Die approbierte Originalversion dieser Diplom-/Masterarbeit ist an der Hauptbibliothek der Technischen Universität Wien aufgestellt (http://www.ub.tuwien.ac.at).

The approved original version of this diploma or master thesis is available at the main library of the Vienna University of Technology (http://www.ub.tuwien.ac.at/englweb/).

DIPLOMARBEIT

Varianten der klassischen Galoisverbindung Operationen-Relationen

Ausgeführt am Institut für
Diskrete Mathematik und Geometrie
der Technischen Universität Wien

unter der Anleitung von
Ao.Univ.Prof. Dipl.-Ing. Dr.techn. Martin Goldstern

durch

Andreas Banlaki
Scheibelreitergasse 14
A-1190 Wien

Contents

Introduction

The basic notion underlying this work is concept of an operation preserving a relation that is an n-ary operation f is said to preserve a relation ρ iff $f(\mathbf{r}_1, \ldots \mathbf{r}_n) \in \rho$ for all $\mathbf{r}_1, \ldots \mathbf{r}_n \in \rho$. This concept will be used to map sets of operations to sets of relations and vice versa. Thereby one obtains a Galois connection between sets of operations and relations. We will be interested in giving an inner characterization of the Galois closed sets of operations and relations in different settings (e.g. we may consider only unary functions and binary relations etc.). In particular we are interested if and in what way the characterization depends on the cardinality of the base set. In general we will see that the characterization becomes more complicated with increasing cardinality and that we need to distinguish between finite, countable and uncountable base set.

Historically one of the first to study the Galois connection arising from functions preserving some relations was Krasner (for the exact citation we refer to [299-304] in [PK79]). Motivated by the idea to generalize the notion of a field by generalizing ordinary Galois theory [1] he only considered unary operations. The Galois closed sets of relations in this case are still named after him.
Another motivation for the study of the Galois connections between operations and relations comes from the study of (local) clones (i.e., sets of operations closed under functional composition and containing all projections), in particular maximal clones[2], and the (local) clone lattice. The basic idea behind this approach is that "large" clones can be described by "small" sets of relations which they preserve and which are easier to characterize. For finite base set for example all precomplete clones have been described with the help of the Galois connection $\mathrm{Pol} - \mathrm{Inv}$ by Rosenberg [Ros70]. For infinite base set Rosenberg and Szabo gave an example of a set of relations whose polymorphisms form a cofinal set in the local clone lattice [**?**].
The Galois correspondences naturally are also closely related to the concrete (and to a certain extent also abstract) characterization of related structures (i.e. the characterization of the subalgebra, congruence lattices, automorphism groups etc. of a given algebra or the characterization up to isomorphism in the abstract case) and also find applications there.
Some Galois correspondences have been used in the description of the reducts of ($\aleph_0$-categorical) structures on countable sets and in the treatment of constraint satisfaction problems on finite sets as the Galois closure corresponds to closure w.r.t. to certain logical operations for such cases.
The results on various Galois connections are numerous (see for example [Pös03]) so that this work cannot claim to be a complete treatment.

[1] Here one considers automorphisms of a field which leave a subfield pointwise invariant.
[2] A. V. Kuznecov was one of the first to use this approach (see [325] in [PK79]).

This work is build up in the following way:

- Chapter 1 introduces the basic concepts and gives an overview of the Galois connections that have been treated.
- Chapter 2 introduces the operations necessary for the characterization of the Galois closure.
- Chapter 3 finally gives the characterization of the different Galois connections.

Notation

We use the following notations.
We will use a fixed base set A through this work. If we need to use a different base set we will note so explicitly.

$\mathscr{O}^{(n)}(A)$ will denote the set of all n-ary operations on A. The set of all operations is defined as

$$\mathscr{O}(A) := \bigcup_{n\in\mathbb{N}, n\geq 1} \mathscr{O}^{(n)}(A).$$

Especially note that we do not include 0-ary operations. Also we may write $\mathscr{O}^{(n)}$, $\mathscr{O}$ if explicit reference to the base set is not needed.
For an arbitrary operation $f \in \mathscr{O}$ if its arity has not been given a name yet we will denote it by n_f.
For the set of all unary operations $\mathscr{O}^{(1)}$ we will also write Tr (which stand for the set of all transformations). The set of all unary injective, surjective and bijective operations will be denoted by $\mathscr{O}^{(1-1)}$, $\mathscr{O}^{(surj.)}$ and $\mathscr{S}$ respectively.

For an operation $f \in \mathscr{O}^{(n)}$ we define the *graph* of f to be the relation

$$f^\bullet := \{(a_1, \ldots, a_n, f(a_1 \ldots, a_n)) | a_1, \ldots, a_n \in A\}.$$

For $f \in \mathscr{O}^{(n)}$, $\rho \subseteq A^m$ we define

$$\begin{aligned} f[\rho] &:= \{f(\mathbf{r}_1, \ldots, \mathbf{r}_n) | \mathbf{r}_i \in \rho, i = 1, \ldots, n\} \\ &:= \{(f(r_{11}, \ldots, r_{n1}), \ldots, f(r_{m1}, \ldots, r_{mn})) | \mathbf{r}_i \in \rho, i = 1, \ldots, n\}. \end{aligned}$$

The set of all n-ary relations on A will be denoted by $\mathscr{R}^{(n)}(A)$ and the set of all relations is

$$\mathscr{R}(A) := \bigcup_{n\in\mathbb{N}} \mathscr{R}^{(n)}(A).$$

Again we may just write $\mathscr{R}^{(n)}$, $\mathscr{R}$ if we do not need to mention the base set explicitly. For the set of all equivalence relations on a set X we will write $\mathrm{Eq}(X)$.
For every $\epsilon \in \mathrm{Eq}(\{1, \ldots, m\})$, $m \in \mathbb{N}$ a relation of the form $\delta_m^\epsilon := \{(x_1, \ldots, x_m) \in A^m | (i,j) \in \epsilon \Rightarrow x_i = x_j\}$ is called *diagonal*. In particular for $\epsilon = \{(i,j) | i,j \in n\}$ we define $\delta_A^{(n)} := \delta_n^\epsilon = \{(a, \ldots, a) | a \in A\} \in \mathscr{R}^{(n)}$ on the base set A. The set of all diagonal relations on A is denoted by D_A.

For a set of relations $\mathsf{Q} \subseteq \mathscr{R}$ and a set O of operations on relations we will write

$$O(\mathsf{Q}) \qquad \text{or} \qquad \langle \mathsf{Q} \rangle_O$$

for the closure of Q w.r.t. to O, i.e., the smallest set of relations $\tilde{\mathsf{Q}}$ containing Q that is closed under the operations of O. We will write $O(.)$ for the closure operator obtained in this way. Similarly for a closure operator $C : \mathscr{R} \mapsto \mathscr{R}$ we write

$$C(\mathsf{Q}) \qquad \text{or} \qquad \langle \mathsf{Q} \rangle_C$$

for the closure of Q w.r.t. to C.

Chapter 1

The Galois connections

We will be interested in Galois connections between certain subsets $\mathbf{E}$ of $\mathscr{O}$ and $\mathbf{R}$ of $\mathscr{R}$. That is we will define certain maps from $\mathscr{P}(\mathbf{E})$ to $\mathscr{P}(\mathbf{R})$ and from $\mathscr{P}(\mathbf{E})$ to $\mathscr{P}(\mathbf{R})$ so that they form Galois connections between these to sets. Consequently going from $\mathscr{P}(\mathbf{E})$ ($\mathscr{P}(\mathbf{R})$) to $\mathscr{P}(\mathbf{R})$ ($\mathscr{P}(\mathbf{E})$) and back again using these maps creates a closure operator on $\mathscr{P}(\mathbf{E})$ ($\mathscr{P}(\mathbf{R})$). We will give an "inner" characterization (one not involving the Galois connection explicitly) of the Galois closed sets. We will especially be interested in the changes in the characterization that occur when the cardinality of the base set changes. We will consider the case of finite, countable and uncountable base set.

The map between sets of relations and sets of functions arises from the notion of a function preserving a relation, that is defined a follows.

Definition 1.1. Let $f \in \mathscr{O}^{(n)}$ be an n-ary operation and $\rho \in \mathscr{R}^{(m)}$ an m-ary relation. Then f is said to *preserve* ρ iff for any n-tuple $\mathbf{r}_i = (r_{i1}, \dots, r_{im}) \in \rho$ $(i = 1, \dots, n)$ of elements of ρ it holds that

$$f(\mathbf{r}_1, \dots, \mathbf{r}_n) := (f(r_{11}, \dots, r_{n1}), \dots, f(r_{m1}, \dots, r_{mn})) \in \rho,$$

i.e., the image of the tuple is again in ρ. In this case we also say f is a *polymorphism* for ρ and ρ is *invariant* for f.
We define f to *strongly preserve* ρ iff f preserves ρ and and its complement ρ^C and say that ρ is *strongly invariant* for f in this case.

Remark 1.2. (i) Note that a unary bijective function f strongly preserves a relation ρ iff both f and f^{-1} preserve ρ.

(ii) In the special case when $\rho = g^\bullet$ for some function $g \in \mathscr{O}^{(m-1)}$, f preserves ρ iff f and g commute, i.e.,

$$\begin{aligned} &f(g(a_{11}, \dots, a_{(m-1)1}), \dots, g(a_{1n}, \dots, a_{(m-1)n})) = \\ &\quad g(f(a_{11}, \dots, a_{1n}), \dots, f(a_{m-11}, \dots, a_{(m-1)n})) \end{aligned}$$

holds for all $a_{11}, \dots, a_{1n}, \dots, a_{(m-1)n} \in A$.

We give some simple examples to illustrates these definitions.

Example 1.3.

1. As one of the simplest examples one may consider fixed points of unary operations as unary, one element invariant relations.

2. For a linear order the set of its unary polymorphisms consists of all monotone functions.

3. Very generally for a given algebra $\mathscr{A} = \langle A, \mathscr{F} \rangle$ the set of its preserved unary relations are exactly the base sets of its subalgebras.
   The set of its preserved equivalence relations are just its congruence relations.
   The set of the n-ary relations preserved by $\mathscr{F}$ corresponds to the base sets of subalgebras of the n-th power of $\mathscr{A}$.

4. In the euclidean plane for each real number r we define a relation to consist of all pairs of points whose distance equals r. The automorphisms of the plane preserving all these relations are then just the isometries.

The Galois connection arises now from the following maps.

Definition 1.4. Let $\mathbf{E} \subseteq \mathscr{O}$ and $\mathbf{R} \subseteq \mathscr{R}$, then we define the map

$$\mathrm{Pol}_E : \mathscr{P}(\mathbf{R}) \to \mathscr{P}(\mathbf{E}), \quad \mathsf{Q} \mapsto \mathrm{Pol}_{\mathbf{E}}\mathsf{Q} \;\; := \;\; \{f \in \mathbf{E} | f \text{ preserves all } \rho \in \mathsf{Q} \;\}$$

which maps a set of relations Q to the set of functions in $\mathbf{E}$ preserving all of them and the map

$$\mathrm{Inv}_R : \mathscr{P}(\mathbf{E}) \to \mathscr{P}(\mathbf{R}), \quad \mathsf{F} \mapsto \mathrm{Inv}_{\mathbf{R}}\mathsf{F} \;\; := \;\; \{\rho \in \mathbf{R} | \rho \text{ is invariant for all } f \in \mathsf{F}\}$$

which maps a set of functions F to the set of relations in $\mathbf{R}$ invariant for all of the functions in F.
In the case that $\mathbf{E}$ consists of unary functions only we in addition define the maps

$$\begin{aligned} \mathrm{sPol}_E : \mathscr{P}(\mathbf{R}) \to \mathscr{P}(\mathbf{E}), \quad \mathsf{Q} \mapsto \mathrm{sPol}_{\mathbf{E}}\mathsf{Q} \;\; &:= \{f \in \mathbf{E} | f \text{ strongly preserves all } \rho \in \mathsf{Q} \;\} \\ \mathrm{sInv}_R : \mathscr{P}(\mathbf{E}) \to \mathscr{P}(\mathbf{R}), \quad \mathsf{F} \mapsto \mathrm{sInv}_{\mathbf{R}}\mathsf{F} \;\; &:= \{\rho \in \mathbf{R} | \rho \text{ is strongly invariant for all } f \in \mathsf{F}\} \end{aligned}$$

which are the natural generalizations for strong invariance and strong preservation.

The pairs

$$\mathrm{Inv}_{\mathbf{R}} - \mathrm{Pol}_{\mathbf{E}} \qquad \text{and} \qquad \mathrm{sInv}_{\mathbf{R}} - \mathrm{sPol}_{\mathbf{E}}$$

form a Galois connection between the sets $\mathscr{P}(\mathbf{R})$ and $\mathscr{P}(\mathbf{E})$, i.e., for sets $X \subseteq X' \subseteq \mathscr{P}(\mathbf{R})$ and $Y \subseteq Y' \subseteq \mathscr{P}(\mathbf{E})$ the following holds:

- $\mathrm{Pol}_{\mathbf{E}}(X) \supseteq \mathrm{Pol}_{\mathbf{E}}(X')$, $\mathrm{Inv}_{\mathbf{R}}(Y) \supseteq \mathrm{Inv}_{\mathbf{R}}(Y')$ (antitonie)

- $X \subseteq \mathrm{Inv}_{\mathbf{R}}\mathrm{Pol}_{\mathbf{E}}(X)$, $Y \subseteq \mathrm{Pol}_{\mathbf{E}}\mathrm{Inv}_{\mathbf{R}}(Y)$ (extensivity)

and equivalently for sPol and sInv. This implies that the maps $\mathrm{Pol}_{\mathbf{E}}\mathrm{Inv}_{\mathbf{R}}, \mathrm{sPol}_{\mathbf{E}}\mathrm{sInv}_{\mathbf{R}}$ as well as $\mathrm{Inv}_{\mathbf{R}}\mathrm{Pol}_{\mathbf{E}}, \mathrm{sInv}_{\mathbf{R}}\mathrm{sPol}_{\mathbf{E}}$ are closure operators. We will be interested in characterizing the Galois closed sets i.e sets $\mathsf{F} \in \mathscr{P}(\mathbf{E}), \mathsf{Q} \in \mathscr{P}(\mathbf{R})$ that fulfill

$$\mathsf{F} = \mathrm{Pol}_{\mathbf{E}}\mathrm{Inv}_{\mathbf{R}}\,\mathsf{F} \;\text{ and }\; \mathsf{Q} = \mathrm{Inv}_{\mathbf{R}}\mathrm{Pol}_{\mathbf{E}}\,\mathsf{Q}$$

for certain choices of **E** and **R** "internally" , that is without explicit reference to the Galois connection. As we will see the Galois closed sets of operations will in all cases be rather easy to characterize and so our prime concern will be the Galois closed sets of relations. We will especially be interested in differences in the characterization that arise from different cardinality of the base set A (at best we will be able to distinguish between finite, countable or uncountable cardinality).
The choices for **E** and $\mathrm{Pol}_{\mathbf{E}}$ we will make are (the (*) indicates strong preservation):

E	Pol_E
$\mathscr{O}$ ... the set of all operations	Pol ... polymorphisms
$\mathscr{O}^{(s)}$ ... the set of all s-ary operations	$\mathrm{Pol}^{(s)}$ ... s-ary polymorphisms
$\mathscr{O}^{(1)}$ ... the set of all unary operations	End ... endomorphisms
	$^{(*)}$sEnd ... strong endomorphisms
$\mathscr{O}^{(1-1)}$ ... the set of all injective unary operations	inj-End ... injective endomorphisms
$\mathscr{O}^{(surj.)}$ ... the set of all surjective unary operations	sur-End ... surjective endomrphisms
$\mathscr{S}$ ... the set of all bijective unary operations	wAut ... weak automorphisms
	$^{(*)}$Aut ... automorphisms

For **R** and $\mathrm{Inv}_{\mathbf{R}}$ our choices will be:

R	Inv_R
$\mathscr{R}$ ... the set of all relations	Inv ... invariant relations
	$^{(*)}$sInv ... strongly invariant relations
$\mathscr{R}^{(s)}$ ... the set of all s-ary relations	$\mathrm{Inv}^{(s)}$ ... s-ary invariant relations
Eq ... the set of equivalence relations	Con ... congruence relations
$\mathscr{R}^{(1)}$ ... the set of all unary relations	$\mathrm{Inv}^{(1)}$ ... subalgebras
$\mathscr{O}$ ... the of all operations	Pol ... polymorphisms
$\mathscr{O}^{(s)}$ ... the of all s-ary operations	Pol ... s-ary polymorphisms

In the following we list two tables. The first gives an overview of the different choices for **E** and **R** (where the $(*)$ again indicates the Galois connection arising from $\mathrm{sInv}_{\mathbf{E}}$ and $\mathrm{sPol}_{\mathbf{R}}$). We list the theorems in which they have been characterized (where the first line corresponds to theorems characterizing the Galois closure $\mathrm{Inv}_{\mathbf{E}}\mathrm{Pol}_{\mathbf{R}}$ and the second line to the theorems treating the Galois closure $\mathrm{Pol}_{\mathbf{R}}\mathrm{Inv}_{\mathbf{E}}$) and give the corresponding references from our bibliography. Cases in which we are only able to give a necessary and a sufficient condition for the Galois closure are marked by $(\sim)$.
The second table contains an overview of the characterizations of Galois closed sets of relations that have been treated in this work, stating the inner characterizations for different arities of the base set, where this is easily possible ($\oplus C$ stands for "adding closure w.r.t C") and giving the theorem where these characterizations can be found in this work as well as the corresponding references in the literature.

$E \setminus R$	$\mathscr{R}$	$\mathscr{R}^{(s)}$	$\mathscr{R}^{(1)}$	Eq	$\mathscr{O}/\mathscr{O}^{(s)}$
$\mathscr{O}$	Pol − Inv : 3.7, 3.9, 3.10 3.12 [PK79, Pös80, Pös03, Sza78]	Pol − Inv$^{(s)}$: 3.7, 3.9, 3.10 3.12 [Pös80, Pös03, Ros79, Ros78, Sza78]	Pol − Inv$^{(1)}$: 3.8 3.12 [Grä68, Sza78]	Pol − Con 3.55 [Grä68, Ihr93, Wer74]	Pol − Pol 3.45 [Pös03, Sza78]
$\mathscr{O}^{(s)}$	Pol$^{(s)}$ − Inv 3.7, 3.9, 3.10 3.12 [Pös80, Pös03]	Pol$^{(s)}$ − Inv$^{(s)}$ 3.7, 3.9, 3.10 3.12	Pol$^{(s)}$ − Inv$^{(1)}$: 3.8 3.12	Pol$^{(s)}$ − Con 3.55 [Grä68, Ihr93, Wer74]	Pol$^{(s)}$Pol 3.45
$\mathscr{O}^{(1)}$	End − Inv : 3.17, 3.22, 3.26 3.29 [PK79, Pös84, Pös80, Pös03, Bör00, Sza78] sEnd − sInv (∗) : 3.68 3.69 [BPV02, BPS]	End − Inv$^{(s)}$: 3.17, 3.22, 3.26 3.29	Inv$^{(1)}$End : 3.28	End − Con : 3.55 [Grä68, Ihr93, Wer74]	EndPol : 3.47 [Sza78] EndPol$^{(s)}$(∼) : 3.48 [Pös80], [Sto75]
$\mathscr{O}^{(1-1)}$	inj-End − Inv : 3.27 3.29 [Pös84] loc. invt. inj-End − Inv$^{(s)}$: 3.27 3.29 [Pös84]	inj-End − Inv$^{(s)}$: 3.27 3.29			
$\mathscr{O}^{(surj)}$	sur-End − Inv : 3.27 3.29 [Pös84]	sur-End − Inv$^{(s)}$: 3.27 3.29			
$\mathscr{S}$	wAut − Inv 3.17, 3.22, 3.26 3.29 [Pös84, PK79, Pös80, Pös03, Bör00] Aut − sInv(∗) 3.17, 3.22, 3.26, 3.43 3.29 [Pös84, PK79, Pös80, Pös03, BGS, Bör00]	wAut − Inv$^{(s)}$: 3.17, 3.22, 3.26 3.29 Aut − sInv$^{(s)}$(∗) : 3.17, 3.22, 3.26, 3.43 3.29			AutPol : 3.51 [Jón72] AutPol$^{((s))}$: 3.49, 3.51 [Jón91, Jón72]

E	**R**	closure	$\lvert A\rvert$			References
			$<\aleph_0$	$=\aleph_0$	$\geq\aleph_0$	
$\mathscr{O}$	$\mathscr{R}$	InvPol	$\mathrm{LOP}(\exists,\wedge,=)(\mathsf{Q})$ 3.9	$\mathrm{LOC}\langle\mathsf{Q}\rangle_{\mathrm{LOP}(\exists,\wedge,=)}$ 3.10	$\oplus$ gSup 3.7	[PK79, Pös80, Pös03, Sza78]
$\mathscr{O}^{(s)}$	$\mathscr{R}$	$\mathrm{InvPol}^{(s)}$	s-$\mathrm{LOC}\langle\mathsf{Q}\rangle_{\mathrm{LOP}(\exists,\wedge,=)}$ 3.9	3.10	$\oplus$ gSup 3.7	[Pös80, Pös03]
$\mathscr{O}^{(1)}$	$\mathscr{R}$	InvEnd	$\mathrm{LOP}(\exists,\wedge,\vee,=)(\mathsf{Q})$ 3.17	$\oplus\bigcup,\bigcap$ 3.22	$\oplus$ gSup 3.26	[PK79, Pös84, Pös80, Pös03], [Pös03, Bör00, Sza78]
$\mathscr{O}^{(1)}$	$\mathscr{R}(*)$	sInvsEnd	$\mathrm{LOP}(\exists,\wedge,\vee)(\mathsf{Q})$ 3.68			[BPV02, BPS]
$\mathscr{O}^{(1-1)}$	$\mathscr{R}$	Invinj-End	InvwAut$\mathsf{Q}$ 3.27	clsd.wr.t InvEnd and $(\nu)\in\mathsf{Q}$ 3.27		[Pös84]
$\mathscr{O}^{loc.inv}$	$\mathscr{R}$	Invinj-End and inj-End$\mathsf{Q}$ is locally invertible	sInvAut$\mathsf{Q}$ 3.27		$\oplus$ gSup 3.27	[Pös84]
$\mathscr{O}^{surj.}$	$\mathscr{R}$	Invsur-End	InvwAut$\mathsf{Q}$ 3.27	clsd. wrt. InvEnd and spSup 3.27		[Pös84]
$\mathscr{S}$	$\mathscr{R}$	InvwAut	$\mathrm{LOP}(\exists,\forall,\wedge,\vee,=,\neq)(\mathsf{Q})=$ sInvAut$\mathsf{Q}$ 3.17	$\oplus\bigcup,\bigcap$ 3.22	$\oplus$ sSup 3.26	[Pös84, PK79, Pös80], [Pös03, Bör00]
$\mathscr{S}$	$\mathscr{R}(*)$	sInvAut	$\mathrm{LOP}(\mathsf{Q})$ 3.17	$\oplus\bigcup,\bigcap$ 3.22	$\oplus$ sSup or $\langle\rangle_{\omega-inv.}$ 3.26,3.43	[Pös84, PK79, Pös80, Pös03], [BGS, Bör00]
$\mathscr{O}$	$\mathscr{R}^{(1)}$	$\mathrm{Inv}^{(1)}\mathrm{Pol}$	algebraic closure system 3.8			[Grä68, Sza78]
$\mathscr{O}^{(1)}$	$\mathscr{R}^{(1)}$	$\mathrm{Inv}^{(1)}\mathrm{End}$	$\langle\mathsf{Q}\rangle_{\Delta}$ 3.28			
$\mathscr{O}^{(1)}$	Eq()	ConPol = ConEnd	complete congruence lattice closed w.r.t. $\mathrm{gSup}^{(2)}_{eq.}$ 3.55			[Grä68, Ihr93, Wer74]
$\mathscr{O}$	$\mathscr{O}$	PolPol	operations def. by finite f.s. 3.45	operations locally. def. f.s. 3.45		[Pös03, Sza78]
$\mathscr{O}$	$\mathscr{O}^{(1)}$	EndPol	$\mathrm{PolPol}\cap\mathscr{O}^{(1)}$ 3.47			[Sza78]
$\mathscr{O}^{(s)}$	$\mathscr{O}^{(1)}$	$\mathrm{EndPol}^{(s)}$	Theorem 3.48			[Pös80, Sto75]
$\mathscr{O}$	$\mathscr{S}$	AutPol	$\langle\rangle_{\mathscr{S}}$ 3.51	$\oplus\ \mathrm{Loc}_0$ 3.51		[Jón72]
$\mathscr{O}^{(s)}$	$\mathscr{S}$	$\mathrm{AutPol}^{(s)}$	Theorem 3.49,Theorem 3.51			[Jón91, Jón72]

Chapter 2

Operations on relations and clones

The sets of relations closed w.r.t. the Galois connections introduced in chapter 1 will be characterized as sets closed w.r.t. certain operations on relations. For Galois closed sets of functions we will also use the concept of a clone. In the following we will introduce and discuss the operations and concepts needed.

2.1 Operations on relations

For the characterization of Galois closed sets of relations we will use operations of the form $F : \mathscr{R}^{(m_1)} \times \cdots \times \mathscr{R}^{(m_k)} \to \mathscr{R}^{(m_0)}$ with $k \in \mathbb{N}$ (we allow for $k = 0$ in which case we simply obtain a constant $\mu \in \mathscr{R}^{(m_0)}$); $(m_1, \ldots, m_k; m_0)$ is called the *signature* of F. A set $\mathsf{Q} \subseteq \mathscr{R}$ is closed under F iff $F(\rho_1, \ldots, \rho_k) \in \mathsf{Q}$ for all $\rho_i \in \mathsf{Q}^{(m_i)}$, $1 \leq i \leq k$.
We start with the definition of a clone of operations on relations.

Definition 2.1. For $S_i : \mathscr{R}^{(m_1)} \times \cdots \times \mathscr{R}^{(m_k)} \to \mathscr{R}^{(n_i)}$ for $i = 1, \ldots, l$ and $T : \mathscr{R}^{(n_1)} \times \cdots \times \mathscr{R}^{(n_l)} \to \mathscr{R}^{(m_0)}$ we define the *superposition* of T and $S_1, \ldots, S_l$ through:

$$\begin{array}{rcl} T[S_1, \ldots, S_l] : \mathscr{R}^{(m_1)} \times \cdots \times \mathscr{R}^{(m_k)} & \to & \mathscr{R}^{(m_0)} \\ (\rho_1, \ldots, \rho_k) & \mapsto & T(S_1(\rho_1, \ldots, \rho_k), \ldots, S_l(\rho_1, \ldots, \rho_k)). \end{array}$$

For $k \in \mathbb{N}^+$, $1 \leq i \leq k$ and $(m_1, \ldots, m_k) \in \mathbb{N}^k$ we define the associated *elementary operation* :

$$\begin{array}{rcl} \mathbf{E}_i^{(m_1, \ldots, m_k)} : \mathscr{R}^{(m_1)} \times \cdots \times \mathscr{R}^{(m_k)} & \to & \mathscr{R}^{(m_i)}, \\ (\rho_1, \ldots, \rho_k) & \mapsto & \rho_i. \end{array}$$

Now a set $\mathcal{C}$ of operations with arities in some set J (i.e., with signature in J^k) is called a *clone of operations on relations with arities in* J if the following conditions are fulfilled:

(1) $\mathcal{C}$ is closed w.r.t. to superposition.

(2) For all $k \in \mathbb{N}^+$, $1 \leq i \leq k$ and $(m_1, \ldots, m_k) \in J^k$ the elementary operations $E_i^{(m_1, \ldots, m_k)}$ is in $\mathcal{C}$.

(3) For every constant $\mu \in \mathcal{C}$ with $\mu \in \mathscr{R}^{(m_0)}$ and for all $m \in J$ there is a unary operation $S : \mathscr{R}^{(m)} \to \mathscr{R}^{(m_0)}$ in $\mathcal{C}$, that has the constant value μ.

For a set of operations $\mathcal{C}_0$ we define

$$\mathbf{opcl}\,\mathcal{C}_0 := \bigcap\{\mathcal{C}|\mathcal{C} \text{ is a clone of operations and } \mathcal{C}_0 \subseteq \mathcal{C}\}$$

to be the smallest clone containing $\mathcal{C}_0$

2.1.1 Boolean Operations and projections

We introduce Boolean operations and projections and dual projections.

Definition 2.2. We define *intersection* $\cap$ and *union* $\cup$ to be map two relations ρ_1, ρ_2 to their intersection $\rho_1 \cap \rho_2$ and their union $\rho_1 \cup \rho_2$.

2.2.1. The next operation we introduce is the *complement* of an m-ary relation ρ defined as:

$$\begin{aligned} \mathbf{C} : \mathscr{R} &\rightarrow \mathscr{R} \\ \rho &\mapsto A^m \setminus \rho = \{\mathbf{a} \in A^m | \mathbf{a} \notin \rho\} \\ &= \{\mathbf{a} \in A^m | \neg(\mathbf{a} \in \rho)\}. \end{aligned}$$

We will also write ρ^C for $\mathbf{C}\rho$.

