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One-Loop Renormalization of a $\mathcal{U}_\star(1)$-Gauge Model

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Abstract

One main aim of modern physics is the description of nature by models. These models, with the exception of gravity, are quantum theory models. A motivation for the work in noncommutative spaces arises from the common opinion that gravity should be quantized and that at very small scales continuous space-time will be noncommutative. A possible representation of this noncommutative space-time is the so-called Moyal-Weyl star product. The introduction of this star product goes hand in hand with the main problem of any noncommutative quantum field theory: UV/IR mixing. Basic quantum field theories suffer from ultraviolet (UV) divergences which can be absorbed by a renormalization procedure while in noncommutative theories the UV divergences are reflected by new singularities in the infrared (IR). The aim of this diploma thesis under the inclusion of the IR divergences, is to compute Feynman rules, results for the vacuum polarization and the one-loop renormalization of the gauge boson propagator, based on a special gauge theory model.

Kurzfassung

Eine wesentliche Hauptaufgabe der modernen Physik ist es, die physikalische Natur der beobachtbaren Welt durch Modelle zu beschreiben. Diese Modelle sind mit Ausnahme der Gravitation Quantenfeldtheorien (Eichfeldtheorien). Aus der im allgemeinen üblichen Vorstellung, dass die Gravitation quantisiert werden muss und der Tatsache, dass bei kleinen Skalen die kontinuierliche Raumzeit in eine quantisierte übergeht, kann die Motivation für eine Beschäftigung mit nichtkommutativen Feldtheorien hergeleitet werden, wenn die Quantisierung als ein Nichtkommutieren der Raumzeit interpretiert wird. Eine mögliche Darstellung bietet die Einführung des sogenannten Moyal-Weyl Sternprodukts, bei welchem das Produkt von Funktionen durch eben jenes Sternprodukt ersetzt wird und somit eine Implementierung im nichtkommutativen (verzerrten) Raum möglich wird. Diese nichtkommutative Behandlung geht jedoch Hand in Hand mit einem Problem, innewohnend jeder nichtkommutativen Feldtheorie: Das Mischen von ultravioletten mit infraroten Divergenzen. Klassische Feldtheorien zeigen Divergenzen im ultravioletten Bereich, die jedoch durch geeignete Regularisierungsmaßnahmen zu einer renormierbaren Theorie führen, während in nichtkommutativen Theorien Divergenzen im ultravioletten Betreich von infraroten begleitet werden. Das Ziel dieser Diplomarbeit, unter Einbeziehung eben jener infraroten Divergenzen, ist die Berechnung von Korrekturen auf Einschleifen-Niveau und die explizite Angabe eines renormierten Eichfeldpropagators im Rahmen eines speziellen Eichmodelles.

Contents

Chapter 1

Introduction

The first chapter of this diploma thesis brings light into the motivation for studying noncommutative quantum field theories. We give a short overview of models from the noncommutative 'universe', leading over to the fundament of this work: The BRSW model.

1.1 Motivation

For the description of the universe, as known up to now we have two basic theories: Einstein's theory of general relativity (GRT) for the macroscopic world and quantum field theories (QFT) for the microscopic one.
It is assumed that at very small scales (or very high energy), the description of the physical world will be a combination of GRT and QFT. At this small scale, QFT with continuous space-time and Lorentz symmetry are considered inappropriate for the description of the universe [1].
Four dimensional QFT suffers from infrared (IR), ultraviolet (UV) divergences and from the divergence of the renormalized perturbation expansion. W. Heisenberg suggested to introduce a fundamental length [2], to handle the UV divergences. In [3, 4] H. Snyder formulated these ideas mathematically and introduced noncommutative geometry. Considering the quantization of gravity, the usual commutative space-time seems inappropriate and therefore space-time has to be noncommutative [5].
According to the statements above, the 'road' of gravity and the 'road' of quantum field theory intersect at the crosspoint of noncommutative geometry.

1.2 Models

In noncommutative quantum field theories, the algebra is defined by commutation relations

$$[\hat{x}^i, \hat{x}^j] = i\Theta^{ij}(\hat{x}), \tag{1.1}$$

where the coordinates are operators $\hat{x}^i$ on some Hilbert space $\mathcal{H}$ and $\Theta^{ij}(\hat{x})$ might be any function of the generators with $\Theta^{ij} = -\Theta^{ji}$, satisfying the Jacobi identity. Usually, the commutator relations (1.1) are chosen to be constant, linear or quadratic in the generators. The constant relations

$$[\hat{x}^i, \hat{x}^j] = i\Theta^{ij} = \text{constant}, \tag{1.2}$$

represent the canonical case. From (1.1) and (1.2), one can see the collapse of commutative space concepts, hence leading to noncommutative space.

1.2.1 Scalar Models

The first models on deformed space were scalar models, where the pointwise product of fields was replaced by the Groenewold-Moyal product [6, 7], corresponding to the commutation relation (1.2).
A noncommutative extension to the scalar ϕ^4 model is given by

$$S = \int d^4x \left(\partial_\mu \phi \star \partial^\mu \phi + m^2 \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right). \tag{1.3}$$

In noncommutative Euclidean space, the propagator is exactly the same as in commutative space [8], while the vertex functions gain phase factors in the momenta. The Feynman graphs, known from commutative space (called planar graphs), exhibit the usual UV divergence, which can be handled by a renormalization procedure.
Explicit one-loop calculations [9, 10, 11, 12, 13], showed that in addition to the planar graphs, so called non-planar graphs appear, which are regularized by phases in their UV section but become singular for small external momenta.
For example, the two-point tadpole graph (in Euclidean space) is given by

$$\tilde{\Pi}(\Lambda, p) \propto \lambda \int d^4k \frac{2 + \cos(k\tilde{p})}{k^2 + m^2} \equiv \tilde{\Pi}^{pl}(\Lambda) + \tilde{\Pi}^{npl}(p), \tag{1.4}$$

where

$$\tilde{p}_\mu := \Theta_{\mu\nu} p_\nu. \tag{1.5}$$

As mentioned above the planar part is quadratically divergent in the UV cutoff Λ,

$$\tilde{\Pi}^{pl}(\Lambda) \sim \Lambda^2, \tag{1.6}$$

while the non-planar part shows

$$\tilde{\Pi}^{npl}(p) \sim \frac{1}{\tilde{p}^2}, \tag{1.7}$$

presenting a new IR divergence when $\tilde{p} \to 0$. This effect, known as 'UV/IR mixing' is the main problem of any noncommutative quantum field theory.
Singularities of arbitrary inverse power can be generated if we insert the non-planar contributions into higher order graphs, hence the theory seems to be non-renormalizable [14].
One of the main aims of noncommutative field theories is now to modify the Lagrangian in order to obtain a damping mechanism in the IR region. The first renormalizable noncommutative scalar field model (in Euclidean space) was introduced by H. Grosse and R. Wulkenhaar [15] by adding an oscillator-like term with parameter

Ω into the Lagrangian. This model suffers from a broken translation invariance.

Another renormalizable model, in the literature referred to as the $\frac{1}{p^2}$ model was presented by the Orsay group around V. Rivasseau [16]. It preserves translation invariance. A non-local term

$$S_{nloc} = -\int d^4x\phi(x) \star \frac{a^2}{\Theta^2 \Box_x} \star \phi(x), \tag{1.8}$$

is added to the action (1.3), where a is a dimensionless constant. The addition of such a term provides a counter term for the expected quadratic IR divergence in the external momentum [17]. The interpretation of the operator $\frac{1}{\Box}$ in coordinate space is difficult because one is confronted with the inverse of a derivative, while in momentum space the operator represents the well known inverse of the scalar function k^2. The new operator is interpreted as the 'Green operator' of $\Box \equiv \partial_\mu \partial_\mu$. The propagator is given by

$$G(k) = \frac{1}{k^2 + m^2 + \frac{a^2}{\tilde{k}^2}}, \tag{1.9}$$

showing a finite behaviour in the IR region as well as in the UV:

$$\lim_{k\to 0} G(k) = \lim_{k\to\infty} G(k) = 0, \quad \forall a \neq 0, \tag{1.10}$$

approving the motivation for the added term (1.8).

Naturally a generalization of these models to noncommutative gauge models must be done.

1.2.2 Gauge Models

For the the Grosse-Wulkenhaar model and its extension to gauge theories, we refer to [18].

Different ways of implementing the damping behaviour from the $\frac{1}{p^2}$ model have been advertised. The quadratic divergence of a noncommutative $U(1)$ gauge theory is of the form

$$\tilde{\Pi}^{IR}_{\mu\nu} \sim \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2}, \tag{1.11}$$

where

$$\tilde{p}_\mu = \Theta_{\mu\nu} k_\nu.$$

One possible implementation of such a term as (1.11) in order to accommodate the IR divergences in the vacuum polarization in a gauge invariant way, is given by [19]:

$$S_{nloc} = \int d^4x F_{\mu\nu} \star \frac{1}{\tilde{D}^2 D^2} \star F_{\mu\nu}, \tag{1.12}$$

where $F_{\mu\nu}$ represents the field strength and $D_\mu \bullet = \partial_\mu \bullet - ig[A_\mu \stackrel{\star}{,} \bullet]$ denotes the noncommutative covariant derivative. The introduction of the inverse of covariant derivatives (including the gauge field) leads to a power counting non-renormalizable theory, due to the fact that they can only be interpreted in terms of infinite series.
However, non-localized terms can be presented 'localized' by coupling them to unphysical auxiliary fields. One example is given by [19]

$$
\begin{aligned}
S_{inv}^{nloc} &= \int d^4x \left\{ \frac{1}{4}\left(F_{\mu\nu} \star F_{\mu\nu} + F_{\mu\nu} \star \frac{a^2}{\tilde{D}^2 D^2} \star F_{\mu\nu} \right) \right\} \\
S_{inv}^{nloc} \rightarrow S_{inv}^{loc} &= \int d^4x \left\{ \frac{1}{4}\left(F_{\mu\nu} \star F_{\mu\nu} + a\mathcal{B} \star F_{\mu\nu} - \mathcal{B} \star \tilde{D}^2 D^2 \star \mathcal{B} \right) \right\}, \qquad (1.13)
\end{aligned}
$$

where a is a dimensionless constant and $\mathcal{B}$ represents an auxiliary real-valued antisymmetric field.
It turned out that the auxiliary field has not been introduced in a physically invariant way [20], hence additional ghosts are required to rebuild the original physical content. Vilar *et al* [21] replaced the field $\mathcal{B}$ by two pairs of complex conjugated fields ($B, \tilde{B}$ and $\chi, \tilde{\chi}$), assigned to appropriate ghost fields. The localized term reads

$$S_{loc} = S_{loc,0} + S_{break}, \qquad (1.14)$$

where $S_{loc,0}$ represents a BRST invariant part. Following the approaches of Zwanziger [22], the additional degrees of freedom can be eliminated by adding a ghost field for each auxiliary field. As a consequence, BRST doublet structures are formed, resulting in a trivial BRST cohomology for $S_{loc,0}$:

$$sS_{loc,0} = 0, \qquad (1.15)$$

which means that the part of the action depending on the auxiliary fields and their associated ghosts can be written as a BRST transformation itself.
The term S_{break} does not show this property but can be constructed in such a way that the mass dimension of the part depending on the gauge field is smaller than the dimension of the underlying Euclidean space, i.e. $\mathbb{R}^4_\theta$ with $dim = 4$. Constructed as mentioned, the breaking term is referred to as 'soft' [23] and does not destroy renormalizability [24]. Moreover, the term S_{break} is able to suppress the UV/IR mixing by modifying the IR sector while the UV part is not affected. This soft breaking 'technique' leads straight to the next model.

The BRSW Model

The following heavily relies on the work of [20, 25, 26].

To avoid the problems discussed above and achieve renormalizability, the action has to be modified. The main ideas for modification are recapitulated from [20]. The model should achieve the following properties:

- A counter term of the form of (1.11) should appear in the tree level action.
- All relevant propagators should be finite in the IR as well as in the UV.
- Auxiliary fields and related ghosts should be decoupled from the gauge sector.

According to [27], it is desirable to remove any explicit appearance of parameters with $dim_{mass} < 0$ from the action to avoid (or restrict) the appearance of dimensionless derivative operators. This is impossible, since the UV/IR mixing leads to divergences (contracted with $\Theta_{\mu\nu}$) which enter the action in the form of counter terms. One possible solution to this problem is to split $\Theta_{\mu\nu}$ into a dimensionless tensor structure $\theta_{\mu\nu} = -\theta_{\nu\mu}$, and a dimensionful scalar parameter ϵ:

$$\Theta_{\mu\nu} \to \epsilon\theta_{\mu\nu}, \tag{1.16}$$

with

$$dim_{mass}(\theta_{\mu\nu}) = 0, \quad dim_{mass}(\epsilon) = -2. \tag{1.17}$$

Therefore, the appearance of the parameter ϵ in the tree level action is reduced by modification of the contractions, given by

$$\tilde{\Box} := \theta_{\alpha\beta}\theta_{\alpha\gamma}\partial_\beta\partial_\gamma, \tag{1.18}$$

and

$$\tilde{k}_\mu := \theta_{\mu\nu}k_\nu. \tag{1.19}$$

As a result, the dimensionful parameter ϵ only appears in the phase factors associated with the star product, while the bilinear parts are unaffected. The field strength tensor $F_{\mu\nu}$ in the soft breaking term can be reduced to its bilinear part

$$f_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\nu, \qquad \tilde{f} \equiv \theta_{\mu\nu} f_{\mu\nu}. \tag{1.20}$$

The soft breaking term, implementing the IR damping mechanism, can now be written with ordinary derivatives. The gauge field in the breaking term is completely decoupled from new fields ($\bar{B}_{\mu\nu}, B_{\mu\nu}$ and associated ghosts $\bar{\psi}_{\mu\nu}, \psi_{\mu\nu}$) which implement the IR damping and BRST invariance. In order to restore the BRST invariance in the UV additional sources $\bar{Q}, Q, \bar{J}, J$ are needed. Those sources vanish or take constant values in the IR. The breaking term is given by [20]

$$\begin{aligned}
S_{break} &= \\
&= \int d^4x \Big\{ (\bar{J}_{\mu\nu\alpha\beta} \star B_{\mu\nu} + J_{\mu\nu\alpha\beta} \star \bar{B}_{\mu\nu}) \star \frac{1}{\tilde{\Box}}(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f}) \\
&\quad - \bar{Q}_{\mu\nu\alpha\beta} \star \psi_{\mu\nu} \star \frac{1}{\tilde{\Box}}(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f}) \\
&\quad - (\bar{Q}_{\mu\nu\alpha\beta} \star B_{\mu\nu} + Q_{\mu\nu\alpha\beta} \star \bar{B}_{\mu\nu}) \star \frac{1}{\tilde{\Box}} s(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f}) \Big\}.
\end{aligned} \tag{1.21}$$

1.3 Acknowledgement

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Chapter 2

Noncommutative Quantum Field Theory (NCQFT)

In the second chapter, we try to show the principal points of constructing a noncommutative field theory. Based on the hypothesis that at very short distances [1, 3, 5, 28], the space coordinates do not commute any more and the aim to use fields which are functions and not operator valued objects, we will introduce the star product.

2.1 Conventions

In this work, we consider an Euclidean space $\mathbb{R}^4$.

We use natural units which means

$$\hbar = c = 1, \tag{2.1}$$

and use the Einstein summation convention

$$\sum_{\mu=1}^{4} V_\mu W_\mu = V_\mu W_\mu = V^\mu W^\mu. \tag{2.2}$$

Furthermore, whenever a vector V_μ is multiplied with another vector W_μ in an exponential we will omit the indices

$$e^{V_\mu W_\mu} = e^{VW}, \qquad \{V_\mu, W_\mu\} \in \mathbb{R}^4, \tag{2.3}$$

as long as we will not underline the mathematical aspects.

2.2 The Algebra of NCQFT

There are three steps to create a satisfying base for NCQFT.

In the first step we replace the ordinary coordinates x_μ by operators $\hat{x}_\mu$ which are self-adjoint.

The next step is to define the algebra (if we do this, $\mathbb{R}^4$ will be modified to $\mathbb{R}^4_\theta$):

$$[\hat{x}_\mu, \hat{x}_\nu] =: \hat{x}_\mu \hat{x}_\nu - \hat{x}_\nu \hat{x}_\mu = i\epsilon\theta_{\mu\nu}, \tag{2.4}$$

$$[\epsilon\theta_{\mu\nu}, \hat{x}_\sigma] = 0. \tag{2.5}$$

Notice that we have $\epsilon\theta_{\mu\nu}$ constant: The antisymmetric matrix $\theta_{\mu\nu}$ of mass dimension zero is multiplied with a real factor ϵ of mass dimension minus two. For our purpose, we will use a $\theta_{\mu\nu}$ of the form:

$$(\theta_{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2.6}$$

We must mention that there are many approaches to noncommutativity, leading to a different variety of Eq. (2.4). For details the work of [29] is suggested.

The last step is to find a multiplication law for commutative functions because we want to use fields and not operator valued objects.

2.3 The Star Product

This new multiplication law is deduced from (2.4) through the so called Moyal-Weyl correspondence [6, 7]:

$$\hat{\phi}(\hat{x}) \Longleftrightarrow \phi(x), \tag{2.7}$$

with $\hat{\phi}(\hat{x})$ as an arbitrary operator valued object.[1] To define the product of two operator valued objects we use the Fourier-Integral theorem

$$\hat{\phi}(\hat{x}) = \int \frac{d^4k}{(2\pi)^4} e^{ik\hat{x}} \tilde{\phi}(k), \tag{2.8}$$

$$\tilde{\phi}(k) = \int d^4x e^{-ik\hat{x}} \hat{\phi}(\hat{x}),$$

where k is a four dimensional real variable[2].

The product of two operator valued objects then reads

$$\hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x}) = \frac{1}{(2\pi)^8} \int d^4k_1 \int d^4k_2 e^{ik_1\hat{x}} e^{ik_2\hat{x}} \tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2). \tag{2.9}$$

Using the Baker-Campbell-Hausdorff-formula,

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}, \qquad \text{if} \qquad [A,[A,B]] = [B,[A,B]] = 0, \tag{2.10}$$

[1] In this work the operator valued objects are fields and sources.
[2] Equation (2.8) already represents the Moyal-Weyl correspondence.

which is applicable due to (2.5):

$$[A,[A,B]] = [A,\epsilon\theta] = [B,[A,B]] = [B,\epsilon\theta] = 0,$$

we receive

$$\begin{aligned}
\hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x}) &= \frac{1}{(2\pi)^8}\int d^4k_1 \int d^4k_2 e^{ik_1\hat{x}} e^{ik_2\hat{x}} \tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2) \\
&= \frac{1}{(2\pi)^8}\int d^4k_1 \int d^4k_2 e^{ik_{1,\mu}\hat{x}_\mu} e^{ik_{2,\nu}\hat{x}_\nu} \tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2) \\
&= \frac{1}{(2\pi)^8}\int d^4k_1 \int d^4k_2 e^{ik_{1,\mu}\hat{x}_\mu + ik_{2,\nu}\hat{x}_\nu} e^{\frac{1}{2}[ik_{1,\mu}\hat{x}_\mu, ik_{2,\nu}\hat{x}_\nu]} \tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2) \\
&= \frac{1}{(2\pi)^8}\int d^4k_1 \int d^4k_2 e^{i(k_{1,\mu}+k_{2,\mu})\hat{x}_\mu} e^{-\frac{1}{2}k_{1,\mu}[\hat{x}_\mu,\hat{x}_\nu]k_{2,\nu}} \tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2) \\
&= \frac{1}{(2\pi)^8}\int d^4k_1 \int d^4k_2 e^{i(k_1+k_2)\hat{x}} e^{-\frac{i}{2}k_{1,\mu}\epsilon\theta_{\mu\nu}k_{2,\nu}} \tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2).
\end{aligned}$$

Introducing the abbreviation

$$k_i\epsilon\theta k_j = k_{i,\mu}\epsilon\theta_{\mu\nu}k_{j,\nu}, \qquad i,j = 1,2,3,4,..., \tag{2.11}$$

we get a compact representation of the product made up by two operator valued objects:

$$\hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x}) = \frac{1}{(2\pi)^8}\int d^4k_1 \int d^4k_2 e^{i(k_1+k_2)\hat{x}} e^{-\frac{i}{2}k_1\epsilon\theta k_2} \tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2). \tag{2.12}$$

If we now define the star product of two functions as:

$$\begin{aligned}
\phi_1(x)\star\phi_2(x) &:= e^{\frac{i}{2}\epsilon\theta_{\mu\nu}\partial^x_\mu\partial^y_\nu}\phi_1(x)\phi_2(y)\Big|_{x=y} \\
&= e^{\frac{i}{2}\epsilon\theta_{\mu\nu}\partial^x_\mu\partial^y_\nu}\frac{1}{(2\pi)^8}\int d^4k_1 \int d^4k_2 e^{i(k_1x+k_2y)}\tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2)\Big|_{x=y} \\
&= \frac{1}{(2\pi)^8}\int d^4k_1 \int d^4k_2 \left(1+\frac{i}{2}\epsilon\theta_{\mu\nu}\partial^x_\mu\partial^y_\nu + ...\right) e^{i(k_1x+k_2y)}\tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2)\Big|_{x=y} \\
&= \frac{1}{(2\pi)^8}\int d^4k_1 \int d^4k_2 \left(1-\frac{i}{2}\epsilon\theta_{\mu\nu}k_{1_\mu}k_{2,\nu} + ...\right) e^{i(k_1x+k_2y)}\tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2)\Big|_{x=y} \\
&= \frac{1}{(2\pi)^8}\int d^4k_1 \int d^4k_2 e^{i(k_1+k_2)x} e^{-\frac{i}{2}k_1\epsilon\theta k_2} \tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2),
\end{aligned} \tag{2.13}$$

we find an expression which looks very similar to Eq. (2.12). The only difference is that (2.12) has operator valued space coordinates in the exponent and Eq. (2.13) not, but the introduction of the star product ensures that the functions on the left hand side of (2.13) are no longer operator valued objects, hence leading to the correspondence (2.7).

To find a correspondence for (2.4) we use the definition of the star product (2.13) applied to the product of two vectors:

$$\begin{aligned}
x_\mu \star x_\nu &= e^{\frac{i}{2}\epsilon\theta_{\mu'\nu'}\partial^x_{\mu'}\partial^y_{\nu'}} x_\mu y_\nu \Big|_{x=y} \\
&= \left(1 + \frac{i}{2}\epsilon\theta_{\mu'\nu'}\partial^x_{\mu'}\partial^y_{\nu'} + ...\right) x_\mu y_\nu \Big|_{x=y} \\
&= x_\mu x_\nu + \frac{i}{2}\epsilon\theta_{\mu'\nu'}\delta_{\mu\mu'}\delta_{\nu\nu'} \\
&= x_\mu x_\nu + \frac{i}{2}\epsilon\theta_{\mu\nu}. \qquad (2.14)
\end{aligned}$$

Interchanging indices ($\mu \leftrightarrow \nu$):

$$x_\nu \star x_\mu = x_\nu x_\mu + \frac{i}{2}\epsilon\theta_{\nu\mu}$$

and combining the last two results will deliver our desired result:

$$\begin{aligned}
x_\mu \star x_\nu - x_\nu \star x_\mu &= x_\mu x_\nu + \frac{i}{2}\epsilon\theta_{\mu\nu} - x_\nu x_\mu - \frac{i}{2}\epsilon\theta_{\nu\mu} \\
&= \frac{i}{2}\epsilon(\theta_{\mu\nu} + \theta_{\mu\nu}) = i\epsilon\theta_{\mu\nu} \\
&= [x_\mu \overset{\star}{,} x_\nu]. \qquad (2.15)
\end{aligned}$$

Summary of the Moyal-Weyl correspondence:

$$\begin{aligned}
\hat{\phi}(\hat{x}) &\Longleftrightarrow \phi(x), \\
\hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x}) &\Longleftrightarrow \phi_1(x) \star \phi_2(x), \qquad (2.16) \\
[\hat{x}_\mu, \hat{x}_\nu] &\Longleftrightarrow [x_\mu \overset{\star}{,} x_\nu],
\end{aligned}$$

with

$$\begin{aligned}
\hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x}) &= \frac{1}{(2\pi)^8}\int d^4k_1 \int d^4k_2 e^{i(k_1+k_2)\hat{x}} e^{-\frac{i}{2}k_1\epsilon\theta k_2} \tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2), \\
\phi_1(x) \star \phi_2(x) &= \frac{1}{(2\pi)^8}\int d^4k_1 \int d^4k_2 e^{i(k_1+k_2)x} e^{-\frac{i}{2}k_1\epsilon\theta k_2} \tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2).
\end{aligned}$$

The most important thing is that on the left side (2.16) we have noncommuting operator valued objects while the right side shows non operator valued objects.

2.3.1 Properties of the Star Product

For practical calculations it will prove wise to have some properties of the star product at hand.

1. The Star Product of two Exponentials:

The definition of the star product (2.13) can be applied to the product of two exponentials:

$$
\begin{aligned}
e^{ikx} \star e^{ik'x} &= e^{\frac{i}{2}\epsilon\theta_{\mu\nu}\partial_\mu^x\partial_\nu^y} e^{ikx} e^{ik'y}\Big|_{x=y} \\
&= \left(1 + \frac{i}{2}\epsilon\theta_{\mu\nu}\partial_\mu^x\partial_\nu^y + \frac{1}{2}(\frac{i}{2})^2\epsilon\theta_{\mu\nu}\partial_\mu^x\partial_\nu^y\epsilon\theta_{\sigma\rho}\partial_\sigma^x\partial_\rho^y + ...\right) e^{ikx}e^{ik'y}\Big|_{x=y} \\
&= e^{i(k+k')x} - \frac{i}{2}\epsilon\theta_{\mu\nu}k_\mu k'_\nu e^{i(k+k')x} + \frac{1}{2}(\frac{i}{2})^2\epsilon\theta_{\mu\nu}k_\mu k'_\nu\epsilon\theta_{\sigma\rho}k_\sigma k'_\rho e^{i(k+k')x} + ... \\
&= e^{i(k+k')x}\left(1 - \frac{i}{2}k\epsilon\theta k' + \frac{1}{2}(\frac{i}{2})^2(k\epsilon\theta k')^2 ...\right) = e^{i(k+k')x}e^{-\frac{i}{2}k\epsilon\theta k'}. \qquad (2.17)
\end{aligned}
$$

2. Integral of Bilinear Expressions:

We find an important feature of the star product, if we integrate a bilinear term, for example

$$\int d^4x\phi_1(x) \star \phi_2(x),$$

and use Definition (2.13):

$$
\begin{aligned}
&\int d^4x\phi_1(x) \star \phi_2(x) = \\
&\quad = \frac{1}{(2\pi)^8}\int d^4x \int d^4k_1 \int d^4k_2 e^{i(k_1+k_2)x} e^{-\frac{i}{2}k_1\epsilon\theta k_2}\tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2) \\
&\quad = \frac{1}{(2\pi)^4}\int d^4k_1 \int d^4k_2 \delta^4(k_1+k_2) e^{-\frac{i}{2}k_1\epsilon\theta k_2}\tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2) \\
&\quad = \frac{1}{(2\pi)^4}\int d^4k_1 \tilde{\phi}_1(k_1)\tilde{\phi}_2(-k_1).
\end{aligned}
$$

The inverse Fourier transformation of $\tilde{\phi}_1(k_1)\tilde{\phi}_2(-k_1)$ shows disappearance of the star product, which means that we can skip the star for bilinear expressions,

$$
\begin{aligned}
&\int d^4x\phi_1(x) \star \phi_2(x) = \\
&\quad = \frac{1}{(2\pi)^4}\int d^4k_1 \int d^4x \int d^4x' e^{-ik_1(x-x')}\phi_1(x)\phi_2(x') \\
&\quad = \int d^4x \int d^4x' \delta^4(x-x')\phi_1(x)\phi_2(x') \\
&\quad = \int d^4x\phi_1(x)\phi_2(x). \qquad (2.18)
\end{aligned}
$$

3. Associativity:

We will introduce the short notation:

$$dk \equiv \frac{1}{(2\pi)^4} d^4k,$$

execute the star product,

$$\begin{aligned}\phi_1(x) \star \phi_2(x) &= \int dk \int dq \tilde{\phi}_1(k)\tilde{\phi}_2(q) e^{ikx} \star e^{iqx} \\ &= \int dk \int dq \tilde{\phi}_1(k)\tilde{\phi}_2(q) e^{i(k+q)x - \frac{i}{2}k\epsilon\theta q}\end{aligned}$$

and show associativity given by

$$[(\phi_1 \star \phi_2) \star \phi_3] = [\phi_1 \star (\phi_2 \star \phi_3)]. \tag{2.19}$$

For the left hand side of the last equation we have:

$$\begin{aligned}&\int dk \int dq \tilde{\phi}_1(k)\tilde{\phi}_2(q) e^{i(k+q)x - \frac{i}{2}k\epsilon\theta q} \star \phi_3(x) = \\ &\quad = \int dk \int dq \int dp \tilde{\phi}_1(k)\tilde{\phi}_2(q)\tilde{\phi}_3(p) e^{-\frac{i}{2}k\epsilon\theta q} (e^{i(k+q)x} \star e^{ipx}) \\ &\quad = \int dk \int dq \int dp \tilde{\phi}_1(k)\tilde{\phi}_2(q)\tilde{\phi}_3(p) e^{-\frac{i}{2}k\epsilon\theta q} e^{i(k+q+p)x} e^{-\frac{i}{2}(k+q)\epsilon\theta p} \\ &\quad = \int dk \int dq \int dp \tilde{\phi}_1(k)\tilde{\phi}_2(q)\tilde{\phi}_3(p) e^{-\frac{i}{2}k\epsilon\theta q - \frac{i}{2}k\epsilon\theta p - \frac{i}{2}q\epsilon\theta p} e^{i(k+q+p)x},\end{aligned} \tag{2.20}$$

while the right hand side shows the same result:

$$\begin{aligned}&\phi_1(x) \star \int dq \int dp \tilde{\phi}_2(q)\tilde{\phi}_3(p) e^{i(q+p)x - \frac{i}{2}q\epsilon\theta p} = \\ &\quad = \int dk \int dq \int dp \tilde{\phi}_1(k)\tilde{\phi}_2(q)\tilde{\phi}_3(p) e^{ikx} \star e^{-\frac{i}{2}q\epsilon\theta p} e^{i(q+p)x} \\ &\quad = \int dk \int dq \int dp \tilde{\phi}_1(k)\tilde{\phi}_2(q)\tilde{\phi}_3(p) e^{-\frac{i}{2}q\epsilon\theta p} (e^{ikx} \star e^{i(q+p)x}) \\ &\quad = \int dk \int dq \int dp \tilde{\phi}_1(k)\tilde{\phi}_2(q)\tilde{\phi}_3(p) e^{-\frac{i}{2}q\epsilon\theta p} e^{i(k+q+p)x} e^{-\frac{i}{2}k\epsilon\theta(q+p)} \\ &\quad = \int dk \int dq \int dp \tilde{\phi}_1(k)\tilde{\phi}_2(q)\tilde{\phi}_3(p) e^{-\frac{i}{2}k\epsilon\theta q - \frac{i}{2}k\epsilon\theta p - \frac{i}{2}q\epsilon\theta p} e^{i(k+q+p)x}.\end{aligned} \tag{2.21}$$

4. Star Product of higher Orders:

The star product can be generalized to higher order if we keep the results of associativity (2.19-2.21) in mind,

$$\begin{aligned}&\phi_1(x) \star \phi_2(x) \star \ldots \star \phi_m(x) = \\ &= \frac{1}{(2\pi)^{4m}} \int d^4k_1 \int d^4k_2 \ldots \int d^4k_m e^{i\sum_{i=1}^{m} k_{i,\mu}x_\mu} e^{-\frac{i}{2}\sum_{i<j}^{m} k_i\epsilon\theta k_j} \tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2)\ldots\tilde{\phi}_m(k_m).\end{aligned} \tag{2.22}$$

Additionally, we have to say that according to (2.18), we can skip one star under the integral

$$\int d^4x\phi_1(x)\star\phi_2(x)\star...\star\phi_m(x) = \int d^4x\phi_1(x)\star\Big(\phi_2(x)\star...\star\phi_m(x)\Big) = \int d^4x\phi_1(x)\Big(\phi_2(x)\star...\star\phi_m(x)\Big). \tag{2.23}$$

5. Cyclic Permutation under the Integral:

The property of cyclic permutation reads:

$$\int d^4x\phi_1(x)\star\phi_2(x)\star...\star\phi_m(x) = \int d^4x\Big(\pm\,\phi_2(x)\star...\star\phi_m(x)\star\phi_1(x)\Big). \tag{2.24}$$

The plus or minus sign may occur if we permutate the fields or sources due to the fact that fermions are Grassmann valued object which means:

$$AB = -BA, \tag{2.25}$$

for two arbitrary Grassmann valued objects A and B. In Appendix A.1.1 is shown the emergence of the plus or minus sign for our model.

However we will show the property of cyclic permutation for $m = 3$, while all other cases follow through generalization:

$$\begin{aligned}
&\int d^4x\phi_1(x)\star\phi_2(x)\star\phi_3(x) = \\
&= \frac{1}{(2\pi)^{12}}\int d^4x\int d^4k_1\int d^4k_2\int d^4k_3\times \\
&\qquad e^{i(k_1+k_2+k_3)x}e^{-\frac{i}{2}(k_1\epsilon\theta k_2+k_1\epsilon\theta k_3+k_2\epsilon\theta k_3)}\tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2)\tilde{\phi}_3(k_3) \\
&= \frac{1}{(2\pi)^8}\int d^4k_1\int d^4k_2\int d^4k_3\delta^4(k_1+k_2+k_3)\times \\
&\qquad e^{-\frac{i}{2}(k_1\epsilon\theta k_2+k_1\epsilon\theta k_3+k_2\epsilon\theta k_3)}\tilde{\phi}_1(k_1)\tilde{\phi}_2(k_2)\tilde{\phi}_3(k_3) \\
&= \frac{1}{(2\pi)^8}\int d^4k_1\int d^4k_2\int d^4k_3\delta^4(k_1+k_2+k_3) \\
&\qquad\times\Big(\pm\, e^{-\frac{i}{2}(k_1\epsilon\theta k_2+k_1\epsilon\theta k_3+k_2\epsilon\theta k_3)}\tilde{\phi}_2(k_2)\tilde{\phi}_3(k_3)\tilde{\phi}_1(k_1)\Big).
\end{aligned}$$

With a renaming of the k-indices:

$$\begin{aligned}
k_1 &\to k_3, \\
k_2 &\to k_1, \\
k_3 &\to k_2,
\end{aligned}$$

follows

$$\int d^4x\phi_1(x)\star\phi_2(x)\star\phi_3(x) = \\
= \frac{1}{(2\pi)^8}\int d^4k_3\int d^4k_1\int d^4k_2\delta^4(k_3+k_1+k_2) \\
\times\Big(\pm e^{-\frac{i}{2}(k_3\epsilon\theta k_1+k_3\epsilon\theta k_2+k_1\epsilon\theta k_2)}\tilde{\phi}_2(k_1)\tilde{\phi}_3(k_2)\tilde{\phi}_1(k_3)\Big).$$

If we keep in mind that we can use the following relations (based on the delta function $\delta^4(k_1+k_2+k_3)$, which we will use to get the exponent in the right "star order"):

$$k_1=-k_2-k_3,\qquad k_2=-k_1-k_3,\qquad k_3=-k_1-k_2,$$

we get

$$\int d^4x\phi_1(x)\star\phi_2(x)\star\phi_3(x) = \\
= \frac{1}{(2\pi)^8}\int d^4k_3\int d^4k_1\int d^4k_2\delta^4(k_1+k_2+k_3) \\
\times\Big(\pm e^{-\frac{i}{2}(k_1\epsilon\theta k_2+k_1\epsilon\theta k_3+k_2\epsilon\theta k_3)}\tilde{\phi}_2(k_1)\tilde{\phi}_3(k_2)\tilde{\phi}_1(k_3)\Big) \\
= \int d^4x\Big(\pm\phi_2(x)\star\phi_3(x)\star\phi_1(x)\Big),$$

which is what we wanted to show.

So we conclude that that every time we want to change the position of a field under the integral (this is absolutely necessary for the functional derivative as we will see later on), we must cyclicly permute and have an eye on the statistic nature of the fields.

Chapter 3

BRSW Model

In this chapter we put forward an action for a noncommutative $U_\star(1)$-gauge theory, introduce the BRST transformation and talk about symmetries which will arise from the action. Especially the action and the following definitions are based on the work of [20, 25, 26].

3.1 Definitions

We introduce definitions which are the base of our whole work.

The field strength tensor for $U_\star(1)$-gauge group:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu \stackrel{\star}{,} A_\nu]. \tag{3.1}$$

The covariant derivative (the fat dot represents a placeholder):

$$D_\mu \bullet = \partial_\mu \bullet - ig[A_\mu \stackrel{\star}{,} \bullet]. \tag{3.2}$$

The bilinear part of our field strength tensor:

$$f_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, \tag{3.3}$$

and

$$\tilde{f} = \theta_{\alpha\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha). \tag{3.4}$$

The square of $\theta_{\mu\nu}$[1]:

$$\theta^2 = \theta_{\alpha\beta}\theta_{\alpha\beta}. \tag{3.5}$$

The definition of $\tilde{\Box}$:

$$\begin{aligned}\tilde{\Box} &= \tilde{\partial}_\mu\tilde{\partial}_\mu = \theta_{\mu\alpha}\partial_\alpha\theta_{\mu\beta}\partial_\beta = \theta_{\mu\alpha}\theta_{\mu\beta}\partial_\alpha\partial_\beta \\ &= -\theta_{\alpha\mu}\theta_{\mu\beta}\partial_\alpha\partial_\beta = -(-\delta_{\alpha\beta})\partial_\alpha\partial_\beta = \delta_{\alpha\beta}\partial_\alpha\partial_\beta = \partial_\alpha\partial_\alpha = \Box.\end{aligned} \tag{3.6}$$

[1]The special choice of $\theta_{\mu\nu}$ in the form of Eq. (2.6) will lead to $\theta^2 = \theta_{\alpha\beta}\theta_{\alpha\beta} = \delta_{\alpha\alpha} = 4$. This result will not be inserted to keep calculations more general.

Definitions (3.4) and (3.6) show the fact that every time a tilde object appears we have our matrix $\theta_{\mu\nu}$ involved, in summary:

$$\tilde{W}_\mu = \theta_{\mu\nu} W_\nu, \qquad W_\nu \in \{\partial_\nu, D_\nu, A_\nu, k_\nu\}. \tag{3.7}$$

3.2 Equivalences

Here we will show some important equivalences for our calculations.

The square of a tilde vector:

$$\begin{aligned}\tilde{k}^2 = \tilde{k}_\mu \tilde{k}_\mu &= \theta_{\mu\alpha} k_\alpha \theta_{\mu\beta} k_\beta = \theta_{\mu\alpha}\theta_{\mu\beta} k_\alpha k_\beta = -\theta_{\alpha\mu}\theta_{\mu\beta} k_\alpha k_\beta \\ &= -(-\delta_{\alpha\beta}) k_\alpha k_\beta = \delta_{\alpha\beta} k_\alpha k_\beta = k_\alpha k_\alpha = k^2.\end{aligned} \tag{3.8}$$

Taking a closer look at (3.4) and renaming $(\alpha \leftrightarrow \beta)$ the first term we receive:

$$\begin{aligned}\tilde{f} &= \theta_{\alpha\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) = \theta_{\beta\alpha}\partial_\beta A_\alpha - \theta_{\alpha\beta}\partial_\beta A_\alpha \\ &= -\theta_{\alpha\beta}\partial_\beta A_\alpha - \theta_{\alpha\beta}\partial_\beta A_\alpha = -2\theta_{\alpha\beta}\partial_\beta A_\alpha.\end{aligned} \tag{3.9}$$

The last equivalence is deduced from the abbreviation (2.11):

$$\begin{aligned}k_{1,\mu}\epsilon\theta_{\mu\nu}k_{2,\nu} &= k_{1,\nu}\epsilon\theta_{\nu\mu}k_{2,\mu} = k_{1,\nu}k_{2,\mu}\epsilon\theta_{\nu\mu} \\ &= k_{2,\mu}k_{1,\nu}\epsilon\theta_{\nu\mu} = k_{2,\mu}\epsilon\theta_{\nu\mu}k_{1,\nu} \\ &= -k_{2,\mu}\epsilon\theta_{\mu\nu}k_{1,\nu}.\end{aligned} \tag{3.10}$$

For a fluent reading of the work some of the equivalences will be recapitulated for our calculations without equation numbers.

The interval of an integral is defined by the following short notation:

$$\int d^4k I(k) \equiv \int_{-\infty}^{+\infty} dk_0 \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \int_{-\infty}^{+\infty} dk_3 I(k) =: \int_{-\infty}^{+\infty} d^4k I(k). \tag{3.11}$$

Last we have to mention that we use this notation (3.11) for integrals if we perform a substitution $k \to -k$:

$$\begin{aligned}\int d^4k I(k) &\overset{k\to -k}{=} \int_{+\infty}^{-\infty} (-dk_0) ... \int_{+\infty}^{-\infty} (-dk_3) I(-k_0, ..., -k_3) \\ &\equiv -\int_{+\infty}^{-\infty} d^4k I(-k) = \int d^4k I(-k), \\ \Rightarrow \int d^4k I(k) &\overset{k\to -k}{=} \int d^4k I(-k).\end{aligned} \tag{3.12}$$

3.3 Functional Derivative

Next we will introduce the functional derivative defined by

$$\frac{\delta\psi_\alpha(x)}{\delta\psi_\beta(y)} := \delta_{\alpha\beta}\delta^4(x-y), \tag{3.13}$$

for two arbitrary fields ψ_α, ψ_β. As we will see later on this definition will be extended to sources associated with their fields.

If we use this definition we must also investigate the consequences of using the functional derivative for our star product. For example, let us take a closer look at the product of three arbitrary fields:

$$\begin{aligned}
&\frac{\delta}{\delta\psi_\mu(y)}\int d^4x\phi_\alpha(x)\star\psi_\beta(x)\star\varphi_\gamma(x) = \qquad (3.14)\\
&\quad = \frac{\delta}{\delta\psi_\mu(y)}\int d^4x\Big(\pm\psi_\beta(x)\star\varphi_\gamma(x)\star\phi_\alpha(x)\Big)\\
&\quad = \int d^4x\Big(\pm\delta_{\beta\mu}\delta^4(x-y)\varphi_\gamma(x)\star\phi_\alpha(x)\Big) = \pm\delta_{\beta\mu}\Big(\varphi_\gamma(y)\star\phi_\alpha(y)\Big).
\end{aligned}$$

The occurring plus or minus sign generated through cyclic permutation depends on the statistic nature of the fields.

We conclude that there are two steps for practical calculation: First cyclic permutation and after this execution of the functional derivative.

The proof of Eq. (3.14) is shown in Appendix B.1.1.

3.4 Partial Integration

In most cases, we use partial integration if we want to swap the derivative from one field to another.

For bilinear expressions we have:

$$\int d^4xA(x)\star\partial_\mu B_\mu(x) = -\int d^4x\partial_\mu A(x)\star B_\mu(x). \tag{3.15}$$

Clearly the star can be skipped and so no problems are expected. But now the question arises: What consequences has the star product or expressions like $\frac{1}{\tilde{\Box}}$ for partial integration? The answer is we can use partial integration without restrictions

$$\int d^4xA(x)\star\frac{1}{\tilde{\Box}}B(x) = \int d^4x\frac{1}{\tilde{\Box}}A(x)\star B(x), \tag{3.16}$$

and

$$\int d^4xA(x)\star B(x)\star\partial_\mu C_\mu(x) = -\int d^4x\partial_\mu(A(x)\star B(x))\star C_\mu(x). \tag{3.17}$$

Note that all surface terms are assumed to disappear in coordinate space (natural boundary conditions).

The proof of Eqs. (3.15)-(3.17) can be found in Appendix B.2.1.

3.5 The Action

The action of the BRSW Model is given by:

$$S = S_{inv} + S_{ghost} + S_{gf} + S_{aux} + S_{break} + S_{ext}. \tag{3.18}$$

Each term has the detailed expression [20]:

$$\begin{aligned}
S_{inv} &= \int d^4x \frac{1}{4} F_{\mu\nu} \star F_{\mu\nu}, \\
S_{ghost} &= \int d^4x (-\bar{c} \star \partial_\mu D_\mu c), \\
S_{gf} &= \int d^4x (b \star \partial_\mu A_\mu - \frac{\alpha}{2} b \star b), \\
S_{aux} &= -\int d^4x s(\bar{\psi}_{\mu\nu} \star B_{\mu\nu}) = \int d^4x (-\bar{B}_{\mu\nu} \star B_{\mu\nu} + \bar{\psi}_{\mu\nu} \star \psi_{\mu\nu}), \\
S_{break} &= \int d^4x s\big\{(\bar{Q}_{\mu\nu\alpha\beta} \star B_{\mu\nu} + Q_{\mu\nu\alpha\beta} \star \bar{B}_{\mu\nu}) \star \frac{1}{\tilde{\Box}} (f_{\alpha\beta} + \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{f})\big\} \\
&= \int d^4x \big\{(\bar{J}_{\mu\nu\alpha\beta} \star B_{\mu\nu} + J_{\mu\nu\alpha\beta} \star \bar{B}_{\mu\nu}) \star \frac{1}{\tilde{\Box}} (f_{\alpha\beta} + \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{f}) \\
&\quad - \bar{Q}_{\mu\nu\alpha\beta} \star \psi_{\mu\nu} \star \frac{1}{\tilde{\Box}} (f_{\alpha\beta} + \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{f}) \\
&\quad - (\bar{Q}_{\mu\nu\alpha\beta} \star B_{\mu\nu} + Q_{\mu\nu\alpha\beta} \star \bar{B}_{\mu\nu}) \star \frac{1}{\tilde{\Box}} s(f_{\alpha\beta} + \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{f})\big\}, \\
S_{ext} &= \int d^4x s(-\Omega^A_\mu \star A_\mu + \Omega^c \star c) = \int d^4x (\Omega^A_\mu \star sA_\mu + \Omega^c \star sc).
\end{aligned} \tag{3.19}$$

The noncommutative generalization of a $U_\star(1)$ gauge field is denoted by A_μ, $\bar{c}$ and c are the (anti-)ghosts and the multiplier field b implements the Landau gauge fixing $\partial_\mu A_\mu = 0$, with α denoted as a real gauge fixing parameter and σ is a new real parameter. Both parameters are dimensionless. Ω^A_μ and Ω^c are external sources introduced for the non-linear BRST transformations sA_μ and sc (with the BRST operator s[2]), which will be defined in the next sections. The complex field $B_{\mu\nu}$ and its conjugate $\bar{B}_{\mu\nu}$ as well as associated ghosts $\psi_{\mu\nu}$ and $\bar{\psi}_{\mu\nu}$ have been introduced into the bilinear part of the action in order to implement the infrared (IR) damping. These new unphysical fields do not interact with the gauge field A_μ.

The additional sources $\bar{Q}, Q, \bar{J}, J$ are needed in order to ensure BRST transformation invariance of S_{break} in the ultraviolet (UV).

In the infrared, these sources take their "physical values"

$$\bar{Q}_{\mu\nu\alpha\beta}\Big|_{phys} = 0, \qquad Q_{\mu\nu\alpha\beta}\Big|_{phys} = 0, \tag{3.20}$$

and

$$\bar{J}_{\mu\nu\alpha\beta}\Big|_{phys} = \frac{\gamma^2}{4}(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}), \qquad J_{\mu\nu\alpha\beta}\Big|_{phys} = \frac{\gamma^2}{4}(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}), \tag{3.21}$$

[2]The BRST operator s and transformations will be introduced in the Section 3.5.2, Eqs. (3.27) and (3.28).

implementing a 'soft breaking' [30, 22, 24], of BRST transformation, where γ is a new real parameter of the theory of mass dimension one.

The following table (3.1) shows all appearing fields and sources of this work.

Fields:	A_μ	$\bar{c}$	c	b	$\bar{B}_{\mu\nu}$	$B_{\mu\nu}$	$\bar{\psi}_{\mu\nu}$	$\psi_{\mu\nu}$
Sources:	Ω^A_μ	Ω^c	$\bar{Q}_{\mu\nu\alpha\beta}$	$Q_{\mu\nu\alpha\beta}$	$\bar{J}_{\mu\nu\alpha\beta}$	$J_{\mu\nu\alpha\beta}$		

Table 3.1: Fields and sources.

3.5.1 The Landau Gauge

After defining the action let us concentrate on the gauge fixing. The sum of S_{ghost} and S_{gf} may be combined to $S_{gf'}$ (from now on we will use this notation for the rest of our work):

$$S_{ghost} + S_{gf} = S_{gf'} \tag{3.22}$$

which leads to

$$\begin{aligned} S_{gf'} &= \int d^4x(b \star \partial_\mu A_\mu - \bar{c} \star \partial_\mu D_\mu c - \frac{\alpha}{2} b \star b) \\ &= \int d^4x s(\bar{c} \star (\partial_\mu A_\mu - \frac{\alpha}{2} b)). \end{aligned} \tag{3.23}$$

Finally we take a closer look at the equation of motion for the field b by using the functional derivative:

$$\begin{aligned} \frac{\delta}{\delta b(y)} S_{gf'} &= \frac{\delta}{\delta b(y)} \int d^4x(b \star \partial_\mu A_\mu - \bar{c} \star \partial_\mu D_\mu c - \frac{\alpha}{2} b \star b) \\ &= \int d^4x \big(\partial_\mu A_\mu - \alpha b\big) \delta^4(x-y) = \big(\partial_\mu A_\mu - \alpha b\big)(y) = 0, \\ \Rightarrow b &= \frac{1}{\alpha} \partial_\mu A_\mu. \end{aligned} \tag{3.24}$$

Note that the field b represents an unphysical field and so it is allowed to set the external source[3] j^b equal zero and use the free equation of motion.

If we now take the limit

$$\alpha \to 0, \tag{3.25}$$

we get the Landau gauge:

$$\partial_\mu A_\mu = 0. \tag{3.26}$$

[3]The external source j^b will become clear in the next sections.

3.5.2 BRST Transformation

From now on we will omit the stars between the product of fields/fields or fields/sources[4], but keep always in mind that the star product is still present.

The obvious gauge invariance of the gauge theory is lost by the introduction of the gauge-fixing term. However, a symmetry of the Lagrangian can be defined again by an extension of the gauge transformation to the ghost fields. The extended gauge transformation is the Becchi-Rouet-Stora-Tyutin (BRST) or BRST transformation [31].

The product rule with s as the fermionic BRST operator reads:

$$s(AB) = (sA)B \pm AsB, \tag{3.27}$$

where the minus sign occurs if A is a fermionic field or source.

In our work the transformations are given by [20]:

$$\begin{aligned}
&sA_\mu = D_\mu c, && sc = igcc, \\
&s\bar{c} = b, && sb = 0, \\
&s\bar{\psi}_{\mu\nu} = \bar{B}_{\mu\nu}, && s\bar{B}_{\mu\nu} = 0, \\
&sB_{\mu\nu} = \psi_{\mu\nu}, && s\psi_{\mu\nu} = 0, \\
&s\bar{Q}_{\mu\nu\alpha\beta} = \bar{J}_{\mu\nu\alpha\beta}, && s\bar{J}_{\mu\nu\alpha\beta} = 0, \\
&sQ_{\mu\nu\alpha\beta} = J_{\mu\nu\alpha\beta}, && sJ_{\mu\nu\alpha\beta} = 0, \\
&s\Omega^A_\mu = 0, && s\Omega^c = 0.
\end{aligned} \tag{3.28}$$

Dimensions, statistic behaviour and ghost numbers $g_\sharp$ of our fields and sources are summarised in the following tables.

Field	**Statistic**	$g_\sharp$	**Mass dim.**
A_μ	b	0	1
$\bar{c}$	f	-1	1
c	f	1	0
b	b	0	2
$\bar{B}_{\mu\nu}$	b	0	2
$B_{\mu\nu}$	b	0	2
$\bar{\psi}_{\mu\nu}$	f	-1	2
$\psi_{\mu\nu}$	f	1	2

Table 3.2: Statistic behaviour of fields. Fermions are denoted by f while bosons are denoted by b.

The BRST operator s is nilpotent:

$$s(sA) = s^2A = 0, \tag{3.29}$$
$$A \in \{A_\mu, c, \bar{c}, b, \bar{B}_{\mu\nu}, B_{\mu\nu}, \bar{\psi}_{\mu\nu}, \psi_{\mu\nu}, \Omega^A_\mu, \Omega^c, \bar{Q}_{\mu\nu\alpha\beta}, Q_{\mu\nu\alpha\beta}, \bar{J}_{\mu\nu\alpha\beta}, J_{\mu\nu\alpha\beta}\}.$$

[4]Sometimes the star will be written to emphasise the noncommutativity.

Source	**Statistic**	$g_\sharp$	**Mass dim.**
Ω_μ^A	f	-1	3
Ω^c	b	-2	4
$\bar{Q}_{\mu\nu}$	f	-1	2
$Q_{\mu\nu}$	f	-1	2
$\bar{J}_{\mu\nu\alpha\beta}$	b	0	2
$J_{\mu\nu\alpha\beta}$	b	0	2

Table 3.3: Statistic behaviour of sources.

Taking a closer look at Eqs. (3.19) and (3.23) we see that all action terms are BRST transformations except the S_{inv} term. Nilpotency is not fulfilled for the whole Lagrangian ($sS \neq \int d^4xs^2\mathcal{A},\ S = \int d^4x\mathcal{L} = \int d^4xs\mathcal{A}$) because of the term S_{inv}, but we must point out that we can nevertheless receive $sS = 0$, which means BRST invariance for the given action. For details take a closer look at Appendix B.3.1.

For the sake of completeness we give the expression of the term $s(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f})$ from S_{break}:

$$\begin{aligned} &s(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f}) = \\ &= (+ig)\{[c, (f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f})] + [\partial_\alpha c, A_\beta] - [\partial_\beta c, A_\alpha] - \sigma\theta_{\alpha\beta}\theta_{\mu\nu}[\partial_\nu c, A_\mu]\}. \end{aligned} \tag{3.30}$$

This term plays an important role in the calculation of the symmetries.

The detailed calculation of (3.30) can be found in Appendix B.3.2.

3.5.3 Symmetries

There are relations between a symmetry of the classical Lagrangian, currents and charge conservation. If we extend these considerations to the path-integral representation of quantum field theory the results will be the so called Ward identities for Green functions. These identities are relations between Green functions resulting from a symmetry of the action [32]. Transferring these considerations to BRST symmetry gives the Ward identities for the generating functional of the Green functions of a non-abelian gauge theory, the so called Slavnov-Taylor identities [33], which are essential for the proof of the renormalizability of gauge theories.

The Slavnov-Taylor identity describing the BRST symmetry content of the model is given by [20]

$$\begin{aligned} \mathcal{B}(S) = \int d^4x \Big[&\frac{\delta S}{\delta\Omega_\mu^A}\frac{\delta S}{\delta A_\mu} + \frac{\delta S}{\delta\Omega^c}\frac{\delta S}{\delta c} + b\frac{\delta S}{\delta\bar{c}} + \bar{J}_{\mu\nu\alpha\beta}\frac{\delta S}{\delta\bar{Q}_{\mu\nu\alpha\beta}} \\ &+ J_{\mu\nu\alpha\beta}\frac{\delta S}{\delta Q_{\mu\nu\alpha\beta}} + \psi_{\mu\nu}\frac{\delta S}{\delta B_{\mu\nu}} + \bar{B}_{\mu\nu}\frac{\delta S}{\delta\bar{\psi}_{\mu\nu}}\Big] = 0, \end{aligned} \tag{3.31}$$

from which one derives the linearized Slavnov-Taylor operator [20]:

$$\mathcal{B}_S = \int d^4x \left[\frac{\delta S}{\delta \Omega^A_\mu} \frac{\delta}{\delta A_\mu} + \frac{\delta S}{\delta A_\mu} \frac{\delta}{\delta \Omega^A_\mu} + \frac{\delta S}{\delta \Omega^c} \frac{\delta}{\delta c} + \frac{\delta S}{\delta c} \frac{\delta}{\delta \Omega^c} + b \frac{\delta}{\delta \overline{c}} \right]. \tag{3.32}$$

Furthermore we have the gauge fixing condition ($\alpha \to 0$)

$$\frac{\delta S}{\delta b} = \partial_\mu A_\mu = 0, \tag{3.33}$$

the ghost equation

$$\mathcal{G}(S) = \partial_\mu \frac{\delta S}{\delta \Omega^A_\mu} + \frac{\delta S}{\delta \overline{c}} = 0, \tag{3.34}$$

with the ghost operator

$$\mathcal{G} := \partial_\mu \frac{\delta}{\delta \Omega^A_\mu} + \frac{\delta}{\delta \overline{c}}. \tag{3.35}$$

The antighost equation

$$\overline{\mathcal{G}}(S) = \int d^4x \frac{\delta S}{\delta c(x)} = 0. \tag{3.36}$$

with the antighost operator

$$\overline{\mathcal{G}} := \int d^4x \frac{\delta}{\delta c(x)}. \tag{3.37}$$

We also have the symmetry $\mathcal{U}(S)$:

$$\mathcal{U}_{\alpha\beta\mu\nu}(S) = 0, \tag{3.38}$$

with the symmetry operator $\mathcal{U}$ given by

$$\begin{aligned} \mathcal{U}_{\alpha\beta\mu\nu} = \int d^4 \Bigg[& B_{\alpha\beta} \frac{\delta}{\delta B_{\mu\nu}} - \bar{B}_{\mu\nu} \frac{\delta}{\delta \bar{B}_{\alpha\beta}} + J_{\alpha\beta\sigma\rho} \frac{\delta}{\delta J_{\mu\nu\rho\sigma}} - \bar{J}_{\mu\nu\rho\sigma} \frac{\delta}{\delta \bar{J}_{\alpha\beta\rho\sigma}} \\ & + \psi_{\alpha\beta} \frac{\delta}{\delta \psi_{\mu\nu}} - \bar{\psi}_{\mu\nu} \frac{\delta}{\delta \bar{\psi}_{\alpha\beta}} + Q_{\alpha\beta\rho\sigma} \frac{\delta}{\delta Q_{\mu\nu\rho\sigma}} - \bar{Q}_{\mu\nu\rho\sigma} \frac{\delta}{\delta \bar{Q}_{\alpha\beta\rho\sigma}} \Bigg]. \end{aligned} \tag{3.39}$$

The trace of $\mathcal{U}$ is connected to the reality of the action, and is hence denoted 'reality charge' $\mathcal{Q}$ [21]:

$$\begin{aligned} \mathcal{Q} \equiv \delta_{\alpha\mu}\delta_{\beta\nu}\mathcal{U}_{\alpha\beta\mu\nu} = & \\ = \int d^4 \Bigg[& B_{\alpha\beta} \frac{\delta}{\delta B_{\alpha\beta}} - \bar{B}_{\alpha\beta} \frac{\delta}{\delta \bar{B}_{\alpha\beta}} + J_{\alpha\beta\sigma\rho} \frac{\delta}{\delta J_{\alpha\beta\rho\sigma}} - \bar{J}_{\alpha\beta\rho\sigma} \frac{\delta}{\delta \bar{J}_{\alpha\beta\rho\sigma}} \\ & + \psi_{\alpha\beta} \frac{\delta}{\delta \psi_{\alpha\beta}} - \bar{\psi}_{\alpha\beta} \frac{\delta}{\delta \bar{\psi}_{\alpha\beta}} + Q_{\alpha\beta\rho\sigma} \frac{\delta}{\delta Q_{\alpha\beta\rho\sigma}} - \bar{Q}_{\mu\nu\rho\sigma} \frac{\delta}{\delta \bar{Q}_{\alpha\beta\rho\sigma}} \Bigg]. \end{aligned} \tag{3.40}$$

Obviously $\mathcal{Q}(S)$ also generates a symmetry of the action (3.18):

$$\mathcal{U}_{\alpha\beta\mu\nu}(S) = 0 \Rightarrow \mathcal{Q}(S) = 0. \tag{3.41}$$

Having defined the operators $\mathcal{B}_S, \bar{\mathcal{G}}$ and $\mathcal{Q}$ we may derive the following graded commutators and anticommutators:

$$
\begin{aligned}
\{\bar{\mathcal{G}},\bar{\mathcal{G}}\} &= 0, & \{\mathcal{B}_S,\mathcal{B}_S\} &= 0, & \{\bar{\mathcal{G}},\mathcal{B}_S\} &= 0, \\
[\bar{\mathcal{G}},\mathcal{Q}] &= 0, & [\mathcal{Q},\mathcal{Q}] &= 0, & [\mathcal{B}_S,\mathcal{Q}] &= 0.
\end{aligned}
\tag{3.42}
$$

which means that these symmetry operators form a closed algebra.

The detailed calculation of all (anti-)commutators (except the ones involving the linearized Slavnov Taylor operator $\mathcal{B}_S$) and the proof of Eqs. (3.34, 3.36) and (3.38) can be found in Appendix B.4.

Chapter 4

Feynman Rules

After having introduced the action (3.18), we want to find out which parts give rise to propagators and which to vertices. In a first step we divide the action S into a bilinear part S' containing only products of two fields and a part which consists of products of more than two fields. The bilinear of the action describes the free propagation of the fields without any interaction and therefore produces propagators. The non-bilinear terms contain interactions of the fields and will therefore be the source for vertices.

4.1 Propagators

The content of the following section represents a summary of [34, 35] .

4.1.1 Technical Remarks on the Calculation of Propagators

The free propagator in Minkowski space is defined as the time ordered expectation value of the free fields

$$\Delta_{ab}(x-y) = \langle 0|T\Psi_a(x)\Psi_b(y)|0\rangle_{(0)}. \tag{4.1}$$

Arbitrary fields are denoted by Ψ_a, Ψ_b. The index (0) denotes that we are looking at free fields and T stands for the time ordering operator. This operator is defined by

$$T\Psi_a(x)\Psi_b(y) = \Theta(t_x - t_y)\Psi_a(x)\Psi_b(y) + \Theta(t_y - t_x)\Psi_b(y)\Psi_a(x). \tag{4.2}$$

In contrast to Minkowski space the propagator in Euclidean space reads

$$\Delta_{ab}(x-y) = \langle 0|\Psi_a(x)\Psi_b(y)|0\rangle_{(0)}. \tag{4.3}$$

Next we introduce the generating functional for all Green functions in an Euclidean space as the vacuum to vacuum transition amplitude

$$Z[J] = \langle 0|e^{-\int d^4x J_a(x)\Psi_a(x)}|0\rangle = \frac{\int \mathcal{D}[\psi]e^{-\int d^4x\mathcal{L}[\psi]-\int d^4x J_a(x)\psi_a(x)}}{\int \mathcal{D}[\psi]e^{-\int d^4x\mathcal{L}[\psi]}}. \tag{4.4}$$

The field $\Psi_a(x)$ represents the full field operator with all quantum corrections while $\psi_a(x)$ represents the classical field.

$Z[J]$ can be expanded as a power series in the sources $J_a(x)$,

$$Z[J] = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int d^4x_1 ... \int d^4x_n J_{a_1}(x_1)...J_{a_n}(x_n) G_{a_1...a_n}(x_1, ..., x_n) \quad (4.5)$$

At this point we have introduced the classical sources (i.e. not quantized) $J(x) = \{J_a(x); a = 1, .., n\}$ of the field $\psi(x) = \{\psi_a(x); a = 1, .., n\}$. The correspondence is shown in Tab. 4.1. Note that all fields and their sources commute, except of the

Fields:	$\psi_a \in$	A_μ	$\bar{c}$	c	b	$\bar{B}_{\mu\nu}$	$B_{\mu\nu}$	$\bar{\psi}_{\mu\nu}$	$\psi_{\mu\nu}$
Sources:	$J^a \in$	j^A_μ	j^c	$j^{\bar{c}}$	j^b	$j^{\bar{B}}_{\mu\nu}$	$j^B_{\mu\nu}$	$j^{\psi}_{\mu\nu}$	$j^{\psi}_{\mu\nu}$

Table 4.1: Correspondence of fields and sources.

ghost fields $\bar{c}, c$ and their sources $j^{\bar{c}}, j^c$ and the associated ghosts $\psi_{\mu\nu}, \bar{\psi}_{\mu\nu}$ and their sources $j^{\psi}_{\mu\nu}, j^{\bar{\psi}}_{\mu\nu}$.

A general n-point Green function $G_{a_1...a_n}(x_1, ..., x_n)$ is then obtained from $Z[J]$ by functional derivation with respect to the sources $J_a(x)$,

$$G_{a_1...a_n}(x_1, ..., x_n) = (-1)^n \frac{\delta^n Z[J]}{\delta J_{a_1}(x_1)...\delta J_{a_n}(x_n)}\bigg|_{J=0}. \quad (4.6)$$

The contribution of a connected graph to a Green function is called a connected Green function. The generating functional $Z^c[J]$ of connected Green functions G^c is related to $Z[J]$ via

$$Z[J] = e^{-Z^c[J]}. \quad (4.7)$$

Its expansion with respect to the sources $J_a(x)$ reads

$$Z^c[J] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \int d^4x_1 ... \int d^4x_n J_{a_1}(x_1)...J_{a_n}(x_n) \langle 0|\psi_{a_1}(x_1)...\psi_{a_n}(x_n)|0\rangle^c. \quad (4.8)$$

The connected n-point Green functions are given by functional derivation of $Z^c[J]$ with respect to the sources $J_a(x)$,

$$G^c_{a_1...a_n}(x_1, ..., x_n) = (-1)^{n-1} \frac{\delta^n Z^c[J]}{\delta J_{a_1}(x_1)...\delta J_{a_n}(x_n)}\bigg|_{J=0}. \quad (4.9)$$

A connected graph with amputated external legs which remains connected after cutting an arbitrary internal line is called an one-particle irreducible (1PI) graph. The contribution of an 1PI graph to an amputated connected Green function is called a n-point vertex function,

$$\begin{aligned} \Gamma_{a_1...a_n}(x_1, ..., x_n) &= - \frac{\delta^n \Gamma[\psi^{cl.}]}{\delta\psi^{cl.}_{a_1}(x_1)...\delta\psi^{cl.}_{a_n}(x_n)}\bigg|_{J=J[\psi^{cl.}]} \\ &= \langle 0|\psi_{a_1}(x_1)...\psi_{a_n}(x_n)|0\rangle^{1PI}, \end{aligned} \quad (4.10)$$

where the classical field $\psi_a^{cl.}(x)$ is defined as

$$\psi_a^{cl.}(x) = \psi_a^{cl.}[J](x) = \frac{\delta Z^c[J]}{\delta J_a(x)}, \tag{4.11}$$

with

$$J_a(x) = J_a[\psi^{cl.}](x) = \pm_b^f \frac{\delta\Gamma[\psi^{cl.}]}{\delta\psi_a^{cl.}(x)}, \tag{4.12}$$

where the minus sign occurs for bosons (denoted by the index b) and the plus sign for fermions (denoted by the index f).

