TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

DIPLOMARBEIT

Hilbert Spaces of Entire Functions in the Hardy Space Setting

Ausgeführt am Institut für

Analysis and Scientific Computing

der Technischen Universität Wien

unter der Leitung von

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Ehrenwörtliche Erklärung

Ich versichere, dass ich die eingereichte Diplomarbeit selbständig verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt und mich auch sonst keiner unerlaubten Hilfsmittel bedient habe. Ich erkläre weiters, dass ich diese Diplomarbeit bisher weder im In- noch im Ausland in irgendeiner Form als wissenschaftliche Arbeit vorgelegt habe.

Wien, August 2011

Günther Koliander

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Preface

In my thesis I give an introduction to the theory of de Branges' Hilbert spaces of entire functions. My approach was to use Hardy space theory as background. Unfortunately, it would have gone beyond the scope of this work to give an introduction to Hardy space theory in all details. Therefore, the first part of my work contains only the definitions and theorems that I will need later on. For the interested reader I recommend the book [5] by M. Rosenblum and J. Rovnyak which I used as reference.

The first main result of my work is Theorem 1.2.2 which characterises Hilbert spaces of entire functions in a Hardy space setting. It is the most general equivalence I could think of, and, particularly, the functions are not required to be entire. Another result of this kind, which characterises Hilbert spaces of entire functions as special subspaces of Hardy spaces, is obtained after this one quite easy. Furthermore, the first chapter contains basic results de Branges' already showed or mentioned in his book [3], although I tried to give easier proofs which often make use of Hardy space theory.

In the second chapter I introduce functions associated with a Hilbert space of entire functions. The first section is again a collection of already known facts in a new context. In the second section I introduce a linear relation M_S which has a strong tie to associated functions. Although M. Kaltenbäck and H. Woracek referred to this relation in [6] in an abstract way, it was, as far as I know, never before stated in this explicit way. I introduce the definition of strongly associated functions and give many equivalences for a function to be strongly associated. Some of these equivalent conditions were already mentioned by de Branges, but without any connection to the linear relation M_S and never this clearly arranged.

From the beginning, the final result I wanted to show, was the ordering theorem for Hilbert spaces of entire functions. When I thought, that everything that remains for me to do, is to understand and formulate de Branges' proof of this theorem, I encountered an error in his work. I'm glad that my supervisor Prof. Michael Kaltenbäck found a workaround, and I could prove the result I longed for. However a minor flaw remained, because I wasn't able to prove it in all cases.

I hope that I managed to formulate and prove everything in a way anyone with basic knowl-

edge about complex analysis understands and that the work serves as a good introduction to this interesting theory.

Finally, I want to thank Michael Kaltenbäck for his patience and advice.

Chapter 1

The Space $H(E)$

In this chapter the Hilbert space of entire functions associated with a function E is defined and useful properties from Hardy space theory are cited.

1.1 Preliminary Results

Starting with the definition and basic properties of harmonic and subharmonic functions, the space $N(\mathbb{C}^+)$ is defined. This space of so called functions of bounded type and the factorisation of these functions will be useful throughout this work. The chapter is based on [5] and more details about Hardy spaces and functions of bounded type can be found there.

Definition 1.1.1. *A complex valued function $f(x+iy)$ defined on an open set $\Omega \subseteq \mathbb{C}$ is called harmonic on Ω if $f \in C^2(\Omega)$ and the Laplacian Δf vanishes, i.e. it is twice partial differentiable with respect to x and y and*

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Definition 1.1.2. *Let $\Omega \subseteq \mathbb{C}$ be open. A function $f : \Omega \to [-\infty, \infty)$ is called subharmonic on Ω if*

(i) f is upper semicontinuous, i.e. if $\{x \in \Omega : f(x) < a\}$ is open for all $a \in \mathbb{R}$.

(ii) For every open set A with compact closure $\overline{A} \subseteq \Omega$ and every continuous function $h : \overline{A} \to \mathbb{R}$ whose restriction to A is harmonic, if $f \leq h$ on ∂A, then $f \leq h$ on A.

Some useful properties of subharmonic and harmonic functions are collected in the following theorem.

Theorem 1.1.3. *Let $\Omega \subseteq \mathbb{C}$ be an open subset.*

(i) *A continuous function* $f : \Omega \to \mathbb{R}$ *is harmonic if and only if for any closed disk* $\{z + re^{i\varphi} : \varphi \in (0, 2\pi], r \in [0, R]\}$ *with center* z *and radius* R *contained in* Ω

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{i\varphi})\, d\varphi.$$

(ii) *An upper semicontinuous function* $f : \Omega \to \mathbb{R}$ *is subharmonic if and only if for any closed disk* $\{z + re^{i\varphi} : \varphi \in (0, 2\pi], r \in [0, R]\}$ *with center* z *and radius* R *contained in* Ω

$$f(z) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{i\varphi})\, d\varphi.$$

In this case

$$0 \leq \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\varphi})\, dt \leq \frac{1}{2\pi} \int_0^{2\pi} f(z + Re^{i\varphi})\, d\varphi$$

holds for all $r < R$.

(iii) *A function* $f : \Omega \to \mathbb{R}$ *in* $C^2(\Omega)$ *is subharmonic if and only if* $\Delta u \geq 0$.

(iv) *Let* f *be an analytic function on a region* Ω. *Then* $f, \overline{f}, \operatorname{Re} f, \operatorname{Im} f$ *are harmonic and*

$$\log^+ |f(z)| = \max\{\log |f(z)|, 0\}$$

is subharmonic on Ω.

Definition 1.1.4. *A harmonic function* h *on a region* Ω *is called a harmonic majorant of a subharmonic function* $f \not\equiv -\infty$, *if* $h \geq f$ *on* Ω.

Theorem 1.1.5. *Let* Ω *be a simply connected region of* $\mathbb{C}$ *and* f *be an analytic function on* Ω. *Then the following assertions are equivalent.*

(i) *There exist analytic and bounded functions* g *and* h *on* Ω *such that* $f = \frac{g}{h}$.

(ii) $\log^+ |f(z)|$ *has a harmonic majorant on* Ω.

Definition 1.1.6. *A function* f *defined and analytic on a simply connected region* Ω *is said to be of bounded type in* Ω, *if it satisfies the equivalent conditions in Theorem 1.1.5. The space of all functions of bounded type on* Ω *is denoted* $N(\Omega)$.

Theorem 1.1.7. *Let* $f \not\equiv 0$ *be in* $N(\mathbb{C}^+)$. *Then*

$$\hat{f}(x) = \lim_{y \to 0+} f(x + iy)$$

exists almost everywhere on $\mathbb{R}$. *This formula defines the boundary function* $\hat{f}$ *of* f.

Definition 1.1.8. *An analytic function f on $\mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is called inner if $|f(z)| < 1$ for all $z \in \mathbb{C}^+$ and the nontangential boundary function satisfies $|f(x)| = 1$ almost everywhere.*

It is called outer if

$$f(z) = \alpha \exp\left(\frac{1}{\pi i}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) \log K(t)\, dt\right)$$

where $|\alpha| = 1$, $K(t) > 0$ and

$$\int_{\mathbb{R}} \frac{|\log K(t)|}{1+t^2}\, dt < \infty.$$

A special kind of inner function is called a Blaschke product:

$$f(z) = \alpha \left(\frac{z-i}{z+i}\right)^n \prod_{j\in J} \frac{\left|z_j^2+1\right|}{z_j^2+1} \frac{z-z_j}{z-\overline{z_j}}$$

where $|\alpha| = 1$, $n \in \mathbb{N}_0$, $z_j \in \mathbb{C}^+ \backslash \{i\}$ for all $j \in J$ and $J \subseteq \mathbb{N}$. An empty product is defined as 1. If the product is infinite the z_j have to satisfy ($z_j = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}$)

$$\sum_{j\in J} \frac{y_j}{x_j^2 + (y_j+1)^2} < \infty.$$

Theorem 1.1.9. *Let $f \not\equiv 0$ be in $N(\mathbb{C}^+)$. Then there exist functions B Blaschke product, G outer and a real number τ such that*

$$f(z) = e^{-i\tau z} B(z) G(z) \frac{S_+(z)}{S_-(z)}$$

where the functions $S_\pm$ have the form

$$S_\pm = \exp\left(-\frac{1}{\pi i}\int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\mu_\pm(t)\right),$$

where $\mu_\pm$ are singular and mutually singular non-negative Borel measures on the real line satisfying

$$\int_{\mathbb{R}} \frac{1}{1+t^2}\, d\mu_\pm(t) < \infty.$$

The functions $S_\pm$ are inner and $e^{-i\tau z}$ is inner for τ nonpositive.

If f has an analytic continuation across some interval I of the real line, then $\mu_\pm|_I = 0$. If f is an inner function the factors $G(z)$ and $S_-(z)$ are constants of modulus one and τ is nonpositive.

Definition 1.1.10. *The number τ in Theorem 1.1.9 is called the mean type of f.*

Theorem 1.1.11. *The mean type τ of $f \in N(\mathbb{C}^+)\backslash\{0\}$ satisfies*

$$\tau = \lim_{R\to\infty} \frac{2}{\pi R} \int_0^\pi \log\left|f(Re^{i\varphi})\right| \sin\varphi\, d\varphi$$

and

$$\tau = \limsup_{y\to\infty} \frac{1}{y} \log|f(iy)|\,.$$

If $\tau \geq 0$, then

$$\tau = \lim_{R\to\infty} \frac{2}{\pi R} \int_0^\pi \log^+ \left|f(Re^{i\varphi})\right| \sin\varphi\, d\varphi.$$

Example 1.1.12. *The mean type of any polynomial $f(z) = a_n z^n + \ldots + a_0$ is zero, because*

$$\left|\limsup_{y\to\infty} \frac{\log|f(iy)|}{y}\right| = \left|\limsup_{y\to\infty} \frac{\log|a_n(iy)^n + \ldots + a_0|}{y}\right| \leq \left|\limsup_{y\to\infty} \frac{\log\left(|a_n|y^n + \ldots + |a_0|\right)}{y}\right| = 0.$$

A useful method to calculate mean types for products or sums of functions in $N(\mathbb{C}^+)$ is the following lemma.

Lemma 1.1.13. *Let $f, g \in N(\mathbb{C}^+)$ and let τ_f and τ_g be the mean types of these functions. Then the mean type of the product $f \cdot g$ is*

$$\tau_{fg} = \tau_f + \tau_g$$

and the mean type of the sum $f + g$ satisfies

$$\tau_{f+g} \leq \max\{\tau_f, \tau_g\}.$$

Definition 1.1.14. *A function f defined and analytic on $\mathbb{C}^+$ is said to belong to the class $N^+(\mathbb{C}^+)$ if*

$$f = \frac{g}{h},$$

where g and h are analytic, bounded by 1 and h is outer.

Theorem 1.1.15. *A function f in $N(\mathbb{C}^+)$ belongs to $N^+(\mathbb{C}^+)$ if and only if $d\mu_- = 0$ in the factorication in Theorem 1.1.9 and it has nonpositive mean type. In particular, for $f \in N^+(\mathbb{C}^+)$ this yields the factorisation $f(z) = A(z)G(z)$ where A is inner and G is outer.*

Lemma 1.1.16.

(i) Polynomials, inner functions and outer functions belong to $N^+(\mathbb{C}^+)$.

(ii) *Every polynomial with no zeros in* $\mathbb{C}^+$ *is outer.*

(iii) *An analytic function satisfying* $\operatorname{Re} f(z) \geq 0$ *in* $\mathbb{C}^+$ *is outer.*

(iv) *Products and quotients of outer functions are outer.*

(v) *If* $f, g \in N^+(\mathbb{C}^+)$ *then* $f+g, fg \in N^+(\mathbb{C}^+)$*. If* $\frac{f}{g}$ *is analytic in* $\mathbb{C}^+$ *then* $\frac{f}{g} \in N(\mathbb{C}^+)$*.*

Remark 1.1.17. *Lemma 1.1.16 and the factorisation result, Theorem 1.1.15, especially yield that* $N^+(\mathbb{C}^+)$ *is the smallest algebra containing all inner and outer functions.*

Lemma 1.1.18. *Let* $f \in N^+(\mathbb{C}^+)$*. Further, let* $w \in \mathbb{C}^+$ *with* $f(w) = 0$*. Then the function* $\frac{f(z)}{z-w}$ *belongs to* $N^+(\mathbb{C}^+)$*.*

Proof. Because the zeros of a function in $N^+(\mathbb{C}^+)$ are the zeros of its Blaschke product B, it must have a factor $\frac{z-\overline{w}}{z-w}$. Hence, the function $\frac{f(z)(z-\overline{w})}{z-w} \in N^+(\mathbb{C}^+)$ because only the Blaschke product and a multiplicative constant change. In this case $(z-\overline{w})$ is an outer function by Lemma 1.1.16. The quotient of a function in $N^+(\mathbb{C}^+)$ and an outer function just changes the outer part. Hence, it belongs to $N^+(\mathbb{C}^+)$ which yields $\frac{f(z)}{z-w} \in N^+(\mathbb{C}^+)$. □

It is possible to define the space $N^+(\Omega)$ for any region Ω, but this is not necessary in what follows. Just one special property of functions in $N^+(\mathbb{D})$, where $\mathbb{D}$ is the open unit circle, is needed. Therefore, only this space has to be defined.

Definition 1.1.19. *An analytic function on the unit circle* $\mathbb{D}$ *belongs to* $N^+(\mathbb{D})$ *if* $f\left(\frac{z-i}{z+i}\right)$ *belongs to* $N^+(\mathbb{C}^+)$*.*

Lemma 1.1.20. *Let* f *ba an analytic function on the open unit circle* $\mathbb{D}$*. If* f *has an analytic continuation to some open set containing the closed unit circle, then* f *belongs to* $N^+(\mathbb{D})$*.*

Theorem 1.1.21. *Let* f *be an analytic function. Then* f *belongs to* $N^+(\mathbb{D})$ *if and only if*

$$\log|f(z)| \leq \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{1-|z|^2}{|e^{i\varphi}-z|^2} \log|f(e^{i\varphi})|\, d\varphi,$$

for all $z \in \mathbb{C}^+$*.*

With the done preliminary work the Hardy space $H^2(\mathbb{C}^+)$ can be defined.

Definition 1.1.22. *The space* $H^2(\mathbb{C}^+)$ *is the space of all functions* $f \in N^+(\mathbb{C}^+)$*, such that the boundary function* $\hat{f}$ *from Theorem 1.1.7 satisfies*

$$\int\limits_{\mathbb{R}} |\hat{f}(t)|^2\, dt < \infty.$$

Theorem 1.1.23 (Cauchy Representation)**.** *Let $f \in H^2(\mathbb{C}^+)$ and $z \in \mathbb{C}^+$. Then the boundary function $\hat{f}(t)$ belongs to $L^2(\mathbb{R})$ and*

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\hat{f}(t)}{t - z}\, dt$$

and

$$0 = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\hat{f}(t)}{t - \overline{z}}\, dt.$$

Conversely, if $g \in L^2(\mathbb{R})$ such that

$$0 = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{g(t)}{t - \overline{z}}\, dt$$

holds for all $z \in \mathbb{C}^+$, then the function f defined on $\mathbb{C}^+$ by

$$f(z) := \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{g(t)}{t - z}\, dt$$

belongs to $H^2(\mathbb{C}^+)$ and $g = \hat{f}$ almost everywhere.

Definition 1.1.24. *The space $H^2(\mathbb{R})$ is the space of all boundary functions of $H^2(\mathbb{C}^+)$.*

Remark 1.1.25. *The Theorems 1.1.23 and 1.1.7 show, that there exists a bijective correspondence between the spaces $H^2(\mathbb{C}^+)$ and $H^2(\mathbb{R})$.*

Theorem 1.1.26. *For every function $f \in L^2(\mathbb{R})$ there exist functions $f_1, f_2 \in L^2(\mathbb{R})$ with $f_1, \overline{f_2} \in H^2(\mathbb{R})$ such that $f(t) = f_1(t) + f_2(t)$ for all $t \in \mathbb{R}$. The spaces $H^2(\mathbb{R})$ and $\{f : \overline{f} \in H^2(\mathbb{R})\}$ are orthogonal with respect to the inner product on $L^2(\mathbb{R})$, i.e.*

$$L^2(\mathbb{R}) = H^2(\mathbb{R}) \oplus \{f : \overline{f} \in H^2(\mathbb{R})\}.$$

Theorem 1.1.27. *$H^2(\mathbb{C}^+)$ is a Hilbert space with respect to the inner product*

$$\langle f, g \rangle_{H^2(\mathbb{C}^+)} = \int_{\mathbb{R}} \hat{f}(t)\overline{\hat{g}(t)}\, dt.$$

Definition 1.1.28. *An entire function f is called of exponential type if*

$$\tau_f = \limsup_{r \to \infty} \max_{\varphi \in (0, 2\pi]} \frac{\log |f(re^{i\varphi})|}{r} < \infty$$

where τ_f is called the exact type of f. It is called of exponential type τ if $\tau_f \leq \tau$.

Remark 1.1.29. *By Liouville's Theorem a function $f \not\equiv 0$ of exponential type must have nonnegative exact type. Otherwise for some $\varepsilon > 0$*

$$\log|f(re^{i\varphi})| \leq -\varepsilon r.$$

Therefore, f would be a constant with nonnegative exact type. This shows that $\log$ *in the definition of τ_f can be exchanged with* $\log^+$ *for any function $f \not\equiv 0$. Hence,*

$$\tau_f = \limsup_{r\to\infty} \max_{\varphi\in(0,2\pi]} \frac{\log^+|f(re^{i\varphi})|}{r}.$$

Theorem 1.1.30 (Kreins Theorem)**.** *Let f be an entire function. Then the following assertions are equivalent:*

(i) f is of exponential type with exact type τ_f and

$$\int_{\mathbb{R}} \frac{\log^+|f(t)|}{1+t^2}\,dt < \infty.$$

(ii) f and $f^{\#}$ are in $N(\mathbb{C}^+)$ with mean types τ_+ and τ_-.

In this case

$$\tau_+ + \tau_- \geq 0$$

and

$$\max\{\tau_+, \tau_-\} = \tau_f.$$

The following lemma is a basic result in measure theory and can be found e.g. in [4], p.148 Theorem 5.8.

Lemma 1.1.31. *Let $(\Omega, \mathfrak{A}, \mu)$ be a measure space, $G \subseteq \mathbb{C}$ an open set and $f : G \times \Omega \to \mathbb{C}$ a function satisfying:*

(i) For all $z \in G$ the function $t \mapsto f(z,t)$ is integrable.

(ii) For all $t \in \Omega\backslash N$ with a zero set $N \in \mathfrak{A}$ the function $z \mapsto f(z,t)$ is analytic.

(iii) If $K \subseteq G$ is compact, then there exists an integrable function $g_K : \Omega \mapsto \overline{\mathbb{R}}$, satisfying $|f(z,t)| \leq g_K(t)$ for all $z \in K$ and $t \in \Omega\backslash N$.

Then the function $F(z) := \int_\Omega f(z,t)\,d\mu(t)$ is analytic in G.

Lemma 1.1.32. *Let μ be a Borel measure on $\mathbb{R}$ and $h : \mathbb{R} \to \mathbb{C}$ be a Borel measureable function. Assume further that $\frac{|h(t)|}{|t|+1}$ belongs to $L^1(\mathbb{R})$. Define a function f on the upper halfplane by*

$$f(z) = \int_{\mathbb{R}} \frac{h(t)}{t-z}\, d\mu(t).$$

Then $f \in N^+(\mathbb{C}^+)$.

Proof. By Lemma 1.1.16 every analytic function on the upper halfplane with nonnegative real part belongs to $N^+(\mathbb{C}^+)$. Therefore, it is sufficient to show that f is a linear combination of such functions. The function $h(t)$ can be seperated into $h(t) = (\operatorname{Re} h(t))^+ + i(\operatorname{Im} h(t))^+ - (\operatorname{Re} h(t))^- - i(\operatorname{Im} h(t))^-$. Here $f^+(t) := \max\{f(t), 0\}$ defines the positive part and $f^-(t) := \max\{-f(t), 0\}$ the negative part of a function f. Hence, the integral is the same as

$$f(z) = \int_{\mathbb{R}} \frac{(\operatorname{Re} h(t))^+ + i(\operatorname{Im} h(t))^+ - (\operatorname{Re} h(t))^- - i(\operatorname{Im} h(t))^-}{t-z}\, d\mu(t).$$

The imaginary part of $\frac{1}{t-z}$ satisfies

$$\operatorname{Im} \frac{1}{t-z} = \operatorname{Im} \frac{t-x+iy}{(t-x)^2+y^2} = \frac{y}{(t-x)^2+y^2} > 0.$$

Therefore, all the integrals in

$$f(z) = \int_{\mathbb{R}} \frac{(\operatorname{Re} h(t))^+}{t-z}\, d\mu(t) + i\int_{\mathbb{R}} \frac{(\operatorname{Im} h(t))^+}{t-z}\, d\mu(t) - \int_{\mathbb{R}} \frac{(\operatorname{Re} h(t))^-}{t-z}\, d\mu(t) - i\int_{\mathbb{R}} \frac{(\operatorname{Im} h(t))^-}{t-z}\, d\mu(t)$$

have nonnegative imaginary part and, hence, f is the linear combination of functions with nonnegative real part. By Lemma 1.1.16 such functions are outer if they are analytic on the upper halfplane. Hence, it remains to show that these integrals converge and are analytic on $\mathbb{C}^+$. Lemma 1.1.31 is used to show this. The first condition of this lemma is, that the function has to be integrable in t for any z. Because the modulus of h is always greater than or equal to the positive or negative part of the imaginary or real part of h it is sufficient to show

$$\frac{|h(t)|}{|t-z|} \leq \tilde{C}\frac{|h(t)|}{|t|+1}.$$

Because $\frac{|t|+1}{|t-z|}$ converges to 1 for $|t| \to \infty$ there exists a $C > 0$ such that

$$\frac{|t|+1}{|t-z|} < 2 \tag{1.1}$$

for $|t| > C$. Because $z \in \mathbb{C}^+$, the imaginary part of z is greater than zero. Hence,

$$|t-z| > \delta$$

for all $t \in \mathbb{R}$. Hence, for $|t| \leq C$ one gets

$$|t| + 1 \leq C + 1 = \delta \frac{C+1}{\delta} < |t - z| \frac{C+1}{\delta}.$$

With (1.1) this yields

$$\frac{|t| + 1}{|t - z|} < \max \left\{ 2, \frac{C+1}{\delta} \right\}$$

and further

$$\frac{|h(t)|}{|t - z|} \leq \max \left\{ 2, \frac{C+1}{\delta} \right\} \frac{|h(t)|}{|t| + 1}.$$

The second condition of Lemma 1.1.31 is, that the function has to be analytic in $z \in \mathbb{C}^+$ for any fixed t, which is obvious. The last condition is, that for any compact subset $K \subset \mathbb{C}^+$ the function must have a nonnegative, integrable majorant $g_K(t)$ independent of $z \in K$. As above it is sufficient to show

$$\frac{|h(t)|}{|t - z|} \leq C_K \frac{|h(t)|}{|t| + 1}$$

for all $z \in K$ with some $C_K \in \mathbb{R}$, because the modulus of h is always greater than or equal to the positive or negative part of the imaginary or real part of h. The imaginary parts of the numbers z in the compact set K satisfy $\operatorname{Im} z > \delta$ for some $\delta > 0$. Therefore, for any $C > 0$

$$|t - z| > \delta$$

for all $t \in \mathbb{R}$. Hence, for $|t| \leq C$ and all $z \in K$

$$|t| + 1 \leq C + 1 = \delta \frac{C+1}{\delta} < |t - z| \frac{C+1}{\delta}.$$

It remains to show that there exists a $C > 0$ such that

$$\frac{|t| + 1}{|t - z|} < 2$$

for all $z \in K$ and all $|t| > C$. Let $\overline{z} := \max_{z \in K} \operatorname{Re} z + i\delta$ and $\underline{z} := \min_{z \in K} \operatorname{Re} z + i\delta$. Then for $t > \max_{z \in K} \operatorname{Re} z$

$$|t - z| = \sqrt{(t - x)^2 + y^2} \geq \sqrt{(t - \max_{z \in K} \operatorname{Re} z)^2 + \delta^2} = |t - \overline{z}|$$

and for $t < \min_{z \in K} \operatorname{Re} z$

$$|t - z| = \sqrt{(t - x)^2 + y^2} \geq \sqrt{(t - \min_{z \in K} \operatorname{Re} z)^2 + \delta^2} = |t - \underline{z}|.$$

For $\overline{z}$ and $\underline{z}$ there exist $\overline{C}$ and $\underline{C}$ such that

$$\frac{|t| + 1}{|t - \overline{z}|} < 2$$

for $|t| > \overline{C}$ and

$$\frac{|t|+1}{|t-\underline{z}|} < 2$$

for $|t| > \underline{C}$. Hence, for $|t| > \max\{\overline{C}, \underline{C}, \max_{z\in K} \operatorname{Re} z, \min_{z\in K} \operatorname{Re} z\}$ one gets

$$\frac{|t|+1}{|t-z|} < 2$$

for all $z \in K$.

□

The following theorem is a special kind of the Phragmen Lindelöf Principle. It can be found e.g. in [1], Theorem 5.

Theorem 1.1.33. *Let f be an analytic function on the closed right halfplane. Define*

$$m(r) := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log^+ |f(re^{i\varphi})| \cos\varphi \, d\varphi.$$

If $|f(iy)| \le 1$ for all $y \in \mathbb{R}$, then $\frac{m(r)}{r}$ is a nondecreasing function.

The next lemma is a reversed version of Fatou's Lemma.

