathrm{I}((A \wedge B)^*)$, $\mathrm{I}'(A \supset B) = \mathrm{I}'(A) \trianglelefteq \mathrm{I}'(B) = \mathrm{I}(A^*) \trianglelefteq \mathrm{I}(B^*) = \mathrm{I}(A^* \supset B^*) = \mathrm{I}((A \supset B)^*)$. Suppose we have $\mathrm{I}'(A(u)) = \mathrm{I}((A(u))^*)$ for all $u \in |I|$, then

$$\mathrm{I}'(\forall x A(x)) = \inf_{u \in |I|} \mathrm{I}'(A(u)) = \inf_{u \in |I|} \mathrm{I}((A(u))^*) = \mathrm{I}(\forall x (A(x))^*)$$

and

$$\mathrm{I}'(\exists x A(x)) = \sup_{u \in |I|} \mathrm{I}'(A(u)) = \sup_{u \in |I|} \mathrm{I}((A(u))^*) = \mathrm{I}(\exists x (A(x))^*).$$

The first part of the proposition now follows by induction on the formula complexity.

For the second part, let $\mathrm{I}''(A) = \mathrm{I}''(B)$ for all classical interpretations I″, and let I be a Gödel interpretation. By the above, there is a classical interpretation I′ such that $\mathrm{I}'(A) = \mathrm{I}(A^*)$ and $\mathrm{I}'(B) = \mathrm{I}(B^*)$, but now $\mathrm{I}'(A) = \mathrm{I}'(B)$ yields $\mathrm{I}(A^*) = \mathrm{I}(B^*)$, as claimed. ♢

In the following definition, the predicate symbol I is always used either negated or double-negated. The double-negation translation of intuitionistic logic works also here to simulate a classical behaviour of I. The purpose of I will be to model an equivalence relation.

§3.3.3 Definition. Given a formula A in $\mathscr{L}$, define the formula $\alpha(A)$ in $\mathscr{L}_{\bigcirc}$ recursively as follows. Let $(P_k)_{k<K}$ denote all the predicate symbols that occur in A, and let R be a fresh unary predicate symbol and I a fresh binary predicate symbol. For each $k < K$, define

$$\begin{aligned} E_k := \forall a_1, b_1, \ldots, a_n, b_n. \\ ((\neg\neg I(a_1, b_1) \wedge \ldots \wedge \neg\neg I(a_n, b_n)) \supset \\ \supset (\neg\neg P_k(a_1, \ldots, a_n) \leftrightarrow \neg\neg P_k(b_1, \ldots, b_n))) \end{aligned}$$

if P_k has some positive arity n, otherwise define $E_k := \top$.

Let

$$
\begin{aligned}
&C_1 := \forall x \neg\neg I(x,x),\\
&C_2 := \forall x \forall y(\neg\neg I(x,y) \supset \neg\neg I(y,x)),\\
&C_3 := \forall x \forall y \forall z((\neg\neg I(x,y) \wedge \neg\neg I(y,z)) \supset \neg\neg I(x,z)),\\
&C_4 := \textstyle\bigwedge_{k<K} E_k,\\
&C_5 := \neg\neg\bigcirc\bot,\\
&C_6 := \forall x \forall y(\neg I(x,y) \supset ((\bigcirc R(x) \supset R(y)) \vee (\bigcirc R(y) \supset R(x)))).
\end{aligned}
$$

Eventually, we define

$$\alpha(A) := (C_1 \wedge C_2 \wedge C_3 \wedge C_4 \wedge C_5 \wedge C_6) \supset (A^* \vee \exists x R(x)).$$

§3.3.4 Lemma. Let A be a formula in $\mathscr{L}$ without free variables. Then the following conditions are equivalent:

(a) There is $r \in [0,1]$ and a Gödel r-interpretation I such that $\mathrm{I}(\alpha(A)) < 1$.

(b) There is a classical interpretation I' such that $\mathrm{I}'(A) = 0$ and $|\mathrm{I}'|$ is finite.

Proof. We will use the notation from §3.3.3 in this proof.

(b)→(a): Suppose I' is a classical interpretation such that $\mathrm{I}'(A) = 0$ and $|\mathrm{I}'| \neq \emptyset$ is finite. Let $d_0, \ldots, d_N$ be an enumeration of $|\mathrm{I}'|$. Take $r := \frac{1}{N+2}$ and define an r-interpretation I by taking the same domain as I' and by $\mathrm{I}(I)(x,y) := 1$ if $x = y$, and $\mathrm{I}(I)(x,y) := 0$ if $x \neq y$, $\mathrm{I}(R)(d_i) := \frac{i}{N+1}$, and $\mathrm{I}(P_k)(\overline{x}) := \mathrm{I}'(P_k)(\overline{x}) \in \{0,1\}$ for all predicate symbols P_k that occur in A. It remains to show $\mathrm{I}(\alpha(A)) < 1$.

We have $\pi_0(\mathrm{I}(P_k)(\overline{x})) = \pi_0(\mathrm{I}'(P_k)(\overline{x})) = \mathrm{I}'(P_k)(\overline{x})$ for all $k < K$, neither R nor I occurs in A, and A is ring-free. Thus, it follows $\mathrm{I}(A^*) = \mathrm{I}'(A) = 0$ by §3.3.2. We obtain $\mathrm{I}(\exists x R(x)) = \max\{\frac{0}{N+1}, \frac{1}{N+1}, \ldots, \frac{N}{N+1}\} = \frac{N}{N+1}$, $\mathrm{I}(C_1) = \mathrm{I}(\forall x \neg\neg I(x,x)) = \inf\{\pi_0(I(u,u))\colon u \in |I|\} = 1$ and in a similar manner also $1 = \mathrm{I}(C_2) = \mathrm{I}(C_3) = \mathrm{I}(C_4) = \mathrm{I}(C_5)$. Since for all d_i, $d_j \in |I|$ with $\mathrm{I}(I)(d_i,d_j) = 0$ we have $|\mathrm{I}(R)(d_i) - \mathrm{I}(R)(d_j)| = \frac{|i-j|}{N} \geq \frac{1}{N} > \frac{1}{N+1} = r$, we see $\mathrm{I}(\neg I(u,v) \supset ((\bigcirc R(u) \supset R(v)) \vee (\bigcirc R(v) \supset R(u)))) = 1$ for all $u, v \in |\mathrm{I}|$ by (B10). Thus $\mathrm{I}(C_6) = 1$ and hence $\mathrm{I}(\alpha(A)) = 1 \trianglelefteq \max\{0, \frac{N}{N+1}\} = \frac{N}{N+1} < 1$.

(a)→(b): Suppose $r \in [0,1]$ and I is a Gödel r-interpretation with $\mathrm{I}(\alpha(A)) < 1$ so that $\max\{\mathrm{I}(A^*), \mathrm{I}(\exists x R(x))\} < \min_{1 \leq i \leq 6} \mathrm{I}(C_i)$ by (A16). Since every atom is put under double negation in A^* and in every C_i, $1 \leq i \leq 5$, we see that $\mathrm{I}(A^*)$ and $\mathrm{I}(C_i)$, $1 \leq i \leq 5$, can take only the values 0 or 1. Thus $\mathrm{I}(A^*) = 0$ and $1 = \mathrm{I}(C_i)$ whenever $1 \leq i \leq 5$. We obtain $r > 0$ from $\mathrm{I}(C_5) = 1$.

For later use, we prove for all a, $b \in |\mathrm{I}|$ that $\mathrm{I}(I)(a,b) = 0$ implies

$$|\mathrm{I}(R)(a) - \mathrm{I}(R)(b)| \geq r.$$

Suppose not, then there are a, $b \in |\mathrm{I}|$ such that $\mathrm{I}(I)(a,b) = 0$ but

$$|\mathrm{I}(R)(a) - \mathrm{I}(R)(b)| < r.$$

We apply the above convention that domain members in formulae stand for their associated constants. By definition of C_6, we have $\mathrm{I}(C_6) \leq \mathrm{I}(\neg I(a,b) \supset ((\bigcirc R(a) \supset R(b)) \vee (\bigcirc R(b) \supset R(a))))$. By (B10), we conclude from $\mathrm{I}(I)(a,b) = 0$ and $|\mathrm{I}(R)(a) - \mathrm{I}(R)(b)| < r$ that $\mathrm{I}(\neg I(a,b) \supset ((\bigcirc R(a) \supset R(b)) \vee (\bigcirc R(b) \supset R(a)))) = 1 \trianglelefteq \max\{\mathrm{I}(R)(a), \mathrm{I}(R)(b)\} = \max\{\mathrm{I}(R)(a), \mathrm{I}(R)(b)\} \leq \sup\{\mathrm{I}(R)(u)\colon\ u \in |\mathrm{I}|\} = \mathrm{I}(\exists x R(x)) \leq \max\{\mathrm{I}(A^*), \mathrm{I}(\exists x R(x))\} < \min_{1 \leq i \leq 6} \mathrm{I}(C_i) \leq \mathrm{I}(C_6)$, which is a contradiction.

For all a, $b \in |\mathrm{I}|$, put $a \sim b :\Leftrightarrow \mathrm{I}(I)(a,b) > 0$. This is an equivalence relation but we only prove transitivity here as symmetry and reflexivity can be similarly dealt with by $\mathrm{I}(C_1) = \mathrm{I}(C_2) = 1$. Suppose we had $a \sim b \sim c$ but $a \not\sim c$ so that $\mathrm{I}(I)(a,b) > 0$, $\mathrm{I}(I)(b,c) > 0$ and $\mathrm{I}(I)(a,c) = 0$. We conclude $1 = \mathrm{I}(C_3) \leq \mathrm{I}((\neg\neg I(a,b) \wedge \neg\neg I(b,c)) \supset \neg\neg I(a,c)) = \max\{\pi_0(\mathrm{I}(I)(a,b)), \pi_0(\mathrm{I}(I)(b,c))\} \trianglelefteq \pi_0(\mathrm{I}(I)(a,c)) = \max\{1,1\} \trianglelefteq 0 = 0$, which is absurd.

We will now prove that there are only finitely many equivalence classes w. r. t. $\sim$. Suppose not, then we can find $(x_i)_{i \in \mathbb{N}}$ such that $x_i \sim x_j$ if and only if $i = j$. For all $i \neq j$, we have $\mathrm{I}(I)(x_i, x_j) = 0$ by definition of $\sim$, and $|\mathrm{I}(R)(x_i) - \mathrm{I}(R)(x_j)| \geq r$ by the above. Due to $\{\mathrm{I}(R)(x_i);\ i \in \mathbb{N}\} \subseteq [0,1]$, this would be only possible if $r = 0$ but this contradicts the earlier proved fact that $r > 0$.

Given equivalent elements $a_1 \sim b_1, \ldots, a_n \sim b_n$ in $|\mathrm{I}|$, we will prove

$$\pi_0(\mathrm{I}(P_k)(a_1,\ldots,a_n)) = \pi_0(\mathrm{I}(P_k)(b_1,\ldots,b_n))$$

for later use. By assumption, we have $\mathrm{I}(I)(a_i,b_i) > 0$ for all i, thus

$$\mathrm{I}(\neg\neg I(a_i,b_i)) = \pi_0(\mathrm{I}(I)(a_i,b_i)) = 1.$$

By $\mathrm{I}(C_4) = 1$ and (B5), we see

$$1 = \mathrm{I}(\neg\neg P_k(a_1,\ldots,a_n) \leftrightarrow \neg\neg P_k(b_1,\ldots,b_n)) =$$
$$= \pi_0(\mathrm{I}(P_k)(a_1,\ldots,a_n)) \bowtie \pi_0(\mathrm{I}(P_k)(b_1,\ldots,b_n)).$$

The claim now follows from (A17).

The property just proved allows us to define a classical interpretation I' as follows: Let its domain $|\mathrm{I}'|$ consist of the finitely many equivalence classes w. r. t. $\sim$ and put $\mathrm{I}'(P_k)([x_1],\ldots,[x_n]) := \pi_0(\mathrm{I}(P_k)(x_1,\ldots,x_n))$ for any predicate symbol P_k with some arity n; here $[x_i]$ denotes the equivalence class containing x_i. Interpret any free variable v by $\mathrm{I}'(v) := [\mathrm{I}(v)]$. We leave the easy task to the reader to repeat the proof of §3.3.2 in the situation of I and I′ to obtain that $\mathrm{I}'(B([x_1],\ldots,[x_n])) = \mathrm{I}(B^*(x_1,\ldots,x_n))$ holds whenever $x_1,\ldots,x_n \in |\mathrm{I}|$ and $B(b_1,\ldots,b_n)$ is a formula in $\mathscr{L}$ with free variables $b_1, \ldots, b_n$. In particular, it follows $\mathrm{I}'(A) = \mathrm{I}(A^*) = 0$, which completes the proof. $\diamondsuit$

By the tools proved above, we can already show that the valid formulae w. r. t. Gödel semantics in $\mathscr{L}_{\bigcirc}$ is not r. e.: By §3.3.4, a formula A in $\mathscr{L}$ without free variables is (1) classically valid in all finite domains if and only if (2) $\alpha(A)$ is valid w.r.t. Gödel semantics in $\mathscr{L}_{\bigcirc}$. The statement follows since the formulae obeying (1) are not r. e. by Trakhtenbrot's theorem [22]. We proceed to sharpen the result to the prenex fragment of $\mathscr{L}_{\bigcirc}$.

§3.3.5 Proposition. Let a_0, b_0, …, a_n, b_n, c_0, d_0, …, c_m, d_m be lists of bound variables, none of them necessarily of positive length, such that their concatenation lists each variable only once. Let

$$K = K(a_0, b_0, \ldots, a_n, b_n)$$

and

$$L = L(c_0, d_0, \ldots, c_m, d_m)$$

be semiformulae such that the indicated variables comprise all occurrences of bound variables not bound by a quantifier; the formulae are allowed to contain all free variables as well as bound variables that are bound by a quantifier. Let U and V be two fresh unary predicate symbols. Then, w. r. t. Gödel semantics,

(1) $(\exists a_0 \forall b_0 \ldots \exists a_n \forall b_n K) \supset \exists c_0 \forall d_0 \ldots \exists c_m \forall d_m L$ is valid

if and only if

(2) $\forall a_0 \exists b_0 \ldots \forall a_n \exists b_n \exists c_0 \forall d_0 \ldots \exists c_m \forall d_m$
$((K \supset U) \vee (U \supset V) \vee (V \supset L))$ is valid. ☡

Proof. Validity and interpretations will always relate to Gödel semantics in this proof. The symbols a_i, b_i, c_i, d_i will, by abuse of language, also represent domain elements $u \in |\mathrm{I}|$; we remind the reader that an occurrence of $u \in |\mathrm{I}|$ in a formula stands for a fresh constant $\underline{u}$ to be interpreted as u. The proof merely unwinds the definitions and uses elementary properties of $\mathbb{R}$. We split it into several parts because of its length.

(a) Definition: If c is a list of bound variables $c_1, \ldots, c_n$, let $\forall c$ and $\exists c$ denote $\forall c_1 \ldots \forall c_n$ and $\exists c_1 \ldots \exists c_n$. If c is a list of $|\mathrm{I}|$-elements $c_1, \ldots, c_n$, let $\inf_c t$ and $\sup_c t$ denote $\inf_{c_1} \ldots \inf_{c_n} t$ and $\sup_{c_1} \ldots \sup_{c_n} t$; if c is the empty list and the term t takes only values in $[0,1]$, we put $\inf_c t := 1$ and $\sup_c t := 0$.

(b) It is well-known that for every set X, every function $f\colon X \to [0,1]$ and every $d \in [0,1]$, the following equivalences hold:

(1) $d < \inf_x f(x)$
$\Leftrightarrow \exists C \in [0,1].\ \forall x \in X.\ d < C < f(x)$
(2) $d < \sup_x f(x)$
$\Leftrightarrow \exists C \in [0,1].\ \exists x \in X.\ d < C < f(x)$
$\Leftrightarrow \exists x \in X.\ d < f(x)$
(3) $\inf_x f(x) < d$
$\Leftrightarrow \exists C \in [0,1].\ \exists x \in X.\ f(x) < C < d$
$\Leftrightarrow \exists x \in X.\ f(x) < d$
(4) $\sup_x f(x) < d$
$\Leftrightarrow \exists C \in [0,1].\ \forall x \in X.\ f(x) < C < d$

(c) Under the conditions stated in the proposition, we claim, for every $e \in [0,1]$ and every interpretation I, that

$$e < \mathrm{I}(\exists a_0 \forall b_0 \ldots \exists a_n \forall b_n K(a_0, b_0, \ldots, a_n, b_n))$$

if and only if

$(*_1)$: $\exists r \in \mathbb{R}\, \exists a_0 \forall b_0 \ldots \exists a_n \forall b_n\ e < r < \mathrm{I}(K(a_0, b_0, \ldots, a_n, b_n))$,
where a_i, b_i, c_i, $d_i \in |\mathrm{I}|$ in the latter condition.

