D I P L O M A R B E I T

Optimal Controls in Models of Economic Growth and the Environment

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Abstract

The usage of economic instruments in environmental policy with the intent to control the present environmental situation has become increasingly important during the past years, especially since climate change has appeared as crucial keyword in the world wide media. Environmental regulation is supposed to reduce or ideally minimize emissions and pollution. However, the questions arise how effective these regulations really are and whether they rather repress innovation and economic growth than they induce a shift towards greener technology. To answer these questions this thesis investigates an endogenous growth model in an environmental context, which is taken from M. Rauscher [Green R & D versus End-of-Pipe Emission Abatement: A Model of Directed Technical Change. Thuenen-Series of Applied Economic Theory, 106, 2009] who addresses this issue by investigating the impact of environmental quality standards on capital accumulation and R&D investments. In his paper Rauscher considers this problem in a rather general formulation without assuming specific model functions. The focus of this thesis now is to investigate various scenarios of this model with different types of production functions and state dynamics by applying optimal control theory to this two-state control model. Further on, also an extended model version is considered, which additionally includes subsidies as economic instrument.

Acknowledgement

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With this thesis I finish my study of Technical Mathematics at the Vienna University of Technology, which would not have been possible without the support of relatives, friends and university colleagues. At this point, sincere thanks to my parents who always support me in everything I do and made it possible to achieve this goal. I also thank all my friends and colleagues for their encouragement during these years. Special thanks to Christian Schmid for all the interesting technical discussions, which often have been a source of inspiration for this thesis.

Contents

Chapter 1

Introduction

In recent years climate change and the possible consequences that human society might have to deal with, if further global warming cannot be stopped, have become one of the most important topics in science, politics and the world wide media. The scientific evidence that many key climate indicators are already moving beyond the patterns of natural variability defines this dramatic change as a world wide concern. Hence, the importance of climate mitigation has become undeniable. These indicators, including global mean surface temperature, global ocean temperature, global average sea level, northern hemisphere snow cover and Arctic sea ice decline as well as extreme climatic events, additionally come along with the risk of abrupt or irreversible climatic shifts, which might have devastating consequences for the entire world population. This underlines how urgent the need of climate actions has become (see Richardson et al. [2009]).

In the 4th Assessment Report by the International Panel on Climate Change (IPCC [2007]), scientific evidence on global warming, its damages and the importance of climate mitigation as well as the reduction of anthropogenic greenhouse gas (GHG) emissions are demonstrated extensively. According to their *Synthesis Report*, the industry sector, besides the energy supply and transport sectors is one of the main sources of anthropogenic GHG emissions with a portion of almost 20% (2004). The majority are CO_2 emissions due to the use of fossil fuels, but also the emissions of other gases like $PFCs$, SF_6, CH_4 and N_2O due to physical and chemical processes yield an essential fraction. Additionally, one has to consider the polluting impact of industrial waste and wastewater. Further on, not only the sources are discussed in IPCC [2007] but also a broad range of mitigation policy measures are suggested, which especially emphasizes the role of technology policies and the increasing need for more R&D efforts. In the *Mitigation of Climate Change Report*, some possible mitigation options for a greener technology are explained, such as fuel switching, including the use of waste material, advanced energy efficiency, the use of bioenergy and material recycling and substitution. As far as according policy instruments are concerned, they consider performance standards, subsidies, tax credits, tradeable permits and voluntary agreements as the most environmentally effective ones.

Although these environmental policy instruments seem to be promising, the question arises how they can be utilized in the most effective way and whether strict environmental regulation has a supporting or repressing impact on innovation and economic growth. To answer these questions I resume in this thesis the analysis of Rauscher [2009] who already addressed this issue in his paper by constructing a simple dynamic environmental-economic model which considers capital accumulation, end-of-pipe emission abatement, R&D investments and knowledge spillovers in an endogenous growth framework. He investigates in a conveniently tractable way whether tighter environmental standards will induce a shift from end-of-pipe emission abatement to a process-integrated one and which impact they have on R&D investments and growth. The model Rauscher employs is kept algebraically simple without specifying concrete functional forms. The focus of this thesis now is to do the same for several scenarios of the model with specific functions and parameters by applying optimal control theory on this two-state control model. Additionally, a modification of the model including subsidies as supplementary policy instrument is regarded.

The thesis is organized as follows. In Chapter 2 I introduce the basic model, first in the general form as used in Rauscher [2009] and then with specified model functions and parameters, where the chosen functional forms and parameter values are justified, while possible difficulties and disadvantages are also discussed. Further on, the basic model is analysed and solved by using Pontryagin's Maximum Principle. To get first insights into the impact of environmental regulation in this basic approach, bifurcation analysis is carried out.

Chapter 3 seizes one of the possible disadvantages of the first approach, namely the use of a Cobb Douglas production function. Hence, the basic model with a CES production function is analysed instead. The resulting differences of the solutions are investigated and the disadvantages due to the previous use of a Cobb Douglas production function are discussed.

The introduction of subsides as additional environmental policy instrument is considered in Chapter 4, where scenarios with two different types of subsidization are investigated. In the first approach R&D investments for green capital are subsidized. The investigation is carried out by using bifurcation analysis. The same is done for a model approach with the total level of green capital being the object of subsidization. As it turns out, the government's choice of the subsidy rate in both approaches is quite crucial. Hence, the trade-off between subsidy payments and achieved results in increasing green R&D investments and green capital, respectively, is considered to investigate the optimal subsidy rate. Finally, the results and achievements of both approaches are compared and discussed.

Chapter 5 deals with a promising modification of the basic model. The idea is to make capital accumulation quite difficult at the beginning, when initial capital levels are very low. Therefore, high R&D investments are necessary to maintain growth until finally a tipping point is reached and capital accumulation suddenly is much faster than in the basic model due to a higher positive feedback of the capital stock. To achieve this behavior, convex-concave state dynamics are introduced. The solution of this approach turns out to be quite problematic due to analytical as well as numerical difficulties. To provide a first insight, the problem is considered under the assumption of constant controls, which lays open the expected multiple equilibria.

In Chapter 6 I revisit the problems of the convex-concave approach and provide a more detailed explanation. Further on, possibilities to evade these difficulties, with regard to possible future investigations, are discussed.

Finally, in Chapter 7 this thesis concludes with summarizing and discussing the results from the analysis carried out in the previous chapters.

Chapter 2

The Basic Model

2.1 General Formulation

The model used as base for the following analysis is taken from Rauscher [2009] who considers a model of directed technical change in an environmental-economics context. On the one hand, this paper deals with the investigation of the environmental regulation's impact on the allocation of available resources to end-of-pipe abatement and to the two types of research and development (R&D), namely *conventional* and *green R&D*. On the other hand, the question arises whether or not stricter emission standards are supportive for green R&D and economic growth.

Consider a competitive market economy consisting of a continuum of identical firms using identical technologies to produce a homogenous GDP good, in which two types of capital are accumulated: first, there is *conventional capital*, also called *brown capital*, which is pollutive and therefore not quite eco-friendly, secondly, non-polluting *green capital* can be chosen. Additionally, the government sets environmental standards which the entrepreneurs are obligated to meet. The necessary abatement effort as well as the abatement costs depend on the stringency of these regulations. Consequently, firms adopting cleaner technologies have to spend less on end-of-pipe abatement. This benefit, however, comes at a cost because the required resources for green R&D's could be invested otherwise profitably in conventional R&D. Many other papers approaching this topic assume different groups of agents, including for example households, which save and consume, capital owners accumulating capital, innovators doing R&D and entrepreneurs combining capital and technology for production. In contrast Rauscher [2009] focuses only on one type of agent in the private sector of the economy, who is a capital-owning entrepreneur doing his/her R&D in-house and who saves and consumes at the same time. In case of perfect competition of the markets on which these agents interact, the simple homogenous-representative-agent model generates the same results as its more elaborated version with heterogeneous agents.

Maximizing his/her own profit, this representative agent has to consider the present value of future utility, given as

$$\int_0^\infty e^{-rt}(\ln(C(t))+u(\varepsilon))dt, \tag{2.1}$$

where $C(t)$ is the consumption or dividend income, $\ln(C(t))$ describes the utility level that one can get out of it and r is the discount rate. Further on, ε specifies the environmental quality determined by the government due to their required standards. It is considered as exogenously given and is an index between 0 and 1, where $\varepsilon = 0$ denotes the laissez-faire scenario considering an economy without any environmental regulation and therefore with bad environmental quality. The opposite case is given by setting $\varepsilon = 1$, where the maximum of attainable environmental quality is given due to a complete ban of pollution. This environmental quality leads to a private sector's utility of $u(\varepsilon)$. Note that $u(.)$ is an increasing and concave utility function.

The entrepreneurs use conventional capital $K(t)$ and/or green capital $G(t)$ to produce an output $F(K(t),G(t))$, with F being a well-behaved, neoclassical production function, satisfying the Inada conditions. One of the central assumptions in this model is that the output is used completely for consumption, for the coverage of opportunity costs due to R&D investments of either type and for end-of-pipe emission abatement, which is summarized in the following budget constraint:

$$F(K(t),G(t)) - C(t) - w(R_K(t)+R_G(t)) - \chi(\varepsilon)K(t) = 0. \tag{2.2}$$

Note that as of here, I will often omit the time argument t for the ease of exposition. R_K and R_G denote the investments for R&D to generate new capital of types K and G, respectively. The parameter $w \in [0,1]$ represents the exogenous opportunity costs. The abatement costs for achieving the binding environment constraints of the government are proportional to the installed conventional capital K. The costs per unit capital is given as $\chi(\varepsilon)$ which is increasing and convex in the stringency of environmental regulation, i.e. $\chi' > 0, \chi'' > 0$.

In this model, economic growth is driven by the accumulation of physical capital and technological know-how. However, physical and knowledge capital are not distinguished in this approach. Instead, both aspects are encompassed in using aggregate variables. The process of accumulation is modeled such, that existing capital or knowledge levels have a positive feedback on the accumulation of new capital or knowledge. The accumulation of conventional and green capital is described by the following differential equations:

$$\dot{K} = A(K,R_K), \tag{2.3}$$

$$\dot{G} = B(G,R_G), \tag{2.4}$$

with $A(.,.)$ and $B(.,.)$ being concave functions, which have positive first derivatives and satisfy the Inada conditions. In Chapter 5, however, a model approach will be considered where $A(.,.)$ and $B(.,.)$ satisfy neither concavity nor the Inada conditions. In Rauscher [2009] also knowledge spillovers in the R&D sector are taken into account which are, for the sake of simplicity, neglected in this analysis. Nevertheless, to ensure a limited growth of capital stocks, decreasing instead of constant returns to scale are considered for the state dynamics.

Figure 2.1 shows the interrelations of the described variables to make the dynamics of the problem more understandable. Starting from a certain capital stock in K and G, the two input factors lead to the according output amount F(K,G), depending on the used production function. Fulfilling the budget constraint (2.2), the decision-maker has to determine the extend of R&D investments that are made for either brown (R_K) or green (R_G) capital or maybe even both of them. These investments in turn influence the growth of the capital stocks K and G, respectively, depending on the functional form again. Additionally, also the existing capital itself has a positive feedback on the stock.

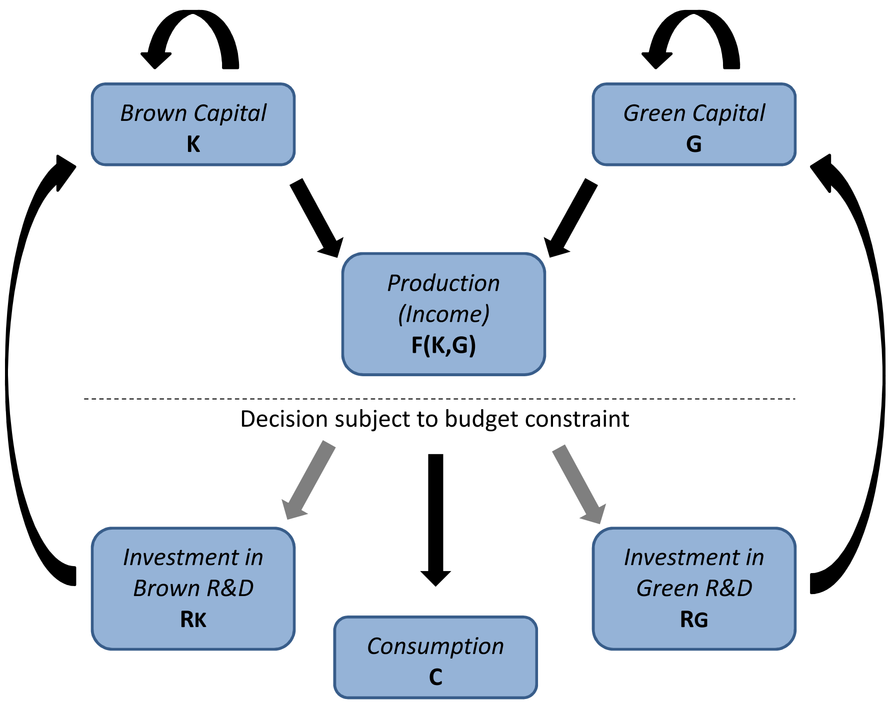


Figure 2.1: Sketch of the dynamics of the model.

Summing up, the model describes an economy in which conventional capital enhances output with the drawback of environmental pollution, implicating necessary end-of-pipe abatement technology at some costs. Alternatively, the representative agent has the possibility to invest in green R&D to accumulate non-polluting green capital as a substitute in production. All decisions about investments in R&D are taken under consideration of opportunity costs and efficiency regarding the government's environmental standard settings.

Solving equation (2.2) for consumption C and substituting the result into the target function in (2.1) leads to an optimal control problem with R_K and R_G as control variables and the two available types of capital as states. Additionally, also a mixed path constraint has to be introduced to ensure that consumption remains non-negative. Further on, to avoid infinite slope in case this mixed path constraint is active, τ is added in the argument of the logarithm in the target function. In the following analysis $\tau = 1$ so that the utility of zero consumption is greater or equal to zero.

The described model is then given as follows:

$$\max_{R_K, R_G} \int_0^\infty e^{-rt} \Big(\ln(\tau + F(K,G) - w(R_K + R_G) - \chi(\varepsilon)K) + u(\varepsilon) \Big) dt \tag{2.5}$$

$$\text{s.t.:} \quad \dot{K} = A(K, R_K) \tag{2.5a}$$

$$\dot{G} = B(G, R_G) \tag{2.5b}$$

$$R_K \geq 0 \tag{2.5c}$$

$$R_G \geq 0 \tag{2.5d}$$

$$0 \leq F(K,G) - w(R_K + R_G) - \chi(\varepsilon)K \tag{2.5e}$$

2.2 Specification of the Model Functions

In the previous section the basic model has been introduced with generally framed functions. The emphasis of this section is to determine specified model functions satisfying the necessary properties from above, as well as parameter values.

2.2.1 Functional Forms

Since a well behaved production function satisfying the Inada conditions is required, a Cobb Douglas production function with K and G as capital inputs will be used in a first approach for the function F. This, however, comes along with the fact that both capital types are essential due to an elasticity of substitution equal to one (see Perman, Ma, McGilvray, and Common [2003]). Therefore unilateral production innately is excluded which might implicate the loss of a solution (an alternative approach dealing with this restriction will be considered in Chapter 3). The partial elasticities of production of conventional and green capital are generally denoted as α_1 and α_2, respectively. Note that both constant returns to scale as well as decreasing returns to scale will be considered:

$$\begin{gathered} F(K,G) = bK^{\alpha_1} G^{\alpha_2} \\ \text{with } b > 0,\, t \in [0,\infty),\, 0 < \alpha_2 \leq \alpha_1 < 1 \text{ and } \alpha_1 + \alpha_2 \leq 1. \end{gathered} \tag{2.6}$$

The function χ describes the abatement costs per unit of capital K and should be increasing and convex in the stringency of environmental regulation ε and consequently is set as

$$\chi(\varepsilon) = a\varepsilon^{\beta} \quad \text{with } a > 0 \text{ and } \beta > 1. \tag{2.7}$$

The utility of the environmental quality for the private sector, $u(\varepsilon)$, should be increasing and concave, and, in the following, is considered as

$$u(\varepsilon) = c\varepsilon^{\gamma} \quad \text{with } c > 0 \text{ and } 0 < \gamma < 1. \tag{2.8}$$

Regarding the state dynamics, again a Cobb Douglas production function is used, this time with decreasing returns to scale and K and R_K, and accordingly G and R_G as input factors. Assuming that the positive feedback of the capital stock on itself is weaker than the contribution of new technology due to R&D, the partial elasticity of production of the capital stock is supposed to be less than the partial elasticity of production of the R&D investments. Additionally, it is more likely that conventional capital is more established in the economy than green one and therefore accumulation is much easier. To take this imbalance into account, the partial elasticities of green capital G should at least not be greater than those of conventional capital K. Further on, one has to consider that capital of either type is subject to depreciation over time due to wear and tear as well as obsolescence. Therefore, I additionally postulate that $A(K,0) < 0$ and $B(G,0) < 0$ for $K,G > 0$, so that in case of zero R&D investments no capital can be accumulated and already existing capital levels will decline. Hence, also depreciation rates ϕ and ψ are considered in the state dynamics which then are given as

$$\dot{K} = dK^{\delta_1}{R_K}^{\delta_2} - \phi K \tag{2.9}$$
$$\dot{G} = eG^{\sigma_1}{R_G}^{\sigma_2} - \psi G \tag{2.10}$$

with $\delta_1 < \delta_2$ and $\sigma_1 < \sigma_2$.

The dashed lines in Figure 2.2 shows the growth paths of conventional capital for three constant levels of R_K. Additionally, also the Cobb Douglas production function and the depreciation term are depicted separately, the difference of which results in the dashed curve representing $\dot{K}$. Naturally, the growth path of green capital is similar, possibly with a flatter slope in the Cobb Douglas function in case of smaller partial elasticities.

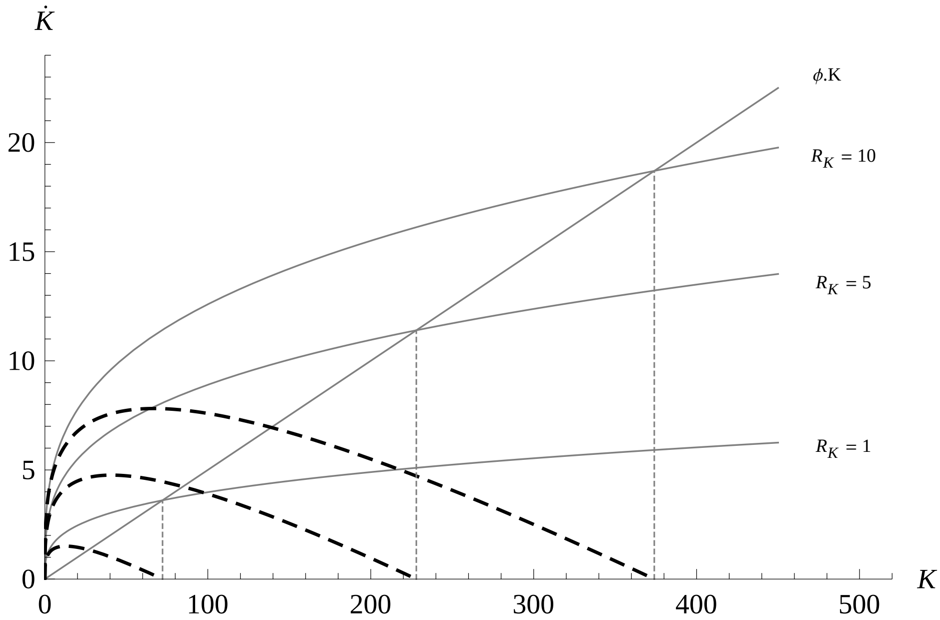


Figure 2.2: Growth path of conventional capital with $\delta_1 = 0.3$, $\delta_2 = 0.5$ and $\phi = 0.05$.

Summing up, the considered optimization model is

$$\max_{R_K, R_G} \int_0^{\infty} e^{-rt} \left(\ln \left(\tau + bK^{\alpha_1} G^{\alpha_2} - w(R_K + R_G) - a\varepsilon^{\beta} K \right) + c\varepsilon^{\gamma} \right) dt \tag{2.11}$$

$$\begin{aligned}
\text{s.t.:} \quad & \dot{K} = dK^{\delta_1} R_K^{\delta_2} - \phi K && (2.11a)\\
& \dot{G} = eG^{\sigma_1} R_G^{\sigma_2} - \psi G && (2.11b)\\
& 0 \le R_K \quad \forall t \ge 0 && (2.11c)\\
& 0 \le R_G \quad \forall t \ge 0 && (2.11d)\\
& 0 \le bK^{\alpha_1} G^{\alpha_2} - w(R_K + R_G) - a\varepsilon^{\beta} K && (2.11e)\\
& 0 \le \varepsilon \le 1 && (2.11f)\\
& 0 < \alpha_1, \alpha_2, \gamma, w < 1 \quad \text{and} \quad \alpha_1 + \alpha_2 \le 1 && (2.11g)\\
& 0 < \delta_1, \delta_2 < 1 \quad \text{and} \quad \delta_1 + \delta_2 < 1 && (2.11h)\\
& 0 < \sigma_1, \sigma_2 < 1 \quad \text{and} \quad \sigma_1 + \sigma_2 < 1 && (2.11i)\\
& 1 < \beta && (2.11j)\\
& 0 < \phi, \psi, a, b, c, d, e, r && (2.11k)\\
& 0 < \tau \le 1. && (2.11l)
\end{aligned}$$

2.2.2 Parameters

In Table 2.1 the parameters used for the analysis in this thesis are summarized, whereas the partial production elasticities α_1 and α_2 of the production function as well as the environmental quality index ε will be a matter of variation for the investigation of different scenarios and therefore are not listed yet.

Parameter	Value	Description
a	1	Constant of proportionality of abatement costs
b	1	Scale parameter of the production function
c	5	Scale parameter describing the utility of environmental quality
d	1	Scale parameter of $\dot{K}$
e	1	Scale parameter of $\dot{G}$
r	0.05	Discount rate
w	0.1	Opportunity cost of research
β	2	Exponent of abatement costs
γ	0.4	Exponent describing the utility of environmental quality
δ_1	0.3	Production elasticity of K in $\dot{K}$
δ_2	0.5	Production elasticity of R_K in $\dot{K}$
σ_1	0.3	Production elasticity of G in $\dot{G}$
σ_2	0.4	Production elasticity of R_G in $\dot{G}$
τ	1	Additive constant of consumption
ϕ	0.05	Depreciation rate of $\dot{K}$
ψ	0.05	Depreciation rate of $\dot{G}$

Table 2.1: Parameter values in the basic model.

2.3 Solving the Problem with Pontryagin's Maximum Principle

2.3.1 Derivation of the Canonical System

Summing up, I consider a discounted autonomous model with infinite planning horizon. To derive the necessary conditions for an optimal solution I consider the Lagrangian $\mathcal{L}$ in current value notation, where $\mathcal{H}$ denotes the Hamiltonian, $\mathcal{C}$ the control and mixed path constraints and μ the vector of Lagrange Multipliers:

$$\begin{aligned}\mathcal{L} = \mathcal{H} + \mu\mathcal{C} \;&=\; \lambda_0(\ln(\tau + F(K,G) - w(R_K+R_G) - \chi(\varepsilon)K) + u(\varepsilon)) + \\ &\quad\; \lambda_1 A(K,R_K) + \lambda_2 B(G,R_G) + \mu_1 R_K + \mu_2 R_G + \\ &\quad\; \mu_3(F(K,G) - w(R_K+R_G) - \chi(\varepsilon)K)\end{aligned}$$

with the co-states $(\lambda_0, \lambda_1, \lambda_2) \neq 0$. Then the first order conditions are

$$\mathcal{L}_{R_K} = \frac{-w\lambda_0}{\tau + F(K,G) - w(R_K+R_G) - \chi(\varepsilon)K} + \lambda_1 A_{R_K} + \mu_1 - w\mu_3 = 0 \tag{2.12}$$

$$\mathcal{L}_{R_G} = \frac{-w\lambda_0}{\tau + F(K,G) - w(R_K+R_G) - \chi(\varepsilon)K} + \lambda_2 B_{R_G} + \mu_2 - w\mu_3 = 0 \tag{2.13}$$

$$\begin{aligned}\dot{\lambda}_1 &= \lambda_1(r - A_K) - \lambda_0 \frac{F_K(K,G) - \chi(\varepsilon)}{\tau + F(K,G) - w(R_K+R_G) - \chi(\varepsilon)K} - \\ &\quad \mu_3(F_K(K,G) - \chi(\varepsilon))\end{aligned} \tag{2.14}$$

$$\begin{aligned}\dot{\lambda}_2 &= \lambda_2(r - B_G) - \lambda_0 \frac{F_G(K,G)}{\tau + F(K,G) - w(R_K+R_G) - \chi(\varepsilon)K} - \\ &\quad \mu_3 F_G(K,G)\end{aligned} \tag{2.15}$$

where subscripts denote partial derivatives of multivariate functions. The complementary slackness conditions are

$$\begin{aligned}\mu_1 &\geq 0 \quad \text{and} \quad 0 = \mu_1 R_K \\ \mu_2 &\geq 0 \quad \text{and} \quad 0 = \mu_2 R_G \\ \mu_3 &\geq 0 \quad \text{and} \quad 0 = \mu_3(F(K,G) - w(R_K+R_G) - \chi(\varepsilon)K).\end{aligned} \tag{2.16}$$

Assumption 1. *Without loss of generality* $\lambda_0 = 1$.

Proof. Let λ_0 be equal to 0. If none of the constraints is active the first order conditions are

$$\begin{aligned}\mathcal{L}_{R_K} &= \lambda_1 A_{R_K}(K,R_K) = 0 \\ \mathcal{L}_{R_G} &= \lambda_2 B_{R_G}(G,R_G) = 0.\end{aligned}$$

Note that A and B are concave functions, which satisfy the Inada conditions and therefore

$$\begin{aligned}\lim_{R_K\to 0} A_{R_K} = \infty \quad &\text{and} \quad \lim_{R_K\to\infty} A_{R_K} = 0 \\ \lim_{R_G\to 0} B_{R_G} = \infty \quad &\text{and} \quad \lim_{R_G\to\infty} B_{R_G} = 0.\end{aligned}$$

Consequently, the first order conditions only hold for

$$\lambda_1 = \lambda_2 = 0$$

which, however, is contradictory to

$$(\lambda_0, \lambda_1, \lambda_2) \neq 0$$

and therefore $\lambda_0 > 0$. Adequate standardization yields

$$\tilde{\lambda_0} = \lambda_0 \frac{1}{\lambda_0} = 1$$

which proofs the assumption within the admissible domain.

To prove the assumption for the boundary arc case one can show that

$$C(t) = bK^{\alpha_1}G^{\alpha_2} - w(R_K + R_G) - a\varepsilon^{\beta}K = 0$$

with $R_K(t) > 0$ or $R_G(t) > 0$ cannot be optimal for $t \in [t_1, t_2)$ with $t_1 < t_2$. Let

$$(K^*, G^*, R_K{}^*, R_G{}^*, \lambda_1{}^*, \lambda_2{}^*) \quad \text{with} \quad R_K > 0, R_G > 0 \quad \text{and} \quad C = 0$$

be steady state of the canonical system,

$$0 \leq t_1 \leq \infty \quad \text{and} \quad (K(.), G(.), R_K(.), R_G(.), \lambda_1(.), \lambda_2(.))$$

be a solution of the according differential algebraic equations (DAEs) on the interval $[t_1, \infty)$ which satisfies

$$\begin{aligned}
(K(t_1), G(t_1), R_K(t_1), R_G(t_1), \lambda_1(t_1), \lambda_2(t_1)) &\neq (K^*, G^*, R_K{}^*, R_G{}^*, \lambda_1{}^*, \lambda_2{}^*) \\
\lim_{t\to\infty}(K(t), G(t), R_K(t), R_G(t), \lambda_1(t), \lambda_2(t)) &= (K^*, G^*, R_K{}^*, R_G{}^*, \lambda_1{}^*, \lambda_2{}^*).
\end{aligned}$$

Then there exists a $t_2 < \infty$ with $0 \leq t_1 < t_2$ and admissible controls

$$\tilde{R_K}(.) = \frac{R_K{}^*(.)}{2} \quad \text{and} \quad \tilde{R_G}(.) = \frac{R_G{}^*(.)}{2}$$

with corresponding $(\tilde{K}(.), \tilde{G}(.))$ and initial states $(\tilde{K}(t_1), \tilde{G}(t_1)) = (K(t_1), G(t_1))$ which satisfy

$$\begin{aligned}
C(\tilde{K}(.), \tilde{G}(.), \tilde{R_K}(.), \tilde{R_G}(.)) &> 0 \\
\int_{t_1}^{t_2} e^{-rt} \ln(C(\tilde{K}(t), \tilde{G}(t), \tilde{R_K}(t), \tilde{R_G}(t)) + \tau) dt &> \\
\int_{t_1}^{\infty} e^{-rt} \ln(C(K(t), G(t), R_K(t), R_G(t)) + \tau) dt &= 0.
\end{aligned}$$

Due to the fact that the value of the target function for the solution

$$(K(.), G(.), R_K(.), R_G(.))$$

is zero,

$$(\tilde{K}(.), \tilde{G}(.), \tilde{R_K}(.), \tilde{R_G}(.))$$

obtains a higher target function value even on the interval $[t_1, t_2]$ which proves that $C(t) = 0$ with $R_K(t) > 0$ or $R_G(t) > 0$ cannot be optimal.

The case $C(t) = R_K(t) = R_G(t) = 0$ for $K(0) > 0$, $G(0) > 0$ contains the necessary conditions

$$\begin{aligned}
\dot{K} &= -\phi K = 0 \\
\dot{G} &= -\psi G = 0 \\
\dot{C} &= -bK^{\alpha_1} G^{\alpha_2} (\alpha_1 \phi + \alpha_2 \psi) + a\varepsilon^{\beta} \phi K = 0.
\end{aligned}$$

There exists only one point of the state-space that satisfies all these conditions, which is $(K,G) = (0,0)$. Inserting the two solutions

$$K(t) = K_0\, e^{-\phi t} \quad \text{and} \quad G(t) = G_0\, e^{-\psi t} \quad \text{with} \quad K_0 = K(0), G_0 = G(0)$$

in $C(t)$ yields

$$C(t) = b(K_0\, e^{-\phi t})^{\alpha_1} (G_0\, e^{-\psi t})^{\alpha_2} - a\varepsilon^{\beta} (K_0\, e^{-\phi t}) = 0$$

for which no solution with $K_0, G_0 > 0$ on a whole time interval (t_1, t_2) can be found. Therefore, this case can be dismissed. There exists no admissible path which starts at an initial value with $K, G > 0$ and leads to this steady state, which consequently proves that $C = 0$ cannot hold on a time interval (t_1, t_2).

On the other hand, if $C > 0$ and one or both of the control constraints $R_K = 0$ and $R_G = 0$ is/are active, one can show that there does not exist a path which leads to the according equilibrium.

If $R_K = R_G = 0$, note that A and B additionally satisfy $A(K,0) < 0$ and $B(G,0) < 0$. Therefore

$$\lim_{t\to\infty} K(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} G(t) = 0$$

and consequently, also

$$\lim_{t\to\infty} C(t) = bK(t)^{\alpha_1} G(t)^{\alpha_2} - a\varepsilon^{\beta} K(t) = 0.$$

However, considering $\dot{C}(t)$, which is given as

$$\dot{C}(t) = -bK(t)^{\alpha_1} G(t)^{\alpha_2} (\alpha_1 \phi + \alpha_2 \psi) + a\varepsilon^{\beta} \phi K(t),$$

then yields

$$\lim_{t\to\infty} \dot{C}(t) > 0. \tag{2.17}$$

This shows, that there cannot exist a path leading to the only equilibrium $K(t) = G(t) = 0$ with $C(t) = 0$.

In case $R_G = 0$ and $R_K > 0$, $B(G,0) < 0$ yields again

$$\lim_{t\to\infty} G(t) = 0.$$

Additionally, to satisfy the budget constraint $C(t) \geq 0$, K has to be bounded because

$$\lim_{K\to\infty} bK^{\alpha_1}G^{\alpha_2} - wR_K - a\varepsilon^{\beta}K = -\infty.$$

Hence,

$$\lim_{t\to\infty} bK^{\alpha_1}G^{\alpha_2} = 0,$$

which finally implies that

$$\lim_{t\to\infty} R_K(t) = \lim_{t\to\infty} C(t) = 0,$$

and therefore (2.17) holds which again excludes the existence of such a path.