The generating functional of the n-point vertex functions, the vertex functional is defined by the Legendre transformation

$$\Gamma[\psi^{cl.}] = Z^c[J] - \int d^4x J_a(x)\psi_a^{cl.}(x)\Bigg|_{J_a=J_a[\psi^{cl.}](x)}. \tag{4.13}$$

With the definition of the vertex functional $\Gamma[\psi^{cl.}]$, the proof of (4.12) can be shown:

$$\begin{aligned}\frac{\delta\Gamma[\psi^{cl.}]}{\delta\psi_a^{cl.}(x)} &= \int d^4y \frac{\delta Z^c[J]}{\delta J_b(y)}\frac{\delta J_b(y)}{\delta\psi_a^{cl.}(x)} - \int d^4y\left(\frac{\delta J_b(y)}{\delta\psi_a^{cl.}(x)}\psi_b^{cl.}(y) \mp_b^f \delta_{ab}\delta^4(y-x)J_b(y)\right) \\ &= \int d^4y\left(\frac{\delta Z^c[J]}{\delta J_b(y)} - \psi_b^{cl.}(y)\right)\frac{\delta J_b(y)}{\delta\psi_a^{cl.}(x)} \pm_b^f J_a(x).\end{aligned} \tag{4.14}$$

With the help of (4.11), the integrand vanishes and in the end we get Eq. (4.12):

$$\frac{\delta\Gamma[\psi^{cl.}]}{\delta\psi_a^{cl.}(x)} = \pm_b^f J_a(x).$$

Additionally, we have to say that the two point vertex functions represent the inverse of the two-point connected Green functions in the following sense (for bosonic statistic)

$$\begin{aligned}\int d^4y\Gamma_{ac}(x,y)G_{cb}^c(y,z) &= \int d^4y \frac{\delta^2\Gamma[\psi^{cl.}]}{\delta\psi_a^{cl.}(x)\delta\psi_c^{cl.}(y)}\frac{\delta^2 Z^c[J]}{\delta J_c(y)\delta J_b(z)} \\ &= -\int d^4y \frac{\delta J_c(y)}{\delta\psi_a^{cl.}(x)}\frac{\delta\psi_b^{cl.}(z)}{\delta J_c(y)} = -\delta_{ab}\delta^4(z-x).\end{aligned} \tag{4.15}$$

The pertubative expansion of the Green functions can be ordered according to the number of loops of the corresponding Feynman graphs which correspond to a formal power series in $\hbar$

$$\Gamma[\psi^{cl.}] = \sum_{n=0}^{\infty}\hbar^n\Gamma^{(n)}[\psi^{cl.}], \tag{4.16}$$

where $\Gamma^{(n)}$ is the contribution of the n-loop 1PI graphs to the vertex functional. $S[\psi^{cl.}]$ equals the zeroth order vertex functional $\Gamma^{(0)}[\psi^{cl.}]$,

$$\Gamma^{(0)}[\psi^{cl.}] = S[\psi^{cl.}]. \tag{4.17}$$

The explicit calculation of the propagators read

$$\begin{aligned}\Delta_{ab}(x-y) := G_{ab}(x,y) = G^c_{ab}(x,y) &= -\left.\frac{\delta^2 Z^c[J]}{\delta J_a(x)\delta J_b(y)}\right|_{J_a=J_b=0}\\ &= -\frac{\delta\psi_b^{cl.}[J](y)}{\delta J_a(x)},\end{aligned} \tag{4.18}$$

which means that we first have to find an expression for the field $\psi_b^{cl.}(y)$ in terms of sources by using Eq. (4.12),

$$\frac{\delta\Gamma[\psi^{cl.}]}{\delta\psi_a^{cl.}(x)} = \pm J_a(x) \xrightarrow{\text{tree level}} \frac{\delta S[\psi^{cl.}]}{\delta\psi_a^{cl.}(x)} = \pm J_a(x), \tag{4.19}$$

and then differentiate the field $\psi_b^{cl.}(y)$ with respect to the source $J_a(x)$.

The proof of (4.18) can be found in Appendix C.1.1.

For the model of this work, Eqs. (4.11) and (4.12) have the appearance

$$\begin{aligned}
\frac{\delta Z^c}{\delta j_\mu(x)} &= A_\mu(x), & \frac{\delta\Gamma^{(0)}}{\delta A_\mu(x)} &= -j^A_\mu(x),\\
\frac{\delta Z^c}{\delta j^{\bar{c}}(x)} &= c(x), & \frac{\delta\Gamma^{(0)}}{\delta c(x)} &= +j^{\bar{c}}(x),\\
\frac{\delta Z^c}{\delta j^{c}(x)} &= \bar{c}(x), & \frac{\delta\Gamma^{(0)}}{\delta \bar{c}(x)} &= +j^{c}(x),\\
\frac{\delta Z^c}{\delta j^{b}(x)} &= b(x), & \frac{\delta\Gamma^{(0)}}{\delta b(x)} &= -j^{b}(x),\\
\frac{\delta Z^c}{\delta j^{\bar{B}}(x)} &= \bar{B}_{\mu\nu}(x), & \frac{\delta\Gamma^{(0)}}{\delta \bar{B}_{\mu\nu}} &= -j^{\bar{B}}(x),\\
\frac{\delta Z^c}{\delta j^{B}(x)} &= B_{\mu\nu}(x), & \frac{\delta\Gamma^{(0)}}{\delta B_{\mu\nu}} &= -j^{B}(x),\\
\frac{\delta Z^c}{\delta j^{\bar{\psi}}(x)} &= \psi_{\mu\nu}(x), & \frac{\delta\Gamma^{(0)}}{\delta \psi_{\mu\nu}} &= +j^{\bar{\psi}}(x),\\
\frac{\delta Z^c}{\delta j^{\psi}(x)} &= \bar{\psi}_{\mu\nu}(x), & \frac{\delta\Gamma^{(0)}}{\delta \bar{\psi}_{\mu\nu}} &= +j^{\psi}(x).
\end{aligned} \tag{4.20}$$

Note that if we calculate propagators we only look at bilinear terms of the action and insert the 'physical values' given from (3.20, 3.21) for our additional sources.

Next we will introduce the following abbreviation for the bilinear part of the action

$$S_i^{bilinear} \equiv S_i', \tag{4.21}$$

with

$$i = inv, gf', aux, break, ext.$$

Due to bilinearity our star product disappears under the integral (2.18), which means in our mathematical formalism (for two arbitrary fields A and B):

$$\frac{\delta}{\delta B(y)}\int d^4x(A\star B)=\frac{\delta}{\delta B(y)}\int d^4x(AB)=\frac{\delta}{\delta B(y)}\int d^4x\Big(\pm(BA)\Big) \tag{4.22}$$
$$=\int d^4x\Big(\pm\delta^4(x-y)A(x)\Big)=\int d^4x\Big(\pm A(x)\delta^4(x-y)\Big)=\pm A(y).$$

Before we start with the calculation of the propagators we must say a few words about notation. The formal notation in this work is given by:

$$G^{12}_{ab}(x,y)=-\frac{\delta}{\delta J^1_a(x)}\frac{\delta}{\delta J^2_b(y)}Z^c. \tag{4.23}$$

4.1.2 The Photon Propagator $\tilde{G}^{AA}$

The first propagator will be calculated in detail to show the mathematical techniques.

According to Eqs. (4.18) and (4.19), the photon propagator has the general form of:

$$\begin{aligned}G^{AA}_{\sigma\epsilon}(x,y)&=-\frac{\delta^2Z^c}{\delta j^A_\sigma(x)\delta j^A_\epsilon(y)}=-\frac{\delta A_\epsilon(y)}{\delta j^A_\sigma(x)}\\&=-\frac{\delta A_\epsilon[j^A_\epsilon(y)]}{\delta j^A_\sigma(x)}.\end{aligned} \tag{4.24}$$

Looking at this expression we see that we have to express the field $A_\epsilon(y)$ in terms of sources. Therefore we must eliminate all other remaining fields.

First we try to find the functional derivative of the whole bilinear action by using Eq. (4.19):

$$\frac{\delta}{\delta A_\epsilon}S^{bilinear}=\frac{\delta}{\delta A_\epsilon}(S'_{inv}+S'_{gf}+S'_{aux}+S'_{break}+S'_{ext})=-j^A_\epsilon, \tag{4.25}$$

and next we take a closer look at each term.

For S'_{inv} we have:

$$\begin{aligned}S_{inv}&=\int d^4x\frac{1}{4}F_{\mu\nu}F_{\mu\nu},\\ \Rightarrow S'_{inv}&=\int d^4x\frac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu)(\partial_\mu A_\nu-\partial_\nu A_\mu)\\&=\int d^4x\frac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu)\partial_\mu A_\nu-\frac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu)\partial_\nu A_\mu\\&=\int d^4x\frac{1}{2}(\partial_\mu A_\nu-\partial_\nu A_\mu)(\partial_\mu A_\nu).\end{aligned}$$

Using partial integration we receive a more compact representation

$$\begin{aligned}S'_{inv}&=-\int d^4x\frac{1}{2}\partial_\mu(\partial_\mu A_\nu-\partial_\nu A_\mu)A_\nu=-\int d^4x\frac{1}{2}\Big((\partial_\alpha\partial_\alpha\delta_{\mu\nu}-\partial_\mu\partial_\nu)A_\mu\Big)A_\nu\\&=\int d^4x\frac{1}{2}\Big((\partial_\mu\partial_\nu-\Box\delta_{\mu\nu})A_\mu\Big)A_\nu.\end{aligned} \tag{4.26}$$

The functional derivative yields:

$$\frac{\delta S'_{inv}}{\delta A_\epsilon(y)} = \frac{\delta}{\delta A_\epsilon(y)} \int d^4x \Big\{ \frac{1}{2} \Big((\partial_\mu \partial_\nu - \Box \delta_{\mu\nu}) A_\mu \Big) A_\nu \Big\}(x)$$
$$= \int d^4x \Big(\frac{1}{2} (\partial_\mu \partial_\nu - \Box \delta_{\mu\nu}) \delta_{\mu\epsilon} \delta^4(x-y) A_\nu + \frac{1}{2} (\partial_\mu \partial_\nu - \Box \delta_{\mu\nu}) A_\mu \delta_{\nu\epsilon} \delta^4(x-y) \Big)$$
$$= \frac{1}{2} \Big((\partial_\mu \partial_\nu - \Box \delta_{\mu\nu}) \delta_{\mu\epsilon} A_\nu + (\partial_\mu \partial_\nu - \Box \delta_{\mu\nu}) A_\mu \delta_{\nu\epsilon} \Big)(y).$$

If we now rename the first term ($\mu \leftrightarrow \nu$) we get:

$$\frac{\delta S'_{inv}}{\delta A_\epsilon(y)} = [(\partial_\epsilon \partial_\mu - \Box \delta_{\epsilon\mu}) A_\mu](y). \tag{4.27}$$

For a fluent reading, we will omit the space coordinates as long as we will not point out the mathematical techniques.

Next we look at the term $S_{gf'}$.

$$S_{gf'} = \int d^4x (b \partial_\mu A_\mu - \frac{\alpha}{2} b^2 - \bar{c} \partial_\mu D_\mu c) = \int d^4x (-\partial_\mu b A_\mu - \frac{\alpha}{2} b^2 - \bar{c} \partial_\mu D_\mu c).$$

The bilinear parts reads:

$$S'_{gf'} = \int d^4x \Big(- \partial_\mu b A_\mu - \frac{\alpha}{2} b^2 - \bar{c} \Box c \Big)(x). \tag{4.28}$$

Executing the functional derivative gives:

$$\frac{\delta S'_{gf'}}{\delta A_\epsilon(y)} = \int d^4x \Big(- \partial_\mu b \delta_{\mu\epsilon} \delta^4(x-y) \Big) = \Big(- \partial_\mu b \delta_{\mu\epsilon} \Big)(y) = -\partial_\epsilon b(y). \tag{4.29}$$

For the other action terms we receive

$$\frac{\delta S'_{aux}}{\delta A_\epsilon} = 0, \qquad S'_{ext} = 0, \tag{4.30}$$

and finally S_{break} gives:

$$S'_{break}\big|_{phys} = \int d^4x \Big((\bar{J}_{\mu\nu\alpha\beta}\Big|_{phys} B_{\mu\nu} + J_{\mu\nu\alpha\beta}\Big|_{phys} \bar{B}_{\mu\nu}) \frac{1}{\tilde{\Box}} (f_{\alpha\beta} + \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{f}) \Big)$$
$$= \int d^4x \Big(\frac{\gamma^2}{4} ((\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}) B_{\mu\nu} + (\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}) \bar{B}_{\mu\nu})$$
$$\times \frac{1}{\tilde{\Box}} (f_{\alpha\beta} + \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{f}) \Big)$$
$$= \int d^4x \Big(\frac{\gamma^2}{2} (\bar{B}_{\alpha\beta} + B_{\alpha\beta}) \frac{1}{\tilde{\Box}} (f_{\alpha\beta} + \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{f}) \Big) =$$
$$= \int d^4x \Big(\frac{\gamma^2}{2} (\bar{B}_{\alpha\beta} + B_{\alpha\beta}) \frac{1}{\tilde{\Box}} (\partial_\alpha A_\beta - \partial_\beta A_\alpha - \sigma \theta_{\alpha\beta} \theta_{\mu\nu} \partial_\nu A_\mu) \Big). \tag{4.31}$$

If we keep in mind that the following equations (remember the Section B.2.1) are valid,

$$\int d^4x A \frac{1}{\tilde{\square}} B = \int d^4x \frac{1}{\tilde{\square}} AB; \qquad \partial_\alpha \frac{1}{\tilde{\square}} = \frac{1}{\tilde{\square}} \partial_\alpha,$$

we get

$$\begin{aligned}
\frac{\delta S'_{break}}{\delta A_\epsilon(y)} &= \\
&= \frac{\gamma^2}{2} \int d^4x \Big(-\partial_\alpha \frac{1}{\tilde{\square}} (B_{\alpha\beta} + \bar{B}_{\alpha\beta}) \delta_{\epsilon\beta} \delta^4(x-y) + \partial_\beta \frac{1}{\tilde{\square}} (B_{\alpha\beta} + \bar{B}_{\alpha\beta}) \delta_{\epsilon\alpha} \delta^4(x-y) \\
&\qquad + \sigma \theta_{\alpha\beta} \theta_{\mu\nu} \delta_{\epsilon\mu} \delta^4(x-y) \partial_\nu \frac{1}{\tilde{\square}} (B_{\alpha\beta} + \bar{B}_{\alpha\beta}) \Big) \\
&= \frac{\gamma^2}{2} \Big(-\partial_\alpha \frac{1}{\tilde{\square}} (B_{\alpha\beta} + \bar{B}_{\alpha\beta}) \delta_{\epsilon\beta} + \partial_\beta \frac{1}{\tilde{\square}} (B_{\alpha\beta} + \bar{B}_{\alpha\beta}) \delta_{\epsilon\alpha} \\
&\qquad + \sigma \theta_{\alpha\beta} \theta_{\mu\nu} \delta_{\epsilon\mu} \partial_\nu \frac{1}{\tilde{\square}} (B_{\alpha\beta} + \bar{B}_{\alpha\beta}) \Big)(y) \\
&= \gamma^2 \Big(-\partial_\alpha \frac{1}{\tilde{\square}} (B_{\alpha\epsilon} + \bar{B}_{\alpha\epsilon}) + \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{\partial}_\epsilon \frac{1}{\tilde{\square}} (B_{\alpha\beta} + \bar{B}_{\alpha\beta}) \Big)(y).
\end{aligned} \tag{4.32}$$

Therefore, follows for the bilinear part of the action (3.18):

$$\begin{aligned}
\frac{\delta S'}{\delta A_\epsilon} &= (\partial_\epsilon \partial_\mu - \square \delta_{\epsilon\mu}) A_\mu - \partial_\epsilon \frac{1}{\alpha} b \\
&\quad + \gamma^2 [-\partial_\alpha \frac{1}{\tilde{\square}} (B_{\alpha\epsilon} + \bar{B}_{\alpha\epsilon}) + \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{\partial}_\epsilon \frac{1}{\tilde{\square}} (B_{\alpha\beta} + \bar{B}_{\alpha\beta})] = -j^A_\epsilon.
\end{aligned} \tag{4.33}$$

Next step is to find an expression for the fields b, $B_{\mu\nu}$ and $\bar{B}_{\mu\nu}$ in terms of A_μ.

We start with the field b,

$$\begin{aligned}
&\frac{\delta S'_{gf'}}{\delta b} = \int d^4x \Big(\partial_\mu A_\mu - \alpha b \Big) \delta^4(x-y) = \Big(\partial_\mu A_\mu - \alpha b \Big)(y) = -j^b(y), \\
&\Rightarrow b = \frac{1}{\alpha}(j^b + \partial_\mu A_\mu).
\end{aligned} \tag{4.34}$$

The fields $B_{\mu\nu}$ and $\bar{B}_{\mu\nu}$ are antisymmetric ($B_{\mu\nu} = -B_{\nu\mu}$), which influences the functional derivative in the following sense:

$$\frac{\delta B_{\alpha\beta}(x)}{\delta B_{\mu\nu}(y)} = \frac{1}{2}(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu})\delta^4(x-y), \tag{4.35a}$$

$$\frac{\delta \bar{B}_{\alpha\beta}(x)}{\delta \bar{B}_{\mu\nu}(y)} = \frac{1}{2}(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu})\delta^4(x-y). \tag{4.35b}$$

Again we try to find the functional derivative for each action term (3.19), involving those fields:

$$
\begin{aligned}
\frac{\delta S'_{break}}{\delta B_{\sigma\rho}(y)} &= \frac{\gamma^2}{2}\int d^4x\bigg(\frac{1}{2}(\delta_{\alpha\sigma}\delta_{\beta\rho}-\delta_{\alpha\rho}\delta_{\beta\sigma})\delta^4(x-y)\frac{1}{\tilde{\square}}(f_{\alpha\beta}+\sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f})\bigg)\\
&= \frac{\gamma^2}{2}\bigg(\frac{1}{2}(\delta_{\alpha\sigma}\delta_{\beta\rho}-\delta_{\alpha\rho}\delta_{\beta\sigma})\frac{1}{\tilde{\square}}(f_{\alpha\beta}+\sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f})\bigg)(y)\\
&= \frac{\gamma^2}{2}\frac{1}{\tilde{\square}}\bigg(f_{\sigma\rho}+\sigma\frac{\theta_{\sigma\rho}}{2}\tilde{f}\bigg)(y),
\end{aligned}
$$

and

$$
\frac{\delta S'_{aux}}{\delta B_{\sigma\rho}(y)} = -\int d^4x\bigg(\bar{B}_{\mu\nu}\frac{1}{2}(\delta_{\mu\sigma}\delta_{\nu\rho}-\delta_{\mu\rho}\delta_{\nu\sigma})\delta^4(x-y)\bigg) = -\bar{B}_{\sigma\rho}(y).
$$

The sum of the last two equations leads to:

$$
\frac{\delta S'}{\delta B_{\sigma\rho}(y)} = -\bar{B}_{\sigma\rho}+\frac{\gamma^2}{2}\frac{1}{\tilde{\square}}(f_{\sigma\rho}+\sigma\frac{\theta_{\sigma\rho}}{2}\tilde{f}) = -j^B_{\sigma\rho}(y).
$$

Due to the same structure of $\bar{B}_{\mu\nu}$ given in the action (3.19) we calculate the source $j^{\bar{B}}_{\sigma\rho}$ in the same way,

$$
\frac{\delta S'}{\delta \bar{B}_{\sigma\rho}} = -B_{\sigma\rho}+\frac{\gamma^2}{2}\frac{1}{\tilde{\square}}(f_{\sigma\rho}+\sigma\frac{\theta_{\sigma\rho}}{2}\tilde{f}) = -j^{\bar{B}}_{\sigma\rho}.
$$

So we arrive at:

$$
\bar{B}_{\sigma\rho} = [j^B_{\sigma\rho}+\frac{\gamma^2}{2}\frac{1}{\tilde{\square}}(f_{\sigma\rho}+\sigma\frac{\theta_{\sigma\rho}}{2}\tilde{f})], \tag{4.36a}
$$

$$
B_{\sigma\rho} = [j^{\bar{B}}_{\sigma\rho}+\frac{\gamma^2}{2}\frac{1}{\tilde{\square}}(f_{\sigma\rho}+\sigma\frac{\theta_{\sigma\rho}}{2}\tilde{f})]. \tag{4.36b}
$$

Inserting Eq. (4.34) and the last two equations in (4.33) using the known identities:

$$
\begin{aligned}
\tilde{f} = \theta_{\mu\nu}f_{\mu\nu} &= \theta_{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu) = -2\theta_{\mu\nu}\partial_\nu A_\mu,\\
\tilde{\partial}_\mu &= \theta_{\mu\nu}\partial_\nu,
\end{aligned}
$$

leads to the following expression for the source j^A_ϵ:

$$
\begin{aligned}
-j^A_\epsilon &= (\partial_\epsilon\partial_\mu-\square\delta_{\epsilon\mu})A_\mu-\partial_\epsilon\frac{1}{\alpha}(j^b+\partial_\mu A_\mu) \qquad (4.37)\\
&+\gamma^2\bigg\{-\partial_\alpha\frac{1}{\tilde{\square}}\bigg([j^{\bar{B}}_{\alpha\epsilon}+\frac{\gamma^2}{2}\frac{1}{\tilde{\square}}(f_{\alpha\epsilon}+\sigma\frac{\theta_{\alpha\epsilon}}{2}\tilde{f})]+[j^B_{\alpha\epsilon}+\frac{\gamma^2}{2}\frac{1}{\tilde{\square}}(f_{\alpha\epsilon}+\sigma\frac{\theta_{\alpha\epsilon}}{2}\tilde{f})]\bigg)\\
&+\sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\epsilon\frac{1}{\tilde{\square}}\bigg([j^{\bar{B}}_{\alpha\beta}+\frac{\gamma^2}{2}\frac{1}{\tilde{\square}}(f_{\alpha\beta}+\sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f})]+[j^B_{\alpha\beta}+\frac{\gamma^2}{2}\frac{1}{\tilde{\square}}(f_{\alpha\beta}+\sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f})]\bigg)\bigg\}.
\end{aligned}
$$

Algebraic manipulation gives:

$$
\begin{aligned}
-j^A_\epsilon &= (\partial_\epsilon\partial_\mu - \Box\delta_{\epsilon\mu})A_\mu - \partial_\epsilon\frac{1}{\alpha}(j^b + \partial_\mu A_\mu)\\
&\quad - \gamma^2\partial_\alpha\frac{1}{\tilde{\Box}}(j^B_{\alpha\epsilon} + j^{\bar{B}}_{\alpha\epsilon}) + \gamma^2\sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\epsilon\frac{1}{\tilde{\Box}}(j^B_{\alpha\beta} + j^{\bar{B}}_{\alpha\beta})\\
&\quad - \gamma^4\partial_\alpha\frac{1}{\tilde{\Box}^2}(f_{\alpha\epsilon} - \sigma\theta_{\alpha\epsilon}\theta_{\mu\nu}\partial_\nu A_\mu) + \gamma^4\sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\epsilon\frac{1}{\tilde{\Box}^2}(f_{\alpha\beta} - \sigma\theta_{\alpha\beta}\theta_{\mu\nu}\partial_\nu A_\mu)\\
&= (\partial_\epsilon\partial_\mu - \Box\delta_{\epsilon\mu})A_\mu - \partial_\epsilon\frac{1}{\alpha}(j^b + \partial_\mu A_\mu) - \gamma^2[\partial_\alpha\delta_{\epsilon\beta} - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\epsilon]\frac{1}{\tilde{\Box}}(j^{\bar{B}}_{\alpha\beta} + j^B_{\alpha\beta})\\
&\quad - \gamma^4\partial_\alpha\frac{1}{\tilde{\Box}^2}(\partial_\alpha A_\epsilon - \partial_\epsilon A_\alpha - \sigma\theta_{\alpha\epsilon}\theta_{\mu\nu}\partial_\nu A_\mu)\\
&\quad + \gamma^4\sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\epsilon\frac{1}{\tilde{\Box}^2}(\partial_\alpha A_\beta - \partial_\beta A_\alpha - \sigma\theta_{\alpha\beta}\theta_{\mu\nu}\partial_\nu A_\mu).
\end{aligned}
$$

The next step is to use the identities:

$$
\begin{aligned}
\theta_{\alpha\epsilon}\partial_\alpha &= -\theta_{\epsilon\alpha}\partial_\alpha = -\tilde{\partial}_\epsilon,\\
\theta_{\mu\nu}\partial_\nu &= \tilde{\partial}_\mu,\\
\theta_{\alpha\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) &= -2\theta_{\alpha\beta}\partial_\beta A_\alpha = -2\tilde{\partial}_\alpha A_\alpha,
\end{aligned}
$$

in order to obtain

$$
\begin{aligned}
-j^A_\epsilon &= (\partial_\epsilon\partial_\mu - \Box\delta_{\epsilon\mu})A_\mu - \partial_\epsilon\frac{1}{\alpha}(j^b + \partial_\mu A_\mu)\\
&\quad - \gamma^2[\partial_\alpha\delta_{\epsilon\beta} - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\epsilon]\frac{1}{\tilde{\Box}}(j^{\bar{B}}_{\alpha\beta} + j^B_{\alpha\beta})\\
&\quad - \gamma^4\frac{1}{\tilde{\Box}^2}(\Box A_\epsilon - \partial_\epsilon\partial_\alpha A_\alpha - \sigma\theta_{\alpha\epsilon}\partial_\alpha\theta_{\mu\nu}\partial_\nu A_\mu)\\
&\quad + \gamma^4\sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\epsilon\frac{1}{\tilde{\Box}^2}(\partial_\alpha A_\beta - \partial_\beta A_\alpha - \sigma\theta_{\alpha\beta}\theta_{\mu\nu}\partial_\nu A_\mu)\\
&= (\partial_\epsilon\partial_\mu - \Box\delta_{\epsilon\mu})A_\mu - \partial_\epsilon\frac{1}{\alpha}(j^b + \partial_\mu A_\mu)\\
&\quad - \gamma^2[\partial_\alpha\delta_{\epsilon\beta} - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\epsilon]\frac{1}{\tilde{\Box}}(j^{\bar{B}}_{\alpha\beta} + j^B_{\alpha\beta})\\
&\quad - \gamma^4\frac{1}{\tilde{\Box}^2}(\Box A_\epsilon - \partial_\epsilon\partial_\mu A_\mu + \sigma\tilde{\partial}_\epsilon\tilde{\partial}_\mu A_\mu)\\
&\quad + \gamma^4\frac{1}{\tilde{\Box}^2}(-\sigma\tilde{\partial}_\epsilon\tilde{\partial}_\alpha A_\alpha - \sigma^2\frac{\theta^2}{2}\tilde{\partial}_\epsilon\tilde{\partial}_\mu A_\mu)\\
&= (1 + \frac{\gamma^4}{\tilde{\Box}^2})(\partial_\epsilon\partial_\mu - \Box\delta_{\epsilon\mu})A_\mu - \partial_\epsilon(\frac{1}{\alpha}(j^b + \partial_\mu A_\mu))\\
&\quad - \gamma^2[\partial_\alpha\delta_{\epsilon\beta} - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\epsilon]\frac{1}{\tilde{\Box}}(j^{\bar{B}}_{\alpha\beta} + j^B_{\alpha\beta})\\
&\quad - \gamma^4[2\sigma + 2(\frac{\sigma}{2})^2\theta^2]\frac{1}{\tilde{\Box}^2}\tilde{\partial}_\epsilon\tilde{\partial}_\mu A_\mu.
\end{aligned}
\tag{4.38}
$$

Looking at the last equation we see that we have to express:

$$\partial_\mu A_\mu, \qquad \tilde{\partial}_\alpha A_\alpha,$$

in terms of the sources.

Keeping in mind

$$\partial_\alpha \tilde{\partial}_\alpha = \theta_{\alpha\beta}\partial_\alpha\partial_\beta = 0, \tag{4.39}$$

we start with the calculation of $\partial_\mu A_\mu$ by differentiating the source j^A_ϵ:

$$-\partial_\epsilon j^A_\epsilon = (1+\frac{\gamma^4}{\tilde{\Box}^2})(\Box\partial_\mu - \Box\partial_\mu)A_\mu - \frac{1}{\alpha}\Box j^b - \frac{1}{\alpha}\Box\partial_\mu A_\mu - \gamma^2\frac{1}{\tilde{\Box}}\partial_\alpha\partial_\beta(j^B_{\alpha\beta} + j^{\bar{B}}_{\alpha\beta}).$$

Rearranging gives

$$\partial_\mu A_\mu = \frac{\alpha}{\Box}\partial_\epsilon j^A_\epsilon - j^b - \gamma^2\frac{\alpha}{\Box}\frac{1}{\tilde{\Box}}\partial_\alpha\partial_\beta(j^{\overline{B}}_{\alpha\beta} + j^B_{\alpha\beta}). \tag{4.40}$$

The following expression $\tilde{\partial}_\alpha A_\alpha$ will be calculated in the same way,

$$\begin{aligned}-\tilde{\partial}_\epsilon j^A_\epsilon = {} & (1+\frac{\gamma^4}{\tilde{\Box}^2})(-\Box\tilde{\partial}_\epsilon A_\epsilon) - \gamma^2[\partial_\alpha\tilde{\partial}_\beta - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\Box}]\frac{1}{\tilde{\Box}}(j^B_{\alpha\beta} + j^{\bar{B}}_{\alpha\beta}) \\ & - \gamma^4[2\sigma + 2(\frac{\sigma}{2})^2\theta^2]\frac{1}{\tilde{\Box}}\tilde{\partial}_\alpha A_\alpha.\end{aligned}$$

We rearrange and obtain[1]:

$$\tilde{\partial}_\alpha A_\alpha = \frac{[-\tilde{\partial}_\epsilon j^A_\epsilon + \gamma^2(\partial_\alpha\tilde{\partial}_\beta - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\Box})\frac{1}{\tilde{\Box}}(j^{\bar{B}}_{\alpha\beta} + j^B_{\alpha\beta})]}{[(1+\frac{\gamma^4}{\tilde{\Box}^2})(-\Box) - \gamma^4(2\sigma + 2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{\Box}}]}. \tag{4.41}$$

Inserting Eqs. (4.40) and (4.41) into (4.38) leads to:

$$\begin{aligned}-j^A_\epsilon = {} & (1+\frac{\gamma^4}{\tilde{\Box}^2})\left(\partial_\epsilon[\frac{\alpha}{\Box}\partial_\beta j^A_\beta - j^b - \gamma^2\frac{\alpha}{\Box}\frac{1}{\tilde{\Box}}\partial_\alpha\partial_\beta(j^B_{\alpha\beta} + j^{\bar{B}}_{\alpha\beta})]\right) \\ & - (1+\frac{\gamma^4}{\tilde{\Box}^2})\Box A_\epsilon - \frac{1}{\alpha}\partial_\epsilon j^b \\ & - \frac{1}{\alpha}\partial_\epsilon[\frac{\alpha}{\Box}\partial_\beta j^A_\beta - j^b - \gamma^2\frac{\alpha}{\Box}\frac{1}{\tilde{\Box}}\partial_\alpha\partial_\beta(j^{\bar{B}}_{\alpha\beta} + j^B_{\alpha\beta})] \\ & - \gamma^2[\partial_\alpha\delta_{\epsilon\beta} - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\epsilon]\frac{1}{\tilde{\Box}}(j^{\bar{B}}_{\alpha\beta} + j^B_{\alpha\beta}) \\ & - \gamma^4[2\sigma + 2(\frac{\sigma}{2})^2\theta^2]\frac{1}{\tilde{\Box}^2}\tilde{\partial}_\epsilon\left\{\frac{[-\tilde{\partial}_\beta j^A_\beta + \gamma^2(\partial_\alpha\tilde{\partial}_\beta - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\Box})\frac{1}{\tilde{\Box}}(j^{\bar{B}}_{\alpha\beta} + j^B_{\alpha\beta})]}{[(1+\frac{\gamma^4}{\tilde{\Box}^2})(-\Box) - \gamma^4(2\sigma + 2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{\Box}}]}\right\},\end{aligned}$$

[1] To shorten expressions we introduce $(...\Box)^{-1}A \equiv \frac{A}{(...\Box)}$.

which means for the field A_ϵ:

$$A_\epsilon(y) = \left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\Big\{j^A_\epsilon + (1+\frac{\gamma^4}{\tilde{\Box}^2})\Big(\partial_\epsilon[\frac{\alpha}{\Box}\partial_\beta j^A_\beta - j^b - \gamma^2\frac{\alpha}{\Box}\frac{1}{\tilde{\Box}}\partial_\alpha\partial_\beta(j^{\bar{B}}_{\alpha\beta} + j^B_{\alpha\beta})]\Big) \tag{4.42}$$

$$\begin{aligned}
&-\frac{1}{\alpha}\partial_\epsilon j^b \\
&-\frac{1}{\alpha}\partial_\epsilon[\frac{\alpha}{\Box}\partial_\beta j^A_\beta - j^b - \gamma^2\frac{\alpha}{\Box}\frac{1}{\tilde{\Box}}\partial_\alpha\partial_\beta(j^{\bar{B}}_{\alpha\beta} + j^B_{\alpha\beta})] \\
&-\gamma^2[\partial_\alpha\delta_{\epsilon\beta} - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\epsilon]\frac{1}{\tilde{\Box}}(j^{\bar{B}}_{\alpha\beta} + j^B_{\alpha\beta}) \\
&-\gamma^4[2\sigma + 2(\frac{\sigma^2}{4})\theta^2]\tilde{\partial}_\epsilon\frac{1}{\tilde{\Box}^2}\Big\{\frac{[-\tilde{\partial}_\beta j^A_\beta + \gamma^2(\partial_\alpha\tilde{\partial}_\beta - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\Box})\frac{1}{\tilde{\Box}}(j^{\bar{B}}_{\alpha\beta} + j^B_{\alpha\beta})]}{[(1+\frac{\gamma^4}{\tilde{\Box}^2})(-\Box) - \gamma^4(2\sigma + 2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{\Box}}]}\Big\}\Big\}(y).
\end{aligned}$$

The functional derivative of A_ϵ with respect to the source j^A_σ (4.24) finally leads to $G^{AA}_{\sigma\epsilon}$:

$$\begin{aligned}
G^{AA}_{\sigma\epsilon}(x,y) &= -\left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\Big\{\delta_{\sigma\epsilon} - \Big(1-\alpha(1+\frac{\gamma^4}{\tilde{\Box}^2})\Big)\frac{1}{\Box}\partial_\epsilon\partial_\beta\delta_{\beta\sigma} \\
&-\gamma^4\frac{1}{\tilde{\Box}^2}\Big\{\frac{[2\sigma + 2(\frac{\sigma^2}{4})\theta^2](-\tilde{\partial}_\epsilon\tilde{\partial}_\beta\delta_{\beta\sigma})}{[(1+\frac{\gamma^4}{\tilde{\Box}^2})(-\Box) - \gamma^4(2\sigma + 2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{\Box}}]}\Big\}\Big\}\delta^4(y-x) \\
&= -\left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\Big\{\delta_{\sigma\epsilon} - \Big(1-\alpha(1+\frac{\gamma^4}{\tilde{\Box}^2})\Big)\frac{1}{\Box}\partial_\sigma\partial_\epsilon \\
&-\gamma^4\frac{1}{\tilde{\Box}^2}\Big\{\frac{[2\sigma + 2(\frac{\sigma^2}{4})\theta^2](-\tilde{\partial}_\sigma\tilde{\partial}_\epsilon)}{[(1+\frac{\gamma^4}{\tilde{\Box}^2})(-\Box) - \gamma^4(2\sigma + 2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{\Box}}]}\Big\}\Big\}\delta^4(y-x).
\end{aligned} \tag{4.43}$$

To get a more descriptive illustration of this propagator we perform a Fourier transformation while using:

$$\delta^4(y-x) = \frac{1}{(2\pi)^4}\int d^4p e^{ip(y-x)}. \tag{4.44}$$

This leads to (keep in mind: $\Box \equiv \Box_y$ and $\partial_\epsilon \equiv \frac{\partial}{\partial y_\epsilon}$):

$$\begin{aligned}
G^{AA}_{\sigma\epsilon}(x,y) &= -\left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\Big\{\delta_{\sigma\epsilon} - \Big(1-\alpha(1+\frac{\gamma^4}{\tilde{\Box}^2})\Big)\frac{1}{\Box}\partial_\sigma\partial_\epsilon \\
&-\gamma^4\frac{1}{\tilde{\Box}^2}\Big\{\frac{[2\sigma + 2(\frac{\sigma^2}{4})\theta^2](-\tilde{\partial}_\sigma\tilde{\partial}_\epsilon)}{[(1+\frac{\gamma^4}{\tilde{\Box}^2})(-\Box) - \gamma^4(2\sigma + 2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{\Box}}]}\Big\}\Big\}\frac{1}{(2\pi)^4}\int d^4p e^{ip(y-x)} \\
&= -\frac{1}{(2\pi)^4}\int d^4p\left(\frac{1}{[1+\frac{\gamma^4}{\tilde{p}^4}](-p^2)}\right)\Big\{\delta_{\sigma\epsilon} + \Big(1-\alpha(1+\frac{\gamma^4}{\tilde{p}^4})\frac{1}{p^2}\Big)(ip_\sigma)(ip_\epsilon) \\
&-\gamma^4\frac{1}{\tilde{p}^4}\Big\{\frac{[2\sigma + 2(\frac{\sigma^2}{4})\theta^2](-(i\tilde{p}_\epsilon)(i\tilde{p}_\sigma))}{[(1+\frac{\gamma^4}{\tilde{p}^4})p^2 - \gamma^4(2\sigma + 2(\frac{\sigma^2}{4})\theta^2)(-\frac{1}{\tilde{p}^2})]}\Big\}\Big\}e^{ip(y-x)}.
\end{aligned} \tag{4.45}$$

The propagator $G^{AA}_{\sigma\epsilon}$ has now already the presentation of a Fourier transformation. The factor $(1/2\pi)^4$ in front of the integral and the exponent are already chosen. So the factor and exponent for the inverse Fourier transformation is fixed.
Due to the fact that our propagator only depends on the difference of $(y-x)$, $G^{AA}_{\sigma\epsilon}$ is translation invariant, thus we can write:

$$G^{AA}_{\sigma\epsilon}(x,y) = G^{AA}_{\sigma\epsilon}(z), \qquad z = y - x. \tag{4.46}$$

The proof of this equation can be found in Appendix C.1.2.

Hence we have:

$$G^{AA}_{\sigma\epsilon}(z) = \frac{1}{(2\pi)^4}\int d^4p\Big(\frac{1}{[1+\frac{\gamma^4}{\tilde{p}^4}]p^2}\Big)\times$$
$$\Big\{\delta_{\sigma\epsilon} - \Big(1-\alpha(1+\frac{\gamma^4}{\tilde{p}^4})\Big)\frac{p_\sigma p_\epsilon}{p^2} - \gamma^4\frac{1}{\tilde{p}^4}\Big\{\frac{[2\sigma+2(\frac{\sigma^2}{4})\theta^2]\tilde{p}_\epsilon\tilde{p}_\sigma}{[(1+\frac{\gamma^4}{\tilde{p}^4})p^2+\gamma^4(2\sigma+2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{p}^2}]}\Big\}\Big\}e^{ipz}.$$

The Fourier transform $\tilde{G}^{AA}_{\sigma\epsilon}$ is now given by:

$$\begin{aligned}
\tilde{G}^{AA}_{\sigma\epsilon}(k) &= \int d^4z e^{-ikz} G^{AA}_{\sigma\epsilon}(z) \qquad (4.47)\\
&= \frac{1}{(2\pi)^4}\int d^4z \int d^4p\Big(\frac{1}{[1+\frac{\gamma^4}{\tilde{p}^4}]p^2}\Big)\times\\
&\Big\{\delta_{\sigma\epsilon} - \Big(1-\alpha(1+\frac{\gamma^4}{\tilde{p}^4})\Big)\frac{p_\epsilon p_\sigma}{p^2} - \gamma^4\Big\{\frac{[2\sigma+2(\frac{\sigma^2}{4})\theta^2]\tilde{p}_\epsilon\tilde{p}_\sigma}{\tilde{p}^4[(1+\frac{\gamma^4}{\tilde{p}^4})p^2+\gamma^4(2\sigma+2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{p}^2}]}\Big\}\Big\}e^{iz(p-k)}\\
&= \int d^4p\Big(\frac{1}{[1+\frac{\gamma^4}{\tilde{p}^4}]p^2}\Big)\times\\
&\Big\{\delta_{\sigma\epsilon} - \Big(1-\alpha(1+\frac{\gamma^4}{\tilde{p}^4})\Big)\frac{p_\epsilon p_\sigma}{p^2} - \gamma^4\Big\{\frac{[2\sigma+2(\frac{\sigma^2}{4})\theta^2]\tilde{p}_\epsilon\tilde{p}_\sigma}{\tilde{p}^4[(1+\frac{\gamma^4}{\tilde{p}^4})p^2+\gamma^4(2\sigma+2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{p}^2}]}\Big\}\Big\}\delta^4(p-k)\\
&= \frac{1}{[1+\frac{\gamma^4}{\tilde{k}^4}]k^2}\Big[\delta_{\sigma\epsilon} - (1-\alpha(1+\frac{\gamma^4}{\tilde{k}^4}))\frac{k_\sigma k_\epsilon}{k^2} - \gamma^4\Big\{\frac{[2\sigma+2(\frac{\sigma^2}{4})\theta^2]\tilde{k}_\sigma\tilde{k}_\epsilon}{\tilde{k}^4[(1+\frac{\gamma^4}{\tilde{k}^4})k^2+\gamma^4(2\sigma+2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{k}^2}]}\Big\}\Big].
\end{aligned}$$

If we now introduce the abbreviation:

$$\bar{\sigma}^4 \equiv 2[\sigma + \frac{\sigma^2}{4}\theta^2]\gamma^4, \tag{4.48}$$

and use the Landau gauge ($\alpha \to 0$) we receive:

$$\tilde{G}^{AA}_{\sigma\epsilon}(k) = \frac{1}{[1+\frac{\gamma^4}{\tilde{k}^4}]k^2}\Big[\delta_{\sigma\epsilon} - \frac{k_\sigma k_\epsilon}{k^2} - \frac{\bar{\sigma}^4}{[(\tilde{k}^2+\frac{\gamma^4}{\tilde{k}^2})k^2+\bar{\sigma}^4]}\frac{\tilde{k}_\sigma\tilde{k}_\epsilon}{\tilde{k}^2}\Big]. \tag{4.49}$$

Behaviour of $\tilde{G}^{AA}_{\sigma\epsilon}$ in the IR/UV-Limit

Of special interest is the behaviour of $\tilde{G}^{AA}_{\sigma\epsilon}$ in the IR-and UV-limit precisely because divergences will arise from these limits.

We start with the IR-limit (given by $\tilde{k}^2 \to 0$) and take a closer look at the overall factor outside the big square brackets and the factor of the third term inside the big brackets of Eq. (4.49). Both factors involve the variable k in the denominator and hence possibly influence the nature of divergence.

The first one leads to

$$\frac{1}{[1+\frac{\gamma^4}{\tilde{k}^4}]k^2} = \frac{1}{[k^2+\frac{\gamma^4}{\tilde{k}^2}]} \approx \frac{\tilde{k}^2}{\gamma^4}, \qquad \tilde{k}^2 \to 0, \tag{4.50}$$

the second one can be approximated by

$$\frac{\bar{\sigma}^4}{[(\tilde{k}^2+\frac{\gamma^4}{\tilde{k}^2})k^2+\bar{\sigma}^4]} \approx \frac{\bar{\sigma}^4}{[\gamma^4+\bar{\sigma}^4]}, \qquad \tilde{k}^2 \to 0. \tag{4.51}$$

Therefore, in the IR-limit the propagator reads

$$\tilde{G}^{AA}_{\sigma\epsilon}(k) \approx \frac{\tilde{k}^2}{\gamma^4}\left[\delta_{\sigma\epsilon} - \frac{k_\epsilon k_\sigma}{k^2} - \frac{\bar{\sigma}^4}{[\gamma^4+\bar{\sigma}^4]}\frac{\tilde{k}_\epsilon\tilde{k}_\sigma}{\tilde{k}^2}\right]. \tag{4.52}$$

The UV-limit (given by $k^2 \to \infty$) shows for the first factor

$$\frac{1}{[1+\frac{\gamma^4}{\tilde{k}^4}]k^2} \approx \frac{1}{k^2}, \qquad \tilde{k}^2 \to \infty, \tag{4.53}$$

for the second one

$$\frac{\bar{\sigma}^4}{[(\tilde{k}^2+\frac{\gamma^4}{\tilde{k}^2})k^2+\bar{\sigma}^4]} \approx 0, \qquad \tilde{k}^2 \to \infty, \tag{4.54}$$

and finally leads to

$$\tilde{G}^{AA}_{\sigma\epsilon}(k) \approx \frac{1}{k^2}\left[\delta_{\epsilon\sigma} - \frac{k_\epsilon k_\sigma}{k^2}\right]. \tag{4.55}$$

The last approximation will play an important role in this work. As we will discuss sufficiently in the next chapters all divergences at one-loop arise from the limit $k \to \infty$ and therefore (4.55) represents the adequate candidate for calculations involving the photon propagator.

4.1.3 The Ghost Propagators $\tilde{G}^{\bar{c}c}$ and $\tilde{G}^{c\bar{c}}$

The ghost fields $\bar{c}$ und c are fermionic.

The bilinear part of the action term

$$S_{gf} = \int d^4x(b\partial_\mu A_\mu - \bar{c}\partial_\mu D_\mu c - \frac{\alpha}{2}b^2),$$

reads

$$S'_{gf} = -\int d^4x(\bar{c}\Box c - \frac{\alpha}{2}b^2). \tag{4.56}$$

The source j^c is given by:

$$\begin{aligned}\frac{\delta S'_{gf}}{\delta\bar{c}(y)} &= -\int d^4x \frac{\delta\bar{c}(x)}{\delta\bar{c}(y)}\Box c = -\int d^4x\delta^4(x-y)\Box c = [-\Box c](y)\\ &= +j^c(y)\\ \Rightarrow c &= -\frac{1}{\Box}j^c \end{aligned}\tag{4.57}$$

The propagator $G^{\bar{c}c}$ is now calculated in the typical way

$$\begin{aligned}G^{\bar{c}c}(x,y) &= -\frac{\delta^2 Z^c}{\delta j^c(x)\delta j^{\bar{c}}(y)} = -\frac{\delta c(y)}{\delta j^c(x)} = \frac{1}{\Box}\delta^4(y-x) \\ &= \frac{1}{(2\pi)^4}\int d^4p\frac{1}{\Box}e^{ip(y-x)} = \frac{1}{(2\pi)^4}\int d^4p(-\frac{1}{p^2})e^{ip(y-x)}.\end{aligned}\tag{4.58}$$

Introducing $z = y - x$ gives:

$$G^{\bar{c}c}(z) = \frac{1}{(2\pi)^4}\int d^4p(-\frac{1}{p^2})e^{ipz}.$$

Hence the Fourier transform reads

$$\begin{aligned}\tilde{G}^{\bar{c}c}(k) &= \frac{1}{(2\pi)^4}\int d^4z\int d^4p(-\frac{1}{p^2})e^{iz(p-k)} = \int d^4p(-\frac{1}{p^2})\delta^4(p-k)\\ &= -\frac{1}{k^2}.\end{aligned}\tag{4.59}$$

The calculation of $\tilde{G}^{c\bar{c}}$ is done in the same way but we have to pay attention to the fermionic character of the fields which shows a special property of propagators derived by fermionic fields,

$$\begin{aligned}\frac{\delta S'_{gf}}{\delta c(y)} &= \int d^4x\frac{\delta c(x)}{\delta c(y)}\Box\bar{c} = \int d^4x\delta^4(x-y)\Box\bar{c} = \Box\bar{c}(y)\\ &= +j^{\bar{c}(y)}\\ \Rightarrow \bar{c} &= \frac{1}{\Box}j^{\bar{c}}\end{aligned}\tag{4.60}$$

The functional derivative of $\bar{c}(y)$ gives the propagator,

$$\begin{aligned}G^{c\bar{c}}(x,y) &= -\frac{\delta^2 Z^c}{\delta j^{\bar{c}}(x)\delta j^c(y)} = -\frac{\delta\bar{c}(y)}{\delta j^{\bar{c}}(x)} = -\frac{1}{\Box}\delta^4(y-x)\\ &= -\frac{1}{(2\pi)^4}\int d^4p\frac{1}{\Box}e^{ip(y-x)}\\ &= -\frac{1}{(2\pi)^4}\int d^4p(-\frac{1}{p^2})e^{ip(y-x)}.\end{aligned}\tag{4.61}$$

Introducing $z = y - x$ leads to

$$G^{c\bar{c}}(z) = \frac{1}{(2\pi)^4} \int d^4p \frac{1}{p^2} e^{ipz},$$

and the Fourier transform finally reads

$$\begin{aligned} \tilde{G}^{c\bar{c}}(k) &= \frac{1}{(2\pi)^4} \int d^4z \int d^4p \frac{1}{p^2} e^{iz(p-k)} = \int d^4p \frac{1}{p^2} \delta^4(k-p) \\ &= \frac{1}{k^2}. \end{aligned} \tag{4.62}$$

Therefore, the fermionic character leads to a minus sign for the propagator generated through the interchange of the sources (they obey the same statistic as their assigned fields) in Eq. (4.58), i.e.

$$\tilde{G}^{\bar{c}c}(k) = -\tilde{G}^{c\bar{c}}(k). \tag{4.63}$$

Note that $\tilde{G}^{\bar{c}c}(k)$ and $\tilde{G}^{c\bar{c}}(k)$ are quadratically divergent in the IR.

4.1.4 Remaining Propagators

This section is intended to state the remaining propagators which can be deduced from the action (3.18, 3.19), but will not contribute to physical results. As we will see from the next section non bilinear terms of the given action give rise to vertices. The non bilinear terms involve only the fields $A_\mu, \bar{c}$ and c.
However, for the sake of completeness, we give the respective expressions for the remaining propagators and their *relatives*. *Relatives* means that for example if the propagator G^{AB} exists, obviously G^{BA} exists too and therefore is a relative of G^{AB} .

First of all we have

$$\tilde{G}^{\overline{\psi}\psi}_{\alpha\beta,\mu\nu}(k) = -\frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}) = -\tilde{G}^{\psi\bar{\psi}}_{\alpha\beta,\mu\nu}(k). \tag{4.64}$$

For the next propagator $G^{\bar{B}B}$ it will be wise to think about the general structure which leads to this propagator. If we recall Eqs. (4.36a) and (4.36b):

$$\begin{aligned} \bar{B}_{\sigma\rho} &= [j^B_{\sigma\rho} + \frac{\gamma^2}{2} \frac{1}{\tilde{\square}}(f_{\sigma\rho} + \sigma\frac{\theta_{\sigma\rho}}{2}\tilde{f})], \\ B_{\sigma\rho} &= [j^{\bar{B}}_{\sigma\rho} + \frac{\gamma^2}{2} \frac{1}{\tilde{\square}}(f_{\sigma\rho} + \sigma\frac{\theta_{\sigma\rho}}{2}\tilde{f})], \end{aligned}$$

the detailed calculation given in Appendix C.1.4 shows that the general structure will consist of

$$\bar{B}_{\sigma\rho} = [j^B_{\sigma\rho} + (...)_{\sigma\rho,\alpha\beta}(j^B_{\alpha\beta} + j^{\bar{B}}_{\alpha\beta})], \tag{4.65a}$$

$$B_{\sigma\rho} = [j^{\bar{B}}_{\sigma\rho} + (...)_{\sigma\rho,\alpha\beta}(j^B_{\alpha\beta} + j^{\bar{B}}_{\alpha\beta})]. \tag{4.65b}$$

Due to the fact that the term $(...)_{\sigma\rho,\alpha\beta}$ in front of $(j^B_{\alpha\beta} + j^{\bar{B}}_{\alpha\beta})$ is the same for the field $\bar{B}_{\sigma\rho}$ and $B_{\sigma\rho}$, the derivation of $\bar{B}_{\sigma\rho}$ with respect to the source $j^{\bar{B}}_{\alpha\beta}$ will be equal to the derivation of $B_{\sigma\rho}$ with respect to the source $j^B_{\alpha\beta}$. Additionally, we have to mention that the structure of $B_{\sigma\rho}$ and $\bar{B}_{\sigma\rho}$ will lead to two new propagators G^{BB} and $G^{\bar{B},\bar{B}}$ which are not obvious at first sight from the action (3.19). These new propagators are given by

$$G^{BB}_{\gamma\delta,\sigma\rho}(x,y) = -\frac{\delta^2 Z^c}{\delta j^B_{\gamma\delta}(x)\delta j^B_{\sigma\rho}(y)} = -\frac{\delta B_{\sigma\rho}(y)}{\delta j^B_{\gamma\delta}(x)} = -\frac{\delta}{\delta j^B_{\gamma\delta}(x)}[(...)_{\sigma\rho,\alpha\beta} j^B_{\alpha\beta})](y), \quad (4.66)$$

$$G^{\bar{B}\bar{B}}_{\gamma\delta,\sigma\rho}(x,y) = -\frac{\delta^2 Z^c}{\delta j^{\bar{B}}_{\gamma\delta}(x)\delta j^{\bar{B}}_{\sigma\rho}(y)} = -\frac{\delta \bar{B}_{\sigma\rho}(y)}{\delta j^{\bar{B}}_{\gamma\delta}(x)} = -\frac{\delta}{\delta j^{\bar{B}}_{\gamma\delta}(x)}[(...)_{\sigma\rho,\alpha\beta} j^{\bar{B}}_{\alpha\beta})](y), \quad (4.67)$$

$$\Rightarrow G^{BB}_{\gamma\delta,\sigma\rho}(x,y) = G^{\bar{B}\bar{B}}_{\gamma\delta,\sigma\rho}(x,y). \quad (4.68)$$

Knowing these propagators, $G^{\bar{B}B}$ and $G^{B\bar{B}}$ read

$$\begin{aligned} G^{\bar{B}B}_{\gamma\delta,\sigma\rho}(x,y) &= -\frac{\delta^2 Z^c}{\delta j^{\bar{B}}_{\gamma\delta}(x)\delta j^B_{\sigma\rho}(y)} = -\frac{\delta B_{\sigma\rho}(y)}{\delta j^{\bar{B}}_{\gamma\delta}(x)} = -\frac{\delta}{\delta j^{\bar{B}}_{\gamma\delta}(x)}[j^{\bar{B}}_{\sigma\rho} + (...)_{\sigma\rho,\alpha\beta} j^{\bar{B}}_{\alpha\beta}](y) \\ &= -\frac{\delta j^{\bar{B}}_{\sigma\rho}(y)}{\delta j^{\bar{B}}_{\gamma\delta}(x)} + G^{\bar{B}\bar{B}}_{\gamma\delta,\sigma\rho}(x,y), \end{aligned} \quad (4.69)$$

and

$$\begin{aligned} G^{B\bar{B}}_{\gamma\delta,\sigma\rho}(x,y) &= -\frac{\delta^2 Z^c}{\delta j^B_{\gamma\delta}(x)\delta j^{\bar{B}}_{\sigma\rho}(y)} = -\frac{\delta \bar{B}_{\sigma\rho}(y)}{\delta j^B_{\gamma\delta}(x)} = -\frac{\delta}{\delta j^B_{\gamma\delta}(x)}[j^B_{\sigma\rho} + (...)_{\sigma\rho,\alpha\beta} j^B_{\alpha\beta}](y) \\ &= -\frac{\delta j^B_{\sigma\rho}(y)}{\delta j^B_{\gamma\delta}(x)} + G^{BB}_{\gamma\delta,\sigma\rho}(x,y), \quad (4.70) \\ &\Rightarrow G^{\bar{B}B}_{\gamma\delta,\sigma\rho}(x,y) = G^{B\bar{B}}_{\gamma\delta,\sigma\rho}(x,y). \quad (4.71) \end{aligned}$$

The explicit expression of the Fourier transform is given by

$$\begin{aligned} &\tilde{G}^{BB}_{\gamma\delta,\sigma\rho}(k) = \frac{\gamma^4}{4}\left[\frac{1}{\tilde{k}^2(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2})k^2}\right]\times \quad (4.72) \\ &\Big\{k_\sigma k_\gamma \delta_{\rho\delta} - k_\sigma k_\delta \delta_{\rho\gamma} - k_\rho k_\gamma \delta_{\sigma\delta} + k_\rho k_\delta \delta_{\sigma\gamma} \\ &+ \sigma\theta_{\gamma\delta}(k_\rho \tilde{k}_\sigma - k_\sigma \tilde{k}_\rho) + \sigma\theta_{\sigma\rho}(k_\delta \tilde{k}_\gamma - k_\gamma \tilde{k}_\delta) + \sigma^2 \theta_{\sigma\rho}\theta_{\gamma\delta}\tilde{k}^2 \\ &+ \frac{\bar{\sigma}^4}{\tilde{k}^2[(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2})k^2 + \bar{\sigma}^4]}\Big(k_\rho k_\gamma \tilde{k}_\sigma \tilde{k}_\delta - k_\rho k_\delta \tilde{k}_\sigma \tilde{k}_\gamma + k_\sigma k_\delta \tilde{k}_\rho \tilde{k}_\gamma - k_\sigma k_\gamma \tilde{k}_\rho \tilde{k}_\delta \\ &+ \sigma\theta_{\gamma\delta}(k_\sigma \tilde{k}_\rho - k_\rho \tilde{k}_\sigma)\tilde{k}^2 + \sigma\theta_{\sigma\rho}(k_\gamma \tilde{k}_\delta - k_\delta \tilde{k}_\gamma)\tilde{k}^2 - \sigma^2\theta_{\sigma\rho}\theta_{\gamma\delta}\tilde{k}^4\Big)\Big\} = \tilde{G}^{\bar{B}\bar{B}}_{\gamma\delta,\sigma\rho}(k). \end{aligned}$$

The calculation of $\tilde{G}^{BB}$ finally leads to the expression of $\tilde{G}^{\bar{B}B}$

$$\tilde{G}^{\bar{B}B}_{\gamma\delta,\sigma\rho}(k) = -\frac{1}{2}(\delta_{\gamma\sigma}\delta_{\delta\rho} - \delta_{\delta\sigma}\delta_{\gamma\rho}) + \tilde{G}^{\bar{B}\bar{B}}_{\gamma\delta,\sigma\rho}(k) = \tilde{G}^{B\bar{B}}_{\gamma\delta,\sigma\rho}(k), \quad (4.73)$$

where we have used the analogon of Eqs. (4.35a) and (4.35b) applied to the sources $j^B, j^{\bar{B}}$:

$$\frac{j^Y_{\sigma\rho}(x)}{j^Y_{\gamma\delta}(y)} = \frac{1}{2}(\delta_{\gamma\sigma}\delta_{\delta\rho} - \delta_{\delta\sigma}\delta_{\gamma\rho})\delta^4(x-y), \qquad Y \in \{B, \bar{B}\}. \quad (4.74)$$

Finally $\tilde{G}^{BA}_{\rho\sigma,\epsilon}$ reads

$$\tilde{G}^{BA}_{\rho\sigma,\epsilon}(k) = i\gamma^2\left(\frac{1}{2[\tilde{k}^2+\frac{\gamma^4}{\tilde{k}^2}]k^2}\right)\times \tag{4.75}$$
$$\left\{[k_\rho\delta_{\epsilon\sigma} - k_\sigma\delta_{\epsilon\rho} - \sigma\theta_{\rho\sigma}\tilde{k}_\epsilon] + \frac{\bar{\sigma}^4[k_\sigma\tilde{k}_\epsilon\tilde{k}_\rho - k_\rho\tilde{k}_\epsilon\tilde{k}_\sigma + \sigma\theta_{\rho\sigma}\tilde{k}_\epsilon\tilde{k}^2]}{\tilde{k}^2[(\tilde{k}^2+\frac{\gamma^4}{\tilde{k}^2})k^2+\bar{\sigma}^4]}\right\}$$
$$= \tilde{G}^{\bar{B}A}_{\rho\sigma,\epsilon}(k) = -\tilde{G}^{AB}_{\epsilon,\rho\sigma}(k) = -\tilde{G}^{A\bar{B}}_{\epsilon,\rho\sigma}(k).$$

At the very end we state, that the calculation of G^{bb} will lead to zero and therefore matches the unphysical character of the field b. The calculation of G^{bb} shows

$$\tilde{G}^{bb} = 0. \tag{4.76}$$

The detailed calculations can be found in Appendix (C.1.3,C.1.4,C.1.5,C.1.6,C.1.7 and C.1.8).

4.2 Vertices

4.2.1 Technical Remarks on the Calculation of Vertices

As we have mentioned in the previous sections only non bilinear terms contain interactions of the fields and will therefore be the source for vertices. We will recapitulate from Section 4.1.1 that the n-point vertex function is given by Eq. (4.10):

$$\Gamma_{a_1\ldots a_n}(x_1,\ldots,x_n) = -\frac{\delta^n\Gamma[\psi^{cl.}]}{\delta\psi^{cl.}_{a_1}(x_1)\ldots\delta\psi^{cl.}_{a_n}(x_n)}\bigg|_{J=J[\psi^{cl.}]}.$$

Looking at our action (3.18) we see that only $S_{gf'}$ and S_{inv} will give rise to three, respectively four-point vertex functions.

The formal notation of this section is given by:

$$S^{abc\ldots} := S^{\text{only non bilinear terms made up by abc..}}, \qquad a,b,c \in \{\bar{c}, c, A_\mu\}$$

and (for example a three-point vertex function):

$$V^{ABC}_{\alpha\beta\gamma}(x_1,x_2,x_3) = -\frac{\delta^3 S[\psi]}{\delta A_\alpha(x_1)\delta B_\beta(x_2)\delta C_\gamma(x_3)}. \tag{4.77}$$

The last equation shows the consequence that at zeroth order the vertex functional $\Gamma[\psi^{cl.}]$ equals $S[\psi^{cl.}]$.

Now the procedure of calculation is clear: First find the relevant part of the given action, use the definition of the star product and then perform the functional derivative of S with respect to the considered fields.
Note that if we will transform the vertex function (again we will look at a three-point vertex function) into momentum space[2], given by:

$$\tilde{V}^{ABC}_{\alpha\beta\gamma}(k_1,k_2,k_3) = -(2\pi)^{12}\frac{\delta^3 S[\tilde{\psi}]}{\delta\tilde{A}_\alpha(-k_1)\delta\tilde{B}_\beta(-k_2)\delta\tilde{C}_\gamma(-k_3)}, \tag{4.78}$$

[2]The transformation into momentum space implies the transformation of $S[\psi] \to S[\tilde{\psi}]$ which is automatically ensured by the definition of the star product (2.13).

a factor $(2\pi)^4$ for each field arises. The minus sign in front of the variable k in the denominator of the functional derivative ensures that the momentum at each vertex points inwards. We must underline that this is only a convention.

