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Star Product on Non(Anti)Commutative Superspace

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Ao.Univ.Prof. Dipl.-Ing. Dr.techn. Maximilian Kreuzer
Institutsnummer: E 136
Institut für Theoretische Physik

eingereicht an der Technischen Universität Wien
Fakultät für Physik

von

Rashid Ahmad
Matrikelnummer: e0625300
Sensengasse 2B 414/A
A-1090 Wien

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Abstract

This thesis is devoted to the explicit calculation of the star product in bosonic space as well as in superspace and its generalization to the non-associative case. Promoting the coordinate functions as elements of a commutative algebra to elements of a non-commutative associative algebra is carefully reviewed and further generalized to a non-associative algebra. The coordinate monomials as basis of the commutative algebra of functions are naturally mapped to Weyl ordered monomials of the generators of the non-commutative algebra. Via this embedding the non-commutative algebra product induces a non-commutative product on the space of functions, namely the star product. Recently an effective method for the explicit calculation of the star product to higher derivative orders has been presented, based on a representation of the non-commutative algebra via polydifferential operators. This approach is reviewed up to third order in the expansion parameter and generalized to the superspace. Comments on a possible extension of the method to the non-associative case are given. In one of the proposed approaches the non-associative star product is calculated to the second order and a cyclicity condition is imposed. Once we have the star product at every order, it is compelling to look for its different properties. Diffeomorphisms on commutative coordinate space are defined with the help of comultiplication of the Hopf algebra and deformed diffeomorphisms are introduced on noncommutative coordinate space by deforming the comultiplication and hence the Hopf algebra. Quantum corrections to the classical transformation of the star product under Lie derivative are proposed via the formality theorem and computed to the second order in the star product expansion parameter. Although all the calculations are done with graded objects, equations look as in the bosonic case due to the use of a graded Einstein summation convention. However, different components of the star product on the non(anti)commutative superspace are explicitly computed at the end. The twist representation of the star product on non(anti)superspace is given in the appendix.

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Chapter 1

Introduction

Historically, Heisenberg was the first to propose a radical idea of the spacetime noncommutativity in late 1930's. However, it somehow got passed to Snyder, see [1] for the story about it, who published the first article in 1947 about the noncommutative spacetime [2]. This noncommutativity of spacetime coordinates x^m, or more generally of any arbitrary space, is expressed by promoting the coordinate functions as elements of a commutative algebra to elements of a non-commutative associative algebra

$$[\hat{x}^m, \hat{x}^n] = i\omega^{mn}(\hat{x}) \tag{1.1}$$

where the parameter of noncommutativity ω^{mn} is an antisymmetric tensor.

On the other hand Weyl in 1927 gave a general rule for associating an operator in the Hilbert space to the function $f(q,p)$ of phase space conjugate variables [3]. Eugene Wigner discovered the inverse to the Weyl rule that maps an operator into what is called symbol of the operator $f(\hat{q},\hat{p})$ [4].

Some years later Blackett assigned the problem of a physical interpretation of the phase space function $f(q,p)$ and symbol $f(\hat{q},\hat{p})$ of a quantum operator $\hat{O}(q,p)$ to his student Moyal who found what is now called Moyal star product [5]. A similar product for symbols of the operator $f(\hat{q},\hat{p})g(\hat{q},\hat{p})$ was found by Groenewold [6]. See [7] for an overview. Although the main purpose of introducing noncommutative spacetime (1.1) was to cure the Quantum Field Theories of divergences, the noncommutativity of quantum phase space operators $\hat{q}^m$ and momentum $\hat{p}_n$ $[\hat{q}^m, \hat{p}_n] = i\hbar\delta^m_n$, served as a motivational example from Quantum Mechanics.

The star product as a tool of deforming the usual ordinary product of functions is a main ingredient of the deformation quantization. The existence of the star product and deformation quantization is established on different spaces and mainfolds. For Poisson manifolds it is established via the formality theorem [8]

However, only very recently an efficient procedure, based on [9], for calculating a star product order by order in a expansion parameter is developed in [10].

During 1970's due to the discovery of supersymmetry, the structure of bosonic spacetime was extended by including fermionic coordinates θ^μ and $\bar{\theta}^{\bar{\mu}}$. The resulting space was called superspace [11] $X^M = (x^m, \theta^\mu, \bar{\theta}^{\bar{\mu}})$. The deformation of this superspace is done in [12] and also from string theory perspective in [13][14] to get non(anti)commutative superspace. In [15] a star product for a class of general superfields is constructed, but the star product is associative only up to second order of the deformation parameter $\hbar$. One of the subjects of this thesis is to extend the procedure of [10] to the non(anti)commutative superspace where the star product is associative to all orders of the deformation parameter. Only after that the possibility to relax the associativity constraint in a controlled manner will be discussed.

In this thesis the notion of non(anti)commutative superspace is introduced by promoting coordinate functions $\varphi^M(X) = X^M$ as elements of a graded commutative algebra on superspace to elements of a graded non-commutative associative algebra with generators $\hat{X}^M = (\hat{x}^m, \hat{\theta}^\mu, \hat{\bar{\theta}}^{\bar{\mu}})$ in the same sense as in the bosonic case

$$[\hat{X}^M, \hat{X}^N] = i\omega^{MN}(\hat{X}) \tag{1.2}$$

implies, at most general, the following relations[1]

$$\begin{aligned}
&[\hat{x}^m, \hat{x}^n] = i\omega^{mn}(\hat{X}), [\hat{x}^m, \hat{\theta}^\mu] = \omega^{m\mu}(\hat{X}), [\hat{x}^m, \hat{\bar{\theta}}^{\bar{\mu}}] = \omega^{m\bar{\mu}}(\hat{X}),\\
&[\hat{\theta}^\mu, \hat{\theta}^\nu] = \omega^{\mu\nu}(\hat{X}), [\hat{\bar{\theta}}^{\bar{\mu}}, \hat{\bar{\theta}}^{\bar{\nu}}] = \omega^{\bar{\mu}\bar{\nu}}(\hat{X}), [\hat{\theta}^\mu, \hat{\bar{\theta}}^{\bar{\nu}}] = \omega^{\mu\bar{\nu}}(\hat{X})
\end{aligned} \tag{1.3}$$

We call such a space with above relations a non(anti)commutative superspace. For a pedagogical introduction of superspace see [16] and for calculus on superspace see for example [17].

This thesis is divided in seven chapters and three appendices. The brief overview of every chapter is as follows.

In chapter 2 we define commutative coordinate space [18] and then we define a noncommutative space of the coordinate operators and generalize it to the nonassociative case. This chapter provides the basis for the whole approach we develop in this thesis. We expand the noncommutativity parameter ω in symmetrically ordered coordinate operators and read off the constraints imposed by the Jacobi identity and do the same for the modified Jacobi identity in the nonassociative case.

We introduce the operator algebra whose element maps the algebra of functions into themselves. We present then a useful polydifferential representation of coordinate operators. We give the explicit expansion of a function in algebra of operator functions in coordinate operators up to second order in the formal expansion parameter.

In chapter 3 we define a map which orders the coordinate operators in the noncommutative space which are in one to one correspondence with coordinates in commutative space . We give different ordering prescriptions for the ordering map namely Weyl ordering, Index-value-ordering and Anti-index-value-ordering. The star product is then defined by mapping an algebra of functions to the noncommutative space and bringing them in particular ordering using the algebra relations 1.2 which is possible due to the Poincare-Birkoff-Witt property of the algebra.

In chapter 4 we explain in detail the procedure developed in [10] for calculating star product order by order. The star product is defined via a polydifferential representation developed in chapter 3. We explicitly calculate the star product to third order and give a consistency check at every order.

In chapter 5 we first briefly review the notion of diffeomorphisms in coordinate space and deformed diffeomorphism in noncommutative coordinate space based on [19]. We define the star product defeomorphisms based on formality theorem and give the results up to second order in the expansion parameter. However, we put the results for our anstaz in to appendix B.

In chapter 6 we introduce the graded structures and define a non(anti)commutative superspace. Graded Einstein Summation Convention give us all the components of a star product on superspace. This allows us to treat the graded star product in the same way as a bosonic star product. The star product at second order of the expansion parameter becomes quite lengthy

[1] We understand (1.2) as a graded commutator. In the first chapters we consider purely bosonic coordinates only later in chapter 6 we introduce the explicit grading and give different components of superspace separately

after splitting into bosonic and fermionic components and we put the results into the appendix C.

In the last chapter 7 we generalize the ordering map defined in chapter 3 to the nonassociative case and define a nonassociative star product by reordering noncommutative nonassociative coordinate operators. We give results to first order expansion and check the cyclicity property of the resulting star product.

Finally we give in appendix A a twist representation of the star product on non(anti)commutative superspace. We begin with brief review of Hopf algebra and define the graded twist element of Hopf algebra. We use then Graded Einstein Summation Convention to get different components of the graded twist and hence of a graded star product via a twisting procedure.

Chapter 2

Coordinate Space

2.1 Coordinate Space

Let F be some arbitrary field such as the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. Later on we will further allow F to be the space of super numbers $\mathbb{S}$. For any positive integer n the space of all n-tuples of elements of F forms an n-dimensional vector space over F which we call a coordinate space and denote it by F^n. Let X^M be the coordinates of an element $X \in F^n$ (this can also be generalized to a manifold by the usual patching). For the moment let us consider X^M to be simply bosonic coordinates. Only later in the chapter 6 we will reinterpret X^M to be superspace coordinates. The algebra of functions on the space F^n

$$A = \{f : F^n \to F\} \tag{2.1}$$

is commutative with the pointwise product, simply because the product in F is commuting:

$$\begin{aligned} \forall f, g \in A: \quad fg(X) \equiv& f(X)g(X) = g(X)f(X) = gf(X) \qquad \forall X \in F^n & (2.2)\\ \Rightarrow fg =& gf \qquad \forall f, g \in A & (2.3) \end{aligned}$$

We denote the unit element of this algebra by e:

$$e(X) = 1, \qquad eg = ge = g \qquad \forall g \in A \tag{2.4}$$

Let further φ^M be the coordinate function

$$\varphi^M(X) \equiv X^M \tag{2.5}$$

The monomials

$$\varphi^{M_1 \ldots M_n}(X) \equiv \varphi^{M_1} \ldots \varphi^{M_n}(X) = X^{M_1} \ldots X^{M_n} \tag{2.6}$$

provide a natural basis for the algebra A. The indices are automatically symmetrized. For the special case $n = 0$, the map is defined to be a a constant map

$$\varphi^{(0)}(X) \equiv 1 \tag{2.7}$$

Thus any function $f \in A$ can be expanded as in the following

$$f = \sum_{n=0}^{\infty} \frac{1}{n!} f_{M_1 \ldots M_n} \varphi^{M_1 \ldots M_n} \tag{2.8}$$

which has pointwise the usual form of the power series

$$f(X) = \sum_{n=0}^{\infty} \frac{1}{n!} f_{M_1...M_n} X^{M_1}...X^{M_n} \tag{2.9}$$

2.2 Non-Commutative Coordinate Space

Let us generalize the commutative algebra A of functions on F^n defined in the previous section to a non-commutative coordinate space $\hat{A}$ [20] by promoting the coordinate functions defined in (2.5) to the noncommutative coordinate $\hat{X}^M$.

$\hat{X}^M$ are now generators of the algebra $\hat{A}$ and obey the following relations

$$[\hat{X}^M, \hat{X}^N] = 2\alpha\omega^{MN}(\hat{X}) \tag{2.10}$$

where α is some formal expansion parameter. If the product in $\hat{A}$ is associative then the Jacobi identity for the above commutator puts some constraints on the form of ω^{MN}

$$[[\hat{X}^{[M}, \hat{X}^N], \hat{X}^{P]}] = 0 \tag{2.11}$$

In order to see these constraints we assume that ω^{MN} depend analytically on $\hat{X}$ with F valued expansion-coefficients. We thus can expand ω^{MN} in a power series

$$\omega^{MN}(\hat{X}) = \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\omega}^{MN}_{K_1...K_n} \hat{X}^{K_1}...\hat{X}^{K_n} \tag{2.12}$$

where the expansion coefficients are now just numbers, i.e. $\tilde{\omega}^{MN}_{K_1...K_n} \in F$. Plugging equation (2.10) into equation (2.11) gives

$$2\alpha[\omega^{[MN}, \hat{X}^{P]}] = 0 \tag{2.13}$$

and turns via (2.12) into

$$2\alpha \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\omega}^{[MN|}_{K_1...K_n} [\hat{X}^{K_1}...\hat{X}^{K_n}, \hat{X}^{|P]}] = 0 \tag{2.14}$$

where now equation (2.14) is a consistency condition on the algebra coefficients $\tilde{\omega}^{MN}_{K_1...K_n}$.

Although a priori the coefficients $\tilde{\omega}^{MN}_{K_1...K_n}$ are not symmetric in $K_1...K_n$, we can symmetrize them by Weyl ordering the monomials $\hat{X}^{K_1}...\hat{X}^{K_n}$ in equation (2.12)[1]. We denote then the symmetrized coefficients without tilde i.e. $\omega^{MN}_{K_1...K_n}$.

$$\omega^{MN}(\hat{X}) = \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\omega}^{MN}_{K_1...K_n} \hat{X}^{K_1}...\hat{X}^{K_n} = \sum_{n=0}^{\infty} \frac{1}{n!} \omega^{MN}_{K_1...K_n} \hat{X}^{(K_1}...\hat{X}^{K_n)} \tag{2.15}$$

[1]We can do so because of the so called PWB property of the algebra explained in the next section. The expansion (2.16) in the symmetrized (or Weyl ordered) basis can also be written as

$$\omega^{MN}(\hat{X}) = \exp(\hat{X}^K \partial_{Y^K}) \omega^{MN}(Y)|_{Y=0}$$

$\omega^{MN}_{K_1\dots K_n}$ is now symmetric in its lower indices and we denote the symmetrized indices by repeated indices at the same vertical position.

$$\omega^{MN}(\hat{X}) = \sum_{n=0}^{\infty} \frac{1}{n!} \omega^{MN}_{(n)K\dots K} (\hat{X}^K)^n \tag{2.16}$$

The non-commutative coordinates are now symmetrically ordered and build what is known as Weyl ordered basis. Let us write the Jacobi identity in equation (2.14) with ω expanded in the Weyl ordered basis as in (2.16):

$$2\alpha \sum_{n=0}^{\infty} \frac{1}{n!} \omega^{[MN|}_{(n)K\dots K} [(\hat{X}^K)^n, \hat{X}^{|P]}] = 0 \tag{2.17}$$

$$4\alpha^2 \sum_{n=1}^{\infty} \frac{1}{n!} \omega^{[MN|}_{(n)K\dots K} \sum_{k=0}^{n-1} (\hat{X}^K)^k \omega^{K|P]}(\hat{X}) (\hat{X}^K)^{n-1-k} = 0 \tag{2.18}$$

Dividing equation (2.18) by $4\alpha^2$ and inserting the expression for $\omega^{K|P]}$ in the Weyl ordered basis, we obtain

$$\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n!m!} \omega^{[MN|}_{(n)K\dots K} \omega^{K|P]}_{(m)L\dots L} \sum_{k=0}^{n-1} (\hat{X}^K)^k (\hat{X}^L)^m (\hat{X}^K)^{n-1-k} = 0 \tag{2.19}$$

Generally, the non-commutative coordinates $\hat{X}$ in the above expression are not in the Weyl ordering. However, for $m = 1$ generators are always in the Weyl ordering

$$\begin{aligned} \sum_{k=0}^{n-1} (\hat{X}^K)^k (\hat{X}^L)^m (\hat{X}^K)^{n-1-k} =& \hat{X}^L \hat{X}^K \dots \hat{X}^K + \hat{X}^K \hat{X}^L \hat{X}^K \dots \hat{X}^K + \dots + \\ &+ \hat{X}^K \dots \hat{X}^K \hat{X}^L = n \hat{X}^{(L} \hat{X}^K \dots \hat{X}^{K)} \end{aligned} \tag{2.20}$$

In principle we can also always write (2.19) in Weyl ordering, but it is very hard to give an explicit expression for it. Let us show this with the simple example.
For $n = 3, m = 2$

$$\sum_{k=0}^{2} (\hat{X}^K)^k (\hat{X}^L)^2 (\hat{X}^K)^{2-k} = \hat{X}^L \hat{X}^L \hat{X}^K \hat{X}^K + \hat{X}^K \hat{X}^L \hat{X}^L \hat{X}^K + \hat{X}^K \hat{X}^K \hat{X}^L \hat{X}^L \tag{2.21}$$

$$\begin{aligned} =& \frac{1}{2} (\hat{X}^L \hat{X}^L \hat{X}^K \hat{X}^K + \hat{X}^K \hat{X}^L \hat{X}^L \hat{X}^K + \hat{X}^K \hat{X}^K \hat{X}^L \hat{X}^L + \\ &+ \hat{X}^K \hat{X}^L \hat{X}^K \hat{X}^L + \hat{X}^L \hat{X}^K \hat{X}^L \hat{X}^K + \hat{X}^L \hat{X}^K \hat{X}^K \hat{X}^L) + \\ &+ \frac{1}{2} (\hat{X}^L \hat{X}^L \hat{X}^K \hat{X}^K + \hat{X}^K \hat{X}^L \hat{X}^L \hat{X}^K + \hat{X}^K \hat{X}^K \hat{X}^L \hat{X}^L - \\ &- \hat{X}^K \hat{X}^L \hat{X}^K \hat{X}^L - \hat{X}^L \hat{X}^K \hat{X}^L \hat{X}^K - \hat{X}^L \hat{X}^K \hat{X}^K \hat{X}^L) \end{aligned} \tag{2.22}$$

$$\begin{aligned} =& \frac{6}{2} \hat{X}^{(L} \hat{X}^L \hat{X}^K \hat{X}^{K)} + \frac{1}{2} \hat{X}^L [\hat{X}^L, \hat{X}^K] \hat{X}^K + \\ &+ \frac{1}{2} [\hat{X}^K \hat{X}^L, \hat{X}^L \hat{X}^K] + \frac{1}{2} \hat{X}^K [\hat{X}^K, \hat{X}^L] \hat{X}^L \end{aligned} \tag{2.23}$$

$$\begin{aligned} =& 3 \hat{X}^{(L} \hat{X}^L \hat{X}^K \hat{X}^{K)} + \alpha \Big(\hat{X}^K \omega^{KL}(\hat{X}) \hat{X}^L - \hat{X}^L \omega^{KL} \hat{X}^K + \\ &+ \omega^{KL}(\hat{X}) \hat{X}^K \hat{X}^L - \hat{X}^K \hat{X}^L \omega^{KL}(\hat{X}) \Big) \end{aligned} \tag{2.24}$$

For this example we see that reordering becomes very quickly quite nasty, and also that one is never done with reordering, this is because the terms of order α are still not in correct ordering. Reordering them leads to the unordered terms of the order α^2 and so on. But, in a formal power series in α, this can be done order by order iteratively. A convergent series in alpha as a real parameter can be obtained only in special cases and for sufficiently small alpha.

As every reordering produces terms of higher order in α, it is also clear that to lowest order in α, we can treat $\hat{X}$ like a commuting variable. Equation (2.19) then becomes

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{n!m!}\omega^{[MN|}_{(n)K\ldots K}n(\hat{X}^K)^{n-1}\omega^{K|P]}_{(m)L\ldots L}(\hat{X}^L)^m + \mathcal{O}(\alpha) = 0 \tag{2.25}$$

$$\omega^{[P|K}\partial_K\omega^{K|P]}(\hat{X}) + \mathcal{O}(\alpha) = 0 \tag{2.26}$$

The antisymmetric 2-vector ω^{MN} is thus required to be a Poisson-structure. Or to be more precise (as Poisson structures are usually defined with commuting coordinates): The expansion coefficients of ω^{MN} are equal to the expansion coefficients of the Poisson structure.