2.2.2. For $n, m \in \mathbb{N}$ and $s : n \to m$ and $\mathbf{a} \in A^m$ define $\mathbf{a} \circ s := (a_{s(0)}, \ldots, a_{s(n-1)})$.
Then we introduce the operation

$$\begin{aligned} \mathbf{W}_s : \mathscr{R}^{(n)} &\rightarrow \mathscr{R}^{(m)} \\ \rho &\mapsto \{\mathbf{a} \in A^m | \mathbf{a} \circ s \in \rho\}. \end{aligned}$$

2.2.3. Further we define the *projections* through

$$\begin{aligned} \mathbf{Pr}^{(m)} : \mathscr{R}^{(m+1)} &\rightarrow \mathscr{R}^{(m)} \\ \rho &\mapsto \{(a_0, \ldots, a_{m-1}) \in A^m | \\ &\quad (\exists a_m \in A)(a_0, \ldots, a_{m-1}, a_m) \in \rho\}. \end{aligned}$$

2.2.4. The *dual projections* are defined by

$$\begin{aligned} \mathbf{Qr}^{(m)} : \mathscr{R}^{(m+1)} &\rightarrow \mathscr{R}^{(m)} \\ \rho &\mapsto \{(a_0, \ldots, a_{m-1}) \in A^m | \\ &\quad (\forall a_m \in A)(a_0, \ldots, a_{m-1}, a_m) \in \rho\} \\ &= \mathbf{C}\mathbf{Pr}^{(m)}(\mathbf{C}\rho). \end{aligned}$$

2.2.5. Intersection, union and complementation together with the constants $\emptyset$ and A^m form the *Boolean operations* on $\mathscr{R}^{(m)}$, $m \in \mathbb{N}$.

2.2.6. A set Q of relations is called a *Boolean system* if it is closed w.r.t to $\mathbf{W}_s$ for all $n, m \in \mathbb{N}$, $s : n \to m$ and all Boolean operations.

2.1.2 Logical Operations

Definition 2.3. Let $\varphi(P_1,\ldots,P_n;x_1,\ldots,x_m)$ be a first order formula with predicate symbols P_i (of arity m_i) and free variables $\{x_i | 1 \leq i \leq m\}$. Then for $\rho_i \in \mathscr{R}^{(m_i)}$ $(i=1,\ldots,n)$ we define

$$L_\varphi(\rho_1,\ldots\rho_n) := \{(a_1,\ldots,a_m) \in A^m | \varphi_A(\rho_1,\ldots\rho_n;a_1,\ldots,a_m)\},$$

where $\varphi_A(\rho_1,\ldots\rho_n;a_1,\ldots,a_m)$ means that φ holds in the structure $\langle A;\rho_1,\ldots,\rho_n\rangle$ for the evaluation $x_j = a_j$, $(1 \leq j \leq m)$. The operation L_φ obtained in this way is called a *logical operation.*

Some simple examples of logical operations are boolean operations like intersection $\cap$ and complementation, defined by the formula $P_1(x_1,\ldots,x_m) \wedge P_2(x_1,\ldots,x_m)$ and $\neg P(x_1,\ldots,x_n)$.

2.3.1. We will write $\mathrm{LOP}(Z_1,\ldots,Z_n)$ for the set of logical operations coming from first order formulas which are made up of the symbols $Z_1,\ldots,Z_n$.
A set $\mathrm{LOP}(Z_1,\ldots,Z_n)$ of logical operations forms a clone of operations. Some of the clones of logical operation that will encounter are:
(i)$\mathrm{LOP}(\exists,\wedge,=)$,
(ii) LOP $(\exists,\wedge,\vee)$,
(iii) $\mathrm{LOP}(\exists,\wedge,\vee,=)$,
(iv) $\mathrm{LOP}(\exists,\forall,\wedge,\vee,=,\neq)$,
(v) LOP.

2.3.2. Sets of relations closed w.r.t. to (ii) will be called sir-algebras, (iii)-(v) will be called *weak Krasneralgebras, pre-Krasneralgebras* and *Krasneralgebras* [Bör00] respectively [1].

2.3.3. All the logical clones (i),(iii)-(v) can be expressed in terms of Boolean operations and projections in the following way [Bör00]:

$$\begin{aligned}
\mathrm{LOP} &= \mathbf{opcl}(\{\emptyset,\cap,\cup,\mathbf{C}\} \cup \{A^m | m \in \mathbb{N}\} \cup \{\mathbf{W}_s | s \in \mathrm{FF}_0\} \cup \{\delta_A^{(2)}\} \cup \{\mathbf{Pr}^{(n)} | n \in \mathbb{N}\}),\\
\mathrm{LOP}(\exists,\forall,\wedge,\vee,=,\neq) &= \mathbf{opcl}(\{\emptyset,\cap,\cup\} \cup \{A^m | m \in \mathbb{N}\} \cup \\
&\quad \{\mathbf{W}_s | s \in \mathrm{FF}_0\} \cup \{\delta_A^{(2)}, (\delta_A^{(2)})^C\} \cup \{\mathbf{Pr}^{(n)}, \mathbf{Qr}^{(n)} | n \in \mathbb{N}\}),\\
\mathrm{LOP}(\exists,\wedge,\vee,=) &= \mathbf{opcl}(\{\emptyset,\cap,\cup\} \cup \{A^m | m \in \mathbb{N}\} \cup \\
&\quad \{\mathbf{W}_s | s \in \mathrm{FF}_0\} \cup \{\delta_A^{(2)}\} \cup \{\mathbf{Pr}^{(n)} | n \in \mathbb{N}\}),\\
\mathrm{LOP}(\exists,\wedge,=) &= \mathbf{opcl}(\{\emptyset,\cap\} \cup \{A^m | m \in \mathbb{N}\} \cup \\
&\quad \{\mathbf{W}_s | s \in \mathrm{FF}_0\} \cup \{\delta_A^{(2)}\} \cup \{\mathbf{Pr}^{(n)} | n \in \mathbb{N}\}),
\end{aligned}$$

where we have introduced $\mathrm{FF}_0 := \{s | (\exists\, n,m \in \mathbb{N})(m=0 \Rightarrow n=0) \text{ and } (s : n \to m)\}$ [2].

As a side note we mention that in particular LOP is finitely generated on finite sets but not finitely generated on infinite sets [Jón91].

[1]Krasneralgebra were first studied by M. Krasner. There algebras closed w.r.t. $\mathrm{LOP}(\exists,\wedge,\vee,=)$ are called Krasner algebras of first kind whereas the algebras closed w.r.t. LOP are called Krasneralgebas of second kind. For the exact citation we refer to [PK79].

[2]FF is supposed to stand for *finite functions*.

2.1.3 Invariant operations

Next we introduce invariant operations and certain subsets of them.

Definition 2.4. An operation $S : \mathscr{R}^{(m_1)} \times \cdots \times \mathscr{R}^{(m_k)} \to \mathscr{R}^{(m_0)}$ is called
a) *invariant* iff for all $f \in \mathscr{S}(A)$ and all relations $\rho_i \in \mathscr{R}^{(m_i)}$ $(i = 1, \ldots, k)$

$$S(f[\rho_1], \ldots, f[\rho_k]) = f[S(\rho_1, \ldots, \rho_k)]$$

holds.
b) *monotone* iff for all $\rho_i, \sigma_i \in \mathscr{R}^{(m_i)}$, $\sigma_i \subseteq \rho_i$ $(i = 1, \ldots, k)$ implies $S(\sigma_1, \ldots, \sigma_k) \subseteq S(\rho_1, \ldots, \rho_k)$.
c) $\mathrm{Tr}(A)$-*permutable* iff for all $f \in \mathscr{O}^{(1)}$, for all $\rho_i \in \mathscr{R}^{(m_i)}$

$$f[S(\rho_1, \ldots, \rho_k)] \subseteq S(f[\rho_1], \ldots, f[\rho_k]).$$

2.4.1. The set of all *invariant* operations is denoted by $\mathbf{IOP}_A$.

2.4.2. The set of all *monotone invariant* operations is denoted by $\mathbf{MIOP}_A$.

2.4.3. The set of all $\mathrm{Tr}(A)$-*permutable invariant* operations is denoted by $\mathbf{MVOP}_A$.

Remark 2.5. Note that $\mathbf{IOP}_A, \mathbf{MIOP}_A$ and $\mathbf{MVOP}_A$ form clones of operations. For the closure w.r.t to **IOP** we will also write $\langle . \rangle_{inv}$

Invariant operations were introduced in [Jón91]. Motivation for studying them can be found in the fact that our base set A does not carry any structure. Invariant operations are just the operations that respect this property, i.e., for any $f \in \mathscr{S}$ and any $\mathsf{Q} \subseteq \mathrm{Rel}(A)$ they do not distinguish between (A, Q) and $(A, f[\mathsf{Q}])$. $\mathrm{Tr}(A)$-permutable operations might be seen to be the natural generalization of this idea to arbitrary unary functions (note that all $\mathrm{Tr}(A)$-permutable operations are indeed invariant operations).
For operations $\mathbf{S}_i$ $(i \in I)$ of equal signature $(m_1, \ldots, m_k; m_0)$ we can introduce their intersection $\bigcap_{i \in I} \mathbf{S}_i$ and union $\bigcup_{i \in I} \mathbf{S}_i$ in the following way

$$\Big(\bigcup_{i \in I} \mathbf{S}_i\Big)(\rho_1, \ldots, \rho_k) := \bigcup_{i \in I} \mathbf{S}_i(\rho_1, \ldots, \rho_k)\ ,$$
$$\Big(\bigcap_{i \in I} \mathbf{S}_i\Big)(\rho_1, \ldots, \rho_k) := \bigcap_{i \in I} \mathbf{S}_i(\rho_1, \ldots, \rho_k)\ .$$

Also we may introduce the complement of an operation $\mathbf{S}$ simply as $\mathbf{CS}$.
With these operations the invariant operations of a fixed signature form a complete and atomic boolean algebra [Jón91]. Explicit expressions for the atoms of signature $(m_1, \ldots, m_k; m_0)$ can be given in the following way. Let $\sigma_i \in \mathscr{R}^{(m_i)}$ $(i = 1, \ldots, k)$ and let $\mathbf{b} \in A^{m_0}$. Then the operations

$$\mathbf{At}_{(A;\sigma_1,\ldots,\sigma_k;\mathbf{b})}(\rho_1, \ldots, \rho_k) := \{f\mathbf{b} | f \in \mathscr{S} \text{ and for } i = 1, \ldots, k \ f[\sigma_i] = \rho_i\}$$

are the "smallest" among the invariant operations $\mathbf{S}$ with $\mathbf{b} \in \mathbf{S}$ and the atoms of the respective boolean algebra.
All logical operations are invariant, the converse however only holds for finite base set A. This can be seen through a simple counting argument as already for signature $(1; 0)$ there is an

infinite number of pairwise distinct atomic invariant operations and so there are at leat $2^{\aleph_0}$ invariant operations [Bör00]. On the other hand the number of logical operations is always countable.
The operations from **MIOP** and **MVOP** of fixed signature are not closed w.r.t complementation but closed under arbitrary unions and intersections and form a complete, distributive lattice, which in general will not be atomic.
The relations between the clones of logical operations introduced so far and **IOP**, **MIOP**, **MVOP** are given by the following lemma.

Lemma 2.6. [Bör00] The following inclusions hold:
(i) LOP $\subseteq$ **IOP**.
(ii) LOP$(\exists, \forall, \wedge, \vee, =) \subseteq$ **MIOP**.
(iii) LOP$(\exists, \wedge, \vee, =) \subseteq$ **MVOP**.
Equality holds iff A is finite.

2.1.4 Operations of infinite arity - formula schemes, gSup, sSup, spSup

For infinite base set we will need some operations of infinite arity, i.e., operations F of the form $F : \prod_{i\in I} \mathscr{R}^{(m_i)} \to \mathscr{R}^m$ with infinite index set I. The simplest such operations will be infinite union and intersection defined in the usual way.

Definition 2.7. Let $m \in \mathbb{N}$ and let $\rho_i \in \mathscr{R}^{(m)}$, $i \in I$ for some index set I. Then we define infinite intersection/union in the following way

$$\bigcap_{i\in I} \rho_i := \{r \in A^m | r \in \rho_i \text{ for all } i \in I\} \qquad \bigcup_{i\in I} \rho_i := \{r \in A^m | r \in \rho_i \text{ for some } i \in I\}.$$

2.7.1. For a set of relations $\mathsf{Q} \subseteq \mathscr{R}$ we write $\langle \mathsf{Q} \rangle_{\bigcap} (\langle \mathsf{Q} \rangle_{\bigcup})$ for the closure of Q w.r.t. arbitrary intersections (unions).

2.7.2. A set $\mathsf{Q} \subseteq \mathscr{P}(A)$ is called an *algebraic closure system* iff Q is closed w.r.t. arbitrary intersections and under unions of directed systems [Grä68].

2.7.3. We call a set $\mathsf{Q} \subseteq \mathscr{R}$ Δ-*complete* iff
(1) $\emptyset \in \mathsf{Q}^{(m)}$ and $A^m \in \mathsf{Q}^{(m)}$,
(2) Q is closed under arbitrary intersections and unions.
We denote the smallest Δ-complete set containing $\mathsf{Q} \subseteq \mathscr{R}$ by $\langle \mathsf{Q} \rangle_\Delta$.

Other operations of infinite arity we will use are introduced with the help of formula schemes, which can seen as generalization of logical formulas.

Definition 2.8. Let $X = \{x_i | i \in I\}$ be a set of variables, Q a set of relations of A, $\rho \in \mathsf{Q}^{(n)}$, $f \in \mathsf{Q} \cap \mathscr{O}^{(m)}$, $g \in \mathsf{Q} \cap \mathscr{O}^{(s)}$. Then

$$\rho(x_{i_1}, \ldots, x_{i_n}) \quad , \quad f(x_{j_1}, \ldots, x_{j_m}) = g(x_{k_1}, \ldots, x_{k_s})$$

are said to be formulas of the variable set X over Q provided $x_{i_1}, \ldots, x_{i_n}, x_{j_1}, \ldots, x_{i_m}, x_{k_1}, \ldots, x_{k_s} \in X$ (as noted in [Sza78] formulas of the first kind would be sufficient but introducing both kinds

makes notation a bit easier later on). A family $(a_i|i \in I) \in A^I$ is said to satisfy above formulas if $\rho(a_{i_1}, \ldots, a_{i_n})$, $f(a_{j_1}, \ldots, a_{j_m}) = g(a_{k_1}, \ldots, a_{k_s})$ holds.
A triple $\Psi = (\Sigma, X, (x_{i_1}, ..., x_{i_n}))$ is called a *formula scheme* over Q when X is a set of variables indexed by I, $(x_{i_1}, ..., x_{i_n}) \in X^n$ and Σ is a set of formulas of the variable set X over Q. We say Ψ is finite if both Σ and X are finite. To Ψ as above we associate a n-ary relation R_Ψ defined by

$$R_\Psi := \{(a_{i_1}, \ldots, a_{i_n}) | (a_i|i \in I) \in A^I \text{ satisfies every member of } \Sigma\}$$

and say R_Ψ is defined by Ψ.

2.8.1. On $\mathscr{R}$ we define the following closure operator

$$\begin{aligned} [\ \ ]_{\text{f.s.}} : \mathscr{R} &\to \mathscr{R} \\ \mathsf{Q} &\mapsto [\mathsf{Q}]_{\text{f.s.}} := \{R_\Psi | \Psi \text{ is a formula scheme over } \mathsf{Q}\} \cup \{\emptyset\} \end{aligned}$$

and say Q is closed w.r.t. formula schemes if $\mathsf{Q} = [\mathsf{Q}]_{\text{f.s.}}$ [3].

2.8.2. We say that $\Psi = (\Sigma, X, (x_{i_1}, \ldots, x_{i_n}, x_{i_{n+1}}))$ defines an n-ary operation f on $B \subseteq A^n$ if for any $(a_1, \ldots, a_n) \in B$, $f(a_1, \ldots, a_n) = a_{n+1}$ for some $a_{n+1} \in A$ iff $R_\Psi(a_1, \ldots, a_n, a_{n+1})$ holds.
A n-ary operation f is said to be *locally definable* by a set of relations $\mathsf{Q} \subseteq \mathscr{R}(A)$ if for every finite $B \subseteq A^n$ there exists a formula scheme over Q defining f on B.

As a next step we show that $[\mathsf{Q}]_{\text{f.s.}}$ is closed w.r.t to formula schemes, i.e., $[[\mathsf{Q}]_{\text{f.s.}}]_{\text{f.s.}} = [\mathsf{Q}]_{\text{f.s.}}$.

Proof. Let $\Psi = (\Gamma, X, (x_{j_1}, \ldots, x_{j_k}))$ be a formula scheme over Q and let

$$\varphi := R_\Psi = \{(b_{j_1}, \ldots, b_{j_k}) | (b_j|j \in J) \in A^J \text{ satisfies } \Gamma\}.$$

We consider the simplest formula scheme including φ namely $\tilde{\Psi} = (\{\varphi(x_{r_1}, \ldots, x_{r_k})\}, X, (x_{i_1}, \ldots, x_{i_n}))$ over $[\mathsf{Q}]_{\text{f.s.}}$ (where w.l.o.g. we assume that the index set I of the variables $X = \{x_i|i \in I\}$) is disjoint from J) and show that $R_{\tilde{\Psi}} \in [\mathsf{Q}]_{\text{f.s.}}$.
$R_{\tilde{\Psi}}$ can be rewritten in the following way:

$$\begin{aligned} R_{\tilde{\Psi}} &= \{(a_{i_1}, \ldots, a_{i_n}) \ \mid \ (a_i|i \in I) \in A^I \text{ satisfies } \{\varphi(x_{r_1}, \ldots, x_{r_k})\}\} \\ &= \{(a_{i_1}, \ldots, a_{i_n}) \ \mid \ (a_i|i \in I) \in A^I \text{ satisfies } \varphi(a_{r_1}, \ldots, a_{r_k})\} \\ &= \{(a_{i_1}, \ldots, a_{i_n}) \ \mid \ (a_i|i \in I) \in A^I \text{ satisfies } \{(b_j|j \in J) \in A^J \\ &\qquad\qquad \text{satisfies } \Gamma \text{ and } a_{r_1} = b_{j_1}, \ldots, a_{r_k} = b_{j_k}\}\} \\ &= \{(a_{i_1}, \ldots, a_{i_n}) \ \mid \ (a_i|i \in I) \cup (b_j|j \in J) \in A^{(I \cup J)} \text{ satisfies } \Gamma \text{ and} \\ &\qquad\qquad a_{r_1} = b_{j_1}, \ldots, a_{r_k} = b_{j_k}\}, \end{aligned}$$

which obviously becomes a formula scheme over Q after relabelling the variables to accommodate for $a_{r_1} = b_{j_1}, \ldots, a_{r_k} = b_{j_k}$. Now one can easily see that the same method also works for a more complicated formula schemes $\tilde{\Psi}$. This then ends the proof. □

[3] Note that the empty set is always an invariant relation as we do not consider nullary functions and so was added here.

The relations obtained through finite f.s. are equivalent to relations obtained from primitive positive formulas as can be seen in the following way.
For a f.s. $\Psi = (\Sigma, X, (x_{i_1}, \ldots, x_{i_n}))$ by adding formulas of the type $x_{i_r} = x_{j_s}$ to Σ we can w.l.o.g assume that $i_k \neq i_l$ for $k \neq l \in \{1, \ldots, n\}$. Calling the new set of formulas obtained in this way $\tilde{\Sigma}$ we rewrite R_Ψ in the following way

$$\begin{aligned} R_\Psi &= \{(a_{i_1}, \ldots, a_{i_n}) \;\mid\; (a_i | i \in I) \in A^I \text{ satisfies } \tilde{\Sigma}\} \\ &= \{(a_{i_1}, \ldots, a_{i_n}) \;\mid\; \exists (a_i | i \in I \setminus \{i_1, \ldots, i_n\}) \in A^{I \setminus \{i_1, \ldots, i_n\}} \text{ satisfies} \\ &\qquad\qquad \bigwedge_{\varphi \in \tilde{\Sigma}} \varphi(x_{r_1}, \ldots, x_{r_k})\}. \end{aligned}$$

For an infinite formula schemes Ψ we can clearly rewrite R_Ψ in the same way. In that sense R_Ψ can be thought to arise from a logical operation belonging to a primitive positive formula which may have an infinite number of bound variables but only a finite number of free variables.

The other operations of infinite arity we introduce are strong, special and general superposition.

Definition 2.9. Let I be an arbitrary index set and $n_i \in \mathbb{N}, \mathbf{b}_i \in A^{n_i} (i \in I)$ and $\rho_i \in \mathscr{R}^{(n_i)} (i \in I)$. Further let $m \in \mathbb{N}^+$ and $\mathbf{a} \in A^m$ then we define the following operations

2.9.1. *strong superposition*: $\mathrm{sSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) := \{g\mathbf{a} | g \in \mathscr{S} \text{ and } (\forall i \in I)\; g\mathbf{b}_i \in \rho_i\}$

2.9.2. *special superposition*: $\mathrm{spSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) := \{g\mathbf{a} | g \in \mathscr{O}^{(surj.)} \text{ and } (\forall i \in I)\, g\mathbf{b}_i \in \rho_i\}$

2.9.3. *general superposition*: $\mathrm{gSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) := \{g\mathbf{a} | g \in \mathrm{Tr} \text{ and } (\forall i \in I)\; g\mathbf{b}_i \in \rho_i\}$.

A set Q of relations is said to be closed w.r.t. gSup / spSup/ sSup iff $\rho_i \in \mathsf{Q}^{(n_i)}$ $(i \in I)$ implies $\mathrm{gSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I})$/ $\mathrm{spSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I})$/ $\mathrm{sSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) \in \mathsf{Q}^{(m)}$.

As in the situation of formula schemes we can rewrite strong, special and general superposition as "infinite" formulas in the following way.
Let $\mathbf{a}, \mathbf{b}_i, n_i, \rho_i, I$ be as above. Then we can rewrite sSup/spSup/gSup in the following way:

$$\begin{aligned} \mathrm{sSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) &:= \{g\mathbf{a} | g \in \mathscr{S} \text{ and } (\forall i \in I)\; g\mathbf{b}_i \in \rho_i\} \\ &= \{(x_{a_1}, \ldots, x_{a_m}) | (\exists x_r)_{r \in A \setminus \{a_1, \ldots, a_m\}} : (\forall y \in A \bigvee_{r \in A} y = x_r) \\ &\qquad \wedge (\bigwedge_{r \neq s, r, s \in A} x_r \neq x_s) \wedge \bigwedge_{i \in I} (x_{b_{i1}}, \ldots, x_{b_{in_i}}) \in \rho_i\}, \\ \mathrm{spSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) &:= \{g\mathbf{a} | g \in \mathscr{O}^{(surj.)} \text{ and } (\forall i \in I)\; g\mathbf{b}_i \in \rho_i\} \\ &= \{(x_{a_1}, \ldots, x_{a_m}) | (\exists x_r)_{r \in A \setminus \{a_1, \ldots, a_m\}} : (\forall y \in A \bigvee_{r \in A} y = x_r) \\ &\qquad \wedge \bigwedge_{i \in I} (x_{b_{i1}}, \ldots, x_{b_{in_i}}) \in \rho_i\}, \\ \mathrm{gSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) &:= \{g\mathbf{a} | g \in \mathrm{Tr} \text{ and } (\forall i \in I)\; g\mathbf{b}_i \in \rho_i\} \\ &= \{(x_{a_1}, \ldots, x_{a_n}) | (\exists x_k)_{k \in A \setminus \{a_1, \ldots, a_m\}} : \bigwedge_{i \in I} (x_{b_{i1}}, \ldots, x_{b_{in_i}}) \in \rho_i\}. \end{aligned}$$

This in particular shows that every general superposition $\mathrm{gSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I})$ is in fact equivalent to some R_Ψ for an appropriate formula scheme Ψ.
Also one can easily check that $\mathsf{Q} \subseteq \mathscr{R}$ is closed w.r.t. sSup iff Q is closed w.r.t. spSup and $\nu := \{(x,y) \in A^2 | x \neq y\} \in \mathsf{Q}$.

We give some examples gSup/spSup/sSup for different choices of $\mathbf{a}, \mathbf{b}_i, \rho_i$:

1. Relational product: Let $|A| \geq 3$, $\rho_1, \rho_2 \in \mathscr{R}^{(2)}$ and let $\mathbf{a} = (a_1, a_2) \in A^2 \setminus \delta_A^{(2)}$ and $b \in A \setminus \{a_1, a_2\}$ be arbitrary but fixed. Then

$$\begin{aligned}\mathrm{gSup}(\mathbf{a}, (a_1, b), (b, a_2), \rho_1, \rho_2) &= \{f\mathbf{a} | f \in \mathrm{Tr} \text{ and } f(a_1, b) \in \rho_1 \wedge f(b, a_2) \in \rho_2\} \\ &= \rho_1 \circ \rho_2 \ ; \\ \mathrm{sSup}(\mathbf{a}, (a_1, b), (b, a_2), \rho_1, \rho_2) &= \{f\mathbf{a} | f \in \mathscr{S} \text{ and } f(a_1, b) \in \rho_1 \wedge f(b, a_2) \in \rho_2\} \\ &= (\rho_1 \setminus \delta_A^{(2)} \circ \rho_2 \setminus \delta_A^{(2)}) \setminus \delta_A^{(2)}.\end{aligned}$$

2. Intersection: Let I be some index set, $\rho_i \in \mathscr{R}^{(n)}$ for $i \in I$ and $n \in \mathbb{N}$. Further choose $\mathbf{a} \in A^n$ and set $\mathbf{b}_i = \mathbf{a}$ for all $i \in I$. Then

$$\begin{aligned}\mathrm{gSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) &= \{f\mathbf{a} | f \in \mathrm{Tr} \text{ and } f\mathbf{a} \in \rho_i \text{ for all } i \in I\} \\ &= \bigcap_{i\in I} \rho_i.\end{aligned}$$

   Assuming $\mathbf{a} \in A^n \setminus \delta_A^{(n)}$ we get for the same choice of $\mathbf{b}_i, \rho_i$ the strong superposition

$$\begin{aligned}\mathrm{sSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) &= \{f\mathbf{a} | f \in \mathscr{S} \text{ and } f\mathbf{a} \in \rho_i \text{ for all } i \in I\} \\ &= (\bigcap_{i\in I} \rho_i) \setminus \delta_A^{(n)}.\end{aligned}$$

3. Diagonal relation: By choosing an arbitrary $\mathbf{a} \in \delta_A^{(n)}$ and an empty index set I we get

$$\begin{aligned}\mathrm{gSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) &= \{f\mathbf{a} | f \in \mathrm{Tr}\} \\ &= \{g\mathbf{a} | g \in \mathscr{S}\} = \mathrm{sSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) = \delta_A^{(n)}.\end{aligned}$$

4. Logical operations: As can be easily seen all logical operations from $\mathrm{LOP}(\exists, \wedge, =)$ can be expressed though an appropriate general superposition.
   Through special superposition the operation $\exists x \rho(x,y)$ can in general not be produced as one might see by choosing $\rho = \{(1,2), (3,3)\}$ on the base set $A = \{1,2,3\}$. Then choosing $\mathbf{b} \in A^2 \setminus \delta_A^{(2)}$ always only leaves one possibility for $g \in \mathscr{S}$ with $g\mathbf{b} \in \rho$, so the appropriate special superposition cannot be $\{2,3\}$ as it should. Choosing $\mathbf{b} \in \delta_A^{(2)}$ however also does not lead to the correct result.
   Also the complement of a relation can in general not be produced through a special superposition as can be seen by choosing the relation $\rho := \{(1,1), (1,2)\}$ on the base set $A = \{1,2\}$.
   For finite base set A let $\rho \in \mathscr{R}^{(2)}$ be such that there is at least one $a \in A$ with $\forall y \rho(y,a)$ holds and choose $\{\mathbf{b}_i | i \in I\} = \{(c,d) \in A^2 | \forall y \rho(y,d)\}$. Then

$$\mathrm{sSup}(a, (\mathbf{b}_i)_{i\in I}, (\rho)_{i\in I}) = \{ga | g \in \mathscr{S} \text{ and } g\mathbf{b}_i \in \rho \forall i \in I\} = \forall y \rho(y,x).$$

For an arbitrary relation this will however no longer be possible, as for example can be seen through the relation $\rho := \{(y, a, a)|y \in A\} \cup \{(y, a, b)|y \in A\}$ for $a \neq b \in A$.
Further for example any relation $\rho \in \mathscr{R}^{(n)}$ with $\rho \cap \delta_A^{(n)} \neq \emptyset$ and $\rho \nsubseteq \delta_A^{(n)}$ cannot be written as a special superposition as a consequence of the injectiveness of the functions g in the definition of sSup.

2.1.5 Local closure

In addition to the closure operators that are obtained from the operations introduced above we need "local" closure operators. The presence of these operators is a direct consequence of the fact that we study relations and functions of finite arity only.