Proof: Clearly,

$$e < \mathrm{I}(\exists a_0 \forall b_0 \ldots \exists a_n \forall b_n K)$$

is equivalent to

$$e < \sup_{a_0} \inf_{b_0} \ldots \sup_{a_n} \inf_{b_n} \mathrm{I}(K).$$

After $2(n+1)$ applications of (b), we see that this is equivalent to

$$\begin{aligned} &\exists a_0 \exists r_0 \forall b_0 (e < r_0 \wedge \\ &\wedge \exists a_1 \exists r_1 \forall b_1 (r_0 < r_1 \wedge \\ &\wedge \exists a_2 \exists r_2 \forall b_2 (r_1 < r_2 \wedge \\ &\ldots \\ &\wedge \exists a_{n-1} \exists r_{n-1} \forall b_{n-1} (r_{n-2} < r_{n-1} \wedge \\ &\wedge \exists a_n \exists r_n \forall b_n (r_{n-1} < r_n < \mathrm{I}(K))) \ldots)). \end{aligned}$$

By classical quantifier shift laws, we obtain after swapping two existential quantifiers that this is equivalent to

$(*_2)$: $\exists r_0 \exists a_0 \forall b_0 \exists a_1 \exists r_1 \forall b_1 \ldots \exists a_n \exists r_n \forall b_n.$
$e < r_0 < r_1 < \ldots < r_n < \mathrm{I}(K)$.

This clearly implies $(*_1)$ by taking $r := r_0$. Conversely, if $(*_1)$ holds, we have $e < r$ and thus we can choose constants $r_0, \ldots, r_n$ such that $e < r_0 < r_1 < \ldots < r_n < r$; this implies $(*_2)$. This completes the proof of the claim.

(d) Under the conditions stated in the proposition, we claim, for every $e \in [0,1]$ and every interpretation I, that

$$\mathrm{I}(\exists c_0 \forall d_0 \ldots \exists c_m \forall d_m\ L(c_0, d_0, \ldots, c_m, d_m)) < e$$

if and only if

$$\exists r \in \mathbb{R}\ \forall c_0 \exists d_0 \ldots \forall c_m \exists d_m\ \mathrm{I}(L(c_0, d_0, \ldots, c_m, d_m)) < r < e,$$

where a_i, b_i, c_i, $d_i \in |\mathrm{I}|$ in the latter condition. (Observe that the quantifiers are now dualised.) The proof is completely analogous to the proof of (c).

(e) For every $e \in [0, 1]$, every interpretation I and all atoms A, B, C, D, we claim that we have

$$\mathrm{I}((A \supset B) \vee (B \supset C) \vee (C \supset D)) < e$$

if and only if

$$\mathrm{I}(D) < \mathrm{I}(C) < \mathrm{I}(B) < \mathrm{I}(A) \wedge \mathrm{I}(B) < e.$$

Proof: By definition, we have $\mathrm{I}((A \supset B) \vee (B \supset C) \vee (C \supset D)) = \max\{\mathrm{I}(A) \trianglelefteq \mathrm{I}(B), \mathrm{I}(B) \trianglelefteq \mathrm{I}(C), \mathrm{I}(C) \trianglelefteq \mathrm{I}(D)\}$. If $\mathrm{I}(D) < \mathrm{I}(C) < \mathrm{I}(B) < \mathrm{I}(A) \wedge \mathrm{I}(B) < e$, then $\mathrm{I}((A \supset B) \vee (B \supset C) \vee (C \supset D)) = \max\{\mathrm{I}(B), \mathrm{I}(C), \mathrm{I}(D)\} = \mathrm{I}(B) < e$, as required. Conversely, suppose that $\mathrm{I}((A \supset B) \vee (B \supset C) \vee (C \supset D)) < e \leq 1$. Thus $\mathrm{I}(C \supset D) < e \leq 1$, $\mathrm{I}(B \supset C) < e \leq 1$ and $\mathrm{I}(B) \leq \mathrm{I}(A) \trianglelefteq \mathrm{I}(B) = \mathrm{I}(A \supset B) < e \leq 1$. Hence $\mathrm{I}(B) < e$ and, by (A16), we find $\mathrm{I}(B) < \mathrm{I}(A)$, $\mathrm{I}(C) < \mathrm{I}(B)$ and $\mathrm{I}(D) < \mathrm{I}(C)$, as claimed. This completes the proof.

(f) We can now prove the proposition. By (A16), condition (1) of the proposition is false if and only if there is some interpretation I such that

$$\mathrm{I}(\exists c_0 \forall d_0 \ldots \exists c_m \forall d_m L) < \mathrm{I}(\exists a_0 \forall b_0 \ldots \exists a_n \forall b_n K),$$

i. e. if and only if

$$\exists e \in [0, 1].\ \mathrm{I}(\exists c_0 \forall d_0 \ldots \exists c_m \forall d_m L) < e < \mathrm{I}(\exists a_0 \forall b_0 \ldots \exists a_n \forall b_n K).$$

By (c) and (d), this is the case if and only if

$$\exists e \in [0, 1].\ ((\exists s \in \mathbb{R}\ \forall c_0 \exists d_0 \ldots \forall c_m \exists d_m\ \mathrm{I}(L(c_0, d_0, \ldots, c_m, d_m)) < s < e) \wedge (\exists r \in \mathbb{R}\ \exists a_0 \forall b_0 \ldots \exists a_n \forall b_n\ e < r < \mathrm{I}(K(a_0, b_0, \ldots, a_n, b_n)))).$$

After shifting quantifiers and eliminating e, it follows that this is in turn equivalent to

$$(*_3):\quad \exists r, s \in \mathbb{R}\ \exists a_0 \forall b_0 \ldots \exists a_n \forall b_n \forall c_0 \exists d_0 \ldots \forall c_m \exists d_m.\ \mathrm{I}(L) < s < r < \mathrm{I}(K).$$

Condition (2) of the proposition is false if and only if there is some interpretation I' such that

$$\mathrm{I}'(\forall a_0 \exists b_0 \ldots \forall a_n \exists b_n \exists c_0 \forall d_0 \ldots \exists c_m \forall d_m.\ ((K \supset U) \vee (U \supset V) \vee (V \supset L))) < 1.$$

An instance of (d) shows that this is equivalent to

$$\exists e \in \mathbb{R}\ \exists a_0 \forall b_0 \ldots \exists a_n \forall b_n \forall c_0 \exists d_0 \ldots \forall c_m \exists d_m.\ \mathrm{I}'((K \supset U) \vee (U \supset V) \vee (V \supset L)) < e < 1.$$

By (e), this is equivalent to

$(*_4)$: $\exists e \in \mathbb{R}\, \exists a_0 \forall b_0 \ldots \exists a_n \forall b_n \forall c_0 \exists d_0 \ldots \forall c_m \exists d_m.$
$$\mathrm{I}'(L) < \mathrm{I}'(V) < \mathrm{I}'(U) < \mathrm{I}'(K) \wedge \mathrm{I}'(U) < e < 1.$$
Observe that $(*_4)$ implies $(*_3)$ because we can put $s := \mathrm{I}'(V)$, $r := \mathrm{I}'(U)$ and $\mathrm{I} := \mathrm{I}'$. Conversely, suppose now that $(*_3)$ holds. In particular, we have $s < r < 1$ so that $r < e < 1$ for $e := \frac{r+1}{2}$. Since U and V do not occur in K and L, we can define an interpretation I' that has the same universe as I and that agrees with I on all atoms except U and V, where $\mathrm{I}'(U) := r$ and $\mathrm{I}'(V) := s$. This implies $(*_4)$ as we have $\mathrm{I}'(U) < e < 1$ and as every formula that neither contains U nor V receives the same value under I and I'.

This proves the equivalence. ◇

§3.3.6 Lemma. Let A be a formula in $\mathscr{L}$ without free variables. Then there is a recursive translation β from $\mathscr{L}$ to $\mathscr{L}_{\bigcirc}$ such that $\beta(A)$ is prenex for all A in $\mathscr{L}$ and such that, w. r. t. Gödel semantics, $\alpha(A)$ is valid if and only if $\beta(A)$ is valid. ☡

Proof. We transform A into a classically equivalent prenex formula K by applying all classically valid quantifier shift rules. Thus K^* has the form
$$Q_1 w_1 \ldots Q_M w_M\, U(w_1, \ldots, w_M),$$
where $Q_i \in \{\forall, \exists\}$, the w_i are bound variables and U is quantifier free. We take the symbols R, I, C_1, …, C_6 and $(P_k)_{k<K}$ from §3.3.3. Let N be the maximum of 1 and of all arities of the P_k. Let F denote the following formula
$$\begin{aligned}
&\forall x, y, z, a_1, b_1, \ldots, a_N, b_N.\\
&((\neg\neg I(x,x)) \wedge\\
&\wedge (\neg\neg I(x,y) \supset \neg\neg I(y,x)) \wedge\\
&\wedge ((\neg\neg I(x,y) \wedge \neg\neg I(y,z)) \supset \neg\neg I(x,z)) \wedge\\
&\wedge \textstyle\bigwedge_{k<K} (\neg\neg I(a_1,b_1) \wedge \ldots \wedge \neg\neg I(a_n,b_n)) \supset\\
&\qquad\qquad \supset (\neg\neg P_k(a_1,\ldots,a_n) \leftrightarrow \neg\neg P_k(b_1,\ldots,b_n)) \wedge\\
&\wedge \neg\neg \bigcirc\bot \wedge\\
&\wedge (\neg I(x,y) \supset ((\bigcirc R(x) \supset R(y)) \vee (\bigcirc R(y) \supset R(x))))),
\end{aligned}$$
and put
$$D := F \supset \exists x Q_1 w_1 \ldots Q_M w_M (U(w_1, \ldots, w_M) \vee R(x)).$$
Out next step is to prove $\mathrm{I}(D) = \mathrm{I}(\alpha(A))$ for every interpretation I of $\mathscr{L}_{\bigcirc}$. It is easy to check that all quantifier shift rules for $\wedge$ and $\vee$ are valid in Gödel logics. (However, some quantifier shift rules fail for $\supset$, but we do not need to shift $\supset$.) In particular, we find
$$\begin{aligned}
&\mathrm{I}(\exists x Q_1 w_1 \ldots Q_M w_M (U(w_1, \ldots, w_M) \vee R(x))) =\\
&= \mathrm{I}(\exists x (Q_1 w_1 \ldots Q_M w_M U(w_1, \ldots, w_M)) \vee R(x)) =\\
&= \mathrm{I}((Q_1 w_1 \ldots Q_M w_M U(w_1, \ldots, w_M)) \vee \exists x R(x)) =
\end{aligned}$$

$= \mathrm{I}(K^* \vee \exists x R(x))$.
Since A and K are classically equivalent, we have $\mathrm{I}(A^*) = \mathrm{I}(K^*)$ by §3.3.2. Since we clearly have $\mathrm{I}(F) = \mathrm{I}(C_1 \wedge C_2 \wedge C_3 \wedge C_4 \wedge C_5 \wedge C_6)$, the above equations yield $\mathrm{I}(D) = \mathrm{I}(\alpha(A))$, as claimed. Therefore, w. r. t. Gödel semantics, D is valid if and only if $\alpha(A)$ is valid.

Since D has the form that is required in condition (1) of §3.3.5, we can define $\beta(A)$ to be the corresponding formula instance in condition (2); thus $\beta(A)$ is valid w. r. t. Gödel semantics if and only if D is valid, i. e. if and only if $\alpha(A)$ is valid. This completes the proof since all steps in the construction of β were effective. ◇

§3.3.7 Theorem. The set of valid prenex formulae w. r. t. Gödel semantics in $\mathscr{L}_{\bigcirc}$ is not r. e. ☡

Proof. By §3.3.4 and §3.3.6, the following conditions are equivalent for any formula A in $\mathscr{L}$ without free variables: (1) A is classically valid in all finite domains, (2) $\alpha(A)$ is valid w. r. t. Gödel semantics in $\mathscr{L}_{\bigcirc}$, (3) $\beta(A)$ is valid w. r. t. Gödel semantics in $\mathscr{L}_{\bigcirc}$. By Trakhtenbrot's theorem [22], the formulae obeying (1) are not r. e. Hence the theorem follows. ◇

§3.3.8 Remark. As the above construction involves a binary predicate I, this method of proof does not imply a statement on the decidability of validity in the monadic fragment of $\mathscr{L}_{\bigcirc}$. This problem is also open for monadic Łukasiewicz logic. As we have transferred Scarpellini's method from Łukasiewicz logic to $\mathscr{L}_{\bigcirc}$, we also expect a connection between the problems for their monadic fragments.—Another open question is the exact complexity class in the arithmetical hierarchy of the validity problem for prenex formulae: We think that it has the same complexity as in Łukasiewicz logic, i. e. Π_2-complete. ☡

§3.4 Propositional proof systems

First, we introduce the proof system whose completeness w.r.t. Gödel semantics for $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$ we will establish. The method of proof will be described in detail in §3.4.9.

§3.4.1 Definition (Proof systems). Let $\mathbf{G}$ be the proof system resulting from extending IPL by the *linearity axiom*

(LIN) $(A \supset B) \vee (B \supset A)$.

Let $\mathbf{H}$ be the proof system resulting from extending IL by (LIN) and by the *quantifier shift*

$$\text{(QS)}\quad \forall x(B \vee A(x)) \supset (B \vee \forall x A(x)),$$

where x is not free in B. ≀

§3.4.2 Proposition. **G** is sound and complete for $\mathscr{L}^{\mathrm{p}}$ w. r. t. Gödel semantics. **H** is sound and complete for $\mathscr{L}$ w. r. t. Gödel semantics. ≀

Proof. The first theorem is due to Dummett [12], the latter is due to Horn [17]. Indeed, from Dummett's paper one can easily read off an algorithm that either yields a **G**-proof or a countermodel for a given formula. $\Diamond$

§3.4.3 Proposition. Let $E[\cdot]$ denote an $\mathscr{L}^{\mathrm{p}}$-context, and let A, B, C be formulae in $\mathscr{L}^{\mathrm{p}}$. Then **G** proves the following:

(G1) $A \prec \top$

(G2) $(\bot \prec A) \vee (\bot \leftrightarrow A)$

(G3) $(A \prec B) \vee (A \leftrightarrow B) \vee (B \prec A)$

(G4) $(A \leftrightarrow B) \supset (E[A] \leftrightarrow E[B])$

(G5) $(A \Box B) \supset (E[A \wedge B] \leftrightarrow E[A])$ for $\Box \in \{\prec, \leftrightarrow\}$

(G6) $(A \Box B) \supset (E[A \vee B] \leftrightarrow E[B])$ for $\Box \in \{\prec, \leftrightarrow\}$

(G7) $(A \Box B) \supset (E[A \supset B] \leftrightarrow E[\top])$ for $\Box \in \{\prec, \leftrightarrow\}$

(G8) $(A \prec B) \supset (E[B \supset A] \leftrightarrow E[A])$

(G9) $E[A \Box A] \leftrightarrow E[A]$ for $\Box \in \{\wedge, \vee\}$

(G10) $E[A \Box B] \leftrightarrow E[B \Box A]$ for $\Box \in \{\wedge, \vee, \leftrightarrow\}$

(G11) $E[(A \Box B) \Box C] \leftrightarrow E[A \Box (B \Box C)]$ for $\Box \in \{\wedge, \vee\}$

(G12) $E[A \Box (B \Diamond C)] \leftrightarrow E[(A \Box B) \Diamond (A \Box C)]$ for $\Box, \Diamond \in \{\wedge, \vee\}$

(G13) $(A \prec A) \leftrightarrow ((A \leftrightarrow A) \wedge (A \leftrightarrow \top))$

(G14) $A \supset A$

(G15) $((A \prec B) \wedge (B \Box A)) \leftrightarrow ((A \leftrightarrow B) \wedge (B \leftrightarrow \top))$ for $\Box \in \{\supset, \prec, \leftrightarrow\}$

(G16) $((A \Box B) \wedge (B \Diamond C) \wedge (C \prec A)) \leftrightarrow ((A \leftrightarrow B) \wedge (B \leftrightarrow C) \wedge (C \leftrightarrow \top))$ for $\Box, \Diamond \in \{\leftrightarrow, \prec\}$

(G17) $E[\top \vee A] \leftrightarrow E[\top]$

(G18) $E[\bot \vee A] \leftrightarrow E[A]$

(G19) $E[\bot \wedge A] \leftrightarrow E[\bot]$

(G20) $E[A \prec \bot] \leftrightarrow E[\bot]$

(G21) $(\top \prec A) \leftrightarrow (\top \leftrightarrow A)$

(G22) $(D \vee E) \leftrightarrow (((A \leftrightarrow A) \wedge D) \vee E)$

(G23) $(((A \leftrightarrow B) \wedge D) \vee E) \leftrightarrow (((A \leftrightarrow B) \wedge (B \leftrightarrow A) \wedge D) \vee E)$

(G24) $(((A \prec \bot) \wedge D) \vee E) \leftrightarrow E$

(G25) $(A \supset (B \leftrightarrow C)) \supset ((A \wedge (B \prec C)) \leftrightarrow (A \wedge (B \leftrightarrow C) \wedge (C \leftrightarrow \top)))$

(G26) $(A \supset (B \prec C)) \supset ((A \wedge (C \prec B)) \leftrightarrow (A \wedge (B \leftrightarrow C) \wedge (C \leftrightarrow \top)))$

(G27) $(A \supset (B \prec C)) \supset$
$\supset ((A \wedge (B \leftrightarrow C) \wedge (C \prec \top)) \leftrightarrow (A \wedge (B \leftrightarrow C) \wedge (C \leftrightarrow \top)))$
(G28) $(B \supset C) \supset ((C \prec B) \leftrightarrow ((B \leftrightarrow C) \wedge (C \leftrightarrow \top)))$

Consider the proof system **G** in the language $\mathscr{L}^{\mathrm{p}}$. Then:

(G29) The deduction theorem holds for **G**.
(G30) The rule $\frac{A \supset B \quad B \supset C}{A \supset C}$ is admissible. ☡

Proof. The reader easily verifies the validity of the claims (G1)–(G28). Then §3.4.2 establishes (effective) provability. For (G3), see also [12, Lemma 3] or [2, Proposition 2.3].