2.3 Non-Associative Non-Commutative Coordinate Space

We can further generalize the coordinate space by relaxing the associativity of the noncommutative coordinates $\hat{X}^M$ (the generators of the algebra $\hat{A}$) as in the following

$$(\hat{X}^M\hat{X}^N)\hat{X}^K - \hat{X}^M(\hat{X}^N\hat{X}^K) = \beta\kappa^{MNK}(\hat{X},\alpha) \tag{2.27}$$

In general the commutator-tensor ω^{MN} will now also depend on the new non-associativity parameter β

$$[\hat{X}^M,\hat{X}^N] = 2\alpha\omega^{MN}(\hat{X},\beta) \tag{2.28}$$

The ansatz is such that for $\alpha \to 0$ we recover a commutative (but not necessarily associative) algebra and for $\beta \to 0$ we recover an associative but not necessarily commutative algebra. Both tensors, κ^{MNK} and ω^{MN} are assumed to depend analytically on the generators $\hat{X}^M$ of the algebra, otherwise the latter would not be proper generators of the algebra. For the explicit expansion in powers of the generators, we have to decide not only about the ordering, but also about the bracketing. We will take again Weyl ordering as reference order of the operators. For the brackets it is natural to start bracketing from one side and work through one's way to the other side. We decide to start from the right, so that the expansion of the tensors in powers of the generators looks as follows (compare to the associative case (2.15):

$$\omega^{MN}(\hat{X},\beta) \equiv \sum_{n=0}^{\infty}\frac{1}{n!}\omega_{K_1\ldots K_n}(\beta)\cdot\hat{X}^{(K_1}(\cdots(\hat{X}^{K_{n-2}}(\hat{X}^{K_{n-1}}\hat{X}^{K_n)}))) \tag{2.29}$$

$$\kappa^{MNK}(\hat{X},\alpha) \equiv \sum_{n=0}^{\infty}\frac{1}{n!}\kappa_{K_1\ldots K_n}(\alpha)\cdot\hat{X}^{(K_1}(\cdots(\hat{X}^{K_{n-2}}(\hat{X}^{K_{n-1}}\hat{X}^{K_n)}))) \tag{2.30}$$

Jacobi identity for a non-associative algebra

The Jacobi identity also gets modified in a non-associative algebra

$$
\begin{aligned}
[[\hat{X}^{[M}, \hat{X}^N], \hat{X}^{P]}] =& 2\Big[(\hat{X}^{[M}, \hat{X}^N), \hat{X}^{P]}\Big] = & (2.31)\\
=& 2(\hat{X}^{[M}\hat{X}^N)\hat{X}^{P]} - 2\hat{X}^{[M}(\hat{X}^N\hat{X}^{P]}) & (2.32)\\
\Rightarrow [[\hat{X}^{[M}, \hat{X}^N], \hat{X}^{P]}] =& 2\beta\kappa^{[MNP]}(\hat{X}, \alpha) & (2.33)
\end{aligned}
$$

Note that in the associative case the Jacobi identity can also be seen as a Leibniz rule of the commutator acting on a product (another commutator)

$$
[[\hat{X}^M, \hat{X}^N], \hat{X}^P] = [\hat{X}^M, [\hat{X}^N, \hat{X}^P] + [[\hat{X}^M, \hat{X}^P], \hat{X}^N] \quad \text{(associative case)} \tag{2.34}
$$

In a non-associative coordinate space, equation (2.34) is due to (2.33) modified to the following form

$$
\Big[[\hat{X}^M, \hat{X}^N], \hat{X}^P\Big] = \Big[\hat{X}^M, [\hat{X}^N, \hat{X}^P]\Big] + \Big[[\hat{X}^M, \hat{X}^P], \hat{X}^N\Big] + 6\beta\kappa^{[MNP]}(\hat{X}, \alpha) \tag{2.35}
$$

We obtain the opposite sign for the non-associativity-term $\beta\kappa^{MNP}$ when we act from the left instead from the right:

$$
\Big[\hat{X}^P, [\hat{X}^M, \hat{X}^N]\Big] = \Big[[\hat{X}^P, \hat{X}^M], \hat{X}^N\Big] + \Big[\hat{X}^M, [\hat{X}^P, \hat{X}^N]\Big] - 6\beta\kappa^{[PMN]}(\hat{X}, \alpha) \tag{2.36}
$$

Although the above equations are just equivalent rewritings of (2.33), they are instructive when comparing to the action of a commutator on a product of generators. In the associative case such an action also follows a Leibniz rule

$$
[\hat{X}^P, \hat{X}^M\hat{X}^N] = [\hat{X}^P, \hat{X}^M]\hat{X}^N + \hat{X}^M[\hat{X}^P, \hat{X}^N] \quad \text{(associative case)} \tag{2.37}
$$

In the non-associative case we expect also this Leibniz rule to be modified. In order to find the deviation, let us first calculate the expressions on the left and on the right side independently. The left side reads

$$
\Big[\hat{X}^P, (\hat{X}^M\hat{X}^N)\Big] = \hat{X}^P(\hat{X}^M\hat{X}^N) - (\hat{X}^M\hat{X}^N)\hat{X}^P \tag{2.38}
$$

Now we can do the calculation from the righthand side

$$
\begin{aligned}
&\Big[\hat{X}^P, \hat{X}^M\Big]\hat{X}^N + \hat{X}^M\Big[\hat{X}^P, \hat{X}^N\Big] = \\
=\ & (\hat{X}^P\hat{X}^M)\hat{X}^N \underbrace{-(\hat{X}^M\hat{X}^P)\hat{X}^N + \hat{X}^M(\hat{X}^P\hat{X}^N)}_{-\beta\kappa^{MPN}(\hat{X},\alpha)} - \hat{X}^M(\hat{X}^N\hat{X}^P) = & (2.39)\\
=\ & \Big[\hat{X}^P, (\hat{X}^M\hat{X}^N)\Big] - \beta\kappa^{MPN}(\hat{X}, \alpha) + \beta\kappa^{PMN}(\hat{X}, \alpha) + \beta\kappa^{MNP}(\hat{X}, \alpha) & (2.40)
\end{aligned}
$$

The modified Leibniz rule therefore reads

$$
\begin{aligned}
\Big[\hat{X}^P, (\hat{X}^M\hat{X}^N)\Big] \;=\; & \Big[\hat{X}^P, \hat{X}^M\Big]\hat{X}^N + \hat{X}^M\Big[\hat{X}^P, \hat{X}^N\Big] + \\
& + \beta\kappa^{MPN}(\hat{X}, \alpha) - \beta\kappa^{PMN}(\hat{X}, \alpha) - \beta\kappa^{MNP}(\hat{X}, \alpha) & (2.41)
\end{aligned}
$$

After completely antisymmetrizing the indices P, M and N, we recover the modified Jacobi-identity in the form (2.36).

In the associative case the Jacobi identity imposes the consistency condition (2.13) on the commutator tensor ω^{MN}. This is also true for the modified Jacobi-identity, but the new consistency condition will of course also contain the associator. Plugging the commutator (2.28) into the modified Jacobi-identity (2.33), we obtain

$$\left[2\alpha\omega^{[MN}(\hat{X},\beta),\hat{X}^{P]}\right] = 2\beta\kappa^{[MNP]}(\hat{X},\alpha) \tag{2.42}$$

We could now plug the expansions (2.29) and (2.30) into this consistency condition, in order to determine conditions on the expansion coefficients. To this end one would need to iteratively use commutator and associator, in order to rewrite both sides of the above equation order by order in the parameters α and β in our reference basis. In the associative case this procedure was sketched starting from equation (2.19) and arriving at lowest order in α at the Poisson-condition on ω^{MN}. In the non-associative case this is getting much more involved and one might expect (apart from modified Poisson-conditions on ω^{MN}) also some conditions on κ^{MNP}.

2.4 Operator Representation of the Algebra

For the associative case operators provide a natural choice for the representation of the non-commutative coordinates $\hat{X}^M$. Let us consider the operator algebra $\hat{\mathbb{O}}$ where an operator $\hat{O} \in \hat{\mathbb{O}}$ is a linear map from the algebra of functions to itself

$$\hat{\mathbb{O}} \ni \hat{O}: \qquad A \ni f \mapsto \hat{O}f \in A \tag{2.43}$$

Let $\hat{m}_{f(X)} \in \hat{\mathbb{O}}$ be the multiplication operator which multiplies an arbitrary function $g \in A$ by an another function $f \in A$.

$$\hat{\mathbb{O}} \ni \hat{m}_f: \qquad A \ni g \mapsto \hat{m}_f g = fg \in A \tag{2.44}$$

This gives us an embedding of the commutative algebra A into the non-commutative algebra $\hat{\mathbb{O}}$

$$f \mapsto \hat{m}_f \Rightarrow A \hookrightarrow \hat{\mathbb{O}} \tag{2.45}$$

in this way A can be thought as a subset of $\hat{\mathbb{O}}$

$$\tag{2.46}$$

2.4.1 Polydifferential Basis

The algebra of operators $\hat{\mathbb{O}}$ can be generated by $\{\hat{m}_{\varphi^M}, \partial_M\}$, where in this case the natural basis is given by $\{\hat{m}_{\varphi^{M_1 \ldots M_n}}, \partial_{K_1}...\partial_{K_p} | n, p \in \mathbb{N}_0\}$. The non-commutative operators $\hat{X}$ can, therefore, be expanded in this basis and it is natural to make an ansatz where higher derivatives come with higher powers of the expansion parameter α

$$\hat{X}^M = X^M + \sum_{k=1}^{\infty} \alpha^k \Gamma^{MK_1 \ldots K_k}(\alpha, X)\partial_{K_1}...\partial_{K_k} \tag{2.47}$$

Here, we roughly follow the notation of [10]. Expression (2.47) can also be written as

$$\hat{X}^M = \sum_{k=0}^{\infty} \alpha^k \Gamma^{MK_1 \ldots K_k}(\alpha, X)\partial_{K_1}...\partial_{K_k} \tag{2.48}$$

if one defines

$$\Gamma^M(\alpha, X) = X^M \tag{2.49}$$

This ansatz assumes that every derivative comes with at least one power of α. In addition we recover X^M for $\alpha \to 0$. This would also be true for a choice of $\Gamma^M(\alpha, X) = X^M + \mathcal{O}(\alpha)$. We have to expand Γ further in powers of α to get an actual expansion of $\hat{X}$ in terms of α

$$\Gamma^{MK_1 \ldots K_k}(\alpha, X) = \sum_{n=0}^{\infty} \alpha^n \Gamma_n^{MK_1 \ldots K_k}(X) \tag{2.50}$$

$\hat{X}$ can then be rewritten as

$$\hat{X}^M = X^M + \sum_{n=1}^{\infty} \alpha^n \sum_{k=0}^{n-1} \Gamma_k^{MK_1 \ldots K_{n-k}}(X) \partial_{K_1} \ldots \partial_{K_{n-k}} \tag{2.51}$$

In order to have a valid representation, the expansion coefficients $\Gamma_k^{MK_1 \ldots K_{n-k}}(X)$ have to be determined in such a way that the algebra (2.10) is satisfied. This will be done order by order in the expansion parameter α. It is, therefore, convenient to introduce a symbol for the coordinate operator whose α-expansion is truncated after the n-th order.

$$\hat{X}_n^M \equiv X^M + \sum_{p=1}^{n} \alpha^p \sum_{k=0}^{p-1} \Gamma_k^{MK_1 \ldots K_{p-k}}(X) \partial_{K_1} \ldots \partial_{K_{p-k}} \tag{2.52}$$

If we further denote the difference that is added when going from order n to order $(n+1)$ by $\alpha^{n+1} \gamma_{n+1}^M$, we obtain

$$\hat{X}_{n+1}^M = \hat{X}_n^M + \alpha^{n+1} \gamma_{n+1}^M \tag{2.53}$$

$$= X^M + \sum_{k=1}^{n+1} \alpha^k \gamma_k^M \tag{2.54}$$

The explicit form of the difference operator γ_n^M is thus given by

$$\gamma_n^M \equiv \sum_{k=0}^{n-1} \Gamma_k^{MK_1 \ldots K_{n-k}}(X) \partial_{K_1} \ldots \partial_{K_{n-k}} \tag{2.55}$$

Assume that we obey algebra (2.10) already up to the nth order. At (n+1)st order we have to fulfil

$$[\hat{X}_{n+1}^M, \hat{X}_{n+1}^N] \stackrel{!}{=} 2\alpha \omega_n^{MN} + \mathcal{O}(\alpha^{n+2}) \tag{2.56}$$

where ω_n^{MN} is defined by

$$\omega^{MN}(\hat{X}) = \omega_n^{MN} + \mathcal{O}(\alpha^{n+1}) \tag{2.57}$$

Note that the α-expansion of $\omega^{MN}(\hat{X})$ is based completely on the α-dependence of $\hat{X}^M$. For a given algebra (with given $\omega_{K \ldots K}^{MN}$) fixing $\hat{X}_n^M$ therefore also fixes ω_n^{MN}. So the commutator in equation (2.56) becomes

$$[\hat{X}_n^M + \alpha^{n+1} \gamma_{n+1}^M, \hat{X}_n^N + \alpha^{n+1} \gamma_{n+1}^N] \stackrel{!}{=} 2\alpha \omega_n^{MN} + \mathcal{O}(\alpha^{n+2}) \tag{2.58}$$

$$[\hat{X}_n^M, \hat{X}_n^N] + [\hat{X}_n^M, \alpha^{n+1} \gamma_{n+1}^N] + [\alpha^{n+1} \gamma_{n+1}^M, \hat{X}_n^N] \stackrel{!}{=} 2\alpha \omega_n^{MN} + \mathcal{O}(\alpha^{n+2}) \tag{2.59}$$

$$[\hat{X}_n^M, \hat{X}_n^N] + 2\alpha^{n+1} [\gamma_{n+1}^{[M|}, \hat{X}_n^{|N]}] \stackrel{!}{=} 2\alpha \omega_n^{MN} + \mathcal{O}(\alpha^{n+2}) \tag{2.60}$$

Let us now define

$$\gamma_{n+1}^{MN} \equiv [\gamma_{n+1}^{[M|}, X^{|N]}] = \sum_{p=0}^{n}(p+1)\Gamma_{n-p}^{[MN]K_1...K_p}(X)\partial_{K_1}...\partial_{K_p} \tag{2.61}$$

and

$$G_{n+1}^{MN} \equiv 2\alpha\omega_n^{MN} - [\hat{X}_n^M, \hat{X}_n^N] + \mathcal{O}(\alpha^{n+2}) \tag{2.62}$$

The $\mathcal{O}(\alpha^{n+2})$ in the definition has to be understood such that G_{n+1} contains maximally (n+1)st order terms and higher order terms coming from the commutator are dropped. Note that the above defined object G_{n+1} at order $n+1$ contains only objects which have already been determined at order n, namely ω_n and $\hat{X}_n$. From (2.53) and (2.55) it is instead visible that γ_{n+1} contains only objects that have not yet been determined at order n. At (n+1)st order we thus obtain the following condition for the above defined γ_{n+1}^{MN} and, therefore, implicitly for the antisymmetric part $\Gamma^{[MN]}$ of our expansion coefficients

$$2\alpha^{n+1}\gamma_{n+1}^{MN} \stackrel{!}{=} G_{n+1}^{MN} \tag{2.63}$$

The operators G_{n+1}^{MN} can be expanded in powers of differential operator as in the following

$$G_{n+1}^{MN} = \sum_{p\geq 0} G_{n+1}^{MNK_1...K_p}\partial_{K_1}\ldots\partial_{K_P} \tag{2.64}$$

Plugging equation (2.61) together with equation (2.64) into the equation (2.63) we obtain

$$2\alpha^{n+1}(p+1)\Gamma_{n-p}^{[MN]K_1...K_p} \stackrel{!}{=} G_{n+1}^{MNK_1...K_p} \tag{2.65}$$

As mentioned above, this should be a condition on the coefficients Γ on the lefthand side and not on G which contains objects that have already been determined at order n. Both sides are antisymmetric in the first two indices and symmetric in the remaining ones. However, on the left-hand side we have the additional symmetry-property $\Gamma^{[MNK_1]K_2...K_p} = 0$. G should obey the same property already from its definition (2.62) via n-th order objects, otherwise (2.65) would be a new condition on the n-th order that we already assume to be complete. Luckily, according to Lemma 1 in [10] this is indeed the case and can be seen as follows:

The anti-symmetrization of the functions $G_{n+1}^{MNK_1...K_p}$ in its first three indices vanishes.

$$G_{n+1}^{[MNK_1]...K_p} = 0 \tag{2.66}$$

In order to prove this statement let us start with

$$[[\hat{X}_{n+1}^{[M}, \hat{X}_{n+1}^{N}], \hat{X}_{n+1}^{P]}] = 0 \tag{2.67}$$

Plugging equation (2.56) in to equation (2.67) we obtain

$$[2\alpha\omega_n^{[MN} + \mathcal{O}(\alpha^{n+2}), \hat{X}_{n+1}^{P]}] = 0 \tag{2.68}$$

$$[2\alpha\omega_n^{[MN}, \hat{X}_n^{P]}] + \mathcal{O}(\alpha^{n+2}) = 0 \tag{2.69}$$

Plugging equation (2.62) into equation (2.69) we get

$$[G_{n+1}^{[MN|} + [\hat{X}_n^{[M}, \hat{X}_n^{N|}], \hat{X}_n^{|P]}] + \mathcal{O}(\alpha^{n+2}) = 0 \tag{2.70}$$

$$[G_{n+1}^{[MN}, \hat{X}_n^{P]}] + [[\hat{X}_n^{[M}, \hat{X}_n^{N}], \hat{X}_n^{P]}] + \mathcal{O}(\alpha^{n+2}) = 0 \tag{2.71}$$

Since

$$[[\hat{X}_n^{[M}, \hat{X}_n^N], \hat{X}_n^{P]}] = 0 \tag{2.72}$$

We obtain

$$[G_{n+1}^{[MN}, \hat{X}_n^{P]}] + \mathcal{O}(\alpha^{n+2}) = 0 \tag{2.73}$$

Now substituting equation (2.64) in equation (2.73) we obtain

$$[G_{n+1}^{[MN|K_1\ldots K_p}\partial_{K_1}\ldots\partial_{K_P}, \hat{X}^{|P]}] + \mathcal{O}(\alpha^{n+2}) = 0 \tag{2.74}$$

and the lowest order in α yields the equation (2.66). Since from equation (2.64) it is clear that $G_{n+1}^{MNK_1\ldots K_p}$ are symmetric in last p indices, therefore, this property holds for all permutations of $[MNK_l]$ where $l = 1, \ldots, p$.

We have now seen that (2.65) is consistent with the symmetries of G_{n+1} and therefore is only a condition on the antisymmetric part of the expansion coefficients $\Gamma_{n-p}^{MNK_1\ldots K_p}$ that contribute at $(n+1)$st order to $\hat{X}_{n+1}^M$. Because of the symmetries of these expansion coefficients, knowing this antisymmetric part determines the complete coefficients already up to the completely symmetric term, as we will argue in the following. To this end, let us spell out the completely symmetrized part of the coefficients and then try to rewrite them in terms of the full ones and the antisymmetrized part:

$$\Gamma_k^{(MNK_1\ldots K_p)} = \frac{1}{(p+2)}\Big(\Gamma_k^{MNK_1\ldots K_p} + \Gamma_k^{NMK_1\ldots K_p} + \Gamma_k^{K_1NMK_2\ldots K_p} + \cdots + \Gamma_k^{K_pN\ldots K_{p-1}M}\Big) \tag{2.75}$$

We can rewrite each of the last $(p+1)$ terms on the righthand side as a term with first index M (like the first term) and a term antisymmetric in two indices.

$$\begin{aligned}\Gamma_k^{(MNK_1\ldots K_p)} =& \frac{1}{(p+2)}\Big((p+2)\Gamma_k^{MNK_1\ldots K_p} + 2\Gamma_k^{[NM]K_1\ldots K_p} + 2\Gamma_k^{[K_1M]K_2\ldots K_pN} + \\ &+ \cdots + 2\Gamma_k^{[K_pM]NK_1\ldots K_{p-1}}\Big) \\ =& \Gamma_k^{MNK_1\ldots K_p} + \frac{2}{(p+2)}\Big(\Gamma_k^{[NM]K_1\ldots K_p} + \Gamma_k^{[K_1M]K_2\ldots K_pN} + \cdots + \\ &+ \Gamma_k^{[K_pM]NK_1\ldots K_{p-1}}\Big)\end{aligned} \tag{2.76}$$

As claimed before, the full coefficients can now be written as a sum of the completely symmetrized part and terms that are antisymmetrized in two indices:

$$\begin{aligned}\Gamma_k^{MNK_1\ldots K_p} =& \Gamma_k^{(MNK_1\ldots K_p)} + \frac{2}{(p+2)}\Big(\Gamma_k^{[MN]K_1\ldots K_p} + \Gamma_k^{[MK_1]K_2\ldots K_pN} + \cdots + \\ &+ \Gamma_k^{[MK_p]NK_1\ldots K_{p-1}}\Big)\end{aligned} \tag{2.77}$$

We can also write equation (2.75) more compactly using the symmetrization brackets and then can anti-symmetrize the last $(p+1)$ terms by writing antisymmetrization bracket explicitly as in the following

$$\Gamma_k^{(MNK_1...K_p)} = \frac{1}{(p+2)}\Big(\Gamma_k^{MNK_1...K_p} + (p+1)\Gamma_k^{(N|M|K_1...K_p)}\Big)$$
$$= \frac{1}{(p+2)}\Big((p+2)\Gamma_k^{MNK_1...K_p} + (p+1)\big(\Gamma_k^{(N|M|K_1...K_p)} - \Gamma_k^{M(NK_1...K_p)}\big)\Big)$$
$$= \Gamma_k^{MNK_1...K_p} + \frac{(p+1)}{(p+2)}\Big(\Gamma_k^{(N|M|K_1...K_p)} - \Gamma_k^{M(NK_1...K_p)}\Big) \tag{2.78}$$

So we obtain the same result as in (2.77), just written in a slightly different way:

$$\Gamma_k^{MNK_1...K_p} = \Gamma_k^{(MNK_1...K_p)} + \frac{(p+1)}{(p+2)}\Big(\Gamma_k^{M(NK_1...K_p)} - \Gamma_k^{(N|M|K_1...K_p)}\Big) \tag{2.79}$$

Now by plugging condition (2.65) into equation (2.77) or (2.79) we obtain the final constraint which the expansion coefficients Γ have to obey such that $\hat{X}$ fulfils the correct algebra:

$$\Gamma_{n-p}^{MNK_1...K_p} \overset{!}{=} \Gamma_{n-p}^{(MNK_1...K_p)} + \frac{\alpha^{-(n+1)}}{(p+2)}\Big(G_{n+1}^{M(NK_1K_2...K_p)}\Big) \tag{2.80}$$
$$\overset{!}{=} \Gamma_{n-p}^{(MNK_1...K_p)} + \frac{\alpha^{-(n+1)}}{(p+1)(p+2)}\Big(G_{n+1}^{MNK_1K_2...K_p} +$$
$$+ G_{n+1}^{MK_1N...K_p} + \cdots + G_{n+1}^{MK_pNK_1...K_{p-1}}\Big) \tag{2.81}$$

The completely symmetric part of Γ is still undetermined and can be chosen in such a way that the calculations simplify. If we set the completely symmetric part of Γ to zero, we arrive at the statement of Lemma 2 in [10].

