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The Keller-Segel Model in $\mathbb{R}^d$: Global Existence in the Case of Linear and Non-Linear Diffusion

Institute for

Analysis and Scientific Computing

Vienna University of Technology

supervised by

Univ.Prof. Dr.rer.nat. Ansgar JÜNGEL

by

Stefan Schuchnigg
Buchengasse 155/150
1100 Wien

Declaration

Unless otherwise indicated in the text or references, this thesis is entirely the product of my own scholarly work. This thesis has not been submitted either in whole or part, for a degree at this or any other university or institution.

Wien, April 2011

Stefan Schuchnigg

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Abstract

The goal of this diploma thesis was to write out and extend the paper [14] by Martin Burger, Marco di Francesco and Yasim Dolak-Struss, dealing with two different versions of the so called Keller-Segel model, describing diffusion and movement of certain cells and a chemoattractant in a liquid. They achieved global-in-time existence under certain restrictions on the parameters and analysed the long time behaviour of the densities for linear and non-linear diffusion in the special case where overcrowding does not occur.
The author of this thesis was able to prove that similar results can be obtained for a wider class of differential equations both in the case of linear and non-linear diffusion, either for subcritical mass or for models where overcrowding is prevented. It was shown that under almost natural restrictions on the sensitivity and diffusivity functions, global-in-time solutions exist.

Contents

1 Introduction

1.1 Introduction to the Keller-Segel Model

The following short introduction to the class of Keller-Segel type differential equations is a brief version of the extensive historical survey of the Keller-Segel model given in [12].
We start by considering a density of cells $\rho(x,t)$ and a density of a chemoattractant $S(x,t)$. Assuming that the total mass of the cells is conserved, we conclude that the change of mass in a small volume $D \in \mathbb{R}^d$ is equal to the total flow through the boundary ∂D of D.
Hence,

$$\frac{\partial}{\partial t}\int_D \rho(x,t)\,dx \;=\; -\int_{\partial D} J_\rho(x,t)\cdot\nu\,ds,$$

where $J_\rho(x,t)$ denotes the flow of the density of cells and ν is the outward pointing unit surface normal. Now the flow should depend linearly on

1. the gradient of ρ (diffusion) and
2. the gradient of S (attraction).

Therefore we obtain

$$J_\rho(x,t) \;=\; -k_1(\rho,S)\nabla\rho + k_2(\rho,S)\nabla S,$$

for non-negative functions $k_1(x), k_2(x)$. Inserting this into the integral equation yields

$$\begin{aligned}\int_D \frac{\partial\rho(x,t)}{\partial t}\,dx &= \int_{\partial D} k_1(\rho,S)\frac{\partial\rho}{\partial\nu} - k_2(\rho,S)\frac{\partial S}{\partial\nu}\,ds \\ &= \int_D \operatorname{div}\left(k_1(\rho,S)\nabla\rho - k_2(\rho,S)\nabla S\right)\,dx,\end{aligned}$$

where we have used Gauss' divergence theorem. This equation is clearly fulfilled if ρ and S satisfy the pointwise differential equation

$$\frac{\partial\rho}{\partial t} \;=\; \operatorname{div}\left(k_1(\rho,S)\nabla\rho - k_2(\rho,S)\nabla S\right). \tag{1.1}$$

For an equation describing the behaviour of the chemoattractant S we shall assume that S diffuses and, taking into account that the chemoattractant is produced by the cells, we have a source term proportional to the density of cells and to a function $k_4(S) > 0$. Since this will lead to a behaviour where the total mass of S increases (for there is a source but no drain) we shall also assume that the chemoattractant is decomposed with a rate proportional to the density S. Therefore the integral equation reads

$$\int_D \frac{\partial S(x,t)}{\partial t}\,dx = \underbrace{\int_D \operatorname{div}\left(k_3(\rho,S)\nabla S\right)\,dx}_{diffusion} + \underbrace{\int_D k_4(S)\rho\,dx}_{source} - \underbrace{R\int_D S\,dx}_{drain},$$

and again, switching to the pointwise differential equation

$$\frac{\partial S}{\partial t} = \operatorname{div}\left(k_3(\rho,S)\nabla S\right) - RS + k_4(S)\rho. \tag{1.2}$$

Therefore we get the famous system of differential equations

$$\begin{cases} \frac{\partial\rho}{\partial t} &= \operatorname{div}\left(k_1(\rho,S)\nabla\rho - k_2(\rho,S)\nabla S\right) \\ \frac{\partial S}{\partial t} &= \operatorname{div}\left(k_3(\rho,S)\nabla S\right) - RS + k_4(S)\rho, \end{cases} \tag{1.3}$$

that was first presented by E. F. Keller and L. A. Segel in [13].
In the following we shall also assume that the diffusion of the chemical is much faster than the motion of the cells (due to diffusion and attraction), which will consequently lead to a parabolic-elliptic system:
If we consider a time intervall $[t_0, t_1]$ that is small compared to the motion of the cells but large compared to the motion of the chemoattractant, we can assume that the change in the density ρ is negligable. The second equation of (1.3) then reads

$$\frac{\partial S}{\partial t} = \operatorname{div}\left(k_3(\rho_0,S)\nabla S\right) - RS + k_4(S)\rho_0. \tag{1.4}$$

where $\rho_0 = \rho(t_0) \approx \rho(t_1)$. Since we assume that the chemical diffuses very fast, we expect that S has *almost* reached the equilibrium state (here we shall also assume that there *is* an equilibrium state for every ρ_0) at time t_1 and therefore

$$0 = \operatorname{div}\left(k_3(\rho(t_1),S(t_1))\nabla S(t_1)\right) - RS(t_1) + k_4(S(t_1))\rho(t_1). \tag{1.5}$$

Now we simply state that our solution shall statisfy this quasi-stationary equation for each time t_1 and obtain the coupled parabolic-elliptic system of non-linear partial differential equations

$$\begin{cases} \frac{\partial\rho}{\partial t} &= \operatorname{div}\left(k_1(\rho,S)\nabla\rho - k_2(\rho,S)\nabla S\right) \\ 0 &= \operatorname{div}\left(k_3(\rho,S)\nabla S\right) - RS + k_4(S)\rho. \end{cases} \tag{1.6}$$

In the following we will analyse two special cases of the system (1.6) for $x \in \mathbb{R}^d$.
In chapter 2 we will fix

$$\left.\begin{aligned} k_1(\rho,S) &= \varepsilon \\ k_2(\rho,S) &= f(\rho) \\ k_3(\rho,S) &= R = 1 \\ k_4(S) &= 1 \end{aligned}\right\} \Longrightarrow \begin{cases} \frac{\partial\rho}{\partial t} &= \varepsilon\Delta\rho - \operatorname{div}\left(f(\rho)\nabla S\right) \\ 0 &= \Delta S - S + \rho, \end{cases}$$

and obtain a model with linear diffusion, whereas in chapter 3 on the other hand we will fix

$$\left.\begin{array}{rcl} k_1(\rho,S) & = & \varepsilon m(\rho) \\ k_2(\rho,S) & = & m(\rho) \\ k_3(\rho,S) & = & R \;=\; 1 \\ k_4(S) & = & 1 \end{array}\right\} \implies \begin{cases} \frac{\partial \rho}{\partial t} & = \operatorname{div}\left(m(\rho)\nabla(\varepsilon\rho - S)\right) \\ 0 & = \Delta S - S + \rho. \end{cases}$$

This choice leads to a degenerated problem at points where $m(\rho(x,t)) = 0$ and the diffusion vanishes. Here, $\varepsilon \in \mathbb{R}^+$ and $f(x)$ and $m(x)$ are functions $\mathbb{R} \to \mathbb{R}$. Similar to the results in [14] (parabolic-elliptic-system) and [15] (parabolic-parabolic-system), where these two cases are studied for $f(x) = m(x) = x(1-x)$, we will achieve global-in-time existence and other properties such as positiviy and conservation of mass under almost natural restrictions on $f(x)$, $m(x)$ and ρ_0.

We will also analyse the existence of stationary solutions for both cases and come to the conclusion that there exist no stationary solutions in the case of linear diffusion if the mass is too small. In the case of non-linear diffusion on the other hand, we will prove existence of stationary solutions with arbitrarily small mass at least for one space-dimension.

Before we proceed, we will shortly recall definitions and basic results from the theory of Sobolev spaces both for further reference and to introduce the notation used in chapter 2 and 3.

1.2 Introduction to Sobolev Spaces

Here, we want to briefly recall some basic definitions and properties of Sobolev spaces. A detailed elaboration, and proofs of the stated propositions can be found in [2, Chapter 5] and [1] for the ordinary Sobolev spaces and [8] for the Sobolev spaces including time.
In the following, let $\Omega \subset \mathbb{R}^d$ be an open set, not necessarily bounded, but $\partial\Omega \in C^1$ if there is a boundary.

Definition 1.1 ($W^{m,p}$-spaces)**.** *Let $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. Then we define the Sobolev space $W^{m,p}(\Omega)$ as the set of functions $u \in L^p(\Omega)$ satisifying,*

$$D^\alpha u \in L^p(\Omega), \text{ for all multiindices } |\alpha| \leq m.$$

Here, D^α denotes the partial derivative in the sense of distributions.

The Sobolevspaces $W^{m,p}$ are reflexive Banachspaces with the following norms

$$\begin{aligned} \|u\|^p_{W^{m,p}(\Omega)} &= \sum_{|\alpha|\leq m} \|D^\alpha u\|^p_{L^p(\Omega)}, \text{ for } p < \infty \\ \|u\|_{W^{m,p}(\Omega)} &= \max_{|\alpha|\leq m} \|D^\alpha u\|_{L^p(\Omega)}, \text{ for } p = \infty. \end{aligned}$$

Definition 1.2 (H^m-spaces)**.** *Fixing $p = 2$ leads to an important class of Hilbert spaces $H^m(\Omega) := W^{m,2}(\Omega)$ with the inner product*

$$< \varphi, \psi >_{H^m} \;=\; \sum_{|\alpha|\leq m} < D^\alpha\varphi, D^\alpha\psi >_{L^2(\Omega)} .$$

Proposition 1.3 (Density)**.** *Let $1 \leq p < \infty$. Then $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.*

Definition 1.4 (H^{-m}-spaces)**.** *By definition $H^{-m}(\Omega) = (H^m(\Omega))'$ is the dual space of H^m.*

H^{-m} is a reflexive Banach space with the operator norm. For $u \in H^{-m}(\Omega)$ and $\varphi \in H^m(\Omega)$ we use the following notation

$$u(\varphi) \;=: < u, \varphi >_{H^{-m}},$$

indicating that for functions with enough regularity, for example $u \in L^2(\Omega)$ the H^{-m} bracket can be seen as the inner product:

$$< u, \varphi >_{H^{-m}} \;=\; u(\varphi) \;= < u, \varphi >_{L^2} .$$

Definition 1.5 ($W_0^{m,p}$-spaces)**.** *Similar to the $W^{m,p}$-spaces (taking proposition 1.3 into account) we define the space $W_0^{m,p}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{m,p}}}$ as the closure of $C_0^\infty(\Omega)$ with respect to the $W^{m,p}(\Omega)$-norm. Again, fixing $p = 2$ leads to a class of Hilbert spaces $H_0^m(\Omega)$ with the inner product given by definition 1.2.*

Since we want to study equations in space and time we have to introduce suitable Sobolev spaces that also inlcude time.

Definition 1.6 ($C^m(0,T,B)$-spaces)**.** *Let B be a reflexive Banach space and $T > 0$ and $m \in \mathbb{N}_0$. The space $C^m(0,T,B)$ defined as the set of all m-times continuous differentiable functions $u : [0,T] \to B$ with the norm*

$$\|u\|_{C^m(0,T,B)} \quad = \quad \sum_{i=0}^{m} \max_{0\le t\le T} \|u^{(i)}(t)\|_B$$

Definition 1.7 ($L^p(0,T,B)$-spaces)**.** *Let B be a reflexive Banach space and $T > 0$ and $p \in [1,\infty]$. The space $L^p(0,T,B)$ is the set of all measureable functions $u : [0,T] \to B$ satisfying*

$$\begin{aligned} \|u\|_{L^p(0,T,B)} \quad &= \quad \left(\int_0^T \|u(t)\|_B^p \, dt\right)^{1/p} \quad < \ \infty, \ \textit{for } p < \infty \\ \|u\|_{L^\infty(0,T,B)} \quad &= \quad \operatorname*{ess\,sup}_{0\le t\le T} \|u(t)\|_B \ < \ \infty. \end{aligned}$$

Finally let us introduce the standard space for dealing with parabolic equations:

Definition 1.8 ($W^{1,2}(0,T,V,H)$-spaces)**.** *Let V be a seperable reflexive Banach space and H a seperable Hilbert space such that there exists a continuous embedding $V \hookrightarrow H$. Then we define*

$$W^{1,2}(0,T,V,H) \quad := \quad \left\{ \psi \in L^2(0,T,V) \mid \frac{\partial}{\partial t}\psi \in L^2(0,T,V') \right\}.$$

Proposition 1.9. *Let the assumptions on V and H be the same as in definition 1.8. Then,*

- *$W^{1,2}(0,T,V,H)$ is a Banach space with the natural norm*

$$\|u\|_{W^{1,2}(0,T,V,H)} := \|u\|_{L^2(0,T,V)} + \|u_t\|_{L^2(0,T,V')}.$$

- *$C^1(0,T,V)$ is dense in $W^{1,2}(0,T,V,H)$.*
- *the embedding $W^{1,2}(0,T,V,H) \hookrightarrow C^0(0,T,H)$ is continuous.*
- *for $u \in W^{1,2}(0,T,V,H)$, the function $t \to \|u(t)\|_H$ is absolutely continuous (especially differentiable almost everywhere) and*

$$\frac{\partial}{\partial t}\|u(t)\|_H^2 \quad = \quad 2 < u_t(t), u(t) >_{V'} .$$

In order to prove boundedness (in the case where overcrowding is prevented), we will need the following two propositions. For a proof we refer to [6, Theorem 1.56 and Theorem 1.57].

Proposition 1.10. *Let $G \in C^1(\mathbb{R})$ with $G' \in L^\infty(\mathbb{R})$ and in addition $G(0) = 0$ if Ω is not bounded. Whenever $u \in W^{1,p}(\Omega)$ (for $1 \leq p < \infty$), $G(u)$ belongs to $W^{1,p}(\Omega)$ with $\nabla G(u) = G'(u)\nabla u$.*
If Ω is bounded and $u \in W_0^{1,p}(\bar{\Omega})$, then $G(u)$ also belongs to $W_0^{1,p}(\bar{\Omega})$.

Proposition 1.11 (Stampaccia's theorem)**.** *For $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega)$ or $u \in W_0^{1,p}(\bar{\Omega})$ (in the case of bounded domains). Then $[u]^+$,$[u]^-$ and $[u-c]^+$ for some constant $c > 0$ belong to $W^{1,p}(\Omega)$ (or $W_0^{1,p}(\bar{\Omega})$ respectively) and the weak derivatives are given by*

$$\begin{aligned} \nabla[u]^+ &= \chi_{\mathbb{R}^+}(u)\nabla u, \\ \nabla[u]^- &= \chi_{\mathbb{R}^-}(u)\nabla u \text{ and} \\ \nabla[u-c]^+ &= \chi_{\mathbb{R}^+}(u-c)\nabla u \end{aligned}$$

where $\chi_I(x)$ denotes the characteristic function statisfying $\chi_I(x) = 0$ if $x \notin I$ and $\chi_I(x) = 1$ if $x \in I$.

More tools and definitions that will be needed, especially in the case of non-linear diffusion, are to be found in the appendix (chapter 5) or directly before their application.

2 Linear Diffusion for General Attraction Terms

In this chapter we consider the Cauchy-problem for the parabolic-elliptic-system

$$\begin{cases} \frac{\partial \rho}{\partial t} = \varepsilon \Delta \rho - \operatorname{div}\left(f(\rho)\nabla S\right) \\ -\Delta S + S = \rho \\ \rho(x,0) = \rho_0(x) \end{cases} \tag{2.7}$$

where $x \in \mathbb{R}^d$, $d \geq 1$, $t \geq 0$, $\varepsilon > 0$, $f : \mathbb{R} \to \mathbb{R}$ locally Lipschitz continuous (with local Lipschitz constant L_M on the intervall $[-M, M]$), and $\rho_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. For reasons of simplicity we shall also assume that $f(0) = 0$. We will see later that this assumption is not necessary to prove existence of a local-in-time solution but it shortens some of the proofs and seems to be a natural restriction. Intuitively, it is evident that the particle flux originates only in diffusion if $\rho(x_0, t_0) = 0$ at a point $(x_0, t_0) \in \mathbb{R}^d \times \mathbb{R}^+$. In order to develop a suitable definition of weak solutions of (2.7) and prove the existence of solutions, we need some expertise on the linear theory first.

2.1 Non-Homogeneous Heat Equation in $\mathbb{R}^d$

We first turn our attention to the non-homogeneous heat equation on the whole space

$$\begin{cases} \frac{\partial \rho}{\partial t} = \varepsilon \Delta \rho + f \\ \rho(x,0) = \rho_0(x) \in L^2(\mathbb{R}^d) \\ (x,t) \in \mathbb{R}^d \times [0,T] \text{ and } f(x,t) \in L^2(\mathbb{R}^d \times [0,T]). \end{cases} \tag{2.8}$$

Luckily, due to the Fourier transform, an explicit representation formula for the solution of (2.8) can be derived at least for $\rho_0(x) \in C(\mathbb{R}^d)$ and $f(x,t) \in C([0,T] \times \mathbb{R}^d)$. We recall the following proposition (see for instance [2, Chapter 2.3]):

Proposition 2.12. *Let $\rho_0(x) \in C(\mathbb{R}^d)$ and $f(x,t) \in C([0,T]\times\mathbb{R}^d)$. Then the strong solution $\rho \in C^\infty((0,T) \times \mathbb{R}^d) \cap C([0,T] \times \mathbb{R}^d)$ of the heat equation is given by*

$$\rho(x,t) = (G * \rho_0)(x,t) + \int_0^t \int_{\mathbb{R}^d} G(x-y, t-s) f(y,s)\, dyds \tag{2.9}$$

wherein $G(x,t) = \frac{1}{(4\pi\varepsilon t)^{d/2}} e^{-\frac{|x|^2}{4\varepsilon t}}$ is called the fundamental solution of the heat equation or the heat convolution kernel.

Since the existence result in proposition 2.12 does not yield integrability we need to prove the following:

Corollary 2.13. *Let $\rho \in C^\infty((0,T)\times\mathbb{R}^d)\cap C([0,T]\times\mathbb{R}^d)$ be the strong solution of the non-homogeneous heat equation provided by proposition 2.12 and let the right hand side $f(x)\in C([0,T]\times\mathbb{R}^d)$ and the inital data $\rho_0\in C(\mathbb{R}^d)$ additionally be in $L^2([0,T]\times\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ respectively. Then, $\rho\in W^{1,2}(0,T,H^1(\mathbb{R}^d),L^2(\mathbb{R}^d))$.*

Proof. Multiplying the heat equation (2.8) by ρ and integrating with respect to x yields

$$\begin{aligned} \int_{\mathbb{R}^d}\frac{\partial\rho}{\partial t}\rho\,dx &= \int_{\mathbb{R}^d}\varepsilon\Delta\rho\cdot\rho+f\rho\,dx \\ \Rightarrow\ \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^d}\rho^2\,dx &= -\varepsilon\int_{\mathbb{R}^d}|\nabla\rho|^2\,dx+\int_{\mathbb{R}^d}f\rho\,dx, \end{aligned}$$

where we have used integration by parts. Using Young's inequality on the last term leads to

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^d}\rho^2\,dx+\varepsilon\int_{\mathbb{R}^d}|\nabla\rho|^2\,dx &\le \frac{\gamma}{2}\int_{\mathbb{R}^d}f^2\,dx+\frac{1}{2\gamma}\int_{\mathbb{R}^d}\rho^2\,dx \\ \Rightarrow\ \frac{\partial}{\partial t}\|\rho(.,t)\|^2_{L^2(\mathbb{R}^d)}+2\|\nabla\rho(.,t)\|^2_{L^2(\mathbb{R}^d)} &\le \|f(.,t)\|^2_{L^2(\mathbb{R}^d)}+\|\rho(.,t)\|^2_{L^2(\mathbb{R}^d)}, \end{aligned}$$

where we have fixed $\gamma=1$. Applying Gronwall's lemma (see for instance [2, Appendix j]) gives us

$$\begin{aligned} \|\rho(.,t)\|^2_{L^2(\mathbb{R}^d)} &\le e^t\cdot\left(\|\rho_0\|^2_{L^2(\mathbb{R}^d)}+\int_0^t\|f(.,s)\|^2_{L^2(\mathbb{R}^d)}\,ds\right) \\ &\le e^t\cdot\left(\|\rho_0\|^2_{L^2(\mathbb{R}^d)}+\|f\|^2_{L^2([0,T]\times\mathbb{R}^d)}\right) \\ \Rightarrow\ \|\rho\|^2_{L^2([0,T]\times\mathbb{R}^d)} &\le \left(e^T-1\right)\cdot\left(\|\rho_0\|^2_{L^2(\mathbb{R}^d)}+\|f\|^2_{L^2([0,T]\times\mathbb{R}^d)}\right), \end{aligned}$$

where we have integrated with respect to t. Therefore, by inserting the $L^2(\mathbb{R}^d)$-estimate of $\rho(.,t)$, we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\|\rho(.,t)\|^2_{L^2(\mathbb{R}^d)}+2\|\nabla\rho(.,t)\|^2_{L^2(\mathbb{R}^d)} &\le \|f(.,t)\|^2_{L^2(\mathbb{R}^d)} \\ &\quad+e^t\cdot\left(\|\rho_0\|^2_{L^2(\mathbb{R}^d)}+\|f\|^2_{L^2([0,T]\times\mathbb{R}^d)}\right). \end{aligned}$$

Integrating with respect to $0\le t\le T$ leads to

$$\|\rho(.,t)\|^2_{L^2([0,T]\times\mathbb{R}^d)}+2\|\nabla\rho\|^2_{L^2([0,T]\times\mathbb{R}^d)} \le e^T\cdot\left(\|\rho_0\|^2_{L^2(\mathbb{R}^d)}+\|f\|^2_{L^2([0,T]\times\mathbb{R}^d)}\right),$$

which implies that $\rho\in L^2(0,T,H^1(\mathbb{R}^d))$.

For the estimate of ρ_t we consider a testfunction $\varphi \in L^2(0,T,H^1(\mathbb{R}^d))$ and obtain similarly

$$\begin{aligned}
\int_0^T < \rho_t, \varphi >_{H^{-1}(\mathbb{R}^d)} \, dt &= \int_{[0,T]\times\mathbb{R}^d} \frac{\partial \rho}{\partial t} \varphi \, dxdt \\
&= \int_{[0,T]\times\mathbb{R}^d} \varepsilon \Delta \rho \cdot \varphi + f\varphi \, dxdt \\
&= -\varepsilon \int_{[0,T]\times\mathbb{R}^d} \nabla \rho \cdot \nabla \varphi \, dxdt + \int_{[0,T]\times\mathbb{R}^d} f \varphi \, dxdt \\
&\leq \varepsilon \|\nabla \rho\|_{L^2([0,T]\times\mathbb{R}^d)} \|\nabla \varphi\|_{L^2([0,T]\times\mathbb{R}^d)} \\
&\qquad + \|f\|_{L^2([0,T]\times\mathbb{R}^d)} \|\varphi\|_{L^2([0,T]\times\mathbb{R}^d)} \\
&\leq \big(\varepsilon \|\nabla \rho\|_{L^2([0,T]\times\mathbb{R}^d)} + \|f\|_{L^2([0,T]\times\mathbb{R}^d)}\big) \cdot \|\varphi\|_{L^2(0,T,H^1(\mathbb{R}^d))},
\end{aligned}$$

where we have used Cauchy Schwarz's inequality. Now the estimate of $\nabla\rho$, which we have already proven, shows that

$$\int_0^T < \rho_t, \varphi >_{H^{-1}(\mathbb{R}^d)} \, dt \leq C(\rho_0, f, T) \|\varphi\|_{L^2(0,T,H^1(\mathbb{R}^d))},$$

and therefore $\rho_t \in L^2(0,T,H^{-1}(\Omega))$.

□

To prove that we can apply this approach for $f(x,t) \in L^2(\mathbb{R}^d \times [0,T])$ and $\rho_0(x) \in L^2(\mathbb{R}^d)$ (but not necessarily continuous), and that the function defined by (2.9) does indeed exist and is a solution of (2.8) in some sense, we need some properties of the heat kernel, which are summarised in the following lemma.