Lemma 1.1.34. *Let μ be a Borel measure on $\mathbb{R}$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of Borel measureable functions. Further, let g be a Borel measureable function satisfying $\int_{\mathbb{R}} g \, d\mu < \infty$. If $f_n \le g$ μ-almost everywhere for all $n \in \mathbb{N}$, then*

$$\limsup_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu \le \int_{\mathbb{R}} \limsup_{n\to\infty} f_n \, d\mu.$$

Proof. The functions $g - f_n$ are nonnegative. Hence, by Fatou's Lemma

$$\int_{\mathbb{R}} \liminf_{n\to\infty} (g - f_n) \, d\mu \le \liminf_{n\to\infty} \int_{\mathbb{R}} (g - f_n) \, d\mu.$$

Because $\int g \, d\mu < \infty$ subtracting this value and changing signs gives

$$\int_{\mathbb{R}} \limsup_{n\to\infty} f_n \, d\mu \ge \limsup_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu.$$

□

Lemma 1.1.35. *Let f be an entire function of exponential type 0. If f is bounded on the imaginary axis, then f is a constant.*

Proof. Without loss of generality assume that f is bounded on the imaginary axis by 1. In the general case $|f(z)| < c$ for all $z \in i\mathbb{R}$ consider the function $\tilde{f} := \frac{f}{c}$. By assumption the exact type of f has to be less than or equal to 0, i.e.

$$\tau_f = \limsup_{r\to\infty} \max_{\varphi\in(0,2\pi]} \frac{\log|f(re^{i\varphi})|}{r} \leq 0. \tag{1.2}$$

Let r_n be any increasing sequence of radii converging to infinity. Define $m(r)$ as in Theorem 1.1.33. Then $\frac{m(r)}{r}$ is a nondecreasing function. Hence, for any fixed $r \in \mathbb{R}_+$

$$\frac{m(r)}{r} \leq \limsup_{n\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\log^+|f(r_n e^{i\varphi})|}{r_n} \cos\varphi \, d\varphi.$$

Because the functions $\frac{\log^+|f(r_n e^{i\varphi})|}{r_n}\cos\varphi$ are bounded by (1.2), the reversed version of Fatous Lemma 1.1.34 gives

$$\limsup_{n\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\log^+|f(r_n e^{i\varphi})|}{r_n} \cos\varphi \, d\varphi \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \limsup_{n\to\infty} \frac{\log^+|f(r_n e^{i\varphi})|}{r_n} \cos\varphi \, d\varphi \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tau_f \cos\varphi \, d\varphi \leq 0.$$

This yields $m(r) \equiv 0$ and, hence, f is bounded on the right halfplane. The same argument for $f(-z)$ shows that f is bounded everywhere and by Liouville's Theorem f reduces to a constant. □

Lemma 1.1.36. *Let f be an entire function such that $f, f^\# \in N(\mathbb{C}^+)$, where $f^\#(z) := \overline{f(\overline{z})}$. If f is bounded on the imaginary axis, then f is a constant.*

Proof. With Lemma 1.1.35 it is sufficient to show that f is of exponential type 0. By Kreins Theorem 1.1.30 the function f is of exponential type and the exact type is smaller than the maximum of the mean types of f and $f^\#$. Hence, it remains to show that these mean types are nonpositive. One of the formulas for mean type in Theorem 1.1.11 is

$$\tau = \limsup_{y\to\infty} \frac{1}{y} \log|f(iy)|.$$

Because f is bounded on the imaginary axis, the mean type is less than or equal to 0. The same holds for $f^\#$ and, hence, f is of exponential type 0 and reduces to a constant. □

1.2 Definition and basic properties of $H(E)$

The space $H(E)$ is a special Hilbert space of entire functions. It is associated with an entire function E. Starting with the original definition of de Branges, a new characterisation of these spaces through Hardy spaces is proved. Further, the functions $K(w,z)$ satisfying

$$\langle f, K(w,.)\rangle = f(w)$$

for all $f \in H(E)$ are defined, where $\langle .,. \rangle$ is the inner product in $H(E)$. At the end of the chapter a characterisation of the spaces $H(E)$ in all Hilbert spaces of entire functions is given.

Definition 1.2.1. *For an entire function E with $|E(z)| > |E(\overline{z})|$ for all $z \in \mathbb{C}^+$, $H(E)$ is the space of all entire functions f such that*

$$\|f\|_E^2 := \int_{\mathbb{R}} \left| \frac{f(t)}{E(t)} \right|^2 dt < \infty$$

and such that $\frac{f}{E}, \frac{f^\#}{E}$ both belong to $N(\mathbb{C}^+)$ and have nonpositive mean type. These spaces will be referred to as de Branges spaces. An inner product is definded by

$$\langle f, g \rangle_E := \int_{\mathbb{R}} \frac{f(t)\overline{g(t)}}{|E(t)|^2}\, dt.$$

The definition above is the original definition by de Branges in [3]. In the context of Hardy spaces the following theorem gives an easier characterisation.

Theorem 1.2.2. *Let $H(E)$ be a de Branges space. Further let f be a complex valued function defined on $\mathbb{C}\backslash\mathbb{R}$. Then f has a continuation to $\mathbb{C}$ which belongs to $H(E)$ if and only if $\frac{f}{E}, \frac{f^\#}{E} \in H^2(\mathbb{C}^+)$ and the boundary functions $\widehat{\frac{f}{E}}, \widehat{\frac{f^\#}{E}} \in H^2(\mathbb{R})$ satisfy $\widehat{\frac{f}{E}}(t)\frac{E(t)}{E^\#(t)} = \overline{\widehat{\frac{f^\#}{E}}(t)}$.*

For the proof the following lemmas are useful.

Lemma 1.2.3. *Let μ be a Borel measure on $\mathbb{R}$ satisfying*

$$\int_{\mathbb{R}} \frac{1}{t^2+1}\, d\mu(t) < \infty. \tag{1.3}$$

Let $\varphi \in L^2(\mu)$ and ψ be a function which is analytic on some open set $G \subseteq \mathbb{C}$ with $\mathbb{R} \subseteq G$ satisfying

$$\int_{\mathbb{R}} \frac{|\psi(t)|^2}{t^2+1}\, d\mu(t) < \infty. \tag{1.4}$$

Then the function

$$\int_{\mathbb{R}} \varphi(t) \frac{\psi(t)-\psi(z)}{t-z}\, d\mu(t)$$

is analytic in G.

Proof. The proof makes use of Lemma 1.1.31 with $f(z,t) = \varphi(t)\frac{\psi(t)-\psi(z)}{t-z}$ and $\Omega = \mathbb{R}$. The first condition of this lemma is that $f(z,t)$ has to be integrable for any fixed $z \in G$. Because $t \mapsto \frac{\psi(t)-\psi(z)}{t-z}$ is an analytic and, hence, continuous function in G, it is bounded on every compact subset of $\mathbb{R} \subseteq G$. Because $|t-z|^2$ and $\frac{1}{2}(t^2+1)$ are both second order polynomials in t and the leading coefficient of $|t-z|^2$ is greater, there exists a $C > 0$ such that

$$|t-z|^2 > \frac{1}{2}(t^2+1)$$

for $|t| > C$. Further,

$$\int_{\mathbb{R}} \left|\varphi(t)\frac{\psi(t)-\psi(z)}{t-z}\right| d\mu(t) = \int_{[-C,C]} \left|\varphi(t)\frac{\psi(t)-\psi(z)}{t-z}\right| d\mu(t) + \int_{\mathbb{R}\setminus[-C,C]} \left|\varphi(t)\frac{\psi(t)-\psi(z)}{t-z}\right| d\mu(t)$$

where the first integral on the right side is finite by the Cauchy-Schwarz inequality. It remains to show that the second integral is finite:

$$\begin{aligned}
\int_{\mathbb{R}\setminus[-C,C]} \left|\varphi(t)\frac{\psi(t)-\psi(z)}{t-z}\right| d\mu(t) &\le \int_{\mathbb{R}\setminus[-C,C]} \left|\varphi(t)\frac{\psi(t)}{t-z}\right| d\mu(t) + \int_{\mathbb{R}\setminus[-C,C]} \left|\varphi(t)\frac{\psi(z)}{t-z}\right| d\mu(t) \\
&\le \int_{\mathbb{R}\setminus[-C,C]} \left|\varphi(t)\frac{\psi(t)}{\sqrt{\frac{1}{2}(t^2+1)}}\right| d\mu(t) \\
&\quad + \int_{\mathbb{R}\setminus[-C,C]} \left|\varphi(t)\frac{\psi(z)}{\sqrt{\frac{1}{2}(t^2+1)}}\right| d\mu(t) \\
&\le \int_{\mathbb{R}} |\varphi(t)|\frac{|\psi(t)|}{\sqrt{\frac{1}{2}(t^2+1)}}\, d\mu(t) + \int_{\mathbb{R}} |\varphi(t)|\frac{|\psi(z)|}{\sqrt{\frac{1}{2}(t^2+1)}}\, d\mu(t) \\
&\le \|\varphi\|_{L^2(\mu,\mathbb{R})} \cdot \left\|\frac{\psi(t)}{\sqrt{\frac{1}{2}(t^2+1)}}\right\|_{L^2(\mu,\mathbb{R})} \\
&\quad + \|\varphi\|_{L^2(\mu,\mathbb{R})} \cdot |\psi(z)| \left\|\frac{1}{\sqrt{\frac{1}{2}(t^2+1)}}\right\|_{L^2(\mu,\mathbb{R})}.
\end{aligned}$$

The norms are finite because of (1.3) and (1.4).

Condition (ii) in Lemma 1.1.31 is that $z \mapsto \frac{\psi(t)-\psi(z)}{t-z}\varphi(t)$ has to be analytic in G for any $t \in \mathbb{R}$. This is true by assumption.

It remains to show (iii). Let $K \subseteq G$ be a compact set. Because K is bounded there exist C_1, such that $|w| < C_1$. Hence, for $|t| > C_1$

$$|t-w|^2 \ge (|t|-|w|)^2 > (|t|-C_1)^2.$$

Therefore, there exists a $C > 0$, such that for any t with $|t| \geq C$

$$|t-w|^2 > (|t| - C_1)^2 > \frac{1}{2}(t^2+1).$$

Hence, for these t and $w \in K$

$$\left|\varphi(t)\frac{\psi(t)-\psi(w)}{t-w}\right| \leq \left|\varphi(t)\frac{\psi(t)-\psi(w)}{\sqrt{\frac{1}{2}(t^2+1)}}\right| \leq \left|\varphi(t)\frac{\psi(t)}{\sqrt{\frac{1}{2}(t^2+1)}}\right| + \sup_{w\in K}|\psi(w)|\left|\varphi(t)\frac{1}{\sqrt{\frac{1}{2}(t^2+1)}}\right|. \quad (1.5)$$

Because ψ is analytic, it is locally Lipschitz continuous in G. Thus, there exists an $L_K > 0$ such that

$$|\psi(t) - \psi(w)| \leq L_K|t-w|$$

and, hence,

$$\frac{|\psi(t)-\psi(w)|}{|t-w|} \leq L_K$$

for t and w in the compact set $K \cup [-C, C] \subseteq G$. This leads to

$$\left|\varphi(t)\frac{\psi(t)-\psi(w)}{t-w}\right| \leq L_K\,|\varphi(t)| \quad (1.6)$$

for $t \in (-C, C)$ and $w \in K$. Let g_K be defined as

$$g_K := \begin{cases} L_K\,|\varphi(t)|\,, & t \in (-C,C) \\ \left|\varphi(t)\frac{\psi(t)}{\sqrt{\frac{1}{2}(t^2+1)}}\right| + \sup_{w\in K}|\psi(w)|\left|\varphi(t)\frac{1}{\sqrt{\frac{1}{2}(t^2+1)}}\right|, & t \in \mathbb{R}\backslash(-C,C). \end{cases}$$

The equations (1.5) and (1.6) show that $|f(z,t)| \leq g_K(t)$ for all $z \in K$. The condition $\varphi \in L^2(\mu, \mathbb{R})$ implies $\varphi \in L^2(\mu, (-C, C)) \subseteq L^1(\mu, (-C, C))$ and the Cauchy-Schwarz inequality shows that the remaining integrals in $\int_{\mathbb{R}} g_K(t)\,d\mu(t)$ converge:

$$\begin{aligned} \int_{\mathbb{R}} g_K(t)\,d\mu(t) = &\int_{(-C,C)} L_K\,|\varphi(t)|\,d\mu(t) \\ &+ \int_{\mathbb{R}\backslash(-C,C)} \left|\varphi(t)\frac{\psi(t)}{\sqrt{\frac{1}{2}(t^2+1)}}\right| + \sup_{w\in K}|\psi(w)|\left|\varphi(t)\frac{1}{\sqrt{\frac{1}{2}(t^2+1)}}\right|\,d\mu(t) \\ &\leq L_K\|\varphi(t)\|_{L^1(\mu,(-C,C))} + \|\varphi\|_{L^2(\mu,\mathbb{R})}\left\|\frac{\psi(t)}{\sqrt{\frac{1}{2}(t^2+1)}}\right\|_{L^2(\mu,\mathbb{R})} \\ &+ \sup_{w\in K}|\psi(w)|\left(\|\varphi\|_{L^2(\mu,\mathbb{R})}\left\|\frac{1}{\sqrt{\frac{1}{2}(t^2+1)}}\right\|_{L^2(\mu,\mathbb{R})}\right). \end{aligned}$$

This shows (iii) of Lemma 1.1.31 and, therefore, $\int_{\mathbb{R}} \varphi(t)\frac{\psi(t)-\psi(z)}{t-z}\,d\mu(t)$ is analytic in G. $\square$

Lemma 1.2.4. *Let $H(E)$ be a de Branges space and $f \in H(E)$. If E has a zero at some $t_0 \in \mathbb{R}$ of multiplicity m_E, then f has a zero at t_0 of multiplicity $m_f \geq m_E$.*

Proof. Argue by contradiction. Assume that the function $\frac{f}{E}$ has a pole of order m at t_0. Then there exists an entire function G with $\frac{f(z)}{E(z)} = \frac{G(z)}{(z-t_0)^m}$ and $G(t_0) \neq 0$. By continuity there are positive δ and ε such that $G(t) > \varepsilon$ for all $t \in (t_0 - \delta, t_0 + \delta)$. Hence,

$$\begin{aligned}
\int_{\mathbb{R}} \left| \frac{f(t)}{E(t)} \right|^2 dt &\geq \int_{(t_0-\delta, t_0+\delta)} \left| \frac{f(t)}{E(t)} \right|^2 dt \\
&= \int_{(t_0-\delta, t_0+\delta)} \left| \frac{G(t)}{(t-t_0)^m} \right|^2 dt \\
&\geq \varepsilon \int_{(t_0-\delta, t_0+\delta)} \frac{1}{(t-s)^{2m}} dt \\
&= \infty.
\end{aligned}$$

This is a contradiction to the integral condition

$$\int_{\mathbb{R}} \left| \frac{f(t)}{E(t)} \right|^2 dt < \infty$$

in Definition 1.2.1. □

Proof of Theorem 1.2.2. Assume first, that $\frac{f}{E}, \frac{f^\#}{E}$ belong to $H^2(\mathbb{C}^+)$. Because of $H^2(\mathbb{C}^+) \subseteq N^+(\mathbb{C}^+) \subseteq N(\mathbb{C}^+)$ the functions are of bounded type and have nonpositive mean type by Theorem 1.1.15. The definition of $H^2(\mathbb{C}^+)$ yields the integral condition

$$\int_{\mathbb{R}} \left| \frac{f(t)}{E(t)} \right|^2 dt < \infty.$$

It remains to show that f is entire. By the Cauchy representation 1.1.23 the following holds:

$$\begin{aligned}
\frac{f(w)}{E(w)} &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\widehat{\frac{f}{E}}(t)}{t-w} dt & 0 &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\widehat{\frac{f}{E}}(t)}{t-\overline{w}} dt & w &\in \mathbb{C}^+ \\
\frac{f^\#(\overline{w})}{E(\overline{w})} &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\widehat{\frac{f^\#}{E}}(t)}{t-\overline{w}} dt & 0 &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\widehat{\frac{f^\#}{E}}(t)}{t-w} dt & w &\in \mathbb{C}^-.
\end{aligned}$$

Taking the conjugate of the equations for $w \in \mathbb{C}^-$ yields

$$f(w) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\widehat{\frac{f}{E}}(t)E(w)}{t-w}\, dt \qquad 0 = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\widehat{\frac{f}{E}}(t)E(\overline{w})}{t-\overline{w}}\, dt \qquad w \in \mathbb{C}^+$$

$$f(w) = -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\overline{\widehat{\frac{f^\#}{E}}(t)}\,\overline{E(\overline{w})}}{t-w}\, dt \qquad 0 = -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\overline{\widehat{\frac{f^\#}{E}}(t)}\,\overline{E(w)}}{t-\overline{w}}\, dt \qquad w \in \mathbb{C}^-.$$

Hence, for any $w \in \mathbb{C}\backslash\mathbb{R}$

$$f(w) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\widehat{\frac{f}{E}}(t)E(w)}{t-w} - \frac{\overline{\widehat{\frac{f^\#}{E}}(t)}\,\overline{E(\overline{w})}}{t-w}\, dt.$$

Then by the assumption made for the boundary functions

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\widehat{\frac{f}{E}}(t)E(w)}{t-w} - \frac{\overline{\widehat{\frac{f^\#}{E}}(t)}\,\overline{E(\overline{w})}}{t-w}\, dt \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\widehat{\frac{f}{E}}(t)E(w)}{t-w} - \frac{\widehat{\frac{f}{E}}(t)\frac{E(t)}{E^\#(t)}\overline{E(\overline{w})}}{t-w}\, dt \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}} \widehat{\frac{f}{E}}(t)E^\#(w)\frac{\frac{E(w)}{E^\#(w)} - \frac{E(t)}{E^\#(t)}}{t-w}\, dt. \end{aligned} \tag{1.7}$$

Now Lemma 1.2.3 with $\psi(z) = \frac{E(z)}{E^\#(z)}$, $\varphi(t) = \widehat{\frac{f}{E}}(t)$ and $\mu = \lambda$ will show that $\frac{f(w)}{E^\#(w)}$ is an analytic function on any set G, on which ψ is analytic. The assumptions of this lemma are, that

$$\int_{\mathbb{R}} \frac{1}{1+t^2}\, d\mu(t) < \infty \tag{1.8}$$

that ψ is meromorphic and satisfies

$$\int_{\mathbb{R}} \frac{|\psi(t)|^2}{1+t^2}\, d\mu(t) < \infty$$

and, finally, that the function φ belongs to $L^2(\mu)$. The condition (1.8) is a well known fact for $\mu = \lambda$. The integral condition for $\psi(t)$ is true because of $\left|\frac{E(t)}{E^\#(t)}\right| = 1$ for all $t \in \mathbb{R}$. Finally $\widehat{\frac{f}{E}}(t)$ belongs to $L^2(\mathbb{R})$ by Theorem 1.1.23. The set G, on which $\frac{E(z)}{E^\#(z)}$ is analytic, contains the closed lower halfplane, because $|E(w)| > |E(\overline{w})|$ for $w \in \mathbb{C}^+$ and so $\left|\frac{E(t)}{E^\#(t)}\right| = 1$. Hence, f has an analytic continuation to the closed lower halfplane. Because $\frac{f}{E} \in H^2(\mathbb{C}^+)$ it is analytic on the upper halfplane too. Thus, it is entire.

For the other implication let f belong to $H(E)$. To show that the functions $\frac{f}{E}$ and $\frac{f^{\#}}{E}$ belong to $H^2(\mathbb{C}^+)$ the integral condition of Definition 1.1.22

$$\int_{\mathbb{R}} \left| \widehat{\frac{f}{E}}(t) \right|^2 dt < \infty.$$

has to be satisfied and the functions have to belong to $N^+(\mathbb{C}^+)$. The integral condition is satisfied by the Definition 1.2.1 of the space $H(E)$. To show that the functions belong to $N^+(\mathbb{C}^+)$ Theorem 1.1.15 is used. That the functions have nonpositive mean type and are of bounded type is again true by the Defintion 1.2.1 of the space $H(E)$. It remains to show that $d\mu_- = 0$ in the factorisation in 1.1.9. By Theorem 1.1.9 this is the case, if for every real interval (a, b) the functions $\frac{f}{E}, \frac{f^{\#}}{E}$ have an analytic continuation across (a, b). Because $f, f^{\#}$ and E are analytic the only problem could be real zeros of E. At these zeros the function f must have a zero of higher multiplicity by Lemma 1.2.4. Hence, the functions are analytic on the closed upper halfplane and $d\mu_- = 0$. $\square$

Equation (1.7) in the proof also shows an interesting property of the function $\frac{E(w)\overline{E(t)}-E(t)\overline{E(\overline{w})}}{2\pi i|E(t)|^2(t-w)}$ as is stated in the following Corollary.

Corollary 1.2.5. *Let $H(E)$ be a de Branges space. For any function $f \in H(E)$*

$$f(w) = \int_{\mathbb{R}} f(t) \frac{1}{2\pi i} \frac{E(w)\overline{E(t)} - E(t)\overline{E(\overline{w})}}{|E(t)|^2(t-w)}\, dt$$

for all $w \in \mathbb{C}$.

Proof. Because f and E are entire functions, the boundary functions $\widehat{\frac{f}{E}}(t)$ and $\widehat{\frac{f^{\#}}{E}}(t)$ are the restrictions of the functions to the real line. Hence, with (1.7)

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\widehat{\frac{f}{E}}(t) E(w)}{t-w} - \frac{\overline{\widehat{\frac{f^{\#}}{E}}(t)}\, \overline{E(\overline{w})}}{t-w}\, dt \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\frac{f(t)}{E(t)} E(w)}{t-w} - \frac{\frac{f(t)}{E^{\#}(t)} E^{\#}(w)}{t-w}\, dt \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)E(w)}{E(t)(t-w)} - \frac{f(t)E^{\#}(w)}{E^{\#}(t)(t-w)}\, dt \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}} f(t) \frac{E^{\#}(t)E(w) - E(t)E^{\#}(w)}{|E(t)|^2(t-w)}\, dt \end{aligned}$$

follows. $\square$

Another characterisation of $H(E)$ makes use of orthogonal complements.

Theorem 1.2.6. *Let $H(E)$ be a de Branges space. A complex valued function f, defined on $\mathbb{R}$, satisfying $\frac{f}{E} \in L^2(\mathbb{R})$ has a representative with a continuation $\tilde{f} \in H(E)$ if and only if there exists a function $g \in \left(H^2(\mathbb{C}^+) \ominus \frac{E^\#}{E} H^2(\mathbb{C}^+)\right)$ such that the boundary function $\widehat{g}(t)$ satisfies*

$$\widehat{g}(t) = \frac{f(t)}{E(t)}.$$

Here $\left(H^2(\mathbb{C}^+) \ominus \frac{E^\#}{E} H^2(\mathbb{C}^+)\right)$ is the orthogonal complement of the space $\left\{\frac{E^\#}{E} f : f \in H^2(\mathbb{C}^+)\right\}$ in $H^2(\mathbb{C}^+)$.

In this context the map $g \mapsto \tilde{f}$ is an isometric isomorphism.

Proof. Let $\tilde{f} \in H(E)$ be a continuation of f. By Theorem 1.2.2 the functions $\frac{\tilde{f}}{E}$ and $\frac{\tilde{f}^\#}{E}$ belong to $H^2(\mathbb{C}^+)$. By Theorem 1.1.26, $H^2(\mathbb{R})$ is orthogonal to $\{f : \overline{f} \in H^2(\mathbb{R})\}$ in $L^2(\mathbb{R})$. Hence, for any $h \in H^2(\mathbb{C}^+)$

$$\int_{\mathbb{R}} \frac{\tilde{f}(t)}{E(t)} \overline{\frac{E^\#(t)}{E(t)} h(t)}\, dt = \int_{\mathbb{R}} \frac{\tilde{f}(t)}{E^\#(t)} \overline{h(t)}\, dt = \int_{\mathbb{R}} \overline{\left(\frac{\tilde{f}^\#(t)}{E(t)}\right)}\, \overline{h(t)}\, dt = 0.$$

Setting $g = \frac{\tilde{f}}{E}$ proves the first direction.

For the other implication let $g \in \left(H^2(\mathbb{C}^+) \ominus \frac{E^\#}{E} H^2(\mathbb{C}^+)\right)$ with $\widehat{g}(t) = \frac{f(t)}{E(t)}$. Define $\tilde{f}(z) = g(z)E(z)$ for $z \in \mathbb{C}^+$. Obviously, $\frac{f^\#(t)}{E(t)} \in L^2(\mathbb{R})$. For $h \in H^2(\mathbb{R})$

$$\int_{\mathbb{R}} \frac{f^\#(t)}{E(t)} h(t)\, dt = \int_{\mathbb{R}} \frac{f^\#(t)}{E^\#(t)} \frac{E^\#(t)}{E(t)} h(t)\, dt = \int_{\mathbb{R}} \overline{\left(\frac{f(t)}{E(t)}\right)} \frac{E^\#(t)}{E(t)} h(t)\, dt = 0.$$

This shows that $\frac{f^\#(t)}{E(t)}$ is orthogonal to $\{f : \overline{f} \in H^2(\mathbb{R})\}$ and, by Theorem 1.1.26, $\frac{f^\#(t)}{E(t)} \in H^2(\mathbb{R})$. Hence, by Definition 1.1.24 there exists a function $g_2 \in H^2(\mathbb{C}^+)$ such that the boundary function satisfies $\widehat{g_2}(t) = \frac{f^\#(t)}{E(t)}$. Define $\tilde{f}(z) = g_2^\#(z) E^\#(z)$ for $z \in \mathbb{C}^-$. Now $\tilde{f}$ is an analytic function defined on $\mathbb{C}\backslash\mathbb{R}$ such that $\frac{\tilde{f}}{E}, \frac{\tilde{f}^\#}{E} \in H^2(\mathbb{C}^+)$. The boundary functions satisfy

$$\widehat{\frac{\tilde{f}}{E}} \frac{E(t)}{E^\#(t)} = \widehat{g}(t) \frac{E(t)}{E^\#(t)} = \frac{f(t)}{E(t)} \frac{E(t)}{E^\#(t)} = \frac{f(t)}{E^\#(t)} = \overline{\widehat{g_2}(t)} = \overline{\widehat{\frac{\tilde{f}^\#}{E}}}(t).$$

Hence, by Theorem 1.2.2 the function $\tilde{f}$ belongs to $H(E)$ and is obviously a continuation of f.

To show that the map $g \mapsto \tilde{f}$ is an isometric isomorphism let $g_1, g_2 \in \left(H^2(\mathbb{C}^+) \ominus \frac{E^\#}{E} H^2(\mathbb{C}^+)\right)$ and $\lambda, \mu \in \mathbb{C}$. As shown above the images $\tilde{f}_1, \tilde{f}_2$ of g_1 and g_2 satisfy $\widehat{g_j}(t) = \frac{\tilde{f}_j(t)}{E(t)}$ for almost every

$t \in \mathbb{R}$. The image $\tilde{f}$ of $\lambda g_1 + \mu g_2$ satisfies $\widehat{\lambda g_1 + \mu g_2}(t) = \frac{\tilde{f}(t)}{E(t)}$ for almost every $t \in \mathbb{R}$. Hence,

$$\begin{aligned}
\frac{\lambda \tilde{f}_1(t) + \mu \tilde{f}_2(t)}{E(t)} &= \lambda \widehat{g_1}(t) + \mu \widehat{g_2}(t) \\
&= \lambda \lim_{y \to 0+} g_1(t + iy) + \mu \lim_{y \to 0+} g_2(t + iy) \\
&= \lim_{y \to 0+} (\lambda g_1(t + iy) + \mu g_2(t + iy)) \\
&= \widehat{\lambda g_1 + \mu g_2}(t) \\
&= \frac{\tilde{f}(t)}{E(t)}
\end{aligned}$$

almost everywhere on $\mathbb{R}$. Because $\tilde{f}_j$ and $\tilde{f}$ are analytic

$$\tilde{f} = \lambda \tilde{f}_1 + \mu \tilde{f}_2$$

follows. Isometry is shown by

$$\|g\|_{H^2(\mathbb{C}^+)} = \int_{\mathbb{R}} |\widehat{g}(t)|^2 \, dt = \int_{\mathbb{R}} \left| \frac{\tilde{f}(t)}{E(t)} \right|^2 dt = \|\tilde{f}\|_{H(E)}.$$

$\square$

Corollary 1.2.7. *Let $H(E)$ be a de Branges space. Then $H(E)$ is a Hilbert space.*

Proof. By Theorem 1.2.6 the space $H(E)$ is isometrically isomorphic to $\left(H^2(\mathbb{C}^+) \ominus \frac{E^\#}{E} H^2(\mathbb{C}^+)\right)$. This is a closed subspace of the Hilbert space $H^2(\mathbb{C}^+)$ and, hence, a Hilbert space. $\square$

A useful property of the function E is given in the following Lemma.