Definition 2.10. Let $\mathsf{Q} \subseteq \mathscr{R}$. Then we define

$$\begin{aligned} s\text{-LOC}\,\mathsf{Q} &:= \{\rho|\forall B \subseteq \rho, |B| \leq s, \exists \sigma \in \mathsf{Q} : B \subseteq \sigma \subseteq \rho\}, \\ \text{LOC}\,\mathsf{Q} &:= \{\rho|\forall B \subseteq \rho, |B| \leq \aleph_0, \exists \sigma \in \mathsf{Q} : B \subseteq \sigma \subseteq \rho\} \\ &= \bigcap_{s \in \mathbb{N}} s\text{-LOC}\,\mathsf{Q}. \end{aligned}$$

(s-)LOCQ is called the (s-)local closure of Q.

Remark 2.11. (i): Note that if a relation ρ is not in the (s-)LOC closure of some set Q of relations there will be a finite set $B \subseteq \rho$ to "witness" that.
(ii): Obviously LOC$\mathsf{Q} = \mathsf{Q}$ always holds for a finite base set. For s-LOC this same statement is only true when $|A| \leq$ s. A typical example of a non-s-LOC closed set of relations in case where s$\leq |A|$ choose $A = \{1, 2, 3\}$ and $\mathsf{Q} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, which is not 2-LOC closed.

The operators s-LOC are ordered in the following way:

$$1\text{-LOC}\,\mathsf{Q} \supseteq 2\text{-LOC}\mathsf{Q} \supseteq .... \supseteq \text{LOC}\,\mathsf{Q}.$$

In particular we note that closure w.r.t 1-LOC just is closure w.r.t arbitrary unions.
Writing Q in the form $\mathsf{Q} = \bigcup_{i \in \mathbb{N}} \mathsf{Q}^{(i)}$ where $\mathsf{Q}^{(i)} := \mathsf{Q} \cap \mathscr{R}^{(i)}$, we see that

$$s\text{-LOC}\,\mathsf{Q} = \bigcup_{i \in \mathbb{N}} s\text{-LOC}\,\mathsf{Q}^{(i)} \text{ and } \text{LOC}\,\mathsf{Q} = \bigcup_{i \in \mathbb{N}} \text{LOC}\,\mathsf{Q}^{(i)},$$

i.e., (s-)LOC does not mix aritys.
We mention that LOC is a topological closure operator, i.e., for $\mathsf{Q}_1, \mathsf{Q}_2 \subseteq \mathscr{R}$ we have

$$\text{LOC}(\text{LOC}\mathsf{Q}_1 \cup \text{LOC}\mathsf{Q}_2) = \text{LOC}\mathsf{Q}_1 \cup \text{LOC}\mathsf{Q}_2.$$

We give some examples for s-LOC closed sets.

Example 2.12. 1.) Any set Q of relations where each relation consist of no more than $s - 1$ elements is always trivially s-LOC closed.
2.) Any set of linear orders of a base set A is always 2-LOC closed.

3.) Any set Q of relations where each relation of Q can be singled out by (at most) $s-1$ specific elements it contains (i.e., for each $\rho \in \mathsf{Q}$ there exists a set $B \subseteq \rho$,$|B| \leq s-1$ s.t. $B \subseteq \sigma \in \mathsf{Q} \Leftrightarrow \sigma = \rho$).
Proof: Suppose $\mathsf{Q} \neq s\text{-LOC}\mathsf{Q}$, i.e., there is a $\rho \in s\text{-LOC}\mathsf{Q} \setminus \mathsf{Q}$. Then we find a $\sigma \in \mathsf{Q}$ s.t. $\sigma \subset \rho$. Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ $(n \leq s-1)$ be elements that characterize σ and choose an element $\mathbf{b} \in \rho \setminus \sigma$. For the set $B = \{\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b}\}$ there can be no $\tilde{\sigma} \in Q$ s.t. $B \subset \sigma \subset \rho$ which shows $\rho \notin s\text{-LOC}\mathsf{Q}$.
4.) As we will see below for $\mathsf{Q} \subseteq \mathscr{R}$ every set of the form $\mathrm{InvPol}^{(s)}\mathsf{Q}$ is s-LOC closed.

For a further characterization of s-LOC we need the notion of s-directed systems defined as generalizations of directed systems.

Definition 2.13. [Pös80] Recall, a set $\mathcal{T}$ of sets is called upwards directed if for all $X, Y \in \mathcal{T}$ there exists a $Z \in \mathcal{T}$ s.t. $X \cup Y \subseteq Z$. We define a set $\mathcal{T}$ of sets to be s-directed ($s \in \mathbb{N}$) if for all $X_1, \ldots, X_s \in \mathcal{T}$ and $r_1 \in X_1, \ldots, r_s \in X_s$ there exists a $Z \in \mathcal{T}$ s.t. $\{r_1, \ldots, r_s\} \subseteq Z$.

For sets of relation that are closed w.r.t. arbitrary intersection closure w.r.t. s-LOC is equivalent to closure w.r.t. union of s-directed systems as stated in the following proposition.

Proposition 2.14. [Pös80] Let Q be a set of relations closed under arbitrary intersections. Then

$$\begin{aligned}(i)\; s\text{-LOC}\mathsf{Q} &= \{\bigcup \mathcal{T} | \emptyset \neq \mathcal{T} \subseteq \mathsf{Q} \text{ and } \mathcal{T} \text{ is } s\text{-directed}\}.\\ (ii)\;\; \mathrm{LOC}\mathsf{Q} &= \{\bigcup \mathcal{T} | \emptyset \neq \mathcal{T} \subseteq \mathsf{Q} \text{ and } \mathcal{T} \text{ is directed}\}.\end{aligned}$$

An additional characterization is given by the following proposition.

Proposition 2.15. [Pös80] Let Q be a set of relations closed under arbitrary intersections. Then the following conditions are equivalent (for a fixed $s \in \mathbb{N}$).
(a) $\mathsf{Q} = s\text{-LOC}\mathsf{Q}$,
(b) $B \in \mathsf{Q}$ iff $\Gamma^{\mathsf{Q}}(X) \subseteq B$ for all $X \subseteq B$ with $|X| \leq s$ [4].

For the proof of both propositions we refer to [Pös80].

For sets of functions we can also introduce local closure operators similar to s-LOC/LOC.

Definition 2.16. Let $\mathsf{F} \subseteq \mathscr{O}$. We define

$$\begin{aligned} s\text{-Loc}\mathsf{F} &= \{g \in \mathscr{O}^{(n)} | \forall B \subseteq A^n, |B| \leq s, \exists f \in \mathsf{F}^{(n)} : g \restriction B = f \restriction B, n \in \mathbb{N}\},\\ \mathrm{Loc}\mathsf{F} &= \{g \in \mathscr{O}^{(n)} | \forall B \subseteq A^n, |B| < \aleph_0, \exists f \in \mathsf{F}^{(n)} : g \restriction B = f \restriction B, n \in \mathbb{N}\}\\ &= \bigcap_{s \in \mathbb{N}} s\text{-Loc}\mathsf{F},\end{aligned}$$

i.e., the set of all functions that can be approximated on all subsets of A^n of size less or equal s and the set of all functions that can be approximated on all finite subsets of A^n.

[4] For the definition of $\Gamma^{\mathsf{Q}}(X)$ see definition 3.1.

2.16.1. For $\mathsf{G} \subseteq \mathscr{S}(A)$ we define the restrictions of Loc/s-Loc to $\mathscr{S}(A)$ in the following way:

$$\mathrm{Loc}_o\mathsf{G} := \mathrm{Loc}\mathsf{G} \cap \mathscr{S}(A) \quad , \quad s\text{-}\mathrm{Loc}_o\mathsf{G} := s\text{-}\mathrm{Loc}\mathsf{G} \cap \mathscr{S}(A).$$

As for s-LOC/LOC, s-Loc/Loc do not mix arities, i.e., $\mathsf{F} \subseteq \mathscr{O}^{(n)}$ (for some $n \in \mathbb{N}$) then also s-Loc$\mathsf{F}$, Loc$\mathsf{F} \subseteq \mathscr{O}^{(n)}$.

Remark 2.17. Note that on a finite base set A all sets of functions are Loc closed. However for $|A| \geq$s they are not necessarily s-Loc-closed.

The operators s-Loc are ordered in the following way:

$$1\text{-}\mathrm{Loc}\mathsf{F} \supseteq 2\text{-}\mathrm{Loc}\mathsf{F} \supseteq \cdots \supseteq \mathrm{Loc}\mathsf{F}.$$

We give some examples to illustrate these definitions.

Example 2.18. 1.)For a set of relations Q, Pol$\mathsf{Q}$, Pol$^{(s)}\mathsf{Q}$, End$\mathsf{Q}$ are all Loc-closed. For $\mathsf{Q} \subseteq \mathscr{R}^{(s)}$ these sets are s-Loc-closed. Aut$\mathsf{Q}$ is a typical example of a Loc$_o$-closed set.
2.)Let $n, k \in \mathbb{N}$ and define $\mathsf{F} := \{f \in \mathscr{O}^{(n)} | f[A^n] \leq k\}$. Then F is $(k+1)$-Loc closed. See also example 2.25.
3.)The set $\mathscr{O}^{(1-1)}$ of all injective functions is 2-Loc closed. In particular Loc$\mathscr{S} = \mathscr{O}^{(1-1)}$ which also illustrates that surjectiveness is not a local property.

We mention that Loc and Loc$_o$ are topological closure operators, i.e., for $\mathsf{F}_1, \mathsf{F}_2 \subseteq \mathscr{O}$

$$\mathrm{Loc}(\mathrm{Loc}\mathsf{F}_1 \cup \mathrm{Loc}\mathsf{F}_2) = \mathrm{Loc}\mathsf{F}_1 \cup \mathrm{Loc}\mathsf{F}_2$$

and an analogous statement holds for Loc$_o$. The topology belonging to Loc is the product topology on A^{A^n} where A, is taken to be discrete. The topology belonging to Loc$_o$ is obtained by restricting the topology of A^A to $\mathscr{S}(A)$.

Let $m \in \mathbb{N}$, $\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b} \in A^m$. Then the sets

$$\mathcal{U}_{\{\mathbf{a}_1,\ldots,\mathbf{a}_n\}\mapsto\mathbf{b}} := \{f \in \mathscr{O}^{(n)} | f(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \mathbf{b}\}$$

form a basis of the topology of Loc. Similarly for $m \in \mathbb{N}$, $\mathbf{a}, \mathbf{b} \in A^m$

$$\mathcal{U}_{\mathbf{a}\mapsto\mathbf{b},\,o} := \{f \in \mathscr{S}(A) | f(\mathbf{a}) = \mathbf{b}\}$$

form the basic open sets of the topology belonging to Loc$_o$.
In fact all these sets are clopen so they as well as their complements

$$\begin{aligned} \mathcal{U}^C_{\mathbf{a}\mapsto\mathbf{b}} &= \{f \in A^{A^n} | f(\mathbf{a}_1, \ldots, \mathbf{a}_n) \neq \mathbf{b}\}, \\ \mathcal{U}^C_{\mathbf{a}\mapsto\mathbf{b},\,o} &= \{f \in \mathscr{S}(A) | f(\mathbf{a}) \neq \mathbf{b}\} \end{aligned}$$

provide examples for Loc- and Loc$_o$-closed sets.
Note that s-Loc is in general not a topological closure operator.

2.2 Clones

At last we will define clones of operations and the closure operator associated with them. First we define functional composition and projections.

Definition 2.19. A *projection* is an operation $f(x_1, \ldots, x_n) \in \mathscr{O}$ that satisfies an identity of the form $f(x_1, \ldots, x_n) = x_k$ for some $1 \leq k \leq n$. We denote the n-ary projection onto the k-th variable by π_k^n.
For $f \in \mathscr{O}^{(n)}$ and $g_1, \ldots, g_n \in \mathscr{O}^{(m)}$ $(n, m \in \mathbb{N})$ the functional composition of f and $g_1, \ldots, g_n$ is defined as the m-ary operation

$$f(g_1, \ldots, g_n) : (x_1, \ldots, x_m) \mapsto f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m)).$$

Finally we call a function $f \in \mathscr{O}^{(n)}$ *essentially unary* iff it depends only on one of its variable, i.e., iff there is a unary function $F \in \mathscr{O}^{(1)}$ and $1 \leq k \leq n$ such that $f = F(\pi_k^n)$.

Definition 2.20. Let $\mathsf{F} \subseteq \mathscr{O}$. We say F is a *clone* if F is closed under functional composition and contains all projections. We denote the clone generated by a set of operations $\mathsf{F} \subseteq \mathscr{O}$, i.e., the smallest clone containing F, by $\langle \mathsf{F} \rangle$.
For $\mathsf{F} \subseteq \mathscr{O}^{(1)}$, $\mathsf{G} \subseteq \mathscr{S}$ we define $\langle \mathsf{F} \rangle_{\mathscr{O}^{(1)}}$ to be the submonoid of $\mathscr{O}^{(1)}$ generated by F and $\langle \mathsf{G} \rangle_{\mathscr{S}}$ to be the subgroup of $\mathscr{S}$ generated by G.

Definition 2.21. A submonoid F of $\mathscr{O}^{(1)}$ is said to be *locally invertible* if for all $n \in \mathbb{N}$, $\mathbf{a} \in A^n$ and all $f \in \mathsf{F}$ there exists a $g \in \mathsf{F}$ such that $g \circ f\mathbf{a} = \mathbf{a}$, i.e., f has an inverse on every finite subset. Note that in this case the functions of F are necessarily injective.

We note that we can also think of clones as subalgebras of a certain algebra with base set $\mathscr{O}$ (see for example [PK79, 1.1.1]). That is we can think of $\langle \mathsf{F} \rangle$ as being generated through certain operations acting on F similar to the logical operations and formula schemes introduced for relations above. In contrast to formula schemes however these operations are always of finite arity.
Also note that for $\mathsf{F} \subseteq \mathscr{O}^{(1)}$, $\langle \mathsf{F} \rangle$ and $\langle \mathsf{F} \rangle_{\mathscr{O}^{(1)}}$ only differ in essentially unary functions and for some needs may be identified with each other.

In the following we give examples of clones and mention some of their basic properties. For more on clones we refer to [PK79, GP08] and references therein.
Some examples of clones are:

1. The full clone $\mathscr{O}$, which is the largest clone and the set $\mathscr{J}$ of all projections, which is the smallest clone.

2. For a given linear order the set of all monotone functions (all functions respecting this order) form a clone.

3. More generally, for a set of relations $\mathsf{Q} \subseteq \mathscr{R}$ the set $\mathrm{Pol}\mathsf{Q}$ of functions preserving it is always a clone. On a finite base set every clone is of this form, for an infinite base sets we have to allow for relations of infinite arity as well to be able describe every clone in such a way [Ros72].

4. For an algebra $\mathcal{X} = (X, \mathcal{F})$ the set of all term operations is a clone. Every clone is of this form.

By ordering all the clones over A by inclusion one obtains the *clone lattice* $\mathrm{Cl}(A)$. Its largest element is the full clone $\mathscr{O}$ and its smallest element the clone consisting of all projections $\mathscr{J}$. In this lattice the meet of two clones is just their intersection while the join of two clones is the smallest clone containing both of them. The clone lattice is complete as arbitrary intersections of clones are again clones. Its compact elements are just the finitely generated clones and since every clone is the supremum of its finitely generated subclones the clone lattice is algebraic.
For an one element base set ($|A| = 1$) the clone lattice only consist of one element, but already for $|A| = 2$ the clone lattice becomes uncountable. For a finite base with three or more elements its cardinality is $2^{\aleph_0}$, i.e., the maximal cardinality possible. For an infinite base set the size of the clone lattice is $2^{2^{|A|}}$.
On a finite base set the clone lattice is atomic as well as dually atomic. While the dual atoms have been described in [Ros70] a list of the finitely many atoms of $\mathrm{Cl}(A)$ for $|A| \geq 3$ does not exist. On an infinite set the clone lattice is no longer atomic as can be seen through the following example.

Example 2.22. [GP08] Let $f \in \mathscr{S}$ be a permutation with only infinite cycles, that is for any iterate f^k (where $k \geq 1$) of f, $f^k(x) \neq x$ for all $x \in A$. The interval $[\mathscr{J}, \langle\{f\}\rangle]$ is isomorphic to the lattice of all submonoids of the monoid $(\mathbb{N}, +, 0)$. In particular it is not atomic.

Under the assumption $|A| = \kappa$ a regular cardinal and $2^\kappa = \kappa^+$ it has been shown by Goldstern and Shelah that $\mathrm{Cl}(A)$ is not dually atomic (for the exact references see [GP08]). If this statement can be proved outright for any infinite base set is however still an open problem.

As we will see we will mostly be interested in (s-)locally closed clones. We will refer to them as *(s-)local clones* and in contrast speak of *global clones* when we want to be sure to mean clones in general, i.e., not necessarily locally closed ones. When speaking of local clone we naturally always assume the base set to be infinite.
The following lemma shows that (s-)local clones are obtained simply through (s-)local closure of ordinary clones (and the same holds for submonoids and subgroups).

Lemma 2.23. [Pös80, Bör00] Let $\mathsf{F} \subseteq \mathscr{O}$, $\mathsf{H} \subseteq \mathscr{O}^{(1)}$, $\mathsf{G} \subseteq \mathscr{S}$ then the following holds:
1.)s-$\mathrm{Loc}\langle\mathsf{F}\rangle$ is a clone of operations.
2.) $\mathrm{Loc}\langle\mathsf{F}\rangle$ is a clone of operations.
3.) $\mathrm{Loc}\langle\mathsf{H}\rangle_{\mathscr{O}^{(1)}}$ is a submonoid of $\mathscr{O}^{(1)}$.
4.) $\mathrm{Loc}_o\langle\mathsf{G}\rangle_{\mathscr{S}}$ is a subgroup of $\mathscr{S}$.

As with global clones local clones can be ordered by inclusion forming a complete lattice $\mathrm{Cl}_{loc}(A)$, which however is not a sublattice of the clone lattice as the clone generated by two local clones needs not to be local as is illustrated by the following example.

Example 2.24. Let $A = \mathbb{Z}$, i.e., the set of integers and let $f \in \mathscr{S}$ be the permutation that exchanges 1 and 0 and is the identity else. Further take $g \in \mathscr{S}$ to be the permutation that maps every $x \in \mathbb{Z}$ to $x + 1$. We call the local clone generated by $\{f\}$, $\mathscr{C}$ and the local clone generated by $\{g, g^{-1}\}$ is called $\mathscr{D}$. Then the only nontrivial unary operation in $\mathscr{C}$ is f, and in

$\mathscr{D}$ the only nontrivial unary operations are g^k and g^{-k}, $k \geq 1$. As one can verify the join of $\mathscr{C}$ and $\mathscr{D}$ in the local clone lattice contains $\mathscr{S}$. In the global clone lattice however this cannot be the case since this join is countable whereas $|\mathscr{S}| = 2^{\aleph_0}$.

Also in contrast to the ordinary clone lattice over an infinite set $\mathrm{Cl}_{loc}(A)$ is not algebraic. The simple reason for this is that the local clone lattice over an infinite set only posses one compact element, the clone of all projections. The proof of this statement and more on the structure of the local clone lattice we again refer to [GP08] and references therein. Further we mention that the local clone lattice is not dually atomic [5] which can be seen through the following example.

Example 2.25. For each cardinal λ with $2 \leq \lambda \leq |A|$ define the set

$$\mathscr{K}_{<\lambda} = \langle \mathscr{O}^1 \rangle \cup \{f : |f[A^{n_f}]| < \lambda\}.$$

One can check that this set is always a clone. For a finite number n we will write $\mathscr{K}_n$ instead of $\mathscr{K}_{<n+1}$ and note that $\mathscr{K}_n$ is a local clone.
Further we will call $f \in \mathscr{O}^{(n)}$ *quasilinear* iff there exists functions $\phi_0 : 2 \to A$ and $\phi_1, \ldots, \phi_n : A \to 2$ such that $f(x_1, \ldots, x_n) = \phi_0(\phi_1(x_1) \dot{+} \ldots \dot{+} \phi_n(x_n))$ where $\dot{+}$ denotes the sum modulo 2. We write $\mathscr{B}$ for the (local) clone of all operations which are either essentially unary or quasilinear; $\mathscr{B}$ is often referred to as BURLEsclone.
Then the interval of nontrivial local clones which contain $\mathscr{O}^{(1)}$ is the following countably infinite chain which ascends to $\mathscr{O}$:

$$\langle \mathscr{O}^{(1)} \rangle \subsetneq \mathscr{B} \subsetneq \mathscr{K}_2 \subsetneq \mathscr{K}_3 \subsetneq \cdots \subsetneq \mathscr{O}$$

For finite $|A|$ the interval of clones above $\mathscr{O}^{(1)}$ is exactly this chain but stops at $\mathscr{K}_{|A|} = \mathscr{O}$.

This example supports the intuition that the local clone lattice is closer to clone lattice on a finite base set than to the global clone lattice on an infinite set.
This is further backed by the fact that the size of the local clone lattice on an infinite set A is $2^{|A|}$. That there can be no more local clones can be easily understood from the fact that a local clone is determined by all restrictions of its operations to finite subsets of A, for which there are only $2^{|A|}$ possibilities.

[5] A cofinal set in the local clone lattice has been given by Rosenberg and Szabo in [?].

Chapter 3

Characterization of the Galois closure

Before we start with the characterization we define the following relations.

Definition 3.1. ([Pös80, 1.8]) For $\mathsf{F} \subseteq \mathbf{E}$, $\mathsf{Q} \subseteq \mathbf{R}$ and $\sigma \subseteq A^m$ ($m \in \mathbb{N}$) we define

$$\begin{aligned} \Gamma^{\mathsf{Q}}(\sigma) &:= \bigcap\{\rho \in \mathscr{R}^{(m)} | \sigma \subseteq \rho \in \mathsf{Q}\}, \\ \Gamma_{\mathsf{F}}(\sigma) &:= \bigcap\{\rho \in \mathscr{R}^{(m)} | \sigma \subseteq \rho \in \mathrm{Inv}_{\mathbf{R}}\mathsf{F}\}. \end{aligned}$$

If $\mathsf{F} \subseteq \mathscr{S}$ then we also define

$$\Gamma^*_{\mathsf{F}}(\sigma) := \bigcap\{\rho \in \mathscr{R}^{(m)} | \sigma \subseteq \rho \in \mathrm{sInv}_{\mathbf{R}}\mathsf{F}\}.$$

For $\mathbf{a} \in A^m$ we will write $\Gamma^{\mathsf{Q}}(\mathbf{a})$ for $\Gamma^{\mathsf{Q}}(\{\mathbf{a}\})$ (and the same for Γ_{F}).

Remark 3.2. Note that obviously for $\mathsf{F} \subseteq \mathscr{O}, \mathsf{G} \subseteq \mathscr{S}$, $\Gamma_{\mathsf{F}}(\sigma) = \Gamma^{\mathrm{Inv}\mathsf{F}}(\sigma)$ and $\Gamma^*_{\mathsf{G}}(\sigma) = \Gamma^{\mathrm{sInv}\mathsf{G}}(\sigma)$ for $\sigma \subseteq A^m$ ($m \in \mathbb{N}$).

In the following lemma we collect some simple properties of $\Gamma^{\mathsf{Q}}(\sigma)$ and $\Gamma_{\mathsf{F}}(\sigma)$.

Lemma 3.3. ([Pös80, 1.8], [BGS, Lemma 2.4]) Let $\mathsf{F} \subseteq \mathbf{E}$, $\mathsf{Q} \subseteq \mathbf{R}$ and $\sigma \subseteq A^m$ ($m \in \mathbb{N}$). Then the following holds:

(i) If $\mathbf{R}$ is closed under arbitrary intersection then $\Gamma^{\mathsf{Q}}(\sigma)$ and $\Gamma_{\mathsf{F}}(\sigma)$ are elements of $\mathbf{R}$ and $\Gamma_{\mathsf{F}}(\sigma) \in \mathrm{Inv}_{\mathbf{R}}\mathsf{F}$. In addition for all $\rho \in \mathrm{Inv}_{\mathbf{R}}\mathsf{F}$

$$\Gamma_{\mathsf{F}}(\rho) = \rho = \bigcup_{\sigma \subseteq \rho,\, |\sigma| < \aleph_0} \Gamma_{\mathsf{F}}(\sigma),$$

holds and in particular

$$\Gamma_{\mathsf{F}}(\rho) = \rho \quad \text{iff} \quad \rho \in \mathrm{Inv}_{\mathscr{R}}\mathsf{F}.$$

(ii) If Q is closed under arbitrary intersections then $\Gamma^{\mathsf{Q}}(\sigma) \in \mathsf{Q}$ and $\Gamma^{\mathsf{Q}}(\sigma)$ is the smallest relation of Q containing σ. If in addition Q is closed under complementation the relations $\Gamma^{\mathsf{Q}}(\mathbf{a})$ with $\mathbf{a} \in A^m$ and fixed $m \in \mathbb{N}$ form a partition of A^m. We will write $\equiv_{\mathsf{Q}}$ for the equivalence relation belonging to this partition (using the same symbol for different m).

(*iii*) Let Q_1, Q_2 be two sets of relations closed under arbitrary intersections. If in addition $\mathsf{Q}_1, \mathsf{Q}_2$ are closed under LOC (s-LOC) then $\mathsf{Q}_1 = \mathsf{Q}_2$ iff $\Gamma^{\mathsf{Q}_1}(B) = \Gamma^{\mathsf{Q}_2}(B)$ for all $m \in \mathbb{N}$ and $B \subseteq A^m$,$|B| \leq \aleph_0$ ($|B| \leq s$).
In particular if $\mathsf{Q}_1, \mathsf{Q}_2$ are closed under arbitrary unions then $\mathsf{Q}_1 = \mathsf{Q}_2$ iff $\Gamma^{\mathsf{Q}_1}(\mathbf{a}) = \Gamma^{\mathsf{Q}_2}(\mathbf{a})$ for all $m \in \mathbb{N}$ and $\mathbf{a} \in A^m$.

The mostly simple proofs are left to the reader and can be found in the citations. We note that the first part of (*iii*) one might use proposition 2.15.

The following lemma gives an "inner characterization" of $\Gamma_{\mathsf{F}}(\sigma)$ in certain situations.

Lemma 3.4. ([PK79, 1.1.19], [Pös80]) Let $\mathbf{R} = \mathscr{R}$, $\sigma \in \mathscr{R}$, then for

1. $\mathsf{F} \subseteq \mathscr{O}$: $\quad \Gamma_{\mathsf{F}}(\sigma) = \{g(\mathbf{r}_1, \ldots, \mathbf{r}_n) | g \in \langle \mathsf{F} \rangle, \{\mathbf{r}_1, \ldots, \mathbf{r}_n\} \subseteq \sigma, n \in \mathbb{N}\}$.
2. $\mathsf{F} \subseteq \mathscr{O}^{(1)}$: $\quad \Gamma_{\mathsf{F}}(\sigma) = \{g(\mathbf{r}) | g \in \langle \mathsf{F} \rangle_{\mathrm{Tr}}, \mathbf{r} \in \sigma\}$.
3. $\mathsf{F} \subseteq \mathscr{S}$: $\Gamma^*_{\mathsf{F}}(\sigma) = \{g(\mathbf{r}) | g \in \langle \mathsf{F} \rangle_{\mathscr{S}}, \mathbf{r} \in \sigma\}$.

Proof. We prove 1. the rest can be proved in an analogous way. Denoting the r.h.s. by γ, i.e., $\gamma := \{g(\mathbf{r}_1, \ldots, \mathbf{r}_n) | g \in \langle \mathsf{F} \rangle, \{r_1, \ldots, \mathbf{r}_n\} \subseteq \sigma, n \in \mathbb{N}\}$, we notice $\gamma \in \mathrm{Inv}_{\mathscr{R}}\mathsf{F}$ and $\sigma \subseteq \gamma$ (as all projections are in $\langle \mathsf{F} \rangle$). By (**ii**), (**iii**) of lemma 3.3 we see that $\Gamma_{\mathsf{F}}(\sigma) \subseteq \gamma$.
For the other inclusion let $g \in \langle \mathsf{F} \rangle$,$\{\mathbf{r}_1, \ldots, \mathbf{r}_n\} \subseteq \sigma$, $n \in \mathbb{N}$, then we see that $\Gamma_{\mathsf{F}}(\sigma) \in \mathrm{Inv}_{\mathscr{R}}\mathsf{F}$ and $\sigma \subseteq \Gamma_{\mathsf{F}}(\sigma)$ imply that $g(\mathbf{r}_1, \ldots, \mathbf{r}_n) \in \Gamma_{\mathsf{F}}(\sigma)$ and all which proves $\gamma \subseteq \Gamma_{\mathsf{F}}(\sigma)$. $\square$

3.1 $\mathbf{Pol} - \mathbf{Inv}, \mathbf{Pol} - \mathbf{Pol}, \mathbf{InvPol}^{(s)}, \mathbf{PolInv}^{(s)}$

In this section we characterize the following cases:

(a) $\mathbf{E} = \mathscr{O}$ and $\mathbf{R} = \mathscr{R}$, i.e the set of all operations and the set of all relations. We write Pol for $\mathrm{Pol}_{\mathscr{O}}$ and Inv for $\mathrm{Inv}_{\mathscr{R}}$.