In the third and last possible case $R_K = 0$ and $R_G > 0$. Due to $A(K,0) < 0$,

$$\lim_{t\to\infty} K(t) = 0$$

holds. Assume that

$$\lim_{t\to\infty} G(t) = \infty.$$

Then R_G has to satisfy

$$R_G > \left(\frac{\psi}{e}G^{1-\sigma_1}\right)^{\frac{1}{\sigma_2}}$$

so that $\dot{G} > 0$. Because $\alpha_2 < \frac{1-\sigma_1}{\sigma_2}$ holds this implies that

$$\lim_{G\to\infty} bK^{\alpha_1}G^{\alpha_2} - w\left(\frac{\psi}{e}G^{1-\sigma_1}\right)^{\frac{1}{\sigma_2}} = -\infty.$$

Hence, G has to be bounded. The same argumentation as in the previous case shows that such a path cannot exist which finally implies that the case $C > 0$ with $R_K = 0$ and/or $R_G = 0$ can be dismissed.

Therefore only

$$C(0) = 0 \quad \text{and} \quad C(t) > 0 \quad \text{for} \quad t > 0$$

or

$$\lim_{t\to\infty} C(t) = 0 \quad \text{with} \quad C(t) > 0 \quad \text{for} \quad t < \infty$$

have to be considered. The first case occurs if $(K(0), G(0))$ initially start at the curve separating the admissible and inadmissible regions in the state space. The latter case yields

$$\begin{aligned} \mathcal{L}_{R_K} &= \lambda_1 A_{R_K}(K, R_K) = 0 \\ \mathcal{L}_{R_G} &= \lambda_2 B_{R_G}(G, R_G) = 0 \end{aligned}$$

for which the assumption already has been proven. □

For this reason, λ_0 can be omitted in the following analysis. For the derivation of the canonical system one has to distinguish between the different cases of an interior arc and a boundary arc. In the first case none of the constraints are active and, due to the complementary slackness conditions in 2.16, $(\mu_1, \mu_2, \mu_3) = 0$. Hence, an optimal control should maximize the current value Hamiltonian, i.e.

$$({R_K}^*, {R_G}^*) = \arg \max_{(R_K, R_G)} \mathcal{H}$$

and therefore

$$\mathcal{L}_{R_K} = \mathcal{H}_{R_K} = 0 \tag{2.18}$$

$$\mathcal{L}_{R_G} = \mathcal{H}_{R_G} = 0 \quad . \tag{2.19}$$

To prove that the Hamiltonian is strict concave, the positivity of the co-states is necessary which can be shown by solving (2.18) and (2.19) for λ_1 and λ_2 respectively. This yields

$$\begin{aligned} \lambda_1 &= \frac{w}{(\tau + F(K,G) - w(R_K + R_G) - a\varepsilon^{\beta} K) A_{R_K}(K, R_K)} > 0 \\ \lambda_2 &= \frac{w}{(\tau + F(K,G) - w(R_K + R_G) - a\varepsilon^{\beta} K) B_{R_G}(G, R_G)} > 0. \end{aligned}$$

Note that $A_{R_K R_K}(K, R_K) < 0$ and $B_{R_G R_G}(G, R_G) < 0$. The Hessian matrix of the Hamiltonian

$$H = \begin{pmatrix} -\frac{w^2}{(\tau+F(K,G)-w(R_K+R_G)-\chi(\varepsilon)K)^2} + \lambda_1 A_{R_K R_K}(K,R_K) & -\frac{w^2}{(\tau+F(K,G)-w(R_K+R_G)-\chi(\varepsilon)K)^2} \\ -\frac{w^2}{(\tau+F(K,G)-w(R_K+R_G)-\chi(\varepsilon)K)^2} & -\frac{w^2}{(\tau+F(K,G)-w(R_K+R_G)-\chi(\varepsilon)K)^2} + \lambda_2 B_{R_G R_G}(G,R_G) \end{pmatrix}$$

therefore is negative definite and the Hamiltonian $\mathcal{H}$ is strict concave.

The optimality conditions in (2.18) and (2.19) allow to derive control functions depending on co-state and state variables (cf. conditions (2.12) and (2.13))

$$\begin{aligned} R_K &= R_K(K, G, \lambda_1, \lambda_2) \\ R_G &= R_G(K, G, \lambda_1, \lambda_2). \end{aligned} \tag{2.20}$$

Substituting these control functions into the state dynamics (2.3) and (2.4) as well as into the adjoint equations (2.14) and (2.15) the canonical system in the state-co-state-space is given as

$$\begin{aligned} \dot{K} &= A(K, R_K(K, G, \lambda_1, \lambda_2)) \\ \dot{G} &= B(G, R_G(K, G, \lambda_1, \lambda_2)) \\ \dot{\lambda_1} &= \lambda_1(r - A_K) - \frac{F_K(K,G) - \chi(\varepsilon)}{\tau + F(K,G) - w(R_K(K,G,\lambda_1,\lambda_2) + R_G(K,G,\lambda_1,\lambda_2)) - \chi(\varepsilon)K} \\ \dot{\lambda_2} &= \lambda_2(r - B_G) - \frac{F_G(K,G)}{\tau + F(K,G) - w(R_K(K,G,\lambda_1,\lambda_2) + R_G(K,G,\lambda_1,\lambda_2)) - \chi(\varepsilon)K}. \end{aligned}$$

However, from an application orientated point of view it is often more convenient to transform the canonical system from the state-co-state-space into the state-control-space. Within this representation immediate interpretation of the results is more convenient (see Grass, Caulkins, Feichtinger, Tragler, and Behrens [2008]). Additionally, inserting the model functions from above, the two controls from (2.12) and (2.13) are given only implicitly. Therefore, the derivation of the canonical system in the state-control space is even necessary. Considering the model functions from above, the first order conditions are

$$\mathcal{H}_{R_K} = -\frac{w}{\tau + bK^{\alpha_1}G^{\alpha_2} - w(R_K+R_G) - a\varepsilon^{\beta}K} + \lambda_1(dK^{\delta_1}\delta_2 {R_K}^{\delta_2-1}) = 0 \quad (2.21a)$$

$$\mathcal{H}_{R_G} = -\frac{w}{\tau + bK^{\alpha_1}G^{\alpha_2} - w(R_K+R_G) - a\varepsilon^{\beta}K} + \lambda_2(eG^{\sigma_1}\sigma_2 {R_G}^{\sigma_2-1}) = 0 \quad (2.21b)$$

$$\dot{\lambda_1} = \lambda_1(r - d\delta_1 K^{\delta_1-1}{R_K}^{\delta_2} + \phi) - \frac{\alpha_1 bK^{\alpha_1-1}G^{\alpha_2} - a\varepsilon^{\beta}}{\tau + bK^{\alpha_1}G^{\alpha_2} - w(R_K+R_G) - a\varepsilon^{\beta}K} \quad (2.21c)$$

$$\dot{\lambda_2} = \lambda_2(r - e\sigma_1 G^{\sigma_1-1}{R_G}^{\sigma_2} + \psi) - \frac{\alpha_2 bK^{\alpha_1}G^{\alpha_2-1}}{\tau + bK^{\alpha_1}G^{\alpha_2} - w(R_K+R_G) - a\varepsilon^{\beta}K}. \quad (2.21d)$$

Solving 2.21a and 2.21b for λ_1 and λ_2 instead of the controls yields

$$\begin{aligned} \lambda_1(K,G,R_K,R_G) &= \frac{w}{(\tau + bK^{\alpha_1}G^{\alpha_2} - w(R_K+R_G) - a\varepsilon^{\beta}K)dK^{\delta_1}\delta_2{R_K}^{\delta_2-1}} \\ \lambda_2(K,G,R_K,R_G) &= \frac{w}{(\tau + bK^{\alpha_1}G^{\alpha_2} - w(R_K+R_G) - a\varepsilon^{\beta}K)eG^{\sigma_1}\sigma_2{R_G}^{\sigma_2-1}}. \end{aligned} \quad (2.22)$$

By using the total time derivatives of the co-states

$$\begin{aligned} \dot{\lambda_1} &= \lambda_{1_K}\dot{K} + \lambda_{1_G}\dot{G} + \lambda_{1_{R_K}}\dot{R_K} + \lambda_{1_{R_G}}\dot{R_G} \\ \dot{\lambda_2} &= \lambda_{2_K}\dot{K} + \lambda_{2_G}\dot{G} + \lambda_{2_{R_K}}\dot{R_K} + \lambda_{2_{R_G}}\dot{R_G} \end{aligned} \quad (2.23)$$

two equations for the control dynamics can be obtained. Together with the adjoint dynamics in (2.21c) and (2.21c) these control dynamics are given as

$$\begin{aligned} \dot{R_K} &= -\frac{\dot{\lambda_2}\lambda_{1_{R_G}} - \dot{\lambda_1}\lambda_{2_{R_G}} + \dot{G}(\lambda_{1_G}\lambda_{2_{R_G}} - \lambda_{1_{R_G}}\lambda_{2_G}) + \dot{K}(\lambda_{1_K}\lambda_{2_{R_G}} - \lambda_{1_{R_G}}\lambda_{2_K})}{\lambda_{1_{R_K}}\lambda_{2_{R_G}} - \lambda_{1_{R_G}}\lambda_{2_{R_K}}} \\ \dot{R_G} &= -\frac{\dot{\lambda_1}\lambda_{2_{R_K}} - \dot{\lambda_2}\lambda_{1_{R_K}} + \dot{G}(\lambda_{1_{R_K}}\lambda_{2_G} - \lambda_{1_G}\lambda_{2_{R_K}}) + \dot{K}(\lambda_{1_{R_K}}\lambda_{2_K} - \lambda_{1_K}\lambda_{2_{R_K}})}{\lambda_{1_{R_K}}\lambda_{2_{R_G}} - \lambda_{1_{R_G}}\lambda_{2_{R_K}}} \end{aligned} \quad (2.24)$$

which yields the canonical system

$$\begin{aligned} \dot{R_K} = {} & \frac{{D_1}^2 D_2^2 {R_G}^2 {R_K}^2 Y^3}{w^2\left(d(\delta_2-1)\delta_2 K^{\delta_1}{R_K}^{\delta_2}\left(D_2{R_G}^2 w - eY(\sigma_2-1)\sigma_2 G^{\sigma_1}{R_G}^{\sigma_2}\right) + D_1 e{R_K}^2 w(\sigma_2-1)\sigma_2 G^{\sigma_1}{R_G}^{\sigma_2}\right)} \\ & \cdot\Bigg\{\Bigg[\left(eY(\sigma_2-1)\sigma_2 G^{\sigma_1}{R_G}^{\sigma_2-2} - D_2 w\right)\left(D_1(a\varepsilon^{\beta} - b\alpha_1 G^{\alpha_2}K^{\alpha_1-1}) + w\left(-d\delta_1 K^{\delta_1-1}{R_K}^{\delta_2} + r + \phi\right)\right) + \\ & D_2 w\left(w\left(-e\sigma_1 G^{\sigma_1-1}{R_G}^{\sigma_2} + r + \psi\right) - bD_2\alpha_2 G^{\alpha_2-1}K^{\alpha_2}\right)\Bigg]\frac{w}{D_1{D_2}^2 Y^3} + \dot{G}T_1 + \dot{K}T_2\Bigg\} \end{aligned}$$

$$\begin{aligned} \dot{R_G} = {} & \frac{{D_1}^2 D_2^2 {R_G}^2 {R_K}^2 Y^3}{w^2\left(d(\delta_2-1)\delta_2 K^{\delta_1}{R_K}^{\delta_2}\left(D_2{R_G}^2 w - eY(\sigma_2-1)\sigma_2 G^{\sigma_1}{R_G}^{\sigma_2}\right) + D_1 e{R_K}^2 w(\sigma_2-1)\sigma_2 G^{\sigma_1}{R_G}^{\sigma_2}\right)} \\ & \cdot\Bigg\{\Bigg[\left(dY(\delta_2-1)\delta_2 K^{\delta_1}{R_K}^{\delta_2-2} - D_1 w\right)\left(w\left(-e\sigma_1 G^{\sigma_1-1}{R_G}^{\sigma_2} + r + \psi\right) - bD_2\alpha_2 G^{\alpha_2-1}K^{\alpha_2}\right) + \\ & D_1 w\left(aD_1\varepsilon^{\beta} - bD_1\alpha_1 G^{\alpha_2}K^{\alpha_1-1} - dw\delta_1 K^{\delta_1-1}{R_K}^{\delta_2} + w(r+\phi)\right)\Bigg]\frac{w}{D_1{D_2}^2 Y^3} + \dot{G}T_3 + \dot{K}T_4\Bigg\} \end{aligned}$$

$$\begin{aligned}\dot{K} &= dK^{\delta_1}R_K{}^{\delta_2} - \phi K \\ \dot{G} &= eG^{\sigma_1}R_G{}^{\sigma_2} - \psi G\end{aligned} \tag{2.25}$$

with

$$\begin{aligned}
T_1 &= \frac{ew^2\sigma_2 G^{\sigma_1-1}R_G{}^{\sigma_2-2}\left(b\alpha_2(\sigma_2-1)G^{\alpha_2}K^{\alpha_1}+R_G w\sigma_1\right)}{D_1 D_2{}^2 Y^3} \\
T_2 &= -\frac{w^2}{D_1{}^2 D_2{}^2 K R_G{}^2 R_K Y^3}\Big(d\delta_1\delta_2 K^{\delta_1}R_K{}^{\delta_2}\left(D_2 R_G{}^2 w - eY(\sigma_2-1)\sigma_2 G^{\sigma_1}R_G{}^{\sigma_2}\right) - \\
&\quad D_1 e R_K(\sigma_2-1)\sigma_2 G^{\sigma_1}R_G{}^{\sigma_2}\left(b\alpha_1 G^{\alpha_2}K^{\alpha_1} - aK\varepsilon^{\beta}\right)\Big) \\
T_3 &= \frac{w^2}{D_1{}^2 D_2{}^2 G R_G R_K{}^2 Y^3}\Big(d(\delta_2-1)\delta_2 K^{\delta_1}R_K{}^{\delta_2}\left(bD_2 R_G\alpha_2 G^{\alpha_2}K^{\alpha_1} + eY\sigma_1\sigma_2 G^{\sigma_1}R_G{}^{\sigma_2}\right) - \\
&\quad D_1 e R_K{}^2 w\sigma_1\sigma_2 G^{\sigma 1}R_G{}^{\sigma_2}\Big) \\
T_4 &= \frac{dw^2\delta_2 K^{\delta_1-1}R_K{}^{\delta_2-2}\left((\delta_2-1)\left(b\alpha_1 G^{\alpha_2}K^{\alpha_1} - aK\varepsilon^{\beta}\right) + R_K w\delta_1\right)}{D_1{}^2 D_2 Y^3} \\
Y &= \tau + bK^{\alpha_1}G^{\alpha_2} - w(R_K+R_G) - a\varepsilon^{\beta}K
\end{aligned}$$

and D_1 and D_2 being the first derivatives of the state dynamics with respect to the corresponding control

$$\begin{aligned}D_1 &= dK^{\delta_1}\delta_2 R_K{}^{\delta_2-1} \\ D_2 &= eG^{\sigma_1}\sigma_2 R_G{}^{\sigma_2-1}.\end{aligned}$$

In the boundary arc case, the optimal controls do not necessarily maximize the Hamiltonian, i.e. $\mathcal{H}_{R_K}=0$ and $\mathcal{H}_{R_G}=0$ might not be fulfilled in the optimum. Hence, the approach to derive the canonical system in the state-control-space, as done in (2.21a)-(2.25), cannot be used. Instead, the optimal controls have to maximize the Lagrangian. Therefore, in case of one or even both control constraints being active, the partial derivatives of the Lagrange function with respect to the controls, $\mathcal{L}_{R_K}=0$ and $\mathcal{L}_{R_G}=0$, together with the active constraint equations yield the corresponding Lagrange multipliers and the control dynamics, while the adjoint equations can be used to calculate the co-states. The state dynamics remain the same just with the according control values inserted, meaning $R_K=0$ and/or $R_G=0$. If, however, the mixed path constraint is fulfilled, the derivation of the according canonical system is more extensive. Assuming that the mixed path constraint is the only constraint being active, meaning that R_K and R_G are positive, the following DAEs have to be solved

$$\begin{aligned}
\dot{K} &= A(K,G,R_K,R_G) \\
\dot{G} &= B(K,G,R_K,R_G) \\
\dot{\lambda}_1 &= \lambda_1(r-A_K) - \frac{F_K(K,G)-\chi(\varepsilon)}{\tau+F(K,G)-w(R_K+R_G)-\chi(\varepsilon)K} - \mu_3(F_K(K,G)-\chi(\varepsilon))
\end{aligned}$$

$$
\begin{aligned}
\dot{\lambda_2} &= \lambda_2(r-B_G) - \frac{F_G(K,G)}{\tau+F(K,G)-w(R_K+R_G)-\chi(\varepsilon)K} - \mu_3 F_G(K,G) \\
\mathcal{L}_{R_K} &= \mathcal{H}_{R_K} + \mu_3 C_{R_K} = 0 \\
\mathcal{L}_{R_G} &= \mathcal{H}_{R_G} + \mu_3 C_{R_G} = 0 \\
0 &= C(K,G,R_K,R_G)
\end{aligned}
$$

where C defines the mixed path constraint and this time $\mu_3 \geq 0$. In order to transform these DAEs into ordinary differential equations (ODEs), total time derivatives have to be considered:

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}_{R_K} &= (\mathcal{H}_{R_K K} + \mu_3 C_{R_K K})\dot{K} + (\mathcal{H}_{R_K G} + \mu_3 C_{R_K G})\dot{G} + (\mathcal{H}_{R_K R_K} + \mu_3 C_{R_K R_K})\dot{R_K} + \\
&\quad (\mathcal{H}_{R_K R_G} + \mu_3 C_{R_K R_G})\dot{R_G} + \dot{\lambda_1}\mathcal{H}_{R_K \lambda_1} + \dot{\lambda_2}\mathcal{H}_{R_K \lambda_2} + \dot{\mu_3} C_{R_K} = 0 \\
\frac{d}{dt}\mathcal{L}_{R_G} &= (\mathcal{H}_{R_G K} + \mu_3 C_{R_G K})\dot{K} + (\mathcal{H}_{R_G G} + \mu_3 C_{R_G G})\dot{G} + (\mathcal{H}_{R_G R_K} + \mu_3 C_{R_G R_K})\dot{R_K} + \\
&\quad (\mathcal{H}_{R_G R_G} + \mu_3 C_{R_G R_G})\dot{R_G} + \dot{\lambda_1}\mathcal{H}_{R_G \lambda_1} + \dot{\lambda_2}\mathcal{H}_{R_G \lambda_2} + \dot{\mu_3} C_{R_G} = 0 \\
\frac{d}{dt}C &= C_K \dot{K} + C_G \dot{G} + C_{R_K}\dot{R_K} + C_{R_G}\dot{R_G} = 0.
\end{aligned} \tag{2.26}
$$

Inserting the according equations for $\dot{K}$,$\dot{G}$,$\dot{\lambda_1}$ and $\dot{\lambda_2}$ and solving the previous equations for $\dot{R_K}$,$\dot{R_G}$ and $\dot{\mu_3}$ yields the equations for the controls. Note, however, that $\dot{\lambda_1}$ and $\dot{\lambda_2}$ include λ_1 and λ_2 respectively, and therefore also $\dot{R_K}$, $\dot{R_G}$ are both dependent on the co-state. For this reason the reduction of the canonical system to four dimensions is not possible anymore and one has to consider all six dimensions which then are given as

$$
\begin{aligned}
\dot{K} &= A(K,G,R_K,R_G) \\
\dot{G} &= B(K,G,R_K,R_G) \\
\dot{\lambda_1} &= r\lambda_1 - T_K - \lambda_1 A_K - \frac{T_{R_K} + \lambda_1 A_{R_K}}{w}(F_K - \chi(\varepsilon)) \\
\dot{\lambda_2} &= r\lambda_2 - T_G - \lambda_2 B_G - \frac{T_{R_G} + \lambda_2 B_{R_G}}{w} F_G \\
\dot{R_K} &= Y(K,G,R_K,R_G,\lambda_1,\lambda_2) \\
\dot{R_G} &= V(K,G,R_K,R_G,\lambda_1,\lambda_2)
\end{aligned} \tag{2.27}
$$

where T denotes the target function

$$
T = \ln(\tau + F(K,G) - w(R_K+R_G) - \chi(\varepsilon)K) + u(\varepsilon)
$$

and Y and V denote the obtained results for the control dynamics, which I omit here because they are very complex and don't allow any immediate insights.

2.3.2 Steady States

According to the maximum principle (see Grass et al. [2008]), in the following the maximization problem (2.11) subject to (2.11a)-(2.11l) will be solved by determining the stable manifolds arising from the canonical system which has been derived in the previous section. The steady states of the canonical system are determined by solving $\dot{K} = 0$, $\dot{G} = 0$, $\dot{R_K} = 0$, $\dot{R_G} = 0$ simultaneously. Considering the two state dynamics, the according roots are obvious immediately:

$$
\begin{aligned}
K_{\dot{K}} &= \left(\frac{\phi}{d{R_K}^{\delta_2}}\right)^{\frac{1}{\delta_1-1}} \\
R_{K_{\dot{K}}} &= \left(\frac{\phi}{dK^{\delta_1-1}}\right)^{\frac{1}{\delta_2}} \\
G_{\dot{G}} &= \left(\frac{\psi}{e{R_G}^{\sigma_2}}\right)^{\frac{1}{\sigma_1-1}} \\
R_{G_{\dot{G}}} &= \left(\frac{\psi}{eG^{\sigma_1-1}}\right)^{\frac{1}{\sigma_2}}
\end{aligned}
\tag{2.28}
$$

where subscripts denote the equation which is set to zero, respectively. Further on, also $K = 0$ and $G = 0$ would obviously be solutions. However, K and G occur in the denominator of $\dot{R_K}$ and $\dot{R_G}$ multiplicatively. Hence, for $K = G = 0$ I find no feasible steady state solution of the canonical system. Anyway, since the intention of environmental policy is not to let complete shut down of production be the only way to cope with introduced environmental standards, the main focus of this thesis lies on the determination of steady states with a positive production output. Inserting the roots in (2.28) together with parameter values into $\dot{R_K}$ and $\dot{R_G}$, the intersection of the isoclines $\dot{R_K} = 0$ and $\dot{R_G} = 0$ determines the steady states. In this first approach only one steady state can be detected, which will be demonstrated in what follows.

2.3.3 Stability

To determine the stability of this steady state, the Jacobian matrix is used, which is given by

$$
J = \begin{pmatrix}
\dot{K}_K & 0 & \dot{K}_{R_K} & 0 \\
0 & \dot{G}_G & 0 & \dot{G}_{R_G} \\
\dot{R}_{K_K} & \dot{R}_{K_G} & \dot{R}_{K_{R_K}} & \dot{R}_{K_{R_G}} \\
\dot{R}_{G_K} & \dot{R}_{G_G} & \dot{R}_{G_{R_K}} & \dot{R}_{G_{R_G}}
\end{pmatrix}, \tag{2.29}
$$

where subscripts denote partial derivatives again. Hence the characteristic polynomial is

$$
P(\mu) = \Big(\dot{K}_{R_K}\dot{R}_{K_K} - (\dot{K}_K - \mu)(\dot{R}_{K_{R_K}} - \mu)\Big)\Big(\dot{G}_{R_G}\dot{R}_{G_G} - (\dot{G}_G - \mu)(\dot{R}_{G_{R_G}} - \mu)\Big), \tag{2.30}
$$

which determines four eigenvalues

$$\begin{aligned}
\mu_{1,2} &= \frac{\dot{K}_K + \dot{R}_{K_{R_K}}}{2} \pm \underbrace{\sqrt{\frac{(\dot{K}_K - \dot{R}_{K_{R_K}})^2}{4} + \dot{K}_{R_K}\dot{R}_{K_K}}}_{X_1} \\
\mu_{3,4} &= \frac{\dot{G}_G + \dot{R}_{G_{R_G}}}{2} \pm \underbrace{\sqrt{\frac{(\dot{G}_G - \dot{R}_{G_{R_G}})^2}{4} + \dot{G}_{R_G}\dot{R}_{G_G}}}_{X_2}.
\end{aligned} \tag{2.31}$$

Considering the sign of the determinant

$$\det J = \underbrace{(\dot{K}_{R_K}\dot{R}_{K_K} - \dot{K}_K\dot{R}_{K_{R_K}})}_{:=Z_1}\underbrace{(\dot{G}_{R_G}\dot{R}_{G_G} - \dot{G}_G\dot{R}_{G_{R_G}})}_{:=Z_2},$$

the cases summarized in Table 2.2 can be distinguished

<table>
<tr><th colspan="2">det(J)</th><th>Discriminant</th><th>Eigenvalues (EV)</th><th>Signs of real part of EV</th><th>Behavior</th></tr>
<tr><td rowspan="4">>0</td><td>$Z_1, Z_2 > 0$</td><td>$X_1, X_2 > 0$</td><td>real with opposite signs</td><td>(+,-,+,-)</td><td>Saddle point</td></tr>
<tr><td rowspan="3">$Z_1, Z_2 < 0$</td><td>$X_1, X_2 > 0$</td><td>real with same signs</td><td>(-,-,+,+)</td><td>Saddle point</td></tr>
<tr><td>$X_1, X_2 < 0$</td><td>complex</td><td>(-,-,-,-)</td><td>Stable</td></tr>
<tr><td>$sgn(X_1) \neq sgn(X_2)$</td><td>real and complex</td><td>(+,+,+,+)</td><td>Repelling</td></tr>
<tr><td rowspan="4"><0</td><td rowspan="2">$Z_1 > 0, Z_2 < 0$</td><td>$X_1, X_2 > 0$</td><td>real</td><td rowspan="2">(+,+,+,-)</td><td rowspan="4">Unstable</td></tr>
<tr><td>$X_1 < 0, X_2 > 0$</td><td>real and complex</td></tr>
<tr><td rowspan="2">$Z_1 < 0, Z_2 > 0$</td><td>$X_1, X_2 > 0$</td><td>real</td><td rowspan="2">(-,-,-,+)</td></tr>
<tr><td>$X_1 > 0, X_2 < 0$</td><td>real and complex</td></tr>
</table>

Table 2.2: Possible cases of stability.

2.3.4 The Laissez-Faire Scenario and the Introduction of Environmental Policy

At first, an economy is considered in which no environmental standards at all are imposed, i.e. $\varepsilon = 0$. In this laissez-faire scenario, the agent does not have to fulfill any environmental restrictions and therefore is completely free of abatement costs. However, this comes at the expense of environmental quality and consequently of the utility it yields. Anyway, as long as the utility of consumption is high enough to compensate for the loss of environmental quality, the agent's capital accumulation is somehow conceivable. Due to the fact that green capital is less productive than brown capital it is obvious that the agent will mainly use the polluting capital as much as possible. However, complete abandonment of green capital is not possible due to the assumption of a Cobb Douglas production function, but the green input factor is expected to be at least comparatively low. Figure 2.3 shows that the single steady state is at $K = 29,160$, $G = 4,126$ with control levels $R_K = 4,453$ and $R_G = 1,187$, which is a saddle point according to the first case in Table 2.2. Obviously K is dominant in production. The colored region in Figure 2.3 corresponds to the admissible region according to the mixed path constraint $C \geq 0$.

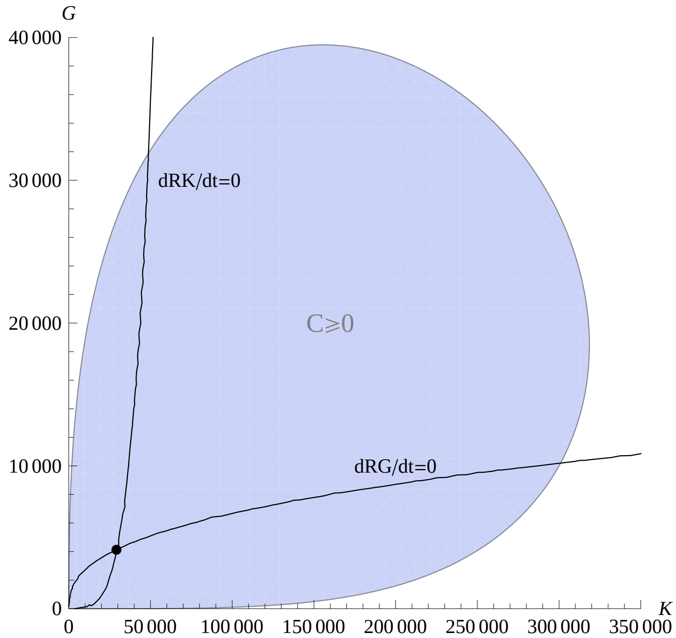


Figure 2.3: Steady state in the laissez-faire scencario for $\alpha_1 = 0.6$ and $\alpha_2 = 0.2$.

In the next step, an economy with a medium environmental quality standard $\varepsilon = 0.4$ is considered. As one can see in Figure 2.4, this causes a big change in the position of the steady state. In this scenario, the saddle point is at $K = 714$, $G = 981$, $R_K = 24$ and $R_G = 96$. Due to the higher abatement cost, brown capital as dominant input factor has become too expensive. Green capital now is an essential substitute, despite its lower productivity. Comparing Figure 2.4 with Figure 2.3 one can see that the admissible region $C \geq 0$ shrinks with increasing ε.

Figure 2.5 finally shows the steady state for the basic model with constant returns to scale (CRS) in the production function, which is at $K = 904,808$, $G = 104,374$, $R_K = 545,908$ and $R_G = 333,154$. One can see that these equilibrium values are quite high, compared to the previous two scenarios. Also the admissible region expands with constant returns instead of decreasing returns to scale.

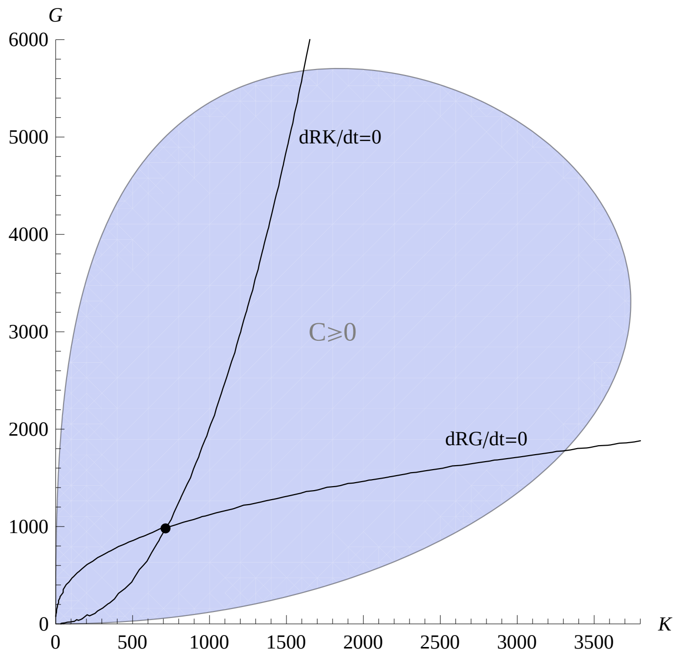


Figure 2.4: Steady state for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $\varepsilon = 0.4$.

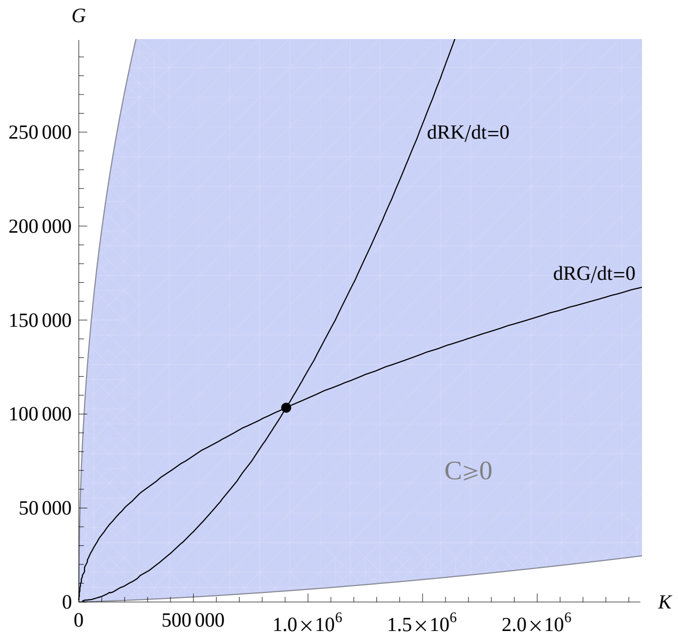


Figure 2.5: Steady state for CRS with $\alpha_1 = 0.7, \alpha_2 = 0.3$ and $\varepsilon = 0.4$.

