Diese Dissertation haben begutachtet:

....

TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

DISSERTATION

Models for cyclic definitizable selfadjoint operators in Kreĭn spaces

ausgeführt zum Zwecke der Erlangung des akademischen Grades eines Doktors der technischen Wissenschaften unter der Leitung von

Ao.Univ.Prof. Dipl.-Ing. Dr. Harald Woracek
E101 - Institut für Analysis und Scientific Computing

eingereicht an der Technischen Universität Wien
bei der Fakultät für Mathematik und Geoinformation

von

Dipl.-Ing. Maximilian Kleinert
Matrikelnummer 9925330
Handelskai 300/3/86
1020 Wien

Datum | Unterschrift

Contents

Introduction

In [JLT] it was shown that a bounded cyclic selfadjoint operator in a Pontryagin space is unitarily equivalent to the operator of multiplication by the independent variable in some space $\Pi(\phi)$, where $\Pi(\phi)$ is a Pontryagin space generated by a distribution ϕ which belongs to a certain class $\mathcal{F}$ of linear functionals. In this thesis we investigate how this result can be generalized to the case of a bounded cyclic definitizable selfadjoint operator in a Kreĭn space.

In Chapter 1 we give a basic overview of distributions and extend the class of linear functionals $\mathcal{F}$, introduced in [JLT], to fit the spectral properties of a bounded definitizable selfadjoint operator in a Kreĭn space. The main difference to the Pontryagin space case is the real part of the spectrum. For a selfadjoint operator in a Pontryagin space with only real spectrum there exists a positive definitizing polynomial, which is no longer true in the situation of a bounded definitizing selfadjoint operator in a Kreĭn space. This reflects in the definition of the class of distributions corresponding to the real part of the spectrum, which is denoted by $\mathcal{F}(\mathbb{R})$. The condition in [JLT] that the distribution corresponding to the real part of the spectrum is a positive measure except on a finite subset M of $\mathbb{R}$ is weakened, to the requirement that on each closed interval which is a subset of $\mathbb{R} \setminus M$ the distribution is either a positive or negative measure, see Definition 1.5. The main result of the first chapter is an integral representation for a distribution of $\mathcal{F}(\mathbb{R})$ with a one element exception set, see Proposition 1.21.

In Chapter 2 we build a model for a distribution in $\varphi \in \mathcal{F}(\mathbb{R})$ with exception set $\{0\}$ using the data from some integral representation of φ. We define an inner product on the space of all polynomials by means of the distribution φ, and an embedding from the set of all polynomials into a space $L^2(\nu) \oplus \mathbb{C}^n$. On this space we explicitly define an indefinite inner product such that this embedding becomes isometric. The space obtained by factorizing out the isotropic part of the closure of the range of this embedding is defined as the model Kreĭn space, cf. Theorem 2.18. The model multiplication operator is defined such that it is compatible with the operator of multiplication by the independent variable on the space of all polynomials. Finally we show that the model spaces and operators constructed from different integral representations of φ are unitarily equivalent, cf. Theorem 2.19.

The main purpose of Chapter 3 is to construct an operator model for an arbitrary distribution of $\mathcal{F}(\mathbb{R})$, and to add possible non-real spectrum. This is a rather technical section. Given a distribution ϕ, we construct a Kreĭn space $\mathcal{K}_\phi$, a bounded cyclic definitizable selfadjoint operator A_ϕ, and an embedding ι_ϕ of the space of all polynomials into $\mathcal{K}_\varphi$, such that ι_ϕ has dense range and A_ϕ is compatible with the multiplication operator on the space of all polynomials, cf. Theorem 3.19 and Corollary 3.20.

In the last chapter, Chapter 4, we investigate the converse question. Given a bounded cyclic definitizable selfadjoint operator A in a Kreĭn space $\mathfrak{K}$, we construct a distribution ϕ. We show that A is weakly unitarily equivalent to the operator A_ϕ, cf. Theorem 4.4. Being weakly unitarily equivalent means that there exists an isometric mapping $U : \mathfrak{K} \to \mathcal{K}_\phi$ with dense domain and dense range, such that $A_\phi U = UA$. We show that under some additional properties on the spectrum of A, namely if each singular critical point of A has finite index, this weak unitary equivalence is bicontinuous, i.e. a strong unitary equivalence. Finally, we show that the distribution constructed from a model operator A_ϕ is nothing but ϕ.

In contrast to [JLT], where the situation in a Pontryagin space is treated, we consider a Kreĭn space and therefore the measure representing the distribution on intervals without critical points is either a positive or negative measure. This affects the result known in the Pontryagin space situation in so far that in general only a weak isometry is obtained. The methods used are mostly the same as in a Pontryagin space except that we make use of the geometry and spectral theory of a Kreĭn space. Further we do not restrict ourselves to a fixed representation for a distribution but show that every representation leads to the same object.

Chapter 1

Distributions of the class $\mathcal{F}$

1.1 Basic Theory of distributions

Let Ω be an open subset of $\mathbb{R}$. By $C^\infty(\Omega)$ we denote the space of complex valued functions on Ω which have derivatives of all orders. As usual we consider on $C^\infty(\Omega)$ the topology generated by the seminorms

$$p_N(f) := \max\{|f^{(n)}(x)| : x \in K_N, n \leq N\}, \quad N \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, f \in C^\infty(\Omega),$$

where K_i, $i \in \mathbb{N}$, are compact subsets of Ω such that $K_i \subseteq \mathring{K}_{i+1}$ and $\Omega = \bigcup_{i\in\mathbb{N}} K_i$. This topology we will denote by τ_∞. For a compact set $K \subseteq \Omega$ we denote by $\mathcal{D}_K(\Omega)$ the space of all $f \in C^\infty(\Omega)$ whose support lies in K. By $\mathcal{D}(\Omega)$ we denote the union of the spaces $\mathcal{D}_K(\Omega)$ where K ranges over all compact subsets of Ω and call it the *test function space*. Consider on each $\mathcal{D}_K(\Omega)$ the subspace topology induced by the topology τ_∞ on $C^\infty(\Omega)$ and denote it by τ_K. Further define

$$\mathfrak{W} := \left\{ W \subseteq \mathcal{D}(\Omega) : \begin{array}{c} W \text{ is convex and balanced} \\ W \cap \mathcal{D}_K \in \tau_K \ \forall K \subseteq \Omega \text{ compact} \end{array} \right\}$$

and let τ be the collection of all unions of sets of the form $f + W$, with $f \in \mathcal{D}(\Omega)$ and $W \in \mathfrak{W}$. Then $\big(\mathcal{D}(\Omega), \tau\big)$ is a locally convex topological vector space and $\mathfrak{W}$ is a local base for τ. A *distribution* on Ω is a continuous linear functional on $\mathcal{D}(\Omega)$ with respect to the topology τ. The space of distributions on Ω will be denoted by $\mathcal{D}'(\Omega)$.
A distribution is called *real* if it takes real values on real test functions.
For $m \in \mathbb{N}_0$ denote by $C_c^m(\Omega)$ the space of complex valued functions on Ω with compact support which have derivatives up to order m. For $f \in C_c^m(\Omega)$, $m \in \mathbb{N}_0$, we introduce the norms

$$\|f\|_m := \max\{|f^{(j)}(x)| : \ x \in \Omega, \ 0 \leq j \leq m\} = \max\{\|f^{(j)}\|_\infty : \ 0 \leq j \leq m\},$$

where $\|.\|_\infty$ denotes the uniform norm.
A linear functional φ on $\mathcal{D}(\Omega)$ belongs to $\mathcal{D}'(\Omega)$ if and only if, for every compact set K in Ω, there exists $N \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and a constant $C_K < \infty$ such that

$$|\varphi(f)| \leq C_K \|f\|_N, \quad \text{for all } f \in \mathcal{D}_K(\Omega). \tag{1.1.1}$$

If φ is such that one N will do for all K, then φ is said to have *finite order*. In this case the smallest number $N \in \mathbb{N}_0$ for which (1.1.1) is satisfied for all K is called the *order* of φ. If no N will do for all K, then φ is said to have *infinite order*. The order of φ is denoted by $\operatorname{ord}(\varphi)$. To characterize the order of a distribution it is useful to introduce further function spaces similar to the spaces $\mathcal{D}_K(\Omega)$ and $\mathcal{D}(\Omega)$, respectively. For a compact subset K of Ω and $m \in \mathbb{N}_0$ we define

$$\mathcal{D}_K^m(\Omega) := \{f \in C^m(\Omega) : \ \operatorname{supp} f \subseteq K\},$$
$$\mathcal{D}^m(\Omega) := \bigcup_{\substack{K \subseteq \Omega \\ K \text{ compact}}} \mathcal{D}_K^m(\Omega).$$

Note that $\mathcal{D}^m(\Omega)$ is precisely the space of functions of class C^m having compact support. Equip the spaces $\mathcal{D}_K^m(\Omega)$ with the norm $\|.\|_m$ and denote the induced topology by τ_K^m. Analogously to the construction of the topology on $\mathcal{D}(\Omega)$ define

$$\mathfrak{W}^m := \left\{ W \subseteq \mathcal{D}^m(\Omega) : \begin{array}{c} W \text{ is convex and balanced} \\ W \cap \mathcal{D}_K^m \in \tau_K^m \ \forall K \subseteq \Omega \text{ compact} \end{array} \right\}$$

and let τ^m be the collection of all unions of sets of the form $f + W$, with $f \in \mathcal{D}^m(\Omega)$ and $W \in \mathfrak{W}^m$. We consider on each space $\mathcal{D}^m(\Omega)$, $m \in \mathbb{N}_0$, the topology τ^m.

The following table ($\Omega \subseteq \mathbb{R}$ open, $m \in \mathbb{N}_0$, $K \subseteq \mathbb{R}$ compact) gives an overview of the introduced spaces and the local bases for the corresponding topologies.

space	top.	local base
$C^\infty(\Omega)$	τ_∞	$\left\{V_N : \ N \in \mathbb{N}\right\}$, with $V_N := \left\{f \in C^\infty(\Omega) : p_N(f) < \frac{1}{N}\right\}$
$\mathcal{D}_K(\Omega)$	τ_K	$\left\{\tilde{V}_N : \ N \in \mathbb{N}\right\}$, with $\tilde{V}_N := \left\{f \in \mathcal{D}_K(\Omega) : \|f\|_N < \frac{1}{N}\right\}$
$\mathcal{D}(\Omega)$	τ	$\left\{W \subseteq \mathcal{D}(\Omega) : \begin{array}{c} W \text{ is convex and balanced} \\ W \cap \mathcal{D}_K \in \tau_K \ \forall K \subseteq \Omega \text{ compact} \end{array}\right\}$
$\mathcal{D}_K^m(\Omega)$	τ_K^m	$\left\{\tilde{V}'_N : \ N \in \mathbb{N}\right\}$, with $\tilde{V}'_N := \left\{f \in \mathcal{D}_K^m(\Omega) : \|f\|_N < \frac{1}{N}\right\}$
$\mathcal{D}^m(\Omega)$	τ^m	$\left\{W \subseteq \mathcal{D}^m(\Omega) : \begin{array}{c} W \text{ is convex and balanced} \\ W \cap \mathcal{D}_K^m \in \tau_K^m \ \forall K \subseteq \Omega \text{ compact} \end{array}\right\}$

The distributions of order at most $m \in \mathbb{N}_0$ can be characterized as follows:

1.1 Proposition. *Let φ be a distribution on Ω and $m \in \mathbb{N}_0$. Then φ has order at most m if and only if φ can be extended to a continuous linear functional on $\mathcal{D}^m(\Omega)$. In this case the extension is unique.*

The proof uses the fact that $\mathcal{D}(\Omega)$ is dense in $\mathcal{D}^m(\Omega)$ and can be found in [HL, Proposition 3.1, p. 268]. By this proposition we can identify the space of distributions of order at most m, $m \in \mathbb{N}_0$, on Ω with the space of continuous linear forms on $\mathcal{D}^m(\Omega)$, which we denote by $\mathcal{D}'^m(\Omega)$.

For a locally integrable complex function g in Ω, define

$$\varphi_g(f) := \int_\Omega fg\,d\lambda, \quad f \in \mathcal{D}(\Omega),$$

where λ denotes the Lebesgue measure. Then for all $f \in \mathcal{D}_K(\Omega)$ it holds that $|\varphi_g(f)| \leq \left(\int_K |g| d\lambda\right) \|f\|_0$, so φ_g is a distribution of order zero. Similarly we can define a distribution of order zero corresponding to a complex or positive Borel measure μ on Ω by

$$\varphi_\mu(f) := \int_\Omega f d\mu.$$

Recall that a Borel measure is a positive measure μ defined on the σ-algebra of all Borel sets in a locally compact Hausdorff space X which satisfies $\mu(K) < \infty$, for every compact set K. If a locally compact Hausdorff space X has a countable base it follows that every Borel measure is regular. Therefore any Borel measure on any subset of $\mathbb{R}$ is regular. If μ is a negative or complex measure defined on the σ-algebra of all Borel sets in a locally compact Hausdorff space X we call it a *Borel measure* if its total variation $|\mu|$ is a Borel measure. By $\mathfrak{M}(\Omega)$ we denote set of all complex Borel measures on Ω and by $\mathfrak{B}(\Omega)$ the Borel σ-algebra on Ω.

The support of a distribution φ is defined as:

$$\operatorname{supp}\varphi := \Omega \setminus \bigcup \big\{\omega \text{ is an open subset of } \Omega : \ \varphi(f) = 0, \ \forall f \in \mathcal{D}(\omega)\big\}.$$

Of special interest for the next section are distributions with compact support. They can be characterized as follows:

1.2 Proposition. *Let φ be a distribution on Ω. Then φ has compact support if and only if φ can be extended to a continuous linear functional on $C^\infty(\Omega)$. In this case the extension is unique.*

A proof can be found in [HL, Proposition 3.3, p. 282].

For $p \in \mathbb{N}_0$ and $\varphi \in \mathcal{D}'(\Omega)$ the formula

$$\varphi^{(p)}(f) := (-1)^p \varphi\big(f^{(p)}\big),$$

defines a linear functional on $\mathcal{D}(\Omega)$, which is called the *p-th (distributional) derivative of φ*. If $|\varphi(f)| \le C\|f\|_N$ for all $f \in \mathcal{D}_K(\Omega)$, then it follows that

$$|\varphi^{(p)}(f)| \le C\|f^{(p)}\|_N \le C\|f\|_{N+p}, \quad f \in \mathcal{D}_K(\Omega),$$

which implies that $\varphi^{(p)}$ is a distribution. Further it holds $\operatorname{supp}\varphi^{(p)} \subseteq \operatorname{supp}\varphi$ for all $p \in \mathbb{N}_0$, c.f. [HL, Proposition 2.4, p. 294]. Moreover if $\varphi \in \mathcal{D}'^m(\Omega)$ it follows that $\varphi^{(p)} \in \mathcal{D}'^{m+p}(\Omega)$.

The following proposition characterizes elements of class $\mathcal{D}'^m$, $m \in \mathbb{N}_0$. A proof can be found in [E, Theorem 4.8.1, p. 318].

1.3 Proposition. *Let $m \in \mathbb{N}_0$ and $\varphi \in \mathcal{D}'^m(\Omega)$. Then there exist distributions $\varphi_j \in \mathcal{D}'^0(\Omega)$, $j = 0, \dots, m$, such that*

$$\varphi = \sum_{j=0}^m \varphi_j^{(j)}. \tag{1.1.2}$$

If φ has compact support K, the distributions φ_j, $j = 1, \dots, m$, may be assumed to have their supports contained in any preassigned neighborhood of K. Conversely, any finite sum of the form (1.1.2) is an element of $\mathcal{D}'^m(\Omega)$.

1.4 Corollary. *Let φ be a distribution on Ω with compact support. Then the order of φ is finite and there exist regular complex Borel measures μ_j, $j = 0, \dots, \operatorname{ord}(\varphi)$, with compact support on Ω such that*

$$\varphi = \sum_{j=0}^{\operatorname{ord}(\varphi)} \mu_j^{(j)}.$$

Proof. By [R, Theorem 6.24, p. 164] a distribution with compact support has finite order. Denote by $m \in \mathbb{N}_0$ the order of the distribution φ and by $K \subseteq \Omega$ its support. Fix compact subsets K', K'' of Ω such that $K \subseteq \mathring{K}' \subseteq K' \subseteq \mathring{K}'' \subseteq K''$. By Proposition 1.3 there exist distributions $\varphi_j \in \mathcal{D}'^0(\Omega)$, $j = 0, \dots, m$, such that $\varphi = \sum_{j=0}^m \varphi_j^{(j)}$ and $\operatorname{supp} \varphi_j \subseteq \mathring{K}'$ for $j = 0, \dots, m$. Every φ_j is an element of $\mathcal{D}'^0(\Omega)$, so every φ_j is a continuous linear functional on $\mathcal{D}^0(\Omega)$. Hence there exist constants C_j, $j = 0, \dots, m$, such that

$$|\varphi_j(f)| \le C_j \|f\|_0, \quad f \in \mathcal{D}^0_{K''}(\Omega).$$

Let $\{\omega_1, \omega_2\}$ be a partition of unity subordinate to the open cover $\{\mathring{K}'', (K')^c\}$. Then for $f \in \mathcal{D}^0(\Omega)$ it holds $\omega_1 f \in \mathcal{D}^0_{K''}$ and $\varphi_j(f) = \varphi_j(\omega_1 f)$, $j = 0, \dots, m$. This yields

$$|\varphi_j(f)| \le C_j \|\omega_1 f\|_0 \le C_j \|f\|_\infty, \quad j = 0, \dots, m, \ f \in \mathcal{D}^0(\Omega).$$

Therefore φ_j, $j = 0, \dots, m$, is a continuous linear functional on $\mathcal{D}^0(\Omega) = C_c(\Omega)$. By the Riesz Representation Theorem there exist regular Borel measures μ_j, $j = 0, \dots, m$, such that

$$\varphi_j(f) = \int_\Omega f \, d\mu_j, \quad j = 0, \dots, m, \ f \in C_c(\Omega).$$

Therefore $\varphi_j = \mu_j$, $j = 0, \dots, m$, in the distributional sense. ❑

1.2 Distributions of the class $\mathcal{F}(\mathbb{R})$

Let φ be a distribution on $\mathbb{R}$ and $I \subseteq \mathbb{R}$ an interval. We say φ restricted to I is a Borel measure if there exists a Borel measure μ supported on a subset of I such that

$$\varphi(f) = \int_I f \, d\mu, \quad f \in \mathcal{D}(\mathbb{R}) \text{ with } \operatorname{supp} f \subseteq I.$$

1.5 Definition. For a finite set $M \subseteq \mathbb{R}$ denote by $\mathcal{F}(\mathbb{R}, M)$ the class of real distributions φ with compact support, such that for all $[a, b] \subseteq \mathbb{R} \setminus M$ the distribution φ restricted to $[a, b]$ is a finite positive or negative Borel measure. Then we define

$$\mathcal{F}(\mathbb{R}) := \bigcup_{\substack{M \subseteq \mathbb{R} \\ M \text{ is finite}}} \mathcal{F}(\mathbb{R}, M).$$

If the cardinality of M equals 1, i.e. $M = \{\alpha\}$, $\alpha \in \mathbb{R}$, we write abbreviatory $\mathcal{F}(\mathbb{R}, \alpha)$ for $\mathcal{F}(\mathbb{R}, \{\alpha\})$.

By Proposition 1.2 every $\varphi \in \mathcal{F}(\mathbb{R})$ has a unique extension to a continuous linear functional on $C^\infty(\Omega)$. In this sense we can apply φ to elements of $C^\infty(\Omega)$.

1.6 Definition. Let $\varphi \in \mathcal{F}(\mathbb{R})$ and M a finite subset of $\mathbb{R}$ such that $\varphi \in \mathcal{F}(\mathbb{R}, M)$. We define $\mathfrak{Z}_M$ as the set of all components of $\mathbb{R} \setminus M$. We call M *minimal* if there exists no $N \subseteq \mathbb{R}$ such that $|N| < |M|$ and $\varphi \in \mathcal{F}(\mathbb{R}, N)$.

Note that for $\varphi \in \mathcal{F}(\mathbb{R})$ a minimal set M such that $\varphi \in \mathcal{F}(\mathbb{R}, M)$ does not have to be unique.

1.7 Proposition. *Let $\varphi \in \mathcal{F}(\mathbb{R})$ and M a finite subset of $\mathbb{R}$ such that $\varphi \in \mathcal{F}(\mathbb{R}, M)$. Then for every $Z \in \mathfrak{Z}_M$ there exists a positive or negative (possibly unbounded) Borel measure μ_Z on Z such that*

$$\varphi(f) = \int_Z f d\mu_Z, \quad f \in \mathcal{D}(\mathbb{R}) \text{ with } \operatorname{supp} f \subseteq Z.$$

Proof. Let $Z \in \mathfrak{Z}_M$ and assume first that Z is bounded. In this case $Z = (a, b)$ with $a, b \in \mathbb{R}$. For every $n \in \mathbb{N}$ define $a_n := a + \frac{1}{n}$ and $b_n := b - \frac{1}{n}$. Then by definition for every $n \in \mathbb{N}$ there exists a finite positive or negative measure μ_n on $[a_n, b_n]$ such that

$$\varphi(f) = \int_{[a_n,b_n]} f d\mu_n, \quad f \in \mathcal{D}(\mathbb{R}) \text{ with } \operatorname{supp} f \subseteq [a_n, b_n].$$

For any natural number m less than n and any test function f with $\operatorname{supp} f \subseteq [a_m, b_m]$ it follows that

$$\int_{[a_m,b_m]} f d\mu_m = \varphi(f) = \int_{[a_n,b_n]} f d\mu_n = \int_{[a_m,b_m]} f d\mu_n.$$

By a density argument this implies $\mu_n(\Delta) = \mu_m(\Delta)$ for all $\Delta \in \mathfrak{B}\big([a_m, b_m]\big)$. Clearly μ_n is a positive (negative) measure if and only if μ_m is a positive (negative) measure. Now we define a measure on (a, b) by

$$\mu(\Delta) := \lim_{n\to\infty} \mu_n\big(\Delta \cap [a_n, b_n]\big), \quad \Delta \in \mathfrak{B}\big((a, b)\big).$$

Since $(\mu_n)_{n\in\mathbb{N}}$ is a monotonic sequence, μ is a well-defined set function from (a, b) to $[0, \infty]$, or $[-\infty, 0]$ respectively. We need to show that μ is a Borel measure on (a, b). Assume that $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of positive (and of course finite) measures. In the case where all measures μ_n, $n \in \mathbb{N}$, are negative just consider the sequence $(-\mu_n)_{n\in\mathbb{N}}$ and the same argument applies. Now consider a disjoint countable collection $(A_i)_{i\in\mathbb{N}}$ of members of $\mathfrak{B}\big((a, b)\big)$ then the monotone convergence theorem shows that

$$\mu\Big(\bigcup_{i=1}^\infty A_i\Big) = \lim_{n\to\infty} \mu_n\Big(\bigcup_{i=1}^\infty A_i \cap [a_n, b_n]\Big) = \lim_{n\to\infty} \sum_{i=1}^\infty \mu_n\big(A_i \cap [a_n, b_n]\big) =$$

$$= \sum_{i=1}^\infty \lim_{n\to\infty} \mu_n\big(A_i \cap [a_n, b_n]\big) = \sum_{i=1}^\infty \mu(A_i).$$

Let K be a compact subset of (a,b), then there exists a natural number n such that $K \subseteq [a_n, b_n]$. Since μ_n is a finite measure it follows that

$$\mu(K) \leq \mu([a_n, b_n]) = \mu_n([a_n, b_n]) < \infty,$$

thus μ is a Borel measure on (a,b). Further for every $f \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp} f \subseteq (a,b)$ it holds that

$$\varphi(f) = \int\limits_{[a_n,b_n]} f d\mu_n = \int\limits_{(a,b)} f d\mu,$$

when n is chosen such that $\operatorname{supp} f \subseteq [a_n, b_n]$. The cases $Z = (-\infty, b)$ or $Z = (a, \infty)$, $a, b \in \mathbb{R}$, can be proven similarly using the fact, that φ has compact support. If the only component is $Z = \mathbb{R}$ the assertion of this proposition follows immediately by definition. ❑

1.8 Definition. Let $\varphi \in \mathcal{F}(\mathbb{R})$ and M a finite subset of $\mathbb{R}$ such that $\varphi \in \mathcal{F}(\mathbb{R}, M)$. For any $\alpha \in M$ denote by $Z_\alpha^\pm$ the components of $\mathbb{R} \setminus M$ such that $\sup Z_\alpha^- = \alpha = \inf Z_\alpha^+$. For $a \in \mathbb{R} \setminus M$ we denote by Z_a the element of $\mathfrak{Z}_M$ such that $a \in Z_a$. For $Z \in \mathfrak{Z}_M$ we denote by μ_Z the measure from Proposition 1.7 corresponding to the component Z.

1.9 Corollary. *Let $\varphi \in \mathcal{F}(\mathbb{R})$ and M a finite subset of $\mathbb{R}$ such that $\varphi \in \mathcal{F}(\mathbb{R}, M)$. Define $\tilde{M} := M \cap \operatorname{supp} \varphi$, then it holds*

$$\operatorname{supp} \varphi = \bigcup_{Z \in \mathfrak{Z}_M} \operatorname{supp} \mu_{Z_M} \cup \tilde{M}.$$

1.10 Remark. Let $\varphi \in \mathcal{F}(\mathbb{R})$ and M a finite subset of $\mathbb{R}$ such that $\varphi \in \mathcal{F}(\mathbb{R}, M)$. Assume that M consists of one element $\alpha \in \mathbb{R}$. If M is not minimal, i.e. $\varphi \in \mathcal{F}(\mathbb{R}, \emptyset)$, then φ is either a positive or negative measure on $\mathbb{R}$ and clearly the support of φ coincides with the support of $\mu_\mathbb{R}$. If $M = \{\alpha\}$ is minimal, then $\mu_{Z_\alpha^+}$ and $\mu_{Z_\alpha^-}$ can have different sign. In this case the following situations can occur:

·) α is an element of $\operatorname{supp} \mu_{Z_\alpha^+}$ and $\operatorname{supp} \mu_{Z_\alpha^-}$,

·) α is an element of $\operatorname{supp} \mu_{Z_\alpha^+}$ or $\operatorname{supp} \mu_{Z_\alpha^-}$,

·) α is not an element of $\operatorname{supp} \mu_{Z_\alpha^+}$ and $\operatorname{supp} \mu_{Z_\alpha^-}$ but an element of $\operatorname{supp} \varphi$,

·) α is not an element of $\operatorname{supp} \mu_{Z_\alpha^+}$, $\operatorname{supp} \mu_{Z_\alpha^-}$ and $\operatorname{supp} \varphi$.

It follows that the set $\operatorname{supp} \varphi \setminus \big(\operatorname{supp} \mu_{Z_\alpha^+} \cup \operatorname{supp} \mu_{Z_\alpha^-}\big)$ is either empty or consists of the element α.

We want to consider the question how we can restrict an element of $\mathcal{F}(\mathbb{R})$ to real intervals. Let $\varphi \in \mathcal{F}(\mathbb{R}, M)$, where M is a finite subset of $\mathbb{R}$. Let Δ be an open, half-open or closed interval of $\mathbb{R}$ such that $\partial\Delta \cap M = \emptyset$ and $\mu_Z(\partial\Delta \cap Z) = 0$ for every $Z \in \mathfrak{Z}_M$. We denote the set of all these intervals with $\mathfrak{D}_{\varphi,M}$. For $\Delta \in \mathfrak{D}_{\varphi,M}$ we define

$$\mathfrak{G}_\Delta := \left\{ \{(-\infty, a), \Omega, (b, \infty)\} : \begin{array}{c} \Omega \text{ is an open interval of } \mathbb{R},\ a, b \in \mathbb{R}, \\ M \cap \Delta \subsetneq [a,b] \subsetneq \Omega \subseteq \Delta \end{array} \right\}.$$

Now we can define the restriction of φ to elements of $\mathfrak{D}_{\varphi,M}$. Let $\Delta \in \mathfrak{D}_{\varphi,M}$ and $\Gamma \in \mathfrak{G}_\Delta$, i.e. there is an open interval Ω and $a,b \in \mathbb{R}$ such that $M \cap \Delta \subsetneq [a,b] \subsetneq \Omega$ and $\Gamma = \{(-\infty,a),\Omega,(b,\infty)\}$. Let $\gamma := \{\omega_l,\omega_0,\omega_r\} \subseteq \mathcal{D}(\mathbb{R})$ be a partition of unity subordinate to the cover Γ, then we define

$$\varphi|_\Delta(f) := \varphi(f\omega_0) + \int\limits_{Z_a} f\omega_l \chi_\Delta d\mu_{Z_a} + \int\limits_{Z_b} f\omega_r \chi_\Delta d\mu_{Z_b}, \quad f \in \mathcal{D}(\mathbb{R}),$$

where χ_Δ denotes the indicator function on Δ. We have to show that this construction is independent of the choice of Γ and γ. This is the assertion of the following lemma:

1.11 Lemma. *Let $\varphi \in \mathcal{F}(\mathbb{R})$ and M a finite subset of $\mathbb{R}$ such that $\varphi \in \mathcal{F}(\mathbb{R},M)$. For each $\Delta \in \mathfrak{D}_{\varphi,M}$ the functional $\varphi|_\Delta$ is independent of the choice of the open cover $\Gamma \in \mathfrak{G}_\Delta$ and the partition of unity γ. Further $\varphi|_\Delta$ is a distribution on $\mathbb{R}$.*

Proof. For $i = 1,2$ let Ω_i be open intervals of $\mathbb{R}$ and $a_i, b_i \in \mathbb{R}$, such that $\Gamma_i := \{(-\infty,a_i),\Omega_i,(b_i,\infty)\} \in \mathfrak{G}_\Delta$ and choose partitions of unity $\gamma_i := \{\omega_{i,l},\omega_{i,0},\omega_{i,r}\}$ subordinate to Γ_i. Now define $a := \max\{a_1,a_2\}$, $b := \min\{b_1,b_2\}$ and $\Omega := \Omega_1 \cup \Omega_2$, then clearly $\Gamma := \{(-\infty,a),\Omega,(b,\infty)\} \in \mathfrak{G}_\Delta$. For a partition of unity $\gamma := \{\omega_l,\omega_0,\omega_r\}$ subordinate to Γ it holds

$$\omega_0 = \omega_{i,l} - \omega_l + \omega_{i,0} + \omega_{i,r} - \omega_r, \quad i = 1,2.$$

Clearly $Z_{a_i} = Z_a$ and $Z_{b_i} = Z_b$ for $i = 1,2$. Further $\operatorname{supp}(\omega_{i,l} - \omega_l) \subseteq [\inf\Omega, a] \subseteq \Delta$ and $\operatorname{supp}(\omega_{i,r} - \omega_r) \subseteq [b, \sup\Omega] \subseteq \Delta$, $i = 1,2$ and therefore for every $f \in \mathcal{D}(\mathbb{R})$ it follows that

$$\varphi(f\omega_0) = \varphi(f\omega_{i,0}) + \varphi(f(\omega_{i,l} - \omega_l)) + \varphi(f(\omega_{i,r} - \omega_r)) =$$
$$= \varphi(f\omega_{i,0}) + \int\limits_{Z_a} f(\omega_{i,l} - \omega_l)\chi_\Delta d\mu_{Z_a} + \int\limits_{Z_b} f(\omega_{i,r} - \omega_r)\chi_\Delta d\mu_{Z_b}, \quad i = 1,2.$$

This implies that the definition of $\varphi|_\Delta$ is independent of the choice of the open cover Γ and the partition of unity γ. It remains to show that $\varphi|_\Delta \in \mathcal{D}'(\mathbb{R})$. Let $K \subset \mathbb{R}$ be a compact set, then there exists a constant $C_K > 0$ and $N \in \mathbb{N}_0$ such that $|\varphi(f)| \leq C_K \|f\|_N$ for $f \in \mathcal{D}_K(\mathbb{R})$, since φ is a distribution. This implies

$$\big|\varphi|_\Delta(f)\big| \leq |\varphi(f\omega_0)| + \int\limits_{Z_a} |f|\omega_l\chi_\Delta d|\mu_{Z_a}| + \int\limits_{Z_b} |f|\omega_r\chi_\Delta d|\mu_{Z_b}| \leq$$
$$\leq C_K\|f\omega_0\|_N + \|f\|_N\Big(|\mu_{Z_a}|\big((-\infty,a]\cap\Delta\big) + |\mu_{Z_b}|\big([b,\infty)\cap\Delta\big)\Big), \ f \in \mathcal{D}_K(\mathbb{R}).$$

By the Leibniz formula it follows that there exists a constant $C' > 0$ such that $\|f\omega_0\|_N \leq C'\|f\|_N$. This gives the following estimate

$$\big|\varphi|_\Delta(f)\big| \leq \Big(C_K C' + |\mu_{Z_a}|\big((-\infty,a]\cap\Delta\big) + |\mu_{Z_b}|\big([b,\infty)\cap\Delta\big)\Big)\|f\|_N, \ f \in \mathcal{D}_K(\mathbb{R}),$$

which shows that $\varphi|_\Delta$ is a distribution on $\mathbb{R}$. ❑

1.12 Definition. Let $\varphi \in \mathcal{F}(\mathbb{R})$ and M a finite subset of $\mathbb{R}$ such that $\varphi \in \mathcal{F}(\mathbb{R},M)$. A system of disjoint ordered intervals $\Delta_i \in \mathfrak{D}_{\varphi,M}$, $i = 1,\dots,n$, i.e. $\Delta_i \leq \Delta_{i+1}$ for $i = 1,\dots,n-1$, such that $\mathbb{R} \setminus \bigcup_{j=1}^n \Delta_j$ is a finite set is called a *φ-M-decomposition of* $\mathbb{R}$.

1.13 Lemma. *Let $\varphi \in \mathcal{F}(\mathbb{R})$ and M a finite subset of $\mathbb{R}$ such that $\varphi \in \mathcal{F}(\mathbb{R}, M)$ and Δ, $\Delta' \in \mathfrak{D}_{\varphi,M}$. Then it holds:*

(*i*) *If $\Delta \cap M = \emptyset$, then there exists $Z \in \mathfrak{Z}_M$ such that $\Delta \subseteq Z$ and*

$$\varphi|_\Delta(f) = \int_\Delta f d\mu_Z, \quad f \in \mathcal{D}(\mathbb{R}).$$

(*ii*) *The support of $\varphi|_\Delta$ is a subset of $\overline{\Delta}$.*

(*iii*) *The functional $\varphi|_\Delta$ belongs to the class $\mathcal{F}(\mathbb{R}, M \cap \Delta)$ and $\varphi|_\Delta(f) = \varphi(f)$ for $f \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp} f \subseteq \overline{\Delta}$.*

(*iv*) $\big(\varphi|_\Delta\big)|_\Delta = \varphi|_\Delta$.

(*v*) $\varphi|_\Delta = \varphi|_{\mathring{\Delta}} = \varphi|_{\overline{\Delta}}$.

(*vi*) *If $\Delta \cap \Delta' = \emptyset$ and $\overline{\Delta \cup \Delta'} \in \mathfrak{D}_{\varphi,M}$ then $\varphi|_{\overline{\Delta \cup \Delta'}} = \varphi|_\Delta + \varphi|_{\Delta'}$.*

(*vii*) *Let $\Delta_1, \ldots, \Delta_n$ be a φ-M-decomposition of $\mathbb{R}$. Then the distribution φ can be written as*

$$\varphi(f) = \sum_{j=1}^n \varphi|_{\Delta_j}(f), \quad f \in \mathcal{D}(\mathbb{R}).$$

Proof.

ad (*i*) : The existence of $Z \in \mathfrak{Z}_M$ with $\Delta \subseteq Z$ is obvious. Choose a partition of unity $\gamma := \{\omega_l, \omega_0, \omega_r\}$ subordinate to the cover $\Gamma := \{(-\infty, a), \Omega, (b, \infty)\} \in \mathfrak{G}_\Delta$. Then $Z_a = Z_b = Z$ and for $f \in \mathcal{D}(\mathbb{R})$ it holds

$$\varphi|_\Delta(f) = \varphi(f\omega_0) + \int_Z f\omega_l\chi_\Delta d\mu_Z + \int_Z f\omega_r\chi_\Delta d\mu_Z = \int_\Delta f d\mu_Z.$$

ad (*ii*) : For any open set $O \subseteq \mathbb{R} \setminus \overline{\Delta}$ it clearly holds $\varphi|_\Delta(f) = 0$ for every $f \in \mathcal{D}(O)$, hence $\operatorname{supp} \varphi|_\Delta \subseteq \overline{\Delta}$.

ad (*iii*) : By Lemma 1.11 $\varphi|_\Delta$ is a distribution and by (*ii*) $\varphi|_\Delta$ has compact support. We have to show that for every $[a, b] \in \mathbb{R} \setminus (M \cap \Delta)$ the distribution $\varphi|_\Delta$ restricted to $[a, b]$ is a finite positive or negative Borel measure.
Let $[a, b] \in \mathbb{R} \setminus (M \cap \Delta)$ and $f \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp} f \subseteq [a, b]$. If $M \cap \Delta = \emptyset$ by (*i*) there exists $Z \in \mathfrak{Z}_M$ such that $\Delta \subseteq Z$ and

$$\varphi|_\Delta(f) = \int_\Delta f d\mu_Z = \int_{[a,b]} f\chi_{\Delta\cap[a,b]} d\mu_Z.$$

Hence $\varphi|_\Delta$ restricted to $[a, b]$ is a finite positive or negative Borel measure.
If $M \cap \Delta \neq \emptyset$, then there exists an element $\alpha \in \mathbb{R}$ such that $M \cap \Delta = \{\alpha\}$. Choose an open interval Ω and $c, d \in \mathbb{R}$ such that $\alpha \in (c, d) \subsetneq [c, d] \subsetneq \Omega \subseteq \Delta$ and $\Omega \cap [a, b] = \emptyset$. Assume first $b < \inf \Omega$ and choose a partition of unity $\gamma := \{\omega_l, \omega_0, \omega_r\}$ subordinate to the cover $\Gamma := \{(-\infty, c), \Omega, (d, \infty)\} \in \mathfrak{G}_\Delta$.

Then $\omega_l \equiv 1$ on $[a,b]$ and ω_0, ω_r vanishes on $[a,b]$. Therefore we have for every $f \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp} f \subseteq [a,b]$

$$\varphi|_\Delta(f) = \int\limits_{Z_c} f\omega_l\chi_\Delta d\mu_{Z_c} = \int\limits_{[a,b]} f\chi_{Z_c\cap\Delta\cap[a,b]} d\mu_{Z_c},$$

which shows that $\varphi|_\Delta$ restricted to $[a,b]$ is a finite positive or negative measure. The case $a > \sup\Omega$ is proven analogously. Therefore $\varphi|_\Delta \in \mathcal{F}(\mathbb{R}, M\cap\Delta)$.
If $f \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp} f \subseteq \overline{\Delta}$, then it holds

$$\varphi|_\Delta(f) = \varphi(f\omega_0) + \int\limits_{Z_c} f\omega_l d\mu_{Z_c} + \int\limits_{Z_d} f\omega_r\mu_{Z_d} = \varphi(f\omega_0) + \varphi(f\omega_l) + \varphi(f\omega_r) = \varphi(f).$$

ad (iv) : Assume $\Delta\cap M = \emptyset$. Choose an element $\Gamma := \{(-\infty,a),\Omega,(b,\infty)\} \in \mathfrak{G}_\Delta$ with corresponding partition of unity $\gamma := \{\omega_l,\omega_0,\omega_r\}$. Let $Z \in \mathfrak{Z}_M$ such that $\Delta \subseteq Z$. Since $\Delta\cap M = \emptyset$ we have $Z_a = Z_b = Z$. By (i) it follows that

$$\big(\varphi|_\Delta\big)|_\Delta(f) = \varphi|_\Delta(f\omega_0) + \int\limits_Z f\omega_l\chi_\Delta d\mu_Z + \int\limits_Z f\omega_r\chi_\Delta d\mu_Z = \int\limits_\Delta f d\mu_Z = \varphi|_\Delta(f).$$

If $\Delta\cap M \neq \emptyset$, then there exists $\alpha \in \mathbb{R}$ such that $\Delta\cap M = \{\alpha\}$. Choose an element $\Gamma := \{(-\infty,a),\Omega,(b,\infty)\} \in \mathfrak{G}_\Delta$ with corresponding partition of unity $\gamma := \{\omega_l,\omega_0,\omega_r\}$. The restriction of φ to Δ is given by

$$\varphi|_\Delta(f) = \varphi(f\omega_0) + \int\limits_{Z_a} f\omega_l\chi_\Delta d\mu_{Z_a} + \int\limits_{Z_b} f\omega_r\chi_\Delta d\mu_{Z_b}, \quad f \in \mathcal{D}(\mathbb{R}),$$

Denote the measures corresponding to $\varphi|_\Delta$ on the elements of $\mathfrak{Z}_{M\cap\Delta}$, i.e. on $\{(-\infty,\alpha),(\alpha,\infty)\}$, by μ_{α^-} and μ_{α^+}. By (iii) it follows that μ_α^- (μ_α^+) and μ_{Z_a} (μ_{Z_b}) coincide on Δ. Therefore for every $f \in \mathcal{D}(\mathbb{R})$ it holds

$$\begin{aligned}\big(\varphi|_\Delta\big)_\Delta(f) &= \varphi|_\Delta(f\omega_0) + \int\limits_{(-\infty,\alpha)} f\omega_l\chi_\Delta d\mu_{\alpha^-} + \int\limits_{(\alpha,\infty)} f\omega_r\chi_\Delta d\mu_{\alpha^+} = \\ &= \varphi(f\omega_0\omega_0) + \int\limits_{Z_a} f(\omega_0\omega_l+\omega_l)\chi_\Delta d\mu_{Z_a} + \int\limits_{Z_b} f(\omega_0\omega_r+\omega_r)\chi_\Delta d\mu_{Z_b} = \varphi|_\Delta(f),\end{aligned}$$

since $\{\omega_0\omega_l+\omega_l, \omega_0\omega_0, \omega_0\omega_r+\omega_r\}$ is a partition of unity subordinate to Γ.

ad (v) : This follows immediately from the definition of the restriction and the fact that $\mu_Z(\partial\Delta\cap Z)$ for every $Z \in \mathfrak{Z}_M$.

ad (vi) : Assume $\Delta < \Delta'$. Since Δ, Δ' and $\overline{\Delta\cup\Delta'}$ are elements of $\mathfrak{D}_{\varphi,M}$ and $\Delta\cap\Delta' = \emptyset$ it follows that $\overline{\Delta}\cap\overline{\Delta'}$ consists exactly of one element which we will denote by x. By (v) we can assume that $x \notin \Delta$ and $x \notin \Delta'$. Let $\Gamma := \{(-\infty,a),\Omega,(b,\infty)\} \in \mathfrak{G}_\Delta$, $\Gamma' := \{(-\infty,a'),\Omega',(b',\infty)\} \in \mathfrak{G}_{\Delta'}$ and choose corresponding partitions of unity $\gamma := \{\omega_l,\omega_0,\omega_r\}$, $\gamma' := \{\omega_l',\omega_0',\omega_r'\}$ subordinate to Γ and Γ', respectively. Further let $\tilde\Gamma := \{(-\infty,a),\tilde\Omega,(b',\infty)\} \in \mathfrak{G}_{\overline{\Delta\cup\Delta'}}$, where $\tilde\Omega$ is the smallest open interval which contains $\Omega\cup\Omega'$. Since $\Delta < \Delta'$ the components Z_b and $Z_{a'}$ coincides. Further $\omega_r\omega_0' = \omega_0'$ and $\omega_r\omega_r' = \omega_r'$. This implies that

$$\omega_0 + \omega_r\omega_l' + \omega_0' + \omega_l + \omega_r' = \omega_0 + \omega_r - \omega_r\omega_0' - \omega_r\omega_r' + \omega_0' + \omega_l + \omega_r' = 1,$$

hence $\{\omega_l, \omega_0 + \omega_r\omega_l' + \omega_0', \omega_r'\}$ is a partition of unity subordinate to the cover $\tilde{\Gamma}$. Then for $f \in \mathcal{D}(\mathbb{R})$ it holds

$$\varphi|_\Delta(f) + \varphi|_{\Delta'}(f) = \varphi(f\omega_0) + \int\limits_{Z_a} f\omega_l\chi_\Delta d\mu_{Z_a} + \int\limits_{Z_b} f\omega_r\chi_\Delta d\mu_{Z_b} +$$

$$+\varphi(f\omega_0') + \int\limits_{Z_{a'}} f\omega_l'\chi_{\Delta'} d\mu_{Z_{a'}} + \int\limits_{Z_{b'}} f\omega_r'\chi_{\Delta'} d\mu_{Z_{b'}} =$$

$$= \varphi(f\omega_0) + \int\limits_{Z_b} f \underbrace{\left(\omega_r\chi_\Delta + \omega_l'\chi_{\Delta'} + \chi_{\{x\}}\right)}_{\omega_r\omega_l'} d\mu_{Z_b} + \varphi(f\omega_0') +$$

$$+ \int\limits_{Z_a} f\omega_l\chi_\Delta d\mu_{Z_a} + \int\limits_{Z_{b'}} f\omega_r'\chi_{\Delta'} d\mu_{Z_{b'}} =$$

$$= \varphi\big(f(\omega_0 + \omega_r\omega_l' + \omega_0')\big) + \int\limits_{Z_a} f\omega_l\chi_\Delta d\mu_{Z_a} + \int\limits_{Z_{b'}} f\omega_r'\chi_{\Delta'} d\mu_{Z_{b'}} = \varphi|_{\overline{\Delta\cup\Delta'}}.$$

ad (vii) : This follows directly from (v) and (vi). ❑

1.14 Lemma. *Let $\beta \in C^\infty(\mathbb{R})$ be a monotone function with $\operatorname{supp}\beta \subseteq (0,\infty)$ such that $\beta|_{(1-\varepsilon,\infty)} \equiv 1$, for some $\varepsilon > 0$. For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ define functions $\beta_{k,\alpha_+}(t) := \beta\big(k(t-\alpha)\big)$, $\beta_{k,\alpha_-}(t) := \beta_{k,\alpha_+}(-t)$ and $\beta_{k,\alpha}(t) := 1 - \beta_{k,\alpha_-}(t) - \beta_{k,\alpha_+}(t)$. Then*

(i) For every compact subset K of $\mathbb{R}$ and $m \in \mathbb{N}_0$ there exists a constant $C_{K,m} > 0$ such that

$$\max\Big\{\Big\|\big((t-\alpha)^m\beta_{k,\alpha_\pm}\big)^{(j)}\Big\|_{\infty,K} : 0 \le j \le m\Big\} \le C_{K,m}, \quad k \in \mathbb{N},$$

where $\|.\|_{\infty,K}$ denotes the uniform norm of a function restricted to K.