For a proof of (G29) see, e. g., [15, p. 99]; this can easily be done by induction on the length of the proof. For (G30), the reader readily checks the validity of $(A \supset B) \supset (B \supset C) \supset (A \supset C)$, applies §3.4.2 to obtain a proof thereof, and then uses (MP) twice. ♢

§3.4.4 Definition. Let $\mathbf{G}_{\bigcirc}$ denote the proof system resulting from extending **G** by the following axiom schemata:

(R1) $(\bot \prec \bigcirc\bot) \supset (A \prec \bigcirc A)$,
(R2) $(\bot \leftrightarrow \bigcirc\bot) \supset (A \leftrightarrow \bigcirc A)$,
(R3) $\bigcirc(A \supset B) \leftrightarrow (\bigcirc A \supset \bigcirc B)$,

where A and B are any formulae from $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$. If A is $\mathbf{G}_{\bigcirc}$-derivable, we write $\mathbf{G}_{\bigcirc} \vdash A$. ☡

For our purposes, it does not matter if **G** is taken to be Hájek's Basic Logic plus Idempotency, or if it is Intuitionistic Logic plus Linearity. We will consider mere provability, but not proof-theoretic properties like the length of a proof.

While the next lemma establishes the soundness of $\mathbf{G}_{\bigcirc}$, the remainder of this section will be devoted to its completeness.

§3.4.5 Lemma. $\mathbf{G}_{\bigcirc}$ is sound w. r. t. Gödel semantics in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$. ☡

Proof. This routine proof runs by induction on formula complexity. Clearly, all $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$-instances of (MP), (LIN) and (IPL1)–(IPL9) are sound, and (R3) is established by (B8), (A17) and (B5).

In order to prove (R1), suppose it was not valid so that $\mathrm{I}((\bot \prec \bigcirc\bot) \supset (A \prec \bigcirc A)) < 1$ for some r-interpretation I, $r \in [0,1]$. From (A16), we obtain $\mathrm{I}(A \prec \bigcirc A) < \mathrm{I}(\bot \prec \bigcirc\bot)$. We have $r > 0$ for otherwise $0 \leq \mathrm{I}(A \prec$

$\bigcirc A) < \mathrm{I}(\bot \prec \bigcirc\bot) = 0 \lhd (r \oplus 0) = 0 \lhd r = 0 \lhd 0 = 0$. We must have $\mathrm{I}(A) < 1$ for otherwise $1 = 1 \lhd 1 = 1 \lhd (r \oplus 1) = \mathrm{I}(A) \lhd (r \oplus \mathrm{I}(A)) = \mathrm{I}(A \prec \bigcirc A) < \mathrm{I}(\bot \prec \bigcirc\bot)$, which was absurd. Together with $r > 0$, we conclude $\mathrm{I}(A) < r \oplus \mathrm{I}(A)$, thus $1 = \mathrm{I}(A) \lhd (r \oplus \mathrm{I}(A)) = \mathrm{I}(A \prec \bigcirc A) < \mathrm{I}(\bot \prec \bigcirc\bot) \leq 1$, which is absurd. Hence (R1) holds.

In order to prove (R2), suppose it was not valid so that $\mathrm{I}((\bot \leftrightarrow \bigcirc\bot) \supset (A \leftrightarrow \bigcirc A)) < 1$ for some r-interpretation I, $r \in [0, 1]$. From (A16), we obtain $\mathrm{I}(A \leftrightarrow \bigcirc A) < \mathrm{I}(\bot \leftrightarrow \bigcirc\bot)$. We must have $0 = r$ for otherwise $0 \leq \mathrm{I}(A \leftrightarrow \bigcirc A) < \mathrm{I}(\bot \leftrightarrow \bigcirc\bot) = 0 \bowtie (r \oplus 0) = 0 \bowtie r = 0$. But now $1 = \mathrm{I}(A) \bowtie \mathrm{I}(A) = \mathrm{I}(A) \bowtie (r \oplus \mathrm{I}(A)) = \mathrm{I}(A \leftrightarrow \bigcirc A) < \mathrm{I}(\bot \leftrightarrow \bigcirc\bot) \leq 1$, which is absurd. Hence (R2) holds. ♢

§3.4.6 Remark. When we will need to give a formal derivation of a formula in a proof system, say, e. g. in $\mathbf{G}_{\bigcirc}$, we will often only indicate how to construct sub-derivations instead of writing them down in full detail: Usually, we will merely state the provability of an $\mathscr{L}^{\mathrm{p}}$-formula A in $\mathbf{G}$ and leave to the reader the inexpensive task of checking its validity in Gödel semantics and of applying §3.4.2 to effectively obtain a $\mathbf{G}$-proof Π of A. Since the language, the axiom schemata and the rule of $\mathbf{G}$ are contained in the ones of $\mathbf{G}_{\bigcirc}$, it is readily proved that a substitution of propositional variables in Π by formulae in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$ is a $\mathbf{G}_{\bigcirc}$-derivation (that neither uses (R1), (R2) nor (R3)), thus, any $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$-instance of A is $\mathbf{G}_{\bigcirc}$-provable. We will use this substitutivity property often tacitly in the remainder. ☡

The next proposition is already an example of the technique in the above remark.

§3.4.7 Proposition. The rule $\frac{A \supset B \quad B \supset C}{A \supset C}$ is admissible in $\mathbf{G}_{\bigcirc}$. ☡

Proof. Since $\mathbf{G}$ proves $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$, also $\mathbf{G}_{\bigcirc}$ proves it for all formulae A, B, C in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$. Using (MP) twice for the given proofs of $A \supset B$ and $B \supset C$, we obtain $A \supset C$ as claimed. ♢

We will prove two well-known variants of the deduction theorem in the next paragraph. However, we will not use it in the sequel because we like to generalise our results to the situation of §3.5.4, where it does not hold.

§3.4.8 Proposition. (a) Let $X \cup \{A\}$ be a set of formulae in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$; here we allow X to be infinite. Then $\mathbf{G}_{\bigcirc} + X + A \vdash B$ if and only if $\mathbf{G}_{\bigcirc} + X \vdash A \supset B$. In fact, a proof of $A \supset B$ can be constructed from a proof of B in $\mathbf{G}_{\bigcirc} + X + A$, and also vice versa.

(b) The deduction theorem holds for $\mathbf{G}_\bigcirc$: Suppose $X \cup \{A, B_0, \ldots, B_{N-1}\}$ is a set of formulae in $\mathscr{L}^{\mathrm{p}}_\bigcirc$, and let $\bigwedge B$ stand for the conjunction of the B_i, with an arbitrary parenthesisation. Then $\mathbf{G}_\bigcirc + X + B_0 + \ldots + B_{N-1} \vdash A$ if and only if $\mathbf{G}_\bigcirc + X \vdash (\bigwedge B) \supset A$. ☡

Proof. (a) Let $\mathbf{G}_\bigcirc + X \vdash A \supset B$, then, by monotonicity, $\mathbf{G}_\bigcirc + X + A \vdash A \supset B$. Since we clearly have $\mathbf{G}_\bigcirc + X + A \vdash A$, we conclude $\mathbf{G}_\bigcirc + X + A \vdash B$ by (MP).

The converse direction is showed by induction on length on derivation. To establish the induction base, we need to show $\mathbf{G}_\bigcirc + X \vdash A \supset B$ if B has an $\mathbf{G}_\bigcirc + X + A$-derivation of length 1, i. e. if (1) $A = B$ or (2) $B \in X$ or (3) B is axiom of $\mathbf{G}_\bigcirc$. In case (1), we use (G14) to obtain $\mathbf{G}_\bigcirc \vdash B \supset B$, thus $\mathbf{G}_\bigcirc + X \vdash B \supset B$, as required. In cases (2) and (3), we have $\mathbf{G}_\bigcirc + X \vdash B$ so that $\mathbf{G}_\bigcirc + X \vdash A \supset B$ by (IPL8) and (MP).

Let Π be a $\mathbf{G}_\bigcirc + X + A$-derivation of B of length n and suppose that we have $\mathbf{G}_\bigcirc + X \vdash A \supset B'$ whenever B' has a $\mathbf{G}_\bigcirc + X + A$-derivation of length $< n$. If (MP) is not the last line of Π, then B has in fact a $\mathbf{G}_\bigcirc + X + A$-derivation of length 1 since $\mathbf{G}_\bigcirc + X + A$ does not have a rule other than (MP). The induction base then yields $\mathbf{G}_\bigcirc + X \vdash A \supset B$, as desired. If (MP) is the last line of Π, Π contains subderivations $\mathbf{G}_\bigcirc + X + A \vdash C$ and $\mathbf{G}_\bigcirc + X + A \vdash C \supset B$ for some C. Since both subderivations have length $< n$ we can construct proofs $\mathbf{G}_\bigcirc + X \vdash A \supset C$ and $\mathbf{G}_\bigcirc + X \vdash A \supset (C \supset B)$ by the induction hypotheses. By (IPL9), we find $\mathbf{G}_\bigcirc + X \vdash A \supset B$, as required. This completes the proof of (a).

(b) This follows from (a) and from the fact that $\mathbf{G} \vdash (B_0 \supset B_1 \supset \ldots \supset B_{N-1} \supset A) \leftrightarrow ((\bigwedge B) \supset A)$, for any parenthesisation of $\bigwedge B$. ◇

Next, we will give some background information on the proof of completeness.

§3.4.9 Remark. As we have remarked in §3.2.10, the completeness of $\mathbf{G}_\bigcirc$ cannot be shown in a typically 'algebraic' way; however, it would be elucidating—for all Gödel logics, not only for $\mathbf{G}_\bigcirc$—to find a method that is as close as possible to the 'algebraic' one. Therefore we had to go back to a refined method of Dummett's original idea of *evaluating* the formula: Whether a formula receives the value 1 under an interpretation depends only on the order of the values that are assigned to the propositional variables. As a formula contains only a finite number of variables, there are only finitely many orderings we need to consider. These orderings are called *chains.* Given a fixed chain, it is clear how to evaluate a formula semantically: We repeatedly simplify some (in principle) innermost formula to a variable, until we are left with $\top$, $\bot$, or a variable. This is, of course, only possible because the equivalence scheme for formula contexts holds. Each step in this semantical evaluation can be turned into a proof of an implication where the antecedent is a formula expressing the chain and the consequent is the equivalence of the previous and the current formula in the loop. If the formula A reduces to $\top$, we join the subproofs of A under all chains by a repeated application of the linearity axiom into a proof of A. If A reduces to $\bot$ or a propositional variable v (and where v is not equal to $\top$ w. r. t. the given ordering), it is easy to find an interpretation such that A receives a value less than 1; the formula A thus cannot be valid. Although it is clear that propositional Gödel logics is decidable (validity and satisfiability both can be translated, e. g., to formulae in the decidable language of the algebraically closed field $\mathbb{R}$), the above method provides a proof-theoretic way of determining validity (and also of satisfiability).

The above procedure can be understood as a generalisation of the effective method in classical propositional logic where a formula is evaluated under all 2^n interpretations of n variables to establish validity (or satisfiability).

The possibility of testing the validity of a formula in Gödel logic by evaluating a formula under only finitely many chains is of proof-theoretical significance: It enables, e. g., interpolation, see [7].

For $\mathbf{G}_{\bigcirc}$, the proof of completeness is more involved: First, we have to prove the properties we mentioned in the paragraph about Gödel logics without $\bigcirc$: In §3.4.10, we will prove, e. g., the equivalence scheme that allows us to substitute formulae 'in depth'. (This feature is remarkable because only a weaker version holds for t-norm based logics.) We will prove also in §3.4.10 that $\bigcirc$ commutes with all binary connectives so that we are able to find a provably equivalent normal form of any formula where $\bigcirc$ occurs only in front of variables and of $\bot$. This allows us in some steps of the proof to treat these ring-powers $\bigcirc^i v$ as if they were indexed variables $v_{(i)}$. A case where this approach alone works, is presented in [4].

However, there are two important differences to Gödel logics without $\bigcirc$: (1) Not all chains over $v_{(i)}$ need to be considered, e. g., if $v_{(i)}$ precedes $w_{(j)}$ in the chain, then $w_{(j+1)}$ cannot precede $v_{(i+1)}$ (in particular, equality in the ordering is possible). If A is a formula that is not valid, say, it reduces to a variable v under a chain C, we need to build a 'countermodel' I of v under C, i. e., such that $\mathrm{I}(C \supset v) < 1$. (2) The construction of this 'countermodel' is technically difficult and takes much more space than in the situation of Gödel logics without $\bigcirc$. However, the geometric intuition behind it is quite simple. Let n be the maximal nesting level of $\bigcirc$ in A. (In particular, n is bounded by the length of A.) Shifted copies of $\{0, \dots, n\}$ in $\mathbb{R}$ have to be arranged such that the shifted copies of 0 fulfil the same ordering as the variables in C. This is done essentially in §3.4.13. A problem here is that the ordering in C does not necessarily force a unique model, which would have allowed just to copy the variable ordering from C to the countermodel. In fact, there is some freedom in arranging the variables and so some effort has to be put into this construction; still, the copies can be arranged sequentially by stepping through the ring powers in C from left to right with a certain anticipation to the next ring power $v_{(i+1)}$ when working at $v_{(i)}$. Finally, in §3.4.15, we will scale these shifted copies by a factor that corresponds to r, i. e. the value that $\bigcirc$ adds, and then we need to cut off all points in $\mathbb{R}$ that exceed a certain value that corresponds to 1 in the truth-value set of the countermodel. This cutting off is typical for Gödel logics in contrast to, say, Łukasiewicz logics, where—roughly said—all the constructions have to be done inside of $[0, 1]$. The lifting lemma and the lack of the expressibility of $\triangle$ suggest that Gödel logics is not capable of exactly localising the truth value 1. In the proof of §3.4.15, one can see that the choice of r and of this cutting point is to some extent arbitrary. ☡

§3.4.10 Proposition. Let $E[\cdot]$ denote an $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$-context, and let A, B, C be formulae in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$. Then $\mathbf{G}_{\bigcirc}$ proves the following formulae:

(S1) $A \supset \bigcirc A$
(S2) $\bigcirc(A \,\square\, B) \leftrightarrow (\bigcirc A \,\square\, \bigcirc B)$ for $\square \in \{\prec, \wedge, \vee, \leftrightarrow\}$
(S3) $(A \leftrightarrow B) \supset (E[A] \leftrightarrow E[B])$ ☡

Proof. (S1): Since $\mathbf{G}$ proves $((\bot \prec X) \supset (Y \prec Z)) \supset (((\bot \leftrightarrow X) \supset (Y \leftrightarrow Z)) \supset (Y \supset Z))$, $\mathbf{G}_{\bigcirc}$ proves $((\bot \prec \bigcirc\bot) \supset (A \prec \bigcirc A)) \supset (((\bot \leftrightarrow \bigcirc\bot) \supset (A \leftrightarrow \bigcirc A)) \supset (A \supset \bigcirc A))$. From (MP) together with (R1) and (R2), we obtain (S1).

(S2)($\prec$): Due to (R3), we have $\mathbf{G}_{\bigcirc} \vdash \bigcirc((B \supset A) \supset B) \leftrightarrow (\bigcirc(B \supset A) \supset \bigcirc B)$ and $\mathbf{G}_{\bigcirc} \vdash \bigcirc(B \supset A) \leftrightarrow (\bigcirc B \supset \bigcirc A)$. Since $\mathbf{G}$ proves $(C \leftrightarrow (D \supset E)) \supset (D \leftrightarrow F) \supset (C \leftrightarrow (F \supset E))$, it follows $\mathbf{G}_{\bigcirc} \vdash \bigcirc((B \supset A) \supset B) \leftrightarrow ((\bigcirc B \supset \bigcirc A) \supset \bigcirc B)$, i. e. (S2).

(S2)($\wedge$): $\mathbf{G}$ proves $\big(((C \wedge D) \supset C) \supset P\big) \supset \big(((E \wedge F) \supset F) \supset Q\big) \supset \big((G \supset (H \supset (G \wedge H))) \supset R\big) \supset \big(P \leftrightarrow (K \supset N)\big) \supset \big(Q \leftrightarrow (K \supset M)\big) \supset \big(R \leftrightarrow (N \supset S)\big) \supset \big(S \leftrightarrow (M \supset K)\big) \supset \big(K \leftrightarrow (N \wedge M)\big)$. We apply (MP) to an appropriate instance of this formula and the (S1)-instances

$((A \wedge B) \supset A) \supset \bigcirc((A \wedge B) \supset A)$, $((A \wedge B) \supset B) \supset \bigcirc((A \wedge B) \supset B)$, $(A \supset (B \supset (A \wedge B))) \supset \bigcirc(A \supset (B \supset (A \wedge B)))$ and the (R3)-instances $\bigcirc((A \wedge B) \supset A) \leftrightarrow (\bigcirc(A \wedge B) \supset \bigcirc A)$, $\bigcirc((A \wedge B) \supset B) \leftrightarrow (\bigcirc(A \wedge B) \supset \bigcirc B)$, $\bigcirc(A \supset (B \supset (A \wedge B))) \leftrightarrow (\bigcirc A \supset \bigcirc(B \supset (A \wedge B)))$, $\bigcirc(B \supset (A \wedge B)) \leftrightarrow (\bigcirc B \supset \bigcirc(A \wedge B))$ to obtain $\bigcirc(A \wedge B) \leftrightarrow (\bigcirc A \wedge \bigcirc B)$.