4.2.2 The Two-Ghost One-Photon Vertex $\tilde{V}_\mu^{cA\bar{c}}$

The first vertex will be calculated in detail and is made up by the fields $\bar{c}, A_\mu, c$ represented by $\tilde{V}_\mu^{cA\bar{c}}(q_1, k_2, q_3)$. The part of the action, which will give rise to this Vertex is given by ($\alpha \to 0$)

$$\begin{aligned} S_{gf'} &= \int d^4x (b\partial_\mu A_\mu - \bar{c}\partial_\mu D_\mu c), \\ \Rightarrow S_{gf'}^{\bar{c}Ac} &= \int d^4x \{+ig\bar{c}\partial_\mu [A_\mu, c]\} = \int d^4x \{-ig\partial_\mu \bar{c}[A_\mu, c]\} \\ &= \int d^4x \{-ig\partial_\mu \bar{c} \star A_\mu \star c + ig\partial_\mu \bar{c} \star c \star A_\mu\}. \end{aligned} \tag{4.79}$$

Inserting the definition of the star product (2.13), the last expression reads

$$\begin{aligned} S_{gf'}^{\bar{c}Ac} &= \frac{1}{(2\pi)^{12}} \int d^4x \int d^4q_1' \int d^4k_2' \int d^4q_3' \times \\ &\{(-ig)iq'_{\mu',1}\tilde{\bar{c}}(q_1')\tilde{A}_{\mu'}(k_2')\tilde{c}(q_3')e^{-\frac{i}{2}q_1'\epsilon\theta k_2' - \frac{i}{2}k_2'\epsilon\theta q_3' - \frac{i}{2}q_1'\epsilon\theta q_3'}e^{i(q_1'+k_2'+q_3')x} \\ &+ (ig)iq'_{\mu',1}\tilde{\bar{c}}(q_1')\tilde{c}(q_3')\tilde{A}_{\mu'}(k_2')e^{-\frac{i}{2}q_1'\epsilon\theta q_3' - \frac{i}{2}q_3'\epsilon\theta k_2' - \frac{i}{2}q_1'\epsilon\theta k_2'}e^{i(q_1'+k_2'+q_3')x}\} \\ &= \frac{1}{(2\pi)^{12}} \int d^4x \int d^4q_1' \int d^4k_2' \int d^4q_3' [e^{-\frac{i}{2}k_2'\epsilon\theta q_3'} - e^{-\frac{i}{2}q_3'\epsilon\theta k_2'}] \\ &\times \{(-ig)iq'_{\mu',1}\tilde{\bar{c}}(q_1')\tilde{A}_{\mu'}(k_2')\tilde{c}(q_3')e^{-\frac{i}{2}q_1'\epsilon\theta k_2' - \frac{i}{2}q_1'\epsilon\theta q_3'}e^{i(q_1'+k_2'+q_3')x}\}. \end{aligned}$$

The use of the identity

$$k_i\epsilon\theta k_j = -k_j\epsilon\theta k_i,$$

leads to

$$\begin{aligned} S_{gf'}^{\bar{c}Ac} &= \frac{1}{(2\pi)^{12}} \int d^4x \int d^4q_1' \int d^4k_2' \int d^4q_3' [e^{+\frac{i}{2}k_2'\epsilon\theta q_3'} - e^{-\frac{i}{2}k_2'\epsilon\theta q_3'}] \\ &\times \{(+ig)iq'_{\mu',1}\tilde{\bar{c}}(q_1')\tilde{A}_{\mu'}(k_2')\tilde{c}(q_3')e^{-\frac{i}{2}q_1'\epsilon\theta k_2' - \frac{i}{2}q_1'\epsilon\theta q_3'}e^{i(q_1'+k_2'+q_3')x}\} \\ &= \frac{1}{(2\pi)^{12}} \int d^4x \int d^4q_1' \int d^4k_2' \int d^4q_3' 2i\sin[\frac{1}{2}k_2'\epsilon\theta q_3'] \\ &\times \{(+ig)iq'_{\mu',1}\tilde{\bar{c}}(q_1')\tilde{A}_{\mu'}(k_2')\tilde{c}(q_3')e^{-\frac{i}{2}q_1'\epsilon\theta k_2' - \frac{i}{2}q_1'\epsilon\theta q_3'}e^{i(q_1'+k_2'+q_3')x}\}. \end{aligned} \tag{4.80}$$

The integration over the space coordinate brings

$$\begin{aligned} S_{gf'}^{\bar{c}Ac} &= \frac{1}{(2\pi)^{8}} \int d^4q_1' \int d^4k_2' \int d^4q_3' \delta^4(q_1' + k_2' + q_3') \\ &\times \{(+ig)iq'_{\mu',1}\tilde{\bar{c}}(q_1')\tilde{A}_{\mu'}(k_2')\tilde{c}(q_3')e^{-\frac{i}{2}q_1'\epsilon\theta k_2' - \frac{i}{2}q_1'\epsilon\theta q_3'}2i\sin[\frac{1}{2}k_2'\epsilon\theta q_3']\}. \end{aligned}$$

As usual we use the delta function, pick out k_2'[3], ($k_2' = -q_1' - q_3'$) and hence get for the exponential function

$$
\begin{aligned}
e^{-\frac{i}{2}q_1'\epsilon\theta k_2' - \frac{i}{2}q_1'\epsilon\theta q_3'}\delta^4(q_1'+k_2'+q_3') &= e^{-\frac{i}{2}q_1'\epsilon\theta(-q_1'-q_3') - \frac{i}{2}q_1'\epsilon\theta q_3'}\delta^4(q_1'+k_2'+q_3') \\
&= e^{-\frac{i}{2}q_1'\epsilon\theta(-q_3') - \frac{i}{2}q_1'\epsilon\theta q_3'}\delta^4(q_1'+k_2'+q_3') \\
&= e^{+\frac{i}{2}q_1'\epsilon\theta q_3' - \frac{i}{2}q_1'\epsilon\theta q_3'}\delta^4(q_1'+k_2'+q_3') = \delta^4(q_1'+k_2'+q_3'),
\end{aligned}
$$

and for the sine function

$$
\sin[\frac{1}{2}k_2'\epsilon\theta q_3'] = \sin[\frac{1}{2}(-q_1'-q_3')\epsilon\theta q_3'] = -\sin[\frac{1}{2}q_1'\epsilon\theta q_3'].
$$

Therefore we have

$$
\begin{aligned}
S_{gf'}^{\bar{c}Ac} = \frac{1}{(2\pi)^8}\int d^4q_1'\int d^4k_2'\int d^4q_3'\delta^4(q_1'+k_2'+q_3') \\
\times \{(-ig)iq'_{\mu',1}\tilde{\bar{c}}(q_1')\tilde{A}_{\mu'}(k_2')\tilde{c}(q_3')2i\sin[\frac{1}{2}q_1'\epsilon\theta q_3']\}.
\end{aligned}
\tag{4.81}
$$

The expression of the vertex is now given by

$$
\begin{aligned}
&\tilde{V}_\mu^{cA\bar{c}}(q_1,k_2,q_3) = -(2\pi)^{12}\frac{\delta}{\delta\tilde{c}(-q_1)}\frac{\delta}{\delta\tilde{A}_\mu(-k_2)}\frac{\delta}{\delta\tilde{\bar{c}}(-q_3)}S^{\bar{c}Ac} \\
&= -(2\pi)^{12}\frac{\delta}{\delta\tilde{c}(-q_1)}\frac{\delta}{\delta\tilde{A}_\mu(-k_2)}\frac{\delta}{\delta\tilde{\bar{c}}(-q_3)}\times \\
&\quad\Bigg(\frac{1}{(2\pi)^8}\int d^4q_1'\int d^4k_2'\int d^4q_3'\delta^4(q_1'+k_2'+q_3') \\
&\qquad\times \{(-ig)iq'_{\mu',1}\tilde{\bar{c}}(q_1')\tilde{A}_{\mu'}(k_2')\tilde{c}(q_3')2i\sin[\frac{1}{2}q_1'\epsilon\theta q_3']\}\Bigg) \\
&= -(2\pi)^4\frac{\delta}{\delta\tilde{c}(-q_1)}\frac{\delta}{\delta\tilde{A}_\mu(-k_2)}\int d^4q_1'\int d^4k_2'\int d^4q_3'\delta^4(q_1'+k_2'+q_3') \\
&\qquad\times\Bigg((-ig)iq'_{\mu',1}\delta^4(q_1'+q_3)\tilde{A}_{\mu'}(k_2')\tilde{c}(q_3')2i\sin[\frac{1}{2}q_1'\epsilon\theta q_3']\Bigg)
\end{aligned}
$$

[3] In this case the selection of k_2' is arbitrary.

$$
\begin{aligned}
&= -(2\pi)^4 \frac{\delta}{\delta\tilde{c}(-q_1)} \int d^4q_1' \int d^4k_2' \int d^4q_3' \delta^4(q_1' + k_2' + q_3') 2i \sin[\frac{1}{2} q_1' \epsilon\theta q_3'] \\
&\qquad \times \Big((-ig) i q'_{\mu',1} \delta^4(q_1' + q_3) \delta_{\mu\mu'} \delta^4(k_2' + k_2) \tilde{c}(q_3') \Big) \\
&= -(2\pi)^4 \int d^4q_1' \int d^4k_2' \int d^4q_3' \delta^4(q_1' + k_2' + q_3') 2i \sin[\frac{1}{2} q_1' \epsilon\theta q_3'] \\
&\qquad \times \Big((-ig) i q'_{\mu',1} \delta^4(q_1' + q_3) \delta_{\mu\mu'} \delta^4(k_2' + k_2) \delta^4(q_3' + q_1) \Big) \\
&= -(2\pi)^4 \int d^4q_1' \int d^4k_2' \delta^4(q_1' + k_2' - q_1) 2i \sin[\frac{1}{2} q_1' \epsilon\theta(-q_1)] \\
&\qquad \times \Big((-ig) i q'_{\mu',1} \delta^4(q_1' + q_3) \delta_{\mu\mu'} \delta^4(k_2' + k_2) \Big) \\
&= -(2\pi)^4 \int d^4q_1' \delta^4(q_1' - k_2 - q_1) 2i \sin[\frac{1}{2} q_1' \epsilon\theta(-q_1)] \\
&\qquad \times \Big((-ig) i g'_{\mu',1} \delta^4(q_1' + q_3) \delta_{\mu\mu'} \Big) \\
&= -(2\pi)^4 \delta^4(-q_3 - k_2 - q_1) 2i \sin[\frac{1}{2}(-q_3) \epsilon\theta(-q_1)](-ig)(-iq_{\mu,3}).
\end{aligned}
$$

Algebraic manipulation finally leads to

$$
\tilde{V}_\mu^{cA\bar{c}}(q_1, k_2, q_3) = -2ig(2\pi)^4 \delta^4(q_1 + k_2 + q_3) q_{\mu,3} \sin[\frac{1}{2} q_1 \epsilon\theta q_3]. \tag{4.82}
$$

4.2.3 The Three-Photon Vertex $\tilde{V}^{3A}_{\alpha\beta\gamma}$

The next vertex we have to calculate arises from

$$
\begin{aligned}
S_{inv} &= \int d^4x \frac{1}{4} F_{\mu\nu} F_{\mu\nu} \\
&= \int d^4x \frac{1}{4} \{(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu])(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu])\} \\
&= \int d^4x \frac{1}{4} \{(\partial_\mu A_\nu - \partial_\nu A_\mu - igA_\mu A_\nu + igA_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu - igA_\mu A_\nu + igA_\nu A_\mu)\}.
\end{aligned}
$$

The part which contains the product of three A_μ fields reads

$$
\begin{aligned}
S^{3A}_{inv} &= \int d^4x \frac{1}{4} ig\{-A_\mu A_\nu \partial_\mu A_\nu + A_\nu A_\mu \partial_\mu A_\nu + A_\mu A_\nu \partial_\nu A_\mu - A_\nu A_\mu \partial_\nu A_\mu \\
&\qquad - \partial_\mu A_\nu A_\mu A_\nu + \partial_\nu A_\mu A_\mu A_\nu + \partial_\mu A_\nu A_\nu A_\mu - \partial_\nu A_\mu A_\nu A_\mu\} \\
&= \int d^4x \frac{1}{4} ig\{(A_\nu A_\mu - A_\mu A_\nu)\partial_\mu A_\nu + (A_\mu A_\nu - A_\nu A_\mu)\partial_\nu A_\mu \\
&\qquad + \partial_\nu A_\mu (A_\mu A_\nu - A_\nu A_\mu) + \partial_\mu A_\nu (A_\nu A_\mu - A_\mu A_\nu)\}.
\end{aligned}
$$

If we use the property of cyclic rotation we get

$$
S^{3A}_{inv} = \int d^4x \frac{1}{2} ig\{(A_\nu A_\mu - A_\mu A_\nu)\partial_\mu A_\nu + (A_\mu A_\nu - A_\nu A_\mu)\partial_\nu A_\mu\}.
$$

The second term will be renamed ($\mu \leftrightarrow \nu$) and therefore we have

$$S^{3A}_{inv} = \int d^4x ig\{(A_\nu A_\mu - A_\mu A_\nu)\partial_\mu A_\nu\}. \tag{4.83}$$

The definition of the star product (2.13) yields

$$\begin{aligned} S^{3A}_{inv} =& \frac{1}{(2\pi)^{12}} ig \int d^4k_1' \int d^4k_2' \int d^4k_3' \int d^4x e^{i(k_1'+k_2'+k_3')x} \\ & \times \{\tilde{A}_\nu(k_1')\tilde{A}_\mu(k_2') ik'_{\mu,3}\tilde{A}_\nu(k_3') e^{-\frac{i}{2}k_1'\epsilon\theta k_2' - \frac{i}{2}k_2'\epsilon\theta k_3' - \frac{i}{2}k_1'\epsilon\theta k_3'} \\ & - \tilde{A}_\mu(k_2')\tilde{A}_\nu(k_1') ik'_{\mu,3}\tilde{A}_\nu(k_3') e^{-\frac{i}{2}k_2'\epsilon\theta k_1' - \frac{i}{2}k_1'\epsilon\theta k_3' - \frac{i}{2}k_2'\epsilon\theta k_3'}\} \\ =& \frac{1}{(2\pi)^{12}} ig \int d^4k_1' \int d^4k_2' \int d^4k_3' \int d^4x e^{i(k_1'+k_2'+k_3')x} \tilde{A}_\nu(k_1')\tilde{A}_\mu(k_2') ik'_{\mu,3}\tilde{A}_\nu(k_3') \\ & \times e^{-\frac{i}{2}k_2'\epsilon\theta k_3' - \frac{i}{2}k_1'\epsilon\theta k_3'}[e^{-\frac{i}{2}k_1'\epsilon\theta k_2'} - e^{-\frac{i}{2}k_2'\epsilon\theta k_1'}]. \end{aligned}$$

The square brackets will give a sine function

$$\begin{aligned} [e^{-\frac{i}{2}k_1'\epsilon\theta k_2'} - e^{-\frac{i}{2}k_2'\epsilon\theta k_1'}] &= +[e^{-\frac{i}{2}k_1'\epsilon\theta k_2'} - e^{+\frac{i}{2}k_1'\epsilon\theta k_2'}] = -[e^{\frac{i}{2}k_1'\epsilon\theta k_2'} - e^{-\frac{i}{2}k_1'\epsilon\theta k_2'}] \\ &= -2i\sin[\frac{1}{2}k_1'\epsilon\theta k_2']. \end{aligned}$$

This means for S^{3A}_{inv}

$$\begin{aligned} S^{3A}_{inv} =& \frac{2}{(2\pi)^{12}} g \int d^4k_1' \int d^4k_2' \int d^4k_3' \int d^4x e^{i(k_1'+k_2'+k_3')x} \\ & \times \tilde{A}_\nu(k_1')\tilde{A}_\mu(k_2') ik'_{\mu,3}\tilde{A}_\nu(k_3') e^{-\frac{i}{2}k_2'\epsilon\theta k_3' - \frac{i}{2}k_1'\epsilon\theta k_3'} \sin[\frac{1}{2}k_1'\epsilon\theta k_2']. \end{aligned} \tag{4.84}$$

The evaluation of the coordinate space integral leads to the delta function which we can use to modify the exponential function

$$\begin{aligned} & \delta^4(k_1'+k_2'+k_3') \rightarrow k_2' = -k_1' - k_3', \\ & e^{-\frac{i}{2}k_2'\epsilon\theta k_3' - \frac{i}{2}k_1'\epsilon\theta k_3'}\delta^4(k_1'+k_2'+k_3') = e^{-\frac{i}{2}(-k_1'-k_3')\epsilon\theta k_3' - \frac{i}{2}k_1'\epsilon\theta k_3'}\delta^4(k_1'+k_2'+k_3') \\ & = e^{+\frac{i}{2}k_1'\epsilon\theta k_3' - \frac{i}{2}k_1'\epsilon\theta k_3'}\delta^4(k_1'+k_2'+k_3') = \delta^4(k_1'+k_2'+k_3'). \end{aligned}$$

Therefore we have

$$\begin{aligned} S^{3A}_{inv} =& \frac{2}{(2\pi)^8} g \int d^4k_1' \int d^4k_2' \int d^4k_3' \delta^4(k_1'+k_2'+k_3') \\ & \times \tilde{A}_\nu(k_1')\tilde{A}_\mu(k_2') ik'_{\mu,3}\tilde{A}_\nu(k_3') \sin[\frac{1}{2}k_1'\epsilon\theta k_2']. \end{aligned} \tag{4.85}$$

The Fourier transform $\tilde{V}^{3A}_{\alpha\beta\gamma}(k_1,k_2,k_3)$ is now given by

$$\begin{aligned} \tilde{V}^{3A}_{\alpha\beta\gamma}(k_1,k_2,k_3) &= -(2\pi)^{12}\frac{\delta}{\delta\tilde{A}_\alpha(-k_1)}\frac{\delta}{\delta\tilde{A}_\beta(-k_2)}\frac{\delta}{\delta\tilde{A}_\gamma(-k_3)}S^{3A}_{inv} \\ &= -2ig(2\pi)^4\delta^4(k_1+k_2+k_3)\sin[\frac{1}{2}k_1\epsilon\theta k_2]\times \\ & \{\delta_{\alpha\gamma}(k_1-k_3)_\beta + \delta_{\alpha\beta}(k_2-k_1)_\gamma + \delta_{\beta\gamma}(k_3-k_2)_\alpha\}. \end{aligned} \tag{4.86}$$

The detailed calculation can be found in Appendix C.2.1.

4.2.4 The Four-Photon Vertex $\tilde{V}^{4A}_{\alpha\beta\gamma\delta}$

The vertex $\tilde{V}^{4A}_{\alpha\beta\gamma\delta}$ comes from

$$S^{4A}_{inv} = \int d^4x \frac{1}{4}(-ig)^2[A_\mu, A_\nu][A_\mu, A_\nu] = \int d^4x \frac{1}{4}(-ig)^2(A_\mu A_\nu - A_\nu A_\mu)[A_\mu, A_\nu].$$

The second term will be renamed ($\mu \leftrightarrow \nu$):

$$-A_\nu A_\mu[A_\mu, A_\nu] = -A_\mu A_\nu[A_\nu, A_\mu] = +A_\mu A_\nu[A_\mu, A_\nu].$$

This leads to

$$\begin{aligned} S^{4A}_{inv} &= \int d^4x \frac{1}{2}(-ig)^2\{A_\mu A_\nu[A_\mu, A_\nu]\} = \int d^4x \frac{1}{2}(-ig)^2\{A_\mu A_\nu(A_\mu A_\nu - A_\nu A_\mu)\} \\ &= \int d^4x \frac{1}{2}(-ig)^2\{A_\mu A_\nu A_\mu A_\nu - A_\mu A_\nu A_\nu A_\mu\}. \end{aligned} \tag{4.87}$$

The star product (2.13) gives

$$\begin{aligned} S^{4A}_{inv} &= \frac{1}{2(2\pi)^{16}}(-ig)^2 \int d^4x \int d^4k'_1 \int d^4k'_2 \int d^4k'_3 \int d^4k'_4 e^{i(k'_1+k'_2+k'_3+k'_4)x} \\ &\times \{\tilde{A}_\mu(k'_1)\tilde{A}_\nu(k'_2)\tilde{A}_\mu(k'_3)\tilde{A}_\nu(k'_4)e^{-\frac{i}{2}k'_1\epsilon\theta k'_2 - \frac{i}{2}k'_1\epsilon\theta k'_3 - \frac{i}{2}k'_1\epsilon\theta k'_4 - \frac{i}{2}k'_2\epsilon\theta k'_3 - \frac{i}{2}k'_2\epsilon\theta k'_4 - \frac{i}{2}k'_3\epsilon\theta k'_4} \\ &\quad - \tilde{A}_\mu(k'_1)\tilde{A}_\nu(k'_2)\tilde{A}_\nu(k'_4)\tilde{A}_\mu(k'_3)e^{-\frac{i}{2}k'_1\epsilon\theta k'_2 - \frac{i}{2}k'_1\epsilon\theta k'_3 - \frac{i}{2}k'_1\epsilon\theta k'_4 - \frac{i}{2}k'_2\epsilon\theta k'_3 - \frac{i}{2}k'_2\epsilon\theta k'_4 - \frac{i}{2}k'_4\epsilon\theta k'_3}\} \\ &= \frac{1}{2(2\pi)^{16}}(-ig)^2 \int d^4x \int d^4k'_1 \int d^4k'_2 \int d^4k'_3 \int d^4k'_4 e^{i(k'_1+k'_2+k'_3+k'_4)x} \\ &\times \Big(\tilde{A}_\mu(k'_1)\tilde{A}_\nu(k'_2)\tilde{A}_\mu(k'_3)\tilde{A}_\nu(k'_4)e^{-\frac{i}{2}k'_1\epsilon\theta k'_2 - \frac{i}{2}k'_1\epsilon\theta k'_3 - \frac{i}{2}k'_1\epsilon\theta k'_4 - \frac{i}{2}k'_2\epsilon\theta k'_3 - \frac{i}{2}k'_2\epsilon\theta k'_4} \\ &\qquad \{e^{-\frac{i}{2}k'_3\epsilon\theta k'_4} - e^{-\frac{i}{2}k'_4\epsilon\theta k'_3}\}\Big). \end{aligned}$$

With

$$\{e^{-\frac{i}{2}k'_3\epsilon\theta k'_4} - e^{-\frac{i}{2}k'_4\epsilon\theta k'_3}\} = -\{e^{+\frac{i}{2}k'_3\epsilon\theta k'_4} - e^{-\frac{i}{2}k'_3\epsilon\theta k'_4}\} = -2i\sin[\frac{1}{2}k'_3\epsilon\theta k'_4],$$

we get

$$\begin{aligned} S^{4A}_{inv} &= \frac{1}{2(2\pi)^{16}}(-ig)^2 \int d^4x \int d^4k'_1 \int d^4k'_2 \int d^4k'_3 \int d^4k'_4 \times \\ &\times \tilde{A}_\mu(k'_1)\tilde{A}_\nu(k'_2)\tilde{A}_\mu(k'_3)\tilde{A}_\nu(k'_4)e^{i(k'_1+k'_2+k'_3+k'_4)x}e^{-\frac{i}{2}k'_1\epsilon\theta k'_2 - \frac{i}{2}k'_1\epsilon\theta k'_3 - \frac{i}{2}k'_1\epsilon\theta k'_4 - \frac{i}{2}k'_2\epsilon\theta k'_3 - \frac{i}{2}k'_2\epsilon\theta k'_4} \\ &\qquad \times (-2i)\sin[\frac{1}{2}k'_3\epsilon\theta k'_4] \\ &= \frac{1}{(2\pi)^{16}}ig^2 \int d^4x \int d^4k'_1 \int d^4k'_2 \int d^4k'_3 \int d^4k'_4 \times \\ &\times \tilde{A}_\mu(k'_1)\tilde{A}_\nu(k'_2)\tilde{A}_\mu(k'_3)\tilde{A}_\nu(k'_4)e^{i(k'_1+k'_2+k'_3+k'_4)x}e^{-\frac{i}{2}k'_1\epsilon\theta k'_2 - \frac{i}{2}k'_1\epsilon\theta k'_3 - \frac{i}{2}k'_1\epsilon\theta k'_4 - \frac{i}{2}k'_2\epsilon\theta k'_3 - \frac{i}{2}k'_2\epsilon\theta k'_4} \\ &\qquad \times \sin[\frac{1}{2}k'_3\epsilon\theta k'_4]. \end{aligned}$$

Integration of the space coordinate leads to

$$\begin{aligned}S^{4A}_{inv} = \frac{1}{(2\pi)^{12}} ig^2 \int d^4k'_1 \int d^4k'_2 \int d^4k'_3 \int d^4k'_4 \delta^4(k'_1 + k'_2 + k'_3 + k'_4) \\ \times \tilde{A}_\mu(k'_1)\tilde{A}_\nu(k'_2)\tilde{A}_\mu(k'_3)\tilde{A}_\nu(k'_4) e^{-\frac{i}{2}k'_1\epsilon\theta k'_2 - \frac{i}{2}k'_1\epsilon\theta k'_3 - \frac{i}{2}k'_1\epsilon\theta k'_4 - \frac{i}{2}k'_2\epsilon\theta k'_3 - \frac{i}{2}k'_2\epsilon\theta k'_4} \\ \times \sin[\frac{1}{2}k'_3\epsilon\theta k'_4]. \end{aligned} \tag{4.88}$$

Next we use the delta function to eliminate k'_3

$$k'_3 = -k'_1 - k'_2 - k'_4,$$

and hence get for the exponential function

$$\begin{aligned}&e^{-\frac{i}{2}k'_1\epsilon\theta k'_2 - \frac{i}{2}k'_1\epsilon\theta k'_3 - \frac{i}{2}k'_1\epsilon\theta k'_4 - \frac{i}{2}k'_2\epsilon\theta k'_3 - \frac{i}{2}k'_2\epsilon\theta k'_4}\delta^4(k'_1 + k'_2 + k'_3 + k'_4) = \\ &= e^{-\frac{i}{2}k'_1\epsilon\theta k'_2 - \frac{i}{2}k'_1\epsilon\theta(-k'_1-k'_2-k'_4) - \frac{i}{2}k'_1\epsilon\theta k'_4 - \frac{i}{2}k'_2\epsilon\theta(-k'_1-k'_2-k'_4) - \frac{i}{2}k'_2\epsilon\theta k'_4}\delta^4(k'_1 + k'_2 + k'_3 + k'_4) \\ &= e^{-\frac{i}{2}k'_1\epsilon\theta k'_2 + \frac{i}{2}k'_1\epsilon\theta k'_2 + \frac{i}{2}k'_1\epsilon\theta k'_4 - \frac{i}{2}k'_1\epsilon\theta k'_4 + \frac{i}{2}k'_2\epsilon\theta k'_1 + \frac{i}{2}k'_2\epsilon\theta k'_4 - \frac{i}{2}k'_2\epsilon\theta k'_4}\delta^4(k'_1 + k'_2 + k'_3 + k'_4) \\ &= e^{-\frac{i}{2}k'_1\epsilon\theta k'_2}\delta^4(k'_1 + k'_2 + k'_3 + k'_4).\end{aligned}$$

Finally we have

$$\begin{aligned}S^{4A}_{inv} =& \frac{1}{(2\pi)^{12}} ig^2 \int d^4k'_1 \int d^4k'_2 \int d^4k'_3 \int d^4k'_4 \delta^4(k'_1 + k'_2 + k'_3 + k'_4) \\ &\times \tilde{A}_\mu(k'_1)\tilde{A}_\nu(k'_2)\tilde{A}_\mu(k'_3)\tilde{A}_\nu(k'_4) e^{-\frac{i}{2}k'_1\epsilon\theta k'_2} \sin[\frac{1}{2}k'_3\epsilon\theta k'_4]. \end{aligned} \tag{4.89}$$

The calculation of $\tilde{V}^{4A}_{\alpha\beta\gamma\delta}$ gives

$$\begin{aligned}\tilde{V}^{4A}_{\alpha\beta\gamma\delta}(k_1, k_2, k_3, k_4) &= -(2\pi)^{16} \frac{\delta}{\delta\tilde{A}_\alpha(-k_1)} \frac{\delta}{\delta\tilde{A}_\beta(-k_2)} \frac{\delta}{\delta\tilde{A}_\gamma(-k_3)} \frac{\delta}{\delta\tilde{A}_\delta(-k_4)} S^{4A}_{inv} \\ &= (2\pi)^4 4g^2 \delta^4(k_1 + k_2 + k_3 + k_4) \times \\ &\Big\{ \quad (\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\beta}\delta_{\gamma\delta}) \sin[\frac{1}{2}k_1\epsilon\theta k_4] \sin[\frac{1}{2}k_2\epsilon\theta k_3] \\ &+ (\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta}) \sin[\frac{1}{2}k_1\epsilon\theta k_2] \sin[\frac{1}{2}k_3\epsilon\theta k_4] \\ &+ (\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}) \sin[\frac{1}{2}k_2\epsilon\theta k_4] \sin[\frac{1}{2}k_1\epsilon\theta k_3] \Big\}. \end{aligned} \tag{4.90}$$

The detailed calculation can be found in Appendix C.2.2.

4.3 Superficial Degree of Divergence d_γ

Before we start to calculate one-loop corrections to the propagators we must discuss the question whether the algebraic expression of a Feynman graph will diverge or not. Several factors need to be considered. Every integral over four dimensional space coordinates gives a contribution of four powers of k in the numerator and therefore raises the degree of divergence. Additionally, the two-ghost one-photon vertex and the three-photon vertex each rise the degree of divergence by one. On the contrary, propagators reduce the degree of divergence. The photon propagator and the ghost propagator each give a contribution of two powers of k to the denominator. In other words we only count the powers (=power counting) of k.

Table 3.2 shows the superficial degree of divergence for the relevant factors[4] of a given Feynman graph[5]. Therefore we can derive the so-called superficial degree of divergence d_γ for an amputated Feynman graph:

Factor in Graph	**Divergence d**
Loop integration	$+4$
$\tilde{V}^{3A}$	$+1$
$\tilde{V}^{4A}$	0
$\tilde{V}^{cA\bar{c}}$	$+1$
$\tilde{G}^{AA}$	-2
$\tilde{G}^{\bar{c}c}$	-2

Table 4.2: Superficial degree of divergence.

$$d_\gamma = 4L - 2I_A - 2I_{\bar{c}c} + V_{3A} + V_{cA\bar{c}}, \tag{4.91}$$

where L denotes the number of loop integrations, $I_A, I_{\bar{c}c}$ are the numbers of internal photon and ghost lines and $V_{3A}, V_{4A}, V_{cA\bar{c}}$ are the number of 3-photon, 4-photon and ghost-photon vertices, respectively. Since there is overall momentum conservation and momentum conservation at each vertex and I internal momenta, the number of independent momenta (represented by L) is given by

$$L = I_A + I_{\bar{c}c} - (V_{cA\bar{c}} + V_{3A} + V_{4A} - 1). \tag{4.92}$$

Finally we need a relation between the number of vertices and the number of external lines. External legs are denoted by $E_{\bar{c}c}, E_A$ for the external ghost and photon lines, respectively. External lines count once whereas internal lines count twice because internal lines are always connected to two vertices. Therefore we have the relations

$$\begin{aligned} E_{\bar{c}c} + 2I_{\bar{c}c} &= 2V_{cA\bar{c}}, \\ E_A + 2I_A &= 3V_{3A} + 4V_{4A} + V_{cA\bar{c}}. \end{aligned} \tag{4.93}$$

In order to get an expression for d_γ without internal lines we eliminate those internal lines.

$$\begin{aligned} d_\gamma &= 4L - 2I_A - 2I_{\bar{c}c} + V_{3A} + V_{cA\bar{c}} \\ &= 4(I_A + I_{\bar{c}c} - (V_{cA\bar{c}} + V_{3A} + V_{4A} - 1)) - 2I_A - 2I_{\bar{c}c} + V_{3A} + V_{cA\bar{c}} \\ &= 2I_a + 2I_{\bar{c}c} - 3V_{cA\bar{c}} - 3V_{3A} - 4V_{4A} + 4 \\ &= (3V_{3A} + 4V_{4A} + V_{cA\bar{c}} - E_A) + (2V_{cA\bar{c}} - E_{\bar{c}c}) - 3V_{cA\bar{c}} - 3V_{3A} - 4V_{4A} + 4 \\ &= 4 - E_A - E_{\bar{c}c}. \end{aligned} \tag{4.94}$$

[4]The relavant factors are algebraic expressions of propagators and vertices, usually stated as Feynman rules.

[5]Feynman graphs are explained in the next chapter.

Note that d_γ is an upper limit for the strength of the divergence. The real divergence of a graph can at most be d_γ. The following table shows d_γ for our arising graphs (only one-loop graphs). Feynman graphs with $d_\gamma = 0, 1, 2, , ...$ are called logarithmic,

Graph	E_A	$E_{\bar{c}c}$	**Divergence** d_γ
Ghost-tadpole	1	0	3
Photon-tadpole	1	0	3
Ghost-loop	2	0	2
Photon-loop	2	0	2
2pt.-Photon-tadpole	2	0	2

Table 4.3: Superficial degree of divergence for Feynman graphs.

linear, quadratic, ... UV divergent.

Chapter 5

One-loop Calculations

This chapter is dedicated to the one-loop correction in the Feynman gauge. We have all the ingredients to do loop calculations. The main difference of the Feynman rules in noncommutative field theory compared to the ones of classical field theory is the existence of additional phases. In this work (BRSW model), the additional phases are sine terms. These terms will only appear in vertices and naturally will enter the Feynman amplitudes of the one-loop graphs. As we will see the sine terms can be written as exponentials and divided into a so called planar and non planar part.

The rules for associating analytic expressions to pieces of diagrams are called Feynman rules. These rules are now the building blocks for Feynman graphs. Since $G(x, y) = G(y, x)$, the following propagators are translation invariant, the direction of the momentum is arbitrary while the momentum at each vertex is drawn inwards according to the momentum conservation represented by positive arguments of the delta functions.

The BRSW model gives rise to two one-point one-loop and three two-point one-loop graphs. Figure 5.1 at the next page shows schematically all possible graphs which contribute to the self-energy of the photon propagator.

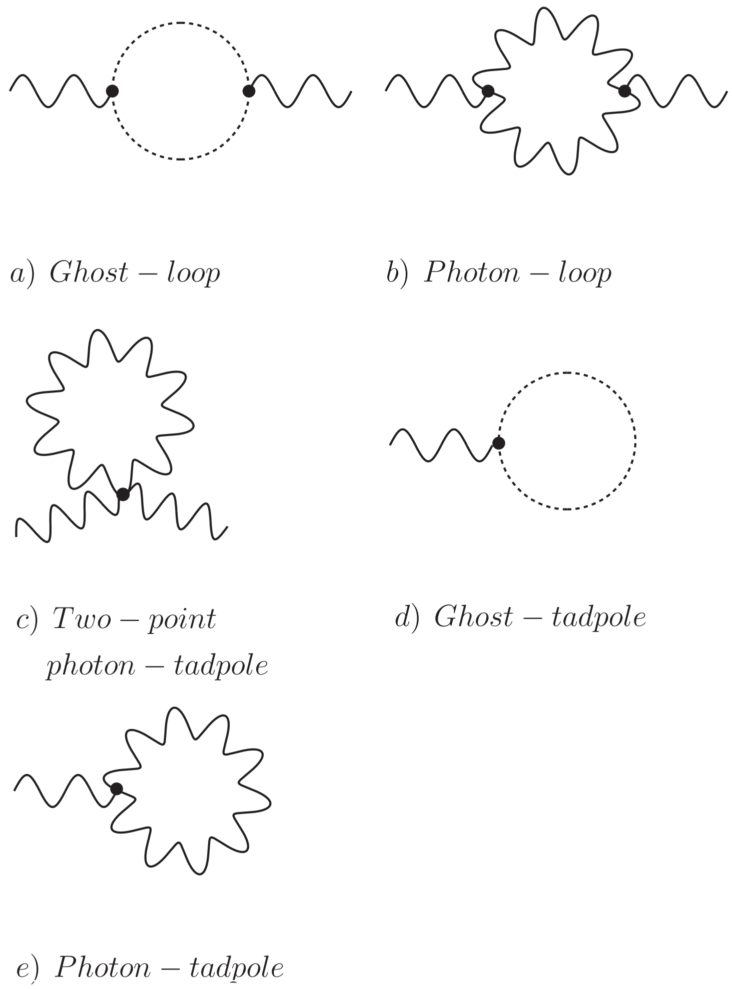
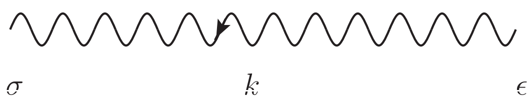


Figure 5.1: One-loop Feynman graphs of the BRSW model.

5.1 Feynman Rules

This section will list all necessary Feynman rules and their corresponding analytic expressions. We start in order of appearance up to now[1].

The photon propagator $\tilde{G}^{AA}_{\sigma\epsilon}$:



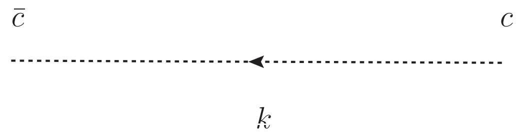
$= \tilde{G}^{AA}_{\sigma\epsilon}(k)$

Figure 5.2: Photon propagator.

with ($\alpha \to 0$)

$$\tilde{G}^{AA}_{\sigma\epsilon}(k) = \frac{1}{[1+\frac{\gamma^4}{\tilde{k}^4}]k^2}\left[\delta_{\sigma\epsilon} - \frac{k_\sigma k_\epsilon}{k^2} - \frac{\bar{\sigma}^4}{[(\tilde{k}^2+\frac{\gamma^4}{\tilde{k}^2})k^2+\bar{\sigma}^4]}\frac{\tilde{k}_\sigma \tilde{k}_\epsilon}{\tilde{k}^2}\right]. \tag{5.1}$$

The ghost propagator $\tilde{G}^{\bar{c}c}$:



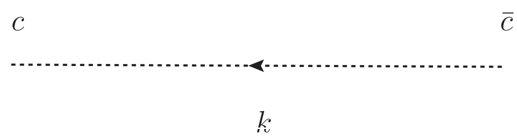
$= \tilde{G}^{\bar{c}c}(k)$

Figure 5.3: Ghost propagator.

with

$$\tilde{G}^{\bar{c}c}(k) = -\frac{1}{k^2}. \tag{5.2}$$

The ghost propagator $\tilde{G}^{c\bar{c}}$:



$= \tilde{G}^{c\bar{c}}(k)$

Figure 5.4: Ghost propagator.

with

$$\tilde{G}^{c\bar{c}}(k) = \frac{1}{k^2}. \tag{5.3}$$

[1]The following expressions equal Eqs. (4.49, 4.59, 4.62, 4.82, 4.86) and (4.90) from the previous chapter.

The two-ghost one-photon vertex $\tilde{V}_{\mu}^{cA\bar{c}}$:

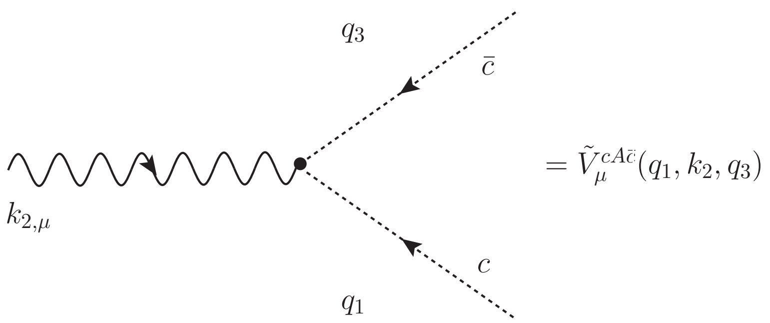


Figure 5.5: Two-ghost one-photon vertex.

with

$$\tilde{V}_{\mu}^{cA\bar{c}}(q_1, k_2, q_3) = -2ig(2\pi)^4\delta^4(q_1 + k_2 + q_3)q_{\mu,3}\sin\big[\frac{1}{2}q_1\epsilon\theta q_3\big]. \tag{5.4}$$

The three-photon vertex $\tilde{V}_{\alpha\beta\gamma}^{3A}$:

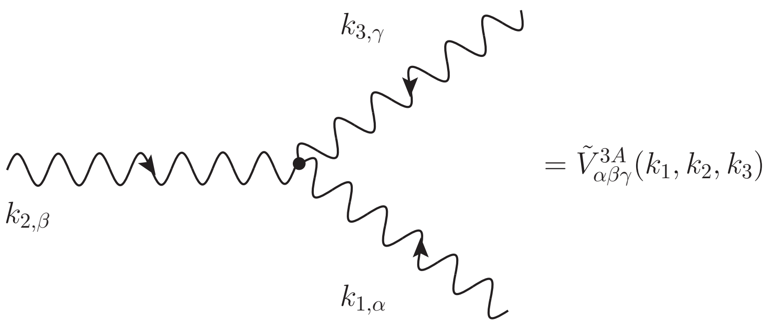


Figure 5.6: Three-photon vertex.

with

$$\begin{aligned}\tilde{V}_{\alpha\beta\gamma}^{3A}(k_1, k_2, k_3) = -2ig(2\pi)^4\delta^4(k_1 + k_2 + k_3)\sin\big[\frac{1}{2}k_1\epsilon\theta k_2\big]\times \\ \times\Big\{\delta_{\alpha\gamma}(k_1 - k_3)_\beta + \delta_{\alpha\beta}(k_2 - k_1)_\gamma + \delta_{\beta\gamma}(k_3 - k_2)_\alpha\Big\}.\end{aligned} \tag{5.5}$$

The four-photon vertex $\tilde{V}^{4A}_{\alpha\beta\gamma\delta}$:

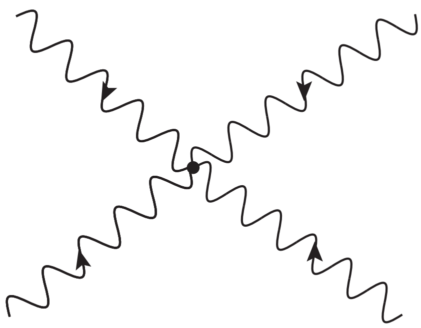


Figure 5.7: Four-photon vertex.

with

$$\begin{aligned}
\tilde{V}^{4A}_{\alpha\beta\gamma\delta}(k_1, k_2, k_3, k_4) = {} & (2\pi)^4 4g^2 \delta^4(k_1 + k_2 + k_3 + k_4)\times \\
& \times \Big\{ \quad (\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\beta}\delta_{\gamma\delta}) \sin\big[\frac{1}{2}k_1\epsilon\theta k_4\big] \sin\big[\frac{1}{2}k_2\epsilon\theta k_3\big] \\
& \quad + (\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta}) \sin\big[\frac{1}{2}k_1\epsilon\theta k_2\big] \sin\big[\frac{1}{2}k_3\epsilon\theta k_4\big] \\
& \quad + (\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}) \sin\big[\frac{1}{2}k_2\epsilon\theta k_4\big] \sin\big[\frac{1}{2}k_1\epsilon\theta k_3\big] \Big\}.
\end{aligned} \tag{5.6}$$

All Feynman graphs given by Fig. 5.1 can be generated through those rules.

5.2 Combinatoric Factor C

There are several possibilities to combine the Feynman rules which all should lead to a given Feynman graph. For example we could have calculated the Vertex $\tilde{V}^{\bar{c}Ac}$ in the way of

$$\begin{aligned}
\tilde{V}^{\bar{c}Ac} &= -(2\pi)^{12} \frac{\delta^3 S^{cA\bar{c}}}{\delta\tilde{\bar{c}}(-k_1)\delta\tilde{A}_\mu(-k_2)\delta\tilde{c}(-k_3)} \\
&= -2ig(2\pi)^4\delta^4(q_1 + k_2 + q_3)q_{\mu,1}\sin[\frac{1}{2}q_1\epsilon\theta q_3],
\end{aligned}$$

differentiate the vertex $\tilde{V}^{4A}_{\alpha\beta\gamma\delta}(k_1, k_2, k_3, k_4)$ in such a way that we get a vertex denoted by $\tilde{V}^{4A}_{\beta\delta\alpha\gamma}(k_1, k_4, k_3, k_2)$ or use the propagator $\tilde{G}^{c\bar{c}}(k)$ instead of $\tilde{G}^{\bar{c}c}(k)$, let the momentum point outwards/inwards and connect the propagator to the vertex in multiple ways.

We pick out one vertex and one propagator, draw the Feynman graph and multiply it with the combinatoric factor C. This factor pays attention to the way of connecting vertices with propagators and is defined as:

$$C_i := \frac{M_i}{S_i}, \qquad i = a, b, c, d, e, \tag{5.7}$$

where a, b, c, d, e denotes the possible Feynman graphs given in the introduction of this chapter, M the multiplicity factor and S the symmetry factor. In order to get the multiplicity factor and the symmetry factor, we give the procedure which leads to the right combinatoric factor.

Step 1:

Start with a given graph. Separate all elements of this graph and draw them in the right position with respect to the graph we started from. This leads to a so called pre-graph.

Step 2:

Count the number of ways the first vertex (we start from the left side) can be connected to the external legs. Now one external leg is connected to the first vertex.

Step 3:

Count the number of ways the next external leg can be connected to the remaining vertices. Connect this leg to the vertex and change to the next external leg. Repeat this until all external legs are connected to vertices.

Step 4:

Take a free leg of a vertex and count the ways to connect it to other vertices. Do this until all internal legs are connected.

The product of all possibilities from step one up to step four gives the multiplicity M.

To get the symmetry factor S we take a closer look at the inner symmetry of the Vertices.

Step 5:

For s identical vertices, we get a factor $s!$ for S.

Step 6:

For a given vertex denoted by the number of $s_{\psi_1}, s_{\psi_2}, ..., s_{\psi_n}$ different fields $\psi_1, \psi_2, ..., \psi_n$ we have a factor $s_{\psi_1}!s_{\psi_2}!...s_{\psi_n}!$ for each vertex.

The product of step five and step six gives the symmetry factor S.

The formal notation of Eq. (5.7) can now be written as

$$C_i = \frac{M_i}{S_i} = \frac{M_i}{[(s!) \times (s_{\psi_1}!s_{\psi_2}!...s_{\psi_n}!)]_i}.$$

To demonstrate this procedure we use the photon-loop as an example. The Feynman graph is given by Fig. 5.8.

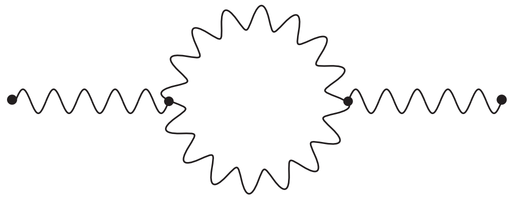


Figure 5.8: Photon-loop with external legs.

Step 1:

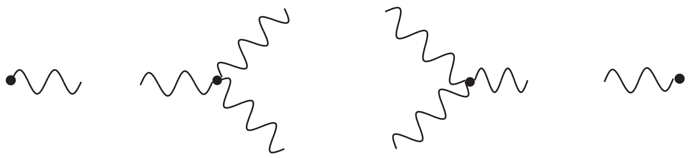


Figure 5.9: Pre-photon-loop.

Step 2:

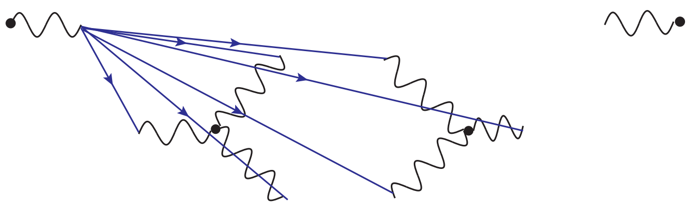


Figure 5.10: All possible connections for the left external line.

Step 3:

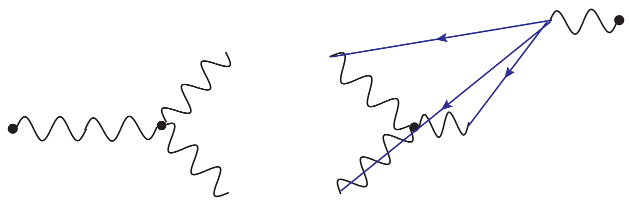


Figure 5.11: All possible connections for the right external line.

Step four is now divided into step 4a and step 4b to make things more transparent.

Step 4a:

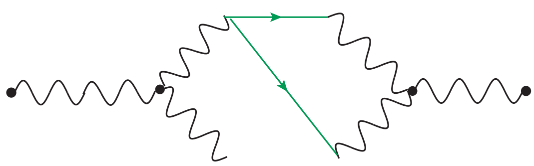


Figure 5.12: All possible connections for one of the left internal lines.

Step 4b:

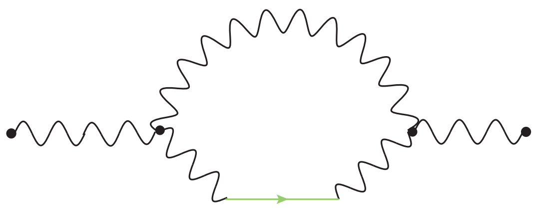


Figure 5.13: All possible connections for the last left internal line.

Step 5:

We have two identical vertices which give a factor 2! for the symmetry factor S.

Step 6:

Each vertex is made up by three identical fields and hence leads to a factor $3! \times 3!$ for S.

Therefore we get

$$C_b = \frac{M_b}{S_b} = \frac{(6 \times 3)^P \times (2 \times 1)^V}{(2!) \times (3! \times 3!)} = \frac{1}{2},$$

where the superscript P, V shall denote the relevant propagator to vertex and vertex to vertex connection.

This procedure is valid for all Feynman graphs in this work. The following table shows each combinatoric factor for a given graph. Before we start the explicit calculation a

Feynman Graph	**M**	**S**	**C**
Ghost-tadpole	$1^P \times 1^V$	$1!$	$C_d = 1$
Photon-tadpole	$3^P \times 1^V$	$3!$	$C_e = \frac{1}{2}$
Ghost-loop	$(2 \times 1)^P \times (1 \times 1)^V$	$2!$	$C_a = 1$
Photon-loop	$(6 \times 3)^P \times (2 \times 1)^V$	$2! \times 3! \times 3!$	$C_b = \frac{1}{2}$
2pt.-Photon-tadpole	$(4 \times 3)^P \times 1^V$	$1! \times 4!$	$C_c = \frac{1}{2}$

Table 5.1: Combinatoric factors.

few words about the notation concerning Feynman graphs. The figure representing a Feynman graph is drawn with the incoming external momentum from the right side, the inner momentum is drawn clockwise and finally the external momentum goes out at the left side[2]. Contrary the explicit propagator and vertex calculations which represent the factors of a given Feynman graph start from the left side, follow the external momentum until it arrives at the left vertex and follow the internal momentum via propagator clockwise to the right vertex[3] and close the internal loop via propagator clockwise. We must underline that this notation is only convention.

Three important points are left to mention:

i) Feynman graphs with a closed ghost loop line will receive an extra overall minus sign.

ii) The following calculations are done with amputated external legs.

iii) It will turn out to be wise not to insert the full analytical expressions for the needed photon-propagators.

5.3 One-Point Loops

We start with the calculation of the one-point tadpoles. As we will see, all those tadpoles will vanish due to momentum conservation. Note that the external momentum is drawn inwards.

[2] A special case are the one-point tadpoles, where the external momentum is drawn from the left side.

[3] Obviously the one-point tadpoles have no second vertex. The internal momentum goes straight back via propagator to the first vertex.

5.3.1 Ghost-Tadpole

The first one-point tadpole we consider is the ghost-tadpole.

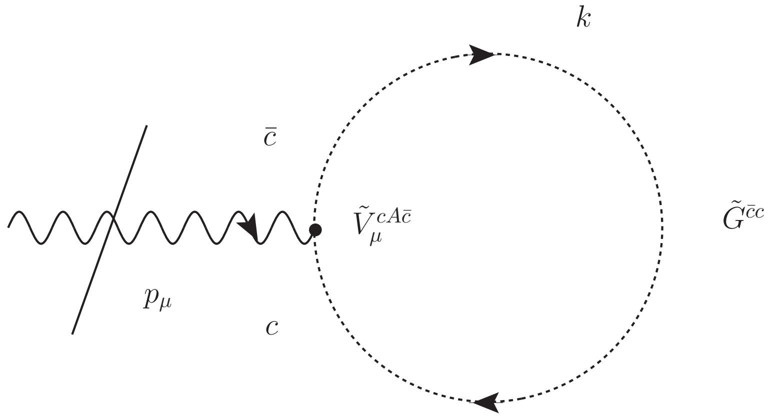


Figure 5.14: Ghost-tadpole.

Refering to Fig. 5.14 the expression for $\tilde{\Pi}_\mu^d(p)$ is given by:

$$\tilde{\Pi}_\mu^d(p) = -C_d \frac{1}{(2\pi)^4} \int d^4k \tilde{V}_\mu^{cA\bar{c}}(k,p) \tilde{G}^{\bar{c}c}(k). \tag{5.8}$$

The building blocks for the depicted graph are given by Eqs. (5.2, 5.4) and denoted with the right momentum description[4]:

$$\tilde{V}_\mu^{cA\bar{c}}(k,p) = -2ig(2\pi)^4 \delta^4(k+p-k)(-k_\mu) \sin\big[\frac{1}{2} k\epsilon\theta(-k)\big],$$

and

$$\tilde{G}^{\bar{c}c}(k) = -\frac{1}{(-k)^2} = -\frac{1}{k^2}.$$

Therefore we have

$$\begin{aligned}
&\tilde{\Pi}_\mu^d(p) = \\
&= -C_d \frac{1}{(2\pi)^4} \int d^4k \Big\{ -2ig(2\pi)^4 \delta^4(k+p-k)(-k_\mu) \sin\big[\frac{1}{2} k\epsilon\theta(-k)\big] \Big(-\frac{1}{k^2} \Big) \Big\} \\
&= -C_d(2ig) \int d^4k \Big\{ \delta^4(k+p-k) k_\mu \sin\big[\frac{1}{2} k\epsilon\theta k\big] \frac{1}{k^2} \Big\}.
\end{aligned} \tag{5.9}$$

The last equation shows the correlation of external and internal momentum in the sense of:

$$\delta^4(k+p-k) \Rightarrow p = k-k = 0, \qquad \sin[\frac{1}{2} k\epsilon\theta k] = 0,$$

[4]The direction of the momentum will not change $\tilde{G}^{AA}(k)$ and $\tilde{G}^{\bar{c}c}(k)$.

hence the momentum conservation is the reason of a vanishing ghost-tadpole

$$\tilde{\Pi}^d_\mu(p) = 0. \tag{5.10}$$

5.3.2 Photon-Tadpole

The calculation of the photon tadpole is done in the same way as the previous tadpole.

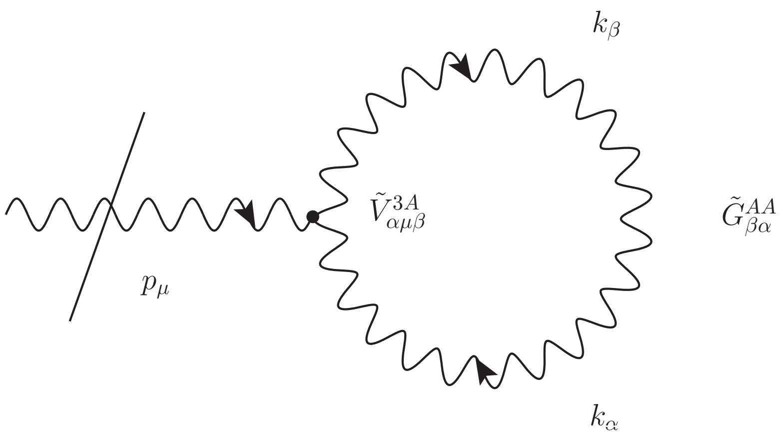


Figure 5.15: Photon-tadpole.

The required Feynman rules are represented by Eqs. (5.1) and (5.5). We connect the first leg $k_{1,\alpha}$ and third leg $k_{3,\beta}$, hence receive $k_1 = k_3 \equiv k$ according to momentum conservation. Considering Fig. 5.15, the vertex and propagator are given by

$$\begin{aligned}\tilde{V}^{3A}_{\alpha\mu\beta}(k,p) =& -2ig(2\pi)^4\delta^4(k+p-k)\sin\big[\frac{1}{2}k\epsilon\theta p\big]\times\\ &\times\big\{\delta_{\alpha\gamma}(k+k)_\beta + \delta_{\alpha\beta}(p-k)_\gamma + \delta_{\beta\gamma}(-k-p)_\alpha\big\},\end{aligned}$$

and

$$\tilde{G}^{AA}_{\beta\alpha}(-k) = \tilde{G}^{AA}_{\beta\alpha}(k),$$

where we use the fact that the photon propagator, represented by Eq. (5.1) is symmetric.

This leads to

$$\begin{aligned}\tilde{\Pi}^e_\mu(p) =& C_e\frac{1}{(2\pi)^4}\int d^4k\tilde{V}^{3A}_{\alpha\mu\beta}(k,p)\tilde{G}^{AA}_{\beta\alpha}(k)\times\\ =& -C_e(2ig)\int d^4k\Big\{\delta^4(k+p-k)\sin\big[\frac{1}{2}k\epsilon\theta p\big]\tilde{G}^{AA}_{\beta\alpha}(k)\\ &\times\big(\delta_{\alpha\gamma}(2k)_\beta + \delta_{\alpha\beta}(p-k)_\gamma + \delta_{\beta\gamma}(-k-p)_\alpha\big)\Big\}.\end{aligned} \tag{5.11}$$

The last equation implies

$$\delta^4(k+p-k) \Rightarrow p = k-k = 0, \qquad \sin\left[\frac{1}{2}k\epsilon\theta p\right] = 0,$$

which gives

$$\tilde{\Pi}^e_\mu(p) = 0. \tag{5.12}$$

We conclude:

All tadpoles of our model vanish due to momentum conservation.

5.4 Two-Point Loops

5.4.1 Ghost-Loop

As usual, the first two-point loop will be calculated in detail. To get rid of the delta functions inherent in the vertex expressions we introduce a so-called auxiliary[5] Feynman graph.

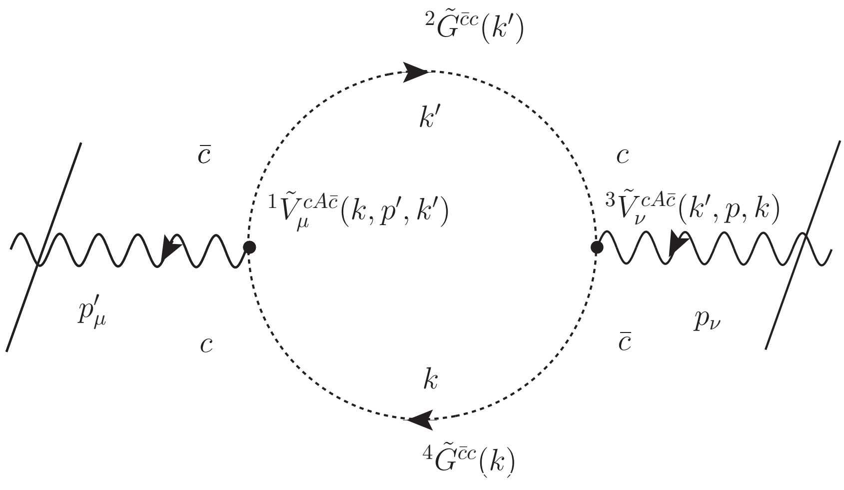


Figure 5.16: Auxiliary ghost-loop graph.

[5]The descriptions *auxiliary* and *final* stand for explicit expressions with/without delta functions (auxiliary/final) representing the Feynman graphs.

According to Fig. 5.16, the necessary building blocks arise from Eqs. (5.2) and (5.4):

$$\begin{aligned}
{}^1\tilde{V}^{cA\bar{c}}_{\mu}(k,p',k') &= (-2ig)(2\pi)^4\delta^4(k-p'-k')(-k'_{\mu})\sin\big[\frac{1}{2}k\epsilon\theta(-k')\big],\\
{}^2\tilde{G}^{\bar{c}c}(k') &= -\frac{1}{(-k')^2} = -\frac{1}{k'^2},\\
{}^3\tilde{V}^{cA\bar{c}}_{\nu}(k',p,k) &= (-2ig)(2\pi)^4\delta^4(k'+p-k)(-k_{\nu})\sin\big[\frac{1}{2}k'\epsilon\theta(-k)\big],\\
{}^4\tilde{G}^{\bar{c}c}(k) &= -\frac{1}{(-k)^2} = -\frac{1}{k^2}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\tilde{\Pi}^a_{\mu\nu}(p) =\\
&= -C_a\frac{1}{(2\pi)^{12}}\int d^4k\int d^4k'\int d^4p'\times\\
&\times\Big\{{}^1\tilde{V}^{cA\bar{c}}_{\mu}(k,p',k')^2\tilde{G}^{\bar{c}c}(k')^3\tilde{V}^{cA\bar{c}}_{\nu}(k',p,k)^4\tilde{G}^{\bar{c}c}(k)\Big\}\\
&= -C_a\frac{1}{(2\pi)^{12}}\int d^4k\int d^4k'\int d^4p'\times\\
&\Big\{(-2ig)(2\pi)^4\delta^4(k-p'-k')(-k'_{\mu})\sin\big[\frac{1}{2}k\epsilon\theta(-k')\big](-\frac{1}{k'^2})\\
&\quad\times(-2ig)(2\pi)^4\delta^4(k'+p-k)(-k_{\nu})\sin\big[\frac{1}{2}k'\epsilon\theta(-k)\big](-\frac{1}{k^2})\Big\}\\
&= -C_a\frac{1}{(2\pi)^4}\int d^4k\int d^4k'\int d^4p'\times\\
&\Big\{(2ig)^2\delta^4(k-p'-k')\delta^4(k'+p-k)\frac{k'_{\mu}k_{\nu}}{k'^2k^2}\sin\big[\frac{1}{2}k\epsilon\theta k'\big]\sin[\frac{1}{2}k'\epsilon\theta k]\Big\}\\
&= -C_a\frac{1}{(2\pi)^4}\int d^4k\int d^4k'\delta^4(k'+p-k)\Big\{-(2ig)^2\frac{k'_{\mu}k_{\nu}}{k'^2k^2}\sin^2\big[\frac{1}{2}k\epsilon\theta k'\big]\Big\}\\
&= -C_a\frac{4g^2}{(2\pi)^4}\int d^4k\Big\{\frac{(k-p)_{\mu}k_{\nu}}{(k-p)^2k^2}\sin^2\big[\frac{1}{2}k\epsilon\theta p\big]\Big\}.
\end{aligned}\tag{5.13}$$

If we now perform the variable shift

$$k\to k+\frac{p}{2},$$

it will not bother the sine function since

$$\sin^2\big[\frac{1}{2}(k+\frac{p}{2})\epsilon\theta p\big] = \sin^2\big[\frac{1}{2}k\epsilon\theta p\big],$$

and hence we get a symmetric expression given by

$$\tilde{\Pi}^a_{\mu\nu}(p) = -C_a\frac{4g^2}{(2\pi)^4}\int d^4k\Big\{\frac{(k-\frac{p}{2})_{\mu}(k+\frac{p}{2})_{\nu}}{(k-\frac{p}{2})^2(k+\frac{p}{2})^2}\sin^2\big[\frac{1}{2}k\epsilon\theta p\big]\Big\}.\tag{5.14}$$

The last equation may be interpreted as the following so-called final ghost-loop graph (Fig. 5.17).

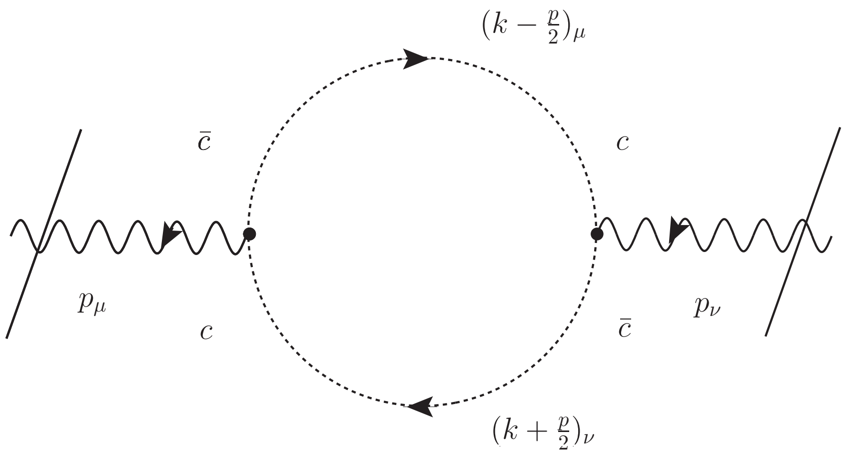


Figure 5.17: Final ghost-loop graph.

5.4.2 Photon-Loop

The following calculations belong to the photon-loop.

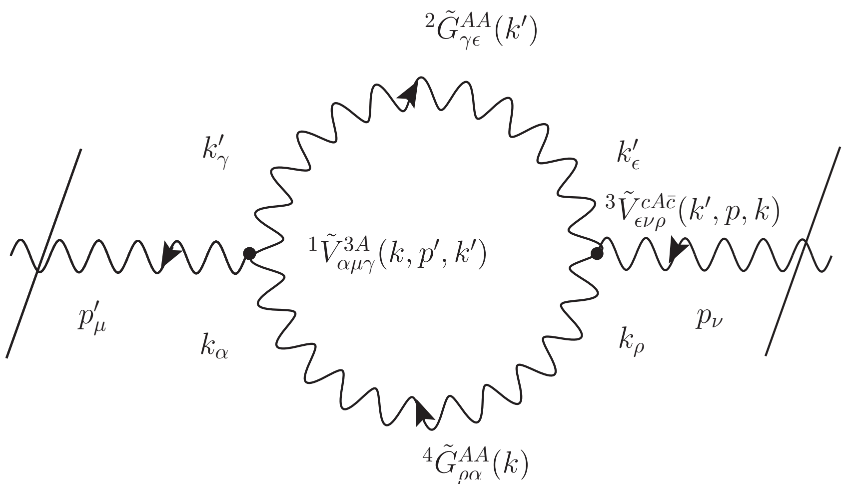


Figure 5.18: Auxiliary photon-loop graph.

Figure 5.18 equals the analytic expression given by

$$\tilde{\Pi}^b_{\mu\nu}(p) = C_b \frac{1}{(2\pi)^{12}} \int d^4k \int d^4k' \int d^4p' \times \tag{5.15}$$
$$\times \left\{ {}^1\tilde{V}^{3A}_{\alpha\mu\gamma}(k,p',k')\, {}^2\tilde{G}^{AA}_{\gamma\epsilon}(k')\, {}^3\tilde{V}^{3A}_{\epsilon\nu\rho}(k',p,k)\, {}^4\tilde{G}^{AA}_{\rho\alpha}(k) \right\},$$

which is with the right modification of (5.1, 5.5):

$$\begin{aligned}
{}^{1}\tilde{V}^{3A}_{\alpha\mu\gamma}(k,p',k') &= -2ig(2\pi)^4\delta^4(k-p'-k')\sin\big[\frac{1}{2}k\epsilon\theta(-p')\big]\times \\
&\times\left(\delta_{\alpha\gamma}(k+k')_\mu+\delta_{\alpha\mu}(-p'-k)_\gamma+\delta_{\mu\gamma}(-k'+p')_\alpha\right), \\
{}^{2}\tilde{G}^{AA}_{\gamma\epsilon}(k'), \\
{}^{3}\tilde{V}^{3A}_{\epsilon\nu\rho}(k',p,k) &= -2ig(2\pi)^4\delta^4(k'+p-k)\sin\big[\frac{1}{2}k'\epsilon\theta p\big]\times \\
&\times\left(\delta_{\epsilon\rho}(k'+k)_\nu+\delta_{\epsilon\nu}(p-k')_\rho+\delta_{\nu\rho}(-k-p)_\epsilon\right), \\
{}^{4}\tilde{G}^{AA}_{\rho\alpha}(k).
\end{aligned}$$

If we now use the last two vertex identities and use the delta functions, we receive:

$$\begin{aligned}
\tilde{\Pi}^b_{\mu\nu}(p) &= C_b\frac{4g^2}{(2\pi)^4}\int d^4k\sin^2\big[\frac{1}{2}k\epsilon\theta p\big]\tilde{G}^{AA}_{\gamma\epsilon}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k) \\
&\times\Big\{\big(\delta_{\alpha\gamma}(2k-p)_\mu+\delta_{\alpha\mu}(-k-p)_\gamma+\delta_{\mu\gamma}(2p-k)_\alpha\big) \\
&\quad\times\big(\delta_{\epsilon\rho}(2k-p)_\nu+\delta_{\epsilon\nu}(2p-k)_\rho+\delta_{\nu\rho}(-k-p)_\epsilon\big)\Big\}.
\end{aligned} \tag{5.16}$$

With a variable shift $k\rightarrow k+\frac{p}{2}$, we get the expression

$$\tilde{\Pi}^b_{\mu\nu}(p) = C_b\frac{4g^2}{(2\pi)^4}\int d^4k\mathcal{A}^b_{\mu\nu,\gamma\epsilon\rho\alpha}(k,p)\tilde{G}_{\gamma\epsilon}(k-\frac{p}{2})\tilde{G}_{\rho\alpha}(k+\frac{p}{2}), \tag{5.17}$$

where $\mathcal{A}^b_{\mu\nu,\gamma\epsilon\rho\alpha}(k,p)$ is given by

$$\begin{aligned}
&\mathcal{A}^b_{\mu\nu,\gamma\epsilon\rho\alpha}(k,p)\equiv \\
&\equiv\sin^2\big[\frac{1}{2}k\epsilon\theta p\big]\Big\{\big(2\delta_{\alpha\gamma}k_\mu+\delta_{\alpha\mu}(-\frac{3}{2}p-k)_\gamma+\delta_{\mu\gamma}(\frac{3}{2}p-k)_\alpha\big) \\
&\qquad\times\big(2\delta_{\epsilon\rho}k_\nu+\delta_{\epsilon\nu}(\frac{3}{2}p-k)_\rho+\delta_{\nu\rho}(-\frac{3}{2}p-k)_\epsilon\big)\Big\}.
\end{aligned} \tag{5.18}$$

The following graph may be interpreted as the graphical expression of Eq. (5.17):

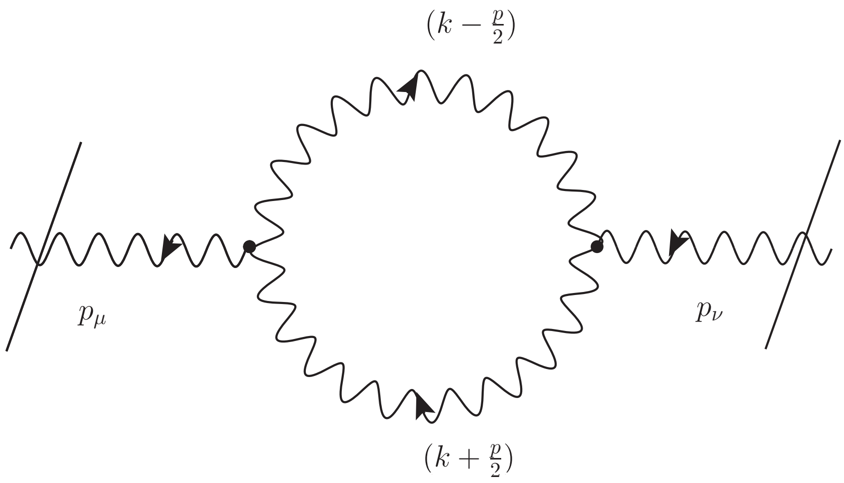


Figure 5.19: Final photon-loop graph.

5.4.3 Two-Point Photon-Tadpole

The next graph is the two-point photon-tadpole.

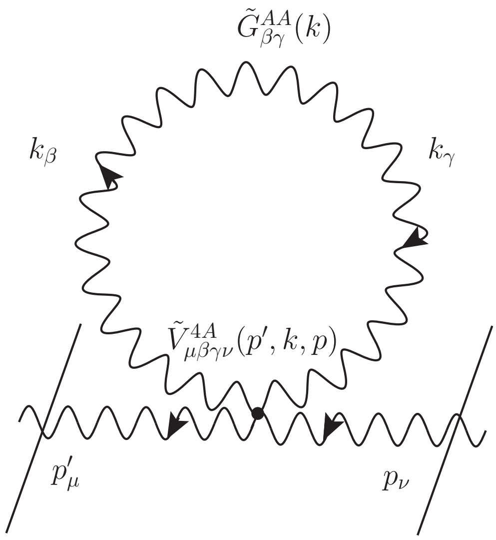


Figure 5.20: Auxiliary two-point photon-tadpole graph.

The Feynman rules (5.1, 5.6), modified according to the depicted graph given by Fig.