(ii) For every $f \in \mathcal{D}(\mathbb{R})$ with $f^{(j)}(\alpha) = 0$, $0 \le j \le m$, there exist constants $\tilde{C}_m > 0$, $k_0 \in \mathbb{N}$ such that

$$\|f\beta_{k,\alpha}\|_m \le \tilde{C}_m, \quad k \ge k_0.$$

(iii) For every $k \in \mathbb{N}$ the family $\{\beta_{k,\alpha_-}, \beta_{k,\alpha}, \beta_{k,\alpha_+}\}$ is a C^∞ partition of unity.

Proof.

ad (i) : Let K be a compact subset of $\mathbb{R}$ and $m' \in \mathbb{N}_0$, $m' \le m$. For $k \in \mathbb{N}$ it follows that

$$\big((t-\alpha)^m\beta_{k,\alpha_+}(t)\big)^{(m')}(t) = \sum_{j=0}^{m'} \binom{m'}{j} \frac{m!}{(m'-j)!}(t-\alpha)^{m-m'+j}k^j\beta^{(j)}\big(k(t-\alpha)\big).$$

For $j=0$ the addend is bounded by $\frac{m!}{m'!}\|(t-\alpha)^{(m-m')}\beta\|_{\infty,K}$. If $j>0$ then $\operatorname{supp}\beta^{(j)}\subseteq[0,1]$, so the addend may be non zero only if $0\le k(t-\alpha)\le 1$. Therefore the j-th summand can be estimated by $\binom{m'}{j}\frac{m!}{(m'-j)!}\|(t-\alpha)^{(m-m')}\beta^{(j)}\|_{\infty,K}$. This implies the existence of constants $C_{K,m'}>0$, $m'=0,\dots,m$, such that

$$\Big\|\big((t-\alpha)^m\beta_{k,\alpha_+}(t)\big)^{(m')}\Big\|_{\infty,K}\le C_{K,m'},\quad k\in\mathbb{N},m'\in\mathbb{N}_0,m'\le m.$$

Now define $C_{K,m}:=\max\{C_{K,m'}:\ m'\in\mathbb{N}_0,m'\le m\}$ which gives the desired estimate. The case for β_{k,α_-} can be proven analogously.

ad (ii) : Let $f\in\mathcal{D}(\mathbb{R})$ such that $f^{(j)}(\alpha)=0$, $0\le j\le m$, and $k\in\mathbb{N}$. Then for every $\eta>0$ there exists a $\delta>0$ such that

$$\big|f^{(j)}(t)\big|\le\eta|t-\alpha|^{m-j},\quad |t-\alpha|\le\delta,\ j=0,\dots,m.\tag{1.2.1}$$

This can be proven by induction. For $j=m$ the assertion follows by continuity. Suppose $1\le j\le m$ and assume $|f^{(j)}(t)|\le\eta|t-\alpha|^{m-j}$. For every $t\in\mathbb{R}$ by the mean value theorem there exists an intermediate value $\xi_t\in\big(\min\{\alpha,t\},\max\{\alpha,t\}\big)$ such that

$$f^{(j)}(\xi_t)=\frac{f^{(j-1)}(t)}{t-\alpha}.$$

If $|t-\alpha|\le\delta$ then $|\xi_t-\alpha|\le\delta$ and our induction hypothesis implies that

$$|f^{(j-1)}(t)|=\frac{|f^{(j)}(\xi_t)|}{|t-\alpha|}\le\eta|t-\alpha|^{m-j+1},\quad |t-\alpha|\le\delta.$$

This shows (1.2.1). Define $\beta_0(t):=1-\beta(t)-\beta(-t)$, then $\beta_{k,\alpha}(t)=\beta_0\big(k(t-\alpha)\big)$, $k\in\mathbb{N}$. It follows that

$$\big(f\beta_{k,\alpha}\big)^{(m')}(t)=\sum_{j=0}^{m'}\binom{m'}{j}f^{(j)}(t)\beta_0^{(m-j)}\big(k(t-\alpha)\big)k^{m-j}.$$

Since $\operatorname{supp}\beta_0\subseteq(-1,1)$ this sum vanishes if $k|t-\alpha|\ge 1$. If $k|t-\alpha|<1$ define $k_0:=[\frac{1}{\delta}]+1$, then it holds

$$|t-\alpha|<\frac{1}{k}\le\delta,\quad k\ge k_0.$$

Using inequality (1.2.1) it follows that

$$\begin{aligned}\big(f\beta_{k,\alpha}\big)^{(m')}(t)&\le\sum_{j=0}^{m'}\binom{m'}{j}\eta|t-\alpha|^{m-j}\beta_0^{(m-j)}\big(k(t-\alpha)\big)\frac{1}{|t-\alpha|^{m-j}}=\\&=\sum_{j=0}^{m'}\binom{m'}{j}\eta\delta^{m-j}\beta_0^{(m-j)}\big(k(t-\alpha)\big),\quad k\ge k_0.\end{aligned}$$

Therefore the j-th summand is bounded by $\binom{m'}{j}\eta\delta^{m-j}\|\beta_0^{(m-j)}\|_\infty$ if $k\ge k_0$. This yields the existence of a constant $\tilde{C}_m$ such that $\|f\beta_{k,\alpha}\|_m\le\tilde{C}_m$ if $k\ge k_0$.

ad (iii) : This assertion is immediate. ❑

1.15 Definition. Let $\varphi \in \mathcal{F}(\mathbb{R})$ and M a finite subset of $\mathbb{R}$ such that $\varphi \in \mathcal{F}(\mathbb{R}, M)$. Choose $T \in \mathfrak{T}$ such that $|M \cap \Delta| \leq 1$ for every $\Delta \in \mathfrak{I}_T$. For an element $\alpha \in M \cap \Delta$ we define the order of the distribution φ at α as

$$\operatorname{ord}(\varphi; \alpha) := \operatorname{ord}(\varphi_\Delta).$$

1.16 Proposition. *Let $\varphi \in \mathcal{F}(\mathbb{R})$ and M a finite subset of $\mathbb{R}$ such that $\varphi \in \mathcal{F}(\mathbb{R}, M)$. For every interval $I \subseteq \mathbb{R}$ with $I \cap M = \{\alpha\}$ and* $\operatorname{dist}(I, M \setminus \{\alpha\}) > 0$ *it holds*

$$|t-\alpha|^{\operatorname{ord}(\varphi;\alpha)} d\mu_{Z_\alpha^\pm}|_{I \cap Z_\alpha^\pm}$$

are finite measures.

Proof. Set $m := \operatorname{ord}(\varphi; \alpha)$ and assume that $\mu_{Z_\alpha^+}$ is a positive measure. If $\mu_{Z_\alpha^+}$ is a negative measure, simply consider $-\mu_{Z_\alpha^+}$. We show that $(t-\alpha)^m d\mu_{Z_\alpha^+}$ is a finite measure. Therefore fix some $a \in \mathbb{R}$ with $a > |\alpha|$ and $\operatorname{supp}\varphi \cap I \subseteq (-a, a)$. For $k \in \mathbb{N}$ let β_{α,k_+} be as in Lemma 1.14. Abbreviatory we will write β_k instead of β_{α,k_+}. Since $\beta_k(t) = 1$ if $t > \alpha + \frac{1}{k}$, for all $k \in \mathbb{N}$, it follows that

$$\Big((t-\alpha)^m d\mu_{Z_\alpha^+}\Big)\big([\alpha + \frac{1}{k}, a]\big) = \int\limits_{\alpha+\frac{1}{k}}^{a} (t-\alpha)^m d\mu_{Z_\alpha^+} = \int\limits_{\alpha+\frac{1}{k}}^{a} \beta_k(t)(t-\alpha)^m d\mu_{Z_\alpha^+} \leq$$

$$\leq \int\limits_{\alpha}^{a} \beta_k(t)(t-\alpha)^m d\mu_{Z_\alpha^+} = \varphi\big((t-\alpha)^m \beta_k\big), \quad k \in \mathbb{N}.$$

By Proposition 1.4 there exist complex Borel measures μ_j, $j = 0, \dots, m$, on $\mathbb{R}$ such that $\varphi = \sum_{j=0}^m \mu_j^{(j)}$ on $I \cap Z_\alpha^+$. This yields for every $k \in \mathbb{N}$

$$|\varphi\big((t-\alpha)^m \beta_k\big)| = \Big|\sum_{j=0}^m \mu_j^{(j)}\big((t-\alpha)^m \beta_k\big)\Big| \leq \sum_{j=0}^m \Big|\mu_j\Big(\big((t-\alpha)^m \beta_k\big)^{(j)}\Big)\Big| \leq$$

$$\leq \sum_{j=0}^m \int\limits_{\alpha}^{a} \Big|\big((t-\alpha)^m \beta_k(t)\big)^{(j)}\Big| d|\mu_j| \leq \sum_{j=0}^m \Big\|\big((t-\alpha)^m \beta_k\big)^{(j)}\Big\|_{\infty,[\alpha,a]} |\mu_j|\big([\alpha, a]\big).$$

According to Lemma 1.14 the terms $\|((t-\alpha)^m \beta_k)^{(j)}\|_{\infty,[\alpha,a]}$, $j = 0, \dots, m$, are uniformly bounded with respect to k and since the measures $|\mu_j|$, $j = 0, \dots, m$, are finite there exists a constant $M > 0$ such that $|\varphi\big((t-\alpha)^m \beta_k\big)| \leq M$ for all $k \in \mathbb{N}$. Now it follows that

$$\Big|\Big((t-\alpha)^m d\mu_{Z_\alpha^+}\Big)\big([\alpha + \frac{1}{k}, a]\big)\Big| \leq M, \quad k \in \mathbb{N},$$

hence $(t-\alpha)^m d\mu_{Z_\alpha^+}$ is a finite measure on (α, a).
The claim that $|t-\alpha|^{\operatorname{ord}(\varphi;\alpha)} d\mu_{Z_\alpha^-}$ is a finite measure can be proven analogously. ❑

1.17 Definition. Let $M \subseteq \mathbb{R}$ be a finite set and $\varphi \in \mathcal{F}(\mathbb{R}, M)$. For $\alpha \in M$ and an interval $I \subseteq \mathbb{R}$ with $I \cap M = \{\alpha\}$ and $\operatorname{dist}(I, M \setminus \{\alpha\}) > 0$ we define

$$k_\alpha^\pm := \min\Big\{m_\alpha^\pm \in \mathbb{N}_0 : \ |t-\alpha|^{m_\alpha^\pm} d\mu_{Z_\alpha^\pm}|_{I \cap Z_\alpha^\pm} \text{ is a finite measure}\Big\},$$

and $k_\alpha := \max\{k_\alpha^+, k_\alpha^-\}$.

1.18 Remark. This definition is independent of the choice of the interval I. Moreover for $\varphi \in \mathcal{F}(\mathbb{R}, \alpha)$, $\alpha \in \mathbb{R}$, Proposition 1.16 shows that $k_\alpha \leq \operatorname{ord}(\varphi; \alpha) \leq \operatorname{ord}(\varphi)$.

1.19 Proposition. *Let $\alpha \in \mathbb{R}$ and $\varphi \in \mathcal{F}(\mathbb{R}, \alpha)$. Further assume that $f \in \mathcal{D}(\mathbb{R})$ is such that $f^{(0)}(\alpha) = \cdots = f^{(\operatorname{ord}(\varphi))}(\alpha) = 0$. Then*

$$\varphi(f) = \int\limits_{Z_\alpha^-} f d\mu_{Z_\alpha^-} + \int\limits_{Z_\alpha^+} f d\mu_{Z_\alpha^+}.$$

Proof. For $k \in \mathbb{N}$ let β_{k,α_-}, $\beta_{k,\alpha}$ and β_{k,α_+} as in Lemma 1.14. Since $f \in \mathcal{D}(\mathbb{R})$ and $\{\beta_{k,\alpha_-}, \beta_{k,\alpha}, \beta_{k,\alpha_+}\}$ is a C^∞ partition of unity for every $k \in \mathbb{N}$ it follows that

$$\begin{aligned}\varphi(f) = \varphi(f\beta_{k,\alpha_-}) + \varphi(f\beta_{k,\alpha}) + \varphi(f\beta_{k,\alpha_+}) = \\ = \int\limits_{Z_\alpha^-} f\beta_{k,\alpha_-} d\mu_{Z_\alpha^-} + \varphi(f\beta_{k,\alpha}) + \int\limits_{Z_\alpha^+} f\beta_{k,\alpha_+} d\mu_{Z_\alpha^+}.\end{aligned}$$

Define $f_{\alpha_+} := (t-\alpha)^{-k_\alpha^+} f$, then f_{α_+} is continuous, and hence bounded on $[\alpha, \infty)$. We have $f_{\alpha_+}\beta_{k,\alpha_+} \to f_{\alpha_+}\chi_{Z_\alpha^+}$ pointwise on Z_{α^+} for $k \to \infty$, and the dominated convergence theorem yields

$$\begin{aligned}\lim_{k\to\infty} \int\limits_{Z_\alpha^+} f\beta_{k,\alpha_+} d\mu_{Z_\alpha^+} = \lim_{k\to\infty} \int\limits_{Z_\alpha^+} f_{\alpha_+}\beta_{k,\alpha_+}(t-\alpha)^{k_\alpha^+} d\mu_{Z_\alpha^+} = \\ = \int\limits_{Z_\alpha^+} f_{\alpha_+}(t-\alpha)^{k_\alpha^+} d\mu_{Z_\alpha^+} = \int\limits_{Z_\alpha^+} f\mu_{Z_\alpha^+}.\end{aligned}$$

Similarly it follows that

$$\lim_{k\to\infty} \int\limits_{Z_\alpha^-} f\beta_{k,\iota} d\mu_{Z_\alpha^-} = \int\limits_{Z_\alpha^-} f d\mu_{Z_\alpha^-}.$$

Set $m := \operatorname{ord}(\varphi)$ and denote by μ_j, $j = 0, \dots, m$, the complex Borel measures on $\mathbb{R}$ as in Proposition 1.4 such that $\varphi = \sum_{j=0}^m \mu_j^{(j)}$. We have

$$\varphi(f\beta_{k,\alpha}) = \sum_{j=0}^m \mu_j^{(j)}(f\beta_{k,\alpha}) = \sum_{j=0}^m (-1)^j \int\limits_{\mathbb{R}} (f\beta_{k,\alpha})^{(j)} d\mu_j.$$

For each $j = 0, \dots, m$, the integrand converges pointwise to zero when k tends to ∞. According to Lemma 1.14 the integrand is uniformly bounded with respect to k for every $j = 0, \dots, m$, hence we can apply the dominated convergence theorem and obtain

$$\lim_{k\to\infty} \varphi(f\beta_{k,\alpha}) = 0,$$

which completes the proof. ❑

1.20 Remark. If $\varphi \in \mathcal{F}(\mathbb{R}, \alpha)$, $\alpha \in \mathbb{R}$, then, by definition, φ has compact support. By Proposition 1.2 the distribution φ can be extended in a unique way to a linear functional on $C^\infty(\mathbb{R})$. Therefore the last proposition implies that

$$\varphi\big((t-\alpha)^j\big) = \int\limits_{Z_\alpha^-} (t-\alpha)^j d\mu_{Z_\alpha^-} + \int\limits_{Z_\alpha^+} (t-\alpha)^j d\mu_{Z_\alpha^+}, \quad j > \operatorname{ord}(\varphi).$$

For a function f which possess a n-th derivative at $t = \alpha$ we use the notation

$$f^{\{\alpha,n\}}(t) := \begin{cases} f(t) & \text{if } n = 0, \\ f(t) - \sum_{i=0}^{n-1} \frac{(t-\alpha)^i}{i!} f^{(i)}(\alpha) & \text{if } n \geq 1. \end{cases}$$

If α is clear from the context we may just write $f^{\{n\}}$.

1.21 Proposition. *If $\alpha \in \mathbb{R}$ and $\varphi \in \mathcal{F}(\mathbb{R}, \alpha)$, then there exist constants $k \in \mathbb{N}_0$, $l \in \mathbb{N}_0 \cup \{-1\}$, $c_0, \dots, c_l \in \mathbb{R}$, and a signed finite Borel measure σ with compact support, $\sigma(\{\alpha\}) = 0$ and $\sigma|_{Z_{\alpha^\pm}}$ has the same sign as $\mu_{Z_{\alpha^\pm}}$, such that*

$$\varphi(f) = \int\limits_{\mathbb{R}} \frac{f^{\{\alpha,2k\}}(t)}{(t-\alpha)^{2k}} d\sigma(t) + \sum_{i=0}^{l} \frac{c_i}{i!} f^{(i)}(\alpha), \quad f \in \mathcal{D}(\mathbb{R}). \tag{1.2.2}$$

The data $k, l, c_0, \dots, c_l, \sigma$ can be chosen such that

(IR-1) $c_l \neq 0$ *if* $l \geq 0$,

(IR-2) *if $k > 0$ the function $t \mapsto \frac{1}{(t-\alpha)^2}$ is not σ-integrable,*

and with these requirements $k, l, c_0, \dots, c_l, \sigma$ are unique. In fact, they can be computed by means of φ as follows

$$k := \begin{cases} \frac{1}{2} k_\alpha, & \text{if } k_\alpha \text{ is even} \\ \frac{1}{2}(k_\alpha + 1), & \text{if } k_\alpha \text{ is odd} \end{cases},$$

σ is the measure corresponding to the distribution $(t-\alpha)^{2k}\varphi$ restricted to $\mathbb{R}\setminus\{\alpha\}$, i.e. $d\sigma(t) = (t-\alpha)^{2k} d\mu_{Z_\alpha^-}(t) + (t-\alpha)^{2k} d\mu_{Z_\alpha^+}(t)$,

$$c_i := \begin{cases} \varphi\big((t-\alpha)^i\big), & i = 0, \dots, 2k-1, \\ \varphi\big((t-\alpha)^i\big) - \\ \quad -\Big(\int_{Z_\alpha^-} (t-\alpha)^i d\mu_{Z_\alpha^-} + \int_{Z_\alpha^+} (t-\alpha)^i d\mu_{Z_\alpha^+}\Big), & i = 2k, 2k+1, \dots, \end{cases}$$

and $l := \max\{j \in \mathbb{N} :\ c_j \neq 0\} \cup \{-1\}$.

Note that by Remark 1.20, $c_i = 0$ if i is larger than the order of φ.

Proof. Define $k, l, c_0, \dots, c_l, \sigma$ as given by means of the distribution φ in the formulation of this proposition. By Remark 1.18 the measure σ is a signed finite Borel measure with compact support and clearly it has no point mass at α. Further define λ as

$$\lambda := \begin{cases} \frac{\operatorname{ord}(\varphi)+1}{2} & \text{if } \operatorname{ord}(\varphi) \text{ is odd}, \\ \frac{\operatorname{ord}(\varphi)+2}{2} & \text{if } \operatorname{ord}(\varphi) \text{ is even}. \end{cases}$$

Now let $f \in \mathcal{D}(\mathbb{R})$ and apply Proposition 1.19 to the function $f^{\{\alpha,2\lambda\}}$, then it follows that

$$\varphi(f) = \varphi(f^{\{\alpha,2\lambda\}}) + \sum_{i=0}^{2\lambda-1} \varphi\big((t-\alpha)^i\big)\frac{f^{(i)}(\alpha)}{i!} =$$

$$= \int_{Z_\alpha^-} f^{\{\alpha,2\lambda\}} d\mu_{Z_\alpha^-} + \int_{Z_\alpha^+} f^{\{\alpha,2\lambda\}} d\mu_{Z_\alpha^+} + \sum_{i=0}^{2\lambda-1} \varphi\big((t-\alpha)^i\big)\frac{f^{(i)}(\alpha)}{i!} =$$

$$= \int_{Z_\alpha^-} f^{\{\alpha,2k\}} d\mu_{Z_\alpha^-} + \int_{Z_\alpha^+} f^{\{\alpha,2k\}} d\mu_{Z_\alpha^+} + \sum_{i=0}^{2\lambda-1} \varphi\big((t-\alpha)^i\big)\frac{f^{(i)}(\alpha)}{i!} -$$

$$- \sum_{i=2k}^{2\lambda-1} \Big(\int_{Z_\alpha^-} (t-\alpha)^i d\mu_{Z_\alpha^-} + \int_{Z_\alpha^+} (t-\alpha)^i d\mu_{Z_\alpha^+}\Big)\frac{f^{(i)}(\alpha)}{i!} =$$

$$= \int_{\mathbb{R}} \frac{f^{\{\alpha,2k\}}}{(t-\alpha)^{2k}} d\sigma(t) + \sum_{i=0}^{l} \frac{c_i}{i!} f^{(i)}(\alpha).$$

This gives the desired integral representation of φ. Condition (IR-1) is satisfied by the definition of l. In order to show that (IR-2) holds let $k > 0$ and assume that the function $t \mapsto \frac{1}{(t-\alpha)^2}$ is σ-integrable. This would imply

$$\int_{\mathbb{R}} \frac{1}{(t-\alpha)^2} d|\sigma|(t) = \int_{Z_\alpha^-} (t-\alpha)^{2k-2} d|\mu_{Z_\alpha^-}|(t) + \int_{Z_\alpha^+} (t-\alpha)^{2k-2} d|\mu_{Z_\alpha^+}|(t) < \infty.$$

This is a contradiction to the minimality of k_α. The uniqueness is immediate. ❑

1.22 Remark. Let $\varphi \in \mathcal{F}(\mathbb{R},\alpha)$, $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}_0$, $l \in \mathbb{N}_0 \cup \{-1\}$, $c_0, \dots, c_l \in \mathbb{R}$ and σ as in the preceding proposition satisfying 1.21 (IR-1), (IR-2). The integral representation (1.2.2) implies that $\alpha \in \operatorname{supp}\varphi$ if $k > 0$ or $l \geq 0$ and $\alpha \in \operatorname{supp}\sigma$ if $k > 0$ (condition 1.21 (IR-2)). By Corollary 1.9 and Remark 1.10 it follows that

$$\operatorname{supp}\varphi = \operatorname{supp}\sigma \quad \text{if} \quad k = 0 \wedge l = -1 \text{ or } k > 0.$$

If $k = 0$ then the situation $\alpha \in \operatorname{supp}\sigma$ and $\alpha \notin \operatorname{supp}\sigma$ can occur. Therefore we have

$$\operatorname{supp}\varphi = \operatorname{supp}\sigma \cup \{\alpha\} \quad \text{if} \quad k = 0 \wedge l \geq 0.$$

1.3 Distributions of the class $\mathcal{F}(\mathbb{C}\setminus\mathbb{R})$

As in [JLT] we introduce a class $\mathcal{F}(\mathbb{C}\setminus\mathbb{R}, B)$ as follows. Let $B \subset \mathbb{C}\setminus\mathbb{R}$ be a finite set such that B is symmetric with respect to the real axis, i.e. $B = \{\beta_1, \dots, \beta_m, \overline{\beta_1}, \dots, \overline{\beta_m}\}$ with $\beta_i \in \mathbb{C}^+$, $i = 1, \dots, m$. Further fix some $\nu_i \in \mathbb{N}$, $i = 1, \dots, m$, and $d_{ij} \in \mathbb{C}$, $i = 1, \dots, m$, $j = 0, \dots, \nu_i - 1$ and define for locally holomorphic functions on B the linear functional

$$\psi(f) = \sum_{i=1}^{m} \sum_{j=0}^{\nu_i-1} \left(\frac{d_{ij}}{j!} f^{(j)}(\beta_i) + \frac{\overline{d_{ij}}}{j!} f^{(j)}(\overline{\beta_i})\right). \tag{1.3.1}$$

We write $\psi \in \mathcal{F}(\mathbb{C} \setminus \mathbb{R}, B)$ if ψ is a linear functional on $H(B)$, the space of all locally holomorphic functions on B, of the form (1.3.1). By a corollary of Runge's Theorem [C2, Corollary 1.15, p.200] every $f \in H(B)$ can be uniformly approximated by polynomials on B, hence every $\psi \in \mathcal{F}(\mathbb{C} \setminus \mathbb{R})$ is uniquely determined by its restriction to $\mathcal{P}$.

The minimal set B such that $\psi \in \mathcal{F}(\mathbb{C} \setminus \mathbb{R}, B)$ is denoted by $\sigma_0(\phi)$.

1.23 Definition. We define $\mathcal{F}(\mathbb{C}\setminus\mathbb{R}) := \cup_B \mathcal{F}(\mathbb{C}\setminus\mathbb{R}, B)$, where B runs through all finite $\mathbb{R}$-symmetric subsets of $\mathbb{C} \setminus \mathbb{R}$, and the class $\mathcal{F}$ of linear functionals by

$$\mathcal{F} := \mathcal{F}(\mathbb{C} \setminus \mathbb{R}) \times \mathcal{F}(\mathbb{R}).$$

Every $\phi \in \mathcal{F}$ can be represented as $\phi = (\varphi, \psi)$ with $\varphi \in \mathcal{F}(\mathbb{R})$ and $\psi \in \mathcal{F}(\mathbb{C} \setminus \mathbb{R})$.

1.24 Lemma. *Let $[a, b] \subseteq \mathbb{R}$, $k \in \mathbb{N}_0$ and $\mathcal{A}$ a subalgebra of $C^k([a, b])$ that separates points, is closed under complex conjugation and is nowhere vanishing. If for every $f \in \mathcal{A}$ and $z \in \mathbb{C}$ there exists an element $g \in \mathcal{A}$ such that $g' = f$ and $g(0) = z$ then $\mathcal{A}$ is dense in $C^k([a, b])$, where $C^k([a, b])$ is endowed with the norm $\|.\|_k$.*

Proof. Let $f \in C^k([a, b])$ and $\varepsilon > 0$. We have to show that there exists an element $g \in \mathcal{A}$ such that $\|f - g\|_k < \varepsilon$. Since $f^{(k)} \in C([a, b])$ by the Stone-Weierstrass theorem there exists an element $g_0 \in \mathcal{A}$ such that $\|f^{(k)} - g_0\|_\infty < \varepsilon$. By assumption there exists an element $g_1 \in \mathcal{A}$ with $g_1' = g_0$ and $g_1(0) = f^{(k-1)}(0)$. For $x \in [a, b]$ it follows that

$$\int_a^x \big(f^{(k)}(t) - g_0(t)\big)dt = f^{(k-1)}(x) - f^{(k-1)}(0) - g_1(x) + g_1(0) = f^{(k-1)}(x) - g_1(x).$$

Without loss of generality assume $b - a \geq 1$, then the last equation implies

$$\|f^{(k-1)} - g_1\|_\infty \leq \sup_{x\in[a,b]} \int_a^x \big|f^{(k)}(t) - g_0(t)\big|dt < \frac{\varepsilon}{(b-a)^{k-1}} \leq \varepsilon.$$

Inductively it follows that there exist elements $g_j \in \mathcal{A}$ with $g_j' = g_{j-1}$, $g_j(0) = f^{(k-j)}(0)$, $j = 2, \dots, k$, such that

$$\|f^{(k-j)} - g_j\|_\infty < \varepsilon, \quad j = 0, \dots, k.$$

Define $g := g_k$, then we have

$$\|f - g\|_k = \max\big\{\|f^{(j)} - g^{(j)}\|_\infty : j = 0, \dots, k\big\} < \varepsilon,$$

which completes the proof. ❑

1.25 Lemma. *The set of all polynomials $\mathbb{C}[x]$ is dense in $C^\infty(\mathbb{R})$.*

Proof. Recall that a local base for the topology on $C^\infty(\mathbb{R})$ is given by

$$\mathfrak{W}(0) = \big\{U_\varepsilon^{\|\cdot\|_{m,[-N,N]}}(0) : \ \varepsilon > 0, N \in \mathbb{N}, m \in \mathbb{N}_0\big\},$$

where $U_\varepsilon^{\|\cdot\|_{m,[-N,N]}}(0) = \{g \in C^\infty(\mathbb{R}) : \|f\|_{m,[-N,N]} < \varepsilon\}$. By Lemma 1.24 for any set $W \subseteq \mathfrak{W}(0)$ and $f \in C^\infty(\mathbb{R})$ it holds $(f+W) \cap \mathbb{C}[x] \neq \emptyset$. This implies that $f \in \operatorname{clos}_{\tau_\infty}(\mathcal{P})$, hence the space of all polynomials is dense in $\big(C^\infty(\mathbb{R}), \tau_\infty\big)$. ❑

Chapter 2

Model spaces for distributions of class $\mathcal{F}(\mathbb{R}, 0)$

2.1 Representations for elements of $\mathcal{F}(\mathbb{R}, 0)$

2.1 Definition. By Θ we denote the set of all tuples $(k, l, \sigma, c_0, \ldots, c_l)$ where

·) $k \in \mathbb{N}_0$ and $l \in \mathbb{N}_0 \cup \{-1\}$,

·) σ is a finite signed measure on $\mathbb{R}$ with compact support and $\sigma(\{0\}) = 0$,

·) each of the the restrictions $\sigma|_{\mathbb{R}^+}$ and $\sigma|_{\mathbb{R}^-}$ is either a positive or a negative measure,

·) $c_0, \ldots, c_l \in \mathbb{R}$.

For $\vartheta \in \Theta$ we define

$$\varphi_\vartheta(f) := \int_{\mathbb{R}} \frac{f^{\{2k\}}(t)}{t^{2k}} d\sigma(t) + \sum_{i=0}^{l} \frac{c_i}{i!} f^{(i)}(0), \quad f \in \mathcal{D}(\mathbb{R}). \tag{2.1.1}$$

Note that if $l = -1$ the constants c does not appear, i.e. $(k, -1, \sigma) \in \Theta$.

2.2 Definition. We call $\vartheta = (k, l, \sigma, c_0, \ldots, c_l) \in \Theta$ a *minimal representation* if

·) $k = 0$ and $l = -1$ or

·) $k > 0$, $\int_{\mathbb{R}} \frac{1}{t^2} d\sigma(t) = \infty$ and $l = -1$ or

·) $k = 0$ and $l = \max\{n \in \mathbb{N}_0 : \ c_n \neq 0\}$ or

·) $k > 0$, $\int_{\mathbb{R}} \frac{1}{t^2} d\sigma(t) = \infty$ and $l = \max\{n \in \mathbb{N}_0 : \ c_n \neq 0\}$.

2.3 Lemma. *If $\vartheta \in \Theta$, then $\varphi_\vartheta \in \mathcal{F}(\mathbb{R}, 0)$.*

Proof. Clearly φ_ϑ is a linear functional on $\mathcal{D}(\mathbb{R})$. Let $\Delta := \operatorname{supp} \sigma$ and define for $f \in \mathcal{D}(\mathbb{R})$

$$g(t) := \frac{f^{\{2k\}}(t)}{t^{2k}}, \quad t \in \Delta \setminus \{0\}.$$

Then by Taylor's theorem there exists for every $t \in \Delta$ an intermediate value $\zeta_t \in (\min\{0,t\}, \max\{0,t\})$ such that $f^{\{2k\}}(t) = f^{(2k)}(\zeta_t)\frac{t^{2k}}{(2k)!}$. It follows that $\lim_{t\to 0} g(t) = \frac{f^{(2k)}(0)}{(2k)!}$. Therefore the function g has a continuous extension to Δ, which we also denote by g. Clearly we have $\|g\|_\infty \le \frac{1}{(2k)!}\|f^{(2k)}\|_\infty$. Let

$$V := \Big\{ f \in \mathcal{D}(\mathbb{R}) : \sup_{\substack{t\in\Delta \\ 0\le i\le \max\{2k,l\}}} |f^{(i)}(t)| \le 1 \Big\},$$

then we have for all $f \in V$

$$|\varphi_\vartheta(f)| \le \Big| \int_{\mathbb{R}} \frac{f^{\{2k\}}(t)}{t^{2k}} d\sigma(t) \Big| + \max_{i=0,\dots,l} |f^{(i)}(0)| \sum_{i=0}^{l} \frac{|c_i|}{i!} \le \int_{\Delta} |g(t)| d|\sigma|(t) +$$

$$+ \max_{i=0,\dots,l} |f^{(i)}(0)| \sum_{i=0}^{l} \frac{|c_i|}{i!} \le \frac{1}{(2k)!} \|f^{(2k)}\|_\infty |\sigma|(\Delta) + \max_{i=0,\dots,l} |f^{(i)}(0)| \sum_{i=0}^{l} \frac{|c_i|}{i!} \le$$

$$\le \frac{1}{(2k)!}|\sigma|(\Delta) + \sum_{i=0}^{l} \frac{|c_i|}{i!} < \infty.$$

Since V is a neighborhood of 0 it follows that φ_ϑ is a distribution and obviously φ_ϑ belongs to $\mathcal{F}(\mathbb{R},0)$. ❑

If $\varphi \in \mathcal{F}(\mathbb{R},0)$, an element $\vartheta \in \Theta$ is called a representation of φ, if $\varphi = \varphi_\vartheta$. We know from Proposition 1.21 that for each $\varphi \in \mathcal{F}(\mathbb{R},0)$ there exists a representation. This representation, however, is not unique. The set of all representations of a fixed distribution $\varphi \in \mathcal{F}(\mathbb{R},0)$ is denoted by Θ_φ, i.e. $\Theta_\varphi := \{\vartheta \in \Theta : \ \varphi_\vartheta = \varphi\}$.

For some density arguments we will need a corollary of the Stone-Weierstrass Theorem (see [C1, Corollary V.8.2, p.146]):

2.4 Corollary. *If X is compact and $\mathcal{A}$ is a closed subalgebra of $C(X)$ that separates the points of X and is closed under complex conjugation, then either $\mathcal{A} = C(X)$ or there is a point $x_0 \in X$ such that $\mathcal{A} = \{f \in C(X) : f(x_0) = 0\}$.*

2.5 Remark. Let ν be a positive Borel measure on $\mathbb{R}$ with compact support such that $\nu(\{0\}) = 0$ and $N \in \mathbb{N}_0$. Then for any compact set X which contains the support of ν the set $\mathcal{A} := \{t \mapsto t^N p(t) : p \in \mathcal{P}\} \subseteq \mathbb{C}^X$ is dense in $L^2(\nu)$. Note that $\mathcal{A}$ is the set of all polynomials on X such that the derivatives up to order $N-1$ vanishes at $t = 0$.
Obviously $\mathcal{A}$ is a subalgebra of $C(X)$. Further it separates point, since the polynomial $p(t) := t^{2N+1}$ is an element of $\mathcal{A}$, and clearly $\mathcal{A}$ is closed under complex conjugation. By Corollary 2.4 it follows that

$$\operatorname{clos}_{\|.\|_\infty}(\mathcal{A}) = \{f \in C(X) : f(0) = 0\}.$$

Since convergence in $\big(C(X), \|.\|_\infty\big)$ implies convergence in $\big(L^2(\nu), \|.\|_{L^2(\nu)}\big)$ we have

$$\{f \in C(X) : f(0) = 0\} = \operatorname{clos}_{\|.\|_\infty}(\mathcal{A}) \subseteq \operatorname{clos}_{\|.\|_{L^2(\nu)}}(\mathcal{A}).$$

Since $\nu(\{0\}) = 0$, the $L^2(\nu)$-closure of the left hand side equals $L^2(\nu)$, hence $\mathcal{A}$ is dense in $L^2(\nu)$.

2.6 Lemma. *Let $\varphi \in \mathcal{F}(\mathbb{R}, 0)$ and let $\vartheta_1 = (k, l, \sigma, c_0, \dots, c_l)$ and $\vartheta_2 = (k, \hat{l}, \hat{\sigma}, \hat{c}_0, \dots, \hat{c}_{\hat{l}})$ be representations of φ with $l \leq \hat{l}$. Then $\sigma = \hat{\sigma}$, $c_i = \hat{c}_i$, $i = 0, \dots, l$, and $\hat{c}_{l+1} = \dots = \hat{c}_{\hat{l}} = 0$.*

Proof. Since $\varphi_{\vartheta_1} = \varphi_{\vartheta_2}$, by relation (2.1.1), for $f \in C^\infty(\mathbb{R})$ it follows that

$$\int_{\mathbb{R}} \frac{f^{\{2k\}}(t)}{t^{2k}} d\sigma(t) + \sum_{i=0}^{l} \frac{c_i}{i!} f^{(i)}(0) = \int_{\mathbb{R}} \frac{f^{\{2k\}}(t)}{t^{2k}} d\hat{\sigma}(t) + \sum_{i=0}^{\hat{l}} \frac{\hat{c}_i}{i!} f^{(i)}(0). \quad (2.1.2)$$

Let X be a compact subset of $\mathbb{R}$ which contains the support of σ and $\hat{\sigma}$ and define $r := \max\{2k - 1, l\}$. Then for every polynomial p whose derivates vanish up to order r at $t = 0$, it holds $p^{\{2k\}} = p$ and (2.1.2) implies

$$\int_{\mathbb{R}} \frac{p(t)}{t^{2k}} d\sigma(t) = \int_{\mathbb{R}} \frac{p(t)}{t^{2k}} d\hat{\sigma}(t).$$

By Remark 2.5 the set of functions $t \mapsto \frac{p(t)}{t^{2k}}$, $t \in X$, is dense in $L^2(\sigma)$ and $L^2(\hat{\sigma})$, therefore $\sigma = \hat{\sigma}$. Hence the integrals in (2.1.2) cancel, and we obtain

$$\sum_{i=0}^{l} \frac{c_i}{i!} f^{(i)}(0) = \sum_{i=0}^{\hat{l}} \frac{\hat{c}_i}{i!} f^{(i)}(0), \quad f \in C^\infty(\mathbb{R}),$$

which implies $c_i = \hat{c}_i$, $i = 0, \dots, l$, and $\hat{c}_{l+1} = \dots = \hat{c}_{\hat{l}} = 0$. ❑

2.7 Definition. Let $\varphi \in \mathcal{F}(\mathbb{R}, 0)$ and let $\vartheta = (k, l, \sigma, c_0, \dots, c_l)$ be a representation of φ. If k is as in the minimal representation we say ϑ is a *representation of φ with minimal k*.

2.8 Definition. For a fixed distribution $\varphi \in \mathcal{F}(\mathbb{R}, 0)$ we define a relation $\preccurlyeq$ on Θ_φ as follows: Let $\vartheta_1 = (k, l, \sigma, c_0, \dots, c_l)$ and $\vartheta_2 = (\hat{k}, \hat{l}, \hat{\sigma}, \hat{c}_0, \dots, \hat{c}_{\hat{l}})$ be representations of φ, then

$$\vartheta_1 \preccurlyeq \vartheta_2 \ :\Leftrightarrow\ l \leq \hat{l} \text{ and } k \leq \hat{k}.$$

2.9 Remark. The relation $\preccurlyeq$ is a partial order on Θ_φ. Further the minimal element in $(\Theta_\varphi, \preccurlyeq)$ is exactly the minimal representation.

2.2 Linear Spaces associated to a representation

Let σ be a finite signed measure on $\mathbb{R}$ with compact support and $\sigma(\{0\}) = 0$ such that the restrictions $\sigma|_{\mathbb{R}^+}$ and $\sigma|_{\mathbb{R}^-}$ are either positive or negative measures. Then the linear space $L^2(|\sigma|)$ endowed with the inner product

$$(f, g)_{|\sigma|} := \int_{\mathbb{R}} f \overline{g} d|\sigma|, \quad f, g \in L^2(|\sigma|),$$

is a Hilbert space. Denote by $P_{\mathbb{R}^\pm}$ the orthogonal projection onto the subspace $L^2(|\sigma|)|_{\mathbb{R}^\pm}$. Then we can write $L^2(|\sigma|) = P_{\mathbb{R}^-} L^2(|\sigma|) \oplus P_{\mathbb{R}^+} L^2(|\sigma|)$. Now consider the operator $J : L^2(|\sigma|) \to L^2(|\sigma|)$ defined by

$$
J := \begin{cases} I & \operatorname{sign}(\sigma|_{\mathbb{R}^+}) = \operatorname{sign}(\sigma|_{\mathbb{R}^-}) = 1, \\ P_{\mathbb{R}^+} - P_{\mathbb{R}^-} & \operatorname{sign}(\sigma|_{\mathbb{R}^+}) = 1 \text{ and } \operatorname{sign}(\sigma|_{\mathbb{R}^-}) = -1, \\ P_{\mathbb{R}^-} - P_{\mathbb{R}^+} & \operatorname{sign}(\sigma|_{\mathbb{R}^+}) = -1 \text{ and } \operatorname{sign}(\sigma|_{\mathbb{R}^-}) = 1, \\ -I & \operatorname{sign}(\sigma|_{\mathbb{R}^+}) = \operatorname{sign}(\sigma|_{\mathbb{R}^-}) = -1. \end{cases} \tag{2.2.1}
$$

Obviously the operator J is a fundamental symmetry, i.e. $J^{-1} = J^* = J$, and $\big(L^2(|\sigma|), (J.,.)_{|\sigma|}\big)$ is a Kreĭn space. The J-inner product is explicitly given by

$$
(f,g)_\sigma := (Jf,g)_{|\sigma|} = \int_{\mathbb{R}} f\overline{g}\,d\sigma, \quad f,g \in L^2(|\sigma|).
$$

Abbreviatory we will write $L^2(\sigma)$ for the Kreĭn space $\big(L^2(|\sigma|), (.,.)_\sigma\big)$. By $(.,.)_{\mathbb{C}^n}$, $n \in \mathbb{N}$, we denote the usual euclidean inner product on $\mathbb{C}^n$. We define an inner product on the space $L^2(|\sigma|) \oplus \mathbb{C}^n$ by

$$
\big((f;\xi),(g;\zeta)\big)_{L^2(|\sigma|)\oplus\mathbb{C}^n} := (f,g)_{|\sigma|} + (\xi,\zeta)_{\mathbb{C}^n}, \quad (f;\xi),(g;\zeta) \in L^2(\sigma) \oplus \mathbb{C}^n.
$$

Clearly $\big(L^2(|\sigma|) \oplus \mathbb{C}^n, (.,.)_{L^2(|\sigma|)\oplus\mathbb{C}^n}\big)$ is a Hilbert space.