(S2)($\vee$): **G** proves $\big(((U \vee V) \supset V) \supset P\big) \supset \big(((U \vee V) \supset U) \supset Q\big) \supset \big((C \supset (C \vee D)) \supset R\big) \supset \big((E \supset (E \vee F)) \supset S\big) \supset \big(P \leftrightarrow (G \supset E)\big) \supset \big(Q \leftrightarrow (G \supset F)\big) \supset \big(R \leftrightarrow (E \supset G)\big) \supset \big(S \leftrightarrow (F \supset G)\big) \supset \big(G \leftrightarrow (E \vee F)\big)$. We apply (MP) to an appropriate instance of this formula and the (S1)-instances $((A \vee B) \supset A) \supset \bigcirc((A \vee B) \supset A)$, $((A \vee B) \supset B) \supset \bigcirc((A \vee B) \supset B)$, $(A \supset (A \vee B)) \supset \bigcirc(A \supset (A \vee B))$, $(B \supset (A \vee B)) \supset \bigcirc(B \supset (A \vee B))$, the (R3)-instance $\bigcirc((A \vee B) \supset A) \leftrightarrow (\bigcirc(A \vee B) \supset \bigcirc A)$, $\bigcirc((A \vee B) \supset B) \leftrightarrow (\bigcirc(A \vee B) \supset \bigcirc B)$, $\bigcirc(A \supset (A \vee B)) \leftrightarrow (\bigcirc A \supset \bigcirc(A \vee B))$, $\bigcirc(B \supset (A \vee B)) \leftrightarrow (\bigcirc B \supset \bigcirc(A \vee B))$ to obtain $\bigcirc(A \vee B) \leftrightarrow (\bigcirc A \vee \bigcirc B)$.

(S2)($\leftrightarrow$): **G** proves $\big(G \leftrightarrow (E \wedge F)\big) \supset \big(E \leftrightarrow (C \supset D)\big) \supset \big(F \leftrightarrow (D \supset C)\big) \supset \big(G \leftrightarrow (C \leftrightarrow D)\big)$. We apply (MP) to an appropriate instance of this formula and the (S2)($\wedge$)-instance $\bigcirc((A \supset B) \wedge (B \supset A)) \leftrightarrow (\bigcirc(A \supset B) \wedge \bigcirc(B \supset A))$ and the (R3)-instances $\bigcirc(A \supset B) \leftrightarrow (\bigcirc A \supset \bigcirc B)$ and $\bigcirc(B \supset A) \leftrightarrow (\bigcirc B \supset \bigcirc A)$ to obtain $\bigcirc((A \supset B) \wedge (B \supset A)) \leftrightarrow ((\bigcirc A \supset \bigcirc B) \wedge (\bigcirc B \supset \bigcirc A))$, i. e. $\bigcirc(A \leftrightarrow B) \leftrightarrow (\bigcirc A \leftrightarrow \bigcirc B)$ as required.

For later use, apply (IPL2) to the last formula to obtain $\bigcirc(A \leftrightarrow B) \supset (\bigcirc A \leftrightarrow \bigcirc B)$ so that, from the (S1)-instance $(A \leftrightarrow B) \supset \bigcirc(A \leftrightarrow B)$, we now see $\mathbf{G}_\bigcirc$ proves $(A \leftrightarrow B) \supset (\bigcirc A \leftrightarrow \bigcirc B)$.

(S3) is proved by induction on the formula complexity of $E[\cdot]$: Clearly, $\mathbf{G}_\bigcirc$ proves $(A \leftrightarrow B) \supset (A \leftrightarrow B)$ and $(A \leftrightarrow B) \supset (C \leftrightarrow C)$ since **G** does. If a $\mathbf{G}_\bigcirc$-proof of $(A \leftrightarrow B) \supset (E[A] \leftrightarrow E[B])$ is given, we use $\mathbf{G}_\bigcirc \vdash (E[A] \leftrightarrow E[B]) \supset (\bigcirc E[A] \leftrightarrow \bigcirc E[B])$ from the preceding remark and conclude $\mathbf{G}_\bigcirc \vdash (A \leftrightarrow B) \supset (\bigcirc E[A] \leftrightarrow \bigcirc E[B])$ as required. Given $\mathbf{G}_\bigcirc$-proofs of $(A \leftrightarrow B) \supset (E[A] \leftrightarrow E[B])$ and $(A \leftrightarrow B) \supset (F[A] \leftrightarrow F[B])$, it is easy to obtain proofs of $(A \leftrightarrow B) \supset ((E[A] \Box F[A]) \leftrightarrow (E[B] \Box F[B]))$ for any $\Box \in \{\wedge, \vee, \supset\}$ since **G** proves $(X \supset (P \leftrightarrow Q)) \supset (X \supset (R \leftrightarrow S)) \supset (X \supset ((P \Box R) \leftrightarrow (Q \Box S)))$. This establishes (S3). $\diamondsuit$

Our next goal is §3.4.15, which constructs a Gödel r-interpretation I with $\mathrm{I}(A) < 1$ for non-valid formulae A of a particular syntactic form.

§3.4.11 Definition (Grid). Let X be a finite, non-empty set. We call $(Y, \ll, \sim)$ a *grid over* X if $Y = \{(x, n);\ x \in X,\ n \leq N(x)\}$ for some $N\colon X \to \mathbb{N}$ such that $\sim$ is an reflexive, symmetric and transitive relation on Y, and $\ll$ is a transitive relation on Y, and for all $a, b, c \in Y$ we have

(T1) either $a \ll b$ or $a \sim b$ or $b \ll a$;
(T2) $a \sim b \ll c \supset a \ll c$,
(T3) $a \ll b \sim c \supset a \ll c$,
(T4) $a+1 \in Y \supset a \ll a+1$,
(T5) $(a+1 \in Y \wedge b+1 \in Y) \supset (a \ll b \Leftrightarrow a+1 \ll b+1)$,
(T6) $(a+1 \in Y \wedge b+1 \in Y) \supset (a \sim b \Leftrightarrow a+1 \sim b+1)$,

here $(x,n)+k := (x,n+k)$ for all n, $k \in \mathbb{N}$ and $x \in X$. X can be thought of as being a subset of Y by virtue of $x \mapsto (x,0)$. We put $\lessapprox := \ll \cup \sim$. ☡

§3.4.12 Lemma. Let $\mathscr{C} = (Y, \ll, \sim)$ be a grid over X. Then there is an algorithm to construct a grid $\mathscr{C}_* = (Y_*, \ll^*, =)$ over X and an $\sigma\colon\ Y \to Y_*$ such that

$$\begin{aligned}
&\forall y, y' \in Y.\ (y \sim y' \Leftrightarrow \sigma(y) = \sigma(y')),\\
&\forall y, y' \in Y.\ (y \ll y' \Leftrightarrow \sigma(y) \ll^* \sigma(y')),\\
&\forall y \in Y.\ (y+1 \in Y \supset \sigma(y+1) = \sigma(y)+1 \in Y_*)
\end{aligned}$$ ☡

Proof. It suffices to prove the following statement: For every grid $\mathscr{C} = (Y, \ll, \sim)$ and p, $q \in Y$ such that $p \sim q$, we can specify a grid $\mathscr{C}' = (Y_*, \ll^*, \sim^*)$ and $t\colon\ Y \to Y_*$ such that $t(p) = t(q)$ and for all $y, y_* \in Y$ holds: (1) $y \sim y_*$ if and only if $t(y) \sim^* t(y_*)$, (2) $y \ll y_*$ if and only if $t(y) \ll^* t(y_*)$, and (3) $t(y+1) \sim^* t(y)+1 \in Y_*$ whenever $y \in Y$ such that $y+1 \in Y$; observe that this properly reduces the number of equivalence classes if $p \neq q$. The claim of the lemma now follows by eliminating all the finitely many equivalence of Y classes iteratively; σ is obtained by concatenating the obtained t's.

Thus, let $\mathscr{C} = (Y, \ll, \sim)$ and $p, q \in Y$ such that $p \sim q$. Then there are a, $b \in X$ and n, $m \in \mathbb{N}$ such that $p = (a,n) \sim q = (b,m)$. W. l. o. g. we assume that $n \geq m$. Thus, by (T5), $a+K \sim b$ for $K := n-m \in \mathbb{N}$. Define $N'(x) := N(x)$ for all $x \in X \setminus \{a,b\}$ and let $N'(a) := \max\{N(a), N(b)+K\}$. Put $Y_* := \{(x,i);\, x \in X \setminus \{b\},\, i < N'(x)\}$ and define $t\colon\ Y \to Y_*$ by $t(b,i) := (a, i+K)$ and $t(x,i) := (x,i)$ for all $x \in X \setminus \{b\}$.

For all (x,n), $(x',n') \in Y$, we will see by distinguishing four cases that $t(x,n) = t(x',n')$ implies $(x,n) \sim (x',n')$: If both $x, x' \in X \setminus \{b\}$ then $(x,n) = t(x,n) = t(x',n') = (x',n')$. If $x = b = x'$ then $(a, n+K) = t(x,n) = t(x',n') = (a, n'+K)$ so that $n = n'$ and thus $(x,n) \sim (x',n')$. If $x = b$ and $x' \in X \setminus \{b\}$ then $(x,n) = (b,n) \sim (a, n+K) = t(x,n) = t(x',n') = (x',n')$. If $x' = b$ and $x \in X \setminus \{b\}$ then $(x,n) = t(x,n) = t(x',n') = (a, n'+K) \sim (b, n') = (x',n')$.

Since t is surjective, the property just proved enables us to define two relations $\ll^*$ and $\sim^*$ on Y_* by $t(c) \ll^* t(d) :\Leftrightarrow c \ll d$ and by $t(c) \sim^* t(d) :\Leftrightarrow c \sim d$. This establishes properties (1) and (2). Clearly, $t(b, i+1) = (a, i+K+1) = t(b, i)+1$ and $t(x, i+1) = (x, i+1) = (x, i)+1 = t(x, i)+1$ for all $x \in X \setminus \{b\}$. Thus property (3) holds, and so it easy to check that $(Y_*, \ll^*, \sim^*)$ is indeed a grid. We also have $t(q) = t(b, m) = (a, m+K) = (a, n) = t(a, n) = t(p)$. $\diamondsuit$

Observe that, if $(Y, \ll, \sim)$ is a grid, every subset Z of Y has a $\lesssim$-minimal element in the sense that there is $u \in Z$ such that $u' \lesssim u$ for all $u' \in Z$; likewise there is a $\lesssim$-maximal element $U \in Z$. It follows that any subset of a grid $(Y, \ll, =)$, with identity as equivalence relation, has a unique $\ll$-minimal and a unique $\ll$-maximal element.

§3.4.13 Lemma. Let $(Y, \ll, =)$ be a finite grid over X and let s be the $\ll$-minimal element of Y. Then we can construct $f\colon\ Y \to [0, \infty) \cap \mathbb{Q}$ such that

$$\begin{aligned}
&f(s) = 0,\\
&\forall y, y' \in Y.\ (y \ll y' \supset f(y) < f(y')), \text{ and}\\
&\forall y \in Y.\ (y+1 \in Y \supset f(y+1) = f(y)+1).
\end{aligned}$$

Proof. We will use the following definitions in the slightly involved iterative construction of f. Let S be the $\ll$-maximal element of Y, cf. the remark before the statement of the lemma. Take $\underline{\ll} := \ll \cup \mathrm{id}_Y$, which is obviously a reflexive and transitive relation on Y. Let $E(a, f)$ abbreviate the condition that $a \in Y$ and f is a function from $\{y \in Y; y \underline{\ll} a\}$ to $[0, \infty) \cap \mathbb{Q}$ such that

(1) $\forall y, y' \in Y\ \big(y \ll y' \underline{\ll} a \supset f(y) < f(y')\big)$,
(2) $\forall y \in Y\ \big((y+1 \in Y \land y+1 \underline{\ll} a) \supset f(y+1) = f(y)+1\big)$,
(3) $\forall y \in Y\ \big((y+1 \in Y \land y \underline{\ll} a \ll y+1) \supset f(a) < f(y)+1\big)$.

We start the construction by putting $f_0(s) := 0$; it is easy to check that indeed $E(s, f_0)$. In the remainder of the proof, we will prove for any given $a \in Y$ and f such that $a \ll S$ and $E(a, f)$ that we can construct $a_* \in Y$ and f_* such that $a \ll a_*$ and $E(a_*, f)$, in particular $a \neq a_*$ then. This clearly suffices to construct an f such that $E(S, f)$ holds; hence this f is total on Y and conditions (1) and (2) establish the claim of the lemma.

Now let $a \in Y$ and f be given such that $a \ll S$ and $E(a, f)$ hold. We will distinguish two cases.

In the first case, we suppose that $\emptyset \neq B := \{b \in Y;\ b \leqq a \ll b+1\}$ holds. Let b be $\leqq$-minimal in B. By definition of B, we have $b+1 \in Y$. By (1) and (3), we see $f(b) \leq f(a) < f(b)+1$, thus $0 < f(b)+1-f(a) \leq 1$.

For $C := \{c \in Y;\ a \ll c \ll b+1\}$, we have $C \subseteq X$ for otherwise $a \ll c \ll b+1$ for some $c \in Y \setminus X$; the latter means that $c = d+1$ for some $d \in Y$, but then $a \ll d+1 \ll b+1$ implies $d \ll b$ by (T5), which contradicts the minimality of b.

Let $c_1 \ll c_2 \ll \ldots \ll c_M$ be an enumeration of C. Extend f to f_* by $f_*(c_m) := f(a) + \frac{m}{M+1}(f(b)+1-f(a))$ and $f_*(b+1) := f(b)+1$ so that f_* is defined for all $y \leqq b+1$ and $f(a) = f_*(a) < f_*(c_1) < f_*(c_2) < \cdots < f_*(c_M) < f_*(b+1)$. Thus, by the definition of C, we see $f_*(a) < f_*(y) < f_*(y') \leq f_*(b+1)$ whenever $a \leqq y \ll y' \leqq b+1$. Now, we will prove $E(b+1, f_*)$. The statements (1), (2), (3) will refer to the conditions in $E(a,f)$.

Given $y, y' \in Y$ with $y \ll y' \leqq b+1$, we need to show $f_*(y) < f_*(y')$. We may assume $y \ll a$ since $a \leqq y$ implies $a \leqq y \ll y' \leqq b+1$ and this yields, as proved above, $f_*(y) < f_*(y')$. From $y \ll a$, we see $f_*(y) = f(y) < f(a) = f_*(a)$ by (1). We may assume also $a \ll y'$ since $y' \leqq a$ implies $y \ll y' \leqq a$ and then $f_*(y) < f_*(y')$ by (1). As observed earlier, we have $f_*(a) < f_*(y')$ and thus $f_*(y) < f_*(a) < f_*(y')$, as required.

Given $y \in Y$ such that $y+1 \in Y$ and $y+1 \leqq b+1$, we need to prove $f_*(y+1) = f_*(y)+1$. We may assume $y+1 \ll b+1$ because $b+1 \leqq y+1$, together with $y+1 \leqq b+1$, yields $b+1 = y+1$ so that $y = b$ and $f_*(b+1) = f_*(b)+1$ by definition of f_*. We may assume $y+1 \leqq a$ for otherwise $a \ll y+1 \in Y$ holds and thus $y+1 \in C$ but this contradicts $C \subseteq X$ and $y \in Y$. Now $y \ll y+1 \leqq a$, and so $f_*(y+1) = f(y+1) = f(y)+1 = f_*(y)+1$ by (2), as required.

In the last three paragraphs, we have proved $E(b+1, f_*)$ under the condition $\emptyset \neq B$.

In the second case, we suppose that $\emptyset = \{y \in Y;\ y \leqq a \ll y+1\}$ holds in addition to $E(a,f)$ and $a \ll S$. Due to $a \ll S$ there is some $c \in Y$ that is $\leqq$-minimal among the $y \in Y$ with $a \ll y$. We must have $c \in X$ for otherwise there is $y \in Y$ with $a \ll c = y+1$, but since $y \leqq a \ll y+1$ is impossible, we obtain $a \ll y$ and thus $a \ll y \ll y+1 = c$, which contradicts the minimality of c. Extend f to f_* by $f_*(c) := f(a) + \varepsilon$ for some $\varepsilon > 0$, e. g. $\varepsilon = 1$. We will prove $E(c, f_*)$.

Given $y, y' \in Y$ such that $y \ll y' \leqq c$, we need to prove $f_*(y) < f_*(y')$. We may assume $a \ll y'$ since otherwise $y' \leqq a$ holds and then $y \ll y' \leqq a$ implies $f_*(y) = f(y) < f(y') = f_*(y')$ by (1). From $a \ll y' \leqq c$, we conclude

$y' = c$ by minimality of c. We may assume that $a \neq y$ since $a = y$ implies $f(y) = f(a) < f(a) + \varepsilon = f_*(c) = f_*(y')$. We cannot have $a \ll y$ since then $a \ll y \ll y' = c$ contradicted the minimality of c. Thus $y \ll a$ and now $f_*(y) = f(y) < f(a) < f(a) + \varepsilon = f_*(c) = f_*(y')$ by (1), as required.

Given $y \in Y$ such that $y+1 \in Y$, $y+1 \underline{\ll} c$, we need to prove $f_*(y+1) = f_*(y)+1$. We have $y \underline{\ll} a$ since otherwise $a \ll y$ holds, which implies $a \ll y \ll y+1 \underline{\ll} c$, but this contradicts the minimality of c. Since $y \underline{\ll} a \ll y+1$ is impossible, we have $y+1 \underline{\ll} a$. By (2), we find $f(y+1) = f(y)+1$, as required.