(b) $\mathbf{E} = \mathscr{O}^{(s)}$ and $\mathbf{R} = \mathscr{R}$, i.e all s-ary operations and all relations. We write $\mathrm{Pol}^{(s)}$ for $\mathrm{Pol}_{\mathscr{O}^{(s)}}$ and Inv for $\mathrm{Inv}_{\mathscr{R}}$.

(c) $\mathbf{E} = \mathscr{O}$ and $\mathbf{R} = \mathscr{R}^{(s)}$, i.e., all operations and s-ary relations. We write Pol for $\mathrm{Pol}_{\mathscr{O}}$ and $\mathrm{Inv}^{(s)}$ for $\mathrm{Inv}_{\mathscr{R}^{(s)}}$.

We will mostly follow [Pös80] however replacing "general superposition" by the "formula schemes" of [Sza78].

First we will describe the Galois closed sets of relations for these situations. As a motivation we may note that for two relations ρ_1, ρ_2 of the same arity n their intersection $\rho_1 \cap \rho_2$ as well as $\tilde{\rho} := \{(x_1, \ldots, x_{n-1}) | \exists x \rho_1(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)\}$ (for $n \geq 2$) and
$\tilde{\sigma} := \{(x_1, x_1, x_3, \ldots, x_n) | \rho_1(x_1, x_1, x_3, \ldots, x_n)\}$ etc. are all members of $\mathrm{InvPol}(\{\rho_1, \rho_2\})$ (indeed

this set is closed w.r.t Lop($\exists, \wedge, =$) as one can easily check). Even more for an arbitrary number of n-ary relations $(\rho_i)_{i\in I}$ the relations

$$\bigcap_{i\in I}\rho_i \quad , \quad \rho := \{(x_1,\dots,x_{n-1})|(\exists y_i)_{i\in I}\bigwedge_{i\in I}\rho_i(y_i,x_1,\dots,x_n)\}$$

are also members of $\mathrm{InvPol}(\{\rho_i|i \in I\})$ and can be thought to arise through certain logical operations of "infinite order". Indeed the relevant operations are just the formula schemes introduced in 2.8.

In addition the fact we only deal with functions of finite arity indicates that only "local" properties of relations should play a role. This will be reflected by the LOC/s-LOC closure of the Galois closed sets.

We formalize our above observations in the following lemmata which form the basis for the characterization of Galois closed sets of relations.
The first lemma shows that the closure operators $[.]_{\text{f.s.}}$ and s-LOC do not lead out of the Galois closure of a set of relations.

Lemma 3.5. (Pöschel [Pös80, 3.9]). Let $\mathsf{Q} \subseteq \mathscr{R}$, $n, s \in \mathbb{N}$. Then
(i) $\mathrm{Pol}^{(n)}\mathsf{Q} = \mathrm{Pol}^{(n)}[\mathsf{Q}]_{\text{f.s.}}$.
(ii) $\mathrm{Pol}^{(n)}\mathsf{Q} = \mathrm{Pol}^{(n)}s\text{-LOC}\mathsf{Q}$ for $n \leq s$.
Thus the closure operators $[.]_{\text{f.s.}}$ and s-LOC only add relations to Q that have at least the same n-ary ($n \leq s$) polymorphisms that Q has.

Proof. (i): Since $\mathsf{Q} \subseteq [\mathsf{Q}]_{\text{f.s.}}$ we immediately get $\mathrm{Pol}^{(n)}[\mathsf{Q}]_{\text{f.s.}} \subseteq \mathrm{Pol}^{(n)}\mathsf{Q}$.
The other inclusion $\mathrm{Pol}^{(n)}\mathsf{Q} \subseteq \mathrm{Pol}^{(n)}[\mathsf{Q}]_{\text{f.s.}}$ can easily be seen from the definition of $[.]_{\text{f.s.}}$.
(ii) : Let $\rho \in s\text{-LOC}\mathsf{Q}$ and $\mathbf{r}_1,\dots,\mathbf{r}_n \in \rho$. Then there exists a $\sigma \in \mathsf{Q}$ such that $\{\mathbf{r}_1,\dots,\mathbf{r}_n\} \subseteq \sigma \subseteq \rho$. So for $f \in \mathrm{Pol}^{(n)}\mathsf{Q}$ we find $f(\mathbf{r}_1,\dots,\mathbf{r}_n) \in \sigma \subseteq \rho$, i.e., f preserves ρ, which proves $f \in \mathrm{Pol}^{(n)}s\text{-LOC}\mathsf{Q}$. The other inclusion is again immediate. $\square$

Whereas LOC/s-LOC only add "big" relations (i.e., supersets of relations that are already in Q), $[.]_{\text{f.s.}}$ also adds subsets of the relations of Q. That indeed the "smallest" invariant relations are added is shown by the following lemma.

Lemma 3.6. (Pöschel [Pös80, 4.3], Szabo [Sza78, Lemma 2]). For $\mathsf{Q} \subseteq \mathscr{R}$ the following holds
(i) $\Gamma_{\mathrm{Pol}\mathsf{Q}}(B) \in [\mathsf{Q}]_{\text{f.s.}}$ for all finite $B \subseteq A^n$ ($n \in \mathbb{N}$).
(ii) $\Gamma_{\mathrm{Pol}^{(s)}\mathsf{Q}}(B) \in [\mathsf{Q}]_{\text{f.s.}}$ for all $B \subseteq A^n$ with $|B| \leq s$ ($s, n \in \mathbb{N}$).
If the set A is finite the formula schemes used to generate $[\mathsf{Q}]_{\text{f.s.}}$ can all be chosen to be finite.

Proof. Since $\Gamma_{\mathrm{Pol}\mathsf{Q}}(B) = \Gamma_{\mathrm{Pol}^{(s)}\mathsf{Q}}(B)$ for all B with $|B| \leq s$ (ii) is an immediate consequence of (i). We prove (i) following [Sza78]. Let $B = \{\mathbf{b}_i = (b_{1i},\dots,b_{ni}) \in A^n | i = 1,\dots,s\}$ be an arbitrary subset of A^n. Since $\mathrm{Pol}\mathsf{Q}$ is a clone by lemma 3.4 $\Gamma_{\mathrm{Pol}\mathsf{Q}}(B)$ can be written in the form $\Gamma_{\mathrm{Pol}\mathsf{Q}}(B) = \{f(\mathbf{b}_1,\dots,\mathbf{b}_s)|f \in \mathrm{Pol}^{(s)}Q\}$. We construct a formula scheme defining $\Gamma_{\mathrm{Pol}\mathsf{Q}}(B)$.
Choose a set of variables $X = \{x_i | i \in A^s\}$ indexed by A^s. Let $\rho \in \mathsf{Q}$ be an m-ary relation and denote by ρ^s the $m \times s$ matrices whose columns are elements of ρ. To each $M \in \rho^s$ associate the formula $\rho(x_{M_1},\dots,x_{M_m})$ where M_i denotes the i-th row of M. A tuple $(a_i | i \in A^s) \in A^{A^s}$ fulfilling these formulas for all $M \in \rho^s$ is equivalent to a function $f : A^s \to A, f(i) = a_i$,

fulfilling $\rho(f(M_1),\ldots,f(M_m))$ [1] for all $M\in\rho^s$, i.e., $f\in\mathrm{Pol}\rho$. Also every $f\in\mathrm{Pol}^{(s)}\rho$ gives rise to the tuple $(f(i)|i\in A^s)$ fulfilling all the above formulas. So we find that $f\in\mathrm{Pol}^{(s)}\mathsf{Q}$ iff $(f(i)|i\in A^s)$ satisfies $\Sigma:=\{\rho(x_{M_1},\ldots,x_{M_m})|\rho\in\mathsf{Q}^{(m)},m\in\mathbb{N}$ and $M\in\rho^s\}$.
Choosing $i_k:=(b_{k1},\ldots,b_{ks})$ for $k=1,\ldots,n$, we define the formula scheme $\Psi:=(\Sigma,X,(x_{i_1},\ldots,x_{i_s}))$ and obtain

$$\begin{aligned}R_\Psi &= \{(a_{i_1},\ldots,a_{i_n})|(a_i|i\in A^s)\text{ satisfies }\Sigma\} =\\ &= \{(f(i_1),\ldots,f(i_k))|f\in\mathscr{O}^s\ ,\ (f(i)|i\in A^s)\text{ satisfies }\Sigma\} =\\ &= \{f(\mathbf{b}_1,\ldots,\mathbf{b}_s)|f\in\mathrm{Pol}^{(s)}\mathsf{Q}\} =\\ &= \Gamma_{\mathrm{Pol}\mathsf{Q}}(B).\end{aligned}$$

For finite A we note that for every s-ary operation f that does not satisfy the formula scheme Σ there exists a formula $\varphi_f\in\Sigma$ such that f does not satisfy φ_f. Define Σ' to be the set of all such formulas, i.e., $\Sigma':=\{\varphi_f|f\in\mathscr{O}^s$ and f does not satisfy $\Sigma\}$ and $\Psi':=(\Sigma',X,\{x_{i_1},\ldots,x_{i_n}\})$ the corresponding formula scheme (arising from Ψ by replacing Σ with Σ'). Then clearly $R_\Psi=R_{\Psi'}$. Since $|X|=|A^s|$ and $|\Sigma'|\leq|\mathscr{O}^s|=|A^{A^s}|$ it follows that X and Σ' are finite, hence Ψ' is a finite formula scheme. □

Now we are ready to give the characterization for Galois closed sets of relations.

Theorem 3.7. (Pöschel [Pös80, 4.2], Szabo [Sza78, Theorem 7] Let $\mathsf{Q}\subseteq\mathscr{R}$. Then the following holds:
(*i*) $\mathrm{InvPol}\mathsf{Q}=\mathrm{LOC}[\mathsf{Q}]_{\text{f.s.}}$, i.e, a set of relations is Galois closed w.r.t. Pol − Inv iff it contains all unions of directed systems of relations defined by formula schemes over Q. [LOC: 2.10; $[.]_{\text{f.s.}}$: 2.8.1]
(*ii*) $\mathrm{InvPol}^{(s)}\mathsf{Q}=s\text{-LOC}[\mathsf{Q}]_{\text{f.s.}}$, i.e., a set of relations is Galois closed w.r.t. $\mathrm{Pol}^{(s)}-\mathrm{Inv}$ iff it contains all unions of s-directed systems of relations defined by a formula scheme over Q.[s-LOC: 2.10; $[.]_{\text{f.s.}}$: 2.8.1]

Proof. Obviously it is sufficient to prove (*ii*). Making use of lemma 3.5 we find $s\text{-LOC}[\mathsf{Q}]_{\text{f.s.}}\subseteq\mathrm{InvPol}^{(s)}\ s\text{-LOC}[\mathsf{Q}]_{\text{f.s.}}=\mathrm{InvPol}^{(s)}\mathsf{Q}$ proving one inclusion.
For the other inclusion let $\rho\in\mathrm{Inv}^{(m)}\mathrm{Pol}^{(s)}\mathsf{Q}$ ($m\in\mathbb{N}$). Then ρ is the union of the s-directed system $\mathscr{T}:=\{\Gamma_{\mathrm{Pol}^{(s)}\mathsf{Q}}(B)|B\subset\rho,|B|\leq s\}$. However by lemma 3.6 $\Gamma_{\mathrm{Pol}^{(s)}\mathsf{Q}}(B)\in[\mathsf{Q}]_{\text{f.s.}}$ and so $\rho\in s\text{-LOC}[\mathsf{Q}]_{\text{f.s.}}$ □

For $\mathsf{Q}\subseteq\mathscr{R}^{(r)}$ we see that $\mathrm{Inv}^{(r)}\mathrm{Pol}^{(s)}\mathsf{Q}=s\text{-LOC}[\mathsf{Q}]_{\text{f.s.}}\cap\mathscr{R}^{(r)}$. In the case of $r=1$ we get the following well known result proved by Birkhoff and O. Frink [BF48], see also [Grä68, Chapter 1 § 9 Theorem 1 and 2].

Corollary 3.8. (Szabo [Sza78, Corollary 11]) Let $\mathsf{Q}\subseteq\mathscr{R}^{(1)}$. Then $\mathrm{Inv}^{(1)}\mathrm{Pol}\mathsf{Q}=\mathsf{Q}$ iff Q is an algebraic closure system. In particular $\mathrm{Inv}^{(1)}\mathrm{Pol}\mathsf{Q}=\mathrm{LOC}\langle\mathsf{Q}\rangle_\bigcap$. Specializing to $s-$ary operations only we find $\mathrm{Inv}^{(1)}\mathrm{Pol}^{(s)}\mathsf{Q}=s\text{-LOC}\langle\mathsf{Q}\rangle_\bigcap$. [algebraic closure. system: 2.7.2; LOC: 2.10; $\langle.\rangle_\bigcap$: 2.7.1]

Proof. The proof is immediate from theorem 3.7 if we note that for $\subseteq\mathscr{R}^{(1)}$, $[\mathsf{Q}]_{f.s.}\cap\mathscr{R}^{(1)}$ is just the closure of Q w.r.t arbitrary intersection. □

For the case of a finite or countable base set we get a slightly simpler characterization from theorem 3.7 as follows.

[1](where $f(M_i)$ is short for $f(M_{1i},\ldots,M_{si})$)

3.1.1 $|A| < \aleph_0$

For finite base set we obtain the following corollary from 3.7 and the comments made after 2.8.

Corollary 3.9. Let $\mathsf{Q} \subseteq \mathscr{R}$. Then

$$\begin{array}{rl} (i) & \mathrm{InvPol}\mathsf{Q} = \mathrm{LOP}(\exists, \wedge, =)(\mathsf{Q}), \\ (ii) & \mathrm{InvPol}^{(s)}\mathsf{Q} = s\text{-}\mathrm{LOC}\langle \mathsf{Q} \rangle_{\mathrm{LOP}(\exists,\wedge,=)}, \end{array}$$

i.e., the Galois closure w.r.t InvPol corresponds to closure w.r.t to primitive positive formulas, for InvPol$^{(s)}$ addition s-LOC-closure must be added. [LOP(.): 2.3.1; s-LOC: 2.10]

3.1.2 $|A| = \aleph_0$

For countable base set we find the following.

Theorem 3.10. Let $\mathsf{Q} \subseteq \mathscr{R}$. Then

$$\begin{array}{rl} (i) & \mathrm{InvPol}\mathsf{Q} = \mathrm{LOC}\langle \mathsf{Q} \rangle_{\mathrm{LOP}(\exists,\wedge,=)} . \\ (ii) & \mathrm{InvPol}^{(s)}\mathsf{Q} = s\text{-}\mathrm{LOC}\langle \mathsf{Q} \rangle_{\mathrm{LOP}(\exists,\wedge,=)} . \end{array}$$

i.e the Galois closure corresponds to closure w.r.t. unions of (s-)directed systems of relations from the primitive positive closure of Q. [LOP(.): 2.3.1; s-LOC, LOC: 2.10]

Proof. The corollary can be proved similar to theorem 3.22. We sketch the proof for (ii).
The inclusion $s\text{-}\mathrm{LOC}\langle \mathsf{Q} \rangle_{\mathrm{LOP}(\exists,\wedge,=)} \subseteq \mathrm{InvPol}^{(s)}\mathsf{Q}$ is immediate from theorem 3.7 .
For the other inclusion let $s\text{-}\mathrm{LOC}\langle \mathsf{Q} \rangle_{\mathrm{LOP}(\exists,\wedge,=)} = \mathsf{Q}$ and let $B = \{\mathbf{a}_1, \ldots, \mathbf{a}_s\} \subseteq A^n$. Then for $\mathbf{b} \in \Gamma^{\mathsf{Q}}(B)$ the following holds:

1. Let $i, j < n$ and $a_{1i} = a_{1j}, \ldots, a_{si} = a_{sj}$ then $b_i = b_j$. This is a consequence of $\delta_n^\epsilon \in \mathsf{Q}$, with $\epsilon = \{(i, j), (j, i)\} \cup \{(k, k) | k \in n\}$.

2. For all $\mathbf{d} \in A^s$ there exists an $e \in A$ such that $(\mathbf{b}, e) \in \Gamma^{\mathsf{Q}}(\{(\mathbf{a}_1, d_1), \ldots, (\mathbf{a}_s, d_s)\})$. This is a consequence of Q being closed w.r.t. $\mathbf{Pr}^{(n)}$.

3. Let $s : m \to n$ then $\mathbf{b} \circ s \in \Gamma^{\mathsf{Q}}(\{\mathbf{a}_1 \circ s, \ldots, \mathbf{a}_s \circ\})$. This is a consequence of Q being closed w.r.t. $\mathbf{W}_s$.

Making use of $(1) - (3)$ we inductively construct an $f \in \mathrm{Pol}^{(s)}\mathsf{Q}$ s.t. $f(\mathbf{a}_1, \ldots, \mathbf{a}_s) = \mathbf{b}$.
Define f_0 to be the partial s-ary function with $\mathrm{dom}(f_0) = \{\mathbf{a}_1, \ldots, \mathbf{a}_s\} = \{(a_{11}, \ldots, a_{s1}), (a_{12}, \ldots, a_{s2}), \ldots, (a_{1n}, \ldots, a_{sn})\} =: \{\mathbf{a}_1^{(0)}, \ldots, \mathbf{a}_s^{(0)}\}$ and $f(\mathbf{a}_1, \ldots, \mathbf{a}_s) = \mathbf{b} =: \mathbf{b}^{(0)}$; f_0 is well defined by (1).
In the induction step choose $\mathbf{d} \in A^s \backslash \mathrm{dom} f_k$. By (2) we find an $e \in A$ s.t. $\mathbf{b}^{(k+1)} := (\mathbf{b}^{(k)}, e) \in \Gamma^{\mathsf{Q}}(\{(\mathbf{a}_1^{(k)}, d_1), \ldots, (\mathbf{a}_s^{(k)}, d_s)\})$ and define $\mathbf{a}_i^{(k+1)} := (\mathbf{a}_i^{(k)}, d_i)$ for $i = 1, \ldots, s$.
We define $f := \bigcup_{k \in \mathbb{N}} f_k$ with $\mathrm{dom} f = A^s$. Property (3) can be used to show that f fulfills

$$f[B'] \in \Gamma^{\mathsf{Q}}(B') \text{ for all } B' \subseteq A^m,\ m \in \mathbb{N}\,, |B'| \leq s.$$

This implies that $f \in \mathrm{Pol}^{(s)}\mathsf{Q}$. From this we get $\mathbf{b} = f(\mathbf{a}_1, \ldots, \mathbf{a}_s) \in \Gamma^{\mathrm{InvPol}^{(s)}\mathsf{Q}}(B)$ and thereby $\Gamma^{\mathrm{InvPol}^{(s)}\mathsf{Q}}(B) \subseteq \Gamma^{Q}(B)$. As the other inclusion always holds true we get $\Gamma^{\mathrm{InvPol}^{(s)}\mathsf{Q}}(B) = \Gamma^{Q}(B)$ for all $B \subseteq A^n, |B| \leq s$, which by Lemma 3.3 (iii) $\mathrm{InvPol}^{(s)}\mathsf{Q} = \mathsf{Q}$ and finishes the proof. □

Now we turn to the characterization of the Galois closed sets of functions which is somewhat simpler.
First we may note that for any set of relations $\mathsf{Q} \subseteq \mathscr{R}$ the set of its polymorphisms $\mathrm{Pol}\mathsf{Q}$ is always closed w.r.t to functional composition and includes all projections, i.e., is a clone.
As for relations the fact that we only work with relations of finite arity leads to a "local"-closure of the set $\mathrm{Pol}\mathsf{Q}$ which will be given by the operators s-Loc/Loc introduced in definition 2.16.
The following lemma shows that these closure operators do not lead out of the Galois closure of a set of functions.

Lemma 3.11. ([Pös80]) Let $\mathsf{F} \subseteq \mathscr{O}$, $n, s \in \mathbb{N}$. Then the following holds:
(i) $\mathrm{Inv}^{(n)}\mathsf{F} = \mathrm{Inv}^{(n)}\langle\mathsf{F}\rangle$.
(ii) $\mathrm{Inv}^{(n)}\mathsf{F} = \mathrm{Inv}^{(n)}s\text{-Loc}\mathsf{F}$ for $1 \leq n \leq s$.

Proof. (i) is immediate from the definitions.
(ii): Let $\rho \in \mathrm{Inv}^{(n)}\mathsf{F}$. Then for any $f \in s\text{-Loc}\mathsf{F}$ and $r_1, \ldots, r_n \in \rho$ there exists a $g \in \mathsf{F}$ such that $f(r_1, \ldots, r_n) = g(r_1, \ldots, r_n) \in \rho$ proving $\rho \in \mathrm{Inv}^{(n)}s\text{-Loc}\mathsf{F}$. So $\mathrm{Inv}^{(n)}\mathsf{F} \subseteq \mathrm{Inv}^{(n)}s\text{-Loc}\mathsf{F}$ and as the other inclusion is immediate (ii) is proved. □

The characterization of the Galois closed sets of functions is given by the following theorem.

Theorem 3.12. (Pöschel [Pös80, 4.2]) Let $\mathsf{F} \subseteq \mathscr{O}$. Then the following holds:
(i) $\mathrm{Loc}\langle\mathsf{F}\rangle = \mathrm{PolInv}\mathsf{F}$.
(ii) $s\text{-Loc}\langle\mathsf{F}\rangle = \mathrm{PolInv}^{(s)}\mathsf{F}$.
$\big[\langle . \rangle$: 2.20; s-Loc, Loc: 2.16 $\big]$

Proof. We start by proving (ii).
By lemma 3.11 we have $s\text{-Loc}\langle\mathsf{F}\rangle \subseteq \mathrm{PolInv}(s\text{-Loc}\langle\mathsf{F}\rangle) \subseteq \mathrm{PolInv}^{(s)}(s\text{-Loc}\langle\mathsf{F}\rangle) = \mathrm{PolInv}^{(s)}\mathsf{F}$.
To show the opposite inclusion let $f \in \mathrm{Pol}^{(n)}\mathrm{Inv}^{(s)}\mathsf{F}$. We will show that $f \in s\text{-Loc}\langle\mathsf{F}\rangle$. Let $B = \{\mathbf{b}_0, \ldots, \mathbf{b}_{t-1}\} \subseteq A^n$, $t \leq s$. Define $\mathbf{r}_i := (b_0(i), \ldots, b_{t-1}(i))$, $i \in n$ and $\sigma := \{\mathbf{r}_i | i \in n\}$. Since $\Gamma_{\mathsf{F}}(\sigma) \in \mathrm{Inv}^{(t)}\mathsf{F}$ and $f \in \mathrm{PolInv}^{(s)}\mathsf{F} \subseteq \mathrm{PolInv}^{(t)}\mathsf{F}$, $f(\mathbf{r}_0, \ldots, \mathbf{r}_{n-1}) \in \Gamma_{\mathsf{F}}(\sigma)$. By lemma 3.4 (1) then there is a $g \in \langle\mathsf{F}\rangle$ s.t. $f(\mathbf{r}_0, \ldots, \mathbf{r}_{n-1}) = g(\mathbf{r}_0, \ldots, \mathbf{r}_{n-1})$ and so $f \restriction B = g \restriction B$ i.e $f \in s\text{-Loc}\langle\mathsf{F}\rangle$.
Now (i) follows from $\mathrm{Loc}\langle\mathsf{F}\rangle = \bigcap_{s\in\mathbb{N}} s\text{-Loc}\langle\mathsf{F}\rangle = \bigcap_{s\in\mathbb{N}} \mathrm{PolInv}^{(s)}\mathsf{F} = \mathrm{Pol}\bigcup_{s\in\mathbb{N}} \mathrm{Inv}^{(s)}\mathsf{F} = \mathrm{PolInv}\mathsf{F}$.
□

We give some examples of Galois closed sets of relations in this setting.

Example 3.13. 1.) The simplest examples of Galois closed sets of functions/relations are the set of all projections $\mathscr{J}$, the set of all operations $\mathscr{O}$,the set of all diagonal relations D_A (together with the empty set) and the set of all relations $\mathscr{R}$,which are related in the following way:

$$\mathrm{Inv}\mathscr{J} = \mathscr{R} \quad , \qquad \mathrm{Pol}\mathscr{R} = \mathscr{J} ,$$
$$\mathrm{Inv}\mathscr{O} = D_A \cup \{\emptyset\} \quad , \qquad \mathrm{Pol}D_A = \mathscr{O} .$$

2.) [PK79, 2.2.2.] Consider a single unary relation $\rho \in \mathscr{R}^{(1)}$. Then it is easy to see that all relations of the form $\rho_1 \times \cdots \times \rho_n$ where $\rho_i \in \{\rho, A\}$, $i = 1, \ldots, n$ are in the InvPol$\{\rho\}$. Adding all diagonal relations and closing under intersections we obtain the complete Galois closure.
Pol$\{\rho\}$ consists of all functions that when restricted to ρ take values only in ρ.

3.) [Ros74, PK79, 2.2.4] For $|A| \geq 3$ we define the relations $\iota_i \in \mathscr{R}^{(i)}$ $(i = 2, \ldots, |A|)$ to be the union of all diagonal relations of arity i, that is

$$\iota_i := \{(a_1, \ldots, a_i) \in A^i | |\{a_1, \ldots, a_i\}| < i\}.$$

Then $\mathrm{Pol}\{\iota_i\} = \mathscr{K}_{i-1}$.
The description of the sets $\mathrm{Inv}^{(j)}\mathscr{K}_{i-1}$ $(j \in \mathbb{N})$ is given in [Ros74]. We first introduce the following notion.
For $G \subseteq \mathrm{Eq}(j)$ let $[G)$ be the least filter on $\mathrm{Eq}(j)$ containing G. Further for $k \leq j$ define $G^k := \{\epsilon \in G | \epsilon$ has at most k equivalence classes$\}$ and $\{G\}^k := [G)^k \setminus G^k$.

Theorem 3.14. [Ros74, Theorem 1] A relation $\rho \in \mathscr{R}^{(j)}$ is invariant under $\mathscr{K}_{i-1}$ iff $\rho = \bigcup\{\delta_j^\epsilon | \epsilon \in G\}$, where $G \subseteq \mathrm{Eq}(j)$ satisfies $\{G\}^{i-1} = \emptyset$.

Note that the theorem implies that a ρ is an elemnet of $\in \mathrm{Inv}^{(j)}\mathscr{K}_{i-1}$ iff it is of the form $\rho = \bigcup\{\delta_j^\epsilon | \epsilon \in G\}$ for some $G \subseteq \mathrm{Eq}(j)$ (this is clear from $\mathscr{O}^{(1)} \subseteq \mathscr{K}_{i-1}$) and further for $\epsilon_1, \ldots, \epsilon_n \in G$, $\epsilon_0 \in \mathrm{Eq}(j)^{i-1}$ with $\epsilon_0 \supseteq \bigcap_{i=1}^n \epsilon_i$

$$\delta_j^{\epsilon_0} \subset \rho.$$

4.) For an ideal I of subsets of A (i.e. a downset of the powerset of A that is closed under finite unions) the set $\mathscr{C}_I$ of all functions f s.t. $f[B^{n_f}] \in I$ for all $B \in I$ is a clone.
Except for the set A (seen as a unary relation), $\mathrm{Inv}^{(1)}\mathscr{C}_I$ contains only the relation $\bigcup I$. This is the case as the functions of $\mathscr{C}_I$ are not restricted in any way outside $\bigcup I$,i.e., for $\rho \in \mathscr{R}^{(1)}$ with $\rho \setminus \bigcup I \neq \emptyset$, $\mathscr{C}_I[\rho] = A$. Also any function which takes a constant value $y \in \bigcup I$ is an element of $\mathscr{C}_I$ (since $\{y\} \in I$), so a relation $\tilde{\rho} \subsetneqq \bigcup I$ cannot be in $\mathrm{Inv}^{(1)}\mathscr{C}_I$.
So the unary invariant relations are of the same type as in example 1.)which leads to the the same result for $\mathrm{Inv}\mathscr{C}_I$. Note that $\mathscr{C}_I$ is in general not a local clone so $\mathrm{PolInv}\mathscr{C}_I = \mathrm{Loc}\mathscr{C}_I \supset \mathscr{C}_I$ in general.

5.) For a filter D on A (i.e a nonempty subset of the power set of A which is upward close and closed under finite intersections , we allow also the improper filter A) we define

$$\mathscr{C}_D := \{f | (\exists B \in D)(\forall x \in B) f(x, \ldots, x) = x\}.$$

Then $\mathscr{C}_D$ is a clone. For special choices of D one may try to specify $\mathrm{Inv}\mathscr{C}_D$.

6.) Let $\mathscr{I}$ be the clone of idempotent functions, i.e., all functions $f \in \mathscr{O}^{(n)}$ $(n \in \mathbb{N})$ that fulfill $f(x, \ldots, x) = x$ for all $x \in A$. Then

$$\mathrm{Inv}^{(1)}\mathscr{I} = \{\rho | |\rho| = 1\} \cup A,$$

which is clear since all these relations are certainly invariant and any other relations cannot be invariant since elements of $\mathscr{I}$ are unrestricted off the diagonal. $\mathrm{Inv}^{(n)}\mathscr{I}$ consists of relations ρ of the form $\rho = \rho_1 \times \cdots \times \rho_n$ with $\rho_i \in \mathrm{Inv}^{(1)}\mathscr{I}$ since (as $\mathrm{Inv}\mathscr{I}$ is closed under $\mathbf{Pr}^{(m)}$) each line must be an invariant relation.