Lemma 2.14. *The heat kernel G fulfils*

$$\|G(.,t)\|_{L^1(\mathbb{R}^d)} = 1 \tag{2.10}$$

$$\|\nabla G(.,t)\|_{L^p(\mathbb{R}^d)} = C(p,\varepsilon,d) t^{-\frac{dp-d+p}{2p}}, \ p \geq 1 \tag{2.11}$$

$$\int_{\mathbb{R}^d} \nabla G(x,t) \, dx = 0 \ \in \mathbb{R}^d \tag{2.12}$$

Proof. The estimates are easily obtained by straightforward computation.

$$\begin{aligned}
\|G(.,t)\|_{L^1} &= \frac{1}{(4\pi\varepsilon t)^{d/2}}\int_{\mathbb{R}^d} e^{-\frac{|x|^2}{4\varepsilon t}}\,dx \;=\; \Big(\frac{1}{(4\pi\varepsilon t)^{1/2}}\int_{\mathbb{R}} e^{-\frac{x^2}{4\varepsilon t}}\,dx\Big)^d \\
&= \left|\begin{array}{ccc} \frac{x}{\sqrt{4\varepsilon t}} & = & \frac{z}{\sqrt{2}} \\ \frac{dx}{\sqrt{4\varepsilon t}} & = & \frac{dz}{\sqrt{2}} \end{array}\right| \;=\; \Big(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{z^2}{2}}\,dz\Big)^d \;=\; 1
\end{aligned}$$

$$\begin{aligned}
\|\nabla G(.,t)\|^p_{L^p(\mathbb{R}^d)} &= \frac{1}{(4\pi\varepsilon t)^{dp/2}}\int_{\mathbb{R}^d}\left|\Big(-\frac{2x}{4\varepsilon t}\Big)e^{-\frac{|x|^2}{4\varepsilon t}}\right|^p dx \\
&= C(p,\varepsilon,d)t^{-\frac{dp}{2}-p}\int_{\mathbb{R}^d}|x|^p e^{-\frac{p|x|^2}{4\varepsilon t}}\,dx \;=\; \left|\begin{array}{ccc} x(\frac{p}{4\varepsilon t})^{1/2} & = & u \\ dx(\frac{p}{4\varepsilon t})^{d/2} & = & du \end{array}\right| \\
&= C(p,\varepsilon,d)t^{-\frac{dp}{2}-p+p/2+d/2}\underbrace{\int_{\mathbb{R}^d}|u|^p e^{-|u|^2}\,du}_{=C'(d,p)<\infty} \\
&= C(p,\varepsilon,d)t^{-p\frac{dp-d+p}{2p}}
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}^d}\frac{\partial}{\partial x_i}G(x,t)\,dx &= C\int_{\mathbb{R}^d} x_i e^{-\frac{|x|^2}{4\varepsilon t}}\,dx \\
&= C'\Big(\int_{-\infty}^{\infty} e^{-\frac{z^2}{4\varepsilon t}}\,dz\Big)^{d-1}\cdot\underbrace{\Big(\int_{-\infty}^{\infty} z e^{-\frac{z^2}{4\varepsilon t}}\,dz\Big)}_{=0} \;=\; 0
\end{aligned}$$

□

Now we have the proper tools to prove

Theorem 2.15. *Let $f(x,t) \in L^2(\mathbb{R}^d \times [0,T])$ and $\rho_0(x) \in L^2(\mathbb{R}^d)$. Then, the representation formula (2.9) yields a weak solution $\rho(x,t) \in W^{1,2}(0,T,H^1(\mathbb{R}^d),L^2(\mathbb{R}^d))$ of the heat equation (2.8) in the following sense:*

- $$\int_0^t < \frac{\partial}{\partial t}\rho, \varphi >_{H^{-1}} \, ds = -\varepsilon \int_0^t \int_{\mathbb{R}^d} \nabla\rho\cdot\nabla\varphi + f\varphi \, dxds$$
  $$\forall \varphi \in L^2(0,T,H^1(\mathbb{R}^d)) \tag{2.13}$$
- $\rho(.,0) = \rho_0(.)$

If, in addition, $f(x,t) \in L^2(0,T,H^m(\mathbb{R}^d))$ and $\rho_0(x) \in H^m(\mathbb{R}^d)$ for some $m \in \mathbb{N}_0$, then there even holds $\rho(x,t) \in L^2(0,T,H^{m+1}(\mathbb{R}^d)) \cap L^\infty(0,T,H^m(\mathbb{R}^d))$.

Proof. Let us start by considering sequences

$$\begin{aligned}
f_n \in C^\infty([0,T]\times\mathbb{R}^d) \cap L^2([0,T]\times\mathbb{R}^d), &\qquad f_n \to f \text{ in } L^2([0,T]\times\mathbb{R}^d) \text{ and} \\
\rho_{0,n} \in C^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d), &\qquad \rho_{0,n} \to \rho_0 \text{ in } L^2(\mathbb{R}^d).
\end{aligned}$$

The existence of these sequences is justified by proposition 1.3, telling us that the space $C^\infty([0,T]\times\mathbb{R}^d)\cap L^2([0,T]\times\mathbb{R}^d)$ is dense in $L^2([0,T]\times\mathbb{R}^d)$. According to proposition 2.12 we now define, for all $n\in\mathbb{N}$, ρ_n as the strong solution of the heat equation (2.8) given by the representation formula (2.9) with source f_n and inital data $\rho_{0,n}$.
Our first aim is to prove that the sequence $\rho_n\in C^\infty((0,T)\times\mathbb{R}^d)\cap C([0,T]\times\mathbb{R}^d)\cap W^{1,2}$ is a Cauchy sequence in $W^{1,2}:=W^{1,2}(0,T,H^1(\mathbb{R}^d),L^2(\mathbb{R}^d))$.
Let, for fixed $m,n\in\mathbb{N}$, $\rho_n-\rho_m=:\bar\rho$, $f_n-f_m=:\bar f$ and $\rho_{0,n}-\rho_{0,m}=:\bar\rho_0$. Subtracting the differential equations for ρ_n and ρ_m yields

$$\frac{\partial}{\partial t}\bar\rho=\varepsilon\Delta\bar\rho+\bar f \tag{2.14}$$

multiplying by $\bar\rho$ and integrating over $\mathbb{R}^d$ leads to

$$\begin{aligned}
\int_{\mathbb{R}^d}\frac{\partial}{\partial t}\bar\rho\cdot\bar\rho\,dx &= \varepsilon\int_{\mathbb{R}^d}\Delta\bar\rho\cdot\bar\rho+\bar f\bar\rho\,dx\\
\Rightarrow\ \frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^d}\bar\rho^2\,dx &= -\varepsilon\int_{\mathbb{R}^d}\nabla\bar\rho\cdot\nabla\bar\rho+\bar f\bar\rho\,dx\\
\Rightarrow\ \frac{\partial}{\partial t}\|\bar\rho\|^2_{L^2(\mathbb{R}^d)} &\le -2\varepsilon\|\nabla\bar\rho\|^2_{L^2(\mathbb{R}^d)}+\|\bar f\|^2_{L^2(\mathbb{R}^d)}+\|\bar\rho\|^2_{L^2(\mathbb{R}^d)} \qquad (2.15)\\
\Rightarrow\ \frac{\partial}{\partial t}\|\bar\rho\|^2_{L^2(\mathbb{R}^d)} &\le \|\bar f\|^2_{L^2(\mathbb{R}^d)}+\|\bar\rho\|^2_{L^2(\mathbb{R}^d)},
\end{aligned}$$

where we have used integration by parts and Young's inequality. Applying Gronwall's lemma gives us

$$\begin{aligned}
\|\bar\rho(.,t)\|^2_{L^2(\mathbb{R}^d)} &\le e^t\cdot(\|\bar\rho_0\|^2_{L^2(\mathbb{R}^d)}+\|\bar f\|^2_{L^2(\mathbb{R}^d\times[0,t])})\\
&\le e^t\cdot(\|\bar\rho_0\|^2_{L^2(\mathbb{R}^d)}+\|\bar f\|^2_{L^2(\mathbb{R}^d\times[0,T])})\\
\Rightarrow\ \|\bar\rho\|^2_{L^2(\mathbb{R}^d\times[0,T])} &\le (e^T-1)\cdot(\|\bar\rho_0\|^2_{L^2(\mathbb{R}^d)}+\|\bar f\|^2_{L^2(\mathbb{R}^d\times[0,T])}). \qquad (2.16)
\end{aligned}$$

For the estimate of the gradient of $\bar\rho$ we take the inequality (2.15) and integrate with respect to t from 0 to T

$$\begin{aligned}
\|\bar\rho(.,T)\|^2_{L^2(\mathbb{R}^d)}-\|\bar\rho_0\|^2_{L^2(\mathbb{R}^d)} &\le -2\varepsilon\|\nabla\bar\rho\|^2_{L^2(\mathbb{R}^d\times[0,T])}+\|\bar f\|^2_{L^2(\mathbb{R}^d\times[0,T])}+\|\bar\rho\|^2_{L^2(\mathbb{R}^d\times[0,T])}\\
\Rightarrow\ 2\varepsilon\|\nabla\bar\rho\|^2_{L^2(\mathbb{R}^d\times[0,T])} &\le \|\bar\rho_0\|^2_{L^2(\mathbb{R}^d)}+\|\bar f\|^2_{L^2(\mathbb{R}^d\times[0,T])}+\|\bar\rho\|^2_{L^2(\mathbb{R}^d\times[0,T])}\\
\Rightarrow\ 2\varepsilon\|\nabla\bar\rho\|^2_{L^2(\mathbb{R}^d\times[0,T])} &\le e^T\cdot(\|\bar\rho_0\|^2_{L^2(\mathbb{R}^d)}+\|\bar f\|^2_{L^2(\mathbb{R}^d\times[0,T])}), \qquad (2.17)
\end{aligned}$$

where we have used the estimate (2.16) for $\|\bar\rho\|_{L^2(\mathbb{R}^d\times[0,T])}$ from above.
To achieve convergence in $W^{1,2}$ we need an additional estimate for $\|\frac{\partial}{\partial t}\bar\rho\|_{L^2(0,T,H^{-1}(\mathbb{R}^d))}$. For this purpose let $\varphi\in L^2(0,T,H^1(\mathbb{R}^d))$, $\|\varphi\|_{L^2(0,T,H^1(\mathbb{R}^d))}\le 1$. Then, multiplying equation

(2.14) by φ and integrating over $\mathbb{R}^d \times [0,T]$ yields

$$\begin{aligned}
\int_0^T < \frac{\partial}{\partial t}\bar{\rho}, \varphi >_{H^{-1}(\mathbb{R}^d)} \, dt &= \int_{\mathbb{R}^d \times [0,T]} \frac{\partial}{\partial t}\bar{\rho}\varphi \, dxdt \\
&= -\varepsilon \int_{\mathbb{R}^d \times [0,T]} \nabla\bar{\rho} \cdot \nabla\varphi + \bar{f}\varphi \, dxdt \\
&\leq \varepsilon \|\nabla\bar{\rho}\|_{L^2(\mathbb{R}^d \times [0,T])} \|\nabla\varphi\|_{L^2(\mathbb{R}^d \times [0,T])} \\
&\qquad + \|\bar{f}\|_{L^2(\mathbb{R}^d \times [0,T])} \|\varphi\|_{L^2(\mathbb{R}^d \times [0,T])},
\end{aligned}$$

where we applied Cauchy-Schwarz's inequality. Taking into account the assumption on φ and the estimate (2.17) for the gradient of $\bar{\rho}$ leads to

$$\begin{aligned}
\int_0^T < \frac{\partial}{\partial t}\bar{\rho}, \varphi >_{H^{-1}(\mathbb{R}^d)} \, dt &\leq \frac{e^T}{2} \cdot (\|\bar{\rho}_0\|^2_{L^2(\mathbb{R}^d)} + \|\bar{f}\|^2_{L^2(\mathbb{R}^d \times [0,T])}) + \|\bar{f}\|_{L^2(\mathbb{R}^d \times [0,T])} \\
&= \frac{e^T}{2} \|\bar{\rho}_0\|^2_{L^2(\mathbb{R}^d)} + (\frac{e^T}{2} + 1) \|\bar{f}\|^2_{L^2(\mathbb{R}^d \times [0,T])}. \qquad (2.18)
\end{aligned}$$

In conclusion we obtain

$$\begin{aligned}
\|\bar{\rho}\|^2_{W^{1,2}} &= \|\bar{\rho}\|^2_{L^2(0,T,L^2(\mathbb{R}^d))} + \|\nabla\bar{\rho}\|^2_{L^2(0,T,L^2(\mathbb{R}^d))} + \|\bar{\rho}\|^2_{L^2(0,T,H^{-1}(\mathbb{R}^d))} \\
&\leq \big(e^T \frac{3\varepsilon + 1}{2\varepsilon} - 1\big) \cdot \|\bar{\rho}_0\|^2_{L^2(\mathbb{R}^d)} + e^T \frac{3\varepsilon + 1}{2\varepsilon} \|\bar{f}\|^2_{L^2(\mathbb{R}^d \times [0,T])} \qquad (2.19)
\end{aligned}$$

Since $f_n \to f$ in $L^2([0,T] \times \mathbb{R}^d)$ and $\rho_{0,n} \to \rho_0$ in $L^2(\mathbb{R}^d)$, the difference $\|\rho_m - \rho_n\|_{W^{1,2}}$ vanishes as $m, n \to \infty$ and therefore ρ_n is a Cauchy sequence. Since $W^{1,2}$ is a Banach space this implies that the sequence converges to $\rho' \in W^{1,2}$ in the sense of $W^{1,2}$. As already used, the functions ρ_n naturally satisfy the weak formulation of the heat equation (2.13) by simply multiplying by φ, integrating over $\mathbb{R}^d \times [0,T]$ and applying integration by parts. Because of the convergence in $W^{1,2}$ we just proved we can pass the limit and establish the weak formulation for ρ'

$$\int_0^t < \frac{\partial}{\partial t}\rho', \varphi >_{H^{-1}} \, ds = -\varepsilon \int_0^t \int_{\mathbb{R}^d} \nabla\rho' \cdot \nabla\varphi + f\varphi \, dxds$$

for all functions $\varphi \in L^2(0,T,H^1(\mathbb{R}^d))$ and $\rho'(.,0) = \rho_0(.)$. In order to complete the proof we have to show that $\rho' = \rho$, wherein ρ denotes the function we obtain by inserting f and ρ_0 into the representation formula (2.9). It is, of course, sufficient to prove that $\|\rho_n -$

$\rho\|_{L^2(\mathbb{R}^d\times[0,T])} \to 0$.

$$\begin{aligned}
\|\rho_n - \rho\|_{L^2(\mathbb{R}^d)} &\le \|G*(\rho_{0,n}-\rho_0)\|_{L^2(\mathbb{R}^d)} + \left\|\int_0^t G(.,t-s)*(f_n(.,s)-f(.,s))\,ds\right\|_{L^2(\mathbb{R}^d)} \\
&\le \|G\|_{L^1(\mathbb{R}^d)}\|(\rho_{0,n}-\rho_0)\|_{L^2(\mathbb{R}^d)} \\
&\qquad + \int_0^t \|G(.,t-s)*(f_n(.,s)-f(.,s))\|_{L^2(\mathbb{R}^d)} ds \\
&\le \|(\rho_{0,n}-\rho_0)\|_{L^2(\mathbb{R}^d)} + \int_0^t \|G(.,t-s)\|_{L^1(\mathbb{R}^d)}\|f_n(.,s)-f(.,s)\|_{L^2(\mathbb{R}^d)}\,ds \\
&\le \|(\rho_{0,n}-\rho_0)\|_{L^2(\mathbb{R}^d)} + \int_0^t \|f_n(.,s)-f(.,s)\|_{L^2(\mathbb{R}^d)}\,ds \\
&\le \|(\rho_{0,n}-\rho_0)\|_{L^2(\mathbb{R}^d)} + t^{1/2}\cdot\|f_n(.,s)-f(.,s)\|_{L^2(\mathbb{R}^d\times[0,t])} \\
&\le \|(\rho_{0,n}-\rho_0)\|_{L^2(\mathbb{R}^d)} + t^{1/2}\cdot\|f_n(.,s)-f(.,s)\|_{L^2(\mathbb{R}^d\times[0,T])},
\end{aligned}$$

where we have used the inequalties of Hölder and Cauchy-Schwarz and $\|G\|_{L^1(\mathbb{R}^d)} = 1$.
By construction of the sequences, we can find, for every $\delta > 0$, an integer $N(\delta) \in \mathbb{N}$ such that $\|(\rho_{0,n}-\rho_0)\|_{L^2(\mathbb{R}^d)} < \delta$ and $\|f_n(.,s)-f(.,s)\|_{L^2(\mathbb{R}^d\times[0,T])} < \delta$. Integrating with respect to t yields

$$\begin{aligned}
\|\rho_n-\rho\|^2_{L^2(\mathbb{R}^d)\times[0,T]} &\le \int_0^T (\delta + t^{1/2}\delta)^2\,dt \\
&\le \frac{T\delta^2}{6}(3T + 8T^{1/2} + 6) \overset{\delta\to 0}{\longrightarrow} 0.
\end{aligned}$$

For the higher regularity we can assume that $f_n \to f$ in $L^2(0,T,H^m(\mathbb{R}^d))$ and $\rho_{0,n} \to \rho_0$ in $H^m(\mathbb{R}^d)$. Just like before, we subtract the strong formulation of the heat equation for ρ_n and ρ_m, respectively. Since $\rho_n, \rho_m \in C^\infty((0,T)\times\mathbb{R}^d)$, we can differentiate the equation

$$\frac{\partial}{\partial t}D^\alpha(\rho_n-\rho_m) = \varepsilon\Delta D^\alpha(\rho_n-\rho_m) + D^\alpha(f_n-f_m)$$

for some multiindex $|\alpha| \le m$. Multiplying by $\bar\rho_\alpha := D^\alpha(\rho_n-\rho_m)$ and integrating with respect to x leads to

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^d}\bar\rho_\alpha^2\,dx &= -\varepsilon\int_{\mathbb{R}^d}|\nabla\bar\rho_\alpha|^2\,dx + \int_{\mathbb{R}^d}\bar\rho_\alpha\bar f_\alpha\,dx \\
\Rightarrow \frac{\partial}{\partial t}\|\bar\rho_\alpha(.,t)\|^2_{L^2(\mathbb{R}^d)} &\le -2\varepsilon\|\nabla\bar\rho_\alpha(.,t)\|^2_{L^2(\mathbb{R}^d)} + \gamma\|\bar\rho_\alpha(.,t)\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{\gamma}\|\bar f_\alpha(.,t)\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Choosing $\gamma = 1$, omitting the norm of $\nabla\bar{\rho}_\alpha$ and applying Gronwall's lemma leads to

$$
\begin{aligned}
\|\bar{\rho}_\alpha(.,t)\|^2_{L^2(\mathbb{R}^d)} &\leq e^t\|\bar{\rho}_\alpha(.,0)\|^2_{L^2(\mathbb{R}^d)} + \int_0^t e^{t-s}\|\bar{f}_\alpha(.,s)\|^2_{L^2(\mathbb{R}^d)}\,ds \\
&\leq e^t\|\bar{\rho}_\alpha(.,0)\|^2_{L^2(\mathbb{R}^d)} + e^t\int_0^t \|\bar{f}_\alpha(.,s)\|^2_{L^2(\mathbb{R}^d)}\,ds \\
&\leq e^t\|\bar{\rho}_\alpha(.,0)\|^2_{L^2(\mathbb{R}^d)} + e^t\|\bar{f}_\alpha\|^2_{L^2([0,t]\times\mathbb{R}^d)} \\
&\leq e^T\|\bar{\rho}_\alpha(.,0)\|^2_{L^2(\mathbb{R}^d)} + e^T\|\bar{f}_\alpha\|^2_{L^2([0,T]\times\mathbb{R}^d)} \\
\Rightarrow\ \|\bar{\rho}_\alpha(.,t)\|_{L^\infty(0,T,L^2(\mathbb{R}^d))} &\to 0,\ \text{for } n,m\to\infty.
\end{aligned}
$$

Here we used that $\|\bar{f}_\alpha\|_{L^2([0,T]\times\mathbb{R}^d)} \leq \|f_n - f_m\|_{L^2(0,T,H^m(\mathbb{R}^d))}$ for $|\alpha| \leq m$ (and the same relation for the inital data). Taking again the inequality from above and integrating with respect to t yields

$$
\begin{aligned}
\|\bar{\rho}_\alpha(.,t)\|^2_{L^2(\mathbb{R}^d)} + 2\varepsilon\|\nabla\bar{\rho}_\alpha(.,t)\|^2_{L^2([0,T]\times\mathbb{R}^d)} \leq\ & \|\bar{\rho}_\alpha(.,0)\|^2_{L^2(\mathbb{R}^d)} + \|\bar{\rho}_\alpha(.,t)\|^2_{L^2([0,T]\times\mathbb{R}^d)} \\
&+\|\bar{f}_\alpha(.,t)\|^2_{L^2([0,T]\times\mathbb{R}^d)}.
\end{aligned}
$$

Together with the previous result we have

$$
\|\nabla\bar{\rho}_\alpha(.,t)\|_{L^2([0,T]\times\mathbb{R}^d)} \to 0,\ \text{for } n,m\to\infty.
$$

Since this holds true for every multiindex $\alpha, |\alpha| \leq m$, we can conclude that ρ_n converges in $L^2(0,T,H^{m+1}(\mathbb{R}^d)) \cap L^\infty(0,T,H^m(\mathbb{R}^d))$. □

Unfortunately, we will see that $f \in L^2(0,T,L^2(\mathbb{R}^d))$ is not wide enough to cover the situation we will have to deal with later on. Therefore, we will prove a more general version of the previous lemma.

Theorem 2.16. *Let $f(x,t) = \operatorname{div}(V(x,t))$, $V \in \left(L^\infty(0,T,L^2(\mathbb{R}^d))\right)^d$. Then, the function ρ defined by*

$$
\rho(x,t) = (G*\rho_0)(x,t) - \int_0^t\int_{\mathbb{R}^d}\nabla G(x-y,t-s)\cdot V(y,s)\,dyds \tag{2.20}
$$

belongs to $W^{1,2} := W^{1,2}(0,T,H^1(\mathbb{R}^d),L^2(\mathbb{R}^d))$ and satisfies, for all $\varphi \in L^2(0,T,H^1(\mathbb{R}^d))$, the following equation

$$
\int_0^t <\frac{\partial}{\partial t}\rho,\varphi>_{H^{-1}}\,ds = -\int_0^t\int_{\mathbb{R}^d}\varepsilon\nabla\rho\cdot\nabla\varphi + V\cdot\nabla\varphi\,dxds. \tag{2.21}
$$

In addition, there holds $\rho(.,0) = \rho_0(.)$ and therefore ρ is called a weak solution of the heat equation.

Proof. The proof will basically be the same as in the previous lemma and we will therefore keep the calculations shorter. In addition, we can assume that the initial data $\rho_0 = 0$. The

solution for arbitrary ρ_0 is then given by the sum of the solution ρ_h of the homogeneous heat equation with inital data ρ_0 (for which the previous lemma can be applied) and ρ_p, the solution of the non-homogeneous heat equation with inital data equal to zero.
We start by taking a sequence $V_n \in \left(C^\infty([0,T]\times\mathbb{R}^d)\cap L^\infty(0,T,L^2(\mathbb{R}^d))\right)^d$ that converges towards V in $\left(L^\infty(0,T,L^2(\mathbb{R}^d))\right)^d$. Then the sequence $f_n = \mathrm{div}(V_n) \in C^\infty([0,T]\times\mathbb{R}^d)$ leads to a sequence $\rho_n \in C^\infty((0,T)\times\mathbb{R}^d)\cap C([0,T]\times\mathbb{R}^d)\cap W^{1,2}$ of solutions of the heat equation (see remark 2.17), such that there holds

$$\rho_n(x,t) = (G*\rho_0)(x,t) + \int_0^t\int_{\mathbb{R}^d} G(x-y,t-s)\ \mathrm{div}(V_n(y,s))\,dyds \tag{2.22}$$

$$= (G*\rho_0)(x,t) - \int_0^t\int_{\mathbb{R}^d} \nabla G(x-y,t-s)\cdot V_n(y,s)\,dyds \tag{2.23}$$

$$\frac{\partial\rho_n}{\partial t} = \varepsilon\Delta\rho_n + \mathrm{div}(V_n). \tag{2.24}$$

Now we consider for fixed $n,m\in\mathbb{N}$, the differences $\bar\rho = \rho_n - \rho_m$ and $\bar V = V_n - V_m$.
Let $\|V_n - V_m\|_{(L^2([0,T]\times\mathbb{R}^d))^d} \le \delta$. Then, by using the same procedure as in the proof of theorem 2.15 we obtain

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|\bar\rho\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} -\varepsilon\nabla\bar\rho\cdot\nabla\bar\rho - \bar V\cdot\nabla\bar\rho\,dx\\ &\le -\varepsilon\|\nabla\bar\rho\|^2_{L^2(\mathbb{R}^d)} + \frac{\alpha}{2}\|\nabla\bar\rho\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2\alpha}\|\bar V\|^2_{(L^2(\mathbb{R}^d))^d}.\end{aligned}$$

And therefore by integrating with respect to t and then fixing $\alpha=\varepsilon$ we obtain in the same way as before

$$\begin{aligned}\|\bar\rho\|_{L^2([0,T]\times\mathbb{R}^d)} &\le C(\varepsilon,T)\|\bar V\|_{(L^\infty(0,T,L^2(\mathbb{R}^d)))^d} \le C(\varepsilon,T)\cdot\delta\\ \|\nabla\bar\rho\|_{L^2([0,T]\times\mathbb{R}^d)} &\le C'(\varepsilon,T)\|\bar V\|_{(L^\infty(0,T,L^2(\mathbb{R}^d)))^d} \le C'(\varepsilon,T)\cdot\delta.\end{aligned}$$

This estimates allow us to easily estimate $\|\frac{\partial}{\partial t}\bar\rho\|_{L^2(0,T,H^{-1}(\mathbb{R}^d))} \le C''(\varepsilon,T)\cdot\delta$.
Therefore ρ_n is a Cauchy sequence in $W^{1,2}$ and converges towards a function $\rho' \in W^{1,2}$. By using the exact same arguments as in theorem 2.15 we can pass the limit in the weak formulation and achieve for all $\varphi\in L^2(0,T,H^1(\mathbb{R}^d))$

$$\int_0^t < \frac{\partial}{\partial t}\rho',\varphi >_{H^{-1}} ds = -\int_0^t\int_{\mathbb{R}^d}\varepsilon\nabla\rho'\cdot\nabla\varphi + V\cdot\nabla\varphi\,dxds.$$

Now we need to identify ρ' with ρ given by the formula (2.20). This can be done easily by looking at the difference between ρ_n and ρ:

$$
\begin{aligned}
\|\rho_n - \rho\|_{L^2(\mathbb{R}^d)} &\leq \left\| \int_0^t \nabla G(.,t-s) * (V_n(.,s) - V(.,s))\, ds \right\|_{L^2(\mathbb{R}^d)} \\
&\leq \int_0^t \|\nabla G(.,t-s)\|_{L^1(\mathbb{R}^d)} \|V_n(.,s) - V(.,s)\|_{(L^2(\mathbb{R}^d))^d}\, ds \\
&\leq C(d,\varepsilon)\ \|V_n - V\|_{(L^\infty(0,T,L^2(\mathbb{R}^d)))^d} \int_0^t (t-s)^{-1/2}\, ds \\
&\leq C(d,\varepsilon)\ \delta\ t^{1/2} \\
&\leq C(d,\varepsilon,T)\ \delta\ \to\ 0.
\end{aligned}
$$

□

Remark 2.17. *For reasons of clarity we skipped a similar result to corollary 2.13 in the case of* $f(x,t) = \operatorname{div}(V(x,t))$, $V \in \left(L^\infty(0,T,L^2(\mathbb{R}^d))\right)^d$. *The proof is mainly the same to the one given above, where* $f \in L^2([0,T] \times \mathbb{R}^d)$.

2.2 A Special Elliptic Equation

We now direct our attention to the elliptic equation

$$
-\Delta S(x) + S(x) = f(x)\ ,\ x \in \mathbb{R}^d \tag{2.25}
$$

for $f \in L^r(\mathbb{R}^d)$, $r \geq 1$. Just as in the previous section, thanks to Fourier transformation, there exists an explicit representation of the solution for smooth $f(x)$ (see for instance [4, Chapter 12, Example 8]).