2.10 Definition. Let $\varphi \in \mathcal{F}(\mathbb{R},0)$ and $\vartheta = (k,l,\sigma,c_0,\cdots,c_l) \in \Theta_\varphi$. We define a linear space $\mathfrak{L}_\vartheta$ and two inner products $(.,.)_\vartheta$ and $[.,.]_\vartheta$ on $\mathfrak{L}_\vartheta$ by

$$
\begin{aligned}
\mathfrak{L}_\vartheta &:= L^2(|\sigma|) \oplus \mathbb{C}^{\max\{l+1,k\}+k}, \\
(.,.)_\vartheta &:= (.,.)_{L^2(|\sigma|)\oplus\mathbb{C}^{\max\{l+1,k\}+k}}, \\
[.,.]_\vartheta &:= (G.,.)_\vartheta,
\end{aligned}
$$

where G is the Gram operator given by

$$
\left(\begin{array}{c|cccc|ccccc|ccc}
J & & & & & & & & & & & & \\ \hline
 & c_0 & c_1 & \cdots & c_{k-1} & c_k & \cdots & c_{2k-2} & \cdots & c_l & 1 & & \\
 & c_1 & & \iddots & & & \iddots & & \iddots & & & \ddots & \\
 & \vdots & \iddots & & & \iddots & & \iddots & & & & & \ddots \\
 & c_{k-1} & & & c_{2k-2} & & \iddots & & & & & & 1 \\ \hline
 & c_k & & \iddots & & \iddots & & & & & & & \\
 & \vdots & \iddots & & \iddots & & & & & & & & \\
 & c_{2k-2} & & \iddots & & & & & & & & & \\
 & \vdots & \iddots & & & & & & & & & & \\
 & c_l & & & & & & & & & & & \\ \hline
 & 1 & & & & & & & & & & & \\
 & & \ddots & & & & & & & & & & \\
 & & & \ddots & & & & & & & & & \\
 & & & & 1 & & & & & & & &
\end{array}\right)
$$

$$
\underbrace{\qquad\qquad\qquad}_{k\text{-columns}} \quad \underbrace{\qquad\qquad\qquad}_{l-k+1\text{-columns}} \quad \underbrace{\qquad\qquad}_{k\text{-columns}}
$$

Note that if $k = 0$ and $l = -1$ then $\mathfrak{L}_\vartheta = L^2(|\sigma|)$ and the inner product $[.,.]_\vartheta$ is just $(.,.)_\sigma$.
If $l \le k-1$ then the finite dimensional part of the Gram operator has the form $\left(\begin{smallmatrix} C & E_k \\ E_k & 0 \end{smallmatrix}\right)$, where E_k denotes the $k \times k$ unit matrix and C the Hankel matrix which contains the constants $c_0, \dots, c_l$ as in the above scheme.
Obviously, it holds that $\mathfrak{L}_\vartheta = L^2(|\sigma|)[\dot{+}]_\vartheta \mathbb{C}^{\max\{l+1,k\}+k}$. Further it is practically to write $\vec{f} \in \mathfrak{L}_\vartheta$ as

$$\vec{f} = (f; a_0, \dots, a_l; b_0, \dots, b_{k-1}) \in L^2(|\sigma|) \oplus \mathbb{C}^{\max\{l+1,k\}} \oplus \mathbb{C}^k.$$

2.11 Remark. Let $\varphi \in \mathcal{F}(\mathbb{R}, 0)$, $\vartheta = (k, l, \sigma, c_0, \dots, c_l) \in \Theta_\varphi$ and $M := \max\{j \in \mathbb{N}_0 : \ c_j \neq 0\} \cup \{-1\}$. The isotropic part of $\mathfrak{L}_\vartheta$ with respect to the inner product $[.,.]_\vartheta$ equals the kernel of G. It is sufficient to consider the restriction of G to its finite dimensional part. Denote this restriction by $\hat{G}$. It follows immediately from the structure of $\hat{G}$ that every element in the kernel must be 0 in the first k entries. If $l < k$ then clearly the $\hat{G}$ is regular. If $l \ge k$ we have to compute the kernel of the matrix

$$\tilde{G} := \left(\begin{array}{ccc|cccc|ccc}
c_k & \cdots & c_{l-k} & c_{l-k+1} & \cdots & \cdots & c_l & 1 & & \\
\vdots & & \vdots & \vdots & & \iddots & & & \ddots & \\
\vdots & & \vdots & \vdots & \iddots & & & & & \ddots \\
c_{2k-1} & \cdots & c_{l-1} & c_l & & & & & & 1 \\
\hline
c_{2k} & \cdots & c_l & & & & & & & \\
\vdots & \iddots & & & & & & & & \\
c_l & & & & & & & & &
\end{array}\right).$$

Clearly $\tilde{G}$ is a $l-k+1 \times l+1$ dimensional matrix. If $M \le k-1$ then an element $\vec{f} \in \mathfrak{L}_\vartheta$ is in the kernel of $\tilde{G}$ if and only if

$$\vec{f} = (0; \underbrace{0, \dots, 0}_{k\text{-times}}, y_0, \dots, y_{l-k}; \underbrace{0, \dots, 0}_{k\text{-times}})^T, \quad y_i \in \mathbb{C}, \ i = 0, \dots, l-k.$$

If $M \ge k$, define $C_{k,M}$ as the Hankel matrix of the form

$$C_{k,M} := \begin{pmatrix} c_{\max\{k,M-k+1\}} & \cdots & c_M \\ \vdots & \iddots & \\ c_M & & 0 \end{pmatrix}. \tag{2.2.2}$$

Then $C_{k,M}$ is $(M-k+1) \times (M-k+1)$ dimensional if $M < 2k-1$ and $k \times k$ dimensional if $M \ge 2k-1$. Since $c_M \neq 0$ it follows that $C_{k,M}$ is regular. To compute the kernel of the above matrix it is useful to write it in block matrix form. For $i, j \in \mathbb{N}_0$ denote by $\mathbf{0}_{i,j} \in \mathbb{C}^{i \times j}$ the $i \times j$ dimensional zero matrix and by $I_i \in \mathbb{C}^{i \times i}$ the unit matrix. Then for $M < 2k-1$ the matrix $\tilde{G}$ writes as

$$\left(\begin{array}{cc|cc}
C_{k,M} & \mathbf{0}_{M-k+1,l-M} & I_{M-k+1} & \mathbf{0}_{M-k+1,2k-M-1} \\
\mathbf{0}_{2k-M-1,M-k+1} & \mathbf{0}_{2k-M-1,l-M} & \mathbf{0}_{2k-M-1,M-k+1} & I_{2k-M-1}
\end{array}\right),$$

and for $M \ge 2k+1$ we have

$$\left(\begin{array}{c|cc|c}
\tilde{C}_{k,M-2k+1} & C_{k,M} & \mathbf{0}_{k,l-M} & I_k \\
\hline
C'_{M-2k+1,M-2k+1} & \mathbf{0}_{M-2k+1,k} & \mathbf{0}_{M-2k+1,l-M} & \mathbf{0}_{M-2k+1,k} \\
\mathbf{0}_{l-M,M-2k+1} & \mathbf{0}_{l-M,k} & \mathbf{0}_{l-M,l-M} & \mathbf{0}_{l-M,k}
\end{array}\right),$$

with

$$\tilde{C} = \begin{pmatrix} c_k & \dots & c_{M-k} \\ \vdots & & \vdots \\ c_{2k-1} & \dots & c_{M-1} \end{pmatrix} \quad \text{and} \quad C' = \begin{pmatrix} c_{2k} & \dots & c_M \\ \vdots & \iddots & \\ c_M & & 0 \end{pmatrix}.$$

Note that C' is regular. Consider the case $M < 2k-1$ first. Let $\vec{\alpha} \in \mathbb{C}^{M-k+1}$, $\vec{\beta} \in \mathbb{C}^{l-M}$, $\vec{\gamma} \in \mathbb{C}^{M-k+1}$ and $\vec{\delta} \in \mathbb{C}^{2k-M-1}$ then $(\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta})^T$ is in the kernel of $\tilde{G}$ if and only if

$$C_{k,M}\vec{\alpha} = -\vec{\gamma} \quad \text{and} \quad \vec{\delta} = \vec{0}_{2k-M-1},$$

where $\vec{0}_n$, $n \in \mathbb{N}$, denotes the n-dimensional zero vector. Therefore the kernel is given by

$$\ker \tilde{G} = \left\{ \left(\vec{\alpha}, \vec{\beta}, -C_{k,M}\vec{\alpha}, \vec{0}_{2k-M-1}\right)^T : \ \vec{\alpha} \in \mathbb{C}^{M-k+1}, \ \vec{\beta} \in \mathbb{C}^{l-M} \right\}.$$

Obviously, it holds that $\dim \ker \tilde{G} = l - k + 1$.
In the case $M \geq 2k-1$ consider $\vec{\alpha} \in \mathbb{C}^{M-2k+1}$, $\vec{\beta} \in \mathbb{C}^k$, $\vec{\gamma} \in \mathbb{C}^{l-M}$ and $\vec{\delta} \in \mathbb{C}^k$, then $(\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta})^T$ is in the kernel of $\tilde{G}$ if and only if

$$\vec{\alpha} = \vec{0}_{M-2k+1} \quad \text{and} \quad C_{k,M}\vec{\beta} = -\vec{\delta},$$

so the kernel in this case is given by

$$\ker \tilde{G} = \left\{ \left(\vec{0}_{M-2k+1}, \vec{\beta}, \vec{\gamma}, -C_{k,M}\vec{\beta}\right)^T : \ \vec{\beta} \in \mathbb{C}^k, \ \vec{\gamma} \in \mathbb{C}^{l-M} \right\},$$

and clearly $\dim \ker \tilde{G} = k + l - M$.

The isotropic part of $\mathfrak{L}_\vartheta$ with respect to the inner product $[.,.]_\vartheta$ is given by

$$\mathfrak{L}_\vartheta^{[\circ]_\vartheta} = \begin{cases} \{(0; \vec{0}_k; \vec{0}_k)^T\}, & l \leq k-1, \\ \{(0; \vec{0}_k, \vec{y}_{l-k+1}; \vec{0}_k)^T\}, & M \leq k-1, l \geq k, \\ \{(0; \vec{0}_k, \vec{x}_{M-k+1}, \vec{y}_{l-M}; -C_{k,M}\vec{x}_{M-k+1}, \vec{0}_{2k-M-1})^T\}, & \begin{array}{l} k-1 < M, \\ M < 2k-1, \end{array} \\ \{(0; \vec{0}_{M-k+1}, \vec{x}_k, \vec{y}_{l-M}; -C_{k,M}\vec{x}_k)^T\}, & k-1 < M, M \geq 2k-1, \end{cases}$$

where $\vec{0}$, $\vec{x}$, $\vec{y}$ are row vectors whose index corresponds to their dimension and $C_{k,M}$ is the regular Hankel matrix defined in (2.2.2).

Note that an element $(a_0, \dots, a_{\max\{l,k-1\}}; b_0, \dots b_{k-1})^T \in \mathbb{C}^{\max\{l+k+1,2k\}}$ is in $\operatorname{ran} \hat{G}$ if and only if there exists $(x_0, \dots, x_{\max\{l,k-1\}}; y_0, \dots, y_{k-1})^T \in \mathbb{C}^{\max\{l+k+1,2k\}}$ such that

$$\begin{aligned} a_j &= \sum_{i=j}^{\max\{l,k-1\}} c_i x_{i-j} + y_j, \quad j = 0, \dots, k-1, \\ a_j &= \sum_{i=j}^{\max\{l,k-1\}} c_i x_{i-j}, \quad j = k, \dots \max\{l, k-1\}, \\ b_j &= x_j, \qquad j = 0, \dots, k-1. \end{aligned}$$

This implies that $\hat{G}$ is injective on $\operatorname{ran} \hat{G}$.

Now we endow the space of all polynomials $\mathcal{P}$ with an inner product which derives from a distribution $\varphi \in \mathcal{F}(\mathbb{R}, 0)$:

2.12 Definition. For a fixed distribution $\varphi \in \mathcal{F}(\mathbb{R}, 0)$ we define an inner product on $\mathcal{P}$, the space of all complex valued polynomials, by

$$[p, q]_\varphi := \varphi(p\overline{q}), \quad p, q \in \mathcal{P}.$$

By A_t, $t \in \mathbb{R}$, we denote the multiplication operator on $\mathcal{P}$, i.e. $A_t(p) = tp(t)$, $p \in \mathcal{P}$.

2.13 Definition. Let $\varphi \in \mathcal{F}(\mathbb{R}, 0)$, $\vartheta = (k, l, \sigma, c_0, \dots, c_l)$ be a representation of φ and $N := \max\{l, k-1\}$. Then we define an operator on $\mathfrak{L}_\vartheta$ by

$$\mathfrak{A}_\vartheta : \begin{cases} \mathfrak{L}_\vartheta \to \mathfrak{L}_\vartheta \\ (f; a_0, \dots, a_N; b_0, \dots, b_{k-1})^T \mapsto \Big(tf + a_{k-1}; 0, a_0, \dots, a_{N-1}; \\ \qquad\qquad b_1, \dots, b_{k-1}, \int\limits_{\mathbb{R}} f(t) d\sigma(t)\Big)^T \end{cases}$$

and call it the *multiplication operator on* $\mathfrak{L}_\vartheta$.

Since the measure σ has compact support, the multiplication operator $\mathfrak{A}_\vartheta$ is everywhere defined and bounded. Also note that if $k = 0$, then $\mathfrak{L}_\vartheta = L^2(|\sigma|) \oplus \mathbb{C}^{l+1}$ and the multiplication operator is understood as

$$\mathfrak{A}_\vartheta\big((f; a_0, \dots, a_l)^T\big) = (tf; 0, a_0, \dots, a_{l-1})^T.$$

The operator $\mathfrak{A}_\vartheta$ admits the following matrix representation

$$\left(\begin{array}{c|cccc|cccc|cccc}
t\cdot & 0 & \dots & 0 & 1 & & & & & & & & \\ \hline
 & 0 & \dots & \dots & 0 & & & & & & & & \\
 & 1 & \ddots & & \vdots & & & & & & & & \\
 & 0 & \ddots & \ddots & \vdots & & & & & & & & \\
 & \vdots & \ddots & 1 & 0 & & & & & & & & \\ \hline
 & 0 & \dots & 0 & 1 & 0 & \dots & \dots & 0 & & & & \\
 & 0 & \dots & 0 & 0 & 1 & \ddots & & \vdots & & & & \\
 & \vdots & & \vdots & \vdots & 0 & \ddots & \ddots & \vdots & & & & \\
 & 0 & \dots & 0 & 0 & \vdots & \ddots & 1 & 0 & & & & \\ \hline
0 & & & & & 0 & \dots & \dots & 0 & 0 & 1 & 0 & \dots \\
\vdots & & & & & \vdots & & & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & & & & & \vdots & & & \vdots & \vdots & & \ddots & 1 \\
(.,1)_\sigma & & & & & 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0
\end{array}\right)$$

$$\underbrace{\hphantom{0 \dots 0 \; 1}}_{k\text{-columns}} \quad \underbrace{\hphantom{0 \dots \dots 0}}_{N-k+1\text{-columns}} \quad \underbrace{\hphantom{0 \; 1 \; 0 \dots}}_{k\text{-columns}}$$

2.14 Definition. Let $\varphi \in \mathcal{F}(\mathbb{R},0)$ and $\vartheta = (k,l,\sigma,c_0,\dots,c_l)$ be a representation of φ. For any polynomial $p \in \mathcal{P}$ we define

$$\tilde{p}_j^{[k,\sigma]} := \int\limits_{\mathbb{R}} \frac{p^{\{2k-j\}}(t)}{t^{2k-j}} d\sigma(t), \quad 0 \le j \le k.$$

The notation $.^{[k,\sigma]}$ is to distinguish between different representation of φ. If there are only representations under consideration with the same k and same σ we just write $\tilde{p}_j$ instead of $\tilde{p}_j^{[k,\sigma]}$.

2.15 Proposition. *Let $\varphi \in \mathcal{F}(\mathbb{R},0)$ and $\vartheta = (k,l,\sigma,c_0,\dots,c_l)$ be a representation of φ. Then the mapping*

$$\iota_\vartheta : \begin{cases} (\mathcal{P},[.,.]_\varphi) \to (\mathfrak{L}_\vartheta,[.,.]_\vartheta) \\ \qquad p \mapsto \Big(\frac{p^{\{k\}}}{t^k}; \frac{p^{(0)}(0)}{0!},\dots,\frac{p^{(\max\{l,k-1\})}(0)}{(\max\{l,k-1\})!}; \tilde{p}_0,\dots,\tilde{p}_{k-1}\Big)^T \end{cases}$$

is isometric and the following diagram commutes

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\iota_\vartheta"] \arrow[d, "A_t"'] & \mathfrak{L}_\vartheta \arrow[d, "\mathfrak{A}_\vartheta"] \\ \mathcal{P} \arrow[r, "\iota_\vartheta"'] & \mathfrak{L}_\vartheta . \end{tikzcd}$$

If $k=0$ then ι_ϑ is understood as the mapping

$$p \mapsto \Big(p, \frac{p^{(0)}(0)}{0!},\dots,\frac{p^{(l)}(0)}{l!}\Big)^T \in L^2(|\sigma|) \oplus \mathbb{C}^{l+1}, \quad p \in \mathcal{P}.$$

In order to proof this Proposition we need a rather technical result first.

2.16 Lemma. *For $p,q \in \mathcal{P}$, $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}_0$ the following identity holds*

$$(p\overline{q})^{\{\alpha,2k\}}(t) = p^{\{\alpha,k\}}(t)\overline{q^{\{\alpha,k\}}(t)} + \sum_{j=0}^{k-1}(t-\alpha)^j \Big(p_j\overline{q^{\{\alpha,2k-j\}}(t)} + \overline{q}_j p^{\{\alpha,2k-j\}}(t)\Big),$$

where $p_i := \frac{p^{(i)}(\alpha)}{i!}$, $i \in \mathbb{N}_0$, and $q_i := \frac{q^{(i)}(\alpha)}{i!}$, $i \in \mathbb{N}_0$.

Proof. If $k=0$ the statement of this lemma writes as $(p\overline{q})(t) = p(t)\overline{q(t)}$ which is trivial. For $k \in \mathbb{N}$ this lemma can be proved using induction on k. Abbreviatory we will write $p^{\{m\}}$ ($q^{\{m\}}$) instead of $p^{\{\alpha,m\}}$ ($q^{\{\alpha,m\}}$), $m \in \mathbb{N}_0$. For $k=1$ the right side of the equation yields

$$\begin{aligned} p^{\{1\}}(t)\overline{q^{\{1\}}(t)} + p_0\overline{q^{\{2\}}(t)} + \overline{q}_0 p^{\{2\}}(t) &= p(t)\overline{q}(t) - p_0\overline{q}(t) - \overline{q}_0 p(t) + p_0\overline{q}_0 + \\ &+ p_0\big(\overline{q}(t) - \overline{q}_0 - (t-\alpha)\overline{q}_1\big) + \overline{q}_0\big(p(t) - p_0 - (t-\alpha)p_1\big) = \\ &= p(t)\overline{q}(t) - p_0\overline{q}_0 - (t-\alpha)(p_0\overline{q}_1 + p_1\overline{q}_0) = (p\overline{q})^{\{2\}}(t), \end{aligned}$$

hence the equation holds in the case $k=1$. Now using the inductive hypothesis we have

$$
\begin{aligned}
(p\overline{q})^{\{2k+2\}}(t) &= (p\overline{q})^{\{2k\}}(t) - \frac{(t-\alpha)^{2k}}{(2k)!}(p\overline{q})^{(2k)}(\alpha) - \frac{(t-\alpha)^{2k+1}}{(2k+1)!}(p\overline{q})^{(2k+1)}(\alpha) = \\
&= p^{\{k\}}(t)\overline{q^{\{k\}}(t)} + \sum_{j=0}^{k-1}(t-\alpha)^j\left(p_j\overline{q^{\{2k-j\}}(t)} + \overline{q}_j p^{\{2k-j\}}(t)\right) - \\
&\quad - (t-\alpha)^{2k}\sum_{j=0}^{2k} p_j\overline{q}_{2k-j} - (t-\alpha)^{2k+1}\sum_{j=0}^{2k+1} p_j\overline{q}_{2k+1-j} = \\
&= p^{\{k+1\}}(t)\overline{q^{\{k+1\}}(t)} + (t-\alpha)^k p_k\overline{q^{\{k+1\}}(t)} + (t-\alpha)^k\overline{q}_k p^{\{k+1\}}(t) + \\
&\quad + (t-\alpha)^{2k}p_k\overline{q}_k + \sum_{j=0}^{k-1}(t-\alpha)^j\Big[p_j\left(\overline{q^{\{2k-j+1\}}(t)} + (t-\alpha)^{2k-j}\overline{q}_{2k-j}\right) + \\
&\quad + \overline{q}_j\left(p^{\{2k-j+1\}}(t) + (t-\alpha)^{2k-j}p_{2k-j}\right)\Big] - (t-\alpha)^{2k}\sum_{j=0}^{2k} p_j\overline{q}_{2k-j} - \\
&\quad - (t-\alpha)^{2k+1}\sum_{j=0}^{2k+1} p_j\overline{q}_{2k+1-j} = \\
&= p^{\{k+1\}}(t)\overline{q^{\{k+1\}}(t)} + \sum_{j=0}^{k}(t-\alpha)^j\left(p_j\overline{q^{\{2k-j+1\}}(t)} + \overline{q}_j p^{\{2k-j+1\}}(t)\right) + \\
&\quad + (t-\alpha)^{2k}p_k\overline{q}_k + (t-\alpha)^{2k}\sum_{j=0}^{k-1}(p_j\overline{q}_{2k-j} + \overline{q}_j p_{2k-j}) - \\
&\quad - (t-\alpha)^{2k}\sum_{j=0}^{2k} p_j\overline{q}_{2k-j} - (t-\alpha)^{2k+1}\sum_{j=0}^{2k+1} p_j\overline{q}_{2k+1-j} = \\
&= p^{\{k+1\}}(t)\overline{q^{\{k+1\}}(t)} + \sum_{j=0}^{k}(t-\alpha)^j\left(p_j\overline{q^{\{2k-j+2\}}(t)} + \overline{q}_j p^{\{2k-j+2\}}(t)\right)
\end{aligned}
$$

and the equation is proved. ❑

Proof (Proposition 2.15). By Lemma 2.16 it holds

$$(p\overline{q})^{\{2k\}}(t) = p^{\{k\}}(t)\overline{q^{\{k\}}(t)} + \sum_{j=0}^{k-1} t^j\left(\frac{p^{(j)}(0)}{j!}\overline{q^{\{2k-j\}}(t)} + \frac{\overline{q^{(j)}(0)}}{j!}p^{\{2k-j\}}(t)\right),$$

therefore we obtain from (2.1.1)

$$[p,q]_\varphi = \varphi(p\overline{q}) = \int_{\mathbb{R}} \frac{(p\overline{q})^{\{2k\}}(t)}{t^{2k}}d\sigma(t) + \sum_{i=0}^{l} c_i\frac{(p\overline{q})^{(i)}(0)}{i!} =$$

$$= \int_{\mathbb{R}}\left(\frac{p^{\{k\}}(t)\overline{q^{\{k\}}(t)}}{t^{2k}} + \sum_{j=0}^{k-1}\frac{p^{(j)}(0)}{j!}\frac{\overline{q^{\{2k-j\}}(t)}}{t^{2k-j}} + \sum_{j=0}^{k-1}\frac{\overline{q^{(j)}(0)}}{j!}\frac{p^{\{2k-j\}}(t)}{t^{2k-j}}\right)d\sigma(t) +$$

$$+\sum_{i=0}^{l} c_i \sum_{h=0}^{i} \frac{p^{(h)}(0)}{h!}\frac{\overline{q^{(i-h)}(0)}}{(i-h)!} =$$

$$= \int_{\mathbb{R}} \frac{p^{\{k\}}(t)\overline{q^{\{k\}}(t)}}{t^{2k}} d\sigma(t) + \sum_{j=0}^{k-1} \frac{p^{(j)}(0)}{j!} \int_{\mathbb{R}} \frac{\overline{q^{\{2k-j\}}(t)}}{t^{2k-j}} d\sigma(t) +$$

$$+\sum_{j=0}^{k-1} \int_{\mathbb{R}} \frac{p^{\{2k-j\}}(t)}{t^{2k-j}} d\sigma(t) \frac{\overline{q^{(j)}(0)}}{j!} + \sum_{i=0}^{l} c_i \sum_{h=0}^{i} \frac{p^{(h)}(0)}{h!}\frac{\overline{q^{(i-h)}(0)}}{(i-h)!}. \tag{2.2.3}$$

Let $N := \max\{l, k-1\}$, then clearly

$$\int_{\mathbb{R}} \frac{p^{\{k\}}(t)\overline{q^{\{k\}}(t)}}{t^{2k}} d\sigma(t) =$$

$$= \left[\left(\frac{p^{\{k\}}}{t^k}; 0,\dots,0; 0,\dots,0\right)^T, \left(\frac{q^{\{k\}}}{t^k}; 0,\dots,0; 0,\dots,0\right)^T \right]_{\vartheta},$$

$$\sum_{j=0}^{k-1} \frac{p^{(j)}(0)}{j!} \int_{\mathbb{R}} \frac{\overline{q^{\{2k-j\}}(t)}}{t^{2k-j}} d\sigma(t) =$$

$$= \left[\left(0; \frac{p^{(0)}(0)}{0!}, \dots, \frac{p^{(N)}(0)}{(N)!}; 0, \dots, 0\right)^T, \left(0; 0, \dots, 0; \overline{\tilde{q}_0}, \dots, \overline{\tilde{q}_{k-1}}\right)^T \right]_{\vartheta},$$

$$\sum_{j=0}^{k-1} \int_{\mathbb{R}} \frac{p^{\{2k-j\}}(t)}{t^{2k-j}} d\sigma(t) \frac{\overline{q^{(j)}(0)}}{j!} =$$

$$= \left[\left(0; 0, \dots, 0; \tilde{p}_0, \dots, \tilde{p}_{k-1}\right)^T, \left(0; \frac{\overline{q^{(0)}(0)}}{0!}, \dots, \frac{\overline{q^{(N)}(0)}}{N!}; 0, \dots, 0\right)^T \right]_{\vartheta}.$$

In the case $l < k-1$ we set $c_n := 0$, $n > l$. Then we have

$$\sum_{i=0}^{l} c_i \sum_{h=0}^{i} \frac{p^{(h)}(0)}{h!}\frac{\overline{q^{(i-h)}(0)}}{(i-h)!} = \sum_{i=0}^{N}\sum_{h=0}^{i} c_i \frac{p^{(h)}(0)}{h!}\frac{\overline{q^{(i-h)}(0)}}{(i-h)!} =$$

$$= \sum_{h=0}^{N}\sum_{i=h}^{N} c_i \frac{p^{(h)}(0)}{h!}\frac{\overline{q^{(i-h)}(0)}}{(i-h)!} = \sum_{h=0}^{N}\sum_{i=0}^{N} c_{i+h} \frac{p^{(h)}(0)}{h!}\frac{\overline{q^{(i)}(0)}}{i!} =$$

$$= \left[\left(0; \frac{p^{(0)}(0)}{0!}, \dots, \frac{p^{(N)}(0)}{N!}; 0, \dots, 0\right)^T, \left(0; \frac{q^{(0)}(0)}{0!}, \dots, \frac{q^{(N)}(0)}{N!}; 0, \dots, 0\right)^T \right]_{\vartheta}.$$

Hence by (2.2.3) it follows that $[p,q]_\varphi = [\iota_\vartheta(p), \iota_\vartheta(q)]_\vartheta$, i.e. ι_ϑ is an isometry. It remains to prove that the operator $\mathfrak{A}_\vartheta$ satisfies the diagram. Let $p \in \mathcal{P}$ and define $q(t) := A_t p(t) = tp(t)$, then we have

$$\frac{q^{(j)}(0)}{j!} = \frac{1}{j!}\big(tp(t)\big)^{(j)}(0) = \frac{1}{j!}\sum_{i=0}^{j}\binom{j}{i} t^{(i)}|_{t=0} p^{(j-i)}(0) = \begin{cases} 0, & j = 0 \\ \frac{p^{(j-1)}(0)}{(j-1)!}, & j \geq 1. \end{cases}$$

Using this relation we can rewrite the regularized term $q^{\{k\}}$. For $k > 0$ it follows that

$$q^{\{k\}}(t) = q(t) - \sum_{i=0}^{k-1} \frac{q^{(i)}(0)}{i!} t^i = tp(t) - \sum_{i=1}^{k-1} \frac{p^{(i-1)}(0)}{(i-1)!} t^i =$$
$$= t\Big(p(t) - \sum_{i=0}^{k-2} \frac{p^{(i)}(0)}{i!} t^i\Big) = tp^{\{k-1\}}(t).$$

Then we have for $j = 0, \dots, k-1$, $k > 0$,

$$\tilde{q}_j = \int_{\mathbb{R}} \frac{q^{\{2k-j\}}(t)}{t^{2k-j}} d\sigma(t) = \int_{\mathbb{R}} \frac{tp^{\{2k-j-1\}}(t)}{t^{2k-j}} d\sigma(t) =$$
$$= \int_{\mathbb{R}} \frac{p^{\{2k-(j+1)\}}(t)}{t^{2k-(j+1)}} d\sigma(t) = \tilde{p}_{j+1}.$$

Now clearly $\iota_\vartheta(q)$ writes as

$$\iota_\vartheta(q) = \begin{cases} \Big(tp; 0, \frac{p^{(0)}(0)}{0!}, \dots, \frac{p^{(l-1)}(0)}{(l-1)!}\Big)^T & k = 0, \\ \Big(\frac{p^{\{k-1\}}}{t^{k-1}}; 0, \frac{p^{(0)}(0)}{0!}, \dots, \frac{p^{(\max\{l,k-1\}-1)}(0)}{(\max\{l,k-1\}-1)!}; \\ \qquad \tilde{p}_1, \dots, \tilde{p}_{k-1}, \int_{\mathbb{R}} \frac{p^{\{k\}}(t)}{t^k} d\sigma(t)\Big)^T & k > 0. \end{cases}$$

Since $\frac{p^{\{k-1\}}(t)}{t^{k-1}} = t\frac{p^{\{k\}}(t)}{t^k} + \frac{p^{(k-1)}(0)}{(k-1)!}$ for $k > 0$ it follows that $\iota_\vartheta(A_t p) = \mathfrak{A}_\vartheta\big(\iota_\vartheta(p)\big)$ for all $p \in \mathcal{P}$.

❑

2.17 Corollary. *Let $\varphi \in \mathcal{F}(\mathbb{R}, 0)$, $\vartheta = (k, l, \sigma, c_0, \dots, c_l) \in \Theta_\varphi$ and $N := \max\{l+1, k\}$. Then*

(*i*) $\mathfrak{A}_\vartheta\big(\overline{\operatorname{ran}\iota_\vartheta}\big) \subseteq \overline{\operatorname{ran}\iota_\vartheta}$,

(*ii*) $[\mathfrak{A}_\vartheta\, x, y]_\vartheta = [x, \mathfrak{A}_\vartheta\, y]_\vartheta, \quad x, y \in \overline{\operatorname{ran}\iota_\vartheta}$,

(*iii*) $\mathfrak{A}_\vartheta\big(\overline{\operatorname{ran}\iota_\vartheta}^{[\circ]_\vartheta}\big) \subseteq \overline{\operatorname{ran}\iota_\vartheta}^{[\circ]_\vartheta}$,

(*iv*) *for all $x \in \mathfrak{L}_\vartheta$ it holds $\epsilon[\mathfrak{A}_\vartheta^{N+1+\nu}\, x, \mathfrak{A}_\vartheta^{N+1}\, x]_\vartheta \geq 0$, where $\epsilon := \operatorname{sign}(\sigma|_{\mathbb{R}^+})$ and $\nu := \frac{1}{2}|\operatorname{sign}(\sigma|_{\mathbb{R}^+}) - \operatorname{sign}(\sigma|_{\mathbb{R}^-})|$.*

Proof. ad (i) : By Proposition 2.15 we have $\mathfrak{A}_\vartheta(\operatorname{ran}\iota_\vartheta) = \operatorname{ran}\iota_\vartheta$. Since the operator $\mathfrak{A}_\vartheta$ is bounded it follows that $\mathfrak{A}_\vartheta\big(\overline{\operatorname{ran}\iota_\vartheta}\big) \subseteq \overline{\operatorname{ran}\iota_\vartheta}$.

ad (ii) : By Proposition 2.15 the mapping $\iota_\vartheta : \mathcal{P} \to \mathfrak{L}_\vartheta$ is an isometry and $\mathfrak{A}_\vartheta \circ \iota_\vartheta = \iota_\vartheta \circ A_t$ on $\mathcal{P}$. Therefore it follows for every $p, q \in \mathcal{P}$

$$[\mathfrak{A}_\vartheta\, \iota_\vartheta(p), \iota_\vartheta(q)]_\vartheta = [(\mathfrak{A}_\vartheta \circ \iota_\vartheta)p, \iota_\vartheta(q))]_\vartheta = [(\iota_\vartheta \circ A_t)p, \iota_\vartheta(q)]_\vartheta = [A_t p, q]_\varphi =$$
$$= [p, A_t q]_\varphi = [\iota_\vartheta(p), (\iota_\vartheta \circ A_t)q]_\vartheta = [\iota_\vartheta(p), (\mathfrak{A}_\vartheta \circ \iota_\vartheta)q]_\vartheta = [\iota_\vartheta(p), \mathfrak{A}_\vartheta\, \iota_\vartheta(q)]_\vartheta.$$

This implies that $\mathfrak{A}_\vartheta$ is symmetric on $\operatorname{ran}\iota_\vartheta$. Since the inner product $[.,.]_\vartheta$ is obtained from the Hilbert space inner product $(.,.)_\vartheta$ via the Gram operator G it follows by continuity that $\mathfrak{A}_\vartheta$ is symmetric on $\overline{\operatorname{ran}\iota_\vartheta}$.

ad (iii) : Let $x \in \overline{\operatorname{ran}\iota_\vartheta}^{[\circ]_\vartheta}$, then $[x,z]_\vartheta = 0$ for all $z \in \overline{\operatorname{ran}\iota_\vartheta}$. By (i) and (ii) it follows that $[\mathfrak{A}_\vartheta\, x, y]_\vartheta = [x, \mathfrak{A}_\vartheta\, y]_\vartheta = 0$ for all $y \in \overline{\operatorname{ran}\iota_\vartheta}$. This implies that $\mathfrak{A}_\vartheta\, x \in \overline{\operatorname{ran}\iota_\vartheta}^{[\circ]_\vartheta}$.

ad (iv) : Let $\vec{f} = (f; a_0, \dots, a_N; b_0, \dots, b_{k-1})^T \in \mathfrak{L}_\vartheta$ and define $g(t) := t^{N+1} f + t^N a_{k-1} + t^{N-1} a_{k-2} + \dots + t^{N-k+1} a_0$, then for some $\vec{b}_1, \vec{b}_2 \in \mathbb{C}^k$ it holds

$$\epsilon[\mathfrak{A}_\vartheta^{N+1+\nu}\, \vec{f}, \mathfrak{A}_\vartheta^{N+1}\, \vec{f}]_\vartheta = \epsilon\big[\big(t^\nu g; 0, \dots, 0; \vec{b}_1\big)^T, \big(g; 0, \dots, 0; \vec{b}_2\big)^T\big]_\vartheta =$$

$$= \epsilon \int_{\mathbb{R}} t^\nu |g(t)|^2 d\sigma(t) = \epsilon \int_{\mathbb{R}^-} t^\nu |g(t)|^2 d\sigma|_{\mathbb{R}^-}(t) + \epsilon \int_{\mathbb{R}^+} t^\nu |g(t)|^2 d\sigma|_{\mathbb{R}^+}(t) \geq 0.$$

This completes the proof. ❑

2.3 Model space for $\mathcal{F}(\mathbb{R},0)$

Our aim in this section is to construct a Kreĭn space to a given distribution $\varphi \in \mathcal{F}(\mathbb{R},0)$ such that the space of all polynomials can be embedded isometrically and that there exists a bounded selfadjoint definitizable operator which is compatible with the multiplication operator on the polynomials.
In the beginning we choose some representation $\vartheta \in \Theta_\varphi$ and consider the space $\overline{\operatorname{ran}\iota_\vartheta}/_{\overline{\operatorname{ran}\iota_\vartheta}^{[\circ]_\vartheta}}$ endowed with the *factor space inner product*[1]. That this factor space is a Kreĭn space is one assertion of the following theorem:

2.18 Theorem. *Let $\varphi \in \mathcal{F}(\mathbb{R},0)$ and $\vartheta \in \Theta_\varphi$. Then $\mathcal{K}_\vartheta := \overline{\operatorname{ran}\iota_\vartheta}/_{\overline{\operatorname{ran}\iota_\vartheta}^{[\circ]_\vartheta}}$, endowed with the factor space inner product $[.,.]_{\mathcal{K}_\vartheta}$, is a Kreĭn space. The operator $\mathfrak{A}_\vartheta$ induces a bounded, selfadjoint, definitizable operator A_ϑ in $\mathcal{K}_\vartheta$. There exists a real definitizing polynomial p for A_ϑ such that $x = 0$ is the only zero of p. There exists an isometry $\hat{\iota}_\vartheta : \mathcal{P} \to \mathcal{K}_\vartheta$ with dense range such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\hat{\iota}_\vartheta} & \mathcal{K}_\vartheta \\ {\scriptstyle A_t}\downarrow & & \downarrow{\scriptstyle A_\vartheta} \\ \mathcal{P} & \xrightarrow[\hat{\iota}_\vartheta]{} & \mathcal{K}_\vartheta. \end{array} \tag{2.3.1}$$

Then we will show that the Kreĭn space $\mathcal{K}_\vartheta$ is independent up to unitary equivalence. This is is the main theorem of this section:

2.19 Theorem. *Let $\varphi \in \mathcal{F}(\mathbb{R},0)$ and $\vartheta_1, \vartheta_2 \in \Theta_\varphi$. Then there exists a unitary*[2] *mapping $U_{\vartheta_1,\vartheta_2} : \mathcal{K}_{\vartheta_1} \to \mathcal{K}_{\vartheta_2}$ such that $U_{\vartheta_1,\vartheta_2} A_{\vartheta_1} = A_{\vartheta_2} U_{\vartheta_1,\vartheta_2}$ and the following diagram commutes*

$$\begin{tikzcd} \mathcal{K}_{\vartheta_1} \arrow[rr, "U_{\vartheta_1,\vartheta_2}"] & & \mathcal{K}_{\vartheta_2} \\ & \mathcal{P}. \arrow[ul, "\hat{\iota}_{\vartheta_1}"] \arrow[ur, "\hat{\iota}_{\vartheta_2}"'] & \end{tikzcd} \tag{2.3.2}$$

[1]Let $(\mathcal{L}, [.,.])$ be an inner product space, $\mathcal{M}$ be a linear subspace of $\mathcal{L}^\circ$, and denote by $\pi : \mathcal{L} \to \mathcal{L}/_{\mathcal{M}}$ the canonical projection. Then an inner product, the *factor space inner product* is well-defined on the factor space by $[\pi x, \pi y]_{\mathcal{L}/_{\mathcal{M}}} := [x,y]$ for $x, y \in \mathcal{L}$.

[2]Let $(\mathfrak{K}_i, [.,.]_{\mathfrak{K}_i})$, $i = 1,2$, be Kreĭn spaces and $U \in \mathcal{B}(\mathfrak{K}_1, \mathfrak{K}_2)$. We say U is unitary if U is surjective and isometric, i.e. $[Ux, Uy]_{\mathfrak{K}_2} = [x,y]_{\mathfrak{K}_2}$ for all $x, y \in \mathfrak{K}_1$.

This justifies the following definition:

2.20 Definition. Let $\varphi \in \mathcal{F}(\mathbb{R}, 0)$ and choose a representation $\vartheta \in \Theta_\varphi$. Define $\mathcal{K}_\varphi := \mathcal{K}_\vartheta$, $A_\varphi := A_\vartheta$ and $\iota_\varphi := \hat{\iota}_\vartheta$. The triple $(\mathcal{K}_\varphi, A_\varphi, \iota_\varphi)$ will be the model for the distribution $\varphi \in \mathcal{F}(\mathbb{R}, 0)$. We refer to $\mathcal{K}_\varphi$ as the *model space*, to A_φ as the *model operator* and to ι_φ as the *model embedding*.

By the main theorem $\mathcal{K}_\varphi$ is well-defined up to unitary equivalence and the following diagram commutes

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\iota_\varphi"] \arrow[d, "\cdot t"'] & \mathcal{K}_\varphi \arrow[d, "A_\varphi"] \\ \mathcal{P} \arrow[r, "\iota_\varphi"'] & \mathcal{K}_\varphi . \end{tikzcd} \tag{2.3.3}$$

Recall that an operator A in a Kreĭn space $(\mathfrak{K}, [.,.]_\mathfrak{K})$ is called *cyclic* if there exists a *generating element* $u_0 \in \mathfrak{K}$ such that $\operatorname{cls}\{A^n u_0 : n \in \mathbb{N}_0\} = \mathfrak{K}$, where cls denotes the closed linear span and the closure is taken with respect to the topology induced by a fundamental decomposition.

2.21 Corollary. *Let $\varphi \in \mathcal{F}(\mathbb{R}, 0)$ and $\mathcal{K}_\varphi$ be the corresponding model space with model operator A_φ. Then A_φ is cyclic with generating element $\iota_\varphi(1)$.*

Proof. By diagram 2.3.3 it follows that $p(A_\varphi)\big(\iota_\varphi(1)\big) = \iota_\varphi(p)$ for every $p \in \mathcal{P}$. Further Theorem 2.18 implies that $\operatorname{ran} \iota_\varphi$ is dense in $\mathcal{K}_\varphi$. This shows that A_φ is cyclic with generating element $\iota_\varphi(1)$. ❑

Before we are able to prove these theorems, we need some auxiliary results.

2.22 Lemma. *Let $\big(X, (.,.)_X\big)$ and $\big(Y, (.,.)_Y\big)$ be Hilbert spaces and $B : X \to Y$ be a bounded operator. Further let $[.,.]_X$ be a $(.,.)_X$-continuous indefinite inner product on X and $[.,.]_Y$ be a $(.,.)_Y$-continuous indefinite inner product on Y. If $B(X^{[\circ]_X}) \subseteq B(Y^{[\circ]_Y})$, then there exists a unique bounded operator $\hat{B} : X/_{X^{[\circ]_X}} \to Y/_{Y^{[\circ]_Y}}$, such that the diagram*

$$\begin{tikzcd} X \arrow[r, "B"] \arrow[d, "\pi_X"'] & Y \arrow[d, "\pi_Y"] \\ X/_{X^{[\circ]_X}} \arrow[r, "\hat{B}"'] & Y/_{Y^{[\circ]_Y}} \end{tikzcd} \tag{2.3.4}$$

commutes, where π_X and π_Y denote the respective quotient maps.
If $X/_{X^{[\circ]_X}}$ and $Y/_{Y^{[\circ]_Y}}$ are a Kreĭn spaces it holds

(i) $\hat{B}$ is isometric if $[Bx, By]_Y = [x, y]_X$ for all $x, y \in X$.

(ii) $\hat{B}$ is unitary if B is surjective and $[Bx, By]_Y = [x, y]_X$ for all $x, y \in X$.

If in addition the spaces X and Y coincide we have

(iii) $\hat{B}$ is selfadjoint if $[Bx, y]_X = [x, By]_X$ for all $x, y \in X$.

Proof. First note, that $X^{[\circ]_X}$ and $Y^{[\circ]_Y}$ are closed, since $[.,.]_X$ and $[.,.]_Y$ are continuous. Define a mapping $\hat{B} : X/_{X^{[\circ]_X}} \to Y/_{Y^{[\circ]_Y}}$ by

$$\hat{B}\big(x + X^{[\circ]_X}\big) := Bx + Y^{[\circ]_Y}, \quad x \in X.$$

If $x + X^{[\circ]_X} = x' + X^{[\circ]_X}$, $x, x' \in X$, then $x - x' \in X^{[\circ]_X}$, and therefore we have $B(x) - B(x') = B(x - x') \subseteq Y^{[\circ]_Y}$. Now it follows that

$$\hat{B}\big(x + X^{[\circ]_X}\big) - \hat{B}\big(x' - X^{[\circ]_X}\big) = 0 + Y^{[\circ]_Y},$$

so $\hat{B}$ is well-defined. Clearly $\hat{B}$ is linear. Note that $\pi_Y \circ B = \hat{B} \circ \pi_X$, and that if $\hat{B}' : X/_{X^{[\circ]_X}} \to Y/_{Y^{[\circ]_Y}}$ is such that $\pi_Y \circ B = \hat{B}' \circ \pi_X$, then

$$\hat{B}'\big(x + X^{[\circ]_X}\big) = \hat{B}'\big(\pi_X(x)\big) = \pi_Y(Bx) = \hat{B}\big(x + X^{[\circ]_X}\big)$$

for each member $x + X^{[\circ]_X}$ of $X/_{X^{[\circ]_X}}$, thus $\hat{B}$ is unique. Since π_X and π_Y are continuous open mappings it follows that $\hat{B}$ is a bounded linear operator.
Now suppose $X/_{X^{[\circ]_X}}$ and $Y/_{Y^{[\circ]_Y}}$ are a Kreĭn spaces.

ad (i) : For $x, y \in X$ it follows

$$\begin{aligned}\big[\hat{B}\big(\pi_X(x)\big), \hat{B}\big(\pi_X(y)\big)\big]_{Y/_{Y^{[\circ]_Y}}} &= \big[\pi_Y(Bx, \pi_Y(By)\big]_{Y/_{Y^{[\circ]_Y}}} = \\ = \big[Bx, By\big]_Y = \big[x, y\big]_X &= \big[\pi_X(x), \pi_Y(y)\big]_{X/_{X^{[\circ]_X}}}.\end{aligned}$$

ad (ii) : Clearly, if B is surjective so is $\hat{B}$ and using part (i) gives (ii).

ad (iii) : Since $X = Y$ denote the quotient map $\pi_X : X \to X/_{X^{[\circ]_X}}$ just by π. For $x, y \in X$ we have

$$\begin{aligned}\big[\hat{B}\big(\pi(x)\big), \pi(y)\big]_{X/_{X^{[\circ]_X}}} &= \big[\pi(Bx), \pi(y)\big]_{X/_{X^{[\circ]_X}}} = \big[Bx, y\big]_X = \\ = \big[x, By\big]_X = \big[\pi(x), \pi(By)\big]_{X/_{X^{[\circ]_X}}} &= \Big[\pi(x), \hat{B}\big(\pi(y)\big)\Big]_{X/_{X^{[\circ]_X}}}.\end{aligned}$$

Since π is surjective the operator B is selfadjoint. ❑

2.23 Lemma. *Let $(X, \|.\|_X)$, $(Y, \|.\|_Y)$ be Banach spaces and $X = X_1 \dot{+} X_2$ such that $\dim X_2 < \infty$ and X_1 is closed. Then it holds:*

(i) *If $P : X \to X$ is the projection with $\ker P = X_2$, then $P(Z)$ is closed in $(X_1, \|.\|_X)$ for every closed subspace $Z \subseteq X$.*

(ii) *If $T \in \mathcal{B}(X,Y)$, $\ker T = X_2$ and $\|Tx\|_Y \geq \gamma \|x\|_X$ for all $x \in X_1$ and a constant $\gamma > 0$, then $T(Z)$ is closed in $(Y, \|.\|_Y)$ for every closed subspace $Z \subseteq X$.*

Proof.

ad (i) : The quotient space $\big(X/_{X_2}, \|.\|_{X/_{X_2}}\big)$ is a Banach space and the topology induced by the quotient norm $\|.\|_{X/_{X_2}}$ coincides with the final topology on $X/_{X_2}$ with respect to the canonical projection $\pi : X \to X/_{X_2}$. The mapping $\pi|_{X_1} : X_1 \to X/_{X_2}$ is bijective and, as a restriction of π, continuous. Since X_1 is closed, $\big(X_1, \|.\|_X\big)$ is a Banach space and the open mapping theorem implies that $\pi|_{X_1}$ is a homeomorphism. Let Z be a closed subspace of X, then we have $\pi|_{X_1}\big(P(Z)\big) = \pi(Z) \subseteq X/_{X_2}$. The space $\pi(Z)$ is closed in $X/_{X_2}$ with respect to the final topology, because $\pi^{-1}\big(\pi(Z)\big) = Z + X_2$ and $\dim X_2 < \infty$. Since $\pi|_{X_1}$ is a homeomorphism it follows that $P(Z) = \pi|_{X_1}^{-1}\big(\pi(Y)\big)$ is closed in $(X_1, \|.\|_X)$.

ad (ii) : Assume first that Z is a closed subspace and $Z \subseteq X_1$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $T(Z)$ with $y_n \to y \in Y$. Then for each $n \in \mathbb{N}$ there exists a $x_n \in Z$ such that $y_n = Tx_n$, $n \in \mathbb{N}$. We have

$$\|x_n - x_m\|_X \le \frac{1}{\gamma}\|y_n - y_m\|_Y, \quad n, m \in \mathbb{N},$$

hence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, thus $(x_n)_{n\in\mathbb{N}}$ converges to an element $x \in X$. Since Z is closed it follows that $x \in Z$. The boundedness of T implies that $y_n = Tx_n \to Tx$. Therefore $y = Tx$ with $x \in Z$, thus $T(Z)$ is closed.
If Z is an arbitrary closed subspace of X, then $T(Z) = T(P(Z))$. By the first part of this lemma $P(Z)$ is closed and a subspace of X_1. By the preceding argument it follows that $T(Z)$ is closed. ❑

Now we are able to prove the first theorem of this section.