5.20 read:

$$\begin{aligned}\tilde{V}^{4A}_{\mu\beta\gamma\nu}(p',k,p) = 4g^2(2\pi)^4\delta^4(-p'-k+k+p)\times \\ \times\Big\{(\delta_{\mu\gamma}\delta_{\beta\nu}-\delta_{\mu\beta}\delta_{\gamma\nu})\sin\big[\frac{1}{2}(-p')\epsilon\theta p\big]\sin\big[\frac{1}{2}(-k)\epsilon\theta k\big] \\ +(\delta_{\mu\nu}\delta_{\beta\gamma}-\delta_{\mu\gamma}\delta_{\beta\nu})\sin\big[\frac{1}{2}(-p')\epsilon\theta(-k)\big]\sin\big[\frac{1}{2}k\epsilon\theta p\big] \\ +(\delta_{\mu\nu}\delta_{\beta\gamma}-\delta_{\mu\beta}\delta_{\gamma\nu})\sin\big[\frac{1}{2}(-k)\epsilon\theta p\big]\sin\big[\frac{1}{2}(-p')\epsilon\theta k\big]\Big\},\end{aligned}$$

and

$$\tilde{G}^{AA}_{\beta\gamma}(k).$$

Therefore, we have

$$\begin{aligned}\tilde{\Pi}^c_{\mu\nu}(p) &= C_c\frac{1}{(2\pi)^8}\int d^4k\int d^4p'\Big\{\tilde{V}^{4A}_{\mu\beta\gamma\nu}(p',k,p)\tilde{G}^{AA}_{\beta\gamma}(k)\Big\} \\ &= C_c\frac{1}{(2\pi)^4}\int d^4k\int d^4p' 4g^2\delta^4(-p'-k+k+p)\tilde{G}^{AA}_{\beta\gamma}(k)\times \\ &\times\Big\{(\delta_{\mu\nu}\delta_{\beta\gamma}-\delta_{\mu\gamma}\delta_{\beta\nu})\sin\big[\frac{1}{2}p'\epsilon\theta k\big]\sin\big[\frac{1}{2}k\epsilon\theta p\big] \\ &+(\delta_{\mu\nu}\delta_{\beta\gamma}-\delta_{\mu\beta}\delta_{\gamma\nu})\sin\big[\frac{1}{2}k\epsilon\theta p\big]\sin\big[\frac{1}{2}p'\epsilon\theta k\big]\Big\}.\end{aligned}$$

The delta function $\delta^4(-p'-k+k+p)$ with $p'=p$ solves the integral over p' and hence we have

$$\begin{aligned}\tilde{\Pi}^c_{\mu\nu}(p) &= -C_c\frac{4g^2}{(2\pi)^4}\int d^4k\sin^2\big[\frac{1}{2}k\epsilon\theta p\big]\times \\ &\times\Big\{(\delta_{\mu\nu}\delta_{\beta\gamma}-\delta_{\mu\gamma}\delta_{\beta\nu})+(\delta_{\mu\nu}\delta_{\beta\gamma}-\delta_{\mu\beta}\delta_{\gamma\nu})\Big\}\tilde{G}^{AA}_{\beta\gamma}(k) \\ &= -C_c\frac{4g^2}{(2\pi)^4}\int d^4k\mathcal{A}^c_{\mu\nu,\beta\gamma}(k,p)\tilde{G}^{AA}_{\beta\gamma}(k),\end{aligned} \tag{5.19}$$

while $\mathcal{A}^c_{\mu\nu,\beta\gamma}(k,p)$ is given by

$$\mathcal{A}^c_{\mu\nu,\beta\gamma}(k,p)\equiv\sin^2\big[\frac{1}{2}k\epsilon\theta p\big]\Big\{(\delta_{\mu\nu}\delta_{\beta\gamma}-\delta_{\mu\gamma}\delta_{\beta\nu})+(\delta_{\mu\nu}\delta_{\beta\gamma}-\delta_{\mu\beta}\delta_{\gamma\nu})\Big\}. \tag{5.20}$$

The final Feynman graph may look like Fig. 5.21.

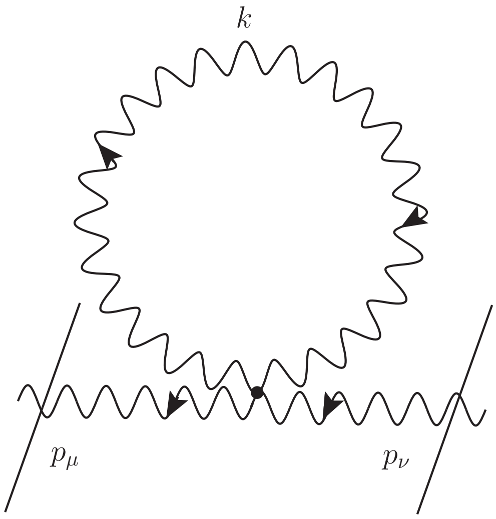


Figure 5.21: Final two-point photon-tadpole graph.

5.5 General Structure of the Remaining Integrals

5.5.1 Introduction to the UV/IR Mixing-Problem

We give a short recapitulation about the findings so far.

The explicit analytic expressions of one-loop level, representing the Feynman graphs of the BRSW model, read (the following expressions equal Eqs. (5.14, 5.16, 5.19, 5.9) and (5.11) from Sections 5.3 and 5.4):

$$\tilde{\Pi}^a_{\mu\nu}(p) = -C_a \frac{4g^2}{(2\pi)^4} \int d^4k \Big\{ \frac{(k-p)_\mu k_\nu}{(k-p)^2 k^2} \sin^2 \big[\frac{1}{2} k\epsilon\theta p\big] \Big\}, \tag{5.21a}$$

$$\begin{aligned} \tilde{\Pi}^b_{\mu\nu}(p) = C_b \frac{4g^2}{(2\pi)^4} \int d^4k \sin^2 \big[\frac{1}{2} k\epsilon\theta p\big] \tilde{G}^{AA}_{\gamma\epsilon}(k-p) \tilde{G}^{AA}_{\rho\alpha}(k) \\ \times \Big\{ \big(\delta_{\alpha\gamma}(2k-p)_\mu + \delta_{\alpha\mu}(-k-p)_\gamma + \delta_{\mu\gamma}(2p-k)_\alpha\big) \\ \times \big(\delta_{\epsilon\rho}(2k-p)_\nu + \delta_{\epsilon\nu}(2p-k)_\rho + \delta_{\nu\rho}(-k-p)_\epsilon\big) \Big\}, \end{aligned} \tag{5.21b}$$

$$\begin{aligned} \tilde{\Pi}^c_{\mu\nu}(p) = -C_c \frac{4g^2}{(2\pi)^4} \int d^4k \sin^2 \big[\frac{1}{2} k\epsilon\theta p\big] \times \\ \times \Big\{ (\delta_{\mu\nu}\delta_{\beta\gamma} - \delta_{\mu\gamma}\delta_{\beta\nu}) + (\delta_{\mu\nu}\delta_{\beta\gamma} - \delta_{\mu\beta}\delta_{\gamma\nu}) \Big\} \tilde{G}^{AA}_{\beta\gamma}(k), \end{aligned} \tag{5.21c}$$

$$\tilde{\Pi}^d = 0, \tag{5.21d}$$

$$\tilde{\Pi}^e = 0, \tag{5.21e}$$

where Tab. 5.2 depicts the Feynman graph. The main aim is now to solve the

Graph	$\tilde{\Pi}^n$
Ghost-loop	$\tilde{\Pi}^a$
Photon-loop	$\tilde{\Pi}^b$
2pt.-Photon-tadpole	$\tilde{\Pi}^c$
Ghost-tadpole	$\tilde{\Pi}^d$
Photon-tadpole	$\tilde{\Pi}^e$

Table 5.2: Summary of Feynman graphs.

remaining integral over k. Obviously the given expressions are complicated functions of internal and external momentum k and p and can not be evaluated in a straight forward way. To get some information about the behaviour of divergence we use power counting as a possibility of information.

Referring to Section 4.3, all Feynman graphs show a divergent behaviour for the region of large momentum ($k \to \pm\infty$). As we have mentioned in the previous sections, the sine term (often called phase factor) will lead to a damping mechanism for this region but leads to a divergence for the region of small external momentum p. This effect inherent in all noncommutative theories with the star product introduced in Section

2.3, Eq. (2.13) is known as the UV/IR mixing-problem [36, 37, 38].

For details we give a short introduction to this problem.

To bring light into the UV/IR mixing-problem we use the easiest possible expression occurring in our work. Therefore we will work with the two-point photon-tadpole expression given by Eq. (5.21c):

$$\begin{aligned}
\tilde{\Pi}^{c}_{\mu\nu}(p) &= -C_c \frac{4g^2}{(2\pi)^4} \int d^4k \sin^2\big[\frac{1}{2}k\epsilon\theta p\big] \times \\
&\quad \times \Big\{ (\delta_{\mu\nu}\delta_{\beta\gamma} - \delta_{\mu\gamma}\delta_{\beta\nu}) + (\delta_{\mu\nu}\delta_{\beta\gamma} - \delta_{\mu\beta}\delta_{\gamma\nu}) \Big\} \tilde{G}^{AA}_{\beta\gamma}(k) \\
&= -C_c \frac{4g^2}{(2\pi)^4} \int d^4k \sin^2\big[\frac{1}{2}k\epsilon\theta p\big] \times \\
&\quad \times \Big\{ (\delta_{\mu\nu}\tilde{G}^{AA}_{\beta\beta}(k) - \tilde{G}^{AA}_{\nu\mu}(k)) + (\delta_{\mu\nu}\tilde{G}^{AA}_{\beta\beta}(k) - \tilde{G}^{AA}_{\mu\nu}(k)) \Big\} \\
&= -C_c \frac{8g^2}{(2\pi)^4} \int d^4k \sin^2\big[\frac{1}{2}k\epsilon\theta p\big] \Big\{ \delta_{\mu\nu}\tilde{G}^{AA}_{\beta\beta}(k) - \tilde{G}^{AA}_{\mu\nu}(k) \Big\},
\end{aligned} \tag{5.22}$$

where we have used

$$\tilde{G}^{AA}_{\mu\nu}(k) = \tilde{G}^{AA}_{\nu\mu}(k).$$

Keeping in mind that all divergences arise from the limit of large internal momenta k [9, 17], it will be wise to use the expression of the propagator $\tilde{G}^{AA}_{\beta\gamma}(k)$ in the limit $k \to \infty$, given by Eq. (4.55):

$$\tilde{G}^{AA}_{\beta\gamma}(k) \approx \frac{1}{k^2}[\delta_{\beta\gamma} - \frac{k_\beta k_\gamma}{k^2}], \qquad \alpha \to 0, k \to \infty$$

Therefore we get

$$\begin{aligned}
&\tilde{\Pi}^{c}_{\mu\nu}(p) \approx \\
&\approx -C_c \frac{8g^2}{(2\pi)^4} \int d^4k \sin^2\big[\frac{1}{2}k\epsilon\theta p\big] \Big\{ \delta_{\mu\nu}\frac{1}{k^2}\Big[\delta_{\beta\beta} - \frac{k_\beta k_\beta}{k^2}\Big] - \frac{1}{k^2}\Big[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big]\Big\} \\
&= -C_c \frac{8g^2}{(2\pi)^2} \int d^4k \sin^2\big[\frac{1}{2}k\epsilon\theta p\big] \Big\{ \frac{1}{k^2}\Big[\delta_{\mu\nu}\delta_{\beta\beta} - \delta_{\mu\nu}\Big] - \frac{1}{k^2}\Big[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big]\Big\} \\
&= -C_c \frac{8g^2}{(2\pi)^2} \int d^4k \sin^2\big[\frac{1}{2}k\epsilon\theta p\big] \Big\{ \frac{1}{k^2}\Big[2\delta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\Big]\Big\} \\
&= -C_c \frac{8g^2}{(2\pi)^2} \int d^4k \sin^2\big[\frac{1}{2}k\epsilon\theta p\big] \Big\{ 2\delta_{\mu\nu}\frac{1}{k^2} + \frac{k_\mu k_\nu}{k^4}\Big\},
\end{aligned} \tag{5.23}$$

where we have used the representation of the Kronecker delta in four dimensions

$$(\delta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Rightarrow \delta_{\beta\beta} = 4. \tag{5.24}$$

The first term of this expression is now used as a concrete example (the pre-factors and Kronecker delta are omitted to avoid confusion) for the mixing-problem and hence denoted by:

$$\tilde{\Pi}^{ex}(p) := \int d^4k \frac{1}{k^2} \sin^2 \big[\frac{1}{2} k\epsilon\theta p\big], \qquad \tilde{\Pi}^{ex} \equiv \tilde{\Pi}^{example}. \tag{5.25}$$

Before we start the explicit calculation of $\tilde{\Pi}^{ex}(p)$ we must admit that we are only interested in the final results which show satisfactorily the occurring problems in a noncommutative theory. The detailed method of evaluation of such integrals (including new concepts like parametrization of integrals, cutoffs, etc.) will be the topic of Section 5.6.

We obtain

$$\begin{aligned}\tilde{\Pi}^{ex}(p) &= \int d^4k \frac{1}{k^2} \sin^2 \big[\frac{1}{2} k\epsilon\theta p\big] \\ &= \int d^4k \frac{1}{k^2} \bigg(\frac{1}{2}\big[1 - e^{+ik\epsilon\theta p}\big]\bigg).\end{aligned} \tag{5.26}$$

The analytic expression $\tilde{\Pi}^{ex}(p)$ is now divided into a so-called planar and non-planar part:

$$\tilde{\Pi}^{ex}(p) = \tilde{\Pi}^{ex,pl} + \tilde{\Pi}^{ex,npl}(p), \tag{5.27}$$

where $\tilde{\Pi}^{ex,pl}$ and $\tilde{\Pi}^{ex,npl}(p)$ are given by

$$\tilde{\Pi}^{ex,pl} = \frac{1}{2} \int d^4k \frac{1}{k^2}, \tag{5.28a}$$

$$\tilde{\Pi}^{ex,npl}(p) = -\frac{1}{2} \int d^4k \frac{1}{k^2} e^{+ik\epsilon\theta p}. \tag{5.28b}$$

Note that from now on the occurring parameter α is introduced to solve parameter integrals and must not be mixed with the gauge fixing parameter α from the previous chapters. As long as we work till the end of this chapter with the photon propagator (4.55), there should not be any confusion.

The detailed calculation of both parts shows:

1) The planar part:

For the planar part, we obtain

$$\tilde{\Pi}^{ex,pl} = \frac{\pi^2}{2} \int_0^\infty d\alpha \frac{1}{\alpha^2}. \tag{5.29}$$

Obviously the last expression will cause problems because of the divergent structure given in the limit of $\alpha \to 0$. Therefore we introduce an ultaviolet cutoff Λ which leads to

$$\tilde{\Pi}^{ex,pl} = \frac{\pi^2}{2} \Lambda^2. \tag{5.30}$$

The planar part shows the usual divergent structure in the limit of large Λ:

$$\tilde{\Pi}^{ex,pl} = \frac{\pi^2}{2}\Lambda^2. \tag{5.31}$$

2) The non-planar part:

The non-planar part reads

$$\tilde{\Pi}^{ex,npl}(p) = -\frac{2\pi^2}{(\epsilon\theta p)^2}. \tag{5.32}$$

The last equation shows the main problem:

Although, all divergences arise from large internal momenta k, the non-planar part generated through the noncommutative treatment shows divergence in the limit of small external momenta p,

$$\tilde{\Pi}^{ex,npl}(p) = -\frac{2\pi^2}{(\epsilon\theta p)^2} \to -\infty, \qquad p \to 0. \tag{5.33}$$

However, we will show in the next chapter that we can treat this problem in a satisfying way.

After this introduction into the UV/IR mixing-problem, the observation of the expressions given by Eqs. (5.21a, 5.21b) and (5.21c) shows that the general structure is given by

$$\begin{aligned}
\tilde{\Pi}^a_{\mu\nu}(p) &= \mathcal{M}_a \int d^4k \mathit{\Pi}^a_{\mu\nu}(k,p)\sin^2\left[\frac{1}{2}k\epsilon\theta p\right],\\
\tilde{\Pi}^b_{\mu\nu}(p) &= \mathcal{M}_b \int d^4k \mathit{\Pi}^b_{\mu\nu}(k,p)\sin^2\left[\frac{1}{2}k\epsilon\theta p\right],\\
\tilde{\Pi}^c_{\mu\nu}(p) &= \mathcal{M}_c \int d^4k \mathit{\Pi}^c_{\mu\nu}(k)\sin^2\left[\frac{1}{2}k\epsilon\theta p\right],
\end{aligned} \tag{5.34}$$

with

$$\begin{aligned}
\mathcal{M}_j &= -C_j\frac{4g^2}{(2\pi)^4}, \qquad j = a, c,\\
\mathcal{M}_b &= +C_b\frac{4g^2}{(2\pi)^4},
\end{aligned}$$

and

$$\mathit{\Pi}^a_{\mu\nu}(k,p) = \frac{(k-p)_\mu k_\nu}{(k-p)^2k^2}, \tag{5.35a}$$

$$\begin{aligned}
\mathit{\Pi}^b_{\mu\nu}(k,p) = {}& \tilde{G}^{AA}_{\gamma\epsilon}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)\\
&\times\Big\{\big(\delta_{\alpha\gamma}(2k-p)_\mu + \delta_{\alpha\mu}(-k-p)_\gamma + \delta_{\mu\gamma}(2p-k)_\alpha\big)\\
&\quad\times\big(\delta_{\epsilon\rho}(2k-p)_\nu + \delta_{\epsilon\nu}(2p-k)_\rho + \delta_{\nu\rho}(-k-p)_\epsilon\big)\Big\},
\end{aligned} \tag{5.35b}$$

$$\mathit{\Pi}^c_{\mu\nu}(k) = \Big\{(\delta_{\mu\nu}\delta_{\beta\gamma} - \delta_{\mu\gamma}\delta_{\beta\nu}) + (\delta_{\mu\nu}\delta_{\beta\gamma} - \delta_{\mu\beta}\delta_{\gamma\nu})\Big\}\tilde{G}^{AA}_{\beta\gamma}(k). \tag{5.35c}$$

5.5.2 Expansion of $\tilde{\Pi}^i_{\mu\nu}(p)$

As mentioned above we are mainly interested in the divergence structure of these expressions in the limit of small external momenta $p \to 0$. Therefore the analytic expressions $\tilde{\Pi}^a_{\mu\nu}(p)$ and $\tilde{\Pi}^b_{\mu\nu}(p)$ are expanded for small momenta p according to (i=a,b):

$$
\begin{aligned}
\tilde{\Pi}^i_{\mu\nu}(p) &= \mathcal{M}_i \int d^4k \mathit{\Pi}^i_{\mu\nu}(k,p) \sin^2\big[\frac{1}{2}k\epsilon\theta p\big] \\
&= \mathcal{M}_i \int d^4k \sin^2\big[\frac{1}{2}k\epsilon\theta p\big] \times \qquad (5.36)\\
&\times \Big[\mathit{\Pi}^i_{\mu\nu}(k,0) + p_\sigma\big[\partial_{p_\sigma}\mathit{\Pi}^i_{\mu\nu}(k,p)\big]_{p\to 0} + \frac{p_\delta p_\sigma}{2}\big[\partial_{p_\delta}\partial_{p_\sigma}\mathit{\Pi}^i_{\mu\nu}(k,p)\big]_{p\to 0} + \mathcal{O}(p^3)\Big],
\end{aligned}
$$

while (5.35c) stays unmodified according to the p independence of $\mathit{\Pi}^c(k)$. The phase factors are not expanded in order to loose not the damping effect of the highly oscillating functions at large momenta k[6].

First of all we introduce the abbreviations:

$$
\mathit{\Pi}^i_{\mu\nu}(k,0) \equiv \mathit{\Pi}^{i,(0)}_{\mu\nu}(k,0), \qquad (5.37a)
$$

$$
p_\sigma\big[\partial_{p_\sigma}\mathit{\Pi}^i_{\mu\nu}(k,p)\big]_{p\to 0} \equiv \mathit{\Pi}^{i,(1)}_{\mu\nu}(k,p), \qquad (5.37b)
$$

$$
\frac{p_\delta p_\sigma}{2}\big[\partial_{p_\delta}\partial_{p_\sigma}\mathit{\Pi}^i_{\mu\nu}(k,p)\big]_{p\to 0} \equiv \mathit{\Pi}^{i,(2)}_{\mu\nu}(k,p), \qquad (5.37c)
$$

representing the zeroth, first and second order ((0), (1) and (2)) of the expansion (5.36). Secondly if any $\tilde{\Pi}^i_{\mu\nu}(k,p)$ contains the the propagator $\tilde{G}^{AA}$ we will use the Landau gauge and the expression of $\tilde{G}^{AA}$ in the limits of large momentum k given by Eq. (4.55):

$$
\tilde{G}^{AA}_{\alpha\beta}(k) \approx \frac{1}{k^2}\left[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right], \qquad \alpha \to 0, k \to \infty.
$$

Therefore (5.35a) reads

$$
\begin{aligned}
\mathit{\Pi}^a_{\mu\nu}(k,p) &= \frac{(k-p)_\mu k_\nu}{(k-p)^2 k^2} \\
&\approx \mathit{\Pi}^{a,(0)}_{\mu\nu}(k,0) + \mathit{\Pi}^{a,(2)}_{\mu\nu}(k,p) + \mathcal{O}(p^3) \\
&= \frac{k_\mu k_\nu}{k^4} + \left\{ -2p_\mu p_\delta \frac{k_\delta k_\nu}{k^6} - p^2\frac{k_\mu k_\nu}{k^6} + 4p_\delta p_\sigma \frac{k_\sigma k_\delta k_\mu k_\nu}{k^8}\right\} + \mathcal{O}(p^3), \qquad (5.38)
\end{aligned}
$$

[6]This point we be discussed in detail in the next section.

for the ghost-loop, while (5.35b) related to the photon-loop gives

$$\begin{aligned}
\Pi^b_{\mu\nu}(k,p) = {} & \tilde{G}^{AA}_{\gamma\epsilon}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)\times \\
& \times\Big\{\big(\delta_{\alpha\gamma}(2k-p)_\mu + \delta_{\alpha\mu}(-k-p)_\gamma + \delta_{\mu\gamma}(2p-k)_\alpha\big) \\
& \times\big(\delta_{\epsilon\rho}(2k-p)_\nu + \delta_{\epsilon\nu}(2p-k)_\rho + \delta_{\nu\rho}(-k-p)_\epsilon\big)\Big\} \\
\approx {} & \Pi^{b,(0)}_{\mu\nu}(k,0) + \Pi^{b,(2)}_{\mu\nu}(k,p) + \mathcal{O}(p^3) \\
= {} & 12\frac{k_\mu k_\nu}{k^4} + \Big\{ -5p_\mu p_\nu \frac{1}{k^4} - 8p_\alpha p_\nu \frac{k_\alpha k_\mu}{k^6} - 8p_\alpha p_\mu \frac{k_\alpha k_\nu}{k^6} \\
& - 16p^2 \frac{k_\mu k_\nu}{k^6} + 52 p_\alpha p_\beta \frac{k_\alpha k_\beta k_\mu k_\nu}{k^8} \\
& + 8p^2 \delta_{\mu\nu}\frac{1}{k^4} - 8p_\alpha p_\beta \delta_{\mu\nu}\frac{k_\alpha k_\beta}{k^6}\Big\} + \mathcal{O}(p^3).
\end{aligned} \tag{5.39}$$

Note that the first order always contains an odd number of the variable k and therefore the symmetric integral over the variable k leads to zero,

$$\int d^4k\,\Pi^{i,(1)}(k,p) = 0, \qquad i = a, b. \tag{5.40}$$

For the two-point photon-tadpole we use the calculations already derived from Section 5.5.1, Eqs. (5.22) and (5.23):

$$\Pi^c_{\mu\nu}(k) = \Big\{\delta_{\mu\nu}\tilde{G}^{AA}_{\beta\beta}(k) - \tilde{G}^{AA}_{\mu\nu}(k)\Big\} \approx \Big\{2\delta_{\mu\nu}\frac{1}{k^2} + \frac{k_\mu k_\nu}{k^4}\Big\}. \tag{5.41}$$

The full analytic expressions for the Feynman graphs are now given by

$$\tilde{\Pi}^a_{\mu\nu}(p) \approx -C_a\frac{4g^2}{(2\pi)^4}\int d^4k\,\sin^2[\frac{1}{2}k\epsilon\theta p] \tag{5.42a}$$
$$\times\Big\{\frac{k_\mu k_\nu}{k^4} + \Big[-2p_\mu p_\delta\frac{k_\delta k_\nu}{k^6} - p^2\frac{k_\mu k_\nu}{k^6} + 4p_\delta p_\sigma \frac{k_\sigma k_\delta k_\mu k_\nu}{k^8}\Big] + \mathcal{O}(p^3)\Big\},$$

$$\tilde{\Pi}^b_{\mu\nu}(p) \approx +C_b\frac{4g^2}{(2\pi)^4}\int d^4k\,\sin^2[\frac{1}{2}k\epsilon\theta p] \tag{5.42b}$$
$$\begin{aligned}
\times\Big\{12\frac{k_\mu k_\nu}{k^4} + \Big[& -5p_\mu p_\nu\frac{1}{k^4} - 8p_\alpha p_\nu\frac{k_\alpha k_\mu}{k^6} - 8p_\alpha p_\mu\frac{k_\alpha k_\nu}{k^6} \\
& - 16p^2\frac{k_\mu k_\nu}{k^6} + 52p_\alpha p_\beta\frac{k_\alpha k_\beta k_\mu k_\nu}{k^8} \\
& + 8p^2\delta_{\mu\nu}\frac{1}{k^4} - 8p_\alpha p_\beta\delta_{\mu\nu}\frac{k_\alpha k_\beta}{k^6}\Big] + \mathcal{O}(p^3)\Big\},
\end{aligned}$$

$$\tilde{\Pi}^c_{\mu\nu}(p) \approx -C_c\frac{8g^2}{(2\pi)^4}\int d^4k\,\sin^2[\frac{1}{2}k\epsilon\theta p]\Big\{2\delta_{\mu\nu}\frac{1}{k^2} + \frac{k_\mu k_\nu}{k^4}\Big\}, \tag{5.42c}$$

where the first term inside the curly brackets of $\tilde{\Pi}^i_{\mu\nu}, i = a, b$ represents the zeroth order of the expansion, while the second order is inside the square brackets.

The detailed calculations of $\tilde{\Pi}^i_{\mu\nu}$ can be found in Appendix E.

5.6 Evaluation of the Remaining Integrals

5.6.1 General Structure

Taking a closer look at the expressions of Eq. (5.42a, 5.42b) and (5.42c), we can conclude that the remaining integrals have the general form of:

A)

$$\tilde{\mathcal{I}}^1(p) = \int d^4k \frac{1}{k^2} \sin^2\big[\frac{1}{2}k\epsilon\tilde{p}\big], \tag{5.43a}$$

$$\tilde{\mathcal{I}}^2(p) = \int d^4k \frac{1}{k^4} \sin^2\big[\frac{1}{2}k\epsilon\tilde{p}\big], \tag{5.43b}$$

B)

$$\tilde{\mathcal{I}}^3_{\mu\nu}(p) = \int d^4k \frac{k_\mu k_\nu}{k^4} \sin^2\big[\frac{1}{2}k\epsilon\tilde{p}\big], \tag{5.43c}$$

$$\tilde{\mathcal{I}}^4_{\mu\nu}(p) = \int d^4k \frac{k_\mu k_\nu}{k^6} \sin^2\big[\frac{1}{2}k\epsilon\tilde{p}\big], \tag{5.43d}$$

$$\tilde{\mathcal{I}}^5_{\alpha\beta\mu\nu}(p) = \int d^4k \frac{k_\alpha k_\beta k_\mu k_\nu}{k^8} \sin^2\big[\frac{1}{2}k\epsilon\tilde{p}\big], \tag{5.43e}$$

where we have omitted the prefactors and used the abbreviation:

$$\tilde{p} \equiv \theta p.$$

Those expressions are divided into a part A and part B because the two integrals of part A will cover all arising solutions: The ones with modified Bessel functions and the ones without.

Obviously the integrals show the same structure:

$$\tilde{\mathcal{I}}^l(p) = \int d^4ski^l(k) \sin^2\big[\frac{1}{2}k\epsilon\tilde{p}\big], \qquad l = 1, 2, 3, 4, 5. \tag{5.44}$$

The sine function can be decomposited in the following way:

$$\sin^2\big[\frac{1}{2}k\epsilon\tilde{p}\big] = \frac{1}{2}\big[1 - \cos(k\epsilon\tilde{p})\big] = \frac{1}{2}\big[1 - \frac{1}{2}\big(e^{+ik\epsilon\tilde{p}} + e^{-ik\epsilon\tilde{p}}\big)\big], \tag{5.45}$$

leading to

$$\tilde{\mathcal{I}}^l(p) = \int d^4kI^l(k)\frac{1}{2}\big[1 - \frac{1}{2}\big(e^{+ik\epsilon\tilde{p}} + e^{-ik\epsilon\tilde{p}}\big)\big]. \tag{5.46}$$

The second exponential function can be written as (remember the short notation 3.12):

$$\int_{-\infty}^{+\infty} d^4kI^l(k)e^{-ik\epsilon\tilde{p}} \overset{k\to -k}{=} -\int_{+\infty}^{-\infty} d^4kI^l(-k)e^{+ik\epsilon\tilde{p}}, \tag{5.47}$$

$$= \int_{-\infty}^{+\infty} d^4kI^l(k)e^{+ik\epsilon\tilde{p}}, \qquad I^l(k) = I^l(-k).$$

Therefore we have

$$\tilde{\mathcal{I}}^l(p) = \int d^4k I^l(k)\frac{1}{2}\big[1 - e^{+ik\epsilon\tilde{p}}\big]. \tag{5.48}$$

The whole expression can be divided into a so-called planar and non-planar part.

$$\begin{aligned}\tilde{\mathcal{I}}^l(p) &= \tilde{\mathcal{I}}^{l,pl} + \tilde{\mathcal{I}}^{l,npl}(p) \qquad (5.49)\\ &= \frac{1}{2}\int\limits_{-\infty}^{+\infty} d^4 I^l(k) - \frac{1}{2}\int d^4k I^l(k)e^{+ik\epsilon\tilde{p}}.\end{aligned}$$

A.1):

We start with the first expression of part A given by Eq. (5.43a):

$$\begin{aligned}\tilde{\mathcal{I}}^1(p) &= \int d^4k\frac{1}{k^2}\sin^2\big[\frac{1}{2}k\epsilon\tilde{p}\big]\\ &= \int d^4k\frac{1}{k^2}\bigg(\frac{1}{2}\big[1 - e^{+ik\epsilon\tilde{p}}\big]\bigg). \qquad (5.50)\end{aligned}$$

The analytic expression $\tilde{\mathcal{I}}^1(p)$ is now divided into the planar and non-planar part:

$$\tilde{\mathcal{I}}^1(p) = \tilde{\mathcal{I}}^{1,pl} + \tilde{\mathcal{I}}^{1,npl}(p), \tag{5.51}$$

where $\tilde{\mathcal{I}}^{1,pl}$ and $\tilde{\mathcal{I}}^{1,npl}(p)$ are given by

$$\tilde{\mathcal{I}}^{1,pl} = \frac{1}{2}\int d^4k\frac{1}{k^2}, \tag{5.52a}$$

$$\tilde{\mathcal{I}}^{1,npl}(p) = -\frac{1}{2}\int d^4k\frac{1}{k^2}e^{+ik\epsilon\tilde{p}}. \tag{5.52b}$$

The parametrization and integration formulae used in the following context can be found in Appendix D.

1) The planar part:

The Schwinger parametrization of $1/k^2$ is now given by Formula (D.2):

$$\frac{1}{k^2} = \frac{1}{\Gamma(1)}\int\limits_0^\infty d\alpha e^{-\alpha k^2} = \int\limits_0^\infty d\alpha e^{-\alpha k^2},$$

which leads to

$$\begin{aligned}\tilde{\mathcal{I}}^{1,pl} &= \frac{1}{2}\int d^4k\frac{1}{k^2} = \frac{1}{2}\int d^4k\int\limits_0^\infty d\alpha e^{-\alpha k^2}\\ &= \frac{1}{2}\int\limits_0^\infty d\alpha\int d^4k e^{-\alpha k^2} = \frac{1}{2}\int\limits_0^\infty d\alpha\frac{\pi^2}{\alpha^2}\\ &= \frac{\pi^2}{2}\int\limits_0^\infty d\alpha\frac{1}{\alpha^2}. \qquad (5.53)\end{aligned}$$

where we have evaluated the integral over the variable k with Formula (D.6).

As we have discussed in Section 5.5.1, the last expression will cause problems because of the divergent structure given in the limit of $\alpha \to 0$. Therefore we introduce an ultaviolet cutoff Λ to regularize the integral, evaluate this parameter integral with Formula (D.7b) and use the definition of the gamma function $\Gamma(N)$ given by Formula (D.1), which together lead to

$$\begin{aligned}\tilde{\mathcal{I}}^{1,pl} &= \frac{\pi^2}{2}\int_0^\infty d\alpha \frac{1}{\alpha^2} e^{-\frac{1}{\Lambda^2 \alpha}} = \frac{\pi^2}{2}\Gamma(1)\left(\frac{1}{\Lambda^2}\right)^{-1} \\ &= \frac{\pi^2}{2}\Lambda^2. \end{aligned} \tag{5.54}$$

The planar part shows the usual divergent structure in the limit of large Λ

$$\tilde{\mathcal{I}}^{1,pl} = \frac{\pi^2}{2}\Lambda^2 \to \infty, \qquad \Lambda \to \infty. \tag{5.55}$$

2) The non-planar part:

The non-planar part reads

$$\tilde{\mathcal{I}}^{1,npl}(p) = -\frac{1}{2}\int d^4k \frac{1}{k^2} e^{+ik\epsilon\tilde{p}} = -\frac{1}{2}\int d^4k \int_0^\infty d\alpha e^{-\alpha k^2 + ik\epsilon\tilde{p}}.$$

First of all we bring the exponent into quadratic form:

$$-\alpha k^2 + ik\epsilon\tilde{p} = \alpha\left(ik + \frac{\epsilon\tilde{p}}{2\alpha}\right)^2 - \frac{(\epsilon\tilde{p})^2}{4\alpha}. \tag{5.56}$$

This leads to

$$\tilde{\mathcal{I}}^{1,npl}(p) = -\frac{1}{2}\int_0^\infty d\alpha \int d^4k \left[e^{\alpha(ik+\frac{\epsilon\tilde{p}}{2\alpha})^2}\right] e^{-\frac{(\epsilon\tilde{p})^2}{4\alpha}}. \tag{5.57}$$

Next, we try to bring the following expression

$$\begin{aligned}\int d^4k \big[e^{\alpha(ik+\frac{\epsilon\tilde{p}}{2\alpha})^2}\big] &= \int d^4k \big[e^{\alpha(-k^2+ik\frac{\epsilon\tilde{p}}{\alpha}+(\frac{\epsilon\tilde{p}}{2\alpha})^2)}\big] \\ &= \int d^4k \big[e^{-\alpha(k^2-ik\frac{\epsilon\tilde{p}}{\alpha}-(\frac{\epsilon\tilde{p}}{2\alpha})^2)}\big] = \int d^4k \big[e^{-\alpha(k-i\frac{\epsilon\tilde{p}}{2\alpha})^2}\big],\end{aligned}$$

into the form of a Gaussian integral

$$\int d^4k' e^{-ak'^2} = \frac{\pi^2}{a^2}.$$

We substitute with new variables:

$$\alpha := a, \qquad k' := k - i\frac{\epsilon\tilde{p}}{2\alpha},$$

thus we have

$$d^4k = d^4k',$$

and now the integral has the desired form and can be evaluated with Formula (D.6):

$$\int d^4k \big[e^{\alpha(ik+\frac{\epsilon\tilde{p}}{2\alpha})^2}\big] = \int d^4k' e^{-ak'^2} = \frac{\pi^2}{a^2} = \frac{\pi^2}{\alpha^2}. \tag{5.58}$$

Therefore only the integral over the parameter α is still left and leads to (evaluation with (D.7b))

$$\begin{aligned}\tilde{\mathcal{I}}^{1,npl}(p) &= -\frac{\pi^2}{2}\int_0^\infty d\alpha \frac{1}{\alpha^2} e^{-\frac{(\epsilon\tilde{p})^2}{4\alpha}} = -\frac{\pi^2}{2}\Gamma(1)\left[\frac{(\epsilon\tilde{p})^2}{4}\right]^{-1} \\ &= -\frac{2\pi^2}{(\epsilon\tilde{p})^2}. \end{aligned} \tag{5.59}$$

Note that there is no need for a ultraviolet cutoff Λ because the exponential function (=phase factor) leads to a natural damping mechanism for values of small α.

As discussed in the previous section the noncommutative treatment shows divergence for the limit of small external momenta p.

$$\tilde{\mathcal{I}}^{1,npl}(p) = -\frac{2\pi^2}{(\epsilon\tilde{p})^2} \to -\infty, \qquad p \to 0. \tag{5.60}$$

A.2):

The second part of A given by Eq. (5.43b) can be written as

$$\begin{aligned}\tilde{\mathcal{I}}^2(p) &= \int d^4k \frac{1}{k^4}\sin^2\big[\frac{1}{2}k\epsilon\tilde{p}\big] = \tilde{\mathcal{I}}^{2,pl} + \tilde{\mathcal{I}}^{2,npl}(p) \\ &= \frac{1}{2}\Big\{\int d^4k \frac{1}{k^4} - \int d^4k \frac{1}{k^4} e^{+ik\epsilon\tilde{p}}\Big\}. \end{aligned} \tag{5.61}$$

The parametrization and integration formulae used in the following context can be found in Appendix D.

1) The planar part:

The Schwinger parametrization of $1/k^4$ is now given by

$$\frac{1}{k^4} = \frac{1}{\Gamma(2)}\int_0^\infty d\alpha \alpha e^{-\alpha k^2} = \int_0^\infty d\alpha \alpha e^{-\alpha k^2}, \tag{5.62}$$

which leads to

$$\begin{aligned}\tilde{\mathcal{I}}^{2,pl} &= \frac{1}{2}\int d^4k \frac{1}{k^4} = \frac{1}{2}\int d^4k \int\limits_0^\infty d\alpha\alpha e^{-\alpha k^2} \\ &= \frac{1}{2}\int\limits_0^\infty d\alpha\alpha \int d^4k e^{-\alpha k^2} = \frac{1}{2}\int\limits_0^\infty d\alpha \frac{\pi^2}{\alpha} \\ &= \frac{\pi^2}{2}\int\limits_0^\infty d\alpha \frac{1}{\alpha}. \end{aligned} \tag{5.63}$$

As distinguished from the planar part of A.1 (5.54), the second one shows a divergent structure not only for the limit of $\alpha \to 0$ but also for the limit $\alpha \to \infty$. Therefore we introduce an ultaviolet cutoff Λ and an infrared cutoff μ to regularize the integral and evaluate this parameter integral with Formula (D.7a):

$$\tilde{\mathcal{I}}^{2,pl} = \frac{\pi^2}{2}\int\limits_0^\infty d\alpha \frac{1}{\alpha} e^{-\frac{1}{\Lambda^2 \alpha} - \mu^2 \alpha} = \frac{\pi^2}{2} K_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right), \tag{5.64}$$

where K_0 represents the modified Bessel function of second kind (D.9b).

2) The non-planar part:

The non-planar part reads

$$\begin{aligned}\tilde{\mathcal{I}}^{2,npl}(p) &= -\frac{1}{2}\int d^4k \frac{1}{k^4} e^{+ik\epsilon\tilde{p}} = -\frac{1}{2}\int d^4k \int\limits_0^\infty d\alpha\alpha e^{-\alpha k^2 + ik\epsilon\tilde{p}} \\ &= -\frac{1}{2}\int\limits_0^\infty d\alpha\alpha \int d^4k \left[e^{\alpha(ik + \frac{\epsilon\tilde{p}}{2\alpha})^2}\right] e^{-\frac{(\epsilon\tilde{p})^2}{4\alpha}}. \end{aligned} \tag{5.65}$$

The evaluation of the Gaussian integral (5.58) leaves us with the integral over the parameter α and hence leads to

$$\begin{aligned}\tilde{\mathcal{I}}^{2,npl}(p) &= -\frac{\pi^2}{2}\int\limits_0^\infty d\alpha \frac{1}{\alpha} e^{-\frac{(\epsilon\tilde{p})^2}{4\alpha} - \mu^2\alpha} = -\pi^2 K_0\left(2\sqrt{\frac{(\epsilon\tilde{p})^2}{4}\mu^2}\right) \\ &= -\pi^2 K_0\left(\sqrt{(\epsilon\tilde{p})^2\mu^2}\right), \end{aligned} \tag{5.66}$$

where we have used Formula (D.7a). Again we have a damping mechanism for small α.

The noncommutative treatment also shows divergence in the limit of small external momenta p.

$$\tilde{\mathcal{I}}^{2,npl}(p) = -\pi^2 K_0\left(\sqrt{(\epsilon\tilde{p})^2\mu^2}\right) \to -\infty, \qquad p \to 0. \tag{5.67}$$

B:

In the following we will give a methodology for calculating the integrals appearing in part B.

1.

The integrals are of the general form

$$\begin{aligned}\tilde{\mathcal{I}}_{\eta_1...\eta_m}(p) &= \int d^4k \frac{k_{\eta_1}...k_{\eta_m}}{k^{2N}} \sin^2\left[\frac{1}{2}k\epsilon\tilde{p}\right] \\ &= \frac{1}{2}\int d^4k \frac{k_{\eta_1}...k_{\eta_m}}{k^{2N}}\left[1 - e^{+ik\epsilon\tilde{p}}\right],\end{aligned} \tag{5.68}$$

where we have splitted the integral explicitly into the planar and non-planar part. The given integrals of part B (5.43c, 5.43d) and (5.43e) generate the following possible combinations, given in Tab. 5.3.

Integral	**N**	**m**
$\tilde{\mathcal{I}}^3_{\mu\nu}$	2	2
$\tilde{\mathcal{I}}^4_{\mu\nu}$	3	2
$\tilde{\mathcal{I}}^5_{\alpha\beta\mu\nu}$	4	4

Table 5.3: Possible combinations of N and m.

2.

For the momenta in the numerator we can write

$$k_\eta = \left[-i\partial_{z_\eta}e^{ikz}\right]_{z=0} \Rightarrow k_{\eta_1}...k_{\eta_m} = (-i)^m\partial_{z_{\eta_1}}...\partial_{z_{\eta_m}}e^{ikz}\big|_{z=0}, \tag{5.69}$$

while the Schwinger parametrization of the denominator is given by

$$\frac{1}{k^{2N}} = \frac{1}{\Gamma(N)}\int\limits_0^\infty d\alpha\alpha^{(N-1)}e^{-\alpha k^2},$$

together leading to

$$\tilde{\mathcal{I}}_{\eta_1...\eta_m}(p) = \frac{(-i)^m}{2\Gamma(N)}\int\limits_0^\infty d\alpha\alpha^{(N-1)}\partial_{z_{\eta_1}}...\partial_{z_{\eta_m}}\int d^4k\left(e^{-\alpha k^2+ikz} - e^{-\alpha k^2+ik(\epsilon\tilde{p}+z)}\right)\Bigg|_{z=0}. \tag{5.70}$$

3.

The quadratic form of the exponential is given by

$$-\alpha k^2 + ikz = \alpha\left(ik + \frac{z}{2\alpha}\right)^2 - \frac{z^2}{4\alpha}, \quad \text{planar,} \tag{5.71a}$$

$$-\alpha k^2 + ik(\epsilon\tilde{p} + z) = \alpha\left(ik + \frac{\epsilon\tilde{p} + z}{2\alpha}\right)^2 - \frac{(\epsilon\tilde{p} + z)^2}{4\alpha}, \quad \text{non-planar.} \tag{5.71b}$$

With application of Formula (D.6), the evaluation of the integral over k leads to a factor π^2/α^2 for the planar and non-planar part (the detailed procedure can be seen in Eqs. (5.56, 5.57) and (5.58)). Therefore we have

$$\tilde{\mathcal{I}}_{\eta_1 \ldots \eta_m}(p) = \frac{(-i)^m \pi^2}{2\Gamma(N)} \int_0^\infty d\alpha \alpha^{(N-3)} \partial_{z_{\eta_1}} \ldots \partial_{z_{\eta_m}} \left(e^{-\frac{z^2}{4\alpha}} - e^{-\frac{(\epsilon\tilde{p}+z)^2}{4\alpha}} \right)\Bigg|_{z=0}. \tag{5.72}$$

4.

The derivations for the non-planar part are given by (according to Tab. 5.3, we have m=2, 4)[7]:

$$\partial_{z_\mu}\partial_{z_\nu} e^{-\frac{(\epsilon\tilde{p}+z)^2}{4\alpha}} = \left\{ -\frac{\delta_{\mu\nu}}{2\alpha} + \frac{(\epsilon\tilde{p}+z)_\mu(\epsilon\tilde{p}+z)_\nu}{(2\alpha)^2} \right\} e^{-\frac{(\epsilon\tilde{p}+z)^2}{4\alpha}}, \quad m = 2 \tag{5.73a}$$

$$\begin{aligned}
&\partial_{z_\alpha}\partial_{z_\beta}\partial_{z_\mu}\partial_{z_\nu} e^{-\frac{(\epsilon\tilde{p}+z)^2}{4\alpha}} = \\
&\quad = \Bigg\{ \frac{\delta_{\alpha\beta}\delta_{\mu\nu} + \delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\beta\mu}\delta_{\alpha\nu}}{(2\alpha)^2} \\
&\quad - \frac{\delta_{\alpha\beta}(\epsilon\tilde{p}+z)_\mu(\epsilon\tilde{p}+z)_\nu + \delta_{\alpha\mu}(\epsilon\tilde{p}+z)_\beta(\epsilon\tilde{p}+z)_\nu + \delta_{\beta\mu}(\epsilon\tilde{p}+z)_\alpha(\epsilon\tilde{p}+z)_\nu}{(2\alpha)^3} \\
&\quad - \frac{\delta_{\alpha\nu}(\epsilon\tilde{p}+z)_\beta(\epsilon\tilde{p}+z)_\mu + \delta_{\beta\nu}(\epsilon\tilde{p}+z)_\alpha(\epsilon\tilde{p}+z)_\mu + \delta_{\mu\nu}(\epsilon\tilde{p}+z)_\alpha(\epsilon\tilde{p}+z)_\beta}{(2\alpha)^3} \\
&\quad + \frac{(\epsilon\tilde{p}+z)_\alpha(\epsilon\tilde{p}+z)_\beta(\epsilon\tilde{p}+z)_\mu(\epsilon\tilde{p}+z)_\nu}{(2\alpha)^4} \Bigg\} e^{-\frac{(\epsilon\tilde{p}+z)^2}{4\alpha}}, \; m = 4
\end{aligned} \tag{5.73b}$$

while the derivations for the planar part can be found similar if we set $p = 0$.

5.

The integrals $\tilde{\mathcal{I}}^3, \tilde{\mathcal{I}}^4$ (5.43c, 5.43d) can now be written in the form of

$$\begin{aligned}
\tilde{\mathcal{I}}(p) &= \tilde{\mathcal{I}}^{pl} + \tilde{\mathcal{I}}^{npl}(p) \\
&= \frac{(-i)^m \pi^2}{2\Gamma(N)} \left\{ A_{m-1} \int_0^\infty d\alpha \alpha^{(N-3-(m-1))} - \sum_{k=m-1,m} A_k \int_0^\infty d\alpha \alpha^{(N-3-k)} e^{-\frac{(\epsilon\tilde{p})^2}{4\alpha}} \right\},
\end{aligned} \tag{5.74}$$

[7] In the following context we use the notation $(\epsilon\tilde{p} + z)_\mu \equiv \epsilon\tilde{p}_\mu + z_\mu$.

while the integral $\tilde{\mathcal{I}}^5$ (5.43e) reads

$$\begin{aligned}\tilde{\mathcal{I}}(p) &= \tilde{\mathcal{I}}^{pl} + \tilde{\mathcal{I}}^{npl}(p) \\ &= \frac{(-i)^m \pi^2}{2\Gamma(N)} \left\{ A_{m-2} \int\limits_0^\infty d\alpha \alpha^{(N-3-(m-2))} - \sum_{k=m-2,m-1,m} A_k \int\limits_0^\infty d\alpha \alpha^{(N-3-k)} e^{-\frac{(\epsilon\tilde{p})^2}{4\alpha}} \right\},\end{aligned} \tag{5.75}$$

where the prefactors A_m, representing expressions not involving the parameter α can be deduced from Eqs. (5.73a) and (5.73b).

6.

Evaluation of the integrals: Insertion of N, m given from Tab. 5.3 and deducing the right prefactors leads to four types of integrals:

$$\tilde{\mathcal{I}}^{i,pl} \sim \int\limits_0^\infty d\alpha \frac{1}{\alpha} e^{-\frac{1}{\Lambda^2\alpha} - \mu^2\alpha}, \qquad i = 4, 5, \tag{5.76a}$$

$$\tilde{\mathcal{I}}^{i,pl} \sim \int\limits_0^\infty d\alpha \frac{1}{\alpha^2} e^{-\frac{1}{\Lambda^2\alpha}}, \qquad i = 3, \tag{5.76b}$$

$$\tilde{\mathcal{I}}^{i,npl} \sim \int\limits_0^\infty d\alpha \frac{1}{\alpha} e^{-\frac{(\epsilon\tilde{p})^2}{4\alpha} - \mu^2\alpha}, \qquad i = 4, 5, \tag{5.76c}$$

$$\tilde{\mathcal{I}}^{i,npl} \sim \int\limits_0^\infty d\alpha \frac{1}{\alpha^j} e^{-\frac{(\epsilon\tilde{p})^2}{4\alpha}}, \qquad j = 2, \quad i = 3, 4, 5, \qquad j = 3 \quad i = 3, 5, \tag{5.76d}$$

where Λ and μ represent the already introduced UV-and IR-cutoff. The integrals (5.76a -5.76d) can be evaluated with Formula (D.7a) and (D.7b). In Appendix E.2.1, we give the explicit calculation of $\tilde{\mathcal{I}}^3(p)$ as an example.

The analytic expressions for the Feynman graphs are now given by

$$\begin{aligned}\tilde{\Pi}^a_{\mu\nu}(p) &\approx -C_a \frac{4g^2}{(2\pi)^4} \times \\ &\times \left\{ \tilde{\mathcal{I}}^3_{\mu\nu}(p) + \left[-2p_\mu p_\delta \tilde{\mathcal{I}}^4_{\delta\nu}(p) - p^2 \tilde{\mathcal{I}}^4_{\mu\nu}(p) + 4p_\delta p_\sigma \tilde{\mathcal{I}}^5_{\sigma\delta\mu\nu}(p) \right] + \mathcal{O}(p^3) \right\},\end{aligned} \tag{5.77a}$$

$$\begin{aligned}\tilde{\Pi}^b_{\mu\nu}(p) \approx{}& +C_b\frac{4g^2}{(2\pi)^4}\times \\ &\times\Bigg\{12\tilde{\mathcal{I}}^3_{\mu\nu}(p)+\Big[-5p_\mu p_\nu\tilde{\mathcal{I}}^2(p)-8p_\alpha p_\nu\tilde{\mathcal{I}}^4_{\alpha\mu}(p)-8p_\alpha p_\mu\tilde{\mathcal{I}}^4_{\alpha\nu}(p) \\ &\quad -16p^2\tilde{\mathcal{I}}^4_{\mu\nu}(p)+52p_\alpha p_\beta\tilde{\mathcal{I}}^5_{\alpha\beta\mu\nu}(p) \\ &\quad +8p^2\delta_{\mu\nu}\tilde{\mathcal{I}}^2(p)-8p_\alpha p_\beta\delta_{\mu\nu}\tilde{\mathcal{I}}^4_{\alpha\beta}(p)\Big]+\mathcal{O}(p^3)\Bigg\},\end{aligned} \tag{5.77b}$$

$$\tilde{\Pi}^c_{\mu\nu}(p) \approx -C_c\frac{8g^2}{(2\pi)^4}\Big\{2\delta_{\mu\nu}\tilde{\mathcal{I}}^1(p)+\tilde{\mathcal{I}}^3_{\mu\nu}(p)\Big\}. \tag{5.77c}$$

5.6.2 Explicit Evaluation

The detailed outcome of each Integral $\tilde{\mathcal{I}}^i$ can be found in Appendix E.2.2, and inserted into (5.77a-5.77c) leads to:

$$\begin{aligned}\tilde{\Pi}^a_{\mu\nu}(p) \approx{}& -C_a g^2\times \\ &\times\Bigg\{\Bigg[\Big(\frac{1}{16\pi^2}\delta_{\mu\nu}\Lambda^2\Big)^{(0)} \\ &-\Bigg(\frac{1}{48\pi^2}p^2\delta_{\mu\nu}K_0\Big(2\sqrt{\frac{\mu^2}{\Lambda^2}}\Big)+\frac{1}{24\pi^2}p_\mu p_\nu K_0\Big(2\sqrt{\frac{\mu^2}{\Lambda^2}}\Big)\Bigg)^{(2)}\Bigg]^{pl} \\ &+\Bigg[\Big(-\frac{1}{4\pi^2}\frac{\delta_{\mu\nu}}{(\epsilon\tilde{p})^2}+\frac{1}{2\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\epsilon\tilde{p}^2)^2}\Big)^{(0)}+\Big((\text{finite terms})'\Big)^{(2)} \\ &+\Bigg(\frac{1}{24\pi^2}p_\mu p_\nu K_0\Big(\sqrt{\mu^2(\epsilon\tilde{p}^2)^2}\Big)+\frac{1}{48\pi^2}p^2\delta_{\mu\nu}K_0\Big(\sqrt{\mu^2(\epsilon\tilde{p})^2}\Big)\Bigg)^{(2)}\Bigg]^{npl} \\ &+\mathcal{O}(p^3)\Bigg\},\end{aligned} \tag{5.78}$$

$$\begin{aligned}\tilde{\Pi}^b_{\mu\nu}(p) \approx{}& +C_b g^2\times \\ &\times\Bigg\{\Bigg[\Big(\frac{3}{4\pi^2}\delta_{\mu\nu}\Lambda^2\Big)^{(0)} \\ &+\Bigg(\frac{50}{48\pi^2}K_0\Big(2\sqrt{\frac{\mu^2}{\Lambda^2}}\Big)p^2\delta_{\mu\nu}-\frac{56}{48\pi^2}K_0\Big(2\sqrt{\frac{\mu^2}{\Lambda^2}}\Big)p_\mu p_\nu\Bigg)^{(2)}\Bigg]^{pl} \\ &+\Bigg[\Big(\frac{6}{\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\epsilon\tilde{p}^2)^2}-\frac{3}{\pi^2}\frac{\delta_{\mu\nu}}{(\epsilon\tilde{p})^2}\Big)^{(0)}+\Big((\text{finite terms})''\Big)^{(2)} \\ &-\Bigg(\frac{50}{48\pi^2}K_0\Big(\sqrt{\mu^2(\epsilon\tilde{p})^2}\Big)p^2\delta_{\mu\nu}-\frac{56}{48\pi^2}K_0\Big(\sqrt{\mu^2(\epsilon\tilde{p})^2}\Big)p_\mu p_\nu\Bigg)^{(2)}\Bigg]^{npl} \\ &+\mathcal{O}(p^3)\Bigg\},\end{aligned} \tag{5.79}$$

$$\tilde{\Pi}^c_{\mu\nu}(p) \approx -C_c g^2\Bigg\{\Big[\frac{5}{8\pi^2}\delta_{\mu\nu}\Lambda^2\Big]^{pl}-\Big[\frac{5}{2\pi^2}\frac{\delta_{\mu\nu}}{(\epsilon\tilde{p})^2}-\frac{1}{\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\epsilon\tilde{p}^2)^2}\Big]^{npl}\Bigg\}. \tag{5.80}$$

Note that the superscript $(0), (2)$ represents the order of our expansion and that we have devided each $\tilde{\Pi}^i_{\mu\nu}(p)$ into the planar and non-planar part. The expression *finite* *terms* refers to all finite terms which will not contribute to the divergent structure in the limit of small p. They are given by

$$
\begin{aligned}
&(\text{finite terms})' = \\
&= -\Bigg[\frac{1}{8\pi^2}\frac{p_\mu p_\delta \tilde{p}_\delta \tilde{p}_\nu}{\tilde{p}^2} + \frac{1}{16\pi^2}\frac{p^2\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2} + \frac{1}{12\pi^2}\frac{p_\delta p_\sigma \tilde{p}_\mu \tilde{p}_\nu \tilde{p}_\delta \tilde{p}_\sigma}{\tilde{p}^4} \\
&\quad - \frac{1}{24\pi^2}(\delta_{\mu\nu} p_\delta p_\sigma \tilde{p}_\delta \tilde{p}_\sigma + p_\mu p_\sigma \tilde{p}_\nu \tilde{p}_\sigma + p_\sigma p_\nu \tilde{p}_\mu \tilde{p}_\sigma \\
&\quad + p_\delta p_\mu \tilde{p}_\nu \tilde{p}_\delta + p_\delta p_\nu \tilde{p}_\mu \tilde{p}_\delta + p^2 \tilde{p}_\mu \tilde{p}_\nu)\frac{1}{\tilde{p}^2}\Bigg], \qquad (5.81) \\
&(\text{finite terms})'' = \\
&= -\Bigg[\frac{1}{2\pi^2}\frac{p_\alpha p_\beta \tilde{p}_\alpha \tilde{p}_\beta}{\tilde{p}^2}\delta_{\mu\nu} + \frac{1}{\pi^2}\frac{p^2\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2} + \frac{1}{2\pi^2}\frac{p_\alpha p_\mu \tilde{p}_\alpha \tilde{p}_\nu}{\tilde{p}^2} + \frac{1}{2\pi^2}\frac{p_\alpha p_\nu \tilde{p}_\alpha \tilde{p}_\mu}{\tilde{p}^2} \\
&\quad - \frac{26}{48\pi^2}(p^2\tilde{p}_\mu\tilde{p}_\nu + p_\mu p_\beta \tilde{p}_\beta \tilde{p}_\nu + p_\alpha p_\mu \tilde{p}_\alpha \tilde{p}_\nu + p_\nu p_\beta \tilde{p}_\beta \tilde{p}_\mu \\
&\quad + p_\alpha p_\mu \tilde{p}_\alpha \tilde{p}_\nu + \delta_{\mu\nu} p_\alpha p_\beta \tilde{p}_\alpha \tilde{p}_\beta)\frac{1}{\tilde{p}^2} + \frac{52}{48\pi^2}\frac{p_\alpha p_\beta \tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^4}\Bigg]. \qquad (5.82)
\end{aligned}
$$

The detailed calculations can be found in Appendix E.2.3.

5.7 One-Loop Correction

The full one-loop correction of the photon propagator is now given by the sum of all analytic expressions $\tilde{\Pi}^j_{\mu\nu}(p)$, representing the particular Feynman graph,

$$
\begin{aligned}
\tilde{\Pi}'(p) &= \tilde{\Pi}^a_{\mu\nu}(p) + \tilde{\Pi}^b_{\mu\nu}(p) + \tilde{\Pi}^c_{\mu\nu}(p) + \tilde{\Pi}^d_{\mu}(p) + \tilde{\Pi}^e_{\mu}(p) \qquad (5.83) \\
&= \tilde{\Pi}^j(p), \qquad j = a, b, c, d, e.
\end{aligned}
$$

We have approximated the photon propagator in the limit of large momenta k. Since the one-point tadpoles vanished ($\tilde{\Pi}^d_\mu(p) = \tilde{\Pi}^e_\mu(p) = 0$), and each $\tilde{\Pi}^{a,b,c}_{\mu\nu}(p)$ contains a planar and non-planar part which are expanded up to the second order (except $\tilde{\Pi}^c_{\mu\nu}(p)$), we may write

$$
\tilde{\Pi}_{\mu\nu}(p) \approx \sum_{i=pl,npl}\left[\sum_{j=a,b}\sum_{k=(0),(2)} \tilde{\Pi}^{i,j,k}_{\mu\nu}(p) + \tilde{\Pi}^{c,i}_{\mu\nu}(p)\right] + \text{finite terms}, \qquad (5.84)
$$

where *finite terms* will contain all finite expressions which appear in our calculations and is explained later on in detail.

Aside from this formal expression it will be wise to take a closer look at each part, collect all (non)-planar parts, seperate each order and bring together the prefactors.

Therefore we get for the planar part of $\tilde{\Pi}_{\mu\nu}(p)$:

$$\tilde{\Pi}^{(0),pl}_{\mu\nu}(p) \approx -\frac{g^2}{(4\pi)^2}\Lambda^2\delta_{\mu\nu}(C_a - 12C_b + 10C_c) = 0, \tag{5.85}$$

$$\begin{aligned}\tilde{\Pi}^{(2),pl}_{\mu\nu}(p) \approx& \\ \approx&\frac{g^2}{3(4\pi)^2}K_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right)\left\{(C_a + 50C_b)p^2\delta_{\mu\nu} + 2(C_a - 28C_b)p_\mu p_\nu\right\} \\ =&\frac{26g^2}{3(4\pi)^2}K_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right)(p^2\delta_{\mu\nu} - p_\mu p_\nu),\end{aligned} \tag{5.86}$$

and for the non-planar part:

$$\begin{aligned}&\tilde{\Pi}^{(0),npl}_{\mu\nu}(p) \approx \\ &\approx \frac{g^2}{(2\pi)^2}\frac{1}{(\epsilon\tilde{p}^2)^2}\left\{(C_a - 12C_b + 10C_c)\tilde{p}^2\delta_{\mu\nu} - 2(C_a - 12C_b + 2C_c)\tilde{p}_\mu\tilde{p}_\nu\right\} \\ &= \frac{2g^2\tilde{p}_\mu\tilde{p}_\nu}{\pi^2(\epsilon\tilde{p}^2)^2},\end{aligned} \tag{5.87}$$

$$\begin{aligned}&\tilde{\Pi}^{(2),npl}_{\mu\nu}(p) \approx \\ &\approx -\frac{g^2}{3(4\pi)^2}K_0\left(\sqrt{(\epsilon\tilde{p})^2\mu^2}\right)\left\{(C_a + 50C_b)p^2\delta_{\mu\nu} + 2(C_a - 28C_b)p_\mu p_\nu\right\} \\ &\qquad + (\text{finite terms})' + (\text{finite terms})'' + \mathcal{O}(p^3) \\ &= -\frac{26g^2}{3(4\pi)^2}K_0\left(\sqrt{(\epsilon\tilde{p})^2\mu^2}\right)(p^2\delta_{\mu\nu} - p_\mu p_\nu) + (\text{finite terms})''' + \mathcal{O}(p^3),\end{aligned} \tag{5.88}$$

where we have used

$$(\text{finite terms})''' = (\text{finite terms})' + (\text{finite terms})'',$$

and the expressions of higher orders caused by the expansion added to the term $\tilde{\Pi}^{(2),npl}_{\mu\nu}(p)$. In the face of calculating a sum the final expression will not be affected. The sum of all expressions with the combinatoric factors $C_a = 1, C_b = \frac{1}{2}, C_c = \frac{1}{2}$ given from Section 5.2, Tab. 5.1 leads to:

$$\begin{aligned}&\tilde{\Pi}_{\mu\nu}(p) \approx \\ &\approx \frac{2g^2}{\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\epsilon\tilde{p}^2)^2} + \frac{26g^2}{3(4\pi)^2}\left\{K_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right) - K_0\left(\sqrt{(\epsilon\tilde{p})^2\mu^2}\right)\right\}(p^2\delta_{\mu\nu} - p_\mu p_\nu) \\ &\qquad + (\text{finite terms})''' + \mathcal{O}(p^3) \\ &\approx \frac{2g^2}{\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\epsilon\tilde{p}^2)^2} + \frac{13g^2}{24\pi^2}\ln(\Lambda)(p^2\delta_{\mu\nu} - p_\mu p_\nu) + \text{finite terms}.\end{aligned} \tag{5.89}$$

This result exhibits a quadratic IR divergence.

In the last expression we have used the expansion of the modified Bessel function of second kind given by Eq. (D.11a). The expression *finite terms* now contains all non-relevant terms which will not contribute to the divergent structure in the limits of $\mu \to 0, \Lambda \to \infty$ and in the light of small external momenta p:

$$\begin{aligned}
\text{finite terms} &\equiv (\text{finite terms})''' + (\text{finite terms})^{IV} \\
&\quad + \{\mathcal{O}(a^4) + \mathcal{O}(b^4)\}(p^2\delta_{\mu\nu} - p_\mu p_\nu) + \mathcal{O}(p^3), \\
a &\equiv 2\sqrt{\frac{\mu^2}{\Lambda^2}}, \\
b &\equiv \sqrt{(\epsilon\tilde{p})^2\mu^2},
\end{aligned} \tag{5.90}$$

while the term $(\text{finite terms})^{IV}$ contains finite expressions for small p caused by the expansion of the Bessel function $K_0(\sqrt{(\epsilon\tilde{p})^2\mu^2})$. For the sake of completeness we give the analytic expression of this term

$$(\text{finite terms})^{IV} = \left\{ \left(1 + \frac{((\epsilon\tilde{p})^2)^2\mu^2}{4}\right)\ln(\epsilon\tilde{p}) - \ln(2) \right\}(p^2\delta_{\mu\nu} - p_\mu p_\nu). \tag{5.91}$$

We also have to mention that each expansion of $K_0(a)$ and $K_0(b)$ leads to a term $1/2\ln(\mu^2)$ which cancel each other. For details take a closer look at Appendix E.3.1.

Finally, the Feynman graph which represents the analytic expression $\tilde{\Pi}_{\mu\nu}(p)$ may look like the following figure (Fig. 5.22).

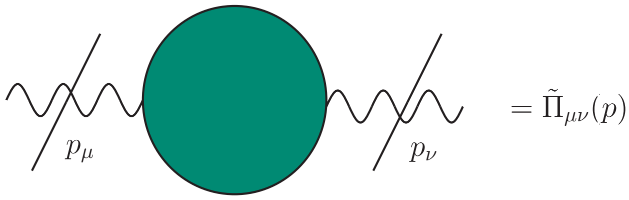


Figure 5.22: Feynman graph representing the sum of one-loop corrections to the photon propagator.

Chapter 6

Renormalization

The last chapter is dedicated to the topic of *Renormalization*. In general renormalization means that the parameters of a given theory will be replaced by renormalized parameters which can absorb the occurring divergences. In the special case of this work, the parameters γ, σ will be replaced by the renormalized parameters γ_r, σ_r. The procedure of renormalization will be discussed step by step.

We start with the one-loop correction (5.89), given by[1]

$$\tilde{\Pi}_{\mu\nu}(k) \approx \frac{2g^2}{\pi^2}\frac{\tilde{k}_\mu \tilde{k}_\nu}{(\epsilon\tilde{k}^2)^2} + \frac{13g^2}{24\pi^2}\ln(\Lambda)(k^2\delta_{\mu\nu} - k_\mu k_\nu) + \text{finite terms},$$

and analyse each term separately.

Before we start the renormalization procedure we underline that the limit $\Lambda \to \infty$ must be considered at the very end.

a) Limit of small momenta k:

As we have mentioned in the previous chapter, the first term is quadratic IR divergent while the second one is finite. The Lagrangian of our BRSW model was constructed in such a way that the soft breaking term will lead to a damping mechanism in the IR region. This can be understood if we take a closer look at the complete or in other words 'dressed' propagator $\tilde{G}^{AA,dress}_{\mu\nu}$. This dressed propagator represents all possible combinations of propagators and one-loop corrections $\tilde{\Pi}_{\mu\nu}$. The essential part of the dressed propagator will be the expression (we give schematic statements, the detailed ones can be found in the next section):

$$\tilde{G}\tilde{\Pi}\tilde{G} \sim k^2 \frac{1}{k^2} k^2, \tag{6.1}$$

which is finite in the limit of small momenta k.

b) Limit of $\Lambda \to \infty$:

The second term of Eq. (5.89) shows a logarithmic divergence in the limit of $\Lambda \to \infty$.

[1] The external momenta p is now denoted by k.

This can be treated satisfyingly if we introduce the renormalized parameter γ_r which absorbs the divergences.

c) Limit $\epsilon \to 0$:

The predication 'Limit of $\epsilon \to 0$' makes only sense we if consider that ϵ can not reach the value $\epsilon = 0$, since ϵ represents the deformation parameter. This parameter must be understood in the way that it can be arbitrarily close to zero and therefore exhibits a quadratic divergence. To underline this aspect we recall Eq. (2.4):

$$\left[\hat{x}_\mu, \hat{x}_\nu\right] = i\epsilon\theta_{\mu\nu},$$

where the noncommutativity is only given for $\epsilon \neq 0$.
As we will see in the next sections this parameter will be included in the renormalized σ_r.

A few words about notation: In the following renormalized quantities are denoted by the the index r or the superscript ren. Any occurring propagator without explicit superscript denotation represents the tree-level two point Green function

$$\tilde{G}^{AA}_{\mu\nu}(k) \equiv \tilde{G}^{AA,tree}_{\mu\nu}(k). \tag{6.2}$$

Any expansion in terms of the coupling constant g is done under the assumption $g^2 < 1$.