2.4.2 Expansion of a Function in the Non-Commutativity Parameter

So far we have found a recursive formula (2.65) or equivalently (2.80) and (2.81) for the expansion coefficients Γ at order $n+1$ in terms of G_{n+1}, which according to its definition (2.62) contains only objects at order n. The latter includes ω_n^{MN}, which is in principle determined, as soon as $\hat{X}_n^M$ is fixed, but we have to perform the expansion of $\omega^{MN}(\hat{X})$ in the parameter α in order to obtain their explicit form. In this section we would like to present the α-expansion of $\omega^{MN}(\hat{X})$ or any other function $f(\hat{X})$ explicitly up to second order by a brute force calculation in order to get a feeling what is going on. Later we will follow [10] and use a more elegant method including the so-called Duhamel formula. This will be presented in section 4. In general we can expand $f(\hat{X})$ as in the following.

$$f(\hat{X}) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}_{K_1...K_n} \hat{X}^{K_1}...\hat{X}^{K_n} \tag{2.82}$$

For the symmetrized basis we can write

$$f(\hat{X}) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}_{K...K} \hat{X}^{K}...\hat{X}^{K} \tag{2.83}$$
$$f(\hat{X}) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}_{K...K} (\hat{X}^{K})^n \tag{2.84}$$

Let us substitute the expansion of $\hat{X}$ (2.47) up to order α^2 and sum up the results once again

$$
\begin{aligned}
f(\hat{X}) =& \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}_{K\dots K} (X^K)^n + \\
&+ \alpha \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}_{K\dots K} \sum_{i=1}^{n} (X^K)^{n-i} \Gamma_0^{KL}(X) \partial_L ((X^K)^{i-1} \dots) + \\
&+ \alpha^2 \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}_{K\dots K} \sum_{i=1}^{n} (X^K)^{n-i} \Gamma_1^{KL}(X) \partial_L ((X^K)^{i-1} \dots) + \\
&+ \alpha^2 \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}_{K\dots K} \sum_{i=1}^{n} (X^K)^{n-i} \Gamma_0^{KLL}(X) \partial_L \partial_L ((X^K)^{i-1} \dots) + \\
&+ \alpha^2 \sum_{n=2}^{\infty} \frac{1}{n!} f^{(n)}_{K\dots K} \sum_{j=2}^{n} \sum_{i=1}^{j-1} (X^K)^{n-j} \Gamma_0^{KL_1}(X) \partial_{L_1} ((X^K)^{j-i-1} \Gamma_0^{KL_2} \partial_{L_2} ((X^K)^{i-1}) \dots) + \\
&+ \mathcal{O}(\alpha^3)
\end{aligned} \tag{2.85}
$$

and consequently we obtain

$$
\begin{aligned}
f(\hat{X}) =& f(X) + \alpha \sum_{n=2}^{\infty} \frac{1}{n!} f^{(n)}_{K\dots K} \sum_{i=2}^{n} (i-1) (X^K)^{n-2} \Gamma_0^{KK}(X) + \\
&+ \alpha \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}_{K\dots K} \sum_{i=1}^{n} (X^K)^{n-1} \Gamma_0^{KL}(X) \partial_L + \\
&+ \alpha^2 \sum_{n=2}^{\infty} \frac{1}{n!} f^{(n)}_{K\dots K} \sum_{i=2}^{n} (i-1) (X^K)^{n-2} \Gamma_1^{KK}(X)) + \\
&+ \alpha^2 \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}_{K\dots K} \sum_{i=1}^{n} (X^K)^{n-1} \Gamma_1^{KL}(X) \partial_L + \\
&+ \alpha^2 \sum_{n=3}^{\infty} \frac{1}{n!} f^{(n)}_{K\dots K} \sum_{i=3}^{n} (i-1)(i-2) (X^K)^{n-3} \Gamma_0^{KKK}(X) + \\
&+ 2\alpha^2 \sum_{n=2}^{\infty} \frac{1}{n!} f^{(n)}_{K\dots K} \sum_{i=2}^{n} (i-1) (X^K)^{n-2} \Gamma_0^{KKL}(X) \partial_L + \\
&+ \alpha^2 \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}_{K\dots K} \sum_{i=1}^{n} (X^K)^{n-1} \Gamma_0^{KLL}(X) \partial_L \partial_L + \\
&+ \alpha^2 \sum_{n=4}^{\infty} \frac{1}{n!} f^{(n)}_{K\dots K} \sum_{j=4}^{n} \sum_{i=2}^{j-1} (i-1)(j-3) (X^K)^{n-4} \Gamma_0^{KK}(X) \Gamma_0^{KK}(X) + \\
&+ \alpha^2 \sum_{n=3}^{\infty} \frac{1}{n!} f^{(n)}_{K\dots K} \sum_{j=3}^{n} \sum_{i=2}^{j-1} (i-1) (X^K)^{n-3} \Gamma_0^{KL}(X) (\partial_L \Gamma_0^{KK}(X) + \Gamma_0^{KK}(X) \partial_L) + \\
&+ \alpha^2 \sum_{n=3}^{\infty} \frac{1}{n!} f^{(n)}_{K\dots K} \sum_{j=3}^{n} \sum_{i=1}^{j-1} (j-2) (X^K)^{n-3} \Gamma_0^{KK}(X) \Gamma_0^{KL}(X) \partial_L) +
\end{aligned}
$$

$$+\alpha^2\sum_{n=2}^{\infty}\frac{1}{n!}f^{(n)}_{K\ldots K}\sum_{j=2}^{n}\sum_{i=1}^{j-1}(X^K)^{n-2}\Gamma_0^{KL_1}(X)(\partial_{L_1}\Gamma_0^{KL_2}(X)\partial_{L_2}+\Gamma_0^{KL_2}(X)\partial_{L_1}\partial_{L_2})+$$
$$+\mathcal{O}(\alpha^3) \tag{2.86}$$

which is simplified to the following expression

$$\begin{aligned}f(\hat{X}) =& f(X)+\frac{\alpha}{2}\Gamma_0^{KK}\partial_K\partial_K f(X)+\alpha\partial_K f(X)\Gamma_0^{KL}\partial_L+\alpha^2\partial_K\partial_K f(X)\Gamma_1^{KK}(X)+\\&+\frac{\alpha^2}{3}\partial_K\partial_K\partial_K f(x)\Gamma_0^{KKK}(X)+\frac{\alpha^2}{8}\partial_K\partial_K\partial_K\partial_K f(X)\Gamma_0^{KK}(X)\Gamma_0^{KK}(X)+\\&+\frac{\alpha^2}{6}\partial_K\partial_K\partial_K f(X)\Gamma_0^{KL}(X)\partial_L\Gamma_0^{KK}(X)+\alpha^2\partial_K f(X)\Gamma_1^{KL}(X)\partial_L+\\&+\alpha^2\partial_K\partial_K f(X)\Gamma_0^{KKL}(X)\partial_L+\frac{\alpha^2}{3}\partial_K\partial_K\partial_K f(X)\Gamma_0^{KK}(X)\Gamma_0^{KL}(X)\partial_L+\\&+\frac{\alpha^2}{2}\partial_K\partial_K f(X)\Gamma_0^{KL_1}(X)\partial_{L_1}\Gamma_0^{KL_2}(X)\partial_{L_2}+\alpha^2\partial_K f(X)\Gamma_0^{KLL}(X)\partial_L\partial_L+\\&+\frac{\alpha^2}{2}\partial_K\partial_K f(X)\Gamma_0^{KL_1}(X)\Gamma_0^{KL_2}(X)\partial_{L_1}\partial_{L_2}+\mathcal{O}(\alpha^3)\end{aligned} \tag{2.87}$$

Expansion of ω: Now let us put $f(\hat{X})=\omega^{MN}(\hat{X})$ and find out the expression for $\omega^{MN}(\hat{X})$ up to order α^2. Equation (2.82) then takes the following form

$$\omega^{MN}(\hat{X})=\sum_{k=0}^{\infty}\frac{1}{k!}\omega^{MN}_{K_1\ldots K_k}\hat{X}^{K_1}\ldots\hat{X}^{K_k} \tag{2.88}$$

and the expansion (2.87) turns into

$$\begin{aligned}\omega^{MN}(\hat{X}) =& \omega^{MN}(X)+\frac{\alpha}{2}\Gamma_0^{KK}\partial_K\partial_K\omega^{MN}(X)+\alpha\partial_K\omega^{MN}(X)\Gamma_0^{KL}\partial_L+\\&+\alpha^2\partial_K\partial_K\omega^{MN}(X)\Gamma_1^{KK}(X)+\frac{\alpha^2}{3}\partial_K\partial_K\partial_K\omega^{MN}(X)\Gamma_0^{KKK}(X)+\\&+\frac{\alpha^2}{8}\partial_K\partial_K\partial_K\partial_K\omega^{MN}(X)\Gamma_0^{KK}(X)\Gamma_0^{KK}(X)+\\&+\frac{\alpha^2}{6}\partial_K\partial_K\partial_K\omega^{MN}(X)\Gamma_0^{KL}(X)\partial_L\Gamma_0^{KK}(X)+\\&+\alpha^2\partial_K\omega^{MN}(X)\Gamma_1^{KL}(X)\partial_L+\alpha^2\partial_K\partial_K\omega^{MN}(X)\Gamma_0^{KKL}(X)\partial_L+\\&+\frac{\alpha^2}{3}\partial_K\partial_K\partial_K\omega^{MN}(X)\Gamma_0^{KK}(X)\Gamma_0^{KL}(X)\partial_L+\\&+\frac{\alpha^2}{2}\partial_K\partial_K\omega^{MN}(X)\Gamma_0^{KL_1}(X)\partial_{L_1}\Gamma_0^{KL_2}(X)\partial_{L_2}+\\&+\frac{\alpha^2}{2}\partial_K\partial_K\omega^{MN}(X)\Gamma_0^{KL_1}(X)\Gamma_0^{KL_2}(X)\partial_{L_1}\partial_{L_2}+\\&+\alpha^2\partial_K\omega^{MN}(X)\Gamma_0^{KLL}(X)\partial_L\partial_L+\mathcal{O}(\alpha^3)\end{aligned} \tag{2.89}$$

As mentioned at the beginning of this subsection, we will later use a more effective tool to do the α-expansion. This will be presented in section 4 along with the calculation of the star product.

2.4.3 Commutator at n-th Level

Equations (2.80) or (2.81) recursively determine (up to the completely symmetric part) the expansion coefficients Γ of the non-commutative coordinates $\hat{X}$ in terms of the object G_{n+1}. G_{n+1} in turn consists of ω_n, which we have discussed in the previous subsection, and the commutator $[\hat{X}_n^M, \hat{X}_n^N]$. The latter commutator is therefore the last missing ingredient for the explicit calculation of $\hat{X}^M$ at order $n+1$ and we are going to calculate it in the following. As we are assuming that the algebra $[\hat{X}_n^M, \hat{X}_n^N] = 2\alpha\omega_{n-1} + \mathcal{O}(\alpha^{n+1})$ is already obeyed up to order n, we need to calculate only the contribution of this commutator to the order $n+1$:

$$[\hat{X}_n^M, \hat{X}_n^N] = \Big[X^M + \sum_{p=1}^{n} \alpha^p \sum_{k=0}^{p-1} \Gamma_k^{MK_1 \ldots K_{p-k}}(X)\partial_{K_1} ... \partial_{K_{p-k}}, X^N + \\ + \sum_{q=1}^{n} \alpha^q \sum_{l=0}^{q-1} \Gamma_l^{NL_1 \ldots L_{q-l}}(X)\partial_{L_1} ... \partial_{L_{q-l}}\Big] \tag{2.90}$$

where we have used equation (2.52) for the expansion of $\hat{X}_n^M$. The commutator of X^M with the sum $\sum_{q=1}^n \alpha^q(\ldots)$ on the other side is maximally of order n and is therefore fully contained in $2\alpha\omega_{n-1}$. The same is true for $[\sum_{p=1}^n \alpha^p(\ldots)\, , X^N]$. In order to extract the order (n+1) from the remaining commutators $[\sum_{p=1}^n \alpha^p(\ldots), \sum_{q=1}^n \alpha^q(\ldots)]$, we need to make a reparametrization of the summation variables such that

$$\tilde{p} \equiv q + p \tag{2.91}$$

(the total power of α) becomes a summation variable. As second summation variable we keep q. Starting from $1 \leq p, q \leq n$, the new summation variables have to obey the following inequalities in the summation

$$1 \leq q \leq \tilde{p}, n \quad , \quad 2 \leq \tilde{p} \leq 2n \tag{2.92}$$

We thus obtain the following equation for the reparametrized summation that is valid for any p, q-dependent summand $C_{p,q}$:

$$\sum_{p=1}^{n} \sum_{q=1}^{n} \alpha^{\overbrace{q+p}^{\tilde{p}}} C_{p,q} = \sum_{\tilde{p}=2}^{2n} \alpha^{\tilde{p}} \sum_{q=1}^{\min\{\tilde{p},n\}} C_{\tilde{p}-q,q} \tag{2.93}$$

We are interested only in the contribution at order $\tilde{p} = n+1$ to the commutator. So applying the above equation to (2.90), using that we know the result $2\alpha\omega_{n-1}$ up to order n, we obtain

$$[\hat{X}_n^M, \hat{X}_n^N] = 2\alpha\omega_{n-1} + \alpha^{n+1} \sum_{q=1}^{n} \sum_{k=0}^{n-q} \sum_{l=0}^{q-1} \times \\ \times \Big[\Gamma_k^{MK_1 \ldots K_{n+1-q-k}}(X)\partial_{K_1} \ldots \partial_{K_{n+1-q-k}}\, , \, \Gamma_l^{NL_1 \ldots L_{q-l}}(X)\partial_{L_1} ... \partial_{L_{q-l}}\Big] + \\ + \mathcal{O}(\alpha^{n+2}) = \tag{2.94}$$

$$= 2\alpha\omega_{n-1} + 2\alpha^{n+1} \sum_{q=1}^{n} \sum_{k=0}^{n-q} \sum_{l=0}^{q-1} \times \\ \times \Gamma_k^{[M|K_1 \ldots K_{n+1-q-k}}(X) \Big[\partial_{K_1} \ldots \partial_{K_{n+1-q-k}}\, , \, \Gamma_l^{|N]L_1 \ldots L_{q-l}}(X)\Big] \partial_{L_1} \ldots \partial_{L_{q-l}} + \\ + \mathcal{O}(\alpha^{n+2}) \tag{2.95}$$

Note that for $n = 0$ the commutator vanishes identically. We can include this case in the above formula, if we use

$$\omega_{-1} \equiv 0, \quad \sum_{q=1}^{0}(\ldots) \equiv 0 \tag{2.96}$$

Remember now the formula (valid in one dimension)

$$\partial^m (A \cdot B) = \sum_{p=0}^{m} \binom{m}{p} \partial^{m-p} A \partial^p B \tag{2.97}$$

This equation implies for the commutator of higher order derivatives with some function that

$$[\partial^m, A](\ldots) = \sum_{p=0}^{m} \binom{m}{p} \partial^{m-p} A \partial^p - A\partial^m = \tag{2.98}$$

$$= \sum_{p=0}^{m-1} \binom{m}{p} \partial^{m-p} A \partial^p \quad \forall m \geq 1 \tag{2.99}$$

In higher dimensions the formula can be used likewise, when the derivatives are totally symmetrized, as it is the case in our commutator (with m replaced by $n+1-q-k$):

$$\begin{aligned}[\hat{X}^M_n, \hat{X}^N_n] =& 2\alpha\omega_{n-1} + 2\alpha^{n+1} \sum_{q=1}^{n} \sum_{k=0}^{n-q} \sum_{l=0}^{q-1} \sum_{p=0}^{n-q-k} \binom{n+1-q-k}{p} \Gamma_k^{[M|K_1\ldots K_{n+1-q-k}}(X) \times \\ & \times \partial_{K_{p+1}} \ldots \partial_{K_{n+1-q-k}} \Gamma_l^{|N]L_1\ldots L_{q-l}}(X) \partial_{K_1} \ldots \partial_{K_p} \partial_{L_1} \ldots \partial_{L_{q-l}} + \mathcal{O}(\alpha^{n+2}) = \end{aligned} \tag{2.100}$$

$$\begin{aligned} =& 2\alpha\omega_{n-1} + 2\alpha^{n+1} \sum_{q=1}^{n} \sum_{k=0}^{n-q} \sum_{l=0}^{q-1} \sum_{p=0}^{n-q-k} \binom{n+1-q-k}{p} \times \\ & \times \Gamma_k^{[M|L_1\ldots L_p K_1 \ldots K_{n+1-q-k-p}}(X) \partial_{K_1} \ldots \partial_{K_{n+1-q-k-p}} \times \\ & \times \Gamma_l^{|N]L_1\ldots L_{q-l}}(X) \partial_{L_1} \ldots \partial_{L_{p+q-l}} + \mathcal{O}(\alpha^{n+2}) \end{aligned} \tag{2.101}$$

Finally we would like to do a last reparametrization of the summation variables, such that $p+q-l$ becomes one of the summation variables. Let us call it m, although it has nothing to do with the m in the general formula (2.97):

$$m \equiv p + q - l \tag{2.102}$$

Let us replace p by it and keep the other variables. The old variables were restricted to $1 \leq q \leq n, \quad 0 \leq k \leq n-q, \quad 0 \leq l \leq q-1, \quad 0 \leq p \leq n-q-k$. The inequality for p gets replaced by $(1 \leq) \quad q-l \leq m \leq n-k-l \quad (\leq n)$.