Proof (Theorem 2.18). Assume $\vartheta = (k, l, \sigma, c_0, \ldots, c_l)$. Let $\mathcal{H} := \overline{\operatorname{ran} \iota_\vartheta}$ and denote by $P_\mathcal{H} : \mathfrak{L}_\vartheta \to \mathcal{H}$ the orthogonal projection onto the closed subspace $\mathcal{H}$. Define $G_\mathcal{H} := P_\mathcal{H} G|_\mathcal{H}$, then $G_\mathcal{H}$ is a bounded and selfadjoint operator on $\mathcal{H}$. The Hermitian sesquilinear form $[.,.]_\mathcal{H} := (G_\mathcal{H}.,.)_\vartheta$ defines an indefinite inner product on $\mathcal{H}$ which satisfies

$$[x,y]_\mathcal{H} = (G_\mathcal{H}x,y)_\vartheta = (P_\mathcal{H}Gx,y)_\vartheta = (Gx,P_\mathcal{H}y)_\vartheta = (Gx,y)_\vartheta = [x,y]_\vartheta, \quad \forall x,y \in \mathcal{H}. \tag{2.3.5}$$

The isotropic part $\mathcal{H}^{[\circ]_\vartheta}$ is given by

$$\mathcal{H}^{[\circ]_\vartheta} = \{x \in \mathcal{H} : \ [x,y]_\mathcal{H} = 0 \ \forall y \in \mathcal{H}\} = \ker G_\mathcal{H}.$$

In order to show that $\mathcal{H}/_{\mathcal{H}^{[\circ]_\vartheta}}$ is a Kreĭn space it is sufficient to show that $\operatorname{ran} G_\mathcal{H}$ is closed (see [AI, 6.13 page 40]). Denote by $\hat{G}$ the restriction of G to the finite dimensional subspace $\mathbb{C}^{\max\{k+l+1,2k\}}$. Define $X_1 := L^2(|\sigma|)\dot{+}\operatorname{ran}\hat{G}$ and $s := \dim \operatorname{ran} \hat{G}$. Then we have $\mathfrak{L}_\vartheta = X_1 \dot{+} \ker \hat{G}$ and clearly $\dim\ker\hat{G} = \max\{k+l+1, 2k\} - s < \infty$. Since $\hat{G}$ is injective on $\operatorname{ran}\hat{G}$ (see Remark 2.11) there exists a constant $\gamma > 0$ such that $\|\hat{G}a\|_{\mathbb{C}^s} \ge \gamma\|a\|_{\mathbb{C}^s}$. Then for $x = (f;a) \in X_1$ it follows

$$\|Gx\|_\vartheta = \|Jf\|_{L^2(|\sigma|)} + \|\hat{G}a\|_{\mathbb{C}^s} \ge \min\{1,\gamma\}\big(\|f\|_{L^2(|\sigma|)} + \|a\|_{\mathbb{C}^s}\big) = \min\{1,\gamma\}\|x\|_\vartheta.$$

By Lemma 2.23 (ii) we obtain that $G(\mathcal{H})$ is closed in $\mathfrak{L}_\vartheta$. Since $\ker P_\mathcal{H}$ is finite dimensional Lemma 2.23 (i) yields that $P_\mathcal{H}\big(G(\mathcal{H})\big)$ is closed. Therefore $\operatorname{ran} G_\mathcal{H}$ is closed.
Since $\big(\mathcal{H}, (.,.)_\vartheta|_{\mathcal{H}\times\mathcal{H}}\big)$ is a Hilbert space and the inner product $[.,.]_\vartheta$ is continuous with respect to the inner product $(.,.)_\vartheta$ we can apply Lemma 2.22. By Corollary 2.17 (iii) and Lemma 2.22 the operator $\mathfrak{A}_\vartheta$ induces a bounded operator A_ϑ on $\mathcal{K}_\vartheta = \mathcal{H}/_{\mathcal{H}^{[\circ]_\vartheta}}$. Denote by $\pi_\mathcal{H} : \mathcal{H} \to \mathcal{K}_\vartheta$ the quotient map and by $\iota_\mathcal{H} : \mathcal{K}_\vartheta \to \mathcal{H}$ the natural embedding. The factor space inner product on $\mathcal{K}_\vartheta$ is given by

$$[\pi_\mathcal{H}x, \pi_\mathcal{H}y]_{\mathcal{K}_\vartheta} = [x,y]_\mathcal{H}, \quad x,y \in \mathcal{H}.$$

By (2.3.5), Corollary 2.17 (i), (ii), Lemma 2.22 (iii) and the fact that $A_\vartheta = \pi_\mathcal{H}\,\mathfrak{A}_\vartheta\,\iota_\mathcal{H}$, for $x, y \in \mathcal{H}$ it holds

$$\begin{aligned}[A_\vartheta x, y]_{\mathcal{K}_\vartheta} &= [\pi_\mathcal{H}\,\mathfrak{A}_\vartheta\,\iota_\mathcal{H} x, \pi_\mathcal{H}\iota_\mathcal{H} y]_{\mathcal{K}_\vartheta} = [\mathfrak{A}_\vartheta\,\iota_\mathcal{H} x, \iota_\mathcal{H} y]_\mathcal{H} = [\mathfrak{A}_\vartheta\,\iota_\mathcal{H} x, \iota_\mathcal{H} y]_\vartheta = \\ &= [\iota_\mathcal{H} x, \mathfrak{A}_\vartheta\,\iota_\mathcal{H} y]_\vartheta = [\iota_\mathcal{H} x, \mathfrak{A}_\vartheta\,\iota_\mathcal{H} y]_\mathcal{H} = [\pi_\mathcal{H}\iota_\mathcal{H} x, \pi_\mathcal{H}\,\mathfrak{A}_\vartheta\,\iota_\mathcal{H} y]_{\mathcal{K}_\vartheta} = [x, A_\vartheta y]_{\mathcal{K}_\vartheta},\end{aligned}$$

hence A_ϑ is selfadjoint.
Since $A_\vartheta = \pi_\mathcal{H}\,\mathfrak{A}_\vartheta\,\iota_\mathcal{H}$ it follows that $A_\vartheta^m = (\pi_\mathcal{H}\,\mathfrak{A}_\vartheta\,\iota_\mathcal{H})^m = \pi_\mathcal{H}\,\mathfrak{A}_\vartheta^m\,\iota_\mathcal{H}$, for $m \in \mathbb{N}$, which implies that

$$[A_\vartheta^m x, x]_{\mathcal{K}_\vartheta} = [\pi_\mathcal{H}\,\mathfrak{A}_\vartheta^m\,\iota_\mathcal{H} x, \pi_\mathcal{H}\iota_\mathcal{H} x]_{\mathcal{K}_\vartheta} = [\mathfrak{A}_\vartheta^m\,\iota_\mathcal{H} x, \iota_\mathcal{H} x]_\vartheta, \quad x \in \mathcal{K}_\vartheta,\ m \in \mathbb{N}.$$

By Corollary 2.17 (iv) it follows immediately that $p(z) = \epsilon z^{2N+2+\nu}$ is a definitizing polynomial for A_ϑ, where $\epsilon := \operatorname{sign}(\sigma|_{\mathbb{R}^+})$m $\nu := \frac{1}{2}|\operatorname{sign}(\sigma|_{\mathbb{R}^+}) - \operatorname{sign}(\sigma|_{\mathbb{R}^-})|$ and $N := \max\{l+1, k\}$. Clearly p is real and $z = 0$ is the only zero of p.
Let $\hat\iota_\vartheta := \pi_\mathcal{H} \circ \iota_\vartheta$, then by Proposition 2.15 it follows that

$$[\hat\iota_\vartheta p, \hat\iota_\vartheta q]_{\mathcal{K}_\vartheta} = [\pi_\mathcal{H}(\iota_\vartheta p), \pi_\mathcal{H}(\iota_\vartheta q)]_{\mathcal{K}_\vartheta} = [\iota_\vartheta p, \iota_\vartheta q]_\mathcal{H} + 0 = [p, q]_\varphi, \quad p, q \in \mathcal{P}.$$

Hence $\hat\iota_\vartheta : \big(\mathcal{P}, [.,.]_\varphi\big) \to \big(\mathcal{K}_\vartheta, [.,.]_{\mathcal{K}_\vartheta}\big)$ is an isometry. Further we have

$$A_\vartheta \circ \hat\iota_\vartheta = A_\vartheta \circ \pi_\mathcal{H} \circ \iota_\vartheta = \pi_\mathcal{H} \circ \mathfrak{A}_\vartheta \circ \iota_\mathcal{H} \circ \pi_\mathcal{H} \circ \iota_\vartheta = \pi_\mathcal{H} \circ \mathfrak{A}_\vartheta \circ \iota_\vartheta = \pi_\mathcal{H} \circ \iota_\vartheta \circ A_t = \hat\iota_\vartheta \circ A_t,$$

thus the diagram commutes. ❑

We now turn to the proof of the main result of this section.

2.24 Lemma. *Let $\varphi \in \mathcal{F}(\mathbb{R},0)$ and let $\vartheta_1 = (k, l, \sigma, c_0, \dots, c_l)$ and $\vartheta_2 = (k, \hat l, \hat\sigma, \hat c_0, \dots, \hat c_{\hat l})$ be representations of φ with $l < \hat l$. Let $N := \max\{l, k-1\}$ and $\hat N := \max\{\hat l, k-1\}$, then the mapping*

$$\Phi_{\mathfrak{L}_{\vartheta_2}, \mathfrak{L}_{\vartheta_1}} : \begin{cases} \big(\mathfrak{L}_{\vartheta_2}, [.,.]_{\vartheta_2}\big) \to \big(\mathfrak{L}_{\vartheta_1}, [.,.]_{\vartheta_1}\big) \\ \big(f; a_0, \dots, a_{\hat N}; b_0, \dots, b_{k-1}\big)^T \mapsto \big(f; a_0, \dots, a_N; b_0, \dots, b_{k-1}\big)^T \end{cases}$$

is a continuous isometry from $\mathfrak{L}_{\vartheta_2}$ onto $\mathfrak{L}_{\vartheta_1}$ and the following diagram commutes

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, "\iota_{\vartheta_1}"'] \arrow[dr, "\iota_{\vartheta_2}"] & \\ \mathfrak{L}_{\vartheta_1} & & \mathfrak{L}_{\vartheta_2}. \arrow[ll, "\Phi_{\mathfrak{L}_{\vartheta_2}, \mathfrak{L}_{\vartheta_1}}"] \end{tikzcd} \tag{2.3.6}$$

Proof. Lemma 2.6 implies $\sigma = \hat\sigma$, $c_i = \hat c_i$, $i = 0, \dots, l$, and $\hat c_{l+1} = \cdots = \hat c_{\hat l} = 0$. Since $N \leq \hat N$ the mapping $\Phi_{\mathfrak{L}_{\vartheta_2}, \mathfrak{L}_{\vartheta_1}}$ is always surjective and clearly $\Phi_{\mathfrak{L}_{\vartheta_2}, \mathfrak{L}_{\vartheta_1}}$ is continuous. In the case that $\hat l < k-1$ the spaces $\mathfrak{L}_{\vartheta_1}$ and $\mathfrak{L}_{\vartheta_2}$ as well as the inner products $(.,.)_{\vartheta_1}$ and $(.,.)_{\vartheta_2}$ coincide. Thus $\Phi_{\mathfrak{L}_{\vartheta_2}, \mathfrak{L}_{\vartheta_1}}$ is the identical map and therefore an isometry.

Now suppose $l < k-1$ and $\hat l \geq k-1$: Let $\vec f = (f; a_0, \dots, a_{\hat l}; b_0, \dots, b_{k-1})^T$ and $\vec g = (g; \alpha_0, \dots, \alpha_{\hat l}; \beta_0, \dots, \beta_{k-1})^T$ be in $\mathfrak{L}_{\vartheta_2}$. By Remark 2.11 it follows that $\vec f^{[\circ]_{\vartheta_2}} := (0; 0, \dots, 0, a_k, \dots, a_{\hat l}; 0, \dots, 0)^T$ and $\vec g^{[\circ]_{\vartheta_2}} := (0, 0, \dots, 0, \alpha_k, \dots, \alpha_{\hat l}; 0, \dots, 0)^T$ are in the isotropic part of $\mathfrak{L}_{\vartheta_2}$. Therefore we have

$$[\vec f, \vec g]_{\vartheta_2} = [\vec f - \vec f^{[\circ]_{\vartheta_2}}, \vec g - \vec g^{[\circ]_{\vartheta_2}}]_{\vartheta_2} = [\Phi_{\mathfrak{L}_{\vartheta_2}, \mathfrak{L}_{\vartheta_1}} \vec f, \Phi_{\mathfrak{L}_{\vartheta_2}, \mathfrak{L}_{\vartheta_1}} \vec g]_{\vartheta_1}.$$

It remains to prove the case $l \geq k-1$. Let $\vec{f}$ and $\vec{g}$ be as in the previous case, then the elements $\vec{f}^{[\circ]\vartheta_2} := (0;0,\dots,0,a_{l+1},\dots,a_{\hat{l}};0,\dots,0)^T$ and $\vec{g}^{[\circ]\vartheta_2} := (0,0,\dots,0,\alpha_{l+1},\dots,\alpha_{\hat{l}};0,\dots,0)^T$ are in the isotropic part of $\mathfrak{L}_{\vartheta_2}$. It follows that

$$[\vec{f},\vec{g}]_{\vartheta_2} = [\vec{f}-\vec{f}^{[\circ]\vartheta_2},\vec{g}-\vec{g}^{[\circ]\vartheta_2}]_{\vartheta_2} = [\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}\vec{f},\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}\vec{g}]_{\vartheta_1}.$$

To prove that $\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}(\iota_{\vartheta_2}(p)) = \iota_{\vartheta_1}$ let $p \in \mathcal{P}$. Then we have

$$\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}(\iota_{\vartheta_2}(p)) = \Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}\left(\left(\frac{p^{\{k\}}}{t^k};\frac{p^{(0)}(0)}{0!},\dots,\frac{p^{\hat{N}}(0)}{\hat{N}!};\tilde{p}_0,\dots,\tilde{p}_{k-1}\right)^T\right) =$$

$$= \left(\frac{p^{\{k\}}}{t^k};\frac{p^{(0)}(0)}{0!},\dots,\frac{p^{N}(0)}{N!};\tilde{p}_0,\dots,\tilde{p}_{k-1}\right)^T = \iota_{\vartheta_1}(p).$$ ❑

2.25 Lemma. *Let $\varphi \in \mathcal{F}(\mathbb{R},0)$ and let $\vartheta_1 = (k,l,\sigma,c_0,\dots,c_l)$ and $\vartheta_2 = (k,\hat{l},\hat{\sigma},$ $\hat{c}_0,\dots,\hat{c}_{\hat{l}})$ be representations of φ with $l < \hat{l}$. Consider the mapping $\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}$ from the previous lemma, then $\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}\big(\overline{\operatorname{ran}\iota_{\vartheta_2}} + \mathfrak{L}_{\vartheta_2}^{[\circ]\vartheta_2}\big) = \overline{\operatorname{ran}\iota_{\vartheta_1}} + \mathfrak{L}_{\vartheta_1}^{[\circ]\vartheta_1}$.*

Proof. By (2.3.6) it follows that $\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}(\operatorname{ran}\iota_{\vartheta_2}) = \operatorname{ran}\iota_{\vartheta_1}$ and by continuity we have $\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}(\overline{\operatorname{ran}\iota_{\vartheta_2}}) \subseteq \overline{\operatorname{ran}\iota_{\vartheta_1}}$. Define a mapping $\Phi' : \mathfrak{L}_{\vartheta_1} \to \mathfrak{L}_{\vartheta_2}$ by

$$\Phi'\big((f;a_0,\dots,a_N;b_0,\dots,b_{k-1})^T\big) := (f;a_0,\dots,a_N,0,\dots,0;b_0,\dots,b_{k-1})^T, \tag{2.3.7}$$

where $N := \max\{l,k-1\}$. Then the mapping Φ' is continuous and clearly $\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}} \circ \Phi' = \operatorname{id}|_{\mathfrak{L}_{\vartheta_1}}$. Further Φ' is isometric with respect to the indefinite inner products $[.,.]_{\vartheta_1}$ and $[.,.]_{\vartheta_2}$. Since $\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}\big(\mathfrak{L}_{\vartheta_2}^{[\circ]\vartheta_2}\big) = \mathfrak{L}_{\vartheta_1}^{[\circ]\vartheta_1}$ it follows that $\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}\big(\operatorname{ran}\iota_{\vartheta_2} + \mathfrak{L}_{\vartheta_2}^{[\circ]\vartheta_2}\big) = \operatorname{ran}\iota_{\vartheta_1} + \mathfrak{L}_{\vartheta_1}^{[\circ]\vartheta_1}$ and continuity of $\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}$ implies

$$\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}\left(\overline{\operatorname{ran}\iota_{\vartheta_2} + \mathfrak{L}_{\vartheta_2}^{[\circ]\vartheta_2}}\right) \subseteq \overline{\operatorname{ran}\iota_{\vartheta_1} + \mathfrak{L}_{\vartheta_1}^{[\circ]\vartheta_1}}. \tag{2.3.8}$$

Further we have $\Phi'\big(\operatorname{ran}\iota_{\vartheta_1} + \mathfrak{L}_{\vartheta_1}^{[\circ]\vartheta_1}\big) \subseteq \operatorname{ran}\iota_{\vartheta_2} + \mathfrak{L}_{\vartheta_2}^{[\circ]\vartheta_2}$ and since Φ' is continuous it holds

$$\Phi'\left(\overline{\operatorname{ran}\iota_{\vartheta_1} + \mathfrak{L}_{\vartheta_1}^{[\circ]\vartheta_1}}\right) \subseteq \overline{\operatorname{ran}\iota_{\vartheta_2} + \mathfrak{L}_{\vartheta_2}^{[\circ]\vartheta_2}}.$$

Let $x \in \overline{\operatorname{ran}\iota_{\vartheta_1} + \mathfrak{L}_{\vartheta_1}^{[\circ]}} \subseteq \mathfrak{L}_{\vartheta_1}$ and define $y := \Phi'(x)$, then $\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta 1}}(y) = x$, hence (2.3.8) yields

$$\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}\left(\overline{\operatorname{ran}\iota_{\vartheta_2} + \mathfrak{L}_{\vartheta_2}^{[\circ]\vartheta_2}}\right) = \overline{\operatorname{ran}\iota_{\vartheta_1} + \mathfrak{L}_{\vartheta_1}^{[\circ]\vartheta_1}}.$$

Obviously the following inclusions holds

$$\operatorname{ran}\iota_{\vartheta_i} + \mathfrak{L}_{\vartheta_i}^{[\circ]\vartheta_i} \subseteq \overline{\operatorname{ran}\iota_{\vartheta_i}} + \mathfrak{L}_{\vartheta_i}^{[\circ]\vartheta_i} \subseteq \overline{\operatorname{ran}\iota_{\vartheta_i} + \mathfrak{L}_{\vartheta_i}^{[\circ]\vartheta_i}}, \quad i=1,2.$$

Since $\mathfrak{L}_{\vartheta_i}^{[\circ]\vartheta_i}$, $i=1,2$, is finite dimensional $\overline{\operatorname{ran}\iota_{\vartheta_i}} + \mathfrak{L}_{\vartheta_i}^{[\circ]\vartheta_i}$ is closed and therefore

$$\overline{\operatorname{ran}\iota_{\vartheta_i} + \mathfrak{L}_{\vartheta_i}^{[\circ]\vartheta_i}} = \overline{\operatorname{ran}\iota_{\vartheta_i}} + \mathfrak{L}_{\vartheta_i}^{[\circ]\vartheta_i}, \quad i=1,2.$$

It follows that $\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}} : \overline{\operatorname{ran}\iota_{\vartheta_2}} + \mathfrak{L}_{\vartheta_2}^{[\circ]\vartheta_2} \to \overline{\operatorname{ran}\iota_{\vartheta_1}} + \mathfrak{L}_{\vartheta_1}^{[\circ]\vartheta_1}$ is surjective. ❑

2.26 Lemma. *Let $\varphi \in \mathcal{F}(\mathbb{R},0)$ and $\vartheta_1 = (k,l,\sigma,c_0,\dots,c_l)$ and $\vartheta_2 = (\hat{k},l,\hat{\sigma},\hat{c}_0,\dots,\hat{c}_l)$ be representations of φ with $k < \hat{k}$ and $l \geq 2\hat{k}+1$. Then it holds $d\hat{\sigma} = t^{2(\hat{k}-k)}d\sigma$ and*

$$\hat{c}_i = \begin{cases} c_i, & i = 0,\dots,2k-1 \\ c_i + (t^{i-2k},1)_\sigma, & i = 2k,\dots,2\hat{k}-1 \\ c_i, & i = 2\hat{k},\dots,l \end{cases},$$

$$\tilde{p}_j^{[\hat{k},\hat{\sigma}]} = \tilde{p}_j^{[k,\sigma]} - \sum_{i=0}^{2(\hat{k}-k)-1} \frac{p^{(2k-j+i)}(0)}{(2k-j+i)!}(t^i,1)_\sigma, \quad j = 0,\dots,\hat{k}.$$

Proof. Since $\vartheta_1,\vartheta_2 \in \Theta_\varphi$ we have by relation (2.1.1) for all $f \in \mathcal{D}(\mathbb{R})$

$$\int_{\mathbb{R}} \frac{f^{\{2k\}}(t)}{t^{2k}}d\sigma(t) + \sum_{i=0}^{l} c_i \frac{f^{(i)}(0)}{i!} = \int_{\mathbb{R}} \frac{f^{\{2\hat{k}\}}(t)}{t^{2\hat{k}}}d\hat{\sigma}(t) + \sum_{i=0}^{l} \hat{c}_i \frac{f^{(i)}(0)}{i!}. \tag{2.3.9}$$

The integral on the left side of this relation can be rewritten as

$$\int_{\mathbb{R}} \frac{f^{\{2k\}}(t)}{t^{2k}}d\sigma(t) = \int_{\mathbb{R}} \frac{f^{\{2\hat{k}\}}(t) + \sum_{i=2k}^{2\hat{k}-1}\frac{f^{(i)}(0)}{i!}t^i}{t^{2k}}\,d\sigma(t) =$$

$$= \int_{\mathbb{R}} \frac{f^{\{2\hat{k}\}}(t)}{t^{2\hat{k}}}t^{2(\hat{k}-k)}d\sigma(t) + \sum_{i=2k}^{2\hat{k}-1}\frac{f^{(i)}(0)}{i!}(t^{i-2k},1)_\sigma.$$

Now relation (2.3.9) writes as

$$\int_{\mathbb{R}} \frac{f^{\{2\hat{k}\}}(t)}{t^{2\hat{k}}}t^{2(\hat{k}-k)}d\sigma(t) + \sum_{i=2k}^{2\hat{k}-1}\frac{f^{(i)}(0)}{i!}(t^{i-2k},1)_\sigma + \sum_{i=0}^{l} c_i\frac{f^{(i)}(0)}{i!} =$$

$$= \int_{\mathbb{R}} \frac{f^{\{2\hat{k}\}}(t)}{t^{2\hat{k}}}d\hat{\sigma}(t) + \sum_{i=0}^{l}\hat{c}_i\frac{f^{(i)}(0)}{i!}.$$

For every $f \in \mathcal{D}(\mathbb{R})$ with $f^{(i)}(0) = 0$, $i = 0,\dots,l$, it follows that

$$\int_{\mathbb{R}} \frac{f^{\{2\hat{k}\}}(t)}{t^{2\hat{k}}}d\hat{\sigma}(t) = \int_{\mathbb{R}} \frac{f^{\{2\hat{k}\}}(t)}{t^{2\hat{k}}}t^{2(\hat{k}-k)}d\sigma(t).$$

Therefore we have $d\hat{\sigma} = t^{2(\hat{k}-k)}d\sigma$. Further we have the following relationship between c_i and $\hat{c}_i$, $i = 1,\dots,l$:

$$\hat{c}_i = \begin{cases} c_i & i = 0,\dots,2k-1 \\ c_i + (t^{i-2k},1)_\sigma, & i = 2k,\dots,2\hat{k}-1 \\ c_i, & i = 2\hat{k},\dots,l \end{cases}.$$

For $j = 0, \ldots, \hat{k}$ we have

$$\tilde{p}_j^{[\hat{k},\hat{\sigma}]} = \int_{\mathbb{R}} \frac{p^{\{2\hat{k}-j\}}(t)}{t^{2\hat{k}-j}} d\hat{\sigma}(t) = \int_{\mathbb{R}} \frac{p^{\{2\hat{k}-j\}}(t)}{t^{2k-j}} d\sigma(t) =$$

$$= \int_{\mathbb{R}} \frac{p^{\{2k-j\}}(t)}{t^{2k-j}} d\sigma(t) - \int_{\mathbb{R}} \sum_{i=2k-j}^{2\hat{k}-j-1} \frac{p^{(i)}(0)}{i!} \frac{d\sigma(t)}{t^{2k-j}} = \tilde{p}_j^{[k,\sigma]} - \sum_{i=0}^{2(\hat{k}-k)-1} \frac{p^{(i)}(0)}{i!} (t^i, 1)_\sigma .$$

This completes the proof. ❑

2.27 Lemma. *Let $\varphi \in \mathcal{F}(\mathbb{R},0)$ and $\vartheta_1 = (k, l, \sigma, c_0, \ldots, c_l)$ and $\vartheta_2 = (k+1, l, \hat{\sigma}, \hat{c}_0, \ldots, \hat{c}_l)$ be representations of φ with $l \geq 2k+1$. Define a mapping*

$$\Psi_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}} : \begin{cases} \big(\mathfrak{L}_{\vartheta_1}, [.,.]_{\vartheta_1}\big) \to \big(\mathfrak{L}_{\vartheta_2}, [.,.]_{\vartheta_2}\big) \\ \big(f; a_0, \ldots, a_l; b_0, \ldots, b_{k-1}\big)^T \mapsto \Big(\frac{1}{t}(f - a_k); a_0, \ldots, a_l; \tilde{b}_0, \ldots, \tilde{b}_{k-1}, \\ \qquad (f,1)_\sigma - a_k(1,1)_\sigma - a_{k+1}(t,1)_\sigma\Big)^T, \end{cases}$$

where $\tilde{b}_j := b_j - a_{2k-j}(1,1)_\sigma - a_{2k+1-j}(t,1)_\sigma$, $j = 0, \ldots, k-1$. Then $\Psi_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}}$ is a continuous isometry and the diagram

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, "\iota_{\vartheta_1}"'] \arrow[dr, "\iota_{\vartheta_2}"] & \\ \mathfrak{L}_{\vartheta_1} \arrow[rr, "\Psi_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}}"'] & & \mathfrak{L}_{\vartheta_2} \end{tikzcd} \tag{2.3.10}$$

commutes.

Proof. By Lemma 2.26 we have $d\hat{\sigma} = t^2 d\sigma$ and

$$\hat{c}_i = \begin{cases} c_i, & i = 0, \ldots, 2k-1 \\ c_{2k} + (1,1)_\sigma, & i = 2k \\ c_{2k+1} + (t,1)_\sigma, & i = 2k+1 \\ c_i, & i \geq 2k+2 \end{cases} .$$

Let $\vec{f} = (f; a_0, \ldots, a_l; b_0, \ldots, b_{k-1})^T \in \mathfrak{L}_{\vartheta_1}$ then it follows that

$$\begin{aligned} \Big\| \frac{1}{t}(f - a_k) \Big\|^2_{L^2(|\hat{\sigma}|)} &= \int_{\mathbb{R}} \frac{|f(t) - a_k|^2}{t^2} d|t^2 d\sigma|(t) = \\ &= \int_{\mathbb{R}} |f(t) - a_k|^2 d|\sigma|(t) = \|f - a_k\|^2_{L^2(|\sigma|)} . \end{aligned}$$

Therefore the mapping $\vec{f} \mapsto \frac{1}{t}(f - a_k)$ from $\mathfrak{L}_{\vartheta_1} \to L^2(|\hat{\sigma}|)$ is continuous. Since the inner product $(.,.)_\sigma$ is continuous with respect to the Hilbert space topology it follows that the mapping $\Psi_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}}$ is continuous.
Denote by G the Gram operator of $\mathfrak{L}_{\vartheta_1}$ and by G' the restriction to the finite

dimensional subspace $\mathbb{C}^{k+l+1}$. Let $\hat{G}$ and $\hat{G}'$ be defined analogously. Hence we have

$$\hat{G}' = \begin{pmatrix} G' & e_{k+1} \\ e_{k+1}^T & 0 \end{pmatrix} + (1,1)_\sigma T_{2k+1} + (t,1)_\sigma T_{2k+2}$$

where $e_{k+1} = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ denotes the $k+1$ unit vector in $\mathbb{C}^{k+l+1}$ and

$$T_n := \begin{pmatrix} \tilde{T}_n & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{k+l+2}$$

with $\tilde{T}_n := (e_n | e_{n-1} | \cdots | e_1) \in \mathbb{C}^{n \times n}$. Abbreviatory for $\Psi_{\mathfrak{L}_{\vartheta_1}, \mathfrak{L}_{\vartheta_2}}$ we just write Ψ. For $\vec{f} = (f; a_0, \ldots, a_l; b_0, \ldots, b_{k-1})^T$, $\vec{g} = (g; \alpha_0, \ldots, \alpha_l; \beta_0, \ldots, \beta_{k-1})^T \in \mathfrak{L}_{\vartheta_1}$ it follows that

$$(\Psi\vec{f}, \Psi\vec{g})_{\mathfrak{L}_{\vartheta_2}} = (\hat{G}\Psi\vec{f}, \Psi\vec{g}) = \Big(\frac{1}{t}(f - a_k), \frac{1}{t}(g - \alpha_k)\Big)_{\hat{\sigma}} + \Big(\hat{G}'(\Psi\vec{f})', (\Psi\vec{g})'\Big), \tag{2.3.11}$$

where $(\Psi\vec{f})'$ denotes the restriction of $\Psi\vec{f}$ to $\mathbb{C}^{k+l+2}$ and $(\Psi\vec{g})'$ respectively. It's useful to define $\widetilde{\Psi\vec{f}} := (a_0, \ldots, a_l; \tilde{b}_0, \ldots, \tilde{b}_{k-1})^T$, then we have

$$(\Psi\vec{f})' = \begin{pmatrix} \widetilde{\Psi\vec{f}} \\ (f,1)_\sigma - a_k(1,1)_\sigma - a_{k+1}(t,1)_\sigma \end{pmatrix}.$$

Analogously we define $\widetilde{\Psi\vec{g}}$. The first term of the right side of relation (2.3.11) leads to

$$\Big(\frac{1}{t}(f - a_k), \frac{1}{t}(g - \alpha_k)\Big)_{\hat{\sigma}} = \int_{\mathbb{R}} \frac{f(t) - a_k}{t} \overline{\frac{g(t) - \alpha_k}{t}} d\hat{\sigma}(t) =$$

$$= \int_{\mathbb{R}} (f(t) - a_k)\overline{(g(t) - \alpha_k)} d\sigma(t) = (f,g)_\sigma - a_k\overline{(g,1)}_\sigma - \overline{\alpha}_k(f,1)_\sigma + a_k\overline{\alpha}_k(1,1)_\sigma.$$

Further we have

$$\begin{aligned} \Big(\hat{G}'(\Psi\vec{f})', (\Psi\vec{g})'\Big) &= \Big(G'\widetilde{\Psi\vec{f}}, \widetilde{\Psi\vec{g}}\Big) + a_k\Big(\overline{(g,1)}_\sigma - \overline{\alpha}_k\overline{(1,1)}_\sigma - \overline{\alpha}_{k+1}\overline{(t,1)}_\sigma\Big) + \\ &+ \overline{\alpha}_k\Big((f,1)_\sigma - a_k(1,1)_\sigma - a_{k+1}(t,1)_\sigma\Big) + (1,1)_\sigma\Big(T_{2k+1}(\Psi\vec{f})', (\Psi\vec{g})'\Big) + \\ &+ (t,1)_\sigma\Big(T_{2k+2}(\Psi\vec{f})', (\Psi\vec{g})'\Big). \end{aligned}$$

Hence equation (2.3.11) writes as

$$\begin{aligned} (\Psi\vec{f}, \Psi\vec{g})_{\mathfrak{L}_{\vartheta_2}} &= (f,g)_\sigma - a_k\overline{\alpha}_k(1,1)_\sigma - a_k\overline{\alpha}_{k+1}(t,1)_\sigma - \overline{\alpha}_k a_{k+1}(t,1)_\sigma + \\ &+ \Big(G'\widetilde{\Psi\vec{f}}, \widetilde{\Psi\vec{g}}\Big) + (1,1)_\sigma\Big(T_{2k+1}(\Psi\vec{f})', (\Psi\vec{g})'\Big) + (t,1)_\sigma\Big(T_{2k+2}(\Psi\vec{f})', (\Psi\vec{g})'\Big). \end{aligned}$$

We define $\mathbf{0}_n := (0, \ldots, 0)^T \in \mathbb{C}^{n \times 1}$ and for $m, n \in \mathbb{N}$, $m > n$ let $\mathbf{a}_{m,n} := (a_m, a_{m-1}, \ldots, a_n)^T \in \mathbb{C}^{m-n+1 \times 1}$.

$$\begin{aligned} \Big(G'\widetilde{\Psi\vec{f}}, \widetilde{\Psi\vec{g}}\Big) &= \Big(G'\vec{f}' - (1,1)_\sigma G'\begin{pmatrix} \mathbf{0}_{l+1} \\ \mathbf{a}_{2k,k+1} \end{pmatrix} - (t,1)_\sigma G'\begin{pmatrix} \mathbf{0}_{l+1} \\ a_{2k+1,k+2} \end{pmatrix}, \widetilde{\Psi\vec{g}}\Big) = \\ &= \Big(G'\vec{f}', \widetilde{\Psi\vec{g}}\Big) - (1,1)_\sigma\Big(\begin{pmatrix} \mathbf{a}_{2k,k+1} \\ \mathbf{0}_{l+1} \end{pmatrix}, \widetilde{\Psi\vec{g}}\Big) - (t,1)_\sigma\Big(\begin{pmatrix} \mathbf{a}_{2k+1,k+2} \\ \mathbf{0}_{l+1} \end{pmatrix}, \widetilde{\Psi\vec{g}}\Big). \end{aligned}$$

Since $T_{2k+1}(\Psi\vec{f})' = (\mathbf{a}_{2k,0}, \mathbf{0}_{l-k+1})^T$ and $T_{2k+1}(\Psi\vec{f})' = (\mathbf{a}_{2k+1,0}, \mathbf{0}_{l-k})^T$ we have

$$\begin{aligned}
&\Big(G'\widetilde{\Psi\vec{f}}, \widetilde{\Psi\vec{g}}\Big) + (1,1)_\sigma\Big(T_{2k+1}(\Psi\vec{f})', (\Psi\vec{g})'\Big) + (t,1)_\sigma\Big(T_{2k+2}(\Psi\vec{f})', (\Psi\vec{g})'\Big) = \\
&= \Big(G'\vec{f}, \widetilde{\Psi\vec{g}}\Big) + (1,1)_\sigma\left(\begin{pmatrix} \mathbf{0}_k \\ \mathbf{a}_{k,0} \\ \mathbf{0}_{l-k+1}\end{pmatrix}, (\Psi\vec{g})'\right) + (t,1)_\sigma\left(\begin{pmatrix} \mathbf{0}_k \\ \mathbf{a}_{k+1,0} \\ \mathbf{0}_{l-k+1}\end{pmatrix}, (\Psi\vec{g})'\right) = \\
&= \Big(G'\vec{f}, \vec{g}\Big) - \overline{(1,1)}_\sigma\left(G'\vec{f}, \begin{pmatrix} \mathbf{0}_{l+1} \\ \boldsymbol{\alpha}_{2k,k+1}\end{pmatrix}\right) - \overline{(t,1)}_\sigma\left(G'\vec{f}, \begin{pmatrix} \mathbf{0}_{l+1} \\ \boldsymbol{\alpha}_{2k+1,k+2}\end{pmatrix}\right) + \\
&\quad + (1,1)_\sigma\left(\begin{pmatrix} \mathbf{0}_k \\ \mathbf{a}_{k,0} \\ \mathbf{0}_{l-k+1}\end{pmatrix}, (\Psi\vec{g})'\right) + (t,1)_\sigma\left(\begin{pmatrix} \mathbf{0}_k \\ \mathbf{a}_{k+1,0} \\ \mathbf{0}_{l-k+1}\end{pmatrix}, (\Psi\vec{g})'\right) = \\
&= \Big(G'\vec{f}, \vec{g}\Big) + (1,1)_\sigma\left[\left(\begin{pmatrix} \mathbf{0}_k \\ \mathbf{a}_{k,0} \\ \mathbf{0}_{l-k+1}\end{pmatrix}, (\Psi\vec{g})'\right) - \left(\vec{f}, \begin{pmatrix} \boldsymbol{\alpha}_{2k,k+1} \\ \mathbf{0}_{l+1}\end{pmatrix}\right)\right] + \\
&\quad + (t,1)_\sigma\left[\left(\begin{pmatrix} \mathbf{0}_k \\ \mathbf{a}_{k+1,0} \\ \mathbf{0}_{l-k+1}\end{pmatrix}, (\Psi\vec{g})'\right) - \left(\vec{f}, \begin{pmatrix} \boldsymbol{\alpha}_{2k+1,k+2} \\ \mathbf{0}_{l+1}\end{pmatrix}\right)\right] = \\
&= \Big(G'\vec{f}, \vec{g}\Big) + (1,1)_\sigma a_k\overline{\alpha}_k + (t,1)_\sigma(a_{k+1}\overline{\alpha}_k + a_k\overline{\alpha}_{k+1}).
\end{aligned}$$

It remains to show that the diagram (2.3.10) commutes. For every $p \in \mathcal{P}$ we have

$$\iota_{\vartheta_1}(p) = \left(\frac{p^{\{k\}}}{t^k}; \frac{p^{(0)}(0)}{0!}, \dots, \frac{p^{(l)}(0)}{l!}; \tilde{p}_0^{[k,\sigma]}, \dots, \tilde{p}_{k-1}^{[k,\sigma]}\right)^T,$$

$$\iota_{\vartheta_2}(p) = \left(\frac{p^{\{k+1\}}}{t^{k+1}}; \frac{p^{(0)}(0)}{0!}, \dots, \frac{p^{(l)}(0)}{l!}; \tilde{p}_0^{[k+1,\hat{\sigma}]}, \dots, \tilde{p}_k^{[k+1,\hat{\sigma}]}\right)^T.$$

By Lemma 2.26 it follows that

$$\tilde{p}_j^{[k+1,\hat{\sigma}]} = \tilde{p}_j^{[k,\sigma]} - \frac{p^{(2k-j)}(0)}{(2k-j)!}(1,1)_\sigma - \frac{p^{(2k+1-j)}(0)}{(2k+1-j)!}(t,1)_\sigma, \quad j = 0, \dots, k,$$

and since

$$\frac{1}{t}\Big(\frac{p^{\{k\}}(t)}{t^k} - \frac{p^{(k)}(0)}{k!}\Big) = \frac{p^{\{k+1\}}(t)}{t^{k+1}}$$

it follows that $\Psi_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}}\big(\iota_{\vartheta_1}(p)\big) = \iota_{\vartheta_2}(p)$ for every $p \in \mathcal{P}$. ❑

2.28 Lemma. *Let $\varphi \in \mathcal{F}(\mathbb{R},0)$ and $\vartheta_1 = (k,l,\sigma,c_0,\dots,c_l)$ and $\vartheta_2 = (k+1, l, \hat{\sigma}, \hat{c}_0, \dots, \hat{c}_l)$ be representations of φ with $l \geq 2k+1$. Denote $\tilde{b}_j' := b_j + a_{2k-j}(1,1)_\sigma + a_{2k+1-j}(t,1)_\sigma$, $j = 0,\dots,k-1$, and define a mapping*

$$\Psi'_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}} : \begin{cases} \big(\mathfrak{L}_{\vartheta_2}, [.,.]_{\vartheta_2}\big) \to \big(\mathfrak{L}_{\vartheta_1}, [.,.]_{\vartheta_1}\big) \\ \big(f; a_0,\dots,a_l; b_0,\dots,b_k\big)^T \mapsto \big(t\cdot f + a_k; a_0,\dots,a_l; \tilde{b}_0',\dots,\tilde{b}_{k-1}'\big)^T. \end{cases}$$

Then $\Psi'_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}} \circ \Psi_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}} = \mathrm{id}_{\mathfrak{L}_{\vartheta_1}}$ and $\Psi'_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}|_{\overline{\operatorname{ran}\iota_{\vartheta_2}}}$ is a continuous isometry with $\Psi'_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}(\overline{\operatorname{ran}\iota_{\vartheta_2}}) = \overline{\operatorname{ran}\iota_{\vartheta_1}}$.

Proof. Clearly we have $\Psi'_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}} \circ \Psi_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}} = \mathrm{id}_{\mathfrak{L}_{\vartheta_1}}$ and $\Psi'_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}$ is continuous. By (2.3.10) we have $\Psi_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}}(\operatorname{ran}\iota_{\vartheta_1}) = \operatorname{ran}\iota_{\vartheta_2}$ and applying $\Psi'_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}$ yields

$\Psi'_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}(\operatorname{ran}\iota_{\vartheta_2}) = \operatorname{ran}\iota_{\vartheta_1}$. By continuity it follows that $\Psi'_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}(\overline{\operatorname{ran}\iota_{\vartheta_2}}) \subseteq \overline{\operatorname{ran}\iota_{\vartheta_1}}$. On the other hand it holds

$$\overline{\operatorname{ran}\iota_{\vartheta_1}} = \Psi'_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}\big(\Psi_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}}(\overline{\operatorname{ran}\iota_{\vartheta_1}})\big) \subseteq \Psi'_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}(\overline{\operatorname{ran}\iota_{\vartheta_2}}),$$

thus $\Psi'_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}(\overline{\operatorname{ran}\iota_{\vartheta_2}}) = \overline{\operatorname{ran}\iota_{\vartheta_1}}$. Again, by (2.3.10), we have $\Psi_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}} \circ \iota_{\vartheta_1} = \iota_{\vartheta_2}$, and the mappings $\iota_{\vartheta_{1,2}} : \mathcal{P} \to \mathfrak{L}_{\vartheta_{1,2}}$ are isometric with respect to the indefinite inner product $[.,.]_{\vartheta_{1,2}}$ on $\mathfrak{L}_{\vartheta_{1,2}}$. Applying $\Psi'_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}$ yields $\iota_{\vartheta_1} = \Psi'_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}} \circ \iota_{\vartheta_2}$ and therefore $\Psi'_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}$ is isometric on $\operatorname{ran}\iota_{\vartheta_2}$. By continuity this property extends to the closure of $\operatorname{ran}\iota_{\vartheta_2}$. Thus, we have proven the lemma. ❑

Let $\vartheta_1 = (k, l, \sigma, c_0, \dots, c_l)$ and $\vartheta_2 = (\hat{k}, l, \hat{\sigma}, \hat{c}_0, \dots, \hat{c}_{\hat{l}})$ be representations of $\varphi \in \mathcal{F}(\mathbb{R},0)$, with $k < \hat{k}$ and $l \geq 2\hat{k}+1$. Let $n := \hat{k} - k$, then there exists numbers $c_0^{(j)}, \dots, c_l^{(j)}$ and signed measures σ_j, $j = 0, \dots, n$, such that $\theta_j := (k+j, l, \sigma_j, c_0^{(j)}, \dots, c_l^{(j)})$ belong to Θ_φ for $j = 0, \dots, n$. Clearly we have

$$\vartheta_0 = \theta_0 \prec \theta_1 \prec \cdots \prec \theta_n = \vartheta_2.$$

The previous lemma suggests to define a mapping from $\mathfrak{L}_{\vartheta_1}$ to $\mathfrak{L}_{\vartheta_2}$ by

$$\Upsilon_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}} := \Psi_{\mathfrak{L}_{\theta_n},\mathfrak{L}_{\theta_{n-1}}} \circ \cdots \circ \Psi_{\mathfrak{L}_{\theta_1},\mathfrak{L}_{\theta_2}} \circ \Psi_{\mathfrak{L}_{\theta_0},\mathfrak{L}_{\theta_1}}. \tag{2.3.12}$$

Obviously $\Upsilon_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}}$ is an isometry and the following diagram commutes

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, "\iota_{\vartheta_1}"'] \arrow[dr, "\iota_{\vartheta_2}"] & \\ \mathfrak{L}_{\vartheta_1} \arrow[rr, "\Upsilon_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}}"'] & & \mathfrak{L}_{\vartheta_2}. \end{tikzcd} \tag{2.3.13}$$

As the composition of continuous mappings $\Upsilon_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}}$ is continuous.