Lemma 1.2.8. *Let $H(E)$ be a de Branges space. Then the function $\frac{E^\#(z)}{E(z)}$ is inner and has a factorisation*

$$\frac{E^\#(z)}{E(z)} = e^{-i\tau z} B(z)$$

where τ is nonpositive and B is a Blaschke product.

Proof. By definition $|E(z)| > |E^\#(z)|$ on $\mathbb{C}^+$ and, hence, $\left|\frac{E^\#(z)}{E(z)}\right| < 1$. Obviously $\frac{|\overline{E(t)}|}{|E(t)|} = 1$ and, hence, the function is inner. The factorisation now follows from Theorem 1.1.9 with the fact that $\frac{E^\#(z)}{E(z)}$ is analytic on the closed upper halfplane. $\square$

Theorem 1.2.9. *Let $H(E)$ be a de Branges space. Define functions A and B by*

$$A(z) := \frac{E(z) + E^{\#}(z)}{2}$$

and

$$B(z) := \frac{E(z) - E^{\#}(z)}{2} i.$$

Then $E(z) = A(z) - iB(z)$. The functions A and B are real for real z and satisfy $\overline{A(w)} = A(\overline{w})$, $\overline{B(w)} = B(\overline{w})$.

For any $w \in \mathbb{C}$ define the function

$$K(w, z) := \frac{B(z)\overline{A(w)} - A(z)\overline{B(w)}}{\pi(z - \overline{w})}.$$

Then $K(w, z) \in H(E)$ as a function of z and it can also be written as

$$K(w, z) = \frac{E(z)\overline{E(w)} - E^{\#}(z)E(\overline{w})}{2\pi(z - \overline{w})} i.$$

In particular, $H(E)$ always contains nonzero functions.

Proof. To show the properties of A and B calculate

$$A(z) - iB(z) = \frac{E(z) + E^{\#}(z)}{2} + \frac{E(z) - E^{\#}(z)}{2} = E(z)$$

and

$$\begin{aligned} A(\overline{w}) &= \frac{E(\overline{w}) + \overline{E(w)}}{2} = \frac{\overline{\overline{E(\overline{w})} + E(w)}}{2} = \overline{A(w)} \\ B(\overline{w}) &= \frac{E(\overline{w}) - \overline{E(w)}}{2} i = \frac{\overline{E(w) - \overline{E(\overline{w})}}}{2} i = \overline{B(w)}. \end{aligned}$$

Because $B(z)\overline{A(w)} - A(z)\overline{B(w)}$ is entire and has a zero at $\overline{w}$ the function $K(w, z)$ is entire. $K(w, z)$ can be rewritten as

$$\begin{aligned} K(w, z) &= \frac{B(z)\overline{A(w)} - A(z)\overline{B(w)}}{\pi(z - \overline{w})} \\ &= \frac{(E(z) - E^{\#}(z))(\overline{E(w) + E^{\#}(w)}) - (E(z) + E^{\#}(z))(\overline{-E(w) + E^{\#}(w)})}{4\pi(z - \overline{w})} i \\ &= \frac{2E(z)\overline{E(w)} - 2E^{\#}(z)\overline{E^{\#}(w)}}{4\pi(z - \overline{w})} i \\ &= \frac{E(z)\overline{E(w)} - E^{\#}(z)E(\overline{w})}{2\pi(z - \overline{w})} i. \end{aligned}$$

In particular,

$$\frac{K(w,z)2\pi(z-\overline{w})}{E(z)i} = \overline{E(w)} - \frac{\overline{E(\overline{z})}}{E(z)}E(\overline{w}).$$

The function $\frac{\overline{E(\overline{z})}}{E(z)}$ is inner by Lemma 1.2.8. By Lemma 1.1.16 constants belong to $N^+(\mathbb{C}^+)$ and sums and products of these functions are again in $N^+(\mathbb{C}^+)$. Hence, $\frac{K(w,z)(z-\overline{w})}{E(z)} \in N^+(\mathbb{C}^+)$. Let $\overline{w} \in \mathbb{C}^+$. By Lemma 1.1.18 the function $\frac{K(w,z)}{E(z)}$ belongs to $N^+(\mathbb{C}^+)$. If $\overline{w} \notin \mathbb{C}^+$ then $(z-\overline{w})$ is outer by Lemma 1.1.16 and again $\frac{K(w,z)}{E(z)} \in N^+(\mathbb{C}^+)$. The same holds for $\frac{K^\#(w,z)}{E(z)}$ because

$$K^\#(w,z) = \overline{K(w,\overline{z})} = \frac{\overline{B(\overline{z})\overline{A(w)} - A(\overline{z})\overline{B(w)}}}{\pi(\overline{z}-\overline{w})} = \frac{B(z)A(w) - A(z)B(w)}{\pi(z-w)} = K(\overline{w},z).$$

It remains to show, that $K(w,t)$ satisfies the integral condition

$$\int_{\mathbb{R}} \left|\frac{K(w,t)}{E(t)}\right|^2 dt < \infty.$$

Firstly,

$$\frac{K(w,t)}{E(t)} = \frac{-\overline{E(w)} + \frac{\overline{E(t)}}{E(t)}E(\overline{w})}{2\pi i(t-\overline{w})}$$

is a continuous function of $t \in \mathbb{R}$ because $-\overline{E(w)} + \frac{\overline{E(t)}}{E(t)}E(\overline{w})$ is continuous and has a zero at $\overline{w}$. Therefore, the integral over bounded subsets of $\mathbb{R}$ is finite and, hence,

$$\begin{aligned}
\int_{\mathbb{R}} \left|\frac{K(w,t)}{E(t)}\right|^2 dt &\leq C + \int_{\{t\in\mathbb{R}:|t-\overline{w}|>1\}} \left|\frac{K(w,t)}{E(t)}\right|^2 dt \\
&\leq C + \int_{\{t\in\mathbb{R}:|t-\overline{w}|>1\}} \left|\frac{-\overline{E(w)} + \frac{\overline{E(t)}}{E(t)}E(\overline{w})}{2\pi(t-\overline{w})}\right|^2 dt \\
&\leq C + \int_{\{t\in\mathbb{R}:|t-\overline{w}|>1\}} \frac{(|E(w)|+|E(\overline{w})|)^2}{4\pi^2|t-\overline{w}|^2} dt \\
&\leq C + \frac{(|E(w)|+|E(\overline{w})|)^2}{4\pi^2} \int_{\{t\in\mathbb{R}:|t-\overline{w}|>1\}} \frac{1}{|t-\overline{w}|^2} dt \\
&< \infty.
\end{aligned}$$

To show that $H(E)$ contains nonzero functions let $w, z \in \mathbb{C}^+$, $w \neq z$. Then

$$|K(w,z)| = \left|\frac{E(z)\overline{E(w)} - E^\#(z)E(\overline{w})}{2\pi(z-\overline{w})}\right| \geq \frac{\big||E(z)||E(w)| - |E(\overline{z})||E(\overline{w})|\big|}{2\pi|z-\overline{w}|} > 0$$

because $|E(z)| > |E(\overline{z})|$ for all $z \in \mathbb{C}^+$. Hence, $K(w,.)$ belongs to $H(E)$ and is not identically zero. $\square$

The functions $K(w,z)$ have a very important property.

Theorem 1.2.10. *Let $H(E)$ be a de Branges space and for $w \in \mathbb{C}$ let $K(w,z)$ be the function defined in Theorem 1.2.9. Then for any $f \in H(E)$*

$$\langle f, K(w,.)\rangle_E = f(w).$$

Spaces with this property are called reproducing kernel Hilbert spaces. The functions $K(.,.)$ are called kernel functions.

Proof. The property is just an easy application of Corollary 1.2.5 and the representation of K in Theorem 1.2.9:

$$\begin{aligned}\langle f, K(w,.)\rangle_E &= \int_{\mathbb{R}} f(t)\frac{\overline{E(t)}E(w) - E(t)\overline{E(\overline{w})}}{2\pi(t-w)|E(t)|^2}(-i)\,dt \\ &= \int_{\mathbb{R}} f(t)\frac{1}{2\pi i}\frac{\overline{E(t)}E(w) - E(t)\overline{E(\overline{w})}}{|E(t)|^2(t-w)}\,dt \\ &= f(w).\end{aligned}$$

□

Theorem 1.2.11. *Let H be a reproducing kernel Hilbert space, consisting of functions defined and analytic on some open set $G \subseteq \mathbb{C}$ with kernel functions $K(.,.)$. Then*

$$K(w,v) = \overline{K(v,w)}.$$

Furthermore, if $M \subseteq G$ is a set with accumulation point in G, then the span of all functions $K(w,z)$ with $w \in M$ is dense in H.

Proof. The first property is shown by

$$K(w,v) = \langle K(w,z), K(v,z)\rangle = \overline{\langle K(v,z), K(w,z)\rangle} = \overline{K(v,w)}.$$

It remains to prove that the linear span is dense. Let f be a function in the orthogonal complement of this span. Then

$$f(w) = \langle f, K(w,.)\rangle = 0$$

for all $w \in M$ and, hence, $f \equiv 0$. □

The following theorem gives a characterisation of de Branges spaces among all Hilbert spaces of entire functions.

Theorem 1.2.12. *A Hilbert space $H \neq \{0\}$ consisting of entire functions such that the sum (scalar multiplication) is just the pointwise sum (scalar multiplication) coincides with a de Branges space $H(E)$ if and only if it satisfies:*

(i) For $f \in H$, $w \in \mathbb{C}\backslash\mathbb{R}$ with $f(w) = 0$ the function $\frac{f(z)(z-\overline{w})}{z-w}$ belongs to H and has the same norm as f.

(ii) For $w \in \mathbb{C}\backslash\mathbb{R}$ the functional $\Phi_w : f \mapsto f(w)$ is continuous.

(iii) For any $f, g \in H$ the functions $f^\#, g^\#$ belong to H and the inner product satisfies $\langle f, g\rangle = \langle g^\#, f^\#\rangle$.

Proof. Let $H(E)$ be a de Branges space. Then (ii) is satisfied because

$$\langle f(t), K(w,t)\rangle_E = f(w).$$

To show (i) first note that $\frac{f(z)(z-\overline{w})}{z-w}$ is entire. The functions $\frac{f}{E}$ and $\frac{f^\#}{E}$ are in $N^+(\mathbb{C}^+)$. Assume that $w \in \mathbb{C}^+$. By Lemma 1.1.18 the function $\frac{f(z)}{E(z)(z-w)}$ belongs to $N^+(\mathbb{C}^+)$ and by Lemma 1.1.16 the polynomial $(z-\overline{w})$ is outer. Hence, $\frac{f(z)(z-\overline{w})}{E(z)(z-w)} \in N^+(\mathbb{C}^+)$ by Lemma 1.1.16. Further, division by $(z-\overline{w})$ just changes the outer part and $(z-w)$ belongs to $N^+(\mathbb{C}^+)$. Hence, $\frac{z-w}{z-\overline{w}} \in N^+(\mathbb{C}^+)$ and $\frac{f^\#(z)(z-w)}{E(z)(z-\overline{w})} \in N^+(\mathbb{C}^+)$ follows. If $w \in \mathbb{C}^-$ then $\frac{f(z)(z-\overline{w})}{E(z)(z-w)} \in N^+(\mathbb{C}^+)$ because division by $(z-w)$ just changes the outer part and $(z-\overline{w}) \in N^+(\mathbb{C}^+)$. The function $\frac{f^\#}{E}$ has a zero at $\overline{w} \in \mathbb{C}^+$ and by Lemma 1.1.18 $\frac{f^\#(z)}{E(z)(z-\overline{w})} \in N^+(\mathbb{C}^+)$. Hence, $\frac{f^\#(z)(z-w)}{E(z)(z-\overline{w})} \in N^+(\mathbb{C}^+)$. In both cases the functions $\frac{f^\#(z)(z-w)}{E(z)(z-\overline{w})}$ and $\frac{f(z)(z-\overline{w})}{E(z)(z-w)}$ belong to $N^+(\mathbb{C}^+)$. The integral condition is trivial because $\left|\frac{t-w}{t-\overline{w}}\right| = 1$ for $t \in \mathbb{R}$. Therefore, in any case $\frac{f(z)(z-\overline{w})}{z-w}$ belongs to H.

Because $\frac{(f^\#)^\#}{E} = \frac{f}{E}$ and $\frac{f^\#}{E}$ are in $N^+(\mathbb{C}^+)$ and $|f^\#(t)| = |f(t)|$, the function $f^\#$ belongs to $H(E)$. Further,

$$\langle f, g\rangle = \int_{\mathbb{R}} \frac{f(t)\overline{g(t)}}{|E(t)|^2}\,dt = \int_{\mathbb{R}} \overline{\frac{f^\#(t)\overline{g^\#(t)}}{|E(t)|^2}}\,dt = \left\langle g^\#, f^\#\right\rangle.$$

Hence, (iii) holds.

Thus, every de Branges space has the properties (i), (ii) and (iii). Now let $H \neq \{0\}$ be any Hilbert space of entire functions satisfying these conditions. By property (ii) and the Riesz–Fischer representation theorem there exist functions $z \mapsto K(w,z)$, $w \in \mathbb{C}\backslash\mathbb{R}$, such that

$$\langle f, K(w,.)\rangle = f(w)$$

for all $f \in H$. By this property

$$0 \leq \|K(\alpha,.)\|^2 = \langle K(\alpha,.), K(\alpha,.)\rangle = K(\alpha,\alpha)$$

follows. Assume that $K(\alpha,\alpha)=0$ for some $\alpha\in\mathbb{C}\backslash\mathbb{R}$. Then $K(\alpha,.)=0$ and, hence,

$$0=\langle f,K(\alpha,.)\rangle=f(\alpha)$$

for all $f\in H$. By property (i) the function $\frac{f(z)(z-\overline{\alpha})}{z-\alpha}$ belongs to H for all f. Therefore,

$$\begin{aligned}\frac{f(z)(z-\overline{\alpha})}{z-\alpha}-f(z)&=\frac{f(z)(z-\overline{\alpha})-f(z)(z-\alpha)}{z-\alpha}\\&=\frac{f(z)z-f(z)\overline{\alpha}-f(z)z+f(z)\alpha}{z-\alpha}\\&=(\alpha-\overline{\alpha})\frac{f(z)}{z-\alpha}\end{aligned}\tag{1.9}$$

shows that $\frac{f(z)}{z-\alpha}\in H$. The same argument gives $\frac{f(z)}{(z-\alpha)^n}\in H$ for any $n\in\mathbb{N}$. This can only be the case if $f\equiv 0$, which contradicts $H\neq\{0\}$. Thus, $K(\alpha,\alpha)>0$ for all $\alpha\in\mathbb{C}\backslash\mathbb{R}$. Other properties of K are

$$K(\alpha,w)=\langle K(\alpha,.),K(w,.)\rangle=\overline{\langle K(w,.),K(\alpha,.)\rangle}=\overline{K(w,\alpha)}$$

and because of $K^{\#}(\alpha,z)\in H$ by property (iii)

$$\begin{aligned}\overline{K(\alpha,\overline{w})}&=\left\langle K^{\#}(\alpha,.),K(w,.)\right\rangle\\&=\overline{\langle K(\alpha,.),K^{\#}(w,.)\rangle}\\&=\left\langle K^{\#}(w,.),K(\alpha,.)\right\rangle\\&=\overline{K(w,\overline{\alpha})}\\&=K(\overline{\alpha},w).\end{aligned}\tag{1.10}$$

To construct a de Branges space which is isometrically equal to H, define for some fixed $\alpha\in\mathbb{C}^+$ the function $E:\mathbb{C}\to\mathbb{C}$ by

$$E(z)=-\frac{2\pi i(\overline{\alpha}-z)K(\alpha,z)}{\sqrt{2\pi i(\overline{\alpha}-\alpha)K(\alpha,\alpha)}}$$

where $i(\overline{\alpha}-\alpha)\in\mathbb{R}^+$ because $\alpha\in\mathbb{C}^+$ and $K(\alpha,\alpha)\in\mathbb{R}^+$ by the calculations above. The aim is to show that this function E produces a de Branges space with kernel K.

By property (i) the map $f(z)\mapsto\frac{f(z)(z-\overline{w})}{z-w}$ is an isometry on the set of all functions with a zero at w. Because $(K(w,z)-\frac{K(\alpha,z)K(w,\alpha)}{K(\alpha,\alpha)})$ belongs to H and has a zero at α, for any $f\in H$ with $f(\overline{\alpha})=0$ the following equations hold by properties (i) and (ii) (here z is the independent variable in the inner product)

$$\begin{aligned}
\left\langle f(z), \left(K(w,z) - \frac{K(\alpha,z)K(w,\alpha)}{K(\alpha,\alpha)}\right)\frac{z-\overline{\alpha}}{z-\alpha}\right\rangle &= \left\langle f(z)\frac{z-\alpha}{z-\overline{\alpha}}, K(w,z) - \frac{K(\alpha,z)K(w,\alpha)}{K(\alpha,\alpha)}\right\rangle \\
&= \left\langle f(z)\frac{z-\alpha}{z-\overline{\alpha}}, K(w,z)\right\rangle \\
&\quad - \overline{\left(\frac{K(w,\alpha)}{K(\alpha,\alpha)}\right)}\left\langle f(z)\frac{z-\alpha}{z-\overline{\alpha}}, K(\alpha,z)\right\rangle \\
&= f(w)\frac{w-\alpha}{w-\overline{\alpha}} \\
&= \left(\langle f(z), K(w,z)\rangle - \frac{\overline{K(w,\overline{\alpha})}}{K(\alpha,\alpha)}\langle f(z), K(\overline{\alpha},z)\rangle\right)\frac{w-\alpha}{w-\overline{\alpha}} \\
&= \left\langle f(z), \left(K(w,z) - \frac{K(\overline{\alpha},z)K(w,\overline{\alpha})}{K(\alpha,\alpha)}\right)\frac{\overline{w}-\overline{\alpha}}{\overline{w}-\alpha}\right\rangle.
\end{aligned}$$

Therefore,

$$\left(K(w,z) - \frac{K(\alpha,z)K(w,\alpha)}{K(\alpha,\alpha)}\right)\frac{z-\overline{\alpha}}{z-\alpha} - \left(K(w,z) - \frac{K(\overline{\alpha},z)K(w,\overline{\alpha})}{K(\alpha,\alpha)}\right)\frac{\overline{w}-\overline{\alpha}}{\overline{w}-\alpha} \in \{f \in H : f(\overline{\alpha}) = 0\}^{\perp}.$$

Due to (1.10) this function also belongs to $\{f \in H : f(\overline{\alpha}) = 0\}$. Hence,

$$\left(K(w,z) - \frac{K(\alpha,z)K(w,\alpha)}{K(\alpha,\alpha)}\right)\frac{z-\overline{\alpha}}{z-\alpha} = \left(K(w,z) - \frac{K(\overline{\alpha},z)K(w,\overline{\alpha})}{K(\alpha,\alpha)}\right)\frac{\overline{w}-\overline{\alpha}}{\overline{w}-\alpha}.$$

Therefore,

$$\begin{aligned}
&(\overline{w}-\alpha)(z-\overline{\alpha})K(\alpha,\alpha)K(w,z) - (\overline{w}-\alpha)(z-\overline{\alpha})K(\alpha,z)K(w,\alpha) = \\
&\qquad = (\overline{w}-\overline{\alpha})(z-\alpha)K(\alpha,\alpha)K(w,z) - (\overline{w}-\overline{\alpha})(z-\alpha)K(\overline{\alpha},z)K(w,\overline{\alpha})
\end{aligned}$$

and, further,

$$\begin{aligned}
&\big((\overline{w}-\alpha)(z-\overline{\alpha}) - (\overline{w}-\overline{\alpha})(z-\alpha)\big)K(\alpha,\alpha)K(w,z) = \\
&\qquad = (\overline{w}-\alpha)(z-\overline{\alpha})K(\alpha,z)K(w,\alpha) - (\overline{w}-\overline{\alpha})(z-\alpha)K(\overline{\alpha},z)K(w,\overline{\alpha}).
\end{aligned}$$

Thus,

$$\begin{aligned}
&(\overline{w}z - \overline{w\alpha} - \alpha z + \alpha\overline{\alpha} - \overline{w}z + \overline{w}\alpha + \overline{\alpha}z - \overline{\alpha}\alpha)K(\alpha,\alpha)K(w,z) = \\
&\qquad = (\overline{w}-\alpha)(z-\overline{\alpha})K(\alpha,z)K(w,\alpha) - (\overline{w}-\overline{\alpha})(z-\alpha)K(\overline{\alpha},z)K(w,\overline{\alpha}).
\end{aligned}$$

Hence,

$$K(w,z) = \frac{(\overline{w}-\alpha)(z-\overline{\alpha})K(\alpha,z)K(w,\alpha) - (\overline{w}-\overline{\alpha})(z-\alpha)K(\overline{\alpha},z)K(w,\overline{\alpha})}{(-\overline{w\alpha} - \alpha z + \overline{w}\alpha + \overline{\alpha}z)\,K(\alpha,\alpha)}$$

and, finally,

$$K(w,z) = \frac{(\overline{w}-\alpha)(z-\overline{\alpha})K(\alpha,z)K(w,\alpha) - (\overline{w}-\overline{\alpha})(z-\alpha)K(\overline{\alpha},z)K(w,\overline{\alpha})}{(z-\overline{w})(\overline{\alpha}-\alpha)K(\alpha,\alpha)}.$$

Hence,

$$\begin{aligned}
\frac{E(z)\overline{E(w)} - E^{\#}(z)E(\overline{w})}{2\pi(z-\overline{w})}i &= \frac{\frac{2\pi i(\overline{\alpha}-z)K(\alpha,z)\overline{(2\pi i(\overline{\alpha}-w)K(\alpha,w))}}{2\pi i(\overline{\alpha}-\alpha)K(\alpha,\alpha)} - \frac{\overline{(2\pi i(\overline{\alpha}-\overline{z})K(\alpha,\overline{z}))}2\pi i(\overline{\alpha}-\overline{w})K(\alpha,\overline{w})}{2\pi i(\overline{\alpha}-\alpha)K(\alpha,\alpha)}}{2\pi(z-\overline{w})}i \\
&= \frac{(\overline{\alpha}-z)K(\alpha,z)(\alpha-\overline{w})K(w,\alpha) - (\alpha-z)K(\overline{\alpha},z)(\overline{\alpha}-\overline{w})K(w,\overline{\alpha})}{(z-\overline{w})(\overline{\alpha}-\alpha)K(\alpha,\alpha)} \\
&= K(w,z).
\end{aligned}$$

To show $|E(z)| > |E(\overline{z})|$ for $z \in \mathbb{C}^+$ set $w = z$ in the equation above. Then

$$0 < K(z,z) = \frac{E(z)\overline{E(z)} - E^{\#}(z)E(\overline{z})}{2\pi(z-\overline{z})}i,$$

where $\frac{i}{z-\overline{z}} \in \mathbb{R}^+$ and, hence,

$$0 < E(z)\overline{E(z)} - \overline{E(\overline{z})}E(\overline{z}) = |E(z)|^2 - |E(\overline{z})|^2.$$

Thus, there exists a Hilbert space $H(E)$ with the same kernel as H.

Let f be any function in H. To prove that f is also in $H(E)$ consider the set $M := \{K(w,.) : w \in \mathbb{C}\backslash\mathbb{R}\}$. The linear span of this set is contained in $H(E)$ and in H. Moreover, all functions in $\operatorname{span} M$ have the same norm in both spaces. Because no nonzero function can be orthogonal to this subspace by Theorem 1.2.11 the closure has to be the whole space in both cases. Hence, there exists a sequence $f_n(z) \in \operatorname{span}(M)$ with $\lim_{n\to\infty} f_n = f$ in H. This sequence is a Cauchy sequence in H and the norm of these functions is the same in $H(E)$. Hence, it is a Cauchy sequence in $H(E)$. By completeness it has to converge in $H(E)$ to some $g \in H(E)$. For any $w \in \mathbb{C}$

$$f(w) = \langle f(.), K(w,.)\rangle = \lim_{n\to\infty} \langle f_n(.), K(w,.)\rangle = \lim_{n\to\infty} \langle f_n(.), K(w,.)\rangle_E = \langle g(.), K(w,.)\rangle_E = g(w).$$

Therefore, $f = g \in H(E)$. The same argument shows that any $g \in H(E)$ is contained in H. Hence, the spaces are equal. $\square$

Chapter 2

Functions associated with $H(E)$

Some entire functions that do not necessarily belong to $H(E)$ are of great interest in the theory of de Branges spaces. These so called associated functions will be introduced in this chapter. Further, some equivalences and other sufficient conditions for a function to be associated will be proved.

2.1 Associated functions

Definition 2.1.1. *Let $H(E)$ be a de Branges space and S be an entire function. Then S is called associated with $H(E)$ if for every $f \in H(E)$ and every $\alpha \in \mathbb{C}$ the function $\frac{f(z)S(\alpha)-f(\alpha)S(z)}{z-\alpha}$ belongs to $H(E)$.*

A useful characterisation of functions associated with $H(E)$ is the following.

Theorem 2.1.2. *Let $H(E)$ be a de Branges space and S be an entire function. Then the following assertions are equivalent:*

(i) S is associated with $H(E)$.

(ii) For some $\alpha \in \mathbb{C}$ and some $f \in H(E)$ with $f(\alpha) \neq 0$ the function $\frac{f(z)S(\alpha)-f(\alpha)S(z)}{z-\alpha}$ belongs to $H(E)$.