We want to see its left part as an inequality for l, i.e. $(q-1 \geq) \, l \geq q-m$ and its right part as an inequality for k, i.e. $(0 \leq) \, k \leq n-m-l$. This implies also that $l \leq n-m$. The new variables thus have to obey

$$\begin{aligned} & 1 \leq m \leq n \quad , \quad 1 \leq q \leq n, \\ & \max\{0, q-m\} \leq l \leq \min\{q-1, n-m\}, \quad 0 \leq k \leq \min\{n-q, n-m-l\} \end{aligned} \tag{2.103}$$

Our commutator can therefore be rewritten as

$$
\begin{aligned}
[\hat{X}_n^M, \hat{X}_n^N] \quad = \quad & 2\alpha\Bigg\{\omega_{n-1} + \alpha^n \sum_{m=1}^{n}\sum_{q=1}^{n}\sum_{l=\max\{0,q-m\}}^{\min\{q-1,n-m\}}\sum_{k=0}^{\min\{n-q,n-m-l\}} \binom{n+1-q-k}{m-q+l} \times \\
& \times \Gamma_k^{[M|L_{q-l+1}\ldots L_m K_1\ldots K_{n+1-m-k-l}}(X)\partial_{K_1}\ldots\partial_{K_{n+1-m-k-l}} \times \\
& \times \Gamma_l^{|N]L_1\ldots L_{q-l}}(X)\partial_{L_1}\ldots\partial_{L_m}\Bigg\} + \mathcal{O}(\alpha^{n+2}) \qquad (2.104)
\end{aligned}
$$

2.4.4 Solving the Recursion Relation at Lowest Orders

Now we are almost ready to solve at lowest orders the recursion relation (2.80) or (2.81) which determines the expansion coefficients Γ^M of $\hat{X}^M$ at order α^{n+1} in terms of objects known from order n. The recursion relation contains the object G_{n+1}^{MN} which is defined in (2.62) and which we can give now explicitly, using (2.104):

$$
\begin{aligned}
G_{n+1}^{MN} \quad = \quad & 2\alpha\omega_n^{MN} - [\hat{X}_n^M, \hat{X}_n^N] + \mathcal{O}(\alpha^{n+2}) = \qquad (2.105) \\
= \quad & 2\alpha\Big\{\omega_n^{MN} - \omega_{n-1} - \alpha^n \sum_{m=1}^{n}\sum_{q=1}^{n}\sum_{l=\max\{0,q-m\}}^{\min\{q-1,n-m\}}\sum_{k=0}^{\min\{n-q,n-m-l\}} \binom{n+1-q-k}{m-q+l} \times \\
& \times \Gamma_k^{[M|L_{q-l+1}\ldots L_m K_1\ldots K_{n+1-m-k-l}}(X)\partial_{K_1}\ldots\partial_{K_{n+1-m-k-l}} \times \\
& \times \Gamma_l^{|N]L_1\ldots L_{q-l}}(X)\partial_{L_1}\ldots\partial_{L_m}\Big\} \qquad (2.106)
\end{aligned}
$$

where $\omega_n^{MN}$was derived up to second order in (2.89) and will (as mentioned before) be derived to higher order along with the calculation of the star product. For the moment let us just remember

$$
\begin{aligned}
\omega_0^{MN} =& \omega^{MN}(X) \qquad (2.107) \\
\omega_1^{MN} - \omega_0^{MN} =& \frac{\alpha}{2}\Gamma_0^{KK}\partial_K\partial_K\omega^{MN}(X) + \alpha\partial_K\omega^{MN}(X)\Gamma_0^{KL}\partial_L \qquad (2.108) \\
\omega_2^{MN} - \omega_1^{MN} =& \alpha^2\partial_K\partial_K\omega^{MN}(X)\Gamma_1^{KK}(X) + \frac{\alpha^2}{3}\partial_K\partial_K\partial_K\omega^{MN}(X)\Gamma_0^{KKK}(X) + \\
& + \frac{\alpha^2}{8}\partial_K\partial_K\partial_K\partial_K\omega^{MN}(X)\Gamma_0^{KK}(X)\Gamma_0^{KK}(X) + \\
& + \frac{\alpha^2}{6}\partial_K\partial_K\partial_K\omega^{MN}(X)\Gamma_0^{KL}(X)\partial_L\Gamma_0^{KK}(X) + \\
& + \alpha^2\partial_K\omega^{MN}(X)\Gamma_1^{KL}(X)\partial_L + \alpha^2\partial_K\partial_K\omega^{MN}(X)\Gamma_0^{KKL}(X)\partial_L + \\
& + \frac{\alpha^2}{3}\partial_K\partial_K\partial_K\omega^{MN}(X)\Gamma_0^{KK}(X)\Gamma_0^{KL}(X)\partial_L + \\
& + \frac{\alpha^2}{2}\partial_K\partial_K\omega^{MN}(X)\Gamma_0^{KL_1}(X)\partial_{L_1}\Gamma_0^{KL_2}(X)\partial_{L_2} + \\
& + \frac{\alpha^2}{2}\partial_K\partial_K\omega^{MN}(X)\Gamma_0^{KL_1}(X)\Gamma_0^{KL_2}(X)\partial_{L_1}\partial_{L_2} + \\
& + \alpha^2\partial_K\omega^{MN}(X)\Gamma_0^{KLL}(X)\partial_L\partial_L \qquad (2.109)
\end{aligned}
$$

- Let us start the recursion at order $\boxed{n=0}$, where we know that $\hat{X}_0 = X$ and where the commutator relation is trivially obeyed ($\omega_{-1} \equiv 0$):

$$
[\hat{X}_0, \hat{X}_0] \quad = \quad [X, X] = 0 \qquad (2.110)
$$

In order to do the step to $n+1$, we need G_1 given according to (2.106) and (2.96) by

$$G_1^{MN} = 2\alpha\omega_0^{MN} = 2\alpha\omega^{MN}(X) \tag{2.111}$$

In order to determine $\hat{X}_1$ we need to make use of the recursion relation (2.80) or (2.81)

$$\begin{aligned}
\Gamma_0^{MN} &\stackrel{!}{=} \Gamma_0^{(MN)} + \tfrac{\alpha^{-1}}{2} G_1^{MN} = & (2.112)\\
&= \Gamma_0^{(MN)} + \omega^{MN}(X) & (2.113)\\
&\stackrel{\Gamma_k^{(K\ldots K)}=0}{=} \omega^{MN}(X) & (2.114)
\end{aligned}$$

- Now we switch to $\boxed{n=1}$. The above condition implies

$$\begin{aligned}
\hat{X}_1^M &= X^M + \alpha\big(\Gamma_0^{(MN)} + \omega^{MN}(X)\big)\partial_N & (2.115)\\
\omega_1^{MN} - \omega_0^{MN} &= \tfrac{\alpha}{2}\Gamma_0^{(KK)}\partial_K\partial_K\omega^{MN}(X) + &\\
&\quad +\alpha\partial_K\omega^{MN}(X)\big(\Gamma_0^{(KL)} + \omega^{KL}(X)\big)\partial_L & (2.116)\\
[\hat{X}_1^M, \hat{X}_1^N] &= 2\alpha\omega^{MN}(X) + \mathcal{O}(\alpha^2) & (2.117)
\end{aligned}$$

In order to go to $n+1$ we need the object $G_{n+1} = G_2$ which is given according to (2.106) by

$$\begin{aligned}
G_2^{MN} &= 2\alpha\Big\{\omega_1^{MN} - \omega_0^{MN} - \alpha\Gamma_0^{[M|K}(X)\partial_K\Gamma_0^{|N]L}(X)\partial_L\Big\} = & (2.118)\\
&= 2\alpha^2\Big\{\tfrac{1}{2}\Gamma_0^{(KK)}\partial_K\partial_K\omega^{MN}(X) + \partial_K\omega^{MN}(X)\left(\Gamma_0^{(KL)} + \omega^{KL}(X)\right)\partial_L &\\
&\quad -\big(\tfrac{1}{2}\big(\Gamma_0^{[M|K} + \Gamma_0^{K[M|}\big) + \omega^{[M|K}(X)\big)\partial_K \times &\\
&\quad \times\big(\tfrac{1}{2}\big(\Gamma_0^{|N]L} + \Gamma_0^{L|N]}\big) + \omega^{|N]L}(X)\big)\partial_L\Big\} & (2.119)
\end{aligned}$$

If we choose the completely symmetrized coefficients $\Gamma_k^{(K\ldots K)}$ to vanish, the expression further simplifies to

$$G_2^{MN} = -2\alpha^2\Big\{\omega^{LK}(X)\partial_K\omega^{MN}(X) + \omega^{[M|K}(X)\partial_K\omega^{|N]L}(X)\Big\}\partial_L \tag{2.120}$$

The recursion relation (2.80) finally tells us that

$$\begin{aligned}
\Gamma_0^{MNL} &\stackrel{!}{=} \Gamma_0^{(MNL)} + \tfrac{\alpha^{-2}}{6}\left(G_2^{MNL} + G_2^{MLN}\right) = & (2.121)\\
&\stackrel{\Gamma_k^{(K\ldots K)}=0}{=} -\tfrac{1}{3}\big(\omega^{LK}(X)\partial_K\omega^{MN}(X) + \omega^{[M|K}(X)\partial_K\omega^{|N]L}(X) + &\\
&\quad +\omega^{NK}(X)\partial_K\omega^{ML}(X) + \omega^{[M|K}(X)\partial_K\omega^{|L]N}(X)\big) = & (2.122)\\
&= -\tfrac{1}{6}\big(\omega^{LK}(X)\partial_K\omega^{MN}(X) + \omega^{NK}(X)\partial_K\omega^{ML}(X)\big) & (2.123)\\
\Gamma_1^{MN} &\stackrel{!}{=} \Gamma_1^{(MN)} + \tfrac{\alpha^{-2}}{2}G_2^{MN} = & (2.124)\\
&\stackrel{\Gamma_k^{(K\ldots K)}=0}{=} 0 & (2.125)
\end{aligned}$$

Chapter 3

Star Product

In this section, we define the star product by ordering the monomials in non-commutative coordinate space. We use three different examples of the ordering prescriptions namely Symmetric or Weyl ordering and two other orderings that we will call index-value-ordering and anti-index-value-ordering. For details on ordering prescriptions in context of quantum mechanics see [21]. We then realize that different orderings correspond to different star products. See [22] for similar considerations in case of a constant Poisson structure and the Moyal-Weyl star product.

3.1 Ordering the Monomials

There exists a one to one map of the algebra A of analytic functions on commutative coordinate space F^n to the non-commutative algebra $\hat{A}$

$$\mathcal{M}^{(O)} : A \mapsto \hat{A} \tag{3.1}$$

if one defines a reference ordering of the generators. $\mathcal{M}^{(O)}$ maps the monomials in the commutative coordinate space to the monomials in the non-commutative coordinate space in a particular ordering, where the superscript indicates some particular ordering prescription. The map $\mathcal{M}^{(O)}$ in equation (3.1) can explicitly be written as

$$X^{M_1} \dots X^{M_p} \overset{\mathcal{M}^{(O)}}{\mapsto} (\hat{X}^{M_1} \dots \hat{X}^{M_p})_{Ordered}$$

$$X^{M_1} \dots X^{M_p} \quad \overset{\mathcal{M}^{(O)}}{\mapsto} (\hat{X}^{M_1} \dots \hat{X}^{M_p})_O \tag{3.2}$$

$$X^{M_1} \dots X^{M_p} \quad \overset{\mathcal{M}^{-1(O)}}{\leftarrow\!\shortmid} \quad (\hat{X}^{M_1} \dots \hat{X}^{M_p})_O \tag{3.3}$$

$$1 \underset{\mathcal{M}^{-1(O)}}{\overset{\mathcal{M}^{(O)}}{\rightleftharpoons}} 1 \tag{3.4}$$

Strictly speaking we have to understand $X^{M_1} \dots X^{M_p}$ on the left hand side as $\varphi^{M_1} \dots \varphi^{M_p}$, where $\varphi^M(X) = X^M$. However, it is quite common and convenient to ignore the distinction between X^M and φ^M.

Weyl-ordering of the monomials : Symmetric or Weyl-ordering is a natural choice for a reference ordering of the monomials. The map $\mathcal{M}^{(O)} \equiv \mathcal{M}^{(W)}$ in equation (3.2) in this representation maps monomials in commutative coordinate space to the symmetrically ordered monomials

$$(\hat{X}^{M_1} \dots \hat{X}^{M_p})_W \equiv \hat{X}^{(M_1} \dots \hat{X}^{M_p)} \tag{3.5}$$

in the non-commutative coordinate space and the explicit form of the map $\mathcal{M}^{(O)} \equiv \mathcal{M}^{(W)}$ in equations (3.2) and (3.3) takes the form

$$X^{M_1} \dots X^{M_p} \quad \overset{\mathcal{M}^{(W)}}{\mapsto} \hat{X}^{(M_1} \dots \hat{X}^{M_p)} \tag{3.6}$$

$$X^{M_1} \dots X^{M_p} \quad \overset{\mathcal{M}^{-1(W)}}{\hookleftarrow} \quad \hat{X}^{(M_1} \dots \hat{X}^{M_p)} \tag{3.7}$$

Index-value-ordering of the monomials : One can also have some different ordering e.g. index-value-ordering of the monomials . By index-value-ordering of the monomials we mean putting X's with higher indices to the right. For example $\hat{X}^{M_2}\hat{X}^{M_1} \overset{\mathcal{M}^{(I)}}{\mapsto} \hat{X}^{M_1}\hat{X}^{M_2}$ if $M_1 \leq M_2$ and similar to the definition (3.5) in this case we define

$$(\hat{X}^{M_1} \dots \hat{X}^{M_p})_I \equiv \hat{X}^{:M_1} \dots \hat{X}^{M_p:} \tag{3.8}$$

Where

$$: M_{P(1)} ... M_{P(p)} :\equiv M_1 ... M_p \quad \forall \quad \text{permutations} \quad \text{P}, \quad \forall \quad \text{M}_1 \leq \dots \leq \text{M}_\text{p} \tag{3.9}$$

Also note that we will use here the notation ": $\cdots$:" for index ordering of the monomials independently of its usual use for normal ordered operators in Quantum mechanics and Quantum Field theory. The map $\mathcal{M}^{(O)}$ in equations (3.2) and (3.3) looks in this case as in the following

$$X^{M_1} \dots X^{M_p} \quad \overset{\mathcal{M}^{(I)}}{\mapsto} \hat{X}^{:M_1} \dots \hat{X}^{M_p:} \tag{3.10}$$

$$X^{M_1} \dots X^{M_p} \quad \overset{\mathcal{M}^{-1(I)}}{\hookleftarrow} \quad \hat{X}^{:M_1} \dots \hat{X}^{M_p:} \tag{3.11}$$

Anti-index-value-ordering of the monomials: One can equally define anti-index-value-ordering by putting X's with higher indices to the very left.
For example $\hat{X}^{M_1}\hat{X}^{M_2} \overset{\mathcal{M}^{(A_I)}}{\mapsto} \hat{X}^{M_2}\hat{X}^{M_1}$ if $M_1 \leq M_2$. Where again similar to the definitions (3.5) and (3.8) we define

$$(\hat{X}^{M_1} \dots \hat{X}^{M_p})_{A_I} \equiv \hat{X}^{;M_1} \dots \hat{X}^{M_p;} \tag{3.12}$$

and

$$; M_{P(p)} ... M_{P(1)} ; \equiv M_p ... M_1 \quad \forall \quad \text{permutations} \quad \text{P}, \quad \forall \quad \text{M}_1 \leq \dots \leq \text{M}_\text{p} \tag{3.13}$$

The map $\mathcal{M}^{(O)}$ in equations (3.2) and (3.3) looks in this case as in the following

$$X^{M_1} \dots X^{M_p} \quad \overset{\mathcal{M}^{(A_I)}}{\mapsto} \hat{X}^{;M_1} \dots \hat{X}^{M_p;} \tag{3.14}$$

$$X^{M_1} \dots X^{M_p} \quad \overset{\mathcal{M}^{-1(A_I)}}{\hookleftarrow} \quad \hat{X}^{;M_1} \dots \hat{X}^{M_p;} \tag{3.15}$$

Poincaré-Birkhoff-Witt (PBW) property: We have now described different ordering prescriptions for the monomials with non-commutative coordinates $\hat{X}$. We will use in practice, to put monomials with noncommutative coordinate operator $\hat{X}$ in some specific ordering, the property of the algebra called Poincaré-Birkhoff-Witt property. This property of the algebra $\hat{A}$ allows one to reorder the elements using the commutator in equation (2.10). Note that it is always possible when α is formal expansion parameter. In contrast, the form of ω and size of α are severely constrained if α is instead a (small) real parameter. For details see [23] [18] [24].

3.2 Star Product Via Isomorphism

Since the map $\mathcal{M}^{(O)}$ defined in the last section is a vector space isomorphism whose inverse $\mathcal{M}^{-1(O)}$ can be used to map the multiplicative structure of $\hat{A}$ to the A. We use this isomorphism to define the star product in the commutative space by the following commutative diagram.

$$\begin{aligned}
f(X), g(X) &\xmapsto{\mathcal{M}} f(\hat{X}), g(\hat{X}) \\
&\qquad\downarrow \text{(Ordinary operator product)} \\
f \star g(X) &\xleftarrow{\mathcal{M}^{-1}} f(\hat{X})g(\hat{X}) \equiv f \star g(\hat{X}) = \mathcal{M}(f \star g(X))
\end{aligned} \tag{3.16}$$

The star product in commutative space is then defined by

$$f \star g(\hat{X}) = f(\hat{X})g(\hat{X}) \tag{3.17}$$

$$\mathcal{M}(f \star g(X)) = \mathcal{M}(f(X))\mathcal{M}(g(X)) \tag{3.18}$$

or

$$\begin{aligned}
f \star g(X) =& \mathcal{M}^{-1}(\mathcal{M}(f(X))\mathcal{M}(g(X))) \\
=& \mathcal{M}^{-1}((f(\hat{X}))(g(\hat{X})))
\end{aligned} \tag{3.19}$$

Assuming that f(X) is an analytical function, we can expand

$$f(X) = \sum_{n=0}^{\infty} f_{M_1 \dots M_n} X^{M_1} \cdots X^{M_n} \tag{3.20}$$

This is mapped with $\mathcal{M}$ to an $f(\hat{X})$ of the form

$$\begin{aligned}
f(\hat{X}) =& \sum_{n=0}^{\infty} f_{M_1 \dots M_n} \mathcal{M}^{(O)}(X^{M_1} \dots X^{M_n}) \\
=& \sum_{n=0}^{\infty} f_{M_1 \dots M_n} (\hat{X}^{M_1} \dots \hat{X}^{M_n})_O
\end{aligned} \tag{3.21}$$

The star product defined in equation (3.17) becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty} (f \star g)_{M_1 \dots M_n} (\hat{X}^{M_1} \dots \hat{X}^{M_n})_O = \\
=& \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} f_{M_1 \dots M_p} g_{M_{p+1} \dots M_{p+q}} (\hat{X}^{M_1} \dots \hat{X}^{M_p})_O (\hat{X}^{M_{p+1}} \dots \hat{X}^{M_{p+q}})_O
\end{aligned} \tag{3.22}$$

In order to get the explicit expression for the star product defined in equation (3.22) one has to put the monomials on the right hand side in the same ordering as on the left hand side. However, as seen in the example (2.24) reordering lower order α terms leave higher order terms unordered and, therefore, to obtain explicit expression for the star product at higher α orders becomes very hard. In the examples below to show the procedure we will give the star product examples ordered only up to order α terms. It is important to note that unit element of A, i.e. the constant map $\varphi^{(0)} \in A$ with $\varphi^{(0)}(X) = 1$ is mapped to the unit element in $\hat{A}$, i.e. $\varphi^{(0)}(\hat{X}) = 1 \in \hat{A}$.

$$\varphi^{(0)} \in A \qquad \xmapsto{\mathcal{M}^{(O)}} \qquad \varphi^{(0)}(\hat{X}) = 1 \in \hat{A} \tag{3.23}$$

and hence the star product of any function $f(X)$ with constant function $(C(X) = c) \quad C = c \cdot \varphi^{(0)}$ coincides with the ordinary product

$$f \star C = C \star f = Cf \tag{3.24}$$

3.2.1 Weyl-Ordered Star Product

When we choose the ordering prescription to be Weyl ordering i.e. $\mathcal{M}^{(O)} = \mathcal{M}^{(W)}$ the star product in equation (3.22) becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty}(f \star g)_{M_1 \dots M_n}(\hat{X}^{M_1} \dots \hat{X}^{M_n})_W = \\
=&\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} f_{M_1 \dots M_p} g_{M_{p+1} \dots M_{p+q}} (\hat{X}^{M_1} ... \hat{X}^{M_p})_W (\hat{X}^{M_{p+1}} \dots \hat{X}^{M_{p+q}})_W
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
&\sum_{n=0}^{\infty}(f \star g)_{M_1 \dots M_n}(\hat{X}^{(M_1} ... \hat{X}^{M_n)}) = \\
=&\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} f_{M_1 \dots M_p} g_{M_{p+1} \dots M_{p+q}} (\hat{X}^{(M_1} \dots \hat{X}^{M_p)})(\hat{X}^{(M_{p+1}} \dots \hat{X}^{M_{p+q})})
\end{aligned} \tag{3.26}$$

Now one needs to reorder the monomials at the right hand side of equation (3.26) into Weyl ordering. This can be done by the use of a commutator. Let us show this with some simple examples.