Proposition 2.18. *Let* $f(x) \in C^\infty(\mathbb{R}^d)$ *with compact support. Then there exists a unique strong solution* $S \in \mathcal{S}(\mathbb{R}^d)$ *of equation (2.25) given by*

$$
S = B * f,\ \text{where } B(x) := \int_0^\infty \frac{e^{-t-\frac{|x|^2}{4t}}}{(4\pi t)^{d/2}}\, dt \tag{2.26}
$$

is called the Bessel potential. Here $\mathcal{S}(\mathbb{R}^d)$ *denotes the Schwartz space or space of rapidly decreasing functions.*

$$
\mathcal{S} := \left\{ f \in C^\infty(\mathbb{R}^d)\ |\ \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta f(x)| < \infty\ \forall \alpha, \beta \right\}, \tag{2.27}
$$

where α *and* β *are arbitrary multi-indices.*

Just as before we need the following properties of B.

Lemma 2.19.

$$
\|B\|_{L^1(\mathbb{R}^d)} = 1 \tag{2.28}
$$

$$
\|\nabla B\|_{L^1(\mathbb{R}^d)} = C < \infty \tag{2.29}
$$

Proof. Straightforward compution will lead to the desired properties.

$$\begin{aligned}
\|B\|_{L^1(\mathbb{R}^d)} &= \frac{1}{(4\pi)^{d/2}} \int_0^\infty \frac{e^{-t}}{t^{d/2}} \underbrace{\int_{\mathbb{R}^d} e^{-\frac{|x|^2}{4t}}\, dx}_{=(4\pi t)^{d/2}}\, dt \\
&= \int_0^\infty e^{-t} dt \;=\; 1
\end{aligned}$$

$$\begin{aligned}
\|\nabla B\|_{L^1(\mathbb{R}^d)} &= \frac{1}{(4\pi)^{d/2}} \int_0^\infty \frac{e^{-t}}{t^{d/2}} \int_{\mathbb{R}^d} \frac{|x|}{2t} e^{-\frac{|x|^2}{4t}}\, dxdt \\
&= \left| \begin{matrix} x(\frac{1}{4t})^{1/2} & = & u \\ dx(\frac{1}{4t})^{d/2} & = & du \end{matrix} \right| \;=\; C(d) \int_0^\infty \frac{e^{-t}}{t^{1/2}} \underbrace{\int_{\mathbb{R}^d} |u| e^{-|u|^2}\, du}_{=C'(d)}\, dt \\
&= C''(d) \underbrace{\int_0^\infty \frac{e^{-t}}{t^{1/2}}\, dt}_{=\sqrt{\pi}} \;=\; \bar{C}(d) < \infty
\end{aligned}$$

□

Now we can introduce a suitable defintion of weak solutions of (2.25) and prove existence and uniqueness.

Theorem 2.20. *For every $f(x) \in L^2(\mathbb{R}^d)$ the function $S(x)$ defined by $S = B * f$, where B is the Bessel potential, is an element of the Sobolev space $H^1(\mathbb{R}^d)$ and satisfies*

$$\int_{\mathbb{R}^d} \nabla S \cdot \nabla \varphi + S\varphi \; dx = \int_{\mathbb{R}^d} f\varphi \; dx \tag{2.30}$$

for all $\varphi \in H^1(\mathbb{R}^d)$. Therefore S is called a weak solution.
In addition, if $\bar{S} \in H^1(\mathbb{R}^d)$ satisfies the weak formulation (2.30), then $\bar{S} = S$.

Proof. Just like before we want to use an approximation argument. Let $f_n \in C_0^\infty(\mathbb{R}^d)$ be a sequence that converges towards f in the sense of L^2. Let S_n be the corresponding sequence defined by $S_n = B * f_n$. The previous proposition tells us that $S_n \in \mathcal{S}(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d)$ are strong solutions of the elliptic equation $-\Delta S_n(x) + S_n(x) = f_n(x)$.
Since $f_n, f \in L^2(\mathbb{R}^d)$, it is easy to see that $S_n, S \in L^2(\mathbb{R}^d)$, where $S = B * f$.

$$\begin{aligned}
\|S_n\|_{L^2(\mathbb{R}^d)} &= \|B * f_n\|_{L^2(\mathbb{R}^d)} \;\le\; \|B\|_{L^1(\mathbb{R}^d)} \|f_n\|_{L^2(\mathbb{R}^d)} \;=\; \|f_n\|_{L^2(\mathbb{R}^d)} \\
\|S\|_{L^2(\mathbb{R}^d)} &= \|B * f\|_{L^2(\mathbb{R}^d)} \;\le\; \|B\|_{L^1(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} \;=\; \|f\|_{L^2(\mathbb{R}^d)}
\end{aligned}$$

Subtracting the differential equations for S_m and S_n from each other, multiplying the result by $S_n - S_m$ and integrating over $\mathbb{R}^d$ yields

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla(S_n - S_m)|^2 + (S_n - S_m)^2\,dx &= \int_{\mathbb{R}^d} (S_n - S_m)(f_n - f_m)\,dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}^d} (S_n - S_m)^2\,dx + \frac{1}{2}\int_{\mathbb{R}^d} (f_n - f_m)^2\,dx,\end{aligned}$$

where we have used Young's inequality. Therefore

$$\begin{aligned}\|S_n - S_m\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\|\nabla(S_n - S_m)\|^2_{L^2(\mathbb{R}^d)} &\leq \frac{1}{2}\|f_n - f_m\|^2_{L^2(\mathbb{R}^d)} \\ \Rightarrow \|S_n - S_m\|^2_{H^1(\mathbb{R}^d)} &\leq \|f_n - f_m\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Hence, S_n converges in $H^1(\mathbb{R}^d)$ towards some $S' \in H^1(\mathbb{R}^d)$. The S_n clearly satisify the weak formulation (2.30) of the differential equation. Due to the convergence in $H^1(\mathbb{R}^d)$ we can pass the limit in the weak formulation and obtain for all $\varphi \in H^1(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \nabla S' \cdot \nabla\varphi + S'\varphi\,dx = \int_{\mathbb{R}^d} f\varphi\,dx.$$

Now we need to identify S' with S. We do so by simply looking at the difference between S_n and S

$$\begin{aligned}\|S_n - S\|_{L^2(\mathbb{R}^d)} &= \|B * (f_n - f)\|_{L^2(\mathbb{R}^d)} \\ &\leq \|B\|_{L^1(\mathbb{R}^d)}\|f_n - f\|_{L^2(\mathbb{R}^d)} \\ &\leq \|f_n - f\|_{L^2(\mathbb{R}^d)} \to 0,\end{aligned}$$

and therefore $S_n \to S = S'$.

In order to prove uniqueness, we subtract the weak formulations for S and $\bar{S}$ and use the difference $\varphi = S - \bar{S} \in H^1(\mathbb{R}^d)$ as testfunction

$$\begin{aligned}\int_{\mathbb{R}^d} \nabla(S - \bar{S}) \cdot \nabla(S - \bar{S}) + (S - \bar{S})^2\,dx &= \int_{\mathbb{R}^d} (f - f)(S - \bar{S})\,dx \\ \Rightarrow \|\nabla(S - \bar{S})\|^2_{L^2(\mathbb{R}^d)} + \|S - \bar{S}\|^2_{L^2(\mathbb{R}^d)} &= 0,\end{aligned}$$

and therefore $S = \bar{S}$. □

Corollary 2.21 (Higher regularity)**.** *Let $f \in H^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then the solution $S(x)$ of (2.30) given by the convolution $S = B * f$ features also improved regularity, namely*

- $\|\nabla S\|_{L^\infty(\mathbb{R}^d)} \leq C\|f\|_{L^\infty(\mathbb{R}^d)}$
- $\|S\|_{H^2(\mathbb{R}^d)} \leq C\|f\|_{H^1(\mathbb{R}^d)}$.

Proof. These two inequalities can easily be obtained by using the representation of S

$$\begin{aligned}\|\nabla S\|_{L^\infty(\mathbb{R}^d)} &= \|\nabla B * f\|_{L^\infty(\mathbb{R}^d)} \\ &\leq \|\nabla B\|_{L^1(\mathbb{R}^d)}\|f\|_{L^\infty(\mathbb{R}^d)} \\ &\leq C\|f\|_{L^\infty(\mathbb{R}^d)},\end{aligned}$$

where we have used Hölder's inequality and estimate (2.29). We continue analogously for the H^2-estimate

$$\begin{aligned}
\|\frac{\partial^2}{\partial x_i x_j} S\|_{L^2(\mathbb{R}^d)} &= \|\frac{\partial B}{\partial x_i} * \frac{\partial f}{\partial x_j}\|_{L^2(\mathbb{R}^d)} \\
&\leq \|\nabla B\|_{L^1(\mathbb{R}^d)}\|\nabla f\|_{L^2(\mathbb{R}^d)} \\
&\leq C\|\nabla f\|_{H^1(\mathbb{R}^d)}.
\end{aligned}$$

□

2.3 Weak Solutions, Local-in-Time Existence

Our aim in this section is to prove existence and uniqueness of a local solution of (2.7). Because we have explicit representations of the non-homogeneous heat equation and the elliptic equation we define a solution of our problem as a fixed point of an operator $\mathcal{T}$. To define $\mathcal{T}$ properly, we need to establish the underlying space first.

Definition 2.22. *Let $T > 0$ and $M > 0$, then we define*

$$\begin{aligned}
X_T &:= L^\infty(0,T,L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)) \\
\|f\|_{X_T} &:= \|f\|_{L^\infty(0,T,L^1(\mathbb{R}^d))} + \|f\|_{L^\infty(0,T,L^\infty(\mathbb{R}^d))} \ \forall f \in X_T \\
X_T^M &:= \{f \in X_T | \ \|f\|_{X_T} \leq M\}.
\end{aligned}$$

Definition 2.23. *Let $\rho_0 \in L^1 \cap L^\infty$, then for $\rho \in X_T^M$ we formally define*

$$\mathcal{T}(\rho)(x,t) := (G * \rho_0)(x,t) + \int_0^t \int_{\mathbb{R}^d} \nabla G(x-y,t-s) \cdot (f(\rho)\nabla(B * \rho))(y,s)\, dyds. \tag{2.31}$$

In order to define $\mathcal{T}$ rigorously (which will be done in lemma 2.25) and identify a fixed point of $\mathcal{T}$ with some kind of weak solution we are looking for, we need the following short lemma.

Lemma 2.24. *Let $\rho \in X_T^M$ for some constants $T, M > 0$. Then the function defined by $g := f(\rho)\nabla(B * \rho)$ is in $\left(L^\infty(0,T,L^2)\right)^d$.*

Proof.

$$\begin{aligned}
\|g\|_{(L^2(\mathbb{R}^d))^d} &= \|f(\rho)\nabla(B * \rho)\|_{(L^2(\mathbb{R}^d))^d} \\
&\leq \|f(\rho)\|_{L^\infty(\mathbb{R}^d)}\|\nabla B * \rho\|_{(L^2(\mathbb{R}^d))^d} \\
&\leq L_M\|\rho\|_{L^\infty(\mathbb{R}^d)}\|\nabla B\|_{L^1(\mathbb{R}^d)}\|\rho\|_{L^2(\mathbb{R}^d)} \\
&\leq CL_M\|\rho\|_{L^\infty(\mathbb{R}^d)}\|\rho\|_{L^2(\mathbb{R}^d)} \\
&= CL_M\|\rho\|_{L^\infty(\mathbb{R}^d)}\|\rho^2\|_{L^1(\mathbb{R}^d)}^{1/2} \\
&\leq CL_M\|\rho\|_{L^\infty(\mathbb{R}^d)}\|\rho\|_{L^1(\mathbb{R}^d)}^{1/2}\|\rho\|_{L^\infty(\mathbb{R}^d)}^{1/2} \\
&= CL_M\|\rho\|_{L^\infty(\mathbb{R}^d)}^{3/2}\|\rho\|_{L^1(\mathbb{R}^d)}^{1/2},
\end{aligned}$$

where we have used $\|\nabla B\|_{L^1(\mathbb{R}^d)} < C$ (corollary 2.21 of the previous section) and Hölder's inequality. Taking the supremum yields

$$\begin{aligned}
\|g\|_{(L^\infty(0,T,L^2(\mathbb{R}^d)))^d} &\leq CL_M\|\rho\|^{3/2}_{L^\infty(0,T,L^\infty(\mathbb{R}^d))}\|\rho\|^{1/2}_{L^\infty(0,T,L^1(\mathbb{R}^d))} \\
&\leq CL_M M^2
\end{aligned}$$

□

Due to theorem 2.16, this allows us to understand $\mathcal{T}(\rho)$ to be the weak solution of the non-homogeneous heat equation (2.7). This will be useful both in the following and in addition a justification to work with $\mathcal{T}$ in the first place.
To prove the existence of a fixed point we will take advantage of the Banach fixed point theorem (also known as the contraction mapping theorem, see for instance [5, Chapter 1]). It provides us with existence and uniqueness of fixed points of contractive self-maps of banach spaces.

Theorem 2.25. *There exist $M > 0$ and $T > 0$ such that $\mathcal{T}$ is a self mapping, i.e.*

$$\mathcal{T}: \ X_T^M \to X_T^M.$$

Proof. Let $\rho \in X_T^M$ and $1 \leq r \leq \infty$, then using estimates (2.10), (2.11) and Hölder's inequality several times, yields

$$\begin{aligned}
\|\mathcal{T}(\rho)\|_{L^r} &\leq \|(G*\rho_0)(x,t)\|_{L^r} + \left\|\int_0^t\int_{\mathbb{R}^d}\nabla G(x-y,t-s)(f(\rho)\nabla(B*\rho))(y,s)\,dyds\right\|_{L^r} \\
&\leq \underbrace{\|G\|_{L^1}}_{=1}\cdot\|\rho_0\|_{L^r} + \int_0^t\|(\nabla G(.,t-s)*(f(\rho)\nabla(B*\rho))(.,s))\|_{L^r}\,ds \\
&\leq \|\rho_0\|_{L^r} + \int_0^t\|(\nabla G)(.,t-s)\|_{L^1}\cdot\|(f(\rho)\nabla(B*\rho))(.,s)\|_{L^r}\,ds \\
&\leq \|\rho_0\|_{L^r} + C(d)\cdot\int_0^t(t-s)^{-1/2}\cdot\|f(\rho(.,s))\|_{L^\infty}\|(\nabla B*\rho)(.,s)\|_{L^r}\,ds \\
&\leq \|\rho_0\|_{L^r} + C(d)\cdot\int_0^t(t-s)^{-1/2}\cdot\|f(\rho(.,s))\|_{L^\infty}\|\nabla B\|_{L^1}\|\rho(.,s)\|_{L^r}\,ds \\
&\leq \|\rho_0\|_{L^r} + C'(d)\cdot\|\rho\|_{L^\infty(0,T,L^r)}\|f(\rho)\|_{L^\infty(0,T,L^\infty)}\int_0^t(t-s)^{-1/2}\,ds \\
&\leq \|\rho_0\|_{L^r} + C_d\cdot\|\rho\|_{L^\infty(0,T,L^r)}\|f(\rho)\|_{L^\infty(0,T,L^\infty)}\cdot t^{1/2}.
\end{aligned}$$

Since $\rho \in X_T^M \ \Rightarrow \|\rho\|_{L^\infty(0,T,L^\infty)} \leq M$ and $f(x)$ is Lipschitz continuous with $f(0) = 0$, we get:

$$\|\mathcal{T}(\rho)\|_{L^\infty(0,T,L^r)} \leq \|\rho_0\|_{L^r} + C_d\cdot\|\rho\|_{L^\infty(0,T,L^r)}\cdot\sup_{|x|\leq M}|f(x)|\cdot T^{1/2} \tag{2.32}$$

$$\leq \|\rho_0\|_{L^r} + C_d M L_M\cdot\|\rho\|_{L^\infty(0,T,L^r)}T^{1/2}. \tag{2.33}$$

In particular if we insert $r = 1, \infty$

$$\begin{aligned}
\|\mathcal{T}(\rho)\|_{L^\infty(0,T,L^1)} &\leq \|\rho_0\|_{L^1} + C_d M^2 L_M T^{1/2} \\
\|\mathcal{T}(\rho)\|_{L^\infty(0,T,L^\infty)} &\leq \|\rho_0\|_{L^\infty} + C_d M^2 L_M T^{1/2}.
\end{aligned}$$

In conclusion we obtain:

$$\|\mathcal{T}(\rho)\|_{X_T} \leq C_1 + T^{1/2} \cdot (C_2 M^2 L_M) \overset{!}{\leq} M$$

with constants $C_1 = \|\rho_0\|_{L^1} + \|\rho_0\|_{L^\infty}$ and C_2, C_3 depending only on the dimension d and the diffusion constant ε. The inequality above is clearly satisified if

$$C_1 \leq M \text{ and } T^{1/2} \leq \frac{M - C_1}{C_2 M^2 L_M}. \tag{2.34}$$

$\square$

Theorem 2.26. *There exist $M > 0$ and $T > 0$ such that $\mathcal{T}$ is a contraction:*

$$\|\mathcal{T}(f) - \mathcal{T}(g)\|_{X_T} < \|f - g\|_{X_T} \quad \forall f, g \in X_T^M.$$

Proof. We calculate the difference between two functions ρ_1 and ρ_2 using similar methods as in the proof of the theorem above. Note that the term including ρ_0 cancels immediately. Let $\rho_1, \rho_2 \in X_T^M$ and $1 \leq r \leq \infty$

$$\begin{aligned}
\|\mathcal{T}(\rho_1) - \mathcal{T}(\rho_2)\|_{L^r} &\leq \\
&\leq \left\| \int_0^t \big(\nabla G(.,t-s)\big) * \Big[\big(f(\rho_1)(\nabla B * \rho_1) - f(\rho_2)(\nabla B * \rho_2)\big)(.,s)\Big]\, ds \right\|_{L^r} \\
&\leq C \cdot t^{1/2} \cdot \|f(\rho_1)(\nabla B * \rho_1) - f(\rho_2)(\nabla B * \rho_2)\|_{L^\infty(0,T,L^r)} \\
&= C \cdot t^{1/2} \cdot \|f(\rho_1)\big(\nabla B * (\rho_1 - \rho_2)\big) + \big(f(\rho_1) - f(\rho_2)\big)(\nabla B * \rho_2)\|_{L^\infty(0,T,L^r)} \\
&\leq C \cdot t^{1/2} \cdot \|\nabla B\|_{L^1} \cdot \|f(\rho_1)\|_{L^\infty(0,T,L^\infty)} \cdot \|\rho_1 - \rho_2\|_{L^\infty(0,T,L^r)} \\
&\qquad + C \cdot t^{1/2} \cdot \|\nabla B\|_{L^1} \cdot \|f(\rho_1) - f(\rho_2)\|_{L^\infty(0,T,L^r)} \cdot \|\rho_2\|_{L^\infty(0,T,L^\infty)} \\
&\leq C' \cdot t^{1/2} \cdot \big(L_M \cdot M + L_M \cdot M\big) \cdot \|\rho_1 - \rho_2\|_{L^\infty(0,T,L^r)} \\
&\leq C'' \cdot t^{1/2} \cdot L_M \cdot M \cdot \|\rho_1 - \rho_2\|_{L^\infty(0,T,L^r)}.
\end{aligned}$$

Here, we have used again that $f(x)$ is Lipschitz continuous and $f(0) = 0$. Similarly to the proof above we now insert $r = 1, \infty$ and obtain

$$\begin{aligned}
\|\mathcal{T}(\rho_1) - \mathcal{T}(\rho_2)\|_{L^\infty(0,T,L^1)} &\leq C \cdot T^{1/2} \cdot L_M \cdot M \cdot \|\rho_1 - \rho_2\|_{L^\infty(0,T,L^1)} \\
\|\mathcal{T}(\rho_1) - \mathcal{T}(\rho_2)\|_{L^\infty(0,T,L^\infty)} &\leq C \cdot T^{1/2} \cdot L_M \cdot M \cdot \|\rho_1 - \rho_2\|_{L^\infty(0,T,L^\infty)}
\end{aligned}$$

and therefore, for some constant $C_3(\varepsilon, d)$,

$$\|\mathcal{T}(\rho_1) - \mathcal{T}(\rho_2)\|_{X_T} \quad \leq \quad C_3 \cdot T^{1/2} \cdot L_M \cdot M \cdot \|\rho_1 - \rho_2\|_{X_T} \overset{!}{\leq} \|\rho_1 - \rho_2\|_{X_T}.$$

Hence, if

$$T^{1/2} \leq (C_3 L_M M)^{-1}, \tag{2.35}$$

then $\mathcal{T}$ is a contraction. □

Now we easily conclude

Corollary 2.27 (Existence and uniqueness of a fixed point)**.** *Let $\rho_0 \in L^1 \cap L^\infty$, then there exists a fixed point $\rho(x,t) \in X_T^M$ of $\mathcal{T}$ for some $T > 0$.*

Proof. If there is a pair M, T that fulfils inequalities (2.34) and (2.35) of the two previous theorems then $\mathcal{T}$ ist a contractive self-mapping on the banach space X_T^M and the banach fixed point theorem provides us with a unique fixed point in X_T^M. Let $M = 2C_1$ and

$$\begin{aligned} T^{1/2} &= \min\left(\frac{1}{C_3 L_M M}, \frac{M - C_1}{C_2 M^2 L_M}\right) \\ &= \min\left(\frac{1}{2C_3 L_M C_1}, \frac{1}{4C_2 C_1 L_M}\right) \\ &= \frac{C_4}{C_1 L_M} \end{aligned}$$

for some constant C_4 depending only on ε and d and $C_1 = \|\rho_0\|_{L^1} + \|\rho_0\|_{L^\infty}$. This choice clearly satisfies the two inequalties and therefore the proof is complete. □

The following corollary will conclude this section and summarise the results.

Corollary 2.28 (Local existence of weak solution)**.** *Let $\rho_0 \in L^1 \cap L^\infty$, then there exists a unique $\rho \in X_T^M \cap W^{1,2}(0, T, H^1(\mathbb{R}^d), L^2(\mathbb{R}^d))$ that satisifies*

- $$\int_0^t < \frac{\partial}{\partial t}\rho, \varphi >_{H^{-1}} \, ds = -\varepsilon \int_0^t \int_{\mathbb{R}^d} \nabla\rho \cdot \nabla\varphi + \big(f(\rho)\nabla(B * \rho)\big) \cdot \nabla\varphi \, dx ds$$
  $$\forall \varphi \in L^2(0, T, H^1(\mathbb{R}^d)) \tag{2.36}$$
- $\rho(.,0) = \rho_0(.)$

for some $T, M > 0$.

Proof. Due to corollary 2.27 there exists a fixed point ρ of $\mathcal{T}$ and due to theorem 2.16 from the previous section we can interpret ρ as a weak solution of the non-homogeneous heat equation with source $\mathrm{div}(f(\rho)\nabla(B * \rho))$ and inital data ρ_0. □

Remark 2.29 (Initial data in L^p)**.**

The strong assumptions on the initial data are not necessary to achieve a fixed point and therefore local existence. In fact, $\rho_0 \in L^p(\mathbb{R}^d)$ for some $p \geq 1$ is sufficient. To see this, we can simply define $\mathcal{T}$ on $L^\infty(0, T, L^p(\mathbb{R}^d))$. If we want to prove that $\mathcal{T}$ is a self-map and a contraction, we have to perform the same steps as in the proof above, fixing $r = p$ instead of $r = 1, \infty$. We will later see that important properties like mass conservation and even global existence do not necessarily depend on the higher regularity we required. We made the extra effort because in that way we can interpret ρ as weak solution of (2.7) and this will be crucial to prove decay estimates as t goes to infinity (section 2.4).

2.4 Properties of the Local Solution and Global Existence

In this section we want to study the long time behaviour of our system. Naturally, the characteristic for large time or even the existence of a global-in-time solution do heavily depend on the behaviour of f. We can however, without any further requirements, prove global existence and strong decay estimates if the inital mass is not to large. Therefore, in this section, we will not impose any further restrictions on f, we shall just assume that f is Lipschitz continuous and $f(0) = 0$.
We start by proving some properties (namely positivity, mass conservation and uniform L^∞-estimates) of the local solution we found in the previous section. These properties will help us find a global solution. The idea will be to take a T-step in time and take $\rho(., T)$ as the new initial value. If we can guarantee that $\|\rho(., t)\|_{L^1} + \|\rho(., t)\|_{L^\infty}$ does not increase, we can take another step with the same stepsize T and so on.

Theorem 2.30 (Positivity)**.** *Let $\rho_0(x) > 0$, then the local weak solutions $\rho(x, t)$ and $S(x, t)$ of (2.7) are positive for all $0 \leq t \leq T$.*

Proof. Let us for a moment consider $\tilde{f}(x) = \chi_{\mathbb{R}^+}(x) f(x)$, where $\chi_{\mathbb{R}^+}(x)$ is the characteristic function of $\mathbb{R}^+$. Since $\tilde{f}(x)$ is clearly Lipschitz continuous ($f(0) = 0$), we can find a weak solution for our new sensitivity function. Using $\varphi = [\rho]^- = \min(0, \rho)$ as testfunction (see proposition 1.11) in the weak formulation (2.36) leads to

$$\begin{aligned}
\int_0^t < \frac{\partial}{\partial t}\rho, [\rho]^- >_{H^{-1}} ds &= -\varepsilon \int_0^t \int_{\mathbb{R}^d} \nabla\rho \cdot \nabla[\rho]^- + \tilde{f}(\rho)\nabla(B * \rho) \cdot \nabla[\rho]^- \, dxds \\
\Rightarrow \; \frac{1}{2}\int_0^t \frac{\partial}{\partial t} \|[\rho]^-\|^2_{L^2(\mathbb{R}^d)} \, ds &= -\varepsilon \|\nabla[\rho]^-\|^2_{L^2(0,T,L^2(\mathbb{R}^d))} \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \underbrace{[\rho]^- \tilde{f}(\rho)}_{=0} \nabla(B * \rho) \cdot \nabla\rho \, dxds \\
&= -\varepsilon \|\nabla[\rho]^-\|^2_{L^2(0,T,L^2(\mathbb{R}^d))}.
\end{aligned}$$

And therefore

$$\|[\rho]^-(.,t)\|^2_{L^2(\mathbb{R}^d)} + 2\varepsilon\|\nabla[\rho]^-\|^2_{L^2(0,T,L^2(\mathbb{R}^d))} \quad = \quad \underbrace{\|[\rho]^-(.,0)\|^2_{L^2(\mathbb{R}^d)}}_{=0}$$

where we have used that $\rho_0(x) > 0$. Clearly, $[\rho]^-(x,t) = 0 \Leftrightarrow \rho(x,t) \geq 0$.
Since $\rho(x) \geq 0$, the change from $f(x)$ to $\tilde{f}(x)$ has not changed anything after all, because clearly $f(x)$ and $\tilde{f}(x)$ are equal on the range of ρ.
For the positivity of S we follow the same procedure. Taking $\varphi = [S]^- = \min(0,S)$ as testfunction in the weak formulation of the elliptic equation yields

$$\begin{aligned}
\int_{\mathbb{R}^d} \nabla S(.,t)\cdot\nabla[S]^-(.,t)\,dx + \int_{\mathbb{R}^d} S(.,t)[S]^-(.,t)\,dx \quad &= \quad \int_{\mathbb{R}^d} \underbrace{\rho(.,t)}_{\geq 0}\underbrace{[S]^-(.,t)}_{\leq 0}\,dx \\
\Rightarrow \quad \|\nabla[S]^-(.,t)\|^2_{L^2(\mathbb{R}^d)} + \|[S]^-(.,t)\|^2_{L^2(\mathbb{R}^d)} \quad &\leq \quad 0
\end{aligned}$$

and therefore $[S]^-(x,t) = 0 \Leftrightarrow S(x,t) \geq 0$. $\square$

Next, we turn our attention to the conservation of mass. Since we can write our model in divergence form $\frac{\partial}{\partial t}\rho = \operatorname{div}(\varepsilon\nabla\rho - f(\rho)\nabla S)$, we expect the mass to be conserved and we will see that this is naturally incorporated in this model.