2.4 Optimal Paths

In this section, the matter of interest is to find trajectories converging towards the equilibrium and to get the corresponding projections that cover a significant part of the (K,G)-plane. For this purpose, the initial value problem approach is used. Hence, initial values for a backward solution of the four-dimensional canonical system need to be constructed first. However, note that only the stable manifold leads directly into the equilibrium. Consequently, this set of starting points has to be very close to the equilibrium, in order to stay on or at least close to the stable manifold. Additionally, also dominant directions in the convergence to the steady state have to be considered. Therefore, an appropriate ellipse around the equilibrium is generated from which these starting points are taken. To take the dominant directions into account, the eigenvectors with negative eigenvalues are used for the calculation which is done by the formula

$$S = E + \frac{e_1}{|e_1|}\cos(\eta) + \frac{e_2}{|e_2|}\sin(\eta) \quad \text{with} \quad \eta \in [0, 2\pi], \tag{2.32}$$

where S is the calculated starting point, E denotes the equilibrium, and e_1 and e_2 are the corresponding eigenvectors. Within this calculation the values of the angle η are close to $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. This comes along with the fact that in those cases $\cos(\eta)$ is close to zero and therefore the dominant directions are weighted less here (cf. Knoll and Zuba [2004]). Based on these constructed initial values the canonical system is solved backwards. The projection of the resulting four-dimensional optimal trajectories onto the (K,G)-plane leads to a phase portrait, from which those trajectories have to be chosen, which correspond to the given initial conditions. In Figure 2.6 the phase portrait for $\varepsilon = 0.4$ is depicted. Here, the crucial and obviously very narrow intervals for the angle η are $[0.4999755\pi, 0.4999756\pi]$ and $[1.500024418\pi, 1.500024419\pi]$.

As one can see in in Figure 2.6, some of the trajectories are divided into two parts. The first part, which is common for all and depicted in gray, corresponds to the backward solution of the system starting from the equilibrium. On the left hand side the trajectories are continued until $K = 0$. On the right hand side, however, continuation aborts when the trajectories reach the boundary of the admissible region subject to the control constraint in (2.5c) where $R_K = 0$. This constraint is depicted in the figure as dashed black line. To enable further continuation of these trajectory paths, R_K is constantly set to zero and calculation continues with the according canonical system where $\dot{R_K} = 0$. These second parts of the trajectories are depicted in black and their continuation is possible until they finally reach the admissible boundary of the mixed path constraint in (2.11e), where consumption, and therefore also utility from consumption, is zero.

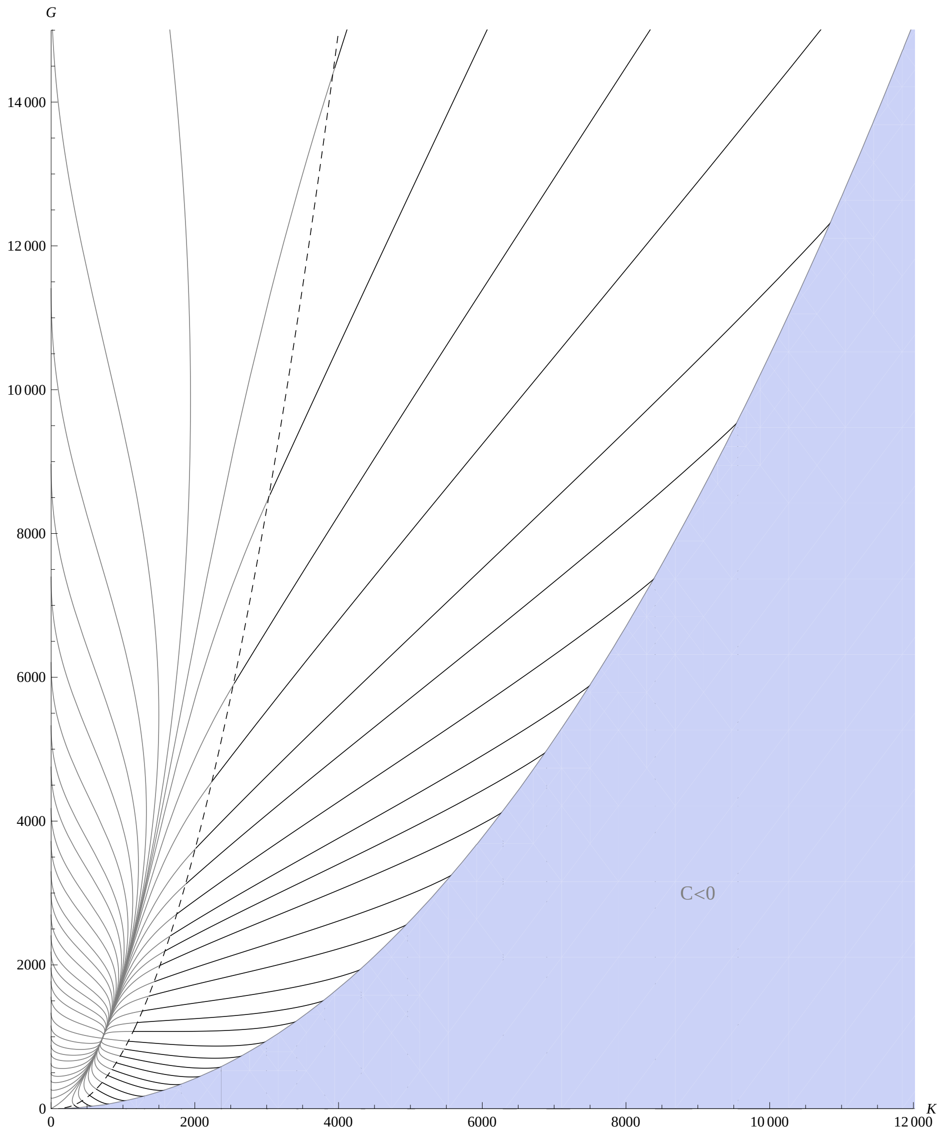


Figure 2.6: Phase portrait in (K,G)-space for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $\varepsilon = 0.4$.

2.4.1 Initial Points with an Equal Level of K and G

Figure 2.7 shows two trajectories from the phase portrait in the (K,G)-plane which both have initial points with almost equal levels of K and G. The first one starts at very low levels of brown and green capital which are smaller than the equilibrium values. Along the path to the equilibrium the levels of both types of capital increase. The second trajectory has its initial point at a high level of brown and green capital above the equilibrium values. Accordingly, the levels of capital decrease along the trajectory while approaching the equilibrium.

Figure 2.8 shows the optimal time paths in K, G, R_K and R_G along the trajectory starting at the lower level of capital. As one can see, the levels of both types of capital increase monotonously while converging towards their equilibrium values, where conventional capital in the beginning is a little bit higher than green capital. Nevertheless, green capital finally gets dominant. Considering the paths of the R&D investments, the levels of R_K and R_G increase very quickly initially.

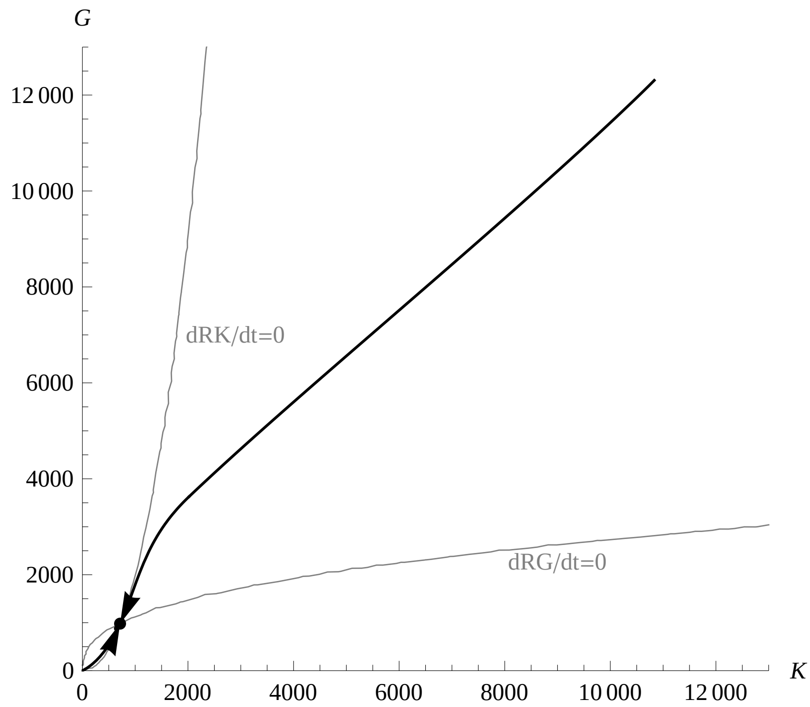
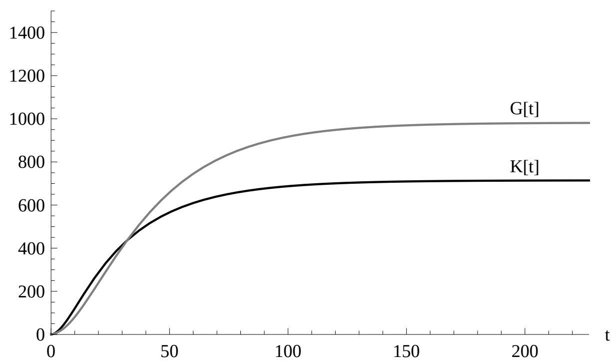


Figure 2.7: Two trajectories for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $\varepsilon = 0.4$ with equal initial capital levels.

Therefore less time is needed to get close to their equilibrium values. In order to cause growth in the capital levels, initially high R&D investments are needed until the positive feedback of the capital stock on itself is effective enough to thwart the negative pressure of depreciation. Note that the level of R_K even decreases after reaching a peak to slow down this positive feedback until growth and depreciation are perfectly balanced close to the equilibrium. Due to the fact that the production elasticity of R_G is less than the one of R_K, the behavior is different here. Higher investments are necessary to achieve the same effects and the R_G level monotonously increases towards the equilibrium value.



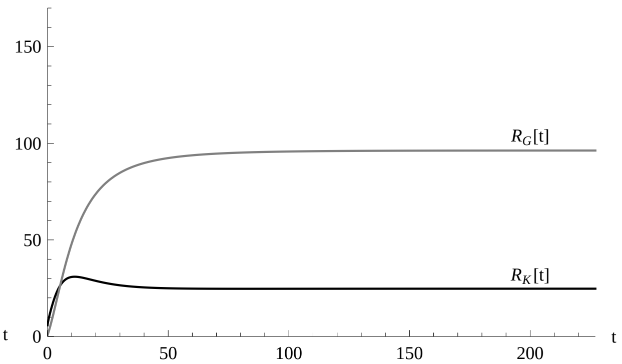


Figure 2.8: Optimal time paths of state and control starting from low capital levels for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $\varepsilon = 0.4$.

In Figure 2.9 the same paths are considered for the trajectory starting at the high capital level. Here the levels of both capitals are decreasing. Due to the almost equal initial level of K and G and the comparatively lower equilibrium level of K, the decline of K is stronger than in green capital. To switch off the positive feedback of K on its own stock completely, and therefore to boost the negative impact of depreciation, R_K initially is even zero and only rises again to stop this decline, but stays at a very low level, though. Due to lower production elasticity the level of green R&D initially rises very quickly up to a peak to stop the negative pressure of depreciation. Then it slightly decreases again to finally remain at a level obviously higher than the one of R_K.

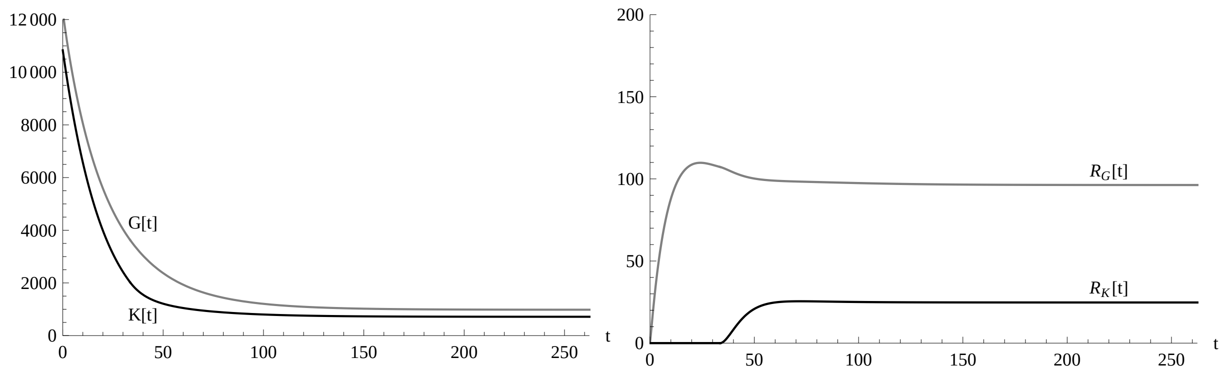


Figure 2.9: Optimal time paths of state and control starting from high capital levels for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $\varepsilon = 0.4$.

2.4.2 Initial Points with One Type of Capital Being Dominant

As mentioned above the initial use of both capital types is assumed due to the use of a Cobb Douglas production function. However, situations in which one type of capital is definitely the dominant input factor, whereas the other one almost equals zero, are certainly a matter of interest. Figure 2.10 shows two trajectories for such initial conditions. One either starts at a green capital-dominated production or in an initial point where K is used almost exclusively as production input. In both cases, the level of the dominant capital lies above the equilibrium values, while the level of the dominated capital is below its equilibrium level.

Figure 2.11 shows the optimal time paths in the case of an initially green capital-dominated production. In contrast to the previous case of an almost balanced initial mix of production, the behavior of the capital levels in this dominated scenario are respectively opposed. Because green capital is dominant here, the level of G decreases while brown capital, starting at a very low level, rises up to the equilibrium value. Considering the R&D investments, the same behavior as in Figure 2.8 can be observed, where R_K rises up to a peak, then falls again and slows down the positive feedback, while R_G increases monotonously. Summarizing this scenario it is interesting to see that R_G is increasing while G is decreasing. In other words, green R&D investments are made so to keep G at a sufficiently high level.

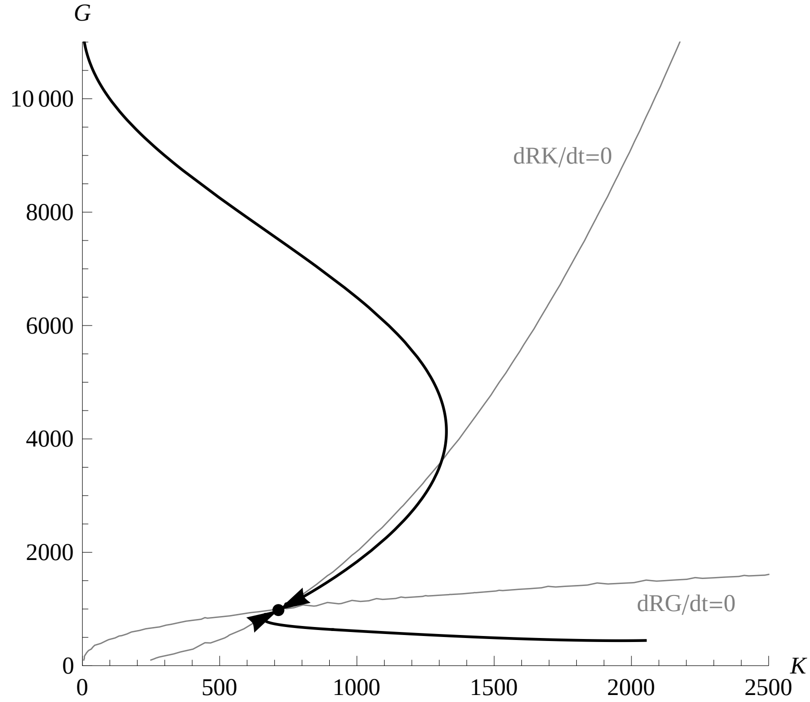


Figure 2.10: Two trajectories starting at a one-capital-type-dominated production for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $\varepsilon = 0.4$.

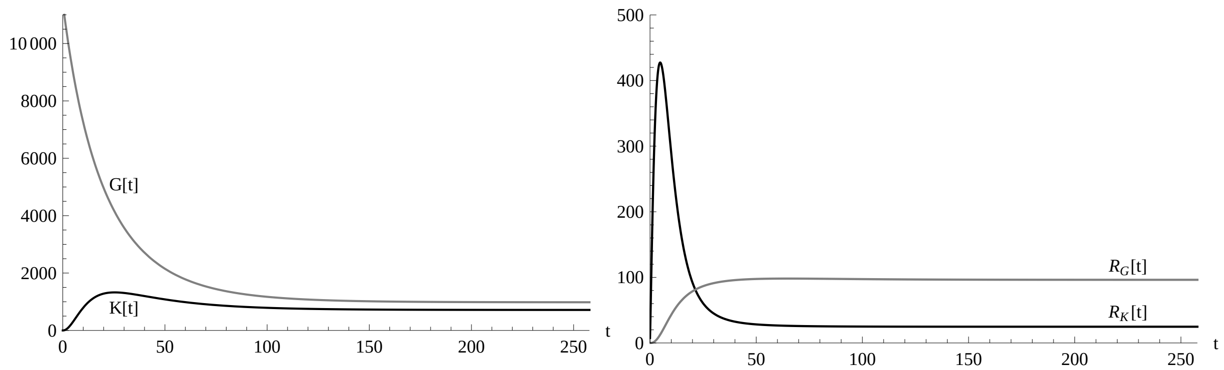


Figure 2.11: Optimal time paths of state and control starting from a definitely green capital-dominated production for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $\varepsilon = 0.4$.

Regarding the case of an initially brown capital-dominated production, the according optimal time paths are depicted in Figure 2.12. Accordingly, in this case K decreases and G rises up to the equilibrium values. Again, R_K is initially zero and rises up to slow down the decline, while R_G rises up to a peak and then slightly decreases.

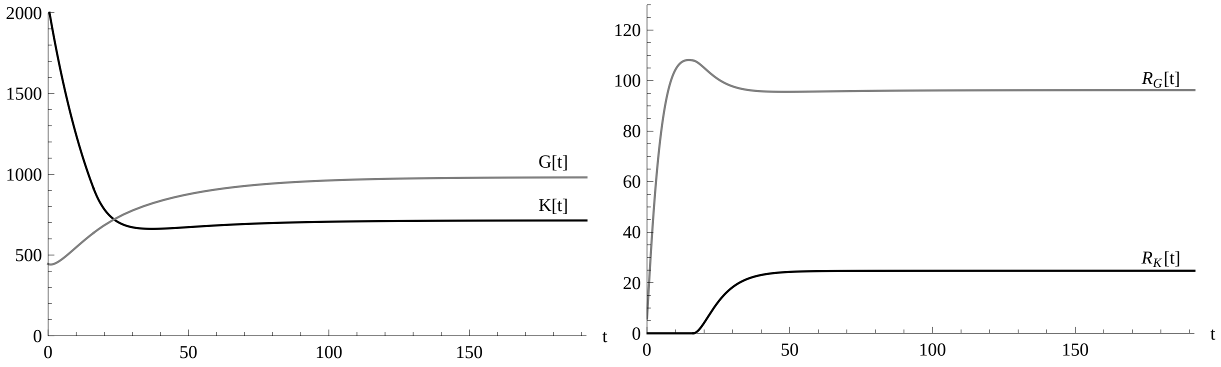


Figure 2.12: Optimal time paths of state and control starting from a definitely brown capital-dominated production for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $\varepsilon = 0.4$.

2.5 Bifurcation Analysis

In the previous sections, equilibria for specific values of ε were considered. However, the main focus of this thesis is the investigation of the influence of the required environmental standards on the capital accumulation and hence on the production. In order to do this, bifurcation analysis is used with ε being the varied parameter. Although only one steady state has been detected so far, and hence the bifurcation diagram for the basic model is quite simple, it gives a first idea about the interrelation of the environmental quality and the usage of both types of capital as input in production.

Figure 2.13 depicts the change of the equilibrium values under the variation of the environmental quality imposed by the government. For $\varepsilon = 0$ (laissez-faire scenario) K is clearly dominant in production as already mentioned above. As one can see, increasing ε results in an immediate decrease of K due to the rising abatement cost per unit of brown capital. Also G decreases with growing environmental quality. This might seem a little bit astonishing at first sight, but comes along with the fact that, due to the Cobb Douglas production function, a complete abandonment of K as production input is impossible, and therefore a sufficiently small level of K has to be used which at the same time has an increasingly absorbing impact on the productivity of G. However, this decrease is much smaller than the one of K. The point of special interest is at $\varepsilon = 0.362$. At this point, abatement gets so expensive that the use of green capital as dominant production input is more advantageous. In Figure 2.14 the changes of the equilibrium values of R_K and R_G over ε are shown. They behave quite similarly. Initially, R_K is dominant until abatement gets too expensive and higher investments in green R&D are optimal. This change happens already at $\varepsilon = 0.263$, i.e. earlier than for the capital stocks.

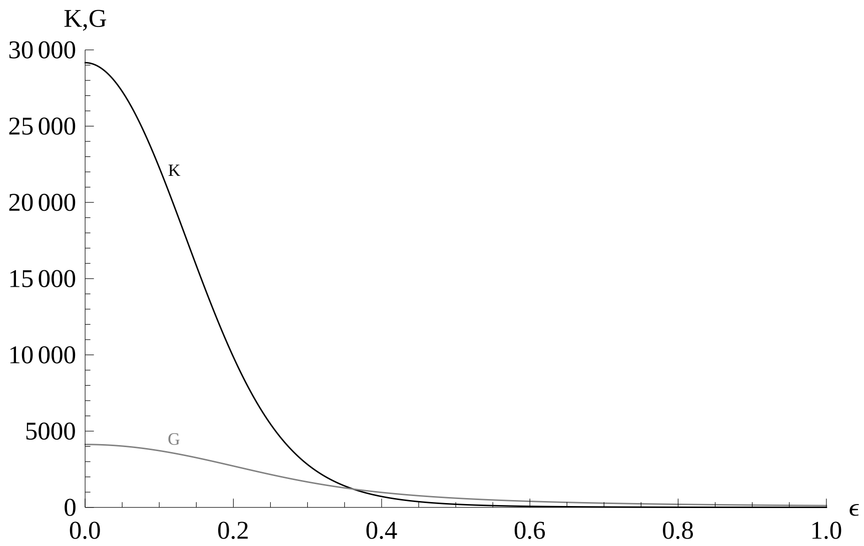


Figure 2.13: Bifurcation diagram for steady state levels of K and G with respect to ε for $\alpha_1 = 0.6$, $\alpha_2 = 0.2$.

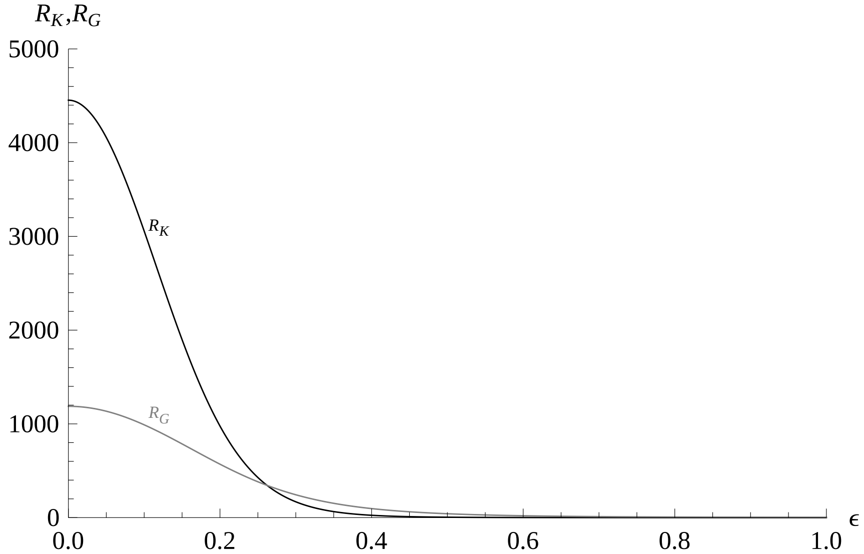


Figure 2.14: Bifurcation diagram for steady state levels of R_K and R_G with respect to ε for $\alpha_1 = 0.6$, $\alpha_2 = 0.2$.

Note, however, that in this basic model increasing environmental standards in general have a diminishing impact on the production inputs, and therefore on production output, and furthermore on economic growth. As one can see in Figure 2.15, the production is strictly monotonously decreasing.

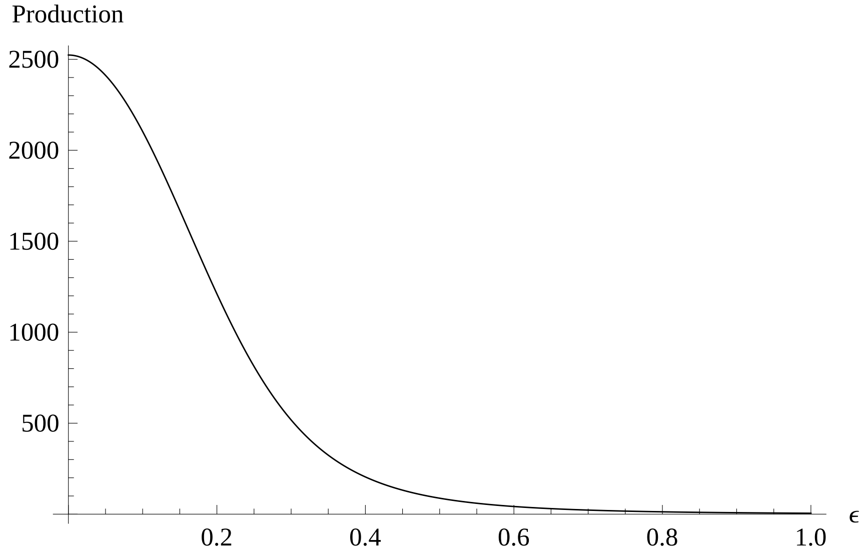


Figure 2.15: Bifurcation diagram of the steady state production output with respect to ε for $\alpha_1 = 0.6$, $\alpha_2 = 0.2$.

In contrast, the utility function as depicted in Figure 2.16 rises up to a peak before it decreases due to the trade-off between consumption and environmental quality. If ε is small enough, a small loss in consumption in return for a slightly better environment is advantageous. The utility-maximizing environmental quality is at $\varepsilon = 0.125$.

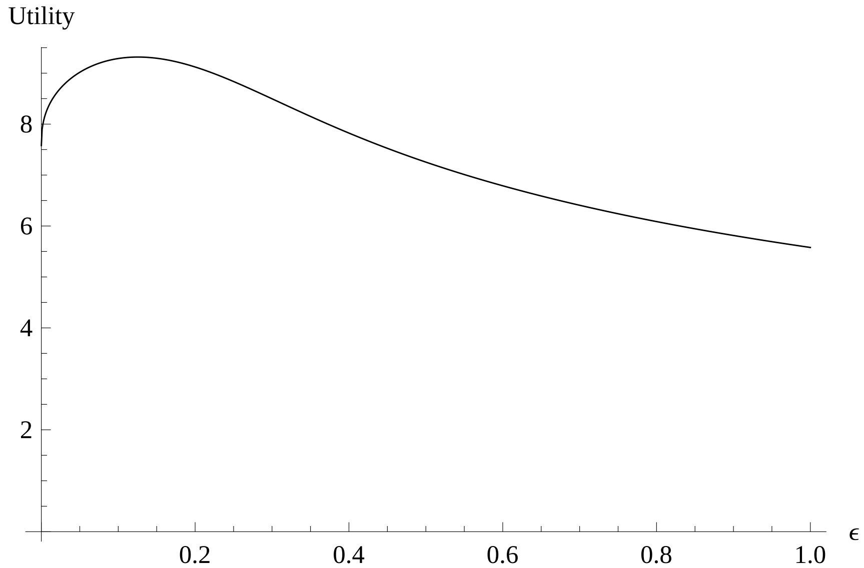


Figure 2.16: Bifurcation diagram of equilibrium utility with respect to ε for $\alpha_1 = 0.6$, $\alpha_2 = 0.2$.

In order to get a more qualitative than a quantitative comparison of the changing usage of K and G with increasing ε, the ratios of both are shown in Figure 2.17. As one can see, the ratio of G follows a convex-concave shape. At the beginning, the usage of G is quite low and does not change much with increasing ε. In this area, the abatement costs are still too low to change the advantage of conventional capital. The inflexion point is at $\varepsilon = 0.362$ where green capital starts to dominate conventional capital. From here on the ratio of G grows quite quickly until it converges to almost 100%. Note however, that 100% can never be reached. Accordingly, the ratio of K follows a concave-convex decrease.

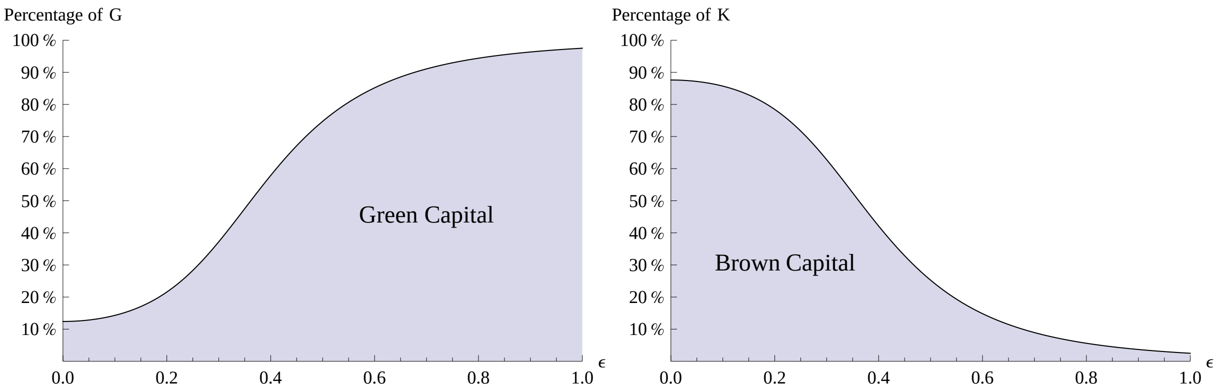


Figure 2.17: Bifurcation diagram of the equilibrium ratios of K and G with respect to ε for $\alpha_1 = 0.6$, $\alpha_2 = 0.2$.

In Figure 2.18 the ratios of the according R&D investments are depicted. Their development is similar, the only difference is the position of the inflexion point which is already at $\varepsilon = 0.263$.

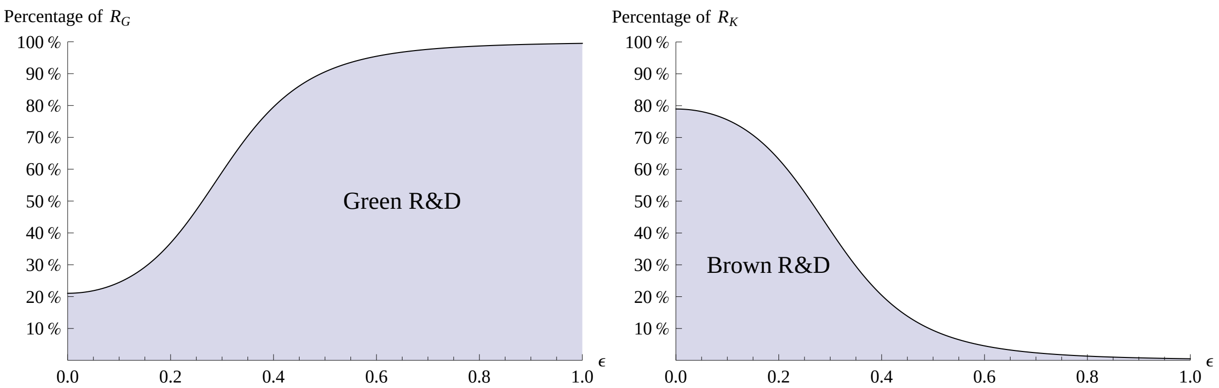


Figure 2.18: Bifurcation diagram of the equilibrium ratios of R_K and R_G with respect to ε for $\alpha_1 = 0.6$, $\alpha_2 = 0.2$.

2.6 Conclusion

To sum up the main insight of this section, the results of this first approach show the impact of environmental quality standards on capital accumulation and production. The higher ε gets, the more green capital is accumulated proportionally in the equilibrium, but in total both capital types as well as the production output decline towards zero. Even the utility function has its maximum at a very low level of ε. One reason for this is the strong interrelationship between brown and green capital due to the usage of a Cobb Douglas production function. Further on, abatement costs as the only negative impact of brown capital seem to be not enough incentive to cause a satisfying switch to a greener production. Therefore, these two aspects can be taken as foundation for modifications of this basic model, which will be the focus of the following chapters.

Chapter 3

CES Production Function

As already mentioned above, one point of criticism in the basic model formulation may be the fact that the assumption of a Cobb Douglas production function postulates the simultaneous use of both types of capital as input factors, i.e. both input factors are essential for a nonzero production output. One possibility to evade this restriction is the usage of a CES production function instead, which will be considered in this chapter as a first modification of the basic model. In this case, inputs can be non-essential, depending on the elasticity of substitution. Suppose that the production function is given as

$$F(K,G) = (\alpha_1 K^{\alpha_2} + (1-\alpha_1)G^{\alpha_2})^{\frac{1}{\alpha_2}}.$$

Then the elasticity of substitution is

$$\omega = \frac{1}{1-\alpha_2} \quad \text{with} \quad \omega \in [0,\infty).$$

Depending on the value of ω, the following three cases can occur (see Perman et al. [2003]):

- $\omega > 1$: none of the inputs are essential

- $\omega < 1$: all inputs are essential

- $\omega = 1$: Cobb Douglas case, all inputs are essential

To obtain the first case, $\alpha_2 \in (0,1)$ has to be satisfied, which is assumed for the following model approach. The higher the elasticity of substitution, the easier it is to switch from a mixed production, meaning a production using both types of capital, to a single input production, where one capital type is completely omitted. Therefore, pure green or brown production is possible.