Examples The simplest example that can still show the basic procedure for calculating the star product is the star product of the coordinates functions $\varphi^M(X) = X^M$.

$$\begin{aligned}
\varphi^{M_1} \star \varphi^{M_2}(\hat{X}) =& \hat{X}^{M_1}\hat{X}^{M_2} = \\
=& \frac{1}{2}\left(\hat{X}^{M_1}\hat{X}^{M_2} + \hat{X}^{M_2}\hat{X}^{M_1}\right) + \alpha\omega^{M_1 M_2}(\hat{X})
\end{aligned} \tag{3.27}$$

This is mapped back to

$$\varphi^{M_1} \star \varphi^{M_2}(X) = X^{M_1}X^{M_2} + \alpha\omega^{M_1 M_2}(X) \quad \xleftarrow{\mathcal{M}^{-1(W)}} \quad \frac{1}{2}\left(\hat{X}^{M_1}\hat{X}^{M_2} + \hat{X}^{M_2}\hat{X}^{M_1}\right) + \alpha\omega^{M_1 M_2}(\hat{X}) \tag{3.28}$$

The star product commutator then coincides with the commutator of the non-commutative coordinates

$$[\varphi^{M_1}, \varphi^{M_2}]_\star = \varphi^{M_1} \star \varphi^{M_2} - \varphi^{M_2} \star \varphi^{M_1} = 2\alpha\omega^{M_1 M_2} \tag{3.29}$$

Another instructive example is the star product of monomials with unity of the algebra i.e.$f(X) = \varphi^{M_1 M_2}(X) \equiv X^{M_1}X^{M_2}$ and $g(X) = e(X) = 1$

$$\begin{aligned}
\varphi^{M_1 M_2} \star e(\hat{X}) =& \hat{X}^{(M_1}\hat{X}^{M_2)} e(\hat{X}) \\
=& \hat{X}^{(M_1}\hat{X}^{M_2)} 1 \\
=& \hat{X}^{(M_1}\hat{X}^{M_2)} = \varphi^{M_1 M_2}(\hat{X})
\end{aligned} \tag{3.30}$$

which is expected because of (3.24) and the result is mapped back to the commutative coordinate space A

$$\varphi^{M_1M_2}\star e(X) = \varphi^{M_1M_2}(X) = X^{M_1}X^{M_2} \quad \overset{\mathcal{M}^{-1(W)}}{\longleftarrow} \quad \hat{X}^{(M_1}\hat{X}^{M_2)} \tag{3.31}$$

A slightly more complicated example would be the star product of functions $f = \varphi^{M_1}(X)$ and $g = \varphi^{M_2M_3}(X)$

$$\begin{aligned}\varphi^{M_1}\star\varphi^{M_2M_3}(\hat{X}) =&\hat{X}^{M_1}\hat{X}^{(M_2}\hat{X}^{M_3)} &(3.32)\\ =&\frac{1}{3}\Big(\hat{X}^{M_1}\hat{X}^{(M_2|}\hat{X}^{|M_3)}+\hat{X}^{(M_2|}\hat{X}^{M_1}\hat{X}^{|M_3)}+\hat{X}^{(M_2|}\hat{X}^{|M_3)}\hat{X}^{M_1}\Big)+\\ &+\frac{1}{3}[\hat{X}^{M_1},\hat{X}^{(M_2|}]\hat{X}^{|M_3)}+\frac{1}{3}[\hat{X}^{M_1},\hat{X}^{(M_2|}\hat{X}^{|M_3)}]\\ =&\hat{X}^{(M_1}\hat{X}^{M_2}\hat{X}^{M_3)}+\frac{2}{3}\alpha\omega^{M_1(M_2|}(\hat{X})\hat{X}^{|M_3)}+\frac{2}{3}\alpha\omega^{M_1(M_2|}(\hat{X})\hat{X}^{|M_3)}+\\ &+\frac{2}{3}\alpha\hat{X}^{(M_2|}\omega^{M_1|M_3)}(\hat{X})\\ =&\hat{X}^{(M_1}\hat{X}^{M_2}\hat{X}^{M_3)}+2\alpha\omega^{M_1(M_2|}(\hat{X})\hat{X}^{|M_3)}+\\ &+\frac{4}{3}\alpha^2\omega^{(M_3|N}(\hat{X})\partial_N\omega^{M_1|M_2)}(\hat{X})+\mathcal{O}(\alpha^3) &(3.33)\end{aligned}$$

Which is mapped back to commutative coordinate space

$$\begin{aligned}\varphi^{M_1}\star\varphi^{M_2M_3}(X) =&X^{M_1}X^{M_2}X^{M_3}+2\alpha\omega^{M_1(M_2|}(X)X^{|M_3)}+\\ &+\frac{4}{3}\alpha^2\omega^{(M_3|N}(X)\partial_N\omega^{M_1|M_2)}(X)+\mathcal{O}(\alpha^3)\end{aligned} \tag{3.34}$$

Exponential representation of the Weyl ordered expansion If we choose an ordering to be the Weyl ordering the function in equation (3.21) takes the form

$$f(\hat{X}) = \sum_{n=0}^{\infty} f_{M_1\ldots M_n}\sum_{p_n}\frac{1}{n!}P_n(\hat{X}^{M_1}\ldots\hat{X}^{M_n}) \tag{3.35}$$

where P_n are all permutations of the of n elements. The second sum is just the definition of the symmetrization brackets, i.e.

$$f(\hat{X}) = \sum_{n=0}^{\infty} f_{M_1\ldots M_n}\hat{X}^{(M_1}\cdots\hat{X}^{M_n)} \tag{3.36}$$

A particular property of the Weyl-ordered basis is that we can drop the symmetrization brackets when we contract the generator-monomials with the completely symmetric expansion coefficients of some analytic function in A:

$$f(\hat{X}) = \sum_{n=0}^{\infty} f_{M_1\ldots M_n}\hat{X}^{M_1}\cdots\hat{X}^{M_n} \tag{3.37}$$

Using this property, we can write $f(\hat{X})$ as

$$\begin{aligned}
f(\hat{X}) =& \sum_{n=0}^{\infty} \frac{1}{n!} \hat{X}^{M_1} \dots \hat{X}^{M_n} \partial_{Y^{M_1}} \dots \partial_{Y^{M_n}} f(Y)|_{Y=0} \\
=& \sum_{n=0}^{\infty} \frac{1}{n!} (\hat{X}^M \partial_{Y^M})^n f(Y)|_{Y=0} \\
f(\hat{X}) =& e^{\hat{X}^M \partial_{Y^M}} f(Y)|_{Y=0} \qquad (3.38)
\end{aligned}$$

and the star product can be written as

$$\begin{aligned}
f \star g(\hat{X}) =& f(\hat{X}) g(\hat{X}) \qquad (3.39) \\
=& e^{\hat{X}^K \partial_{Y^K}} e^{\hat{X}^K \partial_{Z^K}} f(Y) g(Z)|_{Y=Z=0} \qquad (3.40) \\
=& e^{\hat{X}^K \partial_{Y^K} + \hat{X}^K \partial_{Z^K} + \frac{1}{2}[\hat{X}^K, \hat{X}^K] \partial_{Y^K} \partial_{Z^K} + \dots} f(Y) g(Z)|_{Y=Z=0} \qquad (3.41)
\end{aligned}$$

In contrast to [10], we will not use a Fourier-expansion, but will instead work with the representation (3.38), as it generalizes easily to the superspace.

3.2.2 Index-Value-Ordered Star Product

Let us now choose a differnet ordering prescription i.e. index-value-ordering i.e. $\mathcal{M}^{(O)} = \mathcal{M}^{(I)}$ the star product in equation (3.22) becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty} (f \star g)_{M_1 \dots M_n} (\hat{X}^{M_1} \dots \hat{X}^{M_n})_I = \\
=& \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} f_{M_1 \dots M_p} g_{M_{p+1} \dots M_{p+q}} (\hat{X}^{M_1} \dots \hat{X}^{M_p})_I (\hat{X}^{M_{p+1}} \dots \hat{X}^{M_{p+q}})_I \qquad (3.42) \\
&\sum_{n} (f \star g)_{M_1 \dots M_n} (\hat{X}^{:M_1} \dots \hat{X}^{M_n:}) = \\
=& \sum_{p} \sum_{q} f_{M_1 \dots M_p} g_{M_{p+1} \dots M_{p+q}} (\hat{X}^{:M_1} \dots \hat{X}^{M_p:}) (\hat{X}^{:M_{p+1}} \dots \hat{X}^{M_{p+q}:}) \qquad (3.43)
\end{aligned}$$

Now again one needs to reorder the right hand side of equation (3.43) into index-value-ordering. This can again be done by the use of a commutator. Let us repeat this with same examples. The example of the star product of the coordinate functions $\varphi^M(X) = X^M$ in case of index-value-ordering reads then

$$\begin{aligned}
\varphi^{M_1} \star \varphi^{M_2}(\hat{X}) =& \hat{X}^{M_1} \hat{X}^{M_2} = \qquad (3.44) \\
=& \hat{X}^{:M_1} \hat{X}^{M_2:} - \hat{X}^{:M_1} \hat{X}^{M_2:} + \hat{X}^{M_1} \hat{X}^{M_2} \qquad (3.45)
\end{aligned}$$

At this point one needs to make a case distinction. If $M_1 \leq M_2$, the second and third term cancel, while if $M_1 > M_2$, they combine to the commutator. The result can be written without case distinction as follows:

$$\varphi^{M_1} \star \varphi^{M_2}(\hat{X}) = \hat{X}^{:M_1} \hat{X}^{M_2:} + \alpha\omega^{M_1 M_2}(\hat{X}) - \alpha\omega^{:M_1 M_2:}(\hat{X}) \qquad (3.46)$$

This is mapped back to A

$$\varphi^{M_1}\star\varphi^{M_2}(X)=$$
$$= X^{M_1}X^{M_2}+\alpha\omega^{M_1M_2}(X)-\alpha\omega^{:M_1M_2:}(X)\quad\overset{\mathcal{M}^{-1(I)}}{\longleftarrow}\quad \hat{X}^{:M_1}\hat{X}^{M_2:}+\alpha\omega^{M_1M_2}(\hat{X})-\alpha\omega^{:M_1M_2:}(\hat{X}) \tag{3.47}$$

The star product commutator coincides with the the commutator of the non-commutative coordinates

$$[\varphi^{M_1},\varphi^{M_2}]_\star=\varphi^{M_1}\star\varphi^{M_2}-\varphi^{M_2}\star\varphi^{M_1}=2\alpha\omega^{M_1M_2} \tag{3.48}$$

Let us also repeat other two examples to have better feeling of the difference of the two star products. The star product of monomials with unity of the algebra i.e.$f=\varphi^{M_1M_2}(X)$ and $g=e(X)$

$$\begin{aligned}\varphi^{M_1M_2}\star e(\hat{X}) =&\hat{X}^{:M_1}\hat{X}^{M_2:}e(\hat{X})\\ =&\hat{X}^{:M_1}\hat{X}^{M_2:}\cdot 1\\ =&\hat{X}^{:M_1}\hat{X}^{M_2:}\end{aligned} \tag{3.49}$$

which is mapped back to the commutative coordinate space

$$\varphi^{M_1M_2}\star e(X)=X^{M_1}X^{M_2}\quad\overset{\mathcal{M}^{-1(I)}}{\longleftarrow}\quad\hat{X}^{:M_1}\hat{X}^{M_2:} \tag{3.50}$$

Next example is of the star product of $f=\varphi^{M_1M_2}(X)$ and $g=\varphi^{M_3}(X)$

$$\varphi^{M_1M_2}\ \star\ \varphi^{M_3}(\hat{X})=\hat{X}^{:M_1}\hat{X}^{M_2:}\hat{X}^{M_3}= \tag{3.51}$$
$$=\hat{X}^{:M_1}\hat{X}^{M_2}\hat{X}^{M_3:}-\hat{X}^{:M_1}\hat{X}^{M_2}\hat{X}^{M_3:}+\hat{X}^{:M_1}\hat{X}^{M_2:}\hat{X}^{M_3}= \tag{3.52}$$
$$=\begin{cases}\rightarrow\quad \hat{X}^{:M_1}\hat{X}^{M_2}\hat{X}^{M_3:} & \text{if } M_3 \text{ is biggest index}\\ \rightarrow\quad \hat{X}^{:M_1}\hat{X}^{M_2}\hat{X}^{M_3:}+\underbrace{[\hat{X}^{:M_1}\hat{X}^{M_2:},\hat{X}^{M_3}]}_{2\alpha\hat{X}^{:M_1}\omega^{M_2:\,M_3}(\hat{X})+2\alpha\omega^{:M_1|\,M_3}(\hat{X})\hat{X}^{|M_2:}} & \text{if } M_3 \text{ is smallest index}\\ \rightarrow\quad \hat{X}^{:M_1}\hat{X}^{M_2}\hat{X}^{M_3:}+\hat{X}^{:M_1|}\underbrace{[\hat{X}^{|M_2:},\hat{X}^{M_3}]}_{2\alpha\omega^{|M_2:\,M_3}(\hat{X})} & \text{if } M_3 \text{ is between } M_1 \text{ and } M_2\end{cases} \tag{3.53}$$

3.2.3 Anti-Index-Value-Ordered Star Product

Let us now show all our results in yet another ordering prescription i.e. anti-index-value-ordering: $\mathcal{M}^{(O)}=\mathcal{M}^{(A_I)}$. The star product in equation (3.22) becomes

$$\begin{aligned}&\sum_{n=0}^{\infty}(f\star g)_{M_1\ldots M_n}(\hat{X}^{M_1}\ldots\hat{X}^{M_n})_{A_I}=\\ =&\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}f_{M_1\ldots M_p}g_{M_{p+1}\ldots M_{p+q}}(\hat{X}^{M_1}\ldots\hat{X}^{M_p})_{A_I}(\hat{X}^{M_{p+1}}\ldots\hat{X}^{M_{p+q}})_{A_I}\end{aligned} \tag{3.54}$$

$$\begin{aligned}&\sum_{n=0}^{\infty}(f\star g)_{M_{p+q}\ldots M_1}(\hat{X}^{;M_{p+q}}\ldots\hat{X}^{M_1;})=\\ =&\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}f_{M_1\ldots M_p}g_{M_{p+1}\ldots M_{p+q}}(\hat{X}^{;M_1}\ldots\hat{X}^{M_p;})(\hat{X}^{;M_{p+1}}\ldots\hat{X}^{M_{p+q};})\end{aligned} \tag{3.55}$$

Now once again one needs to reorder the right hand side of equation (3.55) into anti-index-value-ordering. This can be done by the use of a commutator. Let us again repeat this with same simple examples of the last two sections. The example of the star product of the coordinates functions $\varphi^M(X) = X^M(X)$ in case of anti-index-value-ordering becomes

$$\varphi^{M_2} \star \varphi^{M_1}(\hat{X}) = \hat{X}^{M_2}\hat{X}^{M_1} = \tag{3.56}$$

$$= \hat{X}^{;M_2}\hat{X}^{M_1;} - \hat{X}^{;M_2}\hat{X}^{M_1;} + \hat{X}^{M_2}\hat{X}^{M_1} \tag{3.57}$$

Again we need to make a case distinction. If $M_1 \leq M_2$, the second and third term cancel, while if $M_1 > M_2$, they combine to the commutator. The result can be written without case distinction as follows:

$$\varphi^{M_2} \star \varphi^{M_1}(\hat{X}) = \hat{X}^{;M_2}\hat{X}^{M_1;} + \alpha\omega^{M_2M_1}(\hat{X}) - \alpha\omega^{;M_2M_1;}(\hat{X}) \tag{3.58}$$

This is mapped back to A

$$\varphi^{M_2} \star \varphi^{M_1}(X) =$$

$$= X^{M_2}X^{M_1} + \alpha\omega^{M_2M_1}(X) - \alpha\omega^{;M_2M_1;}(X) \quad \overset{\mathcal{M}^{-1(I)}}{\longleftarrow} \quad \hat{X}^{;M_2}\hat{X}^{M_1;} + \alpha\omega^{M_2M_1}(\hat{X}) - \alpha\omega^{;M_2M_1;}(\hat{X}) \tag{3.59}$$

The star product commutator coincides with the commutator of the non-commutative coordinates

$$[\varphi^{M_1}, \varphi^{M_2}]_\star = \varphi^{M_1} \star \varphi^{M_2} - \varphi^{M_2} \star \varphi^{M_1} = 2\alpha\omega^{M_1M_2} \tag{3.60}$$

and the example of the star product of monomials with unity of the algebra i.e. $f = \varphi^{M_2M_1}(X)$ and $g = e(X)$ in this case would be

$$\begin{aligned} \varphi^{M_2M_1} \star e(\hat{X}) &= \hat{X}^{;M_2}\hat{X}^{M_1;}e(\hat{X}) \\ &= \hat{X}^{;M_2}\hat{X}^{M_1;} \cdot 1 \\ &= \hat{X}^{;M_2}\hat{X}^{M_1;} \end{aligned} \tag{3.61}$$

which is mapped back to the commutative coordinate space A

$$\varphi^{M_2M_1} \star e(X) = X^{M_2}X^{M_1} \quad \overset{\mathcal{M}^{-1(A_I)}}{\longleftarrow} \quad \hat{X}^{;M_2}\hat{X}^{M_1;} \tag{3.62}$$

and the last example is of the star product of $g = \varphi^{M_3}(X)$ and $f = \varphi^{M_1M_2}(X)$

$$\varphi^{M_1M_2} \star \varphi^{M_3}(\hat{X}) = \hat{X}^{;M_1}\hat{X}^{M_2;}\hat{X}^{M_3} = \tag{3.63}$$

$$= \hat{X}^{;M_1}\hat{X}^{M_2}\hat{X}^{M_3;} - \hat{X}^{;M_1}\hat{X}^{M_2}\hat{X}^{M_3;} + \hat{X}^{;M_1}\hat{X}^{M_2;}\hat{X}^{M_3} = \tag{3.64}$$

$$= \begin{cases} \rightarrow \quad \hat{X}^{;M_1}\hat{X}^{M_2}\hat{X}^{M_3;} & \text{if } M_3 \text{ is smallest index} \\ \rightarrow \quad \hat{X}^{;M_1}\hat{X}^{M_2}\hat{X}^{M_3;} + \underbrace{[\hat{X}^{;M_1}\hat{X}^{M_2;}, \hat{X}^{M_3}]}_{2\alpha\hat{X}^{;M_1}\omega^{M_2;\,M_3}(\hat{X}) + 2\alpha\omega^{;M_1|\,M_3}(\hat{X})\hat{X}^{|M_2;}} & \text{if } M_3 \text{ is biggest index} \\ \rightarrow \quad \hat{X}^{;M_1}\hat{X}^{M_2}\hat{X}^{M_3;} + \hat{X}^{;M_1|}\underbrace{[\hat{X}^{|M_2;}, \hat{X}^{M_3}]}_{2\alpha\omega^{|M_2;\,M_3}(\hat{X})} & \text{if } M_3 \text{ is between } M_1 \text{ and } M_2 \end{cases} \tag{3.65}$$

Chapter 4

Star Product Order by Order

In the last chapter we defined the star product in the commutative space A by reordering the monomails in the non-commutative space $\hat{A}$ and mapping them back to the commutative space. In this section we want to describe a method for calculating the star product order by order. For this reason we are working from now on only with Weyl-ordered star product.