Theorem 2.31 (Mass conservation). *Let $M(t) := \int_{\mathbb{R}^d}\rho(x,t)dx$. Then $M(t) = M(0) =: M_0$ for all $t \leq T$.*

Proof. Using the representation formula of our weak solution leads directly to

$$\begin{aligned}
M(t) \quad &= \quad \int_{\mathbb{R}^d} \rho(x,t)\,dx \\
&= \quad \int_{\mathbb{R}^d} (G*\rho_0)(x,t)\,dx + \int_0^t\int_{\mathbb{R}^d}\underbrace{\int_{\mathbb{R}^d}\nabla G(x-y,t-s)\,dx}_{=\,0}\cdot(f(\rho)\nabla(B*\rho))(y,s)\,dyds \\
&= \quad \int_{\mathbb{R}^d}\rho_0(x,t)\,dx \;=\; M_0.
\end{aligned}$$

$\square$

Knowing that the L^1-norm of $\rho(x,t)$ is conserved (and $0 \leq \rho(x,t)$, assuming that the inital data is non-negative), we still need a global L^∞-bound of ρ. As mentioned before, this can be done by either restricting the initial data or the sensivity function $f(x)$. In the following, we turn our attention to the case of subcritical initial mass, however we will discuss properties of $f(x)$ that will lead to a prevention of overcrowing in section 2.5.
In fact we will prove even more than just global boundedness, but we will derive decay estimates for the $L^p(\mathbb{R}^d)$-norm for $2 \leq p \leq \infty$. Numerical results and the substantial

similarity in the heat equation suggest that the L^∞-norm of $\rho(x,t)$ tends to zero as t goes to infinity. We will see that this is in fact the case if the total mass is small enough.
Let us start by proving L^p decay estimates for $2 \leq p < \infty$.

Lemma 2.32. *Let $f(x) \geq 0$ for $x > 0$, $0 \leq \rho_0(x)$ and $\rho_0(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then, there exists a constant $C(d, \varepsilon, L)$ depending only on the dimension d, the diffusivity ε and on the Lipschitz constant L, such that if the total mass satisifies*

$$\int_{\mathbb{R}^d} \rho_0 \, dx < C(d, \varepsilon, L). \tag{2.37}$$

Then the weak solution $\rho(x,t)$ of (2.7) satisifies the decay estimates

$$\|\rho(t)\|_{L^p(\mathbb{R}^d)} \leq C(t+1)^{-\frac{d(p-1)}{2p}}, 2 \leq p < \infty. \tag{2.38}$$

Proof. Let $\rho \in X_T^M \cap W^{1,2}(0, T, H^1(\mathbb{R}^d), L^2(\mathbb{R}^d))$ be the weak local-in-time solution of (2.36). Our first step is to prove that $\varphi := p\rho^{p-1} \in W^{1,2}$ and we can therefore use it as testfunction in the weak formulation (2.36).

$$\begin{aligned} \|p\rho^{p-1}\|^2_{L^2(\mathbb{R}^d)} &= p^2 \|\rho^{2(p-1)}\|_{L^1(\mathbb{R}^d)} \\ &\leq p^2 \|\rho\|^{2p-3}_{L^\infty(\mathbb{R}^d)} \|\rho\|_{L^1(\mathbb{R}^d)} \\ &\leq p^2 M^{2(p-1)} \end{aligned}$$

$$\begin{aligned} \|p\nabla\rho^{p-1}\|^2_{L^2(\mathbb{R}^d)} &= p^2(p-1)^2 \|\rho^{p-2}\nabla\rho\|^2_{L^2(\mathbb{R}^d)} \\ &\leq p^2(p-1)^2 \|\rho^{p-2}\|^2_{L^2(\mathbb{R}^d)} \|\nabla\rho\|^2_{L^2(\mathbb{R}^d)} \\ &\leq p^2(p-1)^2 M^{2(p-2)} \|\nabla\rho\|^2_{L^2(\mathbb{R}^d)} \ < \ \infty \end{aligned}$$

$$\begin{aligned} < p\frac{\partial}{\partial t}\rho^{p-1}, \psi >_{H^{-1}(\mathbb{R}^d)} &= p(p-1) < \rho^{p-2}\frac{\partial}{\partial t}\rho, \psi >_{H^{-1}(\mathbb{R}^d)} \\ &= p(p-1) < \frac{\partial}{\partial t}\rho, \rho^{p-2}\psi >_{H^{-1}(\mathbb{R}^d)} \\ &= p(p-1) \|\frac{\partial}{\partial t}\rho\|_{H^{-1}(\mathbb{R}^d)} \|\rho^{p-2}\psi\|_{H^1(\mathbb{R}^d)} \ < \ \infty \end{aligned}$$

for some testfunction $\psi \in H^1(\mathbb{R}^d)$, $\|\psi\|_{H^1(\mathbb{R}^d)} \leq 1$. Therefore, we can easily integrate with respect to time and obtain $\varphi := p\rho^{p-1} \in W^{1,2}$. Inserting φ into the weak formulation yields

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^d} \rho^p \, dx &= -\varepsilon p \int_{\mathbb{R}^d} \nabla \rho^{p-1} \cdot \nabla \rho \, dx + p \int_{\mathbb{R}^d} \nabla \rho^{p-1} f(\rho) \cdot \nabla S \, dx \\
&= -\varepsilon p(p-1) \int_{\mathbb{R}^d} \rho^{p-2} \nabla \rho \cdot \nabla \rho \, dx + p(p-1) \int_{\mathbb{R}^d} \rho^{p-2} \nabla \rho f(\rho) \cdot \nabla S \, dx \\
&= -\frac{4\varepsilon(p-1)}{p} \int_{\mathbb{R}^d} |\nabla \rho^{p/2}|^2 \, dx - p(p-1) \int_{\mathbb{R}^d} \left(\int_0^\rho u^{p-2} f(u) du \right) \cdot \Delta S \, dx \\
&= -\frac{4\varepsilon(p-1)}{p} \int_{\mathbb{R}^d} |\nabla \rho^{p/2}|^2 \, dx + p(p-1) \int_{\mathbb{R}^d} \left(\int_0^\rho u^{p-2} f(u) du \right) \cdot (\rho - S) \, dx \\
&\leq -\frac{4\varepsilon(p-1)}{p} \int_{\mathbb{R}^d} |\nabla \rho^{p/2}|^2 \, dx + p(p-1) \int_{\mathbb{R}^d} \left(\int_0^\rho u^{p-2} f(u) du \right) \cdot \rho \, dx,
\end{aligned}$$

where we have used $0 \leq \rho$ and therefore $0 \leq f(\rho)$ and $S \geq 0$. Here, $\int_0^\rho u^{p-2} f(u) du$ is a formal notation denoting the primitive of $u^{p-2} f(u)$ evaluated at ρ. Thanks to the Lipschitz continuity of f we can easily estimate

$$\int_0^\rho u^{p-2} f(u) \, du = \int_0^\rho u^{p-2} (\underbrace{f(u)}_{\geq 0} - \underbrace{f(0)}_{=0}) \, du \leq L \int_0^\rho u^{p-2} (u - 0) \, du \leq \frac{L}{p} \rho^p.$$

Combining the two inequalities above leads to

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^d} \rho^p \, dx \quad \leq \quad -\frac{4\varepsilon(p-1)}{p} \int_{\mathbb{R}^d} |\nabla \rho^{p/2}|^2 \, dx + L(p-1) \int_{\mathbb{R}^d} \rho^{p+1} \, dx.$$

Using a version of the Gagliardo-Nirenberg inequality (see for instance [9])

$$\|\rho^{p+1}\|_{L^1} \leq C(d,p) \|\rho\|_{L^\alpha} \|\nabla \rho^{p/2}\|_{L^2}^2 \leq C(d,p) \|\rho_0\|_{L^1}^{1/\alpha} \|\nabla \rho^{p/2}\|_{L^2}^2$$

where $\alpha = 1$ for $d = 1, 2$ and $\alpha = d/2$ for $d > 2$, leads to

$$\frac{\partial}{\partial t} \|\rho^p\|_{L^1} \leq -\frac{4\varepsilon(p-1)}{p} \|\nabla \rho^{p/2}\|_{L^2}^2 + L(p-1) C(d,p) \|\rho_0\|_{L^1}^{1/\alpha} \|\nabla \rho^{p/2}\|_{L^2}^2.$$

If $4\varepsilon(p-1)p^{-1} > L(p-1)C(d,p)\|\rho_0\|_{L^1}^{1/\alpha} \Leftrightarrow \|\rho_0\|_{L^1} < (\frac{4\varepsilon}{pLC(d,p)})^\alpha$, the inequality can be written as

$$\frac{\partial}{\partial t} \|\rho^p\|_{L^1} + C' \|\nabla \rho^{p/2}\|_{L^2}^2 \leq 0 \quad \text{for some } C' > 0.$$

Due to the following interpolation inequalitiy (see for instance [10])

$$\|\rho\|_{L^p}^{\frac{(d(p-1)+2)p}{d(p-1)}} \leq C'' \|\nabla \rho^{p/2}\|_{L^2}^2 \|\rho\|_{L^1}^{\frac{2p}{d(p-1)}}$$

we can finally conclude

$$\frac{\partial}{\partial t} \|\rho^p\|_{L^1} + \bar{C} \|\rho^p\|_{L^1}^{\frac{d(p-1)+2}{d(p-1)}} \leq 0.$$

Applying Gronwall's lemma leads to the polynomial decay

$$\|\rho(t)\|_{L^p} \leq (C_1 t + C_2)^{-\frac{d(p-1)}{2p}}.$$

□

Lemma 2.33. *Let ρ_0 satisfy the same conditions as in lemma 2.32. Then, the solution $\rho(x,t)$ satisifies the L^∞-decay estimate*

$$\|\rho(t)\|_{L^\infty(\mathbb{R}^d)} \leq C(t+1)^{-\frac{d}{2}}. \tag{2.39}$$

Proof. Using the implicit representation of ρ provided by formula (2.31), we obtain for $t > t_0 > 0$

$$\begin{aligned}
\|\rho(.,2t)\|_{L^\infty(\mathbb{R}^d)} &= \|G(.,t) * \rho(.,t) + \int_t^{2t} \nabla G(.,2t-s) * (f(\rho)\nabla(B*\rho))(.,s)\,ds\|_{L^\infty(\mathbb{R}^d)} \\
&\leq \|G(.,t) * \rho(.,t)\|_{L^\infty(\mathbb{R}^d)} + \\
&\quad + \left\| \int_t^{2t} \nabla G(.,2t-s) * (f(\rho)\nabla(B*\rho))(.,s)\,ds \right\|_{L^\infty(\mathbb{R}^d)} \\
&\leq \|G(t)\|_{L^\infty(\mathbb{R}^d)} \|\rho(t)\|_{L^1(\mathbb{R}^d)} + \\
&\quad + \left\| \int_0^{t} \nabla G(t-s) * (f(\rho)\nabla(B*\rho))(t+s)\,ds \right\|_{L^\infty(\mathbb{R}^d)} \\
&\leq \|G(t)\|_{L^\infty(\mathbb{R}^d)} \|\rho_0\|_{L^1(\mathbb{R}^d)} + \\
&\quad + \int_0^t \|\nabla G(t-s)\|_{L^p(\mathbb{R}^d)} \|f(\rho)\nabla(B*\rho)(t+s)\|_{L^q(\mathbb{R}^d)}\,ds
\end{aligned}$$

where we have taken into account that the mass is conserved and applied Hölder's inequality (with $\frac{1}{p} + \frac{1}{q} = 1$). Inserting the L^∞-estimate for G yields

$$\begin{aligned}
&\|\rho(.,2t)\|_{L^\infty(\mathbb{R}^d)} \leq \\
\leq\ & C' t^{-d/2} \|\rho_0\|_{L^1(\mathbb{R}^d)} + \int_0^t \|\nabla G(t-s)\|_{L^p(\mathbb{R}^d)} \|f(\rho)\nabla(B*\rho)(t+s)\|_{L^q(\mathbb{R}^d)}\,ds \\
\leq\ & C t^{-d/2} + \int_0^t \|\nabla G(t-s)\|_{L^p(\mathbb{R}^d)} \|f(\rho(t+s))\|_{L^u(\mathbb{R}^d)} \|\nabla B\|_{L^v(\mathbb{R}^d)} \|\rho(t+s)\|_{L^w(\mathbb{R}^d)}\,ds \\
\leq\ & C t^{-d/2} + L\|\nabla B\|_{L^v(\mathbb{R}^d)} \int_0^t \|\nabla G(t-s)\|_{L^p(\mathbb{R}^d)} \|\rho(t+s)\|_{L^u(\mathbb{R}^d)} \|\rho(t+s)\|_{L^w(\mathbb{R}^d)}\,ds
\end{aligned}$$

for $\frac{1}{u} + \frac{1}{v} + \frac{1}{w} = \frac{1}{q} + 1$, where we have used Lipschitz continuity of f and again Hölder's inequality. Applying the L^p-estimates proven in lemma 2.32 leads to

$$\begin{aligned}
&\|\rho(x,2t)\|_{L^\infty(\mathbb{R}^d)} \leq \\
\leq\ & C t^{-d/2} + L\|\nabla B\|_{L^v(\mathbb{R}^d)} \int_0^t \|\nabla G(t-s)\|_{L^p(\mathbb{R}^d)} (C_1(t+s)+C_2)^{-\frac{d(u-1)}{2u} - \frac{d(w-1)}{2w}}\,ds \\
\leq\ & C t^{-d/2} + L\|\nabla B\|_{L^v(\mathbb{R}^d)} \int_0^t (t-s)^{-\frac{d(p-1)}{2p} - \frac{1}{2}} (C_1(t+s)+C_2)^{-d+\frac{d}{2}(\frac{1}{u}+\frac{1}{w})}\,ds
\end{aligned}$$

where we have used estimate (2.11) for the convolution kernel G. Choosing $p < \frac{d}{d-1}$ and $v = 1$ implies that the exponent becomes $-d + \frac{d}{2}(\frac{1}{u} + \frac{1}{w}) = -d + \frac{d}{2}\frac{1}{q} = -\frac{d}{2}(\frac{1}{p}+1)$. Therefore

$$
\begin{aligned}
\|\rho(x,2t)\|_{L^\infty(\mathbb{R}^d)} &\le Ct^{-d/2} + C' \int_0^t (t-s)^{-\frac{d(p-1)}{2p}-\frac{1}{2}} (t+s+C'')^{-\frac{d}{2}(\frac{1}{p}+1)}\, ds \\
&\le Ct^{-d/2} + C'(t+C'')^{-\frac{d}{2}(\frac{1}{p}+1)} \int_0^t (t-s)^{-\frac{d(p-1)}{2p}-\frac{1}{2}}\, ds \\
&\le Ct^{-d/2} + C'(t+C'')^{-\frac{d}{2}(\frac{1}{p}+1)} t^{-\frac{d(p-1)}{2p}+\frac{1}{2}} \\
&\le Ct^{-d/2} + C't^{-d+\frac{1}{2}} \\
&\le \bar{C}t^{-d/2}.
\end{aligned}
$$

Here, the last step is justified by $d \ge 1 \Leftrightarrow -d + 1/2 \le -d/2$. Since $\|\rho(t)\|_{L^\infty(\mathbb{R}^d)}$ is bounded by one, we can always find a constant C that provides us with the desired estimate $\|\rho(t)\|_{L^\infty(\mathbb{R}^d)} \le C(t+1)^{-\frac{d}{2}}$. $\square$

As mentioned above, we can now state a corollary including global existence of weak solutions.

Corollary 2.34 (Global existence of weak solutions)**.** *There exists a constant $C(d,\varepsilon)$ such that if*

- $\|\rho_0\|_{L^1(\mathbb{R}^d)} < C(d,\varepsilon),\ \rho_0 \in L^\infty(\mathbb{R}^d),\ \rho_0 \ge 0;$
- $f(x)$ *is Lipschitz continuous,* $f(0) = 0,\ f(x) \ge 0\ \forall x > 0.$

Then, for every $T > 0$ there exists a unique global weak solution ρ of (2.7) satisfying

- $\rho \in W^{1,2}(0,T,H^1(\mathbb{R}^d), L^2(\mathbb{R}^d))$
- $\displaystyle\int_0^t < \frac{\partial}{\partial t}\rho, \varphi >_{H^{-1}}\ ds = -\varepsilon \int_0^t \int_{\mathbb{R}^d} \nabla\rho\cdot\nabla\varphi + \big(f(\rho)\nabla(B*\rho)\big)\cdot\nabla\varphi\, dxds$ $\forall \varphi \in L^2(0,T,H^1(\mathbb{R}^d))$
- $\rho(.,0) = \rho_0(.)$
- $\rho(x,t) \ge 0$
- $\|\rho(t)\|_{L^1} = \|\rho_0\|_{L^1}$
- $\|\rho(t)\|_{L^p} \le (C_1 t + C_2)^{-\frac{d(p-1)}{2p}},\ 2 \le p \le \infty,\ C_1, C_2 \in \mathbb{R}^+.$

Proof. We start by taking a T_0-step in time, where $[0,T_0]$ is the interval of existence of the local solution. Now we want to take another step with initial data $\rho(T_0)$. Since we have a global bound of the L^∞-norm, we can guarantee that the stepsize does not decrease with every step we take. Therefore, we can extend our solution uniquely to any point $T > 0$. The stated properties have already been proven above. $\square$

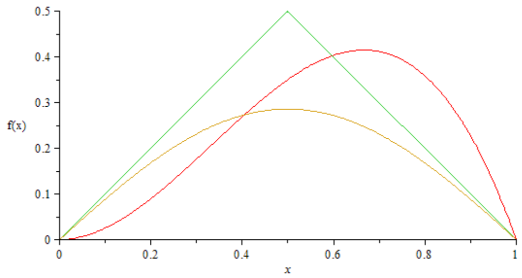


Figure 1: Three different F_1-functions

2.5 Prevention of Overcrowding

In this section we want to find restrictions on $f(x)$ that prevent the system from overcrowding. Therefore we will not need any further assumptions on ρ_0 to achieve global existence.

Definition 2.35. *For further reference we say a function $f(x)$ has property (F_b) if $f(x)$ is Lipschitz continuous, $f(0) = f(b) = 0$ for some $b > 0$ and $f(x) \geq 0$ for $0 \leq x \leq b$.*

Lemma 2.36. *Let f have (F_b), $0 \leq \rho_0 \leq b$ and $\rho \in L^1(\mathbb{R}^d)$. Then, the weak solutions ρ and S are also bounded by b.*

Proof. Without loss of generality we shall assume $f(x) = 0$ for $x \geq b$. For general f we just follow the replacing procedure we already applied proving lemma 2.30.
We start by taking $[\rho(x,t)-b]^+ = \max(\rho(x,t)-b, 0)$ as testfunction in the weak formulation (2.36)

$$\begin{aligned}
\int_0^t < \frac{\partial}{\partial t}\rho, [\rho(x,t)-b]^+ >_{H^{-1}} \, ds &= -\varepsilon \int_0^t \int_{\mathbb{R}^d} \nabla\rho \cdot \nabla[\rho(x,t)-b]^+ \, dxds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \tilde{f}(\rho)\nabla(B*\rho) \cdot \nabla[\rho(x,t)-b]^+ \, dxds \\
\Rightarrow \frac{1}{2}\int_0^t \frac{\partial}{\partial t} \|[\rho(x,t)-b]^+\|^2_{L^2(\mathbb{R}^d)} ds &= -\varepsilon \|\nabla[\rho(x,t)-b]^+\|^2_{L^2(0,T,L^2(\mathbb{R}^d))} \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \underbrace{\chi_{\mathbb{R}^+}[\rho(x,t)-b]\tilde{f}(\rho)}_{=0} \nabla(B*\rho) \cdot \nabla\rho \, dxds \\
&= -\varepsilon \|\nabla[\rho(x,t)-b]^+\|^2_{L^2(0,T,L^2(\mathbb{R}^d))}
\end{aligned}$$

and therefore

$$\|[\rho(x,t)-b]^+(.,t)\|^2_{L^2(\mathbb{R}^d)} + 2\varepsilon \|\nabla[\rho(x,t)-b]^+\|^2_{L^2(0,T,L^2(\mathbb{R}^d))} = \underbrace{\|[\rho(x,t)-b]^+\|^2_{L^2(\mathbb{R}^d)}}_{=0}$$

where we have used that $\rho_0(x) \leq b$. From this it follows that $\rho(x,t) \leq b$ and subsequently that $S(x,t) \leq b$ by using $[S(x,t)-b]^+$ as testfunction in the weak formulation and proceeding exactly as in the proof of positivity. $\square$

Since we have a uniform L^∞-bound, we can state the same result for global existence as in the previous section (of course without the decay estimates):

Corollary 2.37 (Global existence of weak solutions with prevention of overcrowding)**.** *If*

- $0 \leq \rho_0 \leq b, \;\; \rho_0 \in L^1(\mathbb{R}^d)$;
- $f(x)$ *has* (F_b);

then, for every $T > 0$ there exists a unique global weak solution ρ of (2.7) satisfying

- $\rho \in W^{1,2}(0,T,H^1(\mathbb{R}^d), L^2(\mathbb{R}^d))$
- $\displaystyle\int_0^t < \frac{\partial}{\partial t}\rho, \varphi >_{H^{-1}} \, ds = -\varepsilon \int_0^t \int_{\mathbb{R}^d} \nabla\rho \cdot \nabla\varphi + \big(f(\rho)\nabla(B * \rho)\big) \cdot \nabla\varphi \, dxds$
  $\forall \varphi \in L^2(0,T,H^1(\mathbb{R}^d))$
- $\rho(.,0) = \rho_0(.)$
- $0 \leq \rho(x,t) \leq b$
- $\|\rho(t)\|_{L^1} = \|\rho_0\|_{L^1}$.

2.6 Stationary Solutions and Energy of the System

Another interesting question is whether stationary solutions exist. For the case of small initial mass we have already proven that there cannot be any stationary solutions because of the decay estimates (see corollary 2.34).
Our aim in this section is to derive a suitable functional that is non-increasing for every solution and derive characteristics of stationary solutions.
We start by defining an energy function $\mathcal{E}(\rho, S)$:

Definition 2.38 (Energy functional)**.** *Let $g(x) : \mathbb{R}^d \to \mathbb{R}^d$ be a two times differentiable and Lipschitz continuous function satisfying $g(0) = 0$. Then we define the energy function $\mathcal{E}(\rho, S) : \; (H^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d))^2 \longrightarrow \mathbb{R}$ in the following way*

$$\mathcal{E}(\rho, S) := \frac{1}{2}\int_{\mathbb{R}^d} (|\nabla S|^2 + S^2)\, dx - \int_{\mathbb{R}^d} \rho S \, dx + \varepsilon \int_{\mathbb{R}^d} g(\rho)\, dx. \tag{2.40}$$

Since $S \in H^1(\mathbb{R}^d)$ and $\rho \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ (and $g(x)$ is Lipschitz continuous) all three integrals exist and $\mathcal{E}(\rho, S)$ is well defined.

Now we want to analyse what happens if we insert our weak solution into the functional.

Lemma 2.39 (Partial derivatives)**.** *Let ρ, S satisfy the weak formulation of the elliptic equation $-\Delta S + S = \rho$. Then, the partial derivatives of $\mathcal{E}$ in direction $\xi \in H^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ at the point $(\rho, S) \in (H^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d))^2$ exist, and are given by*

$$\begin{aligned}
D_S\mathcal{E}(\rho, S)(\xi) &= 0 \ , \ \forall \xi \in H^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \\
D_\rho\mathcal{E}(\rho, S)(\xi) &= -\int_{\mathbb{R}^d} \xi S \, dx + \varepsilon \int_{\mathbb{R}^d} g'(\rho)\xi \, dx.
\end{aligned}$$

Proof. Since $D_S\mathcal{E}(\rho, S)$ and $D_\rho\mathcal{E}(\rho, S)$ are clearly linear functionals we only need to prove that the difference quotient goes to zero. Let $\xi \in H^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, then we compute

$$\begin{aligned}
\mathcal{E}(\rho, S+\xi) - \mathcal{E}(\rho, S) - D_S\mathcal{E}(\rho, S)(\xi) \ &= \\
&= \frac{1}{2}\int_{\mathbb{R}^d} (|\nabla(S+\xi)|^2 + (S+\xi)^2) \, dx - \int_{\mathbb{R}^d} \rho(S+\xi) \, dx + \varepsilon \int_{\mathbb{R}^d} g(\rho) \, dx \\
&\qquad -\frac{1}{2}\int_{\mathbb{R}^d} (|\nabla S|^2 + S^2) \, dx + \int_{\mathbb{R}^d} \rho S \, dx - \varepsilon \int_{\mathbb{R}^d} g(\rho) \, dx \\
&= \frac{1}{2}\int_{\mathbb{R}^d} 2\nabla S \cdot \nabla \xi + |\nabla \xi|^2 + 2S\xi + \xi^2 \, dx - \int_{\mathbb{R}^d} \rho\xi \, dx \\
&= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla \xi|^2 + \xi^2 \, dx + \underbrace{\int_{\mathbb{R}^d} \nabla S \cdot \nabla \xi + S\xi - \rho\xi \, dx}_{=0} \\
&= \frac{1}{2}\|\nabla \xi\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\|\xi\|^2_{L^2(\mathbb{R}^d)} \\
&= \frac{1}{2}\|\xi\|^2_{H^1(\mathbb{R}^d)} \ \leq \ \|\xi\|^2_{H^1(\mathbb{R}^d)}.
\end{aligned}$$

Where we have used that S satisifies the elliptic equation and $\xi \in H^1(\mathbb{R}^d)$ is a valid test-function. Therefore, it follows

$$\begin{aligned}
\frac{|\mathcal{E}(\rho, S+\xi) - \mathcal{E}(\rho, S) - D_S\mathcal{E}(\rho, S)(\xi)|}{\|\xi\|_{L^1(\mathbb{R}^d)} + \|\xi\|_{H^1(\mathbb{R}^d)}} &\leq \frac{\|\xi\|^2_{H^1(\mathbb{R}^d)}}{\|\xi\|_{L^1(\mathbb{R}^d)} + \|\xi\|_{H^1(\mathbb{R}^d)}} \\
&\leq \frac{\|\xi\|^2_{H^1(\mathbb{R}^d)}}{\|\xi\|_{H^1(\mathbb{R}^d)}} \\
&\leq \|\xi\|_{H^1(\mathbb{R}^d)} \ \stackrel{\xi \to 0}{\longrightarrow} \ 0.
\end{aligned}$$

Here, $\xi \to 0$ means that $\|\xi\|_{L^1(\mathbb{R}^d)} + \|\xi\|_{H^1(\mathbb{R}^d)} \to 0$.