Given $y \in Y$ such that $y+1 \in Y$, $y \underline{\ll} c \ll y+1$, we need to prove $f_*(c) < f_*(y)+1$. We must have $a \ll y+1$ for otherwise we obtain a contradiction from $y+1 \underline{\ll} a$ and $a \ll c \ll y+1$. Since $y \underline{\ll} a \ll y+1$ cannot hold, we must have $a \ll y$. From $y \underline{\ll} c$ and the minimality of c, we conclude that $y = c$. Thus $f_*(c) < f_*(y)+1$ trivially holds.

Thus $E(c, f_*)$ holds, as claimed, also in the second case. This completes the whole proof. ♢

§3.4.14 Definition (Chain). Let $K \in \mathbb{N}$, let $X \subseteq \mathsf{Var}$ be finite and choose two distinct fresh formal symbols $\top$ and $\bot$. Put $Z' := \{(x,k);\, x \in X \cup \{\bot\},\ k \leq K\}$, $Z := \{\top\} \cup Z'$, and $(x,m)+n := (x, m+n)$ for all $x \in X \cup \{\bot\}$, $m, n \in \mathbb{N}$. We understand X as a subset of Z' by the embedding $x \mapsto (x, 0)$. We call $(Z, \prec, \leftrightarrow)$ an (X,K)-*chain* if $\leftrightarrow$ is a reflexive, symmetric, transitive relation on Z, $\prec$ is a transitive relation on Z such that

(U1) $\forall a, b, c \in Z.\quad a \leftrightarrow b \prec c \supset a \prec c$,
(U2) $\forall a, b, c \in Z.\quad a \prec b \leftrightarrow c \supset a \prec c$,
(U3) $\forall a \in Z.\quad$ either $a \prec \top$ or $a \leftrightarrow \top$,
(U4) $\forall a \in Z.\quad$ either $\bot \prec a$ or $\bot \leftrightarrow a$,
(U5) $\forall a, b \in Z.\quad$ either $a \prec b$ or $a \leftrightarrow b$ or $b \prec a$,
(U6) $\forall a \in Z'.\quad (a+1 \in Z \wedge \bot \leftrightarrow \bot+1) \supset a \leftrightarrow a+1$.
(U7) $\forall a \in Z'.\quad (a+1 \in Z \wedge \bot \prec \bot+1) \supset (a \prec a+1 \vee a \leftrightarrow a+1 \leftrightarrow \top)$,
(U8) $\forall a, b \in Z'.\quad (a+1 \in Z \wedge b+1 \in Z \wedge a \leftrightarrow b) \supset a+1 \leftrightarrow b+1$,
(U9) $\forall a, b \in Z'.\quad (a+1 \in Z \wedge b+1 \in Z \wedge \bot \prec \bot+1 \wedge a \prec b) \supset$
$\supset (a+1 \prec b+1 \vee a+1 \leftrightarrow b+1 \leftrightarrow \top)$, ☡

§3.4.15 Lemma. Let $(Z, \prec, \leftrightarrow)$ be an (X,K)-chain. Put $Z' := Z \setminus \{\top\}$. Then we can construct $r \in [0,1]$ and $g\colon Z \to [0,1]$ such that $g(\bot) = 0$, $g(\top) = 1$,

$$\forall a, b \in Z.\ a \prec b \supset g(a) < g(b),$$

$$\forall a, b \in Z.\, a \leftrightarrow b \supset g(a) = g(b),$$
$$\forall a \in Z'.\, a + 1 \in Z \supset g(a+1) = g(a) \oplus r. \qquad ∂$$

Proof. By (U4), either $\bot \leftrightarrow \bot + 1$ or $\bot \prec \bot + 1$ holds. In the case $\bot \leftrightarrow \bot + 1$, we obtain from (U6) that $x \leftrightarrow x + 1 \leftrightarrow \ldots \leftrightarrow x + K$ holds for all $x \in X \cup \{\bot\}$ and $\prec$ is a total order on $X \cup \{\bot, \top\}$; thus we may put $r := 0$ and it is obvious how to choose some v on X. Thus we consider only the case $\bot \prec \bot + 1$ in the remainder of the proof.

For the elements of $Z' := Z \setminus \{\top\}$, we can use the $+$-notation from §3.4.14. Note carefully that $z + k$ with $z \in Z$, $k \in \mathbb{N}$ need not be in Z, however, using a notation like $z \sim w + k$ tacitly implies $w + k \in Z$ since $\sim \subseteq Z \times Z$.

The subsets $Y_0 := \{z \in Z;\, z \prec \top\}$ and $Y_1 := \{z + 1;\, z \in Z',\, z \prec \top \leftrightarrow z + 1\}$ of Z' are disjoint by (U3). For any relation R, let R^T denote the transposed relation. By (U5), the sets $L_0 := \prec \restriction (Y_0 \times Y_0)$, ${L_0}^T$ and $Q_0 := \leftrightarrow \restriction (Y_0 \times Y_0)$ form a pairwise disjoint partition of $Y_0 \times Y_0$. Employing all properties of a chain, particularly (U5), it is easily seen that the sets $L_1 := \{(z+1, z'+1);\, z, z' \in Z',\, z \prec z' \prec \top \leftrightarrow z + 1 \leftrightarrow z' + 1\}$, ${L_1}^T$ and $Q_1 := \{(z+1, z'+1);\, z, z' \in Z',\, z \leftrightarrow z' \prec \top \leftrightarrow z + 1 \leftrightarrow z' + 1\}$ form a pairwise disjoint partition of $Y_1 \times Y_1$. For $Y := Y_0 \cup Y_1$ and $L_2 := Y_0 \times Y_1$, we conclude therefore that the sets L_0, ${L_0}^T$, Q_0, L_1, ${L_1}^T$, Q_1, L_2, ${L_2}^T$ comprise a pairwise disjoint partition of $Y \times Y$. In the following paragraphs, we will prove that $(Y, \ll, \sim)$ is a grid over $X \cup \{\bot\}$ for $\ll := L_0 \cup L_1 \cup L_2$ and $\sim := Q_0 \cup Q_1$. Since $\ll^T = {L_0}^T \cup {L_1}^T \cup {L_2}^T$, it follows that the sets $\ll$, $\sim$, $\ll^T$ comprise a pairwise disjoint partition of $Y \times Y$. Thus (T1) holds for Y. Since Q_0 is reflexive w. r. t. Y_0 and Q_1 w. r. t. Y_1, also $\sim$ is reflexive w. r. t. Y. Since Q_0 and Q_1 are symmetric, so is $\sim$. We see that $\sim$ is transitive since $a \sim b \sim c \in Y_0$ implies $a Q_0 b Q_0 c$ and $a Q_0 c$, and $a \sim b \sim c \in Y_1$ implies $a Q_1 b Q_1 c$ and $a Q_1 c$.

We define $\underset{\leftrightarrow}{\prec} := \prec \cup \leftrightarrow$ and $\underset{\sim}{\ll} := \ll \cup \sim$. Since $\underset{\sim}{\ll} = L_0 \cup L_1 \cup L_2 \cup Q_0 \cup Q_1 \subseteq (Y_0 \times Y_0) \cup (Y_0 \times Y_1) \cup (Y_1 \times Y_1)$, we remark for use in the next paragraph that $d \underset{\sim}{\ll} e \in Y_0$ implies $d \in Y_0$ and that $Y_1 \ni d \underset{\sim}{\ll} e$ implies $e \in Y_1$.

We will now prove that $a \underset{\sim}{\ll} b \ll c$ implies $a \ll c$. Since $Y_0 \times Y_1 \subseteq \ll$, we only need to distinguish the case $a \in Y_1$ and the case $c \in Y_0$. First, suppose $c \in Y_0$ so that $b \in Y_0$ and, in turn, $a \in Y_0$; thus $(a, b) \in \underset{\sim}{\ll} \restriction (Y_0 \times Y_0) = L_0 \cup Q_0$, i. e. $a \underset{\leftrightarrow}{\prec} b$, and $(b, c) \in \ll \restriction (Y_0 \times Y_0) = L_0$, i. e. $b \prec c$, and therefore $a \prec c$ is established by (U1) or by transitivity of $\prec$. Second, suppose $a \in Y_1$ so that $b \in Y_1$ and, in turn, $c \in Y_1$; thus

$(a,b) \in \lesssim\!\!\upharpoonright (Y_1 \times Y_1) = L_1 \cup Q_1$ and $(b,c) \in \ll\!\upharpoonright (Y_1 \times Y_1) = L_1$; hence there are $z_a, z_b, z_c \in Z'$ such that $a = z_a + 1$, $b = z_b + 1$, $c = z_c + 1$ and $z_a \overset{\preceq}{\longleftrightarrow} z_b \prec z_c \prec \top \leftrightarrow z_a + 1 \leftrightarrow z_b + 1 \leftrightarrow z_c + 1$, which yields $a \ll c$ by (U1) or by transitivity of $\prec$.

In a completely symmetrical way, we can prove that $a \ll b \lesssim c$ implies $a \ll c$. This yields that $(Y, \ll, \sim)$ satisfies (T2) and (T3) and that $\ll$ is transitive.

We will prove for later use that any $a \in Y$ such that $a+1 \in Y$ satisfies $a \in Y_0$ and $a \prec a+1$. Since $a+1 \in Y_1$ implies $a \prec \top \sim a+1$ and, in turn, $a \in Y_0$ and $a \prec a+1$, we may assume that $a+1 \notin Y_1$ so that $a+1 \in Y_0$, i. e. $a+1 \prec \top$. By (U7), we have either $a \prec a+1$ or $a \leftrightarrow a+1 \leftrightarrow \top$. From $a+1 \prec \top$ and (U5), we conclude $a \prec a+1 \prec \top$ and hence $a \in Y_0$.

To prove (T4), we suppose $a \in Y$ such that $a+1 \in Y$; we need to show $a \ll a+1$. By the above, we have $a \in Y_0$ and $a \prec a+1$. Since $Y_0 \times Y_1 \subseteq \ll$, we may assume $a+1 \notin Y_1$ so that $a+1 \in Y_0$. Since $L_0 \subseteq \ll$, we have $a \ll a+1$.

The next two paragraphs prepare to prove (T5) and (T6).

Let a, $b \in Y$ such that $a+1$, $b+1 \in Y$ and $a \ll b$; we will prove $a+1 \ll b+1$. As observed above, we have a, $b \in Y_0$, thus $a \prec b$ by $L_0 \subseteq \ll$. From (U9) and (U5), we conclude that either $a+1 \prec b+1$ or $a+1 \sim b+1 \sim \top$. If $b+1 \in Y_0$, then $a+1 \prec b+1 \prec \top$ so that also $a+1 \in Y_0$ and thus $(a+1, b+1) \in L_0 \subseteq \ll$. Therefore, we may assume $b+1 \in Y_1$. If $a+1 \in Y_1$, then $(a+1, b+1) \in L_1 \subseteq \ll$. Therefore, we may assume $a+1 \in Y_0$. Hence $(a+1, b+1) \in Y_0 \times Y_1 \subseteq \ll$ as required.

Let a, $b \in Y$ such that $a+1$, $b+1 \in Y$ and $a \sim b$; we will prove $a+1 \sim b+1$. As observed above, we have a, $b \in Y_0$, thus $a \leftrightarrow b$ by $Q_0 \subseteq \sim$ and therefore $a+1 \leftrightarrow b+1$ by (U8). If $a+1 \in Y_0$ or $b+1 \in Y_0$, then $\{a+1, b+1\} \subseteq Y_0$ and thus $(a+1, b+1) \in Q_0 \subseteq \sim$. Thus we may assume $\{a+1, b+1\} \subseteq Y_1$. Therefore $(a+1, b+1) \in Q_1 \subseteq \sim$ as required.

For any a, $b \in Y$ such that $a+1$, $b+1 \in Y$, the two preceding paragraphs have shown the implications $a \ll b \Rightarrow a+1 \ll b+1$ and $a \sim b \Rightarrow a+1 \sim b+1$ and $b \ll a \Rightarrow b+1 \ll a+1$. By (U5), we see that $a+1 \ll b+1$ implies that neither $a \sim b$ nor $b \ll a$ can hold and thus, again by (U5), we must have $a \ll b$. Similarly, $a+1 \sim b+1$ implies that neither $a \ll b$ nor $b \ll a$ can hold, and thus $a \sim b$ follows. This establishes (T5) and (T6).

Now, we have proved that $(Y, \ll, \sim)$ is indeed a grid over $X \cup \{\bot\}$. By §3.4.12 and §3.4.13, we can construct $f\colon\ Y \to [0, \infty) \cap \mathbb{Q}$ such that $f(\bot) = 0$, $\forall a, b \in Y\ \big(a \ll b \supset f(a) < f(b)\big)$, $\forall a, b \in Y\ \big(a \sim b \supset f(a) = f(b)\big)$, and $\forall a \in Y\ \big(a+1 \in Y \supset f(a+1) = f(a)+1\big)$. Since

$(Y, \ll, \sim)$ is a grid, Y_0 has a $\lesssim$-maximal element u, i. e. $u \in Y_0$ such that $a \lesssim u$ for all $a \in Y_0$, in particular, $f(a) \leq f(u)$. Likewise, there is $U \in Y_1$ such that $U \lesssim b$ for all $b \in Y_1$, in particular $f(U) \leq f(b)$. Since $(u, U) \in Y_0 \times Y_1 \subseteq \ll$, we have $f(u) < f(U)$. Put $r := \frac{2}{f(u)+f(U)}$ and $g(y) := \min\{1, r \cdot f(y)\}$ for all $y \in Y$. Clearly, $g(\bot) = 0$. We observe the two following facts: For all $a \in Y_0$ and $b \in Y_1$, we conclude from $0 \leq f(a) \leq f(u) < \frac{f(u)+f(U)}{2} = \frac{1}{r} < f(U) \leq f(b)$ that $0 \leq g(a) < 1 = g(b)$. For all $a, b \in Y_0$ such that $a \prec b$, we conclude from $(a, b) \in L_0 \subseteq \ll$ that $0 \leq f(a) < f(b) \leq f(u) < \frac{1}{r}$, therefore $0 \leq g(a) < g(b) < 1$.

We extend the domain of g from Y to Z by $g(z) := 1$ for all $z \in Z \setminus Y$. In particular, $g(\top) = 1$. Since $(Z \setminus Y) \cup Y_1 \subseteq (Z \setminus Y_0) \cup Y_1 \subseteq \{z \in Z;\ z \leftrightarrow \top\}$, we see for every $z \in Z$ that $g(z) < 1$ holds if and only if $z \prec \top$.

We need to prove $g(a) < g(b)$ for all $a,\ b \in Z$ with $a \prec b$. If $b \prec \top$, then $a \prec b \prec \top$, thus $a, b \in Y_0$ and, as observed earlier, $g(a) < g(b)$. Thus we may assume $b \leftrightarrow \top$ so that now $g(b) = 1$. We have $a \prec \top$ for otherwise $a \leftrightarrow \top \leftrightarrow b$, which contradicts $a \prec b$. As observed earlier, we have $g(a) < 1 = g(b)$, as required.

We need to prove $g(a) = g(b)$ for all $a,\ b \in Z$ with $a \leftrightarrow b$. If $a \prec \top$, we find $b \prec \top$ and thus $(a, b) \in Q_0 \subseteq \sim$ so that $f(a) = f(b)$ and $g(a) = g(b)$. Thus we may assume $a \leftrightarrow \top$. Now, we see $b \leftrightarrow \top$ and $g(a) = 1 = g(b)$ as required.

We need to prove that $g(a+1) = \min\{1, g(a)+r\}$ for all $a \in Z'$ such that $a+1 \in Z$. In the case of $a \prec a+1 \sim \top$, we find $a \in Y_0$ and $a+1 \in Y_1$ so that $g(a) < 1 = g(a+1)$ as observed earlier; since $\min\{1, r \cdot f(a)\} = g(a) < 1$ and $f(a+1) = f(a) + 1$, we see $r \cdot f(a) = g(a)$ and $g(a+1) = \min\{1, r \cdot (f(a)+1)\} = \min\{1, r \cdot f(a) + r\} = \min\{1, g(a) + r\}$, as required. In the case of $a \prec a+1 \prec \top$, we find $0 \leq g(a) < g(a+1) < 1$ as observed earlier; since $\min\{1, r \cdot f(a)\} = g(a) < 1$, $\min\{1, r \cdot f(a+1)\} = g(a+1) < 1$ and $f(a+1) = f(a) + 1$, we see $1 > g(a+1) = r \cdot f(a+1) = r \cdot f(a) + r = g(a) + r$, thus $g(a+1) = \min\{1, g(a) + r\}$, as required. The remaining case is $a \leftrightarrow \top \leftrightarrow a+1$, due to (U7). We now have $g(a) = 1 = g(a+1)$ and thus $g(a+1) = \min\{1, g(a) + r\}$, as required.