6.1 Renormalization of the Photon Propagator

6.1.1 General Remarks

For the dressed propagator $\tilde{G}^{AA,dress}_{\mu\nu}$, we can give a representation in terms of Feynman graphs (Fig. 6.1) while the analytic expression reads:

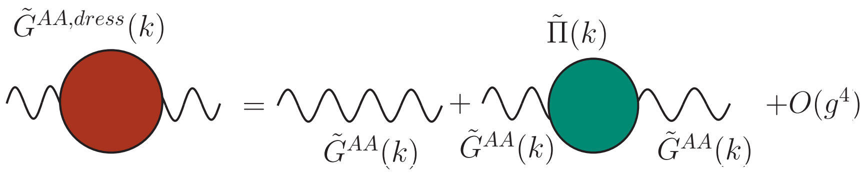


Figure 6.1: Dressed Photon Propagator at one-loop level.

$$\tilde{G}^{AA,dress}_{\mu\nu}(k) = \tilde{G}^{AA}_{\mu\nu}(k) + \tilde{G}^{AA}_{\mu\rho}(k)\tilde{\Pi}_{\rho\sigma}(k)\tilde{G}^{AA}_{\sigma\nu}(k) + \mathcal{O}(g^4). \tag{6.3}$$

Since the dressed propagator represents a sum of tree-level photon propagators $\tilde{G}^{AA}_{\mu\nu}$ in combination with the one-loop correction $\tilde{\Pi}_{\mu\nu}$, we try to show that $\tilde{G}^{AA,dress}$ can be written as a geometric series in the way of (the indices, superscript and the argument

k are omitted to avoid confusion):

$$
\begin{aligned}
\tilde{G}^{dress} &= \tilde{G} + \tilde{G}\tilde{\Pi}\tilde{G} + \tilde{G}\Pi\tilde{G}\Pi\tilde{G} + \mathcal{O}(g^6) \\
&= \tilde{G}\Big\{1 + \tilde{\Pi}\tilde{G} + \tilde{\Pi}\tilde{G}\tilde{\Pi}\tilde{G} + \mathcal{O}(g^6)\Big\} \\
&= \tilde{G}\sum_{n=0}^{\infty}(\tilde{\Pi}\tilde{G})^n = \tilde{G}\frac{1}{1-\tilde{\Pi}\tilde{G}} = \frac{1}{\frac{1}{\tilde{G}} - \tilde{\Pi}},
\end{aligned}
\tag{6.4}
$$

$$
\Rightarrow \tilde{G}^{AA,dress}_{\mu\nu}(k) = \frac{1}{\big(\tilde{G}^{AA}_{\mu\nu}(k)\big)^{-1} - \tilde{\Pi}_{\mu\nu}(k)}. \tag{6.5}
$$

To apply this series we must imply two important requirements:

<u>**1.**</u>

The expansion of this series is only valid for $g^2 < 1$ which is obviously fulfilled since we have postulated this requirement in the introduction. Nevertheless in the light of vanishing momenta k (at this time the parameter Λ must be treated as a finite quantity) the expression $\tilde{\Pi}\tilde{G}$ will not diverge. Due to the fact that we were mainly interested in expressions depending on small k, we approximated $\tilde{\Pi}_{\mu\nu}(k)$ for small values of k and therefore the occurring quadratic IR divergence can now be absorbed by $\tilde{G}^{AA}_{\mu\nu}(k)$ in the limit of small k:

$$
\begin{aligned}
\tilde{\Pi}_{\rho\sigma}(k)\tilde{G}^{AA}_{\sigma\nu}(k) &= \text{finite expression}, \\
\tilde{\Pi}^{\text{first term}}_{\rho\sigma}(k) &\propto \frac{2g^2}{\pi^2}\frac{\tilde{k}_\rho\tilde{k}_\sigma}{(\epsilon\tilde{k}^2)^2}, \quad \text{dominant for small k} \\
\tilde{G}^{AA}_{\sigma\nu}(k) &\approx \frac{\tilde{k}^2}{\gamma^4}\left[\delta_{\sigma\nu} - \frac{k_\sigma k_\nu}{k^2} - \frac{\bar{\sigma}^4}{(\bar{\sigma}^4+\gamma^4)}\frac{\tilde{k}_\sigma\tilde{k}_\nu}{\tilde{k}^2}\right], \text{ for small } k.
\end{aligned}
$$

However the dressed propagator should cover the full range from IR to UV. The next section shows that we must use a more generalized propagator to find an inverse.

<u>**2.**</u>

The inverse of the propagator $\tilde{G}^{AA}_{\mu\nu}$ must be understood in the way of

$$
\tilde{G}^{AA}_{\mu\rho}(\tilde{G}^{AA}_{\rho\nu})^{-1} = (\tilde{G}^{AA}_{\mu\rho})^{-1}\tilde{G}^{AA}_{\rho\nu} = \delta_{\mu\nu}. \tag{6.6}
$$

If we now recall that the inverse of the connected two-point Green function which represents the photon propagator $\tilde{G}^{AA}_{\mu\nu}(k)$ at tree level, equals the two-point vertex function $\tilde{\Gamma}^{AA,tree}_{\mu\nu}(k)$[2], then Eq. (6.5) becomes

$$
\tilde{\Gamma}^{AA,dress}_{\mu\nu}(k) = \tilde{\Gamma}^{AA,tree}_{\mu\nu}(k) - \tilde{\Pi}_{\mu\nu}(k), \tag{6.7}
$$

[2] The superscript *tree* emphasises the fact that the two-point vertex function represents the inverse of the tree-level two-point Green function in the way of $(\tilde{G}^{AA,tree}_{\mu\nu})^{-1} = \tilde{\Gamma}^{AA,tree}_{\mu\nu}$ with $\tilde{G}^{AA}_{\mu\nu} \equiv \tilde{G}^{AA,tree}_{\mu\nu}$. For details take a closer look in Chapter 4, Section 4.1.1 and especially Eq. (4.15).

with

$$\begin{aligned}\frac{1}{\tilde{\Gamma}^{AA,dress}_{\mu\nu}(k)} &= \tilde{G}^{AA,dress}_{\mu\nu}(k),\\ \frac{1}{\tilde{\Gamma}^{AA,tree}_{\mu\nu}(k)} &= \tilde{G}^{AA}_{\mu\nu}(k),\\ \frac{1}{\left(\tilde{G}^{AA}_{\mu\nu}(k)\right)^{-1}-\tilde{\Pi}_{\mu\nu}(k)} &= \frac{1}{\tilde{\Gamma}^{AA,tree}_{\mu\nu}(k)-\tilde{\Pi}_{\mu\nu}(k)}.\end{aligned} \tag{6.8}$$

The inverse of the last expression also has an interpretation in the sense of Eq. (6.6):

$$\left((\tilde{\Gamma}^{AA,tree}-\tilde{\Pi})_{\mu\rho}\right)^{-1}\left(\tilde{\Gamma}^{AA,tree}-\tilde{\Pi})_{\rho\nu}\right)=\delta_{\mu\nu}. \tag{6.9}$$

6.1.2 The Inverse Propagator

The task is now to find the inverse of $\tilde{G}^{AA}_{\mu\nu}(k)$, subtract it from the one-loop correction $\tilde{\Pi}_{\mu\nu}(k)$ to obtain the expression $\tilde{\Gamma}^{AA,dress}_{\mu\nu}(k)$ and invert the latter once again to get the dressed propagator $\tilde{G}^{AA,dress}_{\mu\nu}(k)$.

So far we have worked with the propagator given by Eq. (5.1), where we have used the Landau gauge ($\alpha \to 0$). Since the inverse of this propagator does not exist [25, 26], we will use the more generalized form ($\alpha \neq 0$) given by:

$$\tilde{G}^{AA}_{\mu\nu}(k)=\frac{1}{k^2\mathcal{D}}\left[\delta_{\mu\nu}-(1-\alpha\mathcal{D})\frac{k_\mu k_\nu}{k^2}-\mathcal{F}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right], \tag{6.10}$$

where we have introduced the abbreviations

$$\mathcal{D}(k)\equiv\left(1+\frac{\gamma^4}{(\tilde{k}^2)^2}\right), \tag{6.11}$$

$$\mathcal{F}(k)\equiv\frac{1}{\tilde{k}^2}\frac{\bar{\sigma}^4}{\left(k^2+(\bar{\sigma}^4+\gamma^4)\frac{1}{\tilde{k}^2}\right)}. \tag{6.12}$$

Note that Eq. (6.10) equals (4.47). It has to be remarked that the quadratic IR divergence and the result of the renormalization must be independent of the gauge fixing [39, 40]. In the end we will consider the Landau gauge again.

To get our inverse propagator we make the following ansatz according to Eq. (6.6):

$$\begin{aligned}\delta_{\mu\nu} &= \left(\tilde{G}^{AA}_{\mu\rho}(k)\right)^{-1}\tilde{G}^{AA}_{\rho\nu}(k) \qquad (6.13)\\ &= k^2\mathcal{D}\left[\delta_{\mu\rho}+a_1\frac{k_\mu k_\rho}{k^2}+a_2\frac{\tilde{k}_\mu\tilde{k}_\rho}{\tilde{k}^2}\right]\frac{1}{k^2\mathcal{D}}\left[\delta_{\rho\nu}-(1-\alpha\mathcal{D})\frac{k_\rho k_\nu}{k^2}-\mathcal{F}\frac{\tilde{k}_\rho\tilde{k}_\nu}{\tilde{k}^2}\right],\end{aligned}$$

from which we obtain the coefficients a_1, a_2 by comparison

$$a_1=\frac{1}{\alpha\mathcal{D}}-1, \tag{6.14}$$

$$a_2=\frac{\mathcal{F}}{1-\mathcal{F}}. \tag{6.15}$$

Therefore the two-point vertex function at tree level reads

$$\tilde{\Gamma}^{AA,tree}_{\mu\nu}(k) = \left(\tilde{G}^{AA}_{\mu\nu}(k)\right)^{-1} =$$
$$= k^2\mathcal{D}\left[\delta_{\mu\nu} + \left(\frac{1}{\alpha\mathcal{D}} - 1\right)\frac{k_\mu k_\nu}{k^2} + \frac{\bar{\sigma}^4}{k^2\tilde{k}^2\mathcal{D}}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right]. \tag{6.16}$$

The detailed calculation of the inverse of the tree level propagator can be found in Appendix F.1.1.

Before we insert the last equation in Eq. (6.7), we once again stress that we are mainly interested in expressions depending on small values of k and hence introduce the following abbreviation:

$$\tilde{\Gamma}^{AA.corr}_{\mu\nu}(k) \equiv \tilde{\Pi}_{\mu\nu}(k)\bigg|_{\text{without finite terms}}, \tag{6.17}$$

leading to

$$\tilde{\Gamma}^{AA.corr}_{\mu\nu}(k) = \mathit{\Pi}_a\frac{\tilde{k}_\mu\tilde{k}_\nu}{(\tilde{k}^2)^2} + \mathit{\Pi}_b(k^2\delta_{\mu\nu} - k_\mu k_\nu), \tag{6.18}$$

with

$$\mathit{\Pi}_a = \frac{2g^2}{\pi^2\epsilon^2}, \tag{6.19}$$
$$\mathit{\Pi}_b = \frac{13g^2}{24\pi^2}\ln(\Lambda), \tag{6.20}$$

involving the parameters ϵ and Λ.

6.1.3 Explicit Renormalization

Using the findings so far, the dressed vertex function can now be written as[3]

$$\tilde{\Gamma}^{AA,dress}_{\mu\nu}(k) = \tilde{\Gamma}^{AA,tree}_{\mu\nu}(k) - \tilde{\Gamma}^{AA.corr}_{\mu\nu}(k)$$
$$= k^2(\mathcal{D} - \mathit{\Pi}_b)\left[\delta_{\mu\nu} + \left(\frac{1}{\alpha(\mathcal{D} - \mathit{\Pi}_b)} - 1\right)\frac{k_\mu k_\nu}{k^2} + \frac{\bar{\sigma}^4 - \mathit{\Pi}_a}{k^2\tilde{k}^2(\mathcal{D} - \mathit{\Pi}_b)}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right]. \tag{6.21}$$

The detailed calculation can be found in Appendix F.1.2.

Before we try to find the inverse $\tilde{\Gamma}^{AA,dress}_{\mu\nu}(k)$, representing the dressed propagator which will lead in rapid succession to the renormalized propagator, we must mention some fundamental aspects of renormalization.

i)

The comparison of the tree level two-point vertex function (6.16) with the dressed vertex function (6.21) shows that the latter one differs from the first one by additional

[3]At this point the two point vertex function (6.16) diverges for the Landau gauge ($\alpha \to 0$) but firstly the propagator represents the inverse of this expression and secondly the limit $\alpha \to 0$ will considered at the end.

terms. These terms contain the divergent structure.

ii)

The task is now to *renormalize* the given parameters $(\gamma, \bar{\sigma})$ in such a way that the renormalized parameters (which must be independent of the variable k) absorb the divergences and that finally the now renormalized vertex function can be rewritten in the same form as the tree level expression.

iii)

If this can be achieved, the rewritten renormalized vertex function (and propagator!) shows *stability* of the theory with respect to the one-loop corrections. The postulated stability means that the renormalized propagator must have the same form as the dressed propagator. This allows us to compare the factors of the dressed propagator and of the renormalized one.

Once again we take a closer look at Eq. (6.21) and find that the first term of the dressed vertex function is given by[4]:

$$\tilde{\Gamma}^{AA,dress}_{\mu\nu}(k)^{\text{first term}} = k^2(\mathcal{D} - \mathit{\Pi}_b)\delta_{\mu\nu}.$$

At first sight the expression $\mathcal{D} - \mathit{\Pi}_b$ looks like a candidate for a renormalized $\mathcal{D}_f(\gamma_f^4)$ in the way of

$$\mathcal{D}_f(\gamma_f^4) = \mathcal{D}(\gamma^4) - \mathit{\Pi}_b,$$

but as we will see the renormalized parameter equals the non renormalized one under the constraint $\mathit{\Pi}_b$=0:

$$\mathcal{D}_f(\gamma_f^4) = \left(1 + \frac{\gamma_f^4}{k^2\tilde{k}^2}\right) = \left(1 + \frac{\gamma^4}{k^2\tilde{k}^2}\right) - \mathit{\Pi}_b = \mathcal{D} - \mathit{\Pi}_b,$$
$$\Rightarrow \gamma_f^4 = \gamma^4.$$

Obviously the last expression makes no sense. A way out of this dilemma is to introduce a *wave function renormalization* $A_\mu \to A_\mu = Z_A A^r_\mu$. This implies for the renormalized propagator, already modified with renormalized parameters a multiplication with Z_A^2. The square represents the fact that in general a propagator is represented by a quadratic term in the Langrangian. Therefore we multiply (6.21) with the wave function parameter Z_A^{-2} (which should incorporate $\mathit{\Pi}_b$). According to the announcements ii) and iii) we demand that the renormalized vertex function is cast into the same form as the tree level one:

$$\tilde{\Gamma}^{AA,tree}_{\mu\nu}(k)^{\text{first term}} = k^2(\mathcal{D} - \mathit{\Pi}_b)\delta_{\mu\nu} = \tilde{\Gamma}^{AA,ren}_{\mu\nu}(k)^{\text{first term}} = k^2\frac{\mathcal{D}_r}{Z_A^2}\delta_{\mu\nu} \qquad (6.22)$$

Under the aspect of stability, we define

$$\mathcal{D}_r(\gamma_r^4) := \left(1 + \frac{\gamma_r^4}{k^2\tilde{k}^2}\right), \qquad (6.23)$$

[4]The superscript f emphasises the fact that this renormalized propagator will no be the final result (f stands for false).

and inserting the explicit expressions, we obtain

$$\begin{aligned} \mathcal{D}_r(\gamma_r^4) &= (\mathcal{D} - \mathit{\Pi}_b) Z_A^2, \\ \Rightarrow \left(1 + \frac{\gamma_r^4}{k^2 \tilde{k}^2}\right) &= \left\{ \left(1 + \frac{\gamma^4}{k^2 \tilde{k}^2}\right) - \mathit{\Pi}_b \right\} Z_A^2 \\ &= (1 - \mathit{\Pi}_b) Z_A^2 + \frac{\gamma^4}{k^2 \tilde{k}^2} Z_A^2. \end{aligned} \tag{6.24}$$

Comparing the coefficients we get

$$\gamma_r^4 = \gamma^4 Z_A^2, \tag{6.25}$$

while the wave function parameter reads

$$Z_A = \frac{1}{\sqrt{1 - \mathit{\Pi}_b}}. \tag{6.26}$$

Now the conditional equation (6.25) for γ_r^4 reads

$$\begin{aligned} \gamma_r^4 = \gamma^4 Z_A^2 &= \gamma^4 \big(1 + \frac{1}{2} \mathit{\Pi}_b + \mathcal{O}(g^4)\big) \\ &= \gamma^4 \big(1 + \frac{13 g^2}{48 \pi^2} \ln(\Lambda) + \mathcal{O}(g^4)\big). \end{aligned} \tag{6.27}$$

The treatment of the first term determines the second one under the assumption of the Landau gauge. The approach to renormalize the parameter σ in the third term of Eq. (6.21) proceeds in the same way as the first term:

$$\begin{aligned} \tilde{\Gamma}_{\mu\nu}^{AA,tree}(k)^{\text{third term}} &= (\bar{\sigma}^4 - \mathit{\Pi}_a) \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^4} = \tilde{\Gamma}_{\mu\nu}^{AA,ren}(k)^{\text{third term}} = \frac{\bar{\sigma}_r^4}{Z_a^2} \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^4}, \\ &\Rightarrow (\bar{\sigma}^4 - \mathit{\Pi}_a) Z_A^2 = \bar{\sigma}_r^4. \end{aligned} \tag{6.28}$$

If we define

$$\bar{\sigma}_r^4 := 2\left(\sigma_r + \frac{\theta^2}{4} \sigma_r^2\right) \gamma_r^4, \tag{6.29}$$

we get with the help of Eq. (6.25):

$$2\left(\sigma_r + \frac{\theta^2}{4} \sigma_r^2\right) \gamma^4 Z_A^2 = (\bar{\sigma}^4 - \mathit{\Pi}_a) Z_A^2, \tag{6.30}$$

leading to the conditional equation for σ_r:

$$\sigma_r = \frac{2}{\theta^2} \left\{ -1 \pm \sqrt{\left(1 + \frac{\theta}{2} \sigma\right)^2 - \frac{g^2 \theta^2}{\pi^2 \gamma^4 \epsilon^2}} \right\}. \tag{6.31}$$

The detailed calculation of the last equation can be found in Appendix F.1.3.

The renormalized vertex function is now given by

$$\tilde{\Gamma}_{\mu\nu}^{AA,ren}(k) = \frac{k^2 \mathcal{D}_r}{Z_A^2} \left[\delta_{\mu\nu} + \left(\frac{Z_A^2}{\alpha \mathcal{D}_r} - 1 \right) \frac{k_\mu k_\nu}{k^2} + \frac{\bar{\sigma}_r^4}{k^2 \tilde{k}^2 \mathcal{D}_r} \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2} \right]. \tag{6.32}$$

To get the renormalized propagator we apply the same ansatz as in Eq. (6.13):

$$\begin{aligned}
\delta_{\mu\nu} &= \big(\tilde{\Gamma}^{AA,ren}_{\mu\rho}(k)\big)^{-1}\tilde{\Gamma}^{AA,ren}_{\rho\nu}(k) \qquad (6.33)\\
&= \frac{Z_A^2}{k^2\mathcal{D}_r}\left[\delta_{\mu\rho} + b_1\frac{k_\mu k_\rho}{k^2} + b_2\frac{\tilde{k}_\mu\tilde{k}_\rho}{\tilde{k}^2}\right]\frac{k^2\mathcal{D}_r}{Z_A^2}\left[\delta_{\rho\nu} + \left(\frac{Z_A^2}{\alpha\mathcal{D}_r} - 1\right)\frac{k_\rho k_\nu}{k^2} + \frac{\bar{\sigma}_r^4}{k^2\tilde{k}^2\mathcal{D}_r}\frac{\tilde{k}_\rho\tilde{k}_\nu}{\tilde{k}^2}\right],
\end{aligned}$$

from which we obtain the coefficients b_1, b_2 by comparison

$$b_1 = -\left(\frac{B_1}{1+B_1}\right), \tag{6.34}$$

$$b_2 = -\left(\frac{B_2}{1+B_2}\right), \tag{6.35}$$

where we have introduced the abbreviations

$$\begin{aligned}
B_1 &= \frac{Z_A^2}{\alpha\mathcal{D}_r} - 1, \qquad (6.36)\\
B_2 &= \frac{\bar{\sigma}_r^4}{k^2\tilde{k}^2\mathcal{D}_r}.
\end{aligned}$$

After recasting the coefficients, the renormalized propagator in Landau gauge ($\alpha \to 0$) becomes

$$\tilde{G}^{AA,ren}_{\mu\nu}(k) = \frac{Z_A^2}{k^2\mathcal{D}_r}\left[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - \mathcal{F}_r\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right], \tag{6.37}$$

with the abbreviation $\mathcal{F}_r$

$$\mathcal{F}_r(k) \equiv \frac{1}{\tilde{k}^2}\frac{\bar{\sigma}_r^4}{\left(k^2 + (\bar{\sigma}_r^4 + \gamma_r^4)\frac{1}{\tilde{k}^2}\right)}, \tag{6.38}$$

and the following already named quantities:

$$\begin{aligned}
Z_A^2 &= \frac{1}{1 - \mathit{\Pi}_b},\\
\gamma_r^4 &= \gamma^4 Z_A^2,\\
\mathcal{D}_r &= \left(1 + \frac{\gamma_r^4}{k^2\tilde{k}^2}\right),\\
\bar{\sigma}_r^4 &= 2\left(\sigma_r + \frac{\theta^2}{4}\sigma_r^2\right)\gamma_r^4.
\end{aligned}$$

The detailed calculation of the renormalized propagator (6.37) is shown in Appendix F.1.4.

6.1.4 Renormalization Conditions

In the last section of this work, we provide *renormalization conditions* for the tree level two-point vertex function given by Eq. (6.16):

$$\tilde{\Gamma}^{AA,tree}_{\mu\nu}(k) = k^2\mathcal{D}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \frac{k^2}{\alpha}\frac{k_\mu k_\nu}{k^2} + \frac{\bar{\sigma}^4}{\tilde{k}^2}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}.$$

The last equation can be divided into a transversal (T), longitudinal (L) and non-commutative part (NC). The motivation for the following separation can be seen if we multiply (6.16) with a vector k_μ. The first part gives

$$k^2\mathcal{D}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)k_\mu = k^2\mathcal{D}(k_\nu - k_\nu) = 0, \tag{6.39}$$

leading to the identification

$$\tilde{\Gamma}^{AA,T} = k^2\mathcal{D}. \tag{6.40}$$

The second term

$$\left(\frac{k^2}{\alpha}\frac{k_\mu k_\nu}{k^2}\right)k_\mu = \frac{k^2}{\alpha}k_\nu, \tag{6.41}$$

motivates

$$\tilde{\Gamma}^{AA,L} = \frac{k^2}{\alpha}. \tag{6.42}$$

The last term involves the matrix $\theta_{\mu\nu}$ in the way of

$$\left(\frac{\bar{\sigma}^4}{\tilde{k}^2}\frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2}\right)k_\mu = 0, \qquad k_\mu \tilde{k}_\mu = k_\mu \theta_{\mu\nu} k_\nu = 0, \tag{6.43}$$

and hence explains the superscript NC in the identification

$$\tilde{\Gamma}^{AA,NC} = \frac{\bar{\sigma}^4}{\tilde{k}^2}. \tag{6.44}$$

With respect to the introduced identifications, the tree level two-point vertex function reads

$$\tilde{\Gamma}^{AA,tree}_{\mu\nu}(k) = \tilde{\Gamma}^{AA,T}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \tilde{\Gamma}^{AA,L}\frac{k_\mu k_\nu}{k^2} + \tilde{\Gamma}^{AA,NC}\frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2}, \tag{6.45}$$

and allows us to formulate renormalization conditions given by

$$\left.\frac{(\tilde{k}^2)^2}{k^2}\tilde{\Gamma}^{AA,T}\right|_{k^2=0} = \gamma^4, \tag{6.46a}$$

$$\left.\frac{1}{2k^2}\frac{\partial(k^2\tilde{\Gamma}^{AA,T})}{\partial k^2}\right|_{k^2=0} = 1, \tag{6.46b}$$

$$\left.\tilde{k}^2\Gamma^{AA,NC}\right|_{k^2=0} = \bar{\sigma}^4, \tag{6.46c}$$

$$\left.\tilde{\Gamma}^{AA,L}\right|_{k^2=0} = 0, \tag{6.46d}$$

$$\left.\frac{\partial\Gamma^{AA,L}}{\partial k^2}\right|_{k^2=0} = \frac{1}{\alpha}. \tag{6.46e}$$

With these renormalization conditions, one can in principle determine the qualitative form of the propagator to all loop orders (provided the theory is renormalizable).

Chapter 7

Conclusion

7.1 Summary

The main aim of this work, which is based on the BRSW model, was to compute Feynman rules, results for the vacuum polarization and the one-loop renormalization of the gauge boson propagator. In the first chapter (1), a short overview about the motivation for noncommutative space and a brief review about the main models was presented. The following chapter (2) was dedicated to the algebra and the star product. In the next chapter (3), the action of the gauge field model was stated. The complex fields, as well as the ghost fields, introduced into the bilinear part of the action implement the IR damping. In order to ensure BRST transformations in the UV, additional sources are needed. Those sources implement a soft breaking of BRST in the IR. The computation of the Feynman rules was the main part of chapter four (4). According to the Section 4.1.4, only the photon propagator and the ghost propagator contribute to physical results. In the IR limit, the photon propagator shows the appearance of a term of the same type as the appearing divergent terms in one-loop results. The noncommutativity of the theory entered the vertex functions via phase factors, while the propagators showed the same form as the commutative ones. One-loop calculations of the next chapter (5) showed that from five one-loop Feynman graphs (Fig. 5.1) only three graphs were non-vanishing: The ghost-tadpole and the photon-tadpole vanished due to momentum conservation, pointing out one advantage of the $\frac{1}{p^2}$ model. Explicit calculations under the premise that one-loop corrections to the ghost-loop and photon-loop could be expanded in the limit of small external momenta, exhibited the known quadratic IR divergence in the external momentum and a logarithmic UV divergence in the cutoff Λ of the one-loop corrections of the vacuum polarization. The last chapter (6) pointed straight to the heart of any quantum field theory: Renormalization. Since the dressed propagator contains the product of the tree-level propagator and the one-loop correction, it was shown that as postulated by construction of the action, the inherent quadratic IR divergence could be cured. In the end, the renormalized propagator shows the same form as the tree level expression providing stability of the theory. The renormalized parameters are able to absorb the logarithmic UV divergence given by the cutoff Λ.

7.2 Outlook

The BRSW model may be the first renormalizable noncommutative gauge theory model. Although, only renormalization up to one-loop level was proven, there are strong indications [20] that this can be achieved up to all orders by using Multiscale Analysis. However, aside from this model the soft breaking 'technique' promises to be a powerful tool for other noncommutative quantum field theory models.

Appendix A

Noncommutative Quantum Field Theory

A.1 The Star Product

A.1.1 Cyclic Permutation

First of all we will omit the stars because we only want to show which algebraic sign will occur.
We only look at the product of fermions/bosons and fermions/fermions with the abbreviation f for fermions and b for bosons.

1) Two Fields:

Permutations of 12 are 21; we have

$$\begin{aligned}\int d^4x(b^1f^2) &= +\int d^4x(f^2b^1),\\ \int d^4x(f^1f^2) &= -\int d^4x(f^2f^1).\end{aligned} \tag{A.1}$$

2) Three Fields:

Permutations of 123 are 312 and 231; we have

$$\begin{aligned}\int d^4x(f^1f^2f^3) &= +\int d^4x(f^3f^1f^2) = +\int d^4x(f^2f^3f^1),\\ \int d^4x(f^1f^2b^3) &= +\int d^4x(b^3f^1f^2) = -\int d^4x(f^2b^3f^1),\\ \int d^4x(f^1b^2f^3) &= -\int d^4x(f^3f^1b^2) = -\int d^4x(b^2f^3f^1),\\ \int d^4x(b^1f^2f^3) &= -\int d^4x(f^3b^1f^2) = +\int d^4x(f^2f^3b^1).\end{aligned} \tag{A.2}$$

3) Four Fields:

Permutations of 1234 are 4123, 3412 and 2341; we have

$$\begin{aligned} \int d^4x(f^1b^2f^3b^4) &= + \int d^4x(b^4f^1b^2f^3) \\ &= - \int d^4x(f^3b^4f^1b^2) \\ &= - \int d^4x(b^2f^3b^4f^1), \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} \int d^4x(f^1b^2b^3f^4) &= - \int d^4x(f^4f^1b^2f^3) \\ &= - \int d^4x(b^3f^4f^1b^2) \\ &= - \int d^4x(b^2b^3f^4f^1). \end{aligned} \tag{A.4}$$

All those combinations will arise in our work.

Appendix B

BRSW Model

B.1 Functional Derivative

B.1.1 Functional Derivative and Star Product

We want to calculate

$$
\begin{aligned}
&\frac{\delta}{\delta\psi_\mu(y)}\int d^4x[\phi_\alpha(x)\star\psi_\beta(x)\star\varphi_\gamma(x)] \\
&= \frac{\delta}{\delta\psi_\mu(y)}\int d^4x(\pm)[\psi_\beta(x)\star\varphi_\gamma(x)\star\phi_\alpha(x)] \\
&= \frac{\delta}{\delta\psi_\mu(y)}\frac{1}{(2\pi)^{12}}\int d^4x\int d^4k_1\int d^4k_2\int d^4k_3 e^{i(k_1+k_2+k_3)x}e^{-\frac{i}{2}k_1\epsilon\theta k_2-\frac{i}{2}k_2\epsilon\theta k_3-\frac{i}{2}k_1\epsilon\theta k_3} \\
&\qquad\qquad\times\Big(\pm\tilde{\psi}_\beta(k_1)\tilde{\varphi}_\gamma(k_2)\tilde{\phi}_\alpha(k_3)\Big).
\end{aligned}
$$

where we have a plus sign for bosons and a minus sign for fermions.

The derivation gives

$$
\begin{aligned}
&\frac{\delta}{\delta\psi_\mu(y)}\int d^4x[\phi_\alpha(x)\star\psi_\beta(x)\star\varphi_\gamma(x)] = \\
&= \frac{1}{(2\pi)^{12}}\int d^4x\int d^4k_1\int d^4k_2\int d^4k_3 e^{i(k_1+k_2+k_3)x}e^{-\frac{i}{2}k_1\epsilon\theta k_2-\frac{i}{2}k_2\epsilon\theta k_3-\frac{i}{2}k_1\epsilon\theta k_3} \\
&\qquad\qquad\times\Big(\pm\frac{\delta\tilde{\psi}_\beta(k_1)}{\delta\psi_\mu(y)}\tilde{\varphi}_\gamma(k_2)\tilde{\phi}_\alpha(k_3)\Big) \\
&= \frac{1}{(2\pi)^{8}}\int d^4k_1\int d^4k_2\int d^4k_3\delta^4(k_1+k_2+k_3)e^{-\frac{i}{2}k_1\epsilon\theta k_2-\frac{i}{2}k_2\epsilon\theta k_3-\frac{i}{2}k_1\epsilon\theta k_3} \\
&\qquad\qquad\times\Big(\pm\frac{\delta\tilde{\psi}_\beta(k_1)}{\delta\psi_\mu(y)}\tilde{\varphi}_\gamma(k_2)\tilde{\phi}_\alpha(k_3)\Big).
\end{aligned}
\tag{B.1}
$$

The calculation of the term including a variation leads to:

$$
\frac{\delta\tilde{\psi}_\beta(k_1)}{\delta\psi_\mu(y)} = \int d^4x e^{-ik_1x}\frac{\delta\psi_\beta(x)}{\delta\psi_\mu(y)} = \int d^4x e^{-k_1x}\delta_{\beta\mu}\delta^4(x-y) = \delta_{\beta\mu}e^{-ik_1y}. \tag{B.2}
$$

Inserting the last expression into (B.1) leads to

$$\begin{aligned}
&\frac{\delta}{\delta\psi_\mu(y)}\int d^4x[\phi_\alpha(x)\star\psi_\beta(x)\star\varphi_\gamma(x)] = \\
&= \frac{1}{(2\pi)^8}\int d^4k_1\int d^4k_2\int d^4k_3\delta^4(k_1+k_2+k_3)e^{-\frac{i}{2}k_1\epsilon\theta k_2-\frac{i}{2}k_2\epsilon\theta k_3-\frac{i}{2}k_1\epsilon\theta k_3} \\
&\qquad\qquad \times\Big(\pm\,\delta_{\beta\mu}e^{-ik_1y}\tilde{\varphi}_\gamma(k_2)\tilde{\phi}_\alpha(k_3)\Big).
\end{aligned}$$

The use of $\delta^4(k_1+k_2+k_3)$ for elimination of k_1 ($k_1=-k_2-k_3$) brings

$$\begin{aligned}
&\frac{\delta}{\delta\psi_\mu(y)}\int d^4x[\phi_\alpha(x)\star\psi_\beta(x)\star\varphi_\gamma(x)] = \\
&= \frac{1}{(2\pi)^8}\int d^4k_2\int d^4k_3\Big(\pm\,\delta_{\beta\mu}e^{i(k_2+k_3)y}e^{-\frac{i}{2}k_2\epsilon\theta k_3}\tilde{\varphi}_\gamma(k_2)\tilde{\phi}_\alpha(k_3)\Big) \\
&= \pm\delta_{\beta\mu}\Big(\varphi_\gamma(y)\star\phi_\alpha(y)\Big),
\end{aligned} \tag{B.3}$$

which was the desired result.

B.2 Partial Integration

B.2.1 Partial Integration and Star Product

The first expression is given by (3.15) from the main text. Since it represents a bilinear expression, we can omit the star and have

$$\begin{aligned}
\int d^4x A(x)\star\partial_\mu B_\mu(x) &= \int d^4x A(x)\partial_\mu B_\mu(x) \\
&= \int d^4x\partial_\mu\Big(A(x)B_\mu(x)\Big) - \int d^4x\partial_\mu A(x)B_\mu(x).
\end{aligned}$$

Due to the fact that the first term of the last expression represents a surface term, we get

$$\begin{aligned}
\int d^4x A(x)\star\partial_\mu B_\mu(x) &= -\int d^4x\partial_\mu A(x)\partial_\mu B_\mu(x) \\
&= -\int d^4x\partial_\mu A(x)\star\partial_\mu B_\mu(x).
\end{aligned} \tag{B.4}$$

The second expression is given by Eq. (3.16):

$$\int d^4x A(x)\star\frac{1}{\tilde{\Box}}B(x) = \int d^4x\frac{1}{\tilde{\Box}}A(x)\star B(x).$$

At this point we will stop and restart using new definitions

$$A'(x) := \frac{1}{\tilde{\Box}}A(x), \qquad B'(x) := \frac{1}{\tilde{\Box}}B(x). \tag{B.5}$$

and

$$\frac{\square}{\square} \equiv 1. \tag{B.6}$$

Therefore, we write for Eq. (3.16):

$$\begin{aligned}
&\int d^4x A(x) \star \frac{1}{\tilde{\square}} B(x) = \\
&\quad = \int d^4x \tilde{\square} A'(x) B'(x) = \int d^4x \tilde{\partial}_\alpha \tilde{\partial}_\alpha A'(x) B'(x) \\
&\quad = \int d^4x \tilde{\partial}_\alpha \Big(\tilde{\partial}_\alpha A'(x) B'(x) \Big) - \int d^4x \Big(\tilde{\partial}_\alpha A'(x) \tilde{\partial}_\alpha B'(x) \Big).
\end{aligned}$$

The first term of the last expression represents a surface term and hence vanishes. The second one reads

$$\begin{aligned}
&- \int d^4x \Big(\tilde{\partial}_\alpha A'(x) \tilde{\partial}_\alpha B'(x) \Big) = \\
&\quad = - \int d^4x \tilde{\partial}_\alpha \Big(A'(x) \tilde{\partial}_\alpha B'(x) \Big) + \int d^4x A'(x) \tilde{\partial}_\alpha \tilde{\partial}_\alpha B'(x).
\end{aligned}$$

The surface term vanishes and the insertion of (B.5) leads to

$$\begin{aligned}
&\int d^4x A(x) \star \frac{1}{\tilde{\square}} B(x) = \\
&\quad = \int d^4x A'(x) \tilde{\partial}_\alpha \tilde{\partial}_\alpha B'(x)) = \int d^4x A'(x) \tilde{\square} B'(x) \\
&\quad = \int d^4x \frac{1}{\tilde{\square}} A(x) B(x) = \int d^4x \frac{1}{\tilde{\square}} A(x) \star B(x)
\end{aligned}$$

And finally the last expression is given by Eq. (3.17):

$$\begin{aligned}
&\int d^4x A(x) \star B(x) \star \partial_\mu C_\mu(x) = \int d^4x \Big(A(x) \star B(x) \Big) \partial_\mu C_\mu(x) \\
&\quad = - \int d^4x \partial_\mu \Big(A(x) \star B(x) \Big) C_\mu(x) = - \int d^4x \partial_\mu \Big(A(x) \star B(x) \Big) \star C_\mu(x).
\end{aligned}$$

B.3 BRST Transformation

B.3.1 BRST Transformation and Invariance

We want to show the BRST invariance for our action S. If we can show that our action can be written like:

$$sS = s \int d^4x \mathcal{L} = s \int d^4x s\mathcal{A} = \int d^4x s^2 \mathcal{A} = 0, \tag{B.7}$$

it is easy to show invariance by nilpotency ($s^2 = 0$):

$$\begin{aligned} sS &= s(S_{inv} + S_{ghost} + S_{gf} + S_{aux} + S_{break} + S_{ext}) \\ &= s(S_{inv} + S_{gf'} + S_{aux} + S_{break} + S_{ext}) \\ &= sS_{inv} + sS_{gf'} + sS_{aux} + sS_{break} + sS_{ext}. \end{aligned} \tag{B.8}$$

Taking a closer look at each term of (3.19), we will see that only the term S_{inv} remains

$$\begin{aligned} sS_{inv} &= \int d^4x \frac{1}{4} s(F_{\mu\nu} \star F_{\mu\nu}), \\ sS_{gf'} &= \int d^4x s^2(\bar{c} \star (\partial_\mu A_\mu - \frac{\alpha}{2} b)) = 0, \\ sS_{aux} &= -\int d^4x s^2(\bar{\psi}_{\mu\nu} \star B_{\mu\nu}) = 0, \\ sS_{break} &= \int d^4x s^2 \big\{ (\bar{Q}_{\mu\nu\alpha\beta} \star B_{\mu\nu} + Q_{\mu\nu\alpha\beta} \star \bar{B}_{\mu\nu}) \star \frac{1}{\tilde{\square}} (f_{\alpha\beta} + \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{f}) \big\} = 0, \\ sS_{ext} &= \int d^4x s^2(-\Omega^A_\mu \star A_\mu + \Omega^c \star c) = 0. \end{aligned} \tag{B.9}$$

As we have mentioned above, we must act with the BRST operator s on the term S_{inv} to make sure that this transformation leads to zero,

$$\begin{aligned} sS_{inv} &= s \int d^4x \frac{1}{4} F_{\mu\nu} F_{\mu\nu} \\ &= \int d^4x \frac{1}{4} (sF_{\mu\nu} F_{\mu\nu} + F_{\mu\nu} sF_{\mu\nu}). \end{aligned} \tag{B.10}$$

First of all we calculate the expression:

$$\begin{aligned} & sF_{\mu\nu} = \\ &= s(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]) \\ &= \partial_\mu s A_\nu - \partial_\nu s A_\mu - ig[sA_\mu, A_\nu] - ig[A_\mu, sA_\nu] \\ &= \partial_\mu D_\nu c - \partial_\nu D_\mu c - ig[D_\mu c, A_\nu] - ig[A_\mu, D_\nu c] \\ &= \partial_\mu(\partial_\nu - ig[A_\nu, c]) - \partial_\nu(\partial_\mu - ig[A_\mu, c]) \\ &\quad - ig[(\partial_\mu c - ig[A_\mu, c]), A_\nu] - ig[A_\mu, (\partial_\nu c - ig[A_\nu, c])] \\ &= -ig\partial_\mu[A_\nu, c] + ig\partial_\nu[A_\mu, c] \\ &\quad - ig([\partial_\mu c, A_\nu] - ig[[A_\mu, c], A_\nu]) - ig([A_\mu, \partial_\nu c] - ig[A_\mu, [A_\nu, c]]) \\ &= -ig\partial_\mu[A_\nu, c] + ig\partial_\nu[A_\mu, c] - ig[\partial_\mu c, A_\nu] - ig[A_\mu, \partial_\nu c] \\ &\quad - g^2[[A_\mu, c], A_\nu] - g^2[A_\mu, [A_\nu, c]] \\ &= -ig[\partial_\mu A_\nu, c] + ig[\partial_\nu A_\mu, c] - g^2[[A_\mu, c], A_\nu] - g^2[A_\mu, [A_\nu, c]], \end{aligned}$$

introduce the Jacobi-identity:

$$\begin{aligned} & [[A, B], C] + [[B, C], A] + [[C, A], B] = \\ &= [AB - BA, C] + [BC - CB, A] + [CA - AC, B] \\ &= [AB, C] - [BA, C] + [BC, A] - [CB, A] + [CA, B] - [AC, B] \\ &= ABC - CAB - BAC + CBA + BCA - ABC - CBA + ACB \\ &\quad + CAB - BCA - ACB + BAC = 0, \end{aligned} \tag{B.11}$$

use this identity for the term

$$-g^2[[A_\mu, c], A_\nu] = g^2[[c, A_\nu], A_\mu] + g^2[[A_\nu, A_\mu], c].$$

Therefore we have

$$\begin{aligned}
&sF_{\mu\nu} = \\
&= -ig[\partial_\mu A_\nu, c] + ig[\partial_\nu A_\mu, c] + g^2[[c, A_\nu], A_\mu] + g^2[[A_\nu, A_\mu], c] - g^2[A, [A_\nu, c]] \\
&= -ig[\partial_\mu A_\nu, c] + ig[\partial_\nu A_\mu, c] + g^2[[A_\nu, A_\mu], c] \\
&= -ig[\partial_\mu A_\nu - \partial_\nu A_\mu, c] + g^2[[A_\nu, A_\mu], c] \\
&= -ig[\partial_\mu A_\nu - \partial_\nu A_\mu + \frac{g^2}{(-ig)}[A_\nu, A_\mu], c] \\
&= -ig[\partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\nu, A_\mu], c] \\
&= -ig[\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], c] \\
&= -ig[F_{\mu\nu}, c].
\end{aligned} \tag{B.12}$$

Inserting the last expression into Eq. (B.10) we get

$$\begin{aligned}
sS_{inv} &= \int d^4x \frac{1}{4}(sF_{\mu\nu}F_{\mu\nu} + F_{\mu\nu}sF_{\mu\nu}) \\
&= \int d^4x \frac{1}{4}(-ig[F_{\mu\nu}, c]F_{\mu\nu} - igF_{\mu\nu}[F_{\mu\nu}, c]) \\
&= \int d^4x \frac{1}{4}(-ig)(F_{\mu\nu}cF_{\mu\nu} - cF_{\mu\nu}F_{\mu\nu} + F_{\mu\nu}F_{\mu\nu}c - F_{\mu\nu}cF_{\mu\nu}) \\
&= 0,
\end{aligned} \tag{B.13}$$

which shows that the BRST invariance is fulfilled for the whole action.

B.3.2 Calculation of $s(f_{\alpha\beta}+\sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f})$

$$
\begin{aligned}
&s(f_{\alpha\beta}+\sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f}) = \\
&= s(\partial_\alpha A_\beta - \partial_\beta A_\alpha + \sigma\frac{\theta_{\alpha\beta}}{2}\theta_{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)) \\
&= s(\partial_\alpha A_\beta - \partial_\beta A_\alpha + \sigma\frac{\theta_{\alpha\beta}}{2}(\theta_{\nu\mu}\partial_\nu A_\mu - \theta_{\mu\nu}\partial_\nu A_\mu)) = \\
&= s(\partial_\alpha A_\beta - \partial_\beta A_\alpha - \sigma\theta_{\alpha\beta}\theta_{\mu\nu}\partial_\nu A_\mu) = \\
&= \partial_\alpha sA_\beta - \partial_\beta sA_\alpha - \sigma\theta_{\alpha\beta}\theta_{\mu\nu}\partial_\nu sA_\mu = \\
&= \partial_\alpha D_\beta c - \partial_\beta D_\alpha c - \sigma\theta_{\alpha\beta}\theta_{\mu\nu}\partial_\nu D_\mu c = \\
&= \partial_\alpha(\partial_\beta c - ig[A_\beta, c]) - \partial_\beta(\partial_\alpha c - ig[A_\alpha, c]) - \sigma\theta_{\alpha\beta}\theta_{\mu\nu}\partial_\nu(\partial_\mu c - ig[A_\mu, c]) = \\
&= (-ig)\partial_\alpha(A_\beta c - cA_\beta) + (ig)\partial_\beta(A_\alpha c - cA_\alpha) + (ig)\sigma\theta_{\alpha\beta}\theta_{\mu\nu}\partial_\nu(A_\mu c - cA_\mu) = \\
&= (+ig)\{(c\partial_\alpha A_\beta - \partial_\alpha A_\beta c - c\partial_\beta A_\alpha + \partial_\beta A_\alpha c) + (\partial_\alpha cA_\beta - A_\beta\partial_\alpha c - \partial_\beta cA_\alpha + A_\alpha\partial_\beta c) \\
&\quad - \sigma\theta_{\alpha\beta}\theta_{\mu\nu}(c\partial_\nu A_\mu - \partial_\nu A_\mu c + \partial_\nu cA_\mu - A_\mu\partial_\nu c)\} \\
&= (+ig)[c, \partial_\alpha A_\beta] + (-ig)[c, \partial_\beta A_\alpha] + (+ig)[\partial_\alpha c, A_\beta] + (-ig)[\partial_\beta c, A_\alpha] \\
&\qquad + (-ig)\sigma\theta_{\alpha\beta}\theta_{\mu\nu}\{[c, \partial_\nu A_\mu] + [\partial_\nu c, A_\mu]\} = \\
&= (+ig)[c, \partial_\alpha A_\beta - \partial_\beta A_\alpha] + (+ig)\sigma\frac{\theta_{\alpha\beta}}{2}\theta_{\mu\nu}[c, \partial_\mu A_\nu - \partial_\nu A_\mu] \\
&\qquad + (+ig)[\partial_\alpha c, A_\beta] + (-ig)[\partial_\beta c, A_\alpha] + (-ig)\sigma\theta_{\alpha\beta}\theta_{\mu\nu}[\partial_\nu c, A_\mu] = \\
&= (+ig)[c, (f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f})] \\
&\qquad + (+ig)[\partial_\alpha c, A_\beta] + (-ig)[\partial_\beta c, A_\alpha] + (-ig)\sigma\theta_{\alpha\beta}\theta_{\mu\nu}[\partial_\nu c, A_\mu] = \\
&= (+ig)\{[c, (f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f})] \\
&\qquad + [\partial_\alpha c, A_\beta] - [\partial_\beta c, A_\alpha] - \sigma\theta_{\alpha\beta}\theta_{\mu\nu}[\partial_\nu c, A_\mu]\}.
\end{aligned}
\tag{B.14}
$$

B.4 Symmetries

At first we introduce the definitions:

$$S^a := S^{\text{only terms with a}}, \quad S^{ab} := S^{\text{only terms including the product of ab}}, \tag{B.15}$$

where

$$a, b \in \{A, b, c, \bar{c}, \psi, \bar{\psi}, B, \bar{B}, J, \bar{J}, Q, \bar{Q}, (sA), \Omega^A_\mu, \Omega^c\}.$$

B.4.1 Calculation of the Ghost Equation

We start with (3.34) and insert Eq. (3.18). In the next step it will be wise to use only the relevant part of the action under the use of (B.15).

$$
\begin{aligned}
\mathcal{G}(S) &= \partial_\mu\frac{\delta S}{\delta\Omega^A_\mu} + \frac{\delta S}{\delta\bar{c}} = \partial_\mu\frac{\delta S^{\Omega^A}}{\delta\Omega^A_\mu} + \frac{\delta S^{\bar{c}}}{\delta\bar{c}} \\
&= \partial_\mu(sA_\mu) + (-\partial_\mu D_\mu c) = \partial_\mu D_\mu c - \partial_\mu D_\mu c = 0.
\end{aligned}
\tag{B.16}
$$

B.4.2 Calculation of the Antighost Equation

The antighost equation is given by Eq. (3.36) from the main text:

$$\overline{\mathcal{G}}(S) = \int d^4x \frac{\delta S}{\delta c(x)} = 0.$$

This means for our action (3.19) that we have to look at

$$S^c = \int d^4y \Big[-\bar{c}\partial_\mu D_\mu c + \Omega^A_\mu D_\mu c + ig\Omega^c cc - (\bar{Q}_{\mu\nu\alpha\beta} B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu}) \\ \times \Big((+ig)\{[c, (f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f})] + [\partial_\alpha c, A_\beta] - [\partial_\beta c, A_\alpha] - \sigma\theta_{\alpha\beta}\theta_{\mu\nu}[\partial_\nu c, A_\mu]\} \Big) \Big], \tag{B.17}$$

where we have used Eq. (B.14).
Therefore we have

$$\begin{aligned} S^c &= \\ &= \int d^4y \Big[-\bar{c}\partial_\mu(\partial_\mu c - ig[A_\mu, c]) + \Omega^A_\mu(\partial_\mu c - ig[A_\mu, c]) + ig\Omega^c cc \\ &\quad - (\bar{Q}_{\mu\nu\alpha\beta} B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu}) \\ &\quad \times \Big((+ig)\{c(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f}) - (f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f})c + \partial_\alpha c A_\beta - A_\beta\partial_\alpha c \\ &\quad - \partial_\beta c A_\alpha + A_\alpha\partial_\beta c - \sigma\theta_{\alpha\beta}\theta_{\mu\nu}(\partial_\nu c A_\mu - A_\mu\partial_\nu c)\} \Big) \Big] \\ &= \int d^4y \Big[-\bar{c}\partial_\mu(\partial_\mu c - ig(A_\mu c - cA_\mu)) + \Omega^A_\mu(\partial_\mu c - ig(A_\mu c - cA_\mu)) + ig\Omega^c cc \\ &\quad - (\bar{Q}_{\mu\nu\alpha\beta} B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu}) \\ &\quad \times \Big((+ig)\{c(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f}) - (f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f})c + \partial_\alpha c A_\beta - A_\beta\partial_\alpha c \\ &\quad - \partial_\beta c A_\alpha + A_\alpha\partial_\beta c - \sigma\theta_{\alpha\beta}\theta_{\mu\nu}(\partial_\nu c A_\mu - A_\mu\partial_\nu c)\} \Big) \Big] \\ &= \int d^4y \Big[-\bar{c}\Box c + ig\bar{c}\partial_\mu(A_\mu c - cA_\mu) + \Omega^A_\mu\partial_\mu c - ig\Omega^A_\mu(A_\mu c - cA_\mu) + ig\Omega^c cc \\ &\quad - (\bar{Q}_{\mu\nu\alpha\beta} B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu}) \\ &\quad \times \Big((+ig)\{c(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f}) - (f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f})c + \partial_\alpha c A_\beta - A_\beta\partial_\alpha c \\ &\quad - \partial_\beta c A_\alpha + A_\alpha\partial_\beta c - \sigma\theta_{\alpha\beta}\theta_{\mu\nu}(\partial_\nu c A_\mu - A_\mu\partial_\nu c)\} \Big) \Big]. \end{aligned}$$

The next step involves partial integration and cyclic permutation:

$$
\begin{aligned}
&S^c = \\
&= \int d^4y \Big[c\Box\bar{c} - igc\bar{c}\partial_\mu A_\mu + igc\partial_\mu(\bar{c}A_\mu) - igc\partial_\mu(A_\mu\bar{c}) + igc\partial_\mu A_\mu\bar{c} \\
&\qquad + c\partial_\mu\Omega^A_\mu + igc\Omega^A_\mu A_\mu - igcA^\mu\Omega^A_\mu + igcc\Omega^c \\
&\qquad + (ig)\times\Big(c(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f})(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu}) \\
&\qquad - c(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu})(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f}) \\
&\qquad - c\partial_\alpha(A_\beta(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu})) + c\partial_\alpha((\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu})A_\beta) \\
&\qquad + c\partial_\beta(A_\alpha(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu})) - c\partial_\beta((\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu})A_\alpha) \\
&\qquad + \sigma\theta_{\alpha\beta}\theta_{\mu\nu}c\partial_\nu(A_\mu(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu})) \\
&\qquad - \sigma\theta_{\alpha\beta}\theta_{\mu\nu}c\partial_\nu((\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu})A_\mu)\Big)\Big].
\end{aligned} \tag{B.18}
$$

So we will get for the functional derivative

$$\frac{\delta S^c}{\delta c(x)} = \int d^4y(\Box\bar{c} + \partial_\mu\Omega^A_\mu)\delta^4(y-x) = \Big(\partial_\mu(\partial_\mu\bar{c} + \Omega^A_\mu)\Big)(x), \tag{B.19}$$

and finally have for the antighost equation

$$\overline{\mathcal{G}}(S) = \int d^4x\Big(\partial_\mu(\partial_\mu\bar{c} + \Omega^A_\mu)\Big). \tag{B.20}$$

To make sure the last equation leads to zero we must find an expression for Ω^A_μ. Due to the fact that Ω^A_μ is the external source for sA_μ we will make the ansatz:

$$\frac{\delta S}{\delta(sA_\epsilon)} = 0, \tag{B.21}$$

which is the equation of motion for the “field“ (sA_μ) and shows the unphysical character of (sA_μ), given by

$$s(sA_\mu) = s^2A_\mu = 0. \tag{B.22}$$

With

$$\begin{aligned}
S^{sA} &= \int d^4y\big[-\bar{c}\partial_\mu(sA_\mu)+\Omega^A_\mu(sA_\mu)\\
&\qquad -(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu}+Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu})\frac{1}{\tilde{\square}}(\partial_\alpha sA_\beta-\partial_\beta sA_\alpha-\sigma\theta_{\alpha\beta}\theta_{\gamma\delta}\partial_\delta(sA_\gamma))\big]\\
&= \int d^4y\big[\partial_\mu\bar{c}(sA_\mu)+\Omega^A_\mu(sA_\mu)+\partial_\alpha\frac{1}{\tilde{\square}}(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu}+Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu})(sA_\beta)\\
&\qquad -\partial_\beta\frac{1}{\tilde{\square}}(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu}+Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu})(sA_\alpha)\\
&\qquad -\sigma\theta_{\alpha\beta}\theta_{\gamma\delta}\partial_\delta\frac{1}{\tilde{\square}}(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu}+Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu})(sA_\gamma)\big]\\
&= \int d^4y\big[-(sA_\mu)\partial_\mu\bar{c}-(sA_\mu)\Omega^A_\mu-(sA_\mu)\partial_\alpha\frac{1}{\tilde{\square}}(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu}+Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu})\\
&\qquad +(sA_\mu)\partial_\beta\frac{1}{\tilde{\square}}(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu}+Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu})\\
&\qquad +\sigma\theta_{\alpha\beta}\theta_{\gamma\delta}(sA_\gamma)\partial_\delta\frac{1}{\tilde{\square}}(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu}+Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu})\big].
\end{aligned}\tag{B.23}$$

The functional derivative gives

$$\begin{aligned}
\frac{\delta S^{sA}}{\delta(sA_\epsilon)} &= \big[-\partial_\epsilon\bar{c}-\Omega^A_\epsilon-\partial_\alpha\frac{1}{\tilde{\square}}(\bar{Q}_{\mu\nu\alpha\epsilon}B_{\mu\nu}+Q_{\mu\nu\alpha\epsilon}\bar{B}_{\mu\nu})\\
&\qquad +\partial_\beta\frac{1}{\tilde{\square}}(\bar{Q}_{\mu\nu\epsilon\beta}B_{\mu\nu}+Q_{\mu\nu\epsilon\beta}\bar{B}_{\mu\nu})\\
&\qquad +\sigma\theta_{\alpha\beta}\theta_{\epsilon\delta}\partial_\delta\frac{1}{\tilde{\square}}(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu}+Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu})\big](x)=0.
\end{aligned}\tag{B.24}$$

This leads to

$$\begin{aligned}
\Omega^A_\epsilon &= -\partial_\epsilon\bar{c}-\partial_\alpha\frac{1}{\tilde{\square}}(\bar{Q}_{\mu\nu\alpha\epsilon}B_{\mu\nu}+Q_{\mu\nu\alpha\epsilon}\bar{B}_{\mu\nu})\\
&\quad +\partial_\beta\frac{1}{\tilde{\square}}(\bar{Q}_{\mu\nu\epsilon\beta}B_{\mu\nu}+Q_{\mu\nu\epsilon\beta}\bar{B}_{\mu\nu})+\sigma\theta_{\alpha\beta}\theta_{\epsilon\delta}\partial_\delta\frac{1}{\tilde{\square}}(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu}+Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu}).
\end{aligned}\tag{B.25}$$

The antighost equation can now be written as

$$\overline{\mathcal{G}}(S)=\int d^4x\Big(\partial_\epsilon(\partial_\epsilon\bar{c}+\Omega^A_\epsilon)\Big)=\int d^4x\Big(\square\bar{c}-\square\bar{c}\Big)=0^1.\tag{B.26}$$

B.4.3 Calculation of the Symmetry $\mathcal{U}(S)$

The symmetry $\mathcal{U}_{\alpha\beta\mu\nu}(S)$ is given by Eq. (3.38) from the main text:

$$\begin{aligned}
\mathcal{U}_{\alpha\beta\mu\nu}(S) = \int d^4x\Big[&B_{\alpha\beta}\frac{\delta S}{\delta B_{\mu\nu}}-\bar{B}_{\mu\nu}\frac{\delta S}{\delta\bar{B}_{\alpha\beta}}+J_{\alpha\beta\sigma\rho}\frac{\delta S}{\delta J_{\mu\nu\rho\sigma}}-\bar{J}_{\mu\nu\rho\sigma}\frac{\delta S}{\delta\bar{J}_{\alpha\beta\rho\sigma}}\\
&+\psi_{\alpha\beta}\frac{\delta S}{\delta\psi_{\mu\nu}}-\bar{\psi}_{\mu\nu}\frac{\delta S}{\delta\bar{\psi}_{\alpha\beta}}+Q_{\alpha\beta\rho\sigma}\frac{\delta S}{\delta Q_{\mu\nu\rho\sigma}}-\bar{Q}_{\mu\nu\rho\sigma}\frac{\delta S}{\delta\bar{Q}_{\alpha\beta\rho\sigma}}\Big]=0.
\end{aligned}$$

[1] $\partial_\epsilon\partial_\alpha M_{\mu\nu\alpha\epsilon}=0, M_{\mu\nu\alpha\epsilon}\in\{\bar{Q}_{\mu\nu\alpha\epsilon},Q_{\mu\nu\alpha\epsilon}\}$ and $\theta_{\epsilon\delta}\partial_\delta\partial_\epsilon=0$.

We look at a renamed action, with the renamed terms given by Eq. (3.19), use the Landau gauge ($\alpha \to 0$):

$$
\begin{aligned}
S = \int d^4y \Big[& \frac{1}{4} F_{\gamma\delta} F_{\gamma\delta} + b \partial_\gamma A_\gamma - \bar{c} \partial_\gamma D_\gamma c - \bar{B}_{\gamma\delta} B_{\gamma\delta} + \bar{\psi}_{\gamma\delta} \psi_{\gamma\delta}) \\
& + (\bar{J}_{\gamma\delta\epsilon\lambda} B_{\gamma\delta} + J_{\gamma\delta\epsilon\lambda} \bar{B}_{\gamma\delta}) \frac{1}{\tilde{\square}} (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \\
& - \bar{Q}_{\gamma\delta\epsilon\lambda} \psi_{\gamma\delta} \frac{1}{\tilde{\square}} (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \\
& - (\bar{Q}_{\gamma\delta\epsilon\lambda} B_{\mu\nu} + Q_{\gamma\delta\epsilon\lambda} \bar{B}_{\gamma\delta}) \frac{1}{\tilde{\square}} s (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \\
& + \Omega^A_\gamma s A_\gamma + \Omega^c s c \Big],
\end{aligned}
$$

and insert the following expression into (3.38).

The explicit calculation starts with a closer look at each term of Eq. (3.38).

1)

$$
\begin{aligned}
& \frac{\delta S^B}{\delta B_{\mu\nu}(x)} = \frac{\delta}{\delta B_{\mu\nu}(x)} \times \\
& \int d^4y \Big[- \bar{B}_{\gamma\delta} B_{\gamma\delta} + \bar{J}_{\gamma\delta\epsilon\lambda} B_{\gamma\delta} \frac{1}{\tilde{\square}} (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) - \bar{Q}_{\gamma\delta\epsilon\lambda} B_{\gamma\delta} \frac{1}{\tilde{\square}} s (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \Big](y) \\
& = \frac{\delta}{\delta B_{\mu\nu}(x)} \times \\
& \int d^4y \Big[- B_{\gamma\delta} \bar{B}_{\gamma\delta} + B_{\gamma\delta} \frac{1}{\tilde{\square}} (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \bar{J}_{\gamma\delta\epsilon\lambda} + B_{\gamma\delta} \frac{1}{\tilde{\square}} s (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \bar{Q}_{\gamma\delta\epsilon\lambda} \Big](y) \\
& = \int d^4y \Big[- \frac{1}{2} (\delta_{\gamma\mu} \delta_{\delta\nu} - \delta_{\gamma\nu} \delta_{\delta\mu}) \bar{B}_{\gamma\delta} + \frac{1}{2} (\delta_{\gamma\mu} \delta_{\delta\nu} - \delta_{\gamma\nu} \delta_{\delta\mu}) \frac{1}{\tilde{\square}} (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \bar{J}_{\gamma\delta\epsilon\lambda} \\
& \qquad + \frac{1}{2} (\delta_{\gamma\mu} \delta_{\delta\nu} - \delta_{\gamma\nu} \delta_{\delta\mu}) \frac{1}{\tilde{\square}} s (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \bar{Q}_{\gamma\delta\epsilon\lambda} \Big] \delta^4(y - x) \\
& = [-\bar{B}_{\mu\nu} + \frac{1}{\tilde{\square}} (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \bar{J}_{\mu\nu\epsilon\lambda} + \frac{1}{\tilde{\square}} s (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \bar{Q}_{\mu\nu\epsilon\lambda}](x).
\end{aligned}
\tag{B.27}
$$

2)

$$\frac{\delta S^{\bar{B}}}{\delta \bar{B}_{\alpha\beta}(x)} = \frac{\delta}{\delta \bar{B}_{\alpha\beta}(x)}\times$$
$$\int d^4y \Big[-\bar{B}_{\gamma\delta}B_{\gamma\delta} + J_{\gamma\delta\epsilon\lambda}\bar{B}_{\gamma\delta}\frac{1}{\tilde{\Box}}(f_{\epsilon\lambda}+\sigma\frac{\theta_{\epsilon\lambda}}{2}\tilde{f}) - Q_{\gamma\delta\epsilon\lambda}\bar{B}_{\gamma\delta}\frac{1}{\tilde{\Box}}s(f_{\epsilon\lambda}+\sigma\frac{\theta_{\epsilon\lambda}}{2}\tilde{f})\Big](y)$$
$$= \frac{\delta}{\delta B_{\alpha\beta}(x)}\times$$
$$\int d^4y \Big[-B_{\gamma\delta}\bar{B}_{\gamma\delta} + \bar{B}_{\gamma\delta}\frac{1}{\tilde{\Box}}(f_{\epsilon\lambda}+\sigma\frac{\theta_{\epsilon\lambda}}{2}\tilde{f})J_{\gamma\delta\epsilon\lambda} + \bar{B}_{\gamma\delta}\frac{1}{\tilde{\Box}}s(f_{\epsilon\lambda}+\sigma\frac{\theta_{\epsilon\lambda}}{2}\tilde{f})Q_{\gamma\delta\epsilon\lambda}\Big](y)$$
$$= \int d^4y \Big[-\frac{1}{2}(\delta_{\gamma\mu}\delta_{\delta\nu}-\delta_{\gamma\nu}\delta_{\delta\mu})B_{\gamma\delta} + \frac{1}{2}(\delta_{\gamma\mu}\delta_{\delta\nu}-\delta_{\gamma\nu}\delta_{\delta\mu})\frac{1}{\tilde{\Box}}(f_{\epsilon\lambda}+\sigma\frac{\theta_{\epsilon\lambda}}{2}\tilde{f})J_{\gamma\delta\epsilon\lambda}$$
$$+ \frac{1}{2}(\delta_{\gamma\mu}\delta_{\delta\nu}-\delta_{\gamma\nu}\delta_{\delta\mu})\frac{1}{\tilde{\Box}}s(f_{\epsilon\lambda}+\sigma\frac{\theta_{\epsilon\lambda}}{2}\tilde{f})Q_{\gamma\delta\epsilon\lambda}\Big]\delta^4(y-x)$$
$$= [-B_{\alpha\beta} + \frac{1}{\tilde{\Box}}(f_{\epsilon\lambda}+\sigma\frac{\theta_{\epsilon\lambda}}{2}\tilde{f})J_{\alpha\beta\epsilon\lambda} + \frac{1}{\tilde{\Box}}s(f_{\epsilon\lambda}+\sigma\frac{\theta_{\epsilon\lambda}}{2}\tilde{f})Q_{\alpha\beta\epsilon\lambda}](x). \tag{B.28}$$

3)

For the third term

$$\frac{\delta S^J}{\delta J_{\mu\nu\rho\sigma}(x)} = \frac{\delta}{\delta J_{\mu\nu\rho\sigma}(x)}\int d^4y\Big[J_{\gamma\delta\epsilon\lambda}\bar{B}_{\gamma\delta}\frac{1}{\tilde{\Box}}(f_{\epsilon\lambda}+\sigma\frac{\theta_{\epsilon\lambda}}{2}\tilde{f})\Big], \tag{B.29}$$

$J_{\mu\nu\rho\sigma}$ must be viewed accuratly due to the fact that the source $J_{\mu\nu\rho\sigma}$ is antisymmetric in two indices, which means

$$J_{\mu\nu\rho\sigma} = -J_{\nu\mu\rho\sigma} = +J_{\nu\mu\sigma\rho} = -J_{\mu\nu\sigma\rho}, \tag{B.30}$$

and influences the functional derivative in the following sense

$$\frac{\delta J_{\gamma\delta\epsilon\lambda}(y)}{\delta J_{\mu\nu\rho\sigma}(x)} = \frac{1}{2}(\delta_{\gamma\mu}\delta_{\delta\nu}-\delta_{\gamma\nu}\delta_{\delta\mu})\frac{1}{2}(\delta_{\epsilon\rho}\delta_{\lambda\sigma}-\delta_{\epsilon\sigma}\delta_{\lambda\rho})\delta^4(y-x). \tag{B.31}$$

This leads to

$$\frac{\delta S^J}{\delta J_{\mu\nu\rho\sigma}(x)} =$$
$$= \int d^4y\Big[\frac{1}{4}(\delta_{\gamma\mu}\delta_{\delta\nu}-\delta_{\gamma\nu}\delta_{\delta\mu})(\delta_{\epsilon\rho}\delta_{\lambda\sigma}-\delta_{\epsilon\sigma}\delta_{\lambda\rho})\delta^4(y-x)\bar{B}_{\gamma\delta}\frac{1}{\tilde{\Box}}((f_{\epsilon\lambda}+\sigma\frac{\theta_{\epsilon\lambda}}{2}\tilde{f})\Big]$$
$$= \big[\bar{B}_{\mu\nu}\frac{1}{2\tilde{\Box}}[(f_{\rho\sigma}+\sigma\frac{\theta_{\rho\sigma}}{2}\tilde{f}) - (f_{\sigma\rho}+\sigma\frac{\theta_{\sigma\rho}}{2}\tilde{f})]\big](x)$$
$$= [\bar{B}_{\mu\nu}\frac{1}{\tilde{\Box}}(f_{\rho\sigma}+\sigma\frac{\theta_{\rho\sigma}}{2}\tilde{f})](x). \tag{B.32}$$

4)

$$\frac{\delta S^{\bar{J}}}{\delta \bar{J}_{\alpha\beta\rho\sigma}(x)} = \frac{\delta}{\delta \bar{J}_{\alpha\beta\rho\sigma}} \int d^4y \Big[\bar{J}_{\gamma\delta\epsilon\lambda} B_{\gamma\delta} \frac{1}{\tilde{\Box}} (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \Big]$$
$$= \int d^4y \Big[\frac{1}{4} (\delta_{\gamma\alpha}\delta_{\delta\beta} - \delta_{\gamma\beta}\delta_{\delta\alpha})(\delta_{\epsilon\rho}\delta_{\lambda\sigma} - \delta_{\epsilon\sigma}\delta_{\lambda\rho}) \delta^4(y-x) B_{\gamma\delta} \frac{1}{\tilde{\Box}} ((f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \Big]$$
$$= [B_{\alpha\beta} \frac{1}{\tilde{\Box}} (f_{\rho\sigma} + \sigma \frac{\theta_{\rho\sigma}}{2} \tilde{f})](x). \tag{B.33}$$

5)

$$\frac{\delta S^{\psi}}{\delta \psi_{\mu\nu}(x)} = \frac{\delta}{\delta \psi_{\mu\nu}(x)} \int d^4y \Big[\bar{\psi}_{\gamma\delta}\psi_{\gamma\delta} - \bar{Q}_{\gamma\delta\epsilon\lambda}\psi_{\gamma\delta} \frac{1}{\tilde{\Box}} (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \Big]$$
$$= \frac{\delta}{\delta \psi_{\mu\nu}(x)} \int d^4y \Big[- \psi_{\gamma\delta}\bar{\psi}_{\gamma\delta} + \psi_{\gamma\delta} \frac{1}{\tilde{\Box}} (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \bar{Q}_{\gamma\delta\epsilon\lambda} \Big]$$
$$= \int d^4y \Big[- \frac{1}{2} (\delta_{\gamma\mu}\delta_{\delta\nu} - \delta_{\gamma\nu}\delta_{\delta\mu}) \bar{\psi}_{\gamma\delta} + \frac{1}{2} (\delta_{\gamma\mu}\delta_{\delta\nu} - \delta_{\gamma\nu}\delta_{\delta\mu}) \frac{1}{\tilde{\Box}} (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \bar{Q}_{\gamma\delta\epsilon\lambda} \Big]$$
$$\times \delta^4(y-x)$$
$$= [-\bar{\psi}_{\mu\nu} + \frac{1}{\tilde{\Box}} (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \bar{Q}_{\mu\nu\epsilon\lambda}](x). \tag{B.34}$$

6)

$$\frac{\delta S^{\bar{\psi}}}{\bar{\psi}_{\alpha\beta}(x)} = \frac{\delta}{\bar{\psi}_{\alpha\beta}(x)} \int d^4y \bar{\psi}_{\gamma\delta}\psi_{\gamma\delta} = \psi_{\alpha\beta}(x). \tag{B.35}$$

7)

$$\frac{\delta S^{Q}}{\delta Q_{\mu\nu\rho\sigma}} = \frac{\delta}{\delta Q_{\mu\nu\rho\sigma}} \int d^4y \Big[- Q_{\gamma\delta\epsilon\lambda} \bar{B}_{\gamma\delta} \frac{1}{\tilde{\Box}} s(f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \Big]$$
$$= \int d^4y \Big[- \frac{1}{4} (\delta_{\gamma\mu}\delta_{\delta\nu} - \delta_{\gamma\nu}\delta_{\delta\mu})(\delta_{\epsilon\rho}\delta_{\lambda\sigma} - \delta_{\epsilon\sigma}\delta_{\lambda\rho}) \delta^4(y-x) \bar{B}_{\gamma\delta} \frac{1}{\tilde{\Box}} s(f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \Big]$$
$$= [-\bar{B}_{\mu\nu} \frac{1}{\tilde{\Box}} s(f_{\rho\sigma} + \sigma \frac{\theta_{\rho\sigma}}{2} \tilde{f})](x). \tag{B.36}$$

8)

$$\frac{\delta S^{\bar{Q}}}{\delta \bar{Q}_{\alpha\beta\rho\sigma}} = \frac{\delta}{\delta \bar{Q}_{\alpha\beta\rho\sigma}} \int d^4y \Big[- \bar{Q}_{\gamma\delta\epsilon\lambda}\psi_{\gamma\delta} \frac{1}{\tilde{\Box}} (f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) - \bar{Q}_{\gamma\delta\epsilon\lambda} \bar{B}_{\gamma\delta} \frac{1}{\tilde{\Box}} s(f_{\epsilon\lambda} + \sigma \frac{\theta_{\epsilon\lambda}}{2} \tilde{f}) \Big]$$
$$= [-\psi_{\alpha\beta} \frac{1}{\tilde{\Box}} (f_{\rho\sigma} + \sigma \frac{\theta_{\rho\sigma}}{2} \tilde{f}) - B_{\alpha\beta} \frac{1}{\tilde{\Box}} s(f_{\rho\sigma} + \sigma \frac{\theta_{\rho\sigma}}{2} \tilde{f})](x). \tag{B.37}$$

According to (3.38) and (B.27, B.28, B.32, B.33, B.34, B.35, B.36, B.37), we get

$$\begin{aligned}
&\mathcal{U}_{\alpha\beta\mu\nu}(S) = \\
&= \int d^4x \Big[B_{\alpha\beta}[-\bar{B}_{\mu\nu} + \frac{1}{\tilde{\Box}}(f_{\epsilon\lambda} + \sigma\frac{\theta_{\epsilon\lambda}}{2}\tilde{f})\bar{J}_{\mu\nu\epsilon\lambda} + \frac{1}{\tilde{\Box}}s(f_{\epsilon\lambda} + \sigma\frac{\theta_{\epsilon\lambda}}{2}\tilde{f})\bar{Q}_{\mu\nu\epsilon\lambda}] \\
&\qquad - \bar{B}_{\mu\nu}[-B_{\alpha\beta} + \frac{1}{\tilde{\Box}}(f_{\epsilon\lambda} + \sigma\frac{\theta_{\epsilon\lambda}}{2}\tilde{f})J_{\alpha\beta\epsilon\lambda} + \frac{1}{\tilde{\Box}}s(f_{\epsilon\lambda} + \sigma\frac{\theta_{\epsilon\lambda}}{2}\tilde{f})Q_{\alpha\beta\epsilon\lambda}] \\
&\qquad + J_{\alpha\beta\rho\sigma}[\bar{B}_{\mu\nu}\frac{1}{\tilde{\Box}}(f_{\rho\sigma} + \sigma\frac{\theta_{\rho\sigma}}{2}\tilde{f})] - \bar{J}_{\mu\nu\rho\sigma}[B_{\alpha\beta}\frac{1}{\tilde{\Box}}(f_{\rho\sigma} + \sigma\frac{\theta_{\rho\sigma}}{2}\tilde{f})] \\
&\qquad + \psi_{\alpha\beta}[-\bar{\psi}_{\mu\nu} + \frac{1}{\tilde{\Box}}(f_{\epsilon\lambda} + \sigma\frac{\theta_{\epsilon\lambda}}{2}\tilde{f})\bar{Q}_{\mu\nu\epsilon\lambda}] - \bar{\psi}_{\mu\nu}\psi_{\alpha\beta} \\
&\qquad + Q_{\alpha\beta\rho\sigma}[-\bar{B}_{\mu\nu}\frac{1}{\tilde{\Box}}s(f_{\rho\sigma} + \sigma\frac{\theta_{\rho\sigma}}{2}\tilde{f})] \\
&\qquad - \bar{Q}_{\mu\nu\rho\sigma}[-\psi_{\alpha\beta}\frac{1}{\tilde{\Box}}(f_{\rho\sigma} + \sigma\frac{\theta_{\rho\sigma}}{2}\tilde{f}) - B_{\alpha\beta}\frac{1}{\tilde{\Box}}s(f_{\rho\sigma} + \sigma\frac{\theta_{\rho\sigma}}{2}\tilde{f})]\Big] \\
&\qquad = 0.
\end{aligned} \tag{B.38}$$

For the last expression we have used the property of cyclic permutation.