Slightly more effort is needed to prove the result for the partial derivative with respect to ρ.

$$\begin{aligned}\mathcal{E}(\rho+\xi,S)-\mathcal{E}(\rho,S)-D_\rho\mathcal{E}(\rho,S)(\xi) \;=\; & \\ =\; & \frac{1}{2}\int_{\mathbb{R}^d}(|\nabla S|^2+S^2)\,dx-\int_{\mathbb{R}^d}(\rho+\xi)S\,dx+\int_{\mathbb{R}^d}g(\rho+\xi)\,dx \\ & -\frac{1}{2}\int_{\mathbb{R}^d}(|\nabla S|^2+S^2)\,dx+\int_{\mathbb{R}^d}\rho S\,dx-\varepsilon\int_{\mathbb{R}^d}g(\rho)\,dx \\ & +\int_{\mathbb{R}^d}\xi S dx-\varepsilon\int_{\mathbb{R}^d}g'(\rho)\xi dx \\ =\; & \varepsilon\int_{\mathbb{R}^d}g(\rho+\xi)-g(\rho)-g'(\rho)\xi\,dx.\end{aligned}$$

Since $g(x)$ is differentiable, the operator $G:\psi\to g(\psi):H^1(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)\to L^1(\mathbb{R}^d)$ is clearly Frechet differentiable and its derivative is given by $DG(\psi)(\xi)=g'(\psi)\xi$. Therefore, there exists for every $\epsilon>0$ a $\delta>0$ satisfying $\|G(\psi+\xi)-G(\psi)-g'(\psi)\xi\|_{L^1(\mathbb{R}^d)}\le\epsilon\|\xi\|_{H^1(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)}$ for all ξ with $\|\xi\|_{H^1(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)}<\delta$. Using this reasoning in the equation above yields

$$\begin{aligned}\frac{|\mathcal{E}(\rho+\xi,S)-\mathcal{E}(\rho,S)-D_\rho\mathcal{E}(\rho,S)(\xi)|}{\|\xi\|_{H^1(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)}} &\le \varepsilon\frac{\epsilon\|\xi\|_{H^1(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)}}{\|\xi\|_{H^1(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)}} \\ &\le \varepsilon\epsilon\overset{\epsilon\to 0}{\longrightarrow}0.\end{aligned}$$

□

Now we want to use the results from above and turn our attention to the energy in the evolving system.

Lemma 2.40. *Let (ρ,S) be a pair of weak solutions of (2.7) and $e(t):=\mathcal{E}(\rho(.,t),S(.,t))$. Then its time derivative is given by*

$$\frac{\partial e}{\partial t}=-\int_{\mathbb{R}^d}(\varepsilon g''(\rho)\nabla\rho-\nabla S)\cdot(\varepsilon\nabla\rho-f(\rho)\nabla S)\,dx.$$

Proof. Unfortunately, we cannot differentiate with respect to t by using the chain rule because we would need $\frac{\partial}{\partial t}\rho,\frac{\partial}{\partial t}S\in L^1(\mathbb{R}^d)\cap H^1(\mathbb{R}^d)$. Therefore we start by taking an approximating sequence of functions $\rho_n\in L^2(0,T,L^1(\mathbb{R}^d)\cap H^1(\mathbb{R}^d))\cap C^\infty([0,T]\times\mathbb{R}^d)$ (and the corresponding sequence S_n) such that $\|\rho_n-\rho\|_{W^{1,2}}\to 0$.

Now we can compute

$$\begin{aligned}
\frac{\partial e}{\partial t} &= \frac{\partial}{\partial t}\mathcal{E}(\rho_n(.,t), S_n(.,t)) \\
&= D_S\mathcal{E}(\rho_n, S_n)(\frac{\partial S_n}{\partial t}) + D_\rho\mathcal{E}(\rho_n, S_n)(\frac{\partial \rho_n}{\partial t}) \\
&= \int_{\mathbb{R}^d} \frac{\partial \rho_n}{\partial t}(\varepsilon g'(\rho_n) - S_n)\, dx \\
&= < \frac{\partial \rho_n}{\partial t}, \varepsilon g'(\rho_n) - S_n >_{L^2(\mathbb{R}^d)} \\
&= < \frac{\partial \rho_n}{\partial t}, \varepsilon g'(\rho_n) - S_n >_{L^2(\mathbb{R}^d)} - < \frac{\partial \rho_n}{\partial t}, \varepsilon g'(\rho) - S >_{L^2(\mathbb{R}^d)} \\
&\quad + < \frac{\partial \rho_n}{\partial t}, \varepsilon g'(\rho) - S >_{L^2(\mathbb{R}^d)} - < \frac{\partial \rho}{\partial t}, \varepsilon g'(\rho) - S >_{H^{-1}(\mathbb{R}^d)} \\
&\quad + < \frac{\partial \rho}{\partial t}, \varepsilon g'(\rho) - S >_{H^{-1}(\mathbb{R}^d)} \\
&= \varepsilon < \frac{\partial \rho_n}{\partial t}, g'(\rho_n) - g'(\rho) >_{L^2(\mathbb{R}^d)} + < \frac{\partial \rho_n}{\partial t}, S - S_n >_{L^2(\mathbb{R}^d)} \\
&\quad + < \frac{\partial \rho_n}{\partial t} - \frac{\partial \rho}{\partial t}, \varepsilon g'(\rho) - S >_{H^{-1}(\mathbb{R}^d)} + < \frac{\partial \rho}{\partial t}, \varepsilon g'(\rho) - S >_{H^{-1}(\mathbb{R}^d)},
\end{aligned}$$

where we have inserted several terms and juggled them around a bit. Therefore we get

$$\begin{aligned}
|\frac{\partial}{\partial t}\mathcal{E}(\rho_n(.,t), S_n(.,t)) &- < \frac{\partial \rho}{\partial t}, \varepsilon g'(\rho) - S >_{H^{-1}(\mathbb{R}^d)} | \leq \\
&\leq \varepsilon \| \frac{\partial \rho_n}{\partial t} \|_{L^2(\mathbb{R}^d)} \| g'(\rho_n) - g'(\rho) \|_{L^2(\mathbb{R}^d)} + \| \frac{\partial \rho_n}{\partial t} \|_{L^2(\mathbb{R}^d)} \| S - S_n \|_{L^2(\mathbb{R}^d)} \\
&\quad + \| \frac{\partial \rho_n}{\partial t} - \frac{\partial \rho}{\partial t} \|_{H^{-1}(\mathbb{R}^d)} \| \varepsilon g'(\rho) - S \|_{H^1(\mathbb{R}^d)}.
\end{aligned}$$

Now the assumed convergence implies that $\|\frac{\partial \rho_n}{\partial t}\|_{L^2(\mathbb{R}^d)}$ is uniformly bounded and the norms $\|S - S_n\|_{L^2(\mathbb{R}^d)} \to 0$ and $\|g'(\rho_n) - g'(\rho)\|_{L^2(\mathbb{R}^d)} \to 0$ as n goes to infinity. Therefore we conclude

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{E}(\rho_n(.,t), S_n(.,t)) &= < \frac{\partial \rho}{\partial t}, \varepsilon g'(\rho) - S >_{H^{-1}(\mathbb{R}^d)} \\
&= - < \varepsilon \nabla \rho - f(\rho)\nabla S, \nabla(\varepsilon g'(\rho) - S) >_{(L^2(\mathbb{R}^d))^d} \\
&= - \int_{\mathbb{R}^d} (\varepsilon \nabla \rho - f(\rho)\nabla S) \cdot (\varepsilon g''(\rho)\nabla \rho - \nabla S)\, dx.
\end{aligned}$$

□

Intuitively, we would like to choose $g(x)$ in such a way that $g''(x)f(x) = 1$. But since we only assumed that $f(x)$ is Lipschitz continuous and $f(x) \geq 0$, problems will occur at points $x_0 \in \mathbb{R}$ where $f(x_0) = 0$. To overcome this problem, we define a sequence of functions $g_\alpha(x) \in C^2(\mathbb{R}^d) \cap C^{0,1}(\mathbb{R}^d)$ for $\alpha > 0$ in the following way: $g''_\alpha(x) = \frac{1}{f(x)+\alpha}$. Since $f(x) + \alpha \geq \alpha > 0$, the function $g_\alpha(x)$ is two times differentiable and Lipschitz continuous and we can use the derivations from above. Now we can extract necessary conditions for stationary solutions.

Lemma 2.41 (Stationary solutions)**.** *Let $f(x)$ be Lipschitz continuous, $f(x) \geq 0$, $f(0) = 0$ and $(\rho(x), S(x))$ be a pair of stationary weak solutions of (2.7). Then, there holds*

- $\varepsilon \nabla \rho = f(\rho) \nabla S$ *or* (2.41)
- $f(\rho) = 0.$ (2.42)

Proof. Since we have stationary solutions, every functional is also time independent and therefore its derivative with respect to time is equal to zero. Taking the sequence of functions g_α from above and using the result from lemma 2.40 yields

$$\begin{aligned}
0 &= \frac{\partial}{\partial t} \mathcal{E}_{g_\alpha}(\rho, S) \\
&= -\int_{\mathbb{R}^d} (\varepsilon g_\alpha''(\rho) \nabla \rho - \nabla S) \cdot (\varepsilon \nabla \rho - f(\rho) \nabla S) \, dx \\
&= -\int_{f(\rho(x))=0} (\varepsilon \frac{\nabla \rho}{f(\rho) + \alpha} - \nabla S) \cdot (\varepsilon \nabla \rho - f(\rho) \nabla S) \, dx \\
&\quad - \int_{f(\rho(x))\neq 0} (\varepsilon \frac{\nabla \rho}{f(\rho) + \alpha} - \nabla S) \cdot (\varepsilon \nabla \rho - f(\rho) \nabla S) \, dx \\
&= -\int_{f(\rho(x))=0} (\varepsilon \frac{\nabla \rho}{\alpha} - \nabla S) \cdot \varepsilon \nabla \rho \, dx - \int_{f(\rho(x))\neq 0} \frac{1}{f(\rho)} |\varepsilon \nabla \rho - f(\rho) \nabla S|^2 \, dx \\
&\quad + \int_{f(\rho(x))\neq 0} (\varepsilon \nabla \rho - f(\rho) \nabla S) \cdot \nabla \rho \cdot (\frac{\varepsilon}{f(\rho) + \alpha} - \frac{\varepsilon}{f(\rho)}) \, dx \\
&= -\frac{1}{\alpha} \int_{f(\rho(x))=0} \varepsilon^2 |\nabla \rho|^2 \, dx + \int_{f(\rho(x))=0} \varepsilon \nabla \rho \cdot \nabla S \, dx \\
&\quad - \varepsilon \int_{f(\rho(x))\neq 0} (\varepsilon \nabla \rho - f(\rho) \nabla S) \cdot \nabla \rho \cdot \frac{\alpha}{(f(\rho) + \alpha) f(\rho)} \, dx \\
&\quad - \int_{f(\rho(x))\neq 0} \frac{1}{f(\rho)} |\varepsilon \nabla \rho - f(\rho) \nabla S|^2 \, dx \\
&\leq -\frac{\varepsilon^2}{\alpha} \int_{f(\rho(x))=0} |\nabla \rho|^2 \, dx + \varepsilon \underbrace{\Big(\int_{f(\rho(x))=0} |\nabla \rho|^2 \, dx \Big)^{1/2}}_{=C_1} \underbrace{\Big(\int_{f(\rho(x))=0} |\nabla S|^2 \, dx \Big)^{1/2}}_{=C_2} \\
&\quad + \alpha \varepsilon \int_{f(\rho(x))\neq 0} |\varepsilon \nabla \rho - f(\rho) \nabla S| \frac{|\nabla \rho|}{f^2(\rho)} \, dx - \int_{f(\rho(x))\neq 0} \frac{1}{f(\rho)} |\varepsilon \nabla \rho - f(\rho) \nabla S|^2 \, dx,
\end{aligned}$$

where we have used Cauchy-Schwarz's inequality. Now let $0 < \alpha \leq \frac{C_1}{C_2}$ if $C_1, C_2 > 0$ or $\alpha > 0$ if $C_1 = 0$ or $C_2 = 0$. Then, $-\frac{C_1^2}{\alpha} + C_1 C_2 \leq 0$, and we get

$$0 \;=\; \frac{\partial}{\partial t}\mathcal{E}_{g_\alpha}(\rho), S) \;\leq\; -\int_{f(\rho(x))\neq 0} \frac{1}{f(\rho)}|\varepsilon\nabla\rho - f(\rho)\nabla S|^2\,dx + \alpha\varepsilon\int_{f(\rho(x))\neq 0} |\varepsilon\nabla\rho - f(\rho)\nabla S|\frac{|\nabla\rho|}{f^2(\rho)}\,dx$$

Now passing on to the limit $\alpha \to 0$ in this inequality yields

$$0 \;\leq\; -\int_{f(\rho(x))\neq 0} \frac{1}{f(\rho)}|\varepsilon\nabla\rho - f(\rho)\nabla S|^2\,dx$$

And therefore $\varepsilon\nabla\rho = f(\rho)\nabla S$ or $f(\rho) = 0$.

□

Remark 2.42. *It may feel like we have beaten around the bush here for some time. The main difficulty is that the functional $\mathcal{E}_g$ does not necessarily exist for some $g(x)$ with $g''(x) = \frac{1}{f(x)}$. Though $g(x)$ is always two times differentiable, it does not have to be Lipschitz continuous and we therefore cannot expect the integral $\int_{\mathbb{R}^d} g(\rho)dx$ to be well defined. (See example 2.45).*

Remark 2.43 (Energy dissipation)**.** *If we assume a-priori that the solution is smooth enough, such that the functial exists for each fixed time, the exact same steps as in lemma 2.41 lead to*

$$\frac{\partial}{\partial t}\mathcal{E}_{g_\alpha}(\rho, S) \;\leq\; 0.$$

We therefore say that the system is dissipative.

Now we want to show that the relation between ρ and S given in lemma 2.41 can easily lead to non-existence of stationary solutions.

Example 2.44 (Non-existence of non-trivial continuous stationary solutions)**.** *Let $f(x) = x$ and $(\rho(x), S(x)) \in (H^1(\mathbb{R}^d) \cap C(\mathbb{R}^d))^2$ be a pair of stationary solutions. Due to lemma 2.41 we have $\varepsilon\nabla\rho = \rho\nabla S$ or $\rho = 0$. These equations are equivalent to $\rho = e^{\frac{S+C}{\varepsilon}}$ or $\rho = 0$ for some constant $C \in \mathbb{R}$. Let us assume that $\rho \neq 0$ on a maximal domain Ω. Now since $S(x) \geq 0$, it follows that $\rho \geq e^{\frac{C}{\varepsilon}}$ in Ω and therefore Ω has to be bounded. Because we assumed that $\rho \in C(\mathbb{R}^d)$ and $\rho = 0$ on Ω^c, we conclude that there are no stationary solutions.*

The next example shall emphasize that problems can occur when defining the energy functional for unsuitable choices of $f(x)$.

Example 2.45 (Non-existence of the energy functional). *Let $d = 1$ and $f(x) = 4x^{3/2}$. Then $g(x) = -x^{1/2}$ satisfies $g''(x)f(x) = 1 \ \forall x \in \mathbb{R}^+$. Now, choosing as inital data the function $\rho_0 = \min(|x|^{-3/2}, 1) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, we immediately see that the integral $\int_{\mathbb{R}} g(\rho_0)dx = \int_{-1}^{1}(-1)dx + 2\int_1^\infty(-|x|^{-3/4})dx$ does not exist.*
Still, we can specify conditions for stationary solutions, namely $-2\varepsilon\rho^{-1/2} = S + C$ for some constant $C \in \mathbb{R}$ or $\rho = 0$.

Remark 2.46. *The non-existence of stationary solutions (especially in the case of small mass, where we have derived the decay-estimates) is one of the main reasons for considering the model with non-linear diffusion in the next chapter. We will see that in that case stationary solutions exist for arbitrary mass.*
In chapter 4 we will see that this has also great influence on the behaviour of time dependent solutions. Solutions of the model with linear diffusion will either decay to zero or blow up (in the case with no prevention of overcrowding), solutions of the model with non-linear diffusion on the other hand seem to converge towards a (non-trivial) stationary solution.

3 Non-Linear Diffusion with Prevention of Overcrowding

In this chapter we want to turn our attention to the slightly different non-linear model

$$\begin{cases} \frac{\partial \rho}{\partial t} = \operatorname{div}\left(m(\rho)\nabla(\varepsilon\rho - S)\right) \\ -\Delta S + S = \rho \\ \rho(x,0) = \rho_0(x), x \in \mathbb{R}^d. \end{cases} \tag{3.43}$$

In this chapter we will only consider functions $m(x)$ that imply a prevention of overcrowding similar to section 2.5. Note that this restriction (that leads to global boundedness) is not necessary in order to achieve global existence and can be replaced by an assumption on the initital data ($\|\rho_0\|_{L^1(\mathbb{R}^d)} < C$), which would lead to decay estimates similar to lemma 2.33. Therefore, let $m(x)$ be Lipschitz continuous, differentiable, $m(0) = m(1) = 0$ and let $m(x) > Ax^\beta(1-x)^\beta$ for constants $\beta \geq 0, A > 0$ and for all $x \in (0,1)$. Similar to the case of linear diffusion let $0 \leq \rho_0 \leq 1$ and $\rho_0 \in L^1(\mathbb{R}^d)$.
The main technical difference in comparison to the model in previous chapter is that the non-linearity also affects the diffusion term. Since no representation formula is known (in contrast to the previous chapter where we could obtain one from the non-linear heat equation), we will need a different approach.
Before we derive a suitable defintion of weak solutions, let us briefly recall a well known result from the linear theory of parabolic differential equations, a proof can be found for instance in [2, Chapter 7.1].

Proposition 3.47 (Existence of weak solutions for linear parabolic equations)**.** *Let Ω be a bounded subset of $\mathbb{R}^d$, $a_{ij}, b_i, c \in L^\infty([0,T]\times\Omega)$ and $f \in L^2([0,T]\times\Omega)$. Additionally, let $g \in L^2(\Omega)$ and $\sum_{i,j=1}^d a_{i,j}(x,t)\xi_i\xi_j \geq \theta|\xi|^2$ for all $(x,t) \in \Omega\times[0,T]$, $\xi \in \mathbb{R}^d$ and some $\theta > 0$.*
Then, there exists a unique $u(x,t) \in W^{1,2}(0,T,H_0^1(\Omega),L^2(\Omega))$ satisifying

$$< u_t, \varphi >_{H^{-1}(\Omega)} + \int_\Omega \sum_{i,j=1}^d a_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j} + \sum_{i=1}^d b_i \frac{\partial u}{\partial x_i}\varphi + cu\varphi\,dx \;=\; \int_\Omega f\varphi\,dx, \text{ and}$$
$$u(.,0) = g(.)$$

for all times $0 \leq t \leq T$ and for all $\varphi \in L^2(0,T,H_0^1(\Omega))$. Therefore, u is called the weak solution of the linear parabolic equation.

3.1 Weak Solutions

Now, let us turn our attention back to the system of non-linear equations. By multiplying (3.43) by a $W^{1,2}$-function and integrating by parts one obtains

$$\int_0^t <\rho_t, \varphi>_{H^{-1}} \, ds + \int_0^t \int_{\mathbb{R}^d} [\varepsilon \cdot \nabla M(\rho) - m(\rho) \nabla S] \cdot \nabla \varphi \, dxds = 0 \tag{3.44}$$

$$\int_0^t \int_{\mathbb{R}^d} \nabla S \cdot \nabla \varphi + (S - \rho) \varphi \, dxds = 0 \tag{3.45}$$

Where $\frac{\partial M(x)}{\partial x} = m(x)$, $M(0) = 0$.

Definition 3.48. *A pair of functions (ρ, S) is called weak solution of (3.43) if the following conditions are satisfied:*

1. $\rho_t \in L^2([0,T], H^{-1}(\mathbb{R}^d))$
2. $M(\rho) \in L^2([0,T], H^1(\mathbb{R}^d))$
3. $\rho \in L^\infty([0,T] \times \mathbb{R}^d)$ *and* $0 \leq \rho(x,t) \leq 1$ *almost everywhere in* $[0,T] \times \mathbb{R}^d$.
4. $S \in L^2([0,T], H^1(\mathbb{R}^d))$
5. *For all* $\varphi \in L^2(0,T,H^1(\mathbb{R}^d))$ *the relations (3.44) and (3.45) hold.*
6. $\rho(.,0) = \rho_0(.) \in L^1(\mathbb{R}^d)$ *and* $0 \leq \rho_0 \leq 1$

The subject of the following section is to prove existence of weak solutions. The main technical difficulties are in particular the degeneracy of the diffusion coefficient at the values $\rho = 0, 1$ and the unboundedness of the domain. To overcome this problems we will first consider a non-degenerated-problem on a closed ball with Dirichlet boundary conditions. Then we will pass the limit in the diffusion coefficient and finally send the radius of the ball to infinity.

3.1.1 Regularized Problem on a Ball

Before we proceed we shall introduce some short-hand-notations.

- $\Omega := \left\{ x \in \mathbb{R}^d | \; |x| \leq R \right\}$
- $X := W^{1,2}(0,T,H_0^1(\Omega), L^2(\Omega))$
- $B := L^2(0,T,L^2(\Omega))$
- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \max(\min(x,1),0)$

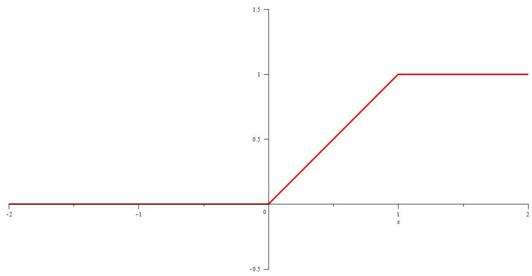


Figure 2: Cut-off-function

By adding $\alpha\varepsilon\nabla\rho$ to the diffiusion term, equation (3.44) becomes

$$\int_0^t < \rho_t, \varphi >_{H^{-1}} \, ds + \int_0^t \int_\Omega [\varepsilon \cdot \nabla M(\rho) + \alpha\varepsilon\nabla\rho - m(\rho)\nabla S] \cdot \nabla\varphi \, dxds = 0 \qquad (3.46)$$

Now we define an Operator $T(\tilde{\rho}) = \rho$ that maps $B \to B$. To keep things clear, we will first write down the plain definition without going into existence details and work on that issue in the following lemma.

Definition 3.49. *Let $\rho_{0,R} := \rho_0|_\Omega$. Then for $\tilde{\rho} \in B$ we shall define the operator T in two steps:*

1. *S is the weak solution of the linear elliptic equation $-\Delta S + S = f(\tilde{\rho})$ for $x \in \Omega$ and $-\Delta S + S = 0$ for $|x| > R$. It is given by $S = \mathcal{B} * (\chi_\Omega f(\tilde{\rho}))$.*

2. *$T(\tilde{\rho}) := \rho$ is the weak solution of the linear parabolic equation*

$$\rho_t - \varepsilon \cdot \operatorname{div}(A\nabla\rho) + b \cdot \nabla\rho \quad = \quad h,$$

with inital data $\rho(.,0) = \rho_{0,R}(.)$ and homogeneous Dirichlet boundary conditions.

for

$$\begin{aligned} A(x,t) &:= m(f(\tilde{\rho}(x,t))) + \alpha \\ b(x,t) &:= m'(f(\tilde{\rho}(x,t)))\nabla S(x,t) \\ h(x,t) &:= -m(f(\tilde{\rho}(x,t))) \cdot \Delta S(x,t) \end{aligned}$$

As mentioned above, first of all we need to make sure that the operator T described above is at least well defined and that it maps $B \to B$.