(iii) $\frac{S}{E}, \frac{S^\#}{E}$ are in $N^+(\mathbb{C}^+)$ and satisfy

$$\int_{\mathbb{R}} \frac{|S(t)|^2}{|E(t)|^2(1+t)^2}\, dt < \infty.$$

Proof. (iii) $\Rightarrow$ (i): To show that S is associated the function $g(z) := \frac{f(z)S(\alpha)-f(\alpha)S(z)}{z-\alpha}$ has to belong to $H(E)$ for any $f \in H(E)$ and $\alpha \in \mathbb{C}$. As $\frac{S}{E}, \frac{f}{E}$ belong to $N^+(\mathbb{C}^+)$ the function $\frac{f(z)S(\alpha)-f(\alpha)S(z)}{E(z)}$

belongs to $N^+(\mathbb{C}^+)$. Assume that $\alpha \in \mathbb{C}^+$. Then $\frac{f(z)S(\alpha)-f(\alpha)S(z)}{E(z)}$ has a zero at α because E has no zeros on the upper halfplane. Therefore, by Lemma 1.1.18 the function $\frac{g(z)}{E(z)} = \frac{f(z)S(\alpha)-f(\alpha)S(z)}{E(z)(z-\alpha)}$ belongs to $N^+(\mathbb{C}^+)$. If $\alpha \in \overline{\mathbb{C}^-}$, the polynomial $(z-\alpha)$ is outer by Lemma 1.1.16. Hence, the function $\frac{g(z)}{E(z)} = \frac{f(z)S(\alpha)-f(\alpha)S(z)}{E(z)(z-\alpha)}$ belongs to $N^+(\mathbb{C}^+)$ in this case, too. As $\frac{S^\#}{E}, \frac{f^\#}{E}$ belong to $N^+(\mathbb{C}^+)$ similar arguments show that $\frac{g^\#(z)}{E(z)} = \frac{f^\#(z)\overline{S(\alpha)}-\overline{f(\alpha)}S^\#(z)}{E(z)(z-\overline{\alpha})}$ belongs to $N^+(\mathbb{C}^+)$. To show that $g(z)$ belongs to $H(E)$, it remains to show the integral condition. Because $g(z) = \frac{f(z)S(\alpha)-f(\alpha)S(z)}{z-\alpha}$ is continuous, it is bounded on the compact set $\{t \in \mathbb{R} : 2|t-\alpha|^2 \leq t^2+1\}$. Hence,

$$\begin{aligned}\int\limits_{\mathbb{R}} \left|\frac{f(t)S(\alpha)-f(\alpha)S(t)}{E(t)(t-\alpha)}\right|^2 dt &\leq C_1 + 2 \int\limits_{\{t\in\mathbb{R}:2|t-\alpha|^2>t^2+1\}} \left|\frac{f(t)S(\alpha)-f(\alpha)S(t)}{E(t)}\right|^2 \frac{1}{t^2+1}\, dt \\ &\leq C_1 + 2C_2 \int\limits_{\mathbb{R}} \left(\frac{|f(t)|^2}{|E(t)|^2(t^2+1)} + \frac{|S(t)|^2}{|E(t)|^2(t^2+1)}\right) dt \qquad (2.1) \\ &< \infty.\end{aligned}$$

This proves the implication.

(i) $\Rightarrow$ (ii) is trivial by the definition of associated functions.

To prove (ii) $\Rightarrow$ (iii) let $\alpha \in \mathbb{C}$ and f belong to $H(E)$ with $f(\alpha) \neq 0$ such that $\frac{f(z)S(\alpha)-f(\alpha)S(z)}{E(z)(z-\alpha)}$ belongs to $H(E)$. Then $\frac{f(z)S(\alpha)-f(\alpha)S(z)}{E(z)(z-\alpha)} \in N^+(\mathbb{C}^+)$ and $\frac{f^\#(z)\overline{S(\alpha)}-\overline{f(\alpha)}S^\#(z)}{E(z)(z-\overline{\alpha})}$ belong to $N^+(\mathbb{C}^+)$. By Lemma 1.1.16 polynomials belong to $N^+(\mathbb{C}^+)$ and products of functions in $N^+(\mathbb{C}^+)$ belong to $N^+(\mathbb{C}^+)$. Hence, $\frac{f(z)S(\alpha)-f(\alpha)S(z)}{E(z)}$ and $\frac{f^\#(z)\overline{S(\alpha)}-\overline{f(\alpha)}S^\#(z)}{E(z)}$ belong to $N^+(\mathbb{C}^+)$. Because $f \in H(E)$, the functions $\frac{S(\alpha)f(z)}{E(z)}$ and $\frac{\overline{S(\alpha)}f^\#(z)}{E(z)}$ belong to $N^+(\mathbb{C}^+)$ and with this $\frac{f(\alpha)S(z)}{E(z)}, \frac{\overline{f(\alpha)}S^\#(z)}{E(z)} \in N^+(\mathbb{C}^+)$. By assumption $f(\alpha) \neq 0$. Hence, $\frac{S(z)}{E(z)}, \frac{S^\#(z)}{E(z)}$ belong to $N^+(\mathbb{C}^+)$. Further,

$$\int\limits_{\mathbb{R}} \left|\frac{f(t)S(\alpha)-f(\alpha)S(t)}{E(t)(t-\alpha)}\right|^2 dt < \infty.$$

This and the fact that $\left|\frac{t-\alpha}{t-i}\right|$ is bounded on $\mathbb{R}$ lead to

$$\begin{aligned}\int\limits_{\mathbb{R}} \left|\frac{|f(t)S(\alpha)|-|f(\alpha)S(t)|}{E(t)}\right|^2 \frac{1}{(1+t^2)}\, dt &\leq \int\limits_{\mathbb{R}} \left|\frac{f(t)S(\alpha)-f(\alpha)S(t)}{E(t)}\right|^2 \frac{1}{(1+t^2)}\, dt \\ &= \int\limits_{\mathbb{R}} \left|\frac{f(t)S(\alpha)-f(\alpha)S(t)}{E(t)(t-i)}\right|^2 dt \\ &< C \int\limits_{\mathbb{R}} \left|\frac{f(t)S(\alpha)-f(\alpha)S(t)}{E(t)(t-\alpha)}\right|^2 dt \\ &< \infty.\end{aligned}$$

Because $\int_{\mathbb{R}} \left|\frac{f(t)}{E(t)}\right|^2 dt < \infty$ the relation

$$\int_{\mathbb{R}} \left|\frac{S(t)}{E(t)}\right|^2 \frac{1}{(1+t^2)}\, dt < \infty$$

follows. □

Corollary 2.1.3. *Let $H(E)$ be a de Branges space. An entire function S is associated with $H(E)$ if and only if there exist $\varphi, \psi \in H(E)$ such that*

$$S(z) = \varphi(z) + z\psi(z).$$

Proof. Let S be associated. Then there exists an $\alpha \in \mathbb{C}$ and an $f \in H(E)$ with $f(\alpha) \neq 0$ such that $g(z) := \frac{f(z)S(\alpha)-f(\alpha)S(z)}{z-\alpha}$ belongs to $H(E)$. It is easy to see that

$$S(z) = \frac{f(z)S(\alpha) - g(z)(z-\alpha)}{f(\alpha)}.$$

With $\varphi(z) := \frac{f(z)S(\alpha)+g(z)\alpha}{f(\alpha)}$ and $\psi(z) := \frac{-g(z)}{f(\alpha)}$ one gets

$$S(z) = \varphi(z) + z\psi(z).$$

For the other implication let $S(z) = \varphi(z) + z\psi(z)$. For any $\alpha \in \mathbb{C}$ with $S(\alpha) \neq 0$ define $f \in H(E)$ by

$$f(z) := \frac{\varphi(z) + \alpha\psi(z)}{S(\alpha)}.$$

Then, obviously, $f(\alpha) = 1$. Hence,

$$\begin{aligned}
\frac{f(z)S(\alpha) - f(\alpha)S(z)}{z-\alpha} &= \frac{\frac{\varphi(z)+\alpha\psi(z)}{S(\alpha)}S(\alpha) - (\varphi(z) + z\psi(z))}{z-\alpha} \\
&= \frac{\varphi(z) + \alpha\psi(z) - \varphi(z) - z\psi(z)}{z-\alpha} \\
&= \frac{(\alpha - z)\psi(z)}{z-\alpha} \\
&= -\psi(z).
\end{aligned}$$

This function belongs to $H(E)$ and, by Theorem 2.1.2, S is associated. □

This gives an immediate result for the real zeros of S.

Corollary 2.1.4. *Let $H(E)$ be a de Branges space and S be associated with $H(E)$. If E has a zero at some $t_0 \in \mathbb{R}$ of multiplicity m_E, then S has a zero at t_0 of multiplicity $m_S \geq m_E$.*

Proof. By Lemma 1.2.4 the same property holds for $f \in H(E)$ in place of S. Corollary 2.1.3 shows that zeros of all functions in $H(E)$ are zeros of S. $\square$

Now some examples of functions associated with $H(E)$ can be given.

Example 2.1.5.

- *Linear combinations of associated functions are again associated by Corollary 2.1.3.*
- *E and $E^{\#}$ are associated by Theorem 2.1.2, because $\frac{E^{\#}}{E}$ is inner and, hence, belongs to $N^+(\mathbb{C}^+)$ and the integral $\int_{\mathbb{R}} \frac{1}{(1+t^2)}\, dt$ is finite.*
- *The functions A and B satisfying $E(z) = A(z) - iB(z)$ as defined in Theorem 1.2.9 are associated, because they are linear combinations of E and $E^{\#}$.*

Corollary 2.1.6. *Let $H(E)$ be a de Branges space. For all $\alpha \in \mathbb{C}$ with $E(\alpha) \neq 0$ there exists some $f \in H(E)$ such that*

$$f(\alpha) \neq 0.$$

Proof. Argue by contradiction. Assume that there exists an $\alpha \in \mathbb{C}$ with $E(\alpha) \neq 0$ such that $f(\alpha)$ vanishes for all $f \in H(E)$. Because E is associated and $E(\alpha) \neq 0$ the function

$$\frac{f(z)E(\alpha) - f(\alpha)E(z)}{z - \alpha} = \frac{f(z)E(\alpha)}{z - \alpha}$$

belongs to $H(E)$. Therefore, $\frac{f(z)}{z-\alpha}$ belongs to $H(E)$ for all $f \in H(E)$. Hence, all $\frac{f(z)}{(z-\alpha)^n}$ for all $n \in \mathbb{N}$ belong to $H(E)$. This can only be true for $f \equiv 0$, which is a contradiction to $H(E) \neq \{0\}$ in Theorem 1.2.9. $\square$

Corollary 2.1.7. *Let $H(E)$ be a de Branges space. For all $\alpha \in \mathbb{C}$ with $E(\alpha) \neq 0$ the function $K(\alpha, z)$ does not vanish at α.*

Proof. By Theorem 1.2.10

$$K(\alpha, \alpha) = \langle K(\alpha, z), K(\alpha, z) \rangle = \|K(\alpha, z)\|^2.$$

By Corollary 2.1.6 there has to exist a function $f \in H(E)$ with $f(\alpha) \neq 0$. Hence,

$$\langle f(z), K(\alpha, z) \rangle = f(\alpha) \neq 0$$

follows. Therefore $K(\alpha, z)$ is not identically zero and

$$\|K(\alpha, z)\| \neq 0.$$

$\square$

Lemma 2.1.8. *Let $H(E)$ be a de Branges space and S, R be entire functions. Let R be associated with $H(E)$. Assume further, that there exists an $\alpha \in \mathbb{C}$ with $R(\alpha) \neq 0$ such that*

$$\frac{R(z)S(\alpha) - R(\alpha)S(z)}{z - \alpha} \in H(E).$$

Then S is associated with $H(E)$.

Proof. With

$$h(z) := \frac{R(z)S(\alpha) - S(z)R(\alpha)}{z - \alpha}$$

one gets

$$S(z) = \frac{1}{R(\alpha)} \left(R(z)S(\alpha) - (z - \alpha)h(z) \right).$$

Applying Corollary 2.1.3 to $R(z)$ yields that the function $S(z)$ satisfies

$$\begin{aligned} S(z) &= \frac{1}{R(\alpha)} \big((\varphi(z) + z\psi(z))S(\alpha) - (z - \alpha)h(z) \big) \\ &= \frac{1}{R(\alpha)} \big(S(\alpha)\varphi(z) + \alpha h(z) + z(S(\alpha)\psi(z) - h(z)) \big). \end{aligned}$$

Therefore, again by Corollary 2.1.3 $S(z)$ is associated with $H(E)$. $\square$

Another useful sufficient condition for a function to be associated, is obtained by an estimate on the imaginary axis:

Theorem 2.1.9. *Let $H(E)$ be a de Branges space and let S be an entire function with $\frac{S(z)}{E(z)}$ and $\frac{S^{\#}(z)}{E(z)} \in N(\mathbb{C}^+)$ and such that $\frac{S(z)}{E(z)}$ has no real singularities. Let μ be a Borel measure on $\mathbb{R}$ such that $H(E)$ is isometrically contained in $L^2(\mu)$, i.e. every function $f \in H(E)$ restricted to the real line belongs to $L^2(\mu)$ and*

$$\|f\|_{H(E)} = \|f|_{\mathbb{R}}\|_{L^2(\mu)}.$$

Assume that there exists no nonzero entire function Q which is associated with $H(E)$ and is zero μ-almost everywhere. If

$$\int_{\mathbb{R}} \frac{|S(t)|^2}{1 + t^2} \, d\mu(t) < \infty,$$

$$\limsup_{y \to \infty} \left| \frac{S(iy)}{E(iy)} \right| < \infty,$$

and

$$\limsup_{y \to \infty} \left| \frac{S(-iy)}{E(iy)} \right| < \infty,$$

then S is associated with $H(E)$.

Proof. The idea is to show that for each function $f \in H(E)$ the function $\frac{f(z)S(w)-S(z)f(w)}{z-w}$ is orthogonal to $L^2(\mu) \ominus H(E)$. Therefore, let h be any function in $L^2(\mu) \ominus H(E)$. Define a function $L(z)$ for $z \in \mathbb{C}$, such that $\frac{S(z)}{E(z)}$ is analytic at z, by

$$E(z)L(z) = \int_{\mathbb{R}} \frac{E(t)S(z) - S(t)E(z)}{t-z} \overline{h(t)}\, d\mu(t). \tag{2.2}$$

With $d\nu := |E(t)|^2 d\mu$ this yields

$$L(z) = \int_{\mathbb{R}} \frac{E(t)\frac{S(z)}{E(z)} - S(t)}{t-z} \overline{h(t)}\, d\mu(t) = \int_{\mathbb{R}} \frac{\frac{S(z)}{E(z)} - \frac{S(t)}{E(t)}}{t-z} E(t)\overline{h(t)}\, d\mu(t) = \int_{\mathbb{R}} \frac{\frac{S(z)}{E(z)} - \frac{S(t)}{E(t)}}{t-z} \frac{\overline{h(t)}}{\overline{E(t)}}\, d\nu(t).$$

In a similar way a function $\tilde{L}(z)$ for $z \in \mathbb{C}$, such that $\frac{S(z)}{E^{\#}(z)}$ is analytic at z, can be defined using $E^{\#}(z)$ in place of $E(z)$:

$$E^{\#}(z)\tilde{L}(z) = \int_{\mathbb{R}} \frac{E^{\#}(t)S(z) - S(t)E^{\#}(z)}{t-z} \overline{h(t)}\, d\mu(t)$$

and

$$\tilde{L}(z) = \int_{\mathbb{R}} \frac{\frac{S(z)}{E^{\#}(z)} - \frac{S(t)}{E^{\#}(t)}}{t-z} \frac{\overline{h(t)}}{E(t)}\, d\nu(t).$$

To prove that L is analytic Lemma 1.2.3 is used with $d\mu = d\nu$, $\psi(z) = \frac{S(z)}{E(z)}$ and $\varphi = \frac{\overline{h(t)}}{\overline{E(t)}}$. The first condition in the lemma is

$$\int_{\mathbb{R}} \frac{1}{1+t^2}\, d\nu(t) < \infty.$$

By definition this is the same as

$$\int_{\mathbb{R}} \frac{|E(t)|^2}{1+t^2}\, d\mu(t) < \infty. \tag{2.3}$$

The function $K(i,z)$ belongs to $H(E)$ and, hence, by assumption to $L^2(\mu)$, i.e. $\|K(i,z)\|_{L^2(\mu)} < \infty$. By the characterisation of $K(w,z)$ in terms of E and $E^{\#}$ in Theorem 1.2.9 and the triangle inequality

$$\begin{aligned}
\infty > \int_{\mathbb{R}} |K(i,z)|^2\, d\mu(t) &= \int_{\mathbb{R}} \left| \frac{E(t)\overline{E(i)} - \overline{E(t)}E(-i)}{2\pi(t+i)} i \right|^2 d\mu(t) \\
&\geq \int_{\mathbb{R}} \frac{\big||E(t)||E(i)| - |E(t)||E(-i)|\big|^2}{4\pi^2(t^2+1)}\, d\mu(t) \\
&= \int_{\mathbb{R}} \frac{|E(t)|^2}{4\pi^2(t^2+1)} \big||E(i)| - |E(-i)|\big|^2\, d\mu(t)
\end{aligned}$$

and, because $|E(i)| > |E(-i)|$, condition (2.3) is verified. The other condition is

$$\int_{\mathbb{R}} \frac{\left|\frac{S(t)}{E(t)}\right|^2}{1+t^2}\, d\nu(t) < \infty.$$

By the definition of $d\nu$ this is equivalent to

$$\int_{\mathbb{R}} \frac{|S(t)|^2}{1+t^2}\, d\mu(t) < \infty,$$

which is true by assumption. It remains to show that $\frac{\overline{h(t)}}{E(t)} \in L^2(\nu)$. This is true by the definition of $d\nu$ and because of $h(t) \in L^2(\mu)$. Now Lemma 1.2.3 states that $L(z)$ is analytic where $\frac{S(z)}{E(z)}$ is analytic. This includes the upper halfplane and the real line. A similar proof shows the same property for $\tilde{L}(z)$ on the closed lower halfplane. To show that these two functions are continuations of each other let f be any function in $H(E)$. Then

$$f(w)\frac{E(z)S(w)-S(z)E(w)}{z-w} = E(w)\frac{f(z)S(w)-S(z)f(w)}{z-w} + S(w)\frac{E(z)f(w)-f(z)E(w)}{z-w} \tag{2.4}$$

where the left hand side belongs to $L^2(\mu)$ because (similar as in (2.1))

$$\int_{\mathbb{R}} \left|\frac{E(t)S(w)-S(t)E(w)}{t-w}\right|^2 d\mu(t) \le C_1 + C_2 \int_{\mathbb{R}} \left(\frac{|E(t)|^2}{t^2+1} + \frac{|S(t)|^2}{t^2+1}\right) d\mu(t) < \infty.$$

The last term on the right hand side of (2.4) belongs to $H(E)$, because E is an associated function. Because of $h(t) \in L^2(\mu) \ominus H(E)$ and the definition of $L(w)$ in (2.2),

$$f(w)E(w)L(w) = \int_{\mathbb{R}} f(w)\frac{E(t)S(w)-S(t)E(w)}{t-w}\overline{h(t)}\, d\mu(t) = \int_{\mathbb{R}} E(w)\frac{f(t)S(w)-S(t)f(w)}{t-w}\overline{h(t)}\, d\mu(t)$$

for any $w \in \overline{\mathbb{C}^+}, E(w) \neq 0$ and, hence,

$$f(w)L(w) = \int_{\mathbb{R}} \frac{f(t)S(w)-S(t)f(w)}{t-w}\overline{h(t)}\, d\mu(t).$$

In the same way

$$f(w)\tilde{L}(w) = \int_{\mathbb{R}} \frac{f(t)S(w)-S(t)f(w)}{t-w}\overline{h(t)}\, d\mu(t)$$

for any $w \in \overline{\mathbb{C}^-}, E^{\#}(w) \neq 0$. By Corollary 2.1.6 there has to be a function $f \in H(E)$ with $f(w) \neq 0$ for all $w \in \mathbb{C}$ with $E(w) \neq 0$. Hence, $L(w) = \tilde{L}(w)$ for all $w \in \mathbb{R}$ with $E(w) \neq 0$. Because both functions are analytic on a neighbourhood of $\mathbb{R}$, they have to coincide there and continue each other to an entire function. On the imaginary axis for $y > 1$ the identity

$$L(iy) = \frac{S(iy)}{E(iy)} \int_{\mathbb{R}} \frac{E(t)\overline{h(t)}}{t - iy}\, d\mu(t) - \int_{\mathbb{R}} \frac{S(t)\overline{h(t)}}{t - iy}\, d\mu(t)$$

holds true. By the Cauchy-Schwarz inequality,

$$\left| \int_{\mathbb{R}} \frac{E(t)\overline{h(t)}}{t - iy}\, d\mu(t) \right|^2 \leq \left(\int_{\mathbb{R}} \frac{|E(t)|^2}{t^2 + y^2}\, d\mu(t) \right) \left(\int_{\mathbb{R}} |\overline{h(t)}|^2\, d\mu(t) \right)$$

where the function in the first integral on the right hand side can be dominated by $\frac{|E(t)|^2}{t^2+1}$. By Lebesgues Theorem it approaches zero as $y \to \infty$. Hence, the left hand side converges to zero. The same argument shows that $\int_{\mathbb{R}} \frac{S(t)\overline{h(t)}}{t-iy}\, d\mu(t)$ approaches zero as y tends to infinity. By assumption

$$\limsup_{y \to \infty} \left| \frac{S(iy)}{E(iy)} \right| < \infty$$

and, hence,

$$\lim_{y \to \infty} |L(iy)| = 0.$$

A similar argument using $\tilde{L}(z)$ and for $y < -1$ the identity

$$L^{\#}(iy) = \frac{S^{\#}(iy)}{E(iy)} \int_{\mathbb{R}} \frac{E(t)h(t)}{t - iy}\, d\mu(t) - \int_{\mathbb{R}} \frac{S^{\#}(t)h(t)}{t - iy}\, d\mu(t)$$

and the assumption

$$\limsup_{y \to \infty} \left| \frac{S(-iy)}{E(iy)} \right| < \infty$$

lead to

$$\lim_{y \to \infty} |L(-iy)| = 0.$$

To show that $L(z)$ and $L^{\#}(z)$ belong to $N(\mathbb{C}^+)$, Lemma 1.1.32 is used. The function L satisfies

$$L(z) = \frac{S(z)}{E(z)} \int_{\mathbb{R}} \frac{E(t)\overline{h(t)}}{t - z}\, d\mu(t) - \int_{\mathbb{R}} \frac{S(t)\overline{h(t)}}{t - z}\, d\mu(t)$$

on the upper halfplane. Because $h(t) \in L^2(\mu)$ and $\int_{\mathbb{R}} \frac{|E(t)|^2}{1+t^2}\, d\mu(t) < \infty$ and $\int_{\mathbb{R}} \frac{|S(t)|^2}{1+t^2}\, d\mu(t) < \infty$ the functions $\frac{|E(t)\overline{h(t)}|}{|t|+1}$ and $\frac{|S(t)\overline{h(t)}|}{|t|+1}$ belong to $L^1(\mu)$ by the Cauchy-Schwarz inequality and because $1 + t^2 \leq (1 + |t|)^2$. Hence, the assumptions of the lemma are satisfied. As $\frac{S(z)}{E(z)} \in N(\mathbb{C}^+)$ and because products of functions of bounded type are of bounded type again, $L(z)$ belongs to $N(\mathbb{C}^+)$. The function $L^\#(z)$ satisfies

$$L^\#(z) = \frac{S^\#(z)}{E(z)} \int_{\mathbb{R}} \frac{E(t)h(t)}{t-z}\, d\mu(t) - \int_{\mathbb{R}} \frac{S^\#(t)h(t)}{t-z}\, d\mu(t)$$

on the upper halfplane and again with Lemma 1.1.32 the function belongs to $N(\mathbb{C}^+)$. The results above show that L is bounded on the imaginary axis and by Lemma 1.1.36 it reduces to a constant. Because the limit on the imaginary axis is zero, L has to vanish identically.

By the definition of $L(w)$ this yields

$$0 = \int_{\mathbb{R}} \frac{E(t)S(w) - S(t)E(w)}{t-w} \overline{h(t)}\, d\mu(t)$$

for all $w \in \mathbb{C}$. Since $h(t)$ was any function in $L^2(\mu) \ominus H(E)$ for some fixed $w_0 \in \mathbb{C}$, the function $\frac{E(z)S(w_0)-S(z)E(w_0)}{z-w_0}$ has to coincide with a function $T(z) \in H(E)$ considered as an element of $L^2(\mu)$. For $w_0 \in \mathbb{C}$ with $E(w_0) \neq 0$ define the entire function

$$\tilde{S}(z) := \frac{-T(z)(z-w_0) + S(w_0)E(z)}{E(w_0)}.$$

Obviously $\tilde{S}(w_0) = S(w_0)$, $\tilde{S}$ and S are μ-equivalent and

$$T(z) = \frac{\tilde{S}(w_0)E(z) - \tilde{S}(z)E(w_0)}{z-w_0}.$$

By Lemma 2.1.8, $\tilde{S}(z)$ is associated with $H(E)$ and, hence, $\frac{\tilde{S}(w)E(z)-\tilde{S}(z)E(w)}{z-w} \in H(E)$ for any $w \in \mathbb{C}$. The function $P(z) := S(z) - \tilde{S}(z)$ is now zero μ-almost everywhere and for a fixed, but arbitrary $w \in \mathbb{C}$ the function $\frac{E(z)P(w)-P(z)E(w)}{z-w}$ has to coincide with some $R_w(z) \in H(E)$ considered as an element of $L^2(\mu)$. As above for $w \in \mathbb{C}$ with $E(w) \neq 0$ the function

$$\tilde{P_w}(z) := \frac{-R_w(z)(z-w) + P(w)E(z)}{E(w)}$$

satisfies $P(w) = \tilde{P_w}(w)$, is μ-equivalent to $P(z)$ and, hence, is zero μ-almost everywhere. It fulfils $\frac{E(z)\tilde{P_w}(w)-\tilde{P_w}(z)E(w)}{z-w} = R_w(z) \in H(E)$ and, thus, is associated. By assumption the function $\tilde{P_w}(z)$ has to vanish everywhere, because it is an associated μ-almost everywhere vanishing entire function. Especially $P(w) = \tilde{P_w}(w) = 0$. By the arbitrariness of the w with $E(w) \neq 0$, the function P vanishes identically and, hence, $S(z) = \tilde{S}(z)$. □

2.2 Strongly associated functions

To study in some kind stronger associated functions the following linear relation is defined:

Definition 2.2.1. *Let $H(E)$ be a de Branges space and S be an entire function. Then a linear relation M_S is defined as*

$$M_S := \{(f; g) \in H(E)^2 : \exists c \in \mathbb{C} : g(z) = zf(z) + cS(z)\}.$$

Elements of M_S will often be written as $(f; zf + cS)$.

Lemma 2.2.2. *Let $H(E)$ be a de Branges space and S be an entire function. Then M_S is a linear relation.*

Proof. Let $(f; zf + c_1S), (g; zg + c_2S) \in M_S$ and $\lambda \in \mathbb{C}$ then

$$(f + \lambda g; zf + c_1S + \lambda(zg + c_2S)) = (f + \lambda g; z(f + \lambda g) + (c_1 + \lambda c_2)S)$$

Because $H(E)$ is a linear space, this pair belongs to M_S. □

Lemma 2.2.3. *Let $H(E)$ be a de Branges space, S be an entire function and $\alpha \in \mathbb{C}$. If $S(\alpha) \neq 0$ then $\ker(M_S - \alpha) = \{0\}$. In particular, $(M_S - \alpha)^{-1} : \operatorname{ran}(M_S - \alpha) \to H(E)$ is an operator.*

Proof. The linear relation $(M_S - \alpha)$ is

$$(M_S - \alpha) = \{(g(z); zg(z) + cS(z) - \alpha g(z)) \in H(E)^2 : c \in \mathbb{C}\}$$

Hence, $g \in \ker(M_S - \alpha)$ if $g \in H(E)$ and there exists a $c \in \mathbb{C}$ such that

$$zg(z) + cS(z) - \alpha g(z) = 0.$$

This is the equivalent to

$$g(z) = -\frac{cS(z)}{z - \alpha}.$$

Because g belongs to $H(E)$ and, thus, is entire, the function $cS(z)$ must have a zero at α. But $S(\alpha) \neq 0$ and, therefore, the constant c has to vanish. Hence,

$$g(z) = -\frac{cS(z)}{z - \alpha} \equiv 0.$$

□

Lemma 2.2.4. *Let $H(E)$ be a de Branges space, S be an entire function and $\alpha \in \mathbb{C}$ such that $S(\alpha) \neq 0$. Then S is associated with $H(E)$ if and only if $\operatorname{ran}(M_S - \alpha) = H(E)$. In this case $(M_S - \alpha)^{-1}$ is the operator satisfying*

$$(M_S - \alpha)^{-1}(f)(z) = \frac{f(z) - \frac{f(\alpha)}{S(\alpha)}S(z)}{z - \alpha}$$

for all $f \in H(E)$.