Obviously follows from the last result (B.38) that

$$\mathcal{Q}(S) = \delta_{\alpha\beta}\delta_{\beta\nu}\mathcal{U}_{\alpha\beta\mu\nu}(S) = 0, \tag{B.39}$$

which is equal to (3.41) from the main text.

B.4.4 Anticommutator $\{\bar{\mathcal{G}}, \bar{\mathcal{G}}\}$

First of all a few words about the calculation of terms like $\{A, B\}$, where A, B are some (symmetric) operators. Such a term makes only sense if $\{A, B\}$ can act on the action S

$$\{A, B\}(S) = AB(S) + BA(S). \tag{B.40}$$

If the operators fulfil the symmetry:

$$A(S) = 0, \quad B(S) = 0, \tag{B.41}$$

we get for the anticommutator

$$\{A, B\}(S) = AB(S) + BA(S) = A(0) + B(0) = 0. \tag{B.42}$$

With the help of Eq. (3.36), the calculation of $\{\bar{\mathcal{G}}, \bar{\mathcal{G}}\}$ shows

$$\begin{aligned}
\{\bar{\mathcal{G}}, \bar{\mathcal{G}}\}S &= \Big\{ \int d^4x \frac{\delta}{\delta c(x)}, \int d^4y \frac{\delta}{\delta c(y)} \Big\} S = \\
&= \int d^4x \int d^4y \frac{\delta}{\delta c(x)} \frac{\delta}{\delta c(y)} S + \int d^4x \int d^4y \frac{\delta}{\delta c(y)} \frac{\delta}{\delta c(x)} S \\
&= \int d^4x \int d^4y \Big(\frac{\delta}{\delta c(x)} \frac{\delta}{\delta c(y)} - \frac{\delta}{\delta c(x)} \frac{\delta}{\delta c(y)} \Big) S \\
&= \bar{\mathcal{G}}\bar{\mathcal{G}}(S) + \bar{\mathcal{G}}\bar{\mathcal{G}}(S) = 0.
\end{aligned} \tag{B.43}$$

B.4.5 Commutators $[\bar{\mathcal{G}}, \mathcal{Q}], [\mathcal{Q}, \mathcal{Q}]$

We start with the commutator $[\bar{\mathcal{G}}, \mathcal{Q}]$ and use the already known results (3.36) and (3.41). Therefore we have

$$[\bar{\mathcal{G}}, \mathcal{Q}]S = \bar{\mathcal{G}}\mathcal{Q}(S) - \mathcal{Q}\bar{\mathcal{G}}(S) = 0. \tag{B.44}$$

The next commutator will give the same result

$$[\mathcal{Q}, \mathcal{Q}](S) = \mathcal{Q}\mathcal{Q}(S) - \mathcal{Q}\mathcal{Q}(S) = 0. \tag{B.45}$$

Appendix C

Feynman Rules

C.1 Propagators

C.1.1 Variation of the Transition Amplitude

We try to show that Eq. (4.18) is fulfilled if we use

$$\left.\frac{\delta^2 Z[J]}{\delta J_a(x)\delta J_b(y)}\right|_{J\to 0} = \langle 0|\psi_a(x)\psi_b(y)|0\rangle_{(0)},$$

in which we will insert

$$Z[J] = e^{-Z^c[J]}. \tag{C.1}$$

This leads to

$$\begin{aligned}
\Delta_{ab}(x-y) &= \left.\frac{\delta^2 Z[J]}{\delta J_a(x)\delta J_b(y)}\right|_{J\to 0} \qquad \text{(C.2)}\\
&= \left.\frac{\delta^2 e^{-Z^c[J]}}{\delta J_a(x)\delta J_b(y)}\right|_{J\to 0}\\
&= \left.\frac{\delta}{\delta J_a(x)}\left(-\frac{\delta Z^c[J]}{\delta J_b(y)}e^{-Z^c[J]}\right)\right|_{J\to 0}\\
&= -\left.\frac{\delta^2 Z^c}{\delta J_a(x)\delta J_b(y)}e^{-Z^c[J]}\right|_{J\to 0} + \left.\frac{\delta Z^c[J]}{\delta J_a(x)}\frac{\delta Z^c[J]}{\delta J_b(y)}e^{-Z^c[J]}\right|_{J\to 0}.
\end{aligned}$$

The last term in this expression would correspond to one-point Green functions, which are called tadpoles. These unphysical parts are dropped at this point. The first factor reduces due to the normalisation of Z^c to

$$\Delta^c_{ab}(x-y) = -\left.\frac{\delta^2 Z^c}{\delta J_a(x)\delta J_b(y)}\right|_{J\to 0}. \tag{C.3}$$

C.1.2 Translation Invariance of Propagators

We want to show that an arbitrary Propagator $G(x,y)$ can be presented translation invariant. A typical propagator of this work looks like

$$G(x,y) \approx -\frac{1}{\Box_y}\delta^4(y-x), \qquad G(x,y) \approx -\partial_{\alpha,y}\delta^4(y-x).$$

Two representatives of these propagators are defined by (we will omit the minus sign to avoid confusion):

$$G_1(x,y) = \frac{1}{\Box_y}\delta^4(y-x), \qquad G_2(x,y) = \partial_{\alpha,y}\delta^4(y-x) \tag{C.4}$$

Now the question arises what $G(x,y)$ means. To solve this question we will omit the space coordinate dependency and introduce possible combinations of (C.4).

Will

$$G_1' = \frac{1}{\Box_y}\delta^4(x-y), \qquad G_2' = \partial_{\alpha,y}\delta^4(x-y), \tag{C.5}$$

$$G_1'' = \frac{1}{\Box_x}\delta^4(x-y), \qquad G_2'' = \partial_{\alpha,x}\delta^4(x-y), \tag{C.6}$$

or

$$G_1''' = \frac{1}{\Box_x}\delta^4(y-x), \qquad G_2''' = \partial_{\alpha,x}\delta^4(y-x), \tag{C.7}$$

give the same results as in Eq. (C.4)?

To solve this problem we start with $G_1(x,y)$:

$$G_1(x,y) = \frac{1}{\Box_y}\delta^4(y-x) = \frac{1}{(2\pi)^4}\int d^4k(-\frac{1}{k^2})e^{ik(y-x)}. \tag{C.8}$$

On the other hand G_1', with the use of (3.12) shows

$$\begin{aligned}
G_1' &= \frac{1}{\Box_y}\delta^4(x-y) = \frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty} d^4k(-\frac{1}{k^2})e^{ik(x-y)} \\
&\overset{k\to -k}{=} -\frac{1}{(2\pi)^4}\int_{+\infty}^{-\infty} d^4k(-\frac{1}{k^2})e^{-ik(x-y)} \\
&= \frac{1}{(2\pi)^4}\int d^4k(-\frac{1}{k^2})e^{ik(y-x)} = G_1(x,y)
\end{aligned} \tag{C.9}$$

We do the same procedure for $G_2(x,y)$:

$$G_2(x,y) = \partial_{\alpha,y}\delta^4(y-x) = \frac{1}{(2\pi)^4}\int d^4k(ik_\alpha)e^{ik(y-x)}, \tag{C.10}$$

and G_2'

$$\begin{aligned}
G_2' &= \partial_{\alpha,y}\delta^4(x-y) = \frac{1}{(2\pi)^4}\int d^4k(-ik_\alpha)e^{ik(x-y)} \\
&\overset{k\to -k}{=} -\frac{1}{(2\pi)^4}\int_{+\infty}^{-\infty} d^4k(ik_\alpha)e^{-ik(x-y)} \\
&= \frac{1}{(2\pi)^4}\int d^4k(ik_\alpha)e^{ik(y-x)} = G_2(x,y).
\end{aligned} \tag{C.11}$$

The remaining propagators are calculated in the same way and show:

$$G''_1 = +G_1(x,y), \qquad G'''_1 = +G_1(x,y) \tag{C.12}$$
$$G''_2 = -G_2(x,y), \qquad G'''_2 = -G_2(x,y).$$

At the end we can conclude that the notation $G(x,y)$ only means that the coordinates x and y are involved and is therefore only the notation and nothing more. We must look at the right side of the equation where we discover that the propagator will only depend on the difference of $(x-y)$ and therefore is translation invariant.

C.1.3 Calculation of the Propagator $\tilde{G}^{\bar{\psi}\psi}_{\alpha\beta,\mu\nu}$

We start with the propagator $G^{\bar{\psi}\psi}_{\alpha\beta,\mu\nu}(x,y)$. The fields $\psi_{\mu\nu}$ and $\bar{\psi}_{\mu\nu}$ are antisymmetric ($\psi_{\mu\nu} = -\psi_{\nu\mu}$) and fermionic. The relevant part of the action (3.19) reads

$$S_{aux} = \int d^4x(-\bar{B}_{\mu\nu}B_{\mu\nu} + \bar{\psi}_{\mu\nu}\psi_{\mu\nu}), \tag{C.13}$$
$$S_{aux} = S'_{aux}.$$

As usual we start our procedure for calculating of the propagators.

$$\begin{aligned}
\frac{\delta S'_{aux}}{\delta\bar{\psi}_{\mu\nu}(y)} &= \int d^4x \frac{\delta\bar{\psi}_{\alpha\beta}(x)}{\delta\bar{\psi}_{\mu\nu}(y)}\psi_{\alpha\beta} \\
&= \int d^4x \frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha})\delta^4(x-y)\psi_{\alpha\beta} = \psi_{\mu\nu}(y) \\
&= +j^{\psi}_{\mu\nu}(y), \\
\Rightarrow \psi_{\mu\nu} &= +j^{\psi}_{\mu\nu}.
\end{aligned} \tag{C.14}$$

The calculation of the propagator leads to

$$\begin{aligned}
G^{\overline{\psi}\psi}_{\alpha\beta,\mu\nu}(x,y) &= -\frac{\delta^2 Z^c}{\delta j^{\psi}_{\alpha\beta}(x)\delta j^{\bar{\psi}}_{\mu\nu}(y)} = -\frac{\delta\psi_{\mu\nu}(y)}{\delta j^{\psi}_{\alpha\beta}(x)} \\
&= -\frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha})\delta^4(y-x).
\end{aligned} \tag{C.15}$$

With the transformation:

$$z = y - x,$$

we get

$$G^{\overline{\psi}\psi}_{\alpha\beta,\mu\nu}(z) = -\frac{1}{2(2\pi)^4}\int d^4p(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha})e^{ipz}.$$

So the Fourier transform reads

$$\begin{aligned}
\tilde{G}^{\overline{\psi}\psi}_{\alpha\beta,\mu\nu}(k) &= -\frac{1}{2(2\pi)^4}\int d^4z \int d^4p(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha})e^{iz(p-k)} \\
&= -\frac{1}{2}\int d^4p(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha})\delta^4(p-k) \\
&= -\frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}).
\end{aligned} \tag{C.16}$$

The next propagator to calculate is $G^{\psi\bar{\psi}}_{\alpha\beta,\mu\nu}(x,y)$:

$$
\begin{aligned}
\frac{\delta S'_{aux}}{\delta\psi_{\mu\nu}(y)} &= -\int d^4x\bar{\psi}_{\alpha\beta}\frac{\delta\psi_{\alpha\beta}(x)}{\delta\psi_{\mu\nu}(y)} \\
&= -\int d^4x\bar{\psi}_{\alpha\beta}\frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta}-\delta_{\mu\beta}\delta_{\nu\alpha})\delta^4(x-y) = -\bar{\psi}_{\mu\nu}(y), \\
&= +j^{\bar{\psi}}_{\mu\nu}(y), \\
\Rightarrow \bar{\psi}_{\mu\nu} &= -j^{\psi}_{\mu\nu}.
\end{aligned} \tag{C.17}
$$

The calculation of the propagator shows

$$
\begin{aligned}
G^{\psi\bar{\psi}}_{\alpha\beta,\mu\nu}(x,y) &= -\frac{\delta^2 Z^c}{\delta j^{\bar{\psi}}_{\alpha\beta}(x)\delta j^{\psi}_{\mu\nu}(y)} = -\frac{\delta\bar{\psi}_{\mu\nu}(y)}{\delta j^{\bar{\psi}}_{\alpha\beta}(x)} \\
&= \frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta}-\delta_{\mu\beta}\delta_{\nu\alpha})\delta^4(y-x), \\
G^{\psi\bar{\psi}}_{\alpha\beta,\mu\nu}(z) &= \frac{1}{2(2\pi)^4}\int d^4p(\delta_{\mu\alpha}\delta_{\nu\beta}-\delta_{\mu\beta}\delta_{\nu\alpha})e^{ipz}, \\
\tilde{G}^{\psi\bar{\psi}}_{\alpha\beta,\mu\nu}(k) &= \frac{1}{2(2\pi)^4}\int d^4z\int d^4p(\delta_{\mu\alpha}\delta_{\nu\beta}-\delta_{\mu\beta}\delta_{\nu\alpha})e^{iz(p-k)} \\
&= \frac{1}{2}\int d^4p(\delta_{\mu\alpha}\delta_{\nu\beta}-\delta_{\mu\beta}\delta_{\nu\alpha})\delta^4(p-k) \\
&= \frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta}-\delta_{\mu\beta}\delta_{\nu\alpha}).
\end{aligned} \tag{C.18}
$$

Therefore we have

$$
\tilde{G}^{\bar{\psi}\psi}_{\alpha\beta,\mu\nu}(k) = -\tilde{G}^{\psi\bar{\psi}}_{\alpha\beta,\mu\nu}(k). \tag{C.19}
$$

C.1.4 Calculation of the Propagator $\tilde{G}^{BB}_{\gamma\delta,\sigma\rho}$

The calculation of $G^{BB}_{\gamma\delta,\sigma\rho}(x,y)$ is given by

$$
\begin{aligned}
G^{BB}_{\gamma\delta,\sigma\rho}(x,y) &= -\frac{\delta B_{\sigma\rho}(y)}{\delta j^B_{\gamma\delta}(x)} = -\frac{\delta[(...)_{\sigma\rho,\alpha\beta}j^B_{\alpha\beta}...]}{\delta j^B_{\gamma\delta}(x)} \qquad \text{(C.20)} \\
&= -[(...)_{\sigma\rho,\alpha\beta}\frac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma})...]\delta^4(y-x).
\end{aligned}
$$

The field $B_{\sigma\rho}$ reads

$$
\begin{aligned}
B_{\sigma\rho} &= [j^{\bar{B}}_{\sigma\rho} + \frac{\gamma^2}{2}\frac{1}{\tilde{\square}}(f_{\sigma\rho}+\sigma\frac{\theta_{\sigma\rho}}{2}\tilde{f})] \\
&= [j^{\bar{B}}_{\sigma\rho} + \frac{\gamma^2}{2}\frac{1}{\tilde{\square}}(\partial_\sigma A_\rho - \partial_\rho A_\sigma - \sigma\theta_{\sigma\rho}\theta_{\mu\nu}\partial_\nu A_\mu)].
\end{aligned} \tag{C.21}
$$

Looking at the last equation we see that we must find expressions for the terms $\partial_\sigma A_\rho$, $\partial_\rho A_\sigma$ und $\partial_\nu A_\mu$. We start with $\partial_\sigma A_\rho$. The derivative of Eq. (4.42) leads to

$$\begin{aligned}
\partial_\sigma A_\rho(y) = &\left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\Big\{\partial_\sigma j^A_\rho + (1+\frac{\gamma^4}{\tilde{\Box}^2})(\partial_\sigma\partial_\rho[\frac{\alpha}{\Box}\partial_\beta j^A_\beta - j_b - \gamma^2\frac{\alpha}{\Box}\frac{1}{\tilde{\Box}}\partial_\alpha\partial_\beta(j^{\overline{B}}_{\alpha\beta}+j^B_{\alpha\beta})]) \\
&-\frac{1}{\alpha}\partial_\sigma\partial_\rho j_b - \frac{1}{\alpha}\partial_\sigma\partial_\rho[\frac{\alpha}{\Box}\partial_\beta j^A_\beta - j_b - \gamma^2\frac{\alpha}{\Box}\frac{1}{\tilde{\Box}}\partial_\alpha\partial_\beta(j^{\overline{B}}_{\alpha\beta}+j^B_{\alpha\beta})] \\
&-\gamma^2\partial_\sigma[\partial_\alpha\delta_{\rho\beta} - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\rho]\frac{1}{\tilde{\Box}}(j^{\overline{B}}_{\alpha\beta}+j^B_{\alpha\beta}) \\
&-\gamma^4[2\sigma+2(\frac{\sigma^2}{4})\theta^2]\partial_\sigma\tilde{\partial}_\rho\frac{1}{\tilde{\Box}^2}\left\{\frac{[-\tilde{\partial}_\beta j^A_\beta + \gamma^2(\partial_\alpha\tilde{\partial}_\beta - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\Box})\frac{1}{\tilde{\Box}}(j^{\overline{B}}_{\alpha\beta}+j^B_{\alpha\beta})]}{[(1+\frac{\gamma^4}{\tilde{\Box}^2})(-\Box) - \gamma^4(2\sigma+2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{\Box}}]}\right\}\Big\}(y).
\end{aligned} \tag{C.22}$$

Furthermore we keep an eye only on the sources $j^B_{\alpha\beta}$ of the field $A_\rho(y)$, use the Landau Gauge ($\alpha \to 0$) and introduce the abbreviation X to shorten calculations:

$$X^{-1} \equiv [(1+\frac{\gamma^4}{\tilde{\Box}^2})(-\Box) - \gamma^4(2\sigma+2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{\Box}}]^{-1}, \tag{C.23a}$$

$$\begin{aligned}
X'^{-1} &\equiv [(1+\frac{\gamma^4}{\tilde{\Box}^2})(-\Box) - \gamma^4(2\sigma+2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{\Box}}]^{-1}\frac{1}{(2\pi)^4}\int d^4p e^{ip(y-x)} \\
&= \frac{1}{(2\pi)^4}\int d^4p[(1+\frac{\gamma^4}{\tilde{p}^4})p^2 + \gamma^4(2\sigma+2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{p}^2}]^{-1}e^{ip(y-x)},
\end{aligned} \tag{C.23b}$$

$$\tilde{X} \equiv [(1+\frac{\gamma^4}{\tilde{p}^4})p^2 + \gamma^4(2\sigma+2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{p}^2}]. \tag{C.23c}$$

Therefore we have

$$\begin{aligned}
\partial_\sigma A_\rho = &\left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\Big\{\gamma^2\frac{1}{\Box\tilde{\Box}}\partial_\sigma\partial_\rho\partial_\alpha\partial_\beta j^B_{\alpha\beta} - \gamma^2\partial_\sigma[\partial_\alpha\delta_{\rho\beta} - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\rho]\frac{1}{\tilde{\Box}}j^B_{\alpha\beta} \\
&-\gamma^4[2\sigma+2(\frac{\sigma^2}{4})\theta^2]\partial_\sigma\tilde{\partial}_\rho\frac{1}{\tilde{\Box}^2}\gamma^2\left\{\frac{[(\partial_\alpha\tilde{\partial}_\beta - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\Box})\frac{1}{\tilde{\Box}}j^B_{\alpha\beta}]}{[(1+\frac{\gamma^4}{\mu^2\tilde{\Box}^2})(-\Box) - \gamma^4(2\sigma+2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{\Box}}]}\right\}\Big\} \\
= &\left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\gamma^2\frac{1}{\tilde{\Box}}\Big\{\frac{1}{\Box}\partial_\sigma\partial_\rho\partial_\alpha\partial_\beta j^B_{\alpha\beta} - \partial_\sigma[\partial_\alpha\delta_{\rho\beta} - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\rho]j^B_{\alpha\beta} \\
&-\gamma^4[2\sigma+2(\frac{\sigma^2}{4})\theta^2]\partial_\sigma\tilde{\partial}_\rho\frac{1}{\tilde{\Box}}\left\{\frac{[(\partial_\alpha\tilde{\partial}_\beta - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\Box})\frac{1}{\tilde{\Box}}j^B_{\alpha\beta}]}{X}\right\}\Big\}.
\end{aligned} \tag{C.24}$$

The other terms can be found by renaming: $\sigma \to \rho, \rho \to \sigma$

$$\begin{aligned}
\partial_\rho A_\sigma = &\left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\gamma^2\frac{1}{\tilde{\Box}}\Big\{\frac{1}{\Box}\partial_\rho\partial_\sigma\partial_\alpha\partial_\beta j^B_{\alpha\beta} - \partial_\rho[\partial_\alpha\delta_{\sigma\beta} - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\sigma]j^B_{\alpha\beta} \\
&-\gamma^4[2\sigma+2(\frac{\sigma^2}{4})\theta^2]\partial_\rho\tilde{\partial}_\sigma\frac{1}{\tilde{\Box}}\left\{\frac{[(\partial_\alpha\tilde{\partial}_\beta - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\Box})\frac{1}{\tilde{\Box}}j^B_{\alpha\beta}]}{X}\right\}\Big\},
\end{aligned} \tag{C.25}$$

and via $\sigma \to \nu, \rho \to \mu$:

$$\partial_\nu A_\mu = \left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\gamma^2 \frac{1}{\tilde{\Box}}\Bigg\{\frac{1}{\Box}\partial_\nu\partial_\mu\partial_\alpha\partial_\beta j^B_{\alpha\beta} - \partial_\nu[\partial_\alpha\delta_{\mu\beta} - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\mu] j^B_{\alpha\beta}$$
$$- \gamma^4[2\sigma + 2(\frac{\sigma^2}{4})\theta^2]\partial_\nu\tilde{\partial}_\mu\frac{1}{\tilde{\Box}}\left\{\frac{[(\partial_\alpha\tilde{\partial}_\beta - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\Box})\frac{1}{\tilde{\Box}}j^B_{\alpha\beta}]}{X}\right\}\Bigg\} \tag{C.26}$$

Inserting these terms, we get for (C.21):

$$B_{\sigma\rho} = \Bigg[j^{\bar{B}}_{\sigma\rho} + \frac{\gamma^4}{2}\left(\frac{1}{[1+\frac{\gamma^4}{\mu^2\tilde{\Box}^2}]\Box}\right)\frac{1}{\tilde{\Box}^2}\times$$
$$\Bigg[\Bigg\{\ \frac{1}{\Box}\partial_\sigma\partial_\rho\partial_\alpha\partial_\beta j^B_{\alpha\beta} - \partial_\sigma[\partial_\alpha\delta_{\rho\beta} - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\rho] j^B_{\alpha\beta}$$
$$- \gamma^4[2\sigma + 2(\frac{\sigma^2}{4})\theta^2]\partial_\sigma\tilde{\partial}_\rho\frac{1}{\tilde{\Box}}\left\{\frac{[(\partial_\alpha\tilde{\partial}_\beta - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\Box})\frac{1}{\tilde{\Box}}j^B_{\alpha\beta}]}{X}\right\}\Bigg\}$$
$$- \Bigg\{\frac{1}{\Box}\partial_\rho\partial_\sigma\partial_\alpha\partial_\beta j^B_{\alpha\beta} - \partial_\rho[\partial_\alpha\delta_{\sigma\beta} - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\sigma] j^B_{\alpha\beta}$$
$$- \gamma^4[2\sigma + 2(\frac{\sigma^2}{4})\theta^2]\partial_\rho\tilde{\partial}_\sigma\frac{1}{\tilde{\Box}}\left\{\frac{[(\partial_\alpha\tilde{\partial}_\beta - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\Box})\frac{1}{\tilde{\Box}}j^B_{\alpha\beta}]}{X}\right\}\Bigg\}$$
$$- \sigma\theta_{\sigma\rho}\theta_{\mu\nu}\Bigg\{\frac{1}{\Box}\partial_\nu\partial_\mu\partial_\alpha\partial_\beta j^B_{\alpha\beta} - \partial_\nu[\partial_\alpha\delta_{\mu\beta} - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\mu] j^B_{\alpha\beta}$$
$$- \gamma^4[2\sigma + 2(\frac{\sigma^2}{4})\theta^2]\partial_\nu\tilde{\partial}_\mu\frac{1}{\tilde{\Box}}\left\{\frac{[(\partial_\alpha\tilde{\partial}_\beta - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\Box})\frac{1}{\tilde{\Box}}j^B_{\alpha\beta}]}{X}\right\}\Bigg\}\Bigg]\Bigg]. \tag{C.27}$$

Next we introduce

$$A \equiv \frac{\gamma^4}{2}\left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\frac{1}{\tilde{\Box}^2}, \tag{C.28a}$$

$$\begin{aligned} A' &\equiv A\delta^4(y-x) \\ &= \frac{\gamma^4}{2}\left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\frac{1}{\tilde{\Box}^2}\frac{1}{(2\pi)^4}\int d^4p e^{ip(y-x)} \\ &= -\frac{\gamma^4}{2(2\pi)^4}\int d^4p \frac{1}{\tilde{p}^2(\tilde{p}^2+\frac{\gamma^4}{\tilde{p}^2})p^2}e^{ip(y-x)}, \end{aligned} \tag{C.28b}$$

$$\tilde{A} = -\frac{\gamma^4}{2}\left[\frac{1}{\tilde{p}^2(\tilde{p}^2+\frac{\gamma^4}{\tilde{p}^2})p^2}\right], \tag{C.28c}$$

and $\bar{\sigma}^4$ given by (4.48) and insert them into (C.27):

$$\begin{aligned}
B_{\sigma\rho}(y) =& \Bigg[j^{\bar{B}}_{\sigma\rho} + A \Bigg[\Bigg\{ -\partial_\sigma [\partial_\alpha \delta_{\rho\beta} - \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{\partial}_\rho] j^B_{\alpha\beta} \\
& \quad - \bar{\sigma}^4 \partial_\sigma \tilde{\partial}_\rho \frac{1}{\tilde{\Box}^2} \Bigg\{ \frac{[(\partial_\alpha \tilde{\partial}_\beta - \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{\Box}) j^B_{\alpha\beta}]}{X} \Bigg\} \Bigg\} \\
& - \Bigg\{ -\partial_\rho [\partial_\alpha \delta_{\sigma\beta} - \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{\partial}_\sigma] j^B_{\alpha\beta} \\
& \quad - \bar{\sigma}^4 \partial_\rho \tilde{\partial}_\sigma \frac{1}{\tilde{\Box}^2} \Bigg\{ \frac{[(\partial_\alpha \tilde{\partial}_\beta - \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{\Box}) j^B_{\alpha\beta}]}{X} \Bigg\} \Bigg\} \\
& - \sigma \theta_{\sigma\rho} \theta_{\mu\nu} \Bigg\{ \frac{1}{\Box} \partial_\nu \partial_\mu \partial_\alpha \partial_\beta j^B_{\alpha\beta} - \partial_\nu [\partial_\alpha \delta_{\mu\beta} - \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{\partial}_\mu] j^B_{\alpha\beta} \\
& \quad - \bar{\sigma}^4 \partial_\nu \tilde{\partial}_\mu \frac{1}{\tilde{\Box}^2} \Bigg\{ \frac{[(\partial_\alpha \tilde{\partial}_\beta - \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{\Box}) j^B_{\alpha\beta}]}{X} \Bigg\} \Bigg\} \Bigg] \Bigg] (y).
\end{aligned}$$

If we collect the source $j^B_{\alpha\beta}$, we get

$$\begin{aligned}
& B_{\sigma\rho}(y) = \\
& = \Bigg[j^{\bar{B}}_{\sigma\rho} + A \Bigg[-\partial_\sigma \partial_\alpha \delta_{\rho\beta} + \sigma \frac{\theta_{\alpha\beta}}{2} \partial_\sigma \tilde{\partial}_\rho - \bar{\sigma}^4 \partial_\sigma \tilde{\partial}_\rho \frac{1}{\tilde{\Box}^2} \Bigg\{ \frac{(\partial_\alpha \tilde{\partial}_\beta - \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{\Box})}{X} \Bigg\} \\
& \quad + \partial_\rho \partial_\alpha \delta_{\sigma\beta} - \sigma \frac{\theta_{\alpha\beta}}{2} \partial_\rho \tilde{\partial}_\sigma + \bar{\sigma}^4 \partial_\rho \tilde{\partial}_\sigma \frac{1}{\tilde{\Box}^2} \Bigg\{ \frac{(\partial_\alpha \tilde{\partial}_\beta - \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{\Box})}{X} \Bigg\} \\
& \quad - \sigma \theta_{\sigma\rho} \theta_{\mu\nu} \Bigg\{ \frac{1}{\Box} \partial_\nu \partial_\mu \partial_\alpha \partial_\beta - \partial_\nu \partial_\alpha \delta_{\mu\beta} + \sigma \frac{\theta_{\alpha\beta}}{2} \partial_\nu \tilde{\partial}_\mu \\
& \quad - \bar{\sigma}^4 \partial_\nu \tilde{\partial}_\mu \frac{1}{\tilde{\Box}^2} \Bigg\{ \frac{(\partial_\alpha \tilde{\partial}_\beta - \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{\Box})}{X} \Bigg\} \Bigg\} \Bigg] j^B_{\alpha\beta}(y) \Bigg]. \qquad \text{(C.29)}
\end{aligned}$$

According to (C.20), the calculation of $G^{BB}_{\sigma\rho,\gamma\delta}(x,y)$ can be started

$$
\begin{aligned}
&G^{BB}_{\gamma\delta,\sigma\rho}(x,y) = \\
&= -A\bigg[-\partial_\sigma\partial_\alpha\delta_{\rho\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\partial_\sigma\tilde{\partial}_\rho - \bar{\sigma}^4\partial_\sigma\tilde{\partial}_\rho\frac{1}{\tilde{\Box}^2}\bigg\{\frac{(\partial_\alpha\tilde{\partial}_\beta - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\Box})}{X}\bigg\} \\
&\quad + \partial_\rho\partial_\alpha\delta_{\sigma\beta} - \sigma\frac{\theta_{\alpha\beta}}{2}\partial_\rho\tilde{\partial}_\sigma + \bar{\sigma}^4\partial_\rho\tilde{\partial}_\sigma\frac{1}{\tilde{\Box}^2}\bigg\{\frac{(\partial_\alpha\tilde{\partial}_\beta - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\Box})}{X}\bigg\} \\
&\quad - \sigma\theta_{\sigma\rho}\theta_{\mu\nu}\bigg\{\frac{1}{\Box}\partial_\nu\partial_\mu\partial_\alpha\partial_\beta - \partial_\nu\partial_\alpha\delta_{\mu\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\partial_\nu\tilde{\partial}_\mu \\
&\quad - \bar{\sigma}^4\partial_\nu\tilde{\partial}_\mu\frac{1}{\tilde{\Box}^2}\bigg\{\frac{(\partial_\alpha\tilde{\partial}_\beta - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\Box})}{X}\bigg\}\bigg\}\bigg]\frac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma})\delta^4(y-x) \\
&= -\frac{A}{2}\bigg[-\partial_\sigma\partial_\gamma\delta_{\rho\delta} + \partial_\sigma\partial_\delta\delta_{\rho\gamma} + \sigma\theta_{\gamma\delta}\partial_\sigma\tilde{\partial}_\rho \\
&\qquad - \frac{\bar{\sigma}^4}{\tilde{\Box}^2 X}(\partial_\sigma\tilde{\partial}_\rho\partial_\gamma\tilde{\partial}_\delta - \partial_\sigma\tilde{\partial}_\rho\partial_\delta\tilde{\partial}_\gamma - \sigma\theta_{\gamma\delta}\partial_\sigma\tilde{\partial}_\rho\tilde{\Box}) \\
&\qquad + \partial_\rho\partial_\gamma\delta_{\sigma\delta} - \partial_\rho\partial_\delta\delta_{\sigma\gamma} - \sigma\theta_{\gamma\delta}\partial_\rho\tilde{\partial}_\sigma \\
&\qquad + \frac{\bar{\sigma}^4}{\tilde{\Box}^2 X}(\partial_\rho\tilde{\partial}_\sigma\partial_\gamma\tilde{\partial}_\delta - \partial_\rho\tilde{\partial}_\sigma\partial_\delta\tilde{\partial}_\gamma - \sigma\theta_{\gamma\delta}\partial_\rho\tilde{\partial}_\sigma\tilde{\Box}) \\
&\qquad - \sigma\theta_{\sigma\rho}\theta_{\mu\nu}[\partial_\nu\partial_\delta\delta_{\mu\gamma} - \partial_\nu\partial_\gamma\delta_{\mu\delta} + \sigma\theta_{\gamma\delta}\partial_\nu\tilde{\partial}_\mu \\
&\qquad - \frac{\bar{\sigma}^4}{\tilde{\Box}^2 X}(\partial_\nu\tilde{\partial}_\mu\partial_\gamma\tilde{\partial}_\delta - \partial_\nu\tilde{\partial}_\mu\partial_\delta\tilde{\partial}_\gamma - \sigma\theta_{\gamma\delta}\partial_\nu\tilde{\partial}_\mu\tilde{\Box})]\bigg]\delta^4(y-x).
\end{aligned}
$$

The Fourier transformation of the delta function leads to

$$
\begin{aligned}
&G^{BB}_{\gamma\delta,\sigma\rho}(x,y) = \\
&= -\frac{A}{2}\bigg[-\partial_\sigma\partial_\gamma\delta_{\rho\delta} + \partial_\sigma\partial_\delta\delta_{\rho\gamma} + \sigma\theta_{\gamma\delta}\partial_\sigma\tilde{\partial}_\rho \\
&\quad - \frac{\bar{\sigma}^4}{\tilde{\Box}^2 X}(\partial_\sigma\tilde{\partial}_\rho\partial_\gamma\tilde{\partial}_\delta - \partial_\sigma\tilde{\partial}_\rho\partial_\delta\tilde{\partial}_\gamma - \sigma\theta_{\gamma\delta}\partial_\sigma\tilde{\partial}_\rho\tilde{\Box}) \\
&\quad + \partial_\rho\partial_\gamma\delta_{\sigma\delta} - \partial_\rho\partial_\delta\delta_{\sigma\gamma} - \sigma\theta_{\gamma\delta}\partial_\rho\tilde{\partial}_\sigma \\
&\quad + \frac{\bar{\sigma}^4}{\tilde{\Box}^2 X}(\partial_\rho\tilde{\partial}_\sigma\partial_\gamma\tilde{\partial}_\delta - \partial_\rho\tilde{\partial}_\sigma\partial_\delta\tilde{\partial}_\gamma - \sigma\theta_{\gamma\delta}\partial_\rho\tilde{\partial}_\sigma\tilde{\Box}) \\
&\quad - \sigma\theta_{\sigma\rho}\theta_{\mu\nu}[\partial_\nu\partial_\delta\delta_{\mu\gamma} - \partial_\nu\partial_\gamma\delta_{\mu\delta} + \sigma\theta_{\gamma\delta}\partial_\nu\tilde{\partial}_\mu \\
&\quad - \frac{\bar{\sigma}^4}{\tilde{\Box}^2 X}(\partial_\nu\tilde{\partial}_\mu\partial_\gamma\tilde{\partial}_\delta - \partial_\nu\tilde{\partial}_\mu\partial_\delta\tilde{\partial}_\gamma - \sigma\theta_{\gamma\delta}\partial_\nu\tilde{\partial}_\mu\tilde{\Box})]\bigg]\frac{1}{(2\pi)^4}\int d^4p e^{ip(y-x)}.
\end{aligned}
$$

Executing the derivatives brings

$$
\begin{aligned}
&G^{BB}_{\gamma\delta,\sigma\rho}(x,y) = \\
&\qquad = -\frac{1}{2(2\pi)^4}\int d^4p\tilde{A}\times \\
&\qquad\quad \Big[p_\sigma p_\gamma \delta_{\rho\delta} - p_\sigma p_\delta \delta_{\rho\gamma} - \sigma\theta_{\gamma\delta} p_\sigma \tilde{p}_\rho \\
&\qquad\quad - \frac{\bar{\sigma}^4}{\tilde{p}^4\tilde{X}}(p_\sigma\tilde{p}_\rho p_\gamma \tilde{p}_\delta - p_\sigma\tilde{p}_\rho p_\delta\tilde{p}_\gamma - \sigma\theta_{\gamma\delta}p_\sigma\tilde{p}_\rho\tilde{p}^2) \\
&\qquad\quad - p_\rho p_\gamma\delta_{\sigma\delta} + p_\rho p_\delta \delta_{\sigma\gamma} + \sigma\theta_{\gamma\delta}p_\rho\tilde{p}_\sigma \\
&\qquad\quad + \frac{\bar{\sigma}^4}{\tilde{p}^4\tilde{X}}(p_\rho\tilde{p}_\sigma p_\gamma\tilde{p}_\delta - p_\rho\tilde{p}_\sigma p_\delta\tilde{p}_\gamma - \sigma\theta_{\gamma\delta}p_\rho\tilde{p}_\sigma\tilde{p}^2) \\
&\qquad\quad - \sigma\theta_{\sigma\rho}\theta_{\mu\nu}[p_\nu p_\gamma \delta_{\mu\delta} - p_\nu p_\delta\delta_{\mu\gamma} - \sigma\theta_{\gamma\delta}p_\nu\tilde{p}_\mu \\
&\qquad\quad - \frac{\bar{\sigma}^4}{\tilde{p}^4\tilde{X}}(p_\nu\tilde{p}_\mu p_\gamma\tilde{p}_\delta - p_\nu\tilde{p}_\mu p_\delta\tilde{p}_\gamma - \sigma\theta_{\gamma\delta}p_\nu\tilde{p}_\mu\tilde{p}^2)]\Big] e^{ip(y-x)}.
\end{aligned} \tag{C.30}
$$

As we have mentioned in the Section 4.1.4, the propagator $G^{\bar{B}\bar{B}}$ gives the same result:

$$
G^{BB}_{\gamma\delta,\sigma\rho}(x,y) = G^{\bar{B}\bar{B}}_{\gamma\delta,\sigma\rho}(x,y) = -\frac{\delta\bar{B}_{\sigma\rho}(y)}{\delta j^{\bar{B}}_{\gamma\delta}(x)}. \tag{C.31}
$$

Finally the Fourier transform reads

$$
\begin{aligned}
\tilde{G}^{BB}_{\gamma\delta,\sigma\rho}(k) = -\frac{1}{2(2\pi)^4}\int d^4z\int d^4p\tilde{A}(k)\times& \\
\Big[p_\sigma p_\gamma \delta_{\rho\delta} - p_\sigma p_\delta \delta_{\rho\gamma} - \sigma\theta_{\gamma\delta} p_\sigma \tilde{p}_\rho& \\
- \frac{\bar{\sigma}^4}{\tilde{p}^4\tilde{X}}(p_\sigma\tilde{p}_\rho p_\gamma \tilde{p}_\delta - p_\sigma\tilde{p}_\rho p_\delta\tilde{p}_\gamma - \sigma\theta_{\gamma\delta}p_\sigma\tilde{p}_\rho\tilde{p}^2)& \\
- p_\rho p_\gamma\delta_{\sigma\delta} + p_\rho p_\delta \delta_{\sigma\gamma} + \sigma\theta_{\gamma\delta}p_\rho\tilde{p}_\sigma& \\
+ \frac{\bar{\sigma}^4}{\tilde{p}^4\tilde{X}}(p_\rho\tilde{p}_\sigma p_\gamma\tilde{p}_\delta - p_\rho\tilde{p}_\sigma p_\delta\tilde{p}_\gamma - \sigma\theta_{\gamma\delta}p_\rho\tilde{p}_\sigma\tilde{p}^2)& \\
- \sigma\theta_{\sigma\rho}\theta_{\mu\nu}[p_\nu p_\gamma \delta_{\mu\delta} - p_\nu p_\delta\delta_{\mu\gamma} - \sigma\theta_{\gamma\delta}p_\nu\tilde{p}_\mu& \\
- \frac{\bar{\sigma}^4}{\tilde{p}^4\tilde{X}}(p_\nu\tilde{p}_\mu p_\gamma\tilde{p}_\delta - p_\nu\tilde{p}_\mu p_\delta\tilde{p}_\gamma - \sigma\theta_{\gamma\delta}p_\nu\tilde{p}_\mu\tilde{p}^2)]\Big] e^{iz(p-k)}.&
\end{aligned}
$$

The integration brings an expression equal to (4.72) from the main text

$$
\begin{aligned}
&= -\frac{\tilde{A}(k)}{2}\Bigg[k_\sigma k_\gamma \delta_{\rho\delta} - k_\sigma k_\delta \delta_{\rho\gamma} - \sigma\theta_{\gamma\delta}k_\sigma \tilde{k}_\rho \\
&\qquad - \frac{\bar{\sigma}^4}{\tilde{k}^4 \tilde{X}(k)}(k_\sigma \tilde{k}_\rho k_\gamma \tilde{k}_\delta - k_\sigma \tilde{k}_\rho k_\delta \tilde{k}_\gamma - \sigma\theta_{\gamma\delta}k_\sigma \tilde{k}_\rho \tilde{k}^2) \\
&\qquad - k_\rho k_\gamma \delta_{\sigma\delta} + k_\rho k_\delta \delta_{\sigma\gamma} + \sigma\theta_{\gamma\delta}k_\rho \tilde{k}_\sigma \\
&\qquad + \frac{\bar{\sigma}^4}{\tilde{k}^4 \tilde{X}(k)}(k_\rho \tilde{k}_\sigma k_\gamma \tilde{k}_\delta - k_\rho \tilde{k}_\sigma k_\delta \tilde{k}_\gamma - \sigma\theta_{\gamma\delta}k_\rho \tilde{k}_\sigma \tilde{k}^2) \\
&\qquad - \sigma\theta_{\sigma\rho}\theta_{\mu\nu}[k_\nu k_\gamma \delta_{\mu\delta} - k_\nu k_\delta \delta_{\mu\gamma} - \sigma\theta_{\gamma\delta}k_\nu \tilde{k}_\mu \\
&\qquad - \frac{\bar{\sigma}^4}{\tilde{k}^4 \tilde{X}(k)}(k_\nu \tilde{k}_\mu k_\gamma \tilde{k}_\delta - k_\nu \tilde{k}_\mu k_\delta \tilde{k}_\gamma - \sigma\theta_{\gamma\delta}k_\nu \tilde{k}_\mu \tilde{k}^2)]\Bigg] \\
&= \frac{\gamma^4}{4}\left[\frac{1}{\tilde{k}^2(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2})k^2}\right]\times \\
&\Big\{k_\sigma k_\gamma \delta_{\rho\delta} - k_\sigma k_\delta \delta_{\rho\gamma} - k_\rho k_\gamma \delta_{\sigma\delta} + k_\rho k_\delta \delta_{\sigma\gamma} \\
&+ \sigma\theta_{\gamma\delta}(k_\rho \tilde{k}_\sigma - k_\sigma \tilde{k}_\rho) + \sigma\theta_{\sigma\rho}(k_\delta \tilde{k}_\gamma - k_\gamma \tilde{k}_\delta) + \sigma^2\theta_{\sigma\rho}\theta_{\gamma\delta}\tilde{k}^2 \\
&+ \frac{\bar{\sigma}^4}{\tilde{k}^2[(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2})k^2 + \bar{\sigma}^4]}\Big(k_\rho k_\gamma \tilde{k}_\sigma \tilde{k}_\delta - k_\rho k_\delta \tilde{k}_\sigma \tilde{k}_\gamma + k_\sigma k_\delta \tilde{k}_\rho \tilde{k}_\gamma - k_\sigma k_\gamma \tilde{k}_\rho \tilde{k}_\delta \\
&+ \sigma\theta_{\gamma\delta}(k_\sigma \tilde{k}_\rho - k_\rho \tilde{k}_\sigma)\tilde{k}^2 + \sigma\theta_{\sigma\rho}(k_\gamma \tilde{k}_\delta - k_\delta \tilde{k}_\gamma)\tilde{k}^2 - \sigma^2\theta_{\sigma\rho}\theta_{\gamma\delta}\tilde{k}^4\Big)\Big\},
\end{aligned}
\tag{C.32}
$$

leading to

$$
\tilde{G}^{BB}_{\gamma\delta,\sigma\rho}(k) = \tilde{G}^{\bar{B}\bar{B}}_{\gamma\delta,\sigma\rho}(k). \tag{C.33}
$$

C.1.5 Calculation of the Propagator $G^{\bar{B}B}_{\gamma\delta,\sigma\rho}$

The calculation of this propagator is given by

$$
G^{\bar{B}B}_{\gamma\delta,\sigma\rho}(x,y) = -\frac{B_{\sigma\rho}(y)}{\delta j^{\bar{B}}_{\gamma\delta}(x)}. \tag{C.34}
$$

According to Eq. (4.70) we have

$$
G^{\bar{B}B}_{\gamma\delta,\sigma\rho}(x,y) = -\frac{1}{2}(\delta_{\gamma\sigma}\delta_{\delta\rho} - \delta_{\delta\sigma}\delta_{\gamma\rho})\delta^4(y-x) + G^{\bar{B}\bar{B}}_{\gamma\delta,\sigma\rho}(x,y). \tag{C.35}
$$

Finally the Fourier transform reads

$$
\tilde{G}^{\bar{B}B}_{\gamma\delta,\sigma\rho}(k) = -\frac{1}{2}(\delta_{\gamma\sigma}\delta_{\delta\rho} - \delta_{\delta\sigma}\delta_{\gamma\rho}) + \tilde{G}^{\bar{B}\bar{B}}_{\gamma\delta,\sigma\rho}(k) = \tilde{G}^{B\bar{B}}_{\gamma\delta,\sigma\rho}(k). \tag{C.36}
$$

C.1.6 Calculation of the Propagator $\tilde{G}^{BA}_{\rho\sigma,\epsilon}$

The calculation of $\tilde{G}^{BA}_{\rho\sigma,\epsilon}(x,y)$ proceeds along the usual way:

$$\begin{aligned} G^{BA}_{\rho\sigma,\epsilon}(x,y) &= -\frac{\delta A_\epsilon(y)}{\delta j^B_{\rho\sigma}(x)} = -\frac{\delta[(...)_{\epsilon,\alpha\beta} j^B_{\alpha\beta}(y)]}{\delta j^B_{\rho\sigma}(x)} \\ &= -[(...)_{\epsilon,\alpha\beta}\frac{1}{2}(\delta_{\alpha\rho}\delta_{\beta\sigma} - \delta_{\alpha\sigma}\delta_{\beta\rho})]\delta^4(y-x). \end{aligned} \tag{C.37}$$

We use our well known abbreviations (C.23a-C.23c), the Landau gauge ($\alpha \to 0$) and derive the field $A_\epsilon(y)$ given by Eq. (4.42) with respect to the source $j^B_{\rho\sigma}$. This leads to

$$\begin{aligned} G^{BA}_{\rho\sigma,\epsilon}(x,y) &= -\left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\times \\ &\Big\{ -\frac{\gamma^2}{2}[\partial_\alpha\delta_{\epsilon\beta} - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\partial}_\epsilon]\frac{1}{\tilde{\Box}}(\delta_{\alpha\rho}\delta_{\beta\sigma} - \delta_{\alpha\sigma}\delta_{\beta\rho}) \\ &\quad -\frac{\gamma^4}{2}[2\sigma + 2(\frac{\sigma^2}{4})\theta^2]\tilde{\partial}_\epsilon\frac{1}{\tilde{\Box}^2}\frac{1}{X}[\gamma^2(\partial_\alpha\tilde{\partial}_\beta - \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{\Box})\frac{1}{\tilde{\Box}}(\delta_{\alpha\rho}\delta_{\beta\sigma} - \delta_{\alpha\sigma}\delta_{\beta\rho})]\Big\}\delta^4(y-x) \\ &= -\left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\times \\ &\Big\{ -\frac{\gamma^2}{2}[\partial_\rho\delta_{\epsilon\sigma} - \sigma\frac{\theta_{\rho\sigma}}{2}\tilde{\partial}_\epsilon]\frac{1}{\tilde{\Box}} - \frac{\gamma^2}{2}[-\partial_\sigma\delta_{\epsilon\rho} + \sigma\frac{\theta_{\sigma\rho}}{2}\tilde{\partial}_\epsilon]\frac{1}{\tilde{\Box}} \\ &\quad -\frac{\gamma^4}{2}[2\sigma + 2(\frac{\sigma^2}{4})\theta^2)]\tilde{\partial}_\epsilon\frac{1}{\tilde{\Box}^3 X}[\gamma^2(\partial_\rho\tilde{\partial}_\sigma - \sigma\frac{\theta_{\rho\sigma}}{2}\tilde{\Box})] \\ &\quad +\frac{\gamma^4}{2}[2\sigma + 2(\frac{\sigma^2}{4})\theta^2)]\tilde{\partial}_\epsilon\frac{1}{\tilde{\Box}^3 X}[\gamma^2(\partial_\sigma\tilde{\partial}_\rho - \sigma\frac{\theta_{\sigma\rho}}{2}\tilde{\Box})]\Big\}\delta^4(y-x) \\ &= -\left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\times \\ &\gamma^2\Big\{[\frac{1}{2}(\partial_\sigma\delta_{\epsilon\rho} - \partial_\rho\delta_{\epsilon\sigma}) + \sigma\frac{\theta_{\rho\sigma}}{2}\tilde{\partial}_\epsilon]\frac{1}{\tilde{\Box}} \\ &\quad -\gamma^4[2\sigma + 2(\frac{\sigma^2}{4})\theta^2]\tilde{\partial}_\epsilon\frac{1}{\tilde{\Box}^3 X}[\frac{1}{2}(\partial_\rho\tilde{\partial}_\sigma - \partial_\sigma\tilde{\partial}_\rho) - \sigma\frac{\theta_{\rho\sigma}}{2}\tilde{\Box}]\Big\}\delta^4(y-x). \end{aligned}$$

The Fourier transformation brings

$$
\begin{aligned}
&= -\left(\frac{1}{[1+\frac{\gamma^4}{\Box^2}]\Box}\right)\times \\
&\gamma^2\Big\{[\frac{1}{2}(\partial_\sigma\delta_{\epsilon\rho} - \partial_\rho\delta_{\epsilon\sigma}) + \sigma\frac{\theta_{\rho\sigma}}{2}\tilde{\partial}_\epsilon]\frac{1}{\tilde{\Box}} \\
&\quad - \gamma^4[2\sigma + 2(\frac{\sigma^2}{4})\theta^2]\tilde{\partial}_\epsilon\frac{1}{\tilde{\Box}^3 X}[\frac{1}{2}(\partial_\rho\tilde{\partial}_\sigma - \partial_\sigma\tilde{\partial}_\rho) - \sigma\frac{\theta_{\rho\sigma}}{2}\tilde{\Box}]\Big\}\frac{1}{(2\pi)^4}\int d^4p e^{ip(y-x)} \\
&= \frac{1}{(2\pi)^4}\int d^4p\left(\frac{1}{[1+\frac{\gamma^4}{\tilde{p}^4}]p^2}\right)\times \\
&\gamma^2 i\Big\{[\frac{1}{2}(p_\rho\delta_{\epsilon\sigma} - p_\sigma\delta_{\epsilon\rho}) - \sigma\frac{\theta_{\rho\sigma}}{2}\tilde{p}_\epsilon]\frac{1}{\tilde{p}^2} \\
&\quad - \gamma^4[2\sigma + 2(\frac{\sigma^2}{4})\theta^2]\tilde{p}_\epsilon\frac{1}{\tilde{p}^6\tilde{X}}[\frac{1}{2}(p_\rho\tilde{p}_\sigma - p_\sigma\tilde{p}_\rho) - \sigma\frac{\theta_{\rho\sigma}}{2}\tilde{p}^2]\Big\}e^{ip(y-x)}.
\end{aligned}
$$

Introducing $z = y - x$ gives

$$
\begin{aligned}
G^{BA}_{\rho\sigma,\epsilon}(z) &= \frac{i}{(2\pi)^4}\int d^4p\left(\frac{1}{2[1+\frac{\gamma^4}{\tilde{p}^4}]p^2}\right)\times \\
&\gamma^2\Big\{[p_\rho\delta_{\epsilon\sigma} - p_\sigma\delta_{\epsilon\rho} - \sigma\theta_{\rho\sigma}\tilde{p}_\epsilon]\frac{1}{\tilde{p}^2} \\
&\quad - \gamma^4[2\sigma + 2(\frac{\sigma^2}{4})\theta^2]\tilde{p}_\epsilon\frac{1}{\tilde{p}^6\tilde{X}}[(p_\rho\tilde{p}_\sigma - p_\sigma\tilde{p}_\rho) - \sigma\theta_{\rho\sigma}\tilde{p}^2]\Big\}e^{ipz}.
\end{aligned}
$$

Therefore the Fourier transform reads

$$
\begin{aligned}
\tilde{G}^{BA}_{\rho\sigma,\epsilon}(k) &= \frac{i}{(2\pi)^4}\int d^4z\int d^4p\left(\frac{1}{2[1+\frac{\gamma^4}{\tilde{p}^4}]p^2}\right)\times \\
&\gamma^2\Big\{[p_\rho\delta_{\epsilon\sigma} - p_\sigma\delta_{\epsilon\rho} - \sigma\theta_{\rho\sigma}\tilde{p}_\epsilon]\frac{1}{\tilde{p}^2} \\
&\quad - \gamma^4[2\sigma + 2(\frac{\sigma^2}{4})\theta^2]\tilde{p}_\epsilon\frac{1}{\tilde{p}^6\tilde{X}}[(p_\rho\tilde{p}_\sigma - p_\sigma\tilde{p}_\rho) - \sigma\theta_{\rho\sigma}\tilde{p}^2]\Big\}e^{iz(p-k)} \\
&= i\gamma^2\int d^4p\left(\frac{1}{2[1+\frac{\gamma^4}{\tilde{p}^4}]p^2}\right)\times \\
&\Big\{[p_\rho\delta_{\epsilon\sigma} - p_\sigma\delta_{\epsilon\rho} - \sigma\theta_{\rho\sigma}\tilde{p}_\epsilon]\frac{1}{\tilde{p}^2} \\
&\quad - \gamma^4[2\sigma + 2(\frac{\sigma^2}{4})\theta^2]\tilde{p}_\epsilon\frac{1}{\tilde{p}^6\tilde{X}}[(p_\rho\tilde{p}_\sigma - p_\sigma\tilde{p}_\rho) - \sigma\theta_{\rho\sigma}\tilde{p}^2]\Big\}\delta^4(p-k) \\
&= i\gamma^2\left(\frac{1}{2[\tilde{k}^2+\frac{\gamma^4}{\tilde{k}^2}]k^2}\right)\times \\
&\Big\{[k_\rho\delta_{\epsilon\sigma} - k_\sigma\delta_{\epsilon\rho} - \sigma\theta_{\rho\sigma}\tilde{k}_\epsilon] + \frac{\bar{\sigma}^4[k_\sigma\tilde{k}_\epsilon\tilde{k}_\rho - k_\rho\tilde{k}_\epsilon\tilde{k}_\sigma + \sigma\theta_{\rho\sigma}\tilde{k}_\epsilon\tilde{k}^2]}{\tilde{k}^2[(\tilde{k}^2+\frac{\gamma^4}{\tilde{k}^2})k^2 + \bar{\sigma}^4]}\Big\}.
\end{aligned} \tag{C.38}
$$

The last expression equals (4.75) from the main text.

C.1.7 Calculation of the Propagator $\tilde{G}^{AB}_{\epsilon,\rho\sigma}$

At first sight we may think that the calculation of the propagator $\tilde{G}^{AB}_{\epsilon,\rho\sigma}$ will give the same result as $\tilde{G}^{BA}_{\rho\sigma,\epsilon}$ but as we will see afterwards the difference will be a minus sign. The two-point Green function is given by

$$G^{AB}_{\epsilon,\rho\sigma}(x,y) = -\frac{\delta^2 Z^c}{\delta j^A_\epsilon(x)\delta j^B_{\rho\sigma}(y)} = -\frac{\delta B_{\rho\sigma}(y)}{\delta j^A_\epsilon(x)}. \tag{C.39}$$

The expression of $B_{\rho\sigma}$ reads (C.21):

$$B_{\rho\sigma}(y) = [j^{\bar{B}}_{\rho\sigma} + \frac{\gamma^2}{2}\frac{1}{\tilde{\Box}}(\partial_\rho A_\sigma - \partial_\sigma A_\rho - \sigma\theta_{\rho\sigma}\theta_{\mu\nu}\partial_\nu A_\mu)](y).$$

Obviously we must find an expression for terms like $\partial_\rho A_\sigma$. We have already calculated such terms in the Section C.1.4. Naturally we only concentrate on terms involving the source j^A. The explicit expression is deduced from Eq. (C.22) with $\alpha \to 0$:

$$\begin{aligned}\partial_\rho A_\sigma(y) = & \left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\Bigg\{\partial_\rho j^A_\sigma - \frac{1}{\alpha}\partial_\rho\partial_\sigma(\frac{\alpha}{\Box}\partial_\beta j^A_\beta) \\ & - \gamma^4[2\sigma + 2(\frac{\sigma^2}{4})\theta^2]\partial_\rho\tilde{\partial}_\sigma\frac{1}{\tilde{\Box}^2}\left\{\frac{(-\tilde{\partial}_\beta j^A_\beta)}{[(1+\frac{\gamma^4}{\tilde{\Box}^2})(-\Box) - \gamma^4(2\sigma + 2(\frac{\sigma^2}{4})\theta^2)\frac{1}{\tilde{\Box}}]}\right\}\Bigg\}(y) \\ = & \left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\left\{\partial_\rho j^A_\sigma - \frac{1}{\Box}\partial_\rho\partial_\sigma\partial_\beta j^A_\beta + \bar{\sigma}^4\frac{1}{\tilde{\Box}^2}\partial_\rho\tilde{\partial}_\sigma\tilde{\partial}_\beta\frac{j^A_\beta}{X}\right\}(y).\end{aligned} \tag{C.40}$$

Therefore we get

$$\begin{aligned}G^{AB}_{\epsilon,\rho\sigma}(x,y) = & -\frac{\gamma^2}{2}\frac{1}{\tilde{\Box}}\left(\frac{1}{[1+\frac{\gamma^4}{\tilde{\Box}^2}]\Box}\right)\times \\ & \left\{\partial_\rho\delta_{\epsilon\sigma} - \partial_\sigma\delta_{\epsilon\rho} - \sigma\theta_{\rho\sigma}\tilde{\partial}_\mu\delta_{\epsilon\mu} + \frac{\bar{\sigma}^4}{\tilde{\Box}^2 X}(\partial_\rho\tilde{\partial}_\sigma\tilde{\partial}_\epsilon - \partial_\sigma\tilde{\partial}_\rho\tilde{\partial}_\epsilon - \sigma\theta_{\rho\sigma}\tilde{\partial}_\mu\tilde{\partial}_\mu\tilde{\partial}_\epsilon)\right\}\delta^4(y-x).\end{aligned}$$

The Fourier transform then gives

$$\begin{aligned}\tilde{G}^{AB}_{\epsilon,\rho\sigma}(k) = & -i\gamma^2\left(\frac{1}{2[\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2}]k^2}\right)\times \\ & \left\{[k_\rho\delta_{\epsilon\sigma} - k_\sigma\delta_{\epsilon\rho} - \sigma\theta_{\rho\sigma}\tilde{k}_\epsilon] + \frac{\bar{\sigma}^4[k_\sigma\tilde{k}_\epsilon\tilde{k}_\rho - k_\rho\tilde{k}_\epsilon\tilde{k}_\sigma + \sigma\theta_{\rho\sigma}\tilde{k}_\epsilon\tilde{k}^2]}{\tilde{k}^2[(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2})k^2 + \bar{\sigma}^4]}\right\}.\end{aligned} \tag{C.41}$$

It is important to say that if we would have calculated the propagator in the way of

$$G^{AB}_{\epsilon,\rho\sigma}(y,x) = -\frac{\delta^2 Z^c}{\delta j^A_\epsilon(y)\delta j^B_{\rho\sigma}(x)} = -\frac{\delta B_{\rho\sigma}(x)}{\delta j^A_\epsilon(y)},$$

the final calculation would read

$$G^{AB}_{\epsilon,\rho\sigma}(y,x) = -G^{AB}_{\epsilon,\rho\sigma}(x,y), \tag{C.42}$$

because the propagator $G^{AB}(x,y)$ shows the structure of

$$G^{AB}(x,y) \approx \partial_{\alpha,y}\delta^4(y-x), \tag{C.43}$$

while the propagator $G^{AB}(y,x)$ shows

$$G^{AB}(y,x) \approx \partial_{\alpha,x}\delta^4(x-y), \tag{C.44}$$

and therefore the plus or minus sign will depend on how we perform the functional derivative. For more details take a closer look in Appendix C.1.2.