Lemma 3.50. *For $\tilde{\rho} \in B$ there exists a unique $\rho \in B$ and S such that the differential equations from definition 3.49 are satisfied.*

Proof. In order to apply the theory for linear parabolic equations, we need to make sure that the coefficients A, b, h satisfy the assumptions of proposition 3.47. Since $0 \leq m(f(x)) \leq L$ we directly obtain

$$\alpha \;\leq\; A(x,t) \;=\; \alpha + m(f(\tilde{\rho}(x,t))) \;\leq\; \alpha + L$$

For the L^∞-estimate of b we recall the properties of $\mathcal{B}$ proven in the first chapter (lemma 2.19)

$$\begin{aligned}
\|b\|_{L^\infty([0,T]\times\Omega)} &= \|m'(f(\tilde{\rho}))\nabla S\|_{L^\infty([0,T]\times\Omega)} \\
&\leq \|m'(f(\tilde{\rho}))\|_{L^\infty([0,T]\times\Omega)}\|\nabla S\|_{L^\infty([0,T]\times\Omega)} \\
&\leq L\|\nabla S\|_{L^\infty([0,T]\times\Omega)} \\
&\leq L\|\nabla S\|_{L^\infty([0,T]\times\mathbb{R}^d)} \\
&\leq L\|\|\nabla\mathcal{B}\|_{L^1(\mathbb{R}^d)}\|\chi_\Omega f(\tilde{\rho})\|_{L^\infty(\mathbb{R}^d)}\|_{L^\infty(0,T)} \\
&\leq LC(d)\|f(\tilde{\rho})\|_{L^\infty([0,T]\times\mathbb{R}^d)} \\
&\leq LC(d) \;=:\; B_0,
\end{aligned}$$

where we have used that $f(x) \leq 1$ and $m'(x) \leq L$. Similarly, we obtain an L^2-estimate for S

$$\begin{aligned}
\|S\|_{L^2(\Omega)} &\leq \|S\|_{L^2(\mathbb{R}^d)} \\
&\leq \|\mathcal{B}\|_{L^1(\mathbb{R}^d)}\|\chi_\Omega f(\tilde{\rho})\|_{L^2(\mathbb{R}^d)} \\
&\leq \|f(\tilde{\rho})\|_{L^2(\Omega)}.
\end{aligned}$$

Since S satisfies the equation $\Delta S = S - \chi_\Omega f(\tilde{\rho})$ in a distributional sense and the right hand side is in $L^2(\Omega)$, ΔS has to be square-integrable, too:

$$\begin{aligned}
\|\Delta S\|_{L^2(\Omega)} &\leq \|S - \chi_\Omega f(\tilde{\rho})\|_{L^2(\Omega)} \\
&\leq \|S\|_{L^2(\Omega)} + \|\chi_\Omega f(\tilde{\rho})\|_{L^2(\Omega)} \\
&\leq \|S\|_{L^2(\Omega)} + \|f(\tilde{\rho})\|_{L^2(\Omega)} \\
&\leq 2\|f(\tilde{\rho})\|_{L^2(\Omega)} \\
&\leq 2C'(d,R).
\end{aligned}$$

Therefore, we can conclude

$$\begin{aligned}
\|h\|_{L^2(\Omega)} &= \|m(f(\tilde{\rho}))\Delta S\|_{L^2(\Omega)} \\
&\leq \|m(f(\tilde{\rho}))\|_{L^\infty(\Omega)}\|\Delta S\|_{L^2(\Omega)} \\
&\leq 2LC'(d,R)
\end{aligned}$$

and finally

$$\|h\|_B \;\leq\; C(d,R,T) \;=:\; H_0.$$

Now, proposition (3.47) provides us with the existence of a unique function ρ belonging to $W^{1,2}(0,T,H_0^1(\Omega),L^2(\Omega))$ satisfying

$$\begin{aligned} <\rho_t,\varphi>_{H^{-1}(\Omega)} + \int_\Omega A\nabla\rho\cdot\nabla\varphi + b\cdot\nabla\rho\varphi\,dx &= \int_\Omega h\varphi\,dx, \\ \text{and } \rho(.,0) = g(.) \end{aligned} \tag{3.47}$$

for all $\varphi \in L^2(0,T,H_0^1(\Omega))$. Therefore, the definition of $T : B \to B$ is justified. □

To prove continuity of T, the coefficients A, b, h need to be continuous with respect to $\tilde{\rho}$ in some sense, which is clarified by

Lemma 3.51. *Let $\tilde{\rho}_k \xrightarrow{B} \tilde{\rho}$ and let A_k, b_k, h_k be the corresponding sequences defined by definition 3.49. Then there exists a subsequence $\tilde{\rho}'_k$ such that $A_{k'} \to A$ and $b_{k'} \to b$ almost everywhere and $h_{k'} \xrightarrow{B} h$. Here A, b, h are the coefficients associated with $\tilde{\rho}$.*

Proof. Since $\tilde{\rho}_k \xrightarrow{B} \tilde{\rho}$ due to proposition 5.67, there exists a subsequence $\tilde{\rho}_{k'}$ that converges almost everywhere. Since $f(x)$, $m(x)$ and $m'(x)$ are continuous functions, we directly obtain

$$\begin{aligned} m(f(\tilde{\rho}_{k'})) &\to m(f(\tilde{\rho})) \text{ almost everywhere, and} \\ m'(f(\tilde{\rho}_{k'})) &\to m'(f(\tilde{\rho})) \text{ almost everywhere.} \end{aligned}$$

Using the representations of $S_{k'}$ and S leads to

$$\begin{aligned} \|\nabla S_{k'} - \nabla S\|_{L^\infty([0,T]\times\Omega)} &\leq \|\nabla S_{k'} - \nabla S\|_{L^\infty([0,T]\times\mathbb{R}^d} \\ &\leq \|\nabla\mathcal{B}\|_{L^\infty(0,T,L^1(\mathbb{R}^d))}\|\chi_\Omega f(\tilde{\rho}_{k'}) - \chi_\Omega f(\tilde{\rho})\|_{L^\infty([0,T]\times\mathbb{R}^d)} \\ &\leq C(d)\|f(\tilde{\rho}_{k'}) - f(\tilde{\rho})\|_{L^\infty([0,T]\times\Omega)} \to 0. \end{aligned}$$

For the convergence of ΔS, we proceed just as before:

$$\begin{aligned} \|\Delta S_{k'} - \Delta S\|_B &\leq \|S_{k'} - S\|_B + \|f(\tilde{\rho}_{k'}) - f(\tilde{\rho})\|_B \\ &\leq \|S_{k'} - S\|_B + \|f(\tilde{\rho}_{k'}) - f(\tilde{\rho})\|_B \\ &\leq \|\mathcal{B}\|_{L^\infty(0,T,L^1(\mathbb{R}^d))}\|f(\tilde{\rho}_{k'}) - f(\tilde{\rho})\|_B + \|f(\tilde{\rho}_{k'}) - f(\tilde{\rho})\|_B \\ &\leq 2\|f(\tilde{\rho}_{k'}) - f(\tilde{\rho})\|_B \to 0. \end{aligned}$$

Now we can combine these convergences and obtain the desired convergences of $A_{k'}, b_{k'}$ and $h_{k'}$. □

Another important ingredient in order to prove existence of a fixed point of T will be the following a-priori estimate:

Lemma 3.52. *The operator T satisfies the a-priori estimate $\|T(\tilde{\rho})\|_X < C_x$, for some constant C_x depending on α, R, d, T and $\|\rho_0\|_{L^2(\Omega)}$.*

Proof. Using $\varphi = \rho$ as testfunction in (3.47) leads to

$$\begin{aligned} < \rho_t, \rho >_{H^{-1}} + \int_\Omega \varepsilon A |\nabla\rho|^2 \, dx &= -\int_\Omega b \cdot \nabla\rho\rho + h\rho \, dx \\ \Rightarrow \frac{1}{2}\frac{\partial}{\partial t}\|\rho(t)\|^2_{L^2(\Omega)} + \varepsilon\alpha \cdot \|\nabla\rho\|^2_{L^2(\Omega)} &\leq B_0 \int_\Omega |\rho||\nabla\rho| \, dx + \int_\Omega h\rho \, dx. \end{aligned}$$

Now applying Young's inequality on both integrals yields

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\|\rho(t)\|^2_{L^2(\Omega)} + \varepsilon\alpha \cdot \|\nabla\rho\|^2_{L^2(\Omega)} &\leq \frac{B_0\gamma}{2}\|\nabla\rho\|^2_{L^2(\Omega)} + \frac{B_0}{2\gamma}\|\rho\|^2_{L^2(\Omega)} \\ &\quad + \frac{1}{2}{H_0}^2 + \frac{1}{2}\|\rho\|^2_{L^2(\Omega)}. \end{aligned}$$

Fixing $\gamma = \frac{2\varepsilon\alpha}{B_0}$ and multiplying by two leads to

$$\frac{\partial}{\partial t}\|\rho(t)\|^2_{L^2(\Omega)} \leq (\frac{B_0^2}{2\varepsilon\alpha} + 1)\|\rho\|^2_{L^2(\Omega)} + H_0^2.$$

Now Gronwall's lemma gives us the first a-priori estimate and we obtain:

$$\|\rho(t)\|^2_B < C(R, d, T, \|\rho_0\|_{L^2(\Omega)}). \tag{3.48}$$

Using again $\varphi = \rho$, but choosing $\gamma = \frac{\varepsilon\alpha}{B_0}$ (so the term including the gradient does not vanish), leads similarly to

$$\|\nabla\rho(t)\|^2_B < C'(\alpha, R, d, T, \|\rho_0\|_{L^2(\Omega)}). \tag{3.49}$$

These two inequalities allow to estimate $< \rho_t, \varphi >_{H^{-1}} \leq C''(\alpha, R, d, T, \|\rho_0\|_{L^2(\Omega)})\|\varphi\|_{H_0^1(\Omega)}$ by simply applying Young's inequality on all terms. Therefore, we obtain the essential a-priori estimate stated above. □

The next step is to prove that T is continuous.

Lemma 3.53. *T is continuous, i.e.*

$$\tilde{\rho}_k \xrightarrow{B} \tilde{\rho} \implies T(\tilde{\rho}_k) \xrightarrow{B} T(\tilde{\rho}).$$

Proof. Let $\tilde{\rho}_k \xrightarrow{B} \tilde{\rho}$ and therefore $\|\tilde{\rho}_k\|_B \leq C$. Now the a-priori estimate from the previous lemma tells us that the sequence $\rho_k := T(\tilde{\rho}_k)$ is bounded in X and therefore, by defintion of the spaces B and X, we have $\|\rho_k\|_B \leq \|\rho_k\|_X \leq C'$. Taking into account that a bounded

sequence is convergent if and only if all convergent subsequences have the same limit (proposition 5.68), we start by taking any convergent subsequence $\rho'_k \xrightarrow{B} \rho'$. Now we want to prove that independently of the choice of ρ'_k we always have $\rho' = T(\tilde{\rho})$ and therefore $\rho_k \xrightarrow{B} T(\tilde{\rho})$. Since ρ'_k is bounded in X, there exists a weakly convergent subsequence and we can therfore assume (without loss of generality) that $\rho'_k \overset{X}{\rightharpoonup} \rho'$ (in addition to the strong convergence in B).
Applying lemma 3.51 and again switching to the subsequence provided by proposition 5.68 we find that

$$\begin{aligned} A_{k'} &\to A \text{ a.e.} \\ b_{k'} &\to b \text{ a.e.} \\ h_{k'} &\xrightarrow{B} h \\ \rho_{k'} &\overset{X}{\rightharpoonup} \rho'. \end{aligned}$$

Therefore, we can pass on to the limit in the weak formulation (3.47) and obtain

$$< \rho'_t, \varphi >_{H^{-1}} + \int_\Omega \varepsilon A \nabla \rho' \cdot \nabla \varphi + b \cdot \nabla \rho' \varphi \, dx = \int_\Omega f \varphi \, dx.$$

Since the solution of the linear parabolic equation is unique, we can conclude that $\rho' = \rho$ for every subsequence ρ'_k. Therefore, T is continuous. $\square$

Now we can make use of Schauder's fixed point theorem (see for instance [7, Theorem 10.1]) and prove the following corollary

Corollary 3.54. *The operator T has a fixed point $\rho \in K$ where $K := \{\varphi \in X \mid \|\varphi\|_X < C_x\}$. C_x denotes the constant given by lemma 3.52.*

Proof. Since K is clearly a subset of B and due to the a-priori estimate from lemma 3.52 we can define $T : K \to K$. Since $X \hookrightarrow B$ is compact, K is a convex compact subset of B. Now we already know that $T : B \to B$ is continuous and therefore especially if we restrict the domain to K. Hence, we can apply Schauder's fixed point theorem and obtain a fixed point $\rho \in K$ $\square$

In the following two corollaries we will derive important estimates for fixed points $\rho = T(\rho)$. These will become crucial for passing the limit $\alpha \to 0$.

Corollary 3.55. *Let ρ be a fixed point of T. Then we have $0 \leq \rho, S \leq 1$. The weak formulation therefore reads*

$$\begin{aligned} < \rho_t, \varphi >_{H^{-1}} + \int_\Omega \varepsilon[m(\rho) + \alpha] \nabla \rho \cdot \nabla \varphi \, dx &= \int_\Omega m(\rho) \nabla S \cdot \nabla \varphi \, dx \\ \int_\Omega \nabla S \cdot \nabla \varphi + (S - \rho) \varphi \, dx &= 0. \end{aligned}$$

Proof. Since $S(.,t) \in H^1(\mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}^d} \nabla S \cdot \nabla \varphi + S\varphi \, dx = \int_{\mathbb{R}^d} \chi_\Omega f(\rho)\varphi \, dx = \int_\Omega f(\rho)\varphi \, dx$$

for all φ in $H^1(\mathbb{R}^d)$, we start by using $\varphi = [S]^-$ as testfunction:

$$\begin{aligned} \int_{\mathbb{R}^d} \nabla S \cdot \nabla [S]^- + S[S]^- \, dx &= \int_\Omega \underbrace{f(\rho)}_{\geq 0} \underbrace{[S]^-}_{\leq 0} \, dx \\ \Rightarrow \; \|\nabla [S]^-\|^2_{L^2(\mathbb{R}^d)} + \|[S]^-\|^2_{L^2(\mathbb{R}^d)} &\leq 0. \end{aligned}$$

and therefore $S \geq 0$. For the upper bound we look at the L^∞-norm

$$\begin{aligned} \|S\|_{L^\infty(\mathbb{R}^d)} &= \|\mathcal{B} * (\chi_\Omega f(\rho))\|_{L^\infty(\mathbb{R}^d)} \\ &\leq \|\mathcal{B}\|_{L^1(\mathbb{R}^d)} \|\chi_\Omega f(\rho)\|_{L^\infty(\mathbb{R}^d)} \\ &\leq \|\mathcal{B}\|_{L^1(\mathbb{R}^d)} \|f(\rho)\|_{L^\infty(\Omega)} \\ &\leq 1. \end{aligned}$$

Similarly, using $\varphi = [\rho]^- \in W^{1,2}$ as testfunction in the weak formulation (3.47) of the parabolic equation leads to

$$\int_0^t < \rho_t, [\rho]^- >_{H^{-1}} \, ds + \int_0^t \int_\Omega \varepsilon[m(f(\rho)) + \alpha] |\nabla [\rho]^-|^2 + m'(f(\rho)) \nabla S \cdot \nabla \rho [\rho]^- \, dxds =$$
$$= - \int_0^t \int_\Omega m(f(\rho)) \cdot \Delta S [\rho]^- \, ds.$$

And therefore

$$\frac{1}{2} \|[\rho]^-(.,t)\|^2_{L^2(\Omega)} + \varepsilon\alpha \|\nabla [\rho]^-\|^2_B = \frac{1}{2} \|[\rho]^-(.,0)\|^2_{L^2(\Omega)} - \int_0^t \int_\Omega m'(0) \nabla S \cdot \nabla \rho [\rho]^- \, dxds,$$

where we have used that $f(x) = 0$ for $x \leq 0$ and $m(0) = 0$. Now, taking into account that $\nabla \rho [\rho]^- = \frac{1}{2} \nabla ([\rho]^{-2})$ and $0 \leq \rho_0 \leq 1$ and due to integration by parts we obtain

$$\begin{aligned} \frac{1}{2} \|[\rho]^-(.,t)\|^2_{L^2(\Omega)} + \varepsilon\alpha \|\nabla [\rho]^-\|^2_B &= \frac{m'(0)}{2} \int_0^t \int_\Omega \Delta S [\rho]^{-2} \, dxds \\ &= \frac{L}{2} \int_0^t \int_\Omega (S - \rho)[\rho]^{-2} \, dxds \\ &\leq \frac{L}{2} \int_0^t \int_\Omega (1 - \rho)[\rho]^{-2} \, dxds \\ &\leq \frac{L}{2} \|[\rho]^-\|^2_B, \end{aligned}$$

and therefore

$$\|[\rho]^-(.,t)\|^2_{L^2(\Omega)} \leq L \|[\rho]^-\|^2_B,$$

which implies that $\|[\rho]^-(.,t)\|_{L^2(\Omega)} = 0$ and therefore $\rho \geq 0$. Analogously, using $\varphi = [\rho - 1]^+$ as testfunction yields $\rho \leq 1$.
Especially the cut-off function $f(\rho)$ is just the identity and we obtain, for fixed points of T, the following equation

$$
\begin{aligned}
< \rho_t, \varphi >_{H^{-1}} + \int_\Omega \varepsilon[m(\rho) + \alpha]\nabla\rho \cdot \nabla\varphi \, dx &= -\int_\Omega m'(\rho)\nabla S \cdot \nabla\rho\varphi + m(\rho)\Delta S\varphi \, dx \\
&= \int_\Omega m(\rho)\nabla S \cdot \nabla\varphi \, dx,
\end{aligned}
$$

which is exactly the weak formulation of the regularized equation (3.46). Similarly, the elliptic equation reads

$$
\int_\Omega \nabla S \cdot \nabla\varphi + (S - \rho)\varphi \, dx = 0
$$

□

Corollary 3.56. *Let ρ be a fixed point of T. Then we have the following estimates*

$$
\|M(\rho)\|_{L^2(0,T,H_0^1(\Omega))} \leq C_1 \tag{3.50}
$$

$$
\sqrt{\alpha}\|\rho\|_{L^2(0,T,H_0^1(\Omega))} \leq C_2 \tag{3.51}
$$

$$
\|\rho_t\|_{L^2(0,T,H_0^{-1}(\Omega))} \leq C_3 \tag{3.52}
$$

with constants C_i independent of α.

Proof. Using $\varphi = \rho$ as testfunction directly leads to

$$
\begin{aligned}
< \rho_t, \rho >_{H^{-1}} + \int_\Omega \varepsilon[m(\rho) + \alpha]\nabla\rho \cdot \nabla\rho \, dx &= \int_\Omega m(\rho)\nabla S \cdot \nabla\rho \, dx \\
\Rightarrow \frac{1}{2}\frac{\partial}{\partial t}\|\rho\|^2_{L^2(\Omega)} + \varepsilon\alpha \int_\Omega |\nabla\rho|^2 \, dx &\leq \int_\Omega M(\rho)(\rho - S) \, dx \\
\Rightarrow \frac{\partial}{\partial t}\|\rho\|^2_{L^2(\Omega)} + 2\varepsilon\alpha\|\nabla\rho\|^2_{L^2(\Omega)} &\leq 2\int_\Omega M(\rho)(\rho - S) \, dx \\
&\leq 2\int_\Omega M(\rho)\rho \, dx \\
&= 2\int_\Omega \left(\int_0^\rho m(u)du\right) \cdot \rho \, dx \\
&\leq 2L\int_\Omega \left(\int_0^\rho u du\right) \cdot \rho \, dx \\
&\leq L\int_\Omega \rho^3 \, dx \\
&\leq L\int_\Omega \rho^2 \, dx,
\end{aligned}
$$

and therefore (by using Gronwall's lemma)

$$\begin{aligned}\frac{\partial}{\partial t}\|\rho\|^2_{L^2(\Omega)} &\leq L\int_\Omega \rho^2\,dx\\ \Rightarrow\ \|\rho(.,t)\|^2_{L^2(\Omega)} &\leq e^{Lt}\|\rho(.,0)\|^2_{L^2(\Omega)}.\end{aligned}$$

Accordingly, we have

$$\begin{aligned}\|\rho(.,T)\|^2_{L^2(\Omega)} + 2\varepsilon\alpha\|\nabla\rho\|^2_B &\leq e^{LT}\|\rho(.,0)\|^2_{L^2(\Omega)}\\ \Rightarrow\ \alpha\|\nabla\rho\|^2_B &\leq \frac{e^{LT}}{2\varepsilon}\|\rho(.,0)\|^2_{L^2(\Omega)}.\end{aligned}$$

Now, using $M(\rho)$ as testfunction (see lemma 5.70), we similarly obtain

$$\begin{aligned}<\rho_t, M(\rho)>_{H^{-1}} + \int_\Omega \varepsilon[m(\rho)+\alpha]\nabla\rho\cdot\nabla M(\rho)\,dx &= \int_\Omega m(\rho)\nabla S\cdot\nabla M(\rho)\,dx\\ \Rightarrow\ \frac{\partial}{\partial t}\|\mathcal{M}(\rho)\|_{L^1(\Omega)} + \varepsilon\int_\Omega |\nabla M(\rho)|^2 + \alpha m(\rho)|\nabla\rho|^2\,dx &= -\int_\Omega\left(\int_0^\rho m^2(u)du\right)\cdot\Delta S\,dx\\ \Rightarrow\ \frac{\partial}{\partial t}\|\mathcal{M}(\rho)\|_{L^1(\Omega)} + \varepsilon\int_\Omega |\nabla M(\rho)|^2\,dx &\leq \int_\Omega\left(\int_0^\rho m^2(u)du\right)\cdot(\rho - S)\,dx\\ \Rightarrow\ \frac{\partial}{\partial t}\|\mathcal{M}(\rho)\|_{L^1(\Omega)} + \varepsilon\|\nabla M(\rho)\|^2_{L^2(\Omega)} &\leq \int_\Omega\left(\int_0^\rho m^2(u)du\right)\cdot\rho\,dx\\ &\leq L^2\int_\Omega\left(\int_0^\rho u^2du\right)\cdot\rho\,dx\\ &= \frac{L^2}{3}\int_\Omega \rho^4\,dx\\ &= \frac{L^2}{3}\int_\Omega \rho^2\,dx\\ &\leq \frac{L^2}{3}e^{Lt}\|\rho(.,0)\|^2_{L^2(\Omega)},\end{aligned}$$

where we have used that $m(x)$ is Lipschitz continuous and $\rho \geq 0$. Integrating with respect to t yields

$$\begin{aligned}\|\mathcal{M}(\rho)(.,T)\|_{L^1(\Omega)} + \varepsilon\|\nabla M(\rho)\|^2_B &\leq \frac{L}{3}(e^{LT}-1)\|\rho(.,0)\|^2_{L^2(\Omega)} + \|\mathcal{M}(\rho)(.,0)\|_{L^1(\Omega)}\\ &= \frac{L}{3}(e^{LT}-1)\|\rho(.,0)\|^2_{L^2(\Omega)} + \|\mathcal{M}(\rho_0)\|_{L^1(\Omega)}\\ &\leq \frac{L}{3}(e^{LT}-1)\|\rho(.,0)\|^2_{L^2(\Omega)} + \frac{L}{6}\|\rho_0^3\|_{L^1(\Omega)}\\ &\leq \frac{L}{6}(2e^{LT}-1)\|\rho(.,0)\|^2_{L^2(\Omega)},\end{aligned}$$

where we have used that $\mathcal{M}(x) \leq L\frac{x^3}{6}$ and $\rho_0 \leq 1$. Due to Poincare's inequality (proposition 5.71) there exists a constant such that $\|\psi\|_{H^1(\Omega)} \leq C\|\nabla\psi\|_{L^2(\Omega)}$ and we have proven the first estimate.

For the estimate of ρ_t we consider a function $\varphi \in H^1(\Omega)$.

$$
\begin{aligned}
< \rho_t, \varphi >_{H^{-1}} &= \int_\Omega \Big(- \varepsilon[m(\rho) + \alpha]\nabla\rho + m(\rho)\nabla S \Big) \cdot \nabla\varphi \, dx \\
&\leq \|\varepsilon[m(\rho) + \alpha]\nabla\rho + m(\rho)\nabla S\|_{L^2(\Omega)} \|\nabla\varphi\|_{L^2(\Omega)} \\
&\leq \|\varepsilon[m(\rho) + \alpha]\nabla\rho + m(\rho)\nabla S\|_{L^2(\Omega)} \|\varphi\|_{H^1(\Omega)} \\
&\leq \Big(\varepsilon\|m(\rho)\nabla\rho\|_{L^2(\Omega)} + \varepsilon\alpha\|\nabla\rho\|_{L^2(\Omega)} + \|m(\rho)\nabla S\|_{L^2(\Omega)}\Big) \cdot \|\varphi\|_{H^1(\Omega)} \\
&\leq \Big(\varepsilon\|\nabla M(\rho)\|_{L^2(\Omega)} + \varepsilon\alpha\|\rho\|_{H^1(\Omega)} + L\|\rho\|_{L^2(\Omega)}\|\nabla S\|_{L^\infty(\Omega)}\Big) \cdot \|\varphi\|_{H^1(\Omega)} \\
&\leq \Big(\varepsilon\|\nabla M(\rho)\|_{L^2(\Omega)} + \varepsilon\sqrt{\alpha_0}\sqrt{\alpha}\|\rho\|_{H^1(\Omega)} + L\|\rho\|_{L^2(\Omega)} C\Big) \cdot \|\varphi\|_{H^1(\Omega)} \\
&\leq C(\rho_0, \alpha_0) \cdot \|\varphi\|_{H^1(\Omega)}
\end{aligned}
$$

where we have used Cauchy-Schwarz's inequality Hölder's inequality and the estimates we have already proven. This holds true for all $\alpha \leq \alpha_0$ and implies that $\|\rho_t\|_{H^{-1}(\Omega)} \leq C$ for some constant independent of α (for α small). $\square$

3.1.2 Limit $\alpha \to 0$

In order to pass on the limit $\alpha \to 0$, we need several terms to converge. Looking at the weak formulation of the regularized equation (3.46), we need to ensure that

- $(\rho_\alpha)_t \rightharpoonup \rho_t$ in $L^2(0, T, H^{-1}(\Omega))$ (3.53)
- $\nabla M(\rho_\alpha) \rightharpoonup \nabla M(\rho)$ in $L^2(0, T, L^2(\Omega))$ (3.54)
- $m(\rho_\alpha)\nabla S_\alpha \rightharpoonup m(\rho)\nabla S$ in $L^2(0, T, L^2(\Omega))$ (3.55)
- $\varepsilon\nabla\rho_\alpha \rightharpoonup 0$ in $L^2(0, T, L^2(\Omega))$ (3.56)
- $\rho_\alpha \rightharpoonup \rho$ in $L^2(0, T, L^2(\Omega))$ (3.57)
- $S_\alpha \rightharpoonup S$ in $L^2(0, T, H^1(\Omega))$. (3.58)

The main difficulty is not to prove that the sequences *converge* but to *identify* the limits with the desired functions. For this purpose it is necessary to have an additional strong convergence of ρ_α in B. This can be done by proving that the sequence ρ_α is uniformly bounded in a function space that is compact in B. The most natural space would be of course $W^{1,2}(0, T, H_0^1(\Omega), L^2(\Omega))$, but unfortunately we can not expect that the $H_0^1(\Omega)$-norm of ρ_α does not increase for $\alpha \to 0$ (although $(\rho_\alpha)_t$ is uniformly bounded, see estimate (3.52)). The idea is to make use of estimate (3.50) which tells us that $\|M(\rho_\alpha)\|_{L^2(0,T,H_0^1(\Omega))} \leq C_1$ and shift the regularity from $M(\rho_\alpha)$ to ρ_α. For this purpose we need to additionally assume that the inverse function $M^{-1}(x)$ is Hölder continuous. The following lemma will give us a sufficient condition.

Lemma 3.57. *Let $m(x) \geq Ax^{\beta}(1-x)^{\beta}$ for some $\beta \geq 0$. Then the inverse function M^{-1} of $M(x)$ exists and satisfies*

$$|M^{-1}(y_2) - M^{-1}(y_1)| < C(A,\beta)|y_2 - y_1|^{\theta}, \ \forall y_2, y_1 \in \{y \in \mathbb{R}| \ \exists x \in [0,1], \ M(x) = y\}$$

for $\theta = (2\beta+1)^{-1} < 1$ and $C = \left(\frac{6^{\beta}}{A}\right)^{\theta} > 0$.