4.1 Polydifferential Representation

It turns out that the Weyl ordered star product given by (3.26) and also in (3.41) simplifies in the polydifferential representation (2.47) if one puts the additional requirement that the completely symmetrized expansion coefficients vanish:

$$\Gamma_k^{(MK_1 \dots K_p)} = 0 \tag{4.1}$$

As we have seen after (2.81), such a choice is still possible without changing the algebra. According to [10], the star product defined in equation (3.17) then takes the useful form

$$f \star g(X) = f(\hat{X}) g(X) \tag{4.2}$$

This has to be understood as the operator $f(\hat{X})$ acting on the function $g(X)$ and not as the operator product of a differential and a multiplication operator! To show this let us compute a simple example. Let us choose $f(\hat{X}) = \varphi^{M_1 M_2}(\hat{X}) = \hat{X}^{(M_1} \hat{X}^{M_2)}$ and $g(X) = e(X) \in A$ and use the polydifferential representation (2.47) of $\hat{X}$.

$$\begin{aligned}
\hat{X}^{(M_1} \hat{X}^{M_2)}(e(X)) =& \hat{X}^{(M_1} X^{M_2)}(1) \\
=& \Big(X^{(M_1|} + \sum_{p=1}^{\infty} \alpha^p \Gamma^{(M_1|K_1 \dots K_p}(\alpha, X) \partial_{K_1} \dots \partial_{K_p} \Big) X^{|M_2)} \\
=& X^{(M_1} X^{M_2)} + \sum_{p=1}^{\infty} \alpha^p \Gamma^{(M_1|K_1 \dots K_p}(\alpha, X) \partial_{K_1} \dots \partial_{K_p} X^{|M_2)} \\
=& X^{(M_1} X^{M_2)} + \sum_{p=1}^{\infty} \alpha^p \Gamma^{(M_1 M_2)}(\alpha, X)
\end{aligned} \tag{4.3}$$

Note that if the above condition (4.1) is not obeyed, we could still use equation (4.2) as a definition of a star product. However, it will be a different star product then, as the one naturally defined in (3.26). In particular it will not be an associative star product any longer! We will discuss this (somewhat artificial) approach to relax associativity in section 7.2. Now in order to calculate the star product using this particular representation (4.2) we have to first calculate $f(\hat{X})$ via equation (3.38). However, to calculate $e^{\hat{X}^M\partial_{Y^M}}$ order by order in α we will use the following formula derived from Duhmael formula in [10].

$$\begin{aligned} e^{A+B} =& e^A\Big\{1+B+\frac{1}{2}[B,A]+\frac{1}{2}B^2+\\ &+\frac{1}{6}[[B,A],A]+\frac{1}{3}[B,A]B+\frac{1}{6}B[B,A]+\frac{1}{6}B^3+\\ &+\frac{1}{24}[[[B,A],A],A]+\frac{1}{8}[[B,A],A]B+\frac{1}{8}[B,A]^2+\\ &+\frac{1}{24}B[[B,A],A]+\frac{1}{8}[B,A]B^2+\frac{1}{12}B[B,A]B+\\ &+\frac{1}{24}B^2[B,A]+\frac{1}{24}B^4\Big\}+\mathcal{O}(\alpha^5) \end{aligned} \tag{4.4}$$

Where $A = X^M\partial_{Y^M}$ is of order of α^0 and $B = (\hat{X}^M - X^M)\partial_{Y^M}$ is of order α^1 and each commutator is at least one order higher, $[B,A] = O(\alpha^2)$, $[[B,A],A] = O(\alpha^3)$, $[[[B,A],A],A] = O(\alpha^4)$. In order to demonstrate the procedure for computing commutators let us compute the lowest order commutator $[B,A]$ in the following

$$A \equiv X^M\partial_{Y^M}, \quad B\equiv(\hat{X}^N - X^N)\partial_{Y^N} \tag{4.5}$$

$$[B,A] = (\hat{X}^N - X^N)\partial_{Y^N}\left(X^M\partial_{Y^M}\dots\right) - X^M\partial_{Y^M}\Big((\hat{X}^N - X^N)\partial_{Y^N}\dots\Big) = \tag{4.6}$$

$$= \Big(\hat{X}^N X^M - X^M\hat{X}^N\Big)\partial_{Y^M}\partial_{Y^N} = \tag{4.7}$$

$$\begin{aligned} =& \sum_{k=1}^{\infty}\alpha^k\Gamma^{N\,K_1\dots K_k}(\alpha,X)\big(\partial_{K_1}\dots\partial_{K_k}(X^M\partial_{Y^M}\partial_{Y^N}\dots)-\\ &-X^M\partial_{K_1}\dots\partial_{K_k}\partial_{Y^M}\partial_{Y^N}\big) = \end{aligned} \tag{4.8}$$

$$= \sum_{k=1}^{\infty}\alpha^k k\Gamma^{(N\,M)K_1\dots K_{k-1}}(\alpha,X)\partial_{K_1}\dots\partial_{K_{k-1}}\partial_{Y^M}\partial_{Y^N} = \tag{4.9}$$

$$= \sum_{k=1}^{\infty}\sum_{l=0}^{k-1}\alpha^k(k-l)\Gamma_l^{(N\,M)K_1\dots K_{k-l-1}}(X)\partial_{K_1}\dots\partial_{K_{k-l-1}}\partial_{Y^M}\partial_{Y^N} \tag{4.10}$$

Writing the result explicitly up to second order in α yields

$$\begin{aligned} [B,A] =& \alpha\underbrace{\Gamma_0^{(N\,M)}(X)}_{=0}\partial_{Y^M}\partial_{Y^N}+\\ &+\alpha^2 2\Gamma_0^{(N\,M)K_1}(X)\partial_{K_1}\partial_{Y^M}\partial_{Y^N}+\alpha^2\underbrace{\Gamma_1^{(N\,M)}(X)}_{=0}\partial_{Y^M}\partial_{Y^N}+\mathcal{O}(\alpha^3) \end{aligned} \tag{4.11}$$

This demonstrates that $[B,A]$ is indeed of order α^2 if $\Gamma_0^{(N\,M)} = 0$.

4.1.1 Zeroth Order Star Product

In the zeroth order ($\alpha \to 0$), the commutator in equation (2.10) reads

$$[\hat{X}^M, \hat{X}^N] = 0 \tag{4.12}$$

so that we have an undeformed commutative algebra, $\hat{X}^M = X^M$, and as expected the star product coincides with ordinary product

$$f \star_0 g = f \cdot g \tag{4.13}$$

4.1.2 First Order Star Product

In order to calculate the first order star product first of all we have to expand the coordinate operator $\hat{X}$ up to the first order in α

$$\hat{X}_1^M = X^M + \alpha \Gamma_0^{MK}(X)\partial_K + \mathcal{O}(\alpha^2) \tag{4.14}$$

Since we know $\Gamma_0^{MK}(X) = \omega^{MN}(X)$ from (2.114) using it we can determine $e^{\hat{X}^M \partial_{Y^M}}$ up to the order α using formula (4.4).

$$\begin{aligned} e^{A+B} =& e^A + e^A B + O(\alpha^2) \\ =& e^{X^M \partial_{Y^M}} + e^{X^M \partial_{Y^M}}((\hat{X}^K - X^K)\partial_{Y^M}) \end{aligned} \tag{4.15}$$

$$\begin{aligned} e^{\hat{X}^M \partial_{Y^M}} =& e^{X^M \partial_{Y^M}} + e^{X^M \partial_{Y^M}}(\alpha \Gamma_0^{KL} \partial_{X^L})\partial_{Y^K} \\ e^{\hat{X}^M \partial_{Y^M}} =& e^{X^M \partial_{Y^M}} + e^{X^M \partial_{Y^M}}(\alpha \omega^{KL}(X) \partial_{X^L})\partial_{Y^K} \end{aligned} \tag{4.16}$$

and by using equation (3.38) we obtain

$$\begin{aligned} f(\hat{X}) =& (e^{X^M \partial_{Y^M}} + e^{X^M \partial_{Y^M}}(\alpha \omega^{KL}(X)\partial_{X^L})\partial_{Y^M}) f(Y)|_{Y=0} \\ =& e^{X^M \partial_{Y^M}} f(Y)|_{Y=0} + e^{X^M \partial_{Y^M}}(\alpha \omega^{KL}(X)\partial_{X^L})\partial_{Y^M}) f(Y)|_{Y=0} \\ =& e^{X^M \partial_{Y^M}} f(Y)|_{Y=0} + \alpha \omega^{KL}(X)\partial_{X^L}(\partial_{X^K}(e^{X^M \partial_{Y^M}}) f(Y)|_{Y=0} \end{aligned} \tag{4.17}$$

$$f(\hat{X}) = f(X) + \alpha \omega^{MN}(X) \partial_{X^M} f(X) \partial_{X^N} \tag{4.18}$$

As we have calculated now the function $f(\hat{X})$ up to the order α, we use definition (4.2) to calculate the star product at this order

$$f \star_1 g = \alpha \omega^{IJ}(X)\partial_{X^I} f(X) \partial_{X^J} g(X) + O(\alpha^2) \tag{4.19}$$

Consistency Condition:
The operator product is an associative product and therefore the commutators of our operators $\hat{X}^M$ obey the Jacobi-identity (2.11). This puts the consistency condition (2.13) on $\omega^{MN}(\hat{X})$, i.e.:

$$[\hat{X}^M, \omega^{NP}(\hat{X})] + cycl.(MNP) = 0 \tag{4.20}$$

It implies the quite complicated condition (2.19) on the expansion coefficients of ω^{MN}. We have derived in (2.26) that at lowest order it forces $\omega^{MN}(X)$ to be a Poisson-structure. At higher orders we obtain corrections, all captured in (2.19). We will recover the lower orders of these corrections, by checking (4.20) explicitly after every step. At order α equation 4.20 becomes

$$X^M \star_1 \omega^{NP} - \omega^{NP} \star_1 X^M + cycl.(MNP) = 0 \tag{4.21}$$

which gives

$$2\alpha \omega^{MJ}(X)\partial_{X^J} \omega^{NP}(X) + cycl.(MNP) = 0 \tag{4.22}$$

This is indeed the integrability condition for a Poisson-structure and thus agrees with (2.26).

4.1.3 Second Order Star Product

We first expand ω using the expansion of the function $f(\hat{X})$ in equation (4.18) in order to determine the constraints on the $\Gamma(X)$'s

$$\omega(\hat{X})^{MN} = \omega(X)^{MN} + \alpha\omega^{KL}(X)\partial_{X^K}\omega(X)^{MN}\partial_{X^L} + \mathcal{O}(\alpha^2) \tag{4.23}$$

which shows that we can not have the first order part of γ_n^M. The coordinate operators $\hat{X}^M$ are now expanded up to the second order in α as in the following.

$$\hat{X}_2^M = X^M + \alpha\omega^{MN}(X)\partial_{X^N} + \alpha^2\Gamma_0^{MNP}(X)\partial_{X^N}\partial_{X^P} \tag{4.24}$$

To use (4.4) we first calculate the commutator

$$\begin{aligned}
[B, A] =& -2\alpha^2\partial_{Y^M}\partial_{Y^N}\Gamma_0^{MNP}(X)\partial_{X^P} + O(\alpha^3).\\
[B, A] =& -\frac{\alpha^2}{3}\partial_{Y^M}\partial_{Y^N}\omega^{KM}(X)\partial_{X^K}\omega^{NL}(X)\partial_{X^L} + O(\alpha^3).
\end{aligned} \tag{4.25}$$

Using $\Gamma_0^{MNL} = -\frac{1}{6}\big(\omega^{LK}(X)\partial_K\omega^{MN}(X) + \omega^{NK}(X)\partial_K\omega^{ML}(X)\big)$ in (2.123) along with above commutator we can now calculate the star product up to second order. From equation (4.4) at second order in α we have

$$e^{A+B} = \frac{1}{2}e^A[B, A] + \frac{1}{2}e^A B^2 \tag{4.26}$$

$$\begin{aligned}
e^{\hat{X}^M\partial_{Y^M}} =& \frac{\alpha^2}{2}e^{X^M\partial_{Y^M}}(\partial_{Y^I}\partial_{Y^K}\omega^{KL}(X)\omega^{IJ}(X)\partial_{X^J}\partial_{X^L}) +\\
&+ \frac{\alpha^2}{3}e^{X^M\partial_{Y^M}}(\partial_{Y^I}\partial_{Y^K}\omega^{KL}(X)\partial_{X^L}\omega^{IJ}(X)\partial_{Y^J}) -\\
&- \frac{\alpha^2}{6}e^{X^M\partial_{Y^M}}\partial_{Y^I}(\omega^{KL}(X)\partial_{X^K}\omega^{IJ}(X) +\\
&+ \omega^{KJ}(X)\partial_{X^K}\omega^{IL}(X))\partial_{X^J}\partial_{X^L} + O(\alpha^3)
\end{aligned} \tag{4.27}$$

and now similar to the first order star product we use equation (4.18) to obtain $f(\hat{X})$ at order α^2

$$\begin{aligned}
f(\hat{X}) =& \Big(\frac{\alpha^2}{2}e^{X^M\partial_{Y^M}}(\partial_{Y^I}\partial_{Y^K}\omega^{KL}(X)\omega^{IJ}(X)\partial_{X^J}\partial_{X^L}) +\\
&+ \frac{\alpha^2}{3}e^{X^M\partial_{Y^M}}(\partial_{Y^I}\partial_{Y^K}\omega^{KL}(X)\partial_{X^L}\omega^{IJ}(X)\partial_{Y^J})\\
&- \frac{\alpha^2}{6}e^{X^M\partial_{Y^M}}\partial_{Y^I}(\omega^{KL}(X)\partial_{X^K}\omega^{IJ}(X) +\\
&+ \omega^{KJ}(X)\partial_{X^K}\omega^{IL}(X))\partial_{X^J}\partial_{X^L}\Big)f(Y)|_{Y=0} + O(\alpha^3)
\end{aligned} \tag{4.28}$$

$$\begin{aligned}
=& \Big(\frac{\alpha^2}{2}e^{X^M\partial_{Y^M}}(\partial_{Y^I}\partial_{Y^K}\omega^{KL}(X)\omega^{IJ}(X)\partial_{X^J}\partial_{X^L})\Big)f(Y)|_{Y=0} +\\
&+ \Big(\frac{\alpha^2}{3}e^{X^M\partial_{Y^M}}(\partial_{Y^I}\partial_{Y^K}\omega^{KL}(X)\partial_{X^L}\omega^{IJ}(X)\partial_{Y^J})\Big)f(Y)|_{Y=0} -\\
&- \Big(\frac{\alpha^2}{6}e^{X^M\partial_{Y^M}}\partial_{Y^I}(\omega^{KL}(X)\partial_{X^K}\omega^{IJ}(X) +\\
&+ \omega^{KJ}\partial_{X^K}\omega^{IL}(X))\partial_{X^J}\partial_{X^L}\Big)f(Y)|_{Y=0} + O(\alpha^3)
\end{aligned} \tag{4.29}$$

$$
\begin{aligned}
=&\frac{\alpha^2}{2}\omega^{KL}(X)\omega^{IJ}(X)\partial_{X^J}\partial_{X^L}(\partial_{X^I}\partial_{X^K}(e^{X^M\partial_{Y^M}}))f(Y)|_{Y=0}+\\
&+\frac{\alpha^2}{3}\omega^{KL}(X)\partial_{X^L}\omega^{IJ}(X)\partial_{X^J}(\partial_{X^I}\partial_{X^K}(e^{X^M\partial_{Y^M}}))f(Y)|_{Y=0}-\\
&-\frac{\alpha^2}{3}\omega^{KL}(X)\partial_{X^K}\omega^{IJ}(X)\partial_{X^J}\partial_{X^L}(\partial_{X^I}(e^{X^M\partial_{Y^M}}))f(Y)|_{Y=0}+O(\alpha^3) \qquad (4.30)\\
=&\frac{\alpha^2}{2}\omega^{KL}(X)\omega^{IJ}(X)\partial_{X^I}\partial_{X^K}f(X)\partial_{X^J}\partial_{X^L}+\\
&+\frac{\alpha^2}{3}\omega^{KL}(X)\partial_{X^L}\omega^{IJ}(X)\partial_{X^I}\partial_{X^K}f(X)\partial_{X^J}-\\
-&\frac{\alpha^2}{3}\omega^{KL}(X)\partial_{X^K}\omega^{IJ}(X)\partial_{X^I}f(X)\partial_{X^J}\partial_{X^L}+O(\alpha^3) \qquad (4.31)
\end{aligned}
$$

and using (4.2) we obtain

$$
\begin{aligned}
f\star_2 g=&\frac{\alpha^2}{2}\omega^{KL}(X)\omega^{IJ}(X)\partial_{X^I}\partial_{X^K}f(X)\partial_{X^J}\partial_{X^L}g(X)+\\
&+\frac{\alpha^2}{3}\omega^{KL}(X)\partial_{X^L}\omega^{IJ}(X)\partial_{X^I}\partial_{X^K}f(X)\partial_{X^J}g(X)-\\
&-\frac{\alpha^2}{3}\omega^{KL}(X)\partial_{X^K}\omega^{IJ}(X)\partial_{X^I}f(X)\partial_{X^J}\partial_{X^L}g(X)+O(\alpha^3) \qquad (4.32)
\end{aligned}
$$

Consistency Condition:
At order α^2 equation (4.20) becomes

$$X^M\star_2\omega^{NP}(X)-\omega^{NP}(X)\star_2 X^M+cycl.(MNP)=0 \qquad (4.33)$$

where we are using the fact that all symmetrized Γ vanish due to equation (4.1). Hence the consistency condition is fulfilled without putting additional restrictions on ω^{MN} at this order in α.

4.1.4 Third Order Star Product

Calculations go in the same way as in the last two sectons. Therefore, in this section we only give the main results. Now again we have to expand ω using the expansion of the function $f(\hat{X})$ in equation (4.31) in order to calculate the constraints on the $\Gamma(X)$'s.

$$
\begin{aligned}
\omega^{MN}(\hat{X})=&-\frac{\alpha^2}{3}(\omega^{IK}(X)\partial_{X^I}\omega^{LJ}(X)+\omega^{IL}(X)\partial_{X^I}\omega^{KJ}(X)\partial_{X^J})\omega^{MN}(X)\partial_{X^K}\partial_{X^L}+\\
&+\frac{\alpha^2}{3}\omega^{IJ}(X)\partial_{X^J}\omega^{KL}(X)\partial_{X^I}\partial_{X^K}\omega^{MN}(X)\partial_{X^L}+\\
&+\frac{\alpha^2}{2}\omega^{IJ}(X)\omega^{KL}(X)\partial_{X^I}\partial_{X^K}\omega^{MN}(X)\partial_{X^J}\partial_{X^L}+O(\alpha^3) \qquad (4.34)
\end{aligned}
$$

which shows that we can not have the zeroth and first order part of γ_n^M in equation (2.55). The coordinate operators $\hat{X}^M$ are now expanded to the third order in α as in the following.

$$
\begin{aligned}
\hat{X}_3^M=&X^M+\alpha\Gamma_0^{MN}(X)\partial_{X^N}+\alpha^2\Gamma_0^{MNP}(X)\partial_{X^N}\partial_{X^P}+\\
&+\alpha^3\Gamma_1^{MNP}(X)\partial_{X^N}\partial_{X^P}+\alpha^3\Gamma_0^{MNPQ}(X)\partial_{X^N}\partial_{X^P}\partial_{X^Q} \qquad (4.35)
\end{aligned}
$$

In order to determine $\Gamma_1^{MNP}(X)$ and $\Gamma_0^{MNPQ}(X)$ we calculate the commutator for $\hat{X}_3^M$ and compare the results with (4.34) at order α^3. As a result we obtain

$$\begin{aligned}\Gamma_1^{MNP}(X) =& \frac{1}{6}\omega^{IL}(X)\partial_{X^L}\omega^{JP}(X)\partial_{X^I}\partial_{X^J}\omega^{MN}(X)+\\ &+\frac{1}{6}\omega^{IL}(X)\partial_{X^L}\omega^{JN}(X)\partial_{X^I}\partial_{X^J}\omega^{MP}(X)\end{aligned} \tag{4.36}$$

and

$$\Gamma_0^{MNPQ}(X) = \frac{1}{12}\Big(G_3^{MNPQ} + G_3^{MPNQ} + G_3^{MQPN}\Big) \tag{4.37}$$

where

$$\begin{aligned}G_3^{MNPQ} =& \alpha^3\Bigg(\omega^{IP}(X)\omega^{JQ}\partial_{X^I}\partial_{X^J}\omega^{MN}(X) + \frac{1}{3}\omega^{PI}\partial_{X^I}\omega^{JQ}(X)\partial_{X^J}\omega^{MN}(X)+\\ &+\frac{1}{3}\omega^{QI}(X)\partial_{X^I}\omega^{PJ}(X)\partial_{X^J}\omega^{MN}(X) + \frac{1}{6}\omega^{NI}(X)\partial_{X^I}\omega^{JQ}(X)\partial_{X^J}\omega^{MP}(X)+\\ &+\frac{1}{6}\omega^{NI}(X)\omega^{JQ}(X)\partial_{X^I}\partial_{X^J}\omega^{MP}(X) + \frac{1}{6}\omega^{NI}(X)\partial_{X^I}\omega^{JP}(X)\partial_{X^J}\omega^{MQ}(X)+\\ &+\frac{1}{6}\omega^{NI}(X)\omega^{JP}(X)\partial_{X^I}\partial_{X^J}\omega^{MQ}(X) - \frac{1}{6}\omega^{MI}(X)\partial_{X^I}\omega^{JQ}(X)\partial_{X^J}\omega^{NP}(X)-\\ &-\frac{1}{6}\omega^{MI}(X)\omega^{JQ}(X)\partial_{X^I}\partial_{X^J}\omega^{NP}(X) - \frac{1}{6}\omega^{MI}(X)\partial_{X^I}\omega^{JP}(X)\partial_{J}\omega^{NQ}(X)-\\ &-\frac{1}{6}\omega^{MI}(X)\omega^{JP}(X)\partial_{X^I}\partial_{X^J}\omega^{NQ}(X) + \frac{1}{6}\omega^{IJ}(X)\partial_{X^J}\omega^{NP}(X)\partial_{J}\omega^{MQ}(X)+\\ &+\frac{1}{6}\omega^{IP}(X)\partial_{X^I}\omega^{NJ}(X)\partial_{X^J}\omega^{MQ}(X) + \frac{1}{6}\omega^{IJ}(X)\partial_{X^I}\omega^{NQ}(X)\partial_{X^J}\omega^{MP}(X)+\\ &+\frac{1}{6}\omega^{IQ}(X)\partial_{X^I}\omega^{NJ}(X)\partial_{X^J}\omega^{MP}(X) - \frac{1}{6}\omega^{IJ}(X)\partial_{X^J}\omega^{MP}(X)\partial_{X^J}\omega^{NQ}(X)-\\ &-\frac{1}{6}\omega^{IP}(X)\partial_{X^I}\omega^{MJ}(X)\partial_{X^J}\omega^{NQ}(X) - \frac{1}{6}\omega^{IJ}(X)\partial_{X^J}\omega^{MQ}(X)\partial_{X^J}\omega^{NP}(X)-\\ &-\frac{1}{6}\omega^{IQ}(X)\partial_{X^I}\omega^{MJ}(X)\partial_{X^J}\omega^{NP}(X)\Bigg)\end{aligned} \tag{4.38}$$