C.1.8 Calculation of the Propagator $\tilde{G}^{bb}$

The calculation of $G^{bb}(x,y)$ is given by

$$G^{bb}(x,y) = -\frac{\delta^2 Z^c}{\delta j^b(x)\delta j^b(y)} = -\frac{\delta b(y)}{\delta j^b(x)}. \tag{C.45}$$

The expression of b is known from the main text (4.34):

$$b = \frac{1}{\alpha}(\partial_\mu A_\mu + j^b).$$

Obviously we must find an expression for $\partial_\mu A_\mu$ in terms of the source j^b. Equation (4.40) containing the sources of j^b reads

$$\partial_\mu A_\mu = \frac{\alpha}{\Box}\partial_\epsilon j^A_\epsilon - j_b - \gamma^2 \frac{\alpha}{\Box}\frac{1}{\tilde{\Box}}\partial_\alpha\partial_\beta(j^{\overline{B}}_{\alpha\beta} + j^B_{\alpha\beta}). \tag{C.46}$$

Therefore we have

$$\begin{aligned}
b &= \frac{1}{\alpha}(\frac{\alpha}{\Box}\partial_\epsilon j^A_\epsilon - j_b - \gamma^2 \frac{\alpha}{\Box}\frac{1}{\tilde{\Box}}\partial_\alpha\partial_\beta(j^{\overline{B}}_{\alpha\beta} + j^B_{\alpha\beta}) + j^b) \\
&= \frac{1}{\alpha}(\frac{\alpha}{\Box}\partial_\epsilon j^A_\epsilon - \gamma^2 \frac{\alpha}{\Box}\frac{1}{\tilde{\Box}}\partial_\alpha\partial_\beta(j^{\overline{B}}_{\alpha\beta} + j^B_{\alpha\beta})), \\
\Rightarrow G^{bb}(x,y) &= -\frac{\delta b(y)}{\delta j^b(x)} = 0.
\end{aligned} \tag{C.47}$$

The outcome shows the unphysical character of b.

C.2 Vertices

C.2.1 Calculation of $\tilde{V}^{3A}_{\alpha\beta\gamma}$

We start with (4.86) from the main text, where we insert (4.85):

$$
\begin{aligned}
&\tilde{V}^{3A}_{\alpha\beta\gamma}(k_1,k_2,k_3) = -(2\pi)^{12}\frac{\delta}{\delta\tilde{A}_\alpha(-k_1)}\frac{\delta}{\delta\tilde{A}_\beta(-k_2)}\frac{\delta}{\delta\tilde{A}_\gamma(-k_3)}S^{3A}_{inv}\\
&= -(2\pi)^{12}\frac{\delta}{\delta\tilde{A}_\alpha(-k_1)}\frac{\delta}{\delta\tilde{A}_\beta(-k_2)}\frac{\delta}{\delta\tilde{A}_\gamma(-k_3)}\frac{2}{(2\pi)^8}g\int d^4k'_1\int d^4k'_2\int d^4k'_3\times\\
&\qquad\qquad \times\,\delta^4(k'_1+k'_2+k'_3)\tilde{A}_\nu(k'_1)\tilde{A}_\mu(k'_2)ik'_{\mu,3}\tilde{A}_\nu(k'_3)\sin[\frac{1}{2}k'_1\epsilon\theta k'_2]\\
&\qquad\qquad\qquad \tilde{A}_\nu(k'_1)\tilde{A}_\mu(k'_2)ik'_{\mu,3}\tilde{A}_\nu(k'_3)\sin[\frac{1}{2}k'_1\epsilon\theta k'_2]\\
&= -2(2\pi)^4\frac{\delta}{\delta\tilde{A}_\alpha(-k_1)}\frac{\delta}{\delta\tilde{A}_\beta(-k_2)}g\int d^4k'_1\int d^4k'_2\int d^4k'_3\delta^4(k'_1+k'_2+k'_3)\\
&\{\delta_{\gamma\nu}\delta(k'_1+k_3)\tilde{A}_\mu(k'_2)ik'_{\mu,3}\tilde{A}_\nu(k'_3)+\tilde{A}_\nu(k'_1)\delta_{\gamma\mu}\delta(k'_2+k_3)ik'_{\mu,3}\tilde{A}_\nu(k'_3)+\\
&\tilde{A}_\nu(k'_1)\tilde{A}_\mu(k'_2)ik'_{\mu,3}\delta_{\gamma\nu}\delta(k'_3+k_3)\}\sin[\frac{1}{2}k'_1\epsilon\theta k'_2].
\end{aligned}
$$

The last derivative leads to

$$
\begin{aligned}
&\tilde{V}^{3A}_{\alpha\beta\gamma}(k_1,k_2,k_3) =\\
&= -2(2\pi)^4\frac{\delta}{\delta\tilde{A}_\alpha(-k_1)}g\int d^4k'_1\int d^4k'_2\int d^4k'_3\delta^4(k'_1+k'_2+k'_3)\sin[\frac{1}{2}k'_1\epsilon\theta k'_2]\\
&\{\delta_{\gamma\nu}\delta(k'_1+k_3)\delta_{\beta\mu}\delta(k'_2+k_2)ik'_{\mu,3}\tilde{A}_\nu(k'_3)+\delta_{\gamma\nu}\delta(k'_1+k_3)\tilde{A}_\mu(k'_2)ik'_{\mu,3}\delta_{\beta\nu}\delta(k'_3+k_2)\\
&+\delta_{\beta\nu}\delta(k'_1+k_2)\delta_{\gamma\mu}\delta(k'_2+k_3)ik'_{\mu,3}\tilde{A}_\nu(k'_3)+\tilde{A}_\nu(k'_1)\delta_{\gamma\mu}\delta(k'_2+k_3)ik'_{\mu,3}\delta_{\beta\nu}\delta(k'_3+k_2)\\
&+\delta_{\beta\nu}\delta(k'_1+k_2)\tilde{A}_\mu(k'_2)ik'_{\mu,3}\delta_{\gamma\nu}\delta(k'_3+k_3)+\tilde{A}_\nu(k'_1)\delta_{\beta\mu}\delta(k'_2+k_2)ik'_{\mu,3}\delta_{\gamma\nu}\delta(k'_3+k_3)\}\\
&= -2(2\pi)^4g\int d^4k'_1\int d^4k'_2\int d^4k'_3\delta^4(k'_1+k'_2+k'_3)\sin[\frac{1}{2}k'_1\epsilon\theta k'_2]\\
&\{\delta_{\gamma\nu}\delta(k'_1+k_3)\delta_{\beta\mu}\delta(k'_2+k_2)ik'_{\mu,3}\delta_{\alpha\nu}\delta(k'_3+k_1)\\
&+\delta_{\gamma\nu}\delta(k'_1+k_3)\delta_{\alpha\mu}\delta(k'_2+k_1)ik'_{\mu,3}\delta_{\beta\nu}\delta(k'_3+k_2)\\
&+\delta_{\beta\nu}\delta(k'_1+k_2)\delta_{\gamma\mu}\delta(k'_2+k_3)ik'_{\mu,3}\delta_{\alpha\nu}\delta(k'_3+k_1)\\
&+\delta_{\alpha\nu}\delta(k'_1+k_1)\delta_{\gamma\mu}\delta(k'_2+k_3)ik'_{\mu,3}\delta_{\beta\nu}\delta(k'_3+k_2)\\
&+\delta_{\beta\nu}\delta(k'_1+k_2)\delta_{\alpha\mu}\delta(k'_2+k_1)ik'_{\mu,3}\delta_{\gamma\nu}\delta(k'_3+k_3)\\
&+\delta_{\alpha\nu}\delta(k'_1+k_1)\delta_{\beta\mu}\delta(k'_2+k_2)ik'_{\mu,3}\delta_{\gamma\nu}\delta(k'_3+k_3)\}.
\end{aligned}
$$

The integration brings

$$
\begin{aligned}
&\tilde{V}^{3A}_{\alpha\beta\gamma}(k_1,k_2,k_3) = \\
&= -2(2\pi)^4 g \int d^4k'_1 \int d^4k'_2 \sin[\frac{1}{2}k'_1\epsilon\theta k'_2] \\
&\{\delta_{\alpha\gamma}\delta(k'_1+k_3)\delta(k'_2+k_2)(-ik_{\beta,1})\delta^4(k'_1+k'_2-k_1) \\
&+\delta_{\beta\gamma}\delta(k'_1+k_3)\delta(k'_2+k_1)(-ik_{\alpha,2})\delta^4(k'_1+k'_2-k_2) \\
&+\delta_{\alpha\beta}\delta(k'_1+k_2)\delta(k'_2+k_3)(-ik_{\gamma,1})\delta^4(k'_1+k'_2-k_1) \\
&+\delta_{\alpha\beta}\delta(k'_1+k_1)\delta(k'_2+k_3)(-ik_{\gamma,2})\delta^4(k'_1+k'_2-k_2) \\
&+\delta_{\beta\gamma}\delta(k_1+k_2)\delta(k'_2+k_1)(-ik_{\alpha,3})\delta^4(k'_1+k'_2-k_3) \\
&+\delta_{\alpha\gamma}\delta(k'_1+k_1)\delta(k'_2+k_2)(-ik_{\beta,3})\delta^4(k'_1+k'_2-k_3)\} \\
&= -2(2\pi)^4 g\delta^4(-k_1-k_2-k_3) \\
&\{\delta_{\alpha\gamma}(-ik_{\beta,1})\sin[\frac{1}{2}k_3\epsilon\theta k_2]+\delta_{\beta\gamma}(-ik_{\alpha,2})\sin[\frac{1}{2}k_3\epsilon\theta k_1] \\
&+\delta_{\alpha\beta}(-ik_{\gamma,1})\sin[\frac{1}{2}k_2\epsilon\theta k_3]+\delta_{\alpha\beta}(-ik_{\gamma,2})\sin[\frac{1}{2}k_1\epsilon\theta k_3] \\
&+\delta_{\beta\gamma}(-ik_{\alpha,3})\sin[\frac{1}{2}k_2\epsilon\theta k_1]+\delta_{\alpha\gamma}(-ik_{\beta,3})\sin[\frac{1}{2}k_1\epsilon\theta k_2]\}.
\end{aligned}
\tag{C.48}
$$

Elimination of $k_3,\ k_3=-k_1-k_2$ gives for each of the above terms (1-6):

$$
\begin{aligned}
&1. \quad \sin[\frac{1}{2}k_3\epsilon\theta k_2] = -\sin[\frac{1}{2}k_1\epsilon\theta k_2], \\
&2. \quad \sin[\frac{1}{2}k_3\epsilon\theta k_1] = +\sin[\frac{1}{2}k_1\epsilon\theta k_2], \\
&3. \quad \sin[\frac{1}{2}k_2\epsilon\theta k_3] = +\sin[\frac{1}{2}k_1\epsilon\theta k_2], \\
&4. \quad \sin[\frac{1}{2}k_1\epsilon\theta k_3] = -\sin[\frac{1}{2}k_1\epsilon\theta k_2], \\
&5. \quad \sin[\frac{1}{2}k_2\epsilon\theta k_1] = -\sin[\frac{1}{2}k_1\epsilon\theta k_2], \\
&6. \quad \sin[\frac{1}{2}k_1\epsilon\theta k_2] = +\sin[\frac{1}{2}k_1\epsilon\theta k_2].
\end{aligned}
\tag{C.49}
$$

Therefore, we have

$$
\begin{aligned}
&\tilde{V}^{3A}_{\alpha\beta\gamma}(k_1,k_2,k_3) = -2g(2\pi)^4\delta^4(-k_1-k_2-k_3)\sin[\frac{1}{2}k_1\epsilon\theta k_2] \\
&\{\delta_{\alpha\gamma}(+ik_{\beta,1})+\delta_{\beta\gamma}(-ik_{\alpha,2})+\delta_{\alpha\beta}(-ik_{\gamma,1})+\delta_{\alpha\beta}(+ik_{\gamma,2})+\delta_{\beta\gamma}(+ik_{\alpha,3})+\delta_{\alpha\gamma}(-ik_{\beta,3})\} \\
&= -2gi(2\pi)^4\delta^4(k_1+k_2+k_3)\sin[\frac{1}{2}k_1\epsilon\theta k_2] \\
&\{\delta_{\alpha\gamma}(k_1-k_3)_\beta+\delta_{\alpha\beta}(k_2-k_1)_\gamma+\delta_{\beta\gamma}(k_3-k_2)_\alpha\}.
\end{aligned}
\tag{C.50}
$$

The last expression equals the second line of (4.86) from the main text.

C.2.2 Calculation of $\tilde{V}^{4A}_{\alpha\beta\gamma\delta}$

The calculation of the Fourier transform $\tilde{V}^{4A}_{\alpha\beta\gamma\delta}$ starts with

$$\tilde{V}^{4A}_{\alpha\beta\gamma\delta}(k_1,k_2,k_3,k_4) = -(2\pi)^{16}\frac{\delta}{\delta\tilde{A}_\alpha(-k_1)}\frac{\delta}{\delta\tilde{A}_\beta(-k_2)}\frac{\delta}{\delta\tilde{A}_\gamma(-k_3)}\frac{\delta}{\delta\tilde{A}_\delta(-k_4)}S^{4A}_{inv}. \quad \text{(C.51)}$$

The structure of $\tilde{V}^{4A}_{\alpha\beta\gamma\delta}(k_1,k_2,k_3,k_4)$ shows that the functional derivative will lead to 4! permutations.

If we now insert Eq. (4.89) we get for the first derivative:

$$\begin{aligned}
&\tilde{V}^{4A}_{\alpha\beta\gamma\delta}(k_1,k_2,k_3,k_4) = \qquad\qquad \text{(C.52)}\\
&= -(2\pi)^4 ig^2\int d^4k'_1\int d^4k'_2\int d^4k'_3\int d^4k'_4\delta^4(k'_1+k'_2+k'_3+k'_4)\\
&e^{-\frac{i}{2}k'_1\epsilon\theta k'_2}\sin[\frac{1}{2}k'_3\epsilon\theta k'_4]\frac{\delta}{\delta\tilde{A}_\alpha(-k_1)}\frac{\delta}{\delta\tilde{A}_\beta(-k_2)}\frac{\delta}{\delta\tilde{A}_\gamma(-k_3)}\\
&\{\delta_{\delta\mu}\delta^4(k'_1+k_4)\tilde{A}_\nu(k'_2)\tilde{A}_\mu(k'_3)\tilde{A}_\nu(k'_4)+\tilde{A}_\mu(k'_1)\delta_{\delta\nu}\delta^4(k'_2+k_4)\tilde{A}_\mu(k'_3)\tilde{A}_\nu(k'_4)\\
&+\tilde{A}_\mu(k'_1)\tilde{A}_\nu(k'_2)\delta_{\delta\mu}\delta^4(k'_3+k_4)\tilde{A}_\nu(k'_4)+\tilde{A}_\mu(k'_1)\tilde{A}_\nu(k'_2)\tilde{A}_\mu(k'_3)\delta_{\delta\nu}\delta^4(k'_4+k_4)\}.
\end{aligned}$$

The second one leads to

$$\begin{aligned}
&\tilde{V}^{4A}_{\alpha\beta\gamma\delta}(k_1,k_2,k_3,k_4) =\\
&= -(2\pi)^4 ig^2\int d^4k'_1\int d^4k'_2\int d^4k'_3\int d^4k'_4\delta^4(k'_1+k'_2+k'_3+k'_4)\\
&e^{-\frac{i}{2}k'_1\epsilon\theta k'_2}\sin[\frac{1}{2}k'_3\epsilon\theta k'_4]\frac{\delta}{\delta\tilde{A}_\alpha(-k_1)}\frac{\delta}{\delta\tilde{A}_\beta(-k_2)}\\
&\{\delta_{\delta\mu}\delta^4(k'_1+k_4)\delta_{\gamma\nu}\delta^4(k'_2+k_3)\tilde{A}_\mu(k'_3)\tilde{A}_\nu(k'_4)+\delta_{\delta\mu}\delta^4(k'_1+k_4)\tilde{A}_\nu(k'_2)\delta_{\gamma\mu}\delta^4(k'_3+k_3)\tilde{A}_\nu(k'_4)\\
&+\delta_{\delta\mu}\delta^4(k'_1+k_4)\tilde{A}_\nu(k'_2)\tilde{A}_\mu(k'_3)\delta_{\gamma\nu}\delta^4(k'_4+k_3)+\delta_{\gamma\mu}\delta^4(k'_1+k_3)\delta_{\delta\nu}\delta^4(k'_2+k_4)\tilde{A}_\mu(k'_3)\tilde{A}_\nu(k'_4)\\
&+\tilde{A}_\mu(k'_1)\delta_{\delta\nu}\delta^4(k'_2+k_4)\delta_{\gamma\mu}\delta^4(k'_3+k_3)\tilde{A}_\nu(k'_4)+\tilde{A}_\mu(k'_1)\delta_{\delta\nu}\delta^4(k'_2+k_4)\tilde{A}_\mu(k'_3)\delta_{\gamma\nu}\delta^4(k'_4+k_3)\\
&+\delta_{\gamma\mu}\delta^4(k'_1+k_3)\tilde{A}_\nu(k'_2)\delta_{\delta\mu}\delta^4(k'_3+k_4)\tilde{A}_\nu(k'_4)+\tilde{A}_\mu(k'_1)\delta_{\gamma\nu}\delta^4(k'_2+k_3)\delta_{\delta\mu}\delta^4(k'_3+k_4)\tilde{A}_\nu(k'_4)\\
&+\tilde{A}_\mu(k'_1)\tilde{A}_\nu(k'_2)\delta_{\delta\mu}\delta^4(k'_3+k_4)\delta_{\gamma\nu}\delta^4(k'_4+k_3)+\delta_{\gamma\mu}\delta^4(k'_1+k_3)\tilde{A}_\nu(k'_2)\tilde{A}_\mu(k'_3)\delta_{\delta\nu}\delta^4(k'_4+k_4)\\
&+\tilde{A}_\mu(k'_1)\delta_{\gamma\nu}\delta^4(k'_2+k_3)\tilde{A}_\mu(k'_3)\delta_{\delta\nu}\delta^4(k'_4+k_4)+\tilde{A}_\mu(k'_1)\tilde{A}_\nu(k'_2)\delta_{\gamma\mu}\delta^4(k'_3+k_3)\delta_{\delta\nu}\delta^4(k'_4+k_4)\}.
\end{aligned}$$

The third to

$$
\begin{aligned}
&\tilde{V}^{4A}_{\alpha\beta\gamma\delta}(k_1,k_2,k_3,k_4) = \\
&= -(2\pi)^4 i g^2 \int d^4k'_1 \int d^4k'_2 \int d^4k'_3 \int d^4k'_4 \delta^4(k'_1+k'_2+k'_3+k'_4) \\
&e^{-\frac{i}{2}k'_1\epsilon\theta k'_2} \sin[\frac{1}{2}k'_3\epsilon\theta k'_4] \frac{\delta}{\delta\tilde{A}_\alpha(-k_1)} \\
&\{\delta_{\delta\mu}\delta^4(k'_1+k_4)\delta_{\gamma\nu}\delta^4(k'_2+k_3)\delta_{\beta\mu}\delta^4(k'_3+k_2)\tilde{A}_\nu(k'_4) \\
&+ \delta_{\delta\mu}\delta^4(k'_1+k_4)\delta_{\gamma\nu}\delta^4(k'_2+k_3)\tilde{A}_\mu(k'_3)\delta_{\beta\nu}\delta^4(k'_4+k_2) \\
&+ \delta_{\delta\mu}\delta^4(k'_1+k_4)\delta_{\beta\nu}\delta^4(k'_2+k_2)\delta_{\gamma\mu}\delta^4(k'_3+k_3)\tilde{A}_\nu(k'_4) \\
&+ \delta_{\delta\mu}\delta^4(k'_1+k_4)\tilde{A}_\nu(k'_2)\delta_{\gamma\mu}\delta^4(k'_3+k_3)\delta_{\beta\nu}\delta^4(k'_4+k_2) \\
&+ \delta_{\delta\mu}\delta^4(k'_1+k_4)\delta_{\beta\nu}\delta^4(k'_2+k_2)\tilde{A}_\mu(k'_3)\delta_{\gamma\nu}\delta^4(k'_4+k_3) \\
&+ \delta_{\delta\mu}\delta^4(k'_1+k_4)\tilde{A}_\nu(k'_2)\delta_{\beta\mu}\delta^4(k'_3+k_2)\delta_{\gamma\nu}\delta^4(k'_4+k_3) \\
&\delta_{\gamma\mu}\delta^4(k'_1+k_3)\delta_{\delta\nu}\delta^4(k'_2+k_4)\delta_{\beta\mu}\delta^4(k'_3+k_2)\tilde{A}_\nu(k'_4) \\
&+ \delta_{\gamma\mu}\delta^4(k'_1+k_3)\delta_{\delta\nu}\delta^4(k'_2+k_4)\tilde{A}_\mu(k'_3)\delta_{\beta\nu}\delta^4(k'_4+k_2) \\
&+ \delta_{\beta\mu}\delta^4(k'_1+k_2)\delta_{\delta\nu}\delta^4(k'_2+k_4)\delta_{\gamma\mu}\delta^4(k'_3+k_3)\tilde{A}_\nu(k'_4) \\
&+ \tilde{A}_\mu(k'_1)\delta_{\delta\nu}\delta^4(k'_2+k_4)\delta_{\gamma\mu}\delta^4(k'_3+k_3)\delta_{\beta\nu}\delta^4(k'_4+k_2) \\
&+ \delta_{\beta\mu}\delta^4(k'_1+k_2)\delta_{\delta\nu}\delta^4(k'_2+k_4)\tilde{A}_\mu(k'_3)\delta_{\gamma\nu}\delta^4(k'_4+k_3) \\
&+ \tilde{A}_\mu(k'_1)\delta_{\delta\nu}\delta^4(k'_2+k_4)\delta_{\beta\mu}\delta^4(k'_3+k_2)\delta_{\gamma\nu}\delta^4(k'_4+k_3) \\
&\delta_{\gamma\mu}\delta^4(k'_1+k_3)\delta_{\beta\nu}\delta^4(k'_2+k_2)\delta_{\delta\mu}\delta^4(k'_3+k_4)\tilde{A}_\nu(k'_4) \\
&+ \delta_{\gamma\mu}\delta^4(k'_1+k_3)\tilde{A}_\nu(k'_2)\delta_{\delta\mu}\delta^4(k'_3+k_4)\delta_{\beta\nu}\delta^4(k'_4+k_2) \\
&+ \delta_{\beta\mu}\delta^4(k'_1+k_2)\delta_{\gamma\nu}\delta^4(k'_2+k_3)\delta_{\delta\mu}\delta^4(k'_3+k_4)\tilde{A}_\nu(k'_4) \\
&+ \tilde{A}_\mu(k'_1)\delta_{\gamma\nu}\delta^4(k'_2+k_3)\delta_{\delta\mu}\delta^4(k'_3+k_4)\delta_{\beta\nu}\delta^4(k'_4+k_2) \\
&+ \delta_{\beta\mu}\delta^4(k'_1+k_2)\tilde{A}_\nu(k'_2)\delta_{\delta\mu}\delta^4(k'_3+k_4)\delta_{\gamma\nu}\delta^4(k'_4+k_3) \\
&+ \tilde{A}_\mu(k'_1)\delta_{\beta\nu}\delta^4(k'_2+k_2)\delta_{\delta\mu}\delta^4(k'_3+k_4)\delta_{\gamma\nu}\delta^4(k'_4+k_3) \\
&\delta_{\gamma\mu}\delta^4(k'_1+k_3)\delta_{\beta\nu}\delta^4(k'_2+k_2)\tilde{A}_\mu(k'_3)\delta_{\delta\nu}\delta^4(k'_4+k_4) \\
&+ \delta_{\gamma\mu}\delta^4(k'_1+k_3)\tilde{A}_\nu(k'_2)\delta_{\beta\mu}\delta^4(k'_3+k_2)\delta_{\delta\nu}\delta^4(k'_4+k_4) \\
&+ \delta_{\beta\mu}\delta^4(k'_1+k_2)\delta_{\gamma\nu}\delta^4(k'_2+k_3)\tilde{A}_\mu(k'_3)\delta_{\delta\nu}\delta^4(k'_4+k_4) \\
&+ \tilde{A}_\mu(k'_1)\delta_{\gamma\nu}\delta^4(k'_2+k_3)\delta_{\beta\mu}\delta^4(k'_3+k_2)\delta_{\delta\nu}\delta^4(k'_4+k_4) \\
&+ \delta_{\beta\mu}\delta^4(k'_1+k_2)\tilde{A}_\nu(k'_2)\delta_{\gamma\mu}\delta^4(k'_3+k_3)\delta_{\delta\nu}\delta^4(k'_4+k_4) \\
&+ \tilde{A}_\mu(k'_1)\delta_{\beta\nu}\delta^4(k'_2+k_2)\delta_{\gamma\mu}\delta^4(k'_3+k_3)\delta_{\delta\nu}\delta^4(k'_4+k_4)\},
\end{aligned}
$$

The last derivative brings

$$
\begin{aligned}
&\tilde{V}^{4A}_{\alpha\beta\gamma\delta}(k_1,k_2,k_3,k_4) = \\
&= -(2\pi)^4 ig^2 \int d^4k'_1 \int d^4k'_2 \int d^4k'_3 \int d^4k'_4 \delta^4(k'_1+k'_2+k'_3+k'_4) \\
&e^{-\frac{i}{2}k'_1\epsilon\theta k'_2} \sin[\frac{1}{2}k'_3\epsilon\theta k'_4] \\
&\{\delta_{\delta\mu}\delta^4(k'_1+k_4)\delta_{\gamma\nu}\delta^4(k'_2+k_3)\delta_{\beta\mu}\delta^4(k'_3+k_2)\delta_{\alpha\nu}\delta^4(k'_4+k_1) \\
&+\delta_{\delta\mu}\delta^4(k'_1+k_4)\delta_{\gamma\nu}\delta^4(k'_2+k_3)\delta_{\alpha\mu}\delta^4(k'_3+k_1)\delta_{\beta\nu}\delta^4(k'_4+k_2) \\
&+\delta_{\delta\mu}\delta^4(k'_1+k_4)\delta_{\beta\nu}\delta^4(k'_2+k_2)\delta_{\gamma\mu}\delta^4(k'_3+k_3)\delta_{\alpha\nu}\delta^4(k'_4+k_1) \\
&+\delta_{\delta\mu}\delta^4(k'_1+k_4)\delta_{\alpha\nu}\delta^4(k'_2+k_1)\delta_{\gamma\mu}\delta^4(k'_3+k_3)\delta_{\beta\nu}\delta^4(k'_4+k_2) \\
&+\delta_{\delta\mu}\delta^4(k'_1+k_4)\delta_{\beta\nu}\delta^4(k'_2+k_2)\delta_{\alpha\mu}\delta^4(k'_3+k_1)\delta_{\gamma\nu}\delta^4(k'_4+k_3) \\
&+\delta_{\delta\mu}\delta^4(k'_1+k_4)\delta_{\alpha\nu}\delta^4(k'_2+k_1)\delta_{\beta\mu}\delta^4(k'_3+k_2)\delta_{\gamma\nu}\delta^4(k'_4+k_3) \\
&\delta_{\gamma\mu}\delta^4(k'_1+k_3)\delta_{\delta\nu}\delta^4(k'_2+k_4)\delta_{\beta\mu}\delta^4(k'_3+k_2)\delta_{\alpha\nu}\delta^4(k'_4+k_1) \\
&+\delta_{\gamma\mu}\delta^4(k'_1+k_3)\delta_{\delta\nu}\delta^4(k'_2+k_4)\delta_{\alpha\mu}\delta^4(k'_3+k_1)\delta_{\beta\nu}\delta^4(k'_4+k_2) \\
&+\delta_{\beta\mu}\delta^4(k'_1+k_2)\delta_{\delta\nu}\delta^4(k'_2+k_4)\delta_{\gamma\mu}\delta^4(k'_3+k_3)\delta_{\alpha\nu}\delta^4(k'_4+k_1) \\
&+\delta_{\alpha\mu}\delta^4(k'_1+k_1)\delta_{\delta\nu}\delta^4(k'_2+k_4)\delta_{\gamma\mu}\delta^4(k'_3+k_3)\delta_{\beta\nu}\delta^4(k'_4+k_2) \\
&+\delta_{\beta\mu}\delta^4(k'_1+k_2)\delta_{\delta\nu}\delta^4(k'_2+k_4)\delta_{\alpha\mu}\delta^4(k'_3+k_1)\delta_{\gamma\nu}\delta^4(k'_4+k_3) \\
&+\delta_{\alpha\mu}\delta^4(k'_1+k_1)\delta_{\delta\nu}\delta^4(k'_2+k_4)\delta_{\beta\mu}\delta^4(k'_3+k_2)\delta_{\gamma\nu}\delta^4(k'_4+k_3) \\
&\delta_{\gamma\mu}\delta^4(k'_1+k_3)\delta_{\beta\nu}\delta^4(k'_2+k_2)\delta_{\delta\mu}\delta^4(k'_3+k_4)\delta_{\alpha\nu}\delta^4(k'_4+k_1) \\
&+\delta_{\gamma\mu}\delta^4(k'_1+k_3)\delta_{\alpha\nu}\delta^4(k'_2+k_1)\delta_{\delta\mu}\delta^4(k'_3+k_4)\delta_{\beta\nu}\delta^4(k'_4+k_2) \\
&+\delta_{\beta\mu}\delta^4(k'_1+k_2)\delta_{\gamma\nu}\delta^4(k'_2+k_3)\delta_{\delta\mu}\delta^4(k'_3+k_4)\delta_{\alpha\nu}\delta^4(k'_4+k_1) \\
&+\delta_{\alpha\mu}\delta^4(k'_1+k_1)\delta_{\gamma\nu}\delta^4(k'_2+k_3)\delta_{\delta\mu}\delta^4(k'_3+k_4)\delta_{\beta\nu}\delta^4(k'_4+k_2) \\
&+\delta_{\beta\mu}\delta^4(k'_1+k_2)\delta_{\alpha\nu}\delta^4(k'_2+k_1)\delta_{\delta\mu}\delta^4(k'_3+k_4)\delta_{\gamma\nu}\delta^4(k'_4+k_3) \\
&+\delta_{\alpha\mu}\delta^4(k'_1+k_1)\delta_{\beta\nu}\delta^4(k'_2+k_2)\delta_{\delta\mu}\delta^4(k'_3+k_4)\delta_{\gamma\nu}\delta^4(k'_4+k_3) \\
&\delta_{\gamma\mu}\delta^4(k'_1+k_3)\delta_{\beta\nu}\delta^4(k'_2+k_2)\delta_{\alpha\mu}\delta^4(k'_3+k_1)\delta_{\delta\nu}\delta^4(k'_4+k_4) \\
&+\delta_{\gamma\mu}\delta^4(k'_1+k_3)\delta_{\alpha\nu}\delta^4(k'_2+k_1)\delta_{\beta\mu}\delta^4(k'_3+k_2)\delta_{\delta\nu}\delta^4(k'_4+k_4) \\
&+\delta_{\beta\mu}\delta^4(k'_1+k_2)\delta_{\gamma\nu}\delta^4(k'_2+k_3)\delta_{\alpha\mu}\delta^4(k'_3+k_1)\delta_{\delta\nu}\delta^4(k'_4+k_4) \\
&+\delta_{\alpha\mu}\delta^4(k'_1+k_1)\delta_{\gamma\nu}\delta^4(k'_2+k_3)\delta_{\beta\mu}\delta^4(k'_3+k_2)\delta_{\delta\nu}\delta^4(k'_4+k_4) \\
&+\delta_{\beta\mu}\delta^4(k'_1+k_2)\delta_{\alpha\nu}\delta^4(k'_2+k_1)\delta_{\gamma\mu}\delta^4(k'_3+k_3)\delta_{\delta\nu}\delta^4(k'_4+k_4) \\
&+\delta_{\alpha\mu}\delta^4(k'_1+k_1)\delta_{\beta\nu}\delta^4(k'_2+k_2)\delta_{\gamma\mu}\delta^4(k'_3+k_3)\delta_{\delta\nu}\delta^4(k'_4+k_4)\}. \qquad \text{(C.53)}
\end{aligned}
$$

Next step is to solve the integrals with the help of the delta functions under the following relations:

$$e^{-\frac{i}{2}(-k_i)\epsilon\theta(-k_j)} = e^{-\frac{i}{2}k_i\epsilon\theta k_j}, \qquad i,j=1,2,3,$$

and

$$\sin[\frac{1}{2}(-k_i)\epsilon\theta(-k_j)] = \sin[\frac{1}{2}k_i\epsilon\theta k_j] = \sin[-\frac{1}{2}k_j\epsilon\theta k_i] = -\sin[\frac{1}{2}k_j\epsilon\theta k_i].$$

Therefore, we get

$$\begin{aligned}
&\tilde{V}^{4A}_{\delta\gamma\beta\alpha}(k_1,k_2,k_3,k_4) = \\
&= -(2\pi)^4 i g^2 \delta^4(-k_1-k_2-k_3-k_4) \\
&\{\delta_{\alpha\gamma}\delta_{\beta\delta}e^{+\frac{i}{2}k_3\epsilon\theta k_4}(-1\sin[\frac{1}{2}k_1\epsilon\theta k_2]) + \delta_{\alpha\delta}\delta_{\beta\gamma}e^{+\frac{i}{2}k_3\epsilon\theta k_4}(+1\sin[\frac{1}{2}k_1\epsilon\theta k_2]) \\
&+ \delta_{\alpha\beta}\delta_{\gamma\delta}e^{+\frac{i}{2}k_2\epsilon\theta k_4}(-1\sin[\frac{1}{2}k_1\epsilon\theta k_3]) + \delta_{\alpha\beta}\delta_{\gamma\delta}e^{+\frac{i}{2}k_1\epsilon\theta k_4}(-1\sin[\frac{1}{2}k_2\epsilon\theta k_3]) \\
&+ \delta_{\alpha\delta}\delta_{\beta\gamma}e^{+\frac{i}{2}k_2\epsilon\theta k_4}(+1\sin[\frac{1}{2}k_1\epsilon\theta k_3]) + \delta_{\alpha\gamma}\delta_{\beta\delta}e^{+\frac{i}{2}k_1\epsilon\theta k_4}(+1\sin[\frac{1}{2}k_2\epsilon\theta k_3]) \\
&+ \delta_{\alpha\delta}\delta_{\beta\gamma}e^{-\frac{i}{2}k_3\epsilon\theta k_4}(-1\sin[\frac{1}{2}k_1\epsilon\theta k_2]) + \delta_{\alpha\gamma}\delta_{\beta\delta}e^{-\frac{i}{2}k_3\epsilon\theta k_4}(+1\sin[\frac{1}{2}k_1\epsilon\theta k_2]) \\
&+ \delta_{\alpha\delta}\delta_{\beta\gamma}e^{-\frac{i}{2}k_2\epsilon\theta k_4}(-1\sin[\frac{1}{2}k_1\epsilon\theta k_3]) + \delta_{\alpha\gamma}\delta_{\beta\delta}e^{-\frac{i}{2}k_1\epsilon\theta k_4}(-1\sin[\frac{1}{2}k_2\epsilon\theta k_3]) \\
&+ \delta_{\alpha\beta}\delta_{\delta\gamma}e^{-\frac{i}{2}k_2\epsilon\theta k_4}(+1\sin[\frac{1}{2}k_1\epsilon\theta k_3]) + \delta_{\alpha\beta}\delta_{\delta\gamma}e^{-\frac{i}{2}k_1\epsilon\theta k_4}(+1\sin[\frac{1}{2}k_2\epsilon\theta k_3]) \\
&+ \delta_{\alpha\beta}\delta_{\gamma\delta}e^{+\frac{i}{2}k_2\epsilon\theta k_3}(-1\sin[\frac{1}{2}k_1\epsilon\theta k_4]) + \delta_{\alpha\beta}\delta_{\gamma\delta}e^{+\frac{i}{2}k_1\epsilon\theta k_3}(-1\sin[\frac{1}{2}k_2\epsilon\theta k_4]) \\
&+ \delta_{\alpha\gamma}\delta_{\beta\delta}e^{-\frac{i}{2}k_2\epsilon\theta k_3}(-1\sin[\frac{1}{2}k_1\epsilon\theta k_4]) + \delta_{\alpha\delta}\delta_{\beta\gamma}e^{-\frac{i}{2}k_1\epsilon\theta k_3}(-1\sin[\frac{1}{2}k_2\epsilon\theta k_4]) \\
&+ \delta_{\alpha\gamma}\delta_{\beta\delta}e^{+\frac{i}{2}k_1\epsilon\theta k_2}(-1\sin[\frac{1}{2}k_3\epsilon\theta k_4]) + \delta_{\alpha\delta}\delta_{\beta\gamma}e^{-\frac{i}{2}k_1\epsilon\theta k_2}(-1\sin[\frac{1}{2}k_3\epsilon\theta k_4]) \\
&+ \delta_{\alpha\gamma}\delta_{\beta\delta}e^{+\frac{i}{2}k_2\epsilon\theta k_3}(+1\sin[\frac{1}{2}k_1\epsilon\theta k_4]) + \delta_{\alpha\delta}\delta_{\beta\gamma}e^{+\frac{i}{2}k_1\epsilon\theta k_3}(+1\sin[\frac{1}{2}k_2\epsilon\theta k_4]) \\
&+ \delta_{\alpha\beta}\delta_{\gamma\delta}e^{-\frac{i}{2}k_2\epsilon\theta k_3}(+1\sin[\frac{1}{2}k_1\epsilon\theta k_4]) + \delta_{\alpha\beta}\delta_{\gamma\delta}e^{-\frac{i}{2}k_1\epsilon\theta k_3}(+1\sin[\frac{1}{2}k_2\epsilon\theta k_4]) \\
&+ \delta_{\alpha\delta}\delta_{\beta\gamma}e^{+\frac{i}{2}k_1\epsilon\theta k_2}(+1\sin[\frac{1}{2}k_3\epsilon\theta k_4]) + \delta_{\alpha\gamma}\delta_{\beta\delta}e^{-\frac{i}{2}k_1\epsilon\theta k_2}(+1\sin[\frac{1}{2}k_3\epsilon\theta k_4])\}. \qquad \text{(C.54)}
\end{aligned}$$

Next we sum up the Kronecker deltas

$$\begin{aligned}
&\tilde{V}^{4A}_{\alpha\beta\gamma\delta}(k_1,k_2,k_3,k_4) = -(2\pi)^4 ig^2\delta^4(k_1+k_2+k_3+k_4)\\
&\{\delta_{\alpha\gamma}\delta_{\beta\delta}(e^{+\frac{i}{2}k_1\epsilon\theta k_4}\sin[\frac{1}{2}k_2\epsilon\theta k_3] - e^{+\frac{i}{2}k_3\epsilon\theta k_4}\sin[\frac{1}{2}k_1\epsilon\theta k_2]\\
&\qquad + e^{-\frac{i}{2}k_3\epsilon\theta k_4}\sin[\frac{1}{2}k_1\epsilon\theta k_2] - e^{-\frac{i}{2}k_1\epsilon\theta k_4}\sin[\frac{1}{2}k_2\epsilon\theta k_3]\\
&\qquad - e^{-\frac{i}{2}k_2\epsilon\theta k_3}\sin[\frac{1}{2}k_1\epsilon\theta k_4] - e^{+\frac{i}{2}k_1\epsilon\theta k_2}\sin[\frac{1}{2}k_3\epsilon\theta k_4]\\
&\qquad + e^{+\frac{i}{2}k_2\epsilon\theta k_3}\sin[\frac{1}{2}k_1\epsilon\theta k_4] + e^{-\frac{i}{2}k_1\epsilon\theta k_2}\sin[\frac{1}{2}k_3\epsilon\theta k_4])\\
&+ \delta_{\alpha\delta}\delta_{\beta\gamma}(+e^{+\frac{i}{2}k_3\epsilon\theta k_4}\sin[\frac{1}{2}k_1\epsilon\theta k_2] + e^{+\frac{i}{2}k_2\epsilon\theta k_4}\sin[\frac{1}{2}k_1\epsilon\theta k_3]\\
&\qquad - e^{-\frac{i}{2}k_3\epsilon\theta k_4}\sin[\frac{1}{2}k_1\epsilon\theta k_2] - e^{-\frac{i}{2}k_2\epsilon\theta k_4}\sin[\frac{1}{2}k_1\epsilon\theta k_3]\\
&\qquad - e^{-\frac{i}{2}k_1\epsilon\theta k_3}\sin[\frac{1}{2}k_2\epsilon\theta k_4] - e^{-\frac{i}{2}k_1\epsilon\theta k_2}\sin[\frac{1}{2}k_3\epsilon\theta k_4]\\
&\qquad + e^{-\frac{i}{2}k_1\epsilon\theta k_3}\sin[\frac{1}{2}k_2\epsilon\theta k_4] + e^{+\frac{i}{2}k_1\epsilon\theta k_2}\sin[\frac{1}{2}k_3\epsilon\theta k_4])\\
&+ \delta_{\alpha\beta}\delta_{\gamma\delta}(-e^{+\frac{i}{2}k_2\epsilon\theta k_4}\sin[\frac{1}{2}k_1\epsilon\theta k_3] - e^{+\frac{i}{2}k_1\epsilon\theta k_4}\sin[\frac{1}{2}k_2\epsilon\theta k_3]\\
&\qquad + e^{-\frac{i}{2}k_2\epsilon\theta k_4}\sin[\frac{1}{2}k_1\epsilon\theta k_3] + e^{-\frac{i}{2}k_1\epsilon\theta k_4}\sin[\frac{1}{2}k_2\epsilon\theta k_3]\\
&\qquad - e^{+\frac{i}{2}k_2\epsilon\theta k_3}\sin[\frac{1}{2}k_1\epsilon\theta k_4] - e^{+\frac{i}{2}k_1\epsilon\theta k_3}\sin[\frac{1}{2}k_2\epsilon\theta k_4]\\
&\qquad + e^{-\frac{i}{2}k_2\epsilon\theta k_3}\sin[\frac{1}{2}k_1\epsilon\theta k_2] + e^{-\frac{i}{2}k_1\epsilon\theta k_3}\sin[\frac{1}{2}k_2\epsilon\theta k_4]\}.
\end{aligned}\tag{C.55}$$

Collecting the sine functions leads to

$$
\begin{aligned}
&\tilde{V}^{4A}_{\alpha\beta\gamma\delta}(k_1,k_2,k_3,k_4) = \\
&\quad = -(2\pi)^4 i g^2 \delta^4(k_1+k_2+k_3+k_4) \\
&\quad \Big\{ \delta_{\alpha\gamma}\delta_{\beta\delta}\big\{ (e^{+\frac{i}{2}k_1\epsilon\theta k_4} - e^{-\frac{i}{2}k_1\epsilon\theta k_4}) \sin[\frac{1}{2}k_2\epsilon\theta k_3] \\
&\qquad\qquad + (e^{-\frac{i}{2}k_3\epsilon\theta k_4} - e^{+\frac{i}{2}k_3\epsilon\theta k_4}) \sin[\frac{1}{2}k_1\epsilon\theta k_2] \\
&\qquad\qquad + (e^{+\frac{i}{2}k_2\epsilon\theta k_3} - e^{-\frac{i}{2}k_2\epsilon\theta k_3}) \sin[\frac{1}{2}k_1\epsilon\theta k_4] \\
&\qquad\qquad + (e^{-\frac{i}{2}k_1\epsilon\theta k_2} - e^{+\frac{i}{2}k_1\epsilon\theta k_2}) \sin[\frac{1}{2}k_3\epsilon\theta k_4]\big\} \\
&\quad + \delta_{\alpha\delta}\delta_{\beta\gamma}\big\{ (e^{+\frac{i}{2}k_3\epsilon\theta k_4} - e^{-\frac{i}{2}k_3\epsilon\theta k_4}) \sin[\frac{1}{2}k_1\epsilon\theta k_2] \\
&\qquad\qquad + (e^{\frac{i}{2}k_2\epsilon\theta k_4} - e^{-\frac{i}{2}k_2\epsilon\theta k_4}) \sin[\frac{1}{2}k_1\epsilon\theta k_3] \\
&\qquad\qquad + (e^{+\frac{i}{2}k_1\epsilon\theta k_3} - e^{-\frac{i}{2}k_1\epsilon\theta k_3}) \sin[\frac{1}{2}k_2\epsilon\theta k_4] \\
&\qquad\qquad + (e^{+\frac{i}{2}k_1\epsilon\theta k_2} - e^{-\frac{i}{2}k_1\epsilon\theta k_2}) \sin[\frac{1}{2}k_3\epsilon\theta k_4]\big\} \\
&\quad + \delta_{\alpha\beta}\delta_{\gamma\delta}\big\{ (e^{-\frac{i}{2}k_2\epsilon\theta k_4} - e^{+\frac{i}{2}k_2\epsilon\theta k_4}) \sin[\frac{1}{2}k_1\epsilon\theta k_3] \\
&\qquad\qquad + (e^{-\frac{i}{2}k_1\epsilon\theta k_4} - e^{+\frac{i}{2}k_1\epsilon\theta k_4}) \sin[\frac{1}{2}k_2\epsilon\theta k_3] \\
&\qquad\qquad + (e^{\frac{i}{2}k_2\epsilon\theta k_3} - e^{+\frac{i}{2}k_2\epsilon\theta k_3}) \sin[\frac{1}{2}k_1\epsilon\theta k_4] \\
&\qquad\qquad + (e^{-\frac{i}{2}k_1\epsilon\theta k_3} - e^{+\frac{i}{2}k_1\epsilon\theta k_3}) \sin[\frac{1}{2}k_2\epsilon\theta k_4]\big\}\Big\}.
\end{aligned}
$$

Again we sum up the exponential function and therefore find a sine function. Each sine function gives a factor (2i) but we have each sinus function twice. Hence we have an overall factor (4i):

$$
\begin{aligned}
&\tilde{V}^{4A}_{\alpha\beta\gamma\delta}(k_1,k_2,k_3,k_4) = -(2\pi)^4(4i)ig^2\delta^4(k_1+k_2+k_3+k_4) \\
&\Big\{\delta_{\alpha\gamma}\delta_{\beta\delta}\ \big\{+\sin[\frac{1}{2}k_1\epsilon\theta k_4]\sin[\frac{1}{2}k_2\epsilon\theta k_3]-\sin[\frac{1}{2}k_1\epsilon\theta k_2]\sin[\frac{1}{2}k_3\epsilon\theta k_4]\big\} \\
&+\delta_{\alpha\delta}\delta_{\beta\gamma}\big\{+\sin[\frac{1}{2}k_1\epsilon\theta k_2]\sin[\frac{1}{2}k_3\epsilon\theta k_4]+\sin[\frac{1}{2}k_2\epsilon\theta k_4]\sin[\frac{1}{2}k_1\epsilon\theta k_3]\big\} \\
&+\delta_{\alpha\beta}\delta_{\gamma\delta}\big\{-\sin[\frac{1}{2}k_1\epsilon\theta k_3]\sin[\frac{1}{2}k_2\epsilon\theta k_4]-\sin[\frac{1}{2}k_1\epsilon\theta k_4]\sin[\frac{1}{2}k_2\epsilon\theta k_3]\big\}\Big\} \\
&= (2\pi)^4 4g^2\delta^4(k_1+k_2+k_3+k_4) \\
&\Big\{\quad (\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\beta}\delta_{\gamma\delta})\sin[\frac{1}{2}k_1\epsilon\theta k_4]\sin[\frac{1}{2}k_2\epsilon\theta k_3] \\
&+(\delta_{\alpha\delta}\delta_{\beta\gamma}-\delta_{\alpha\gamma}\delta_{\beta\delta})\sin[\frac{1}{2}k_1\epsilon\theta k_2]\sin[\frac{1}{2}k_3\epsilon\theta k_4] \\
&+(\delta_{\alpha\delta}\delta_{\beta\gamma}-\delta_{\alpha\beta}\delta_{\gamma\delta})\sin[\frac{1}{2}k_2\epsilon\theta k_4]\sin[\frac{1}{2}k_1\epsilon\theta k_3]\Big\}.
\end{aligned}
\tag{C.56}
$$

The last expression represents (4.90).

Appendix D

Useful Formulae

D.1 Schwinger Parametrization

With the Gamma function $\Gamma(N)$:

$$\Gamma(N) = (N-1)!, \qquad N > 0, n \in \mathbb{N} \tag{D.1}$$

one gets

$$\frac{1}{k^{2N}} = \frac{1}{\Gamma(N)} \int_0^\infty d\alpha \alpha^{N-1} e^{-\alpha k^2}, \qquad \forall N \in \mathbb{N}, Re(k^2) > 0. \tag{D.2}$$

The general form reads

$$\frac{1}{(k^2+\beta)^N} = \frac{1}{\Gamma(N)} \int_0^\infty d\alpha \alpha^{N-1} e^{-\alpha(k^2+\beta)}, \qquad \forall N \in \mathbb{N}, Re(k^2+\beta) > 0. \tag{D.3}$$

D.2 Integration Formulae

D.2.1 Quadratic Form and Gaussian Integral

The integration of quadratic forms ([35], p.179 (5A.3)) is given by:

$$\int_{-\infty}^{+\infty} dx e^{(-\alpha x^2+\beta x+\gamma)} = \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} e^{\left(\frac{\beta^2}{4\alpha}+\gamma\right)}, \tag{D.4}$$

leading to the Gaussian integral with $\beta = \gamma = 0$

$$\int_{-\infty}^{+\infty} dx e^{(-\alpha x^2)} = \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}, \qquad \alpha > 0, \tag{D.5}$$

in one dimension and

$$\int d^4x e^{(-\alpha x^2)} = \frac{\pi^2}{\alpha^2}, \tag{D.6}$$

for four dimensions.

D.2.2 Parameter Integrals

Parameter integrals ([41] 3.471; K_ν are the modified Bessel functions of second kind):

$$\int_0^\infty dx x^{\nu-1} e^{-\frac{\beta}{x}-\gamma x} = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\beta\gamma}), \tag{D.7a}$$

$$\int_0^\infty dx \frac{1}{x^\nu} e^{-\frac{\alpha^2}{x}} = \Gamma(\nu-1)(\alpha^2)^{(1-\nu)}, \qquad \nu > 1, \alpha^2 > 0. \tag{D.7b}$$

D.2.3 Modified Bessel Functions

The modified Bessel differential equation is given by

$$x^2 \frac{d^2 y(x)}{dx^2} + x\frac{dy(x)}{dx} - (x^2+\alpha^2)y(x) = 0, \qquad \alpha, x \in \mathbb{C}, \tag{D.8}$$

where α defines the order of the equation. Two linearly independent solutions of this equation deliver the modified Bessel functions of first kind $I_\alpha(x)$ and second kind $K_\alpha(x)$:

$$I_\alpha(x) = \sum_{m=0}^\infty \frac{1}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}, \tag{D.9a}$$

$$K_\alpha(x) = \frac{\pi}{2} \frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)}. \tag{D.9b}$$

In this work only the second kind with α, beeing an integer will be of interest, i.e. $K_n(x), n \in \mathbb{Z}$. These functions are exponentially decaying and singular at the origin.

From equation (D.9b) follows:

$$K_n(x) = K_{-n}(x). \tag{D.10}$$

D.2.4 Expansion of the modified Bessel Function

The modified Bessel functions of second kind can be expanded for small arguments in the following form:

$$K_0(x) \approx \ln\frac{2}{x} - \gamma_E - \frac{x^2}{4}\left(\gamma_E - 1 + \ln\frac{x}{2}\right) + \mathcal{O}(x^4), \tag{D.11a}$$

$$K_1(x) \approx \frac{1}{x} + \frac{x}{2}\left(\gamma_E - \frac{1}{2} + \ln\frac{x}{2}\right) + \frac{x^3}{16}\left(\gamma_E - \frac{5}{4} + \ln\frac{x}{2}\right) + \mathcal{O}(x^4), \tag{D.11b}$$

$$K_2(x) \approx \frac{2}{x^2} - \frac{1}{2} - \frac{x^2}{8}\left(\gamma_E - \frac{3}{4} + \ln\frac{x}{2}\right) + O(x^4), \tag{D.11c}$$

$$K_3(x) \approx \frac{8}{x^3} - \frac{1}{x} + \frac{x}{8} + \frac{x^3}{48}\left(\gamma_E - \frac{11}{12} + \ln\frac{x}{2}\right) + \mathcal{O}(x^4), \tag{D.11d}$$

$$K_4(x) \approx \frac{48}{x^4} - \frac{4}{x^2} + \frac{1}{4} - \frac{x^2}{48} + \mathcal{O}(x^4), \tag{D.11e}$$

where $\gamma_E = 0,57721...$ represents the Euler-Mascheroni constant.

Appendix E

One-Loop Calculations

E.1 Expansion of $\tilde{\Pi}^i_{\mu\nu}(p)$

Before we start the explicit calculations, we have to mention that the expansion of $\tilde{\Pi}^i_{\mu\nu}(p)$ must be understood in the way of (5.36), together with the abbreviations (5.37a-5.37c).

E.1.1 Expansion of $\tilde{\Pi}^a_{\mu\nu}(p)$

The analytic expression for the ghost-loop is given by (5.21a) from the main text

$$
\begin{aligned}
\tilde{\Pi}^a_{\mu\nu}(p) &= -C_a \frac{4g^2}{(2\pi)^4} \int d^4k \left\{ \frac{(k-p)_\mu k_\nu}{(k-p)^2 k^2} \sin^2\left[\frac{1}{2} k\epsilon\theta p\right] \right\} \\
&= -C_a \frac{4g^2}{(2\pi)^4} \int d^4k \sin^2\left[\frac{1}{2} k\epsilon\theta p\right] \mathit{\Pi}^a_{\mu\nu} \\
&\approx -C_a \frac{4g^2}{(2\pi)^4} \int d^4k \sin^2\left[\frac{1}{2} k\epsilon\theta p\right] \left\{ \mathit{\Pi}^{a,(0)}_{\mu\nu}(k,0) + \mathit{\Pi}^{a,(2)}_{\mu\nu}(k,p) + \mathcal{O}(p^3) \right\},
\end{aligned}
$$

with

$$
\mathit{\Pi}^a_{\mu\nu}(k,p) = \frac{(k-p)_\mu k_\nu}{(k-p)^2 k^2}. \tag{E.1}
$$

The zeroth order $\mathit{\Pi}^{a,(0)}(k,0)$

According to (5.37a), the zeroth order reads

$$
\mathit{\Pi}^{a,(0)}(k,0) = \frac{k_\mu k_\nu}{k^4}. \tag{E.2}
$$

The second order $\mathit{\Pi}^{a,(2)}_{\mu\nu}(k,p)$

With (5.37c), The second order reads

$$\begin{aligned}
\mathit{\Pi}^{a,(2)}_{\mu\nu}(k,p) &= \frac{p_\delta p_\sigma}{2}\Big[\partial_{p_\delta}\partial_{p_\sigma}\mathit{\Pi}^{a}_{\mu\nu}(k,p)\Big]_{p\to 0} \\
&= \frac{p_\delta p_\sigma}{2}\Big[\partial_{p_\delta}\partial_{p_\sigma}\frac{(k-p)_\mu k_\nu}{(k-p)^2k^2}\Big]_{p\to 0} \\
&= \frac{p_\delta p_\sigma}{2}\Big[\partial_{p_\delta}\Big(\frac{-\delta_{\sigma\mu}k_\nu(k-p)^2k^2+2(k-p)_\mu k_\nu(k-p)_\sigma k^2}{(k-p)^4k^4}\Big)\Big]_{p\to 0} \\
&= \frac{p_\delta p_\sigma}{2}\Big[\partial_{p_\delta}\Big(\frac{-\delta_{\sigma\mu}k_\nu(k-p)^2+2(k-p)_\mu k_\nu(k-p)_\sigma}{(k-p)^4k^2}\Big)\Big]_{p\to 0} \\
&= \frac{p_\delta p_\sigma}{2}\Big[\frac{2\delta_{\sigma\mu}k_\nu(k-p)_\delta(k-p)^4k^2-2\delta_{\delta\mu}(k-p)_\sigma k_\nu(k-p)^4k^2}{(k-p)^8k^4} \\
&\qquad+\frac{-2\delta_{\delta\sigma}k_\nu(k-p)_\mu(k-p)^4k^2-4\delta_{\sigma\mu}k_\nu(k-p)^4(k-p)_\delta k^2}{(k-p)^8k^4} \\
&\qquad+\frac{8(k-p)_\mu(k-p)_\sigma k_\nu(k-p)^2(k-p)_\delta k^2}{(k-p)^8k^4}\Big]_{p\to 0} \\
&= \frac{p_\delta p_\sigma}{2}\Big[\frac{2\delta_{\sigma\mu}k_\nu(k-p)_\delta-2\delta_{\delta\mu}(k-p)_\sigma k_\nu}{(k-p)^4k^2} \\
&\qquad+\frac{-2\delta_{\delta\sigma}k_\nu(k-p)_\mu-4\delta_{\sigma\mu}k_\nu(k-p)_\delta}{(k-p)^4k^2} \\
&\qquad+\frac{8(k-p)_\mu(k-p)_\sigma k_\nu(k-p)_\delta k^2}{(k-p)^6k^2}\Big]_{p\to 0} \\
&= \frac{p_\delta p_\sigma}{2}\Big[2\delta_{\sigma\mu}\frac{k_\nu k_\delta}{k^6}-2\delta_{\delta\mu}\frac{k_\sigma k_\nu}{k^6}-2\delta_{\delta\sigma}\frac{k_\mu k_\nu}{k^6}-4\delta_{\sigma\mu}\frac{k_\nu k_\delta}{k^6}+8\frac{k_\mu k_\sigma k_\nu k_\delta}{k^8}\Big] \\
&= \Big[\cancel{p_\mu p_\delta\frac{k_\nu k_\delta}{k^6}}-\cancel{p_\sigma p_\mu\frac{k_\sigma k_\nu}{k^6}}-p_\sigma p_\sigma\frac{k_\mu k_\nu}{k^6}-2p_\delta p_\mu\frac{k_\nu k_\delta}{k^6}+4p_\delta p_\sigma\frac{k_\mu k_\sigma k_\nu k_\delta}{k^8}\Big] \\
&= \Big[-p^2\frac{k_\mu k_\nu}{k^6}-2p_\delta p_\mu\frac{k_\nu k_\delta}{k^6}+4p_\delta p_\sigma\frac{k_\mu k_\sigma k_\nu k_\delta}{k^8}\Big].
\end{aligned} \tag{E.3}$$

Therefore we have

$$\begin{aligned}
\tilde{\Pi}^a_{\mu\nu}(p) = -C_a\frac{4g^2}{(2\pi)^4}\int d^4k\sin^2[\frac{1}{2}k\epsilon\theta p]\times \\
\times\Big\{\frac{k_\mu k_\nu}{k^4}+\Big[-2p_\mu p_\delta\frac{k_\delta k_\nu}{k^6}-p^2\frac{k_\mu k_\nu}{k^6}+4p_\delta p_\sigma\frac{k_\sigma k_\delta k_\mu k_\nu}{k^8}\Big]+\mathcal{O}(p^3)\Big\},
\end{aligned} \tag{E.4}$$

equal to (5.42a).

E.1.2 Expansion of $\tilde{\Pi}^{b}_{\mu\nu}(p)$

Before we start with the expansion of $\Pi^{b}_{\mu\nu}(p)$ given by Eq. (5.15) we have to insert expressions for propagators and vertices,

$$\tilde{\Pi}^{b}_{\mu\nu}(p) = C_b \frac{1}{(2\pi)^{12}} \int d^4k \int d^4k' \int d^4p' \times \\ \times \left\{ {}^1\tilde{V}^{3A}_{\alpha\mu\gamma}(k,p',k')\, {}^2\tilde{G}^{AA}_{\gamma\epsilon}(k')\, {}^3\tilde{V}^{3A}_{\epsilon\nu\rho}(k',p,k)\, {}^4\tilde{G}^{AA}_{\rho\alpha}(k) \right\},$$

with

$$\begin{aligned} {}^1\tilde{V}^{3A}_{\alpha\mu\gamma}(k,p',k') &= -2ig(2\pi)^4\delta^4(k-p'-k')\sin\left[\frac{1}{2}k\epsilon\theta(-p')\right]\times \\ &\quad\times\left((\delta_{\alpha\gamma}(k+k')_\mu + \delta_{\alpha\mu}(-p'-k)_\gamma + \delta_{\mu\gamma}(-k'+p')_\alpha\right), \\ {}^2\tilde{G}^{AA}_{\gamma\epsilon}(k'), & \\ {}^3\tilde{V}^{3A}_{\epsilon\nu\rho}(k',p,k) &= -2ig(2\pi)^4\delta^4(k'+p-k)\sin\left[\frac{1}{2}k'\epsilon\theta p\right]\times \\ &\quad\times\left((\delta_{\epsilon\rho}(k'+k)_\nu + \delta_{\epsilon\nu}(p-k')_\rho + \delta_{\nu\rho}(-k-p)_\epsilon\right), \\ {}^4\tilde{G}^{AA}_{\rho\alpha}(k). & \end{aligned} \tag{E.5}$$

If we now insert (E.5) into $\tilde{\Pi}^{b}_{\mu\nu}(p)$, we get

$$\begin{aligned} \tilde{\Pi}^{b}_{\mu\nu}(p) &= C_b \frac{1}{(2\pi)^{12}} \int d^4k \int d^4k' \int d^4p' \times \\ &\times \Big[-2ig(2\pi)^4\delta^4(k-p'-k')\sin\left[\frac{1}{2}k\epsilon\theta(-p')\right] \\ &\quad\times\left((\delta_{\alpha\gamma}(k+k')_\mu + \delta_{\alpha\mu}(-p'-k)_\gamma + \delta_{\mu\gamma}(-k'+p')_\alpha\right) {}^2\tilde{G}^{AA}_{\gamma\epsilon}(k')\Big] \\ &\times \Big[-2ig(2\pi)^4\delta^4(k'+p-k)\sin\left[\frac{1}{2}k'\epsilon\theta p\right] \\ &\quad\times\left((\delta_{\epsilon\rho}(k'+k)_\nu + \delta_{\epsilon\nu}(p-k')_\rho + \delta_{\nu\rho}(-k-p)_\epsilon\right) {}^4\tilde{G}^{AA}_{\rho\alpha}(k)\Big]. \end{aligned} \tag{E.6}$$

The delta functions imply the correlation of p and p' in the following sense

$$\delta^4(k-p'-k') \to p' = k-k', \\ \delta^4(k'+p-k) \to p = k-k',$$

leading to

$$p' = k - k' = p.$$

Next we solve the integral over p' and receive

$$\tilde{\Pi}^b_{\mu\nu}(p) = C_b \frac{(-2ig)^2}{(2\pi)^4} \int d^4k \int d^4k' \delta^4(k'+p-k) \tilde{G}^{AA}_{\gamma\epsilon}(k') \tilde{G}^{AA}_{\rho\alpha}(k) \times$$
$$\times \Bigg[\sin\big[\frac{1}{2} k\epsilon\theta(-k+k')\big]$$
$$\times \Bigg((\delta_{\alpha\gamma}(k+k')_\mu + \delta_{\alpha\mu}(-2k+k')_\gamma + \delta_{\mu\gamma}(-2k'+k)_\alpha \Bigg) \Bigg]$$
$$\times \Bigg[-2ig(2\pi)^4 \delta^4(k'+p-k) \sin\big[\frac{1}{2} k'\epsilon\theta p\big]$$
$$\times \Bigg((\delta_{\epsilon\rho}(k'+k)_\nu + \delta_{\epsilon\nu}(p-k')_\rho + \delta_{\nu\rho}(-k-p)_\epsilon \Bigg) \Bigg],$$

while the delta function $\delta^4(k'+p-k)$ leads to $k' = k-p$ for the integral over k'

$$= C_b \frac{(-2ig)^2}{(2\pi)^4} \int d^4k \tilde{G}^{AA}_{\gamma\epsilon}(k-p) \tilde{G}^{AA}_{\rho\alpha}(k) \times$$
$$\times \Bigg[\sin\big[\frac{1}{2} k\epsilon\theta(-p)\big]$$
$$\times \Bigg((\delta_{\alpha\gamma}(k+k-p)_\mu + \delta_{\alpha\mu}(-2k+k-p)_\gamma + \delta_{\mu\gamma}(-2k+2p+k)_\alpha \Bigg) \Bigg]$$
$$\times \Bigg[\sin\big[\frac{1}{2}(k-p)\epsilon\theta p\big]$$
$$\times \Bigg((\delta_{\epsilon\rho}(k-p+k)_\nu + \delta_{\epsilon\nu}(p-k+p)_\rho + \delta_{\nu\rho}(-k-p)_\epsilon \Bigg) \Bigg]$$
$$= C_b \frac{(-2ig)^2}{(2\pi)^4} \int d^4k \tilde{G}^{AA}_{\gamma\epsilon}(k-p) \tilde{G}^{AA}_{\rho\alpha}(k) \times$$
$$\times \Bigg[\sin\big[\frac{1}{2} k\epsilon\theta(-p)\big] \Bigg((\delta_{\alpha\gamma}(2k-p)_\mu + \delta_{\alpha\mu}(-k-p)_\gamma + \delta_{\mu\gamma}(-k+2p)_\alpha \Bigg) \Bigg]$$
$$\times \Bigg[\sin\big[\frac{1}{2} k\epsilon\theta p\big] \Bigg((\delta_{\epsilon\rho}(2k-p)_\nu + \delta_{\epsilon\nu}(2p-k)_\rho + \delta_{\nu\rho}(-k-p)_\epsilon \Bigg) \Bigg]$$
$$= C_b \frac{4g^2}{(2\pi)^4} \int d^4k \sin^2\big[\frac{1}{2} k\epsilon\theta p\big] \tilde{G}^{AA}_{\gamma\epsilon}(k-p) \tilde{G}^{AA}_{\rho\alpha}(k) \times$$
$$\times \Big\{ \big(\delta_{\alpha\gamma}(2k-p)_\mu + \delta_{\alpha\mu}(-k-p)_\gamma + \delta_{\mu\gamma}(2p-k)_\alpha\big)$$
$$\times \big(\delta_{\epsilon\rho}(2k-p)_\nu + \delta_{\epsilon\nu}(2p-k)_\rho + \delta_{\nu\rho}(-k-p)_\epsilon\big) \Big\}. \tag{E.7}$$

The last result represents (5.21b).