The modified model now is

$$\max_{R_K,R_G} \int_0^\infty e^{-rt}\left(\ln\left(\tau+(\alpha_1 K^{\alpha_2}+(1-\alpha_1)G^{\alpha_2})^{\frac{1}{\alpha_2}}-w(R_K+R_G)-a\varepsilon^\beta K\right)+c\varepsilon^\gamma\right)dt \tag{3.1}$$

$$\begin{aligned}
\text{s.t.:}\quad & \dot{K}=dK^{\delta_1}R_K^{\delta_2}-\phi K && \text{(3.1a)}\\
& \dot{G}=eG^{\sigma_1}R_G^{\sigma_2}-\psi G && \text{(3.1b)}\\
& 0\le R_K \quad \forall t\ge 0 && \text{(3.1c)}\\
& 0\le R_G \quad \forall t\ge 0 && \text{(3.1d)}\\
& 0\le (\alpha_1 K^{\alpha_2}+(1-\alpha_1)G^{\alpha_2})^{\frac{1}{\alpha_2}}-w(R_K+R_G)-a\varepsilon^\beta K && \text{(3.1e)}\\
& 0\le\varepsilon\le 1 && \text{(3.1f)}\\
& 0<\alpha_1,\gamma,w<1 && \text{(3.1g)}\\
& 0<\alpha_2\le 1 && \text{(3.1h)}\\
& 0<\delta_1,\delta_2<1 \quad \text{and} \quad \delta_1+\delta_2<1 && \text{(3.1i)}\\
& 0<\sigma_1,\sigma_2<1 \quad \text{and} \quad \sigma_1+\sigma_2<1 && \text{(3.1j)}\\
& 1<\beta && \text{(3.1k)}\\
& 0<\phi,\psi,a,b,c,d,e,r && \text{(3.1l)}\\
& 0<\tau\le 1. && \text{(3.1m)}
\end{aligned}$$

3.1 Steady States

Again the approach for the derivation of the canonical system as in (2.21a)-(2.25) is used, which yields

$$\begin{aligned}
\dot{R_K} \quad = \quad & \frac{{D_1}^2 D_2^2 {R_G}^2 {R_K}^2 Y^3}{w^2\left(d(\delta_2-1)\delta_2 K^{\delta_1}{R_K}^{\delta_2}\left(D_2 {R_G}^2 w-eY(\sigma_2-1)\sigma_2 G^{\sigma_1}{R_G}^{\sigma_2}\right)+D_1 e {R_K}^2 w(\sigma_2-1)\sigma_2 G^{\sigma_1}{R_G}^{\sigma_2}\right)}\\
& \cdot\Bigg\{\Bigg[\Bigg(eY(\sigma_2-1)\sigma_2 G^{\sigma_1}{R_G}^{\sigma_2-2}-D_2 w\Bigg)\Bigg(D_1\Bigg(a\varepsilon^\beta-\alpha_1 K^{\alpha_2-1}\left(\alpha_1 K^{\alpha_2}-(\alpha_1-1)G^{\alpha_2}\right)^{\frac{1}{\alpha_2}-1}\Bigg)+\\
& w\Bigg(-d\delta_1 K^{\delta_1-1}{R_K}^{\delta_2}+r+\phi\Bigg)\Bigg)+D_2 w\Bigg(w\left(-e\sigma_1 G^{\sigma_1-1}{R_G}^{\sigma_2}+r+\psi\right)-\\
& D_2\left(\alpha_1 K^{\alpha_2}-(\alpha_1-1)G^{\alpha_2}\right)^{\frac{1}{\alpha_2}-1}\Bigg)\Bigg]\frac{w}{D_1 {D_2}^2 Y^3}+\dot{G}T_1+\dot{K}T_2\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\dot{R_G} \quad = \quad & \frac{{D_1}^2 D_2^2 {R_G}^2 {R_K}^2 Y^3}{w^2\left(d(\delta_2-1)\delta_2 K^{\delta_1}{R_K}^{\delta_2}\left(D_2 {R_G}^2 w-eY(\sigma_2-1)\sigma_2 G^{\sigma_1}{R_G}^{\sigma_2}\right)+D_1 e {R_K}^2 w(\sigma_2-1)\sigma_2 G^{\sigma_1}{R_G}^{\sigma_2}\right)}\\
& \cdot\Bigg\{\Bigg[\Bigg(dY(\delta_2-1)\delta_2 K^{\delta_1}{R_K}^{\delta_2-2}-D_1 w\Bigg)\Bigg(w\left(-e\sigma_1 G^{\sigma_1-1}{R_G}^{\sigma_2}+r+\psi\right)-\\
& D_2\left(\alpha_1 K^{\alpha_2}-(\alpha_1-1)G^{\alpha 2}\right)^{\frac{1}{\alpha_2}-1}\Bigg)+D_1 w\Bigg(D_1\Bigg(a\varepsilon^\beta-\alpha_1 K^{\alpha_2-1}\left(\alpha_1 K^{\alpha_2}-(\alpha_1-1)G^{\alpha_2}\right)^{\frac{1}{\alpha_2}-1}\Bigg)+\\
& w\Bigg(-d\delta_1 K^{\delta_1-1}{R_K}^{\delta_2}+r+\phi\Bigg)\Bigg)\Bigg]\frac{w}{{D_1}^2 D_2 Y^3}+\dot{G}T_3+\dot{K}T_4\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
\dot{K} &= dK^{\delta_1} R_K^{\delta_2} - \phi K \\
\dot{G} &= eG^{\sigma_1} R_G^{\sigma_2} - \psi G
\end{aligned}
\tag{3.2}
$$

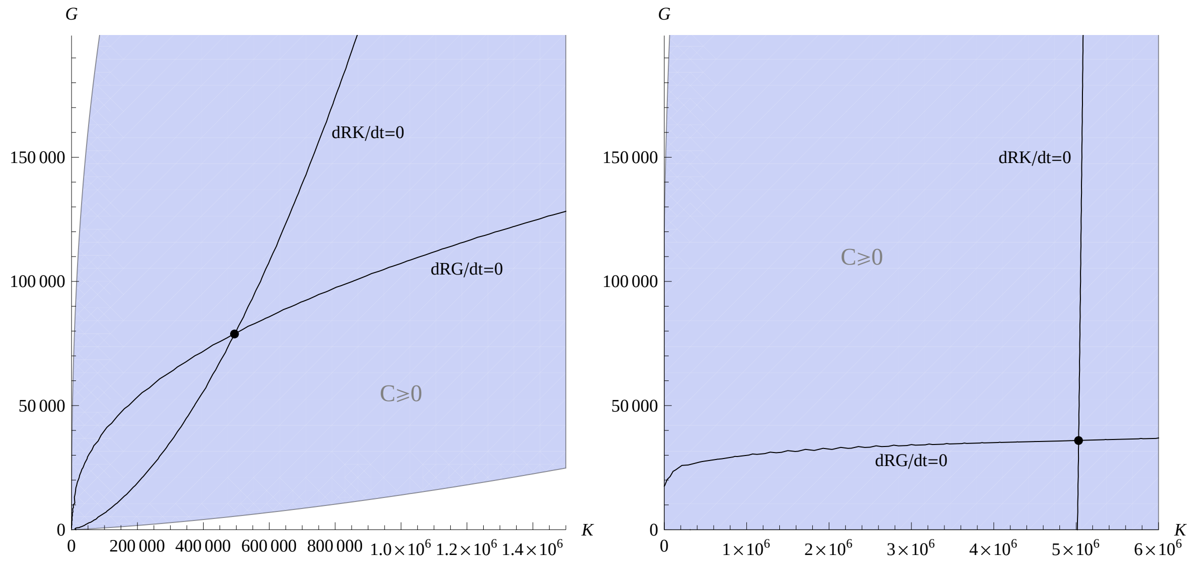
with

$$
\begin{aligned}
T_1 &= \frac{ew^2 \sigma_2 G^{\sigma_1 - 1} R_G{}^{\sigma_2 - 2} \left(b\alpha_2(\sigma_2 - 1)G^{\alpha_2} K^{\alpha_1} + R_G w \sigma_1\right)}{D_1 D_2{}^2 Y^3} \\
T_2 &= \frac{w^2}{D_1{}^2 D_2{}^2 K R_G{}^2 R_K Y^3} \Big(D_1 e R_K (\sigma_2 - 1)\sigma_2 G^{\sigma_1} R_G{}^{\sigma_2} \left(b\alpha_1 G^{\alpha_2} K^{\alpha_1} - aK\varepsilon^{\beta}\right) - \\
&\quad d\delta_1 \delta_2 K^{\delta_1} R_K{}^{\delta_2} \left(D_2 R_G{}^2 w - eY(\sigma_2 - 1)\sigma_2 G^{\sigma_1} R_G{}^{\sigma_2}\right)\Big) \\
T_3 &= \frac{w^2}{D_1{}^2 D_2{}^2 G R_G R_K{}^2 Y^3} \Big(d(\delta_2 - 1)\delta_2 K^{\delta_1} R_K{}^{\delta_2} \left(bD_2 R_G \alpha_2 G^{\alpha_2} K^{\alpha_1} + eY\sigma_1 \sigma_2 G^{\sigma_1} R_G{}^{\sigma_2}\right) - \\
&\quad D_1 e R_K{}^2 w \sigma_1 \sigma_2 G^{\sigma_1} R_G{}^{\sigma_2}\Big) \\
T_4 &= \frac{dw^2 \delta_2 K^{\delta_1 - 1} R_K{}^{\delta_2 - 2} \left((\delta_2 - 1)\left(b\alpha_1 G^{\alpha_2} K^{\alpha_1} - aK\varepsilon^{\beta}\right) + R_K w \delta_1\right)}{D_1{}^2 D_2 Y^3} \\
Y &= \tau + (\alpha_1 K^{\alpha_2} + (1 - \alpha_1)G^{\alpha_2})^{\frac{1}{\alpha_2}} - w(R_K + R_G) - a\varepsilon^{\beta} K
\end{aligned}
$$

and D_1 and D_2 again being the first derivatives of the state dynamics with respect to the corresponding control

$$
\begin{aligned}
D_1 &= dK^{\delta_1} \delta_2 R_K{}^{\delta_2 - 1} \\
D_2 &= eG^{\sigma_1} \sigma_2 R_G{}^{\sigma_2 - 1}.
\end{aligned}
$$

Although the CES function enables pure green or brown production, one can see that $K = 0$ and/or $G = 0$ are not feasible in the canonical system. Therefore, as in the basic model, only one single steady state can be detected, which is again a saddle point according to the first case in Table 2.2. To investigate scenarios with different substitution elasticities, I consider one with $\alpha_2 = 0.1$ and another one with $\alpha_2 = 0.9$. Figure 3.1a shows the first case, where a steady state can be found for $\varepsilon = 0.4$ at $K = 494{,}558$, $G = 78{,}829$, $R_K = 234{,}339$ and $R_G = 20{,}734$. The second scenario is depicted in Figure 3.1b. Again, for a medium high environmental quality standard of $\varepsilon = 0.4$, the steady state lies at $K = 5{,}028{,}762$, $G = 35{,}927$, $R_K = 6{,}025{,}395$ and $R_G = 52{,}412$. Comparing these two results, one can see that a higher elasticity of substitution is quite supportive for brown capital, and consequently brown R&D. This means that the easier a switch to a single input dominated production, the greater the brown portion of production. This comes along with the higher productivity of brown capital.



(a) $\alpha_1 = 0.6, \alpha_2 = 0.1$ and $\varepsilon = 0.4$

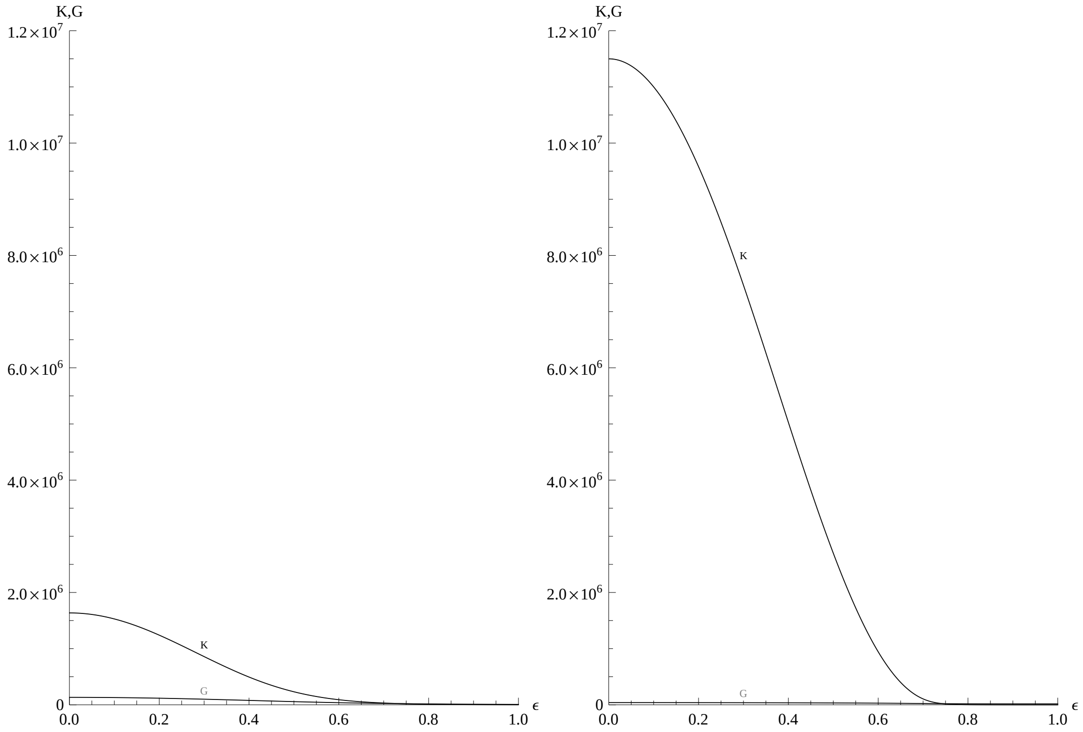
(b) $\alpha_1 = 0.6, \alpha_2 = 0.9$ and $\varepsilon = 0.4$

Figure 3.1: Steady state under the assumption of a CES production function.

3.2 Bifurcation Analysis

To investigate a possible change of tendency towards green production with increasing environmental quality standard, bifurcation analysis is used. Figure 3.2 shows the results for the equilibrium values of G and K. In both scenarios, as in the basic model, brown capital is clearly dominant for small values of ε. Note, however, that the gap between the used input levels is even bigger, especially in the second case with a high substitution elasticity as shown in Figure 3.2b, where the level of K is immensely high and is even the six-fold of the level in the first scenario in Figure 3.2a. As the value of ε increases, the levels of K and G decline in both scenarios. However, compared to the basic model, this time it takes even longer until green capital gets dominant over brown capital, which happens at $\varepsilon = 0.745$ in the first scenario. An interesting fact is that in the scenario with a higher substitution elasticity this point lies at $\varepsilon = 0.746$ which is almost the same. This means that, although the initial tendency for brown capital is very strong, a green capital dominated production gets advantageous if environmental quality standards are high enough. This happens interestingly at the same threshold as in the first scenario. Therefore, a higher elasticity of substitution shows its impact in form of a higher tendency for brown production only as long as the usage of brown capital as input factor is advantageous. As soon as the increasing abatement costs due to higher environmental standards make green capital advantageous, the behavior in both scenarios is the same.

Considering the equilibrium values of R_K and R_G with increasing values of ε, similar results can be observed as shown in Figure 3.3. For very low environmental quality standards, brown R&D investments are clearly dominating. In the second scenario with a higher substitution elasticity, immense brown investments are made. With increasing ε, investments decrease again, but regarding the point at which green R&D investments turn out to be dominating, an interesting

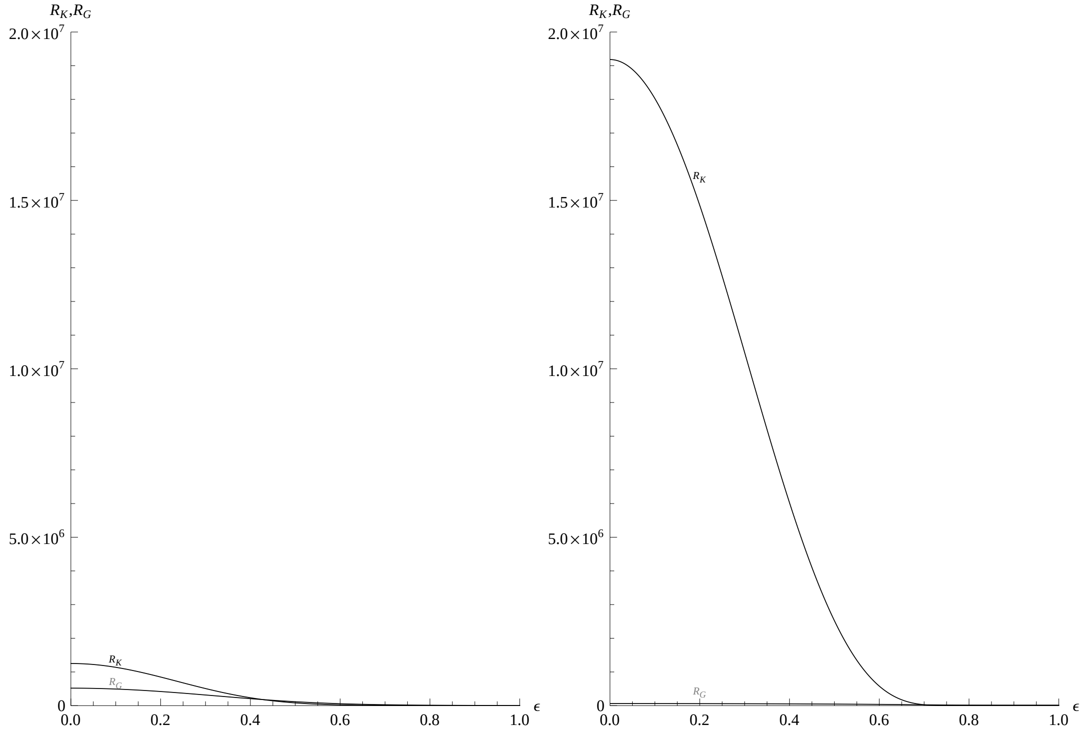


(a) $\alpha_1 = 0.6, \alpha_2 = 0.1$ and $\varepsilon = 0.4$ **(b)** $\alpha_1 = 0.6, \alpha_2 = 0.9$ and $\varepsilon = 0.4$

Figure 3.2: Bifurcation diagram for steady state levels of K and G with respect to ε in the CES-case for a low substitution elasticity (a) and a high substitution elasticity (b).

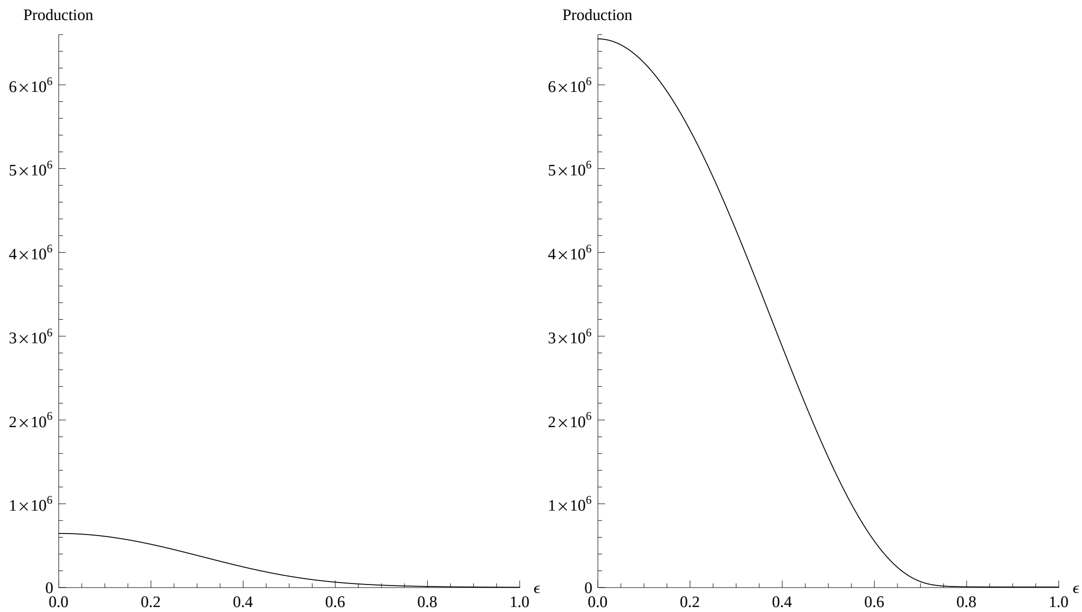
observation can be made. While the threshold, at which the switch happens, has been almost the same for the capital levels, it is different now for the R&D investments. In the first scenario, this critical point is at $\varepsilon = 0.428$, while it lies at $\varepsilon = 0.703$ in the second scenario. Due to the fact that the capital types switch at the same quality standard, it follows that the higher the substitution elasticity, the more flexible is the production. In the first scenario, green R&D already has been dominant for a while when finally the consequences for the capital levels get obvious. In the second scenario, however, the switch happens almost simultaneously.

As far as production outputs are concerned, the general behavior is quite similar to the results for the basic model. Due to the initially higher capital levels, especially in the case of a high substitution elasticity, the production outputs, of course, differ quantitatively. In both scenarios I find an inner utility-maximizing environmental quality, which is for the first one at $\varepsilon = 0.266$, and for the second one at $\varepsilon = 0.29$. Note however, that in the scenario with a higher substitution elasticity, additional to the inner utility maximum also a local maximum occurs at $\varepsilon = 1$.



(a) $\alpha_1 = 0.6, \alpha_2 = 0.1$ and $\varepsilon = 0.4$ **(b)** $\alpha_1 = 0.6, \alpha_2 = 0.9$ and $\varepsilon = 0.4$

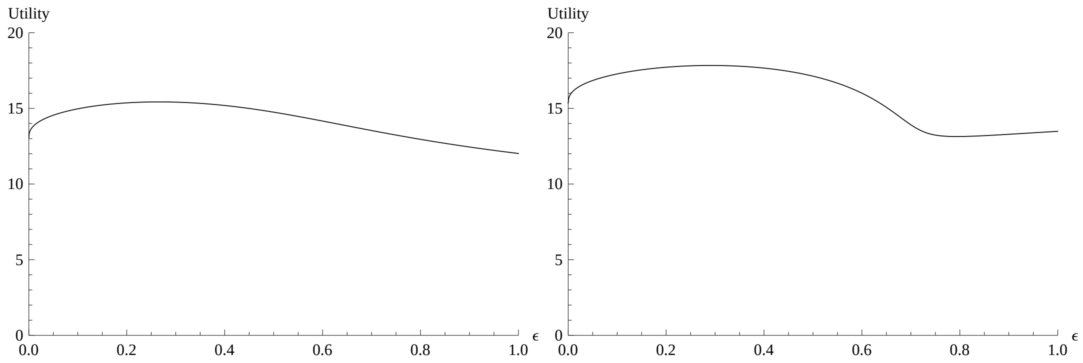
Figure 3.3: Bifurcation diagram for steady state levels of K and G with respect to ε in the CES-case for a low substitution elasticity (a) and a high substitution elasticity (b).



(a) $\alpha_1 = 0.6, \alpha_2 = 0.1$ and $\varepsilon = 0.4$ **(b)** $\alpha_1 = 0.6, \alpha_2 = 0.9$ and $\varepsilon = 0.4$

Figure 3.4: Bifurcation diagram of the steady state production output with respect to ε in the CES-case for a low substitution elasticity (a) and a high substitution elasticity (b).

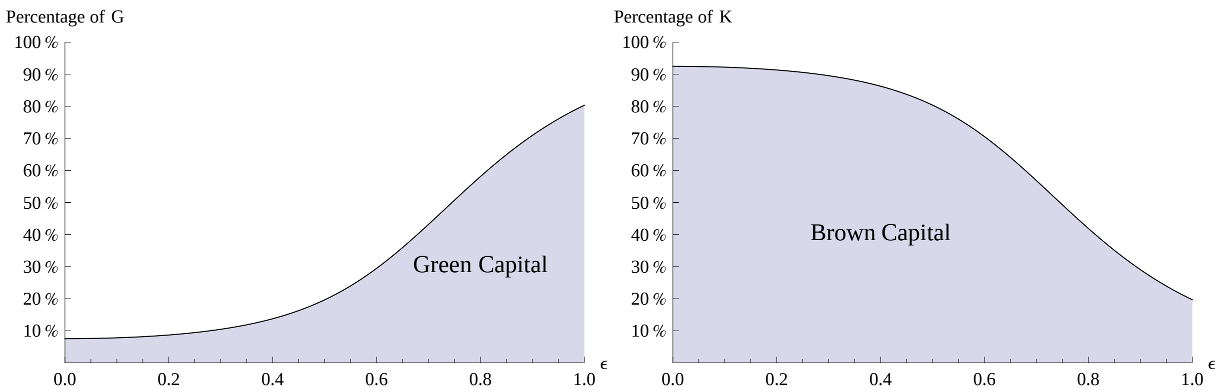
To compare the obtained results also in a qualitative way, Figure 3.6 shows again the equilibrium ratios of green and brown capital under the variation of ε, which stresses again the even higher advantage of brown capital in the CES-case, compared to the basic model. Considering the shares of the two CES scenarios, one can immediately see the difference in flexibility. While



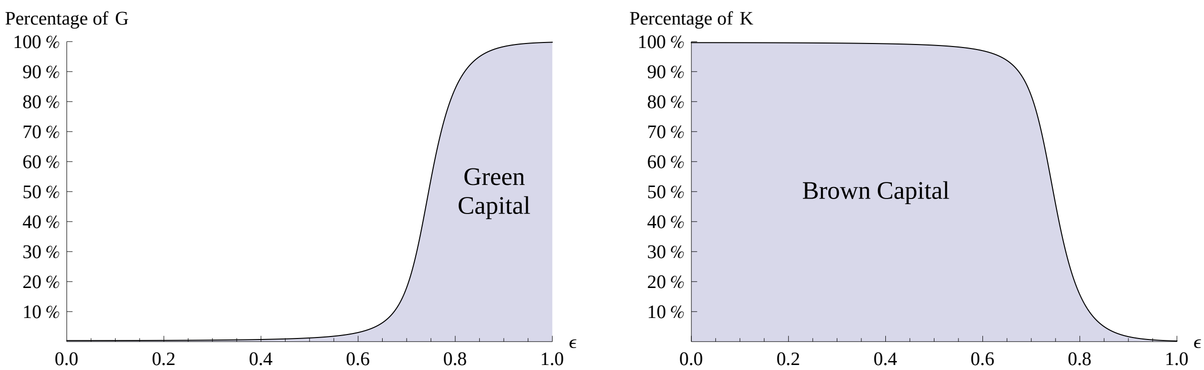
(a) $\alpha_1 = 0.6, \alpha_2 = 0.1$ and $\varepsilon = 0.4$ **(b)** $\alpha_1 = 0.6, \alpha_2 = 0.9$ and $\varepsilon = 0.4$

Figure 3.5: Bifurcation diagram of equilibrium utility with respect to ε in the CES-case for a low substitution elasticity (a) and a high substitution elasticity (b).

the level of green capital slowly increases in the first case, brown capital is used in production as long as possible in the second case, until the threshold of ε is reached at which the two capital types change their position in production. The same applies for the R&D investments (see Figure 3.7). However, the inflexion point in the second scenario is higher than in the first scenario, as already mentioned above.

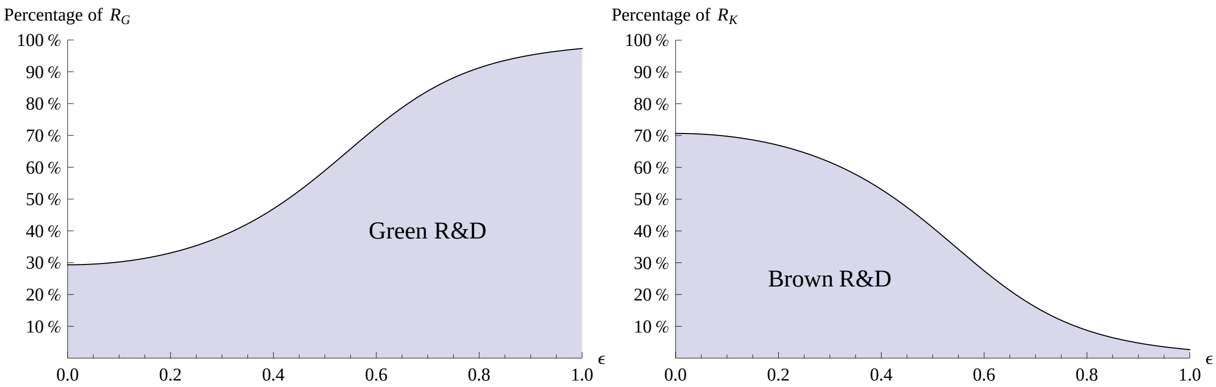


(a) $\alpha_1 = 0.6$ and $\alpha_2 = 0.1$

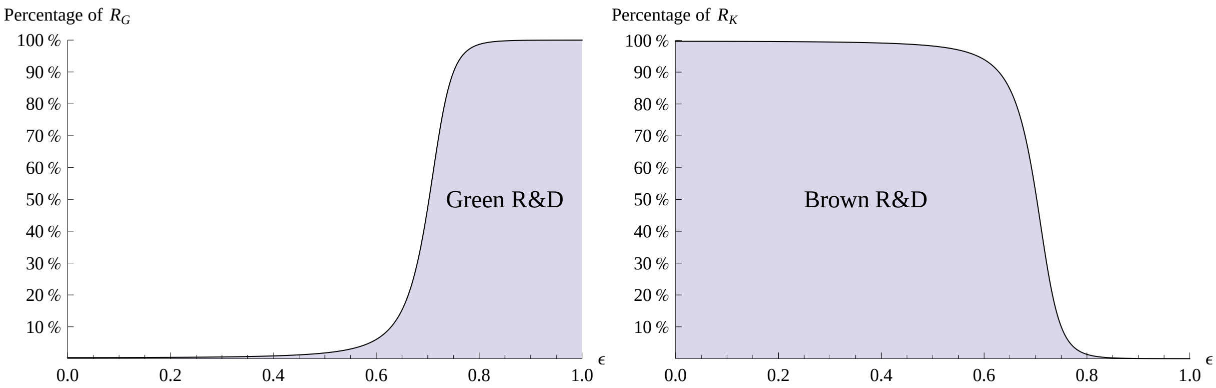


(b) $\alpha_1 = 0.6$ and $\alpha_2 = 0.9$

Figure 3.6: Bifurcation diagram of the equilibrium ratios of K and G with respect to ε in the CES-case for a low substitution elasticity (a) and a high substitution elasticity (b).



(a) $\alpha_1 = 0.6$ and $\alpha_2 = 0.1$



(b) $\alpha_1 = 0.6$ and $\alpha_2 = 0.9$

Figure 3.7: Bifurcation diagram of the equilibrium ratios of R_K and R_G with respect to ε in the CES-case for a low substitution elasticity (a) and a high substitution elasticity (b).

3.3 Conclusion

Summing up, the results derived in this chapter show that the assumption of a CES production function entails higher flexibility in production, which therefore reduces the effectiveness of environmental quality standard constraints beneath the "switching point". Beyond the "switching point", however, the behavior is quite similar to the results for the basic model. Further on, the usage of a CES function leads to an initially higher tendency for brown production, which fades again as soon as the "switch point" in ε is reached. Additionally, a local maximum occurs at $\varepsilon = 1$, if the elasticity of substitution is high enough. This shows that, starting at a necessarily high environmental quality standard, a further increase in quality would even be advantageous for the agent. The quintessence about the impact of the chosen environmental quality standard, however, in general remains the same. Increasing standards reduce more or less rapidly the levels of both types of capital and consequently also the level of production output, while the utility rises up to a global maximum at a quite low environmental quality level and then, except in the case of a high substitution elasticity, also declines. Therefore, the initial assumption of a Cobb Douglas production function is not as restrictive as assumed previously. Hence, for the sake of simplification, a Cobb Douglas production function is postulated again in the following approaches.