Proof. Assume that S is associated. The linear relation $(M_S - \alpha)$ is

$$(M_S - \alpha) = \left\{ \big(g(z); zg(z) + cS(z) - \alpha g(z)\big) \in H(E)^2 : c \in \mathbb{C} \right\}.$$

For any $f \in H(E)$ the function $g(z) := \frac{f(z)S(\alpha)-f(\alpha)S(z)}{z-\alpha}$ belongs to $H(E)$ by the definition of associated functions. Therefore,

$$\begin{aligned} zg(z) - \alpha g(z) + f(\alpha)S(z) &= g(z)(z-\alpha) + f(\alpha)S(z) \\ &= \frac{f(z)S(\alpha) - f(\alpha)S(z)}{z-\alpha}(z-\alpha) + f(\alpha)S(z) \\ &= f(z)S(\alpha) - f(\alpha)S(z) + f(\alpha)S(z) \\ &= f(z)S(\alpha). \end{aligned}$$

Hence, $zg(z) - \alpha g(z) + f(\alpha)S(z) \in H(E)$ and

$$\big(g(z); zg(z) - \alpha g(z) + f(\alpha)S(z)\big) = \big(g(z); f(z)S(\alpha)\big) \in (M_S - \alpha).$$

Because $\operatorname{ran}(M_S - \alpha)$ is a linear space and $S(\alpha) \neq 0$ the function $f(z)$ belongs to $\operatorname{ran}(M_S - \alpha)$. Therefore, $\operatorname{ran}(M_S - \alpha) = H(E)$.

For the other implication let $\operatorname{ran}(M_S - \alpha) = H(E)$. Let f be any nonzero function in $H(E) = \operatorname{ran}(M_S - \alpha)$. Then there exists a function $g \in H(E)$ such that $(g; f) \in (M_S - \alpha)$. Hence,

$$f(z) = zg(z) + cS(z) - \alpha g(z).$$

In particular, $f(\alpha) = cS(\alpha)$. Because $S(\alpha) \neq 0$, $g(z)$ satisfies

$$g(z) = \frac{f(z) - \frac{f(\alpha)}{S(\alpha)}S(z)}{z-\alpha}, \tag{2.5}$$

and, hence, the function $\frac{f(z)S(\alpha)-f(\alpha)S(z)}{z-\alpha}$ belongs to $H(E)$. By Corollary 2.1.4 $E(\alpha)$ cannot vanish and by Corollary 2.1.6 there exists an $f \in H(E)$ such that $f(\alpha) \neq 0$. Therefore, S is associated with $H(E)$ by Theorem 2.1.2.

Equation (2.5) shows the desired property for the operator $(M_S - \alpha)^{-1}$. □

Theorem 2.2.5. *For a nonzero entire function S and a de Branges space $H(E)$ with $E(z) = A(z) - iB(z)$ as in Theorem 1.2.9 the following assertions are equivalent:*

(i) $S(z) = uA(z) + vB(z)$ *with* $u\overline{v} = \overline{u}v$.

(ii) S *is associated with* $H(E)$ *and the following equation holds for* $f, g \in H(E)$ *and* $\alpha, \beta \in \mathbb{C}\backslash\mathbb{R}$ *with* $S(\alpha), S(\beta) \neq 0$:

$$\begin{aligned} 0 = &\left\langle f(t)S(\alpha), \frac{g(t)S(\beta) - S(t)g(\beta)}{t-\beta} \right\rangle - \left\langle \frac{f(t)S(\alpha) - S(t)f(\alpha)}{t-\alpha}, g(t)S(\beta) \right\rangle \\ &+ (\alpha - \overline{\beta}) \left\langle \frac{f(t)S(\alpha) - S(t)f(\alpha)}{t-\alpha}, \frac{g(t)S(\beta) - S(t)g(\beta)}{t-\beta} \right\rangle \end{aligned} \tag{2.6}$$

(iii) For some $w \in \mathbb{C} \backslash \mathbb{R}$ and $z \in \mathbb{C}$

$$\frac{K(w,z)S(w) - K(w,w)S(z)}{z-w} = \frac{K(\overline{w},z)S(\overline{w}) - K(\overline{w},\overline{w})S(z)}{z-\overline{w}}$$

and $|S(w)|^2 = |S(\overline{w})|^2$.

(iv) M_S is selfadjoint.

Proof. (i)⇔(iv): Assume that M_S is selfadjoint. Then

$$\left((M_S - w)^{-1}\right)^* = (M_S - \overline{w})^{-1}$$

and, hence,

$$\left\langle (M_S - w)^{-1}(K(w,z)), K(\beta,z)\right\rangle = \left\langle K(w,z), (M_S - \overline{w})^{-1}(K(\beta,z))\right\rangle$$

holds true for $\beta, w \in \mathbb{C}$, $S(w) \neq 0$ and $S(\overline{w}) \neq 0$. By Lemma 2.2.4 this is equivalent to

$$\left\langle \frac{K(w,z) - \frac{K(w,w)}{S(w)}S(z)}{z-w}, K(\beta,z)\right\rangle = \left\langle K(w,z), \frac{K(\beta,z) - \frac{K(\beta,\overline{w})}{S(\overline{w})}S(z)}{z-\overline{w}}\right\rangle$$

Hence, by Theorem 1.2.10,

$$\frac{K(w,\beta) - \frac{K(w,w)}{S(w)}S(\beta)}{\beta - w} = \frac{K(w,\beta) - \frac{K(\overline{w},\beta)}{S(\overline{w})}\overline{S(w)}}{\overline{w} - w}.$$

The definition of $K(w,z)$ in Theorem 1.2.9,

$$K(w,z) = \frac{B(z)\overline{A(w)} - A(z)\overline{B(w)}}{\pi(z-\overline{w})},$$

leads to

$$\frac{\frac{B(\beta)\overline{A(w)} - A(\beta)\overline{B(w)}}{\pi(\beta-\overline{w})} - \frac{\frac{B(w)\overline{A(w)} - A(w)\overline{B(w)}}{\pi(w-\overline{w})}}{S(w)}S(\beta)}{\beta - w} = \frac{\frac{B(\beta)\overline{A(w)} - A(\beta)\overline{B(w)}}{\pi(\beta-\overline{w})} - \frac{\frac{B(\beta)\overline{A(\overline{w})} - A(\beta)\overline{B(\overline{w})}}{\pi(\beta-w)}}{S(\overline{w})}\overline{S(w)}}{\overline{w} - w}.$$

Multiplication with $\pi(\beta - \overline{w})(w - \overline{w})(\beta - w)S(w)\overline{S(\overline{w})}$ yields

$$\begin{aligned}&\overline{S(\overline{w})}S(w)(w-\overline{w})(B(\beta)\overline{A(w)} - A(\beta)\overline{B(w)}) - \overline{S(\overline{w})}(\beta - \overline{w})(B(w)\overline{A(w)} - A(w)\overline{B(w)})S(\beta) = \\ &= -S(w)\overline{S(\overline{w})}(\beta - w)(B(\beta)\overline{A(w)} - A(\beta)\overline{B(w)}) + S(w)(\beta-\overline{w})(B(\beta)\overline{A(\overline{w})} - A(\beta)\overline{B(\overline{w})})\overline{S(w)}.\end{aligned}$$

This is equivalent to

$$\begin{aligned}&\overline{S(\overline{w})}S(w)(\beta-\overline{w})(B(\beta)\overline{A(w)} - A(\beta)\overline{B(w)}) - \overline{S(\overline{w})}(\beta - \overline{w})(B(w)\overline{A(w)} - A(w)\overline{B(w)})S(\beta) = \\ &\qquad = S(w)(\beta-\overline{w})(B(\beta)\overline{A(\overline{w})} - A(\beta)\overline{B(\overline{w})})\overline{S(w)}.\end{aligned}$$

For $\beta \neq \overline{w}$

$$\begin{aligned}\overline{S(\overline{w})}S(w)(B(\beta)\overline{A(w)} - A(\beta)\overline{B(w)}) - \overline{S(\overline{w})}(B(w)\overline{A(w)} - A(w)\overline{B(w)})S(\beta) = \\ = S(w)(B(\beta)\overline{A(\overline{w})} - A(\beta)\overline{B(\overline{w})})\overline{S(w)}\end{aligned}$$

follows. Hence, for $w \in \mathbb{C}$ with $(B(w)\overline{A(w)} - A(w)\overline{B(w)}) \neq 0$ and $\overline{S(\overline{w})} \neq 0$

$$S(\beta) = \frac{\overline{S(\overline{w})}S(w)(B(\beta)\overline{A(w)} - A(\beta)\overline{B(w)}) - S(w)(B(\beta)\overline{A(\overline{w})} - A(\beta)\overline{B(\overline{w})})\overline{S(w)}}{\overline{S(\overline{w})}(B(w)\overline{A(w)} - A(w)\overline{B(w)})}. \tag{2.7}$$

Because for all $w \in \mathbb{C}\backslash\mathbb{R}$ the value $K(w, w) \neq 0$ by Corollary 2.1.7 and the definition of $K(w, z)$

$$(B(w)\overline{A(w)} - A(w)\overline{B(w)}) \neq 0$$

for $w \in \mathbb{C}\backslash\mathbb{R}$. If S does not vanish identically, there has to be a $w \in \mathbb{C}\backslash\mathbb{R}$ satisfying $\overline{S(\overline{w})} \neq 0$. Hence, there exists a $w \in \mathbb{C}$ such that (2.7) holds for all $\beta \in \mathbb{C}\backslash\{\overline{w}\}$. Because both sides are entire functions the equation holds throughout $\mathbb{C}$. The right hand side is a linear combination of A and B. Therefore, $S(z) = uA(z) + vB(z)$ for some $u, v \in \mathbb{C}$.

It remains to show that with $S(z) = uA(z) + vB(z)$ the linear relation M_S is selfadjoint if and only if $u\overline{v} \in \mathbb{R}$.

The linear relation M_S is selfadjoint if and only if $\left((M_S - w)^{-1}\right)^* = (M_S - \overline{w})^{-1}$ for some $w \in \mathbb{C}$ with $S(w), S(\overline{w}) \neq 0$. Because the linear span of all functions $K(w, z)$ is dense in $H(E)$, this is equivalent to

$$\left\langle (M_S - w)^{-1}(K(\alpha, z)), K(\beta, z)\right\rangle = \left\langle K(\alpha, z), (M_S - \overline{w})^{-1}(K(\beta, z))\right\rangle \tag{2.8}$$

for all $\alpha, \beta \in \mathbb{C}$. By Example 2.1.5 the function S is associated. Hence, the characterisation for $(M_S - w)^{-1}$ in Lemma 2.2.4 can be used. Therefore, (2.8) is equivalent to

$$\left\langle \frac{K(\alpha, z) - \frac{K(\alpha,w)}{S(w)}S(z)}{z - w}, K(\beta, z)\right\rangle = \left\langle K(\alpha, z), \frac{K(\beta, z) - \frac{K(\beta,\overline{w})}{S(\overline{w})}S(z)}{z - \overline{w}}\right\rangle.$$

By the property of $K(w, z)$ in Theorem 1.2.10 this equation holds if and only if

$$\frac{K(\alpha, \beta) - \frac{K(\alpha,w)}{S(w)}S(\beta)}{\beta - w} = \frac{K(\alpha, \beta) - \frac{K(\overline{w},\beta)}{\overline{S(\overline{w})}}\overline{S(\alpha)}}{\overline{\alpha} - w}$$

and, further, if and only if

$$(\overline{\alpha} - \beta)K(\alpha, \beta) - \frac{(\overline{\alpha} - w)K(\alpha, w)}{S(w)}S(\beta) = -\frac{(\beta - w)K(\overline{w}, \beta)}{\overline{S(\overline{w})}}\overline{S(\alpha)}.$$

With $S(z) = uA(z) + vB(z)$ and $K(w,z) = \frac{B(z)\overline{A(w)}-A(z)\overline{B(w)}}{\pi(z-\overline{w})}$ this is equivalent to

$$(-B(\beta)\overline{A(\alpha)} + A(\beta)\overline{B(\alpha)}) - \frac{-B(w)\overline{A(\alpha)} + A(w)\overline{B(\alpha)}}{uA(w) + vB(w)}(uA(\beta) + vB(\beta)) =$$
$$= \frac{-B(\beta)\overline{A(\overline{w})} + A(\beta)\overline{B(\overline{w})}}{\overline{u}A(w) + \overline{v}B(w)}(\overline{u}\overline{A(\alpha)} + \overline{v}\overline{B(\alpha)}).$$

This can be rewritten as

$$\frac{(-B(\beta)\overline{A(\alpha)} + A(\beta)\overline{B(\alpha)})(uA(w) + vB(w)) + (B(w)\overline{A(\alpha)} - A(w)\overline{B(\alpha)})(uA(\beta) + vB(\beta))}{uA(w) + vB(w)} =$$
$$= \frac{(-B(\beta)A(w) + A(\beta)B(w))(\overline{u}\overline{A(\alpha)} + \overline{v}\overline{B(\alpha)})}{\overline{u}A(w) + \overline{v}B(w)}.$$

Computing the left hand side leads to

$$\frac{-B(\beta)\overline{A(\alpha)}uA(w) + A(\beta)\overline{B(\alpha)}vB(w) + B(w)\overline{A(\alpha)}uA(\beta) - A(w)\overline{B(\alpha)}vB(\beta)}{uA(w) + vB(w)} =$$
$$= \frac{(-B(\beta)A(w) + A(\beta)B(w))(\overline{u}\overline{A(\alpha)} + \overline{v}\overline{B(\alpha)})}{\overline{u}A(w) + \overline{v}B(w)}.$$

This is obviously equivalent to

$$\frac{(\overline{A(\alpha)}u + \overline{B(\alpha)}v)(-B(\beta)A(w) + B(w)A(\beta))}{uA(w) + vB(w)} = \frac{(-B(\beta)A(w) + A(\beta)B(w))(\overline{u}\overline{A(\alpha)} + \overline{v}\overline{B(\alpha)})}{\overline{u}A(w) + \overline{v}B(w)}.$$

For $(-B(\beta)A(w) + B(w)A(\beta)) \neq 0$ this is the same as

$$\frac{(\overline{A(\alpha)}u + \overline{B(\alpha)}v)}{uA(w) + vB(w)} = \frac{(\overline{u}\overline{A(\alpha)} + \overline{v}\overline{B(\alpha)})}{\overline{u}A(w) + \overline{v}B(w)}. \tag{2.9}$$

Because $S(w), \overline{S(\overline{w})} \neq 0$ this is equivalent to

$$\overline{B(\alpha)}v\overline{u}A(w) + \overline{A(\alpha)}u\overline{v}B(w) = \;= \overline{v}\overline{B(\alpha)}uA(w) + \overline{u}\overline{A(\alpha)}vB(w),$$

and, finally, to

$$(u\overline{v} - v\overline{u})(\overline{A(\alpha)}B(w) - \overline{B(\alpha)}A(w)) = 0.$$

Hence, $S(z) = uA(z) + vB(z)$ is selfadjoint if and only if $u\overline{v} \in \mathbb{R}$.

(iv) and (i) $\Rightarrow$(iii): With $S(z) = uA(z)+vB(z)$ equation (2.9) follows. Choosing $\alpha = w$ it reads

$$\frac{(\overline{A(w)}u + \overline{B(w)}v)}{uA(w) + vB(w)} = \frac{(\overline{u}\overline{A(w)} + \overline{v}\overline{B(w)})}{\overline{u}A(w) + \overline{v}B(w)}.$$

The definition of S leads to

$$\frac{S(\overline{w})}{S(w)} = \frac{\overline{S(w)}}{\overline{S(\overline{w})}}$$

and, further, to

$$\frac{\overline{S(w)}}{S(\overline{w})} = \frac{\overline{S(\overline{w})}}{S(w)}.$$

With this one gets

$$\frac{\overline{S(w)}}{S(\overline{w})} \frac{f(w)S(\overline{w}) - f(\overline{w})S(w)}{w - \overline{w}} = \frac{\overline{S(\overline{w})}}{S(w)} \frac{f(\overline{w})S(w) - f(w)S(\overline{w})}{\overline{w} - w}, \qquad f \in H(E).$$

Hence,

$$\overline{S(w)} \frac{f(w) - \frac{f(\overline{w})}{S(\overline{w})}S(w)}{w - \overline{w}} = \overline{S(\overline{w})} \frac{f(\overline{w}) - \frac{f(w)}{S(w)}S(\overline{w})}{\overline{w} - w}, \qquad f \in H(E).$$

Because $(M_S - \overline{w})^{-1}$ has to be bijective by assumption, Lemma 2.2.4 leads to

$$\overline{S(w)} \left((M_S - \overline{w})^{-1}\right)(f)(w) = \overline{S(\overline{w})} \left((M_S - w)^{-1}\right)(f)(\overline{w}), \qquad f \in H(E).$$

By the property of $K(w, z)$ in Theorem 1.2.10 this is the same as

$$\overline{S(w)} \left\langle \left((M_S - \overline{w})^{-1}\right)(f)(z), K(w, z) \right\rangle = \overline{S(\overline{w})} \left\langle \left((M_S - w)^{-1}\right)(f)(z), K(\overline{w}, z) \right\rangle, \qquad f \in H(E). \tag{2.10}$$

By (iv) the linear relation M_S is selfadjoint. Therefore,

$$\left((M_S - w)^{-1}\right)^* = \left((M_S^* - \overline{w})^{-1}\right) = \left((M_S - \overline{w})^{-1}\right).$$

Hence, (2.10) is equivalent to

$$\left\langle f(z), S(w) \left((M_S - w)^{-1}\right)(K(w, .))(z) \right\rangle = \left\langle f(z), S(\overline{w}) \left((M_S - \overline{w})^{-1}\right)(K(\overline{w}, .))(z) \right\rangle \quad \forall f \in H(E).$$

This yields

$$S(w) \left((M_S - w)^{-1}\right)(K(w, .))(z) = S(\overline{w}) \left((M_S - \overline{w})^{-1}\right)(K(\overline{w}, .))(z)$$

and again with Lemma 2.2.4 the desired equation

$$\frac{K(w, z)S(w) - K(w, w)S(z)}{z - w} = \frac{K(\overline{w}, z)S(\overline{w}) - K(\overline{w}, \overline{w})S(z)}{z - \overline{w}}.$$

(iii)⇒(i): Let w be as in the assumption. By definition in 1.2.9

$$K(w, z) := \frac{B(z)\overline{A(w)} - A(z)\overline{B(w)}}{\pi(z - \overline{w})}$$

Hence, assumption (iii) is equivalent to

$$\frac{(B(z)A(\overline{w}) - A(z)B(\overline{w}))S(w)}{\pi(z - \overline{w})(z - w)} - \frac{(B(w)A(\overline{w}) - A(w)B(\overline{w}))S(z)}{\pi(w - \overline{w})(z - w)} =$$
$$= \frac{(B(z)A(w) - A(z)B(w))S(\overline{w})}{\pi(z - w)(z - \overline{w})} - \frac{(B(\overline{w})A(w) - A(\overline{w})B(w))S(z)}{\pi(\overline{w} - w)(z - \overline{w})}.$$

This can be rewritten as

$$\begin{aligned}
&S(z)\frac{A(w)B(\overline{w})-B(w)A(\overline{w})}{\pi(w-\overline{w})}\left(\frac{1}{z-w}-\frac{1}{z-\overline{w}}\right)=\\
&\qquad=\frac{A(w)B(z)S(\overline{w})-A(z)B(w)S(\overline{w})-A(\overline{w})B(z)S(w)+A(z)B(\overline{w})S(w)}{\pi(z-\overline{w})(z-w)}
\end{aligned}$$

and, further,

$$\begin{aligned}
&S(z)\frac{A(w)B(\overline{w})-B(w)A(\overline{w})}{\pi(w-\overline{w})(z-w)(z-\overline{w})}(z-\overline{w}-z+w)=\\
&\qquad=\frac{A(w)B(z)S(\overline{w})-A(z)B(w)S(\overline{w})-A(\overline{w})B(z)S(w)+A(z)B(\overline{w})S(w)}{\pi(z-\overline{w})(z-w)}.
\end{aligned}$$

Multiplying with $\pi(z-\overline{w})(z-w)$ yields that this is equivalent to

$$\begin{aligned}
&S(z)\big(A(w)B(\overline{w})-B(w)A(\overline{w})\big)=\\
&\qquad=A(w)B(z)S(\overline{w})-A(z)B(w)S(\overline{w})-A(\overline{w})B(z)S(w)+A(z)B(\overline{w})S(w).
\end{aligned}$$

Hence, $S(z)$ calculates as

$$S(z)=\frac{A(w)B(z)S(\overline{w})-A(z)B(w)S(\overline{w})-A(\overline{w})B(z)S(w)+A(z)B(\overline{w})S(w)}{A(w)B(\overline{w})-B(w)A(\overline{w})}$$

or, equivalently,

$$S(z)=A(z)\frac{-B(w)S(\overline{w})+B(\overline{w})S(w)}{A(w)B(\overline{w})-B(w)A(\overline{w})}+B(z)\frac{A(w)S(\overline{w})-A(\overline{w})S(w)}{A(w)B(\overline{w})-B(w)A(\overline{w})}.$$

It remains to show that $u\overline{v}=\overline{u}v$ with

$$u=\frac{-B(w)S(\overline{w})+B(\overline{w})S(w)}{A(w)B(\overline{w})-B(w)A(\overline{w})}$$

and

$$v=\frac{A(w)S(\overline{w})-A(\overline{w})S(w)}{A(w)B(\overline{w})-B(w)A(\overline{w})}.$$

Because $\overline{A(w)}=A(\overline{w})$, $\overline{B(w)}=B(\overline{w})$ and $S(w)\overline{S(w)}=S(\overline{w})\overline{S(\overline{w})}$

$$\begin{aligned}
u\overline{v}&=\frac{-B(w)S(\overline{w})+B(\overline{w})S(w)}{A(w)B(\overline{w})-B(w)A(\overline{w})}\cdot\frac{A(\overline{w})\overline{S(\overline{w})}-A(w)\overline{S(w)}}{A(\overline{w})B(w)-B(\overline{w})A(w)}\\
&=\frac{B(w)S(\overline{w})A(\overline{w})\overline{S(\overline{w})}-B(w)S(\overline{w})A(w)\overline{S(w)}-B(\overline{w})S(w)A(\overline{w})\overline{S(\overline{w})}+B(\overline{w})S(w)A(w)\overline{S(w)}}{(A(w)B(\overline{w})-B(w)A(\overline{w}))^2}\\
&=\frac{B(w)S(w)A(\overline{w})\overline{S(w)}-B(w)S(\overline{w})A(w)\overline{S(w)}-B(\overline{w})S(w)A(\overline{w})\overline{S(\overline{w})}+B(\overline{w})S(\overline{w})A(w)\overline{S(\overline{w})}}{(A(w)B(\overline{w})-B(w)A(\overline{w}))^2}\\
&=\frac{-B(\overline{w})\overline{S(\overline{w})}+B(w)\overline{S(w)}}{A(\overline{w})B(w)-B(\overline{w})A(w)}\cdot\frac{A(w)S(\overline{w})-A(\overline{w})S(w)}{A(w)B(\overline{w})-B(w)A(\overline{w})}\\
&=\overline{u}v.
\end{aligned}$$

(ii)⇒(iv): Due to Lemma 2.2.4 for $S(\alpha), S(\beta) \neq 0$ equation (2.6) reads as

$$0 = \left\langle f(t)S(\alpha), \left((M_S - \beta)^{-1}\right)(g)(t)S(\beta)\right\rangle - \left\langle \left((M_S - \alpha)^{-1}\right)(f)(t)S(\alpha), g(t)S(\beta)\right\rangle + (\alpha - \overline{\beta})\left\langle \left((M_S - \alpha)^{-1}\right)(f)(t)S(\alpha), \left((M_S - \beta)^{-1}\right)(g)(t)S(\beta)\right\rangle .$$

With $\tilde{f}(t) := f(t)S(\alpha)$ and $\tilde{g}(t) := g(t)S(\beta)$ this yields

$$0 = \left\langle \tilde{f}(t), \left((M_S - \beta)^{-1}\right)(\tilde{g})(t)\right\rangle - \left\langle \left((M_S - \alpha)^{-1}\right)(\tilde{f})(t), \tilde{g}(t)\right\rangle + (\alpha - \overline{\beta})\left\langle \left((M_S - \alpha)^{-1}\right)(\tilde{f})(t), \left((M_S - \beta)^{-1}\right)(\tilde{g})(t)\right\rangle .$$

For $\alpha = \overline{\beta}$ one gets

$$0 = \left\langle \tilde{f}(t), \left((M_S - \beta)^{-1}\right)(\tilde{g})(t)\right\rangle - \left\langle \left((M_S - \overline{\beta})^{-1}\right)(\tilde{f})(t), \tilde{g}(t)\right\rangle .$$

This is true for all $\tilde{f}, \tilde{g} \in H(E)$. Because $(M_S - \beta)^{-1}$ is an operator by Lemma 2.2.4

$$(M_S - \overline{\beta})^{-1} = ((M_S - \beta)^{-1})^*.$$

This yields

$$(M_S - \overline{\beta})^{-1} = (M_S^* - \overline{\beta})^{-1}$$

and, finally,

$$M_S = M_S^*.$$

(iv) and (i) ⇒ (ii): By Example 2.1.5 $S(z) = uA(z) + vB(z)$ is associated. It remains to show the equality

$$0 = \left\langle f(t)S(\alpha), \frac{g(t)S(\beta) - S(t)g(\beta)}{t - \beta}\right\rangle - \left\langle \frac{f(t)S(\alpha) - S(t)f(\alpha)}{t - \alpha}, g(t)S(\beta)\right\rangle + (\alpha - \overline{\beta})\left\langle \frac{f(t)S(\alpha) - S(t)f(\alpha)}{t - \alpha}, \frac{g(t)S(\beta) - S(t)g(\beta)}{t - \beta}\right\rangle .$$

Since $S(\alpha), S(\beta) \neq 0$, this is equivalent to

$$0 = \left\langle f(t), \frac{g(t) - \frac{g(\beta)}{S(\beta)}S(t)}{t - \beta}\right\rangle - \left\langle \frac{f(t) - \frac{f(\alpha)}{S(\alpha)}S(t)}{t - \alpha}, g(t)\right\rangle + (\alpha - \overline{\beta})\left\langle \frac{f(t) - \frac{f(\alpha)}{S(\alpha)}S(t)}{t - \alpha}, \frac{g(t) - \frac{g(\beta)}{S(\beta)}S(t)}{t - \beta}\right\rangle .$$