Proof. Since $m(x) > 0$, it follows that $M(x)$ is strictly increasing and therefore invertible on its range. Now let $y_1 = M(x_1)$ and $y_2 = M(x_2)$ and without loss of generality $x_2 > x_1$.

$$\begin{aligned}
C(A,\beta)|y_2 - y_1|^{\theta} &= C(A,\beta)|M(x_2) - M(x_1)|^{\theta} \\
&= C(A,\beta)\left|\int_{x_1}^{x_2} m(u)\,du\right|^{\theta} \\
&\geq C(A,\beta)\left(\int_{x_1}^{x_2} Au^{\beta}(1-u)^{\beta}\,du\right)^{\theta} \\
&= C(A,\beta)A^{\theta}\|u(1-u)\|^{\beta\theta}_{L^{\beta}([x_1,x_2])} \\
&= 6^{\beta\theta}\|u(1-u)\|^{\beta\theta}_{L^{\beta}([x_1,x_2])}\underbrace{\|1\|^{\beta\theta}_{L^{q}([x_1,x_2])}(x_2-x_1)^{-\frac{\beta\theta}{q}}}_{=1}.
\end{aligned}$$

For $\frac{1}{\beta} + \frac{1}{q} = 1 \ \Leftrightarrow \ \frac{\beta}{q} = \beta - 1$ we can apply Hölder's inequality (reversed) and obtain

$$\begin{aligned}
C(A,\beta)|y_2 - y_1|^{\theta} &\geq 6^{\beta\theta}\|u(1-u)\|^{\beta\theta}_{L^{1}([x_1,x_2])}(x_2-x_1)^{\theta(1-\beta)} \\
&= 6^{\beta\theta}\Big(\frac{-x_2^3 + x_1^3}{3} + \frac{x_2^2 - x_1^2}{2}\Big)^{\beta\theta}(x_2-x_1)^{\theta(1-\beta)} \\
&= 6^{\beta\theta}(x_2-x_1)^{\beta\theta}\Big(-\frac{x_2^2 + x_1x_2 + x_1^2}{3} + \frac{x_1+x_2}{2}\Big)^{\beta\theta}(x_2-x_1)^{\theta(1-\beta)} \\
&= 6^{\beta\theta}(x_2-x_1)^{\theta}\Big(\frac{x_2^2 - 2x_1x_2 + x_1^2}{6} + \frac{-3x_2^2 - 3x_1^2}{6} + \frac{x_1+x_2}{2}\Big)^{\beta\theta} \\
&= 6^{\beta\theta}(x_2-x_1)^{\theta}\Big(\frac{(x_2-x_1)^2}{6} + \underbrace{\frac{(x_2 - x_2^2) + (x_1 - x_1^2)}{2}}_{\geq 0}\Big)^{\beta\theta} \\
&\geq 6^{\beta\theta}(x_2-x_1)^{\theta}\Big(\frac{(x_2-x_1)^2}{6}\Big)^{\beta\theta} \\
&= (x_2-x_1)^{2\beta\theta+\theta} \\
&= M^{-1}(y_2) - M^{-1}(y_1).
\end{aligned}$$

□

As mentioned above, we will use this to prove that ρ_{α} is uniformly bounded in a functional space that is compact in B. For this purpose we briefly recall the lemma of Chavent-Chaffre, the proof of which is straightforward and will be omitted.

Proposition 3.58 (Chavent-Chaffre). *Let Ω be an open subset of $\mathbb{R}^d$, $0 < s < 1$ and $1 < p < \infty$. In addition, let $f : \mathbb{R} \to \mathbb{R}$, $f(0) = 0$ be a Hölder continuous function with exponent $\theta \in (0,1)$ and constant $c_H > 0$. Then, there holds*

$$\|f(u)\|_{W^{s\theta,p/\theta}(\Omega)} \leq c_H \|u\|^{\theta}_{W^{s,p}(\Omega)}$$

for all $u \in W^{s,p}(\Omega)$.

Applying the proposition with $s \in (0,1)$, $p = 2$, $f = M^{-1}$ to the function $u = M(\rho) \in W^{s,p}(\Omega) \subset W^{1,p}(\Omega)$ leads to

$$\begin{aligned}\|\rho\|_{W^{s\theta,2/\theta}(\Omega)} &= \|M^{-1}(M(\rho))\|_{W^{s\theta,2/\theta}(\Omega)} \\ &\leq c_H \|M(\rho)\|^{\theta}_{W^{s,2}(\Omega)} \\ &\leq c_H \|M(\rho)\|^{\theta}_{W^{1,2}(\Omega)}\end{aligned}$$

for $\theta = (2\beta+1)^{-1}$ and $c_H = \left(\frac{6^\beta}{A}\right)^{\theta}$. Taking the a-priori estimate (3.50) into account yields

$$\|\rho\|_{L^2(0,T,W^{s\theta,2/\theta}(\Omega))} \leq c_H \|M(\rho)\|^{\theta}_{L^2(0,T,W^{1,2}(\Omega))} \leq c_H C_1^{\theta}.$$

Furthermore, the compact embedding $W^{s\theta,2/\theta}(\Omega) \hookrightarrow L^{2/\theta}(\Omega)$ (proposition 5.73) allows us to apply the lemma of Aubin (proposition 5.74) and we deduce $L^2(0,T,W^{s\theta,2/\theta}(\Omega), L^2(\Omega))$ is compact in B.
Hence, there exists a strongly convergent subsequence $\rho_{\alpha'} \stackrel{B}{\to} \rho$. Therefore, proposition 5.67 tells us that we can achieve that $\rho_{\alpha'}$ converges pointwise almost everywhere by switching again to a subsequence.
Since $m(x)$ and $M(x)$ are continuous bounded functions, we can now apply lemma 5.69 and it follows that $m(\rho_{\alpha'}) \stackrel{B}{\to} m(\rho)$ and $M(\rho_{\alpha'}) \stackrel{B}{\to} M(\rho)$. We already know that $M(\rho_{\alpha'})$ and $(\rho_{\alpha'})_t$ converge weakly in $L^2(0,T,H_0^1(\Omega))$ and $L^2(0,T,H^{-1}(\Omega))$, respectively. Because of the strong convergence in B we can identify the limits and deduce

- $(\rho_{\alpha'})_t \rightharpoonup \rho_t$ in $L^2(0,T,H^{-1}(\Omega))$ (3.59)
- $\nabla M(\rho_{\alpha'}) \rightharpoonup \nabla M(\rho)$ in $L^2(0,T,L^2(\Omega))$ (3.60)
- $m(\rho_{\alpha'}) \to m(\rho)$ in $L^2(0,T,L^2(\Omega))$ (3.61)
- $\rho_{\alpha'} \to \rho$ in $L^2(0,T,L^2(\Omega)) \cap L^\infty(0,T,L^\infty(\Omega))$ (3.62)

Because of the strong convergence in B of $\rho_{\alpha'}$ it is easy to see that $S_{\alpha'} = \mathcal{B} * \rho_{\alpha'}$ converges towards S in $L^2(0,T,H_0^1(\Omega))$ and $\nabla S_{\alpha'} \to \nabla S$ almost everywhere.
Therefore we finally arrive at the following existence

Corollary 3.59 (Existence for the degenerate equation). *Let $m(x)$ satisfy the assumptions of lemma 3.57 and $0 \leq \rho_0 \leq 1$. Then, there exists a solution satisifying definition 3.48 (replacing $\mathbb{R}^d$ with Ω).*

Proof. Since

$$\begin{aligned}\varepsilon\alpha' \int_0^t \int_\Omega \nabla\rho_{\alpha'} \cdot \nabla\varphi \, dxds &\leq \sqrt{\alpha'} \|\sqrt{\alpha'}\nabla\rho_{\alpha'}\|_{L^2(0,T,L^2(\Omega))} \|\varphi\|_{L^2(0,T,H_0^1(\Omega))} \\ &\leq \sqrt{\alpha'} C_2 \|\varphi\|_{L^2(0,T,H_0^1(\Omega))} \to 0,\end{aligned}$$

where we have used the estimate (3.51), we can pass the limit in the weak formulation. Since a (weakly) convergent sequence is uniformly bounded, regularity follows directly from the convergences. □

3.1.3 Limit $R \to \infty$

In this section we want to finally send the radius of the ball to infintiy. Let ρ_R, S_R be the weak solutions of the degenerated problem for some fixed radius R given by corollary 3.59. Again, we will need a-priori estimates independent of the radius R. Let us recall what we have already proven:

- $0 \leq \rho_R, S_R \leq 1$
- $\|\rho_R(.,t)\|^2_{L^2(\Omega)} \leq e^{Lt}\|\rho_{R,0}\|^2_{L^2(\Omega)} \leq e^{Lt}\|\rho_0\|^2_{L^2(\mathbb{R}^d)}$
- $\|M(\rho_R)\|_{L^2(0,T,H^1(\Omega))} \leq C(T)\|\rho_{R,0}\|_{L^2(\Omega)} \leq C(T)\|\rho_0\|_{L^2(\mathbb{R}^d)}$
- $\|(\rho_R)_t\|_{L^2(0,T,H^{-1}(\Omega))} \leq C'(T)\|\rho_{R,0}\|_{L^2(\Omega)} \leq C'(T)\|\rho_0\|_{L^2(\mathbb{R}^d)}$.

And therefore (regarding the solutions ρ_R from the previous section on a ball with radius R as functions in $\mathbb{R}^d$ with $\rho_R(\mathbb{R}^d \backslash \Omega) = 0$)

$$\begin{aligned}\|\rho_R(.,t)\|^2_{L^2(\mathbb{R}^d)} &\leq e^{Lt}\|\rho_0\|^2_{L^2(\mathbb{R}^d)} \\ \|M(\rho_R)\|_{L^2(0,T,H^1(\mathbb{R}^d))} &\leq C(T)\|\rho_0\|_{L^2(\mathbb{R}^d)} \\ \|(\rho_R)_t\|_{L^2(0,T,H^{-1}(\mathbb{R}^d))} &\leq C'(T)\|\rho_0\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Therefore, we can easily conclude uniform bounds for S_R

$$\begin{aligned}\|S_R\|_{L^2(0,T,L^2(\mathbb{R}^d))} &\leq \|\mathcal{B} * \chi_{\Omega_R}\rho_R\|_{L^2(0,T,L^2(\mathbb{R}^d))} \\ &\leq \|\mathcal{B}\|_{L^\infty(0,T,L^1(\mathbb{R}^d))}\|\chi_{\Omega_R}\rho_R\|_{L^2(0,T,L^2(\mathbb{R}^d))} \\ &\leq \|\rho_R\|_{L^2(0,T,L^2(\Omega_R))} \\ &\leq C(T)\|\rho_0\|_{L^2(\mathbb{R}^d)} \\ \|\nabla S_R\|_{L^2(0,T,L^2(\mathbb{R}^d))} &\leq \|\nabla\mathcal{B} * \chi_{\Omega_R}\rho_R\|_{L^2(0,T,L^2(\mathbb{R}^d))} \\ &\leq \|\nabla\mathcal{B}\|_{L^\infty(0,T,L^1(\mathbb{R}^d))}\|\chi_{\Omega_R}\rho_R\|_{L^2(0,T,L^2(\mathbb{R}^d))} \\ &\leq C\|\rho_R\|_{L^2(0,T,L^2(\Omega_R))} \\ &\leq C'(T)\|\rho_0\|_{L^2(\mathbb{R}^d)},\end{aligned}$$

and similarly

$$\begin{aligned}
\|\nabla S_R\|_{L^\infty(0,T,L^\infty(\mathbb{R}^d))} &\leq \|\nabla\mathcal{B} * \chi_{\Omega_R}\rho_R\|_{L^\infty(0,T,L^\infty(\mathbb{R}^d))}\\
&\leq \|\nabla\mathcal{B}\|_{L^\infty(0,T,L^1(\mathbb{R}^d))}\|\chi_{\Omega_R}\rho_R\|_{L^\infty(0,T,L^\infty(\mathbb{R}^d))}\\
&\leq C\|\rho_R\|_{L^\infty(0,T,L^\infty(\Omega_R))}\\
&\leq C.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\|m(\rho_R)\nabla S\|_{L^2(0,T,L^2(\mathbb{R}^d))} &\leq \|\nabla S\|_{L^\infty(0,T,L^\infty(\mathbb{R}^d))}\|m(\rho_R)\|_{L^2(0,T,L^2(\mathbb{R}^d))}\\
&\leq C\|m(\rho_R)\|_{L^2(0,T,L^2(\mathbb{R}^d))}\\
&\leq CL\|\rho_R\|_{L^2(0,T,L^2(\mathbb{R}^d))}\\
&\leq C'(T)\|\rho_0\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

If we look at a sequence $R_n \to \infty$ and the corresponding sequence ρ_n, S_n we can deduce that there exist subsequences such that

$$(\rho_n)_t \rightharpoonup g_1 \text{ in } L^2(0,T,H^{-1}(\mathbb{R}^d)) \tag{3.63}$$
$$\nabla M(\rho_n) \rightharpoonup g_2 \text{ in } L^2(0,T,L^2(\mathbb{R}^d)) \tag{3.64}$$
$$m(\rho_n)\nabla S_n \rightharpoonup g_3 \text{ in } L^2(0,T,L^2(\mathbb{R}^d)) \tag{3.65}$$
$$\rho_n \rightharpoonup \rho \text{ in } L^2(0,T,L^2(\mathbb{R}^d)) \tag{3.66}$$
$$S_n \rightharpoonup g_5 \text{ in } L^2(0,T,H^1(\mathbb{R}^d)). \tag{3.67}$$

These are exactly the desired convergences necessary to perform the limit in the weak formulation. Just as before, the remaining problem is to identify the limits. Unfortunately, we cannot directly use the same approach as in the previous section, because the compact embeddings we applied do not work in the case of unbounded domains. In fact, we cannot expect a strong convergence at all and we will therefore pursue a different strategy.
We start by proving the following

Lemma 3.60. *For each fixed R', there exists a subsequence $R_{n'} \subset R_n$ such that $\rho'_n \to \rho$ in $L^\infty([0,T] \times \Omega_{R'})$.*

Proof. Since

$$\|M(\rho_n)\|_{L^2(0,T,H^1(\Omega_{R'}))} \leq \|M(\rho_n)\|_{L^2(0,T,H^1(\mathbb{R}^d))} \leq C(T,d)$$

due to the lemma of Chavent-Chaffre (see proposition 3.58 and the following) and the Hölder continuity of M^{-1} (see lemma 3.57), we can deduce that

$$\|\rho_n\|_{L^2(0,T,W^{s\theta,2/\theta}(\Omega_{R'}))} \leq C(\theta, c_H, T, d).$$

Again, by taking into account that the embedding $W^{s\theta,2/\theta}(\Omega_{R'}) \hookrightarrow L^2(\Omega_{R'})$ is compact (proposition 5.73), we conclude that there exists a subsequence ρ_n' satisifying $\rho_n' \to \rho$ in $L^2([0,T] \times \Omega_{R'})$.
Using proposition 5.67 once again leads to the desired subsequence that converges in L^∞. $\square$

Now we can easily prove

Corollary 3.61. *Let $\rho_n \rightharpoonup \rho$ in $L^2(0,T,L^2(\mathbb{R}^d))$ and $S = \mathcal{B} * \rho$. Then, there exists a subsequence $\rho_{n'}$ satisfying*

$$\begin{aligned} (\rho_{n'})_t &\to \rho_t \ \textit{in}\ \mathcal{D}'([0,T]\times\mathbb{R}^d) \\ \nabla M(\rho_{n'}) &\to \nabla M(\rho) \ \textit{in}\ \mathcal{D}'([0,T]\times\mathbb{R}^d) \\ m(\rho_{n'})\nabla S_{n'} &\to m(\rho)\nabla S \ \textit{in}\ \mathcal{D}'([0,T]\times\mathbb{R}^d) \\ S_{n'} &\to S \ \textit{in}\ \mathcal{D}'([0,T]\times\mathbb{R}^d). \end{aligned}$$

Proof. We start by choosing a testfunction $\psi \in \mathcal{D}([0,T]\times\mathbb{R}^d)$. Since ψ has compact support K, there exists a radius R' such that K is a subset of $\Omega_{R'}$. Now we can easily estimate

$$\begin{aligned} |< (\rho_n)_t - \rho_t, \psi >_{\mathcal{D}'}| &= |< \rho - \rho_n, \psi_t >_{\mathcal{D}'}| \\ &= \left| \int_{[0,T]\times\mathbb{R}^d} (\rho - \rho_n)\cdot \psi_t \, dxdt \right| \\ &= \left| \int_{[0,T]\times\Omega_{R'}} (\rho - \rho_n)\cdot \psi_t \, dxdt \right| \\ &\leq \|\rho - \rho_n\|_{L^\infty([0,T]\times\Omega_{R'})} \left| \int_{[0,T]\times\Omega_{R'}} \psi_t \, dxdt \right|. \end{aligned}$$

Now the previous lemma 3.60 tells us that there exists a subsequence $\rho_{n'}$ that converges almost everywhere and therefore

$$|< (\rho_{n'})_t - \rho_t, \psi >_{\mathcal{D}'}| \to 0,$$

for every fixed $\psi \in \mathcal{D}(\mathbb{R}^d)$ $\Rightarrow$ $(\rho_{n'})_t \rightharpoonup \rho_t$ in $\mathcal{D}'([0,T]\times\mathbb{R}^d)$.
Now starting with the subsequence $\rho_{n'}$ from above, we similarly get for $\psi \in \big(\mathcal{D}([0,T]\times\mathbb{R}^d)\big)^d$

$$\begin{aligned} < m(\rho_{n'})\nabla S_{n'} - m(\rho)\nabla S, \psi >_{\mathcal{D}'} &= \int_{[0,T]\times\mathbb{R}^d} (m(\rho_{n'})\nabla S_{n'} - m(\rho)\nabla S)\cdot\psi \, dxdt \\ &= \int_{[0,T]\times\mathbb{R}^d} (m(\rho_{n'})\nabla S_{n'} - m(\rho)\nabla S_{n'})\cdot\psi \, dxdt \\ &\quad + \int_{[0,T]\times\mathbb{R}^d} (m(\rho)\nabla S_{n'} - m(\rho)\nabla S)\cdot\psi \, dxdt \\ &= \int_{[0,T]\times\mathbb{R}^d} (m(\rho_{n'}) - m(\rho))\nabla S_{n'}\cdot\psi \, dxdt \\ &\quad + \int_{[0,T]\times\mathbb{R}^d} m(\rho)(\nabla S_{n'} - \nabla S)\cdot\psi \, dxdt, \end{aligned}$$

and therefore, using that $m(x)$ is Lipschitz continuous and applying Hölder's inequality,

$$\begin{aligned}
|< m(\rho_{n'})\nabla S_{n'} - m(\rho)\nabla S, \psi >_{\mathcal{D}'}| \ \le & \\
\le\ & L\|\rho_{n'} - \rho\|_{L^\infty([0,T]\times\Omega_{R'})}\|\nabla S_{n'}\|_{L^\infty([0,T]\times\Omega_{R'})} \int_{[0,T]\times\mathbb{R}^d} |\psi|\, dxdt \\
& + L\|\nabla S_{n'} - \nabla S\|_{L^\infty([0,T]\times\Omega_{R'})} \int_{[0,T]\times\mathbb{R}^d} |\psi|\, dxdt \\
\le\ & 2LC(d)\|\rho_{n'} - \rho\|_{L^\infty([0,T]\times\Omega_{R'})} \int_{[0,T]\times\mathbb{R}^d} |\psi|\, dxdt \ \to\ 0,
\end{aligned}$$

where we have used that $\rho_{n'} \to \rho$ almost everywhere. The proof of the remaining convergences follows the exact same pattern and will be omitted. □

Now we are well prepared to state the final theorem of this section.

Theorem 3.62 (Existence of weak solutions)**.** *Let* $\rho_0 \in L^\infty(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)$*,* $0 \le \rho_0 \le 1$*,* $m(x)$ *Lipschitz continuous,* $m(0) = m(1) = 0$*,* $m(x) > Ax^\beta(1-x)^\beta$ *for some constants* $A > 0$ *and* $\beta \ge 0$ *and for all* $x \in (0,1)$*.*
Then, there exists a weak solution of (3.43) in the sense of definition 3.48 for any time $T > 0$*.*

Proof. Since $(\rho'_n)_t \rightharpoonup g_1$ in $L^2(0,T,H^{-1}(\mathbb{R}^d))$, it also converges in a weaker sense, namely $(\rho'_n)_t \rightharpoonup g_1$ in $\mathcal{D}'([0,T]\times\mathbb{R}^d)$. Now corollary 3.61 tells us that there exists a subsequence that satisifies

$$(\rho_{n''})_t \quad \to \quad \rho_t \text{ in } \mathcal{D}'([0,T]\times\mathbb{R}^d).$$

Since we started with a $\mathcal{D}'([0,T]\times\mathbb{R}^d)$-convergent sequence $(\rho_{n'})_t$ in the first place, the subsequence $(\rho_{n''})_t$ has to have the same limit, and therefore $\rho_t = g_1$.

The exact same argument can be applied for all needed convergences (3.63)-(3.67) and we finally deduce

$$\begin{aligned}
(\rho_n)_t &\ \rightharpoonup\ \rho_t \text{ in } L^2(0,T,H^{-1}(\mathbb{R}^d)) \\
\nabla M(\rho_n) &\ \rightharpoonup\ \nabla M(\rho) \text{ in } L^2(0,T,L^2(\mathbb{R}^d)) \\
m(\rho_n)\nabla S_n &\ \rightharpoonup\ m(\rho)\nabla S \text{ in } L^2(0,T,L^2(\mathbb{R}^d)) \\
\rho_n &\ \rightharpoonup\ \rho \text{ in } L^2(0,T,L^2(\mathbb{R}^d)) \\
S_n &\ \rightharpoonup\ S \text{ in } L^2(0,T,H^1(\mathbb{R}^d)).
\end{aligned}$$

Therefore we can pass the limit in the weak formulation and obtain

$$\int_0^t < \rho_t, \varphi >_{H^{-1}} \; ds + \int_0^t \int_{\mathbb{R}^d} [\varepsilon \cdot \nabla M(\rho) - m(\rho)\nabla S] \cdot \nabla\varphi \, dxds = 0$$
$$\int_0^t \int_{\mathbb{R}^d} \nabla S \cdot \nabla\varphi + (S-\rho)\varphi \, dxds = 0$$

for all $\varphi \in L^2(0,T,H_0^1(\Omega_{R'}))$ for some $R' > 0$.
Since $\bigcup_{i\in\mathbb{N}} L^2(0,T,H_0^1(\Omega_i)) \supset \mathcal{D}([0,T]\times\mathbb{R}^d)$ is dense in $L^2(0,T,H^1(\mathbb{R}))$ we can also admit $\varphi \in L^2(0,T,H^1(\mathbb{R}))$. The required properties of ρ and S in the defintion 3.48 result directly from the weak convergences.

□

3.2 Energy Dissipation

In this section we will define an energy functional and formally prove the non-positivity of its derivative with respect to time. The proof can be made rigorous by applying a similar approximation procedure we have already employed in section 2.6 (energy dissipation for linear diffusion).

Definition 3.63 (Energy functional)**.** *For $\rho \in L^1(\mathbb{R}^d)$ and $0 \le \rho \le 1$, we define the following energy functional*

$$\mathcal{E}(\rho) := \int_{\mathbb{R}^d} \rho(\varepsilon\rho - \mathcal{B} * \rho)dx \tag{3.68}$$

Lemma 3.64 (Dissipation of energy and stationary solutions)**.** *Let $\rho(x,t)$ be a solution of (3.43). Then*

$$\frac{\partial}{\partial t}\mathcal{E}(\rho(.,t)) \le 0.$$

In addition, if $\frac{\partial}{\partial t}\mathcal{E}(\rho(.,t)) = 0$, then $m(\rho) = 0$ or $\nabla(\varepsilon\rho - \mathcal{B} * \rho) = 0$.

Proof. As mentioned above we will only do a formal computation, assuming ρ is sufficiently smooth.