3.2 End – Inv, wAut – Inv, Aut – sInv

In this section we will study the Galois connection between unary operations and relations. More precisely we study the following situation.

(a) $\mathbf{E} = \mathscr{O}^{(1)}$ and $\mathbf{R} = \mathscr{R}$, i.e., general unary operations and arbitrary relations. The polymorphisms are called endomorphisms and we write End for $\mathrm{Pol}_{\mathscr{O}^{(1)}}$.

(b) $\mathbf{E} = \mathscr{S}$ and $\mathbf{R} = \mathscr{R}$, i.e., bijective unary operations and arbitrary relations. The polymorphisms are called weak automorphisms and we write wAut for $\mathrm{Pol}_{\mathscr{S}}$.

(c) $\mathbf{E} = \mathscr{S}$ and $\mathbf{R} = \mathscr{R}$, i.e., the same as in (**b**) however in this case we deal with strong invariance and strong preservation. The polymorpishms are called automorphisms and we write Aut for $\mathrm{sPol}_{\mathscr{S}}$ and sInv for $\mathrm{sInv}_{\mathscr{R}}$.

We start by characterizing the Galois closed sets of relations. We will make use of some of the properties of the characterization of Inv – Pol and treat the case of finite, countable and uncountable base set separate from each other. As we will see the three different cases (a), (b), (c) are quite similar to each other and following [Bör00] we treat their characterization parallel to each other. For the case of an uncountable base set in subsection 3.2.5 we will follow [BGS] to give an improved characterization (compared to the characterization given in theorem 3.26) of the closure operator sInvAut.
In the following lemma we collect some simple properties of the the three closure operators.

Lemma 3.15. Let $\mathsf{G} \subseteq \mathscr{S}$ and $\mathsf{Q} \subseteq \mathscr{R}$. Then the following holds:
(i) If G contains for every function also its inverse then $\mathrm{Inv}\mathsf{G} = \mathrm{sInv}\mathsf{G}$. In particular $\mathrm{sInvAut}\mathsf{Q} = \mathrm{InvAut}\mathsf{Q}$ always holds.
(ii) $\mathrm{AutsInv}\mathsf{G} = \mathrm{wAutsInv}\mathsf{G}$.
(iii) $\mathrm{InvEnd}\mathsf{Q} \subseteq \mathrm{InvwAut}\mathsf{Q} \subseteq \mathrm{sInvAut}\mathsf{Q}$.

Proof. (i): This an immediate consequence of remark 1.2
(ii): This is immediate from the definition.
(iii): As $\mathrm{wAut}\mathsf{Q} \subseteq \mathrm{End}\mathsf{Q}$ the first inclusion is immediate. Using (i) the second inclusion follows from $\mathrm{InvwAut}\mathsf{Q} \subseteq \mathrm{InvAut}\mathsf{Q} = \mathrm{sInvAut}\mathsf{Q}$. □

3.2.1 $|A| < \aleph_0$

For a finite base set (b) and (c) are identical as every weak automorphism is always an automorphism and every relation that is preserved by an automorphism is strongly preserved by that automorphism. The characterizations of (a) and (b,c) can be found in [PK79] and makes use of our knowledge of the characterization of InvPol.
As $\mathrm{InvPol}\mathsf{Q} \subseteq \mathrm{InvEnd}\mathsf{Q}$ we can simply add new operations to the ones used to characterize InvPol. Indeed when we specialize to unary functions the union of two invariant relations of the same arity is again invariant and so we can safely add this operation to the ones already used. When studying automorphisms and strong invariant relations we see that $\rho \in \mathrm{sInvAut}\mathsf{Q}$ implies that $\rho^C \in \mathrm{sInvAut}\mathsf{Q}$ so in this case we will also want to add $(.)^C$ (which corresponds to

the the logical operation $\neg$) as an operation. These observations already suffice to characterize the Galois closed sets of relations in the finite case as we will show.
The following lemma shows that the transition from general polymorphisms to unary and to unary injective ones can be characterized by certain relations.

Lemma 3.16. (Pöschel, Kalužnin [PK79, Lemma 1.3.1])(This lemma holds for arbitrary $|A|$.) Define the two diagonal relations $d_1 := \{(a,a,b,c)|a,b,c \in A\}$ and $d_2 := \{(a,b,c,c)|a,b,c \in A\}$ and their union $\pi = d_1 \cup d_2$. Then $\mathrm{Pol}\,\pi = \langle \mathscr{O}^{(1)} \rangle$.
Further let $\nu := \{(x,y) \in A^2 | x \neq y\}$ be the inequality relation. Then $\mathrm{Pol}\{\nu, \pi\} = \langle \mathscr{O}^{(1-1)} \rangle$. For $3 \leq |A| < \aleph_0$ even $\mathrm{Pol}\,\nu = \langle \mathscr{O}^{(1-1)} \rangle$ holds.

Proof. Since diagonal relations are invariant for any function and since for any set of unary functions $\mathsf{F} \subseteq \mathscr{O}^{(1)}$, $\mathrm{Inv}\mathsf{F}$ is closed under the union of relations we see $\mathscr{O}^{(1)} \subseteq \mathrm{Pol}\,\pi$ and so also $\langle \mathscr{O}^{(1)} \rangle \subseteq \mathrm{Pol}\,\pi$. On the other hand when $f \in \mathrm{Pol}^{(n)}\pi$ $(n > 1)$ is not essentially unary, i.e., it depends on two different indices i, k we can find n-tuples $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$ differing only in the i-th component and $\mathbf{a}' = (a'_1, \ldots, a'_n)$, $\mathbf{b}' = (b'_1, \ldots, b'_n)$ differing only in the k- th component so that $f(a_1, \ldots, a_n) \neq f(b_1, \ldots, b_n)$ and $f(a'_1, \ldots, a'_n) \neq f(b'_1, \ldots, b'_n)$. From this it follows that $f\begin{pmatrix} a_1, \ldots, a_n \\ b_1, \ldots, b_n \\ a'_1, \ldots, a'_n \\ b'_1, \ldots, b'_n \end{pmatrix} \notin \pi$ but since each row of $(\mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}')$ is an element of π this implies $f \notin \mathrm{Pol}\,\pi$, a contradiction that leads to $\mathrm{Pol}\,\pi \subseteq \mathscr{O}^{(1)}$.
The proof of the first part of the second statement is immediate. The proof of the second part uses the following fact (called Jablonskij main lemma [PK79, 1.1.6.]).

Let $3 \leq |A| < \aleph_0$ and let $f \in \mathscr{O}^{(n)} \setminus \langle \mathscr{O}^{(1)} \rangle$ be an operation on A that takes $l > 2$ values. Then there are sets $K_i \subseteq A$, $|K_i| < l$ $(i = 1, \ldots, n)$ such that f takes all its l values on $K_1 \times \cdots \times K_n$.

Now let $f \in \mathrm{Pol}^{(n)}\,\nu$ then f takes $k := |A|$ different values on the n-tuples $(c, \ldots, c) \in A^n$. If $f \notin \langle \mathscr{O}^{(1)} \rangle$ then by Jablonskij main lemma there exists a n-tuple $\mathbf{c} := (c_1, \ldots, c_n)$ s.t. f takes all of its k-values on $M := \{(a_1, \ldots, a_n) | a_i \neq c_i, i = 1, \ldots, n\}$. This however implies that $f\mathbf{a} = f\mathbf{c}$ for some $\mathbf{a} \in M$ which contradicts $f \in \mathrm{Pol}\,\nu$ (since $(a_i, c_i) \in \nu$ for $i = 1, \ldots, n$). $\square$

For finite base set the characterization of sets of relations closed w.r.t InvEnd /InvAut is now given by the following theorem.

Theorem 3.17. (Pöschel, Kaluznin [PK79, 1.3.5.]) Let $\mathsf{Q} \subseteq \mathscr{R}$, $|A| < \aleph_0$. Then
(i) $\mathrm{InvEnd}\mathsf{Q} = \mathrm{LOP}(\exists, \wedge, \vee, =)(\mathsf{Q})$, i.e., the Galois closed sets are just the sets of relations closed w.r.t. logical operations arising from primitive positive formula with union.[LOP(.): 2.3.1]

(ii) $\mathrm{InvAut}\mathsf{Q} = \mathrm{LOP}(\mathsf{Q})$, i.e., the Galois closed sets are just the sets of relations closed w.r.t. all logical operations.[LOP(.): 2.3.1]

Proof. "$\supseteq$": This follows immediately from $\mathrm{InvPol}\mathsf{Q} \subseteq \mathrm{InvEnd}\mathsf{Q}$ and the fact that $\mathrm{InvEnd}\mathsf{Q}$ is closed under unions.
"$\subseteq$": From

$$\mathrm{InvPol}\mathsf{Q} = \mathrm{LOP}(\exists, \wedge, =)(\mathsf{Q}) \subseteq \mathrm{LOP}(\exists, \wedge, \vee, =)(\mathsf{Q})$$

we see that LOP$(\exists, \wedge, \vee, =)(\mathsf{Q})$ contains all diagonal relations over A. In particular we find that the union $\pi = d_1 \cup d_2$ of lemma 3.16 is in LOP$(\exists, \wedge, \vee, =)(\mathsf{Q})$. So
InvEnd$\mathsf{Q} \subseteq$ InvEnd(LOP$(\exists, \wedge, \vee, =)(\mathsf{Q})$) = InvPol(LOP$(\exists, \wedge, \vee, =)(\mathsf{Q})$) $\subseteq$ LOP$(\exists, \wedge, \vee, =)(\mathsf{Q})$.

The second part of the theorem can be proved in a similar manner, see [PK79]. □

3.2.2 $|A| = \aleph_0$

When going from finite base set to countable infinite base set we should be aware of two things. First the cases (b) and (c) become two separate cases. In particular we note that for a set F of injective unary functions Inv$\mathsf{F}$ is closed w.r.t. to logical operations including $\neq$ (in addition to $\exists, \wedge, \vee, =$) while likewise for a set F of surjective unary functions Inv$\mathsf{F}$ is always closed under logical operations including $\forall$. To see that this not true if F consists of non surjective functions choose, e.g., $A = \{1, 2, 3\}$ and define $f : A \to A, f(1) = 2, f(2) = 1, f(3) = 2$. Then $\rho := \{(1, 2, 1), (1, 2, 2)(1, 2, 3), (2, 1, 1), (2, 1, 2)\}$ is invariant w.r.t f however $\sigma := \{(x, y) | \forall z\, (x, y, z) \in \rho\} = \{(1, 2)\}$ is not.
Secondly we notice that arbitrary (infinite) unions and intersections of invariant relations are again invariant and so we may need to add these operations. That this is indeed necessary is illustrated by example 3.21.

As a first step we will characterize the algebraic part of InvEnd, InvwAut and sInvAut as was done in [Bör00]. We will then use this knowledge to show that these operators are not algebraic.
Indeed we will see that the algebraic parts $(\mathrm{InvEnd})^{\mathrm{alg}}, (\mathrm{InvwAut})^{\mathrm{alg}}, (\mathrm{sInvAut})^{\mathrm{alg}}$ are given by the closure w.r.t. the clones of invariant operations defined in 2.4.1. The characterization is based on the following two lemmata.

Lemma 3.18. Let $S : \mathscr{R}^{(m_1)} \times \cdots \times \mathscr{R}^{(m_k)} \to \mathscr{R}^{(m_0)}$ be an operation and let $\rho_i \in \mathscr{R}^{(m_i)}$ (i=1,...,k). Then the following implications hold:
(i) $S \in \mathbf{MVOP} \Rightarrow S(\rho_1, \dots, \rho_k) \in \mathrm{InvEnd}\{\rho_1, \dots, \rho_k\}$,
(ii) $S \in \mathbf{MIOP} \Rightarrow S(\rho_1, \dots, \rho_k) \in \mathrm{InvwAut}\{\rho_1, \dots, \rho_k\}$,
(iii) $S \in \mathbf{IOP} \Rightarrow S(\rho_1, \dots, \rho_k) \in \mathrm{sInvAut}\{\rho_1, \dots, \rho_k\}$,
i.e., the Galois closure w.r.t. InvEnd/InvwAut/sInvAut of a finite set $\{\rho_1, \dots, \rho_n\}$ of relations is closed under all operations from **MVOP**/**MIOP**/**IOP**.

Proof. The statements follow easily from the definitions. □

Lemma 3.19. For $i = 1, \dots, k$ let $m_i \in \mathbb{N}$ and $\rho_i \in \mathscr{R}^{(m_i)}$. Then the following holds:
(i) For any $\rho_0 \in \mathrm{InvEnd}\{\rho_1, \dots, \rho_k\}$, there exists an operation $S \in \mathbf{MVOP}_A$ such that $\rho_0 = S(\rho_1, \dots, \rho_k)$.
(ii) For any $\rho_0 \in \mathrm{InvwAut}\{\rho_1, \dots, \rho_k\}$, there exists an operation $S \in \mathbf{MIOP}_A$ such that $\rho_0 = S(\rho_1, \dots, \rho_k)$.
(iii) For any $\rho_0 \in \mathrm{sInvAut}\{\rho_1, \dots, \rho_k\}$, there exists an operation $S \in \mathbf{IOP}_A$ such that $\rho_0 = S(\rho_1, \dots, \rho_k)$.

Proof. *(i)*: We explicitly define the operation S through

$$S(\sigma_1, \dots, \sigma_k) := \bigcup \{f[\rho_0] | f \in \mathscr{O}^{(1)} \text{ and } (\forall i = 1, \dots, k)\ f[\rho_i] \subseteq \sigma_i\}^{\cdot}$$

Then $S(\rho_1, \ldots, \rho_k) = \rho_0$. $S \in \mathbf{MVOP}_A$ follows through direct calculation.

For the other two cases the slightly changed definition of S is given below.
(ii):
$$S(\sigma_1, \ldots, \sigma_k) := \bigcup \{g[\rho_0] | g \in \mathscr{S} \text{ and } (\forall i = 1, \ldots, k)\ g[\rho_i] \subseteq \sigma_i\}$$
(iii):
$$S(\sigma_1, \ldots, \sigma_k) := \bigcup \{g[\rho_0] | g \in \mathscr{S} \text{ and } (\forall i = 1, \ldots, k)\ g[\rho_i] = \sigma_i\}$$
□

The following theorem is now an immediate consequence.

Theorem 3.20. The algebraic parts of the closure operators InvEnd, InvwAut, sInvAut are given by:
(i) $(\mathrm{InvEnd})^{\mathrm{alg}} = \mathbf{MVOP}_A(.)$.
(ii) $(\mathrm{InvwAut})^{\mathrm{alg}} = \mathbf{MIOP}_A(.)$.
(ii) $(\mathrm{sInvAut})^{\mathrm{alg}} = \mathbf{IOP}_A(.)$.
$\big[$ $\mathbf{MVOP}_A(.)$: 2.4.3; $\mathbf{MIOP}_A(.)$: 2.4.2; $\mathbf{IOP}_A(.)$: 2.4.1 $\big]$

Proof. The proof is immediate from lemma 3.18 and 3.19 . □

Making use of this explicit form of the algebraic parts of the closure operators the following example shows that InvEnd/InvwAut/sInvAut are not algebraic (see [Bör00, 2.2.7]), i.e., the characterization as given in theorem 3.20 is not complete.

Example 3.21. Let A be an infinite set and define Q to be the set of all finite or cofinite subsets of A, i.e.,
$$\mathsf{Q} := \{\rho \in \mathrm{Rel}^{(1)}(A) | \rho \text{ is finite or cofinite}\}.$$
As Q contains all one element sets we see that $\mathrm{End}\mathsf{Q} = \mathrm{wAut}\mathsf{Q} = \mathrm{Aut}\mathsf{Q} = \{\mathrm{id}\}$ and so $\mathrm{InvEnd}\mathsf{Q} = \mathrm{InvwAut}\mathsf{Q} = \mathrm{sInvAut}\mathsf{Q} = \mathrm{Rel}(A)$. In particular we find $\mathrm{Inv}^{(1)}\mathrm{End}\mathsf{Q} = \mathrm{Inv}^{(1)}\mathrm{wAut}\mathsf{Q} = \mathrm{sInv}^{(1)}\mathrm{Aut}\mathsf{Q} = \mathscr{P}A$.
Further we notice that Q is closed w.r.t. all Boolean operations and Q is also closed w.r.t. $(\mathrm{sInvAut})^{\mathrm{alg}}$. To see the last statement let $\{\rho_1, \ldots, \rho_k\} \subseteq \mathsf{Q}$. The set $\mathrm{Aut}\{\rho_1, \ldots, \rho_k\}$ consists of all permutations that strongly preserve all the (at most 2^k many) atoms of the boolean algebra generated by $\rho_1, \ldots, \rho_k$. The relations strongly preserved by all those permutations are then exactly all relations obtained as (finite) unions of the atoms and so are again in Q.
All in all we find $((\mathrm{sInvAut})^{\mathrm{alg}}\mathsf{Q})^{(1)} = \mathsf{Q} \neq \mathscr{P}A = \mathrm{sInv}^{(1)}\mathrm{Aut}\mathsf{Q}$ showing that sInvAut is not algebraic. As InvEnd and InvwAut are "weaker" than sInvAut (see lemma 3.15) we find $((\mathrm{InvEnd})^{\mathrm{alg}}\mathsf{Q})^{(1)} = \mathsf{Q} \neq \mathscr{P}A = \mathrm{Inv}^{(1)}\mathrm{End}\mathsf{Q}$ and $((\mathrm{InvwAut})^{\mathrm{alg}}\mathsf{Q})^{(1)} = \mathsf{Q} \neq \mathscr{P}A = \mathrm{Inv}^{(1)}\mathrm{wAut}\mathsf{Q}$, i.e., InvEnd and InvwAut are not algebraic either.

Indeed the non algebraic operations that we need are arbitrary unions and intersections as is proven in the following theorem.

Theorem 3.22. (Börner [Bör00, 2.4.4]) Let $\mathsf{Q} \subseteq \mathscr{R}$, $|A| = \aleph_0$. Then the following holds:
(i) Q is closed w.r.t. sInvAut iff Q is closed w.r.t. LOP and is Δ-complete.
(ii) Q is closed wr.t. InvwAut iff Q is closed w.r.t. $\mathrm{LOP}(\exists, \forall, \wedge, \vee, =, \neq)$ and is Δ-complete.

(iii) Q is closed w.r.t InvEnd iff Q is closed wr.t. LOP$(\exists, \wedge, \vee, =)$ and is Δ-complete.
[LOP(.): 2.3.1; Δ-complete: 2.7.3]

Proof. (i): Suppose that sInvAut$\mathsf{Q} = \mathsf{Q}$ then from theorem 3.17 it follows that Q is closed w.r.t. LOP and as we already noted is also Δ-complete.
For the other implication let Q be closed w.r.t. to LOP and be Δ-complete. Then for $n \in \mathbb{N}^+$ fixed, $\Gamma_{\mathsf{Q}}(\mathbf{a}) \in \mathsf{Q}$ for all $\mathbf{a} \in A^n$. As sInvAut$\mathsf{Q} \supseteq \mathsf{Q}$ we have $\Gamma_{\text{sInvAut}\mathsf{Q}}(\mathbf{a}) \subseteq \Gamma_{\mathsf{Q}}(\mathbf{a})$.
The opposite inclusion $\Gamma_{\text{sInvAut}\mathsf{Q}}(\mathbf{a}) \supseteq \Gamma_{\mathsf{Q}}(\mathbf{a})$ follows from the fact that for every $\mathbf{b} \in \Gamma_{\mathsf{Q}}(\mathbf{a})$ we can find a $g \in$ Aut$\mathsf{Q}$ so that $g(\mathbf{a}) = \mathbf{b}$, as we will show below.
Altogether we obtain $\Gamma_{\text{sInvAut}\mathsf{Q}}(\mathbf{a}) = \Gamma_{\mathsf{Q}}(\mathbf{a})$ for all $\mathbf{a} \in A^n$, $n \in \mathbb{N}^+$, which implies sInvAut$\mathsf{Q} = \mathsf{Q}$ by lemma 3.4.

For the construction of the function g we start by noticing that for a $\mathbf{b} \in \Gamma_{\mathsf{Q}}(\mathbf{a})$ the following properties are a consequence of the LOP-closure of Q:
(1) for all $i, j < n$, $a_i = a_j$ iff $b_i = b_j$, as all diagonal relations and complements thereof are in Q.
(2) for all $c \in A$ there exits a $d \in A$ such that $(\mathbf{b}, d) \in \Gamma_{\mathsf{Q}}((\mathbf{a}, c))$, as Q is closed w.r.t to all projection $\mathbf{Pr}^{(n)}$.
(3) for all $d \in A$ there exists $c \in A$ such that $(\mathbf{b}, d) \in \Gamma_{\mathsf{Q}}((\mathbf{a}, c))$, as Q is closed w.r.t. all dual projections $\mathbf{Q}^{(r)}$.
(4) for $s : m \to n$, $\mathbf{b} \circ s \in \Gamma_{\mathsf{Q}}(\mathbf{a} \circ s)$, as Q is closed w.r.t to $\mathbf{W}_s$.
The construction of the function $g \in$ Aut$\mathsf{Q}$ with $g(\mathbf{a}) = \mathbf{b}$ can now be performed through a back and forth argument and can be found in detail in [Bör00]. We state the most important steps.
We construct a series of partial functions g_k on A such that $g = \bigcup_{k \in \mathbb{N}} g_k$ and

- $g_k \subset g_{k+1}$, $\forall k \in \mathbb{N}$;
- for all $\{d'_0, \dots, d'_{h-1}\} \subseteq$dom$g_k$, $g_k(\mathbf{d}') \in \Gamma_{\mathsf{Q}}(\mathbf{d}')$.

To start with we choose some fixed well ordering of the base set A and define g_0 through

$$g_0 : \{a_0, \dots, a_{n-1}\} \to \{b_0, \dots, b_{n-1}\} , \quad g(\mathbf{a}) = \mathbf{b},$$

which is well defined by (1). For the induction step assume that we have already constructed g_k with dom$g_k = \{d_0, \dots, d_{m-1}\}$ and $g_k(d_i) = e_i$, $(i = 1, \dots, m-1)$. Let $d_m =$min$(A \backslash$dom$g_k)$. By our assumption $\mathbf{e} \in \Gamma_{\mathsf{Q}}(\mathbf{d})$ and by (2) we find an element $e_m \in A$ such that $(\mathbf{e}, e_m) \in \Gamma_{\mathsf{Q}}(\mathbf{d}, d_m)$. Thus we define $g_{k+1} : \{d_0, \dots, d_m\} \to \{e_0, \dots, e_m\}$, $g(d_i) = e_i$ $(i = 0, \dots, m)$. In the next step we choose $e_{m+1} =$min$(A \backslash$rang$_{k+1})$ and by (3) find a $d_{m+1} \in A$ such that $(\mathbf{e}, e_m, e_{m+1}) \in \Gamma_{\mathsf{Q}}((\mathbf{d}, d_m, d_{m+1}))$. Defining $g_{m+2} : \{d_0, \dots, d_{m+1}\} \to \{e_0, \dots, e_{m+1}\}$, $g_{m+2}(d_i) = e_i$, $(i = 1, \dots, m+1)$, we have completed the induction step.
The so constructed function $g := \bigcup_{k \in \mathbb{N}} g_k$ is obviously a bijection of A and indeed also preserves all $\Gamma_{\mathsf{Q}}(\mathbf{d})$ $(\mathbf{d} \in A^n)$ (which can be seen with the help of (4)), i.e., it is an element of Aut$\mathsf{Q}$.

The proof of $(ii), (iii)$ can be performed in a similar manner with only slight changes. For details see [Bör00]. □

With the help of this theorem we can easily prove the following lemma.

Lemma 3.23. [Bör00] Let A be countable and let Q be a countable Krasneralgebra (Pre-Krasneralgebra/weak Krasneralgebra). Then Q is closed w.r.t. sInvAut (InvwAut/InvEnd) iff $\mathsf{Q}^{(n)}$ is finite for all $n \in \mathbb{N}$.

Proof. If all $\mathsf{Q}^{(n)}$ are finite then Q is trivially Δ-complete and by theorem 3.22 Galois closed. On the other hand suppose Q is Galois closed and let $n \in \mathbb{N}$. Then Q is Δ-complete and $\Gamma^{\mathsf{Q}}(\mathbf{a}) \in \mathsf{Q}$ for all $\mathbf{a} \in A^n$. If the set $\{\Gamma(\mathbf{a})|\mathbf{a} \in \mathsf{Q}^{(n)}\}$ is finite for a fixed n then $\mathsf{Q}^{(n)}$ is also finite by lemma 3.3. When however this set is infinite then $\mathsf{Q}^{(n)}$ already has cardinality $2^{\aleph_0}$ contradicting Q being countable. □

3.2.3 $|A| > \aleph_0$

We now consider an uncountable base set A. We will start by giving examples that show that our characterization so far is not complete. One may already guess that from the fact that the back and forth construction used in the proof of theorem 3.22 breaks down in the uncountable case. Also we may notice that our characterization so far has mainly used logical operations of finite arity (together with the closure under arbitrary intersections and unions). However with logical operations it is not possible to distinguish between sets of different infinite cardinalities which also hints to the fact that our characterization is not complete.
We give the following examples from [Bör00] and [BGS].

Example 3.24. [Bör00] Let A be an uncountable set, $C \subset A$ a countable subset and $B = A\backslash C$. Let $\chi : A \to \{0, 1\}$ denote the characteristic function of C (that is $\chi(a) = 1$ if $a \in C$ and 0 else) and for $\mathbf{a} \in A^n$ define $\chi(\mathbf{a}) := (\chi(a_0), \dots, \chi(a_{n-1}))$. For each $\mathbf{a} \in A^n$ we define the following relation $\zeta(\mathbf{a})$

$$\zeta(\mathbf{a}) := \{\mathbf{b}|\chi(\mathbf{b}) = \chi(\mathbf{a}) \text{ or } \chi(\mathbf{b}) = 1 - \chi(\mathbf{a}) \text{ and } (\forall i, j < n)(a_i = a_j \Leftrightarrow b_i = b_j)\}.$$

As is easily seen the $\zeta(\mathbf{a})$ ($\mathbf{a} \in A^n$) form a partition of A^n. We now define our set $\mathsf{Q} \subseteq \mathscr{R}$ to be

$$\mathsf{Q}^{(n)} := \{\bigcup_{\mathbf{a}\in\sigma} \zeta(\mathbf{a})|\sigma \subseteq A^n\} \ , \ \mathsf{Q} := \bigcup \mathsf{Q}^{(n)}.$$

It is immediate clear that Q is a Δ-complete Boolean System and that $\Gamma^{\mathsf{Q}}(\mathbf{a}) = \zeta(\mathbf{a})$. In particular note that $\zeta(a, a) = \delta_A^{(2)}$ and that $\mathsf{Q}^{(n)}$ is a finite set for all $n \in \mathbb{N}$, as there are only finitely many $\zeta(\mathbf{a})$ for $\mathbf{a} \in A^n$.
Without too much work one finds ([Bör00]) that Q is also closed w.r.t. all logical operations, i.e., it is a Krasneralgebra[2].
We note that $\mathsf{Q}^{(1)} = \{\emptyset, A\}$ and $(B \times C \cup C \times B) \in \mathsf{Q}^{(2)}$. The second statement implies

$$\mathrm{Aut}\mathsf{Q} = \{g \in \mathscr{S}|\, g[B] = B \text{ and } g[C] = C\}.$$

from which follows

$$\mathrm{sInv}^{(1)}\mathrm{Aut}\mathsf{Q} = \{\emptyset, B, C, A\} \neq \mathsf{Q}^{(1)},$$

which shows that Q is not closed w.r.t. sInvAut.

[2] For a base set A of at most countable cardinality and sets B and C ($A = B \cup C$) of unequal cardinality this no longer holds true.

Example 3.25. [BGS] More generally consider the following countable structures:
(i) $(\mathbb{Q}, <)$ (the rational numbers with the linear order).
(ii) The full countable bipartite graph:$(A \cup B; \rho)$ where A and B are disjoint countable sets and $\rho := (A \times B) \cup (B \times A)$.
(iii) The countable random graph.(See e.g. [Hod97], 6.4.4)

Each of these structures $(\underline{M}, \rho)$ has the following properties:
(a) $Th(\underline{M})$, the first order theory of $\underline{M}$ is ω-categorical.
(b) All unary first order formulas $\varphi(x)$ are equivalent (mod $Th(\underline{M})$) to $x = x$ or to $x \neq x$, i.e. the only subsets of $\underline{M}$ that are first order definable are the empty set and the whole model.
(c) For any uncountable cardinal κ there is a model $\underline{M}_\kappa$ of cardinality κ such that the set

$$\rho^* := \{x : \text{ The set } \{y : \rho(x, y)\} \text{ is countable}\}$$

is neither empty nor the full model.
For each of these models let Q be the set of first order definable relations. It is a Δ-complete Kranser algebra, since by Ryll-Nardzewskis theorem, for any k there are only finitely many k-ary relations in Q.
In each model $\underline{M}_\kappa$ however the set ρ^* is a (higher order) definable subset of M_κ, hence $\rho^* \in \mathrm{sInvAut}(\mathsf{Q}) \setminus \mathsf{Q}$.
This shows that Q is not Galois closed.