Chapter 4

Subsidization for Environmental-Friendly Production

Until now the only inducement for the agent to switch from brown capital dominated production to green capital dominated production is given through the abatement costs which are growing with higher imposed environmental quality. However, there are many kinds of environmental policy instruments that can be chosen. In some papers on this topic (i.e. Ricci [2007] and Acemoglu, Aghion, Bursztyn, and Hemous [2009]) charges, for example in form of emission taxes, are considered. In general, charges are perceived as financial burden or penalty in addition to the occurring costs which are involved in carrying out pollution abatement to meet regulatory standards. In the industry sector, however, this might result in a loss of competitiveness, especially, if industry in competitor countries is not subject to the same charges. Nevertheless, not only negative inducements in the form of cost or disprofit are possible. To compensate for the just mentioned negative aspects of charges on the competitiveness, many schemes involve subsidy or a draw back of the charge to support investments in the short term. As soon as pollution control is introduced, the level of charge falls and hence the implementation of a more efficient and less costly pollution control equipment is induced in the long term. Such combinations, generally called *Charge cum Subsidy Schemes*, provide a double incentive to reduce environmental damaging activities and support technological change. The emphasis of this chapter lies on the introduction of subsidies as a single economic instrument. In general, they are incompatible with the *Polluter-Pays Principle* except under specific circumstances. Anyhow, subsidies and premia are in the forefront of economic instruments in industry because they are often easier to target according to specific regional or sectoral conditions (see Organisation for Economic Co-operation and Development [1991]).

In Mittnik, Semmler, Kato, and Samaan [2010], subsidization of environmental-friendly production as policy instrument is discussed, especially wage and product subsidies. Motivated by this work I will introduce subsidies in the subsequent as another modification of the basic model. However, in contrast to the mentioned paper, I will consider these subsidies not as wage or product benefits, but as additional financial payment to generate a compensation for the abatement costs. Further on, instead of constant subsidies, I assume subsidies which are proportionally growing with the object of subsidization.

4.1 Subsidization of Green R&D

First, I consider an approach in which the R&D investments for green capital are object of subsidization. This means that every effort in R&D towards a greener production is rewarded and supported by the government, whereas the level of green capital which is already used in production is not taken into account. One disadvantage of this approach might be that entrepreneurs, who already use high levels of green capital as production input, are treated equal to those who still produce predominantly with brown capital but are willing to change towards a greener production. On the other hand this kind of subsidization might be a stronger inducement for exactly those who haven‘t accumulated high levels of green capital yet. Anyway, due to the assumed perfect competition these aspects are not further relevant.

In what follows, this kind of subsidization is regarded as financial support for the partial coverage of the arising opportunity costs due to green R&D. The parameter $s \in [0,1]$ represents the subsidy rate the government is willing to pay for green R&D investments. Additionally to ε, s is now the second exogenously given control parameter of the government. The modified model then is given by

$$\max_{R_K, R_G} \int_0^\infty e^{-rt} \left(\ln \left(\tau + bK^{\alpha_1} G^{\alpha_2} - w(R_K + R_G) - a\varepsilon^\beta K + swR_G \right) + c\varepsilon^\gamma \right) dt \tag{4.1}$$

$$\begin{aligned}
\text{s.t.:} \quad & \dot{K} = dK^{\delta_1} R_K^{\delta_2} - \phi K && \text{(4.1a)}\\
& \dot{G} = eG^{\sigma_1} R_G^{\sigma_2} - \psi G && \text{(4.1b)}\\
& 0 \le R_K \quad \forall t \ge 0 && \text{(4.1c)}\\
& 0 \le R_G \quad \forall t \ge 0 && \text{(4.1d)}\\
& 0 \le bK^{\alpha_1} G^{\alpha_2} - w(R_K + R_G) - a\varepsilon^\beta K + swR_G && \text{(4.1e)}\\
& 0 \le \varepsilon \le 1 && \text{(4.1f)}\\
& 0 < \alpha_1, \alpha_2, \gamma, w, s < 1 \quad \text{and} \quad \alpha_1 + \alpha_2 \le 1 && \text{(4.1g)}\\
& 0 < \delta_1, \delta_2 < 1 \quad \text{and} \quad \delta_1 + \delta_2 < 1 && \text{(4.1h)}\\
& 0 < \sigma_1, \sigma_2 < 1 \quad \text{and} \quad \sigma_1 + \sigma_2 < 1 && \text{(4.1i)}\\
& 1 < \beta && \text{(4.1j)}\\
& 0 < \phi, \psi, a, b, c, d, e, r && \text{(4.1k)}\\
& 0 < \tau \le 1. && \text{(4.1l)}
\end{aligned}$$

4.1.1 Steady States

The canonical system for this model, derived by using the approach as in (2.21a)-(2.25), then is

$$
\begin{aligned}
\dot{R_K} &= -\frac{(1-s)^{-1}w^{-2}D_1{}^2D_2{}^2Y^4}{\left(dY(\delta_2-1)\delta_2K^{\delta_1}R_K{}^{\delta_2-2}-D_1w\right)\left(D_2(s-1)w+eY(\sigma_2-1)\sigma_2G^{\sigma_1}R_G{}^{\sigma_2-2}\right)+D_1D_2(s-1)w^2}\\
&\cdot\Bigg\{\Bigg[(1-s)\Big(D_2(s-1)w+eY(\sigma_2-1)\sigma_2G^{\sigma_1}R_G{}^{\sigma_2-2}\Big)\Big(aD_1\varepsilon^\beta-bD_1\alpha_1G^{\alpha_2}K^{\alpha_1-1}+\\
&w\left(-d\delta_1K^{\delta_1-1}R_K{}^{\delta 2}+r+\phi\right)\Big)+D_2(s-1)w\Big(bD_2\alpha_2G^{\alpha_2-1}K^{\alpha_2}+ew\sigma_1G^{\sigma_1-1}R_G{}^{\sigma_2}-w(r+\psi)\Big)\Bigg]\cdot\\
&\frac{w}{D_1D_2{}^2Y^3}+\dot{G}T_1+\dot{K}T_2\Bigg\}\\
\dot{R_G} &= -\frac{(1-s)^{-1}w^{-2}D_1{}^2D_2{}^2Y^4}{\left(\left(dY(\delta_2-1)\delta_2K^{\delta_1}R_K{}^{\delta_2-2}-D_1w\right)\left(D_2(s-1)w+eY(\sigma_2-1)\sigma_2G^{\sigma_1}R_G{}^{\sigma_2-2}\right)+D_1D_2(s-1)w^2\right)}\\
&\cdot\Bigg\{\Bigg[\Big(dY(\delta_2-1)\delta_2K^{\delta_1}R_K{}^{\delta_2-2}-D_1w\Big)\Big(w\left(-e\sigma_1G^{\sigma_1-1}R_G{}^{\sigma_2}+r+\psi\right)-bD_2\alpha_2G^{\alpha_2-1}K^{\alpha_2}\Big)+\\
&D_1(s-1)w\Big(bD_1\alpha_1G^{\alpha_2}K^{\alpha 1-1}+dw\delta_1K^{\delta_1-1}R_K{}^{\delta_2}-\Big(aD_1\varepsilon^\beta+w(r+\phi)\Big)\Big)\Bigg]\frac{w}{D_1{}^2D_2Y^3}+\dot{G}T_3+\dot{K}T_4\Bigg\}\\
\dot{K} &= dK^{\delta_1}R_K{}^{\delta_2}-\phi K\\
\dot{G} &= eG^{\sigma_1}R_G{}^{\sigma_2}-\psi G
\end{aligned}
\tag{4.2}
$$

with

$$
\begin{aligned}
T_1 &= -\frac{(s-1)w^2\left(bD_2R_G{}^2sw\alpha_2G^{\alpha_2}K^{\alpha_1}+eY\sigma_2G^{\sigma_1}R_G{}^{\sigma_2}\left(b\alpha_2(\sigma_2-1)G^{\alpha_2}K^{\alpha_1}+R_Gw\sigma_1\right)\right)}{D_1D_2{}^2GR_G{}^2Y^4}\\
T_2 &= \frac{w^2}{D_1{}^2D_2{}^2Y^4}\Big((1-s)\left(D_2(s-1)w+eY(\sigma_2-1)\sigma_2G^{\sigma_1}R_G{}^{\sigma_2-2}\right)\left(bD_1\alpha_1G^{\alpha_2}K^{\alpha_1-1}-\right.\\
&\left.aD_1\varepsilon^\beta+dY\delta_1\delta_2K^{\delta_1-1}R_K{}^{\delta_2-1}\right)+D_1D_2(s-1)w\big(a\varepsilon^\beta-b\alpha_1G^{\alpha_2}K^{\alpha_1-1}\big)\Big)\\
T_3 &= \frac{w^2}{D_1{}^2D_2{}^2GR_GR_K{}^2Y^4}\Big(dY(\delta_2-1)\delta_2K^{\delta_1}R_K{}^{\delta_2}\left(bD_2R_G\alpha_2G^{\alpha_2}K^{\alpha_1}+eY\sigma_1\sigma_2G^{\sigma_1}R_G{}^{\sigma_2}\right)-\\
&D_1R_K{}^2w\big(bD_2R_Gs\alpha_2G^{\alpha_2}K^{\alpha_1}+eY\sigma_1\sigma_2G^{\sigma_1}R_G{}^{\sigma_2}\big)\Big)\\
T_4 &= \frac{w^2}{D_1{}^2D_2KR_K{}^2Y^4}\Big(dY\delta_2K^{\delta_1}R_K{}^{\delta_2}\left(R_K(s-1)w\delta_1-aK(\delta_2-1)\varepsilon^\beta\right)+aD_1KR_K{}^2sw\varepsilon^\beta+\\
&b\alpha_1G^{\alpha_2}K^{\alpha_1}\left(dY(\delta_2-1)\delta_2K^{\delta_1}R_K{}^{\delta_2}-D_1R_K{}^2sw\right)\Big)\\
Y &= \tau+bK^{\alpha_1}G^{\alpha_2}-w(R_K+R_G)-a\varepsilon^\beta K+swR_G
\end{aligned}
$$

and again D_1 and D_2 being the first derivatives of the state dynamics with respect to the corresponding control

$$
\begin{aligned}
D_1 &= dK^{\delta_1}\delta_2R_K{}^{\delta_2-1}\\
D_2 &= eG^{\sigma_1}\sigma_2R_G{}^{\sigma_2-1}.
\end{aligned}
$$

This system yields one steady state which for $\varepsilon = 0.4$ and $s = 0.5$ lies at $K = 921$, $G = 1,694$, $R_K = 35$ and $R_G = 250$ and is a saddle point corresponding again to the first case in Table 2.2 (see Figure 4.1). Comparing this equilibrium with the results in the basic model one can immediately see that the subsidies indeed have positive impact on the green R&D investments which are now more than twice as high as in the basic model. Accordingly, also the level of green capital used in production raises, but at a lower rate. The marginal changes in the levels of brown capital and R&D are a result of the production's dependency on both types of capital due to the Cobb Douglas production function.

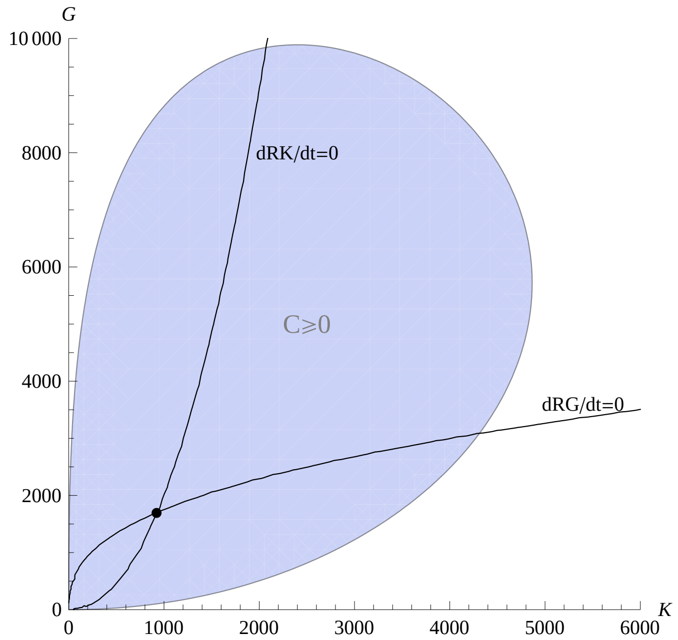


Figure 4.1: Steady state under the subsidization of R_G with $\alpha_1 = 0.6, \alpha_2 = 0.2$, $\varepsilon = 0.4$ and $s = 0.5$.

4.1.2 Optimal Paths

The IVP (Initial Value Problem) approach is again used for the calculation of a phase portrait. As already explained in section 2.4, initial values close enough to the equilibrium have to be constructed first in order to start the backward calculation of the trajectories to get finally a phase portrait. Here, the crucial intervals for the angle η are $[0.49998807\pi, 0.49998808\pi]$ and $[0.500002614853\pi, 0.500002614854\pi]$. Similar to the phase portrait of the basic model, the trajectories in Figure 4.2 are again divided into three parts. The first one corresponds to the continuation from the equilibrium to the boundary of the control constraint in (4.1c), the second one relates to the continuation with $R_K = 0$ until the boundary of the mixed path constraint in (4.1e) is reached, and the third one is again the continuation along this admissible boundary until consumption finally gets negative.

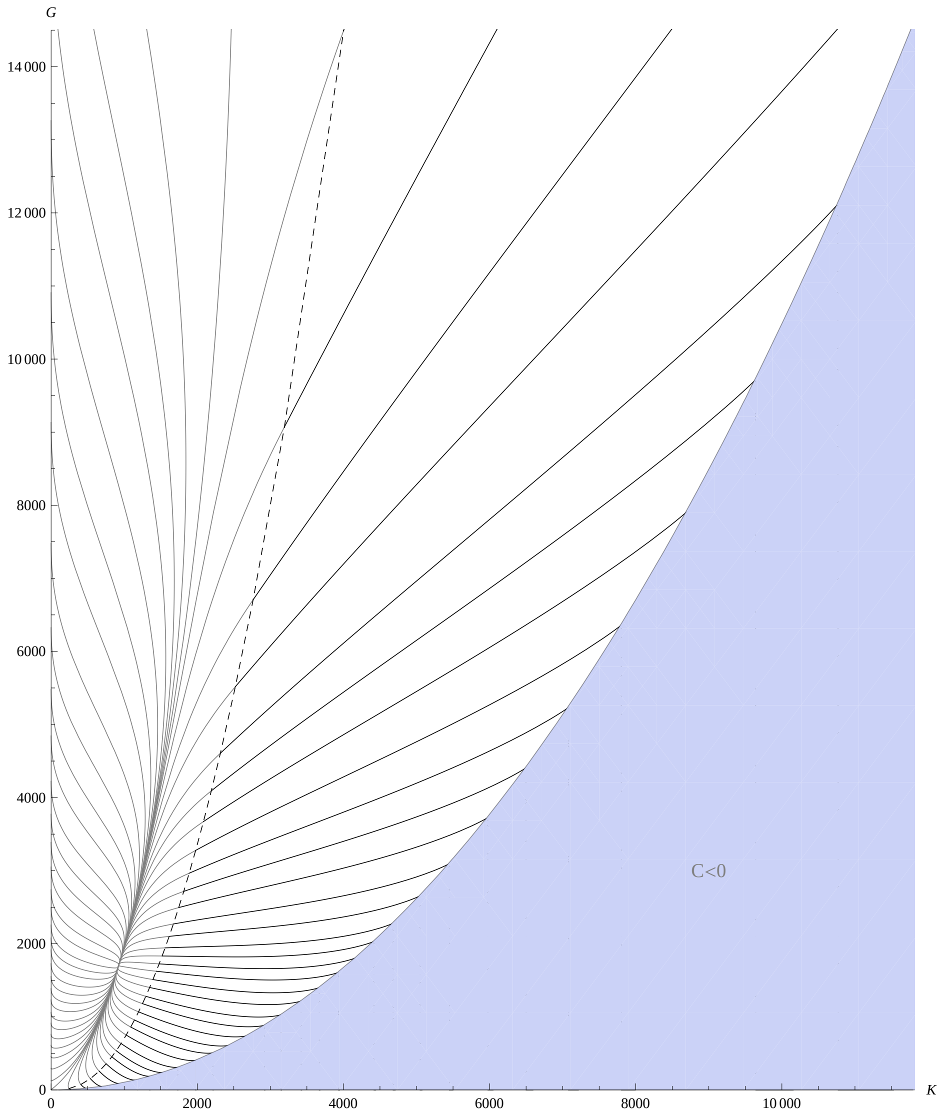


Figure 4.2: Phase portrait under the subsidization of R_G with $\alpha_1 = 0.6, \alpha_2 = 0.2$, $\varepsilon = 0.4$ and $s = 0.5$.

Initial Points with an Equal Level of K and G

In Figure 4.3, as before, two trajectories taken from the phase portrait in Figure 4.2 are depicted which both have initial values with balanced levels of brown and green capital. To point out the differences and similarities of this approach compared with the basic model, Figures 4.4 and 4.5 show the levels of green and brown capital and the according levels of R&D investments over time. Comparing these with the results in Figures 2.8 and 2.9 one can see that the general behavior is the same but the increase of G and R_G along the trajectory starting at the low initial capital levels is much higher than in the basic model. Therefore the gap between conventional and green capital and R&D is also bigger. Considering the optimal time paths along the trajectory starting at the high levels, the behavior of green capital and R&D is quite similar to the base case, but obviously R_G flattens at a higher level.

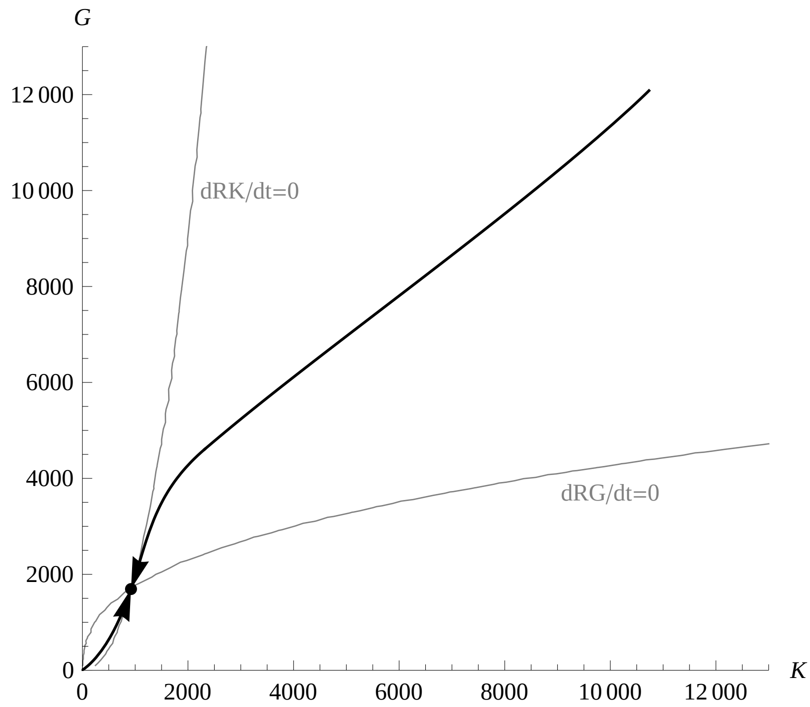


Figure 4.3: Two trajectories under the subsidization of R_G for $\alpha_1 = 0.6, \alpha_2 = 0.2$, $\varepsilon = 0.4$ and $s = 0.5$ with equal initial capital levels.

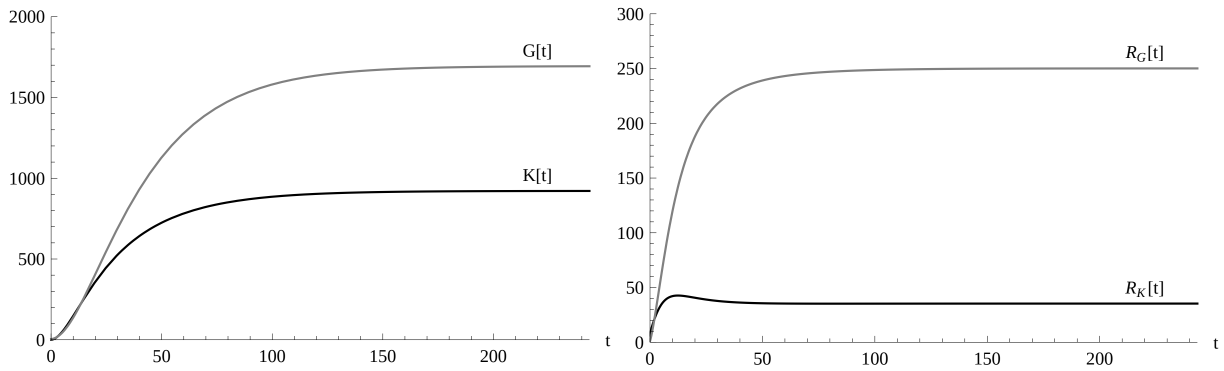


Figure 4.4: Optimal time paths of state and control starting from low capital levels under the subsidization of R_G with $\alpha_1 = 0.6, \alpha_2 = 0.2$, $\varepsilon = 0.4$ and $s = 0.5$.

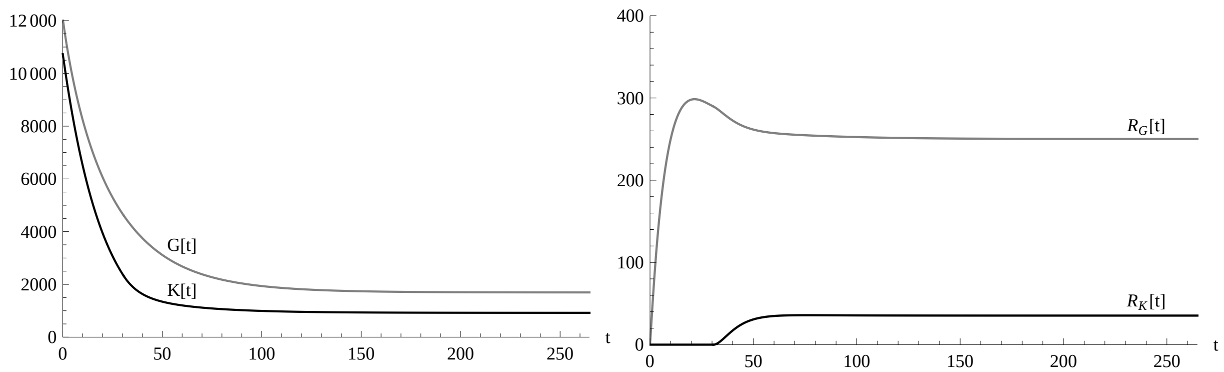


Figure 4.5: Optimal time paths of state and control starting from high capital levels under the subsidization of R_G with $\alpha_1 = 0.6, \alpha_2 = 0.2$, $\varepsilon = 0.4$ and $s = 0.5$.

Initial Points with One Type of Capital Being Dominant

In contrast to Figure 4.3, Figure 4.6 shows two trajectories of the phase portrait which start at a production clearly dominated by one type of capital. Considering the values of K, G, R_K and R_G along these trajectories, a similar behavior as in the basic model with an emphasis on green capital accumulation can be observed. Especially in Figure 4.8, in which the optimal time paths starting at a brown dominated production are shown, one can see that the switch to a green capital dominated one happens earlier than in the basic model. This stresses the positive impact of the introduced subsidy on the accumulation of green capital.

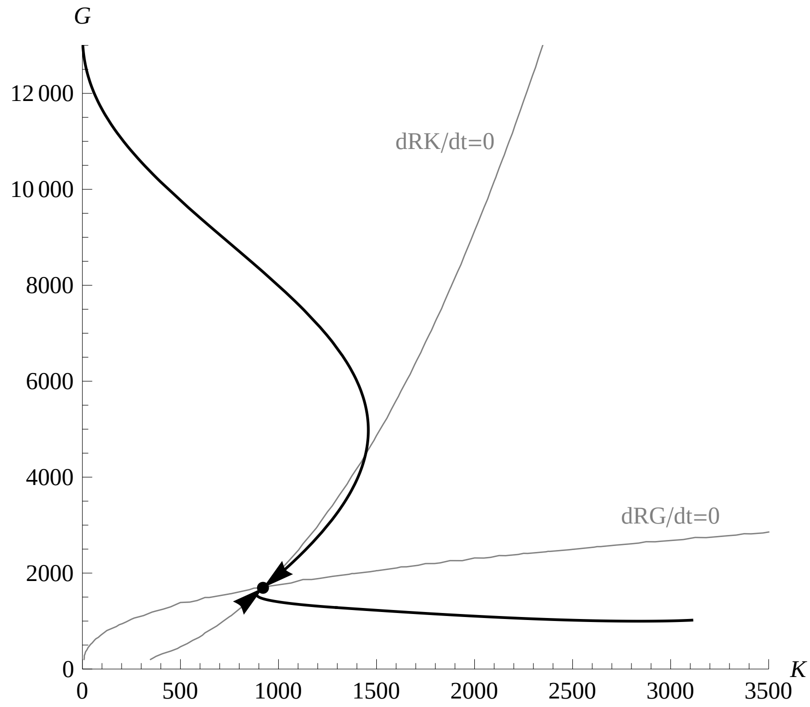


Figure 4.6: Two trajectories starting at a one-capital-type-dominated production under the subsidization of R_G for $\alpha_1 = 0.6, \alpha_2 = 0.2$, $\varepsilon = 0.4$ and $s = 0.5$.

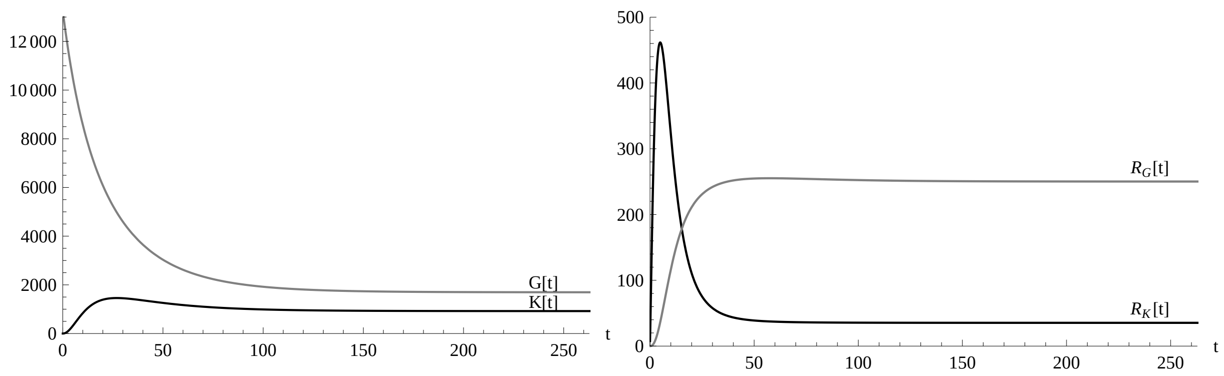


Figure 4.7: Optimal time paths of state and control starting from a definitely green capital-dominated production under the subsidization of R_G for $\alpha_1 = 0.6, \alpha_2 = 0.2$, $\varepsilon = 0.4$ and $s = 0.5$.

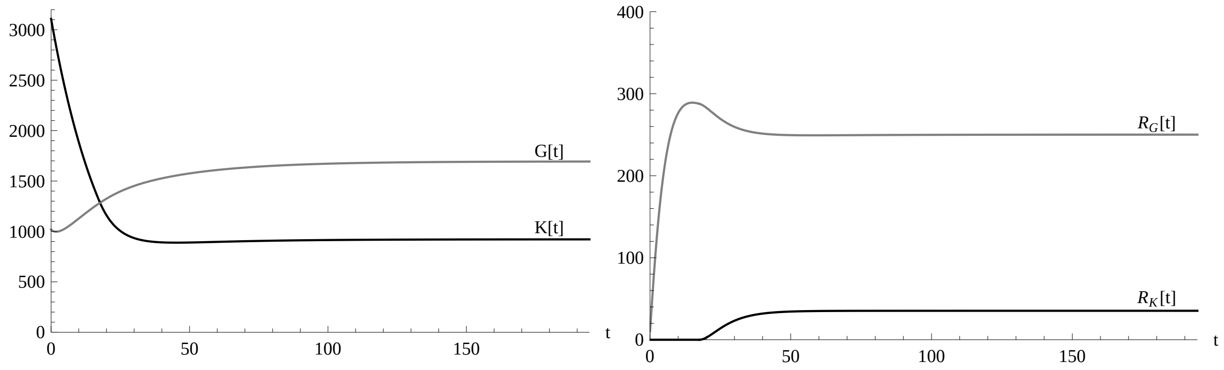


Figure 4.8: Optimal time paths of state and control starting from a definitely brown capital-dominated production under the subsidization of R_G for $\alpha_1 = 0.6, \alpha_2 = 0.2$, $\varepsilon = 0.4$ and $s = 0.5$.

4.1.3 Bifurcation Analysis

To point out the differences of this model compared to the basic model, bifurcation analysis is used to investigate the impact of the introduced subsidy under varying ε.

Figure 4.9 shows the equilibrium values of K and G under increasing environmental quality ε. Similar to the basic model, brown capital is dominant for lower values of ε until production switches to a green dominated one at a certain point. This already happens at $\varepsilon = 0.326$ and therefore green capital gets advantageous at lower environmental standards. This is even more obvious in Figure 4.10 showing the equilibrium values of R_K and R_G, where green R&D is already dominant at $\varepsilon = 0.184$ compared to $\varepsilon = 0.263$ in the basic model. The qualitative behavior itself is similar. The capital levels as well as the R&D investments are declining with increasing ε.

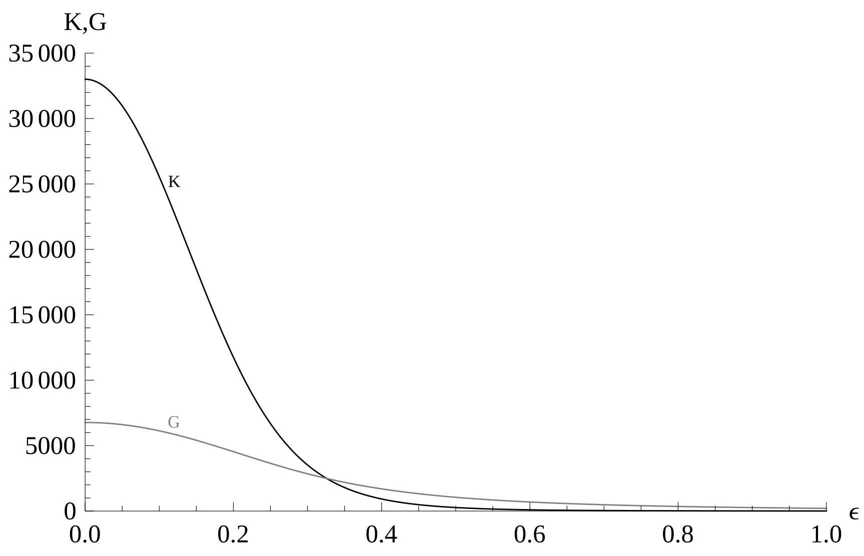


Figure 4.9: Bifurcation diagram for steady state levels of K and G with respect to ε under the subsidization of R_G for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $s = 0.5$.

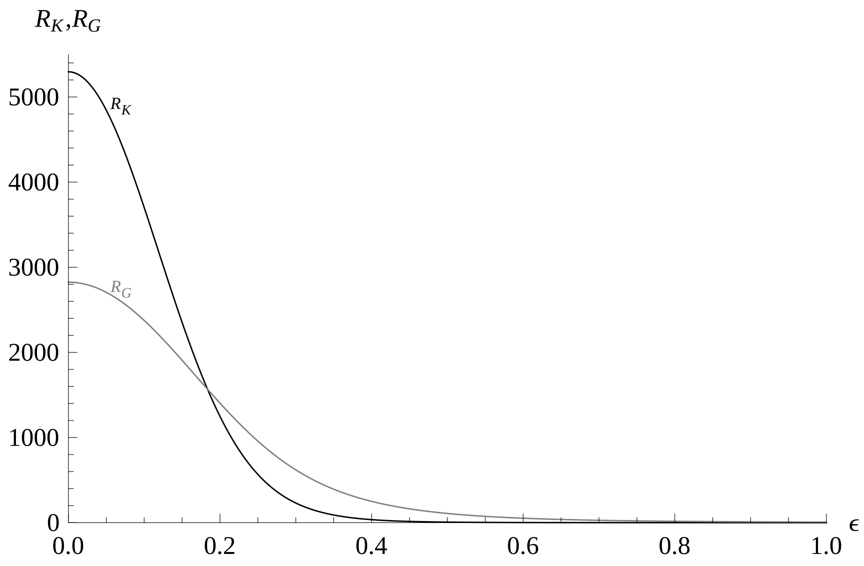


Figure 4.10: Bifurcation diagram for steady state levels of R_K and R_G with respect to ε under the subsidization of R_G for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $s = 0.5$.

Accordingly, the production decreases, where for small values of ε initially more is produced than in the basic model due to the higher values of K and G (see Figure 4.11).