2.29 Corollary. *Let $\varphi \in \mathcal{F}(\mathbb{R},0)$ and $\vartheta_1 = (k, l, \sigma, c_0, \dots, c_l)$ and $\vartheta_2 = (\hat{k}, l, \hat{\sigma}, \hat{c}_0, \dots, \hat{c}_l)$ be representations of φ with $k < \hat{k}$ and $l \geq 2\hat{k}+1$. Then there exists a linear mapping $\Upsilon' : \mathfrak{L}_{\vartheta_2} \to \mathfrak{L}_{\vartheta_1}$ such that $\Upsilon'|_{\overline{\operatorname{ran}\iota_{\vartheta_2}}}$ is a continuous isometry and $\Upsilon'(\overline{\operatorname{ran}\iota_{\vartheta_2}}) = \overline{\operatorname{ran}\iota_{\vartheta_1}}$. Further the following diagram*

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, "\iota_{\vartheta_1}"'] \arrow[dr, "\iota_{\vartheta_2}"] & \\ \mathfrak{L}_{\vartheta_1} & & \mathfrak{L}_{\vartheta_2} \arrow[ll, "\Upsilon'"] \end{tikzcd}$$

commutes.

Proof. This follows immediately from the considerations above and Lemma 2.28. ❑

2.30 Lemma. *Let $\varphi \in \mathcal{F}(\mathbb{R},0)$ and $\vartheta_1 = (k,l,\sigma,c_0,\dots,c_l)$ and $\vartheta_2 = (\hat{k},l,\hat{\sigma},\hat{c}_0,\dots,\hat{c}_l)$ be representations of φ with $k < \hat{k}$ and $l \geq 2\hat{k}+1$. Denote*

$$\tilde{b}_j := b_j - \sum_{i=2k-j}^{2\hat{k}-j-1} a_i\big(t^{i-(2k-j)},1\big)_{\sigma}, \quad j = 0,\dots,k-1,$$

$$\tilde{d}_j := \Big(f,\frac{1}{t^{k-j}}\Big)_{\sigma} - \sum_{i=k}^{2\hat{k}-j-1} a_i\big(t^{i-(2k-j)},1\big)_{\sigma}, \quad j = k,\dots,\hat{k}-1.$$

Then the mapping $\Upsilon_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}}$ is explicitly given by

$$\Upsilon_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}}: \begin{cases} \big(\mathfrak{L}_{\vartheta_1},[.,.]_{\vartheta_1}\big) \to \big(\mathfrak{L}_{\vartheta_2},[.,.]_{\vartheta_2}\big) \\ \big(f;a_0,\dots,a_l;b_0,\dots,b_{k-1}\big)^T \mapsto \Big(\frac{1}{t^{\hat{k}-k}}\big(f - \sum_{i=k}^{\hat{k}-1} a_i t^{i-k}\big);a_0,\dots,a_l; \\ \qquad \tilde{b}_0,\dots,\tilde{b}_{k-1},\tilde{d}_k,\dots,\tilde{d}_{\hat{k}-1}\Big)^T. \end{cases}$$

Proof. Let $n := \hat{k}-k$ and $\theta_j := (k+j,l,\sigma_j,c_0^{(j)},\dots,c_l^{(j)})$, $j = 0,\dots,n$, be representations of φ such that

$$\vartheta_0 = \theta_0 \prec \theta_1 \prec \cdots \prec \theta_n = \vartheta_2.$$

We prove this lemma by induction. For $n = 1$ we have $\Upsilon_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}} = \Psi_{\mathfrak{L}_{\theta_0},\mathfrak{L}_{\theta_1}}$. For the induction step $n-1 \mapsto n$ let $\vec{f} = (f;a_0,\dots,a_l;b_0,\dots,b_{k-1})^T \in \mathfrak{L}_{\theta_0}$. Then we have

$$\Upsilon_{\mathfrak{L}_{\theta_0},\mathfrak{L}_{\theta_{n-1}}}(\vec{f}) = \Big(\frac{1}{t^{\hat{k}-1-k}}\big(f - \sum_{i=k}^{\hat{k}-2} a_i t^{i-k}\big);a_0,\dots,a_l;\beta_0,\dots,\beta_{k-1},\delta_k,\dots,\delta_{\hat{k}-2}\Big)^T,$$

where $\beta_j := b_j - \sum_{i=2k-j}^{2\hat{k}-j-3} a_i(t^{i-(2k-j)},1)_{\sigma_0}$, $j = 0,\dots,k-1$ and $\delta_j := (f,\frac{1}{t^{k-j}})_{\sigma_0} - \sum_{i=k}^{2\hat{k}-j-3} a_i(t^{i-(2k-j)},1)_{\sigma_0}$, $j = k,\dots,\hat{k}-2$. Using the induction hypothesis it follows that

$$\Psi_{\mathfrak{L}_{\theta_n},\mathfrak{L}_{\theta_{n-1}}} \circ \cdots \circ \Psi_{\mathfrak{L}_{\theta_1},\mathfrak{L}_{\theta_2}} \circ \Psi_{\mathfrak{L}_{\theta_0},\mathfrak{L}_{\theta_1}}(\vec{f}) = \Psi_{\mathfrak{L}_{\theta_n},\mathfrak{L}_{\theta_{n-1}}} \circ \Upsilon_{\mathfrak{L}_{\theta_0},\mathfrak{L}_{\theta_{n-1}}}(\vec{f}) =$$
$$= \Psi_{\mathfrak{L}_{\theta_n},\mathfrak{L}_{\theta_{n-1}}}\big(\Upsilon_{\mathfrak{L}_{\theta_0},\mathfrak{L}_{\theta_{n-1}}}(\vec{f})\big) = \Bigg(\frac{1}{t}\Big(\frac{1}{t^{\hat{k}-1-k}}\big(f - \sum_{i=k}^{\hat{k}-2} a_i t^{i-k}\big) - a_{\hat{k}-1}\Big);$$
$$a_0,\dots,a_l;\tilde{\beta}_0,\dots,\tilde{\beta}_{k-1},\tilde{\delta}_k,\tilde{\delta}_{\hat{k}-2},$$
$$\Big(\frac{1}{t^{\hat{k}-1-k}}\big(f - \sum_{i=k}^{\hat{k}-2} a_i t^{i-k}\big),1\Big)_{\sigma_{n-1}} - a_{\hat{k}-1}(1,1)_{\sigma_{n-1}} - a_{\hat{k}}(t,1)_{\sigma_{n-1}}\Bigg)^T,$$

where $\tilde{\beta}_j := \beta_j - a_{2\hat{k}-2-j}(1,1)_{\sigma_{n-1}} - a_{2\hat{k}-1-j}(t,1)_{\sigma_{n-1}}$, $j = 0,\dots,k-1$ and $\tilde{\delta}_j := \delta_j - a_{2\hat{k}-2-j}(1,1)_{\sigma_{n-1}} - a_{2\hat{k}-1-j}(t,1)_{\sigma_{n-1}}$, $j = k,\dots,\hat{k}-2$. The first term on the right side of this relation can be rewritten as

$$\frac{1}{t}\Big(\frac{1}{t^{\hat{k}-1-k}}\big(f-\sum_{i=k}^{\hat{k}-2}a_it^{i-k}\big)-a_{\hat{k}-1}\Big)=\frac{1}{t^{\hat{k}-k}}\Big(f-\sum_{i=k}^{\hat{k}-2}a_it^{i-k}\Big)-\frac{a_{\hat{k}-1}t^{\hat{k}-1-k}}{t^{\hat{k}-k}}=$$
$$=\frac{1}{t^{\hat{k}-k}}\Big(f-\sum_{i=k}^{\hat{k}-1}a_it^{i-k}\Big).$$

Since $(1,1)_{\sigma_{n-1}}=(t^{2(\hat{k}-1-k)},1)_{\sigma_0}$ and $(t,1)_{\sigma_{n-1}}=(t^{2(\hat{k}-1-k)+1},1)_{\sigma_0}$ it follows that $\tilde{b}_j=\tilde{\beta}_j$ for $j=0,\dots,k$ and $\tilde{d}_j=\tilde{\delta}_j$ for $j=k,\dots,\hat{k}-2$. Further we have

$$\Big(\frac{1}{t^{\hat{k}-1-k}}\big(f-\sum_{i=k}^{\hat{k}-2}a_it^{i-k}\big),1\Big)_{\sigma_{n-1}}-a_{\hat{k}-1}(1,1)_{\sigma_{n-1}}-a_{\hat{k}}(t,1)_{\sigma_{n-1}}=$$
$$=\Big(f,\frac{1}{t^{k-(\hat{k}-1)}}\Big)_{\sigma_0}-\sum_{i=k}^{\hat{k}-2}a_i\Big(t^{i-(2k-(\hat{k}-1))},1\Big)_{\sigma_0}-$$
$$-a_{\hat{k}-1}(t^{2(\hat{k}-1-k)},1)_{\sigma_0}-a_{\hat{k}}(t^{2(\hat{k}-1-k)+1},1)_{\sigma_0}=\tilde{d}_{\hat{k}-1}.$$

Thus we have shown that

$$\Psi_{\mathfrak{L}_{\theta_n},\mathfrak{L}_{\theta_{n-1}}}\circ\cdots\circ\Psi_{\mathfrak{L}_{\theta_1},\mathfrak{L}_{\theta_2}}\circ\Psi_{\mathfrak{L}_{\theta_0},\mathfrak{L}_{\theta_1}}=\Upsilon_{\mathfrak{L}_{\vartheta_1},\mathfrak{L}_{\vartheta_2}}.$$ ❑

Now it is possible to give the proof of Theorem 2.19.

Proof (Theorem 2.19). Let $\vartheta_1=(k_1,l_1,\sigma_1,c_0,\dots,c_{l_1})$ and $\vartheta_2=(k_2,l_2,\sigma_2,\hat{c}_0,\dots,\hat{c}_{l_2})$. Without loss of generality we can assume $k_1\le k_2$. If $k_1=k_2$ and $l_2>l_1$ we can apply Lemma 2.24. Therefore there exists a continuous surjective isometry $\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}:\big(\mathfrak{L}_{\vartheta_2},[.,.]_{\vartheta_2}\big)\to\big(\mathfrak{L}_{\vartheta_1},[.,.]_{\vartheta_1}\big)$. Owing to Lemma 2.25 it holds $\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}\big(\overline{\operatorname{ran}\iota_{\vartheta_2}}+\mathfrak{L}_{\vartheta_2}^{[\circ]\vartheta_2}\big)=\overline{\operatorname{ran}\iota_{\vartheta_1}}+\mathfrak{L}_{\vartheta_1}^{[\circ]\vartheta_1}$. Since $\big(\overline{\operatorname{ran}\iota_{\vartheta_i}}+\mathfrak{L}_{\vartheta_i}^{[\circ]\vartheta_i}\big)^{[\circ]\vartheta_i}=\overline{\operatorname{ran}\iota_{\vartheta_i}}^{[\circ]\vartheta_i}+\mathfrak{L}_{\vartheta_i}^{[\circ]\vartheta_i}$ for $i=1,2$ and

$$\Phi_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}\big(\overline{\operatorname{ran}\iota_{\vartheta_2}}^{[\circ]\vartheta_2}+\mathfrak{L}_{\vartheta_2}^{[\circ]\vartheta_2}\big)\subseteq\overline{\operatorname{ran}\iota_{\vartheta_1}}^{[\circ]\vartheta_1}+\mathfrak{L}_{\vartheta_1}^{[\circ]\vartheta_1}$$

we can apply Lemma 2.22. Denote by $\hat{\Phi}_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}$ the induced mapping between the factor spaces with respect to the isotropic parts, then by Lemma 2.22 (ii) the mapping $\hat{\Phi}_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}$ is unitary.
For $i=1,2$ there exists unitary mappings

$$T_{\vartheta_i}:\overline{\operatorname{ran}\iota_{\vartheta_i}}/_{\overline{\operatorname{ran}\iota_{\vartheta_i}}^{[\circ]\vartheta_i}}\to\overline{\operatorname{ran}\iota_{\vartheta_i}}+\mathfrak{L}_{\vartheta_i}^{[\circ]\vartheta_i}/_{\overline{\operatorname{ran}\iota_{\vartheta_i}}^{[\circ]\vartheta_i}+\mathfrak{L}_{\vartheta_i}^{[\circ]\vartheta_i}}.$$

Define $U_{\vartheta_2,\vartheta_1}:=T_{\vartheta_1}^{-1}\circ\hat{\Phi}_{\mathfrak{L}_{\vartheta_2},\mathfrak{L}_{\vartheta_1}}\circ T_{\vartheta_2}$, then $U_{\vartheta_2,\vartheta_1}$ is a unitary mapping from $\mathcal{K}_{\vartheta_2}$ onto $\mathcal{K}_{\vartheta_1}$. The inverse $U_{\vartheta_1,\vartheta_2}$ of $U_{\vartheta_2,\vartheta_1}^{-1}$ is also unitary.

Now consider the case $k_1<k_2$. Let $\tilde{\vartheta}_1:=(k_1,\tilde{l},\tilde{\sigma}_1,\tilde{c}_1,\dots,\tilde{c}_{\tilde{l}})$ and $\tilde{\vartheta}_2:=(k_2,\tilde{l},\tilde{\sigma}_2,\tilde{\hat{c}}_1,\dots,\tilde{\hat{c}}_{\tilde{l}})$ such that $\tilde{l}\ge 2k_2+1$. By the first part of the proof there exists unitary operators $V_{\vartheta_1,\tilde{\vartheta}_1}:\mathcal{K}_{\vartheta_1}\to\mathcal{K}_{\tilde{\vartheta}_1}$ and $V_{\tilde{\vartheta}_2,\vartheta_2}:\mathcal{K}_{\tilde{\vartheta}_2}\to\mathcal{K}_{\vartheta_2}$. According to Corollary 2.29 there exists a continuous surjective isometry $\Upsilon':\mathfrak{L}_{\tilde{\vartheta}_2}\to\mathfrak{L}_{\tilde{\vartheta}_1}$.

Lemma 2.22 (ii) ensures the existence of a unitary operator $\hat{\Upsilon}' : \mathcal{K}_{\tilde{\vartheta}_2} \to \mathcal{K}_{\tilde{\vartheta}_1}$. Now define

$$U_{\vartheta_1,\vartheta_2} := V_{\tilde{\vartheta}_2,\vartheta_2} \circ \hat{\Upsilon}'^{-1} \circ V_{\vartheta_1,\tilde{\vartheta}_1} : \mathcal{K}_{\vartheta_1} \to \mathcal{K}_{\vartheta_2},$$

then $U_{\vartheta_1,\vartheta_2}$ is unitary.

Combining the diagrams (2.3.6), (2.3.1), (2.3.4), and using continuity, it follows that

$$\begin{tikzcd}
 & \mathcal{K}_{\vartheta_1} \arrow[r, "A_{\vartheta_1}"] \arrow[d, "U_{\vartheta_1,\vartheta_2}"'] & \mathcal{K}_{\vartheta_1} \arrow[d, "U_{\vartheta_1,\vartheta_2}"] & \\
\mathcal{P} \arrow[ur, "\iota_{\vartheta_1}"] \arrow[dr, "\iota_{\vartheta_2}"'] \arrow[rrr, bend left=60, "A_t"] \arrow[rrr, bend right=60, "A_t"'] & & & \mathcal{P}. \arrow[ul, "\iota_{\vartheta_1}"'] \arrow[dl, "\iota_{\vartheta_2}"] \\
 & \mathcal{K}_{\vartheta_2} \arrow[r, "A_{\vartheta_2}"'] & \mathcal{K}_{\vartheta_2} &
\end{tikzcd} \tag{2.3.14}$$

Therefore $U_{\vartheta_1,\vartheta_2} A_{\vartheta_1} = A_{\vartheta_2} U_{\vartheta_1,\vartheta_2}$ and the diagram (2.3.2) commutes. ❏

2.4 Notes about the minimal representation

Let $\varphi \in \mathcal{F}(\mathbb{R}, 0)$ and $\vartheta \in \Theta_\varphi$ be a representation of φ. In this section we will show that the embedding ι_ϑ has dense range if ϑ is representation of φ with minimal k.

2.31 Definition. Let ν be a positive finite measure on $\mathbb{R}$ and $k \in \mathbb{N}_0$. Then we define a seminorm on the space of all polynomials by

$$m_{\nu,k}(p) := \left(\int_{\mathbb{R}} \frac{|p^{\{k\}}(t)|^2}{t^{2k}} d\nu(t) \right)^{\frac{1}{2}} \quad (p \in \mathcal{P}).$$

If not otherwise stated in the rest of this section ν will denote a positive finite measure and k a nonnegative integer.

2.32 Lemma. *If g is a linear functional on $\mathcal{P}$ which is continuous with respect to the seminorm $m_{\nu,k}$, then there exists an element $v \in L_2(\nu)$ such that*

$$g(p) = \int_{\mathbb{R}} \frac{p^{\{k\}}(t)}{t^k} \overline{v(t)} d\nu(t),$$

for all $p \in \mathcal{P}$.

Proof. Consider on $\mathcal{P}$ the inner product

$$(p, q)_{\mathcal{P}} := \int_{\mathbb{R}} \frac{p^{\{k\}}(t)}{t^k} \overline{\frac{q^{\{k\}}(t)}{t^k}} d\nu(t),$$

then the mapping

$$\Phi : \begin{cases} \left(\mathcal{P},(.,.)_{\mathcal{P}}\right) \to \left(L_2(\nu),(.,.)_{L_2(\nu)}\right) \\ \qquad p \mapsto \frac{p^{\{k\}}}{t^k} \end{cases}$$

is an isometry. Since g is a linear functional on $\mathcal{P}$ which is continuous with respect to the seminorm $m_{\nu,k}$ there exists a constant $C>0$ such that $|g(p)| \leq Cm_{\nu,k}(p) = C(p,p)_{\mathcal{P}}^{1/2}$ for every $p \in \mathcal{P}$. Define on $\Phi(\mathcal{P})$ a linear functional $\tilde{g}$ by

$$\tilde{g} : \begin{cases} \Phi(\mathcal{P}) \to \mathbb{C} \\ \Phi(p) \mapsto g(p), \end{cases}$$

then we have $|\tilde{g}(\Phi(p))| \leq C(p,p)_{\mathcal{P}}^{1/2} = C(\Phi p, \Phi p)_{L_2(\nu)}^{1/2}$, in particular $\tilde{g}$ is well-defined. By the Hahn-Banach theorem $\tilde{g}$ can be extended to a continuous linear functional on $L_2(\nu)$ and hence there exists an element $v \in L_2(\nu)$, such that

$$g(p) = \tilde{g}(\Phi p) = \int_{\mathbb{R}} \frac{p^{\{k\}}(t)}{t^k} \overline{v(t)} d\nu(t),$$

for every $p \in \mathcal{P}$. ❑

2.33 Lemma. *Let $n \in \mathbb{N}$ and $f_1, \ldots, f_n$ be linear functionals on $\mathcal{P}$ such that no linear combination of them is continuous with respect to the seminorm $m_{\nu,k}$. Then the mapping*

$$\iota : \begin{cases} \mathcal{P} \to L_2(\nu) \oplus \mathbb{C}^n \\ p \mapsto \left(\frac{p^{\{k\}}}{t^k}; f_1(p), \ldots, f_n(p)\right)^T \end{cases},$$

has dense range. Here $L_2(\nu) \oplus \mathbb{C}^n$ is understood as the Hilbert space endowed with the sum inner product of $(.,.)_{L^2(\nu)}$ and the euclidean inner product on $\mathbb{C}^n$.

Proof. To show that ι has dense range assume the converse. Then there exists an element $(y;\xi_1,\ldots,\xi_n)^T \in L_2(\nu) \oplus \mathbb{C}^n$, which is not equal $(0;0,\ldots,0)^T$, such that $\iota(\mathcal{P}) \perp (y;\xi_1,\ldots,\xi_n)^T$. Since the polynomials are dense in $L_2(\nu)$ and every polynomial can be written as $\frac{p^{\{k\}}}{t^k}$, $p \in \mathcal{P}$, this implies that $(\xi_1,\ldots,\xi_n)^T \neq (0,\ldots,0)^T$. For every $p \in \mathcal{P}$ it holds

$$\left(\frac{p^{\{k\}}}{t^k}, y\right)_{L^2(\nu)} + \sum_{i,j=1}^{n} f_j(p)\overline{\xi_i} = 0, \tag{2.4.1}$$

which yields

$$\Big|\sum_{i,j=1}^{n} f_j(p)\overline{\xi_i}\Big| = \Big|\left(\frac{p^{\{k\}}}{t^k}, -y\right)_{L^2(\nu)}\Big| \leq \Big\|\frac{p^{\{k\}}}{t^k}\Big\|_{L^2(\nu)} \|y\|_{L^2(\nu)} =$$
$$= \|y\|_{L^2(\nu)} \left(\int_{\mathbb{R}} \frac{|p^{\{k\}}(t)|^2}{t^{2k}} d\nu(t)\right)^{\frac{1}{2}} = \|y\|_{L^2(\nu)} m_{\nu,k}(p),$$

for every $p \in \mathcal{P}$. This would imply that there exists a non trivial linear combination of the functionals $f_1,\ldots,f_n$ which is continuous with respect to the seminorm $m_{\nu,k}$. This is a contradiction and therefore ι has dense range. ❑

2.34 Lemma. *Assume ν has compact support, $\nu(\{0\}) = 0$ and that if $k > 0$ the function $t \mapsto \frac{1}{t^2}$ is not ν-integrable. For $i \in \mathbb{N}_0$ define linear functionals on $\mathcal{P}$ by $\xi_i(p) := \frac{p^{(i)}(0)}{i!}$, and if $k > 0$,*

$$\tau_j : p \mapsto \int_{\mathbb{R}} \epsilon(t) \frac{p^{\{2k-j\}}(t)}{t^{2k-j}} d\nu(t), \quad j = 0, \dots, k-1,$$

where $\epsilon(t)$ is a function on $\mathbb{R}$ such that each of the restrictions $\epsilon|_{\mathbb{R}^+}$ and $\epsilon|_{\mathbb{R}^-}$ is either $+1$ or -1. Then no finite linear combination of the functionals ξ_i and, if $k > 0$, τ_j is continuous with respect to the seminorm $m_{\nu,k}$.

Proof. Assume there exists a finite linear combination of the functionals ξ_i and τ_i that is continuous with respect to the seminorm $m_{\nu,k}$. Then, by Lemma 2.32, there would exist an element $v \in L_2(\nu)$ and constants $N \in \mathbb{N}_0$, λ_i, $i = 0, \dots, N$, and μ_j, $j = 0, \dots, k-1$, not all equal to zero, such that

$$\sum_{i=0}^{N} \lambda_i \xi_i(p) + \sum_{j=0}^{k-1} \mu_j \tau_j(p) = \int_{\mathbb{R}} \frac{p^{\{k\}}(t)}{t^k} \overline{v(t)} d\nu(t), \quad p \in \mathcal{P}. \tag{2.4.2}$$

Let p be a polynomial of degree less than k, then we have $p^{\{k\}} = 0$, $\xi_i(p) = 0$ for $i = k, k+1, \dots$ and $\tau_j(p) = 0$ for $j = 0, \dots, k-1$. Therefore (2.4.2) implies $\lambda_i = 0$ for $i = 0, \dots, k-1$.

Denote by S the set of polynomials whose derivatives vanish at $t = 0$ up to the order $r := \max\{N, 2k\}$. For $p \in S$ it holds $\xi_k(p) = \xi_{k+1}(p) = \cdots = \xi_r(p) = 0$ and $p^{\{2k-j\}} = p^{\{k\}} = p$, $j = 0, \dots, k-1$, hence it follows

$$\sum_{j=0}^{k-1} \mu_j \int_{\mathbb{R}} \epsilon(t) \frac{p(t)}{t^k} t^{-k+j} d\nu(t) = \int_{\mathbb{R}} \frac{p(t)}{t^k} \overline{v(t)} d\nu(t). \tag{2.4.3}$$

Let X be a compact subset of $\mathbb{R}$ such that $\operatorname{supp} \nu \subseteq X$. We show that the set $\{t \mapsto \epsilon(t) \frac{p(t)}{t^k} :\ p \in S\} \subseteq \mathbb{C}^X$ is dense in $L_2(\nu)$. Since $p \in S$ it is equivalent to show that the set

$$\tilde{S} := \left\{t \mapsto \epsilon(t) t^{r-k+1} p(t) :\ p \in \mathcal{P}\right\} \subseteq \mathbb{C}^X$$

is dense in $L^2(\nu)$. Let $f \in L^2(\nu)$ and define $\tilde{f}(t) := \epsilon(t) f(t)$. By Remark 2.5 there exists polynomials p_n, $n \in \mathbb{N}$, such that for $q_n(t) := t^{r-k+1} p_n(t)$, $n \in \mathbb{N}$, it holds

$$\lim_{n \to \infty} \|\tilde{f} - q_n\|_{L^2(\nu)} = 0.$$

Define $\tilde{q}_n(t) := \epsilon(t) q_n(t)$, $n \in \mathbb{N}$, then $\tilde{q}_n \in \tilde{S}$, $n \in \mathbb{N}$, and since $\epsilon^2 = 1$ ν-a.e. it follows that for $n \in \mathbb{N}$ it holds

$$\|f - \tilde{q}_n\|^2_{L^2(\nu)} = \|\epsilon^2 f - \epsilon q_n\|^2_{L^2(\nu)} = \int_{\mathbb{R}} |\epsilon^2(t) f(t) - \epsilon(t) q_n(t)|^2 d\nu(t) =$$

$$= \int_{\mathbb{R}} |\epsilon(t) f(t) - q_n(t)|^2 d\nu(t) = \|\tilde{f} - q_n\|^2_{L^2(\nu)}.$$

This shows that $\tilde{S}$ is dense in $L^2(\nu)$. Hence and since $\epsilon^2 = 1$ ν-a.e. equation (2.4.3) implies that

$$\sum_{j=0}^{k-1} \mu_j t^{-k+j} = \epsilon\overline{v} \in L_2(\nu).$$

Since t^{-2} is not ν-integrable it follows $\mu_j = 0$, $j = 0, \dots, k-1$. To complete the proof put $\tilde{p}(t) := t^{-k} p^{\{k\}}(t) = \sum_{i=k}^{\deg p} i!^{-1} t^{i-k} p^{(i)}(0)$. Then we have $\tilde{p}^{(n)}(0) = n!(k+n)!^{-1} p^{(k+n)}(0)$, $n = 0, \dots, \max\{l, k-1\}$, and it follows

$$\sum_{i=k}^{N} \lambda_i \xi_i(p) = \sum_{i=0}^{N-k} \lambda_{i+k} \frac{p^{(i+k)}(0)}{(i+k)!} = \sum_{i=0}^{N-k} \lambda_{i+k} \frac{\tilde{p}^{(i)}(0)}{i!} = \int_{\mathbb{R}} \tilde{p}(t)\overline{v(t)} d\nu(t).$$

This yields $\lambda_j = 0$, $j = k, \dots, N$, because $\nu(\{0\}) = 0$. ❑

2.35 Corollary. *Let $\varphi \in \mathcal{F}(\mathbb{R}, 0)$ and $\vartheta = (k, l, \sigma, c_0, \dots, c_l) \in \Theta_\varphi$ be a representation of φ with minimal k. Then $\overline{\operatorname{ran} \iota_\vartheta} = \mathfrak{L}_\vartheta$.*

Proof. By Lemma 2.34 and 2.33 it follows that $\operatorname{ran} \iota_\vartheta$ is dense in $\mathfrak{L}_\vartheta$. ❑

Note that if $k = 0$ and $l = -1$ then $\mathcal{K}_\vartheta = L^2(\sigma)$, A_ϑ is the usual multiplication operator on $L^2(\sigma)$, i.e. $A_\vartheta(f)(t) = tf(t)$, $f \in L^2(\sigma)$, and $\hat{\iota}_\vartheta$ is the canonical embedding $\mathcal{P} \hookrightarrow L^2(\sigma)$.

2.5 The spectrum of A_φ

2.36 Lemma. *Let $\varphi \in \mathcal{F}(\mathbb{R}, 0)$ and $\vartheta = (k, l, \sigma, c_0, \dots, c_l) \in \Theta_\varphi$ be the minimal representation of φ. Then 0 is an eigenvalue of A_φ and a maximal Jordan chain at 0 is given by*

$$\begin{aligned} z_0 &= (0; 0, \dots, 0; 1, 0, 0, \dots, 0)^T, \\ z_1 &= (0; 0, \dots, 0; 0, 1, 0, \dots, 0)^T, \\ &\vdots \\ z_{k-1} &= (0; 0, \dots, 0; 0, 0, \dots, 0, 1)^T, \end{aligned}$$

if $l \le 2k$, and

$$\begin{aligned} z_0 &= (0; 0, \dots, 0, 0, 1; 0, \dots, 0)^T, \\ z_1 &= (0; 0, \dots, 0, 1, 0; 0, \dots, 0)^T, \\ &\vdots \\ z_{l-k} &= (0; \underbrace{0, \dots, 0}_{k\text{-times}}, 1, 0, \dots, 0; 0, \dots, 0)^T, \end{aligned}$$

if $l \ge 2k+1$.

Proof. Assume an element $(f; a_0, \dots, a_N; b_0, \dots, b_{k-1})^T \in \mathfrak{L}_\vartheta$, $N = \max\{l, k-1\}$, is in the kernel of $\mathfrak{A}_\vartheta$, then it must hold that $tf + a_{k-1} = 0$. For $f \neq 0$ it follows that $f = -\frac{1}{t} a_{k-1}$, but this is not possible since $\frac{1}{t} \notin L^2(\sigma)$ if ϑ is the minimal representation. Now it follows that the kernel of $\mathfrak{A}_\vartheta$ is given by $\ker \mathfrak{A}_\vartheta = \{(0;0,\dots,x;y,0,\dots,0)^T : x,y \in \mathbb{C}\}$. Therefore two maximal Jordan chains of $\mathfrak{A}_\vartheta$ are given by

$$\begin{aligned}
x_0 &= (0;0,\dots,0,0,1;0,\dots,0)^T, & y_0 &= (0;0,\dots,0;1,0,0,\dots,0)^T,\\
x_1 &= (0;0,\dots,0,1,0;0,\dots,0)^T, & y_1 &= (0;0,\dots,0;0,1,0,\dots,0)^T,\\
&\vdots & &\vdots\\
&\vdots & y_{k-1} &= (0;0,\dots,0;0,0,\dots,0,1)^T.\\
x_{N-k} &= (0;\underbrace{0,\dots,0}_{k\text{-times}},1,0,\dots,0;0,\dots,0)^T.
\end{aligned}$$

Since ϑ is minimal, by Corollary 2.35 we have that $\mathfrak{L}_\vartheta = \overline{\operatorname{ran} \iota_\vartheta}$. Therefore by Theorem 2.18 the operator $\mathfrak{A}_\vartheta : \mathfrak{L}_\vartheta \to \mathfrak{L}_\vartheta$ induces an operator $A_\vartheta : \mathfrak{L}_\vartheta/_{\mathfrak{L}_\vartheta^{[\circ]_\vartheta}} \to \mathfrak{L}_\vartheta/_{\mathfrak{L}_\vartheta^{[\circ]_\vartheta}}$. Clearly we have that

$$\dim\big(\mathfrak{L}_\vartheta^{[\circ]_\vartheta} \cap \operatorname{ran} \mathfrak{A}_\vartheta\big) = \dim \mathfrak{L}_\vartheta^{[\circ]_\vartheta} - 1 \quad \text{and} \quad \dim\big(\mathfrak{L}_\vartheta^{[\circ]_\vartheta} \cap \ker \mathfrak{A}_\vartheta\big) = 1.$$

Since $\mathfrak{A}_\vartheta\big(\mathfrak{L}_\vartheta^{[\circ]_\vartheta}\big) \subseteq \mathfrak{L}_\vartheta^{[\circ]_\vartheta}$ it follows that $\operatorname{ran}\big(\mathfrak{A}_\vartheta|_{\mathfrak{L}_\vartheta^{[\circ]_\vartheta}}\big) \subseteq \mathfrak{L}_\vartheta^{[\circ]_\vartheta} \cap \operatorname{ran} \mathfrak{A}_\vartheta$. Because of

$$\dim \operatorname{ran}\Big(\mathfrak{A}_\vartheta|_{\mathfrak{L}_\vartheta^{[\circ]_\vartheta}}\Big) = \dim \mathfrak{L}_\vartheta^{[\circ]_\vartheta} - 1 = \dim\big(\mathfrak{L}_\vartheta^{[\circ]_\vartheta} \cap \operatorname{ran} \mathfrak{A}_\vartheta\big),$$

we have $\operatorname{ran}\big(\mathfrak{A}_\vartheta|_{\mathfrak{L}_\vartheta^{[\circ]_\vartheta}}\big) = \mathfrak{L}_\vartheta^{[\circ]_\vartheta} \cap \operatorname{ran} \mathfrak{A}_\vartheta$. Now it follows that

$$\ker A_\vartheta^n = \big(\mathfrak{A}_\vartheta^n\big)^{-1}\Big(\mathfrak{L}_\vartheta^{[\circ]_\vartheta}\Big)/_{\mathfrak{L}_\vartheta^{[\circ]_\vartheta}} = \ker \mathfrak{A}_\vartheta^n /_{\mathfrak{L}_\vartheta^{[\circ]_\vartheta}}.$$

This shows that 0 is an eigenvalue of A_ϑ and that a maximal Jordan chain of A_ϑ is given by $y_0,\dots,y_{k-1}$ if $l \le 2k$ and by $x_0,\dots,x_{l-k}$ if $l \ge 2k+1$. Due to unitary equivalence this assertion follows for A_φ. ❑

2.37 Proposition. *Let $\varphi \in \mathcal{F}(\mathbb{R},0)$, then the spectrum of the multiplication operator A_φ is given by $\sigma(A_\varphi) = \operatorname{supp}\varphi \cup \{0\}$.*

Before we launch into the proof, we need some basic facts about the essential spectrum. More can be found in [GGK, Chapter XI]. For $A \in \mathcal{B}(X)$, where X is a Banach space, the *essential spectrum* of A, denoted by $\sigma_{ess}(A)$, is by definition the set of all $\lambda \in \mathbb{C}$ such that $A-\lambda$ is not a Fredholm operator. Recall that $A \in \mathcal{B}(X)$ is a Fredholm operator if $\ker A$ and $X/\operatorname{ran} A$ are finite dimensional. Note that the condition $X/\operatorname{ran} A$ is finite dimensional implies that $\operatorname{ran} A$ is closed. The essential spectrum is invariant under compact perturbations. Further $\sigma_{ess}(A)$ is compact and if $\mathbb{C} \setminus \sigma_{ess}(A)$ is connected, then $\sigma(A) \setminus \sigma_{ess}(A)$ consists of eigenvalues of finite type only, cf. [GGK, Corollary XI.8.5, p. 204].

Proof (Proposition 2.37). Let $\vartheta = (k,l,\sigma,c_0,\dots,c_l) \in \Theta_\varphi$ be the minimal representation and consider the operator $\mathfrak{A}_\vartheta$, the multiplication operator on $\mathfrak{L}_\vartheta$. Assume that $k > 0$. Recall that for $\vec{f} = (f;a_0,\dots,a_N;b_0,\dots,b_{k-1})$, $N := \max\{l,k-1\}$, the multiplication operator is given by

$$\mathfrak{A}_\vartheta(\vec{f}) = \Big(tf + a_{k-1};0,a_0,\dots,a_{N-1};b_1,\dots,b_{k-1},\int_{\mathbb{R}} f(t)d\sigma(t)\Big),$$

Clearly $(0;0,\dots,0;1,0,\dots,0)^T \in \ker \mathfrak{A}_\vartheta$, so $0\in\sigma_p(\mathfrak{A}_\vartheta)$ and since this vector is not in the isotropic part of $\mathfrak{L}_\vartheta$ it follows that $0\in\sigma_p(A_\vartheta)$, the point spectrum of the factor operator. For $\lambda\neq 0$ and $\sigma(\{\lambda\})=0$ the equation $(\mathfrak{A}_\vartheta-\lambda)\vec{f}=\vec{y}\in \mathfrak{L}_\vartheta^{[\circ]_\vartheta}$ yields

$$\begin{aligned} tf + a_{k-1} - \lambda f &= 0,\\ 0-\lambda a_0 &= 0,\\ a_{i-1}-\lambda a_i &= 0, \quad i=1,\dots,l-k,\\ \begin{pmatrix} a_{l-k}-\lambda a_{l-k+1}\\ \vdots\\ a_{l-1}-\lambda a_l\end{pmatrix} &= \begin{pmatrix} x_1\\ \vdots\\ x_k\end{pmatrix},\\ \begin{pmatrix} b_1-\lambda b_0\\ \vdots\\ b_{k-1}-\lambda b_{k-2}\\ \int_{\mathbb{R}} f\,d\sigma - \lambda b_{k-1}\end{pmatrix} &= -C_{k,l}\begin{pmatrix} x_1\\ \vdots\\ x_{k-1}\\ x_k\end{pmatrix}. \end{aligned}$$

Therefore a_i, $i=0,\dots,l-k$, are equal to zero. This implies $(t-\lambda)f=0$, thus $f=0$ since $\sigma(\{\lambda\})=0$. Now we have

$$\begin{pmatrix}0\\ a_{l-k+1}\\ \vdots\\ a_{l-1}\end{pmatrix} - \lambda\begin{pmatrix} a_{l-k+1}\\ \vdots\\ \vdots\\ a_l\end{pmatrix} = \begin{pmatrix}x_1\\ \vdots\\ \vdots\\ x_k\end{pmatrix},\quad \begin{pmatrix}b_1\\ \vdots\\ b_{k-1}\\ 0\end{pmatrix} - \lambda\begin{pmatrix}b_0\\ \vdots\\ \vdots\\ b_{k-1}\end{pmatrix} = -C_{k,l}\begin{pmatrix}x_1\\ \vdots\\ \vdots\\ x_k\end{pmatrix}.$$

Define linear mappings L_λ, R_λ by

$$L_\lambda := \begin{pmatrix} -\lambda & 0 & \dots & 0\\ 1 & \ddots & \ddots & \vdots\\ & \ddots & \ddots & 0\\ 0 & & 1 & -\lambda\end{pmatrix},\quad R_\lambda := \begin{pmatrix} -\lambda & 1 & & 0\\ 0 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1\\ 0 & \dots & 0 & -\lambda\end{pmatrix},$$

then these conditions can be written as $L_\lambda\vec{a}=\vec{x}$, $R_\lambda\vec{b}=-C_{k,l}\vec{x}$, where $\vec{a}=(a_{l-k+1},\dots,a_l)^T$, $\vec{b}=(b_0,\dots,b_{k-1})^T$ and $\vec{x}=(x_1,\dots,x_k)^T$. This implies that

$$-C_{k,l}L_\lambda\vec{a} = R_\lambda\vec{b} \;\Leftrightarrow\; -R_\lambda^{-1}C_{k,l}L_\lambda\vec{a}=\vec{b}.$$

Easy computation gives $C_{k,l}L_\lambda = R_\lambda C_{k,l}$ and therefore it holds $\vec{b}=-C_{k,l}\vec{a}$. It follows that if $\vec{f}\in\mathfrak{L}_\vartheta$ such that $(\mathfrak{A}_\vartheta-\lambda)\vec{f}\in\mathfrak{L}_\vartheta^{[\circ]_\vartheta}$ then $\vec{f}\in\mathfrak{L}_\vartheta^{[\circ]_\vartheta}$. This shows that the factor operator $(A_\vartheta-\lambda)$ is injective if and only if $\lambda\neq 0$ and $\sigma(\{\lambda\})=0$. Therefore $\sigma_p(A_\vartheta)=\{\lambda\in\operatorname{supp}\sigma : \sigma(\{\lambda\})\neq 0\}\cup\{0\}$.
Now define linear operators $M,T:\mathfrak{L}_\vartheta\to\mathfrak{L}_\vartheta$ by

$$\begin{aligned} M\big((f;a_0,\dots,a_N;b_0,\dots,b_{k-1})\big) &:= (t\cdot f;0,\dots,0),\\ T\big((f;a_0,\dots,a_N;b_0,\dots,b_{k-1})\big) &:= \Big(a_{k-1};0,a_0,\dots,a_{N-1};b_1,\dots,b_{k-1},\int_{\mathbb{R}} f\,d\sigma\Big). \end{aligned}$$

Then we can write the multiplication operator as $\mathfrak{A}_\vartheta = M+T$. Since T is a finite rank operator it follows that T is compact. Therefore $\sigma_{ess}(\mathfrak{A}_\vartheta) = \sigma_{ess}(M)$. Let $\lambda \in \mathbb{C}$ and $\vec{f} = (f; a_0, \ldots, a_N; b_0, \ldots, b_{k-1})$, then it follows that $(M-\lambda)\vec{f} = 0$ if and only if $(t-\lambda)f = 0$. Hence $\sigma_p(M) = \{\lambda \in \mathbb{C} : \sigma(\{\lambda\}) \neq 0\}$. For $\lambda \in \operatorname{supp}(\sigma)$ and $\sigma(\{\lambda\}) = 0$ define $S_n := (\lambda - \frac{1}{n}, \lambda + \frac{1}{n})$, $n \in \mathbb{N}$, then $|\sigma|(S_n) > 0$ for all $n \in \mathbb{N}$ and $|t-\lambda| < \frac{1}{n}$ for all $t \in S_n$, $n \in \mathbb{N}$. Denote by χ_{S_n} the characteristic function on S_n, then for $n \in \mathbb{N}$ it follows that

$$\|(M-\lambda)\chi_{S_n}\|^2 = \|(t-\lambda)\chi_{S_n}\|^2 = \int\limits_{S_n} |t-\lambda|^2 d|\sigma|(t) \leq \frac{1}{n^2}|\sigma|(S_n) = \frac{1}{n^2}\|\chi_{S_n}\|^2.$$

Therefore $M - \lambda$ is not boundedly invertible. We show that $\operatorname{ran}(M-\lambda)$ is dense in $L^2(|\sigma|)$. Let $f \in L^2(|\sigma|)$, then the functions

$$f_n(t) := \frac{1}{t-\lambda}\chi_{\mathbb{R}\setminus S_n}(t) f(t), \quad n \in \mathbb{N},$$

belong to $L^2(|\sigma|)$. By the Lebesgue dominated convergence theorem $(M-\lambda)f_n \to f$ in $L^2(|\sigma|)$. Therefore it follows that $\lambda \in \sigma_c(M)$. If $\lambda \notin \operatorname{supp}\sigma$ then clearly $\lambda \in \rho(M)$. Summarizing we have shown that

$$\begin{aligned} \sigma_p(M) &= \{\lambda \in \mathbb{C} :\ \sigma(\{\lambda\}) \neq 0\}, \\ \sigma_c(M) &= \{\lambda \in \mathbb{C} :\ \lambda \in \operatorname{supp}\sigma,\ \sigma(\{\lambda\}) = 0\}, \\ \rho(M) &= \{\lambda \in \mathbb{C} :\ \lambda \notin \operatorname{supp}\sigma\}. \end{aligned}$$

For $\lambda \in \sigma_c(M)$ the range $\operatorname{ran}(M-\lambda)$ can not be closed and therefore $\sigma_c(M) \subseteq \sigma_{ess}(M)$. Since the spectrum of M is a subset of the real line and the essential spectrum is compact it follows that $\mathbb{C} \setminus \sigma_{ess}(M)$ is connected. Therefore $\mathbb{C} \setminus \sigma_{ess}(\mathfrak{A}_\vartheta)$ is also connected. It follows that $\sigma(\mathfrak{A}_\vartheta) \setminus \sigma_{ess}(\mathfrak{A}_\vartheta)$ consists of eigenvalues of finite type only. We have already shown that $\sigma_p(\mathfrak{A}_\vartheta) = \{\lambda \in \operatorname{supp}\varphi :\ \sigma(\{\lambda\}) \neq 0\} \cup \{0\}$, hence $\sigma(\mathfrak{A}_\vartheta) = \operatorname{supp}(\sigma) \cup \{0\}$. By Remark 1.22 we have $0 \in \operatorname{supp}\sigma$ and $\operatorname{supp}\sigma = \operatorname{supp}\varphi$, hence $\sigma(A_\varphi) = \sigma(\mathfrak{A}_\vartheta) = \operatorname{supp}\varphi$.
If $k = 0$ and ϑ is the minimal representation $\mathfrak{L}_\vartheta$ is non degenerated (see Remark 2.11), hence by Corollary 2.35 we have that $\mathcal{K}_\varphi = \mathfrak{L}_\vartheta$ and $A_\varphi = \mathfrak{A}_\vartheta$. If $k = 0$ and $l = -1$, then the multiplication operator $\mathfrak{A}_\vartheta$ is just the usual multiplication operator on the Kreĭn space $\mathfrak{L}_\vartheta = L^2(\sigma)$. Therefore the spectrum of $\mathfrak{A}_\vartheta$ equals the support of σ. By Remark 1.22 we have $\operatorname{supp}\varphi = \operatorname{supp}\sigma$, hence $\sigma(A_\varphi) = \operatorname{supp}\varphi$.
If $k = 0$ and $l = 0$ then the multiplication operator is given by

$$\mathfrak{A}_\vartheta\big((f; a_0)\big) = (tf; 0).$$

Clearly $(0;1) \in \ker \mathfrak{A}_\vartheta$ and the same reasoning as above gives $\sigma(\mathfrak{A}_\vartheta) = \operatorname{supp}\sigma \cup \{0\}$. By Remark 1.22 it follows that $\sigma(A_\varphi) = \operatorname{supp}\varphi$.
If $k = 0$ and $l > 0$ then the multiplication operator is given by

$$\mathfrak{A}_\vartheta\big((f; a_0, \ldots, a_l)\big) = (tf; 0, a_0, \ldots, a_{l-1}).$$

Again the argument that the essential spectrum is invariant under compact perturbations yields $\sigma(\mathfrak{A}_\vartheta) = \operatorname{supp}\sigma$. ❑

2.38 Definition. Let $(\mathfrak{K}, [.,.])$ be a Kreĭn space and $A \in \mathcal{B}(\mathfrak{K})$ be a selfadjoint definitizable operator. Denote by $\mathfrak{d}(A)$ the set of all definitizing polynomials for A and by $N(p)$ the zero set of the polynomial p. The set of the (finite) *critical points* of A is defined as

$$c(A) := \bigcap_{p \in \mathfrak{d}(A)} N(p) \cap \sigma(A) \cap \mathbb{R}.$$

Further denote for $p \in \mathfrak{d}(A)$ by Ω_p the semi-ring of all Borel subsets of $\mathbb{R}$ the boundaries of which are bounded away from $N(p)$ and by $\Omega_p^{\times}$ the sets $\Delta \subseteq \Omega_p$ such that $N(p) \cap \overline{\Delta} = \emptyset$.