By the characterisation of the operator $(M_S - w)^{-1}$ in Lemma 2.2.4 this can be rewritten as

$$0 = \left\langle f(t), (M_S - \beta)^{-1}(g)(t)\right\rangle - \left\langle (M_S - \alpha)^{-1}(f)(t), g(t)\right\rangle + (\alpha - \overline{\beta})\left\langle (M_S - \alpha)^{-1}(f)(t), (M_S - \beta)^{-1}(g)(t)\right\rangle .$$

Using adjoint operators this is equivalent to

$$\begin{aligned}0 = &\left\langle f(t), (M_S-\beta)^{-1}(g)(t)\right\rangle - \left\langle f(t), \left((M_S-\alpha)^{-1}\right)^* (g)(t)\right\rangle \\ &+ (\alpha-\overline{\beta})\left\langle f(t), \left((M_S-\alpha)^{-1}\right)^* (M_S-\beta)^{-1}(g)(t)\right\rangle.\end{aligned}$$

Since this equation has to hold for all f, g belonging to $H(E)$, it is equivalent to the operator equation

$$(M_S-\beta)^{-1} - \left((M_S-\alpha)^{-1}\right)^* + (\overline{\alpha}-\beta)\left((M_S-\alpha)^{-1}\right)^*(M_S-\beta)^{-1} = 0.$$

Because M_S is selfadjoint this operator is equal to

$$(M_S-\beta)^{-1} - (M_S-\overline{\alpha})^{-1} + (\overline{\alpha}-\beta)(M_S-\overline{\alpha})^{-1}(M_S-\beta)^{-1}.$$

Thus, it vanishes by the resolvent identity. □

Definition 2.2.6. *Let $H(E)$ be a de Branges space and S be an entire function. S is called strongly associated with $H(E)$ if it satisfies the equivalent conditions in Theorem 2.2.5.*

Theorem 2.2.7. *Let $H(E)$ be a de Branges space and let Q be associated with $H(E)$. Further, let μ be a Borel measure on $\mathbb{R}$ with $Q = 0$ μ-almost everywhere and such that $H(E)$ is isometrically contained in $L^2(\mu)$. Then Q is strongly associated, $H(E)$ fills $L^2(\mu)$, $\mu((a,b)) = 0$ if Q has no zeros in (a,b) and $\mu(\{t\}) = \frac{1}{K(t,t)}$ if $Q(t) = 0$.*

Proof. By Theorem 2.2.5 the function Q is strongly associated if

$$\begin{aligned}0 = &\left\langle f(t)Q(\alpha), \frac{g(t)Q(\beta)-Q(t)g(\beta)}{t-\beta}\right\rangle_{H(E)} - \left\langle \frac{f(t)Q(\alpha)-Q(t)f(\alpha)}{t-\alpha}, g(t)Q(\beta)\right\rangle_{H(E)} \\ &+ (\alpha-\overline{\beta})\left\langle \frac{f(t)Q(\alpha)-Q(t)f(\alpha)}{t-\alpha}, \frac{g(t)Q(\beta)-Q(t)g(\beta)}{t-\beta}\right\rangle_{H(E)}.\end{aligned}$$

Because $H(E)$ is isometrically contained in $L^2(\mu)$ this is equivalent to

$$\begin{aligned}0 = &\left\langle f(t)Q(\alpha), \frac{g(t)Q(\beta)-Q(t)g(\beta)}{t-\beta}\right\rangle_{L^2(\mu)} - \left\langle \frac{f(t)Q(\alpha)-Q(t)f(\alpha)}{t-\alpha}, g(t)Q(\beta)\right\rangle_{L^2(\mu)} \\ &+ (\alpha-\overline{\beta})\left\langle \frac{f(t)Q(\alpha)-Q(t)f(\alpha)}{t-\alpha}, \frac{g(t)Q(\beta)-Q(t)g(\beta)}{t-\beta}\right\rangle_{L^2(\mu)}.\end{aligned}$$

As $Q = 0$ μ-almost everywhere the right hand side calculates as

$$\begin{aligned}
&\left\langle f(t)Q(\alpha), \frac{g(t)Q(\beta)}{t-\beta}\right\rangle_{L^2(\mu)} - \left\langle \frac{f(t)Q(\alpha)}{t-\alpha}, g(t)Q(\beta)\right\rangle_{L^2(\mu)} \\
&+ (\alpha - \overline{\beta})\left\langle \frac{f(t)Q(\alpha)}{t-\alpha}, \frac{g(t)Q(\beta)}{t-\beta}\right\rangle_{L^2(\mu)} \\
&= \int_{\mathbb{R}} \frac{f(t)Q(\alpha)\overline{g(t)Q(\beta)}}{t-\overline{\beta}}\,d\mu(t) - \int_{\mathbb{R}} \frac{f(t)Q(\alpha)\overline{g(t)Q(\beta)}}{t-\alpha}\,d\mu(t) + (\alpha-\overline{\beta})\int_{\mathbb{R}} \frac{f(t)Q(\alpha)\overline{g(t)Q(\beta)}}{(t-\alpha)(t-\overline{\beta})}\,d\mu(t) \\
&= \int_{\mathbb{R}} \frac{(t-\alpha)f(t)Q(\alpha)\overline{g(t)Q(\beta)} - (t-\overline{\beta})f(t)Q(\alpha)\overline{g(t)Q(\beta)} + (\alpha-\overline{\beta})f(t)Q(\alpha)\overline{g(t)Q(\beta)}}{(t-\alpha)(t-\overline{\beta})}\,d\mu(t) \\
&= 0.
\end{aligned}$$

Hence, Q is strongly associated. The property that $\mu((a,b)) = 0$ if Q has no zeros on (a,b) is obviously true since $Q = 0$ μ-almost everywhere. Let $(t_n)_{n=1}^N$ with $N \in \mathbb{N} \cup \{\infty\}$ be the set of real zeros of $Q(z)$. Then

$$K(t_j,t_j) = \langle K(t_j,.), K(t_j,.)\rangle_{H(E)} = \langle K(t_j,.), K(t_j,.)\rangle_{L^2(\mu)} = \sum_{i=1}^N \mu(\{t_i\})\ |K(t_j,t_i)|^2. \tag{2.11}$$

Because Q is strongly associated, $Q(z) = uA(z) + vB(z)$ with $u\overline{v} = \overline{u}v$. Hence, the zeros t_i of Q satisfy $A(t_i) = -\frac{v}{u}B(t_i)$. With the representation $K(w,z) = \frac{B(z)\overline{A(w)} - A(z)\overline{B(w)}}{\pi(z-\overline{w})}$ and the fact that $A(z)$ and $B(z)$ are real for real z one gets

$$(t_j - t_i)\pi K(t_i,t_j) = B(t_j)\overline{A(t_i)} - A(t_j)\overline{B(t_i)} = -B(t_j)\frac{v}{u}B(t_i) + \frac{v}{u}B(t_j)B(t_i) = 0$$

for any $i \neq j$. Hence, in (2.11)

$$K(t_j,t_j) = \mu(\{t_j\})\ |K(t_j,t_j)|^2.$$

This yields the desired identity $\mu(\{t_j\}) = \frac{1}{K(t_j,t_j)}$. It remains to show that $H(E)$ fills $L^2(\mu)$. Assume that $f \in L^2(\mu) \ominus H(E)$. Then

$$\begin{aligned}
0 = \langle f, K(t_j,.)\rangle_{L^2(\mu)} &= \int_{\mathbb{R}} f(t)\overline{K(t_j,t)}\,d\mu(t) = \\
&= \sum_{i=1}^N f(t_i)K(t_j,t_i)\mu(\{t_i\}) = f(t_j)K(t_j,t_j)\frac{1}{K(t_j,t_j)} = f(t_j).
\end{aligned}$$

Hence, $f = 0$ μ-almost everywhere. This means $f = 0$ in $L^2(\mu)$. ☐

Theorem 2.2.8. *Let $H(E)$ be a de Branges space. A function $S \in H(E)$ is strongly associated if and only if it is orthogonal to the domain of multiplication by the independent variable z.*

Proof. Let S be strongly associated and let f be in the domain of multiplication by z. Because $(z-w)f(z) \in H(E)$ with a zero at w one gets for any $w \in \mathbb{C}\backslash\mathbb{R}$

$$\begin{aligned}\langle (t-w)f(t), S(t)\rangle K(w,w) &= \langle (t-w)f(t), S(t)K(w,w)\rangle - \langle (t-w)f(t), K(w,t)S(w)\rangle \\ &= \langle (t-w)f(t), S(t)K(w,w) - K(w,t)S(w)\rangle .\end{aligned}$$

With property (iii) in Theorem 2.2.5

$$\frac{S(z)K(w,w) - K(w,z)S(w)}{z-w} = \frac{S(z)K(\overline{w},\overline{w}) - K(\overline{w},z)S(\overline{w})}{z-\overline{w}}$$

and, hence,

$$\begin{aligned}\langle (t-w)f(t), S(t)\rangle K(w,w) &= \langle (t-w)f(t), S(t)K(w,w) - K(w,t)S(w)\rangle \\ &= \left\langle (t-w)f(t), (S(t)K(\overline{w},\overline{w}) - K(\overline{w},t)S(\overline{w}))\frac{t-w}{t-\overline{w}}\right\rangle\end{aligned}$$

Because multiplication with $\frac{t-w}{t-\overline{w}}$ is an isometry

$$\begin{aligned}\langle (t-w)f(t), S(t)\rangle K(w,w) &= \langle (t-\overline{w})f(t), S(t)K(\overline{w},\overline{w}) - K(\overline{w},t)S(\overline{w})\rangle \\ &= \langle (t-\overline{w})f(t), S(t)\rangle K(\overline{w},\overline{w}).\end{aligned}$$

Since $K(w,w) = K(\overline{w},\overline{w}) \neq 0$,

$$\langle wf(t), S(t)\rangle = \langle \overline{w}f(t), S(t)\rangle .$$

This is only possible if S is orthogonal to f.

For the other direction let S be orthogonal to the domain of multiplication. Obviously, any function in $H(E)$ is associated with $H(E)$. To show that S is strongly associated it remains to prove the equality

$$\begin{aligned}0 = \left\langle f(t)S(\alpha), \frac{g(t)S(\beta) - S(t)g(\beta)}{t-\beta}\right\rangle - \left\langle \frac{f(t)S(\alpha) - S(t)f(\alpha)}{t-\alpha}, g(t)S(\beta)\right\rangle \\ + (\alpha - \overline{\beta})\left\langle \frac{f(t)S(\alpha) - S(t)f(\alpha)}{t-\alpha}, \frac{g(t)S(\beta) - S(t)g(\beta)}{t-\beta}\right\rangle\end{aligned}$$

for any $f, g \in H(E)$ and $\alpha, \beta \in \mathbb{C}$. Because with S and g the linear combination $g(z)S(\beta) - S(z)g(\beta)$ belongs to $H(E)$, the ratio $\frac{g(z)S(\beta)-S(z)g(\beta)}{z-\beta}$ belongs to the domain of multiplication by $z-\beta$ and, hence, to the domain of multiplication by z. The same applies for $\frac{f(z)S(\alpha)-S(z)f(\alpha)}{z-\alpha}$. Hence,

$$\begin{aligned}
&\left\langle f(t)S(\alpha), \frac{g(t)S(\beta)-S(t)g(\beta)}{t-\beta}\right\rangle - \left\langle \frac{f(t)S(\alpha)-S(t)f(\alpha)}{t-\alpha}, g(t)S(\beta)\right\rangle \\
&+(\alpha-\overline{\beta})\left\langle \frac{f(t)S(\alpha)-S(t)f(\alpha)}{t-\alpha}, \frac{g(t)S(\beta)-S(t)g(\beta)}{t-\beta}\right\rangle \\
&=\left\langle f(t)S(\alpha)-S(t)f(\alpha), \frac{g(t)S(\beta)-S(t)g(\beta)}{t-\beta}\right\rangle - \left\langle \frac{f(t)S(\alpha)-S(t)f(\alpha)}{t-\alpha}, g(t)S(\beta)-S(t)g(\beta)\right\rangle \\
&+(\alpha-\overline{\beta})\left\langle \frac{f(t)S(\alpha)-S(t)f(\alpha)}{t-\alpha}, \frac{g(t)S(\beta)-S(t)g(\beta)}{t-\beta}\right\rangle \\
&=-\left\langle \frac{f(t)S(\alpha)-S(t)f(\alpha)}{t-\alpha}, g(t)S(\beta)-S(t)g(\beta)\right\rangle \\
&+\left\langle \frac{(t-\alpha)(f(t)S(\alpha)-S(t)f(\alpha))+(\alpha-\overline{\beta})(f(t)S(\alpha)-S(t)f(\alpha))}{t-\alpha}, \frac{g(t)S(\beta)-S(t)g(\beta)}{t-\beta}\right\rangle \\
&=-\left\langle \frac{f(t)S(\alpha)-S(t)f(\alpha)}{t-\alpha}, g(t)S(\beta)-S(t)g(\beta)\right\rangle \\
&+\left\langle \frac{(t-\overline{\beta})(f(t)S(\alpha)-S(t)f(\alpha))}{t-\alpha}, \frac{g(t)S(\beta)-S(t)g(\beta)}{t-\beta}\right\rangle \\
&=-\left\langle \frac{f(t)S(\alpha)-S(t)f(\alpha)}{t-\alpha}, \frac{(t-\beta)(g(t)S(\beta)-S(t)g(\beta))+\beta(g(t)S(\beta)-S(t)g(\beta))}{t-\beta}\right\rangle \\
&+\left\langle \frac{t(f(t)S(\alpha)-S(t)f(\alpha))}{t-\alpha}, \frac{g(t)S(\beta)-S(t)g(\beta)}{t-\beta}\right\rangle \\
&=-\left\langle \frac{f(t)S(\alpha)-S(t)f(\alpha)}{t-\alpha}, \frac{t(g(t)S(\beta)-S(t)g(\beta))}{t-\beta}\right\rangle \\
&+\left\langle \frac{t(f(t)S(\alpha)-S(t)f(\alpha))}{t-\alpha}, \frac{g(t)S(\beta)-S(t)g(\beta)}{t-\beta}\right\rangle \\
&=-\int_{\mathbb{R}} \frac{f(t)S(\alpha)-S(t)f(\alpha)}{t-\alpha}\,\overline{\frac{t(g(t)S(\beta)-S(t)g(\beta))}{t-\beta}}\,\frac{1}{|E(t)|^2}\,dt \\
&+\int_{\mathbb{R}} \frac{t(f(t)S(\alpha)-S(t)f(\alpha))}{t-\alpha}\,\overline{\frac{g(t)S(\beta)-S(t)g(\beta)}{t-\beta}}\,\frac{1}{|E(t)|^2}\,dt \\
&=0.
\end{aligned}$$

□

Chapter 3

Ordering Theorem

The main result of this chapter is an Ordering Theorem for de Branges spaces: Under some minor assumptions all de Branges spaces contained in some space $L^2(\mu)$ are totally ordered with respect to inclusion. The first part of this chapter contains preliminary results that are needed for the proof. In the second part the Ordering Theorem is stated and proved.

3.1 Preliminary results

Lemma 3.1.1. *Let $g(t) \in L^2(a,b)$ where $(a,b) \subset \mathbb{R}$ is a finite interval. If*

$$\int_a^b g(t)\,dt = 0$$

and

$$G(t) = \int_a^t g(s)\,ds,$$

then

$$\pi^2 \int_a^b |G(t)|^2\,dt \leq (b-a)^2 \int_a^b |g(t)|^2\,dt.$$

Proof. Firstly, consider the case $a = 0, b = 2\pi$. It is well known that the functions $\sin(nt)$, $\cos(nt)$, $n \in \mathbb{Z}$ are a complete orthogonal set of $L^2(0,2\pi)$. With this it is easy to see that the functions $\frac{1}{\sqrt{2\pi}} e^{it(n+\frac{1}{2})}$ are a complete orthonormal set of $L^2(0,2\pi)$. Integration by parts yields

$$\int_0^{2\pi} g(t) \frac{1}{\sqrt{2\pi}} e^{it(n+\frac{1}{2})}\,dt = -\int_0^{2\pi} G(t) i \left(n + \frac{1}{2}\right) \frac{1}{\sqrt{2\pi}} e^{it(n+\frac{1}{2})}\,dt.$$

The norm of G can be calculated in terms of inner products. Hence,

$$\|G\|^2_{L^2(0,2\pi)} = \sum_{n\in\mathbb{Z}} \left| \int_0^{2\pi} G(t) \frac{1}{\sqrt{2\pi}} e^{it(n+\frac{1}{2})}\, dt \right|^2 = \sum_{n\in\mathbb{Z}} \left| \frac{i}{n+\frac{1}{2}} \int_0^{2\pi} g(t) \frac{1}{\sqrt{2\pi}} e^{it(n+\frac{1}{2})}\, dt \right|^2 \leq 4\|g\|^2_{L^2(0,2\pi)}.$$

This proves the special case. A simple substitution leads to the general case $a, b \in \mathbb{R}$. $\square$

Lemma 3.1.2. *Let $f(z)$ be a continuous, nonnegative, subharmonic function on $\mathbb{C}$ which is periodic of period $2\pi i$. Then there exist nonnegative, subharmonic functions f_n, such that*

(i) $f_n \overset{n\to\infty}{\longrightarrow} f$ *converges uniformly on every compact set.*

(ii) $f_n(x+iy)$ *has continuous partial second derivatives with respect to x and y.*

(iii) $f_n(u+iv) = 0$ *if and only if f vanishes almost everywhere in the square*

$$u - \frac{2}{n} \leq x \leq u + \frac{2}{n}, v - \frac{2}{n} \leq y \leq v + \frac{2}{n}.$$

Proof. For any subharmonic, continuous, nonnegative, function g on $\mathbb{C}$ which is periodic of period $2\pi i$ the following functions can be defined

$$g_n(z) := n^2 \int_{-\frac{1}{n}}^{\frac{1}{n}} g(z+t)|t|\, dt.$$

Obviously, these functions are continuous, nonnegative, of period $2\pi i$ and are zero at $u+iv$ if and only if $g(x+iv)$ vanishes almost everywhere for $u - \frac{1}{n} \leq x \leq u + \frac{1}{n}$, because g is nonnegative. To show that they are subharmonic, let $\{w + ae^{it} : t \in (0, 2\pi]\}$ be any circle in the complex plane with center $w \in \mathbb{C}$ and radius $a \in \mathbb{R}^+$. With Theorem 1.1.3 and Fubini's Theorem

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} g_n(w + ae^{is})\, ds &= \frac{1}{2\pi} \int_0^{2\pi} n^2 \int_{-\frac{1}{n}}^{\frac{1}{n}} g(w + ae^{is} + t)|t|\, dt\, ds \\
&= \int_{-\frac{1}{n}}^{\frac{1}{n}} n^2 |t| \frac{1}{2\pi} \int_0^{2\pi} g(w + ae^{is} + t)\, ds\, dt \\
&\geq \int_{-\frac{1}{n}}^{\frac{1}{n}} n^2 |t| g(w+t)\, dt \\
&= g_n(w).
\end{aligned}$$

All conditions except differentiability have been proved. Generally, these functions will not be differentiable twice in both variables. Because the integral in the definition of g_n is taken over a compact set, differentiation and integration can be interchanged if g is differentiable. Hence, if $\frac{\partial^{k+m} g}{\partial x^k \partial y^m}$ exists for some $k, m \in \mathbb{N}$ so does $\frac{\partial^{k+m} g_n}{\partial x^k \partial y^m}$. The derivative is then given by

$$\frac{\partial^{k+m} g_n}{\partial x^k \partial y^m}(z) = n^2 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \frac{\partial^{k+m} g}{\partial x^k \partial y^m}(z+t)|t|\, dt.$$

Furthermore,

$$\begin{aligned}
n^2 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \frac{\partial^{k+m} g}{\partial x^k \partial y^m}(x+iy+t)|t|\, dt &= n^2 \int\limits_{x-\frac{1}{n}}^{x+\frac{1}{n}} \frac{\partial^{k+m} g}{\partial x^k \partial y^m}(iy+s)|s-x|\, ds \\
&= n^2 \int\limits_{x-\frac{1}{n}}^{x} \frac{\partial^{k+m} g}{\partial x^k \partial y^m}(iy+s)(-s+x)\, ds \\
&\quad + n^2 \int\limits_{x}^{x+\frac{1}{n}} \frac{\partial^{k+m} g}{\partial x^k \partial y^m}(iy+s)(s-x)\, ds \\
&= n^2 \int\limits_{x-\frac{1}{n}}^{x} \frac{\partial^{k+m} g}{\partial x^k \partial y^m}(iy+s)(-s)\, ds + xn^2 \int\limits_{x-\frac{1}{n}}^{x} \frac{\partial^{k+m} g}{\partial x^k \partial y^m}(iy+s)\, ds \\
&\quad + n^2 \int\limits_{x}^{x+\frac{1}{n}} \frac{\partial^{k+m} g}{\partial x^k \partial y^m}(iy+s)s\, ds - xn^2 \int\limits_{x}^{x+\frac{1}{n}} \frac{\partial^{k+m} g}{\partial x^k \partial y^m}(iy+s)\, ds,
\end{aligned}$$

where the right hand side is differentiable with respect to x. Hence, g_n is in fact $(k+1)$ times differentiable with respect to x.

In a similar way functions $\tilde{g}_n$ can be constructed using

$$\tilde{g}_n(z) := n^2 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} g(z+it)|t|\, dt.$$

As above one can show that they are $(m+1)$ times differentiable with respect to y if g is m times differentiable with respect to y, and are zero at $u+iv$ if $g(u+iy)$ vanishes for $v - \frac{1}{n} \le y \le v + \frac{1}{n}$. They are continuous, nonnegative, of period $2\pi i$ and subharmonic. To get the functions f_n just repeat the process above and define

$$f_n(z) := n^2 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} n^2 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} n^2 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} n^2 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} f(z + it_1 + it_2 + t_3 + t_4)|t_1|\,dt_1|t_2|\,dt_2|t_3|\,dt_3|t_4|\,dt_4.$$

It remains to show that these functions converge uniformly to f on compact sets. Using

$$n^2 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} |t|\,dt = 1$$

one shows

$$\begin{aligned}
|f(z) - f_n(z)| = \Bigg| f(z)\, n^2 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} n^2 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} n^2 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} n^2 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} |t_1|\,dt_1|t_2|\,dt_2|t_3|\,dt_3|t_4|\,dt_4 \\
- n^2 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} n^2 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} n^2 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} n^2 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} f(z + it_1 + it_2 + t_3 + t_4)|t_1|\,dt_1|t_2|\,dt_2|t_3|\,dt_3|t_4|\,dt_4 \Bigg| \\
\le n^8 \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} |f(z) - f(z + it_1 + it_2 + t_3 + t_4)|\; |t_1|\,dt_1|t_2|\,dt_2|t_3|\,dt_3|t_4|\,dt_4 \\
\le \sup_{u \in [x - \frac{2}{n}, x + \frac{2}{n}],\ v \in [y - \frac{2}{n}, y + \frac{2}{n}]} |f(x + iy) - f(u + iv)|.
\end{aligned}$$

Because f is continuous and, hence, uniformly continuous on compact sets, this yields locally uniform convergence. $\square$

Lemma 3.1.3. *Let f be a nonnegative, continuous, convex function on a halfline $[a, \infty)$. If*

$$\liminf_{x \to \infty} \frac{f(x)}{x} = 0,$$

then f is bounded by $f(a)$.

Proof. Because f is nonnegative, it is sufficient to show that $f(x) \le f(a)$. Assume that there exists $b > a$ such that $f(a) < f(b)$. By convexity for any $x > b$ the following holds

$$f(b) \le \frac{b-a}{x-a} f(x) + \frac{x-b}{x-a} f(a).$$

Hence,

$$f(x) \ge \frac{x-a}{b-a} f(b) - \frac{x-b}{b-a} f(a) \ge (x - b) \frac{f(b) - f(a)}{b-a}$$

and, further,

$$\frac{f(x)}{x} \geq \frac{f(b)-f(a)}{b-a} - \frac{b}{x}\frac{f(b)-f(a)}{b-a}$$

where the second summand on the right hand side goes to zero for $x \to \infty$ and, hence, the limit inferior of the left hand side is nonzero. $\square$

Lemma 3.1.4. *Let $f(z)$ be an entire function. Assume further that there exists a real number $r_0 > 0$ such that for any $r \geq r_0$ the set $\{\varphi \in (0, 2\pi] : |f(re^{i\varphi})| < 1\}$ is not empty. Define a function Q by*

$$2\pi Q(r)^2 = \int_0^{2\pi} \left(\log^+ |f(re^{i\varphi})|\right)^2 d\varphi.$$

Let $2\pi P(r) := \lambda(\{\varphi \in (0, 2\pi] : |f(re^{i\varphi})| \geq 1\})$, and define a function $\beta : [r, +\infty) \to \mathbb{R}$ by

$$\beta(x) := \int_r^x \frac{\exp\left(\int_r^s \frac{1}{tP(t)}\,dt\right)}{s}\,ds.$$

If $r_0 < r < t$ and if $Q(r) > 0$, then the function $Q(\beta^{-1}(\xi))$ is convex on $[\beta(r), \beta(t)]$.

Proof. Define a function $g(z) := \log^+ |f(\exp(z))|$. By Theorem 1.1.3 this function is subharmonic. Obviously, it is nonnegative, continuous and of period $2\pi i$. Hence, a sequence g_n converging uniformly to g on compact sets of nonnegative, subharmonic, $2\pi i$ periodic and twice differentiable functions exists by Lemma 3.1.2. For any $x \in [\log r, \log t]$ there exists a y_x such that $|f(\exp(x + iy_x))| < 1$ by the assumptions made for f. Because f is continuous, there has to be some ε_x such that $|f(\exp(z))| < 1$ for all $z \in M_x := \{u + iv : u \in (x - 2\varepsilon_x, x + 2\varepsilon_x), v \in (y_x - 2\varepsilon_x, y_x + 2\varepsilon_x)\}$. Further,

$$\bigcup_{x\in[\log r,\log t]} (x - \varepsilon_x, x + \varepsilon_x) \supseteq \bigcup_{x\in[\log r,\log t]} \{x\} \supseteq [\log r, \log t].$$

Because $[\log r, \log t]$ is compact, there has to be an $N \in \mathbb{N}$ and $x_1, x_2, \ldots, x_N \in [\log r, \log t]$ such that

$$\bigcup_{k=1}^{N} (x_k - \varepsilon_{x_k}, x_k + \varepsilon_{x_k}) \supseteq [\log r, \log t].$$

Define

$$\delta := \min\{\varepsilon_{x_k} : k = 1, \ldots, N\}.$$

Any $x \in [\log r, \log t]$ belongs to some $(x_k - \varepsilon_{x_k}, x_k + \varepsilon_{x_k})$. The square $\{u + iv : u \in (x - \delta, x + \delta), v \in (y_{x_k} - \delta, y_{x_k} + \delta)\}$ is contained in M_{x_k} and $f(\exp(u+iv))$ is bounded by 1 there. Hence, the function g vanishes there. By Lemma 3.1.2 the functions g_n are zero at $x + y_{x_{k(x)}}$ if $\frac{2}{n} < \delta$. Thus, for all sufficiently large n the functions g_n have a zero at some $(x + iy)$ for all $x \in [\log r, \log t]$.