From equation (4.4) at third order in α we have

$$e^{A+B} = \frac{1}{6}[[B,A],A] + \frac{1}{3}[B,A]B + \frac{1}{6}B[B,A] + \frac{1}{6}B^3 \tag{4.39}$$

and calculations similar to the first and second order star product gives the following results

$$\begin{aligned}f\star_3 g =& \alpha^3\Bigg(\frac{1}{3}\omega^{NL}\partial_{X^L}\omega^{MK}\partial_{X^N}\partial_M\omega^{IJ}\Big(\partial_{X^I}f\partial_{X^J}\partial_K g - \partial_{X^I}g\partial_{X^J}\partial_{X^K}f\Big)+\\ &+\frac{1}{6}\omega^{NL}\partial_{X^N}\omega^{JM}\partial_{X^M}\omega^{IL}\Big(\partial_{X^I}\partial_{X^J}f\partial_{X^K}\partial_{X^L}g - \partial_{X^I}\partial_{X^J}g\partial_{X^K}\partial_{X^L}f\Big)+\\ &+\frac{1}{3}\omega^{LN}\partial_{X^L}\omega^{JM}\omega^{IK}\Big(\partial_{X^I}\partial_{X^J}f\partial_{X^K}\partial_{X^N}\partial_M g - \partial_{X^I}\partial_{X^J}g\partial_{XK}\partial_{X^N}\partial_M f\Big)+\\ &+\frac{1}{6}\omega^{JL}\omega^{IM}\omega^{KN}\Big(\partial_{X^I}\partial_{X^J}\partial_{X^K}f\partial_{X^L}\partial_{X^N}\partial_{X^M}g\Big)+\end{aligned}$$

$$+\frac{1}{6}\omega^{NK}\omega^{ML}\partial_{X^N}\partial_{X^M}\omega^{IJ}\Big(\partial_{X^I}f\partial_{X^J}\partial_{X^K}\partial_{X^L}g-\partial_{X^I}g\partial_{X^J}\partial_{X^K}\partial_{X^L}f\Big)\Bigg)$$

Consistency Condition:
At the order α^3 equation 4.20 becomes

$$X^R\star_3\omega^{ST}(X)-\omega^{RS}(X)\star_3 X^T+cycl.(RST)=0 \tag{4.40}$$

which gives

$$\begin{aligned}=&\alpha^3\Bigg(\frac{2}{3}\omega^{NL}(X)\partial_{X^L}\omega^{MK}(X)\partial_{X^N}\partial_{X^M}\omega^{RJ}(X)\partial_{X^J}\partial_{X^K}\omega^{ST}(X)+\\&+\frac{1}{3}\omega^{NK}(X)\omega^{ML}(X)\partial_{X^N}\partial_{X^M}\omega^{RJ}(X)\partial_{X^J}\partial_{X^K}\partial_{X^L}\omega^{ST}(X)\Bigg)+cycl.(RST)=0\end{aligned} \tag{4.41}$$

where again we are using the fact that all symmetrized Γ vanish due to equation (4.1).
Corrections to Star product:
In order to fulfil the consistency condition we have to correct the ω according to the following equation

$$\omega=\omega_0+\alpha^{n-1}\omega_{n-1} \tag{4.42}$$

At order α^3 this has to be

$$\omega=\omega_0+\alpha^2\omega_2 \tag{4.43}$$

So the star product becomes at order α^3

$$f\tilde{\star}_3 g=f\star_3 g+\alpha^3\partial_{X^I}f(X)\omega_2^{IJ}(X)\partial_{X^J}g(X) \tag{4.44}$$

where $\tilde{\star}$ means the corrected star product. Correction to the ω in equation (4.43) will change $\hat{X}_3$ in (4.35) to the following

$$\begin{aligned}\hat{X}_3^M=&X^M+\alpha\Gamma_0^{MN}(X)\partial_{X^N}+\alpha^2\Gamma_0^{MNP}(X)\partial_{X^N}\partial_{X^P}+\alpha^2\Gamma_2^{MN}(X)\partial_{X^N}+\\&+\alpha^3\Gamma_1^{MNP}(X)\partial_{X^N}\partial_{X^P}+\alpha^3\Gamma_0^{MNPQ}(X)\partial_{X^N}\partial_{X^P}\partial_{X^Q}\end{aligned} \tag{4.45}$$

where

$$\Gamma_2^{[MN]}(X)=2\omega_2^{MN}(X)\Rightarrow\Gamma_2^{MN}(X)=\omega_2^{MN}(X) \tag{4.46}$$

Now the ω_2 has to be determined from the consistency condition which in this case becomes

$$X^R\star_3\omega_0^{ST}(X)-\omega_0^{RS}\star_3 X^T+\alpha^2\Big(X^R\star_1\omega_2^{ST}(X)-\omega_2^{RS}(X)\star_1 X^T+cycl.(RST)\Big)=0 \tag{4.47}$$

$$\begin{aligned}=&\Bigg(2\omega_0^{RU}(X)\partial_{X^U}\omega_2^{ST}(X)+2\omega_2^{RU}(X)\partial_{X^U}\omega_0^{ST}(X)+\\&+\frac{2}{3}\omega_0^{NL}(X)\partial_{X^L}\omega_0^{MK}(X)\partial_{X^N}\partial_{X^M}\omega_0^{RJ}(X)\partial_{X^J}\partial_{X^K}\omega_0^{ST}(X)+\\&+\frac{1}{3}\omega_0^{NK}(X)\omega_0^{ML}(X)\partial_{X^N}\partial_{X^M}\omega_0^{RJ}(X)\partial_{X^J}\partial_{X^K}\partial_{X^L}\omega_0^{ST}(X)\Bigg)+cycl.(RST)\Big)=0\end{aligned} \tag{4.48}$$

where $\omega_2^{ST}(X)$ determined in [10] is given by

$$\begin{aligned}\omega_2^{ST}(X) =& c_1 \partial_{X^M} \omega_0^{NL}(X) \partial_{X^N} \omega_0^{MK}(X) \partial_{X^L} \partial_{X^K} \omega_0^{ST}(X) + \\ &+ c_2 \partial_{X^K} \omega_0^{SM}(X) \partial_{X^L} \omega_0^{TN}(X) \partial_{X^N} \partial_{X^M} \omega_0^{KL}(X) + \\ &+ c_3 \partial_{X^N} \partial_{X^K} \omega_0^{SM}(X) \partial_{X^M} \partial_{X^L} \omega_0^{TN}(X) \omega_0^{KL}(X)\end{aligned} \tag{4.49}$$

with $c_1 = -\frac{1}{12}, c_2 = 0$ and $c_3 = \frac{1}{6}$,

Chapter 5

Diffeomorphisms

In this section we construct the diffeomorphisms on the noncommutative space using the formality theorem. For description of the formality theorem see [25] [9]. In [26] diffeomorphism covariant star product is constructed with covariant derivatives. Here, we give quantum corrections to the classical transformations of the star product under formal Lie derivative. In the following two sections we give a brief review of the diffeomorphisms and deformed diffeomorphisms based on [19].

5.1 Diffeomorphisms on CS

On the coordinate space (CS) F^n, introduced in the section 2.1, diffeomorphisms are generated by vector fields ξ. These vector fields are represented as a linear combination of differential operators $\xi = \xi^M(X)\partial_M$ on F^n and under the Lie bracket multiplication these vector fields form a Lie algebra $\mathcal{G}$.

$$[\xi, \eta] = \xi \times \eta \tag{5.1}$$

where $\xi \times \eta$ is again a vector field defined by its action on the function $f(X) \in A$

$$(\xi \times \eta)(f(X)) = \big(\xi^M(\partial_M \eta^N)\partial_N - \eta^M(\partial_M \xi^N)\partial_N\big) f(X) \tag{5.2}$$

The Lie algebra of vector fields can be extended to the Universal enveloping algebra $\mathcal{U}(\mathcal{G})$ which is an associative algebra and can further be extended to the Hopf algebra. Comultiplication of the Hopf algebra (A.3) can be realized on vector fields as

$$\Delta(\xi) := \xi \otimes 1 + 1 \otimes \xi \tag{5.3}$$

under the infinitesimal coordinate transformations

$$X \longrightarrow X + \delta X \tag{5.4}$$

A function $f(X) \in A$ transforms as

$$f(X) \longrightarrow f(X) + \delta_\xi f(X) + \mathcal{O}(\xi^2) \tag{5.5}$$

where

$$\delta_\xi f(X) = \mathcal{L}_\xi f(X) = \xi^R \partial_R f(X) \tag{5.6}$$

$\mathcal{L}_\xi$ is the Lie derivative and the product of two functions transforms with the Leibniz rule

$$\delta_\xi(f(X)g(X)) = (\delta_\xi f(X))g(X) + f(X)(\delta_\xi g(X)) = \xi^M \partial_M(f(X)g(X)) \tag{5.7}$$

so that product of two functions transforms again as a function. Whereas for (co)vector fields transformation reads

$$\delta_\xi V^M(X) = \mathcal{L}_\xi V^M(X) = \xi^P(\partial_P V^M(X)) + (\partial_P \xi^M)V^P(X) \tag{5.8}$$

$$\delta_\xi V_M(X) = \mathcal{L}_\xi V_M(X) = \xi^P(\partial_P V_M(X)) - (\partial_P \xi_M)V^P(X) \tag{5.9}$$

This can be easily generalized to arbitrary tensor fields

$$\begin{aligned}\delta_\xi T^{M_1\dots M_k}_{N_1\dots N_k}(X) =& \mathcal{L}_\xi T^{M_1\dots M_k}_{N_1\dots N_k}(X) = \xi^P(\partial_P T^{M_1\dots M_k}_{N_1\dots N_k}(X)) + (\partial_P \xi^{M_1})T^{P\dots M_k}_{N_1\dots N_k}(X) + \\ &+ \dots + \partial_P \xi^{M_k}(T^{M_1\dots P}_{N_1\dots N_k}(X)) - (\partial_{N_1}\xi^P)T^{M_1\dots M_k}_{P\dots N_k}(X) - \\ &- \dots - \xi^P(\partial_{N_k} T^{M_1\dots M_k}_{N_1\dots P}(X))\end{aligned} \tag{5.10}$$

The infinitesimal transformations δ_ξ satisfy the relation

$$[\delta_\xi, \delta_\eta] = \delta_{\xi\times\eta} \tag{5.11}$$

and the Hopf algebra A $\mathcal{U}(\mathcal{G})$ acts on a product of fields via comultiplication

$$\Delta(\delta_\xi) = \delta_\xi \otimes 1 + 1 \otimes \delta_\xi \tag{5.12}$$

5.2 Deformed Diffeomorphisms on NCS

Let us denote the infinitesimal transformations on the non-commutative coordinate space (NCS) introduced in chapter 2.2 by $\hat{\delta}_\xi$. The Lie bracket of such transformations is left undeformed

$$[\hat{\delta}_\xi, \hat{\delta}_\eta] = \hat{\delta}_{\xi\times\eta} \tag{5.13}$$

while we deform the comultiplication (5.12)

$$\Delta(\hat{\delta}_\xi) = e^{-\alpha\omega^{MN}\hat{\partial}_M\otimes\hat{\partial}_N}(\hat{\delta}_\xi \otimes 1 + 1 \otimes \hat{\delta}_\xi)e^{\alpha\omega^{MN}\hat{\partial}_M\otimes\hat{\partial}_N} \tag{5.14}$$

where $\hat{\partial}_M = \partial_{\hat{X}^M}$ and $[\hat{\partial}_M, \hat{\delta}_\xi] = \hat{\delta}_{\partial_{(M\xi)}}$ and ω is constant. The deformed comultiplication (5.14) reduces to the undeformed comultiplication (5.12) in the limit $\omega \to 0$.

Now to represent the deformed Hopf algebra $\mathcal{U}(\hat{\mathcal{G}})$ on the non-commutative algebra $\hat{A}$ we introduce the differential operator

$$\hat{\mathcal{D}}_\xi := \sum_{n=0}^{\infty} \frac{1}{n!}\alpha\omega^{M_1N_1}\dots\omega^{M_kN_k}(\hat{\partial}_{M_1}\dots\hat{\partial}_{M_k}\xi^P)\hat{\partial}_P\hat{\partial}_{N_1}\dots\hat{\partial}_{N_k} \tag{5.15}$$

Then

$$[\hat{\mathcal{D}}_\xi, \hat{\mathcal{D}}_\eta] = \hat{\mathcal{D}}_{\xi\times\eta} \tag{5.16}$$

and, therefore, we can define the transformation $\hat{\delta}_\xi$ for functions $f(\hat{X}) \in \hat{A}$ by

$$\hat{\delta}_\xi f(\hat{X}) = \hat{\mathcal{D}}_\xi f(\hat{X}) \tag{5.17}$$

See [18] for details. $\hat{\mathcal{D}}_\xi$ act on product of two functions via deformed Liebniz rule

$$\hat{\mathcal{D}}_\xi(f(\hat{X})g(\hat{X})) = m \circ \left(e^{-\alpha\omega^{MN}\hat{\partial}_M \otimes \hat{\partial}_N}(\hat{\delta}_\xi \otimes 1 + 1 \otimes \hat{\delta}_\xi)e^{\alpha\omega^{MN}\hat{\partial}_M \otimes \hat{\partial}_N} f(\hat{X}) \otimes g(\hat{X})\right) \tag{5.18}$$

where we have used (A.1). We can see that the deformed Leibniz rule of the differential operator $\hat{\mathcal{D}}_\xi$ is the same as induced by the comultiplication (5.14). On the product of two functions it looks as

$$\hat{\delta}_\xi(f(\hat{X})g(\hat{X})) = e^{-\alpha\omega^{MN}\hat{\partial}_M \otimes \hat{\partial}_N}(\hat{\delta}_\xi \otimes 1 + 1 \otimes \hat{\delta}_\xi)e^{\alpha\omega^{MN}\hat{\partial}_M \otimes \hat{\partial}_N}(f(\hat{X})g(\hat{X})) \tag{5.19}$$

So, therefore, deformed Hopf algebra $\mathcal{U}(\hat{\mathcal{G}})$ is represented by the differential operator $\hat{\mathcal{D}}_\xi$. Where as on vector and tensor fields it is represented by

$$\hat{\delta}_\xi \hat{V}^M(X) = \hat{\mathcal{L}}_\xi \hat{V}^M(X) = (\hat{\mathcal{D}}_\xi \hat{V}^M(X)) + (\hat{\mathcal{D}}_\xi \xi^M)\hat{V}^P(X) \tag{5.20}$$

$$\hat{\delta}_\xi \hat{V}_M(X) = (\hat{\mathcal{D}}_\xi \hat{V}_M(X)) - (\hat{\mathcal{D}}_\xi \xi_M)\hat{V}_P(X) \tag{5.21}$$

and for arbitrary tensor fields

$$\begin{aligned}\hat{\delta}_\xi \hat{T}^{M_1\ldots M_k}_{N_1\ldots N_k}(X) =& \hat{\mathcal{L}}_\xi \hat{T}^{M_1\ldots M_k}_{N_1\ldots N_k}(X) = (\hat{\mathcal{D}}_\xi \hat{T}^{M_1\ldots M_k}_{N_1\ldots N_k}(X)) + (\hat{\mathcal{D}}_{(\partial_P \xi^{M_1})})\hat{T}^{P\ldots M_k}_{N_1\ldots N_k}(X) + \\ &+ \cdots + \hat{\mathcal{D}}_{(\partial_P \xi^{M_k})}(\hat{T}^{M_1\ldots P}_{N_1\ldots N_k}(X)) - (\hat{\mathcal{D}}_{(\partial_P \xi^{N_1})})\hat{T}^{M_1\ldots M_k}_{P\ldots N_k}(X) - \\ &- \cdots - \hat{\mathcal{D}}_{(\partial_P \xi^{N_k})}\hat{T}^{M_1\ldots M_k}_{N_1\ldots P}(X))\end{aligned} \tag{5.22}$$

5.3 Diffeomorphisms of the Star Product

Position dependent Poisson structure $\omega(X)$ also transform tensorially

$$\omega^{MN}(X) \longrightarrow \omega^{MN}(X) + \delta_\xi \omega^{MN}(X) + \mathcal{O}(\xi^2) \tag{5.23}$$

where the transformation is

$$\delta_\xi \omega^{MN}(X) = \mathcal{L}_\xi \omega^{MN}(X) = \xi^R \partial_R \omega^{MN}(X) - \partial_R \xi^M \omega^{RN}(X) - \omega^{MR}(X)\partial_R \xi^N \tag{5.24}$$

However, the star product does not transform as a function

$$\delta_\xi(f \star g)(X) = \xi^R \partial_R (f \star g)(X) \neq (\xi^R \partial_R f) \star g(X) + f \star (\xi^R \partial_R g)(X) \tag{5.25}$$

According to Kontsevich's formality theorem this can be repaired by a similarity transformation on C^∞

$$f(X) \longrightarrow f(X) + D_\xi f(X) + \mathcal{O}(\xi^2) \tag{5.26}$$

which is of the order $\omega^2(X)$ as the first order expression of star product is manifestly covariant. Here the Poisson structure $\omega(X)$ continues to transform covariantly. D_ξ is the sum of classical and quantum transformations. So that the transformation of $f(X)$ is modified by a "quantum correction" D'_ξ

$$D_\xi f(X) = \delta_\xi f(X) + D'_\xi f(X) \tag{5.27}$$

where $D_\xi(\omega^2(X))$ is a formal sum of the differential operators of unbounded order. Since the total infinitesimal change of $f \star g(X)$ is a sum of three terms, namely the variations of the $f(X)$

and $g(X)$ and the change $\star_\xi$ of the star product that is due to the transformation (5.24) of the $\omega(X)$, we impose

$$D_\xi(f \star g)(X) = (D_\xi f(X)) \star g + f(X) \star (D_\xi g) + f(X) \star_\xi g(X) \tag{5.28}$$

In order to split off the (classically) covariant part of the transformation of the building blocks of the star product we define the operator

$$\Omega_\xi = \delta_\xi - \mathcal{L}_\xi^{formal} \tag{5.29}$$

The formal Lie derivative $\mathcal{L}_\xi^{formal}$ acts on partial derivatives of $f(X)$ and $\omega(X)$ according to their index structure as if they were tensors. In particular

$$\Omega_\xi(\partial_M f(X)) = \delta_\xi(\partial_M f(X)) - \mathcal{L}_\xi^{formal}(\partial_M f(X)) \tag{5.30}$$
$$= \partial_M \delta_\xi f(X) + (\partial_M \delta_\xi) f(X) - \mathcal{L}_\xi^{formal}(\partial_M f(X)) = 0 \tag{5.31}$$

and repeated differentiation of (5.6) gives

$$\Omega_\xi(\partial_M \partial_N f(X)) = \delta_\xi\big(\partial_M \partial_N f(X)\big) - \mathcal{L}_\xi^{formal}\big(\partial_M \partial_N f(X)\big) \tag{5.32}$$
$$= \partial_M \partial_N \delta_\xi f(X) + \partial_M \partial_N \xi^R \partial_R f(X) + 2\partial_{(M} \xi^R \partial_{N)} \partial_R f(X) - $$
$$- \mathcal{L}_\xi^{formal}\big(\partial_M \partial_N f(X)\big) \tag{5.33}$$
$$\Omega_\xi(\partial_M \partial_N f(X)) = \partial_M \partial_N \xi^R \partial_R f(X) \tag{5.34}$$

and so on. While repeated differentiation of (5.24) gives

$$\Omega_\xi(\partial_P \omega^{MN}(X)) = -\partial_P \partial_K \xi^M \omega^{KN}(X) - \omega^{MK}(X) \partial_P \partial_K \xi^N \tag{5.35}$$
$$\Omega_\xi(\partial_P \partial_Q \omega^{MN}(X)) = \partial_R \omega^{MN}(X) \partial_P \partial_Q \xi^R - 2\partial_R \partial_{(P} \xi^M \partial_{Q)} \omega^{RN} - 2\partial_{(P} \omega^{MR} \partial_{Q)} \partial_R \xi^N -$$
$$- \partial_P \partial_Q \partial_R \xi^M \omega^{RN} - \omega^{MR} \partial_P \partial_Q \partial_R \xi^N \tag{5.36}$$

Now since D_ξ on $f \star g(X)$ should actually read as

$$D_\xi(f \star g(X)) = (\mathcal{L}_\xi + D'_\xi)(f \star g(X)) \tag{5.37}$$

where

$$\mathcal{L}_\xi = \xi^R \partial_R (f \star g(X)) \tag{5.38}$$

The covariance condition (5.28) implies that

$$D'_\xi(f \star g(X)) - ((D'_\xi f) \star g)(X) - (f \star (D'_\xi g))(X) =$$
$$= \delta_\xi f \star g + f \star \delta_\xi g + f \star_\xi g - \xi^R \partial_R (f \star g(X)) \tag{5.39}$$
$$= \delta_\xi f \star g + f \star \delta_\xi g + f \star_\xi g - \delta_\xi(f \star g(X)) + \Omega_\xi(f \star g(X)) \tag{5.40}$$
$$= \Omega_\xi(f \star g(X)) \tag{5.41}$$

as the classical variation of the of the $\delta_\xi(f \star g(X))$ is the sum of the first three terms on the r.h.s of the (5.40). So, therefore, equation (5.37) can be written in the most general form as

$$\sum_{m=0}^{n}\sum_{p=0}^{2m} D_\xi^{(m)}(\partial)^{p+1}(f\star g(X))^{n-m} = \mathcal{L}_\xi(f\star g(X))^n + \sum_{m=0}^{n} D_\xi'^{(m)}(f\star g(X))^{n-m} \tag{5.42}$$

and this becomes

$$\begin{aligned}\sum_{m=0}^{n}\sum_{p=0}^{2m} D_\xi^{(m)}(\partial)^{p+1}(f\star g(X))^{n-m} &= \mathcal{L}_\xi(f\star g(X))^n + \Omega_\xi(f\star g(X))^n + \\ &+ \sum_{m=0}^{n}((D_\xi'^m f)\star g(X))^{n-m} + \sum_{m=0}^{n}(f\star(D_\xi'^m g(X))^{n-m}\end{aligned} \tag{5.43}$$

As by now we already know how to calculate the star product order by order. We will use it to calculate the diffeomorphisms at every order.