The following formulation of the spectral theorem is due to M.A. Dritschel, see [D, Theorem 18, p. 100].

2.39 Theorem. *Let $\mathfrak{K}$ be a Kreĭn space and $T \in \mathcal{B}(\mathfrak{K})$ be a selfadjoint operator with definitizing polynomial p, and assume that the set $\mathcal{Z}$ of zeros of p is contained in the real line $\mathbb{R}$. Then there exists a unique spectral function $E : \Omega_p \to \mathcal{B}(\mathfrak{K})$ with the following properties:*

(i) For $\Delta \in \Omega_p$, $E(\Delta) \in \mathcal{B}(\mathfrak{K})$ is an orthogonal projection.

(ii) $E(\emptyset) = 0$ and $E(\mathbb{R}) = I$.

(iii) For $\Delta, \Delta' \in \Omega_p$, $E(\Delta \cap \Delta') = E(\Delta)E(\Delta')$.

(iv) If $\{\Delta_n\}_{n=1}^{\infty} \subseteq \Omega_p$ are pairwise disjoint and $\bigcup_{n=1}^{\infty} \Delta_n \in \Omega_p$, then

$$E\left(\bigcup_{n=1}^{\infty} \Delta_n\right) = \sum_{n=1}^{\infty} E(\Delta_n).$$

(v) Let $\Delta \in \Omega_p$ and $\mathcal{E}(\Delta) = \operatorname{ran} E(\Delta)$. If for all $t \in \Delta$ we have $p(t) > 0$, then $\mathcal{E}(\Delta)$ is uniformly positive, and if $p(t) < 0$, then $\mathcal{E}(\Delta)$ is uniformly negative.

(vi) For $\Delta \in \Omega_p$, $E(\Delta) \in \{T\}''$, the double commutant of T.

(vii) If $\Delta \in \Omega_p$ and $\mathcal{E}(\Delta) = \operatorname{ran} E(\Delta)$, then $\sigma\big(T|_{\mathcal{E}(\Delta)}\big) \subseteq \overline{\Delta}$.

Moreover, if ϕ is a Borel measurable function which is bounded on $\sigma(T)$, then the integral

$$\int_{\mathcal{X}} \phi(\lambda)p(\lambda)dE(\lambda)$$

is a strongly convergent improper integral, where $\mathcal{X} = \mathbb{R} \setminus \mathcal{Z}$. Finally, the operator $\phi(T)p(T)$ may be expressed as

$$\phi(T)p(T) = \int_{\mathcal{X}} \phi(\lambda)p(\lambda)dE(\lambda) + \sum_{\nu \in \mathcal{Z}} \phi(\nu)N_\nu,$$

where $N_\nu \in \mathcal{B}(\mathfrak{K})$ is a positive selfadjoint operator, $N_\nu^2 = 0$, and $E(\Delta)N_\nu = N_\nu E(\Delta) = 0$ for every $\Delta \in \Omega_p$ and $\nu \in \mathcal{Z}$, $\nu \notin \Delta$.

Recall that a subspace $\mathfrak{L}$ of a Kreĭn space $(\mathfrak{K}, [.,.])$ is called *uniformly positive* (*uniformly negative*) if there is a number $\delta > 0$ such that

$$[x,x] \geq \delta \|x\|^2, \quad x \in \mathfrak{L}, \qquad \big([x,x] \leq -\delta\|x\|^2, \quad x \in \mathfrak{L}\big)$$

where $\|.\|$ is a norm on $\mathfrak{K}$ which is induced by some fundamental decomposition. A subspace $\mathfrak{L}$ of $(\mathfrak{K}, [.,.])$ is uniformly positive (uniformly negative) if and only if $\mathfrak{K}$ admits a fundamental decomposition $\mathfrak{K} = \mathfrak{K}_+[\dot{+}]\mathfrak{K}_-$ such that $\mathfrak{L} \subseteq \mathfrak{K}_+$ ($\mathfrak{L} \subseteq \mathfrak{K}_-$), see [B, Theorem V.5.6, p. 108]. So a closed subspace $\mathfrak{L}$ of $(\mathfrak{K}, [.,.])$ is uniformly positive (uniformly negative) if and only if it is a Hilbert (anti-Hilbert) space with the inner product $[.,.]$ of $\mathfrak{K}$. Since orthogonal projections in a Kreĭn space have closed range these considerations yield:

2.40 Corollary. *Let $(\mathfrak{K}, [.,.])$ be a Kreĭn space and $A \in \mathcal{B}(\mathfrak{K})$ be a selfadjoint operator with a real definitizing polynomial p with only real zeros and $\Delta \in \Omega_p$ such that* $\operatorname{sign} p|_\Delta$ *is either $+1$ or -1. Further denote by E the spectral function of A according to Theorem 2.39. Then the decomposition*

$$\mathfrak{K} = E(\Delta)\mathfrak{K}[\dot{+}]\big(I - E(\Delta)\big)\mathfrak{K}$$

reduces[3] *A, and the restriction $A_\Delta := A|_{E(\Delta)\mathfrak{K}}$ is a linear bounded selfadjoint operator in the Hilbert space $\big(E(\Delta)\mathfrak{K}, (.,.)_\Delta\big)$, where the inner product is defined as $(.,.)_\Delta :=$* $\operatorname{sign} p|_\Delta \cdot [.,.]|_{E(\Delta)\mathfrak{K}\times E(\Delta)\mathfrak{K}}$.

2.41 Definition. Let A be a selfadjoint definitizing operator in a Kreĭn space $(\mathfrak{K}, [.,.])$ and E the spectral function of A. If $\alpha \in c(A)$ and for arbitrary $\lambda_0, \lambda_1 \in \mathbb{R} \setminus c(A)$ the limits

$$\lim_{\lambda \nearrow \alpha} E([\lambda_0, \lambda]) \quad \text{and} \quad \lim_{\lambda \searrow \alpha} E([\lambda, \lambda_1])$$

exist in the strong operator topology, α is called a *regular critical point* of A, otherwise α is called a *singular critical point.* Denote by $c_r(A)$, $c_s(A)$ the sets of regular critical points, singular critical points of A, respectively. A point $\alpha \in c(A)$ is said to be a critical point of *finite index* of A if there exists an open interval Δ containing α such that $E(\Delta)\mathfrak{K}$ is a Pontryagin space. The set of all critical points of finite index, singular critical points of finite index, is denoted by $c_f(A)$, $c_{sf}(A)$, respectively.

2.42 Remark. These limits always exists if $\alpha \notin c(A)$, see [L, Section II.5, p. 39].

2.43 Proposition. *Let $\varphi \in \mathcal{F}(\mathbb{R}, 0)$ and $(k, l, \sigma, c_0, \ldots, c_l) \in \Theta_\varphi$ be the minimal representation of φ. Then*

$$c(A_\varphi) = \{0\} \ \textit{if} \ \begin{cases} k = 0, l = -1 \ \wedge \ \{\operatorname{sign}\sigma|_{\mathbb{R}^+}, \operatorname{sign}\sigma|_{\mathbb{R}^-}\} = \{+1, -1\}, \\ k = 0, l = 0 \quad \wedge \ \{\operatorname{sign}\sigma|_{\mathbb{R}^+}, \operatorname{sign}\sigma|_{\mathbb{R}^-}, \operatorname{sign} c_0\} \supseteq \{+1, -1\}, \\ k = 0, l > 0, \\ k > 0, \end{cases}$$

[3]Let $(\mathfrak{K}, [.,.])$ be a Kreĭn space, $\mathcal{L}$ a linear subspace of $\mathfrak{K}$ and $A \in \mathcal{B}(\mathfrak{K})$. If $A\mathcal{L} \subseteq \mathcal{L}$ and $A\mathcal{L}^{[\perp]} \subseteq \mathcal{L}^{[\perp]}$ then we say $\mathcal{L}$ *reduces* A. If $\mathcal{L}_1, \mathcal{L}_2$ are linear subspaces of $\mathfrak{K}$ such that $\mathfrak{K} = \mathcal{L}_1[\dot{+}]\mathcal{L}_2$ and $A\mathcal{L}_i \subseteq \mathcal{L}_i$, $i = 1, 2$, then we say that this decomposition *reduces* A.

and $c(A_\varphi) = \emptyset$ otherwise. For $\Delta \in \Omega_p^\times$ the spectral function $E(\Delta)$ has the following matrix representation:

$$\left(\begin{array}{c|cccc|c|c}
\chi_\Delta \cdot & t^{-k}\chi_\Delta & t^{-k+1}\chi_\Delta & \cdots & t^{-1}\chi_\Delta & 0 & 0 \\
\hline
0 & \multicolumn{4}{c|}{0} & 0 & 0 \\
\hline
0 & \multicolumn{4}{c|}{0} & 0 & 0 \\
\hline
(.,t^{-k}\chi_\Delta)_\sigma & (1,t^{-2k}\chi_\Delta)_\sigma & (1,t^{-2k+1}\chi_\Delta)_\sigma & \cdots & (1,t^{-k-1}\chi_\Delta)_\sigma & & \\
(.,t^{-k+1}\chi_\Delta)_\sigma & (1,t^{-2k+1}\chi_\Delta)_\sigma & (1,t^{-2k+2}\chi_\Delta)_\sigma & \cdots & (1,t^{-k}\chi_\Delta)_\sigma & & \\
\vdots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\
(.,t^{-1}\chi_\Delta)_\sigma & (1,t^{-k-1}\chi_\Delta)_\sigma & (1,t^{-k-2}\chi_\Delta)_\sigma & \cdots & (1,t^{-2}\chi_\Delta)_\sigma & &
\end{array}\right).$$

If $c(A_\varphi) = \{0\}$, then the point 0 is a regular critical point of A_φ if and only if $k = 0$.

Proof. By Theorem 2.18 there exists a real definitizing polynomial p such that $x = 0$ is the only zero of p. By Proposition 2.37 we have $0 \in \sigma(A_\varphi)$ and therefore it holds $c(A_\varphi) \subseteq \{0\}$.

Let p be a real definitizing polynomial of A_φ with only zero $x = 0$, see Theorem 2.18, then Theorem 2.39 implies the existence of a unique spectral function $E : \Omega_p \to \mathcal{B}(\mathcal{K}_\varphi)$. It is sufficient to show the matrix representation of the spectral function for bounded real intervals such that 0 is not in the closure. So let Δ be a bounded real interval with $0 \notin \overline{\Delta}$. Without loss of generality assume $p > 0$ on Δ, and hence $p > 0$ on the half real axis containing Δ. Then by Corollary 2.40 for any closed interval $\tilde{\Delta}$ such that $0 \notin \tilde{\Delta}$ and $\Delta \subseteq \tilde{\Delta}$ the space $\mathcal{H}_{\tilde{\Delta}} := E(\tilde{\Delta})\mathcal{K}_\varphi$ endowed with the inner product $(.,.)_{\tilde{\Delta}} := [.,.]|_{E(\tilde{\Delta})\mathcal{K}_\varphi \times E(\tilde{\Delta})\mathcal{K}_\varphi}$ is a Hilbert space and $A_\varphi|_{\mathcal{H}_{\tilde{\Delta}}}$ is a bounded linear selfadjoint operator on $\mathcal{H}_{\tilde{\Delta}}$. Hence there exists a unique spectral measure F on the Borel subsets of $\sigma(A_\varphi|_{\mathcal{H}_{\tilde{\Delta}}})$ such that

$$F(\Delta) = \int\limits_{\sigma(A_\varphi|_{\mathcal{H}_{\tilde{\Delta}}})} \chi_\Delta dF = \Phi_F(\chi_\Delta),$$

where Φ_F denotes the $*$-homeomorphism from $B(\sigma(A_\varphi|_{\mathcal{H}_{\tilde{\Delta}}}), \mathcal{A})^4 \to \mathcal{B}(\mathcal{H}_\Delta)$, $\phi \mapsto \int \phi dF$, see [C1, Theorem 2.3, p. 264]. Let $n \in \mathbb{N}$, n is odd, and consider the set

$$\mathcal{P}_n := \{p \in \mathbb{C}[x] : \ \exists q \in \mathbb{C}[x] : p(x) = q(x^n) \ x \in \tilde{\Delta}\}.$$

Then $\mathcal{P}_n$ is a linear subspace of the space of bounded complex valued functions on Δ. Further $\mathcal{P}_n$ is nowhere vanishing, separates point and is closed under complex conjugation, hence by the Stone-Weierstrass Theorem $\mathcal{P}_n$ is dense in $C(\tilde{\Delta})$.
Clearly there exists a sequence of continuous functions $(f_k)_{k\in\mathbb{N}}$ which converges pointwise to χ_Δ on $\tilde{\Delta}$. Since $\mathcal{P}_n$ is dense in $C(\tilde{\Delta})$ for each $k \in \mathbb{N}$ there exists a sequence $(g_l^{(k)})_{l\in\mathbb{N}}$ in $\mathcal{P}_n$ which converges uniformly to f_k. Then by a diagonal argument the sequence $p_m := g_m^{(m)}$ converges pointwise to χ_Δ on $\tilde{\Delta}$. Further the sequence $(p_m)_{m\in\mathbb{N}}$ is uniformly bounded, i.e. $\sup_{m\in\mathbb{N}} \|p_m\|_\infty < \infty$. By the definition of $\mathcal{P}_n$ there exists a sequence of polynomials $q_m \in \mathbb{C}[x]$, $m \in \mathbb{N}$, such that $p_m(x) = q_m(x^n)$, $x \in \tilde{\Delta}$. In particular it holds $\lim_{m\to\infty} q_m(x^n) = \chi_\Delta(x)$, $x \in \tilde{\Delta}$. By the properties of Φ_F it follows that $\big(\Phi_F(p_m)\big)_{m\in\mathbb{N}}$ converges to

[4] $\mathcal{A}$ is the Borel σ-algebra on $\sigma(A_\varphi|_{\mathcal{H}_{\tilde{\Delta}}})$ and $B(\sigma(A_\varphi|_{\mathcal{H}_{\tilde{\Delta}}}), \mathcal{A})$ is the set of $\mathcal{A}$-measurable complex valued functions on $\sigma(A_\varphi|_{\mathcal{H}_{\tilde{\Delta}}})$.

$\Phi_F(\chi_\Delta)$ in the strong operator topology. By the spectral theorem it holds $\Phi_F(p_m) = \int p_m dF = p_m(A_{\varphi|\mathcal{H}_{\tilde{\Delta}}})$. This implies that

$$\lim_{m\to\infty} q_m\big((A_\varphi|_{\mathcal{H}_{\tilde{\Delta}}})^n\big)g = \Phi_F(\chi_\Delta)g = F(\Delta)g, \quad g \in \mathcal{H}_{\tilde{\Delta}}.$$

Hence we can obtain $F(\Delta)$ as a strong limit of a sequence of polynomials in $(A_\varphi|_{\mathcal{H}_{\tilde{\Delta}}})^n$, n odd. For sufficiently large $n \in \mathbb{N}$ we have

$$\mathfrak{A}_\vartheta^n = \left(\begin{array}{c|cccc|c|c} t^n\cdot & t^{n-k} & t^{n-k+1} & \dots & t^{n-1} & 0 & 0 \\ \hline 0 & & & 0 & & 0 & 0 \\ \hline 0 & & & 0 & & 0 & 0 \\ \hline (.,t^{n-k})_\sigma & (1,t^{n-2k})_\sigma & (1,t^{n-2k+1})_\sigma & \dots & (1,t^{n-k-1})_\sigma & & \\ (.,t^{n-k+1})_\sigma & (1,t^{n-2k+1})_\sigma & (1,t^{n-2k+2})_\sigma & \dots & (1,t^{n-k})_\sigma & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ (.,t^{n-1})_\sigma & (1,t^{n-k-1})_\sigma & (1,t^{n-k-2})_\sigma & \dots & (1,t^{n-2})_\sigma & & \end{array}\right),$$

with respect to the decomposition $\mathfrak{L}_\vartheta = L^2(\sigma)\oplus\mathbb{C}^k\oplus\mathbb{C}^{l-k+1}\oplus\mathbb{C}^k$. This operator leaves the isotropic part $\mathfrak{L}_\vartheta^{[\circ]_\vartheta}$ (see Remark 2.11) invariant, hence the operator A_φ^n and its restriction to $\mathcal{H}_{\tilde{\Delta}}$ admits a similar matrix representation. In fact it is given by the same scheme except, that this time the decompositions is $L^2(\sigma) \oplus \mathbb{C}^k \oplus \mathbb{C}^r \oplus \mathbb{C}^k$, where

$$r := \begin{cases} 0, & l < 2k-1 \\ l-2k+1, & l \geq 2k-1 \end{cases}.$$

By the linearity of the inner product and since in each term of the matrix representation we can isolate the factor t^n and $q_m(t^n)$ converges to χ_Δ we obtain the following representation for $F(\Delta)$

$$\left(\begin{array}{c|cccc|c|c} \chi_\Delta\cdot & t^{-k}\chi_\Delta & t^{-k+1}\chi_\Delta & \dots & t^{-1}\chi_\Delta & 0 & 0 \\ \hline 0 & & & 0 & & 0 & 0 \\ \hline 0 & & & 0 & & 0 & 0 \\ \hline (.,t^{-k}\chi_\Delta)_\sigma & (1,t^{-2k}\chi_\Delta)_\sigma & (1,t^{-2k+1}\chi_\Delta)_\sigma & \dots & (1,t^{-k-1}\chi_\Delta)_\sigma & & \\ (.,t^{-k+1}\chi_\Delta)_\sigma & (1,t^{-2k+1}\chi_\Delta)_\sigma & (1,t^{-2k+2}\chi_\Delta)_\sigma & \dots & (1,t^{-k}\chi_\Delta)_\sigma & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 & 0 \\ (.,t^{-1}\chi_\Delta)_\sigma & (1,t^{-k-1}\chi_\Delta)_\sigma & (1,t^{-k-2}\chi_\Delta)_\sigma & \dots & (1,t^{-2}\chi_\Delta)_\sigma & & \end{array}\right).$$

Let E be the Kreĭn space spectral function and Δ_0 a bounded real interval such that $0 \notin \overline{\Delta}_0$. Then $\tilde{F}(\Delta) := E(\Delta\cap\Delta_0)$, $\Delta \in \Omega_p$, is a spectral measure for the selfadjoint operator $A_\varphi|_{\mathcal{H}_{\Delta_0}}$ in the Hilbert space $\mathcal{H}_{\Delta_0}$. By the uniqueness of the spectral measure it follows that $\tilde{F} = F$. This proves the matrix representation for $E(\Delta)$, $\Delta \in \Omega_p^\times$.
If $k > 0$, then for $\vec{x} = (0;1,0,\dots,0)^T$ we have

$$\|E(\Delta)\| \geq \|E(\Delta)\vec{x}\| \geq |(1,t^{-2k}\chi_\Delta)_\sigma| = \left|\int_\Delta \frac{1}{t^{2k}}d\sigma(t)\right|, \quad \Delta \in \Omega_p.$$

By the minimality of k the function $t \mapsto \frac{1}{t^{2k}}$ is not σ-integrable on $\mathbb{R}$ and therefore the limits $\lim_{\lambda\nearrow 0} E([-1,\lambda])$ and $\lim_{\lambda\searrow 0} E([\lambda,1])$ does not exist in the

strong operator topology. This implies that $0 \in c(A_\varphi)$ (see Remark 2.42) and that 0 is a singular critical point.
It remains to consider the case $k=0$. Note that if $k=0$ and ϑ is the minimal representation $\mathfrak{L}_\vartheta$ is non degenerated (see Remark 2.11), hence by Corollary 2.35 we have that $\mathcal{K}_\varphi = \mathfrak{L}_\vartheta$ and $A_\varphi = \mathfrak{A}_\vartheta$. According to [L, Proposition 4.2, p.35] 0 is a critical point of A_φ if and only if for each $\Delta \in \Omega_p$ with $0 \in \Delta$ the scalar product $[.,.]_{\mathcal{K}_\varphi}$ is indefinite on $E(\Delta)\mathcal{K}_\varphi$. So let $\Delta \in \Omega_p$ such that $0 \in \Delta$. Since $\mathbb{R} \in \Omega_p$ and because of the semi-ring property of Ω_p there exists $\Delta_j \in \Omega_p$, $j = 1,\dots,n$, pairwise disjoint, such that

$$\Delta^c = \mathbb{R} \setminus \Delta = \bigcup_{j=1}^n \Delta_j.$$

Since $0 \in \Delta$ it follows that $0 \notin \Delta_j$, $j=1,\dots,n$, hence $\Delta_j \in \Omega_p^\times$, $j=1,\dots,n$. Now Theorem 2.39 implies that

$$E(\Delta) = E(\mathbb{R}) - E(\Delta^c) = I - \sum_{j=1}^n E(\Delta_j) = \left(\begin{array}{c|ccc} \chi_{\Delta^c} & 0 & \dots & 0 \\ \hline 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{array}\right) \in L^2(\sigma) \oplus \mathbb{C}^{l+1}.$$

This shows that, in the case $l > 0$, the scalar product $[.,.]_{\mathcal{K}_\varphi}$ is indefinite on $E(\Delta)\mathcal{K}_\varphi$, hence $0 \in c(A_\varphi)$. If $l = -1$, then the spectral function $E(\Delta)$ is simply the multiplication with the characteristic function χ_{Δ^c} on $L^2(\sigma)$ and therefore the scalar product is indefinite on $E(\Delta)\mathcal{K}_\varphi$ if and only if $\{\operatorname{sign} \sigma|_{\mathbb{R}^+}, \operatorname{sign} \sigma|_{\mathbb{R}^-}\} = \{+1,-1\}$, i.e. both signs were attained. If $l = 0$ elements of $E(\Delta)\mathcal{K}_\varphi$ are of the form $(f\chi_{\Delta^c}; a_0)$ with $f \in L^2(\sigma)$ and $a_0 \in \mathbb{C}$ and the scalar product computes as

$$\left[\begin{pmatrix} f\chi_{\Delta^c} \\ a_0 \end{pmatrix}, \begin{pmatrix} f\chi_{\Delta^c} \\ a_0 \end{pmatrix} \right]_{\mathcal{K}_\varphi} = \int_{\Delta^c} |f|^2 d\sigma + c_0 |a_0|^2.$$

Which implies that the scalar product on $E(\Delta)\mathcal{K}_\varphi$ is indefinite if and only if $\{\operatorname{sign} \sigma|_{\mathbb{R}^+}, \operatorname{sign} \sigma|_{\mathbb{R}^-}, \operatorname{sign} c_0\} = \{+1,-1\}$.
Finally assume $k=0$ and $0 \in c(A_\varphi)$. Then it holds

$$\|E(\Delta)\| \leq |\sigma|(\Delta^c) + 1 \leq |\sigma|(\mathbb{R}), \quad \Delta \in \Omega_p, 0 \in \Delta,$$

and since σ is a finite measure by [L, Proposition 5.6, p.40] it follows that 0 is a regular critical point of A_φ. This completes the proof. ❑

2.44 Remark. Note that the above Proposition shows that for $\varphi \in \mathcal{F}(\mathbb{R},0)$ it holds that $c(A_\varphi) = \emptyset$ if and only if $\varphi \in \mathcal{F}(\mathbb{R},\emptyset)$.

Chapter 3

Model spaces for distributions of class $\mathcal{F}$

In this chapter we construct the model space for distributions of class $\mathcal{F}$. We start by defining a model for distributions of class $\mathcal{F}(\mathbb{R})$. Therefore we will make use of Lemma 1.13 (*vii*), which states that if $\varphi \in \mathcal{F}(\mathbb{R})$ and M a finite subset of $\mathbb{R}$ such that $\varphi \in \mathcal{F}(\mathbb{R}, M)$ and $\Delta_1, \dots, \Delta_n$, $n \in \mathbb{N}$, be a φ-M-decomposition of $\mathbb{R}$, then we can write the distribution φ as

$$\varphi(f) = \sum_{j=1}^{n} \varphi|_{\Delta_j}(f), \quad f \in \mathcal{D}(\mathbb{R}).$$

We can choose the φ-M-decomposition of $\mathbb{R}$ such that $|M \cap \Delta_j| \leq 1$, $j = 1, \dots, n$. By Lemma 1.13 (*iii*) it follows that $\varphi|_{\Delta_j} \in \mathcal{F}(\mathbb{R}, \alpha_j)$, for some $\alpha_j \in \mathbb{R}$, or $\varphi|_{\Delta_j} \in \mathcal{F}(\mathbb{R}, \emptyset)$. So we will construct model spaces for distributions of class $\mathcal{F}(\mathbb{R}, \alpha)$, $\alpha \in \mathbb{R}$, and $\mathcal{F}(\mathbb{R}, \emptyset)$.

3.1 Model for distributions of class $\mathcal{F}(\mathbb{R}, \alpha)$

In order to construct the model space for distributions of class $\mathcal{F}(\mathbb{R}, \alpha)$, $\alpha \in \mathbb{R}$, we consider the translation operator on C^∞ which is defined by

$$T_\alpha f(t) := f(t - \alpha), \quad f \in C^\infty(\mathbb{R}).$$

3.1 Definition. For $\varphi \in \mathcal{F}(\mathbb{R}, \alpha)$, $\alpha \in \mathbb{R}$, we define a mapping $\tau_\alpha : \mathcal{F}(\mathbb{R}, \alpha) \to \mathcal{F}(\mathbb{R}, 0)$ by $(\tau_\alpha \varphi)(f) = \varphi(T_\alpha f)$, $f \in \mathbb{C}^\infty(\mathbb{R})$.

First we show that τ_α is well-defined. According to Proposition 1.21 there exist constants $k, l \in \mathbb{N}_0$, $c_0, \dots, c_l \in \mathbb{R}$ and a signed finite Borel measure σ with compact support, $\sigma(\{\alpha\}) = 0$ and $\sigma|_{Z_{\alpha\pm}}$ has the same sign as $\mu_{Z_{\alpha\pm}}$, such that

$$\varphi(f) = \int_{\mathbb{R}} \frac{f^{\{\alpha, 2k\}}(t)}{(t - \alpha)^{2k}} d\sigma(t) + \sum_{i=0}^{l} \frac{c_i}{i!} f^{(i)}(\alpha), \quad f \in \mathcal{D}(\mathbb{R}).$$

For $t \in \mathbb{R}$ set $\tilde{\sigma}(t) := \sigma(t+\alpha)$, then it holds $\tilde{\sigma}(\{0\}) = 0$, $\tilde{\sigma}|_{\mathbb{R}^+}$ and $\tilde{\sigma}|_{\mathbb{R}^-}$ are either positive or negative measures. Now, for $f \in \mathcal{D}(\mathbb{R})$, it follows that

$$(\tau_\alpha \varphi)(f) = \varphi(T_\alpha f) = \int_{\mathbb{R}} \frac{(T_\alpha f)^{\{\alpha,2k\}}(t)}{(t-\alpha)^{2k}} d\sigma(t) + \sum_{i=0}^{l} \frac{c_i}{i!}(T_\alpha f)^{(i)}(\alpha) =$$

$$= \int_{\mathbb{R}} \frac{f(t-\alpha) - \sum_{i=0}^{2k-1} \frac{(t-\alpha)^i}{i!} f^{(i)}(0)}{(t-\alpha)^{2k}} d\sigma(t) + \sum_{i=0}^{l} \frac{c_i}{i!} f^{(i)}(0) =$$

$$= \int_{\mathbb{R}} \frac{f^{\{0,2k\}}(s)}{s^{2k}} d\tilde{\sigma}(s) + \sum_{i=0}^{l} \frac{c_i}{i!} f^{(i)}(0).$$

Lemma 2.3 implies that $\tau_\alpha \varphi$ is an element of $\mathcal{F}(\mathbb{R}, 0)$.
We are ready to define the model space and model operator for $\varphi \in \mathcal{F}(\mathbb{R}, \alpha)$.

3.2 Definition. Let $\varphi \in \mathcal{F}(\mathbb{R}, \alpha)$, then we define

$$\mathcal{K}_\varphi := \mathcal{K}_{\tau_\alpha \varphi}, \quad A_\varphi := A_{\tau_\alpha \varphi} + \alpha I \quad \text{and} \quad \iota_\varphi := \iota_{\tau_\alpha \varphi} \circ T_{-\alpha}.$$

3.3 Lemma. *Let $\varphi \in \mathcal{F}(\mathbb{R}, \alpha)$, then $\sigma(A_\varphi) = \operatorname{supp} \varphi \cup \{\alpha\}$ and the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\iota_\varphi} & \mathcal{K}_\varphi \\ \downarrow{\scriptstyle \cdot t} & & \downarrow{\scriptstyle A_\varphi} \\ \mathcal{P} & \xrightarrow[\iota_\varphi]{} & \mathcal{K}_\varphi. \end{array} \tag{3.1.1}$$

Proof. The assertion about the spectrum of A_φ is an immediate consequence of Proposition 2.37. To see that the diagram commutes, let $p \in \mathcal{P}$, then we have

$$A_\varphi\big(\iota_\varphi(p)\big) = A_{\tau_\alpha\varphi}\iota_\varphi(p) + \alpha\iota_\varphi(p) = A_{\tau_\alpha\varphi}\iota_{\tau_\alpha\varphi}\big(p(t+\alpha)\big) + \alpha\iota_{\tau_\alpha\varphi}\big(p(t+\alpha)\big).$$

On the other hand it holds for every $p \in \mathcal{P}$

$$\iota_\varphi\big(tp(t)\big) = \iota_{\tau_\alpha\varphi}\big((t+\alpha)p(t+\alpha)\big) = \iota_{\tau_\alpha\varphi}\big(tp(t+\alpha)\big) + \alpha\iota_{\tau_\alpha\varphi}\big(p(t+\alpha)\big).$$

Since $\tau_\alpha\varphi \in \mathcal{F}(\mathbb{R}, 0)$ diagram (2.3.3) yields

$$A_{\tau_\alpha\varphi}\iota_{\tau_\alpha\varphi}\big(p(t+\alpha)\big) = \iota_{\tau_\alpha\varphi}\big(tp(t+\alpha)\big),$$

which implies (3.1.1). ❑

3.2 Model for distributions of class $\mathcal{F}(\mathbb{R}, \emptyset)$

Let $\varphi \in \mathcal{F}(\mathbb{R}, \emptyset)$ and choose an arbitrary element $\alpha \in \mathbb{R} \setminus \operatorname{supp} \varphi$. Then we can understand the distribution φ as an element of $\mathcal{F}(\mathbb{R}, \alpha)$, and we define the model space, model operator and model embedding simply as the corresponding triple for the distribution φ considered as an element of $\mathcal{F}(\mathbb{R}, \alpha)$. Because $\alpha \notin \operatorname{supp} \varphi$ this definition does not depend on the choice of α.
Since $\varphi \in \mathcal{F}(\mathbb{R}, \emptyset)$ there exists a Borel measure σ on $\mathbb{R}$ such that $\varphi(f) = \int_{\mathbb{R}} f d\sigma$, $f \in \mathcal{D}(\mathbb{R})$. Hence the model space is explicitly given by $L^2(\sigma)$, the model operator is the multiplication operator on $L^2(\sigma)$, and the model embedding is the canonical embedding $\mathcal{P} \hookrightarrow L^2(\sigma)$.

3.3 Model for distributions of class $\mathcal{F}(\mathbb{C}\setminus\mathbb{R})$

Let $\psi \in \mathcal{F}(\mathbb{C}\setminus\mathbb{R})$ and assume $\sigma_0(\phi) = \{\beta, \overline{\beta}\}$. For $p \in \mathcal{P}$ the distribution ψ admits the following representation (see equation (1.3.1))

$$\psi(p) = \sum_{j=0}^{\nu-1} \left(\frac{d_j}{j!} p^{(j)}(\beta) + \frac{\overline{d}_j}{j!} p^{(j)}(\overline{\beta}) \right),$$

where $d_j \in \mathbb{C}$, $j = 0, \dots, \nu-1$ and $d_{\nu-1} \neq 0$. We equip $\mathcal{P}$ with the inner product $[p,q]_\psi := \Psi(p\overline{q})$. Further define

$$W := \begin{pmatrix} 0 & \hat{W}^* \\ \hat{W} & 0 \end{pmatrix}, \quad \text{with} \quad \hat{W} := \begin{pmatrix} d_0 & d_1 & \cdots & d_{\nu-1} \\ d_1 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ d_{\nu-1} & 0 & \cdots & 0 \end{pmatrix}.$$

3.4 Definition. Let $\psi \in \mathcal{F}(\mathbb{C}\setminus\mathbb{R})$ and $\sigma_0(\varphi) = \{\beta, \overline{\beta}\}$. We define an operator on $\mathbb{C}^{2\nu}$ by

$$A_\psi : \begin{cases} \mathbb{C}^{2\nu} \to \mathbb{C}^{2\nu} \\ (a_0, \dots, a_{\nu-1}; b_0, \dots, b_{\nu-1}) \mapsto (\beta a_0, a_0 + \beta a_1, \dots, a_{\nu-2} + \beta a_\nu; \\ \qquad \overline{\beta} b_0, b_0 + \overline{\beta} b_1, \dots, b_{\nu-2} + \overline{\beta} b_{\nu-1}) \end{cases} \tag{3.3.1}$$

and call it the *multiplication operator on* $\mathbb{C}^{2\nu}$.

3.5 Remark. The multiplication operator A_ψ admits the following matrix representation

$$A_\psi = \begin{pmatrix} \hat{B} & 0 \\ 0 & \overline{\hat{B}} \end{pmatrix}, \text{ with } \hat{B} := \begin{pmatrix} \beta & 0 & 0 & \cdots & 0 \\ 1 & \beta & 0 & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \beta \end{pmatrix}.$$

3.6 Theorem. *Let $\psi \in \mathcal{F}(\mathbb{C}\setminus\mathbb{R})$ and $\sigma_0(\varphi) = \{\beta, \overline{\beta}\}$. Then the mapping*

$$\iota_\psi : \begin{cases} \mathcal{P} \to \mathbb{C}^{2\nu} \\ p \mapsto \left(\frac{p^{(0)}(\beta)}{0!}, \dots, \frac{p^{(\nu-1)}(\beta)}{(\nu-1)!}; \frac{p^{(0)}(\overline{\beta})}{0!}, \dots, \frac{p^{(\nu-1)}(\overline{\beta})}{(\nu-1)!} \right)^T \end{cases}$$

is an isometry of $(\mathcal{P}, [.,.]_\psi)$ onto the finite dimensional non-degenerated inner product space $(\mathbb{C}^{2\nu}, (W.,.)_{\mathbb{C}^{2\nu}})$ and therefore induces an isometric isomorphism of $(\mathcal{P}/\mathcal{P}^0, [.,.]_\psi)$ onto $(\mathbb{C}^{2\nu}, (W.,.)_{\mathbb{C}^{2\nu}})$. Further the following diagram commutes

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\iota_\psi"] \arrow[d, "A_t"'] & \mathbb{C}^{2\nu} \arrow[d, "A_\psi"] \\ \mathcal{P} \arrow[r, "\iota_\psi"'] & \mathbb{C}^{2\nu}. \end{tikzcd}$$

Proof. Since $d_{\nu-1} \neq 0$ the matrix W is regular which implies that the linear space $\big(\mathbb{C}^{2\nu}, (W.,.)_{\mathbb{C}^{2\nu}}\big)$ is non-degenerated. For $n \in \mathbb{N}$ denote by $\mathcal{P}_n$ the space of polynomials with degree less or equal than n. Clearly $\mathcal{P}_{2\nu-1}$ is a 2ν dimensional subspace of $\mathcal{P}$. By the fundamental theorem of algebra it follows that the mapping $\iota_\psi|_{\mathcal{P}_{2\nu-1}}$ is injective. This implies that $\iota_\psi|_{\mathcal{P}_{2\nu-1}}$ and therefore ι_ψ is surjective.
Let $p, q \in \mathcal{P}$, then we have

$$\begin{aligned}
[p,q]_\psi &= \sum_{j=0}^{\nu-1} \frac{1}{j!}\Big(d_j (p\overline{q})^{(j)}(\beta) + \overline{d}_j (p\overline{q})^{(j)}(\overline{\beta})\Big) = \\
&= \sum_{j=0}^{\nu-1}\left(d_j \sum_{i=0}^{j} \frac{p^{(i)}(\beta)}{i!}\frac{\overline{q^{(j-i)}}(\beta)}{(j-i)!} + \overline{d}_j \sum_{i=0}^{j}\frac{p^{(i)}(\overline{\beta})}{i!}\frac{\overline{q^{(j-i)}(\beta)}}{(j-i)!}\right) = \\
&= \sum_{i=0}^{\nu-1}\left(p^{(i)}(\beta)\sum_{j=i}^{\nu-1}\frac{d_j}{i!}\frac{\overline{q^{(j-i)}}(\beta)}{(j-i)!} + p^{(i)}(\overline{\beta})\sum_{j=i}^{\nu-1}\frac{\overline{d}_j}{i!}\frac{\overline{q^{(j-i)}(\beta)}}{(j-i)!}\right) = \\
&= (W\iota_\psi p, \iota_\psi q)_{\mathbb{C}^{2\nu}}.
\end{aligned}$$

Since $\big(\mathbb{C}^{2\nu}, (W.,.)_{\mathbb{C}^{2\nu}}\big)$ is non-degenerated there exists an isometric isomorphism of $\big(\mathcal{P}/\mathcal{P}^0, [.,.]_\psi\big)$ onto $\big(\mathbb{C}^{2\nu}, (W.,.)_{\mathbb{C}^{2\nu}}\big)$.
It remains to show that the operator A_ψ satisfies the diagram. Let $p \in \mathcal{P}$ and define $q(t) := A_t p(t) = tp(t)$, then we have

$$\frac{q^{(j)}(\beta)}{j!} = \frac{1}{j!}\sum_{i=0}^{j}\binom{j}{i} t^{(i)}|_{t=\beta}\, p^{(j-i)}(\beta) = \begin{cases} \beta p(\beta), & j = 0 \\ \beta\frac{p^{(j)}(\beta)}{j!} + \frac{p^{(j-1)}(\beta)}{(j-1)!}, & j \geq 1. \end{cases}$$

Hence $\iota_\psi(q)$ is given by

$$\begin{aligned}
\iota_\psi(q) = \Big(&\beta\frac{p^{(0)}(\beta)}{0!}, \beta\frac{p^{(1)}(\beta)}{1!} + \frac{p^{(0)}(\beta)}{0!}, \dots, \beta\frac{p^{(\nu-1)}(\beta)}{(\nu-1)!} + \frac{p^{(\nu-2)}(\beta)}{(\nu-2)!}; \\
&\beta\frac{p^{(0)}(\overline{\beta})}{0!}, \overline{\beta}\frac{p^{(1)}(\overline{\beta})}{1!} + \frac{p^{(0)}(\overline{\beta})}{0!}, \dots, \overline{\beta}\frac{p^{(\nu-1)}(\overline{\beta})}{(\nu-1)!} + \frac{p^{(\nu-2)}(\overline{\beta})}{(\nu-2)!}\Big).
\end{aligned}$$

This implies that $A_\psi\big(\iota_\psi(p)\big) = \iota_\psi\big(A_t(p)\big)$. ❑

3.7 Definition. For $\psi \in \mathcal{F}(\mathbb{C}\setminus\mathbb{R})$ with $\sigma_0(\psi) = \{\beta, \overline{\beta}\}$ we define the model space $\mathcal{K}_\psi$ as the space $\mathbb{C}^{2\nu}$ with the inner product $[.,.]_{\mathcal{K}_\psi} := (W.,.)_{\mathbb{C}^{2\nu}}$, the model operator A_ψ as in (3.3.1) and the embedding ι_ψ as in Theorem 3.6.

3.8 Proposition. *Let $\psi \in \mathcal{F}(\mathbb{C}\setminus\mathbb{R})$ and $\sigma_0(\psi) = \{\beta, \overline{\beta}\}$. Then the operator A_ψ is bounded, selfadjoint and definitizable in $\mathcal{K}_\psi$. Further the spectrum of A_ψ is given by $\sigma(A_\psi) = \{\beta, \overline{\beta}\}$.*

Proof. Clearly, A_ψ is a bounded operator. In order to show that A_ψ is selfadjoint with respect to the indefinite inner product $(W.,.)_{\mathbb{C}^{2\nu}}$ it is sufficient to show that $WB = B^*W$. Since $\hat{W}^T = \hat{W}$ it follows that

$$WB = \begin{pmatrix} 0 & \overline{\hat{W}\hat{B}} \\ \hat{W}\hat{B} & 0 \end{pmatrix}, \quad \overline{B}^T W = \begin{pmatrix} 0 & \overline{(\hat{W}\hat{B})^T} \\ (\hat{W}\hat{B})^T & 0 \end{pmatrix}.$$

For $\hat{W}\hat{B}$ we have

$$\hat{W}\hat{B} = \beta\hat{W} + \begin{pmatrix} d_1 & d_2 & \cdots & d_{\nu-1} & 0 \\ d_2 & d_3 & \cdots & 0 & \\ \vdots & & 0 & & \\ d_{\nu-1} & 0 & & & \\ 0 & & & & \end{pmatrix},$$

which implies that $(\hat{W}\hat{B})^T = \hat{W}\hat{B}$. Therefore A_ψ is selfadjoint with respect to the inner product $(W.,.)_{\mathbb{C}^{2\nu}}$. By the theorem of Cayley-Hamilton it follows that A_ψ is definitizable.
By definition of the operator A_ψ its spectrum coincides with $\sigma_0(\psi)$. ❑

3.9 Definition. Let $\psi \in \mathcal{F}(\mathbb{C}\setminus\mathbb{R})$ and $\sigma_0(\psi) = \{\beta_1, \ldots, \beta_n, \overline{\beta}_1, \ldots, \overline{\beta}_n\}$. For $i = 1, \ldots, n$, denote by $(\mathcal{K}_{\psi_i}, A_{\psi_i}, \iota_{\psi_i})$ the model space, model operator and the embedding corresponding to the distribution ψ_i with $\sigma_0(\psi_i) = \{\beta_i, \overline{\beta}_i\}$. Then we define

$$\mathcal{K}_\psi := \bigoplus_{i=1}^n \mathcal{K}_{\psi_i}, \quad A_\psi := \bigoplus_{i=1}^n A_{\psi_i} \quad \text{and} \quad \iota_\psi = \bigoplus_{i=1}^n \iota_{\psi_i}.$$

3.10 Proposition. *Let $\psi \in \mathcal{F}(\mathbb{C}\setminus\mathbb{R})$ and $\sigma_0(\psi) = \{\beta_1, \ldots, \beta_n, \overline{\beta}_1, \ldots, \overline{\beta}_n\}$. Then the operator A_ψ is bounded, selfadjoint and definitizable in $\mathcal{K}_\psi$. Further the spectrum of A_ψ is given by $\sigma(A_\psi) = \sigma_0(\psi)$.*

Proof. As the orthogonal sum of bounded and selfadjoint operators A_ψ is a bounded and selfadjoint operator in $\mathcal{K}_\psi$. Since $\mathcal{K}_\psi$ is finite dimensional by the theorem of Cayley-Hamilton there exists a polynomial p such that $p(A_\psi) = 0$. Therefore A_ψ is definitizable. The assertion about the spectrum is immediate. ❑

3.4 Model for distributions of class $\mathcal{F}(\mathbb{R})$

3.11 Definition. Let $\varphi \in \mathcal{F}(\mathbb{R})$ and M a finite subset of $\mathbb{R}$ such that $\varphi \in \mathcal{F}(\mathbb{R}, M)$. Further let $\Delta_1, \ldots, \Delta_n$, $n \in \mathbb{N}$, be a φ-M-decomposition of $\mathbb{R}$ such that $|M \cap \Delta_i| \le 1$, $1 \le i \le n$. Now we define the *model space*, the *model operator* and the *model embedding* for the distribution φ as

$$\mathcal{K}_\varphi := \bigoplus_{i=1}^n \mathcal{K}_{\varphi|_{\Delta_i}}, \quad A_\varphi := \bigoplus_{i=1}^n A_{\varphi|_{\Delta_i}} \quad \text{and} \quad \iota_\varphi := \bigoplus_{i=1}^n \iota_{\varphi|_{\Delta_i}}.$$

We have to show that this definition is (up to unitary equivalence) independent of the φ-M-decomposition of $\mathbb{R}$. This is the assertion of the next lemma.