Define functions q_n by

$$2\pi q_n(x)^2 = \int_0^{2\pi} g_n(x+iy)^2\,dy. \tag{3.1}$$

Define $2\pi p_n(x) := \lambda(\{y \in (0, 2\pi] : g_n(x+iy) > 0\})$. Differentiating (3.1) leads to

$$4\pi q_n(x) q_n'(x) = \int_0^{2\pi} 2 g_n(x+iy) \frac{\partial g_n}{\partial x}(x+iy)\,dy \tag{3.2}$$

where differentiation and integration can be interchanged because $[0, 2\pi]$ is a compact set and g_n has a continous partial derivative with respect to x. By the Cauchy-Schwarz inequality

$$q_n(x)^2 q_n'(x)^2 \leq \frac{1}{4\pi^2}\left(\int_0^{2\pi} g_n(x+iy)^2\,dy\right)\left(\int_0^{2\pi} \frac{\partial g_n}{\partial x}(x+iy)^2\,dy\right) = \frac{1}{2\pi} q_n(x)^2 \int_0^{2\pi} \frac{\partial g_n}{\partial x}(x+iy)^2\,dy.$$

Hence, for $q_n(x) \neq 0$

$$q_n'(x)^2 \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial g_n}{\partial x}(x+iy)^2\,dy. \tag{3.3}$$

Differentiating (3.2) again yields

$$2\pi(q_n'(x)^2 + q_n(x) q_n''(x)) = \int_0^{2\pi} \left(\frac{\partial g_n}{\partial x}(x+iy)^2 + g_n(x+iy)\frac{\partial^2 g_n}{\partial x^2}(x+iy)\right)\,dy.$$

Thus, for $q_n(x) \neq 0$ with (3.3)

$$q_n(x) q_n''(x) \geq \frac{1}{2\pi}\int_0^{2\pi} g_n(x+iy)\frac{\partial^2 g_n}{\partial x^2}(x+iy)\,dy$$

follows. Because g_n is subharmonic,

$$\frac{\partial^2 g_n}{\partial x^2}(x+iy) + \frac{\partial^2 g_n}{\partial y^2}(x+iy) \geq 0$$

by Theorem 1.1.3. Therefore,

$$q_n(x) q_n''(x) \geq -\frac{1}{2\pi}\int_0^{2\pi} g_n(x+iy)\frac{\partial^2 g_n}{\partial y^2}(x+iy)\,dy.$$

Because g_n is of period $2\pi i$, integration by parts leads to

$$q_n(x) q_n''(x) \geq \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial g_n}{\partial y}(x+iy)^2\,dy.$$

Using Lemma 3.1.1 with $g(t) = \frac{\partial g_n}{\partial y}(x+it)$ for any interval $y \in (s_1, s_2)$ where $g_n(x+iy) \neq 0$ and $g_n(x+is_1) = g_n(x+is_2) = 0$ yields

$$\int\limits_{s_1}^{s_2} \frac{\partial g_n}{\partial y}(x+iy)^2 \, dy \geq \frac{\pi^2}{(s_2-s_1)^2} \int\limits_{s_1}^{s_2} (g_n(x+iy) - g_n(x+is_1))^2 \, dy \geq \frac{1}{4p_n(x)^2} \int\limits_{s_1}^{s_2} g_n(x+iy)^2 \, dy$$

Because there is some s with $g_n(x+is) = 0$ thanks to

$$\int\limits_0^{2\pi} \frac{\partial g_n}{\partial y}(x+iy)^2 \, dy = \int\limits_s^{2\pi+s} \frac{\partial g_n}{\partial y}(x+iy)^2 \, dy$$

one can start integrating at this s. Summation over all integrals as above yields

$$\int\limits_0^{2\pi} \frac{\partial g_n}{\partial y}(x+iy)^2 \, dy \geq \frac{1}{4p_n(x)^2} \int\limits_0^{2\pi} g_n(x+iy)^2 \, dy = \frac{\pi q_n(x)^2}{2p_n(x)^2}.$$

This and the inequality of arithmetic and geometric means show that

$$\left[q_n(x)q_n'(x)\right]' = q_n'(x)^2 + q_n(x)q_n''(x) \geq q_n'(x)^2 + \frac{q_n(x)^2}{4p_n(x)^2} \geq 2q_n(x)\frac{q_n'(x)}{2p_n(x)} = \frac{q_n(x)q_n'(x)}{p_n(x)}.$$

Because of

$$\begin{aligned}
\left[\frac{q_n(x)q_n'(x)}{\exp(\int_{\log r}^x \frac{1}{p_n(t)}\,dt)}\right]' &= \frac{[q_n(x)q_n'(x)]' \exp(\int_{\log r}^x \frac{1}{p_n(t)}\,dt) - q_n(x)q_n'(x)\exp(\int_{\log r}^x \frac{1}{p_n(t)}\,dt)\frac{1}{p_n(x)}}{\exp(2\int_{\log r}^x \frac{1}{p_n(t)}\,dt)} \\
&= \frac{[q_n(x)q_n'(x)]' - q_n(x)q_n'(x)\frac{1}{p_n(x)}}{\exp(\int_{\log r}^x \frac{1}{p_n(t)}\,dt)} \\
&\geq 0,
\end{aligned}$$

the function $\frac{q_n(x)q_n'(x)}{\exp(\int_{\log r}^x \frac{1}{p_n(t)}\,dt)}$ is monotonically increasing for $x \in [\log r, \log t]$. Define a function α_n as

$$\alpha_n(x) := \int\limits_{\log r}^{x} \exp\left(\int_{\log r}^{s} \frac{1}{p_n(t)}\,dt\right) ds.$$

Due to

$$\alpha_n'(x) = \exp\left(\int_{\log r}^{x} \frac{1}{p_n(t)}\,dt\right) > 0$$

$\alpha_n(x)$ is strictly increasing. Hence, the inverse function exists and satisfies ($\xi = \alpha_n(x)$)

$$(\alpha_n^{-1})'(\xi) = \frac{1}{\exp(\int_{\log r}^{\alpha_n^{-1}(\xi)} \frac{1}{p_n(t)}\,dt)}.$$

Now the following holds

$$\begin{aligned}\frac{d}{d\xi}\left[\frac{q_n^2\big(\alpha_n^{-1}(\xi)\big)}{2}\right] &= q_n\big(\alpha_n^{-1}(\xi)\big)q_n'\big(\alpha_n^{-1}(\xi)\big)\frac{1}{\exp(\int_{\log r}^{\alpha_n^{-1}(\xi)}\frac{1}{p_n(t)}\,dt)}\\ &= q_n(x)q_n'(x)\frac{1}{\exp(\int_{\log r}^{x}\frac{1}{p_n(t)}\,dt)}.\end{aligned}$$

The right hand side is an increasing function. Hence, the function $\frac{q_n^2\left(\alpha_n^{-1}(\xi)\right)}{2}$ is convex for $\xi \in [\alpha_n(\log r), \alpha_n(\log t)]$. For any $\alpha_n(\log r) \le \xi_1 < \xi_2 < \xi_3 \le \alpha_n(\log t)$ this yields

$$q_n^2\big(\alpha_n^{-1}(\xi_2)\big)(\xi_3-\xi_1) \le q_n^2\big(\alpha_n^{-1}(\xi_1)\big)(\xi_3-\xi_2) + q_n^2\big(\alpha_n^{-1}(\xi_3)\big)(\xi_2-\xi_1).$$

Therefore, for any $\log r \le x_1 < x_2 < x_3 \le \log t$ one gets

$$q_n^2(x_2)(\alpha_n(x_3)-\alpha_n(x_1)) \le q_n^2(x_1)(\alpha_n(x_3)-\alpha_n(x_2)) + q_n^2(x_3)(\alpha_n(x_2)-\alpha_n(x_1)). \tag{3.4}$$

In order to show convergence for $n \to \infty$ define some functions.

$$2\pi q(x)^2 := \int_0^{2\pi} g(x+iy)^2\,dy.$$

Further, set $2\pi p(x) := \lambda(\{y \in (0,2\pi] : g_k(x+iy) > 0 \quad \forall k \in \mathbb{N}\})$ and

$$\alpha(x) := \int_{\log r}^{x} \exp\left(\int_{\log r}^{s} \frac{1}{p(t)}\,dt\right) ds.$$

Because the limit $g_n \to g$ is uniform on compact sets, $q_n(x)^2$ converges to $q(x)^2$. By definition the function $2\pi p_n(x)$ is the measure of the set $\{y \in (0,2\pi] : g_n(x+iy) > 0\}$. The function $g_n(u+iv)$ is zero if and only if $g(x+iy)$ vanishes almost everywhere in the square $u - \frac{2}{n} \le x \le u + \frac{2}{n}, v - \frac{2}{n} \le y \le v + \frac{2}{n}$. Hence, if g_n vanishes at some point $u + iv \in \mathbb{C}$, then all g_m have to vanish at $u+iv$ for $m > n$. Therefore,

$$\{y \in (0,2\pi] : g_m(x+iy) > 0\} \subseteq \{y \in (0,2\pi] : g_n(x+iy) > 0\} \tag{3.5}$$

for $m > n$. Because these sets are of finite measure and the Lebesgue measure is continuous from above

$$\begin{aligned}2\pi p(x) &= \lambda(\{y \in (0,2\pi] : g_n(x+iy) > 0 \quad \forall n \in \mathbb{N}\})\\ &= \lambda\left(\bigcap_{n\in\mathbb{N}}\{y \in (0,2\pi] : g_n(x+iy) > 0\}\right)\\ &= \lim_{n\to\infty} \lambda(\{y \in (0,2\pi] : g_n(x+iy) > 0\})\\ &= \lim_{n\to\infty} 2\pi p_n(x).\end{aligned}$$

Equation (3.5) shows that the functions p_n decrease in n. With this the functions $\frac{1}{p_n}$ are increasing. By the monotone convergence theorem the inner integral in the definition of α_n and the limit can be interchanged:

$$\int\limits_{\log r}^{x} \exp\left(\int_{\log r}^{s} \lim_{n\to\infty} \frac{1}{p_n(t)}\, dt\right) ds = \int\limits_{\log r}^{x} \exp\left(\lim_{n\to\infty} \int_{\log r}^{s} \frac{1}{p_n(t)}\, dt\right) ds$$

Because exp is continuous and the functions

$$\exp\left(\int_{\log r}^{x} \frac{1}{p_n(t)}\, dt\right)$$

are again monotonically increasing, the monotone convergence theorem can be used once more to show that

$$\begin{aligned}
\alpha(x) &= \int\limits_{\log r}^{x} \exp\left(\int_{\log r}^{s} \frac{1}{p(t)}\, dt\right) ds \\
&= \int\limits_{\log r}^{x} \exp\left(\int_{\log r}^{s} \lim_{n\to\infty} \frac{1}{p_n(t)}\, dt\right) ds \\
&= \lim_{n\to\infty} \int\limits_{\log r}^{x} \exp\left(\int_{\log r}^{s} \frac{1}{p_n(t)}\, dt\right) ds \\
&= \lim_{n\to\infty} \alpha_n(x).
\end{aligned}$$

Hence, the limit in equation (3.4) leads to

$$q^2(x_2)(\alpha(x_3) - \alpha(x_1)) \leq q^2(x_1)(\alpha(x_3) - \alpha(x_2)) + q^2(x_3)(\alpha(x_2) - \alpha(x_1)),$$

which shows that $q^2\big(\alpha^{-1}(\xi)\big)$ is convex for $\xi \in [\alpha(\log r), \alpha(\log t)]$. By their definitions Q and q satisfy

$$\begin{aligned}
2\pi q(x)^2 &= \int\limits_{0}^{2\pi} g(x+iy)^2\, dy \\
&= \int\limits_{0}^{2\pi} \big(\log^+ |f(\exp(x+iy))|\big)^2\, dy \\
&= \int\limits_{0}^{2\pi} \big(\log^+ |f(\exp(x)\exp(iy))|\big)^2\, dy \\
&= 2\pi Q(\exp(x))^2.
\end{aligned}$$

Hence, the function $Q^2\big(\exp(\alpha^{-1}(\xi))\big)$ is convex for $\xi \in [\alpha(\log r), \alpha(\log t)]$. To show a similar connection between p and P consider the sets in their definitions

$$2\pi p(x) = \lambda(\{y \in (0, 2\pi] : g_n(x+iy) > 0 \quad \forall n \in \mathbb{N}\})$$

and

$$2\pi P(r) = \lambda(\{\varphi \in (0, 2\pi] : |f(re^{i\varphi})| \geq 1\}).$$

The functions $g_n(u+iv)$ vanish if and only if $g(x+iy) = \log^+ |f(\exp(x+iy))|$ vanishes almost everywhere in the square

$$u - \frac{2}{n} \leq x \leq u + \frac{2}{n}, v - \frac{2}{n} \leq y \leq v + \frac{2}{n}.$$

Hence, $|f(\exp(x+iy))|$ has to be less than or equal to 1 almost everywhere on this square. Because f is analytic, it cannot attain its maximum inside the square without reducing to a constant. Therefore, f has to be of modulus less than 1 particularly for $\exp(u+iv)$. Hence,

$$\{\varphi \in (0, 2\pi] : |f(re^{i\varphi})| \geq 1\} \subseteq \{y \in (0, 2\pi] : g_n(x+iy) > 0 \quad \forall n \in \mathbb{N}\}.$$

For the other inclusion assume that $|f(\exp(u+iv))| < 1$. Then by continuity $f(\exp(x+iy))$ has to be of modulus less than 1 on some square

$$u - \frac{2}{k} \leq x \leq u + \frac{2}{k}, v - \frac{2}{k} \leq y \leq v + \frac{2}{k}.$$

Hence, g_n vanishes for $n \geq k$ and one gets

$$\{\varphi \in (0, 2\pi] : |f(re^{i\varphi})| \geq 1\} \supseteq \{y \in (0, 2\pi] : g_n(x+iy) > 0 \quad \forall n \in \mathbb{N}\}.$$

Obviously, $p(x) = P(\exp(x))$ follows. Now the function α can be considered. Simple substitutions yield

$$\begin{aligned}
\alpha(x) &= \int\limits_{\log r}^{x} \exp\left(\int_{\log r}^{s} \frac{1}{p(t)}\, dt\right) ds \\
&= \int\limits_{\log r}^{x} \exp\left(\int_{\log r}^{s} \frac{1}{P(\exp(t))}\, dt\right) ds \\
&= \int\limits_{\log r}^{x} \exp\left(\int_{r}^{\exp s} \frac{1}{P(\tilde{t})\tilde{t}}\, d\tilde{t}\right) ds \\
&= \int\limits_{r}^{\exp x} \exp\left(\int_{r}^{\tilde{s}} \frac{1}{P(\tilde{t})\tilde{t}}\, d\tilde{t}\right) \frac{1}{\tilde{s}}\, d\tilde{s}.
\end{aligned}$$

The function β now satisfies $\beta(\exp(x)) = \alpha(x)$ and, hence, $Q^2\big(\beta^{-1}(\xi)\big)$ is convex for $\xi \in [\beta(r), \beta(t)]$.

□

Lemma 3.1.5. *Let f_1, f_2 be entire functions of exponential type 0. If*

$$\min\left\{|f_1(x+iy)|, |f_2(x+iy)|\right\} \leq \frac{1}{|y|},$$

then at least one of the functions f_1 or f_2 vanishes identically.

Proof. Two cases have to be discussed. First let the assumptions of Lemma 3.1.4 be satisfied for both functions f_j. Define $P_j(r)$ and $Q_j(r)$ as in Lemma 3.1.4 for the functions f_j. With $z = x + iy = re^{i\varphi}$ the hypotheses reads as

$$\min\{|f_1(re^{i\varphi})|, |f_2(re^{i\varphi})|\} \leq \frac{1}{r|\sin\varphi|}.$$

For $|\sin\varphi| > \frac{1}{r}$ this yields

$$\min\{|f_1(re^{i\varphi})|, |f_2(re^{i\varphi})|\} < 1.$$

Hence, only for $\varphi \in (0, \arcsin\frac{1}{r}) \cup (\pi - \arcsin\frac{1}{r}, \pi + \arcsin\frac{1}{r}) \cup (2\pi - \arcsin\frac{1}{r}, 2\pi]$ both functions can be greater than or equal to 1. For the functions P_j this leads to

$$P_1(r) + P_2(r) \leq 1 + \frac{4\arcsin\frac{1}{r}}{2\pi}$$

By the Taylor series expansion of arcsin

$$\arcsin(x) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^{2k+1}}{4^k(2k+1)}, \tag{3.6}$$

it satisfies

$$\arcsin\frac{1}{r} \leq \frac{\pi}{2}\frac{1}{r}$$

for sufficiently large r. Hence,

$$P_1(r) + P_2(r) \leq 1 + \frac{1}{r}.$$

With the inequality of geometric and arithmetic means

$$(P_1(r) + P_2(r))\left(1 + \frac{1}{r}\right) \geq (P_1(r) + P_2(r))^2 \geq 4P_1(r)P_2(r)$$

holds true. Hence,

$$\frac{1}{P_1(r)} + \frac{1}{P_2(r)} = \frac{P_1(r) + P_2(r)}{P_1(r)P_2(r)} \geq \frac{4}{1+\frac{1}{r}} = \frac{4r}{r+1} \tag{3.7}$$

follows. Assume that both Q_j are unbounded. By Lemma 3.1.4 the functions $Q_j^2(\beta_j^{-1}(\xi))$ are convex. Hence, by Lemma 3.1.3

$$\liminf_{\xi\to\infty} \frac{Q_j^2(\beta_j^{-1}(\xi))}{\xi} = 2c_j > 0.$$

Set $c := \min\{c_1, c_2\}$. With $\xi = \beta_j(x)$

$$Q_j^2(x) \geq c\beta_j(x)$$

follows for sufficiently large x. The definition of β_j, inequality (3.7) and the convexity of the exponential function lead to

$$\begin{aligned}
\frac{Q_1^2(x)+Q_2^2(x)}{2} &\geq c\left(\int\limits_r^x \frac{\exp\left(\int_r^s \frac{1}{tP_1(t)}\,dt\right)}{2s}\,ds + \int\limits_r^x \frac{\exp\left(\int_r^s \frac{1}{tP_2(t)}\,dt\right)}{2s}\,ds\right) \\
&\geq c\int\limits_r^x \frac{\exp\left(\int_r^s \frac{1}{2t}\left(\frac{1}{P_1(t)}+\frac{1}{P_2(t)}\right)\,dt\right)}{s}\,ds \\
&\geq c\int\limits_r^x \frac{\exp\left(\int_r^s \frac{2}{t+1}\,dt\right)}{s}\,ds \\
&= c\int\limits_r^x \frac{(s+1)^2}{(r+1)^2 s}\,ds \\
&\geq c\int\limits_r^x \frac{s}{(r+1)^2}\,ds \\
&= \frac{c}{2}\frac{x^2}{(r+1)^2} - \frac{c}{2}\frac{r^2}{(r+1)^2}.
\end{aligned} \tag{3.8}$$

Because f_1 and f_2 are of exponential type 0

$$\tau_{f_j} = \limsup_{x\to\infty} \max_{t\in(0,2\pi]} \frac{\log^+|f_j(xe^{it})|}{x} = 0.$$

By the reversed version of Fatou's lemma 1.1.34 for any monotone increasing sequence of radii x_n converging to infinity

$$\begin{aligned}
\limsup_{n\to\infty} \frac{2\pi Q_j^2(x_n)}{x_n^2} &= \limsup_{n\to\infty} \int\limits_0^{2\pi} \left(\frac{\log^+|f_j(x_n e^{i\varphi})|}{x_n}\right)^2 d\varphi \\
&\leq \int\limits_0^{2\pi} \limsup_{n\to\infty} \left(\frac{\log^+|f_j(x_n e^{i\varphi})|}{x_n}\right)^2 d\varphi \\
&= 0.
\end{aligned}$$

This contradicts (3.8) and, hence, the assumption that both Q_j are unbounded is false. Without loss of generality assume that Q_1 is bounded. Then with the Cauchy-Schwarz inequality

$$\int\limits_0^{2\pi} \log^+|f_1(xe^{i\varphi})|\,d\varphi \leq \sqrt{2\pi}\sqrt{\int_0^{2\pi} \left(\log^+|f_1(xe^{i\varphi})|\right)^2 d\varphi} \leq c$$

follows for some $c > 0$. For any fixed $R \in (0, +\infty)$ the function $f_1(Rz)$ is an analytic function on the unit circle with analytic continuation to the whole complex plane. Hence, by Lemma 1.1.20 $f_1(Rz)$ belongs to $N^+(\mathbb{D})$. By Theorem 1.1.21

$$\log |f_1(Rz)| \le \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{1-|z|^2}{|e^{i\varphi}-z|^2} \log |f_1(Re^{i\varphi})| \, d\varphi$$

follows for any $z \in \mathbb{D}$. With $w = Rz$ this leads to

$$\begin{aligned}
\log |f_1(w)| &\le \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{R^2-|w|^2}{|Re^{i\varphi}-w|^2} \log |f_1(Re^{i\varphi})| \, d\varphi \\
&\le \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{R^2-|w|^2}{|R-|w||^2} \log |f_1(Re^{i\varphi})| \, d\varphi \\
&= \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{R+|w|}{R-|w|} \log |f_1(Re^{i\varphi})| \, d\varphi \\
&\le \frac{1}{2\pi} \frac{R+|w|}{R-|w|} c
\end{aligned}$$

where the second inequality holds by the reversed triangle inequality. For $R \to \infty$ this shows that f_1 is bounded. By Liouville's theorem $f = C$ for some constant C. If it is not zero, then $|f_2(\pm iy)| < \frac{1}{|y|}$ for $|y| > \frac{1}{C}$. By Lemma 1.1.35 the function f_2 reduces to a constant. This constant has to be zero by the inequality just mentioned.

It remains to prove the case where at least one f_j does not satisfy the assumptions of Lemma 3.1.4. Without loss of generality let f_1 be this function. This gives a series of radii r_n tending to infinity with

$$\min_{\varphi \in [0,2\pi)} |f_1(r_n e^{i\varphi})| \ge 1.$$

By the assumption for the minimum of f_1 and f_2 the function f_2 has to satisfy

$$|f_2(x+iy)| \le \frac{1}{2}$$

for $|y| \ge 2$ and $x^2 + y^2 = r_n^2$. Hence, $\log^+ |f_2(x+iy)|$ has to vanish at these points. Because of $y = r_n \sin\varphi$ the condition $|y| \ge 2$ is equivalent to

$$|\sin\varphi| \ge \frac{2}{r_n}.$$

Therefore, the set $\{\varphi \in (0, 2\pi] : log^+|f_2(r_n e^{i\varphi})| \neq 0\}$ must be contained in $(0, \arcsin \frac{2}{r_n}) \cup (\pi - \arcsin \frac{2}{r_n}, \pi + \arcsin \frac{2}{r_n}) \cup (2\pi - \arcsin \frac{2}{r_n}, 2\pi]$. Thus,

$$\lambda\left(\{\varphi \in (0, 2\pi] : \log^+ |f_2(r_n e^{i\varphi})| \neq 0\}\right) \le 4 \arcsin \frac{2}{r_n}.$$

The Taylor series expansion of arcsin (3.6) shows that arcsin $\frac{2}{r_n}$ can be dominated by $\frac{\pi}{r_n}$ for sufficiently large r_n. Hence,

$$\lambda\left(\{\varphi \in (0, 2\pi] : \log^+ |f_2(r_n e^{i\varphi})| \neq 0\}\right) \leq 4\frac{\pi}{r_n}.$$

Let $\varepsilon > 0$ be given. Because f_2 is of exponential type 0 the maximum on the radii satisfies

$$\max_{\varphi \in [0,2\pi)} \log^+ |f_2(r_n e^{i\varphi})| \leq \varepsilon r_n$$

for sufficiently large n. By Theorem 1.1.21

$$\begin{aligned}
\log |f_2(w)| &\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{r_n^2 - |w|^2}{|r_n e^{i\varphi} - w|^2} \log |f_2(r_n e^{i\varphi})| \, d\varphi \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{r_n^2 - |w|^2}{|r_n - |w||^2} \log |f_2(r_n e^{i\varphi})| \, d\varphi \\
&\leq \frac{1}{2\pi} \int\limits_{\{\varphi \in (0,2\pi] : \log^+ |f_2(r_n e^{i\varphi})| \neq 0\}} \frac{r_n + |w|}{r_n - |w|} \log^+ |f_2(r_n e^{i\varphi})| \, d\varphi \\
&\leq \frac{1}{2\pi} \int\limits_{\{\varphi \in (0,2\pi] : \log^+ |f_2(r_n e^{i\varphi})| \neq 0\}} \frac{r_n + |w|}{r_n - |w|} \varepsilon r_n \, d\varphi \\
&\leq \frac{1}{2\pi} \frac{4\pi}{r_n} \frac{r_n + |w|}{r_n - |w|} \varepsilon r_n \\
&= 2 \frac{r_n + |w|}{r_n - |w|} \varepsilon
\end{aligned}$$

follows. For sufficiently large n this shows that f_2 is bounded and, hence, a constant. As above this yields that either f_1 or f_2 has to vanish and the lemma is proved. □

3.2 Proof of the Ordering Theorem

Theorem 3.2.1. *Let $H(E_1)$ and $H(E_2)$ be de Branges spaces which are isometrically contained in a space $L^2(\mu)$, where μ is a Borel measure on $\mathbb{R}$. Assume for $H(E_1)$ and $H(E_2)$ that their associated functions do not vanish μ-almost everywhere. If $\frac{E_1(z)}{E_2(z)}$ is of bounded type in the upper halfplane and has no real zeros or singularities, then $H(E_1)$ is contained in $H(E_2)$ or $H(E_2)$ is contained in $H(E_1)$.*

Proof. Assume first that the mean type of $\frac{E_2(z)}{E_1(z)}$ is not zero. Without loss of generality assume that it is negative. Otherwise, change the roles of E_1 and E_2: Because $\frac{E_1(z)}{E_2(z)}$ has no zeros in the upper halfplane, $\frac{E_2(z)}{E_1(z)}$ is of bounded type and the mean type of the product $\frac{E_1(z)}{E_2(z)}\frac{E_2(z)}{E_1(z)} = 1$ is zero. Therefore, by Lemma 1.1.13 the mean type of these functions must have different signs.