This completes the proof of all claimed properties. $\diamondsuit$

§3.4.16 Definition. Let $X \subseteq \mathsf{Var}$ be finite and $K \in \mathbb{N}$. Let $h_{uv} \in \{\prec, \leftrightarrow\}$ be given for all $u,\ v \in Z$; here $Z := \{\top\} \cup \{(x, k);\ x \in X \cup \{\bot\}, k \leq K\}$ as in §3.4.14. Let ι map Z to formulae in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$ by $(x, k) \mapsto \bigcirc^k x$, $\top \mapsto \top$, $\bot \mapsto \bot$. Define $R_{\prec} := \{(u, v);\ h_{uv} = \prec\}$ and $R_{\leftrightarrow} := \{(u, v);\ h_{uv} = \leftrightarrow\}$. We call an $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$-conjunction $\bigwedge_{u \in Z, v \in Z} \iota(u)\, h_{uv}\, \iota(v)$, regardless of parenthesisation and order, an (X,K)-*chain formula* if $(Z, R_{\prec}, R_{\leftrightarrow})$ is an (X,K)-chain. In this case, we define write $\iota(u) \prec_C \iota(v)$ whenever $h_{uv} = \prec$, and $\iota(u) \leftrightarrow_C \iota(v)$ whenever $h_{uv} = \leftrightarrow$. ⸙

§3.4.17 Corollary. Suppose C is an (X,K)-*chain formula*. Let $Z_* := \{\top\} \cup \{\bigcirc^k x;\, x \in X \cup \{\bot\},\, k \leq K\}$. Then there is a Gödel r-interpretation $\mathrm{I}\colon \mathsf{Var} \to [0,1]$ such that $\mathrm{I}(C) = 1$ and for all a, $b \in Z_*$ we have: $\mathrm{I}(a) < \mathrm{I}(b)$ whenever $a \prec_C b$; and $\mathrm{I}(a) = \mathrm{I}(b)$ whenever $a \leftrightarrow_C b$. ⸙

Proof. We use the notation of §3.4.16. By §3.4.15, we can construct $r \in [0,1]$ and $g\colon Z \to [0,1]$ such that (1) $g(\bot) = 0$, (2) $g(\top) = 1$, (3) $\forall u, v \in Z.\ (h_{uv} = \prec) \supset g(u) < g(v)$, (4) $\forall u, v \in Z.\ (h_{uv} = \leftrightarrow) \supset g(u) = g(v)$, (5) $\forall u \in Z'.\ u + 1 \in Z \supset g(u+1) = r \oplus g(u)$.

Let $\mathrm{I}(x) := g(x)$ for all $x \in X$ and $\mathrm{I}(x) := 0$ for all $x \in \mathsf{Var} \setminus X$, and extend I to all formulae in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$ such that I is a Gödel r-interpretation.

We claim $\mathrm{I}(\iota(u)) = g(u)$ for all $u \in Z$. For $u \in \{\bot, \top\}$, this follows from (1) and (2). It remains to check $\mathrm{I}(\iota(x,k)) = \mathrm{I}(\bigcirc^k x) = g(x,k)$ for all $x \in X$ and $k \leq K$. We see that $\mathrm{I}(\bigcirc^k x) = (k \cdot r) \oplus \mathrm{I}(x)$ holds by definition of I and by (A6). Using (5) for $k-1$ times, we find $g(x+k) = (k \cdot r) \oplus g(x) = (k \cdot r) \oplus \mathrm{I}(x)$. This establishes the claim.

We claim $\mathrm{I}(\iota(u)\, h_{uv}\, \iota(v)) = 1$ for all u, $v \in Z$. We have to distinguish two cases: If $(h_{uv} = \prec)$, then $\mathrm{I}(\iota(u) \prec \iota(v)) = \mathrm{I}(\iota(u)) \lhd \mathrm{I}(\iota(v)) = 1$ by (A12) and (3). If $(h_{uv} = \leftrightarrow)$, then $\mathrm{I}(\iota(u) \prec \iota(v)) = \mathrm{I}(\iota(u)) \bowtie \mathrm{I}(\iota(v)) = 1$ by (A17) and (4). This proves that $\mathrm{I}(C) = 1$.

The other properties are immediate consequences of (3) and (4). ◇

§3.4.18 Example. Since the relations $\prec$ and $\leftrightarrow$ of a chain must fulfil transitivity, (U1) and (U2), we need not specify h_{uv} in detail. As is done in the following example, it suffices to string the elements of a chain and insert $\prec$ and $\leftrightarrow$ between them. Still, the result needs to be checked to be a chain; but this is easy for the following $(\{d,e,f,g,h\},3)$-chain C given by $(\bot,0) \leftrightarrow_C (d,0) \prec_C (e,0) \prec_C (f,0) \prec_C (\bot,1) \leftrightarrow_C (d,1) \prec_C (e,1) \prec_C (g,0) \prec_C (f,1) \prec_C (\bot,2) \leftrightarrow_C (d,2) \prec_C (h,0) \prec_C (e,2) \prec_C (g,1) \prec_C (f,2) \prec_C (\bot,3) \leftrightarrow_C (d,3) \prec_C (h,1) \prec_C (e,3) \prec_C (g,2) \leftrightarrow_C (f,3) \leftrightarrow_C (h,2) \leftrightarrow_C (g,3) \leftrightarrow_C (h,3) \leftrightarrow_C \top$. Then §3.4.17 says that there is an r-Gödel interpretation I such that $\mathrm{I}(C) = 1$ and $0 = \mathrm{I}(\bot) = \mathrm{I}(d) < \mathrm{I}(e) < \mathrm{I}(f) < \mathrm{I}(\bigcirc\bot) = \mathrm{I}(\bigcirc d) < \mathrm{I}(\bigcirc e) < \mathrm{I}(g) < \mathrm{I}(\bigcirc f) < \mathrm{I}(\bigcirc\bigcirc\bot) = \mathrm{I}(\bigcirc\bigcirc d) < \mathrm{I}(h) < \mathrm{I}(\bigcirc\bigcirc e) < \mathrm{I}(\bigcirc g) < \mathrm{I}(\bigcirc\bigcirc f) < \mathrm{I}(\bigcirc\bigcirc\bigcirc\bot) = \mathrm{I}(\bigcirc\bigcirc\bigcirc d) < \mathrm{I}(\bigcirc h) < \mathrm{I}(\bigcirc\bigcirc\bigcirc e) < \mathrm{I}(\bigcirc\bigcirc g) = \mathrm{I}(\bigcirc\bigcirc\bigcirc f) = \mathrm{I}(\bigcirc\bigcirc h) = \mathrm{I}(\bigcirc\bigcirc\bigcirc g) = \mathrm{I}(\bigcirc\bigcirc\bigcirc h) = 1$. ⸙

Item (c) of the following theorem establishes the completeness of $\mathbf{G}_\bigcirc$ for validity in $\mathscr{L}^{\mathrm{p}}_\bigcirc$ w. r. t. Gödel $\bigcirc$-semantics.

§3.4.19 Theorem. Suppose $X \subseteq \mathsf{Var}$ is finite and $K \in \mathbb{N}$. Let $Z_* := \{\top\} \cup \{\bigcirc^k x;\ x \in X \cup \{\bot\},\ k \leq K\}$.

(a) Then we can construct a set $\mathscr{C}$ of (X,K)-chain formulae and a $\mathbf{G}_\bigcirc$-proof of $\bigvee_{C\in\mathscr{C}} C$.

(b) For any (X,K)-chain C and any formula F with $\mathsf{Var}(F) \subseteq X$ and $\mathsf{rdp}(F) \leq K$, we can construct a (not necessarily unique) $z \in Z_*$ and a $\mathbf{G}_\bigcirc$-proof of $C \supset (F \leftrightarrow z)$. We will say that C *evaluates* F to z.

(c) If F in $\mathscr{L}^{\mathrm{p}}_\bigcirc$ is valid, we can construct a $\mathbf{G}_\bigcirc$-proof of F; thus F is valid if and only if $\mathbf{G}_\bigcirc \vdash F$. ≀

Proof. Since the case of $K = 0$, i. e. without rings, is contained in [12], we will stipulate $K \neq 0$ to obviate trivialities. Still, a proof for $K = 0$ can be easily read off from the ideas presented here.

We would like to remind the reader of §3.4.6.

(a) We will tacitly treat the abbreviations $\prec$, $\leftrightarrow$ and $\top$ as if they were connectives in their own right when we will match a formula against a condition, e. g., a formula presented as $a \leftrightarrow b$ is not meant to undergo a transformation applied to all formulae with top symbol $\wedge$.

By (G3), we have $\mathbf{G}_\bigcirc \vdash (a \prec b) \vee (a \leftrightarrow b) \vee (b \prec a)$ for all $a,\ b \in Z_*$. The conjunction of these formulae is clearly $\mathbf{G}_\bigcirc$-provable. Applying (G12) and (S3) repeatedly to it, we obtain a $\mathbf{G}_\bigcirc$-proof of a disjunctive normal form $\bigvee_m C^0_m$. Now, each disjunct C^0_m has the property $(*)$: it is a conjunction that consists only of conjuncts $a \,\square\, b$ with $a,\ b \in Z_*$, $\square \in \{\leftrightarrow, \prec\}$ and that, moreover, contains for each pair $a,\ b \in Z_*$ at least one conjunct of the form $a \prec b$, $a \leftrightarrow b$, $b \leftrightarrow a$ or $b \prec a$.

In the next paragraph, we will specify an iterative procedure that turns $\bigvee_m C^0_m$ into the required disjunction of chains. We leave the easy task to the reader to verify that any cycle of the iteration outputs a disjunctive normal form E'' with property $(*)$ if its input has been a disjunctive normal form E' with property $(*)$. After the introduction of the procedure, we will indicate how to construct a $\mathbf{G}_\bigcirc$-proof of $E' \leftrightarrow E''$. In the rules of the procedure, we will not care about the order or parenthesisation of formulae in a disjunction or conjunction, since (G10), (G11) and (S3) enable us to provide a $\mathbf{G}_\bigcirc$-proof of $E' \leftrightarrow E'_1$ for any re-parenthesisation and re-ordering E'_1 of E'. We will tacitly assume that an appropriate re-parenthesisation and re-ordering of any such input E' is done before each rule to allow matching if possible at all.

Take $\bigvee_m C_m^0$ and repeatedly apply the first matching rule of the following list until none of the rules matches:

(1) Contract equal conjuncts, i. e. replace $e \wedge e$ by e.
(2) Contract disjuncts that are equal up to the order of their contained conjuncts, i. e. replace $C \vee C$ by C.
(3) Remove some disjunct that contains a conjunct $a \prec \bot$.
(4) Replace $\top \prec a$ by $\top \leftrightarrow a$.
(5) Replace a conjunct $a \prec a$ by $(a \leftrightarrow a) \wedge (a \leftrightarrow \top)$.
(6) Replace $(a \prec b) \wedge (b \,\square\, a)$, where $\square \in \{\leftrightarrow, \prec\}$,
by $(a \prec b) \wedge (b \leftrightarrow \top)$.
(7) Replace $(a \,\square\, b) \wedge (b \lozenge c) \wedge (c \prec a)$, where $\square, \lozenge \in \{\leftrightarrow, \prec\}$,
by $(a \leftrightarrow b) \wedge (b \leftrightarrow c) \wedge (c \leftrightarrow a) \wedge (a \leftrightarrow \top)$,
(8) Replace $(a \leftrightarrow b) \wedge (\bigcirc a \prec \bigcirc b)$
by $(a \leftrightarrow b) \wedge (\bigcirc a \leftrightarrow \bigcirc b) \wedge (\bigcirc b \leftrightarrow \top)$.
(9) Replace $(a \prec b) \wedge (\bigcirc b \prec \bigcirc a)$
by $(a \prec b) \wedge (\bigcirc a \leftrightarrow \bigcirc b) \wedge (\bigcirc b \leftrightarrow \top)$.
(10) Replace $(a \prec b) \wedge (\bigcirc a \leftrightarrow \bigcirc b) \wedge (\bigcirc b \prec \top)$
by $(a \prec b) \wedge (\bigcirc a \leftrightarrow \bigcirc b) \wedge (\bigcirc b \leftrightarrow \top)$.
(11) Replace $(\bot \leftrightarrow \bigcirc \bot) \wedge (a \prec \bigcirc a)$
by $(\bot \leftrightarrow \bigcirc \bot) \wedge (a \leftrightarrow \bigcirc a) \wedge (\bigcirc a \leftrightarrow \top)$.
(12) Replace $(\bigcirc a \prec a)$ by $(a \leftrightarrow \bigcirc a) \wedge (\bigcirc a \leftrightarrow \top)$.
(13) Replace $(\bot \prec \bigcirc \bot) \wedge (a \leftrightarrow \bigcirc a) \wedge (a \prec \top)$
by $(\bot \prec \bigcirc \bot) \wedge (a \leftrightarrow \bigcirc a) \wedge (a \leftrightarrow \top)$.
(14) If a disjunct does not contain the conjunct $a \leftrightarrow a$, $a \in Z$, add it.
(15) If a disjunct contains the conjunct $a \leftrightarrow b$ but not the conjunct $b \leftrightarrow a$, add $b \leftrightarrow a$.

An inspection of all rules yields that the number of disjuncts cannot increase. We obtain an upper bound on the number of conjuncts in a disjunct by property $(*)$ in connection with the facts that any rule can increase the number of conjuncts at most by 1 and that rule (1) immediately removes any conjunct occurring more than once. The number of Z_*-pairs that are joined by $\prec$ properly decreases in the rules (3)–(13) and does not increase in the other rules; thus the rules (3)–(13) can succeed only a bounded number of times. The system with the rules (3)–(13) removed is easily seen to be terminating. Hence the procedure is terminating.

Let $\bigvee_m C_m^1$ denote the result of the procedure. Given a $\mathbf{G}_\bigcirc$-proof of $\bigvee_m C_m^0$, we will iteratively construct a $\mathbf{G}_\bigcirc$-proof of $\bigvee_m C_m^1$: If rule (1) transforms the disjunctive normal form $E[e \wedge e]$ to $E[e]$, extend the

$\mathbf{G}_{\bigcirc}$-proof of $E[e \wedge e]$ by the (G9)-instance $(e \wedge e) \leftrightarrow e$ and the (S3)-instance $((e \wedge e) \leftrightarrow e) \supset (E[e \wedge e] \leftrightarrow E[e])$ to obtain a proof of $E[e]$. A proof for the application of rule (2) is similar. For the other rules, we just briefly indicate the derivations and we tacitly apply (S3): Use (G24) for rule (3), (G21) for (4), (G13) for (5), (G15) for (6), (G16) for (7). For rule (8), apply (MP) to the (S3)-instance $(a \leftrightarrow b) \supset (\bigcirc a \leftrightarrow \bigcirc b)$ and an appropriate instance of (G25). For rule (9), first use the (S1)-instance $(a \prec b) \supset \bigcirc(a \prec b)$ and the (S2)-instance $\bigcirc(a \prec b) \leftrightarrow (\bigcirc a \prec \bigcirc b)$ to find a proof of $(a \prec b) \supset (\bigcirc a \prec \bigcirc b)$; apply (MP) to the latter and an appropriate instance of (G26). Use (G27) instead of (G26) for rule (10); use (R2) and (G25) for (11); (G28) for (12); (R1) and (G27) for (13); use (G22) for (14); use (G23) for (15).

Observe that rule (3) never yields an empty disjunction for otherwise the provable and hence valid input disjunction consisted only of one conjunction containing the non-satisfiable conjunct $a \prec \bot$. Hence there is at least one disjunct in $\bigvee_m C^1_m$.

We will now prove that each disjunct C^1_m of $\bigvee_m C^1_m$ is an (X,K)-formula. We use the notation of §3.4.16, in particular, $Z = \{\top\} \cup \{(x,k);\, x \in X \cup \{\bot\},\, k \leq K\}$. For u, $v \in Z$, let $h_{uv} := \prec$ if $\iota(u) \prec \iota(v)$ is contained in C^1_m and $h_{uv} := \leftrightarrow$ if $\iota(u) \leftrightarrow \iota(v)$ is contained in C^1_m. It remains to show that $(Z, R_{\prec}, R_{\leftrightarrow})$ fulfills properties (U1)–(U9).

By construction, none of the above rules is applicable to $\bigvee_m C^1_m$. From property $(*)$ and the fact that rules (6) and (15) do not apply, we see that either $h_{uv} = \prec$ or $h_{uv} = \leftrightarrow$ or $h_{vu} = \prec$ holds for any u, $v \in Z$; this proves property (U5). In particular, $h_{uu} = \prec$ or $h_{uu} = \leftrightarrow$ holds for any $u \in Z$. Since rule (5) does not apply but $h_{uu} = \prec$ would trigger it, $R_{\leftrightarrow}$ is reflexive. Similarly, rule (15) causes the symmetry of $R_{\leftrightarrow}$. For the transitivity of $R_{\leftrightarrow}$, suppose we have $aR_{\leftrightarrow}bR_{\leftrightarrow}c$ and hence $cR_{\leftrightarrow}bR_{\leftrightarrow}a$; since both $aR_{\leftrightarrow}bR_{\leftrightarrow}cR_{\prec}a$ and $cR_{\leftrightarrow}bR_{\leftrightarrow}aR_{\prec}c$ would trigger rule (7), which is impossible, we neither have $cR_{\prec}a$ nor $aR_{\prec}c$, thus $aR_{\leftrightarrow}c$ follows, as required. In a similar way, transitivity of $R_{\prec}$ and properties (U1), (U2) follow from (7); (U3) from (4); (U4) from (3); (U6) from (11) and (12); (U7) from (12), (13) and (4); (U8) from (8); (U9) from (9), (10) and (4).

(b) The abbreviations $\prec$ and $\leftrightarrow$ are meant to have been unwound in F. In contrast, we will always treat any occurrence of $\bot \supset \bot$ as the symbol $\top$, which is contained in Z_*. We will construct a finite sequence $(F_n)_n$ such that the formula complexity will strictly decrease (considering $\top$ as a nullary connective), such that $\mathbf{G}_{\bigcirc} \vdash C \supset (F \leftrightarrow F_n)$ and $\mathsf{rdp}(F_n) \leq K$ for all n, and such that $F_N \in Z_*$ for some N.