These examples show that we will have to add new operations to the closure operators used so far. In [Bör00] the question was raised if it might be sufficient to add the set of invariant operations for the characterization of sInvAut as the structure in example 3.24 is not closed under all invariant operations (as follows from the finiteness of $\mathsf{Q}^{(n)}$ and theorem 3.20). In [BGS] it was shown that this is not sufficient, i.e., there are sets of relations closed w.r.t. all invariant operations and Δ-complete which are not closed under sInvAut (we will take a look at the construction of this counterexample in subsection 3.2.5).

For InvEnd the additional operations needed for the characterization in the case of an uncountable base set can be taken directly from our characterization of InvPol, i.e. we will use the operations obtained through formula schemes however in the form 2.9.3 as we follow [Bör00]. For the characterization of InvwAut and sInvAut we will add the operations sSup given in 2.9.1, which can be seen as being the natural replacements for gSup when we limit ourselves to bijective functions.

With the help of these additional operations the characterization of the Galois closed sets of relations is straightforward.

Theorem 3.26. (Börner [Bör00, Theorem 2.5.4]) Let $\mathsf{Q} \subseteq \mathscr{R}$ then the following holds:
(i) Q is closed w.r.t. sInvAut iff Q is a Δ-complete Krasneralgebra closed w.r.t. strong superposition. [Δ-complete: 2.7.3; Kranseralgebra: 2.3.2; strong superposition: 2.9.1]
(ii) Q closed w.r.t InvwAut iff Q is a Δ-complete Pre-Krasneralgebra closed w.r.t. strong superposition. [Δ-complete: 2.7.3; Pre - Kranseralgebra: 2.3.2; strong superposition: 2.9.1]
(iii) Q is closed w.r.t InvEnd iff Q is a Δ-complete weak Krasneralgebra closed w.r.t. general superposition. [Δ-complete: 2.7.3; weak Kranseralgebra: 2.3.2; general superposition: 2.9.3]

Proof. (i): Suppose $\mathrm{sInvAut}\mathsf{Q} = \mathsf{Q}$ then it easily seen that Q fulfills the closure properties stated.

For the other implication suppose Q is a Δ-complete Krasneralgebra closed w.r.t. strong superposition. We prove that for each $\mathbf{a} \in A^m$ $(m \in \mathbb{N})$ $\Gamma^{\mathrm{sInvAut}\mathsf{Q}}(\mathbf{a}) = \Gamma^{\mathsf{Q}}(\mathbf{a})$.
For this let I be some index set with cardinality $|A|$. We choose a family $(\mathbf{b}_i)_{i\in I}$ so that for every $n \in \mathbb{N}^+$ every element of A^n is listed once and further let ρ_i be $\Gamma^{\mathsf{Q}}(\mathbf{b}_i)$ for all $i \in I$. The strong superposition $\sigma \in \mathsf{Q}^{(m)}$ resulting from this choice of "parameters" can be written as

$$\begin{aligned}\sigma := \mathrm{sSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) &= \{g\mathbf{a} | g \in \mathscr{S} \text{ and } (\forall n \in \mathbb{N}^+)(\forall \mathbf{b} \in A^n)\ g\mathbf{b} \in \Gamma^{\mathsf{Q}}(\mathbf{b})\} \\ &= \{g\mathbf{a} | g \in \mathrm{Aut}\mathsf{Q}\}.\end{aligned}$$

The second line is a direct consequence of the equality $\rho = \bigcup_{\mathbf{b}\in\rho} \Gamma^{\mathsf{Q}}(\mathbf{b})$ (for $\rho \in \mathsf{Q}$) and the closure of Q under $\mathbf{C}$. It implies that $\sigma = \Gamma^{\mathrm{sInvAut}\mathsf{Q}}(\mathbf{a})$. Since $\mathrm{id} \in \mathrm{Aut}\mathsf{Q}$ we have $\mathbf{a} \in \sigma$ which implies $\Gamma^{\mathsf{Q}}(\mathbf{a}) \subseteq \sigma = \Gamma^{\mathrm{sInvAut}\mathsf{Q}}(\mathbf{a})$. As the opposite inclusion always holds we find that (for arbitrary $\mathbf{a}$) $\Gamma^{\mathsf{Q}}(\mathbf{a}) = \Gamma^{\mathrm{sInvAut}\mathsf{Q}}(\mathbf{a})$ which proves $\mathsf{Q} = \mathrm{sInvAut}\mathsf{Q}$.
The proof of $(ii), (iii)$ can be done in a similar manner. For details see [Bör00]. □

The results of theorem 3.29 might seem a bit dissatisfying as the operations used for the characterization are of arity $|A|$. We will give an improved result for sInvAut in subsection 3.2.5 below. First however we will use the theorem to treat some related characterizations.
We start by characterizing sets of relations closed w.r.t. Invinj-End, Invsur-End and sets of relations closed w.r.t Invinj-End such that their injective endomorphisms form a locally invertible set of functions. For Invinj-End we recall that $\mathrm{End}\nu = \{f \in \mathscr{O}^{(1)} | f \text{ is injective}\}$. For Invsur-End we obtain a characterization for the infinite case by replacing the operation sSup by spSup.

Theorem 3.27. (Pöschel [Pös84, Theorem 3.3]) Let $\mathsf{Q} \subseteq \mathscr{R}$. Then
(i) $\mathsf{Q} = \text{Invinj-End}\mathsf{Q}$ iff $\nu := \{(x, y) \in A^2 | x \neq y\} \in \mathsf{Q}$ and $\mathsf{Q} = \mathrm{InvEnd}\mathsf{Q}$.
(ii) $\mathsf{Q} = \text{Invinj-End}\mathsf{Q}$ and inj-End$\mathsf{Q}$ is locally invertible iff $\mathsf{Q} = \mathrm{InvEnd}\mathsf{Q}$ and Q is closed w.r.t. complementation. In particular for a base set A that is at most countable this equivalent to Q being closed w.r.t. sInvAut. [complementation: 2.2.1]
(iii) $\mathsf{Q} = \text{Invsur-End}\mathsf{Q}$ iff $\mathsf{Q} = \mathrm{InvEnd}\mathsf{Q}$ and closed w.r.t. spSup. [spSup: 2.9.2]

Proof. (i) follow easily from our comments above and the definitions.
(ii) The case of a finite base set is trivial. For countable base set the equivalence to sInvAut is a consequence of the positive answer to question (I) in section 3.2.4 for the countable case. (iii) Note that similar to the proof of 3.26 (where we expressed $\Gamma^{\mathrm{sInvAut}\mathsf{Q}}(\mathbf{a})$ as a strong superposition) we can write $\Gamma^{\text{Invsur-End}\mathsf{Q}}(\mathbf{a})$ as a special superposition for arbitrary $\mathbf{a} \in A^n$, $n \in \mathbb{N}$. This guarantees that the closure w.r.t. spSup and InvEnd is sufficient. Observing that Invsur-End$\mathsf{Q}$ is always closed under spSup finishes the proof. □

The characterization of InvEnd can easily be specialized to unary relations, i.e., choose $\mathsf{Q} \subseteq \mathscr{R}^{(1)}$ and replace by $\mathrm{Inv}^{(1)}$.

Corollary 3.28. Let $\mathsf{Q} \subseteq \mathscr{R}^{(1)}$. Then $\mathsf{Q} = \mathrm{Inv}^{(1)}\mathrm{End}\mathsf{Q}$ iff Q is Δ-complete. [Δ-complete: 2.7.3]

Finally we also give the characterization for the Galois closed sets of functions.

Theorem 3.29. (Pöschel [Pös84, 3.4], [Pös80])
(i) Let $\mathsf{F} \subseteq \mathscr{O}^{(1)}$. Then $\mathsf{F} = \mathrm{EndInv}\mathsf{F}$ iff F is a locally closed submonoid of Tr, i.e., $\mathsf{F} = \mathrm{Loc}\langle\mathsf{F}\rangle_{\mathrm{Tr}}$. In particular $\mathrm{EndInv}\mathsf{F} = \mathrm{Loc}\langle\mathsf{F}\rangle_{\mathrm{Tr}}$ and $\mathrm{EndInv}^{(s)}\mathsf{F} = s\text{-}\mathrm{Loc}\langle\mathsf{F}\rangle_{\mathrm{Tr}}$ holds.
(ii) Let $\mathsf{F} \subseteq \mathscr{S}$. Then $\mathsf{F} = \mathrm{wAutInv}\mathsf{F}$ iff F is a Loc_o-closed submonoid of $\mathscr{S}$, i.e., $\mathsf{F} = \mathrm{Loc}_o\langle\mathsf{F}\rangle_{\mathrm{Tr}}$. In particular $\mathrm{wAutInv}\mathsf{F} = \mathrm{Loc}_o\langle\mathsf{F}\rangle_{\mathrm{Tr}}$ and $\mathrm{wAutInv}^{(s)}\mathsf{F} = s\text{-}\mathrm{Loc}_o\langle\mathsf{F}\rangle_{\mathrm{Tr}}$
(iii) Let $\mathsf{F} \subseteq \mathscr{S}$. Then $\mathsf{F} = \mathrm{AutsInv}\mathsf{F}$ iff F is a Loc_o-closed subgroup of $\mathscr{S}$, i.e., $\mathsf{F} = \mathrm{Loc}_o\langle\mathsf{F}\rangle_{\mathscr{S}}$. In particular $\mathrm{AutsInv}\mathsf{F} = \mathrm{Loc}_o\langle\mathsf{F}\rangle_{\mathscr{S}}$ and $\mathrm{AutsInv}^{(s)}\mathsf{F} = s\text{-}\mathrm{Loc}_o\langle\mathsf{F}\rangle_{\mathscr{S}}$.
(iv) Let $\mathsf{F} \subseteq \mathscr{O}^{(1-1)}$. Then $\mathsf{F} = \text{inj-}\mathrm{EndInv}\mathsf{F}$ iff F is a locally closed submonoid of Tr, i.e., $\mathsf{F} = \mathrm{Loc}\langle\mathsf{F}\rangle_{\mathrm{Tr}}$. In particular $\text{inj-}\mathrm{EndInv}\mathsf{F} = \mathrm{Loc}\langle\mathsf{F}\rangle_{\mathrm{Tr}}$, for $s \geq 2$, $\text{inj-}\mathrm{EndInv}^{(s)}\mathsf{F} = s\text{-}\mathrm{Loc}\langle\mathsf{F}\rangle_{\mathrm{Tr}}$ and $\text{inj-}\mathrm{EndInv}^{(1)}\mathsf{F} = \mathscr{O}^{(1-1)} \cap 1\text{-}\mathrm{Loc}\langle\mathsf{F}\rangle_{\mathrm{Tr}}$.
(v) Let $\mathsf{F} \subseteq \mathscr{O}^{(1-1)}$ be a locally invertible set of functions. Then $\mathsf{F} = \text{inj-}\mathrm{EndInv}\mathsf{F}$ iff F is a Loc closed submonoid of Tr, i.e., $\mathsf{F} = \mathrm{Loc}\langle\mathsf{F}\rangle_{\mathrm{Tr}}$.
(vi) Let $\mathsf{F} \subseteq \mathscr{O}^{(1)}$ be a set of surjective functions. Then $\mathsf{F} = \text{sur-}\mathrm{EndInv}\mathsf{F}$ iff $\mathsf{F} = \mathrm{Loc}\langle\mathsf{F}\rangle_{\mathrm{Tr}} \cap \mathscr{O}^{(surj.)}$. In particular $\text{sur-}\mathrm{EndInv}\mathsf{F} = \mathrm{Loc}\langle\mathsf{F}\rangle_{\mathrm{Tr}} \cap \mathscr{O}^{(surj.)}$ and $\text{sur-}\mathrm{EndInv}^{(s)}\mathsf{F} = s\text{-}\mathrm{Loc}\langle\mathsf{F}\rangle_{\mathrm{Tr}} \cap \mathscr{O}^{(surj.)}$

Proof. The proof of (i)-(iii) is analogous to theorem 3.12 and is left to the reader. $(iv) - (vi)$ follow almost directly from (i). $\square$

3.2.4 sSup≠gSup ?

With the help of our characterization so far we can easily see that closure w.r.t to sSup, spSup and gSup in general do not coincide. As an example let $A = \mathbb{Z}$. Then first choose $\mathsf{Q} = \{\{i\} | i \in \mathbb{Z} \setminus 0\}$. Taking $I = \mathbb{Z} \setminus 0$ and $\mathbf{a} = 0$, $\mathbf{b}_i = i$, $\rho_i = \{i\}$, $i \in I$ we find

$$\begin{aligned}\{0\} = \mathrm{spSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) &\neq \\ \mathrm{gSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) &= \mathbb{Z}.\end{aligned}$$

Indeed one easily sees $\mathrm{Inv}^{(1)}\text{sur-}\mathrm{End}\mathsf{Q} = \langle\{\{i\}|i \in \mathbb{Z}\}\rangle_{\cup} \cup \{\mathbb{Z}\}$ and $\mathrm{Inv}^{(1)}\mathrm{End}\mathsf{Q} = \langle\{\{i\}|i \in \mathbb{Z} \setminus \{0\}\rangle_{\cup} \cup \{\mathbb{Z}\}$.
Now we choose $\mathsf{Q} = \{\{i\}|i \in 2\mathbb{Z}\}$. Then $\{i|i \in 2\mathbb{Z}+1\} \in \mathrm{sInv}^{(1)}\mathrm{Aut}\mathsf{Q}$ but $\{i|i \in 2\mathbb{Z}+1\} \notin \mathrm{Inv}^{(1)}\text{sur-}\mathrm{End}\mathsf{Q}$. Indeed $\mathrm{sSup}(1, (i)_{i\in 2\mathbb{Z}}, (\{i\})_{i\in 2\mathbb{Z}}) = 2\mathbb{Z}+1$ and
$\mathrm{spSup}(1, (i)_{i\in 2\mathbb{Z}}, (\{i\})_{i\in 2\mathbb{Z}}) = \mathbb{Z}$.

The question if a Δ-closed Krasner algebra that is closed w.r.t. gSup is also closed w.r.t. sSup was raised in [Pös84] and in slightly different manner in [Bör00].
This question can be rephrased in different ways. To do this we will start with some lemmata.

Lemma 3.30. (Börner [Bör00, Satz 2.6.2]) Let $\mathsf{Q} \subseteq \mathscr{R}$. Then the following are equivalent.:
(i) There exists a locally invertible monoid $\mathsf{F} \subseteq \mathrm{Tr}$ s.t. $\mathsf{Q} = \mathrm{Inv}\mathsf{F}$.
(ii) Q is a Δ-complete Krasneralgebra closed w.r.t gSup.

Proof. $(i) \Rightarrow (ii)$: Let F be a locally invertible monoid and $\mathsf{Q} = \mathrm{Inv}\mathsf{F}$. Then by 3.26 (iii) Q is a Δ-complete weak Krasner algebra closed under gSup. Thus for (ii) it is sufficient to show that Q is closed w.r.t $\mathbf{C}$.
Let $\rho \in Q$ and $\mathbf{a} \in \mathbf{C}\rho$. For every function $f \in \mathsf{F}$ there exists a $g \in \mathsf{F}$ such that $g \circ f(\mathbf{a}) = \mathbf{a}$. This implies $f(\mathbf{a}) \in \mathbf{C}\rho$ and shows $\mathbf{C}\rho \in \mathrm{Inv}\mathsf{F} = \mathsf{Q}$. Thus Q is closed under $\mathbf{C}$, i.e., is a Krasner

algebra.
$(ii) \Rightarrow (i)$: By 3.26(iii) $\mathsf{Q} = \mathrm{InvEnd}\mathsf{Q}$ so we only have to show that $\mathsf{F} := \mathrm{Inv}\mathsf{Q}$ is locally invertible. Let $f \in \mathsf{F}$ and $\mathbf{a} \in A^n$ ($n \in \mathbb{N}$) then we need to show that there exists a $g \in \mathrm{Inv}\mathsf{Q}$ s.t $g(f(\mathbf{a})) = \mathbf{a}$. Suppose there exists no such g. Then there is a $\sigma \in \mathsf{Q}$ with $f(\mathbf{a}) \in \sigma$ but $\mathbf{a} \in \mathbf{C}\sigma$. Since Q is closed w.r.t to $\mathbf{C}$ this contradicts the fact that $f \in \mathrm{Inv}\mathsf{Q}$. □

Lemma 3.31. (Börner [Bör00, Lemma 2.6.3]) Let $\mathsf{F} \subseteq \mathrm{Tr}$ be a locally invertible monoid. Then the following are equivalent.
(i) There is a permutationgroup $\mathsf{G} \subseteq \mathscr{S}$ s.t. $\mathrm{Inv}\mathsf{F} = \mathrm{sInv}\mathsf{G}$.
(ii) $\mathsf{F} \subseteq \mathrm{Loc}(\mathscr{S} \cap \mathrm{Loc}\mathsf{F})$ holds.
(iii) $\mathrm{Inv}\mathsf{F}$ is closed w.r.t. sSup.

Proof. $(i) \Rightarrow (ii)$: Since $\mathrm{Loc}\mathsf{F} = \mathrm{EndInv}\mathsf{F} = \mathrm{EndsInv}\mathsf{G} \overset{3.15}{=} \mathrm{EndInv}\mathsf{G} = \mathrm{Loc}\mathsf{G}$ we find $\mathsf{F} \subseteq \mathrm{Loc}\mathsf{F} = \mathrm{Loc}\mathsf{G} = \mathrm{Loc}(\mathscr{S} \cap \mathrm{Loc}\mathsf{G}) = \mathrm{Loc}(\mathscr{S} \cap \mathrm{Loc}\mathsf{F})$.
$(ii) \Rightarrow (iii)$: As $\mathsf{F} = \langle \mathsf{F} \rangle_{\mathrm{Tr}}$ making use of theorem 3.29 we can rewrite (ii) in the form $\mathsf{F} \subseteq \mathrm{EndInv}(\mathrm{wAutInv}\mathsf{F})$. This leads to

$$\mathrm{Inv}\mathsf{F} \supseteq \mathrm{InvEndInv}\ \mathrm{wAutInv}\mathsf{F} = \mathrm{Inv}\ \mathrm{wAutInv}\mathsf{F} \supseteq \mathrm{Inv}\mathsf{F},$$

i.e., $\mathrm{Inv}\mathsf{F} = \mathrm{Inv}\ \mathrm{wAut}(\mathrm{Inv}\mathsf{F})$. This shows that $\mathrm{Inv}\mathsf{F}$ is closed w.r.t. InvwAut and so is closed under sSup by 3.26.
$(iii) \Rightarrow (i)$: By 3.30 $\mathrm{Inv}\mathsf{F}$ is a Δ-complete Krasnerlgebra which by assumption is closed w.r.t. sSup. By 3.26 this immediately yields the existence of $\mathsf{G} \leq \mathscr{S}$ such that $\mathrm{Inv}\mathsf{F} = \mathrm{sInv}\mathsf{G}$. □

We can now restate our initial question in three equivalent ways (the equivalence is easily seen with the help of the two lemmata above).

(I) Is every Δ-closed Krasner algebra which is closed w.r.t. gSup also closed w.r.t. sSup ?
(II) For very locally invertible $\mathsf{F} \leq \mathrm{Tr}$ does there exist a $\mathsf{G} \subseteq \mathscr{S}$ s.t. $\mathrm{sInv}\mathsf{G} = \mathrm{Inv}\mathsf{F}$?
(III) Does $\mathsf{F} \subseteq \mathrm{Loc}(\mathscr{S} \cap \mathrm{Loc}\mathsf{F})$ hold for every locally invertible $\mathsf{F} \leq \mathrm{Tr}$?

To answer these question we will use the formulation (III).
In fact (III) does hold for a countable baseset A. The proof is based on a back and fourth construction is due to J. Kollár and can be found in [Pös84]. However in the case of an uncountable base set the answer is negative as was proven in [DKMP01] through an explicit counterexample. We state the very similar counterexample given in [Bör00].

Example 3.32. Let $\mathbf{R}$ be the set of real numbers and $<$ be the usual total order on $\mathbf{R}$. Further let $\mathbf{R}_0$ be an isomorphic copy of $\mathbf{R} \setminus \{0\}$ disjoint from $\mathbf{R}$ and let $<_0$ be the usual total order on $\mathbf{R}_0$. Let $A = \mathbf{R} \cup \mathbf{R}_0$ and define the following two relations on A

$$\rho := < \cup <_0 , \quad \sigma := \mathbf{R} \times \mathbf{R}_0 \cup \mathbf{R}_0 \times \mathbf{R}.$$

Define $\mathsf{F} := \mathrm{End}\{\rho, \sigma\}$. For a function f being an endomorphism of σ implies that $f[\mathbf{R}] \subseteq [\mathbf{R}]$ and $f[\mathbf{R}_0] \subseteq \mathbf{R}_0$ or $f[\mathbf{R}] \subseteq [\mathbf{R}_0]$ and $f[\mathbf{R}_0] \subseteq \mathbf{R}$. This means F can be divided into the following two disjoint sets

$$\begin{aligned} \mathsf{F}_1 &:= \{f \in \mathsf{F} | f[\mathbf{R}] \subseteq \mathbf{R} \text{ and } f[\mathbf{R}_0] \subseteq \mathbf{R}_0\}, \\ \mathsf{F}_2 &:= \{f \in \mathsf{F} | f[\mathbf{R}] \subseteq \mathbf{R}_0 \text{ and } f[\mathbf{R}_0] \subseteq \mathbf{R}\}. \end{aligned}$$

Then $\mathsf{F} = \mathsf{F}_1 \cup \mathsf{F}_2$ and neither F_1 nor F_2 is nonempty. Being an endomorphism of ρ on the other hand implies for every function $f \in \mathsf{F}$ that both $f \upharpoonright \mathbf{R}$ and $f \upharpoonright \mathbf{R}_0$ have to be strictly monotone. As a consequence for every set $\{a_1, \ldots, a_n\} \subseteq A$ there is a local inverse for any $f \in \mathsf{F}$, i.e., F is locally invertible.
As $\langle \mathbf{R}, < \rangle$ and $\langle \mathbf{R}_0, <_0 \rangle$ are not isomorphic F_2 does not contain any permutations. This together with the fact that both F and F_1 are locally closed shows

$$\mathrm{Loc}(\mathscr{S}(A) \cap \mathrm{Loc}\mathsf{F}) = \mathrm{Loc}(\mathscr{S}(A) \cap \mathsf{F}_1) \subseteq \mathsf{F}_1$$

and so $\mathsf{F} \nsubseteq \mathrm{Loc}(\mathscr{S}(A) \cup \mathrm{Loc}\mathsf{F})$.
So we have seen that in the case of $|A| = 2^{\aleph_0}$ the answer to question (III) is negative (for arbitrary cardinality see [DKMP01]).
By lemma 3.31 $\mathrm{Inv}\mathsf{F}$ is not closed w.r.t. sSup. We can see that explicitly by using the special superposition that was used in the proof of 3.26, choosing $\mathsf{Q} = \mathrm{Inv}\mathsf{F}$ and $\mathbf{a} = a \in \mathbf{R}$. Thereby since the local invertibility of F implies the closure of Q under complementation we get

$$\begin{aligned}
\mathrm{sSup}(a, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) &= \{ga | g \in \mathrm{AutInv}\mathsf{F}\} \\
&= \{ga | g \in \mathscr{S} \cap \mathrm{EndInv}\mathsf{F} \text{ and } g^{-1} \in \mathscr{S} \cap \mathrm{EndInv}\mathsf{F}\} \\
&= \{ga | g \in \mathscr{S} \cap \mathsf{F} \text{ and } g^{-1} \in \mathscr{S} \cap \mathsf{F}\} \\
&= \{ga | g \in \mathsf{F}_1 \text{ and } g^{-1} \in \mathsf{F}_1\} \\
&= \mathbf{R}.
\end{aligned}$$

However since F_2 is not empty $\mathbf{R}$ (seen as a unary relation) is not an element of $\mathrm{Inv}\mathsf{F}$.
Even more $\mathrm{Inv}\mathsf{F}$ is not closed w.r.t. **IOP**. This can be seen from the fact that since $\mathbf{R} \in \mathrm{sInvAut}\{\rho, \sigma\}$ (as $\mathrm{Aut}\{\rho, \sigma\} \cap \mathsf{F} \subseteq \mathsf{F}_1$) and by theorem 3.20 there exists an invariant operation $S : \mathscr{R}^2 \times \mathscr{R}^2 \to \mathscr{R}^1$ s.t. $S(\rho, \sigma) = \mathbf{R}$.

3.2.5 sInvAut revisited

In subsection 3.2.3 gave a characterization of those sets of relations that are closed w.r.t. to InvEnd, InvwAut, sInvAut. The operations sSup and gSup used for this characterization were chosen to be of arity $|A|$. We can improve this result in the case of sInvAut. Especially we will show that it will be sufficient to consider operations with at most countable arity.
As we have already noted above in [Bör00] the question was raised if there is any Δ-complete Krasner algebra closed under all invariant operations but not closed under sInvAut. The authors of [BGS] showed through a model theoretic construction that the answer to this question is indeed negative (i.e., adding all invariant operations is in general not sufficient for obtaining the Galois closure). However adding all invariant operations of countable arity does suffice as was also shown in [BGS]. In the following we give a quick overview of these results.
Keeping with the notation of [BGS] we will write ω for $\mathbb{N}$ in this section.
We start with a definition.

Definition 3.33. A partial automorphism f of a relation set $\mathsf{Q} \subseteq \mathscr{R}$ (or equivalently of the structure $\underline{A} = (A; (\sigma)_{\sigma\in\mathsf{Q}}))$ with domain dom $f = A_1 \subseteq A$ and image im $f = A_2 \subseteq A$ is a bijective function $f : A_1 \to A_2$, such that for all $\sigma \in \mathsf{Q}$, m =arity(σ) and all
$a_1, \ldots, a_m \in$dom f, we have: $\sigma(a_1, \ldots, a_m) \Leftrightarrow \sigma(f(a_1), \ldots, f(a_m))$.
A set $\mathsf{Q} \subseteq \mathscr{R}$ (or the structure $\underline{A} = (A; (\sigma)_{\sigma\in\mathsf{Q}}))$ is said to be *homogeneous*, if every finite partial automorphism can be extended to an automorphism of Q.

The closure of a homogeneous set of relations under sInvAut can be characterized in the following way.

Lemma 3.34. If $\mathsf{Q} \subset \mathscr{R}$ is a homogeneous set of relations, then $\langle \mathsf{Q} \rangle_{\mathrm{LOP},\bigcap} = \mathrm{sInvAut}\mathsf{Q}$.

For the proof we will use the following (lemma which essentially already has been used in the proof of 3.22 and follows easily with the help of the lemmata 3.3 and 3.4).

Lemma 3.35. Let $\mathsf{Q} \subseteq \mathscr{R}$ be $\bigcap$-closed and closed under complementation. Then $\mathsf{Q} = \mathrm{sInvAut}\mathsf{Q}$ iff for all m and all $\mathbf{a}, \mathbf{b} \in A^m$ with $\mathbf{a} \equiv_{\mathsf{Q}} \mathbf{b}$ there exists an automorphism $g \in \mathrm{Aut}\mathsf{Q}$ with $\mathbf{b} = g(\mathbf{a})$.

Now we can prove lemma 3.34 .

Proof. Note that since $\mathsf{Q} \subseteq \langle \mathsf{Q} \rangle_{\mathrm{LOP},\bigcap} \subseteq \mathrm{sInvAut}\mathsf{Q}$ and $\mathrm{Aut}\mathsf{Q} = \mathrm{Aut}\langle \mathsf{Q} \rangle_{\mathrm{LOP},\bigcap} = \mathrm{Aut}(\mathrm{sInvAut}\mathsf{Q})$ homogeneity of Q implies homogeneity of $\langle \mathsf{Q} \rangle_{\mathrm{LOP},\bigcap}$ and $\mathrm{sInvAut}\mathsf{Q}$. Thus wlog we put $\mathsf{Q} = \langle \mathsf{Q} \rangle_{\mathrm{LOP},\bigcap}$.
Let $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$. Then assuming $\mathbf{a} \equiv_{\mathsf{Q}} \mathbf{b}$ we define a partial map f on A through $f(a_i) = b_i$ for all $i = 1, \ldots, m$. Since $a_i = a_j$ implies $b_i = b_j$ f is well defined and is even a partial automorphism of Q since no relations separate $\mathbf{a}$ and $\mathbf{b}$. By the homogeneity of Q f can be extended to a full automorphisms and so by 3.35 the lemma is proved. ☐

To construct an example of a Δ-complete Krasner algebra closed under all invariant operations but not closed under sInvAut the authors of [BGS] consider relational models of the form $\underline{M} = (M; (\rho_m)_{1 \leq m \in \omega})$, where $\rho_m \in \mathrm{Rel}^{(m)}(A)$ for all m, thus the langauge $\mathcal{L}$ has exactly one relation symbol for every arity m.
Further let $\underline{M}^{(m)} := (M; \rho_1, \ldots, \rho_m)$ denote the reduct of $\underline{M}$ to the relations $\rho_1, \ldots, \rho_m$.
The basic idea of the construction of [BGS] is given by the following lemma.