The aim is, to show with Theorem 2.1.9 that all functions $f \in H(E_2)$ and the function E_2 are associated with the space $H(E_1)$. To use this theorem the functions $\frac{f(z)}{E_1(z)}$, $\frac{f^\#(z)}{E_1(z)}$, $\frac{E_2(z)}{E_1(z)}$ and $\frac{E_2^\#(z)}{E_1(z)}$ have to be of bounded type and satisfy

$$\limsup_{y\to\infty}\left|\frac{f(\pm iy)}{E_1(iy)}\right| < \infty$$

and

$$\limsup_{y\to\infty}\left|\frac{E_2(\pm iy)}{E_1(iy)}\right| < \infty,$$

respectively. Further, there must not be any function Q_1 which is zero μ-almost everywhere and associated with $H(E_1)$, and the integrals $\int_{\mathbb{R}} \frac{|f(t)|^2}{t^2+1}\,d\mu(t)$ and $\int_{\mathbb{R}} \frac{|E_2(t)|^2}{t^2+1}\,d\mu(t)$ have to be finite. $\frac{f(z)}{E_1(z)}$ and $\frac{f^\#(z)}{E_1(z)}$ are of bounded type, because $\frac{E_2(z)}{E_1(z)}$, $\frac{f(z)}{E_2(z)}$ and $\frac{f^\#(z)}{E_2(z)}$ are of bounded type. $\frac{E_2^\#(z)}{E_1(z)}$ is of bounded type because of $|E_2^\#(z)| < |E_2(z)|$ for all $z \in \mathbb{C}$. The non–existence of a function Q_1 is another assumption. The estimates on the imaginary axis are obtained by a formula for mean type from Theorem 1.1.11

$$\tau = \limsup_{y\to\infty}\frac{1}{y}\log\left|\frac{f(iy)}{E_1(iy)}\right|. \tag{3.9}$$

By the assumption that the mean type of $\frac{E_2(z)}{E_1(z)}$ is negative and because $\frac{f(z)}{E_2(z)}$ has nonpositive mean type, the mean type of $\frac{f(z)}{E_1(z)}$ has to be negative by Lemma 1.1.13. Hence, with (3.9) the relations

$$\limsup_{y\to\infty}\left|\frac{f(iy)}{E_1(iy)}\right| = 0$$

and

$$\limsup_{y\to\infty}\left|\frac{E_2(iy)}{E_1(iy)}\right| = 0$$

follow. The same convergence for $f(-iy)$ follows with $f^\#$ in place of f. The convergence for $E_2(-iy)$ follows with $|E_2(-iy)| < |E_2(iy)|$ for $y > 0$. The integral condition follows from

$$\int_{\mathbb{R}} \frac{|f(t)|^2}{t^2+1}\,d\mu(t) \le \int_{\mathbb{R}} |f(t)|^2\,d\mu(t) = \|f\|^2_{L^2(\mu)} = \|f\|^2_{H(E_2)} < \infty$$

for f and from

$$\begin{aligned}
\infty > \int_{\mathbb{R}} |K_2(i,z)|^2\ d\mu(t) &= \int_{\mathbb{R}} \left|\frac{E_2(t)\overline{E_2(i)} - \overline{E_2(t)}E_2(-i)}{2\pi(t+i)} i\right|^2\ d\mu(t) \\
&\ge \int_{\mathbb{R}} \frac{\big||E_2(t)||E_2(i)| - |E_2(t)||E_2(-i)|\big|^2}{2\pi(t^2+1)}\,d\mu(t) \\
&= \int_{\mathbb{R}} \frac{|E_2(t)|^2}{2\pi(t^2+1)}\big||E_2(i)| - |E_2(-i)|\big|^2\,d\mu(t)
\end{aligned} \tag{3.10}$$

for E_2. Hence, $E_2(z)$ and $f \in H(E_2)$ are associated with $H(E_1)$. Therefore, also $E_2^{\#}(z)$ is associated with $H(E_1)$ by Corollary 2.1.3. Thus, for any $g \in H(E_1)$ and $w \in \mathbb{C}$ with $g(w) \neq 0$ the linear combination

$$\frac{E_2^{\#}(w)}{g(w)} \frac{E_2(z)g(w) - g(z)E_2(w)}{z-w} - \frac{E_2(w)}{g(w)} \frac{E_2^{\#}(z)g(w) - g(z)E_2^{\#}(w)}{z-w} = \frac{E_2^{\#}(w)E_2(z) - E_2(w)E_2^{\#}(z)}{z-w}$$

belongs to $H(E_1)$. This is up to a factor $K_2(w,z)$. Let f be any function belonging to $H(E_2)$. Because the span of the functions $K_2(w,z)$ is dense in $H(E_2)$ by Theorem 1.2.11, there exists a sequence f_n in $\operatorname{span}\{K_2(w,z) : w \in \mathbb{C}\backslash\mathbb{R}\} \subseteq H(E_1) \cap H(E_2)$ converging to f in $H(E_2)$. Hence, it is in particular a Cauchy sequence with respect to the $H(E_2)$ norm. Because both spaces $H(E_k)$ are contained isometrically in $L^2(\mu)$, the $L^2(\mu)$ norm and the $H(E_k)$ norm coincide on $H(E_k)$. Therefore,

$$\|f_n - f_m\|_{H(E_2)} = \|f_n - f_m\|_{L^2(\mu)} = \|f_n - f_m\|_{H(E_1)}.$$

This yields that f_n is also a Cauchy sequence with respect to the $H(E_1)$ norm and converges to some $\tilde{f} \in H(E_1)$. By the isometric inclusion

$$\|f - \tilde{f}\|_{L^2(\mu)} \leq \lim_{n\to\infty} (\|f - f_n\|_{L^2(\mu)} + \|f_n - \tilde{f}\|_{L^2(\mu)}) = \lim_{n\to\infty} (\|f - f_n\|_{H(E_2)} + \|f_n - \tilde{f}\|_{H(E_1)}) = 0.$$

Hence, $f - \tilde{f}$ is zero μ-almost everywhere. Because f is associated with $H(E_1)$ by the calculations above and $\tilde{f}$ is associated, too, the function $f - \tilde{f}$ is an associated μ-almost everywhere vanishing function. Therefore, it has to vanish identically by assumption.

Assume now that the mean type of $\frac{E_1(z)}{E_2(z)}$ is zero. Let $R(z)$ be a function in $L^2(\mu) \ominus H(E_2)$ with norm 1. Let further be $f \in H(E_1)$. Define a function $L_f(z)$ for $z \in \mathbb{C}$ such that $\frac{f(z)}{E_2(z)}$ is analytic by

$$L_f(z) := \int\limits_{\mathbb{R}} \frac{\frac{f(t)}{E_2(t)} - \frac{f(z)}{E_2(z)}}{t-z} \overline{R(t)} E_2(t)\, d\mu(t) = \int\limits_{\mathbb{R}} \frac{\frac{f(t)}{E_2(t)} - \frac{f(z)}{E_2(z)}}{t-z} \frac{\overline{R(t)}}{\overline{E_2(t)}}\, d\nu_2(t)$$

with $d\nu_2 = |E_2(t)|^2 d\mu$ and a function $\tilde{L}_f(z)$ for $z \in \mathbb{C}$ such that $\frac{f(z)}{E_2^{\#}(z)}$ is analytic by

$$\tilde{L}_f(z) := \int\limits_{\mathbb{R}} \frac{\frac{f(t)}{E_2^{\#}(t)} - \frac{f(z)}{E_2^{\#}(z)}}{t-w} \frac{\overline{R(t)}}{E_2(t)}\, d\nu_2(t).$$

To show that these functions are analytic Lemma 1.2.3 is used. The condition $\int_{\mathbb{R}} \frac{1}{t^2+1}\, d\nu_2(t) < \infty$ is equivalent to $\int_{\mathbb{R}} \frac{|E_2(t)|^2}{t^2+1}\, d\mu(t) < \infty$ which is proved as (3.10). Again by the definition of $d\nu_2$ and because $f \in L^2(\mu)$,

$$\int_{\mathbb{R}} \frac{\left|\frac{f(t)}{E_2(t)}\right|^2}{t^2+1}\, d\nu_2(t) = \int_{\mathbb{R}} \frac{|f(t)|^2}{t^2+1}\, d\mu(t) < \infty.$$

Finally $\frac{\overline{R(t)}}{E_2(t)} \in L^2(\nu_2)$ because $R(t) \in L^2(\mu)$. Therefore, $L_f(z)$ is analytic for $E_2(z) \neq 0$. In the same way the analyticity of $\tilde{L}_f(z)$ for $E_2^{\#}(z) \neq 0$ is proved. Let g be any function in $H(E_2)$ then

$$g(w)\frac{f(z)E_2(w)-f(w)E_2(z)}{z-w} = f(w)\frac{g(z)E_2(w)-g(w)E_2(z)}{z-w} - E_2(w)\frac{g(z)f(w)-g(w)f(z)}{z-w} \quad (3.11)$$

where the left hand side belongs to $L^2(\mu)$ by (similar as in (2.1))

$$\int_{\mathbb{R}} \left|\frac{f(t)E_2(w)-f(w)E_2(t)}{t-w}\right|^2 d\mu(t) \leq C_1 + C_2 \int_{\mathbb{R}} \left(\frac{f(t)}{t^2+1} + \frac{E_2(t)}{t^2+1}\right) d\mu(t) < \infty.$$

The first summand on the right hand side of (3.11) belongs to $H(E_2)$ because E_2 is associated with $H(E_2)$. With $R(z) \in L^2(\mu) \ominus H(E_2)$ this yields

$$\begin{aligned} g(w)E_2(w)L_f(w) &= g(w)\int_{\mathbb{R}} \frac{f(t)E_2(w)-E_2(t)f(w)}{t-w}\overline{R(t)}\, d\mu(t) \\ &= E_2(w)\int_{\mathbb{R}} \frac{g(w)f(t)-g(t)f(w)}{t-w}\overline{R(t)}\, d\mu(t) \end{aligned}$$

and further

$$g(w)L_f(w) = \int_{\mathbb{R}} \frac{g(w)f(t)-g(t)f(w)}{t-w}\overline{R(t)}\, d\mu(t). \quad (3.12)$$

A similar equation for $E_2^{\#}$ leads to

$$g(w)\tilde{L}_f(w) = \int_{\mathbb{R}} \frac{g(w)f(t)-g(t)f(w)}{t-w}\overline{R(t)}\, d\mu(t) = g(w)L_f(w).$$

With this $\tilde{L}_f(w) = L_f(w)$ except on the zeros of E and $E^{\#}$. By analyticity they are equal everywhere. Hence, they continue each other to an entire function. In the upper halfplane

$$L_f(z) = \int_{\mathbb{R}} \frac{f(t)\overline{R(t)}}{t-z}\, d\mu(t) - \frac{f(z)}{E_2(z)}\int_{\mathbb{R}} \frac{E_2(t)\overline{R(t)}}{t-z}\, d\mu(t)$$

where the integrals are in $N^+(\mathbb{C}^+)$ by Lemma 1.1.32 and by the fact that $\frac{f(z)}{E_2(z)} = \frac{f(z)}{E_1(z)}\frac{E_1(z)}{E_2(z)}$ is the product of two functions with nonpositive meantype. Thus $L_f(z)$ has nonpositive mean type. A similar argument shows that $L_f^{\#}(z)$ has nonpositive mean type. By Kreins Theorem 1.1.30, $L_f(z)$ is of exponential type with exact type 0.

The same procedure with $P \in L^2(\mu) \ominus H(E_1)$, $\|P\|_{L^2(\mu)} = 1$ and $g \in H(E_2)$ gives an entire function $L_g(z)$ which is of bounded type, exponential type zero and fulfils

$$f(w)L_g(w) = \int_{\mathbb{R}} \frac{f(w)g(t)-f(t)g(w)}{t-w}\overline{P(t)}\, d\mu(t) \quad (3.13)$$

for any $f \in H(E_1)$. The equations (3.12), (3.13) and the Cauchy-Schwarz inequality lead to $(z = x + iy)$

$$|g(z)L_f(z)| \leq \frac{|g(z)| \, \|f(t)\| + \|g(t)\| \, |f(z)|}{|y|}$$

and

$$|f(z)L_g(z)| \leq \frac{|f(z)| \, \|g(t)\| + \|f(t)\| \, |g(z)|}{|y|}.$$

This yields

$$|y| \, |g(z)L_f(z)| \leq |g(z)| \, \|f(t)\| + \|g(t)\| \, |f(z)| \leq 2 \max\{|g(z)| \, \|f(t)\|, \|g(t)\| \, |f(z)|\}$$

and

$$|y| \, |f(z)L_g(z)| \leq |f(z)| \, \|g(t)\| + \|f(t)\| \, |g(z)| \leq 2 \max\{|g(z)| \, \|f(t)\|, \|g(t)\| \, |f(z)|\}.$$

If the maximum attains $|g(z)| \, \|f(t)\|$, then $|y| \leq 2\frac{\|f(t)\|}{|L_f(z)|}$ otherwise $|y| \leq 2\frac{\|g(t)\|}{|L_g(z)|}$. In both cases

$$\min\left\{\frac{|L_f(z)|}{2\|f(t)\|}, \frac{|L_g(z)|}{2\|g(t)\|}\right\} \leq \frac{1}{|y|}.$$

By Lemma 3.1.5 either $L_f(z) \equiv 0$ or $L_g(z) \equiv 0$. If $L_f(z)$ does not vanish for one $f \in H(E_1)$ and one $R \in L^2(\mu) \ominus H(E_2)$ with norm 1, then $L_g(z) \equiv 0$ for all $g \in H(E_2)$ and $P \in L^2(\mu) \ominus H(E_1)$ with norm 1. Otherwise, $L_f(z) \equiv 0$ for all $f \in H(E_1)$ and $R \in L^2(\mu) \ominus H(E_2)$ with norm 1. Without loss of generality assume $L_g(z) \equiv 0$ and, hence,

$$0 = \int_{\mathbb{R}} \frac{E_1(w)g(t) - E_1(t)g(w)}{t - w} \overline{P(t)} \, d\mu(t)$$

for any $g \in H(E_2)$ and $P \in L^2(\mu) \ominus H(E_1)$ of norm 1. By (3.13), $\frac{f(w)g(t) - f(t)g(w)}{t-w}$ is orthogonal to $L^2(\mu) \ominus H(E_1)$. With this the function has to coincide with a function of $H(E_1)$ μ-almost everywhere. The equality

$$f(w)\frac{g(z)E_1(w) - g(w)E_1(z)}{z - w} = g(w)\frac{f(z)E_1(w) - f(w)E_1(z)}{z - w} - E_1(w)\frac{f(z)g(w) - f(w)g(z)}{z - w}$$

yields that also $\frac{g(z)E_1(w) - g(w)E_1(z)}{z-w}$ has to coincide with a function $T(z) \in H(E_1)$ in $L^2(\mu)$. Define the entire function

$$\tilde{g}(z) := \frac{-T(z)(z - w_0) + g(w_0)E_1(z)}{E_1(w_0)}.$$

Obviously $\tilde{g}(w_0) = g(w_0)$, $\tilde{g}$ and g are μ-equivalent and

$$T(z) = \frac{\tilde{g}(w_0)E_1(z) - \tilde{g}(z)E_1(w_0)}{z - w_0}.$$

Hence, the function $\tilde{g}(z)$ is associated with $H(E_1)$ by Lemma 2.1.8. Therefore, $\frac{\tilde{g}(w)E_1(z)-\tilde{g}(z)E_1(w)}{z-w} \in H(E_1)$ for any $w \in \mathbb{C}\backslash\mathbb{R}$. The function $P(z) := g(z) - \tilde{g}(z)$ is now zero μ-almost everywhere and $\frac{E_1(z)P(w)-P(z)E_1(w)}{z-w}$ has to coincide with a function $R_w(z) \in H(E_1)$ in $L^2(\mu)$. As above the function

$$\tilde{P_w}(z) := \frac{-R_w(z)(z-w) + P(w)E_1(z)}{E_1(w)}$$

fulfils $P(w) = \tilde{P_w}(w)$, is μ-equivalent to $P(z)$ and, hence, is zero μ-almost everywhere. It satisfies $\frac{E_1(z)\tilde{P_w}(w)-\tilde{P_w}(z)E_1(w)}{z-w} = R_w(z) \in H(E_1)$ and, hence, is associated by Lemma 2.1.8. By assumption this function has to vanish. Especially $P(w) = \tilde{P_w}(w) = 0$. By the arbitrariness of w the function P vanishes and, hence, $g(z) = \tilde{g}(z)$. Therefore, all functions $g(z) \in H(E_2)$ are associated with $H(E_1)$.

If $H(E_2)$ is contained in $H(E_1)$ as subsets of $L^2(\mu)$, any function $f \in H(E_2)$ coincides with a function $\tilde{f} \in H(E_1)$ μ-almost everywhere. As shown above f is associated with $H(E_1)$ and so is $\tilde{f}$. Therefore, $f - \tilde{f}$ is a μ-almost everywhere vanishing associated function and has to vanish identically by assumption. If $H(E_2)$ is not contained in $H(E_1)$ as subsets of $L^2(\mu)$, there has to be a function $D(z) \in H(E_2)$ that does not coincide with a function belonging to $H(E_1)$ in $L^2(\mu)$. Then there exists a $P(z) \in L^2(\mu) \ominus H(E_1)$ of norm 1 satisfying $\langle D, P\rangle_{L^2(\mu)} > 0$. With the equality

$$\frac{zD(z)E_1(w) - E_1(z)wD(w)}{z-w} = D(z)E_1(w) + w\frac{D(z)E_1(w) - E_1(z)D(w)}{z-w},$$

where the right summand on the right hand side belongs to $H(E_1)$,

$$\int_{\mathbb{R}} \frac{tD(t)E_1(w) - E_1(t)wD(w)}{t-w}\overline{P(t)}\, d\mu(t) = \int_{\mathbb{R}} D(t)E_1(w)\overline{P(t)}\, d\mu(t)$$

follows. Choose $w = iy$ with $y > 0$. Then

$$\begin{aligned} 0 &< \int_{\mathbb{R}} D(t)\overline{P(t)}\, d\mu(t) \\ &= \int_{\mathbb{R}} \frac{tD(t)\overline{P(t)}}{t-iy} - \frac{E_1(t)iyD(iy)\overline{P(t)}}{E_1(iy)(t-iy)}\, d\mu(t) \\ &= \int_{\mathbb{R}} \frac{tD(t)\overline{P(t)}}{t-iy}\, d\mu(t) - \frac{\sqrt{y}D(iy)}{E_2(iy)}\frac{i\sqrt{y}E_2(iy)}{E_1(iy)}\int_{\mathbb{R}} \frac{E_1(t)\overline{P(t)}}{t-iy}\, d\mu(t). \end{aligned} \tag{3.14}$$

With the Cauchy-Schwarz inequality

$$\left|\int_{\mathbb{R}} \frac{tD(t)\overline{P(t)}}{t-iy}\, d\mu(t)\right|^2 \leq \int_{\mathbb{R}} \frac{t^2|D(t)|^2}{t^2+y^2}\, d\mu(t)$$

and

$$\left| \int_{\mathbb{R}} \frac{E_1(t)\overline{P(t)}}{t - iy}\, d\mu(t) \right| \leq \int_{\mathbb{R}} \frac{|E_1(t)|^2}{t^2 + y^2}\, d\mu(t).$$

By the Lebesgue dominated convergence theorem this yields

$$\lim_{y\to\infty} \int_{\mathbb{R}} \frac{tD(t)\overline{P(t)}}{t - iy}\, d\mu(t) = 0, \tag{3.15}$$

and

$$\lim_{y\to\infty} \int_{\mathbb{R}} \frac{E_1(t)\overline{P(t)}}{t - iy}\, d\mu(t) = 0. \tag{3.16}$$

To show that $\frac{\sqrt{y}|D(iy)|}{|E_2(iy)|}$ cannot approach infinity as $y \to \infty$ calculate

$$\begin{aligned}
|D(iy)|^2 &= |\left\langle K_2(iy,.), D\right\rangle_{H(E_2)}|^2 \\
&\leq \|K_2(iy,.)\|^2_{H(E_2)}\ \|D\|^2_{H(E_2)} \\
&= \langle K_2(iy,.), K_2(iy,.)\rangle_{H(E_2)}\ \|D\|^2_{H(E_2)} \\
&= K_2(iy,iy)\ \|D\|^2_{H(E_2)} \\
&= \frac{E_2(iy)\overline{E_2(iy)} - \overline{E_2(-iy)}E_2(-iy)}{2\pi\big(iy - (-iy)\big)} i\ \|D\|^2_{H(E_2)} \\
&= \frac{|E_2(iy)|^2 - |E_2(-iy)|^2}{4\pi y}\ \|D\|^2_{H(E_2)}.
\end{aligned} \tag{3.17}$$

Hence,

$$\frac{y|D(iy)|^2}{|E_2(iy)|^2} \leq \frac{4\pi y|D(iy)|^2}{|E_2(iy)|^2 - |E_2(-iy)|^2} \leq \|D\|^2_{H(E_2)} \tag{3.18}$$

follows because $|E_2(iy)| > |E_2(-iy)|$. By (3.15) and (3.14)

$$0 < -\lim_{y\to\infty} \frac{\sqrt{y}D(iy)}{E_2(iy)}\, \frac{i\sqrt{y}E_2(iy)}{E_1(iy)} \int_{\mathbb{R}} \frac{E_1(t)\overline{P(t)}}{t - iy}\, d\mu(t)$$

Because by (3.16) the integral goes to zero and by (3.18) $\left|\frac{\sqrt{y}D(iy)}{E_2(iy)}\right|$ is bounded, the factor $\left|\frac{\sqrt{y}E_2(iy)}{E_1(iy)}\right|$ has to approach infinity:

$$\lim_{y\to\infty} \frac{\sqrt{y}|E_2(iy)|}{|E_1(iy)|} = \infty.$$

The calculation (3.17) for any $f \in H(E_1)$ in place of D and E_1 in place of E_2 yields

$$\limsup_{y\to\infty} \frac{\sqrt{y}|f(iy)|}{|E_1(iy)|} < \infty.$$

Hence,

$$\lim_{y\to\infty}\frac{f(iy)}{E_2(iy)}=0.$$

The same applies to $f^{\#}$. To use Theorem 2.1.9 the functions $\frac{f(z)}{E_2(z)}$ and $\frac{f^{\#}(z)}{E_2(z)}$ have to belong to $N(\mathbb{C}^+)$. Because $f, f^{\#}\in H(E_1)$ the functions $\frac{f(z)}{E_1(z)}$ and $\frac{f^{\#}(z)}{E_1(z)}$ are in $N(\mathbb{C}^+)$. By assumption $\frac{E_1(z)}{E_2(z)}\in N(\mathbb{C}^+)$ and because products are again in $N(\mathbb{C}^+)$, the functions $\frac{f(z)}{E_2(z)}$ and $\frac{f^{\#}(z)}{E_2(z)}$ belong to $N(\mathbb{C}^+)$. Hence, by Theorem 2.1.9 all functions $f\in H(E_1)$ are associated with $H(E_2)$. But it was also shown above that all functions $\frac{f(w)g(z)-f(z)g(w)}{z-w}$ with $f\in H(E_1)$ and $g\in H(E_2)$ are in $H(E_1)$. By the assumption that $H(E_2)$ is not contained in $H(E_1)$ there has to be a nonzero function $Q\in H(E_2)$ which is orthogonal to all these functions $\frac{f(w)g(z)-f(z)g(w)}{z-w}$ with $f\in H(E_1)$ and $g\in H(E_2)$. Let g be any function in $H(E_2)$ which belongs to the domain of multiplication by the independent variable in this space. Then $(z-w)g(z)\in H(E_2)$ and, hence, for any function $f\in H(E_1)$ and any w satisfying $f(w)\neq 0$

$$0=\left\langle Q(z),\frac{f(w)(z-w)g(z)-f(z)(w-w)g(w)}{z-w}\right\rangle_{H(E_2)}=\langle Q(z),f(w)g(z)\rangle_{H(E_2)}.$$

Because there has to be a function $f\in H(E_1)$ with $f(w)\neq 0$ by Corrolary 2.1.6 for all $w\in\mathbb{C}\backslash\mathbb{R}$, the function $Q(z)$ is orthogonal to domain of multiplication by z in $H(E_2)$. Theorem 2.2.8 yields that $Q(z)$ is strongly associated and, hence, $Q(z)=uA(z)+vB(z)$ with $u\overline{v}=\overline{u}v$. This shows that the orthogonal complement of all functions $\frac{f(w)g(z)-f(z)g(w)}{z-w}$ with $f\in H(E_1)$ and $g\in H(E_2)$ in $H(E_2)$ contains only the linear span of Q. On the other hand the orthogonal comlement of Q in $H(E_2)$ is the closed linear span of all functions $\frac{f(w)g(z)-f(z)g(w)}{z-w}$ and, hence, is contained in $H(E_1)$. By the assumption that $H(E_2)$ is not contained in $H(E_1)$ the function Q cannot belong to $H(E_1)$. Assume that P is a function in $H(E_1)$ which is orthogonal to all functions $\frac{f(w)g(z)-f(z)g(w)}{z-w}$. Especially P and Q are orthogonal to $\frac{P(w)Q(z)-P(z)Q(w)}{z-w}$ within $L^2(\mu)$. Hence,

$$\begin{aligned}
(w-\overline{w})&\left\|\frac{P(w)Q(z)-P(z)Q(w)}{z-w}\right\|^2_{L^2(\mu)}=\\
&=\int_{\mathbb{R}}(w-\overline{w})\frac{P(w)Q(t)-P(t)Q(w)}{t-w}\overline{\frac{P(w)Q(t)-P(t)Q(w)}{t-\overline{w}}}\,d\mu(t)\\
&=\int_{\mathbb{R}}\Bigg(\frac{(P(w)Q(t)-P(t)Q(w))(w-t)}{t-w}\overline{\frac{P(w)Q(t)-P(t)Q(w)}{t-\overline{w}}}\\
&\quad-\frac{P(w)Q(t)-P(t)Q(w)}{t-w}\frac{(\overline{P(w)Q(t)-P(t)Q(w)})(\overline{w}-t))}{t-\overline{w}}\Bigg)\,d\mu(t)\\
&=\int_{\mathbb{R}}\Bigg(-(P(w)Q(t)-P(t)Q(w))\overline{\frac{P(w)Q(t)-P(t)Q(w)}{t-\overline{w}}}
\end{aligned}$$

$$+ \frac{P(w)Q(t) - P(t)Q(w)}{t - w} \overline{(P(w)Q(t) - P(t)Q(w))} \Bigg) \, d\mu(t)$$
$$= 0$$

holds for any $w \in \mathbb{C}\backslash\mathbb{R}$. Hence, the function $P(z)$ belongs to the linear span of $Q(z)$ but does belong to $H(E_1)$. Because $Q(z)$ is not in the space $H(E_1)$ the function P has to vanish. Therefore, the orthogonal complement of all functions $\frac{f(w)g(z)-f(z)g(w)}{z-w}$ in the space $H(E_1)$ is $\{0\}$ and, hence, the closed linear span of these functions is the whole space. Because these functions belong to $H(E_2)$, the space $H(E_1)$ is isometrically contained in $H(E_2)$. $\square$

The Theorem remains true without the assumption of the nonexistence of an associated μ-almost everywhere vanishing function. At first glance this case seems quite easy to prove, because by Theorem 2.2.7 the existance of such a function Q_2 associated with $H(E_2)$ implies that $H(E_2)$ has to fill $L^2(\mu)$. Because $H(E_1)$ is isometrically contained in $L^2(\mu)$, for each function $f \in H(E_1)$ there must exist a function $\tilde{f} \in H(E_2)$ that is μ-almost everywhere equal to f. But it is hard to prove that these functions coincide on the whole complex plane. Actually, to prove this case some information about subspaces of de Brange spaces is needed that would go beyond the scope of this work.

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