The expansion of $\Pi^b(k,p)$ reads

$$\tilde{\Pi}^b_{\mu\nu}(p) = C_b \frac{4g^2}{(2\pi)^4} \int d^4k \sin^2\big[\frac{1}{2} k\epsilon\theta p\big] \Pi^b_{\mu\nu}(k,p) \tag{E.8}$$
$$= C_b \frac{4g^2}{(2\pi)^4} \int d^4k \sin^2\big[\frac{1}{2} k\epsilon\theta p\big] \Big\{ \Pi^{a,(0)}_{\mu\nu}(k,0) + \Pi^{a,(2)}_{\mu\nu}(k,p) + \mathcal{O}(p^3) \Big\},$$

with $\Pi^b_{\mu\nu}(k,p)$ given by

$$
\begin{aligned}
\Pi^b_{\mu\nu}(k,p) &= \tilde{G}^{AA}_{\gamma\epsilon}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)\times \\
&\times \Big\{ \delta_{\alpha\gamma}\delta_{\epsilon\rho}(2k-p)_\mu(2k-p)_\nu + \delta_{\alpha\gamma}\delta_{\epsilon\nu}(2k-p)_\mu(2p-k)_\rho \\
&\quad + \delta_{\alpha\gamma}\delta_{\nu\rho}(2k-p)_\mu(-k-p)_\epsilon + \delta_{\alpha\mu}\delta_{\epsilon\rho}(-k-p)_\gamma(2k-p)_\nu \\
&\quad + \delta_{\alpha\mu}\delta_{\epsilon\nu}(-k-p)_\gamma(2p-k)_\rho + \delta_{\alpha\mu}\delta_{\nu\rho}(-k-p)_\gamma(-k-p)_\epsilon \\
&\quad + \delta_{\mu\gamma}\delta_{\epsilon\rho}(2p-k)_\alpha(2k-p)_\nu + \delta_{\mu\gamma}\delta_{\epsilon\nu}(2p-k)_\alpha(2p-k)_\rho \\
&\quad + \delta_{\mu\gamma}\delta_{\nu\rho}(2p-k)_\alpha(-k-p)_\epsilon \Big\} \\
&= \Big\{ \tilde{G}^{AA}_{\alpha\rho}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)(2k-p)_\mu(2k-p)_\nu \\
&\quad + \tilde{G}^{AA}_{\alpha\nu}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)(2k-p)_\mu(2p-k)_\rho \\
&\quad + \tilde{G}^{AA}_{\alpha\epsilon}(k-p)\tilde{G}^{AA}_{\nu\alpha}(k)(2k-p)_\mu(-k-p)_\epsilon \\
&\quad + \tilde{G}^{AA}_{\gamma\rho}(k-p)\tilde{G}^{AA}_{\rho\mu}(k)(-k-p)_\gamma(2k-p)_\nu \\
&\quad + \tilde{G}^{AA}_{\gamma\nu}(k-p)\tilde{G}^{AA}_{\rho\mu}(k)(-k-p)_\gamma(2p-k)_\rho \\
&\quad + \tilde{G}^{AA}_{\gamma\epsilon}(k-p)\tilde{G}^{AA}_{\nu\mu}(k)(-k-p)_\gamma(-k-p)_\epsilon \\
&\quad + \tilde{G}^{AA}_{\mu\rho}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)(2p-k)_\alpha(2k-p)_\nu \\
&\quad + \tilde{G}^{AA}_{\mu\nu}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)(2p-k)_\alpha(2p-k)_\rho \\
&\quad + \tilde{G}^{AA}_{\mu\epsilon}(k-p)\tilde{G}^{AA}_{\nu\alpha}(k)(2p-k)_\alpha(-k-p)_\epsilon \Big\}
\end{aligned}
\tag{E.9}
$$

The zeroth order $\Pi^{b,(0)}_{\mu\nu}(k,0)$

If we look at $p=0$ in Eq. (E.9), we get

$$
\begin{aligned}
&\Pi^{b,(0)}_{\mu\nu}(k,0) = \\
&= \Big\{ 4k_\mu k_\nu \tilde{G}^{AA}_{\alpha\rho}(k)\tilde{G}^{AA}_{\rho\alpha}(k) - 2k_\mu k_\rho \tilde{G}^{AA}_{\alpha\nu}(k)\tilde{G}^{AA}_{\rho\alpha} - 2k_\mu k_\epsilon \tilde{G}^{AA}_{\alpha\epsilon}(k)\tilde{G}^{AA}_{\nu\alpha}(k) \\
&\quad - 2k_\gamma k_\nu \tilde{G}^{AA}_{\gamma\rho}(k)\tilde{G}^{AA}_{\rho\mu}(k) + k_\gamma k_\rho \tilde{G}^{AA}_{\gamma\nu}(k)\tilde{G}^{AA}_{\rho\mu}(k) + k_\gamma k_\epsilon \tilde{G}^{AA}_{\gamma\epsilon}(k)\tilde{G}^{AA}_{\nu\mu}(k) \\
&\quad - 2k_\alpha k_\nu \tilde{G}^{AA}_{\mu\rho}(k)\tilde{G}^{AA}_{\rho\alpha}(k) + k_\alpha k_\rho \tilde{G}^{AA}_{\mu\nu}(k)\tilde{G}^{AA}_{\rho\alpha}(k) + k_\alpha k_\epsilon \tilde{G}^{AA}_{\mu\epsilon}(k)\tilde{G}^{AA}_{\nu\alpha}(k) \Big\}.
\end{aligned}
\tag{E.10}
$$

As we have discussed in the main text, we insert in the last equation the expression of $\tilde{G}^{AA}(k)$ in the limit of large internal momenta k and use the Landau gauge:

$$
\tilde{G}^{AA}_{\alpha\beta}(k) \approx \frac{1}{k^2}\left[\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right], \qquad k\to\infty, \alpha\to 0.
$$

The expression of $\Pi^{b,(0)}_{\mu\nu}(k,0)$ is made up by nine terms and we look at each term separately.

The first term:

The first one reads

$$
\begin{aligned}
4k_\mu k_\nu \tilde{G}^{AA}_{\alpha\rho}(k)\tilde{G}^{AA}_{\rho\alpha}(k) = \\
&= 4\left\{k_\mu k_\nu \frac{1}{k^2}\left[\delta_{\alpha\rho} - \frac{k_\alpha k_\rho}{k^2}\right]\frac{1}{k^2}\left[\delta_{\alpha\rho} - \frac{k_\rho k_\alpha}{k^2}\right]\right\} \\
&= 4\left\{\frac{1}{k^4}\left[4 - \frac{k_\alpha k_\alpha}{k^2} - \frac{k_\alpha k_\alpha}{k^2} + \frac{k_\alpha k_\alpha k_\rho k_\rho}{k^4}\right]\right\} \\
&= 4\left\{k_\mu k_\nu \frac{1}{k^4}\left[4 - 1 - 1 + 1\right]\right\} = 12\frac{k_\mu k_\nu}{k^4}. \qquad \text{(E.11)}
\end{aligned}
$$

As we will see the remaining eight terms will disappear because of their given structure. However, we will show this for the fourth term as an example.

The fourth term:

The fourth one reads

$$
\begin{aligned}
-2k_\gamma k_\nu \tilde{G}^{AA}_{\gamma\rho}(k)\tilde{G}^{AA}_{\rho\mu}(k) = \\
&= -2k_\gamma k_\nu\left\{\frac{1}{k^2}\left[\delta_{\gamma\rho} - \frac{k_\gamma k_\rho}{k^2}\right]\frac{1}{k^2}\left[\delta_{\rho\mu} - \frac{k_\rho k_\mu}{k^2}\right]\right\} \\
&= -2k_\gamma k_\nu\left\{\frac{1}{k^4}\left[\delta_{\gamma\rho}\delta_{\rho\mu} - \frac{k_\gamma k_\mu}{k^2} - \frac{k_\gamma k_\mu}{k^2} + \frac{k_\gamma k_\rho k_\rho k_\mu}{k^4}\right]\right\} \\
&= -2\frac{1}{k^4}\left[\cancel{k_\mu k_\nu} - \cancel{k_\mu k_\nu} - \cancel{k_\mu k_\nu} + \cancel{k_\mu k_\nu}\right] = 0. \qquad \text{(E.12)}
\end{aligned}
$$

At the end we can give the zeroth order expansion:

$$
\Pi^{b,(0)}_{\mu\nu}(k,0) = 12\frac{k_\mu k_\nu}{k^4}. \qquad \text{(E.13)}
$$

The second order $\Pi^{b,(2)}_{\mu\nu}(k,p)$

The second order reads

$$
\begin{aligned}
\Pi^{b,(2)}_{\mu\nu}(k,p) &= \frac{p_\delta p_\sigma}{2}\left[\partial_{p_\delta}\partial p_\sigma \Pi^{b}_{\mu\nu}(k,p)\right]_{p\to 0} \\
&= \frac{p_\delta p_\sigma}{2}\Bigg\{\partial_{p_\delta}\partial_{p_\sigma}\Big[\tilde{G}^{AA}_{\alpha\rho}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)(2k-p)_\mu(2k-p)_\nu \\
&\quad + \tilde{G}^{AA}_{\alpha\nu}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)(2k-p)_\mu(2p-k)_\rho \\
&\quad + \tilde{G}^{AA}_{\alpha\epsilon}(k-p)\tilde{G}^{AA}_{\nu\alpha}(k)(2k-p)_\mu(-k-p)_\epsilon \\
&\quad + \tilde{G}^{AA}_{\gamma\rho}(k-p)\tilde{G}^{AA}_{\rho\mu}(k)(-k-p)_\gamma(2k-p)_\nu \\
&\quad + \tilde{G}^{AA}_{\gamma\nu}(k-p)\tilde{G}^{AA}_{\rho\mu}(k)(-k-p)_\gamma(2p-k)_\rho \\
&\quad + \tilde{G}^{AA}_{\gamma\epsilon}(k-p)\tilde{G}^{AA}_{\nu\mu}(k)(-k-p)_\gamma(-k-p)_\epsilon \\
&\quad + \tilde{G}^{AA}_{\mu\rho}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)(2p-k)_\alpha(2k-p)_\nu \\
&\quad + \tilde{G}^{AA}_{\mu\nu}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)(2p-k)_\alpha(2p-k)_\rho \\
&\quad + \tilde{G}^{AA}_{\mu\epsilon}(k-p)\tilde{G}^{AA}_{\nu\alpha}(k)(2p-k)_\alpha(-k-p)_\epsilon\Big]\Bigg\}_{p\to 0}.
\end{aligned}
\tag{E.14}
$$

The first term of the last expression is used to bring light into the structure of the calculation.

<u>The first term:</u>

The first one reads

$$
\begin{aligned}
&\frac{p_\delta p_\sigma}{2}\Bigg\{\partial_{p_\delta}\partial_{p_\sigma}\Big[\tilde{G}^{AA}_{\alpha\rho}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)(2k-p)_\mu(2k-p)_\nu\Big]\Bigg\}_{p\to 0} \\
&= \frac{p_\delta p_\sigma}{2}\Bigg\{\partial_{p_\delta}\partial_{p_\sigma}\tilde{G}^{AA}_{\alpha\rho}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)(2k-p)_\mu(2k-p)_\nu \\
&\quad - \delta_{\sigma\mu}\partial_{p_\delta}\tilde{G}^{AA}_{\alpha\rho}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)(2k-p)_\nu \\
&\quad - \delta_{\sigma\nu}\partial_{p_\delta}\tilde{G}^{AA}_{\alpha\rho}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)(2k-p)_\mu \\
&\quad - \delta_{\delta\mu}\partial_{p_\sigma}\tilde{G}^{AA}_{\alpha\rho}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)(2k-p)_\nu \\
&\quad - \delta_{\delta\nu}\partial_{p_\sigma}\tilde{G}^{AA}_{\alpha\rho}(k-p)\tilde{G}^{AA}_{\rho\alpha}(k)(2k-p)_\mu \\
&\quad + \delta_{\delta\nu}\delta_{\sigma\mu}\tilde{G}^{AA}_{\alpha\rho}(k-p)\tilde{G}^{AA}_{\rho\alpha} \\
&\quad + \delta_{\delta\mu}\delta_{\sigma\nu}\tilde{G}^{AA}_{\alpha\rho}(k-p)\tilde{G}^{AA}_{\rho\alpha}\Bigg\}_{p\to 0}.
\end{aligned}
\tag{E.15}
$$

We conclude that each term produces seven new terms and therefore we have as whole an expression containing 63 terms. The best approach to calculate such terms which show the same structure but only differ in the indices, is to use a calculation programm like *MATHEMATICA*®.

The detailed outcome shows

$$
\begin{aligned}
&\mathit{\Pi}^{b,(2)}_{\mu\nu}(k,p) = \\
&= \Big\{ -5p_\mu p_\nu \frac{1}{k^4} - 8p_\alpha p_\nu \frac{k_\alpha k_\mu}{k^6} - 8p_\alpha p_\mu \frac{k_\alpha k_\nu}{k^6} - 16p^2 \frac{k_\mu k_\nu}{k^6} \\
&\quad + 52 p_\alpha p_\beta \frac{k_\alpha k_\beta k_\mu k_\nu}{k^8} + 8p^2 \delta_{\mu\nu} \frac{1}{k^4} - 8 p_\alpha p_\beta \delta_{\mu\nu} \frac{k_\alpha k_\beta}{k^6} \Big\}.
\end{aligned}
\tag{E.16}
$$

Therefore we have

$$
\begin{aligned}
\tilde{\Pi}^b_{\mu\nu}(p) \approx & -C_b \frac{4g^2}{(2\pi)^4} \int d^4k \sin^2[\frac{1}{2} k\epsilon\theta p] \\
& \times \Big\{ 12 \frac{k_\mu k_\nu}{k^4} + \Big[-5p_\mu p_\nu \frac{1}{k^4} - 8p_\alpha p_\nu \frac{k_\alpha k_\mu}{k^6} - 8p_\alpha p_\mu \frac{k_\alpha k_\nu}{k^6} \\
& \quad - 16p^2 \frac{k_\mu k_\nu}{k^6} + 52 p_\alpha p_\beta \frac{k_\alpha k_\beta k_\mu k_\nu}{k^8} \\
& \quad + 8p^2 \delta_{\mu\nu} \frac{1}{k^4} - 8 p_\alpha p_\beta \delta_{\mu\nu} \frac{k_\alpha k_\beta}{k^6} \Big] + \mathcal{O}(p^3) \Big\}
\end{aligned}
\tag{E.17}
$$

equal to (5.42b).

E.2 Evaluation of the remaining Integrals

E.2.1 Calculation of the Example $\tilde{\mathcal{I}}^3(p)$

The integral $\tilde{\mathcal{I}}^3(p)$ is given by Eq. (5.43c) from the main text

$$
\tilde{\mathcal{I}}^3(p) = \int d^4k \frac{k_\mu k_\nu}{k^4} \sin^2 \big[\frac{1}{2} k\epsilon\tilde{p}\big].
\tag{E.18}
$$

The use of expression (5.74) with the correct prefactors deduced from (5.73a) and $N = 2, m = 2$ gives

$$
\begin{aligned}
\tilde{\mathcal{I}}^3(p) = \frac{(-i)^2\pi^2}{2\Gamma(2)} \Big\{ & -\frac{1}{2} \int_0^\infty d\alpha \alpha^{-2} \delta_{\mu\nu} + \frac{1}{2} \int_0^\infty d\alpha \alpha^{-2} \delta_{\mu\nu} e^{-\frac{(\epsilon\tilde{p})^2}{4\alpha}} \\
& - \frac{(\epsilon\tilde{p})_\mu (\epsilon\tilde{p})_\nu}{4} \int_0^\infty d\alpha \alpha^{-3} e^{-\frac{(\epsilon\tilde{p})^2}{4\alpha}} \Big\}.
\end{aligned}
\tag{E.19}
$$

The planar part reads

$$
\begin{aligned}
\tilde{\mathcal{I}}^{3,pl} &= \frac{(-i)^2\pi^2}{2\Gamma(2)} \Big\{ -\frac{1}{2} \int_0^\infty d\alpha \alpha^{-2} \delta_{\mu\nu} \Big\} \\
&= \frac{\pi^2}{4\Gamma(2)} \delta_{\mu\nu} \Big\{ \int_0^\infty d\alpha \alpha^{-2} e^{-\frac{1}{\Lambda^2 \alpha}} \Big\},
\end{aligned}
$$

where we have introduced a UV-cutoff Λ to regularize the integral. Evaluation of the last expression by using Formula (D.7b) leads to

$$\tilde{\mathcal{I}}^{3,pl} = \frac{\pi^2}{4\Gamma(2)}\delta_{\mu\nu}\Gamma(1)\left(\frac{1}{\Lambda^2}\right)^{-1} = \frac{1}{4}\pi^2\delta_{\mu\nu}\Lambda^2. \tag{E.20}$$

The non-planar part yields (evaluation with Formula (D.7b):

$$\begin{aligned}\tilde{\mathcal{I}}^{3,npl}(p) &= \frac{(-i)^2\pi^2}{2\Gamma(2)}\left\{\frac{1}{2}\int\limits_0^\infty d\alpha\alpha^{-2}\delta_{\mu\nu}e^{-\frac{(\epsilon\tilde{p})^2}{4\alpha}} - \frac{\epsilon^2\tilde{p}_\mu\tilde{p}_\nu}{4}\int\limits_0^\infty d\alpha\alpha^{-3}e^{-\frac{(\epsilon\tilde{p})^2}{4\alpha}}\right\}\\ &= \frac{(-i)^2\pi^2}{4\Gamma(2)}\Gamma(1)\delta_{\mu\nu}\left(\frac{(\epsilon\tilde{p})^2}{4}\right)^{-1} - \frac{(-i)^2\pi^2}{8\Gamma(2)}\Gamma(2)\epsilon^2\tilde{p}_\mu\tilde{p}_\nu\left(\frac{(\epsilon\tilde{p})^2}{4}\right)^{-2}\\ &= \pi^2\left\{-\frac{\delta_{\mu\nu}}{(\epsilon\tilde{p})^2} + 2\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\epsilon\tilde{p}^2)^2}\right\}.\end{aligned} \tag{E.21}$$

E.2.2 Detailed Outcome of the Integrals $\tilde{\mathcal{I}}^i$

We give a list of the detailed outcome of the integrals appearing in Section 5.6.1, Eqs. (5.43a-5.43e). The outcome is divided into the planar and non-planar part.

A)

$$\tilde{\mathcal{I}}^1(p) = \int d^4k \frac{1}{k^2} \sin^2\left[\frac{1}{2}k\epsilon\tilde{p}\right]$$

$$= +\frac{\pi^2}{2}\Lambda^2, \qquad planar \tag{E.22a}$$

$$= -2\frac{\pi^2}{(\epsilon\tilde{p})^2}, \qquad non-planar \tag{E.22b}$$

$$\tilde{\mathcal{I}}^2(p) = \int d^4k \frac{1}{k^4} \sin^2\left[\frac{1}{2}k\epsilon\tilde{p}\right]$$

$$= +\pi^2 K_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right), \qquad planar \tag{E.22c}$$

$$= -\pi^2 K_0\left(\sqrt{\mu^2(\epsilon\tilde{p})^2}\right), \qquad non-planar \tag{E.22d}$$

B)

$$\tilde{\mathcal{I}}^3_{\mu\nu}(p) = \int d^4k \frac{k_\mu k_\nu}{k^4} \sin^2\left[\frac{1}{2}k\epsilon\tilde{p}\right]$$

$$= +\frac{\pi^2}{4}\delta_{\mu\nu}\Lambda^2, \qquad planar \tag{E.22e}$$

$$= -\pi^2\left\{\frac{\delta_{\mu\nu}}{(\epsilon\tilde{p})^2} - 2\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\epsilon\tilde{p}^2)^2}\right\}, \quad non-planar \tag{E.22f}$$

$$\tilde{\mathcal{I}}^4_{\mu\nu}(p) = \int d^4k \frac{k_\mu k_\nu}{k^6} \sin^2\left[\frac{1}{2}k\epsilon\tilde{p}\right]$$

$$= +\frac{\pi^2}{4}\delta_{\mu\nu}K_0\left(\sqrt{\frac{\mu^2}{\Lambda^2}}\right), \qquad planar \tag{E.22g}$$

$$= -\frac{\pi^2}{4}\left\{\delta_{\mu\nu}K_0\left(\sqrt{\mu^2(\epsilon\tilde{p})^2}\right) - \frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}\right\}, \quad non-planar \tag{E.22h}$$

$$\tilde{\mathcal{I}}^5_{\alpha\beta\mu\nu}(p) = \int d^4k \frac{k_\alpha k_\beta k_\mu k_\nu}{k^8} \sin^2\left[\frac{1}{2}k\epsilon\tilde{p}\right]$$
$$= +\frac{\pi^2}{24} K_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right)\left[\delta_{\alpha\beta}\delta_{\mu\nu} + \delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\beta\mu}\delta_{\alpha\nu}\right], planar \tag{E.22i}$$
$$= -\frac{\pi^2}{24}\left\{K_0\left(\sqrt{\mu^2(\epsilon\tilde{p})^2}\right)\left[\delta_{\alpha\beta}\delta_{\mu\nu} + \delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\beta\mu}\delta_{\alpha\nu}\right]\right. \tag{E.22j}$$
$$- \frac{1}{\tilde{p}^2}\left[\delta_{\alpha\beta}\tilde{p}_\mu\tilde{p}_\nu + \delta_{\alpha\mu}\tilde{p}_\beta\tilde{p}_\nu + \delta_{\beta\mu}\tilde{p}_\alpha\tilde{p}_\nu + \delta_{\alpha\nu}\tilde{p}_\beta\tilde{p}_\mu\right.$$
$$\left.\left. + \delta_{\beta\nu}\tilde{p}_\alpha\tilde{p}_\mu + \delta_{\mu\nu}\tilde{p}_\alpha\tilde{p}_\beta\right] + 2\frac{\tilde{p}_\alpha\tilde{p}_\beta\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^4}\right\}, \quad non-planar.$$

E.2.3 Explicit Evaluation of $\tilde{\Pi}^i_{\mu\nu}(p)$

Evaluation of $\tilde{\Pi}^a_{\mu\nu}(p)$

We start with Eq. (5.77a) from the main text

$$\tilde{\Pi}^a_{\mu\nu}(p) \approx -C_a\frac{4g^2}{(2\pi)^4}\times$$
$$\times\left\{\tilde{\mathcal{I}}^3_{\mu\nu}(p) + \left[-2p_\mu p_\delta\tilde{\mathcal{I}}^4_{\delta\nu}(p) - p^2\tilde{\mathcal{I}}^4_{\mu\nu}(p) + 4p_\delta p_\sigma\tilde{\mathcal{I}}^5_{\sigma\delta\mu\nu}(p)\right] + \mathcal{O}(p^3)\right\},$$

insert the outcome for each $\tilde{\mathcal{I}}^j(p), j = 3,4,5$ (the list can be found in Appendix E.2.2), divide into the (non)-planar part and get

$$\tilde{\Pi}^{a,pl}_{\mu\nu}(p) = -C_a g^2\times \tag{E.23}$$
$$\times\left\{\frac{1}{16\pi^2}\Lambda^2\delta_{\mu\nu} + \left[\left((\frac{1}{12\pi^2} - \frac{1}{8\pi^2})p_\mu p_\nu + (\frac{1}{24\pi^2} - \frac{1}{16\pi^2})p^2\delta_{\mu\nu}\right)K_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right)\right]\right\},$$
$$\tilde{\Pi}^{a,npl}_{\mu\nu}(p) \approx -C_a g^2\times$$
$$\times\left\{-\frac{1}{4\pi^2}\frac{1}{(\epsilon\tilde{p})^2}\delta_{\mu\nu} + \frac{1}{2\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\epsilon\tilde{p}^2)^2}\right.$$
$$+\left[\left((\frac{1}{8\pi^2} - \frac{1}{12\pi^2})p_\mu p_\nu + (\frac{1}{16\pi^2} - \frac{1}{24\pi^2})p^2\delta_{\mu\nu}\right)K_0\left(\sqrt{\mu^2(\epsilon\tilde{p})^2}\right)\right] \tag{E.24}$$
$$-\left[\frac{1}{8\pi^2}\frac{p_\mu p_\delta\tilde{p}_\delta\tilde{p}_\nu}{\tilde{p}^2} + \frac{1}{16\pi^2}\frac{p^2\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2} + \frac{1}{12\pi^2}\frac{p_\delta p_\sigma\tilde{p}_\mu\tilde{p}_\nu\tilde{p}_\delta\tilde{p}_\sigma}{\tilde{p}^4}\right.$$
$$-\frac{1}{24\pi^2}(\delta_{\mu\nu}p_\delta p_\sigma\tilde{p}_\delta\tilde{p}_\sigma + p_\mu p_\sigma\tilde{p}_\nu\tilde{p}_\sigma + p_\sigma p_\nu\tilde{p}_\mu\tilde{p}_\sigma$$
$$\left.\left.+ p_\delta p_\mu\tilde{p}_\nu\tilde{p}_\delta + p_\delta p_\nu\tilde{p}_\mu\tilde{p}_\delta + p^2\tilde{p}_\mu\tilde{p}_\nu)\frac{1}{\tilde{p}^2}\right]^{finite} + \mathcal{O}(p^3)\right\},$$

where the first term inside the curly brackets represents the zeroth (0) order, while the second order (2) is represented by the square brackets. Additionally we have a so-called finite term which was generated by the second order but will give a finite expression in the limit of small external momenta p. If we split explicitly into the (non)-planar terms and the order of expansion we find

$$
\begin{aligned}
&\tilde{\Pi}^a_{\mu\nu}(p) \approx -C_a g^2 \times \\
&\times \Bigg\{ \Bigg[\left(\frac{1}{16\pi^2}\delta_{\mu\nu}\Lambda^2 \right)^{(0)} \\
&\quad - \left(\frac{1}{48\pi^2} p^2 \delta_{\mu\nu} K_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right) + \frac{1}{24\pi^2} p_\mu p_\nu K_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right)\right)^{(2)} \Bigg]^{pl} \\
&\quad + \Bigg[\left(-\frac{1}{4\pi^2}\frac{\delta_{\mu\nu}}{(\epsilon\tilde{p})^2} + \frac{1}{2\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\epsilon\tilde{p}^2)^2} \right)^{(0)} + \Big((\text{finite terms})' \Big)^{(2)} \\
&\quad + \left(\frac{1}{24\pi^2} p_\mu p_\nu K_0\left(\sqrt{\mu^2(\epsilon\tilde{p}^2)^2}\right) + \frac{1}{48\pi^2} p^2 \delta_{\mu\nu} K_0\left(\sqrt{\mu^2(\epsilon\tilde{p})^2}\right) \right)^{(2)} \Bigg]^{npl} \\
&\quad + \mathcal{O}(p^3) \Bigg\},
\end{aligned}
\tag{E.25}
$$

where we have

$$
\begin{aligned}
(\text{finite terms})' = -\Bigg[&\frac{1}{8\pi^2}\frac{p_\mu p_\delta \tilde{p}_\delta \tilde{p}_\nu}{\tilde{p}^2} + \frac{1}{16\pi^2}\frac{p^2 \tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2} + \frac{1}{12\pi^2}\frac{p_\delta p_\sigma \tilde{p}_\mu \tilde{p}_\nu \tilde{p}_\delta \tilde{p}_\sigma}{\tilde{p}^4} \\
&- \frac{1}{24\pi^2}(\delta_{\mu\nu} p_\delta p_\sigma \tilde{p}_\delta \tilde{p}_\sigma + p_\mu p_\sigma \tilde{p}_\nu \tilde{p}_\sigma + p_\sigma p_\nu \tilde{p}_\mu \tilde{p}_\sigma \\
&+ p_\delta p_\mu \tilde{p}_\nu \tilde{p}_\delta + p_\delta p_\nu \tilde{p}_\mu \tilde{p}_\delta + p^2 \tilde{p}_\mu \tilde{p}_\nu)\frac{1}{\tilde{p}^2} \Bigg]^{finite},
\end{aligned}
\tag{E.26}
$$

equal to (5.78) and (5.81).

Evaluation of $\tilde{\Pi}^b_{\mu\nu}(p)$

In order to start our calculation, we require Eq. (5.77b) from the main text:

$$
\begin{aligned}
\tilde{\Pi}^b_{\mu\nu}(p) \approx C_b &\frac{4g^2}{(2\pi)^4} \times \\
\times \Bigg\{ &12\tilde{\mathcal{I}}^3_{\mu\nu}(p) + \Bigg[-5p_\mu p_\nu \tilde{\mathcal{I}}^2(p) - 8p_\alpha p_\nu \tilde{\mathcal{I}}^4_{\alpha\mu}(p) - 8p_\alpha p_\mu \tilde{\mathcal{I}}^4_{\alpha\nu}(p) \\
&- 16p^2 \tilde{\mathcal{I}}^4_{\mu\nu}(p) + 52 p_\alpha p_\beta \tilde{\mathcal{I}}^5_{\alpha\beta\mu\nu}(p) \\
&+ 8p^2 \delta_{\mu\nu} \tilde{\mathcal{I}}^2(p) - 8p_\alpha p_\beta \delta_{\mu\nu} \tilde{\mathcal{I}}^4_{\alpha\beta}(p) \Bigg] + \mathcal{O}(p^3) \Bigg\},
\end{aligned}
$$

insert the outcome for each integral $\tilde{\mathcal{I}}^j(p), j = 2, 3, 4, 5$ (Appendix E.2.2), divide into the (non)-planar part and get

$$\begin{aligned}
\tilde{\Pi}^{b,pl}_{\mu\nu}(p) &= C_b g^2 \Big\{ \frac{3}{4\pi^2}\Lambda^2 \delta_{\mu\nu} \\
&\quad + \Big[\Big((\frac{2}{\pi^2} + \frac{26}{48\pi^2} - \frac{1}{2\pi^2} - \frac{1}{\pi^2}) p^2 \delta_{\mu\nu} + (\frac{52}{48\pi^2} - \frac{9}{4\pi^2}) p_\mu p_\nu \Big) K_0\Big(2\sqrt{\frac{\mu^2}{\Lambda^2}} \Big) \Big\},
\end{aligned} \tag{E.27}$$

$$\begin{aligned}
&\tilde{\Pi}^{b,npl}_{\mu\nu}(p) \approx C_b g^2 \times \\
&\times \Big\{ -\frac{3}{\pi^2}\frac{1}{(\epsilon\tilde{p})^2}\delta_{\mu\nu} + \frac{6}{\pi^2}\frac{\tilde{p}_\mu \tilde{p}_\nu}{(\epsilon\tilde{p}^2)^2} \\
&+ \Big[\Big((\frac{1}{2\pi^2} - \frac{2}{\pi^2} - \frac{26}{48\pi^2} + \frac{1}{\pi^2}) p^2 \delta_{\mu\nu} + (\frac{1}{\pi^2} + \frac{5}{4\pi^2} - \frac{52}{48\pi^2}) p_\mu p_\nu \Big) K_0\Big(\sqrt{\mu^2(\epsilon\tilde{p})^2} \Big) \Big] \\
&- \Big[\frac{1}{2\pi^2}\frac{p_\alpha p_\beta \tilde{p}_\alpha \tilde{p}_\beta}{\tilde{p}^2}\delta_{\mu\nu} + \frac{1}{\pi^2}\frac{p^2 \tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2} + \frac{1}{2\pi^2}\frac{p_\alpha p_\mu \tilde{p}_\alpha \tilde{p}_\nu}{\tilde{p}^2} + \frac{1}{2\pi^2}\frac{p_\alpha p_\nu \tilde{p}_\alpha \tilde{p}_\mu}{\tilde{p}^2} \\
&\qquad - \frac{26}{48\pi^2}(p^2 \tilde{p}_\mu \tilde{p}_\nu + p_\mu p_\beta \tilde{p}_\beta \tilde{p}_\nu + p_\alpha p_\mu \tilde{p}_\alpha \tilde{p}_\nu + p_\nu p_\beta \tilde{p}_\beta \tilde{p}_\mu \\
&\qquad + p_\alpha p_\mu \tilde{p}_\alpha \tilde{p}_\nu + \delta_{\mu\nu} p_\alpha p_\beta \tilde{p}_\alpha \tilde{p}_\beta)\frac{1}{\tilde{p}^2} + \frac{52}{48\pi^2}\frac{p_\alpha p_\beta \tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^4} \Big]^{finite} + \mathcal{O}(p^3) \Big\}.
\end{aligned} \tag{E.28}$$

The explicit (non)-planar parts depending on the order of expansion reads

$$\begin{aligned}
&\tilde{\Pi}^{b}_{\mu\nu}(p) \approx +C_b g^2 \times \\
&\times \Big\{ \Big[\Big(\frac{3}{4\pi^2}\delta_{\mu\nu}\Lambda^2 \Big)^{(0)} \\
&\quad + \Big(\frac{50}{48\pi^2} K_0\Big(2\sqrt{\frac{\mu^2}{\Lambda^2}} \Big) p^2 \delta_{\mu\nu} - \frac{56}{48\pi^2} K_0\Big(2\sqrt{\frac{\mu^2}{\Lambda^2}} \Big) p_\mu p_\nu \Big)^{(2)} \Big]^{pl} \\
&\quad + \Big[\Big(\frac{6}{\pi^2}\frac{\tilde{p}_\mu \tilde{p}_\nu}{(\epsilon\tilde{p}^2)^2} - \frac{3}{\pi^2}\frac{\delta_{\mu\nu}}{(\epsilon\tilde{p})^2} \Big)^{(0)} + \Big((\text{finite terms})'' \Big)^{(2)} \\
&\quad - \Big(\frac{50}{48\pi^2} K_0\Big(\sqrt{\mu^2(\epsilon\tilde{p})^2} \Big) p^2 \delta_{\mu\nu} - \frac{56}{48\pi^2} K_0\Big(\sqrt{\mu^2(\epsilon\tilde{p})^2} \Big) p_\mu p_\nu \Big)^{(2)} \Big]^{npl} \\
&\quad + \mathcal{O}(p^3) \Big\},
\end{aligned} \tag{E.29}$$

where we have

$$\begin{aligned}
&(\text{finite terms})'' = \\
&= -\Big[\frac{1}{2\pi^2}\frac{p_\alpha p_\beta \tilde{p}_\alpha \tilde{p}_\beta}{\tilde{p}^2}\delta_{\mu\nu} + \frac{1}{\pi^2}\frac{p^2 \tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2} + \frac{1}{2\pi^2}\frac{p_\alpha p_\mu \tilde{p}_\alpha \tilde{p}_\nu}{\tilde{p}^2} + \frac{1}{2\pi^2}\frac{p_\alpha p_\nu \tilde{p}_\alpha \tilde{p}_\mu}{\tilde{p}^2} \\
&\quad - \frac{26}{48\pi^2}(p^2 \tilde{p}_\mu \tilde{p}_\nu + p_\mu p_\beta \tilde{p}_\beta \tilde{p}_\nu + p_\alpha p_\mu \tilde{p}_\alpha \tilde{p}_\nu + p_\nu p_\beta \tilde{p}_\beta \tilde{p}_\mu \\
&\quad + p_\alpha p_\mu \tilde{p}_\alpha \tilde{p}_\nu + \delta_{\mu\nu} p_\alpha p_\beta \tilde{p}_\alpha \tilde{p}_\beta)\frac{1}{\tilde{p}^2} + \frac{52}{48\pi^2}\frac{p_\alpha p_\beta \tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^4} \Big]^{finite}.
\end{aligned} \tag{E.30}$$

which equal expressions (5.79) and (5.82).

Evaluation of $\tilde{\Pi}^c_{\mu\nu}(p)$

We start with Eq. (5.77c) from the main text

$$\tilde{\Pi}^c_{\mu\nu}(p) \approx -C_c \frac{8g^2}{(2\pi)^4}\left\{2\delta_{\mu\nu}\tilde{\mathcal{I}}^1(p) + \tilde{\mathcal{I}}^3_{\mu\nu}(p)\right\},$$

insert $\tilde{\mathcal{I}}^j(p), j = 1, 3$ (Appendix E.2.2) and get

$$\begin{aligned} \tilde{\Pi}^{c,pl}_{\mu\nu}(p) &\approx -C_c g^2 \left\{\frac{5}{8\pi^2}\Lambda^2\delta_{\mu\nu}\right\}, \\ \tilde{\Pi}^{c,npl}_{\mu\nu}(p) &\approx -C_c g^2 \left\{\frac{1}{\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\epsilon\tilde{p}^2)^2} - \frac{5}{2\pi^2}\frac{\delta_{\mu\nu}}{(\epsilon\tilde{p})^2}\right\}, \end{aligned} \tag{E.31}$$

leading to

$$\tilde{\Pi}^c_{\mu\nu}(p) \approx -C_c g^2 \left\{\left[\frac{5}{8\pi^2}\delta_{\mu\nu}\Lambda^2\right]^{pl} - \left[\frac{5}{2\pi^2}\frac{\delta_{\mu\nu}}{(\epsilon\tilde{p})^2} - \frac{1}{\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\epsilon\tilde{p}^2)^2}\right]^{npl}\right\}, \tag{E.32}$$

which is equal to expression (5.80).

E.3 One-Loop Correction

E.3.1 Calculation of $\tilde{\Pi}_{\mu\nu}(p)$

The calculation of $\tilde{\Pi}_{\mu\nu}(p)$ starts with the first line of expression (5.89) given by:

$$\begin{aligned} &\tilde{\Pi}_{\mu\nu}(p) \approx \\ &\approx \frac{2g^2\tilde{p}_\mu\tilde{p}_\nu}{\pi^2(\epsilon\tilde{p}^2)^2} + \frac{26g^2}{3(4\pi)^2}\left\{K_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right) - K_0\left(\sqrt{(\epsilon\tilde{p})^2\mu^2}\right)\right\}(p^2\delta_{\mu\nu} - p_\mu p_\nu) \\ &\quad + (\text{finite terms})''' + \mathcal{O}(p^3). \end{aligned} \tag{E.33}$$

Keeping in mind that we can expand the Bessel function of second kind for small arguments in the way of:

$$K_0(x) \approx \ln\frac{2}{x} - \gamma_E - \frac{x^2}{4}\left(\gamma_E - 1 + \ln\frac{x}{2}\right) + \mathcal{O}(x^4), \tag{E.34}$$

which equals Eq. (D.11a), we have to observe each relevant K_0 from $\tilde{\Pi}_{\mu\nu}$ and define primarily

$$a \equiv 2\sqrt{\frac{\mu^2}{\Lambda^2}}, \tag{E.35}$$

$$b \equiv \sqrt{(\epsilon\tilde{p})^2\mu^2}. \tag{E.36}$$

1. $K_0(a)$:

If we now apply (E.34) we receive

$$
\begin{aligned}
&K_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right) \approx \\
&\quad \approx \ln\left(\frac{2}{2\sqrt{\frac{\mu^2}{\Lambda^2}}}\right) - \gamma_E - \frac{\mu^2}{\Lambda^2}\left\{\gamma_E - 1 + \ln\left(\frac{2\sqrt{\frac{\mu^2}{\Lambda^2}}}{2}\right)\right\} + \mathcal{O}(a^4) \\
&\quad = \ln\left(\sqrt{\frac{\Lambda^2}{\mu^2}}\right) - \gamma_E - \frac{\mu^2}{\Lambda^2}\left\{\gamma_E - 1 + \ln\left(\sqrt{\frac{\mu^2}{\Lambda^2}}\right)\right\} + \mathcal{O}(a^4) \\
&\quad = \frac{1}{2}\ln(\Lambda^2) - \frac{1}{2}\ln(\mu^2) - (1 + \frac{\mu^2}{\Lambda^2})\gamma_E + \frac{\mu^2}{\Lambda^2} - \frac{1}{2}\frac{\mu^2}{\Lambda^2}\ln(\mu^2) + \frac{1}{2}\frac{\mu^2}{\Lambda^2}\ln(\Lambda^2) \\
&\qquad + \mathcal{O}(a^4).
\end{aligned} \tag{E.37}
$$

In the limits: $\mu^2 \to 0, \Lambda^2 \to \infty$, $K_0(a)$ reads

$$
K_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right) \approx \frac{1}{2}\ln(\Lambda^2) - \frac{1}{2}\ln(\mu^2) - \gamma_E + \mathcal{O}(a^4). \tag{E.38}
$$

2. $K_0(b)$:

We apply (E.34) and get

$$
\begin{aligned}
&K_0\left(\sqrt{(\epsilon\tilde{p})^2\mu^2}\right) \approx \\
&\quad \approx \ln\left(\frac{2}{\sqrt{(\epsilon\tilde{p})^2\mu^2}}\right) - \gamma_E - \frac{((\epsilon\tilde{p})^2\mu^2)^2}{4}\left\{\gamma_E - 1 + \ln\left(\frac{\sqrt{(\epsilon\tilde{p})^2\mu^2}}{2}\right)\right\} + \mathcal{O}(b^4) \\
\tilde{\Pi}^b_{\mu\nu}(p) &= \ln(2) - \frac{1}{2}\ln((\epsilon\tilde{p})^2\mu^2) - \gamma_E - \frac{((\epsilon\tilde{p})^2\mu^2)^2}{4}\gamma_E + \frac{((\epsilon\tilde{p})^2\mu^2)^2}{4} \\
&\quad - \frac{((\epsilon\tilde{p})^2\mu^2)^2}{8}\ln((\epsilon\tilde{p})^2\mu^2) + \frac{((\epsilon\tilde{p})^2\mu^2)^2}{4}\ln(2) + \mathcal{O}(b^4) \\
&= \ln(2) - \ln(\epsilon\tilde{p}) - \frac{1}{2}\ln(\mu^2) - \gamma_E - \frac{((\epsilon\tilde{p})^2\mu^2)^2}{4}\gamma_E + \frac{((\epsilon\tilde{p})^2\mu^2)^2}{4} \\
&\quad - \frac{((\epsilon\tilde{p})^2\mu^2)^2}{4}\ln(\epsilon\tilde{p}) - \frac{((\epsilon\tilde{p})^2\mu^2)^2}{4}\ln(\mu^2) + \frac{((\epsilon\tilde{p})^2\mu^2)^2}{4} + \mathcal{O}(b^4).
\end{aligned} \tag{E.39}
$$

In the limit: $\mu^2 \to 0$, $K_0(b)$ reads

$$
K_0\left(\sqrt{(\epsilon\tilde{p})^2\mu^2}\right) \approx \ln(2) - \frac{1}{2}\ln(\mu^2) - \gamma_E - \left(1 + \frac{((\epsilon\tilde{p})^2\mu^2)^2}{4}\right)\ln(\epsilon\tilde{p}) + \mathcal{O}(b^4). \tag{E.40}
$$

Therefore we have

$$\begin{aligned}
&\left\{K_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right) - K_0\left(\sqrt{(\epsilon\tilde{p})^2\mu^2}\right)\right\}(p^2\delta_{\mu\nu} - p_\mu p_\nu) \approx \\
&\quad \approx \left\{\frac{1}{2}\ln(\Lambda^2) - \cancel{\frac{1}{2}\ln(\mu^2)} - \cancel{\gamma_E} - \ln(2) + \cancel{\frac{1}{2}\ln(\mu^2)} + \cancel{\gamma_E}\right. \\
&\quad + \left.\left(1 + \frac{((\epsilon\tilde{p})^2\mu^2)^2}{4}\right)\ln(\epsilon\tilde{p}) + \mathcal{O}(a^4) + \mathcal{O}(b^4)\right\}(p^2\delta_{\mu\nu} - p_\mu p_\nu) \\
&\quad \approx \left\{\frac{1}{2}\ln(\Lambda^2) + \mathcal{O}(a^4) + \mathcal{O}(b^4)\right\}(p^2\delta_{\mu\nu} - p_\mu p_\nu) + (\text{finite terms})^{IV}, \qquad \text{(E.41)}
\end{aligned}$$

where the IR-cutoff μ cancels, and the expression (finite terms)^{IV} represents finite terms in the limit of small external momenta p:

$$(\text{finite terms})^{IV} \equiv \left\{\left(1 + \frac{((\epsilon\tilde{p})^2\mu^2)^2}{4}\right)\ln(\epsilon\tilde{p}) - \ln(2)\right\}(p^2\delta_{\mu\nu} - p_\mu p_\nu)$$

At the end we have the final expression which represents Eq. (5.89) from the main text:

$$\tilde{\Pi}_{\mu\nu}(p) \approx \frac{2g^2\tilde{p}_\mu\tilde{p}_\nu}{\pi^2(\epsilon\tilde{p}^2)^2} + \frac{13g^2}{24\pi^2}\ln(\Lambda)(p^2\delta_{\mu\nu} - p_\mu p_\nu) + \text{finite terms},$$

where

$$\begin{aligned}
\text{finite terms} \equiv\ & (\text{finite terms})''' + (\text{finite terms})^{IV} \qquad \text{(E.42)} \\
& + \{\mathcal{O}(a^4) + \mathcal{O}(b^4)\}(p^2\delta_{\mu\nu} - p_\mu p_\nu) + \mathcal{O}(p^3),
\end{aligned}$$

contains all finite expressions.

Appendix F

Renormalization

F.1 Renormalization of the Photon Propagator

F.1.1 Calculation of $\tilde{\Gamma}^{AA}_{\mu\nu}$

Before we start the explicit calculations, we once again must mention that the two-point vertex function at tree level $\tilde{\Gamma}^{AA,tree}_{\mu\nu}(k)$ represents the inverse of the two-point Green function at tree level $\tilde{G}^{AA}_{\mu\nu}(k)$ in the way of:

$$\tilde{\Gamma}^{AA,tree}_{\mu\nu}(k) = \left(\tilde{G}^{AA}_{\mu\nu}(k)\right)^{-1},$$

which is explained in detail in the main text (see (4.15) and (6.7)).

The calculation of this inverse starts with the photon propagator in the form of (6.10) with $\alpha \neq 0$:

$$\tilde{G}^{AA}_{\mu\nu}(k) = \frac{1}{k^2\mathcal{D}}\left[\delta_{\mu\nu} - (1-\alpha\mathcal{D})\frac{k_\mu k_\nu}{k^2} - \mathcal{F}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right],$$

where we have introduced the abbreviations:

$$\begin{aligned}\mathcal{D}(k) &\equiv \left(1 + \frac{\gamma^4}{(\tilde{k}^2)^2}\right),\\ \mathcal{F}(k) &\equiv \frac{1}{\tilde{k}^2}\frac{\bar{\sigma}^4}{\left(k^2 + (\bar{\sigma}^4+\gamma^4)\frac{1}{\tilde{k}^2}\right)}.\end{aligned}$$

To get the inverse we make the ansatz (6.13):

$$
\begin{aligned}
\delta_{\mu\nu} &= \left(\tilde{G}^{AA}_{\mu\rho}(k)\right)^{-1}\tilde{G}^{AA}_{\rho\nu}(k) \\
&= k^2\mathcal{D}\left[\delta_{\mu\rho} + a_1\frac{k_\mu k_\rho}{k^2} + a_2\frac{\tilde{k}_\mu\tilde{k}_\rho}{\tilde{k}^2}\right]\frac{1}{k^2\mathcal{D}}\left[\delta_{\rho\nu} - (1-\alpha\mathcal{D})\frac{k_\rho k_\nu}{k^2} - \mathcal{F}\frac{\tilde{k}_\rho\tilde{k}_\nu}{\tilde{k}^2}\right] \\
&= \delta_{\mu\rho}\left[\delta_{\rho\nu} - (1-\alpha\mathcal{D})\frac{k_\rho k_\nu}{k^2} - \mathcal{F}\frac{\tilde{k}_\rho\tilde{k}_\nu}{\tilde{k}^2}\right] \\
&\quad + a_1\frac{k_\mu k_\rho}{k^2}\left[\delta_{\rho\nu} - (1-\alpha\mathcal{D})\frac{k_\rho k_\nu}{k^2} - \mathcal{F}\frac{\tilde{k}_\rho\tilde{k}_\nu}{\tilde{k}^2}\right] \\
&\quad + a_2\frac{\tilde{k}_\mu\tilde{k}_\rho}{\tilde{k}^2}\left[\delta_{\rho\nu} - (1-\alpha\mathcal{D})\frac{k_\rho k_\nu}{k^2} - \mathcal{F}\frac{\tilde{k}_\rho\tilde{k}_\nu}{\tilde{k}^2}\right] \\
&= \delta_{\mu\nu} - (1-\alpha\mathcal{D})\frac{k_\mu k_\nu}{k^2} - \mathcal{F}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2} \\
&\quad + a_1\frac{k_\mu k_\nu}{k^2} - a_1(1-\alpha\mathcal{D})\frac{k_\mu k_\nu}{k^2} - a_1\mathcal{F}\cancel{\frac{k_\mu k_\rho\tilde{k}_\rho\tilde{k}_\nu}{k^2\tilde{k}^2}} \\
&\quad + a_2\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2} - a_2(1-\alpha\mathcal{D})\cancel{\frac{k_\nu k_\rho\tilde{k}_\rho\tilde{k}_\mu}{k^2\tilde{k}^2}} - a_2\mathcal{F}\frac{\tilde{k}_\mu\tilde{k}_\rho\tilde{k}_\rho\tilde{k}_\nu}{\tilde{k}^4},
\end{aligned}
\tag{F.1}
$$

where the marked terms vanish through the following fact:

$$
\begin{aligned}
k_\rho\tilde{k}_\rho &= k_\rho\theta_{\rho\alpha}k_\alpha = k_\alpha\theta_{\alpha\rho}k_\rho = k_\rho\theta_{\alpha\rho}k_\alpha = -k_\rho\theta_{\rho\alpha}k_\alpha = -k_\rho\tilde{k}_\rho, \\
&\Rightarrow k_\rho\tilde{k}_\rho = 0.
\end{aligned}
$$

This leads to

$$
\delta_{\mu\nu} = \delta_{\mu\nu} + \left\{a_1 - a_1(1-\alpha\mathcal{D}) - (1-\alpha\mathcal{D})\right\}\frac{k_\mu k_\nu}{k^2} + \left\{a_2(1-\mathcal{F}) - \mathcal{F}\right\}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2},
$$

where the expressions inside the curly brackets must obey the equation

$$
\begin{aligned}
\left\{a_1 - a_1(1-\alpha\mathcal{D}) - (1-\alpha\mathcal{D})\right\} &= 0, \\
\Rightarrow a_1 &= \frac{1}{\alpha\mathcal{D}} - 1,
\end{aligned}
\tag{F.2}
$$

and

$$
\begin{aligned}
\left\{a_2(1-\mathcal{F}) - \mathcal{F}\right\} &= 0, \\
\Rightarrow a_2 &= \frac{\mathcal{F}}{1-\mathcal{F}}
\end{aligned}
\tag{F.3}
$$

The inverse now reads

$$
\begin{aligned}
\tilde{\Gamma}^{AA,tree}_{\mu\nu}(k) = \big(\tilde{G}^{AA}_{\mu\nu}(k)\big)^{-1} &= \\
&= k^2\mathcal{D}\left[\delta_{\mu\rho} + a_1\frac{k_\mu k_\rho}{k^2} + a_2\frac{\tilde{k}_\mu\tilde{k}_\rho}{\tilde{k}^2}\right] \\
&= k^2\mathcal{D}\left[\delta_{\mu\nu} + \left(\frac{1}{\alpha\mathcal{D}} - 1\right)\frac{k_\mu k_\nu}{k^2} + \left(\frac{\mathcal{F}}{1-\mathcal{F}}\right)\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right] \\
&= k^2\mathcal{D}\left[\delta_{\mu\nu} + \left(\frac{1}{\alpha\mathcal{D}} - 1\right)\frac{k_\mu k_\nu}{k^2} + \left(\frac{\bar{\sigma}^4}{k^2\tilde{k}^2\mathcal{D}}\right)\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right],
\end{aligned}
\tag{F.4}
$$

where we have written $\mathcal{F}/(1-\mathcal{F})$ in a term depending on $\mathcal{D}$:

$$
\frac{\mathcal{F}}{1-\mathcal{F}} = \frac{\frac{\bar{\sigma}^4}{k^4+(\bar{\sigma}^4+\gamma^4)}}{1 - \frac{\bar{\sigma}^4}{k^4+(\bar{\sigma}^4+\gamma^4)}} = \frac{\bar{\sigma}^4}{k^4+\gamma^4} = \frac{\bar{\sigma}^4}{k^2\tilde{k}^2\mathcal{D}}. \tag{F.5}
$$

The inverse propagator (F.4) equals Eq. (6.16).

F.1.2 Calculation of $\tilde{\Gamma}^{AA,dress}_{\mu\nu}$

The dressed vertex function is given by the first line of Eq. (6.21), where we insert Eqs. (6.17-6.20) from the main text

$$
\begin{aligned}
&\tilde{\Gamma}^{AA,dress}_{\mu\nu}(k) = \tilde{\Gamma}^{AA,tree}_{\mu\nu}(k) - \tilde{\Gamma}^{AA.corr}_{\mu\nu}(k) \\
&= k^2\mathcal{D}\left[\delta_{\mu\nu} + \left(\frac{1}{\alpha\mathcal{D}} - 1\right)\frac{k_\mu k_\nu}{k^2} + \left(\frac{\bar{\sigma}^4}{k^2\tilde{k}^2\mathcal{D}}\right)\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right] \\
&\quad - \mathit{\Pi}_a\frac{\tilde{k}_\mu\tilde{k}_\nu}{(\tilde{k}^2)^2} - \mathit{\Pi}_b k^2\delta_{\mu\nu} + \mathit{\Pi}_b k_\mu k_\nu \\
&= k^2(\mathcal{D} - \mathit{\Pi}_b)\delta_{\mu\nu} + \left\{\mathcal{D}\left(\frac{1}{\alpha\mathcal{D}} - 1\right) + \mathit{\Pi}_b\right\}k_\mu k_\nu + \left\{\bar{\sigma}^4 - \mathit{\Pi}_a\right\}\frac{\tilde{k}_\mu\tilde{k}_\nu}{(\tilde{k}^2)^2} \\
&= k^2(\mathcal{D} - \mathit{\Pi}_b)\left[\delta_{\mu\nu} + \left(\frac{1}{\alpha(\mathcal{D} - \mathit{\Pi}_b)} - 1\right)\frac{k_\mu k_\nu}{k^2} + \frac{\bar{\sigma}^4 - \mathit{\Pi}_a}{k^2\tilde{k}^2(\mathcal{D} - \mathit{\Pi}_b)}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right].
\end{aligned}
\tag{F.6}
$$

The last expression equals the second line from Eq. (6.21).

F.1.3 Calculation of σ_r

The calculation of σ_r starts with Eq. (6.29) from the main text:

$$
2\left(\sigma_r + \frac{\theta^2}{4}\sigma_r^2\right)\gamma^4 Z_A^2 = (\bar{\sigma}^4 - \mathit{\Pi}_a)Z_A^2,
$$

where we insert Eq. (6.19) leading to

$$
\begin{aligned}
\left(\sigma_r + \frac{\theta^2}{4}\sigma_r^2\right) &= \left\{\left(\sigma + \frac{\theta^2}{4}\sigma^2\right) - \frac{g^2}{\pi^2\gamma^4\epsilon^2}\right\}, \\
\rightarrow \sigma_r^2 + \frac{4}{\theta^2}\sigma_r - \frac{4}{\theta^2}&\left\{\left(\sigma + \frac{\theta^2}{4}\sigma^2\right) - \frac{g^2}{\pi^2\gamma^4\epsilon^2}\right\} = 0, \\
\Rightarrow \sigma_r = -\frac{2}{\theta^2} &\pm \sqrt{\left(\frac{2}{\theta^2}\right)^2 + \frac{4}{\theta^2}\left\{\left(\sigma + \frac{\theta^2}{4}\sigma^2\right) - \frac{g^2}{\pi^2\gamma^4\epsilon^2}\right\}} \\
&= \frac{2}{\theta^2}\left\{-1 \pm \sqrt{\left(1 + \frac{\theta}{2}\sigma\right)^2 - \frac{g^2\theta^2}{\pi^2\gamma^4\epsilon^2}}\right\},
\end{aligned}
\tag{F.7}
$$

which equals Eq. (6.31).

F.1.4 Calculation of $\tilde{G}^{AA,ren}_{\mu\nu}$

We start with Eq. (6.33):

$$
\begin{aligned}
\delta_{\mu\nu} &= \left(\tilde{\Gamma}^{AA,ren}_{\mu\rho}(k)\right)^{-1}\tilde{\Gamma}^{AA,ren}_{\rho\nu}(k) \\
&= \frac{Z_A^2}{k^2\mathcal{D}_r}\left[\delta_{\mu\rho} + b_1\frac{k_\mu k_\rho}{k^2} + b_2\frac{\tilde{k}_\mu\tilde{k}_\rho}{\tilde{k}^2}\right]\frac{k^2\mathcal{D}_r}{Z_A^2}\left[\delta_{\rho\nu} + \left(\frac{Z_A^2}{\alpha\mathcal{D}_r} - 1\right)\frac{k_\rho k_\nu}{k^2} + \frac{\bar{\sigma}_r^4}{k^2\tilde{k}^2\mathcal{D}_r}\frac{\tilde{k}_\rho\tilde{k}_\nu}{\tilde{k}^2}\right] \\
&= \delta_{\mu\nu} + \left(\frac{Z_A^2}{\alpha\mathcal{D}_r} - 1\right)\frac{k_\mu k_\nu}{k^2} + \frac{\bar{\sigma}_r^4}{k^2\tilde{k}^2\mathcal{D}_r}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2} \\
&\quad + b_1\frac{k_\mu k_\rho}{k^2}\left[\delta_{\rho\nu} + \left(\frac{Z_A^2}{\alpha\mathcal{D}_r} - 1\right)\frac{k_\rho k_\nu}{k^2} + \frac{\bar{\sigma}_r^4}{k^2\tilde{k}^2\mathcal{D}_r}\frac{\tilde{k}_\rho\tilde{k}_\nu}{\tilde{k}^2}\right] \\
&\quad + b_2\frac{k_\mu k_\rho}{k^2}\left[\delta_{\rho\nu} + \left(\frac{Z_A^2}{\alpha\mathcal{D}_r} - 1\right)\frac{k_\rho k_\nu}{k^2} + \frac{\bar{\sigma}_r^4}{k^2\tilde{k}^2\mathcal{D}_r}\frac{\tilde{k}_\rho\tilde{k}_\nu}{\tilde{k}^2}\right] \\
&= \delta_{\mu\nu} + \left(\frac{Z_A^2}{\alpha\mathcal{D}_r} - 1\right)\frac{k_\mu k_\nu}{k^2} + \frac{\bar{\sigma}_r^4}{k^2\tilde{k}^2\mathcal{D}_r}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2} \\
&\qquad + b_1\frac{k_\mu k_\nu}{k^2} + b_1\left(\frac{Z_A^2}{\alpha\mathcal{D}_r} - 1\right)\frac{k_\mu k_\nu}{k^2} + b_2\frac{k_\mu k_\nu}{k^2} + b_2\frac{\bar{\sigma}_r^4}{k^2\tilde{k}^2\mathcal{D}_r}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2} \\
&= \delta_{\mu\nu} + \left[\left(\frac{Z_A^2}{\alpha\mathcal{D}_r} - 1\right) + b_1\left(1 + \left(\frac{Z_A^2}{\alpha\mathcal{D}_r} - 1\right)\right)\right]\frac{k_\mu k_\nu}{k^2} \\
&\qquad + \left[\frac{\bar{\sigma}_r^4}{k^2\tilde{k}^2\mathcal{D}_r} + b_2\left(1 + \frac{\bar{\sigma}_r^4}{k^2\tilde{k}^2\mathcal{D}_r}\right)\right]\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}.
\end{aligned}
\tag{F.8}
$$

By comparison, we obtain for the coefficients b_1, b_2

$$
b_1 = -\left(\frac{\frac{Z_A^2}{\alpha\mathcal{D}_r} - 1}{1 + \frac{Z_A^2}{\alpha\mathcal{D}_r} - 1}\right), \tag{F.9a}
$$

$$
b_2 = -\left(\frac{\frac{\bar{\sigma}_r^4}{k^2\tilde{k}^2\mathcal{D}_r}}{1 + \frac{\bar{\sigma}_r^4}{k^2\tilde{k}^2\mathcal{D}_r}}\right). \tag{F.9b}
$$

For the first coefficient b_1 we consider the Landau gauge ($\alpha \to 0$) and get

$$b_1 = -\left(\frac{\frac{Z_A^2}{\alpha \mathcal{D}_r} - 1}{1 + \frac{Z_A^2}{\alpha \mathcal{D}_r} - 1}\right)\Bigg|_{\alpha \to 0} = -\left(\frac{\alpha \mathcal{D}_r (Z_A^2 - \alpha \mathcal{D}_r)}{\alpha \mathcal{D}_r Z_A^2}\right)\Bigg|_{\alpha \to 0} = -1. \tag{F.10}$$

The second one b_2 can be written as

$$b_2 = -\left(\frac{\frac{\bar{\sigma}_r^4}{k^2 \tilde{k}^2 \mathcal{D}_r}}{1 + \frac{\bar{\sigma}_r^4}{k^2 \tilde{k}^2 \mathcal{D}_r}}\right) = -\frac{\bar{\sigma}_r^4}{(k^2 \tilde{k}^2 \mathcal{D}_r + \bar{\sigma}_r^4)}. \tag{F.11}$$

With the help of $\mathcal{D}_r$, given by Eq. (6.23):

$$\mathcal{D}_r = \left(1 + \frac{\gamma_r^4}{k^2 \tilde{k}^2}\right),$$

the coefficient reads

$$b_2 = -\frac{\bar{\sigma}_r^4}{(k^2 \tilde{k}^2 + \gamma_r^4 + \bar{\sigma}_r^4)} = -\frac{1}{\tilde{k}^2} \frac{\bar{\sigma}_r^4}{\left(k^2 + (\bar{\sigma}_r^4 + \gamma_r^4)\frac{1}{\tilde{k}^2}\right)}. \tag{F.12}$$

Therefore the renormalized propagator has the form of

$$\begin{aligned}
\tilde{G}_{\mu\nu}^{AA,ren}(k) &= \left(\tilde{\Gamma}_{\mu\nu}^{AA,ren}(k)\right)^{-1} \\
&= \frac{Z_A^2}{k^2 \mathcal{D}_r}\left[\delta_{\mu\rho} + b_1 \frac{k_\mu k_\rho}{k^2} + b_2 \frac{\tilde{k}_\mu \tilde{k}_\rho}{\tilde{k}^2}\right] \\
&= \frac{Z_A^2}{k^2 \mathcal{D}_r}\left[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - \mathcal{F}_r \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2}\right],
\end{aligned} \tag{F.13}$$

with the abbreviation $\mathcal{F}_r$

$$\mathcal{F}_r(k) \equiv \frac{1}{\tilde{k}^2} \frac{\bar{\sigma}_r^4}{\left(k^2 + (\bar{\sigma}_r^4 + \gamma_r^4)\frac{1}{\tilde{k}^2}\right)}. \tag{F.14}$$

The last two equations equal (6.37) and (6.38), respectively.

Bibliography

[1] E. Schrödinger, *Über die Unanwendbarkeit der Geometrie im Kleinen*, *Die Naturwiss.* **22** (1934) 518–520.

[2] W. Heisenberg, *Über die in der Theorie der Elementarteilchen auftretende universelle Länge*, *Ann. Phys.* **32** (1938) 20–33.

[3] H. S. Snyder, *Quantized space-time*, *Phys. Rev.* **71** (1947) 38–41.

[4] H. S. Snyder, *The Electromagnetic Field in Quantized Space-Time*, *Phys. Rev.* **72** (1947) 68.

[5] S. Doplicher, K. Fredenhagen and J. E. Roberts, *The Quantum structure of space-time at the Planck scale and quantum fields*, *Commun. Math. Phys.* **172** (1995) 187–220, [arXiv:hep-th/0303037].

[6] H. J. Groenewold, *On the Principles of elementary quantum mechanics*, *Physica* **12** (1946) 405–460.

[7] J. E. Moyal, *Quantum mechanics as a statistical theory*, *Proc. Cambridge Phil. Soc.* **45** (1949) 99–124.

[8] T. Filk, *Divergencies in a field theory on quantum space*, *Phys. Lett.* **B376** (1996) 53–58.

[9] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, *JHEP* **02** (2000) 020, [arXiv:hep-th/9912072].

[10] A. Matusis, L. Susskind and N. Toumbas, *The IR/UV connection in the non-commutative gauge theories*, *JHEP* **12** (2000) 002, [arXiv:hep-th/0002075].

[11] A. Micu and M. M. Sheikh Jabbari, *Noncommutative* Φ^4 *theory at two loops*, *JHEP* **01** (2001) 025, [arXiv:hep-th/0008057].

[12] I. Y. Aref'eva, D. M. Belov, A. S. Koshelev and O. A. Rytchkov, *UV/IR mixing for noncommutative complex scalar field theory. II: Interaction with gauge fields*, *Nucl. Phys. Proc. Suppl.* **102** (2001) 11–17, [arXiv:hep-th/0003176].

[13] H. Bozkaya et al., *Space/time noncommutative field theories and causality*, *Eur. Phys. J.* **C29** (2003) 133–141, [arXiv:hep-th/0209253].

[14] I. Chepelev and R. Roiban, *Convergence theorem for non-commutative Feynman graphs and renormalization*, *JHEP* **03** (2001) 001, [arXiv:hep-th/0008090].

[15] H. Grosse and R. Wulkenhaar, *Renormalisation of* ϕ^4 *theory on noncommutative* $\mathbb{R}^2$ *in the matrix base*, *JHEP* **12** (2003) 019, [arXiv:hep-th/0307017].

[16] R. Gurau, J. Magnen, V. Rivasseau and A. Tanasa, *A translation-invariant renormalizable non-commutative scalar model*, *Commun. Math. Phys.* **287** (2009) 275–290, [arXiv:0802.0791].

[17] D. N. Blaschke, F. Gieres, E. Kronberger, T. Reis, M. Schweda and R. I. P. Sedmik, *Quantum Corrections for Translation-Invariant Renormalizable Non-Commutative Φ^4 Theory*, *JHEP* **11** (2008) 074, [arXiv:0807.3270].

[18] D. N. Blaschke, E. Kronberger, R. I. P. Sedmik and M. Wohlgenannt, *Gauge Theories on Deformed Spaces*, *SIGMA* **6** (2010) 062, [arXiv:1004.2127].

[19] D. N. Blaschke, F. Gieres, E. Kronberger, M. Schweda and M. Wohlgenannt, *Translation-invariant models for non-commutative gauge fields*, *J. Phys.* **A41** (2008) 252002, [arXiv:0804.1914].

[20] D. N. Blaschke, A. Rofner, R. I. P. Sedmik and M. Wohlgenannt, *On Non-Commutative $U_\star(1)$ Gauge Models and Renormalizability*, *J. Phys.* **A43** (2010) 425401, [arXiv:0912.2634].

[21] L. C. Q. Vilar, O. S. Ventura, D. G. Tedesco and V. E. R. Lemes, *On the Renormalizability of Noncommutative U(1) Gauge Theory — an Algebraic Approach*, *J. Phys.* **A43** (2010) 135401, [arXiv:0902.2956].

[22] D. Zwanziger, *Local and Renormalizable Action from the Gribov Horizon*, *Nucl. Phys.* **B323** (1989) 513–544.

[23] L. Baulieu and S. P. Sorella, *Soft breaking of BRST invariance for introducing non-perturbative infrared effects in a local and renormalizable way*, *Phys. Lett.* **B671** (2009) 481, [arXiv:0808.1356].

[24] D. Zwanziger, *Renormalizability of the critical limit of lattice gauge theory by BRS invariance*, *Nucl. Phys.* **B399** (1993) 477–513.

[25] A. Rofner, *Investigations on the renormalizability of a non-commutative $U(1)$ gauge theory*, Ph.D. thesis, Vienna University of Technology, 2009, URL http://media.obvsg.at/AC07806222.

[26] R. I. P. Sedmik, *On the Development of Non-Commutative Translation-Invariant Quantum Gauge Field Models*, Ph.D. thesis, Vienna University of Technology, 2009, URL http://media.obvsg.at/AC07806220.

[27] D. N. Blaschke, E. Kronberger, A. Rofner, M. Schweda, R. I. P. Sedmik and M. Wohlgenannt, *On the Problem of Renormalizability in Non-Commutative Gauge Field Models — A Critical Review*, *Fortschr. Phys.* **58** (2010) 364, [arXiv:0908.0467].

[28] A. Connes, *Noncommutative geometry and reality*, *J. Math. Phys.* **36** (1995) 6194–6231.

[29] J. Madore, S. Schraml, P. Schupp and J. Wess, *Gauge theory on noncommutative spaces*, *Eur. Phys. J.* **C16** (2000) 161–167, [arXiv:hep-th/0001203].

[30] V. N. Gribov, *Quantization of non-Abelian gauge theories*, *Nucl. Phys.* **B139** (1978) 1.

[31] C. Becchi, A. Rouet and R. Stora, *Renormalization of gauge theories*, *Annals Phys.* **98** (1976) 287–321.

[32] J. C. Ward, *An Identity in Quantum Electrodynamics*, *Phys. Rev.* **78**, no. 2 (1950) 182.

[33] J. C. Taylor, *Ward identities and charge renormalization of the Yang-Mills field*, *Nuclear Physics B* **33**, no. 2 (1971) 436 – 444, URL http://www.sciencedirect.com/science/article/B6TVC-472T48T-4D/2/e6c19613be1c2e1593894ba548ffbdd5.

[34] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory*, Cambridge, Mass.: Perseus Books, 1995.

[35] L. H. Ryder, *Quantum Field Theory*, Cambridge: University Press, second edition, 1996.

[36] M. R. Douglas and N. A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2001) 977–1029, [arXiv:hep-th/0106048].

[37] V. Rivasseau, *Non-commutative renormalization,* in *Quantum Spaces — Poincaré Seminar 2007*, B. Duplantier and V. Rivasseau eds., Birkhäuser Verlag, [arXiv:0705.0705].

[38] A. Tanasa, *Scalar and gauge translation-invariant noncommutative models*, *Rom. J. Phys.* **53** (2008) 1207–1212, [arXiv:0808.3703].

[39] M. Hayakawa, *Perturbative analysis on infrared aspects of noncommutative QED on* $\mathbb{R}^4$, *Phys. Lett.* **B478** (2000) 394–400, [arXiv:hep-th/9912094].

[40] F. R. Ruiz, *Gauge-fixing independence of IR divergences in non-commutative U(1), perturbative tachyonic instabilities and supersymmetry*, *Phys. Lett.* **B502** (2001) 274–278, [arXiv:hep-th/0012171].

[41] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, seventh edition, 2007.