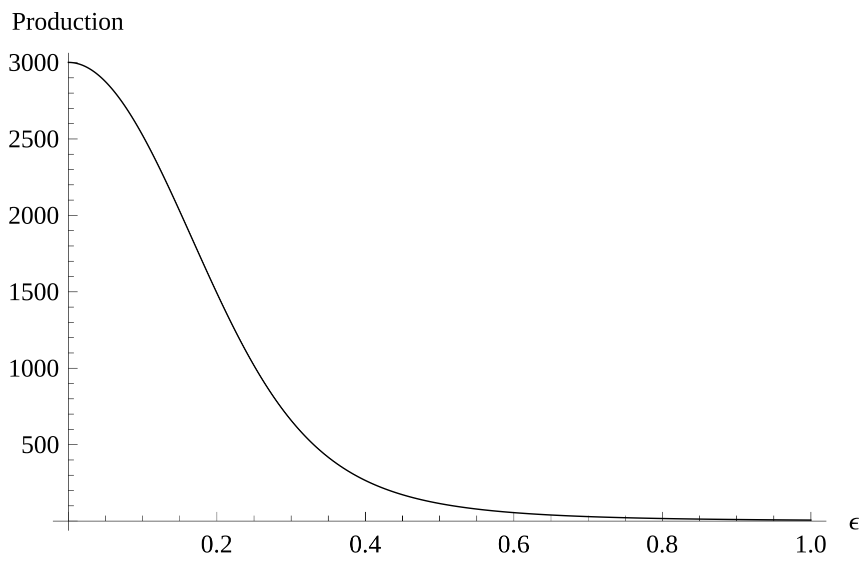


Figure 4.11: Bifurcation diagram of the steady state production output with respect to ε under the subsidization of R_G for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $s = 0.5$.

Considering the equilibrium utility in Figure 4.12, the behavior is quite similar to the utility function in the basic model. The utility maximizing environmental quality is reached at $\varepsilon = 0.129$, which is a little bit higher than before but not further mentionable.

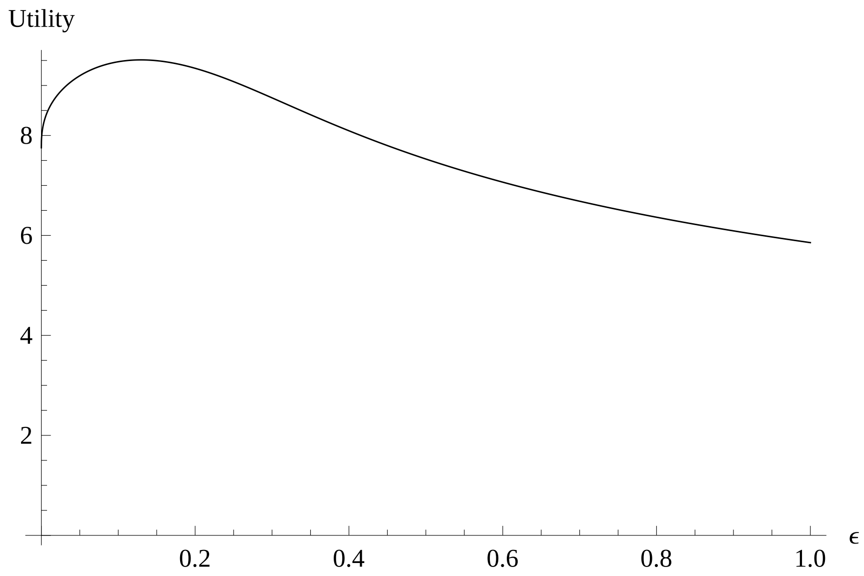


Figure 4.12: Bifurcation diagram of equilibrium utility with respect to ε under the subsidization of R_G for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $s = 0.5$.

Figures 4.13 and 4.14 show the ratios of brown and green capital and brown and green R&D investments, respectively. One can see that the initial ratios of G and R_G are higher than in the basic model, as already mentioned above. Due to the lower values of ε at which green capital and green R&D start to be advantageous, the inflexion points of the convex-concave curves are accordingly closer to the origin, but the behavior generally remains the same.

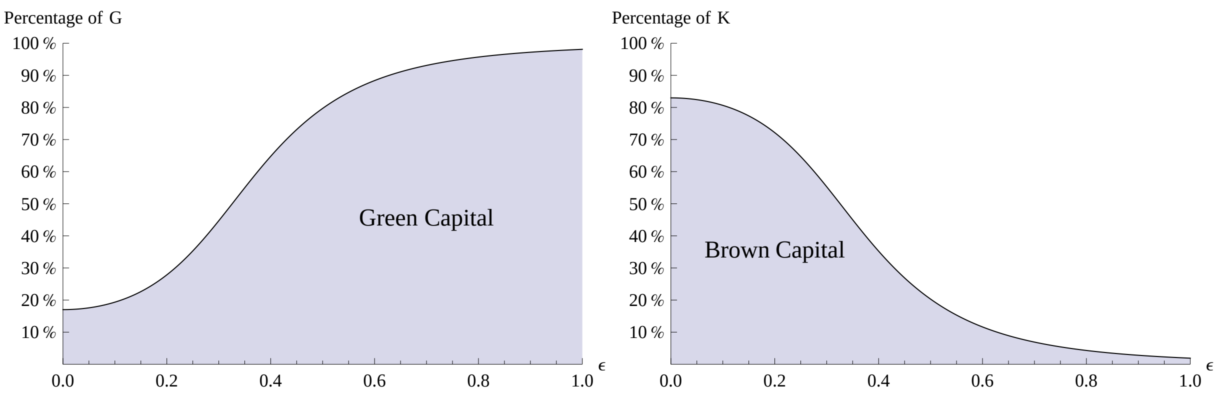


Figure 4.13: Bifurcation diagram of the equilibrium ratios of K and G with respect to ε under the subsidization of R_G for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $s = 0.5$.

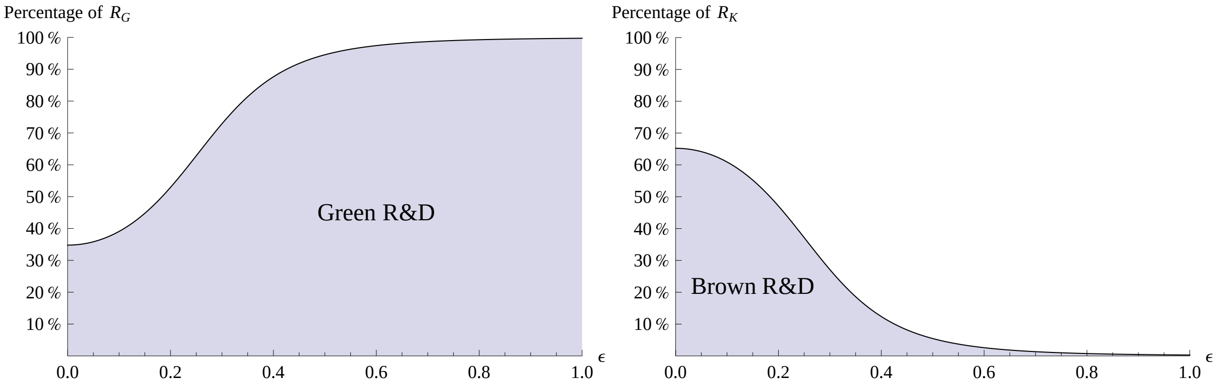


Figure 4.14: Bifurcation diagram of the equilibrium ratios of R_K and R_G with respect to ε under the subsidization of R_G for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $s = 0.5$.

4.2 Subsidization of Green Capital

In contrast to the previous approach considering a subsidy for green R&D, the amount of green capital used in production will be the object of subsidization in this section. Therefore, the greener the agent produces, the higher is the subsidization he/she gets. In this approach not the efforts for greener production but the success in accumulating green capital is rewarded. The parameter s in the subsequent model describes again the subsidy rate the government is willing to pay for green production input. The modified model reads as

$$\max_{R_K,R_G} \int_0^\infty e^{-rt} \left(\ln \left(\tau + bK^{\alpha_1} G^{\alpha_2} - w(R_K + R_G) - a\varepsilon^\beta K + sG \right) + c\varepsilon^\gamma \right) dt$$

$$\begin{aligned}
\text{s.t.:} \quad & \dot{K} = dK^{\delta_1} R_K^{\delta_2} - \phi K && \text{(4.2a)}\\
& \dot{G} = eG^{\sigma_1} R_G^{\sigma_2} - \psi G && \text{(4.2b)}\\
& 0 \le R_K \quad \forall t \ge 0 && \text{(4.2c)}\\
& 0 \le R_G \quad \forall t \ge 0 && \text{(4.2d)}\\
& 0 \le bK^{\alpha_1} G^{\alpha_2} - w(R_K + R_G) - a\varepsilon^\beta K + sG && \text{(4.2e)}\\
& 0 \le \varepsilon \le 1 && \text{(4.2f)}\\
& 0 < \alpha_1, \alpha_2, \gamma, w, s < 1 \quad \text{and} \quad \alpha_1 + \alpha_2 \le 1 && \text{(4.2g)}\\
& 0 < \delta_1, \delta_2 < 1 \quad \text{and} \quad \delta_1 + \delta_2 < 1 && \text{(4.2h)}\\
& 0 < \sigma_1, \sigma_2 < 1 \quad \text{and} \quad \sigma_1 + \sigma_2 < 1 && \text{(4.2i)}\\
& 1 < \beta && \text{(4.2j)}\\
& 0 < \phi, \psi, a, b, c, d, e, r && \text{(4.2k)}\\
& 0 < \tau \le 1. && \text{(4.2l)}
\end{aligned}$$

4.2.1 Steady States

Using the approach to derive the canonical system as in (2.21a)-(2.25), yields

$$
\begin{aligned}
\dot{R_K} &= \frac{D_1{}^2 D_2^2 R_G{}^2 R_K{}^2 Y^3}{w^2\left(d(\delta_2-1)\delta_2 K^{\delta_1} R_K{}^{\delta_2}\left(D_2 R_G{}^2 w - eY(\sigma_2-1)\sigma_2 G^{\sigma_1} R_G{}^{\sigma_2}\right) + D_1 e R_K{}^2 w(\sigma_2-1)\sigma_2 G^{\sigma_1} R_G{}^{\sigma_2}\right)} \\
&\cdot\Bigg\{\Bigg[\left(eY(\sigma_2-1)\sigma_2 G^{\sigma_1} R_G{}^{\sigma_2-2} - D_2 w\right)\left(aD_1\varepsilon^{\beta} - bD_1\alpha_1 G^{\alpha_2} K^{\alpha_1-1} + w\left(-d\delta_1 K^{\delta_1-1} R_K{}^{\delta_2} + r + \phi\right)\right) + \\
&D_2 w\left(w\left(-e\sigma_1 G^{\sigma_1-1} R_G{}^{\sigma_2} + r + \psi\right) - D_2\left(b\alpha_2 G^{\alpha_2-1} K^{\alpha_2} + s\right)\right)\Bigg]\frac{w}{D_1 D_2{}^2 Y^3} + \dot{G}T_1 + \dot{K}T_2\Bigg\} \\
\dot{R_G} &= \frac{D_1{}^2 D_2^2 R_G{}^2 R_K{}^2 Y^3}{w^2\left(d(\delta_2-1)\delta_2 K^{\delta_1} R_K{}^{\delta_2}\left(D_2 R_G{}^2 w - eY(\sigma_2-1)\sigma_2 G^{\sigma_1} R_G{}^{\sigma_2}\right) + D_1 e R_K{}^2 w(\sigma_2-1)\sigma_2 G^{\sigma_1} R_G{}^{\sigma_2}\right)} \\
&\cdot\Bigg\{\Bigg[\left(dY(\delta_2-1)\delta_2 K^{\delta_1} R_K{}^{\delta 2-2} - D_1 w\right)\left(w\left(-e\sigma_1 G^{\sigma_1-1} R_G{}^{\sigma_2} + r + \psi\right) - D_2\left(b\alpha_2 G^{\alpha_2-1} K^{\alpha_2} + s\right)\right) + \\
&D_1 w\left(aD_1\varepsilon^{\beta} - bD_1\alpha_1 G^{\alpha_2} K^{\alpha_1-1} - dw\delta_1 K^{\delta_1-1} R_K{}^{\delta_2} + w(r+\phi)\right)\Bigg]\frac{w}{D_1{}^2 D_2 Y^3} + \dot{G}T_3 + \dot{K}T_4\Bigg\} \\
\dot{K} &= dK^{\delta_1} R_K^{\delta_2} - \phi K \\
\dot{G} &= eG^{\sigma_1} R_G^{\sigma_2} - \psi G
\end{aligned}
\tag{4.3}
$$

with

$$
\begin{aligned}
T_1 &= \frac{ew^2\sigma_2 G^{\sigma_1-1} R_G{}^{\sigma_2-2}\left(b\alpha_2(\sigma_2-1)G^{\alpha_2} K^{\alpha_1} + R_G w\sigma_1\right)}{D_1 D_2{}^2 Y^3} \\
T_2 &= \frac{w^2}{D_1{}^2 D_2{}^2 K R_G{}^2 R_K Y^3}\Big(D_1 e R_K(\sigma_2-1)\sigma_2 G^{\sigma_1} R_G{}^{\sigma_2}\left(b\alpha_1 G^{\alpha_2} K^{\alpha_1} - aK\varepsilon^{\beta}\right) - \\
&\quad d\delta_1\delta_2 K^{\delta_1} R_K{}^{\delta_2}\left(D_2 R_G{}^2 w - eY(\sigma_2-1)\sigma_2 G^{\sigma_1} R_G{}^{\sigma_2}\right)\Big) \\
T_3 &= \frac{w^2}{D_1{}^2 D_2{}^2 G R_G R_K{}^2 Y^3}\Big(d(\delta_2-1)\delta_2 K^{\delta_1} R_K{}^{\delta_2}\left(bD_2 R_G\alpha_2 G^{\alpha_2} K^{\alpha_1} + eY\sigma_1\sigma_2 G^{\sigma_1} R_G{}^{\sigma_2}\right) - \\
&\quad D_1 e R_K{}^2 w\sigma_1\sigma_2 G^{\sigma_1} R_G{}^{\sigma_2}\Big) \\
T_4 &= \frac{dw^2\delta_2 K^{\delta_1-1} R_K{}^{\delta_2-2}\left((\delta_2-1)\left(b\alpha_1 G^{\alpha_2} K^{\alpha_1} - aK\varepsilon^{\beta}\right) + R_K w\delta_1\right)}{D_1{}^2 D_2 Y^3} \\
Y &= \tau + bK^{\alpha_1} G^{\alpha_2} - w(R_K + R_G) - a\varepsilon^{\beta} K + sG
\end{aligned}
$$

and D_1 and D_2 again being the first derivatives of the state dynamics with respect to the corresponding control

$$
\begin{aligned}
D_1 &= dK^{\delta_1}\delta_2 R_K{}^{\delta_2-1} \\
D_2 &= eG^{\sigma_1}\sigma_2 R_G{}^{\sigma_2-1}.
\end{aligned}
$$

This system yields again a single saddle point according to the first case in Table 2.2 which for $\varepsilon = 0.4$ and $s = 0.1$ lies at $K = 1,364$, $G = 4,021$, $R_K = 61$ and $R_G = 1,135$ (see Figure 4.15). Compared to the basic model the level of green capital quadruples and, because R&D investments are necessary to accumulate capital, R_G also highly increases.

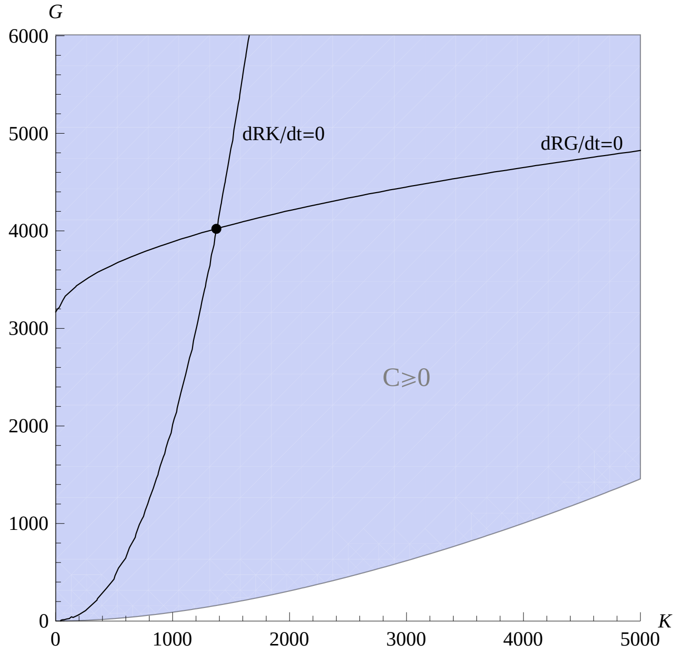


Figure 4.15: Steady state under the subsidization of G with $\alpha_1 = 0.6, \alpha_2 = 0.2$, $\varepsilon = 0.4$ and $s = 0.1$.

4.2.2 Optimal Paths

In Figure 4.16 the phase portrait of the present model for $\varepsilon = 0.4$ and a subsidization rate of 10 percent is shown. For this portrait, the crucial intervals for the angle η are $[0.500002261485398\pi$, $0.500002261485399\pi]$ and $[1.4999973843957\pi, 1.4999973843958\pi]$. As in the previous phase portraits, the trajectories are divided into the three parts according to the continuation within the admissible region, the continuation along admissible boundary of the control constraint in (4.2c) and the continuation along the admissible boundary of the mixed path constraint in (4.2e), until finally consumption gets zero.

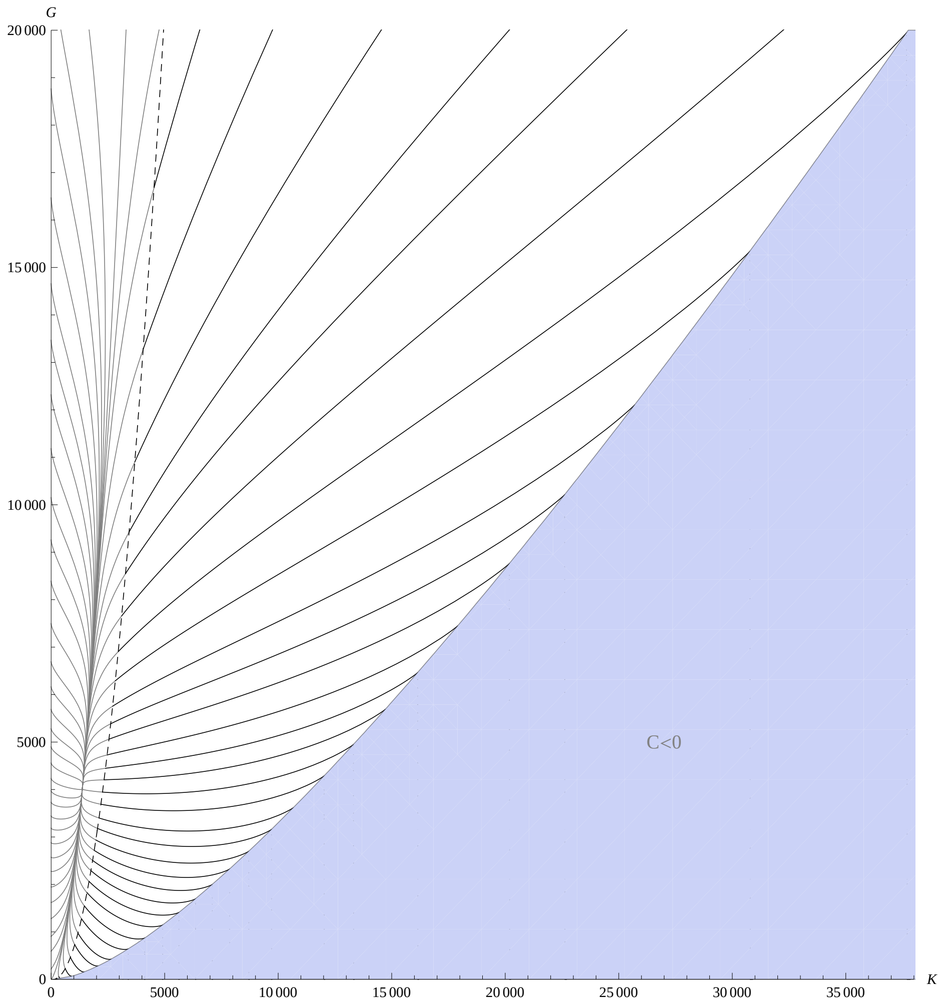


Figure 4.16: Phase portrait under the subsidization of G with $\alpha_1 = 0.6, \alpha_2 = 0.2$, $\varepsilon = 0.4$ and $s = 0.1$.

Initial Points with an Equal Level of K and G

Figure 4.17 shows two trajectories from the phase portrait in Figure 4.16. They both have initial values with almost equal levels of the two types of capital. To point out the differences and equalities compared to the basic model, the values of the capital levels K and G as well as of the R&D investments R_K and R_G along these trajectories are considered. In Figure 4.18 the state and control values along the trajectory with the lower initial values are depicted. Comparing these results with the results of the basic model in Figure 2.8, one can see that the qualitative behavior remains the same but the slope of G and R_G is much steeper than in the basic model. When they finally converge to their equilibrium values, G is more than twice as high as K, and R_G is even more than eighteen times higher than R_K. Likewise, as shown in Figure 4.19, the behavior of the according values along the trajectory with the higher initial levels is similar to the results of the basic model, but, with green capital and R&D investments being more dominating.

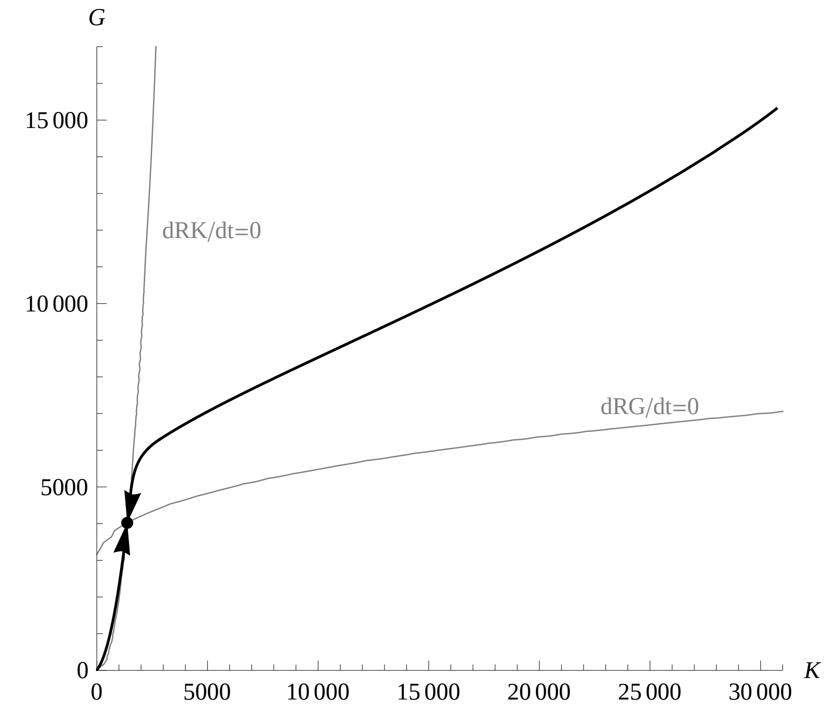


Figure 4.17: Two trajectories with equal initial capital levels under the subsidization of G for $\alpha_1 = 0.6, \alpha_2 = 0.2$, $\varepsilon = 0.4$ and $s = 0.1$.

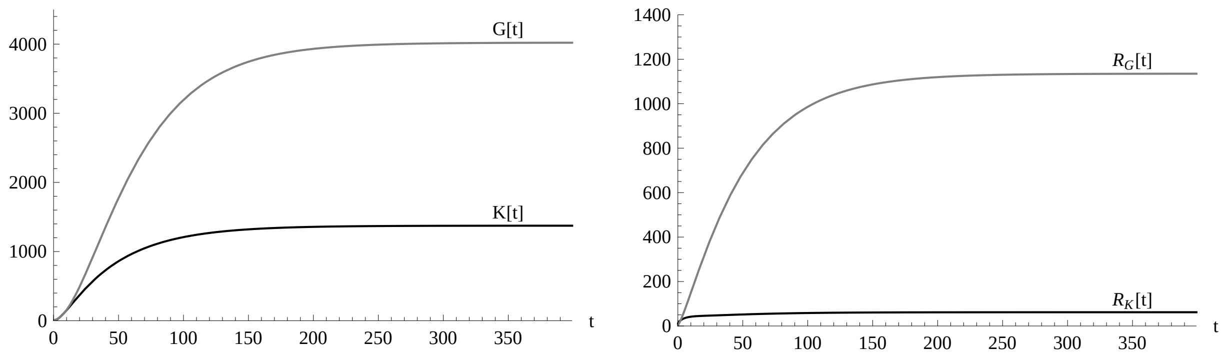


Figure 4.18: Optimal time paths of state and control starting from low capital levels under the subsidization of G with $\alpha_1 = 0.6, \alpha_2 = 0.2$, $\varepsilon = 0.4$ and $s = 0.1$.

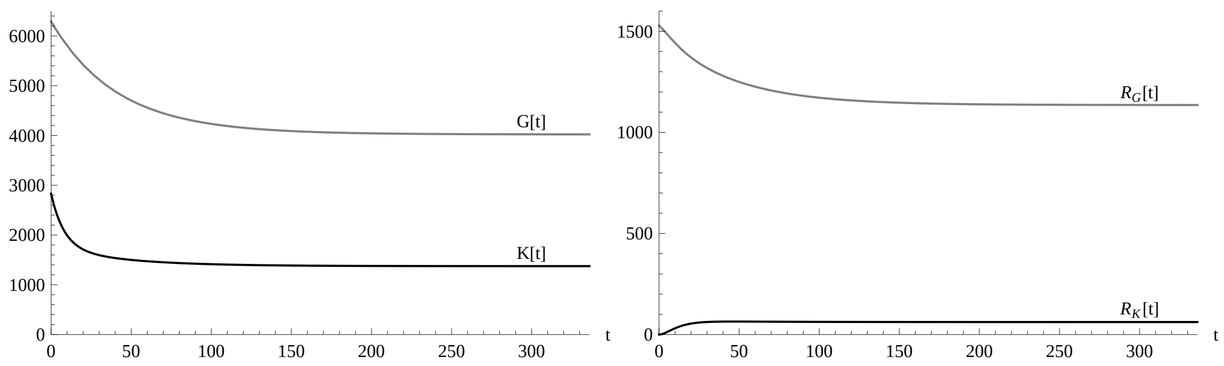


Figure 4.19: Optimal time paths of state and control starting from high capital levels under the subsidization of G with $\alpha_1 = 0.6, \alpha_2 = 0.2$, $\varepsilon = 0.4$ and $s = 0.1$.

Initial Points with One Type of Capital Being Dominant

Figure 4.20 shows two trajectories of Figure 4.16 with a production obviously dominated by one type of capital as initial state. Remember that a complete unilateral production with only one type of capital is not feasible. Figures 4.21 and 4.22 show again the values of K, G, R_K and R_G along these two trajectories. As in the previous, case the qualitative behavior is quite similar to the results of the basic model, but with an obviously higher use of green capital and R&D.

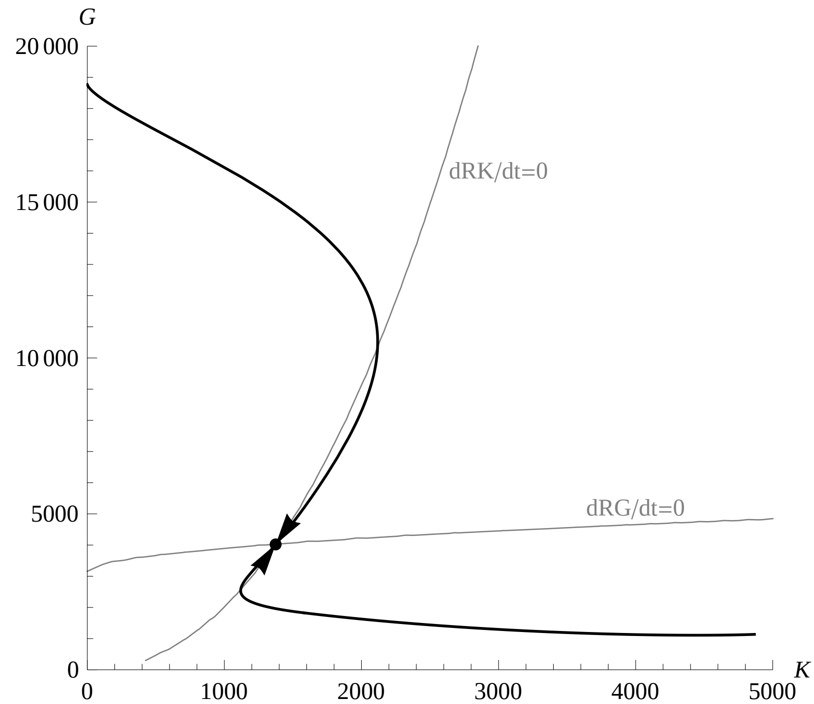


Figure 4.20: Two trajectories starting at a one-capital-type-dominated production under the subsidization of G for $\alpha_1 = 0.6, \alpha_2 = 0.2$, $\varepsilon = 0.4$ and $s = 0.1$.

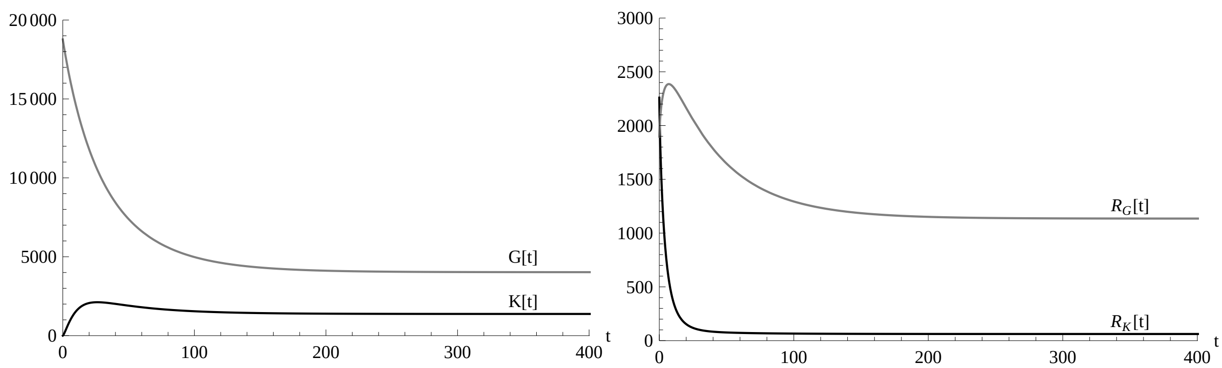


Figure 4.21: Optimal time paths of state and control starting from a definitely green capital-dominated production under the subsidization of G for $\alpha_1 = 0.6, \alpha_2 = 0.2$, $\varepsilon = 0.4$ and $s = 0.1$.

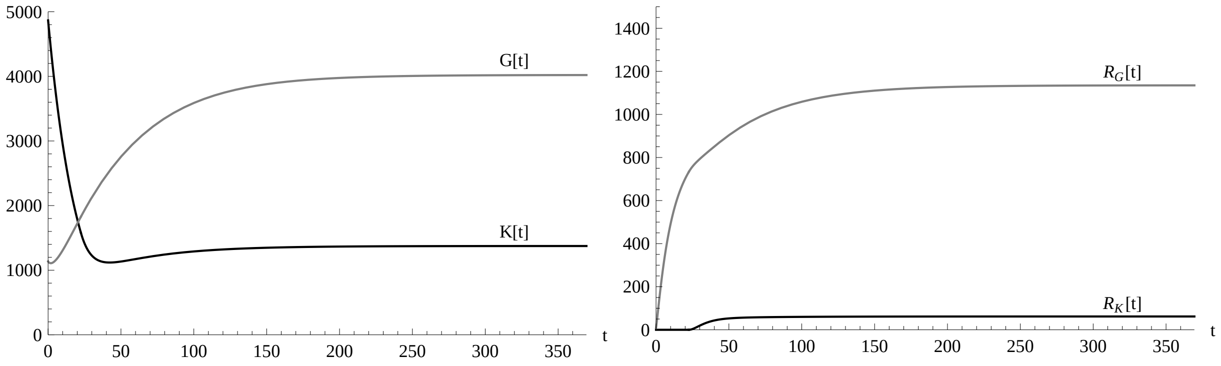


Figure 4.22: Optimal time paths of state and control starting from a definitely brown capital-dominated production under the subsidization of G for $\alpha_1 = 0.6, \alpha_2 = 0.2$, $\varepsilon = 0.4$ and $s = 0.1$.

4.2.3 Bifurcation Analysis

While the phase portrait and the optimal time paths seem to be quite similar to those of the basic model, bifurcation analysis reveals that the introduction of subsidies for green capital has definitely an impact on the results. Remember that in the basic model an increasing value of ε on the one hand causes a shift to more green capital in production, while on the other hand the levels of both capital types, and therefore the production, dramatically decrease. In this model, the shift to more green capital is boosted by the introduced subsidies and, as one can see in Figure 4.23, this causes an interesting change. While the level of brown capital is still declining with higher ε, the level of green capital in the beginning slightly decreases, but then flattens out at a value way higher than K. The point at which G starts to dominate K is at $\varepsilon = 0.292$. The same applies for R_G, which gets obvious in Figure 4.24. Additionally, the investments in green R&D are comparably high even for very small values of ε. The point at which green R&D gets dominant is at $\varepsilon = 0.151$. Compared to the results of the basic model, a smaller value of ε is necessary to let green capital and R&D becoming dominant.