3.12 Lemma. *Let $\varphi \in \mathcal{F}(\mathbb{R})$ and M a finite subset of $\mathbb{R}$ such that $\varphi \in \mathcal{F}(\mathbb{R}, M)$. If $\Delta_1, \ldots, \Delta_n$, $n \in \mathbb{N}$, and $\tilde{\Delta}_1, \ldots, \tilde{\Delta}_m$, $m \in \mathbb{N}$, are φ-M-decompositions of $\mathbb{R}$ such that $|M \cap \Delta_i| \le 1$, $1 \le i \le n$, and $|M \cap \tilde{\Delta}_j| \le 1$, $1 \le j \le m$, then there exists a unitary mapping $U : \bigoplus_{i=1}^n \mathcal{K}_{\varphi|_{\Delta_i}} \to \bigoplus_{i=1}^m \mathcal{K}_{\varphi|_{\tilde{\Delta}_i}}$ such that $U \bigoplus_{i=1}^n A_{\varphi|_{\Delta_i}} = \bigoplus_{i=1}^m A_{\varphi|_{\tilde{\Delta}_i}} U$.*

Proof. Without loss of generality we can assume that $\Delta_1, \dots, \Delta_n, \tilde{\Delta}_1, \dots, \tilde{\Delta}_m$ are open intervals, see Lemma 1.13 (v), and that for every $i \in \{1, \dots, n\}$ there exists $r_i \in \mathbb{N}$, $j_1, \dots, j_{r_i} \in \{1, \dots, m\}$, $j_1 < \dots < j_{r_i}$, such that

$$\overline{\tilde{\Delta}_{j_1} \cup \dots \cup \tilde{\Delta}_{j_{r_i}}} = \overline{\Delta_i}.$$

It is sufficient to show, that there exists a unitary mapping

$$U : \mathcal{K}_{\varphi|_{\Delta_i}} \to \bigoplus_{j=j_1}^{j_{r_i}} \mathcal{K}_{\varphi|_{\tilde{\Delta}_j}}. \tag{3.4.1}$$

Assume $r_i = 2$ and let $\hat{\Delta}_1 := \tilde{\Delta}_{j_1} = (a,b)$, $\hat{\Delta}_2 := \tilde{\Delta}_{j_2} = (b,c)$, then $\Delta := \Delta_i = (a,c)$. If $\Delta \cap M = \emptyset$ Consider the case $\Delta \cap M = \{\alpha\}$, $\alpha \in \mathbb{R}$, and $\alpha \in \hat{\Delta}_1$. The case $\alpha \in \hat{\Delta}_2$ is proven analogously. We start with the special case $\alpha = 0$. Choose a representation $\vartheta = (k, l, \sigma, c_0, \dots, c_l)$ for the distribution $\varphi|_\Delta$ such that $l \geq 2k$. For the distribution $\varphi|_{\hat{\Delta}_1}$ fix a representation $\vartheta_1 = (k, l, \hat{\sigma}, c_0, \dots, c_l)$ and clearly $\hat{\sigma} = \sigma|_{(a,b)}$. Since $\varphi|_{\hat{\Delta}_2}$ is a bounded measure and $\sigma|_{(b,c)}(\{0\}) = 0$ the triple $\vartheta_2 = (0, 0, \overline{\sigma})$ with $\overline{\sigma} = \sigma|_{(b,c)}$ is a representation for $\varphi|_{\hat{\Delta}_2}$. Let $\vec{x} := (x; y_0, \dots, y_l; z_0, \dots, z_{k-1})$ and

$$\tilde{z}_j := z_j - \int\limits_{\mathbb{R}} \frac{1}{t^{k-j}} \Big(x(t) - \sum_{i=k}^{2k-j-1} y_i t^{i-k} \Big) d\overline{\sigma}(t), \quad j = 0, \dots, k-1.$$

Now we can define a mapping ι from $\mathfrak{L}_\vartheta$ onto $\mathfrak{L}_{\vartheta_1} \oplus \mathfrak{L}_{\vartheta_2}$ by

$$\iota(\vec{x}) = \left(x|_{\operatorname{supp} \hat{\sigma}}; y_0, \dots, y_l; \tilde{z}_0, \dots, \tilde{z}_{k-1}; \Big(x(t) t^k + \sum_{i=0}^{k-1} y_i t^i \Big)\Big|_{\operatorname{supp} \overline{\sigma}} \right).$$

First note that ι is surjective. To see this, let $\vec{g} \in \mathfrak{L}_{\vartheta_1} \oplus \mathfrak{L}_{\vartheta_2}$ and assume $\vec{g} = (f_1; a_0, \dots, a_l; b_0, \dots, b_{k-1}; f_2)$. Define

$$f := f_1 \chi_{\hat{\Delta}_1} + \frac{f_2 - \sum_{i=0}^{k-1} a_i t^i}{t^k} \chi_{\hat{\Delta}_2},$$

$$z_j := b_j + \int\limits_{\mathbb{R}} \frac{1}{t^{k-j}} \Big(f(t) - \sum_{i=k}^{2k-j-1} a_i t^{i-k} \Big) d\overline{\sigma}(t).$$

then $f \in L^2(\hat{\sigma})$ and $\iota\big((f; a_0, \dots, a_l; z_0, \dots, z_{k-1})\big) = \vec{g}$.
Further ι is injective. Let $\vec{f} = (f; y_0, \dots, y_l, z_0, \dots, z_{k-1}) \in \mathfrak{L}_\vartheta$, $\vec{f} \neq 0$, and assume $\iota(\vec{f}) = 0$. It follows that $x|_{\hat{\Delta}_1} = 0$ and $y_0 = \ldots = y_l = 0$. This implies that $x(t) t^k|_{\hat{\Delta}_2} = 0$, hence $x = 0$ and therefore $z_0 = \dots = z_{k-1} = 0$. This shows that ι is bijective.
Let $\vec{x} := (x; y_0, \dots, y_l; z_0, \dots, z_{k-1}) \in \mathfrak{L}_\vartheta$ such that $\|\vec{x}\|_{\mathfrak{L}_\vartheta} = 1$. Since $\frac{1}{t^m}$, $m \in \mathbb{N}$, is bounded on $[b,c]$ there exists constants $C_1, C_2 > 0$ such that for $j = 0, \dots, k-1$ and $C := 1 + C_1 + C_2$ it holds

$$|\tilde{z}_j| \leq |z_j| + \int\limits_{\mathbb{R}} \Big| \frac{1}{t^{k-j}} x(t) \Big| d|\overline{\sigma}|(t) + \sum_{i=k}^{2k-j-1} |y_i| \int\limits_{\mathbb{R}} \Big| \frac{1}{t^{2k-j-i}} \Big| d|\overline{\sigma}|(t) \leq$$

$$\leq |z_j| + C_1 \|x\|_{L^2(|\overline{\sigma}|)} + C_2 (k-j) \max_{i \in \{k, \dots, 2k-j-1\}} |y_i| \leq 1 + C_1 + C_2(k-j) \leq C.$$

Further there exists a constant $D > 0$ such that

$$\|xt^k + \sum_{i=0}^{k-1} y_i t^i\|^2_{\mathfrak{L}_{\vartheta_2}} \leq \int_{\mathbb{R}} \left|x(t)t^k\right| d|\overline{\sigma}|(t) + \sum_{i=0}^{k-1} |y_i| \int_{\mathbb{R}} |t^i| d|\overline{\sigma}|(t) \leq D$$

Therefore it follows that

$$\|\iota(\vec{x})\|^2_{\mathfrak{L}_{\vartheta_1} \oplus \mathfrak{L}_{\vartheta_2}} = \|(x; y_0, \ldots, y_l; \tilde{z}_0, \ldots, \tilde{z}_{k-1})\|^2_{\mathfrak{L}_{\vartheta_1}} + \|xt^k + \sum_{i=0}^{k-1} y_i t^i\|^2_{\mathfrak{L}_{\vartheta_2}} =$$

$$= \|x\|^2_{L^2(|\hat{\sigma}|)} + \|(y_0, \ldots, y_l)\|^2_{\mathbb{C}^{l+1}} + \|(\tilde{z}_0, \ldots, \tilde{z}_{k-1})\|^2_{\mathbb{C}^k} + \|xt^k + \sum_{i=0}^{k-1} y_i t^i\|^2_{\mathfrak{L}_{\vartheta_2}} \leq$$

$$\leq 1 + (l+1) + kC + D.$$

This implies that ι is a bounded linear operator from $\mathfrak{L}_\vartheta$ onto $\mathfrak{L}_{\vartheta_1} \oplus \mathfrak{L}_{\vartheta_2}$. Since ι is bijective it follows that ι^{-1} is bounded.
Further the mapping ι satisfies the following diagram

$$\begin{tikzcd}
\mathfrak{L}_\vartheta \arrow[rr, "\iota"] & & \mathfrak{L}_{\vartheta_1} \oplus \mathfrak{L}_{\vartheta_2} \\
& \mathcal{P} \arrow[ul, "\iota_\vartheta"] \arrow[ur, "\iota_{\vartheta_1} \oplus \iota_{\vartheta_2}"'] &
\end{tikzcd} \tag{3.4.2}$$

To see this, let $p \in \mathcal{P}$, then

$$\iota_\vartheta(p) = \Big(\frac{p^{\{k\}}}{t^k}; \frac{p^{(0)}(0)}{0!}, \ldots, \frac{p^{(\max\{l,k-1\})}(0)}{(\max\{l,k-1\})!}; \tilde{p}_0, \ldots, \tilde{p}_{k-1}\Big)^T,$$

with $\tilde{p}_j := \tilde{p}_j^{[k,\sigma]} = \int_{\mathbb{R}} \frac{p^{\{2k-j\}}(t)}{t^{2k-j}} d\sigma(t)$, and

$$\iota_{\vartheta_1}(p) = \Big(\frac{p^{\{k\}}}{t^k}|_{\hat{\Delta}_1}; \frac{p^{(0)}(0)}{0!}, \ldots, \frac{p^{(\max\{l,k-1\})}(0)}{(\max\{l,k-1\})!}; \tilde{p}_0, \ldots, \tilde{p}_{k-1}\Big)^T,$$

$$\iota_{\vartheta_2}(p) = p|_{\hat{\Delta}_2}.$$

Since

$$\frac{p^{\{2k-1\}}(t)}{t^{2k-j}} = \frac{1}{t^{k-j}}\Big(\frac{p^{\{k\}}(t)}{t^k} - \sum_{i=k}^{2k-j-1} \frac{p^{(i)}(0)}{i!} t^{i-k}\Big), \quad j = 0, \ldots, k-1.$$

it follows that

$$\iota(\iota_\vartheta(p)) = \Big(\frac{p^{\{k\}}}{t^k}\Big|_{\hat{\Delta}_1}; \frac{p^{(0)}(0)}{0!}, \ldots, \frac{p^{(\max\{l,k-1\})}(0)}{(\max\{l,k-1\})!}; \tilde{z}_0, \ldots, \tilde{z}_{k-1}; p|_{\hat{\Delta}_2}\Big),$$

which implies diagram (3.4.2).
Now let $\vec{x}, \vec{y} \in \operatorname{ran} \iota_\vartheta$ then there exists elements $p, q \in \mathcal{P}$ such that $\vec{x} = \iota_\vartheta(p)$ and $\vec{y} = \iota_\vartheta(q)$. By diagram (3.4.2) and Lemma 1.13 (v), (vi) it follows that

$$[\vec{x}, \vec{y}]_\vartheta = [\iota_\vartheta(p), \iota_\vartheta(q)]_\vartheta = [p, q]_{\varphi|_\Delta} = [p, q]_{\varphi|_{\hat{\Delta}_1}} + [p, q]_{\varphi|_{\hat{\Delta}_2}} =$$

$$= [\iota_{\vartheta_1}(p), \iota_\vartheta(q)]_{\vartheta_1} + [\iota_{\vartheta_2}(p), \iota_{\vartheta_2}(q)]_{\vartheta_2} = [\iota(p), \iota(q)]_{\mathfrak{L}_{\vartheta_1} \oplus \mathfrak{L}_{\vartheta_2}}.$$

Therefore ι is an isometry from $(\operatorname{ran}\iota_\vartheta, [.,.]_\vartheta)$ onto $(\operatorname{ran}(\iota_{\vartheta_1}\oplus\iota_{\vartheta_2}), [.,.]_{\mathfrak{L}_{\vartheta_1}\oplus\mathfrak{L}_{\vartheta_2}})$. Since ι and ι^{-1} are continuous this property extends to the closure, i.e. the mapping

$$\iota : \left(\overline{\operatorname{ran}\iota_\vartheta}, [.,.]_\vartheta\right) \to \left(\overline{\operatorname{ran}(\iota_{\vartheta_1}\oplus\iota_{\vartheta_2})}, [.,.]_{\mathfrak{L}_{\vartheta_1}\oplus\mathfrak{L}_{\vartheta_2}}\right)$$

is an isometric isomorphism. By Lemma 2.22 (ii) ι induces a unitary operator U between the factor spaces

$$U : \overline{\operatorname{ran}\iota_\vartheta}/_{\overline{\operatorname{ran}\iota_\vartheta}^{[\circ]_\vartheta}} \to \overline{\operatorname{ran}(\iota_{\vartheta_1}\oplus\iota_{\vartheta_2})}/_{\overline{\operatorname{ran}(\iota_{\vartheta_1}\oplus\iota_{\vartheta_2})}^{[\circ]_{\mathfrak{L}_{\vartheta_1}\oplus\mathfrak{L}_{\vartheta_2}}}}.$$

It remains to show that

$$\overline{\operatorname{ran}(\iota_{\vartheta_1}\oplus\iota_{\vartheta_2})}/_{\overline{\operatorname{ran}(\iota_{\vartheta_1}\oplus\iota_{\vartheta_2})}^{[\circ]_{\mathfrak{L}_{\vartheta_1}\oplus\mathfrak{L}_{\vartheta_2}}}} = \overline{\operatorname{ran}\iota_{\vartheta_1}}/_{\overline{\operatorname{ran}\iota_{\vartheta_1}}^{[\circ]_\vartheta}} \oplus L^2(\overline{\sigma}). \tag{3.4.3}$$

First note that $\operatorname{ran}(\iota_{\vartheta_1}\oplus\iota_{\vartheta_2}) \subseteq \operatorname{ran}\iota_{\vartheta_1} \oplus L^2(\overline{\sigma})$ and therefore $\overline{\operatorname{ran}(\iota_{\vartheta_1}\oplus\iota_{\vartheta_2})} \subseteq \overline{\operatorname{ran}\iota_{\vartheta_1}} \oplus L^2(\overline{\sigma})$. To prove the other inclusion let f be a continuous function with $\operatorname{supp} f \subseteq (b,c]$ and $N := \max\{l+1, 2k\}$. Then the function

$$\tilde{f}(t) := \begin{cases} 0 & t \in [a,b] \\ \frac{f(t)}{t^N} & t \in (b,c] \end{cases}$$

is continuous on $[a,c]$. Now we can approximate the function $\tilde{f}$ uniformly on $[a,c]$ by polynomials, i.e. there exists $\tilde{p}_n \in \mathbb{C}[z]$ such that $\tilde{p}_n \to \tilde{f}$ uniformly on $[a,c]$. Now it follows that $p_n := t^N\tilde{p}_n \to t^N\tilde{f} = f$ uniformly on $[a,c]$. Clearly all derivatives of p_n up to the order $N-1$ vanishes at $t=0$. Hence

$$(\iota_{\vartheta_1}\oplus\iota_{\vartheta_2})p_n = \Big(p_n|_{[a,b]}; 0,\dots,0; \int_a^b \frac{p_n(t)}{t^{2k}}d\sigma(t),\dots,\int_a^b \frac{p_n(t)}{t^{k+1}}d\sigma(t)\Big) \oplus (p_n|_{(b,c]}).$$

Since p_n converges uniformly to f and $\operatorname{supp} f \subseteq (b,c]$ we obtain $(\iota_{\vartheta_1}\oplus\iota_{\vartheta_2})(p_n) \to \vec{0}\oplus f \in \overline{\operatorname{ran}(\iota_{\vartheta_1}\oplus\iota_{\vartheta_2})}$. If $f \in L^2(\overline{\sigma})$ then we can approximate f by continuous functions in the L^2 sense. This means that there exists continuous f_n with $\operatorname{supp} f_n \subseteq (b,c]$ such that $f_n \to f$ with respect to the L^2 norm and therefore $\vec{0}\oplus f_n \to \vec{0}\oplus f$ in $\mathfrak{L}_{\vartheta_1}\oplus\mathfrak{L}_{\vartheta_2}$. This yields

$$\{\vec{0}\} \oplus L^2(\overline{\sigma}) \subseteq \overline{\operatorname{ran}(\iota_{\vartheta_1}\oplus\iota_{\vartheta_2})}.$$

If $f \in (\operatorname{ran}\iota_{\vartheta_1}) \oplus L^2(\overline{\sigma})$ then there exists a polynomial $p \in \mathcal{P}$ and an element $g \in L^2(\overline{\sigma})$ such that $f = \iota_{\vartheta_1}p \oplus g$. Further it holds

$$f = (\iota_{\vartheta_1}\oplus\iota_{\vartheta_2})p + \big(\vec{0}\oplus(g - \iota_{\vartheta_2}p|_{(b,c]})\big) \in \operatorname{ran}(\iota_{\vartheta_1}\oplus\iota_{\vartheta_2}) + \{\vec{0}\}\oplus L^2(\overline{\sigma}) \subseteq \overline{\operatorname{ran}(\iota_{\vartheta_1}\oplus\iota_{\vartheta_2})}.$$

Now clearly $\overline{\operatorname{ran}\iota_{\vartheta_1}} \oplus L^2(\overline{\sigma}) \subseteq \overline{\operatorname{ran}(\iota_{\vartheta_1}\oplus\iota_{\vartheta_2})}$, thus we have

$$\overline{\operatorname{ran}\iota_{\vartheta_1}} \oplus L^2(\overline{\sigma}) = \overline{\operatorname{ran}(\iota_{\vartheta_1}\oplus\iota_{\vartheta_2})}. \tag{3.4.4}$$

Since $\left(\overline{\operatorname{ran}\iota_{\vartheta_1}} \oplus L^2(\overline{\sigma})\right)^{[\circ]_{\mathfrak{L}_{\vartheta_1}\oplus\mathfrak{L}_{\vartheta_2}}} = \left(\overline{\operatorname{ran}\iota_{\vartheta_1}}\right)^{[\circ]_{\vartheta_1}} \oplus \{0\}$ equation (3.4.3) follows. This shows the existence of a unitary operator $U : \mathcal{K}_\vartheta \to \mathcal{K}_{\vartheta_1}\oplus\mathcal{K}_{\vartheta_2}$.
Owing to Theorem 2.18 there exists isometries $\hat{\iota}_{\vartheta,\vartheta_{1,2}} : \mathcal{P} \to \mathcal{K}_{\vartheta,\vartheta_{1,2}}$ with dense

range such that diagram (2.3.1) holds. Therefore we have $(A_{\vartheta_1} \oplus A_{\vartheta_2})\big((\hat\imath_{\vartheta_1} \oplus \hat\imath_{\vartheta_2})(p)\big) = (\hat\imath_{\vartheta_1} \oplus \hat\imath_{\vartheta_2})\big(A_t(p)\big)$ and it follows that

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\hat\imath_\vartheta"] \arrow[d, "A_t"'] & \mathcal{K}_\vartheta \arrow[r, "U"] \arrow[d, "A_\vartheta"'] & \mathcal{K}_{\vartheta_1} \oplus \mathcal{K}_{\vartheta_2} \arrow[d, "A_{\vartheta_1} \oplus A_{\vartheta_2}"] & \mathcal{P} \arrow[l, "\hat\imath_{\vartheta_1} \oplus \hat\imath_{\vartheta_2}"'] \arrow[d, "A_t"] \\
\mathcal{P} \arrow[r, "\hat\imath_\vartheta"'] & \mathcal{K}_\vartheta \arrow[r, "U"'] & \mathcal{K}_{\vartheta_1} \oplus \mathcal{K}_{\vartheta_2} & \mathcal{P} \arrow[l, "\hat\imath_{\vartheta_1} \oplus \hat\imath_{\vartheta_2}"]
\end{tikzcd}$$

By continuity we obtain $(A_{\vartheta_1} \oplus A_{\vartheta_2})U = A_\vartheta U$.
Now consider the case $\alpha \neq 0$. By definition $\mathcal{K}_{\varphi|_\Delta} = \mathcal{K}_{\tau_\alpha \varphi|_\Delta}$ and similar $\mathcal{K}_{\varphi|_{\hat\Delta_1}} = \mathcal{K}_{\tau_\alpha \varphi|_{\hat\Delta_1}}$. Since $M \cap \hat\Delta_2 = \emptyset$ there exists a signed measure σ with compact support that coincides with $\varphi|_{\hat\Delta_2}$ on $\hat\Delta_2$ and the space $\mathcal{K}_{\varphi|_{\hat\Delta_2}}$ can be identified with $L^2(\sigma)$. Clearly the space $\mathcal{K}_{\tau_\alpha\varphi|_{\hat\Delta_2}}$ can be identified with $L^2(\tilde\sigma)$ where $\tilde\sigma(t) = \sigma(t+\alpha)$. The existence of a unitary mapping U between $\mathcal{K}_{\varphi|_\Delta}$ and $\mathcal{K}_{\varphi|_{\hat\Delta_1}} \oplus \mathcal{K}_{\varphi|_{\hat\Delta_2}}$ is obtained by the considerations above and special case $\alpha = 0$. The general case for $r_i > 2$ follows by induction. ❑

3.13 Remark. Let $\varphi \in \mathcal{F}(\mathbb{R})$, M a finite subset of $\mathbb{R}$ such that $\varphi \in \mathcal{F}(\mathbb{R}, M)$ and $\Delta_1, \dots, \Delta_n$, $n \in \mathbb{N}$, be a φ-M-decomposition of $\mathbb{R}$. Since $\varphi|_{\Delta_i} \in \mathcal{F}(\mathbb{R})$, $1 \leq i \leq n$, it is clarified what model we associate to this distribution. Therefore we can define a model for φ-M-decompositions without the additional requirement $|M \cap \Delta_i| \leq 1$, $1 \leq 1 \leq n$, analogously to Definition 3.11.

3.14 Lemma. *Let $\big(\mathfrak{K}_i, [.,.]_{\mathfrak{K}_i}\big)$, $i = 1, \dots, n$, be Kreĭn spaces and $A_i \in \mathcal{L}(\mathfrak{K}_i)$, $i = 1, \dots, n$, definitizable selfadjoint operators with respective definitizing polynomials p_i, $i = 1, \dots, n$. Denote by $N(p_i)$ the zero set of the polynomial p_i, $i = 1, \dots, n$, and set $m_i := \inf N(p_i)$, $M_i := \sup N(p_i)$, $i = 1, \dots, n$, and $M_0 := -\infty$, $m_{n+1} := +\infty$. If*

(i) $M_{i-1} < m_i$, $i = 1, \dots, n$,

(ii) $M_{i-1} < \min \sigma(A_i) \leq \max \sigma(A_i) < m_{i+1}$, $i = 1, \dots, n$, *and*

(iii) $\operatorname{sign} p_i|_{(M_i, m_{i+1})} = \operatorname{sign} p_{i+1}|_{(M_i, m_{i+1})}$, $i = 1, \dots, n-1$,

then the operator $\bigoplus_{i=1}^n A_i \in \mathcal{L}\big(\bigoplus_{i=1}^n \mathfrak{K}_i\big)$ is definitizable with definitizing polynomial $p := \delta \cdot \prod_{i=1}^n p_j$, where $\delta := \operatorname{sign} p_n|_{(-\infty, m_n)}$.

Proof. We prove the existence of a definitizing polynomial p for $A_1 \oplus A_2$. Then we show that the operator $A_1 \oplus A_2$ with definitizing polynomial p satisfies the requirements (i)-(iii) of the lemma. The general statement will follow by induction.
Let $\delta := \operatorname{sign} p_2|_{(-\infty, m_2)}$, and define $p := \delta p_1 \cdot p_2$, then for $x = x_1 + x_2 \in \mathfrak{K}_1 \oplus \mathfrak{K}_2$ it follows that

$$\begin{aligned}
[p(A_1 \oplus A_2)x, x]_{\mathfrak{K}_1 \oplus \mathfrak{K}_2} &= \left[\begin{pmatrix} p(A_1) & 0 \\ 0 & p(A_2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right]_{\mathfrak{K}_1 \oplus \mathfrak{K}_2} = \\
&= \big[p_1(A_1) \cdot \delta p_2(A_1) x_1, x_1\big]_{\mathfrak{K}_1} + \big[\delta p_1(A_2) \cdot p_2(A_2) x_2, x_2\big]_{\mathfrak{K}_2}.
\end{aligned}$$

The function $z \mapsto \delta p_2(z)$ is positive on $(-\infty, m_2)$. By (ii) we have $\sigma(A_1) < m_2$ and therefore there exists a open set $U \supseteq \sigma(A_1)$ such that $f : z \mapsto \sqrt{\delta p_2(z)}$ is holomorphic on U. By the Riesz-Dunford functional calculus there exists a linear operator B_1 such that $B_1^2 = \delta p_2(A_1)$. Now we have

$$\big[p_1(A_1) \cdot \delta p_2(A_1)x_1, x_1\big]_{\mathfrak{K}_1} = \big[p_1(A_1)B_1^2 x_1, x_1\big]_{\mathfrak{K}_1} = \big[p_1(A_1)B_1 x_1, B_1 x_1\big]_{\mathfrak{K}_1} \geq 0,$$

since p_1 is definitizing for A_1. By (iii) it follows that $\operatorname{sign} p_1|(M_1, \infty) = \delta$. Therefore the function $z \mapsto \delta p_1(z)$ is positive on (M_1, ∞). Again there exists an open set $U \supseteq \sigma(A_2)$ such that $f : z \mapsto \sqrt{\delta p_1(z)}$ is holomorphic on U. The Riesz-Dunford functional calculus ensures the existence of an operator B_2 such that $B_2^2 = \delta p_1(A_2)$. This yields

$$\big[\delta p_1(A_2) \cdot p_2(A_2)x_2, x_2\big]_{\mathfrak{K}_2} = \big[B_2^2 p_2(A_2)x_2, x_2\big]_{\mathfrak{K}_2} = \big[p_2(A_2)B_2 x_2, B_2 x_2\big]_{\mathfrak{K}_2} \geq 0,$$

since p_2 is definitizing for A_2. Now we have shown that

$$[p(A_1 \oplus A_2)x, x]_{\mathfrak{K}_1 \oplus \mathfrak{K}_2} \geq 0, \quad x \in \mathfrak{K}_1 \oplus \mathfrak{K}_2.$$

Therefore $A_1 \oplus A_2$ is definitizable with definitizing polynomial $p = \delta p_1 \cdot p_2$. Clearly $\min N(p) = m_1$, $\max N(p) = M_2$ and $\sigma(A_1) \leq \sigma(A_1 \oplus A_2) \leq \sigma(A_2)$. Since δp_1 is positive on (M_1, ∞) by (iii) it follows that

$$\operatorname{sign} p|_{(M_2, m_3)} = \operatorname{sign} \delta p_1 p_2|_{(M_2, m_3)} = \operatorname{sign} p_2|_{(M_2, m_3)} = \operatorname{sign} p_3|_{(M_2, m_3)}.$$

We have shown that the operator $A_1 \oplus A_2$ with definitizing polynomial p satisfies the requirements (i)-(iii). ❑

3.15 Lemma. *Let $(\mathfrak{K}_1, [.,.]_{\mathfrak{K}_1})$, $(\mathfrak{K}_2, [.,.]_{\mathfrak{K}_2})$ be Kreĭn spaces and $A_i \in \mathcal{B}(\mathfrak{K}_i)$, $i = 1, 2$, be definitizable selfadjoint operators with only real spectrum such that the operator $A := A_1 \oplus A_2 \in \mathcal{B}(\mathfrak{K}_1 \oplus \mathfrak{K}_2)$ is definitizing with real definitizing polynomial p. Further assume that there exists a closed interval $\Delta \in \Omega_p$ such that $\sigma(A_1) \subseteq \Delta$ and $\sigma(A_2) \subseteq \mathbb{R} \setminus \mathring{\Delta}$ and $E(\partial\Delta) = 0$, where E denotes the spectral function of A. If A_1 and A_2 are cyclic with generating elements u_1 and u_2, respectively, then A is cyclic with generating element $u := u_1 \oplus u_2$.*

Proof. Clearly, A is selfadjoint in the Kreĭn space $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ and has only real spectrum. Therefore there exists a spectral function E, see Theorem 2.39. Let $\mathfrak{L} := \operatorname{cls}\{A^n u : n \in \mathbb{N}_0\}$. Since A is a bounded operator it holds that

$$\mathfrak{L} = \operatorname{cls}\{(A - z)^{-1}u : z \in \rho(A)\},$$

Further $E(\Delta)$ is the strong limit of a contour integral over the resolvent and therefore it holds $E(\Delta)\mathfrak{L} \subseteq \mathfrak{L}$. By the same reasoning we have that $E(\mathbb{R}\setminus\mathring{\Delta})\mathfrak{L} \subseteq \mathfrak{L}$. Since $E(\partial\Delta) = 0$ it follows that

$$\mathfrak{L} = E(\Delta)\mathfrak{L}[\dot{+}]_{\mathfrak{K}_1 \oplus \mathfrak{K}_2}\big(I - E(\Delta)\big)\mathfrak{L}.$$

Then we have

$$\begin{aligned} E(\Delta)\mathfrak{L} &= E(\Delta)\operatorname{cls}\{A^n u : n \in \mathbb{N}_0\} = \operatorname{cls}\{E(\Delta)A^n u : n \in \mathbb{N}_0\} = \\ &= \operatorname{cls}\{A_1^n u_1 : n \in \mathbb{N}_0\} = \mathfrak{K}_1. \end{aligned}$$

Analogously it follows that $\big(I - E(\Delta)\big)\mathfrak{L} = \mathfrak{K}_2$. This implies that $\mathfrak{L} = \mathfrak{K}_1 \oplus \mathfrak{K}_2$, i.e. A is cyclic with generating element u. ❑

3.16 Remark. If X is a Banach space and $T_1, T_2 \in \mathcal{B}(X)$ are cyclic operators with generating element u_1 and u_2, respectively, such that their spectra can be separated, i.e. there exists closed disjoint subsets Δ_1, Δ_2 of $\mathbb{C}$ such that $\sigma(T_i) \subseteq \Delta_i$, $i = 1, 2$, then $T_1 \oplus T_2$ is cyclic with generating element $u_1 \oplus u_2$. The proof works in the same way, just the spectral function E is replaced by the *Riesz Idempotent*, see [C1, p. 210].

3.17 Proposition. *Let $\varphi \in \mathcal{F}(\mathbb{R})$ and $\mathcal{K}_\varphi$ the corresponding model space with model operator A_φ. Then A_φ is bounded selfadjoint and definitizing in $\mathcal{K}_\varphi$. Further A_φ is cyclic with generating element $\iota_\varphi(1)$.*

Proof. Let M be a finite subset of $\mathbb{R}$ such that $\varphi \in \mathcal{F}(\mathbb{R}, M)$ and $\Delta_1, \cdots, \Delta_n$, $n \in \mathbb{N}$, be a φ-M-decomposition of $\mathbb{R}$. By definition we have $A_\varphi = \bigoplus_{i=1}^n A_{\varphi|_{\Delta_i}}$. As an orthogonal sum of bounded selfadjoint operators A_φ is bounded and selfadjoint in $\mathcal{K}_\varphi$. In order to prove that A_φ is definitizable we have to show that the operators $A_{\varphi|_{\Delta_i}}$, $i = 1, \ldots, n$, satisfy the requirements of Lemma 3.14. Owing to Proposition 2.37 it follows that $\sigma(A_{\varphi|_{\Delta_i}}) \subseteq \Delta_i$ for $i = 1, \ldots, n$.
For $i \in \{1, \ldots, n\}$ denote by σ_i the measure with $\operatorname{supp}(\sigma_i) \subseteq \Delta_i$ corresponding to the distribution $\varphi|_{\Delta_i}$. If $\Delta_i \cap M = \alpha$, $\alpha \in \mathbb{R}$, then by definition $A_{\varphi_{\Delta_i}} = A_{\tau_\alpha \varphi|_{\Delta_i}} + \alpha I$. Therefore by Theorem 2.18 the polynomial $p(z) = \epsilon(z-\alpha)^{2N+2+\nu}$, with $\epsilon = \operatorname{sign}(\sigma_i|_{(\alpha,\infty)})$, $\nu = \frac{1}{2}|\operatorname{sign}(\sigma_i|_{(\alpha,\infty)}) - \operatorname{sign}(\sigma_i|_{(-\infty,\alpha)})|$ and $N \in \mathbb{N}_0$, is definitizing for A_{φ_i}.
In the case $\Delta_i \cap M = \emptyset$ either $p(z) = 1$ or $p(z) = -1$ is a definitizing polynomial for $A_{\varphi|_{\Delta_i}}$.
It remains to show that A_φ is cyclic. Let $A := A_{\varphi_1} \oplus A_{\varphi_2}$, then clearly A is selfadjoint and has real spectrum. By Lemma 3.14 A is definitizing and since A_{φ_1} and A_{φ_2} have real definitizing polynomials there exists a real definitizing polynomial p for A. By Lemma 3.3 and the choice of the φ-M-decomposition of $\mathbb{R}$ there exists a closed interval $\Delta \in \Omega_p$ such that $\sigma(A_{\varphi_1}) \subseteq \Delta$ and $\sigma(A_{\varphi_2}) \subseteq \mathbb{R} \setminus \mathring{\Delta}$. For example $\Delta := \overline{\Delta}_{\varphi_1}$ is a possible choice. Denote by E the spectral function of A, then $E(\partial\Delta) = 0$, since the measure corresponding to the distribution φ has no mass at $\partial\Delta$. By Corollary 2.21 $\iota_{\varphi_i}(1)$ is a generating element for A_{φ_i}, $i = 1, 2$. Now Lemma 3.15 implies that A is cyclic with generating element $\iota_{\varphi_1}(1) \oplus \iota_{\varphi_2}(1) = \iota_{\varphi_1 \oplus \iota_{\varphi_2}}(1)$. Repeating this argument with A and A_{φ_3} etc., it follows that that A_φ is cyclic with generating element $\iota_\varphi(1)$. ❑

3.5 Model for distributions of class $\mathcal{F}$

Now we are able to define the model space and model operator for a distribution $\phi \in \mathcal{F}$.

3.18 Definition. Let $\phi \in \mathcal{F}$ with $\phi = (\varphi, \psi)$ where $\varphi \in \mathcal{F}(\mathbb{R})$ and $\psi \in \mathcal{F}(\mathbb{C} \setminus \mathbb{R})$. Denote by $(\mathcal{K}_\varphi, A_\varphi, \iota_\varphi)$ and $(\mathcal{K}_\psi, A_\psi, \iota_\psi)$ the model space, model operator and embedding corresponding to the distribution φ and ψ respectively. Then we define

$$\mathcal{K}_\phi := \mathcal{K}_\varphi \oplus \mathcal{K}_\psi, \quad A_\phi := A_\varphi \oplus A_\psi, \quad \iota_\phi := \iota_\varphi \oplus \iota_\psi.$$

3.19 Theorem. *Let $\phi \in \mathcal{F}$ then $\mathcal{K}_\phi$ is a Kreĭn space and A_ϕ is a bounded, selfadjoint and definitizable operator on $\mathcal{K}_\phi$. Further A_ϕ is cyclic with generating* *element* $\iota_\phi(1)$ *and the following diagram commutes*

$$\begin{tikzcd} \mathcal{K}_\phi \arrow[r, "A_\phi"] & \mathcal{K}_\phi \\ \mathcal{P} \arrow[u, "\iota_\phi"] \arrow[r, "A_t"'] & \mathcal{P}. \arrow[u, "\iota_\phi"'] \end{tikzcd}$$

Proof. We can write the distribution ϕ as (φ, ψ) where $\varphi \in \mathcal{F}(\mathbb{R})$ and $\psi \in \mathcal{F}(\mathbb{C} \setminus \mathbb{R})$. Denote by $\mathcal{K}_\varphi$, A_φ, ι_φ and $\mathcal{K}_\psi$, A_ψ, ι_ψ the Kreĭn space, model operator and model embedding corresponding to φ and ψ respectively according to definition 3.11 and 3.9. The space $\mathcal{K}_\phi$ is the orthogonal sum of Kreĭn spaces and therefore a Kreĭn space. By Proposition 3.17 and 3.10 the operators A_φ and A_ψ are bounded, selfadjoint and definitizable. Clearly the operator $A_\phi = A_\varphi \oplus A_\psi$ is bounded and selfadjoint in $\mathcal{K}_\varphi \oplus \mathcal{K}_\psi$.
To show that A_ϕ is definitizable let p_1 be a definitizable polynomial for A_φ. Since $\mathcal{K}_\psi$ is finite dimensional there exists a polynomial p_2 such that $p_2(A_\psi) = 0$ and clearly p_2 is definitizing for A_ψ. Then for $x \in \mathcal{K}_\phi$ with $x = x_1 \oplus x_2 \in \mathcal{K}_\varphi \oplus \mathcal{K}_\psi$ it follows that

$$\begin{aligned} \big[(p_1 p_2 \overline{p}_2)(A_\phi)x, x\big]_{\mathcal{K}_\phi} &= [(p_1 p_2 \overline{p}_2)(A_\varphi)x_1, x_1]_{\mathcal{K}_\varphi} + [(p_1 p_2 \overline{p}_2)(A_\psi)x_2, x_2]_{\mathcal{K}_\psi} = \\ &= \underbrace{\big[p_1(A_\varphi)p_2(A_\varphi)x_1, p_2(A_\varphi)x_1\big]_{\mathcal{K}_\varphi}}_{\geq 0} + \underbrace{\big[p_1(A_\psi)p_2(A_\psi)x_2, p_2(A_\psi)x_2\big]_{\mathcal{K}_\psi}}_{=0} \geq 0, \end{aligned}$$

since p_1 is definitizing for A_φ. Therefore $p(z) := p_1(z)p_2(z)\overline{p}_2(z)$ is a definitizing polynomial for A_ϕ.
Since the real and non-real spectrum of a selfadjoint operator in a Kreĭn space can be separated, it follows from Remark 3.16 that A_ϕ is cyclic with generating element $\iota_\phi(1)$. ❑

3.20 Corollary. *Let* $\phi \in \mathcal{F}$*, then the model embedding* ι_ϕ *has dense range in the model space* $\mathcal{K}_\phi$*.*

Proof. By Theorem 3.19 it follows that A_ϕ is cyclic with generating element $\iota_\phi(1)$ and that

$$\iota_\phi(t \mapsto t^n) = A_\phi^n \iota_\phi(1), \quad n \in \mathbb{N}_0.$$

This implies that $\iota_\phi(\mathcal{P}) = \operatorname{span}\{A_\phi^n \iota_\phi(1) : \ n \in \mathbb{N}_0\}$ and since A_ϕ is cyclic it follows that $\iota_\phi(\mathcal{P})$ is dense in $\mathcal{K}_\phi$. ❑

Chapter 4

Cyclic definitizable selfadjoint operators in Kreĭn spaces

In order to prove the main assertions of this chapter, some basic definitions of spectral theory are needed. Let X be a Banach space and $T \in B(X)$. For a closed and relatively open subset Δ of $\sigma(A)$ denote by $E_R(\Delta)$ the *Riesz Idempotent*, i.e.

$$E_R(\Delta) = E_R(\Delta; A) = \frac{1}{2\pi i} \int_\Gamma (z - A)^{-1} dz,$$

where Γ is a positively oriented Jordan system[1] such that Δ is in the inside[2] of Γ and $\sigma(A) \setminus \Delta$ is in the outside[3] of Γ (see [C1, p. 210]). For $\lambda \in \mathbb{C}$ the number

$$\nu(\lambda) := \inf\{n \in \mathbb{N}_0 : \ (T - \lambda)^n E_R(\{\lambda\}) = 0\} \in \mathbb{N}_0 \cup \{+\infty\}$$

is called the *Riesz index* corresponding to λ with respect to T. A point with finite positive Riesz index is necessarily an eigenvalue, but it could have infinite multiplicity.

The Riesz Idempotent is a useful tool to separate the real and non-real spectrum of a definitizable selfadjoint operator in a Kreĭn space. The non-real part of the spectrum of an operator A is denoted by $\sigma_0(A)$. A proof of the following proposition (even in the unbounded case) can be found in [J, Lemma 1, p. 122]:

4.1 Proposition. *Let A be a bounded definitizable selfadjoint operator in a Kreĭn space $(\mathfrak{K}, [.,.]_{\mathfrak{K}})$. Then for a non-real number z_0 we have $z_0 \in \sigma(A)$ if and only if z_0 is a zero of each real definitizing polynomial of A.*
In particular, $\sigma_0(A)$ consists of no more than a finite number of points, symmetric with respect to the real axis.

[1] If $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ is a collection of closed rectifiable curves, then Γ is *positive oriented* if (a) $\{\gamma_i\} \cap \{\gamma_j\} = \emptyset$, $i \neq j$; (b) for $a \in \mathbb{C} \setminus \bigcup_{j=1}^m \{\gamma_j\}$, $n(\Gamma; a) := \sum_{j=1}^m n(\gamma_j; a)$ is either 0 or 1, where $n(\gamma; a)$ denotes the winding number of a closed rectifiable curve γ in $\mathbb{C}$; (c) each γ_j is a simple curve.

[2] The *inside* of Γ, $\operatorname{ins}\Gamma$, is defined by $\operatorname{ins}\Gamma := \{a : n(\Gamma; a) = 1\}$.

[3] The *outside* of Γ, $\operatorname{out}\Gamma$, is defined by $\operatorname{out}\Gamma := \{a : n(\Gamma; a) = 0\}$.

Moreover, every real definitizing polynomial p of A has a real definitizing divisor p_0 such that the set of non-real zeros of p_0 coincides with $\sigma_0(A)$ and the orders of the non-real zeros of p_0 coincide with their Riesz indices with respect to A, respectively.

4.2 Remark. Let A be a bounded definitizable selfadjoint operator in a Kreĭn space $(\mathfrak{K}, [.,.]_{\mathfrak{K}})$ and denote by E_R the Riesz Idempotent, i.e. the mapping defined for a closed and relatively open subset Δ of $\sigma(A)$ by According to Proposition 4.1 the decomposition

$$\mathfrak{K} = \big(I - E_R(\sigma_0(A))\big)[\dot{+}]_{\mathfrak{K}} E_R\big(\sigma_0(A)\big)\mathfrak{K}$$

reduces the operator A. The operator $A|_{E_R(\sigma_0(A))\mathfrak{K}}$ is bounded and its spectrum consists of a finite number of eigenvalues with finite Riesz index. The operator $A|_{I-E_R(\sigma_0(A))\mathfrak{K}}$ is bounded selfadjoint with only real spectrum and there exists a real definitizing polynomial for $A|_{I-E_R(\sigma_0(A))\mathfrak{K}}$ with only real zeros.