Lemma 3.36. Fix a vocabulary of infinitely many relation symbols $\{\rho_m | m \in \omega\}$. Let $\underline{A} = (A; (\rho)_{1 \leq m \in \omega})$ be an infinite model. Let $\underline{A}^{[m]} = (A; \rho_1, \ldots, \rho_m)$. Assume that the following hold:
(1)The theory of $Th(\underline{A})$ is ω-categorical.
(2)For all m, the reduct $\underline{A}^{[m]}$ is homogeneous.
(3)$\underline{A}$ is rigid, i.e., $\mathrm{Aut}\underline{A} = \{\mathrm{id}_A\}$.
Then, letting $\mathsf{Q} := \langle \rho_1, \rho_2, \ldots \rangle_{\mathrm{LOP}}$ being the set of first order definable relations in $\underline{A}$, we have:

$$\langle \rho_1, \ldots, \rho_m \rangle_{\mathrm{LOP}} = \mathrm{sInvAut}\{\rho_1, \ldots, \rho_m\}$$

and $\mathsf{Q} = \bigcup_m \langle \rho_1, \ldots, \rho_m \rangle_{\mathrm{LOP}}$, but

$$\mathsf{Q} = \langle \mathsf{Q} \rangle_{inv,\bigcap} \subsetneqq \mathrm{sInvAut}\mathsf{Q}\,.$$

Proof. Since $Th(\underline{A})$ is ω-categorical by Ryll-Nardzewskiss theorem it follows that $\mathsf{Q}^{(k)}$ is finite for every k implying it is closed w.r.t. arbitrary intersection.
Trivially we have

$$\langle \rho_1, \ldots, \rho_m \rangle_{\mathrm{LOP}} = \langle \rho_1, \ldots, \rho_m \rangle_{\mathrm{LOP},\bigcap}$$

and (using lemma 2.6 and theorem 3.20)

$$\langle \rho_1, \ldots, \rho_m \rangle_{\mathrm{LOP}} \subseteq \langle \rho_1, \ldots, \rho_m \rangle_{inv} \subseteq \mathrm{sInvAut}\{\rho_1, \ldots, \rho_m\}.$$

By (2) and lemma 3.34 we get

$$\langle \rho_1, \ldots, \rho_m \rangle_{\mathrm{LOP},\bigcap} = \mathrm{sInvAut}\{\rho_1, \ldots, \rho_m\}$$

and so

$$\langle \rho_1, \ldots, \rho_m \rangle_{\mathrm{LOP},\bigcap} = \langle \rho_1, \ldots, \rho_m \rangle_{inv}.$$

As $\langle ... \rangle_{\mathrm{LOP}}$ and $\langle ... \rangle_{inv}$ are algebraic we find

$$\mathsf{Q} = \langle \rho_1, \rho_2, \ldots \rangle_{\mathrm{LOP}} = \langle \rho_1, \rho_2, \ldots \rangle_{inv}$$

hence $\mathsf{Q} = \langle \mathsf{Q} \rangle_{inv,\bigcap}$.
On the other hand $\mathrm{Aut}\mathsf{Q} = \{\mathrm{id}\}$ and so $\mathrm{sInvAut}\mathsf{Q} = \mathscr{R}$ which is an uncountable set whereas Q is clearly countable, i.e., $\mathrm{sInvAut}\mathsf{Q} \neq \langle \mathsf{Q} \rangle$ □

The main part of [BGS] is taken up by the construction of a structure $\underline{A}$ fulfilling above lemma. We will only make some brief comments on this construction.
To define the theory that $\underline{A}$ has to fulfill we start by defining the notion of a clause.

Definition 3.37. A *literal in the variables* $x_0, \ldots, x_n$ $(n \in \omega)$ is a formula of the form

$$\rho_m(x_{i_1}, \ldots, x_{i_m}) \text{ (unnegated) or } \neg\rho_m(x_{i_1}, \ldots, x_{i_m}) \text{ (negated)}$$

such that $1 \leq m \leq n+1$, $\{i_1, \ldots, i_m\} \subseteq \{0, \ldots, n\}$, the $i_1, \ldots, i_m$ are pairwise distinct and $0 \in \{i_1, \ldots, i_m\}$.
A *clause in* $x_0, \ldots, x_n$ is a conjunction K of literals, such that no literal will appear twice, and no literal appears in negated and unnegated form.

The theory $\mathcal{T}$ is now introduced in the following way.

Definition 3.38. $\mathcal{T}$ consist of (the universal closure of) the following formulas: Firstly, for all $1 \leq m \in \omega$ we have:
(T1) $\rho_m(x_1, \ldots, x_m) \to \bigwedge_{1 \leq i < j \leq m} x_i \neq x_j$.
Secondly, for all $n \in \omega$ and all clauses $K = K(x_0, \ldots, x_n)$ in $x_0, \ldots, x_n$ we take the formula:
(T2) $\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \to (\exists x_0) K(x_0, \ldots, x_n)$

This theory $\mathcal{T}$ has the following properties (see [BGS, lemma 3.8]):
(1) $\mathcal{T}$ is consistent and has no finite models.
(2) $\mathcal{T}$ has the property of elimination of quantifiers.
(3) $\mathcal{T}$ is complete.
(4) $\mathcal{T}$ is ω-categorical.

Thereby especially property (1) of lemma 3.36 is guaranteed for every model of $\mathcal{T}$. The authors of [BGS] construct $\underline{A} := \bigcup_{i \in \omega_1} \underline{M}_i$ as the union of models $\underline{M}_i$ of $\mathcal{T}$ with the following properties:

(i) $\underline{M}_i$ is an elementary submodel of $\underline{M}_{i+1}$ and $\underline{M}_j = \bigcup_{i<j} \underline{M}_i$ in the case of a limit ordinal $j \in \omega_1$.

(ii) $i \in M_i$ for all $i \in \omega_1$.

(iii) For every $s \in \omega$ and every $i < j \in \omega_1$, every finite partial automorphism π of $\underline{M}_i^{[s]}$ can be extended to a partial automorphism π_j of $\underline{M}_j^{[s]}$ with $M_i \subseteq$dom π_j and $M_i \subseteq$im π_j. When π_j, π_k are two such extensions with $j \leq k$ then π_k extends π_j.

(**i**) guarantees that $\underline{A}$ is a model of $\mathcal{T}$ and (**ii**) implies that $A = \omega_1$. Finally (**iii**) ensures that $\underline{A}^{[s]}$ is homogeneous for very $s \in \omega$.
The construction in [BGS] is further done in such a way that for "sufficient long" tuples $(x_1, \dots, x_n) \in A^n$, $\rho(x_1, \dots, x_n)$ must hold if $x_1 < x_2 < \dots < x_n$ and must not hold otherwise (for details see [BGS]). This allows to define the well ordering $(\omega_1, <)$ in $\underline{A}$ by a formula of higher order logic which in turn makes $\underline{A}$ rigid (i.e Aut$\underline{A} = \{\mathrm{id}_A\}$). This ends our comments on the construction of $\underline{A}$, for all the details we refer to [BGS].

After having argued that invariant operations together with Δ-closure are not sufficient to obtain Galois closed sets we follow [BGS] to show that closure w.r.t invariant operations of countable arity (as defined below) and Δ-completeness leads to Galois closed sets. The following lemma is of central importance for proving this fact.

Lemma 3.39. ([BGS, Lemma 4.1], [Bör00]) Let $m \in \omega \setminus \{0\}$ and $\mathsf{Q} \subseteq \mathscr{R}^{(m)}$ be closed under complementation. Then there exists a $\rho \in \langle \mathsf{Q} \rangle_{\mathrm{LOP},\Delta}^{(2m)}$ such that $\mathrm{Aut}\{\rho\} = \mathrm{Aut}\mathsf{Q}$.

Proof. As Q is closed w.r.t complementation by lemma 3.3 the set $M := \{\Gamma^{\mathsf{Q}}(\mathbf{a}) | \mathbf{a} \in A^m\}$ forms a partition of A^m. Choose some well ordering on M and let $(\gamma_i)_{i<\kappa}$ be the corresponding enumeration of M (for some cardinal κ). Now we form the new relation

$$\rho := \bigcup_{i \leq j < \kappa} \gamma_i \times \gamma_j$$

which is obviously an element of $\langle \mathsf{Q} \rangle_{\mathrm{LOP},\Delta}^{(2m)}$ and so fulfills $\mathrm{Aut}\{\rho\} \supseteq \mathrm{Aut}\mathsf{Q}$.
We will now show the opposite inclusion. For this let $g \in \mathrm{Aut}\{\rho\}$ and $\mathbf{a}, \mathbf{b} \in \gamma_i$ for some $i < \kappa$. Then $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{a})$ are both elements of ρ and therefore so are $(g(\mathbf{a}), g(\mathbf{b}))$, $(g(\mathbf{b}), g(\mathbf{a}))$. This implies that there exists a single index $l < \kappa$ so that $g(\mathbf{a})$ and $g(\mathbf{b})$ are elements of γ_l. As $i, \mathbf{a}, \mathbf{b}$ were arbitrary we can define a function $g_0 : \kappa \to \kappa$ through $g[\gamma_i] \subseteq \gamma_{g_0(i)}$. Repeating this argument for $g^{-1} \in \mathrm{Aut}\rho$ we find $(g^{-1})_0 = (g_0)^{-1}$ which shows that g_0 is a bijection. Even more g_0 is order preserving as $i < j$ implies $g[\gamma_i] \times g[\gamma_j] = \gamma_{g_0(i)} \times \gamma_{g_0(j)} \subseteq \rho$ which is only possible for $g_0(i) < g_0(j)$. The well ordering of κ finally implies that g_0 is the identity on κ.
This shows that $g[\gamma_i] \subseteq \gamma_i$ and $g^{-1}[\gamma_i] \subseteq \gamma_i$, i.e., $g(\gamma_i) = \gamma_i$ for all $i < \kappa$ and so $g \in \mathrm{Aut}\mathsf{Q}$. □

An immediate consequence is the following lemma.

Lemma 3.40. ([BGS, Lemma 4.2] , [Bör00]) For any set of relations $\mathsf{Q} \subseteq \mathscr{R}$ there exists a countable set $\mathsf{Q}_0 \subseteq \mathscr{R}$ so that $\mathrm{Aut}\mathsf{Q} = \mathrm{Aut}\mathsf{Q}_0$. In particular if $\mathsf{Q} = \langle \mathsf{Q} \rangle_{\mathrm{LOP},\Delta}$, Q_0 can be chosen to be a subset of Q.

Proof. Using lemma 3.39 for every $m \in \mathbb{N}^+$ we find a relation ρ_m such that $\mathrm{Aut}\mathsf{Q}^{(m)} = \mathrm{Aut}\{\rho_m\}$. We define $\mathsf{Q}_0 := \{\rho_m | m \in \mathbb{N}^+\}$ and observe

$$\mathrm{Aut}\mathsf{Q} = \bigcap_{m \in \mathbb{N}^+} \mathrm{Aut}\mathsf{Q}^{(m)} = \bigcap_{m \in \mathbb{N}^+} \mathrm{Aut}\{\rho_m\} = \mathrm{Aut}\mathsf{Q}_0.$$

If $\mathsf{Q} = \langle \mathsf{Q} \rangle_{\mathrm{LOP},\Delta}$ lemma 3.39 tells us that $\rho_m \in \mathsf{Q}$ for all $m \in \mathbb{N}^+$ and so $\mathsf{Q}_0 \subseteq \mathsf{Q}$ which proves the lemma. □

As the next step we define invariant operations of countable arity.

Definition 3.41. [BGS] An operation $S : \prod_{1 \leq i \in \omega} \mathscr{R}^{(m_i)} \to \mathscr{R}^{(m_0)}$ is called *ω-invariant* iff for all $g \in \mathscr{S}$, $\rho_i \in \mathscr{R}^{(m_i)}$ $(1 \leq i \leq \omega)$,

$$S(g[\rho_i])_{1 \leq i \in \omega} = g[S(\rho_i)_{1 \leq i \in \omega}]$$

holds.
For $\mathsf{Q} \subseteq \mathscr{R}$ the smallest set closed under all ω-invariant operations will be called $\langle \mathsf{Q} \rangle_{\omega-inv}$.

We quickly observe the following two facts (which can be proved similar to lemmata 3.18 and 3.19).

Lemma 3.42. ([BGS, Lemma 4.4]) Let $\mathsf{Q} \subseteq \mathscr{R}$.
(1) If $\mathsf{Q} = \mathrm{sInvAut}\mathsf{Q}$ then $\mathsf{Q} = \langle \mathsf{Q} \rangle_{\omega-inv,\Delta}$.
(2) If Q is at most countable then $\mathrm{sInvAut}\mathsf{Q} = \langle \mathsf{Q} \rangle_{\omega-inv,\Delta}$.

Now we can give the simplified characterization of the sets closed with respect to sInvAut.

Theorem 3.43. ([BGS, Theorem 4.5]) Let $\mathsf{Q} \subseteq \mathscr{R}$. Then $\mathrm{sInvAut}\mathsf{Q} = \langle \mathsf{Q} \rangle_{\omega-inv,\Delta}$. [$\langle\cdot\rangle_{\omega-inv}$: 3.41; $\langle\cdot\rangle_{\Delta}$: 2.7.3
]

Proof. We consider the following statements:
(1) $\mathrm{sInvAut}\mathsf{Q} = \mathsf{Q}$
(2) Q is Δ-closed and for all at most countable subsets Q_0 of Q : $\mathrm{sInvAut}\mathsf{Q}_0 \subseteq \mathsf{Q}$
(3) Q is Δ-complete and for all at most countable subsets Q_0 of Q : $\langle \mathsf{Q}_0 \rangle_{\omega-inv,\Delta} \subseteq \mathsf{Q}$
(4) $\langle \mathsf{Q} \rangle_{\omega-inv,\Delta} = \mathsf{Q}$.

They are indeed all equivalent. The equivalence (1) ⇔ (2) is a consequence of lemma 3.40, (2) ⇔ (3) follows from lemma 3.42 and (3) ⇔ (4) is immediate from the definitions. This proves the theorem. □

3.3 Operations only

In this section we will consider Galois connetions between sets of operations only. That is we treat the following situations.

(a) $\mathbf{E} = \mathscr{O}$ and $\mathbf{R} = \mathscr{O}$, i.e., all operations and all operations.

(b) $\mathbf{E} = \mathscr{O}^{(1)}$ and $\mathbf{R} = \mathscr{O}$, i.e., unary operations and all operations.

(b) $\mathbf{E} = \mathscr{S}$ and $\mathbf{R} = \mathscr{O}$, i.e., bijective functions and operations.

As a first step in this section we will characterize the sets of the form PolPol$\mathsf{Q}$ for some set Q of relations or operations.
As mentioned before for a set Q of operations the set of its polymorphisms Pol$\mathsf{Q}$ is equivalent to the set of functions which permute with all the functions of Q [3]. To characterize the Galois closed sets we make use of the "local definability" of a function introduced in definition 2.8. Sets of the form PolPol$\mathsf{Q}$ ($\mathsf{Q} \subseteq \mathrm{Rel}(A)$) are characterized by the following lemma

Lemma 3.44. (Szabo [Sza78, Lemma 5]) Let $\mathsf{Q} \subseteq \mathrm{Rel}(A)$. Then $f \in \mathrm{PolPol}\mathsf{Q}$ iff f can be defined by Q locally. If A is finite we can choose the formula schemes used to define f to be finite.

Proof. For the one implication let f be an n-ary operation defined by Q locally. Let $g \in \mathrm{Pol}^{(m)}\mathsf{Q}$ and $M = (a_{kl})_{m\times n} \in A^{m\times n}$. Then there is a formula scheme Ψ defining f on the set

$$B = \{(a_{k1}, \ldots, a_{kn})|k = 1, \ldots, n\} \cup \{(g(a_{11}, \ldots, a_{m1}), \ldots, g(a_{1n}, \ldots, a_{mn}))\}.$$

This means $R_\Psi(a_{k1}, \ldots, a_{kn}, f(a_{k1}, \ldots, a_{kn}))$ holds for $k = 1, \ldots, m$. Since g is a polymorphism of Q and by lemma 3.5 also
$R_\Psi(g(a_{11}, \ldots, a_{m1}), \ldots, g(a_{1n}, \ldots, a_{mn}), g(f(a_{11}, \ldots, a_{1n}), \ldots, f(a_{m1}, \ldots, a_{mn})))$ holds. This implies

$$f(g(a_{11}, \ldots, a_{m1}), \ldots, g(a_{1n}, \ldots, a_{mn})) = g(f(a_{11}, \ldots, a_{1n}), \ldots, f(a_{m1}, \ldots, a_{mn})),$$

i.e., f and g commute and so $f \in \mathrm{PolPol}\mathsf{Q}$.
For the other implication we start with an $f \in \mathrm{Pol}^{(n)}\mathrm{Pol}\mathsf{Q}$. Seen as a relation f is an element of $\mathrm{Inv}^{(n+1)}\mathrm{Pol}\mathsf{Q}$ and by theorem 3.7 can be written as $f = \bigcup_{i\in I} R_i$ where $(R_i|i \in I)$ is a directed system of $(n+1)$-ary relations defined by formula schemes over Q. Now let $B \subseteq A^n$ be a finite set. Then $f|B \subseteq R_{i_0}$ for some $i_0 \in I$ as $(R_i|i \in I)$ is a directed system. The formula scheme Ψ over Q defining R_{i_0} also defines f on B as $f|B \subseteq R_{i_0} \subseteq f$ implies

$$f|B = \{(a_1, \ldots, a_n, a_{n+1})|(a_1, \ldots, a_n) \in B \text{ and } (a_1, \ldots, a_n, a_{n+1}) \in R_{i_0} = R_\Psi\}.$$

This finishes the proof. $\square$

As an immediate consequence we get the following theorem.

Theorem 3.45. (Szabo [Sza78, Theorem 13]) Let $\mathsf{F} \subseteq \mathscr{O}$ be a set of operations then the set PolPol$\mathsf{F}$ consists of all functions that can be defined by F locally. If A is finite it consists of all functions that can be defined by finite formula schemes.[locally definable: 2.8.2; defined by finite formula scheme: 2.8.1]

To see that closure of some set of functions F w.r.t. finite formula schemes is in general really more than the clone generated by F, i.e., $\mathrm{PolPol}\mathsf{F} \supsetneq \mathrm{PolInv}\mathsf{F}$, we give the following example.

Example 3.46. Consider the base set $A = \{1, 2, 3\}$. Let $g_1 \in \mathscr{O}^{(2)}$ be the constant function $g_1(x_1, x_2) \equiv 1$ and let $g_2 \in \mathscr{O}^{(2)}$ be defined by $g_2(1, 2) = g_2(2, 3) = g_2(3, 1) = 1$ and $g_2(x_1, x_2) = 2$ else. Then let $f \in \mathscr{O}^{(1)}$ be defined by

$$f^\bullet := \{(x_1, x_2)|g_1(x_1, x_2) = g_2(x_1, x_2)\} = \{(1, 2), (2, 3), (3, 1)\},$$

[3] As a note we recall the notion of an *entropic algebra*, which is an algebra whose functions all commute with each other.

in particular we note that $f \in [\{g_1, g_2\}]_{f.s.}$. However since $g_1, g_2 \in \mathrm{Pol}\,(\{(1,1),(2,2)\}) \supseteq \langle\{g_1, g_2\}\rangle$ and $f \notin \mathrm{Pol}\,(\{(1,1),(2,2)\})$ we find $f \notin \langle\{g_1, g_2\}\rangle$.
For infinite base set $\tilde{A} := A \cup B$ with $A \cap B = \emptyset$ and $|B| \geq \aleph_0$ we might extend the example in the following way. Let $\tilde{g}_1 \in \mathscr{O}^{(2)}$ be constant $\tilde{g}_1 \equiv 1$ and let $\tilde{g}_2 \in \mathscr{O}^{(2)}$ be the extension of g_2 defined by

$$\tilde{g}_2 \upharpoonright A^2 = g_2 \upharpoonright A^2 \ , \ \tilde{g}_2 \upharpoonright (\tilde{A}^2 \setminus A^2) = \begin{cases} 1, & \text{for } x_1 = x_2 \\ 2, & \quad \text{else} \end{cases} .$$

Then defining $\tilde{f}$ analogous to f, i.e.,

$$\tilde{f}^\bullet := \{(x_1, x_2) | \tilde{g}_1(x_1, x_2) = \tilde{g}_2(x_1, x_2)\},$$

above we see that $\tilde{f} \in \mathscr{O}^{(1)}$ and $\tilde{f} \in [\{\tilde{g}_1, \tilde{g}_2\}]_{\text{f.s.}}$ and $\tilde{f} \upharpoonright A = f \upharpoonright A$, $\tilde{f} \upharpoonright B = \mathrm{id}_B$. As before $\tilde{g}_1, \tilde{g}_2 \in \mathrm{Pol}\,(\{(1,1),(2,2)\})$ and $\tilde{f} \notin \mathrm{Pol}\,(\{(1,1),(2,2)\})$ which shows $\tilde{f} \notin \mathrm{Loc}\langle\{g_1, g_2\}\rangle$.

As a specialization of above theorem we get.

Theorem 3.47. (Szabo [Sza78, Theorem 15]) Let $\mathsf{F} \subseteq \mathscr{O}^{(1)}$ then $\mathsf{F} = \mathrm{EndPol}\mathsf{F}$ iff F contains every transformation defined by F locally.[locally definable: 2.8.2]

In the case that $\mathsf{F} \subseteq \mathscr{O}^{(1)}$ is monoid that consists of s-locally invertible and constant functions only we can give a necessary and a sufficient condition for F being closed w.r.t. $\mathrm{EndPol}^{(s)}$.
First for every $a \in A$ we define the constant function $c_a : x \mapsto a$. Then to every $\mathsf{F} \subseteq \mathscr{O}^{(1)}$ we associate the following set $\kappa(\mathsf{F})$ of constant functions

$$\kappa(\mathsf{F}) := \{c_a | \forall b \in A, b \neq a, \exists f, g \in \mathsf{F} : f(a) = g(a) \text{ and } f(b) \neq g(b)\}.$$

Theorem 3.48. (Pöschel [Pös80, Theorem 9.6], [Sto75]) Let $s \in \mathbb{N}$ and let $\mathsf{F} = \mathsf{G} \cup K$ be a monoid where $\mathsf{G} \subseteq \mathscr{O}^{(1)}$ is a s-locally invertible monoid of functions and K is a set of constant maps on A. Consider the following conditions:
(i) $\kappa(\mathsf{F}) \subseteq \mathsf{F}$ and F is s-locally closed.
(ii) $\mathsf{F} = \mathrm{EndPol}^{(s)}\mathsf{F}$.
(iii) $\kappa(\mathsf{F}) \subseteq \mathsf{F}$ and F is $(s+1)$-locally closed.
Then $(i) \Rightarrow (ii) \Rightarrow (iii)$. [$s$-locally closed: 2.16]

Proof. The implication $(ii) \Rightarrow (iii)$ is straightforward.
The proof of $(i) \Rightarrow (ii)$ can be done similar to the proof of 3.51 (i) below. We refer to [Pös80] for the details. □

At last we turn to the case of bijective functions.

Theorem 3.49. (Jónsson [Jón91, Theorem 1 and 2]) For $\mathsf{G} \subseteq \mathscr{S}$, $s \in \mathbb{N}$, $s \geq 2$, consider the conditions
(i) $\mathsf{G} = s\text{-}\mathrm{Loc}_o\langle\mathsf{G}\rangle_{\mathscr{S}}$, i.e., G is a s-Loc_o closed group of permutations.
(ii) $\mathsf{G} = \mathrm{AutPol}^{(s)}\mathsf{G}$, i.e., G is closed w.r.t. $\mathrm{AutPol}^{(s)}$.
(iii) $\mathsf{G} = (s+1)\text{-}\mathrm{Loc}_o\langle\mathsf{G}\rangle_{\mathscr{S}}$, i.e., G is a $(s+1)$-Loc_o closed group of permutations.
Then $(i) \Rightarrow (ii) \Rightarrow (iii)$. [$s$-$\mathrm{Loc}_o$: 2.16.1]

Proof. The implication $(ii) \Rightarrow (iii)$ is straightforward.
For the implication $(i) \Rightarrow (ii)$ define $K := \{\mathbf{x} | \mathbf{x} \in A^s,\ x_i \neq x_j \text{ for } i, j \leq s, i \neq j\}$, i.e., the set of all one to one sequences of elements of A having s terms. Then for each $\mathbf{x} \in K$ define a $|\mathbf{x}|$-ary operation $F_{\mathbf{x}}$ by

$$F_{\mathbf{x}}(\mathbf{y}) := \begin{cases} y_1, \text{ if } \mathbf{y} \in \mathsf{G}\mathbf{x} \\ y_2, \text{ if } \mathbf{y} \notin \mathsf{G}\mathbf{x} \end{cases},$$

where $\mathsf{G}\mathbf{x} := \{g\mathbf{x} | g \in \mathsf{G}\}$. Now it is mostly straightforward to show that $\mathsf{G} = \mathrm{Aut}\{F_{\mathbf{x}} | \mathbf{x} \in K\}$. □

The following example shows that for $s = 1$, (i) does not imply (ii).

Example 3.50. (Jónsson [Jón91, Example 1]) Let A be the disjoint union of of the sets B and C, each with at leat three elements. Define $\mathsf{G} := \{g \in \mathscr{S} | g(B) = B \text{ and } g(C) = C\}$ which clearly fulfills 1-Loc$\mathsf{G} = \mathsf{G}$. Now for a unary operation f that is not the identity,i.e., there exists $x \in A$ with $f(x) = y \neq x$ choose $g, g' \in \mathsf{G}$ so that $g(x) = g'(x)$ and $g(y) \neq g'(y)$. Then f is not preserved by both g and g' as $g(y) = f(g(x)) = f(g\prime(x)) = g'(y)$ would lead to a contradiction. This shows that $\mathrm{Pol}^{(1)}\mathsf{G} = \{\mathrm{id}\}$.

An example that the inverse implications in above proposition do not hold in general can also be found in [Jón91].
Finally a characterization of sets of permutations closed w.r.t. AutPol and $\mathrm{AutPol}^{(s)}$ is given by the following theorem.

Theorem 3.51. (Jónsson [Jón72, 2.4.3, 2.4.1]) Let $\mathsf{G} \subseteq \mathscr{S}$. Then
(i) $\mathsf{G} = \mathrm{AutPol}\mathsf{G}$ iff $\mathsf{G} = \mathrm{Loc}_o\langle\mathsf{G}\rangle_{\mathscr{S}}$, i.e., G is closed w.r.t AutPol iff G is a locally closed group of permutations. $\left[\mathrm{Loc}_o\text{: 2.16.1; } \langle\rangle_{\mathscr{S}}\text{: 2.20}\right]$
(ii) For $s \in \mathbb{N}$, $s \geq 2$, $\mathsf{G} = \mathrm{AutPol}^{(s)}\mathsf{G}$ iff $\mathsf{G} = \langle\mathsf{G}\rangle_{\mathscr{S}}$ and $f \in \mathscr{S}$ belongs to G, whenever for all $B \subseteq A$ with at most s elements there exists a $g \in \mathsf{G}$ that agrees with f on $\Gamma_{\mathrm{Pol}^{(|B|)}\mathsf{G}}(B)$.
For the case $s = 1$ we find: $\mathsf{G} = \mathrm{AutEnd}\mathsf{G}$ iff $\mathsf{G} = \langle\mathsf{G}\rangle_{\mathscr{S}}$ and for all bijective functions f

$$(\forall a \in A : \Gamma_{\mathrm{End}\mathsf{G}}(\{a\}) \neq \{a\} \text{ or } \Gamma_{\mathrm{End}\mathsf{G}}(\{f(a)\}) \neq \{f(a)\} \Rightarrow \exists g \in \mathsf{G} : g \restriction \Gamma_{\mathrm{End}\mathsf{G}}(\{a\}) = f \restriction \Gamma_{\mathrm{End}\mathsf{G}}(\{a\}))$$

implies $f \in \mathsf{G}$.