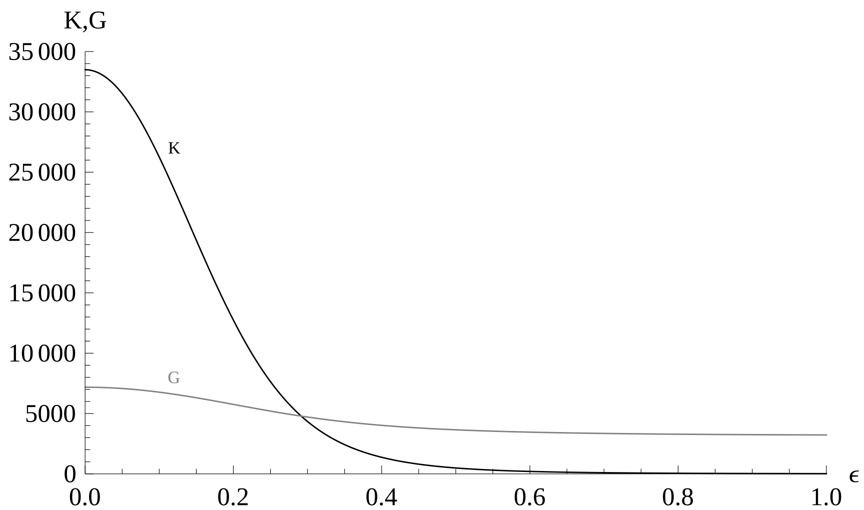


Figure 4.23: Bifurcation diagram for steady state levels of K and G with respect to ε under the subsidization of G for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $s = 0.1$.

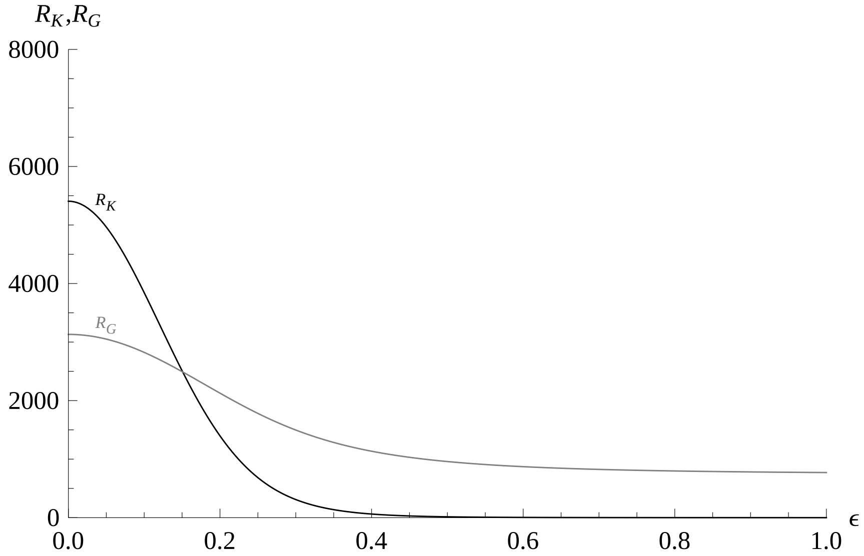


Figure 4.24: Bifurcation diagram for steady state levels of R_K and R_G with respect to ε under the subsidization of G for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $s = 0.1$.

Considering the equilibrium production in Figure 4.25 one can see that despite the fact that G remains on a clearly positive level the production decreases due to the very small values of K.

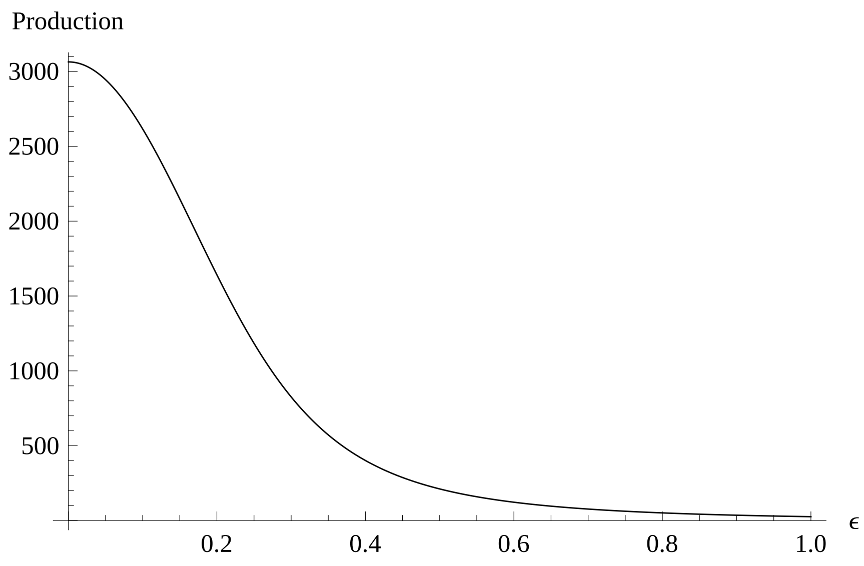


Figure 4.25: Bifurcation diagram the steady state production output with respect to ε under the subsidization of G for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $s = 0.1$.

Regarding the equilibrium utility in Figure 4.26, a remarkable change can be observed. While the utility-maximizing environmental quality is at $\varepsilon = 0.125$ in the basic model and at $\varepsilon = 0.129$ in the previous section with subsidization of green R&D, now the highest utility is reached at $\varepsilon = 1$. This means that the best case scenario for the agent is a completely clean environment, which is a stunning result. Additionally, there is also a local maximum at $\varepsilon = 0.162$.

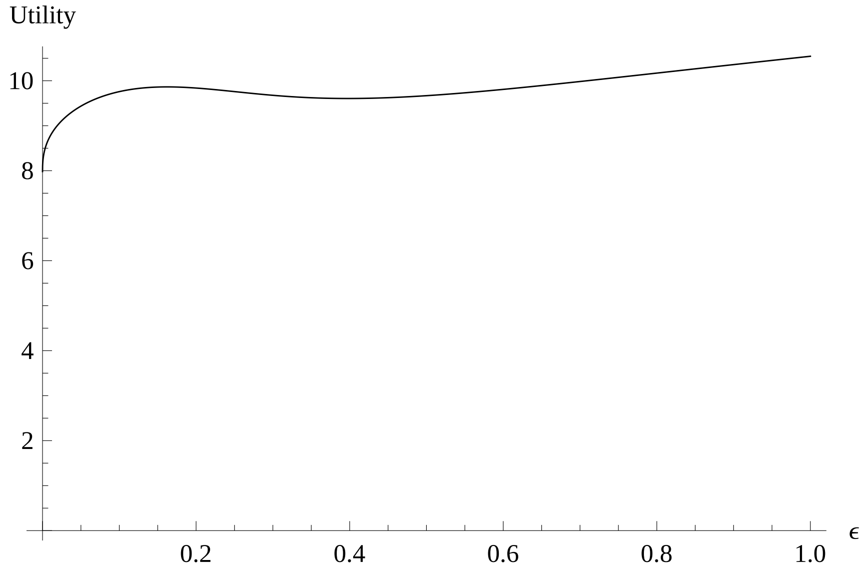


Figure 4.26: Bifurcation diagram of equilibrium utility with respect to ε under the subsidization of G for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $s = 0.1$.

To provide again a qualitative instead of a quantitative comparison, the shares of green and brown capital as well as green and brown R&D are investigated. In Figures 4.27 and 4.28 one can see that the equilibrium ratios of G and R_G are generally higher, compared to the basic model. Due to the lower value of ε at which green capital gets dominant, the inflexion points of the convex-concave curves of the ratios are lower, namely at $\varepsilon = 0.292$ for the capital levels and at $\varepsilon = 0.151$ for the R&D investments.

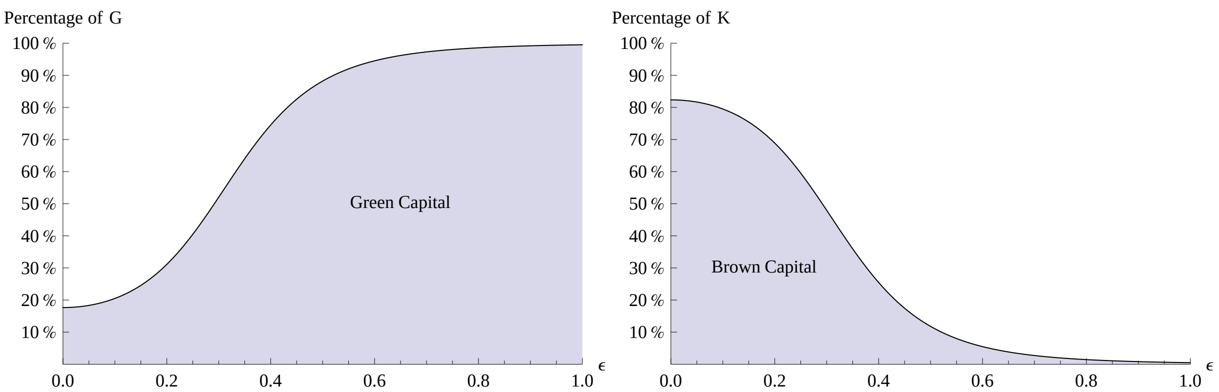


Figure 4.27: Bifurcation diagram of the equilibrium ratios of K and G with respect to ε under the subsidization of G for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $s = 0.1$.

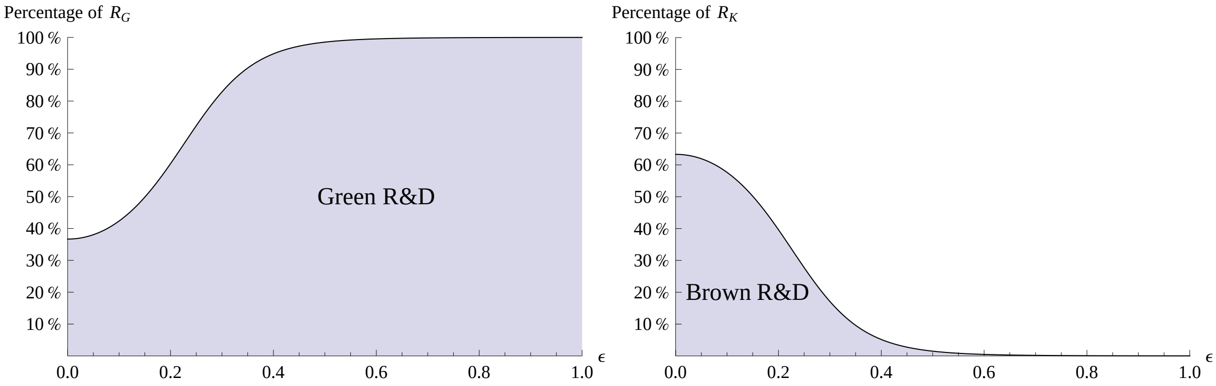


Figure 4.28: Bifurcation diagram of the equilibrium ratios of R_K and R_G with respect to ε under the subsidization of G for $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $s = 0.1$.

4.3 The Optimal Choice of the Subsidy Rate

In the previous two sections, the subsidy rate has been arbitrarily chosen to give an idea about the impacts of subsidization. Due to the fact that the subsidy rate, as well as the environmental quality, is exogenously given by the government, the presented results change quantitatively if the subsidy rate increases. However, considering this problem from the government's point of view, the choice of the optimal subsidization rate is a crucial aspect for the proper use of environmental policy instruments. Naturally, the "right" choice of the subsidization rate depends on the desired degree of changes in production, but further on also on the substitution elasticities. Ideally, the subsidy rate should reflect the marginal social benefits caused by the desired changes. However, in practice, this is quite difficult to achieve (see Organisation for Economic Co-operation and Development [1991]).

To investigate the optimal choice of the subsidization rate s in the present model, bifurcation analysis is used to give an idea about the trade-off which the government may want to consider for this decision.

4.3.1 Optimal Subsidy Rate for Green Capital

Suppose that the government's benefit of subsidization is reflected by a utility function which includes on the one hand the achievements in raising the level of green capital used in production and on the other hand the subsidy payments which come along with this subsidization. Then this utility function is given as

$$U_s^G := \Delta G(s) - sG(s), \tag{4.4}$$

where $G(s)$ denotes the corresponding equilibrium value of green capital under the assumption

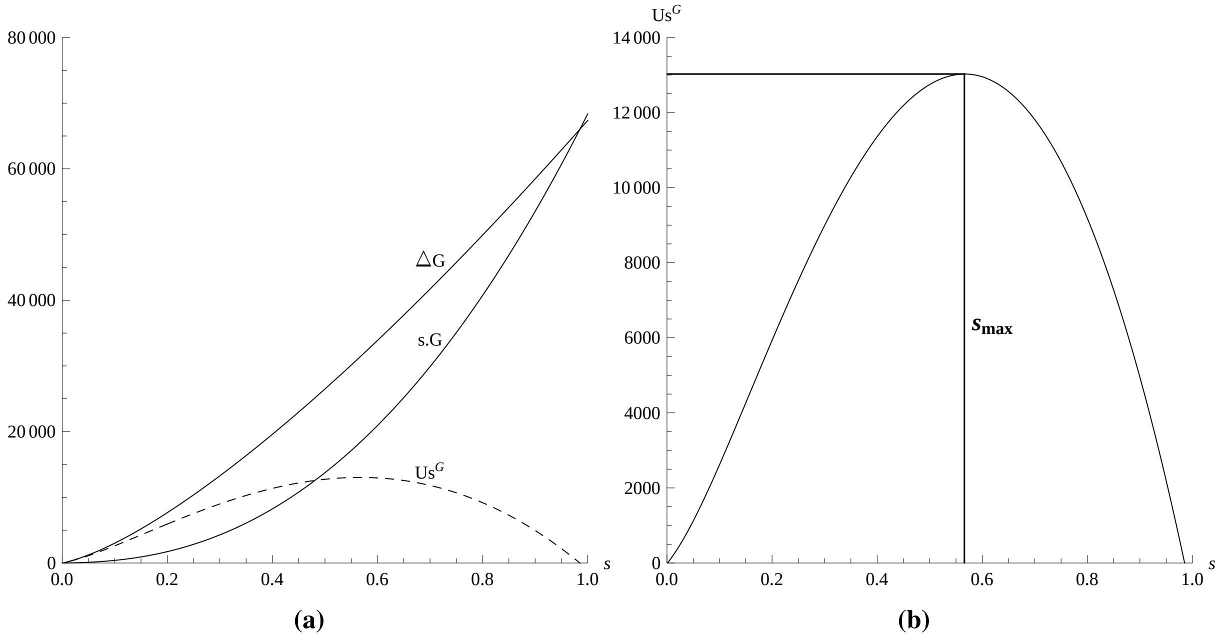


Figure 4.29: Choice of optimal subsidy rate for the subsidization of green capital.

of a subsidy rate of s, $sG(s)$ therefore are the according subsidy payments and $\Delta G(s)$ describes the increase in green capital due to subsidization which is defined as

$$\Delta G(s) := G(s) - G(0), \tag{4.5}$$

where $G(0)$ corresponds to the scenario in the basic model where no subsidization exists at all.

Figure 4.29 shows U_s^G, which is depicted as dashed line in the left panel, the subsidy costs and $\Delta G(s)$. It immediately gets clear that s is optimal when it yields the highest increases in G at the lowest possible costs. For the case in 4.2 with $\alpha_1 = 0.6$, $\alpha_2 = 0.2$ and $\varepsilon = 0.4$ this optimal subsidization rate is $s = 56.6\%$, which seems quite high at first sight, but considering the fact that an increase of more than $12,000$ units of green capital is achieved with this subsidization, the result becomes convincing.

4.3.2 Optimal Subsidy Rate for Green R&D

To do the same investigations for the subsidy for green R&D investments one may consider the utility function in (4.4) for R_G, which then is given as

$$U_s^{R_G} := \Delta R_G(s) - swR_G(s), \tag{4.6}$$

where $\Delta R_G(s) := R_G(s) - R_G(0)$ is again the achieved increase in green R&D investments compared to the basic case with s being zero. $R_G(s)$ accordingly is the equilibrium value under a subsidy rate of s. Note that in this approach, s is the share of the opportunity costs which is covered by the subsidy, and not the share of the whole R&D investments. Therefore w appears

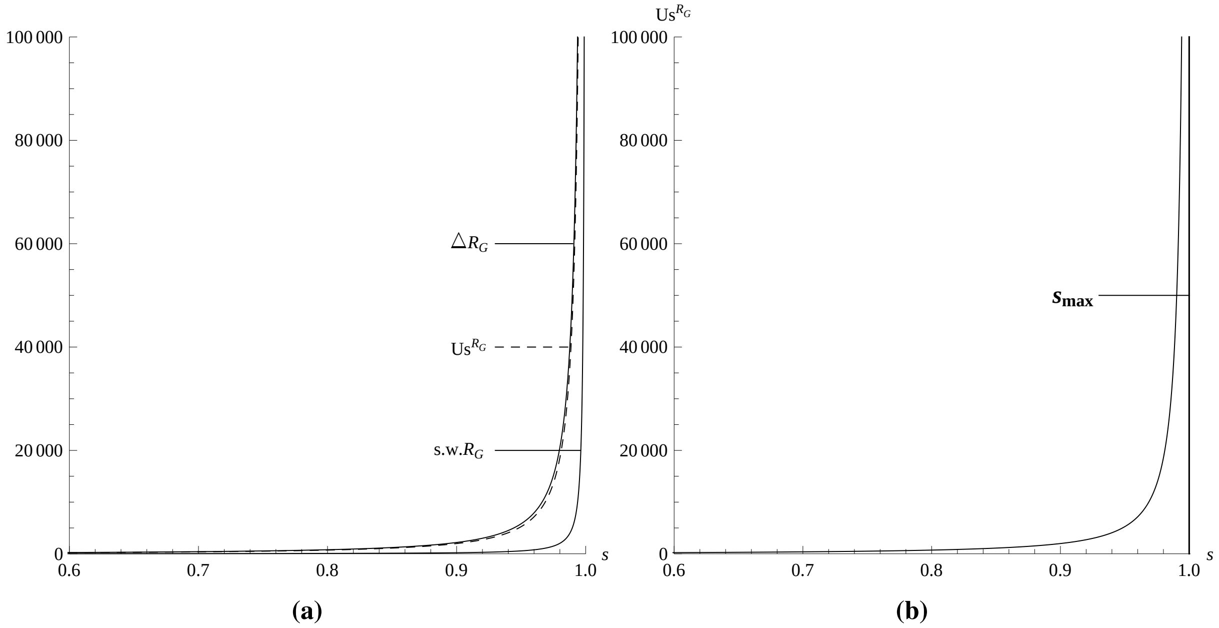


Figure 4.30: Choice of optimal subsidy rate for the subsidization of green R&D investments.

in the function. Maximizing $U_s^{R_G}$ yields again the optimal subsidy rate, which is depicted in Figure 4.30. Obviously, compared with the results for green capital, the behavior of this utility function is different. While an inner maximum occurred in the previous case, which even is expected in this context, the optimal subsidy rate in this approach is $s = 100\%$. The reason for this stunning result, however, can easily be traced back to the model structure itself. While subsidization of green capital needs to be incentive enough to make the agent accept the less productivity of green capital as well as the higher efforts required for the accumulation, the only negative aspects of higher R&D investments for green capital is given through the opportunity costs in the target function. For this reason, only subsidization proportional to the opportunity costs is possible. Subsidization with a rate higher than w is not feasible in this approach. Consider the values of R_G in Figure 4.30a. For very low rates s, the achievments in rising R_G are almost zero. But suddenly the utility of a marginal subsidy rate unit increases, until finally, close to 100%, the values of R_G rise immensely. Here the negative aspects of R&D investments are almost compensated. Higher investments get more and more advantageous and therefore the utility further increases. Note however, that the maximizing subsidy rate of 100% itself is not feasible, as already mentioned. This structural constraint also explains why the subsidization of green capital in this model is more effective than the subsidization of green R&D investments, at least for realistic values.

4.4 Conclusion

The results of this chapter were supposed to show how effective subsidization can be, when it is used properly. A subsidy rate of only 10% for green capital already causes a tremendous change in behavior and, as the previous section shows, this rate is far away from being optimal. In this certain approach, the subsidization of green capital is more effective than the subsidization of green R&D investments leading to the accumulation of green capital, at least for realistic values of the subsidy rate. But, as already mentioned, this only comes along with the fact that for structural reasons no further subsidization than the full coverage of opportunity costs is possible. Anyway, the obtained results show that this kind of subsidization has supporting potential. It also might be interesting to consider a mixture of these two types of subsidies. For example green R&D investments are supported until a certain limit of green capital is reached, like a start-up aid for entrepreneurs who hardly use green capital at the beginning. When this level is reached, the subsidization for R&D investments stops and the subsidization of green capital starts instead. Another interesting aspect could be the use of the share of green capital or green R&D investments instead of the total amount as object of subsidization. If perfect competition is not assumed, this would avoid that an entrepreneur from a big concern with a very high production level and therefore a high need of capital input, from which however only a fraction is green, receives the same subsidy amount as an entrepreneur of a small concern who mainly produces with green capital. Nevertheless, subsidization is besides taxing one of the most important economic instruments in environmental issues whose incentive impact already has become obvious in this approach.

Chapter 5

Convex-Concave Growth Function in Brown and Green Capital

In the basic model, a Cobb Douglas production function is used for the state dynamics which satisfies the Inada conditions. Therefore

$$\lim_{K \longrightarrow 0} \frac{\partial \dot{K}}{\partial K} = \infty \quad ,$$

i.e. a sufficiently small capital level of K already has an immense positive feedback on itself (for G respectively). However, in reality capital is not accumulated that easily. It is more likely that at the beginning of capital accumulation this positive feedback is almost zero. Lots of R&D investments are necessary to raise the capital level, which still grows very slowly until suddenly the momentum of the capital starts to be effective and boosts the capital to a very high level at which the effectiveness of an additional marginal capital unit falls again and the slope flattens. Mathematically, this behavior is perfectly described by a convex-concave growth function which will be considered in this section as another modification of the basic model.

Starting from the Cobb Douglas function, the terms K^{δ_1} and G^{σ_1}, describing this positive feedback will be replaced by a convex-concave function, while the terms describing the impact of the R&D investments, ${R_K}^{\delta_2}$ and ${R_G}^{\sigma_2}$ remain unchanged. The inserted convex-concave function is taken from Maeler, Xepapades, and De Zeeuw [2003], where it is used for the modeling of eutrophication processes in shallow lakes. Further on, it is also used in Heijdra and Heijnen [2009] to model the stock of pollution in an emission equation. The main characteristic of shallow lake dynamics is the so-called *hysteresis effect*, where the transgression of a certain flip point leads to an unstoppable growth of algae, pollution or in my case of capital stock. In order to obtain a better flexibility concerning the inflexion point and limit value, additional scale parameters are introduced. Note, that the state dynamics with this new growth function satisfy neither concavity nor the Inada conditions which are required in the basic model.

The modified model now is

$$\max_{R_K,R_G} \int_0^\infty e^{-rt}\left(\ln\left(\tau + bK^{\alpha_1}G^{\alpha_2} - w(R_K+R_G) - a\varepsilon^\beta K\right) + c\varepsilon^\gamma\right)dt \tag{5.1}$$

$$\text{s.t.:} \quad \dot{K} = \frac{d_1K^2}{d_2K^2+v}R_K^\delta - \phi K \tag{5.1a}$$

$$\dot{G} = \frac{e_1G^2}{e_2G^2+z}R_G^\sigma - \psi G \tag{5.1b}$$

$$0 \le R_K \quad \forall t \ge 0 \tag{5.1c}$$

$$0 \le R_G \quad \forall t \ge 0 \tag{5.1d}$$

$$0 \le bK^{\alpha_1}G^{\alpha_2} - w(R_K+R_G) - a\varepsilon^\beta K \tag{5.1e}$$

$$0 \le \varepsilon \le 1 \tag{5.1f}$$

$$0 < \alpha_1, \alpha_2, \gamma, w < 1 \quad \text{and} \quad \alpha_1 + \alpha_2 \le 1 \tag{5.1g}$$

$$0 < \delta < 1 \tag{5.1h}$$

$$0 < \sigma < 1 \tag{5.1i}$$

$$1 < \beta \tag{5.1j}$$

$$0 < \phi, \psi, a, b, c, d_1, d_2, e_1, e_2, r, v, z \tag{5.1k}$$

$$0 < \tau \le 1. \tag{5.1l}$$

The dashed line in Figure 5.1 shows the growth path of brown capital under this convex-concave function and for two fixed levels of R_K. Note that even negative growth occurs, if the capital level is small enough, which presents the initial difficulties of capital accumulation. Naturally, the growth path of G looks similar.

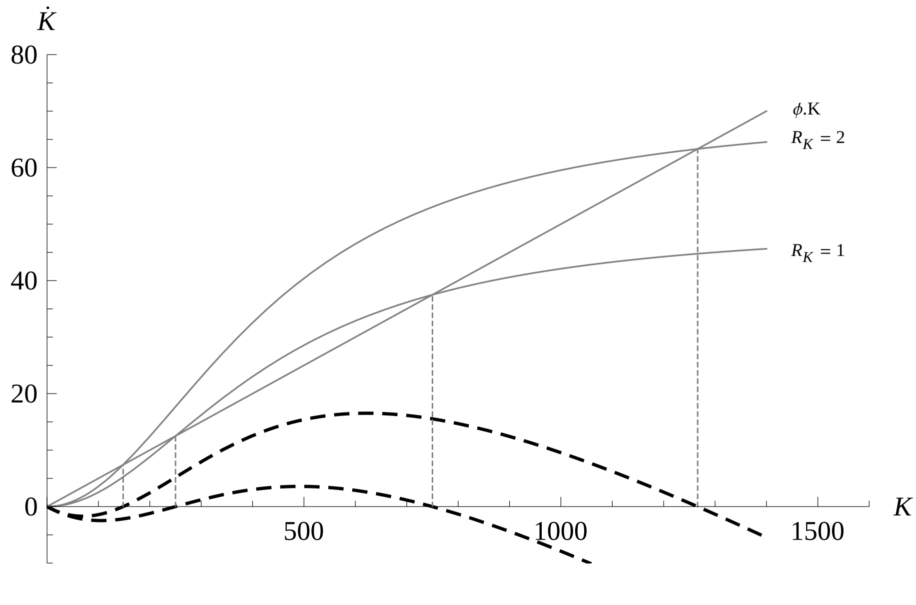


Figure 5.1: Growth path of conventional capital under a convex-concave production with $d_1 = 1$, $d_2 = 0.2$ and $v = 3750$.

Additionally to the parameter values in Table 2.1, the values of the new parameters used for the following analysis are summarized in Table 5.1. In order to obtain comparable results to the basic model, the parameter values are chosen in a way that the inflexion point of the growth path is half the height of the equilibrium value in the basic laissez-faire scenario, while the limit value of the growth function is set twice as high.

Parameter	Value	Description
d_1	1	Scale parameter of $\dot{K}$
d_2	0.02381	Scale parameter of $\dot{K}$
e_1	1	Scale parameter of $\dot{G}$
e_2	0.04167	Scale parameter of $\dot{G}$
v	15184028.57	Scale parameter of $\dot{K}$
z	531996.13	Scale parameter of $\dot{G}$
δ	0.5	Production elasticity of $\dot{K}$
σ	0.4	Production elasticity of $\dot{G}$

Table 5.1: Additional parameter values for the convex-concave model.

5.1 Steady States

Using once more the approach to derive the canonical system as in (2.21a)-(2.25), one obtains

$$
\begin{aligned}
\dot{R_K} &= -\frac{D_1{}^2 D_2{}^2 Y^4}{w^2\left(\left(\frac{d_1 K^2 Y(\delta-1)\delta R_K{}^{\delta-2}}{d_2K^2+v}-D_1w\right)\left(\frac{e_1G^2Y(\sigma-1)\sigma R_G{}^{\sigma-2}}{e_2G^2+z}-D_2w\right)-D_1D_2w^2\right)} \\
&\cdot\Bigg\{\Bigg[\Big(\frac{e_1G^2Y(\sigma-1)\sigma R_G{}^{\sigma-2}}{e_2G^2+z}-D_2w\Big)\Big(aD_1\varepsilon^\beta-bD_1\alpha_1G^{\alpha_2}K^{\alpha_1-1}-\frac{2d_1KvwR_K{}^\delta}{(d_2K^2+v)^2}+w(r+\phi)\Big) \\
&+D_2w\Big(-bD_2\alpha_2G^{\alpha_2-1}K^{\alpha_2}-\frac{2e_1GwzR_G{}^\sigma}{(e_2G^2+z)^2}+w(r+\psi)\Big)\Bigg]\frac{w}{D_1D_2{}^2Y^3}+\dot{G}T_1+\dot{K}T_2\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\dot{R_G} &= -\frac{D_1{}^2 D_2{}^2 Y^4}{w^2\left(\left(\frac{d_1 K^2 Y(\delta-1)\delta R_K{}^{\delta-2}}{d_2K^2+v}-D_1w\right)\left(\frac{e_1G^2Y(\sigma-1)\sigma R_G{}^{\sigma-2}}{e_2G^2+z}-D_2w\right)-D_1D_2w^2\right)} \\
&\cdot\Bigg\{\Bigg[\Big(\frac{d_1K^2Y(\delta-1)\delta R_K{}^{\delta-2}}{d_2K^2+v}-D_1w\Big)\Big(-bD_2\alpha_2G^{\alpha_2-1}K^{\alpha_2}-\frac{2e_1GwzR_G{}^\sigma}{(e_2G^2+z)^2}+rw+w\psi\Big) \\
&+D_1w\Big(aD_1\varepsilon^\beta-bD_1\alpha_1G^{\alpha_2}K^{\alpha_1-1}-\frac{2d_1KvwR_K{}^\delta}{(d_2K^2+v)^2}+rw+w\phi\Big)\Bigg]\frac{w}{D_1{}^2D_2Y^3}+\dot{G}T_3+\dot{K}T_4\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\dot{K} &= \frac{d_1K^2}{d_2K^2+v}R_K{}^\delta-\phi K \\
\dot{G} &= \frac{e_1G^2}{e_2G^2+z}R_G{}^\sigma-\psi G \qquad (5.2)
\end{aligned}
$$

with

$$
\begin{aligned}
T_1 &= \frac{e_1 G w^2 \sigma {R_G}^{\sigma-2} \left(b\alpha_2(\sigma-1)G^{\alpha_2}K^{\alpha_1} + 2R_G w z\right)}{D_1 {D_2}^2 Y^3 \left(e_2 G^2 + z\right)} \\
T_2 &= \frac{w^2}{{D_1}^2 {D_2}^2 Y^4}\Bigg(\Big(\frac{e_1 G^2 Y(\sigma-1)\sigma {R_G}^{\sigma-2}}{e_2 G^2 + z} - D_2 w\Big)\Big(-aD_1\varepsilon^{\beta} + bD_1\alpha_1 G^{\alpha_2}K^{\alpha_1-1} + \\
&\quad \frac{2d_1 K v Y \delta {R_K}^{\delta-1}}{d_2 K^2 + v}\Big) + D_1 D_2 w\Big(b\alpha_1 G^{\alpha_2}K^{\alpha_1-1} - a\varepsilon^{\beta}\Big)\Bigg) \\
T_3 &= \frac{w^2}{{D_1}^2 {D_2}^2 Y^4}\Bigg(\Big(\frac{d_1 K^2 Y(\delta-1)\delta {R_K}^{\delta-2}}{d_2 K^2 + v} - D_1 w\Big)\Big(bD_2\alpha_2 G^{\alpha_2-1}K^{\alpha_1} + \frac{2e_1 G Y z \sigma {R_G}^{\sigma-1}}{e_2 G^2 + z}\Big) + \\
&\quad bD_1 D_2 w \alpha_2 G^{\alpha_2-1}K^{\alpha_1}\Bigg) \\
T_4 &= -\frac{d_1 K w^2 \delta {R_K}^{\delta-2}\Big(-(\delta-1)\Big(b\alpha_1 G^{\alpha_2}K^{\alpha_1} - aK\varepsilon^{\beta}\Big) - 2R_K v w\Big)}{{D_1}^2 D_2 Y^3 \left(d_2 K^2 + v\right)} \\
Y &= \tau + bK^{\alpha_1}G^{\alpha_2} - w(R_K + R_G) - a\varepsilon^{\beta}K
\end{aligned}
$$

and D_1 and D_2 being the first derivatives of the state dynamics for the corresponding control

$$
\begin{aligned}
D_1 &= \frac{d_1 K^2}{d_2 K^2 + v}\delta {R_K}^{\delta-1} \\
D_2 &= \frac{e_1 G^2}{e_2 G^2 + z}\sigma {R_G}^{\sigma-1}.
\end{aligned}
$$

In contrast to the previous models, where steady states have only occurred inside the admissible area, one can see in Figure 5.2 that now there are also some equilibria which lie outside and therefore are not feasible. For this reason, the investigation of potential steady states at the boundary of the admissible area gets important.