Take $F_0 := F$ for the induction basis so that $\mathsf{rdp}(F_0) \leq K$ holds by assumption and, clearly, we have $\mathbf{G}_{\bigcirc} \vdash C \supset (F \leftrightarrow F_0)$. For the induction step, assume $\mathbf{G}_{\bigcirc} \vdash C \supset (F \leftrightarrow F_n)$ and $\mathsf{rdp}(F_n) \leq K$. If $F_n \in Z_*$, the iteration is done; We may assume $F_n \notin Z_*$ for otherwise the iteration is done; in particular, $F_n \neq \top = (\bot \supset \bot)$.

We conclude $F_n = E[a \mathbin{\square} b]$ for some context $E[\cdot]$, a, $b \in Z_*$ and $\square \in \{\wedge, \vee, \supset\}$. For the case of $\square = \supset$, we distinguish three sub-cases: In the case of $b \prec_C a$, we use $\mathsf{rdp}(a) \leq \mathsf{rdp}(F_n) \leq K$, $\mathsf{rdp}(b) \leq \mathsf{rdp}(F_n) \leq K$, (IPL2) and (IPL3) to obtain $\mathbf{G}_{\bigcirc} \vdash C \supset (b \prec a)$; this together with the (G8)-instance $\mathbf{G}_{\bigcirc} \vdash (b \prec a) \supset ((a \supset b) \leftrightarrow b)$, the (S3)-instance $\mathbf{G}_{\bigcirc} \vdash ((a \wedge b) \leftrightarrow b) \supset (F_n \leftrightarrow E[b])$ and the assumption $\mathbf{G}_{\bigcirc} \vdash C \supset (F \leftrightarrow F_n)$ yields $\mathbf{G}_{\bigcirc} \vdash C \supset (F \leftrightarrow E[b])$; hence we put $F_{n+1} := E[b]$ then. In the case of $a \prec_C b$, we use $\mathbf{G}_{\bigcirc} \vdash C \supset (a \prec b)$, the (G7)-instance $\mathbf{G}_{\bigcirc} \vdash (a \prec b) \supset ((a \supset b) \leftrightarrow \top)$, the (S3)-instance $\mathbf{G}_{\bigcirc} \vdash ((a \supset b) \leftrightarrow \top) \supset (F_n \leftrightarrow E[\top])$ and the assumption $\mathbf{G}_{\bigcirc} \vdash C \supset (F \leftrightarrow F_n)$ to obtain $\mathbf{G}_{\bigcirc} \vdash C \supset (F \leftrightarrow E[\top])$; hence we put $F_{n+1} := E[\top]$ then. In the case of $a \leftrightarrow_C b$, we use $\mathbf{G}_{\bigcirc} \vdash (a \leftrightarrow b) \supset ((a \supset b) \leftrightarrow \top)$ to conclude in a similar way that $\mathbf{G}_{\bigcirc} \vdash C \supset (F \leftrightarrow E[\top])$; hence we put $F_{n+1} := E[\top]$ also then. Therefore, we find in all sub-cases that $\mathbf{G}_{\bigcirc} \vdash C \supset (F \leftrightarrow F_{n+1})$ and $\mathsf{rdp}(F_{n+1}) \leq \mathsf{rdp}(F_n) \leq K$ hold and that F_{n+1} has a properly lower formula complexity than F_n, as required. The other cases of $\square = \wedge$ and $\square = \vee$ can be treated similarly by (G5), (G6) and (G7).

(c) Soundness has been proved in §3.4.5. For the converse direction, let now F be valid, put $X := \mathsf{Var}(F)$ and $K := \mathsf{rdp}(F)$ and stipulate that $\mathscr{C}$ has the properties as described in (a); we have to construct a $\mathbf{G}_{\bigcirc}$-proof of F.

First, we will construct a $\mathbf{G}_{\bigcirc}$-proof of $C \supset F$ for every $C \in \mathscr{C}$. By (b), C evaluates F to some $z \in Z_*$, i. e. $\mathbf{G}_{\bigcirc} \vdash C \supset (F \leftrightarrow z)$. In particular, $C \supset (F \leftrightarrow z)$ is valid by soundness. If we had $z \prec_C \top$, then §3.4.17 provided a Gödel r-interpretation $\mathrm{I}\colon X \to [0,1]$ with $\mathrm{I}(C) = 1$ and $\mathrm{I}(z) < \mathrm{I}(\top) = 1$ so that $\mathrm{I}(C \supset (z \leftrightarrow \top)) = \mathrm{I}(C) \trianglelefteq (\mathrm{I}(z) \bowtie 1) = 1 \trianglelefteq \mathrm{I}(z) = \mathrm{I}(z) < 1$, but this contradicts the validity of $C \supset (F \leftrightarrow z)$. By (U3), we conclude $z \leftrightarrow_C \top$ and therefore we can construct a $\mathbf{G}_{\bigcirc}$-proof of $C \supset (z \leftrightarrow \top)$ by (IPL2) and (IPL3). Since $\mathbf{G} \vdash (U \supset (V \leftrightarrow W)) \supset (U \supset W) \supset (U \supset V)$, we conclude from $\mathbf{G}_{\bigcirc} \vdash C \supset (F \leftrightarrow z)$ and $\mathbf{G}_{\bigcirc} \vdash C \supset (z \leftrightarrow \top)$ that $\mathbf{G}_{\bigcirc} \vdash C \supset F$.

Having constructed $\mathbf{G}_{\bigcirc}$-proofs of $C \supset F$ for every $C \in \mathscr{C}$, we can join them by multiple use of (IPL7) to obtain $\mathbf{G}_{\bigcirc} \vdash (\bigvee_{C \in \mathscr{C}} C) \supset F$. Since $\mathbf{G}_{\bigcirc} \vdash \bigvee_{C \in \mathscr{C}} C$ by (a), we find $\mathbf{G}_{\bigcirc} \vdash F$, as claimed. ♢

As a side note, we prove a variant of the lifting lemma, which is typical for logics with Gödel semantics.

§3.4.20 Lemma. Let I be a Gödel r-interpretation in $\mathscr{L}_{\bigcirc}^{\mathrm{p}}$, and let $\varphi \geq 1$. Define $r' := r \cdot \varphi$ and define a Gödel r'-interpretation by $\mathrm{I}'(A) := h(\mathrm{I}(A))$ for all variables A; here $h(x) := \min\{1, \varphi \cdot x\}$. Then $\mathrm{I}'(A) := h(\mathrm{I}(A))$ for all formulae A. ≀

Proof. Clearly, h induces a monotone function $[0,1] \to [0,1]$, i. e. we have $h(x) \leq h(y)$ whenever $0 \leq x \leq y \leq 1$. The claim will be proved by induction on formula complexity: For the induction beginning, we see $\mathrm{I}'(A) = h(\mathrm{I}(A))$ for all variables A by definition, and $h(\mathrm{I}(\bot)) = \min\{1, \varphi \cdot 0\} = 0 = \mathrm{I}'(\bot)$.

For the induction step, we first observe the following for all formulae A and B:

$r' \oplus h(\mathrm{I}(A)) = \min\{1,\ r' + \min\{1,\ \varphi \cdot \mathrm{I}(A)\}\} = \min\{1,\ \min\{r' + 1,\ r' + \varphi \cdot \mathrm{I}(A)\}\} = \min\{1,\ r' + \varphi \cdot \mathrm{I}(A)\}$; $h(\mathrm{I}(\bigcirc A)) = \min\{1,\ \varphi \cdot \min\{1,\ r + \mathrm{I}(A)\}\} = \min\{1, \min\{\varphi \cdot 1,\ \varphi \cdot (r + \mathrm{I}(A))\}\} = \min\{1,\ \varphi \cdot r + \varphi \cdot \mathrm{I}(A)\} = \min\{1,\ r' + \varphi \cdot \mathrm{I}(A)\} = r' \oplus h(\mathrm{I}(A))$; $h(\mathrm{I}(A \wedge B)) = \min\{1,\ \varphi \cdot \min\{\mathrm{I}(A), \mathrm{I}(B)\}\} = \min\{1,\ \min\{\varphi \cdot \mathrm{I}(A), \varphi \cdot \mathrm{I}(B)\}\} = \min\{\min\{1,\ \varphi \cdot \mathrm{I}(A)\},\ \min\{1,\ \varphi \cdot \mathrm{I}(B)\}\} = \min\{h(\mathrm{I}(A),\ h(\mathrm{I}(B)))\}$; $h(\mathrm{I}(A \vee B)) = \min\{1,\ \varphi \cdot \max\{\mathrm{I}(A), \mathrm{I}(B)\}\} = \max\{\min\{1,\ \varphi \cdot \mathrm{I}(A)\},\ \min\{1,\ \varphi \cdot \mathrm{I}(B)\}\} = \max\{h(\mathrm{I}(A),\ h(\mathrm{I}(B)))\}$.

By a case distinction, we will prove $h(\mathrm{I}(A \supset B)) = h(\mathrm{I}(A)) \trianglelefteq h(\mathrm{I}(B))$:

Let $h(\mathrm{I}(B)) < h(\mathrm{I}(A))$. We have $\mathrm{I}(B) < \mathrm{I}(A)$ for otherwise $\mathrm{I}(A) \leq \mathrm{I}(B)$ and then $h(\mathrm{I}(A)) \leq h(\mathrm{I}(B))$ by monotonicity of h. Thus $h(\mathrm{I}(A \supset B)) = h(\mathrm{I}(B)) = h(\mathrm{I}(A)) \trianglelefteq h(\mathrm{I}(B))$.—We will prove that $h(\mathrm{I}(A)) \leq h(\mathrm{I}(B))$ implies $h(\mathrm{I}(A \supset B)) = h(\mathrm{I}(A)) \trianglelefteq h(\mathrm{I}(B))$. Suppose this would not hold, then we have $h(\mathrm{I}(A \supset B)) < 1$ since $h(\mathrm{I}(A)) \leq h(\mathrm{I}(B))$ implies $h(\mathrm{I}(A)) \trianglelefteq h(\mathrm{I}(B)) = 1$. It follows $\mathrm{I}(B) < \mathrm{I}(A)$ for otherwise $\mathrm{I}(A) \leq \mathrm{I}(B)$ and then $h(\mathrm{I}(A \supset B)) = h(1) = 1$. We now have $1 > h(\mathrm{I}(A \supset B)) = h(\mathrm{I}(B))$. From the monotonicity of h, we find $h(\mathrm{I}(B)) \leq h(\mathrm{I}(A))$ so that $h(\mathrm{I}(B)) = h(\mathrm{I}(A)) = \min\{1,\ \varphi \cdot \mathrm{I}(B)\} = \min\{1,\ \varphi \cdot \mathrm{I}(A)\}$. Since $\varphi \cdot \mathrm{I}(B) < \varphi \cdot \mathrm{I}(A)$, we find $1 = h(\mathrm{I}(B))$, which is absurd.

In the remainder, let A, B be formulae such that $\mathrm{I}'(A) = h(\mathrm{I}(A))$ and $\mathrm{I}'(B) = h(\mathrm{I}(B))$ have already been established. By the above, we find $\mathrm{I}'(A \wedge B) = h(\mathrm{I}(A \wedge B))$, $\mathrm{I}'(A \vee B) = h(\mathrm{I}(A \vee B))$, $\mathrm{I}'(A \supset B) = h(\mathrm{I}(A \supset B))$, and $\mathrm{I}'(A) = h(\mathrm{I}(\bigcirc A))$. This proves the induction step. ♢

§3.4.21 Definition. A formula F in $\mathscr{L}_{\bigcirc}^{\mathrm{p}}$ is 1-*satisfiable* if there is $r \in [0,1]$ and a Gödel r-interpretation I such that $\mathrm{I}(F) = 1$. A formula F in $\mathscr{L}_{\bigcirc}^{\mathrm{p}}$ is *positively satisfiable* if there is $r \in [0,1]$ and a Gödel r-interpretation I such that $\mathrm{I}(F) > 0$. ≀

There is no need to distinguish these two notions as the following lemma states.

§3.4.22 Lemma. A formula F in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$ is 1-satisfiable if and only if it is positively satisfiable. ☡

Proof. If F is 1-satisfiable, it is trivially positively satisfiable. Conversely, let F be positively satisfiable such that there is $r \in [0,1]$ and a Gödel r-interpretation I with $\mathrm{I}(F) > 0$. Put $\varphi := 1/\mathrm{I}(F)$ and apply §3.4.20 to obtain a Gödel r'-interpretation I′ such that $r' = r \cdot \varphi$ and $\mathrm{I}'(A) = \min\{1, \varphi \cdot \mathrm{I}(A)\} = 1$. Thus A is 1-satisfiable. ♢

§3.4.23 Theorem. Suppose F is a formula in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$. Obtain F' from F by repeatedly replacing all formulae of the form $\bigcirc G$, G any formula, by $\top$. Obtain F'' from F by repeatedly replacing all formulae of the form $\bigcirc G$, G any formula, by G. Then F is 1-satisfiable if and only if there is a classical interpretation satisfying F' or F''. In particular, satisfiability in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$ is NP-complete. ☡

Proof. If F' has a satisfying classical interpretation I, then extend it to a Gödel r-interpretation I′ by putting $r := 1$; clearly, $\mathrm{I}'(F) = 1$.

If F'' has a satisfying classical interpretation I, then extend it to a Gödel r-interpretation I′ by putting $r := 0$; clearly, $\mathrm{I}'(F) = 1$.

If F has a satisfying Gödel r-interpretation I such that $r > 0$, take φ as the maximum of 1, of $\frac{1}{r}$ and of all $\frac{1}{\mathrm{I}(A)}$, where A is a variable in F such that $\mathrm{I}(A) > 0$. Taking h, r' and I′ as in §3.4.20, we see $h(0) = 0$, $h(x) = 1$ for all $x > 0$, $r' \geq 1$, $\mathrm{I}'(A) = 1$ for all variables A in F such that $\mathrm{I}(A) > 0$, $\mathrm{I}'(A) = 0$ for all variables A in F such that $\mathrm{I}(A) = 0$. Moreover, we find $\mathrm{I}'(\bigcirc G) = \min\{1, r' + \mathrm{I}'(G)\} = 1$ for all subformulae of F. It is then easy to see that I′ is a classical interpretation satisfying F''.

If F has a satisfying Gödel r-interpretation I such that $r = 0$, then F also satisfies F'' in $\mathscr{L}$. The lifting lemma for Gödel logic without $\bigcirc$, see e. g. Lemma 3.4 in [5], yields as classical interpretation for F''.

We now prove the statement about the complexity.

By §3.4.20, a classical formula A is satisfiable if and only if it is satisfiable w. r. t. Gödel semantics in $\mathscr{L}^{\mathrm{p}}$. As the evaluation of a $\bigcirc$-free formula does not depend on the presence of $\bigcirc$, we see that this is the case if and only if A is satisfiable w. r. t. Gödel semantics in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$. Thus classical satisfiability is a subproblem of satisfiability in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$.

By the above, we know that F is 1-satisfiable if and only if there is a classical interpretation satisfying F' or F'' (or both). As the Cook-Levin theorem says that classical satisfiability is NP-complete, we see that satisfiability in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$ is NP-complete: We can join the two non-deterministic machines that guess an assumed interpretation for F' or F''. $\diamondsuit$

§3.4.24 Remark. The first-order fragment inherits the problem of the $\forall$-quantifier from Gödel logics without $\bigcirc$ so that the lifting lemma does not hold in general for predicate logic. ☡

§3.5 The propositional fragment with $\triangle$

The Baaz-Takeuti-Titani $\triangle$-operator has a long history and is known by different names in different branches of research. A sound and complete proof system for Gödel logic with $\triangle$ is given by Baaz [2]; we will extend it for our purposes. For further connections of the $\triangle$ operator to witnessed Gödel logic, see [3].

§3.5.1 Definition. Let $\mathbf{G}_{\bigcirc,\triangle}$ denote the proof system of $\mathbf{G}_{\bigcirc}$ extended by the axiom schemata

($\triangle$1) $\triangle A \supset A$
($\triangle$2) $\triangle A \supset \triangle\triangle A$
($\triangle$3) $\triangle A \vee \neg\triangle A$
($\triangle$4) $\triangle(A \vee B) \supset (\triangle A \vee \triangle B)$
($\triangle$5) $\triangle(A \supset B) \supset \triangle A \supset \triangle B$

and the rule ($\triangle$N) $\frac{A}{\triangle A}$. ☡

We will prove in §3.5.4 that $\mathbf{G}_{\bigcirc,\triangle}$ characterises validity in $\mathscr{L}^{\mathrm{p}}_{\bigcirc,\triangle}$. Clearly, $\mathbf{G}_{\bigcirc,\triangle}$ is substitutive, cf. §3.4.6. By [2, Theorem 3.1], $\mathbf{G}_{\triangle}$ proves all validities in $\mathscr{L}^{\mathrm{p}}_{\triangle}$.

§3.5.2 Proposition. $\mathbf{G}_{\bigcirc,\triangle}$ proves

(D1) $\triangle A \supset \triangle\bigcirc A$,
(D2) $\triangle(A \supset B) \supset \triangle(\bigcirc A \supset \bigcirc B)$
(D3) $(\triangle A \wedge \triangle B) \leftrightarrow \triangle(A \wedge B)$
(D4) $(\triangle A \vee \triangle B) \leftrightarrow \triangle(A \vee B)$
(D5) $\triangle(A \supset B) \supset \triangle(\triangle A \supset \triangle B)$
(D6) $\triangle(A \leftrightarrow B) \supset \triangle(\triangle A \leftrightarrow \triangle B)$
(D7) $\triangle(A \leftrightarrow B) \supset \triangle(E[A] \leftrightarrow E[B])$

(D8) $\triangle(A \leftrightarrow B) \supset (E[A] \leftrightarrow E[B])$

here $E[\cdot]$ denotes a $\mathscr{L}^{\mathrm{p}}_{\bigcirc,\triangle}$-context. ☡

Proof. (D1) follows from $\mathbf{G}_{\bigcirc,\triangle} \vdash A \supset \bigcirc A$, ($\triangle$N) and ($\triangle$5).