Proof. (i) is a direct consequence of theorem 3.49.
We will however give an alternative proof for (i), as was done in [Pös80], since it illustrates the use of general superposition in this situation. We content ourselves with showing that $\mathsf{G} = \mathrm{Loc}_o\langle\mathsf{G}\rangle_{\mathscr{S}}$ implies that G is closed w.r.t AutPol since the other direction is obvious.
Making use of theorem 3.7 this we show $\mathscr{S}^\bullet \cap \mathrm{LOC}[\mathsf{G}^\bullet]_{\text{f.s.}} \subseteq (\mathrm{Loc}_o\langle\mathsf{G}\rangle_{\mathscr{S}})^\bullet$. First we notice that $\langle\mathsf{G}\rangle_{\mathscr{S}}^\bullet \subseteq [\mathsf{G}^\bullet]_{\text{f.s.}}$, so w.l.o.g. we may assume $\mathsf{G} = \langle\mathsf{G}\rangle_{\mathscr{S}}$. Now we consider an $f^\bullet \in \mathscr{S}^\bullet \cap \mathrm{LOC}[\mathsf{G}^\bullet]_{\text{f.s.}}$ and show that $f \in \mathrm{Loc}_o\mathsf{G}$.
By definition for every finite subset $\bar{B} \subseteq f^\bullet$ there is a general superposition (a formula scheme) $\sigma \in [\mathsf{G}^\bullet]_{\text{f.s.}}$ s.t. $\bar{B} \subseteq \sigma \subseteq f^\bullet$. Note that the sets $\bar{B}$ are in one-to-one correspondence with finite sets $B \subseteq A$ through $B = \{x | (x, y) \in B^\bullet\}$ and $\bar{B} = (f \restriction B)^\bullet$.
Since σ is a general superposition of $\mathsf{G}^\bullet$ we find $g_i \in \mathsf{G}$, $\mathbf{b}_i \in A^2$, $(i \in I)$ such that

$$(x_{a_1}, x_{a_2}) \in \sigma \Leftrightarrow (\exists x_k)_{k \in A \setminus \{a_1, a_2\}} : \bigwedge_{i \in I} (x_{b_{i1}}, x_{b_{i2}}) \in g_i^\bullet.$$

We will consider the quantifier free part of this formula as a labelled graph with vertex set $V := \{x_j | j \in A\}$ and for $(b_{i1}, b_{i2}) = (t, t')$ we draw an edge going from x_t to $x_{t'}$ with label g_i $(i \in I)$. There are two cases to be considered.
Case (1): The vertices x_{a_1} and x_{a_2} are connected, i.e., there exist distinct vertices $x_{a_1} = x_{t_0}, x_{t_1}, \ldots, x_{t_{n-1}}, x_{t_n} = x_{a_2}$ such that there is an edge from x_{t_j} to $x_{t_{j+1}}$ or vice versa with label g_{i_j} $(j \in n)$. We put $g'_{i_j} := g_{i_j}$ in the first case and $g'_{i_j} := g_{i_j}^{-1}$ in the second case. Then

$$\sigma \subseteq \{(a_{t_0}, a_{t_n}) | \exists a_{t_1}, \ldots, a_{t_{n-1}} : g'_{i_0}(a_{t_0}) = a_{t_1}, g'_{i_1}(a_{t_1}) = a_{t_2}, \ldots g'_{i_{n-1}}(a_{t_{n-1}}) = a_{t_n}\},$$

i.e., $\sigma \subseteq g^\bullet$, where $g := g'_{i_0} g'_{i_1} \ldots g'_{i_{n-1}} \in \mathsf{G}$. Since $\bar{B} \subseteq \sigma \subseteq g^\bullet$ we find $(g \restriction B)^\bullet = \bar{B}$, i.e., $g \restriction B = f \restriction B$.
Case (2): The vertices x_{a_1}, x_{a_2} are not connected. Let w.l.o.g. (x, y) and (x', y') be two distinct elements of $\bar{B} \subseteq \sigma$. The disconnectedness of the vertices x_{a_1}, x_{a_2} implies that (x, y') is also an element of σ. Since $\sigma \subseteq f^\bullet$ and $f \in \mathscr{S}$ this leads to a contradiction, so only case (1) occurs, which finishes the proof.
For the proof of (ii) we refer to [Jón72]. □

3.4 Congruence relations

In this section we will characterize the Galois closed sets of equivalence relations.

(a) $\mathbf{E} = \mathscr{O}^{(1)}$ and $\mathbf{R} = \mathrm{Eq}(A)$, i.e., all unary functions and all equivalence relations.
The endomorphisms in this case are called dilatations and we write End for $\mathrm{Pol}_{\mathscr{O}^{(1)}}$ (some authors use $\mathcal{D}$). The preserved equivalence relations are called congruence relations and we write Con for $\mathrm{Inv}_{\mathrm{Eq}(A)}$.

We start with a lemma that shows that considering unary operations only is indeed no restriction at all. First we need a definition.

Definition 3.52. Let $\mathsf{F} \subseteq \mathscr{O}$. For $f \in \mathsf{F}^{(n)}$, $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in A$, $n \in \mathbb{N}$, every mapping of the form

$$x \mapsto f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$$

is called a translation of F.

Lemma 3.53. Let $\mathsf{F} \subseteq \mathscr{O}$. Then $\Theta \in \mathrm{Con}\mathsf{F}$ iff Θ is invariant for all translations of F.

Proof. If $\Theta \in \mathrm{Con}\mathsf{F}$ it is immediate clear that it is invariant for all translations.
For the other direction let $f \in \mathsf{F}^{(n)}$ and $a_1 \Theta b_1, \ldots, a_n \Theta b_n$. Then invariance w.r.t. the translations coming from f leads to

$$\begin{aligned} f(a_1, a_2, \ldots, a_n) \;&\Theta\; f(b_1, a_2, \ldots, a_n) \\ &\Theta\; f(b_1, b_2, a_3, \ldots, a_n) \\ &\vdots \\ &\Theta\; f(b_1, b_2, \ldots, b_n), \end{aligned}$$

which proofs the theorem. □

For a binary relation ρ define $[\rho]_{eq} := \bigcap\{\Theta | \Theta \in \mathrm{Eq}(A) \text{ and } \Theta \supseteq \rho\}$. The equivalence relations $\mathrm{Eq}(A)$ on an given base set A naturally form a lattice with the meet $\wedge$ and the join $\vee$ defined by

$$\begin{aligned} \Theta_1 \wedge \Theta_2 &:= \Theta_1 \cap \Theta_2, \\ \Theta_1 \vee \Theta_2 &:= [\Theta_1 \cup \Theta_2]_{eq}, \end{aligned}$$

for $\Theta_1, \Theta_2 \in \mathrm{Eq}(A)$.
We note that for any algebra (A, F) its congruence lattice is a complete sublattice of $\mathrm{Eq}(A)$. Indeed it is even an algebraic lattice [Grä68].

For our characterization we also modify the general superposition in the following way.

Definition 3.54. For $\mathbf{a}, \mathbf{b}_i \in A^2$, $\rho_i \in \mathrm{Eq}(A)$, for all i in some index set I, define

$$\mathrm{gSup}_{eq}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) := [\mathrm{gSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I})]_{eq}.$$

We note that modified in this way the general superposition corresponds to the graphical composition introduced by H. Werner [Wer74]. We write $\mathrm{gSup}^{(2)}$ for all general superpositions where $\mathbf{a} \in A^2$, i.e., which correspond to binary relations.

We follow [Ihr93] for the proof of the following characterization.

Theorem 3.55. Let L be a complete sublattice of $\mathrm{Eq}(A)$. Then the following are equivalent:
(a) L is the congruence lattice of some algebra.
(b) L is closed w.r.t $\mathrm{gSup}^{(2)}_{eq}$.
[gSup^2_{eq}:3.54,2.9.3]

Proof. $(a) \Rightarrow (b)$: Let $L = \mathrm{Con}(A, \mathsf{F})$ for some set of unary functions $\mathsf{F} \subseteq \mathscr{O}^{(1)}$. Further for an index set I let $\mathbf{a}, \mathbf{b}_i \in A^2$, $\rho_i \in L$, for all $i \in I$ and let $(x, y) \in \mathrm{gSup}\,^{(2)}_{eq}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I})$, i.e., in the transitive, symmetric closure of $\mathrm{gSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I})$. Then there exist $x_0, x_1, \ldots, x_n \in A$ such that $x = x_0$, $y = x_n$ and for $j = 0, \ldots, n-1$ either (x_j, x_{j+1}) or (x_{j+1}, x_j) is in $\mathrm{gSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I})$. This implies that for every $j = 0, \ldots, n-1$ there is an $f_j \in \mathscr{O}^{(1)}$ with $f_j\mathbf{b}_i \in \rho_i$ for all $i \in I$ and w.l.o.g. $f_j\mathbf{a} = (x_j, x_{j+1})$. For $f \in \mathsf{F}$ we find that then also $f \circ f_j\mathbf{a} = (fx_j, fx_{j+1}) \in \mathrm{gSup}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I})$ and thereby also $(fx, fy) \in \mathrm{gSup}^{(2)}_{eq}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I})$. This proofs that $\mathrm{gSup}^{(2)}_{eq}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) \in \mathrm{Con}(A, \mathsf{F}) = L$.

$(b) \Rightarrow (a)$: Let $\Theta \in \mathrm{ConEnd}L$. Then we can write Θ as

$$\begin{aligned} \Theta &= \bigcup_{\mathbf{a}\in\Theta} \{f\mathbf{a} | f \in \mathrm{End}L\} \\ &= \bigcup_{\mathbf{a}\in\Theta} \mathrm{gSup}^{(2)}_{eq}(\mathbf{a}, (\mathbf{b}_i)_{i\in I}, (\rho_i)_{i\in I}) \end{aligned}$$

by choosing $\mathbf{b}_i, \rho_i$ as we did in the proof of 3.26(i). Since by assumption L is closed w.r.t. $\mathrm{gSup}^{(2)}_{eq}$ this finishes the proof. $\square$

3.5 sInv – sEnd

Making use of our the characterizations given so far (especially of InvEnd and sInvAut) we turn to the following Galois connection (studied in [BPV02, BPS]):

(a) $\mathbf{E} = \mathscr{O}^{(1)}$ and $\mathbf{R} = \mathscr{R}$, i.e., all unary functions and all relations together with strong invariance and strong preservation.
We call the strongly preserving endomorphisms strong endomorphisms and write sEnd for $\mathrm{sPol}_{\mathscr{O}^{(1)}}$. We call strongly preserved relations strongly invariant relations and as above we write sInv for $\mathrm{sInv}_{\mathscr{R}}$.

We will only give the characterization for *finite* base set A.
We start by defining *sir-algebras* [4]. They will play the same role as for example Krasner algebras did in the case of sInv – Aut, i.e., they will be the Galois closed sets in the case of a finite base set. Their definition can be motivated by noting that compared to the closure w.r.t. InvEnd we should add closure w.r.t complementation for strong invariance however at the same time discard closure w.r.t. logical formulas involving "=", as else we would end up with bijective functions only.

Definition 3.56. Let $\mathsf{Q} \subseteq \mathscr{R}$. If $\mathsf{Q} = \mathrm{LOP}(\exists, \wedge, \vee, \neg)(\mathsf{Q})$ then Q is called a sir-algebra. The sir-algebra generated by some set of relations $\mathsf{Q} \subseteq \mathscr{R}$ will be denoted by $[\mathsf{Q}]_{\mathrm{sir}}$.

An immediate connection to Krasner algebras is given by the following lemma.

Lemma 3.57. Let Q be a sir-algebra. If the diagonal relation d_A is in Q then Q is a Krasner algebra.

Definition 3.58. Consider an equivalence relation Θ on our base set A and define $L := A/\Theta := \{a/\Theta | a \in A\}$. For every relation $\rho \in \mathscr{R}_L^{(m)}$ we associate a relation $\rho^\Theta \in \mathscr{R}_A^{(m)}$ and for every relation $\sigma \in \mathscr{R}_A^{(m)}$ a relation $\sigma/\Theta \in \mathscr{R}_L^{(m)}$ in the following (natural) way:

$$\begin{aligned} \rho^\Theta &:= \{(a_1, \ldots, a_m) \in A^m | (a_1/\Theta, \ldots, a_m/\Theta) \in \rho\}, \\ \sigma/\Theta &:= \{(a_1/\Theta, \ldots, a_m/\Theta) | (a_1, \ldots, a_m) \in \sigma\}. \end{aligned}$$

For $\mathsf{Q} \subseteq \mathscr{R}_L$ we define $\mathsf{Q}^\Theta := \{\rho^\Theta | \rho \in \mathsf{Q}\}$ and call it the Θ-extension of Q.
On the other hand for $\mathsf{R} \subseteq \mathscr{R}_A$ we define $\mathsf{R} := \{\rho/\Theta | \rho \in \mathsf{R}\} \subseteq \mathscr{R}_L$.

The property of being a sir-algebra is conserved under the maps $\mathsf{Q} \mapsto \mathsf{Q}^\Theta$ and $\mathsf{R} \mapsto \mathsf{R}/\Theta$ as is expressed in the following lemma.

Lemma 3.59. Let Θ and L be as above.
(1) If $\mathsf{Q} \subseteq \mathscr{R}_L$ is a sir-algebra Q^Θ is a sir-algebra.
(2) If $\mathsf{R} \subseteq \mathscr{R}_A$ is a sir-algebra the R/Θ is sir- algebra.

Proof. The lemma can be proved by directly checking the necessary conditions. □

[4] The name sir-algebra is derived from strongly invariant relations .

The following theorem now gives the important relationship between n sir-algebra and Krasner algebras through Θ extensions.

Theorem 3.60. Every sir-algebra is a Θ-extension of some Krasner-algebra. In particular, if $\mathsf{R} \subseteq \mathscr{R}_A$ is a sir-algebra and Θ is the least equivalence relation in $\mathsf{Q}^{(2)}$, $L = A/\Theta$ then R/Θ is a Krasner algebra and $\left(\mathsf{R}/\Theta\right)^{\Theta} = \mathsf{R}$. Conversely if $\mathsf{Q} \subseteq \mathscr{R}_L$ is a Krasneralgebra and $\Theta \in \mathrm{Eq}(A)$ then $\mathsf{Q}^{\Theta}$is a sir-algebra and trivially $\mathsf{Q}^{\Theta}/\Theta = \mathsf{Q}$.

Remark 3.61. The existence of a least equivalence in R is a consequence of the closure of R w.r.t. intersection and the finiteness of our base set A.

For the proof we will we use the following lemma.

Lemma 3.62. Let $\mathbf{R} \subseteq \mathscr{R}$ be a sir-algebra and let Θ be the least equivalence relation in $\mathbf{R}^{(2)}$. Let $\mathbf{a}, \mathbf{b} \in A^{(m)}$ then $\mathbf{a}\Theta\mathbf{b}$ implies

$$\mathbf{a} \in \rho \Longleftrightarrow \mathbf{b} \in \rho$$

for all $\rho \in \mathbf{R}$.

Proof. For every $\rho \in \mathbf{R}^{(m)}$, every $i \in m$ we define an equivalence relation in the following way:

$$\begin{aligned} F_i(\rho) := \{(a, b) \in A^2 | \forall x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{m-1}\, (x_0, \ldots, x_{i-1}, a, x_i + 1, \ldots, x_{m-1}) \in \rho \\ \Longleftrightarrow (x_0, \ldots, x_{i-1}, b, x_i + 1, \ldots, x_{m-1}) \in \rho\}. \end{aligned}$$

By construction $F_i(\rho) \in \mathrm{LOP}(\exists, \wedge, \vee, \neg)(\rho) \subseteq \mathrm{LOP}(\exists, \wedge, \vee, \neg)(\mathbf{R}) = \mathbf{R}$ (since $\mathbf{R}$ is a sir-algebra). This implies $\Theta \subseteq F_i(\rho)$.
Now let $\mathbf{a}, \mathbf{b} \in A^{(m)}$ with $\mathbf{a}\Theta\mathbf{b}$. Then $(a_i, b_i) \in F_i(\rho)$ for all $i \in m$ and thus

$$\begin{aligned} (a_0, \ldots, a_{m-1}) \in \rho &\iff (b_0, a_1, a_2 \ldots, a_{m-1}) \in \rho \\ &\iff (b_0, b_1, a_2, \ldots, a_{m-1}) \in \rho \\ &\ \vdots \\ &\iff (b_0, b_1, , b_2 \ldots, b_{m-1}) \in \rho \end{aligned}$$

which proves the lemma. ∎

The proof of the theorem now follows easily.

Proof. By lemma 3.59 R/Θ is a sir-algebra. Further we have $d_L = \Theta/\Theta$ so the the diagonal relation on L is in R, i.e., R is a Krasner algebra by lemma 3.57.
It remains to prove that every $\sigma \in \mathsf{R}$ is of the form ρ^{Θ} for some $\rho \in \mathsf{R}/\Theta$. We show $\sigma = \rho^{\Theta}$ for $\rho = \sigma/\Theta$. Obviously $\sigma \subseteq \rho^{\Theta}$. For the other inclusion let $(b_1, \ldots, b_m) \in \rho^{\Theta}$. Then there is a tuple $(a_1, \ldots, a_m) \in \sigma$ with $a_i \Theta b_i$ for all $i = 1, \ldots, m$. By lemma 3.62 this implies $(b_1, \ldots, b_m) \in \sigma$ which finishes the proof. ∎

For functions we can do a similar constructions as we did for relations.

Definition 3.63. Let $f \in \mathrm{Tr}_A$ be a unary function preserving an equivalence relation $\Theta \in [A]_{eq}$. Then we define

$$f/\Theta : L \to L, a/\Theta \mapsto f(a)/\Theta,$$

which is well defined function on $L = A/\Theta$ as can easily be checked. For a set of functions F preserving some equivalence Θ we define $\mathsf{F}/\Theta := \{f/\Theta | f \in \mathsf{F}\}$.

The following lemma shows the importance of this construction if f strongly preserves Θ.

Lemma 3.64. Let $f \in \mathrm{Tr}_A$ be a unary function strongly preserving a equivalence relation $\Theta \in [A]_{eq}$. Then f/Θ is an injective function. For $\rho \in \mathscr{R}_L$ we have

$$f \text{ strongly preserves } \rho^{\Theta} \Leftrightarrow f/\Theta \text{ is an automorphism of } \rho\,.$$

The proof is straightforward and can be found in [BPS]. As an immediate consequence we get the following corollary.

Corollary 3.65. Let Θ be an equivalence relation on A and let $\mathsf{Q} \subseteq \mathscr{R}_L$ be a Krasneralgebra on $L = A/\Theta$. Then

$$\mathrm{sEnd}(\mathsf{Q}^{\Theta}) = \{f \in \mathrm{Tr}_A | f/\Theta \in \mathrm{Aut}\mathsf{Q}\}.$$

The following lemma is the first step towards the characterization.

Lemma 3.66. Let $\mathsf{F} \subseteq \mathscr{O}^{(1)}$. Then $\mathrm{sInv}\mathsf{F}$ is a sir-algebra.

The proof can be found in [BPV02] and is mostly straight forward.
Different from the situation for e.g. $\mathrm{Inv} - \mathrm{End}$ the lemma does no longer hold for infinite base set as can be seen by the following example.

Example 3.67. Let $A = \mathbb{N}$ and define $\rho = \{(a+1, a) | a \in \mathbb{N}\}$. Then $f : \mathbb{N} \to \mathbb{N}, n \mapsto n+1$ strongly preserves ρ but $\sigma := \{a | \exists x (a, x) \in \rho\} = \mathbb{N} \setminus \{0\}$ is not strongly invariant.

Now we can give the characterization for Galois closed sets of relations.

Theorem 3.68. ([BPV02, Theorem 7.8], [BPS, Theorem 3.11]) Let A be finite and $\mathsf{Q} \subseteq \mathscr{R}_A$. The following are equivalent.

(1) $\mathsf{R} = \mathrm{sInvsEnd}\mathsf{R}$.

(2) $\mathsf{R} = \mathrm{sInv}H$ for some $H \subseteq \mathscr{O}^{(1)}$.

(3) R is a sir-algebra.[sir-algebra: 3.56]

(4) There exists an equivalence relation Θ in R and a Krasneralgebra Q on $L = A/\Theta$ so that $\mathsf{R} = \mathsf{Q}^{\Theta}$. [$()^{\Theta}$: 3.58]

Furthermore for all $\mathsf{Q} \subseteq \mathscr{R}$ we have $[\mathsf{Q}]_{sir} = \mathrm{sInvsEnd}\mathsf{Q}$.

Proof. (1) $\Rightarrow$ (2): trivial
(2) $\Rightarrow$ (3): follows from 3.66
(3) $\Rightarrow$ (4): is the content of 3.60
(4) $\Rightarrow$ (1): We will show that Q^Θ is closed under sInvsEnd, i.e., we will proof that $\mathsf{Q}^\Theta \subseteq$ sInvsEndQ^Θ (as the other inclusion always holds). Let $\sigma \in$ sInvsEndQ^Θ and define $\rho := \sigma/\Theta$. Then we show that $\sigma = \rho^\Theta$ and $\rho \in \mathsf{Q}$.
For the first statement let $t : A \to L, a \mapsto a/\Theta$ and define $u : L \to A$ to be some "choice function" so that $u(a/\Theta) \in a/\Theta$. Then $(u \circ t)/\Theta$ is the identity on L and so is an element of $\mathrm{Aut}_L\mathsf{Q}$. By lemma 3.64 this implies that $u \circ t \in \mathrm{sEnd}_A\mathsf{Q}^\Theta$, from which it easily follows that $\sigma = \rho^\Theta$.
For the second statement we show that $\rho \in$ sInvAut$\mathsf{Q} = \mathsf{Q}$. Let $g \in$ Aut$\mathsf{Q}$ then we find an $f \in \mathrm{Tr}_A$ that strongly preserves Θ and $g = f/\Theta$. From 3.64 we see that $f \in$ sEndQ^Θ. Since $\sigma \in$ sInvsEndQ^Θ, ρ is strongly invariant under f/Θ which finishes the proof.

The last statement of the theorem now follows easily. □

The characterization of Galois closed sets of functions can be done in a similar way as the characterization of Galois closed sets of relations was done. That is we will make use of a relation between sets of the form sEnd$\mathsf{Q}$ and Aut$\mathsf{R}$ (where Q, R will be sets of relations on different base sets related through an equivalence relation).

Theorem 3.69. ([BPS, Theorem 3.18], [BPV02, Proposition 7.9]) A set $\mathsf{F} \subseteq \mathrm{Tr}_A$ is Galois closed, i.e., $\mathsf{F} =$ sEndsInv$\mathsf{F}$ iff there exists a equivalence relation Θ on A and a permutation group $\mathsf{G} \subseteq \mathscr{S}_L$ on $L := A/\Theta$ such that

$$\mathsf{F} = \{f \in \mathscr{O}_A^{(1)} | f/\Theta \in \mathsf{G}\}.$$

Indeed let Θ be the least congruence in sInv$\mathsf{F}$ and $L := A/\Theta$. Then we have:

$$\mathrm{sEndsInv}\mathsf{F} = \{f \in \mathscr{O}_A^{(1)} | f/\Theta \in \langle \mathsf{F}/\Theta \rangle_{\mathscr{O}_L^{(1)}}\}\,.\ \left[./\Theta\text{: 3.63; } \langle . \rangle_{\mathscr{O}_L^{(1)}}\text{: 2.20}\right]$$

Proof. Let $\mathsf{F} =$ sEndsInv$\mathsf{F}$ then by theorem 3.68 there exists a Krasneralgebra Q such that sInv$\mathsf{F} = \mathsf{Q}^\Theta$. With the help of corollary 3.65 we find

$$\mathsf{F} = \mathrm{sEndsInv}\mathsf{F} = \{f \in \mathrm{Tr}_A | f \in \mathrm{sEnd}\mathsf{Q}^\Theta\} = \{f \in \mathrm{Tr}_A | f/\Theta \in \mathrm{Aut}\mathsf{Q}\},$$

which proves one implication.
For the other direction let $\mathsf{F} = \{f \in \mathrm{Tr}_A | f/\Theta \in \mathsf{G}\}$ for some group of permutations G on $L = A/\Theta$. Then by theorem 3.29 we know that $\mathsf{G} = \mathrm{Aut}_L\mathrm{sInv}_L\mathsf{G}$ and by lemma 3.64 we find $\mathsf{F} = \mathrm{sEnd}(\mathrm{sInv}\mathsf{G})^\Theta$.
The proof of the last statement of the theorem is now straightforward and is left to the reader.
□

The following table lists the operations and names that were introduced.

$\mathbf{C}$	2.2.1
$\mathbf{W}_s$	2.2.2
$\mathbf{Pr}^{(m)}$	2.2.3
$\mathbf{Qr}^{(m)}$	2.2.4
Boolean operations	2.2.5
Boolean system	2.2.6
LOP	2.3.1
weak Krasneralgebras, pre-Krasneralgebras and *Krasneralgebras*	2.3.2
sir - algebra	3.56
$\mathbf{IOP}_A$	2.4.1
$\mathbf{MIOP}_A$	2.4.2
$\mathbf{MVOP}_A$	2.4.3
$\langle . \rangle_{\omega-inv}$	3.41
$\langle \mathsf{Q} \rangle_{\bigcap} (\langle \mathsf{Q} \rangle_{\bigcup})$	2.7.1
algebraic closure system	2.7.2
Δ-complete	2.7.3
$[\ \ ]_{\text{f.s.}}$	2.8.1
locally definable	2.8.2
sSup	2.9.1
spSup	2.9.2
gSup	2.9.3
s-LOC,LOC	2.10
s-Loc,Loc	2.16
s-Loc$_o$, Loc$_o$	2.16.1
locally invertible	2.21
$\langle . \rangle, \langle . \rangle_{\mathscr{O}^{(1)}}, \langle . \rangle_{\mathscr{S}}$	2.20

Bibliography

[BF48] G. Birkhoff and O. Frink. Representation of lattices by sets. *Trans. Amer. Math. Soc.*, 64(MR 10):279, 1948.

[BGS] F. Börner, M. Goldstern, and S. Shelah. Automorphisms and strongly invariant relations. *preprint.*

[Bör00] F. Börner. *Krasneralgebren.* Habilitationsschrift, Universität Potsdam 1999. Logos-Verlag, 2000.

[BPS] F. Börner, R. Pöschel, and V. Sushchansky. sir-algebras and Galois connections between operations and relations. *preprint.*

[BPV02] F. Börner, R. Pöschel, and V.Sushchansky. Boolean system of relations and Galois connections. *Acta Sci. Math. (Szeged)*, 68:535–560, 2002.

[DKMP01] M. Droste, D. Kuske, R. McKenzie, and R. Pöschel. Complementary closed relational clones are not always krasner clones. *Algebra Universalis*, 45:155–160, 2001.

[GP08] M. Goldstern and M. Pinsker. A survey of clones on infinte sets. *Coll. Math.*, 59:365–403, 2008.

[Grä68] G. Grätzer. *Universal algebra.* D. Van Nostrand, 1968.

[Hod97] W. Hodges. *A shorter model theory.* Cambridge University press, 1997.

[Ihr93] Th. Ihringer. *Algemeine Algebra.* B. G. Teubner Stuttgart, 1993.

[Jón72] B. Jónsson. *Topics in universal algebra, Vol. 250.* Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[Jón91] B. Jónsson. The theory of binary relations. *in H.Andreka, D. Monk, I. B. Jónsson. (Eds.), Algebraic Logic. Pap. Colloq., Budapest 1988. (Colloq. Math. Soc. J. Bolyai 54), North Holland*, pages 245–292, 1991.

[PK79] R. Pöschel and L.A. Kaluznin. *Funktionen- und Relationenalgebren.* Deutscher Verlag der Wissenschaften, Berlin, 1979.

[Pös80] R. Pöschel. *A general Galois theory for operations and relations and concrete charcterization of related algebraic structures.* Report R-01/80. Zentralinstitut für Math. und Mech.,Berlin, 1980.

[Pös84] R. Pöschel. Closure properties for relational sytems with given endomorphism structure. *Beiträge zur Algebra und Geometrie*, 18:153–166, 1984.

[Pös03] R. Pöschel. *Galois connections for Operations and Relations.* In: K. Dnecke, M. Ern and S.L. Wismath (eds.), Galois connections and applications, Kluwer, Dordrecht,, 2003.

[Ros70] I.G. Rosenberg. Über die funktionale Vollständigkeit in den mehrwertigen Logiken. *Rozpravy Českoslonesnké Akad. věd, Ser. Math. Nat. Sci.*, 80:3–93, 1970.

[Ros72] I.G. Rosenberg. A classification of universal algebras by infinitary relations. *Algebra Universalis*, 1:350–354, 1972.

[Ros74] I.G. Rosenberg. Universal algebras with all operations of bounded range. *Colloq. Math.*, 30:177–185, 1974.

[Ros78] I.G. Rosenberg. The subalgebra system of direct powers. *Algebra Universalis*, 8:221–227, 1978.

[Ros79] I.G. Rosenberg. On a Galois connection between algebras and relations and its applications. *Contributions to general algebra (Proc. of the Klagenfurt conference 1978; H.Kautschtitsch, W.B. Müller, W. Nöbauer, eds.) Verlag J. Heyn Klagenfurt*, pages 273–289, 1979.

[Sto75] M.G. Stone. On endomorphism structure for algebras over a fixed set. *Coll. Math.*, 33:41–45, 1975.

[Sza78] L. Szabó. Conrete representation of related structures of universal algebras. *Acta. Sci. Math. (Szeged)*, 40:175–184., 1978.

[Wer74] H. Werner. Which partition lattices are congruence lattices?. *Coll. Math. Soc, Janos Bolyai, 14 Lattice theory, Szeged (Hungary)*, 1974.