4.3 Proposition. *Let A be a bounded definitizable selfadjoint operator in a Kreĭn space $(\mathfrak{K}, [.,.]_{\mathfrak{K}})$. Then the linear functional*

$$\zeta_u : \begin{cases} \mathcal{P} \to \mathbb{C} \\ p \mapsto [p(A)u, u]_{\mathfrak{K}} \end{cases}, \quad u \in \mathfrak{K}, \tag{4.0.1}$$

induces a unique element $\phi_u := (\zeta_{\mathbb{R},u}, \zeta_{\mathbb{C}\setminus\mathbb{R},u}) \in \mathcal{F}$, such that

$$\zeta_u(p) = \zeta_{\mathbb{R},u}(p) + \zeta_{\mathbb{C}\setminus\mathbb{R},u}(p), \quad p \in \mathcal{P}.$$

In particular $\zeta_{\mathbb{R},u} \in \mathcal{F}(\mathbb{R}, c(A))$ and $\zeta_{\mathbb{C}\setminus\mathbb{R},u} \in \mathcal{F}(\mathbb{C}\setminus\mathbb{R}, \sigma_0(A))$, and the sets $c(A)$ and $\sigma_0(A)$ are minimal (see Definition 1.6).
Denote by E the spectral function for $A|_{E_R(\sigma(A)\cap\mathbb{R})\mathfrak{K}}$ and by μ_Z the Borel measure on a component Z of $\mathbb{R}\setminus c(A)$ corresponding to $\zeta_{\mathbb{R},u}$ (see Proposition 1.7). Then $\mu_Z(\Delta) = [E(\Delta)u, u]_{\mathfrak{K}} =: E_{u,u}(\Delta)$ for $\Delta \in \mathfrak{B}(\mathbb{R})$, $\Delta \subseteq Z$. Therefore we have

$$\zeta_{\mathbb{R},u}(f) = \int_Z f dE_{u,u}, \quad f \in \mathcal{D}(\mathbb{R}) \text{ with } \operatorname{supp} f \subseteq Z. \tag{4.0.2}$$

Proof. According to Remark 4.2 for $p \in \mathcal{P}$ it holds

$$[p(A)u, u]_{\mathfrak{K}} = [p(A)E_R(\sigma(A)\cap\mathbb{R})u, u]_{\mathfrak{K}} + [p(A)E_R(\sigma_0(A))u, u]_{\mathfrak{K}}.$$

The latter inner product can be written as

$$[p(A)E_R(\sigma_0(A))u, u]_{\mathfrak{K}} = \sum_{\beta\in\sigma(A)\cap\mathbb{C}^+} \Big([p(A)E_R(\{\beta\})u, u]_{\mathfrak{K}} + [p(A)E_R(\{\overline{\beta}\})u, u]_{\mathfrak{K}}\Big).$$

Recall that the non-real spectrum of A consists only of eigenvalues with finite Riesz index, hence for every $\beta \in \sigma(A)\setminus\mathbb{R}$ there exists $k_\beta \in \mathbb{N}_0$ such that $(A-\beta I)^{k_\beta} E_R(\{\beta\}) = 0$. Note that $k_\beta = k_{\overline{\beta}}$. Every polynomial $p \in \mathcal{P}$ can be written as

$$p(x) = \sum_{j=0}^{\deg p} \frac{p^{(j)(\beta)}}{j!}(x-\beta)^j,$$

which implies that

$$[p(A)E_R(\{\beta\})u,u]_{\mathfrak{K}} = \sum_{j=0}^{k_\beta-1} \frac{p^{(j)}(\beta)}{j!}[(A-\beta I)^j E_R(\{\beta\})u,u]_{\mathfrak{K}}, \quad p \in \mathcal{P}.$$

Since for $\beta \in \sigma_0(A)$ it holds $E_R(\{\beta\})A = AE_R(\{\beta\})$, $E_R(\{\beta\})^* = E_R(\{\overline{\beta}\})$ and A is selfadjoint it follows that

$$[(A-\overline{\beta}I)^j E_R(\{\overline{\beta}\})u,u]_{\mathfrak{K}} = \overline{[(A-\beta I)^j E_R(\{\beta\})u,u]}_{\mathfrak{K}}.$$

This gives

$$[p(A)E_R(\sigma_0(A))u,u]_{\mathfrak{K}} = \sum_{\beta\in\sigma(A)\cap\mathbb{C}^+} \sum_{j=0}^{k_\beta-1} \Big(\frac{p^{(j)}(\beta)}{j!}[(A-\beta I)^j E_R(\{\beta\})u,u]_{\mathfrak{K}} + \\ + \frac{p^{(j)}(\overline{\beta})}{j!}\overline{[(A-\beta I)^j E_R(\{\beta\})u,u]}_{\mathfrak{K}} \Big).$$

Since the non-real spectrum of A is a finite set this equation implies that the functional

$$p \mapsto [p(A)E_R(\sigma_0(A))u,u]_{\mathfrak{K}}, \quad p \in \mathcal{P}, \tag{4.0.3}$$

is of the form (1.3.1). Since every element of $\mathcal{F}(\mathbb{C}\setminus\mathbb{R})$ is uniquely determined by its restriction to $\mathcal{P}$ it follows that this functional induces a unique element $\zeta_{\mathbb{C}\setminus\mathbb{R},u} \in \mathcal{F}(\mathbb{C}\setminus\mathbb{R},\sigma_0(A))$ such that

$$\zeta_{\mathbb{C}\setminus\mathbb{R},u}(p) = [p(A)E_R(\sigma_0(A))u,u]_{\mathfrak{K}}, \quad p \in \mathcal{P}.$$

Clearly there is no proper subset N of $\sigma_0(A)$ such that $\zeta_{\mathbb{C}\setminus\mathbb{R},u} \in \mathcal{F}(\mathbb{C}\setminus\mathbb{R},N)$. It remains to show that the functional

$$p \mapsto [p(A)E_R(\sigma(A)\cap\mathbb{R})u,u]_{\mathfrak{K}}, \quad p \in \mathcal{P}, \tag{4.0.4}$$

belongs to $\mathcal{F}(\mathbb{R},c(A))$. Define $A_0 := A|_{E_R(\sigma(A)\cap\mathbb{R})\mathfrak{K}}$, then A_0 is a bounded definitizing selfadjoint operator in $E_R(\sigma(A)\cap\mathbb{R})\mathfrak{K}$. Set $u_0 := E_R(\sigma(A)\cap\mathbb{R})u$, then it is sufficient to show that the functional

$$\tilde{\zeta}_{u_0} : \begin{cases} \mathcal{P} \to \mathbb{C} \\ \ p \mapsto [p(A_0)u_0,u_0]_{\mathfrak{K}} \end{cases},$$

induces a unique element $\tilde{\zeta}_{\mathbb{R},u_0} \in \mathcal{F}(\mathbb{R},c(A))$, such that $\tilde{\zeta}_{\mathbb{R},u_0}(p) = \tilde{\zeta}_{u_0}(p)$ for $p \in \mathcal{P}$. Denote by p_0 a real definitizing polynomial for A_0 with only real zeros $\alpha_1,\dots,\alpha_n$ of orders $\mu_1,\dots,\mu_n$. By uniqueness of the partial fraction expansion of $\frac{1}{p_0(t)}$ there exists unique real constants c_{ij} such that

$$\frac{1}{p_0(t)} = \sum_{i=1}^{n}\sum_{j=1}^{\mu_i} \frac{c_{ij}}{(t-\alpha_i)^j}, \quad t \in \mathbb{R}\setminus\{\alpha,\dots,\alpha_n\}.$$

For arbitrary $p \in \mathcal{P}$ we define the polynomials

$$g(t;p) := p_0(t)\sum_{i=1}^{n}\sum_{j=1}^{\mu_i} c_{ij}(t-\alpha_i)^{-j}\left(p(\alpha_i) + \dots + \frac{p^{(j-1)}(\alpha_i)}{(j-1)!}(t-\alpha_i)^{j-1} \right),$$

$$h(t;p) := \sum_{i=1}^{n}\sum_{j=1}^{\mu_i} c_{ij}(t-\alpha_i)^{-j} p^{\{\alpha_i,j\}}(t).$$

Therefore we have

$$p(t) - p_0(t)h(t;p) = p(t) - p_0(t)\sum_{i=1}^{n}\sum_{j=1}^{\mu_i} c_{ij}(t-\alpha_i)^{-j}\cdot$$
$$\cdot\left(p(t) - \sum_{k=0}^{j-1}\frac{p^{(k)}(\alpha_i)}{k!}(t-\alpha_i)^k\right) = g(t;p),$$

and hence any $p \in \mathcal{P}$ can be written as

$$p(t) = g(t;p) + p_0(t)h(t;p).$$

Choose a bounded open interval Δ such that $\sigma(A_0) \subseteq \Delta$ and define the set S by

$$S := \{p \in \mathcal{P} : \sup_{\substack{t\in\Delta \\ 0\le k\le\mu}} |p^{(k)}(t)| \le 1\},$$

where $\mu := \max\{\mu_1, \ldots, \mu_n\}$. The polynomial $g(.;p)$ depends on p only through the numbers $p^{(j)}(\alpha_i),\ i = 1, \ldots, n,\ j = 0, \ldots, \mu_i - 1$, therefore $\|g(A_0;p)\| < \infty$ for $p \in S$ which implies that

$$\sup\{|[g(A_0;p)u_0, u_0]_{\mathfrak{K}}| :\ p \in S\} < \infty.$$

To estimate the remaining terms $|[p_0(A_0)h(A_0;p)u_0, u_0]|$ we introduce on $\mathfrak{K}$ the inner product $\{.,.\} := [p_0(A_0).,.]_{\mathfrak{K}}$. This inner product is nonnegative, since p_0 is a definitizing polynomial for A_0 and further A_0 is symmetric with respect to $\{.,.\}$. Therefore $\mathcal{H} := \overline{(\mathfrak{K}/\mathfrak{K}^0, \{.,.\})}$ is a Hilbert space and A_0 induces a bounded selfadjoint operator $\tilde{A}_0$ in $\mathcal{H}$. Since for $\lambda \in \rho(A_0)$ the operator $R_\lambda(A_0) := (A_0 - \lambda I)^{-1}$ is bounded on $\mathfrak{K}$ it can be extended to an operator $\tilde{R}_\lambda(A_0)$ on $\mathcal{H}$. Further it holds

$$\tilde{R}_\lambda(A_0)(\tilde{A}_0 - \lambda I) = (\tilde{A}_0 - \lambda I)\tilde{R}_\lambda(A_0) = I$$

on a dense subset of $\mathcal{H}$ and it follows $\lambda \in \rho(\tilde{A}_0)$, hence $\rho(A_0) \subseteq \rho(\tilde{A}_0)$. Moreover $\sigma(\tilde{A}_0)$ is contained in Δ. By the spectral theorem there exists a measure ν supported on Δ, such that for $q \in \mathcal{P}$ it holds

$$\{q(A_0)u_0, u_0\} = [p_0(A_0)q(A_0)u_0, u_0]_{\mathfrak{K}} = \int_\Delta q(t)d\nu(t).$$

Using Taylor's formula we obtain for any $p \in \mathcal{P}$

$$p^{\{\alpha_i,j\}}(t) = p(t) - \sum_{k=0}^{j-1}\frac{(t-\alpha_i)^k}{k!}p^{(k)}(\alpha_i) =$$
$$= \sum_{k=j}^{\mu_i-1}\frac{(t-\alpha_i)^k}{k!}p^{(k)}(\alpha_i) + \frac{(t-\alpha_i)^{\mu_i}}{\mu_i!}p^{(\mu_i)}(\xi_i),$$

for a proper intermediate value ξ_i. This yields

$$|h(t;p)| = \Big|\sum_{i=1}^{n}\sum_{j=1}^{\mu_i} c_{ij}(t-\alpha_i)^{-j}p^{\{\alpha_i,j\}}(t)\Big| \le$$
$$\le \Big|\sum_{i=1}^{n}\sum_{j=1}^{\mu_i} c_{ij}\sum_{k=j}^{\mu_i-1}\frac{(t-\alpha_i)^{k-j}}{k!}p^{(k)}(\alpha_i)\Big| + \Big|\sum_{i=1}^{n}\sum_{j=1}^{\mu_i} c_{ij}\frac{(t-\alpha_i)^{\mu_i-j}}{\mu_i!}p^{(\mu_i)}(\xi_i)\Big|,$$

therefore the polynomials $h(.;p),\ p \in S$, are uniformly bounded on Δ and hence

$$\sup\{|[p_0(A_0)h(A_0;p)u_0, u_0]| :\ p \in S\} < \infty.$$

It follows that

$$\sup\{\tilde{\zeta}_{u_0}(p) :\ p \in S\} < \infty.$$

Because S is a neighborhood of 0 in the subspace topology of $C^\infty(\mathbb{R})$ on $\mathcal{P}$, the functional $\tilde{\zeta}_{u_0}$ is continuous. By Lemma 1.25 the polynomials are dense in $C^\infty(\mathbb{R})$, hence $\tilde{\zeta}_{u_0}$ has a unique continuous extension to a linear functional on $C^\infty(\mathbb{R})$ which induces a distribution with compact support $\tilde{\zeta}_{\mathbb{R},u_0}$. We will denote the extension to $C^\infty(\mathbb{R})$ also with $\tilde{\zeta}_{\mathbb{R},u_0}$.
Clearly $\tilde{\zeta}_{\mathbb{R},u_0}$ is real. Let $M = \{\alpha_1, \dots, \alpha_n\} \subseteq \mathbb{R}$, the zeros of the definitizable polynomial for A_0, and $f \in C^\infty$ with $\operatorname{supp} f \subseteq [a,b] \subseteq \mathbb{R} \setminus M$. According to Lemma 1.25 there exists a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ such that p_n converges to f in $\big(C^\infty(\mathbb{R}), \tau_\infty\big)$. Now we have

$$\tilde{\zeta}_{\mathbb{R},u_0}(f) = \lim_{n\to\infty} \tilde{\zeta}_{\mathbb{R},u_0}(p_n) = \lim_{n\to\infty} [p_n(A_0)u_0, u_0]_{\mathfrak{K}} = \lim_{n\to\infty} \Big[\int p_n dEu_0, u_0\Big]_{\mathfrak{K}} =$$
$$= \Big[\int f dEu_0, u_0\Big]_{\mathfrak{K}} = \int\limits_{[a,b]} f dE_{u_0,u_0}.$$

Since E_{u_0,u_0} is a positive or negative measure it follows that $\tilde{\zeta}_{\mathbb{R},u_0} \in \mathcal{F}(\mathbb{R}, M)$. Clearly $c(A) \subseteq M$, hence if there is an index i_0 with $1 \le i_0 \le n$ such that $\alpha_{i_0} \notin c(A)$, then there exists a real definitizing polynomial p_{i_0} with $p_{i_0}(\alpha_{i_0}) \neq 0$. Using the same construction as for the definitizing polynomial p_0 and the argumentation above shows that $\tilde{\zeta}_{\mathbb{R},u_0} \in \mathcal{F}(\mathbb{R}, M \setminus \{\alpha_{i_0}\})$. Repeating this process yields $\tilde{\zeta}_{\mathbb{R},u_0} \in \mathcal{F}(\mathbb{R}, c(A))$. The characterization of a critical point (see [L, Proposition 4.2, p.35]) states that for any real interval Δ containing a critical point of A the scalar product $[.,.]_{\mathfrak{K}}$ is indefinite on $E(\Delta)\mathfrak{K}$, where E denotes the spectral function of A_0. This shows that there exists no proper subset $\tilde{M}$ of $c(A)$ such that $\tilde{\zeta}_{\mathbb{R},u_0} \in \mathcal{F}(\mathbb{R}, \tilde{M})$. ❑

Let $(\mathfrak{K}_1, [.,.]_{\mathfrak{K}_1})$ and $(\mathfrak{K}_2, [.,.]_{\mathfrak{K}_2})$ be Kreĭn spaces and $A_i \in \mathcal{B}(\mathfrak{K}_i)$, $i = 1, 2$. We say that A_1 is *weakly unitarily equivalent* to A_2 if there exists a linear isometry $U : D \subseteq \mathfrak{K}_1 \to \mathfrak{K}_2$ with dense domain and dense range such that $A_1(D) \subseteq D$ and $UA_1 = A_2U$ on D.

4.4 Theorem. *Let A be a cyclic bounded definitizable selfadjoint operator in a Kreĭn space $(\mathfrak{K}, [.,.]_{\mathfrak{K}})$ with generating element $u \in \mathfrak{K}$. Denote by ϕ_u the unique element of $\mathcal{F}$ induced by the functional ζ_u as in* (4.0.1), *Proposition 4.3. Then A is weakly unitarily equivalent to the operator A_{ϕ_u} in $\mathcal{K}_{\phi_u}$. A weak unitary equivalence is given by the mapping*

$$U : \begin{cases} \big(\iota_{\phi_u}(\mathcal{P}), [.,.]_{\mathcal{K}_{\phi_u}}\big) \to \big(\mathfrak{K}, [.,.]_{\mathfrak{K}}\big) \\ \qquad\qquad \iota_{\phi_u}(p) \mapsto p(A)u \end{cases}. \tag{4.0.5}$$

Proof. Write ϕ_u as $(\varphi_u, \psi_u) \in \mathcal{F}$, then by Theorem 3.19 and Proposition 4.3 for $p, q \in \mathcal{P}$ it holds

$$\begin{aligned}[\iota_{\phi_u}(p), \iota_{\phi_u}(q)]_{\mathcal{K}_{\phi_u}} &= [p,q]_{\phi_u} = \varphi_u(p\overline{q}) + \psi_u(p\overline{q}) = \\ &= [p\overline{q}(A)u, u]_{\mathfrak{K}} = [p(A)u, q(A)u]_{\mathfrak{K}}.\end{aligned} \tag{4.0.6}$$

If $q \in \mathcal{P}$ such that $\iota_{\phi_u}(q) = 0$, then the last equation implies that

$$[p(A)u, q(A)u]_{\mathfrak{K}} = 0, \quad p \in \mathcal{P}.$$

Since u is a generating element, it follows that $\{p(A)u : p \in \mathcal{P}\}$ is dense in $\mathfrak{K}$. Further for every $y \in \mathfrak{K}$ each of the functionals $[.,y]_{\mathfrak{K}}$ is continuous and therefore it holds $[x, q(A)u]_{\mathfrak{K}} = 0$, $x \in \mathfrak{K}$. This shows that $q(A)u$ is isotropic and because as a Kreĭn space $\mathfrak{K}$ is non-degenerated we have $q(A)u = 0$. On the other hand if $q \in \mathcal{P}$ such that $q(A)u = 0$, similar argumentation, using the fact that ran ι_{ϕ_u} is dense in $\mathcal{K}_{\phi_u}$ (see Theorem 3.19), yields $\iota_{\phi_u}(q) = 0$. Thus we have shown that for $q \in \mathcal{P}$ it holds $\iota_{\phi_u}(q) = 0$ if and only if $q(A)u = 0$.
Therefore the mapping

$$U : \begin{cases} \left(\iota_{\phi_u}(\mathcal{P}), [.,.]_{\mathcal{K}_{\phi_u}}\right) \to \left(\mathfrak{K}, [.,.]_{\mathfrak{K}}\right) \\ \qquad\qquad \iota_{\phi_u}(p) \mapsto p(A)u \end{cases}$$

is well-defined. Clearly U is linear and equation (4.0.6) shows that U is isometric. Further the domain of U is dense in $\mathcal{K}_{\phi_u}$ and the range of U is dense in $\mathfrak{K}$.
By Theorem 3.19 we have $A_{\phi_u} \circ \iota_{\phi_u}(\mathcal{P}) = \iota_{\phi_u} \circ A_t(\mathcal{P})$ and therefore we have $A_{\phi_u}\big(\iota_{\phi_u}(\mathcal{P})\big) \subseteq \iota_{\phi_u}(\mathcal{P})$. Further for $p \in \mathcal{P}$ it holds

$$UA_{\phi_u}\big(\iota_{\phi_u}(p)\big) = U\iota_{\phi_u}A_t(p) = Ap(A)u = AU\big(\iota_{\phi_u}(p)\big).$$

This shows that A is weakly unitarily equivalent to A_{ϕ_u}. ❑

This theorem raises the question under what conditions the weak unitary equivalence can be be extended to the whole space, i.e. whether the isometry U constructed in proof is continuous and its continuation is surjective or not. We give sufficient conditions depending on the type of critical points.

4.5 Remark. It is sufficient to consider bounded definitizable selfadjoint operators in a Kreĭn space with only real spectrum and only critical point 0. To see this, let A be a bounded definitizing selfadjoint operator in the Kreĭn space $(\mathfrak{K}, [.,.])$ and denote by E the spectral function of A. According to Remark 4.2 we can assume that A has only real spectrum. Let $\alpha \in c(A)$ and choose an open interval Δ containing α but no other critical point of A. Then the decomposition

$$\mathfrak{K} = E(\Delta)\mathfrak{K}[\dot{+}](I - E(\Delta))\mathfrak{K}$$

reduces A, and for the restriction $A_\Delta := A|_{E(\Delta)\mathfrak{K}}$ we have $c(A_\Delta) = \{\alpha\}$. Replacing A by $A - \alpha$ we can assume $\alpha = 0$.
Due to [J, Lemma 3, p. 128] there exists a real definitizing polynomial p_0 of A_Δ whose zeros are real such that

$$c(A_\Delta) = N(p_0) \cap \sigma(A_\Delta),$$

where $N(p_0)$ denotes the zeros of p_0. If Δ is a sufficiently small interval containing α but no other zeros of p_0 we can assume that the polynomial p_0 is of the form $p_0(z) = \pm z^q$, $q \in \mathbb{N}_0$. Therefore we can restrict ourselves to a bounded definitizable selfadjoint operator A in a Kreĭn space with only real spectrum such that $c(A) = \{0\}$ and $p_0(z) = \pm z^q$, $q \in \mathbb{N}_0$ is a definitizing polynomial or $c(A) = \emptyset$.

The next proposition ensures the existence of a unitary equivalence under sufficient conditions on the spectrum of the operator.

4.6 Proposition. *Let A be a cyclic bounded definitizable selfadjoint operator in a Kreĭn space $(\mathfrak{K}, [.,.]_{\mathfrak{K}})$ with generating element $u \in \mathfrak{K}$. Denote by ϕ_u the unique element of $\mathcal{F}$ induced by the functional ζ_u as in* (4.0.1)*, Proposition 4.3. If $c_s(A) = c_{sf}(A)$ then A is unitarily equivalent to the operator A_{ϕ_u} in $\mathcal{K}_{\phi_u}$.*

Proof. According to Remark 4.2 we can divide the proof into the cases that A has only complex and only real spectrum. By Remark 4.5 the latter case can be reduced to the case that $c(A) = \{0\}$ with definitizing polynomial $p_0(z) = \pm z^q$, $q \in \mathbb{N}_0$, and the case $c(A) = \emptyset$. The case that A has a critical point at zero will be distinguished further into the case that 0 is a singular critical point of finite index, case 2a, and the case that 0 is a regular critical point, case 2b. Write ϕ_u as $(\varphi_u, \psi_u) \in \mathcal{F}$ and note that in these cases it is sufficient to consider $\mathcal{K}_{\psi_u}$ and $\mathcal{K}_{\varphi_u}$ respectively. Denote by U the weak unitary mapping which establishes the weak unitarily equivalence as in (4.0.5).

Case 1, $(\sigma(A) = \sigma_0(A))$: Clearly, the mapping $U : \iota_{\psi_u}(\mathcal{P}) \to \mathcal{K}_{\psi_u}$ is already unitary.

Case 2a, $(c(A) = c_f(A) = \{0\},\ \sigma_0(A) = \emptyset)$: Since 0 is the only critical point of A and it is of finite index it follows that $(\mathfrak{K}, [.,.]_{\mathfrak{K}})$ is a Pontryagin space. Further the domain and the range of U are dense, hence their closures are non-degenerated. Following [B, Theorem 3.1, p. 188] this implies that U is invertible and that U and U^{-1} are continuous. This yields the existence of a unitary extension $\hat{U}$ of U from $\mathcal{K}_{\varphi_u}$ onto $\mathfrak{K}$ such that $\hat{U} A_{\varphi_u} = A\hat{U}$.

Case 2b, $(c(A) = c_r(A) = \{0\},\ \sigma_0(A) = \emptyset)$: Since 0 is the only critical point of A it follows that $\varphi_u \in \mathcal{F}(\mathbb{R}, 0)$.
Equation (4.0.2) and the fact that 0 is a regular critical point implies that $k = 0$ in the minimal representation.
Denote by p_0 a real definitizing polynomial for A and by E the spectral function of A as in Theorem 2.39. Further let $\lambda(p_0)$ be the order of the zero 0 of the definitizing polynomial p_0. By Remark 4.5 we can assume that $p_0(z) = \pm z^{\lambda(p_0)}$.
Let $\vartheta = (0, l, \sigma, c_0, \ldots, c_l) \in \Theta_{\varphi_u}$ be a representation of φ_u such that $l \geq \lambda(p_0)$.
Recall that the mapping $\iota_\vartheta : \mathcal{P} \to \mathfrak{L}_\vartheta$, see Proposition 2.15, is given by

$$\iota_\vartheta(p) = \Big(p; \frac{p^{(0)}(0)}{0!}, \ldots, \frac{p^{(l)}(0)}{l!}\Big)^T, \quad p \in \mathcal{P}.$$

Corollary 2.35 implies that $\overline{\operatorname{ran} \iota_\vartheta} = \mathfrak{L}_\vartheta$. The space $\mathfrak{L}_\vartheta$ can be decomposed as

$$\mathfrak{L}_\vartheta = L^2(\sigma_-)[\dot{+}]_\vartheta \mathbb{C}^{l+1}[\dot{+}]_\vartheta L^2(\sigma_+),$$

where $\sigma_\pm = |\sigma|\big|_{\mathbb{R}^\pm}$. Similar to Theorem 4.4 define a mapping $\tilde{U}$ by

$$\tilde{U} : \begin{cases} \big(\iota_\vartheta(\mathcal{P}), [.,.]_{\mathfrak{L}_\vartheta}\big) \to \big(\mathfrak{K}, [.,.]_{\mathfrak{K}}\big) \\ \qquad\qquad \iota_\vartheta(p) \mapsto p(A)u \end{cases}.$$

As in the proof of Theorem 4.4, using Proposition 2.15, it follows that $\tilde{U}$ is linear, isometric and has dense domain and dense range.
For $n \in \mathbb{N}$ define the intervals $\Delta_n^- := (-\infty, -\frac{1}{n}]$, $\Delta_n := (-\frac{1}{n}, \frac{1}{n})$ and $\Delta_n^+ :=$

$[\frac{1}{n}, +\infty)$, then they are elements of Ω_{p_0} and their union is $\mathbb{R}$, hence Theorem 2.39 (iv) yields

$$E\big(\Delta_n\big) = I - E\big(\Delta_n^-\big) - E\big(\Delta_n^+\big).$$

Since A is bounded and 0 is a regular critical point of A the limits $E_- := \lim_{n\to\infty} E\big(\Delta_n^-\big)$ and $E_+ := \lim_{n\to\infty} E\big(\Delta_n^+\big)$ exists in the strong operator topology. Therefore $E_0 := \lim_{n\to\infty} E\big(\Delta_n\big)$ exists in the strong operator topology. By Theorem 2.39 (iii) the family of bounded projections $\big(E(\Delta_n)\big)_{n\in\mathbb{N}}$ is monotone decreasing[4]. Therefore for $m, n \in \mathbb{N}$, $m \le n$, we have that $\operatorname{ran} E(\Delta_n) \subseteq \operatorname{ran} E(\Delta_m)$ which implies that

$$\operatorname{ran} E_0 \subseteq \bigcap_{n\in\mathbb{N}} \operatorname{ran} E(\Delta_n).$$

This shows that $E_0\mathfrak{K} \subseteq \bigcap_{\substack{\Delta\in\Omega_p \\ 0\in\Delta}} E(\Delta)\mathfrak{K} =: S_0$.
Consider the mappings

$$\Lambda_{\substack{\pm\\0}} : \begin{cases} \iota_\vartheta(\mathcal{P}) & \to \mathfrak{K} \\ \Big(p; \frac{p^{(0)}(0)}{0!}, \dots, \frac{p^{(l)}(0)}{l!}\Big)^T & \mapsto p(A)E_{\substack{\pm\\0}} u \end{cases},$$

then $\tilde{U}$ writes as $\tilde{U} = \Lambda_- + \Lambda_0 + \Lambda_+$.
Denote by $\|.\|$ the Hilbert space norm on $\mathfrak{K}$ induced by some fundamental decomposition and let $n \in \mathbb{N}$. Then for $p \in \mathcal{P}$ it holds

$$\begin{aligned} \|\Lambda_-\big(\iota_\vartheta(p)\big)\| &= \|p(A)E_- u\| \le \\ &\le \big\|p(A)\big(E_- - E(\Delta_n^-)\big)u\big\| + \big\|p(A)E(\Delta_n^-)u\big\|. \end{aligned} \tag{4.0.7}$$

Define on $\mathfrak{K}_- := E(\Delta_n^-)\mathfrak{K}$ an inner product by $(.,.)_- := \operatorname{sign} p_0|_{\mathbb{R}^-}$, then $(\mathfrak{K}_-, (.,.)_-)$ is a Hilbert space. Setting $u_n := E(\Delta_n^-)u$ the second norm can be computed using the spectral theorem for a bounded selfadjoint operator in a Hilbert space by

$$\big\|p(A)E(\Delta_n^-)u\big\|^2 = \|p(A)u_n\|^2 = \int\limits_{\Delta_n^-} |p|^2 dE_{u_n,u_n}, \quad p \in \mathcal{P},$$

where $E_{g,h}(\Delta) := \big(E(\Delta)g, h\big)_-$ for $g, h \in \mathfrak{K}_-$ and Δ a Borel set of $\sigma(A|_{\mathfrak{K}_-})$. Since $E_{u_n,u_n} = \sigma_-$ on Δ_n^- it follows that

$$\big\|p(A)E(\Delta_n^-)u\big\| \le \|p\|_{L^2(\sigma_-)}, \quad p \in \mathcal{P}.$$

Since $n \in \mathbb{N}$ was arbitrary and E_- is the limit of $E(\Delta_n^-)$ in the strong operator topology, equation (4.0.7) implies that

$$\|\Lambda_-\big(\iota_\vartheta(p)\big)\| \le \|p\|_{L^2(\sigma_-)} \le \|\iota_\vartheta(p)\|_\vartheta, \quad p \in \mathcal{P},$$

where $\|.\|_\vartheta$ denotes the norm induced by the inner product $(.,.)_\vartheta$. This shows that the mapping $\Lambda_- : \iota_\vartheta(\mathcal{P}) \to \mathfrak{K}$ is continuous. Similar it follows that the

[4]Two projections E_1, E_2 in a Banach space X are said to be *ordered in their natural order* $E_1 \le E_2$ if $E_1E_2 = E_2E_1 = E_1$, see [DS, Definition 4, p. 481]. This definition requires that $E_1X \subseteq E_2X$ and $(I - E_1)X \supseteq (I - E_2)X$.

mapping $\Lambda_+ : \iota_\vartheta(\mathcal{P}) \to \mathfrak{K}$ is continuous.
By [L, Proposition 5.1, p.37] S_0 is the algebraic eigenspace of A corresponding to the eigenvalue 0 and moreover $A^{\lambda(p_0)+1}S_0 = \{0\}$. Therefore there exists $u_0, \dots, u_{\lambda(p_0)} \in S_0$ and $\alpha_0, \dots, \alpha_{\lambda(p_0)} \in \mathbb{C}$ such that $E_0 u = \sum_{i=0}^{\lambda(p_0)} \alpha_i u_i$. By Taylor's theorem any polynomial $p \in \mathcal{P}$ can be written as $p(x) = \sum_{j=0}^{d} \frac{p^{(j)}(0)}{j!} x^j$. Hence for $p \in \mathcal{P}$ it holds

$$p(A)E_0 u = \sum_{j=0}^{d} \frac{p^{(j)}(0)}{j!} A^j \sum_{i=0}^{\lambda(p_0)} \alpha_i u_i = \sum_{j=0}^{\lambda(p_0)} \frac{p^{(j)}(0)}{j!} \sum_{i=0}^{\lambda(p_0)} \alpha_i A^j u_i.$$

Since $l \geq \lambda(p_0)$ and by Hölder's inequality the last equation yields the following estimate for $p \in \mathcal{P}$

$$\begin{aligned} \|\Lambda_0\big(\iota_\vartheta(p)\big)\| = \|p(A)E_0 u\| &\leq \sum_{j=0}^{\lambda(p_0)} \left|\frac{p^{(j)}(0)}{j!}\right| \sum_{i=0}^{\lambda(p_0)} |\alpha_i| \|A\|^j \|u_i\| \leq \\ &\leq \big(\lambda(p_0)+1\big) \left(\sum_{j=0}^{\lambda(p_0)} \left| \frac{p^{(j)}(0)}{j!} \right|^2 \right)^{\frac{1}{2}} \underbrace{\sum_{i=0}^{\lambda(p_0)} |\alpha_i| \|A\|^j \|u_i\|}_{=:\delta_j} \leq \\ &\leq \big(\lambda(p_0)+1\big) \max_{j=0,\dots,\lambda(p_0)} \delta_j \cdot \|\iota_\vartheta(p)\|_\vartheta. \end{aligned}$$

This shows that the mapping $\Lambda_0 : \iota_\vartheta(\mathcal{P}) \to \mathfrak{K}$ is continuous. Now it follows that the mapping $\tilde{U} = \Lambda_- + \Lambda_0 + \Lambda_+$ is continuous. Further the domain of U is dense in the space $\mathfrak{L}_\vartheta$ hence there exists a continuation by continuity $\hat{U} : \mathfrak{L}_\vartheta \to \mathfrak{K}$. Clearly $\hat{U}$ is isometric and since $(\mathfrak{K}, [.,.]_\mathfrak{K})$ is as a Kreĭn space nondegenerated it follows that $\hat{U}\big(\mathfrak{L}_\vartheta^{[\circ]_\vartheta}\big) = \{0\}$. Since $\overline{\operatorname{ran} \iota_\vartheta} = \mathfrak{L}_\vartheta$, Theorem 2.18 shows that $\mathfrak{L}_\vartheta/_{\mathfrak{L}_\vartheta^{[\circ]_\vartheta}} = \mathcal{K}_{\varphi_u}$. According to Lemma 2.22 $\hat{U}$ induces an continuous isometric operator with dense range on the factor space, i.e. $U : \mathcal{K}_{\varphi_u} \to \mathfrak{K}$. Following [B, Lemma 3.9, p. 127] implies that U^{-1} is continuous. This yields the existence of a unitary extension $\hat{U}$ of U from $\mathcal{K}_{\varphi_u}$ onto $\mathfrak{K}$ such that $\hat{U}A_{\varphi_u} = A\hat{U}$.

Case 3, ($\sigma_0(A) = \emptyset$, $c(A) = \emptyset$): In this situation we have a Hilbert or anti Hilbert space, hence the assertion is well known. ❑

Recall that for $\phi \in \mathcal{F}$ by Theorem 3.19 the model operator A_ϕ is cyclic with generating element $\iota_\phi(1)$.

4.7 Corollary. *Let $\phi \in \mathcal{F}$ and A_ϕ be the model operator in the model space $\mathcal{K}_\phi$. Denote by $\tilde{\phi}$ the unique element of $\mathcal{F}$ induced by the functional $\tilde{\zeta}(p) := [p(A_\phi)\iota_\phi(1), \iota_\phi(1)]_{\mathcal{K}_\phi}$, $p \in \mathcal{P}$, i.e. the distribution corresponding to A_ϕ and generating element $\iota_\phi(1)$ as in* (4.0.1)*, Proposition 4.3. Then $\tilde{\phi} = \phi$.*

Proof. Write ϕ as $(\varphi, \psi) \in \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{C} \setminus \mathbb{R})$ and denote by $\tilde{\zeta}_\mathbb{R}$ and $\tilde{\zeta}_{\mathbb{C}\setminus\mathbb{R}}$ the real and non-real part of $\tilde{\phi}$ as in Proposition 4.3. Let B be a subset of $\mathbb{C} \setminus \mathbb{R}$ which is symmetric with respect to the real axis such that $\sigma_0(\psi) = B$, i.e. $\psi \in \mathcal{F}(\mathbb{C} \setminus \mathbb{R}, B)$. By Proposition 3.10 it follows that $\sigma(A_\psi) = B$. Since $\sigma(A_\psi) = \sigma_0(A_\phi)$ Proposition 4.3 implies that $\tilde{\zeta}_{\mathbb{C}\setminus\mathbb{R}} \in \mathcal{F}\big(\mathbb{C} \setminus \mathbb{R}, \sigma(A_\psi)\big)$, thus $\tilde{\zeta}_{\mathbb{C}\setminus\mathbb{R}} = \psi$.

If φ is an element of $\mathcal{F}(\mathbb{R},\emptyset)$ then by Proposition 4.3 $\tilde{\zeta}_{\mathbb{R}}$ is a positive or negative measure on $\mathbb{R}$ and equation (4.0.2) implies that $\tilde{\zeta}_{\mathbb{R}} = \varphi$. Otherwise let $n \in \mathbb{N}$ and $M := \{\alpha_1, \dots, \alpha_n\}$ a n-element subset of $\mathbb{R}$ such that $\varphi \in \mathcal{F}(\mathbb{R}, M)$ and M is minimal. Using Lemma 1.13 we can write φ as sum of distributions $\varphi_{\alpha_i} \in \mathcal{F}(\mathbb{R}, \alpha_i)$, $i = 1, \dots, n$. Therefore and according to the construction of A_φ we can assume that $\varphi \in \mathcal{F}(\mathbb{R}, 0) \setminus \mathcal{F}(\mathbb{R}, \emptyset)$. Let $\vartheta = (k, l, \sigma, c_0, \dots, c_l) \in \Theta_\varphi$ be the minimal representation of φ. By Proposition 2.43 and Remark 2.44 it holds $c(A_\varphi) = \{0\}$ and by Proposition 4.3 it follows that $\tilde{\zeta}_{\mathbb{R}} \in \mathcal{F}(\mathbb{R}, 0) \setminus \mathcal{F}(\mathbb{R}, \emptyset)$. Denote by $k, l, c_0, \dots, c_l$ and σ the unique data satisfying (IR-1) and (IR-2) from the integral representation as in Proposition 1.21 of φ and by $\tilde{k}, \tilde{l}, \tilde{c}_0, \dots, \tilde{c}_{\tilde{l}}$ the unique data of $\tilde{\zeta}_{\mathbb{R}}$. Then it holds

$$\varphi(f) = \int\limits_{\mathbb{R}} \frac{f^{\{2k\}}(t)}{t^{2k}} d\sigma(t) + \sum_{i=0}^{l} \frac{c_i}{i!} f^{(i)}(0), \quad f \in \mathcal{D}(\mathbb{R}),$$

and

$$\tilde{\zeta}_{\mathbb{R}}(f) = \int\limits_{\mathbb{R}} \frac{f^{\{2\tilde{k}\}}(t)}{t^{2\tilde{k}}} d\tilde{\sigma}(t) + \sum_{i=0}^{\tilde{l}} \frac{\tilde{c}_i}{i!} f^{(i)}(0), \quad f \in \mathcal{D}(\mathbb{R}).$$

Denote by E the spectral function of A_φ and by $\mu_{\mathbb{R}^\pm}$ the measure corresponding to φ as in Proposition 1.7. Then by Proposition 4.3 it follows that $\mu_{\mathbb{R}^\pm}(\Delta) = E_{1,1}(\Delta)$ for $\Delta \in \mathfrak{B}(\mathbb{R})$, $\Delta \subseteq \mathbb{R}^\pm$. Therefore equation (4.0.2) yields that

$$\varphi(f) = \int\limits_{\mathbb{R}^\pm} f d\mu_{\mathbb{R}^\pm} = \int\limits_{\mathbb{R}^\pm} f dE_{1,1} = \tilde{\zeta}_{\mathbb{R}}(f), \quad f \in \mathcal{D}(\mathbb{R}) \text{ with } \operatorname{supp} f \subseteq \mathbb{R}^\pm.$$

This shows that $\mu_{\mathbb{R}^\pm} = E_{1,1}|_{\mathbb{R}^\pm}$ which implies that $k = \tilde{k}$ and $\sigma = \tilde{\sigma}$. Setting $p_j(t) := t^j$, $j \in \mathbb{N}_0$, we have since $k = \tilde{k}$ and $\sigma = \tilde{\sigma}$ that

$$\tilde{c}_i = \begin{cases} \tilde{\zeta}_{\mathbb{R}}(p_i), & i = 0, \dots, 2k-1, \\ \tilde{\zeta}_{\mathbb{R},1}(p_i) - (t^{i-2k}, 1)_\sigma, & i = 2k, 2k+1, \dots . \end{cases}$$

Let $\pi_\vartheta : \mathfrak{L}_\vartheta \to \mathcal{K}_\varphi$ the canonical projection and $\vec{x} := \pi_\vartheta^{-1}\big(\iota_\varphi(1)\big)$. Note that $\vec{x} = (0; 1, 0, \dots, 0; 0, \dots, 0)^T$ if $k > 0$ and that $\vec{x} = (1; 1, 0, \dots, 0)^T$ if $k = 0$. Since $A_\varphi^i \iota_\varphi(1) = \pi_\vartheta\big(\mathfrak{A}_\vartheta^i \, \vec{x}\big)$, $i \in \mathbb{N}_0$, we have

$$\begin{aligned} \tilde{\zeta}_{\mathbb{R}}(p_i) &= [p_i(A_\varphi)\iota_\varphi(1), \iota_\varphi(1)]_{\mathcal{K}_\varphi} = [A_\varphi^i \iota_\varphi(1), \iota_\varphi(1)]_{\mathcal{K}_\varphi} = \\ &= \big[\pi_\vartheta\big(\mathfrak{A}_\vartheta^i \, \vec{x}\big), \pi_\vartheta \vec{x}\big]_{\mathcal{K}_\varphi} = \big[\mathfrak{A}_\vartheta^i \, \vec{x}, \vec{x}\big]_\vartheta . \end{aligned}$$

Easy computation gives that

$$\big[\mathfrak{A}_\vartheta^i \, \vec{x}, \vec{x}\big]_\vartheta = \begin{cases} c_i, & i = 0, \dots, 2k-1, \\ c_i + (t^{i-2k}, 1)_\sigma, & i = 2k, 2k+1, \dots, \end{cases}$$

where $c_i = 0$ if $i > l$. This shows that $c_i = \tilde{c}_i$, $i = 0, \dots, l$ and $l = \tilde{l}$. Therefore the integral representations of φ and $\tilde{\zeta}_{\mathbb{R}}$ coincide and by the uniqueness of the data satisfying (IR-1) and (IR-2) it follows that $\varphi = \tilde{\zeta}_{\mathbb{R}}$. Thus we have shown that $\tilde{\phi} = \phi$. ❑

Bibliography

[AI] T. Ya. Azizov I. S. Iokhvidov, *Linear operators in spaces with an indefinite metric*, Pure and Applied Mathematics (New York), John Wiley & Sons Ltd., Chichester, 1989. Translated from the Russian by E. R. Dawson; A Wiley-Interscience Publication.

[B] J. Bognár, *Indefinite inner product spaces*, Springer-Verlag, New York, 1974.

[C1] J. B. Conway, *A course in functional analysis*, 2nd ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990.

[C2] J. B. Conway, *Functions of one complex variable*, 2nd ed., Graduate Texts in Mathematics, vol. 11, Springer-Verlag, New York, 1978.

[D] M. A. Dritschel, *A method for constructing invariant subspaces for some operators on Kreĭn spaces*, (Timişoara, 1992), Oper. Theory Adv. Appl., vol. 61, Birkhäuser, Basel, 1993, pp. 85–113.

[DS] N. Dunford J. T. Schwartz, *Linear operators. Part I*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1988. General theory; With the assistance of William G. Bade and Robert G. Bartle; Reprint of the 1958 original; A Wiley-Interscience Publication.

[E] R. E. Edwards, *Functional analysis. Theory and applications*, Holt, Rinehart and Winston, New York, 1965.

[GGK] I. Gohberg, S. Goldberg, M. A. Kaashoek, *Classes of linear operators. Vol. I*, Operator Theory: Advances and Applications, vol. 49, Birkhäuser Verlag, Basel, 1990.

[HL] F. Hirsch G. Lacombe, *Elements of functional analysis*, Graduate Texts in Mathematics, vol. 192, Springer-Verlag, New York, 1999. Translated from the 1997 French original by Silvio Levy.

[J] P. Jonas, *On the functional calculus and the spectral function for definitizable operators in Kreĭn space*, Beiträge Anal. **16** (1981), 121–135.

[JLT] P. Jonas, H. Langer, B. Textorius, *Models and unitary equivalence of cyclic selfadjoint operators in Pontrjagin spaces*, Operator theory and complex analysis (Sapporo, 1991), 1992, pp. 252–284.

[L] H. Langer, *Spectral functions of definitizable operators in Kreĭn spaces*, Functional analysis (Dubrovnik, 1981), 1982, pp. 1–46.

[R] W. Rudin, *Functional analysis*, 2nd ed., International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991.

Curriculum vitae

Name:	DI Maximilian Kleinert
Date of birth:	May 27, 1980
Place of birth:	Vienna
Citizenship:	Austria
1986-1990	VS Zedlitzgasse, Vienna
1990-1998	AHS Stubenbastei, Vienna
1998-1999	Conscription
1999-2005	Vienna University of Technology *Technical Mathematics with a concentration in Mathematical Computer Science*
2005	Master's degree in mathematics from Vienna UT *Thesis 'Summierung von Jacobipolynomreihen durch ein Gronwallverfahren'*
2006	FWF Project *Indefinite Verallgemeinerungen von kanonischen Systemen*
2008-2010	Project assistant at Vienna UT Institute of Analysis and Scientific Computing