Apply ($\triangle$N) and ($\triangle$5) to (R3) to obtain (D2).

(D3): Apply ($\triangle$N) and ($\triangle$5) to $\mathbf{G}_{\bigcirc,\triangle} \vdash (A \wedge B) \supset A$ to obtain $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle(A \wedge B) \supset \triangle A$. Similarly, $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle(A \wedge B) \supset \triangle B$ holds and thus $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle(A \wedge B) \supset (\triangle A \wedge \triangle B)$. To show the converse direction, apply ($\triangle$N) and ($\triangle$5) to $\mathbf{G}_{\bigcirc,\triangle} \vdash A \supset B \supset (A \wedge B)$ so that $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle A \supset \triangle(B \supset (A \wedge B)))$. Since $\triangle(B \supset (A \wedge B)) \supset \triangle B \supset \triangle(A \wedge B)$ by ($\triangle$5), we have $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle A \supset \triangle B \supset \triangle(A \wedge B)$. Thus (D3) follows.

(D4): One direction is ($\triangle$4). Apply ($\triangle$N) and ($\triangle$5) to $A \supset (A \vee B)$ to obtain $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle A \supset \triangle(A \vee B)$. Similarly, $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle B \supset \triangle(A \vee B)$. Now, (D4) follows by (IPL3).

(D5): By ($\triangle$5), $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle(A \supset B) \supset \triangle A \supset \triangle B$ holds. Applying ($\triangle$N) and ($\triangle$5), we see $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle\triangle(A \supset B) \supset \triangle(\triangle A \supset \triangle B)$. Since $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle(A \supset B) \supset \triangle\triangle(A \supset B)$ by ($\triangle$2), we obtain (D5).

(D6) follows from instances of (D5) and (D3).

(D7) is proved by induction on the complexity of the context. Clearly, $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle(A \leftrightarrow B) \supset \triangle(A \leftrightarrow B)$ and $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle(A \leftrightarrow B) \supset \triangle(C \leftrightarrow C)$. Suppose we already have $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle(A \leftrightarrow B) \supset \triangle(E[A] \leftrightarrow E[B])$, then $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle(A \leftrightarrow B) \supset \triangle(\triangle E[A] \leftrightarrow \triangle E[B])$ by (D6) and also $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle(A \leftrightarrow B) \supset \triangle(\bigcirc E[A] \leftrightarrow \bigcirc E[B])$ from instances of (D2) and (D3). Suppose we already have $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle(A \leftrightarrow B) \supset \triangle(E[A] \leftrightarrow E[B])$ and $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle(A \leftrightarrow B) \supset \triangle(F[A] \leftrightarrow F[B])$; by applying ($\triangle$N) and ($\triangle$5) several times to $\mathbf{G} \vdash (E[A] \leftrightarrow E[B]) \supset (F[A] \leftrightarrow F[B]) \supset ((E[A] \,\square\, F[A]) \leftrightarrow (E[B] \,\square\, F[B]))$ for any $\square \in \{\wedge, \vee, \supset\}$ and by using the assumptions, we find $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle(A \leftrightarrow B) \supset \triangle((E[A] \,\square\, F[A]) \leftrightarrow (E[B] \,\square\, F[B]))$.

(D8) follows from (D7) and ($\triangle$1). ◊

§3.5.3 Definition. An (X,K)-*chain formula* in $\mathscr{L}^{\mathrm{p}}_{\bigcirc,\triangle}$ has the form

$$\triangle S \wedge \neg \triangle N$$

where S is an (X,K)-chain formula in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$ and N is a conjunction $\bigwedge_i a_i$ with $a_i \in Y := \{\bigcirc^k x;\ x \in X \cup \{\bot\},\ k \le K\}$ such that either S contains $a \leftrightarrow \top$ or N contains a. ☡

§3.5.4 Theorem. Let F be a formula in $\mathscr{L}^{\mathrm{p}}_{\bigcirc,\triangle}$. Then F is valid if and only if $\mathbf{G}_{\bigcirc,\triangle} \vdash F$. ☡

Proof. Verifying soundness is a matter of routine. We will only sketch the proof of completeness and focus on the underlying ideas because it is very similar to the one of §3.4.19. We will use the notation of §3.5.3; put $X := \mathsf{Var}(F)$, $K := \mathsf{rdp}(F)$ and $Z := Y \cup \{\top\}$ in addition.

The first step is to construct a set $\mathscr{C}$ of (X,K)-chain formulae in $\mathscr{L}^{\mathrm{p}}_{\bigcirc,\triangle}$ such that $\mathbf{G}_{\bigcirc,\triangle}$ proves $\bigvee_{C\in\mathscr{C}} C$. It is routine to show for all a, $b \in Z$ that $\mathbf{G}_{\bigcirc,\triangle}$ proves $\triangle(a \prec b) \vee \triangle(a \leftrightarrow b) \vee \triangle(b \prec a)$ and $\triangle(a \leftrightarrow \top) \vee \neg\triangle a$, hence also their conjunction. Use (G12) and (D3) to bring this conjunction into disjunctive normal form D. It is easy to see that each conjunction in D contains $\triangle(a \leftrightarrow \top)$ or $\neg\triangle a$ or both, for each $a \in Z$. If the following points are incorporated, the algorithm presented in the proof of part (a) of §3.4.19 achieves the transformation of D into the required form. Due to ($\triangle$N) and (D8), we can replace subformulae by provably equivalent ones, in particular, conjunctions/disjunctions can be re-parenthesised/re-ordered and $\triangle$ can be moved across them by (D3) and (D4). Since $\mathbf{G}_{\bigcirc,\triangle} \vdash (\triangle(a \leftrightarrow \top) \wedge \neg\triangle a) \leftrightarrow \bot$, we are also able to eliminate all conjunctions in D that simultaneously contain $\triangle(a \leftrightarrow \top)$ and $\neg\triangle a$, as required by §3.5.3.

Also part (b) of §3.4.19 straightforwardly generalises to our situation. If $C = \triangle S \wedge \neg\triangle N$ is a chain with S and N as in §3.5.3, we can construct some $z \in Z$ such that $\mathbf{G}_{\bigcirc,\triangle} \vdash C \supset (F \leftrightarrow z)$ by an iteration. For the start, we use of $\mathbf{G}_{\bigcirc,\triangle} \vdash C \supset (F \leftrightarrow F)$. For the iteration step, we suppose $\mathbf{G}_{\bigcirc,\triangle} \vdash C \supset (F \leftrightarrow F')$ and we will replace a "small" subformula U in $F' = E[U]$ by an even simpler one in every run; here small means that $U \neq (\bot \supset \bot)$ and that the head of U is the only connective different from $\bigcirc$ and $\bot$. We distinguish two cases: (1) If $U = a \square b$ for some $\square \in \{\wedge,\vee,\supset\}$ and a, $b \in Z$, then S contains either $a \prec b$ or $a \leftrightarrow b$ or $b \prec a$. We will discuss only $b \prec a$ because $a \leftrightarrow b$ and $a \prec b$ can be treated similarly. It follows $\mathbf{G}_{\bigcirc,\triangle} \vdash C \supset \triangle(b \prec a)$. If $\square = \wedge$, we apply ($\triangle$N) and ($\triangle$5) to the $\mathbf{G}$-provable formula $(V \prec W) \supset ((W \wedge V) \leftrightarrow V)$ to obtain $\mathbf{G}_{\bigcirc,\triangle} \vdash \triangle(b \prec a) \supset \triangle((a \wedge b) \leftrightarrow b)$; it follows from (D8) that $\mathbf{G}_{\bigcirc,\triangle} \vdash C \supset (F' \leftrightarrow E[b])$ and hence $\mathbf{G}_{\bigcirc,\triangle} \vdash C \supset (F \leftrightarrow E[b])$. If $\square = \vee$, we similarly find $\mathbf{G}_{\bigcirc,\triangle} \vdash C \supset (F \leftrightarrow E[a])$. If $\square = \supset$, we find $\mathbf{G}_{\bigcirc,\triangle} \vdash C \supset (F \leftrightarrow E[b])$. Observe that $E[a]$ and $E[b]$ have lower complexity than F'. (2) In the remaining case, U has the form $\triangle a$ for some $a \in Z$ so that $F' = E[\triangle a]$. If N contains a, we find $\mathbf{G}_{\bigcirc,\triangle} \vdash C \supset \neg\triangle a$ and thus $\mathbf{G}_{\bigcirc,\triangle} \vdash C \supset (F' \leftrightarrow E[\bot])$. If N does not, S contains $\triangle(a \leftrightarrow \top)$, thus $\mathbf{G}_{\bigcirc,\triangle} \vdash C \supset \triangle a$ and hence $\mathbf{G}_{\bigcirc,\triangle} \vdash C \supset (F' \leftrightarrow E[\top])$. Treating $\top$ as a nullary connective as we have done in the proof of part (b) of §3.4.19,

$E[\top]$ and $E[\bot]$ also have lower complexity than F'.

Also part (c) of §3.4.19 is easily generalised so that it suffices to prove $\mathbf{G}_{\bigcirc,\triangle} \vdash C \supset (F \leftrightarrow \top)$ for every chain C. With the notation and assumptions from (b), we have $\mathbf{G}_{\bigcirc,\triangle} \vdash (\triangle S \wedge \neg \triangle N) \supset (F \leftrightarrow z)$, $z \in Z$. Since S is a chain in $\mathscr{L}_{\bigcirc}$, we may use the notations $z \prec_S \top$ and $z \leftrightarrow_S \top$. First, we will prove $z \leftrightarrow_S \top$. Suppose not, then we have $z \prec_S \top$ so that §3.4.17 provides a Gödel r-interpretation $\mathrm{I}\colon X \to [0,1]$ such that $\mathrm{I}(S) = 1$ and such that $\mathrm{I}(a) < \mathrm{I}(b)$ whenever $a \prec_S b$, in particular, $\mathrm{I}(z) < \mathrm{I}(\top) = 1$. For all a in N, we have $a \not\leftrightarrow_S \top$, thus $a \prec_S \top$ so that $\mathrm{I}(a) < \mathrm{I}(1)$ and therefore $\mathrm{I}(\triangle a) = 0$. Now that $\mathrm{I}(\neg \triangle N) = 1$, we find $\mathrm{I}((\triangle S \wedge \neg \triangle N) \supset (F \leftrightarrow z)) = (\max\{1,1\}) \trianglelefteq (1 \bowtie \mathrm{I}(z)) = \mathrm{I}(z) < 1$, contradicting $\mathbf{G}_{\bigcirc,\triangle} \vdash (\triangle S \wedge \neg \triangle N) \supset (F \leftrightarrow z)$. Thus we see $z \leftrightarrow_S \top$, as claimed. It follows $\mathbf{G}_{\bigcirc} \vdash \triangle S \supset (z \leftrightarrow \top)$ so that $\mathbf{G}_{\bigcirc,\triangle} \vdash (\triangle S \wedge \neg \triangle N) \supset (F \leftrightarrow \top)$. Therefore we have $\mathbf{G}_{\bigcirc,\triangle} \vdash (\triangle S \wedge \neg \triangle N) \supset F$, as required. ◇

§3.5.5 Remark. It is somewhat astonishing that the axioms we added for $\bigcirc$ and for $\triangle$ do not interfere with each other. One of the reasons for this is that the countermodels in the construction for the fragment with $\bigcirc$ alone can be used as countermodels for the fragment with $\bigcirc$ and $\triangle$. ≀

§3.5.6 Remark. We conclude with a remark on conservativity: In §3.2.7, we remarked that validity (or an interpretation) of a formula F in $\mathscr{L}^{\mathrm{p}}_{\bigcirc,\triangle}$ does not depend on the language in the following sense: If F is actually contained in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$, or $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$ or $\mathscr{L}^{\mathrm{p}}$, validity (or the interpretation) does not change. The next statement becomes obvious by completeness:

If F is a formula in $\mathscr{L}^{\mathrm{p}}_{\bigcirc}$ [or $\mathscr{L}^{\mathrm{p}}_{\triangle}$, or $\mathscr{L}^{\mathrm{p}}$] such that $\mathbf{G}_{\bigcirc,\triangle} \vdash F$, then $\mathbf{G}_{\bigcirc} \vdash F$ [or $\mathbf{G}_{\triangle} \vdash F$, or $\mathbf{G} \vdash F$]. ≀

Chapter 4

Conclusion

§4.1 Summary

The main results are: §3.2.9 (non-compactness of entailment), §3.4.19 (completeness and soundness of $\mathbf{G}_{\bigcirc}$), §3.5.4 (completeness and soundness of $\mathbf{G}_{\bigcirc,\triangle}$), §3.4.23 (satisfiability in the propositional fragment).

This thesis is the basis for further research, as described below.

§4.2 Future work

A trend in the current development in mathematical fuzzy logic is to investigate the features of fuzzy logics that result from adjoining modal operators, see e. g. [10] or [11]. This thesis provides an example of the intractability phenomena that can occur even when the semantics is confined to the unit interval and Kripke semantics are excluded. However, the semantics of the modality operator we used is not very typical for modal logics, e. g., the 'dual' operator $\neg\bigcirc\neg$ has the property

$$\mathrm{I}(\neg\bigcirc\neg A) = \begin{cases} 1 & \text{if } r = 0 < \mathrm{I}(A) \\ 0 & \text{otherwise.} \end{cases}$$

if I is a Gödel r-interpretation.

In view of the initial question for the borderline between Gödel and Łukasiewicz logics, it was natural to investigate whether the following variant would show the same intractability phenomenon for validity. The paper [4], coauthored by the writer of this thesis, generalises the adding operator to a larger class of monotone operators, i. e., we define $\mathrm{I}(\bigcirc A) := f(\mathrm{I}(A))$, where $f\colon [0,1] \to [0,1]$ is a function such that $\forall x \in [0,1].\ x \leq f(x)$ and $\forall x, y \in [0,1].\ x < y \Rightarrow (f(x) < f(y) \vee f(y) = 1)$. The addition of a constant is obviously such a function. The validity of a formula A is analogously defined by the property that A evaluates to 1 under all such functions. In the propositional fragment, validity can be easily described by a simple axiom system and it shares with Gödel logic the deduction theorem, the lifting lemma, and the agreement of entailment and 1-entailment. In fact, the situation there is much simpler than the one presented in this thesis because no complicated construction of a countermodel as in §3.4.17 is needed; a mere application of Dummett's original completeness theorem suffices together with an a-posteriori mending of the values assigned to ringed formulae. In the first-order fragment, however, the behaviour of quantifiers at limit points causes difficulties so that only a partial answer to axiomatisability was given in [4] for a very restricted part of the prenex fragment. This generalisation is unsatisfactory also for another reason, which will be explained in the following.

It is well-known and easy to show that Gödel interpretations in the propositional fragment commute with order-preserving functions on the truth value set, i. e. $[0, 1]$, that preserve 0. In general, this is referred to as the property of Gödel logics that only the order of the truth values assigned to the propositional variables determines the interpretation of a formula. The lifting lemma is an extension of this principle: Given $d \in [0, 1]$, we project all truth values above a given value d to 1 by the function $h_d\colon [0, 1] \to [0, 1]$, $h(x) := x$ for $x \leq d$, $h(x) := 1$ for $x > d$. Every Gödel interpretation I commutes with h_d so that $\mathrm{I}'\colon [0, 1] \to [0, 1]$, $A \mapsto h_d(\mathrm{I}(A))$ is an interpretation with the property that $\mathrm{I}'(F) = 1$ holds for any formula such that $\mathrm{I}(F) = 1$. However, there can be formulae F with $\mathrm{I}'(F) = 1 > \mathrm{I}(F)$ so that validity is preserved only in one way. The lifting lemma is also one of the key properties of Gödel logic because it says that the truth value 1 cannot be crisply distinguished from values close to it, in other words, the Δ-operator is not expressible. As the exact ordering preservation by a condition such as $\forall x, y \in [0,1].\ x < y \Rightarrow f(x) < f(y)$ infringes the lifting property, a natural choice seems to be the condition $\forall x, y \in [0,1].\ x < y \Rightarrow (f(x) < f(y) \vee f(y) = 1)$ chosen above. The condition $\forall x \in [0,1].\ x \leq f(x)$ therefore is superfluous if one is willing to give the deduction theorem and the equality scheme up, which do not hold in many modal logics anyway.

In a follow-up paper, the author of this thesis will consider the conditions $\forall x, y \in [0,1].\ x < y \Rightarrow (f(x) < f(y) \vee f(y) = 1)$ and $f(1) = 1$ for the functions that provide an interpretation for the $\bigcirc$-operator. It is easy to see that neither $A \supset \bigcirc A$ nor $\bigcirc A \supset A$ nor $(A \leftrightarrow B) \supset (\bigcirc A \leftrightarrow \bigcirc B)$ can be valid for this semantics. The deduction theorem neither holds: The rule $\frac{A}{\bigcirc A}$ expresses the condition $f(1) = 1$. The price one has to pay for this further generalisation is that it seems necessary to admit schemata of axiom schemata, where the extra parameter runs through all natural numbers n in order to express all ring powers $\bigcirc^n A$. However, in a proof of a valid formula F only those axiom schemata are involved that have at most the same ring depth; this constitutes a kind of compactness or uniformity w. r. t. the $\bigcirc$-operator. As in the case where the condition $\forall x \in [0,1].\ x \leq f(x)$ is stipulated, the behaviour of quantifiers in the first-order fragment remains as an open question.

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