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{E}(\rho(.,t)) &= \int_{\mathbb{R}^d} \rho_t(\varepsilon\rho - \mathcal{B} * \rho) + \rho(\varepsilon\rho_t - \mathcal{B} * \rho_t)\, dx \\
&= \int_{\mathbb{R}^d} \rho_t(2\varepsilon\rho - \mathcal{B} * \rho) - \rho(\mathcal{B} * \rho_t)\, dx.
\end{aligned}$$

Since $\mathcal{B}$ is even, the second term can be rewritten in the following way:

$$\begin{aligned}
\int_{\mathbb{R}^d} \rho(\mathcal{B} * \rho_t)\, dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x,t)\mathcal{B}(x-y)\rho_t(y,t)\, dydx \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x,t)\mathcal{B}(y-x)\rho_t(y,t)\, dxdy \\
&= \int_{\mathbb{R}^d} (\rho * \mathcal{B})(y,t)\rho_t(y,t)\, dy \\
&= \int_{\mathbb{R}^d} (\rho * \mathcal{B})\rho_t\, dx,
\end{aligned}$$

and therefore we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{E}(\rho(.,t)) &= 2\int_{\mathbb{R}^d} \rho_t(\varepsilon\rho - \mathcal{B} * \rho)\, dx \\
&= 2\int_{\mathbb{R}^d} \operatorname{div}\left(m(\rho)\nabla(\varepsilon\rho - \mathcal{B} * \rho)\right)(\varepsilon\rho - \mathcal{B} * \rho)\, dx \\
&= -2\int_{\mathbb{R}^d} m(\rho)\left|\nabla(\varepsilon\rho - \mathcal{B} * \rho)\right|^2\, dx \ \leq \ 0.
\end{aligned}$$

□

In contrary to chapter 2, where solutions with small mass decay to zero, we can construct solutions with arbitrarily small mass:

Theorem 3.65. *Let $0 < \varepsilon < 1$ and $d = 1$. Then, for every $M_0 < K(\varepsilon)$, there exists at least one pair of stationary solutions $\rho(x), S(x)$ of (3.43) satisfying $0 \leq \rho \leq 1$ and $\int_{\mathbb{R}} \rho(x)dx = M_0$.*

Proof. As suggested by lemma 3.64, we say that our pair of functions shall satisfy

$$\begin{cases} \varepsilon\rho = S + C & \text{for } |x| < R, \\ \rho = 0 & \text{for } |x| \geq R, \end{cases}$$

for constants $R > 0$ and $C \in \mathbb{R}$. Inserting $S = \varepsilon\rho - C$ into the elliptic equation yields

$$\begin{aligned}
-\varepsilon\rho'' + \varepsilon\rho - \varepsilon C &= \rho \\
\Rightarrow\ \rho'' &= -\underbrace{\frac{1-\varepsilon}{\varepsilon}}_{=:\alpha^2}(\rho - C').
\end{aligned}$$

This differential equation can be easily solved and we obtain for $\rho'(0) = 0$ and $\rho(R) = 0$

$$\begin{cases} \rho(x) = C' \cdot \left(1 - \frac{\cos(\alpha x)}{\cos(\alpha R)}\right) & \text{for } |x| < R, \\ \rho(x) = 0 & \text{for } |x| \geq R. \end{cases}$$

The differential equation $-S''(x) + S(x) = \rho(x)$ for $x \in \mathbb{R}$ and $\varepsilon\rho(x) - S(x) = \textit{constant}$ for $x < R$ leads to

$$\begin{cases} S(x) = C' \cdot \left(1 - \varepsilon \frac{\cos(\alpha x)}{\cos(\alpha R)}\right) & \text{for } |x| < R, \\ S(x) = C' \cdot (1-\varepsilon) \cdot e^{R-|x|} & \text{for } |x| \geq R. \end{cases}$$

The remaining constants C' and R will be determined from $S'(R^-) = S'(R^+)$ (i.e. S is continuously differentiable) and $\int_{\mathbb{R}} \rho(x)dx = M_0$, which yields

$$\begin{aligned} C'\varepsilon\alpha \frac{\sin(\alpha R)}{\cos(\alpha R)} &= -C' \cdot (1-\varepsilon) \\ \Rightarrow \tan(\alpha R) &= -\frac{1}{\alpha} \underbrace{\frac{1-\varepsilon}{\varepsilon}}_{=\alpha^2} = -\alpha. \end{aligned}$$

Since $R, \alpha > 0$, we use the second branch of the arctan and obtain $R = \frac{\pi - \arctan(\alpha)}{\alpha}$. For the last constant we compute

$$\begin{aligned} M_0 &= \int_{\mathbb{R}} \rho(x)dx = 2\int_0^R \rho(x)dx = 2C' \cdot \left(R - \frac{\tan(\alpha R)}{\alpha}\right) = 2C' \cdot (R+1) \\ &\Rightarrow C' = \frac{M_0}{2 \cdot (R+1)}. \end{aligned}$$

Since we only searched for solutions $\rho < 1$ and ρ is decreasing by construction, we get an upper bound for C', namely

$$\rho(0) = C' \cdot \left(1 - \frac{1}{\cos(\alpha R)}\right) \leq 1,$$

and therefore after inserting R from above

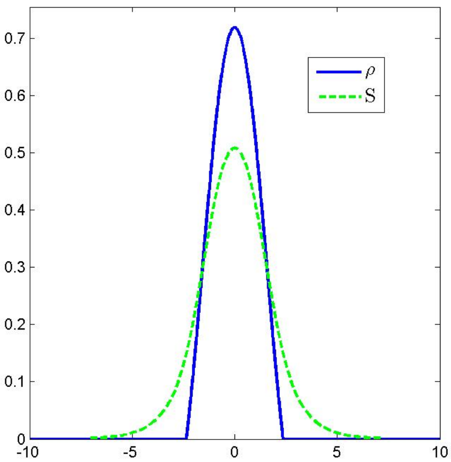
$$M_0 \leq \frac{2}{\alpha} \cdot \frac{\alpha + \pi + \arctan(\alpha)}{\sqrt{\alpha^2+1}+1} =: K(\varepsilon).$$

$\square$

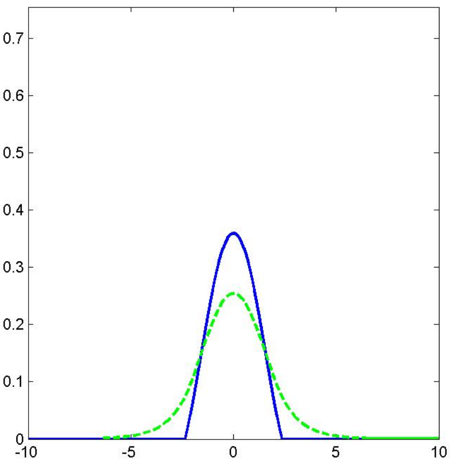
Remark 3.66 (Stationary solutions with larger mass)**.** *A similar approch with*

$$\begin{cases} \rho = 1 & \textit{for } |x| \leq R_1 \\ \varepsilon\rho = S + C & \textit{for } R_1 < |x| < R_0, \\ \rho = 0 & \textit{for } |x| \geq R_0, \end{cases}$$

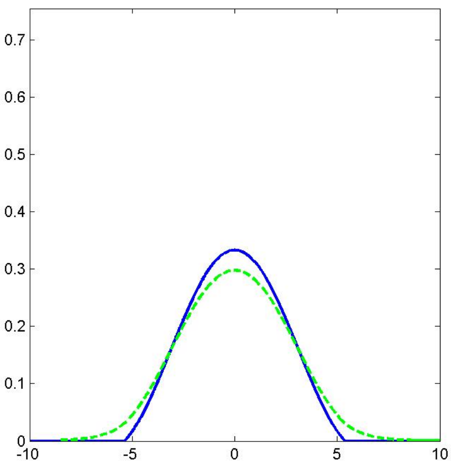
for $0 < R_1 < R_0$ *will (in combination with theorem 3.65) lead to the existence of stationary solutions with arbitrary mass. Since the calculations of the occuring constants are rather lengthy we will omit the derivation.*



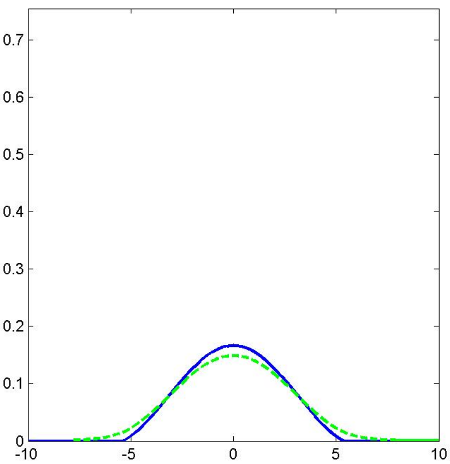
(a) $M_0 = 2$, $\varepsilon = 0.5$



(b) $M_0 = 1$, $\varepsilon = 0.5$



(c) $M_0 = 2$, $\varepsilon = 0.8$

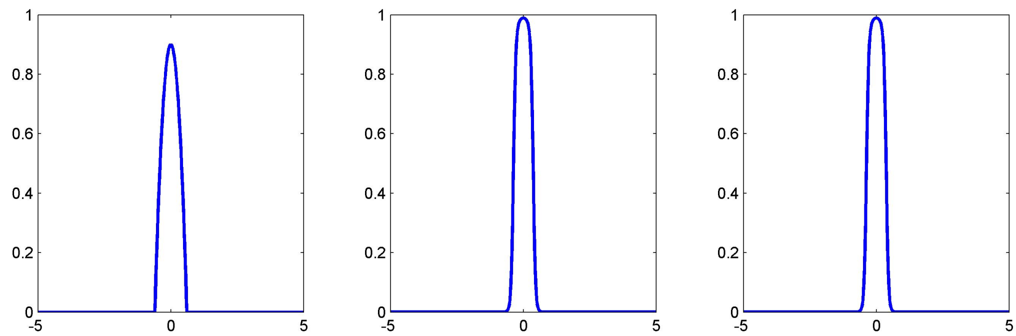


(d) $M_0 = 1$, $\varepsilon = 0.8$

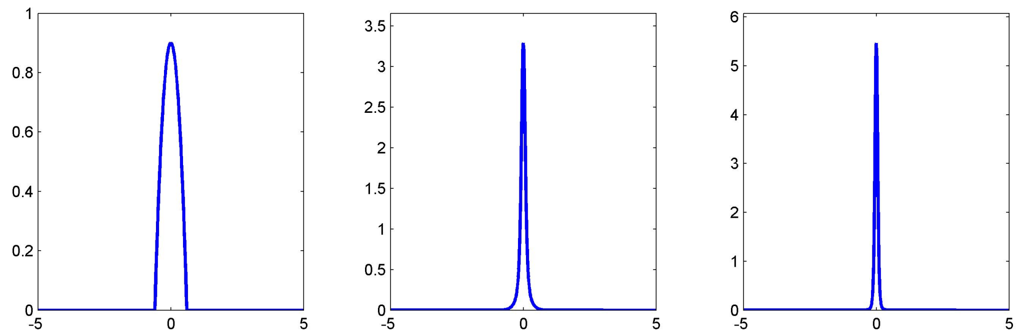
Figure 3: Stationary solutions of theorem 3.65 for different values of M_0 and ε

4 Numerical Results

In this chapter we want to discuss numerical solutions for different choices of $f(x)$ (for the linear model) and $m(x)$ (for the non-linear model) in one dimension, on an intervall with homogeneous Neumann boundary conditions. For reasons of simplicity we use finite differences in space and then solve the resulting non-linear system of ordinary differential equations (in time) by making use of an implicit one-step formula. Note that the use of an implicit scheme is indispensable since numerical results show that we cannot expect the absolute values of the eigenvalues of the Jacobian to be (stay) small.
We start by comparing the effects of different choices of ε and $f(x)$ in the case of linear diffusion.



(a) $f(x) = x(1-x)$, $\varepsilon = 0.01$, $t = 0$ (b) $t = 100$ (c) $t = 1000$

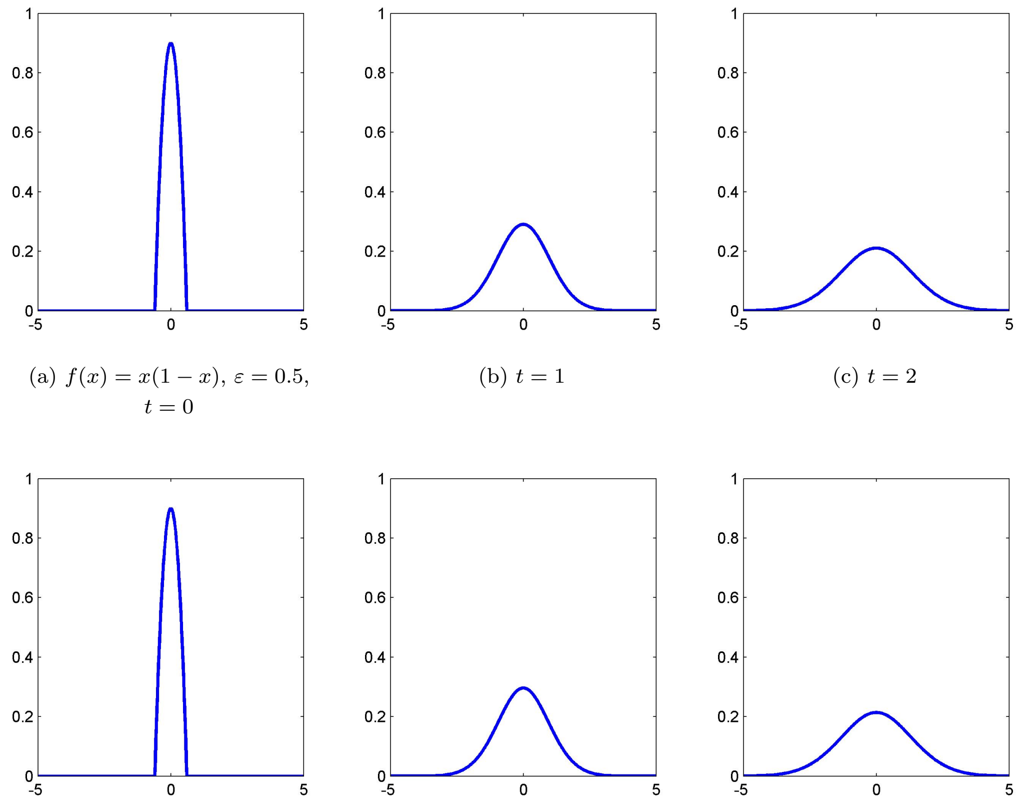


(d) $f(x) = x$, $\varepsilon = 0.01$, $t = 0$ (e) $t = 2$ (f) $t = 4$

Figure 4: Behaviour for supercritical mass

Since ε is small compared to the total mass, we can not expect our solution ρ to decay in

L^∞ for $t \to \infty$. In the first case ($f(x) = x(1-x)$) we observe global boundedness (i.e. prevention of overcrowding, see section 2.5), whereas in the second case a typical behaviour for Keller-Segel models with no prevention of overcrowding, a so called blow-up, occurs. (see for example [12] for details about blow-up phenomena in Keller-Segel-type equations).
A completly different situation is observed in the case of subcritical mass (see lemma 2.33) or equivalently for large ε:



(a) $f(x) = x(1-x)$, $\varepsilon = 0.5$, $t = 0$

(b) $t = 1$

(c) $t = 2$

(d) $f(x) = x$, $\varepsilon = 0.5$, $t = 0$

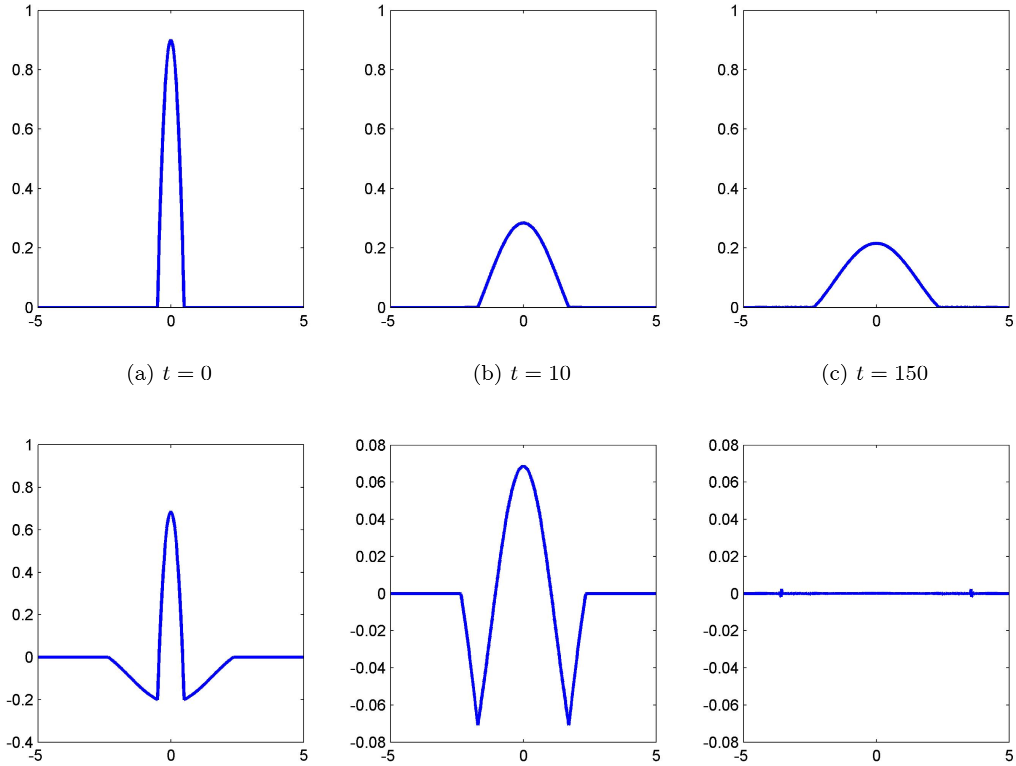
(e) $t = 1$

(f) $t = 2$

Figure 5: Behaviour for subcritical mass

Obviously, the diffusion term is dominant for both functions $f(x)$ and therefore $\rho \to \frac{M_0}{b-a}$ as $t \to \infty$, where M_0 denotes the total mass and $x \in [a, b]$. Therefore, if we consider this numerical solution as an approximation for the solution on $\mathbb{R}$, we obtain, just as expected, that $\rho \to 0$ for $[a, b] \to \mathbb{R}$ and $t \to \infty$.
If we insert the same initial data and $\varepsilon = 0.5$ into the second model, we expect a completely

different behaviour. Since stationary solutions exist, we suppose that the solutions do not decay to zero but they tend towards a stationary state. As shown in the following, numerical results emphasise our expectations. Here, the second set of plots illustrates the difference between the numerical solution (shown in the first set), and the stationary solution given by theorem 3.65.

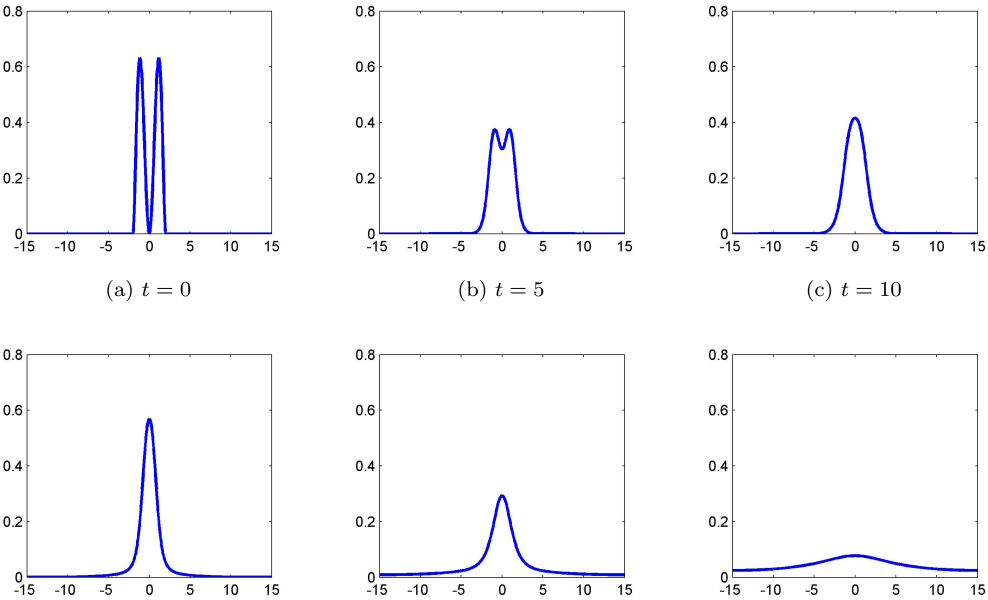


(a) $t = 0$ (b) $t = 10$ (c) $t = 150$

(d) $t = 0$ (e) $t = 10$ (f) $t = 150$

Figure 6: Behaviour in the case of non-linear diffusion

Another interesting difference between linear and non-linear diffusion is the completely different behaviour considering the speed of propagation. As proven in [14], for $f(x) = x(1 - x)$,m ass spreads with infinite speed in the linear model whereas the non-linear model has a finite rate of dispersion. This is not surprising when comparing the differential equations to other similar models like the heat equation and the porous medium equation, respectively. Finally, we want to study a choice of initial data that visualises the effect of attraction. Therefore, we consider a function ρ_0 with two peaks next to each other and choose the diffusion constant not too large in order to prevent pure diffusion. Again, we will first look at the model with linear diffusion:

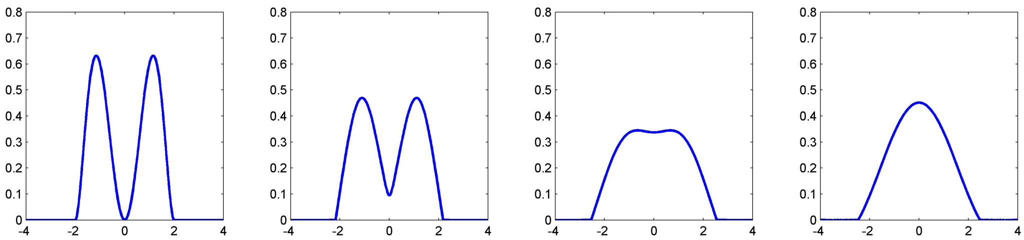


(a) $t = 0$ (b) $t = 5$ (c) $t = 10$

(d) $t = 100$ (e) $t = 1000$ (f) $t = 2000$

Figure 7: Linear diffusion for $f(x) = x(1-x)$ and $\varepsilon = 0.07$

Here we see that initially, due to attraction, the cells tend to clump together. Later however, since the total mass is not large enough, the diffusion becomes dominant and $\rho \to 0$. In the case of non-linear diffusion on the contrary, since the diffusion depends on the density ρ and becomes zero for $\rho = 0$, the solutions tends (just like before) to a stationary solution of the model.



(a) $t = 0$ (b) $t = 1$ (c) $t = 5$ (d) $t = 50$

Figure 8: Non-linear diffusion for $f(x) = x(1-x)$ and $\varepsilon = 0.5$

5 Appendix

The following proposition is one of the most fundamental tools in the theory of non-linear partial differential equations (and of course many other fields). A proof can be found, for instance, in [3, Théorème IV.9].

Proposition 5.67 (Dominated convergence)**.** *Let $\Omega \subset \mathbb{R}^d$ an open set and $f_k \in L^p(\Omega)$ a sequence.*

1. *($1 \le p < \infty$) If $f_k \to f$ almost everywhere in Ω for $k \to \infty$ and there exists a function $g \in L^p(\Omega)$ satisfying $|f_k| \le g$ in Ω for every $k \in \mathbb{N}$, then $f \in L^p(\Omega)$ and $f_k \to f$ in $L^p(\Omega)$.*
2. *($1 \le p \le \infty$) If $f_k \to f$ in $L^p(\Omega)$, then there exists a subsequence $f_{k'}$ and a function $g \in L^p(\Omega)$ satisfying $f_{k'} \to f$ almost everywhere in Ω and $|f_{k'}| \le g, \forall k'$.*

Another standard tool for proving convergence is the following

Proposition 5.68. *Let B be a reflexive Banach space and $u_k \in B$ a bounded sequence.*

1. *u_k is (weakly) convergent towards u if and only if all (weakly) convergent subsequences $u_{k'}$ have the same limit u.*
2. *There exists a weakly convergent subsequence $u_{k'} \overset{B}{\rightharpoonup} u$ and $\|u\|_B \le \liminf \|u_{k'}\|_B$.*

A proof can be found, for instance, in [8, Proposition 21.23]
The proof of the following useful lemma is a straightforward application of proposition 5.67 and 5.68 and will be omitted.

Lemma 5.69. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $f : \mathbb{R} \to \mathbb{R}$ continuous and bounded. Then there holds*

$$u_k \overset{B}{\to} u \quad \Rightarrow \quad f(u_k) \overset{B}{\to} f(u),$$

for $B = L^p(\Omega)$ and $1 \le p < \infty$.

The next lemma is a generalisation of proposition 1.9 and will be crucial in order to prove estimates for fixed points.

Lemma 5.70. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $u \in W^{1,2}(0, T, H_0^1(\Omega), L^2(\Omega))$, $F : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous, differentiable and $F(0) = 0$. Then there holds*

1. *$F(u) \in L^2(0, T, H_0^1(\Omega))$,*
2. *$\nabla F(u) = f(u)\nabla u$, where $F'(x) = f(x)$,*

3. *$t \mapsto \|\mathcal{F}(u(.,t))\|_{L^1(\Omega)}$, where $\mathcal{F}'(x) = F(x)$, $\mathcal{F}(0) = 0$, is an absolute continuous function and*

4. *$< u_t, F(u) >_{H^{-1}} = \frac{\partial}{\partial t}\|\mathcal{F}(u(.,t))\|_{L^1(\Omega)}$.*

Proof. We start by simply estimating and using the Lipschitz continuity of $F(x)$ and proposition 1.10 for the identification $\nabla F(u) = F'(u)\nabla u$

$$\begin{aligned} \|F(u)\|^2_{L^2(0,T,L^2(\Omega))} &= \int_{\Omega\times[0,T]} |F(u)|^2 \, dxdt \\ &\leq L^2 \int_{\Omega\times[0,T]} |u|^2 \, dxdt \ < \ \infty \text{ and} \end{aligned}$$

$$\begin{aligned} \|\nabla F(u)\|^2_{L^2(0,T,L^2(\Omega))} &= \int_{\Omega\times[0,T]} |\nabla F(u)|^2 \, dxdt \\ &= \int_{\Omega\times[0,T]} |F'(u)\nabla u|^2 \, dxdt \\ &\leq L^2 \int_{\Omega\times[0,T]} |\nabla u|^2 \, dxdt \ < \ \infty. \end{aligned}$$

The proof of the remaining properties is similar to the the proof of proposition 1.9, which can be found in [2, Chapter 8, Theorem 3]. □

The following inequality is widely used in the theory of partial differential equations on bounded domains and a proof can be found in [7, Theorem 7.17].

Proposition 5.71 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $\partial\Omega \in C^1$ and $1 \leq p < \infty$. Then there is a constant $C_P > 0$ satisfying*

$$\|u\|_{L^p(\Omega)} < C_P \|\nabla u\|_{L^p(\Omega)},$$

for all functions $u \in W^{1,p}_0(\Omega)$.

Definition 5.72 (Fractional Sobolev spaces)**.** *Let $0 < s < 1 < p < \infty$ and Ω be an open subset of $\mathbb{R}^d$. Then we define:*

$$\begin{aligned} W^{s,p}(\Omega) &:= \left\{ u \in L^p(\Omega) \mid \|u\|_{W^{s,p}(\Omega)} < \infty \right\}, \text{ where} \\ \|u\|^p_{W^{s,p}(\Omega)} &:= \int_{\Omega^2} \frac{|u(x)-u(y)|^p}{|x-y|^{ps}} \, dxdy. \end{aligned}$$

These spaces are reflexive Banach spaces. Another important property is the following embedding into Lebesgue spaces (see for instance [1, chapter VII]).

Proposition 5.73. *Let* $1 > s' > s > 0$, Ω *bounded with* $\partial\Omega \in C^1$ *and*

$$\begin{cases} p \leq r \leq np/(n-sp) & \text{for } n > sp, \\ p \leq r < \infty & \text{for } n = sp \text{ and} \\ p \leq r \leq \infty & \text{for } n < sp \end{cases}$$

then $W^{s',p}(\Omega) \hookrightarrow L^r(\Omega)$ *and* $W^{1,p}(\Omega) \hookrightarrow W^{s',p}(\Omega)$ *are both compact embeddings.*

The following proposition, known as Aubin's lemma, will allow us to achieve strong convergence for a sequence that is bounded in a fractional Sobolev space. A proof of this result can be found for instance in [11, Corollary 4].

Proposition 5.74 (Aubin's lemma)**.** *Let* V *be a seperable reflexive Banach space and* H *a seperable Hilbert space such that there exists a compact embedding* $V \hookrightarrow H$*. Furthermore let* $1 < p < \infty$*. Then the embedding* $W^{1,p}(0,T,V,H) \hookrightarrow L^p(0,T,H)$ *is compact.*

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