5.1.1 Inner Equilibria

The admissible steady state for $\varepsilon = 0.185$ in Figure 5.2 at $K = 21,917$, $G = 6,202$, $R_K = 3,688$ and $R_G = 1,228$ is a saddle point according to the first case in Table 2.2. Considering the change of the equilibrium's position for other values of ε, the continuation for less environmental quality shows that both capital levels rise as in the previous cases. When continuing the equilibrium for higher values, however, an interesting observation can be made. While continuation at the beginning is still successful, the equilibrium suddenly disappears at $\varepsilon > 0.2$. Further investigation shows that this comes along with a difficulty of the model structure itself which also is present in the previous model approaches but hasn't caused serious problems yet. Remember the inadmissible area depicted in the phase portraits in the Figures 2.6, 4.2 and 4.16. Inside this area the abatement costs are higher than the production output and therefore no feasible

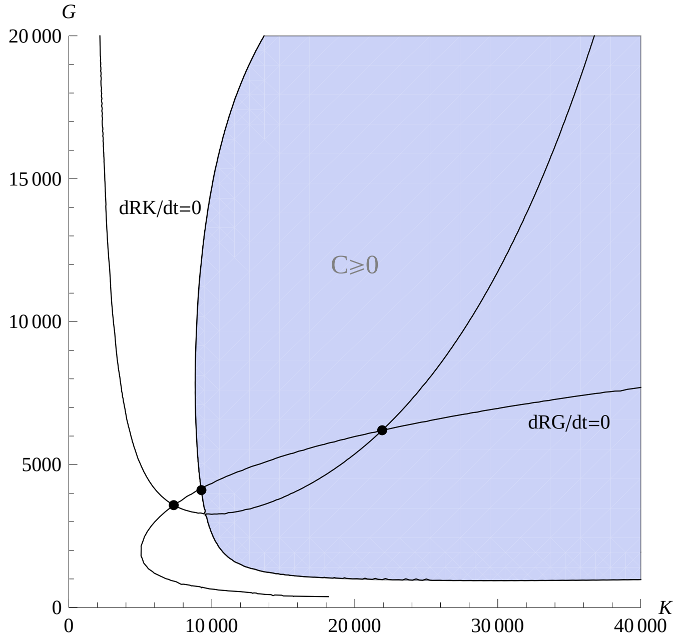


Figure 5.2: Steady states of the convex-concave model with $\alpha_1 = 0.6, \alpha_2 = 0.2$ and $\varepsilon = 0.185$.

pair of control variables can be found to ensure that consumption remains non-negative. In the mentioned phase portraits, this domain represents inadmissible initial values which means that starting from there no admissible path to the equilibrium exists. In the previous model approaches, the equilibria themselves always remain inside the admissible area for all values of ε. This, however, is different now. Figure 5.3 shows the mentioned inner equilibrium for several values of ε. As one can see, the higher ε gets, the closer the equilibrium is to the inadmissible domain until finally, at $\varepsilon = 0.2$, it lies exactly on the boundary. If ε grows further the equilibrium disappears. This means that there exists a threshold for environmental quality where no inner equilibrium occurs, which obviously is problematic.

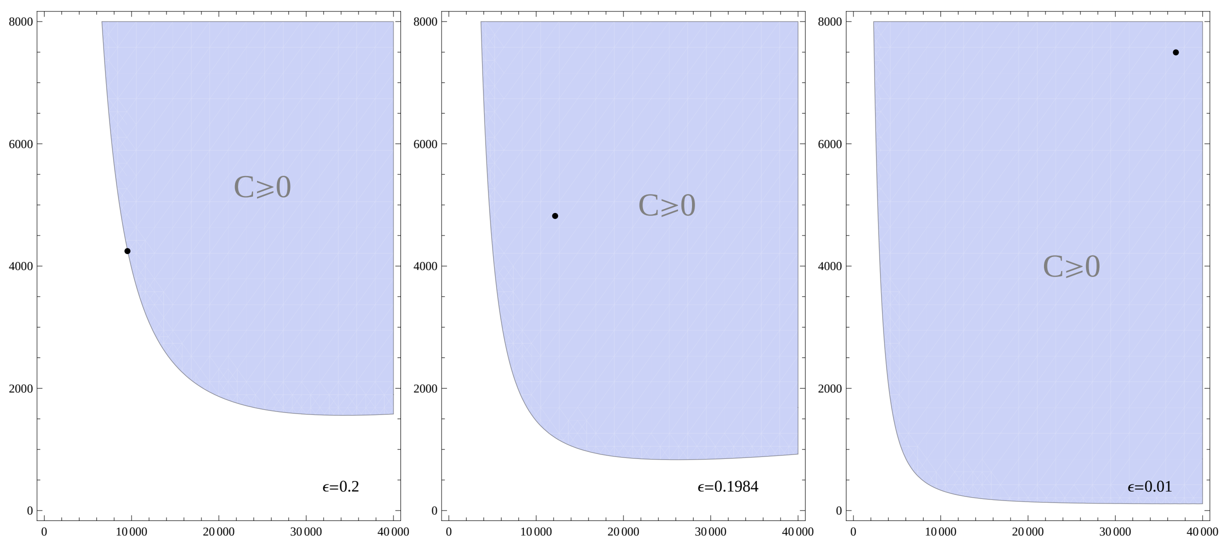


Figure 5.3: Threshold for existence of the inner equilibrium.

5.1.2 Boundary Equilibria in the Case $C = 0$

To investigate the present model for equilibria at the boundary of the admissible area subject to the mixed control path, one has to consider the corresponding canonical system as derived in (2.27). However, here the search for steady states turns out to be problematic. The reasons for this are analytical as well as numerical difficulties, which mainly occur because of the particular model structure and will be presented in more detail in Chapter 6. Because dealing with these problems comes along with extensive calculation methods as well as possible changes of the model structure, further investigations at this point would go beyond the scope of this thesis and therefore remain matter of future work.

However, to give at least some ideas about this indeed interesting approach, I will present some expectations about the occurrence of boundary equilibria in this convex-concave model. As already mentioned in Proof 2.3.1, one can show that the admissible boundary of the mixed path constraint $C(t) = 0$ cannot be optimal for $t \in [t_1, t_2)$ with $t_1 < t_2$ because one can always find another admissible pair of controls

$$\tilde{R_K}(.) = \frac{{R_K}^*(.)}{2} \quad \text{and} \quad \tilde{R_G}(.) = \frac{{R_G}^*(.)}{2}$$

where C no longer is zero and a higher target value can be obtained. Additionally, it is shown in Proof 2.3.1 that the case $C(t) = R_K(t) = R_G(t) = 0$ for $K(0) > 0$, $G(0) > 0$ can be dismissed since no admissible path to the only steady state $K(t) = G(t) = R_K(t) = R_G(t) = 0$ can be found.

Consequently, only the case

$$\lim_{t \to \infty} C(t) = 0 \quad \text{with} \quad C(t) > 0 \quad \text{for} \quad t < \infty$$

has to be considered for the search of possible boundary equilibria. The fact that the controls are unique and the mixed path constraint satisfies the constraint qualification, meaning that the partial derivatives of the constraint with respect to the controls is not singular

$$\frac{\partial C}{\partial R_K} = \frac{\partial C}{\partial R_G} = -w < 0,$$

yields the continuity of the controls and of the Lagrange multiplier. Therefore, $\mu_3 = 0$ has to be satisfied as soon as the mixed path constraint gets active. For the calculation this is equal to the canonical system inside the admissible domain expanded with the additional constraint $C = 0$, which yields the system

$$
\begin{aligned}
\mathcal{L}_{R_K} &= -\frac{w}{\tau} + \lambda_1 \frac{d_1 K^2}{d_2 K^2 + v} \delta R_K{}^{\delta - 1} = 0 \\
\mathcal{L}_{R_G} &= -\frac{w}{\tau} + \lambda_2 \frac{e_1 G^2}{e_2 G^2 + z} \sigma R_G{}^{\sigma - 1} = 0 \\
\dot{K} &= \frac{d_1 K^2}{d_2 K^2 + v} R_K^{\delta} - \phi K \\
\dot{G} &= \frac{e_1 G^2}{e_2 G^2 + z} R_G^{\sigma} - \psi G \\
\dot{\lambda}_1 &= \lambda_1 \left(r - \frac{2 d_1 K v}{(d_2 K^2 + v)^2} R_K{}^{\delta} + \phi \right) - b \alpha_1 K^{\alpha_1 - 1} G^{\alpha_2} + a \varepsilon^{\beta} \\
\dot{\lambda}_2 &= \lambda_2 \left(r - \frac{2 e_1 G z}{(e_2 G^2 + z)^2} R_G{}^{\sigma} + \psi \right) - b K^{\alpha_1} \alpha_2 G^{\alpha_2 - 1} \\
C(K, G, R_K, R_G) &= b K^{\alpha_1} G^{\alpha_2} - w(R_K + R_G) - a \varepsilon^{\beta} K = 0.
\end{aligned}
\tag{5.3}
$$

5.2 Constant Controls

Although an optimal solution of the present model cannot be obtained in the course of this thesis, the impact of the convex-concave growth functions in the state dynamics can be illustrated in part by investigating the model under the assumption of constant controls. Therefore I consider a system of *ODEs* given as

$$
\dot{K} = \frac{d_1 K^2}{d_2 K^2 + v} R_{K_{const}}{}^{\delta} - \phi K \tag{5.4}
$$

$$
\dot{G} = \frac{e_1 G^2}{e_2 G^2 + z} R_{G_{const}}{}^{\sigma} - \psi G. \tag{5.5}
$$

Note that there does not exist an interaction between green and brown capital. Thus these two differential equations are decoupled.

To start with, I assume symmetric controls. Solving the system yields altogether nine steady states from which four are stable, four are saddle points and one is repelling. Figure 5.4 shows the intersections of the isoclines for $R_{K_{const}} = R_{G_{const}} = 10,000$. Stable points are depicted in black, saddle points in gray and the only repelling point is gray with black surrounding. The linearity of the isoclines comes along with the decoupled state dynamics.

In Figure 5.5 the phase portrait for $R_{K_{const}} = R_{G_{const}} = 10,000$ is pictured. As one can see, the one-dimensional stable manifolds of the saddle points, which are also known as *Separatrix*, separates the domains of attraction of the four stable equilibria into seven parts. Due to the constant controls it only depends on the initial capital composition which equilibrium is reached. The economical interpretation of these seven domains is easily explained. As a result of the convex-concave growth function, capital accumulation with initially low capital levels is very

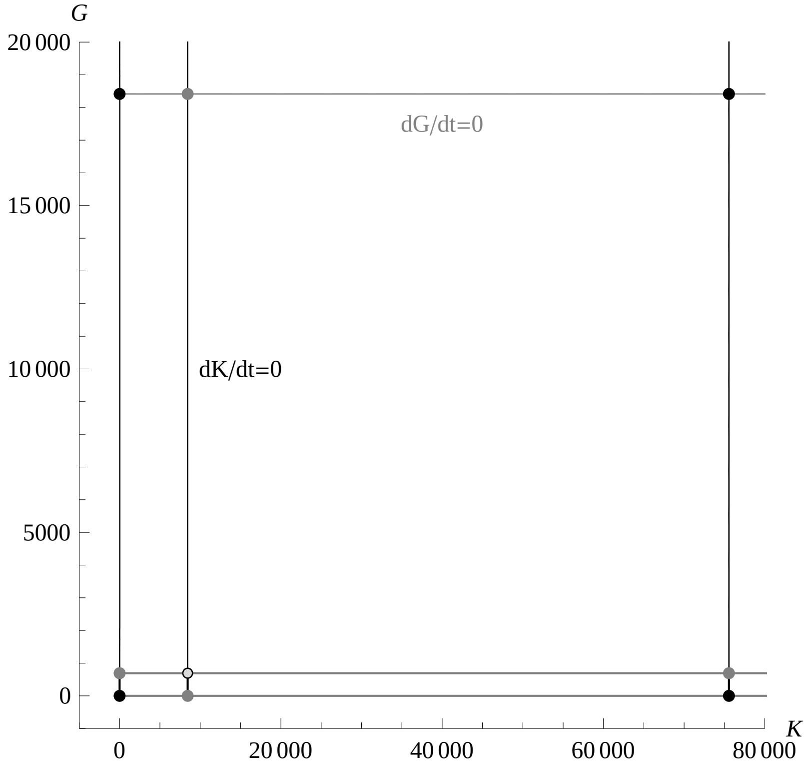


Figure 5.4: Steady states under the assumption of constant controls.

extensive, as shown in Figure 5.1. Consequently, if the initial capital level lies beneath the inflexion point of the growth function, the fixed R&D investments are not high enough to achieve further accumulation and therefore the diminishing impact of depreciation forces the capital level towards zero. If, however, the initial capital level lies beyond the inflexion point but still has a positive growth function, the fixed R&D investments are high enough to maintain further growth and the momentum of the capital drives the level towards the high equilibrium. If the initial capital level is already too high so that the negative force of depreciation gets again dominant, the capital level declines from above towards the high equilibrium. The consideration of both capital types at the same time yields exactly these seven domains. If both capital levels are beneath their inflexion point, the system converges towards the equilibrium in $(0,0)$. If the initial capital levels of both types are high enough and lie beyond their inflexion points the trajectories rise up or decline towards the high equilibrium, depending on the sign of the growth rates. In Figure 5.5 this high equilibrium is at $K = 75,558, G = 18,414$. In case of asymmetric initial capital levels, where one lies above and the other one beneath its inflexion point, the first one rises or declines toward its high equilibrium value, and the second one converges towards zero.

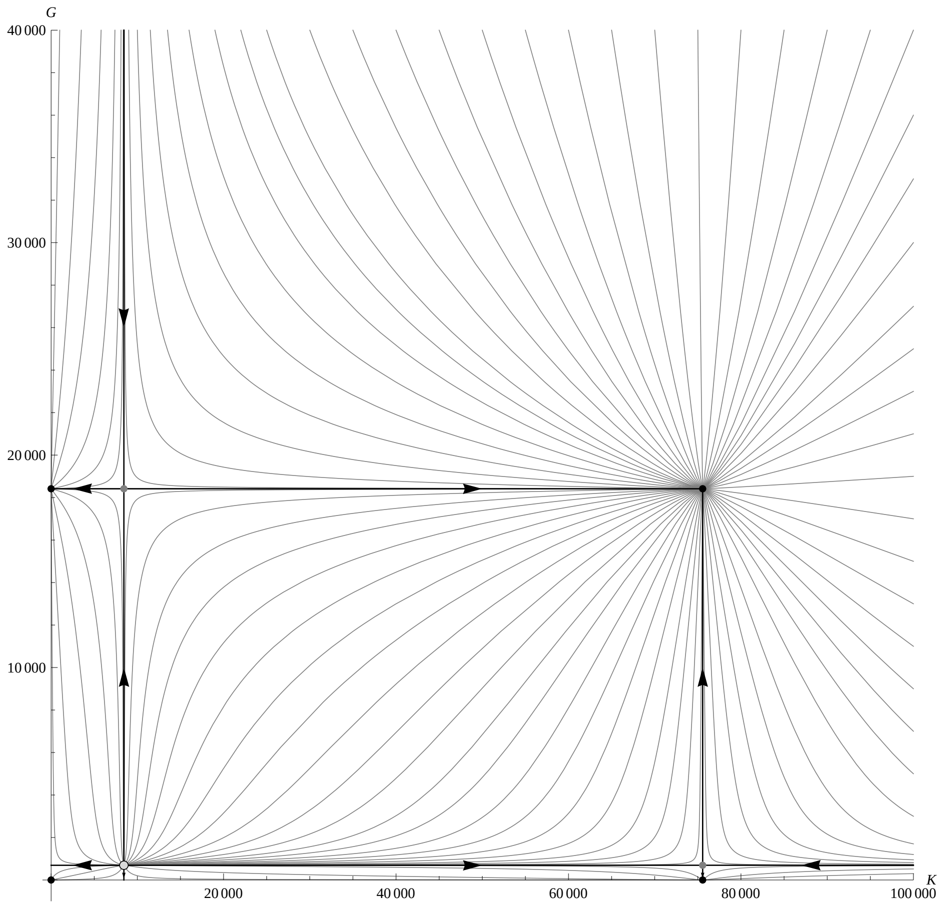
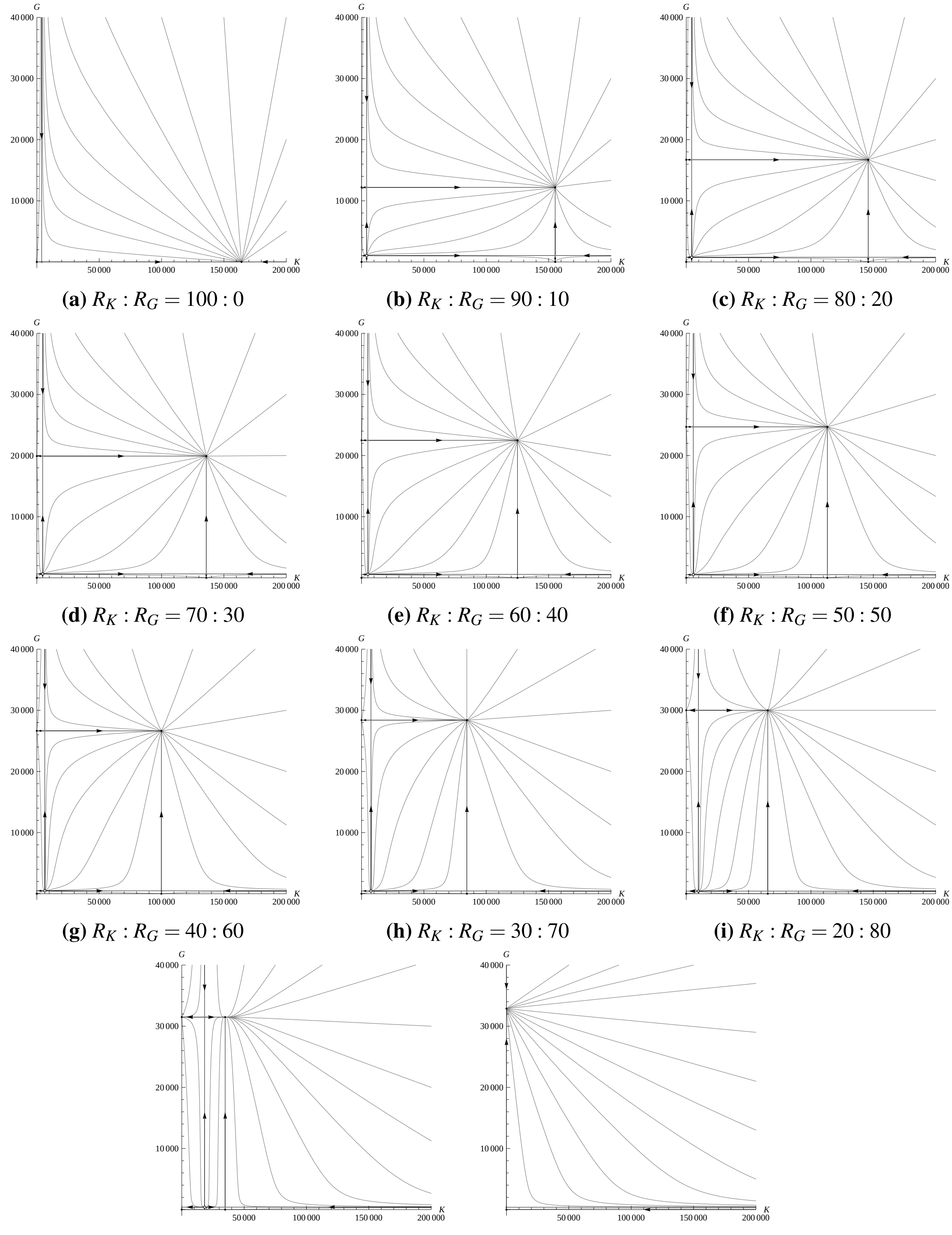


Figure 5.5: Phase portrait under the assumption of constant controls with $R_K = R_G = 10,000$.

Note that the phase portrait in Figure 5.5 is calculated for symmetric controls. A matter of special interest, however, is, how these domains of attraction change with different shares of R_K and R_G. In order to investigate this, I assume a fixed total amount of R&D investments of $R_K + R_G = 40,000$. Starting with the complete amount invested in brown R&D, so that the share of green R&D is 0%, I then change the shares in steps of 10% and investigate the corresponding phase portrait. The results are shown in Figure 5.6. In case of investments made exclusively for one type of capital, the other type of capital naturally converges towards zero no matter how high the inital level is. Because no R&D investments are made, nothing can compensate depreciation. In this scenario, only three steady states can be detected, namely the two stable equilibria of the supported capital type and a saddle point, whose stable manifold again seperates the two domains of attraction. This is the case in Figures 5.6a and 5.6k. As one can see in Figure 5.6, the high equilibrium declines in K and rises in G for an increasing share of R_G, while the unstable equilibrium changes comparatively slowly. However, this slight change is enough to make obvious that the higher the share of R_G is, the more expanded is the attraction domain of the green asymmetric equilibrium with $K = 0$, and the smaller is the attraction domain of the brown asymmetric equilibrium with $G = 0$. Note that in Figure 5.6j the attraction domains with rising trajectories of the high equilibrium and the green asymmetric equilibrium are almost of the same size.



(a) $R_K : R_G = 100 : 0$ **(b)** $R_K : R_G = 90 : 10$ **(c)** $R_K : R_G = 80 : 20$

(d) $R_K : R_G = 70 : 30$ **(e)** $R_K : R_G = 60 : 40$ **(f)** $R_K : R_G = 50 : 50$

(g) $R_K : R_G = 40 : 60$ **(h)** $R_K : R_G = 30 : 70$ **(i)** $R_K : R_G = 20 : 80$

(j) $R_K : R_G = 10 : 90$ **(k)** $R_K : R_G = 0 : 100$

Figure 5.6: Phase portraits with varying shares of R_G and fixed investments of $R_K + R_G = 40,000$.

Chapter 6

Unsolved Problems

Particularly in the previous chapter, which considers the convex-concave growth function, it gets obvious that the analysis of the present model approaches goes along with some analytical as well as numerical problems. These difficulties are mainly of structural nature and therefore can be traced back to the configuration of the model itself. To provide an overview and discuss possible suggestions of improvement, these problems are discussed separately in this chapter.

6.1 Implicit Controls

One of the main problems in the previous analysis is the occurrence of implicit controls. This comes along with the use of a Cobb Douglas production function in the state dynamics, or rather the assumption of decreasing returns to scale. Consider an autonomous optimal control problem with infinite time horizon, one state x and one control u as explained in Feichtinger and Hartl (p.88-p.96)

$$\max_{u(t)} \mathcal{J} = \int_0^\infty e^{-rt} F(x(t), u(t))dt \tag{6.1}$$

subject to

$$\dot{x} = f(x(t), u(t)) \tag{6.1a}$$

$$x(0) = x_0 \tag{6.1b}$$

$$u(t) \in \Omega. \tag{6.1c}$$

The Hamiltonian and the first order conditions then are

$$\mathcal{H} = F(x(t), u(t)) + \lambda f(x(t), u(t)) \tag{6.2}$$

$$\mathcal{H}_u = F_u + \lambda f_u = 0 \tag{6.3}$$

$$\dot{\lambda} = r\lambda - \mathcal{H}_x = (r - f_x)\lambda - F_x. \tag{6.4}$$

The usual approach now would be to solve (6.3) for the control u to obtain the explicit function

$$u = u(x, \lambda) \tag{6.5}$$

which then yields the canonical system

$$\begin{aligned} \dot{x}(x,\lambda) &= f(x, u(x,\lambda)) && (6.6)\\ \dot{\lambda}(x,\lambda) &= (r - f_x(x, u(x,\lambda)))\lambda - F_x(x, u(x,\lambda)). && (6.7) \end{aligned}$$

However, due to the Cobb Douglas function the controls in my approach in general are only implicitly included in (6.3), and therefore the derivation of a function as in (6.5) is not possible. The alternative way to derive the canonical system in the state-control-space is to use the total derivation of (6.3) with respect to time in order to get a differential equation for u:

$$\dot{H}_u = H_{ux}\dot{x} + H_{uu}\dot{u} + f_u\dot{\lambda} = 0 \quad \longrightarrow \quad \dot{u} = -\frac{H_{ux}\dot{x} + f_u\dot{\lambda}}{H_{uu}}. \tag{6.8}$$

Further on, (6.3) yields

$$\lambda = -\frac{F_u}{f_u} \quad \text{with} \quad f_u \neq 0. \tag{6.9}$$

Using (6.9), (6.4) and (6.1a) for (6.8) yields the canonical system in the state-control space

$$\begin{aligned} \dot{x} &= f(x,u) && (6.10)\\ \dot{u} &= -\frac{H_{ux}f(x,u) - F_u(r - f_x) - F_x f_u}{H_{uu}}. && (6.11) \end{aligned}$$

Note that this alternative derivation of the canonical system is only possible as long as the number of states equals the number of controls and if H_{uu} is not singular which exactly corresponds to the

Legendre-Clebsch-Condition: *Let the Hamiltonian $\mathcal{H}(x(t), u(t), \lambda(t), \lambda_0, t)$ be twice continuously differentiable with respect to u and $(x^*(.), u^*(.))$ be an optimal solution. Then for every $t \in [0, T]$ where $u^*(t)$ lies in the interior of the control region $\Omega(x^*(t), t)$ the Hessian matrix with respect to u*

$$\frac{\partial}{\partial u} H_u(x^*(t), u^*(t), \lambda(t), \lambda_0, t) \quad \textit{is negative semidefinite.}$$

(Grass et al. [2008], p.109)

For this reason the usage of the alternative approach is often not unproblematic and solutions might be lost. A possibility to evade this problem might be to set the production elasticity of the controls as 0.5 or 1. If this were the case, the controls would no longer be implicitly given and the derivation of the canonical system in the state-co-state-space would be possible. In the second case, however, the assumption of decreasing returns to scale would be violated, while the assumption of production elasticities of 0.5 for both R&D investment types would ruin the desired effect of brown R&D being more efficient.

6.2 Mixed Path Constraint and Inadmissible Region $C < 0$

Due to the fact that $\ln(C)$ in this model describes the utility of consumption and C is given by the budget constraint in (2.2) it is postulated that C remains non-negative and therefore the mixed path constraint

$$C(K,G,R_K,R_G) = F(K,G) - w(R_K + R_G) - \chi(\varepsilon) \geq 0 \tag{6.12}$$

has to be fulfilled. Naturally, negative consumption would not make sense or at least would be hard to interpret. Therefore, this restriction is definitely reasonable, but the mixed path constraint that comes along is quite problematic as one can see especially in Chapter 5.

In the phase portraits in Figures 2.6, 4.2 and 4.16 the corresponding inadmissible regions are depicted. In such a region the abatement costs are higher than the production output and therefore no admissible pair of controls can be found to let consumption remain non-negative. On the one hand, this inadmissible region represents inadmissible initial values in the phase portraits which is not necessarily problematic. On the other hand, however, it can be the reason for the disappearance of equilibria while being continued in the course of bifurcation analysis, as one can see in Figure 5.3. As soon as the equilibrium slides into the inadmissible domain, it really gets problematic. The only possibility to get solutions in this case would be to consider the optimization problem for a finite time horizon with variable end time. Then for example the convex-concave model would look like

$$\max_{R_K,R_G,T} \int_0^T e^{-rt} \ln\left(\tau + bK^{\alpha_1} G^{\alpha_2} - w(R_K + R_G) - a\varepsilon^{\beta} K\right) + c\varepsilon^{\gamma} dt \tag{6.13}$$

$$\text{s.t.:} \qquad \dot{K} = \frac{d_1 K(t)^2}{d_2 K(t)^2 + v} R_K(t)^{\delta} - \phi K(t) \tag{6.13a}$$

$$\dot{G} = \frac{e_1 G(t)^2}{e_2 G(t)^2 + z} R_G(t)^{\sigma} - \psi G(t) \tag{6.13b}$$

$$0 \leq R_K(t) \quad \forall t \geq 0 \tag{6.13c}$$

$$0 \leq R_G(t) \quad \forall t \geq 0 \tag{6.13d}$$

$$0 \leq bK(t)^{\alpha_1} G(t)^{\alpha_2} - w(R_K(t) + R_G(t)) - a\varepsilon^{\beta} K(t). \tag{6.13e}$$

In this extended problem not only the optimal policy has to be found over the time interval $[0,T]$, but also the end time T has to be considered as controlled variable such that the objective function is maximized on an optimal time interval $[0,T^*]$ (see Grass et al. [2008]). Therefore, the optimal T will be chosen in a way, that optimization along the path to the equilibrium of the original model stops before the inadmissible domain is reached. For a finite optimal end time T^*, the following additional conditions have to be satisfied

$$\lambda_1(T^*) = \lambda_2(T^*) = 0 \tag{6.14}$$

$$\mathcal{H}(K(T^*), G(T^*), R_K(T^*), R_G(T^*), \lambda_1(T^*), \lambda_2(T^*)) = 0. \tag{6.15}$$

Although the usage of a variable end time might be a possibility to evade the problems with the inadmissible domain, it is quite questionable to use this approach in the present model, especially when the emphasis is on pollution control. If this alternative way is used, an optimal solution on a limited time interval is obtained. But then the question arises what happens afterwards. To get the overall picture, long term solutions are required. Short time controls are rather insufficient for environmental issues.

Therefore the only way to evade these problems possibly is the restructuring of the model. The best solution would be to generate a dynamic that drives the decision maker away from the inadmissible region as soon as he/she gets too close. In the present approach, the main reason for the difficulties with the inadmissible domain is the fact that the R&D investments are chosen completely independently from the budget constraint. If, however, the possible extent of R&D investments is adjusted to the remaining financial scope in a budget constraint, such a dynamic can be obtained. As soon as the decision maker comes too close to the inadmissible domain, the possible financial scope for the investments necessarily diminishes which results in lower investments. This, in turn, reduces the capital levels because R&D investments are no longer able to compensate depreciation. Further on, the declining capital levels cause a movement away from the inadmissible region. For this reason, a steady state might occur exactly on the boundary but never inside the inadmissible region.

Chapter 7

Conclusion and Discussion

The subject of this thesis is to investigate how environmental regulation influences economic growth as well as R&D investments and whether or not they induce a shift to a greener technology.

As far as economic growth is concerned, it already becomes obvious in the basic model that increasing stringency of environmental regulation causes a decline in both types of capital and consequently also in production output. Therefore it rather represses than supports economic growth. This conclusion can also be drawn from the results of the other approaches considered in this thesis.

However, the carried out analysis shows, that increasing environmental regulation indeed has a positive impact on the accumulation of green capital and on the increase of green R&D investments. This can especially be seen when the shares of capital levels and R&D investments under varying stringency of environmental standards are considered. Although both capital levels decline, increasing abatement costs even accelerate the decrease of brown capital levels so that in total production turns out to be greener the higher environmental quality standards are. Same applies for R&D investments.

An interesting aspect in the CES approach is that production is more flexible with increasing elasticity of substitution. Here, the production is kept brown as long as possible until the abatement gets too costly so that a switch to a greener production is inevitable. Therefore, the influence of environmental regulation beneath the flipping point at which green capital and R&D gets advantageous, is comparatively small. Hence, in this approach only higher environmental standards equal or beyond the flipping point make sense to achieve the desired effect.

The results of Chapter 4, in which subsidies are included additionally, underline the fact that performance standards as economic policy instrument might not be incentive enough to induce a shift to a greener technology. Especially in the case of subsidization of green capital one can see that subsidization in general can depreciate the repressing impact of environmental standards and therefore enables that green capital levels even out at a significantly positive value. This is a very significant observation. As already mentioned, the less effective performance of subsidization of green R&D investments for realistic values of the subsidy rate can be traced back to the concrete model structure itself, but in general the results show that this kind of subsidy has potential. Therefore, a combination of these two types of subsidization indeed could be a promising instrument of environmental policy.

If one considers environmental quality together with the utility maxima from the agent's point of view, the best scenario is given in case of subsidization of green capital. While maximal utility in the other approaches is reached for quite low environmental standards, here maximal utility is achieved at maximal environmental quality. A similar result occurred in the CES model, where at least a local maximum occurs at maximal environmental quality, but only for a high substitution elasticity.

As far as the model approach with convex-concave state dynamics is concerned, only limited conclusions can be drawn. Due to the mentioned complications, the required analysis mostly had to be left undone but will be a matter of future work. Anyway, as far as one can deduce from the investigation of the model under the assumption of constant controls, the expectation that multiple equilibria may occur has be fulfilled. As it is the case in this scenario, it is expected that the convex-concave behavior in the optimal control problem leads to thresholds which separate the domains of attraction of multiple equilibria, depending on whether further capital accumulation is profitable or not.

To sum up, environmental regulation can cause a shift to greener production but only at the cost of repressed economic growth. However, if environmental economic instruments are wisely used, considering a trade-off between loses in growth and achievements towards a greener technology, a compromise may be found which yields satisfying results.

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Bibliography

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