5.4 Zeroth and First Order Diffeomorphisms

At zeroth and first order the star product transform tensorially and does not need any quantum corrections. i.e

$$(f\star g(X))^{(0)} = fg \tag{5.44}$$

and

$$(f\star g(X))^{(1)} = \alpha\omega^{MN}(X)\partial_M f\partial_N g \tag{5.45}$$

5.5 Second Order Diffeomorphisms

At second order the formal Lie derivative of the star product (4.32) is

$$\mathcal{L}_\xi\big((f\star g)(X)\big)^{(2)} = \xi^R\partial_R((f\star g)(X)\big)^{(2)} \tag{5.46}$$

$$\begin{aligned}&= \xi^R\partial_R\Bigg(\frac{\alpha^2}{2}\omega^{KL}(X)\omega^{IJ}(X)\partial_{X^I}\partial_{X^K}f(X)\partial_{X^J}\partial_{X^L}g(X) + \\ &+ \frac{\alpha^2}{3}\omega^{KL}(X)\partial_{X^L}\omega^{IJ}(X)\partial_{X^I}\partial_{X^K}f(X)\partial_{X^J}g(X) - \\ &- \frac{\alpha^2}{3}\omega^{KL}(X)\partial_{X^K}\omega^{IJ}(X)\partial_{X^I}f(X)\partial_{X^J}\partial_{X^L}g(X) + O(\alpha^3)\Bigg)\end{aligned} \tag{5.47}$$

We use (5.34) and (5.36) to calculate $\big((f\star g)(X)\big)^{(2)}$

$$\begin{aligned}\Omega_\xi\big((f\star g)(X)\big)^{(2)} &= \frac{\alpha^2}{2}\omega^{KL}(X)\omega^{IJ}(X)\partial_I\partial_K\xi^R\Big(\partial_J\partial_L f\partial_R g + \partial_R f\partial_J\partial_L g\Big) + \\ &+ \frac{\alpha^2}{3}\omega^{KL}(X)\big(\omega^{RJ}\partial_R\partial_L\xi^I + \omega^{IR}(X)\partial_R\partial_L\xi^J\big)\Big(\partial_I\partial_K f\partial_J g + \partial_I f\partial_J\partial_K g\Big) + \\ &+ \frac{\alpha^2}{3}\omega^{KL}(X)\partial_K\omega^{IJ}(X)\partial_J\partial_L\xi^R\Big(\partial_R f\partial_I g + \partial_I f\partial_R g\Big)\end{aligned} \tag{5.48}$$

So, therefore, we can write

$$D_\xi^{\prime(2)} =\mathcal{J}^{JLR}\partial_J\partial_L\partial_R + \mathcal{K}^{IKJ}\partial_I\partial_L\partial_J + \mathcal{L}^{RI}\partial_R\partial_I \tag{5.49}$$

where

$$\mathcal{J}^{JLR} =\frac{\alpha^2}{2}\omega^{KL}(X)\omega^{IJ}(X)\partial_I\partial_K\xi^R \tag{5.50}$$

$$\mathcal{K}^{IKJ} =\frac{\alpha^2}{3}\omega^{KL}(X)\big(\omega^{RJ}\partial_R\partial_L\xi^I + \omega^{IR}(X)\partial_R\partial_L\xi^J\big) \tag{5.51}$$

$$\mathcal{L}^{RI} =\frac{\alpha^2}{3}\omega^{KL}(X)\partial_K\omega^{IJ}(X)\partial_J\partial_L\xi^R \tag{5.52}$$

Ansatz for the Gauge Operator: The most general operator that gives the second order ω terms when it acts on $f \star g(X)$ is given by

$$\begin{aligned} D_\xi =& D_\xi^{(0)P}\partial_P + D_\xi^{(1)P}\partial_P + D_\xi^{(1)PQ}\partial_P\partial_Q + D_\xi^{(1)PQR}\partial_P\partial_Q\partial_R + \\ &+ D_\xi^{(2)P}\partial_P + D_\xi^{(2)PQ}\partial_P\partial_Q + D_\xi^{(2)PQR}\partial_P\partial_Q\partial_R + \\ &+ D_\xi^{(2)PQRS}\partial_P\partial_Q\partial_R\partial_S + D_\xi^{(2)PQRST}\partial_P\partial_Q\partial_R\partial_S\partial_T \end{aligned} \tag{5.53}$$

We have

$$D_\xi^{(0)P} = \xi^P \tag{5.54}$$

$$D_\xi^{(1)P} = \partial_R\omega^{PQ}\partial_Q\xi^R \tag{5.55}$$

$$D_\xi^{(1)PQ} = \omega^{PR}\partial_R\xi^Q \tag{5.56}$$

$$\begin{aligned} D_\xi^{(2)P} =& \partial_R\omega^{MN}\partial_M\omega^{RQ}\partial_N\partial_Q\xi^P + \partial_Q\partial_R\omega^{MN}\partial_N\omega^{PQ}\partial_M\xi^R + \\ &+ \partial_Q\omega^{MN}\partial_R\partial_N\omega^{PQ}\partial_M\xi^R + \omega^{MN}\partial_N\partial_R\omega^{PQ}\partial_M\partial_Q\xi^R + \\ &+ \partial_R\omega^{MN}\partial_M\omega^{PQ}\partial_N\partial_Q\xi^R + \partial_R\omega^{PN}\partial_M\omega^{RQ}\partial_N\partial_Q\xi^M \end{aligned} \tag{5.57}$$

$$\begin{aligned} D_\xi^{(2)(PQ)} =& \omega^{MN}\partial_M\omega^{RP}\partial_N\partial_R\xi^Q + \partial_R\omega^{MN}\partial_N\omega^{RP}\partial_M\xi^Q \\ &+ \partial_M\omega^{PN}\partial_R\omega^{MQ}\partial_N\xi^R + \partial_N\partial_R\omega^{MP}\partial_M\omega^{NQ}\xi^R \\ &+ \omega^{MP}\partial_N\omega^{RQ}\partial_M\partial_R\xi^N + \omega^{MP}\partial_M\partial_R\omega^{NQ}\partial_N\xi^R \end{aligned} \tag{5.58}$$

$$D_\xi^{(2)(PQR)} =\omega^{PM}\partial_M\omega^{QN}\partial_N\xi^R + \partial_N\omega^{PM}\partial_M\omega^{QN}\xi^R + \omega^{PM}\omega^{QN}\partial_M\partial_N\xi^R \tag{5.59}$$

Where $D_\xi^{(1)PQR} = 0, D_\xi^{(2)PQRS} = 0, D_\xi^{(2)PQRST} = 0.$

We have put the detailed calculation of the equation (5.43) in to the appendix B. At third and higher orders the star diffeomorphisms can be computed by straight forward extension of the calculations at second order star diffeomorphisms. The above anstaz for the gauge operator can also be represented graphically with Konstsevich-type graphs with D acting on the star product as in the following. See[25] [27] [28] [29] [30] [31]for details of graphical representation of the star product.

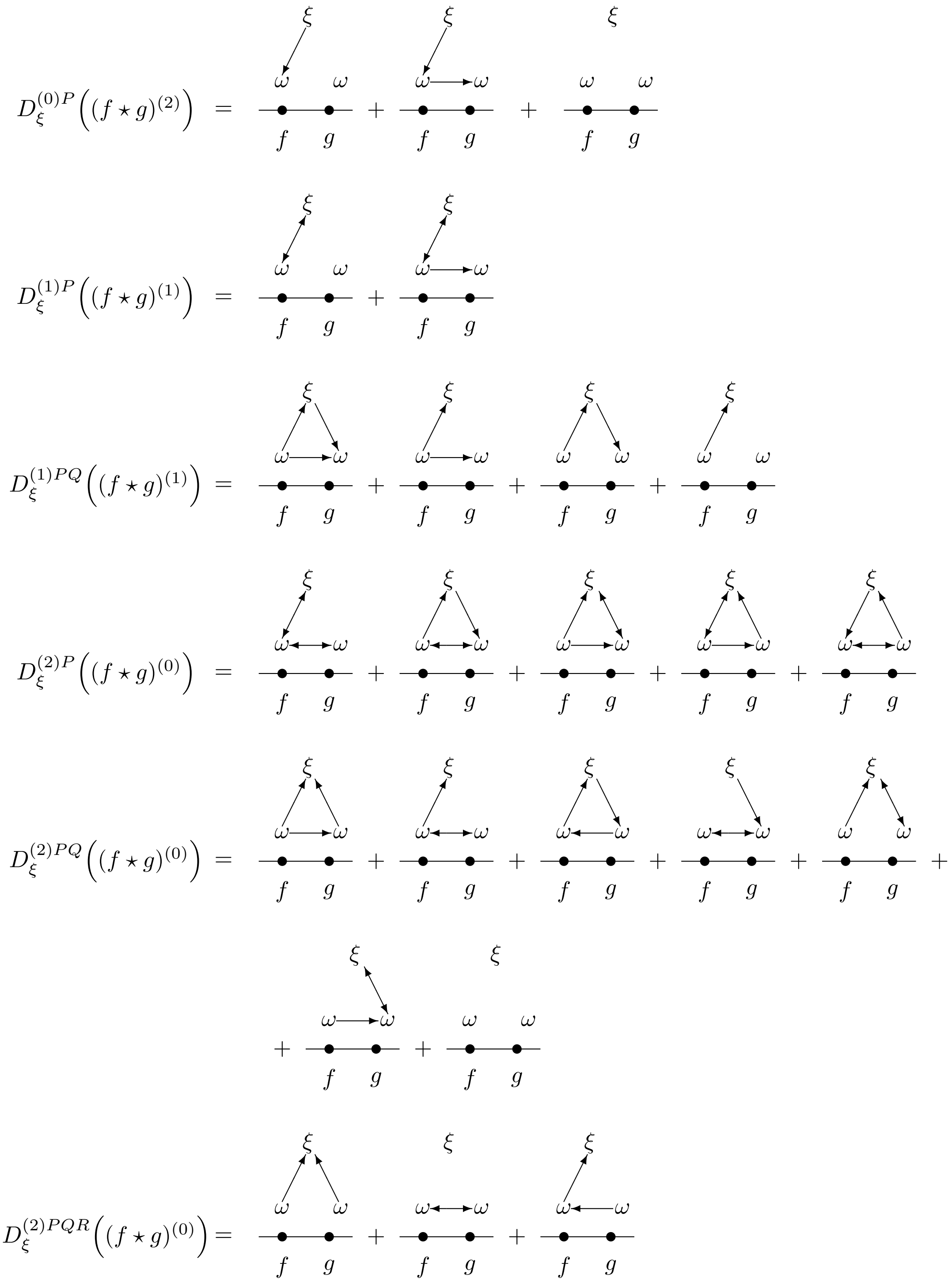

Figure 5.1:Graphical representation of the gauge operator acting on the star product up to second order. Where we have not shown the derivatives on functions.

Chapter 6

Star Product On Non(Anti)Commutative Superspace

All the calculations so far have been performed as if our coordinates X^M were just bosonic. However, all these calculations can be reinterpreted as superspace equations, if we make use of the conventions developed in [32]. When choosing a capital index M for the coordinates we therefore had already in mind to split it into bosonic and fermionic part. We will write the star product using functions of graded coordinates and then obtain its different components on the non(anti)commutative superspace by using graded Einstein summation convention described below.

6.1 Graded Structures

6.1.1 Graded Einstein Summation Convention

As we know, superspace $X^M = (x^m, \theta^\mu, \bar{\theta}^{\bar{\mu}})$ has bosonic and fermionic components where bosonic coordinates commute among themselves and with fermionic coordinates

$$x^m x^n = x^n x^m \tag{6.1}$$
$$x^m \theta^\nu = \theta^\nu x^m \tag{6.2}$$
$$x^m \bar{\theta}^{\bar{\nu}} = \bar{\theta}^{\bar{\nu}} x^m \tag{6.3}$$

and fermionic coordinates anti-commute among themselves

$$\theta^\mu \theta^\nu = -\theta^\nu \theta^\mu \tag{6.4}$$
$$\bar{\theta}^{\bar{\mu}} \bar{\theta}^{\bar{\nu}} = -\bar{\theta}^{\bar{\nu}} \bar{\theta}^{\bar{\mu}} \tag{6.5}$$
$$\theta^\mu \bar{\theta}^{\bar{\nu}} = -\bar{\theta}^{\bar{\nu}} \theta^\mu \tag{6.6}$$

Here we assign a grading to the index of our superspace coordinates according to following rule

$$|X^M| \equiv |M| \equiv \begin{cases} 0 & \text{for } M = m \\ 1 & \text{for } M = \mu, \bar{\mu} \end{cases} \tag{6.7}$$

We can also assign a grading to the body (rumpf) of X^M by the following rule. It is helpful when superforms are involved. Assume that we have given the following one-form

$$a^M = dX^M \tag{6.8}$$

The exterior derivative d turns commuting functions into anti-commuting one-forms and therefore carries itself the grading 1. It is thus clear that the grading of a^M differs by 1 from the grading of X^M. As we want the grading that is assigned to the index M to stay unique, we have to think of the additional grading to sit not in the index, but in the rumpf of a^M:

$$|a^M| \equiv |a| + |M| \equiv 1 + |M| \equiv \begin{cases} 1 & \text{for } M = m \\ 0 & \text{for } M = \mu, \bar{\mu} \end{cases} \tag{6.9}$$

We can use now (6.7) to combine (6.1), (6.2), (6.3) and (6.4), (6.5),(6.6) into a single equation $X^M X^N = (-)^{MN} X^N X^M$ for the graded commuting superspace coordinates X^M. If we forget about the specific example (6.8) and take two general objects a^M and b^M with arbitrary rumpf-grading we can further generalize this equation to arbitrary graded commuting objects:

$$a^M b^N = (-)^{(a+M)(b+N)} b^N a^M \tag{6.10}$$

There are two possibilities while defining Graded Einstein Summation Convetion. $a^M b_M$ (Northwest-Southeast NW for short) and $a_M b^M$ (Northeast-Southwest NE for short).

In NW convention contraction is chosen in such a way that there is no additional sign if the contraction of indices is from upper left to the lower right; while in NE there is no sign for contracting from lower right to the upper right. One can of course choose mixed convention different for different index set.

Here now we grade the Einstein summation convention: repeated indices in opposite vertical position are summed over their complete range taking into account the additional signs corresponding to either NE,NW or mixed conventions.

$$a^M b_M = \begin{cases} \sum_M (-)^{bM} a^M b_M & \text{for NW,} \\ \sum_M (-)^{bM+M} a^M b_M & \text{for NE} \end{cases} \tag{6.11}$$

$$b_M a^M = \begin{cases} \sum_M (-)^{aM+M} b_M a^M & \text{for NW,} \\ \sum_M (-)^{aM} b_M a^M & \text{for NE} \end{cases} \tag{6.12}$$

6.1.2 Graded Equal Sign

If we have a naked index in the equation it can produce some bad signs. To avoid such a situation we introduce the graded equal sign which states that the equality holds if for each term a mismatch in some common ordering of the indices is taken care of by an appropriate sign factor. Consider the following equation where we have considered only bosonic rumpfs for simplicity.

$$X^M Y^N Y_N X_M - X^M Y^N X_M Y_N = 0 \tag{6.13}$$
$$\forall \text{graded commuting} X^M, Y^N, Y_N, X_M$$
$$\Rightarrow X^M Y^N (Y_N X_M - X_M Y_N) = 0 \tag{6.14}$$
$$\forall \text{graded commuting} X^M, Y^N, Y_N, X_M$$

In order to avoid wrong assumptions like $Y_N X_M - X_M Y_N = 0$, we introduce the graded equal sign in the following.

$$Y_N X_M - X_M Y_N =_g 0 :\Longleftrightarrow Y_N X_M - (-)^{MN} X_M Y_N = 0 \tag{6.15}$$

We can now include the graded rumpfs in the definition of the graded equal sign.

$$\begin{aligned}
& a^M b^N b_N a_M - (-)^{ab} a^M b^N a_M b_N = 0 && (6.16)\\
& \quad \forall \text{graded commuting} a^M, b^N, b_N, a_M \\
\Rightarrow & a^M b^N (b_N a_M - (-)^{ab} a_M b_N) = 0 && (6.17)\\
& \quad \forall \text{graded commuting} a^M, b^N, b_N, a_M
\end{aligned}$$

which suggests the following definition:

$$b_N a_M - (-)^{ab} a_M b_N =_g 0, :\Longleftrightarrow (-)^{Na} b_N a_M - (-)^{MN+Mb} (-)^{ab} a_M b_N = 0 \tag{6.18}$$

All the equations in the previous chapters have to be understood with this graded equal sign $=_g$. In order to keep the notation simpler, we have not put this subscript in these chapters. This should not lead to any ambiguity, because (as just mentioned) all these equations should be understood with graded equal sign (and graded summation convention).

6.2 Star Product on Non(Anti)Commutative Superspace

Here, we give both the Moyal-Weyl (spacetime independent Poisson structure) and the Kontsevich (spacetime dependent Poisson structure) star products up to the second order of the expansion parameter α. Both zeroth order star products coincide with the ordinary point wise product and at the first order of α both star products are of similar form, therefore, we give results only for the Moyal-Weyl star product. At second order the Kontsevich star product has more terms than the Moyal-Weyl star product so, therefore, we give results for only the Kontsevich star product.

6.2.1 Moyal-Weyl Star Product

For spacetime independent ω the star product is the Moyal-Weyl star product given by the closed formula.

$$f \star_M g = \left(e^{\alpha\omega^{MN}\partial_M\partial_N}\right) f(X) g(X) \tag{6.19}$$

Expanding (6.19) up to second order in α, we obtain

$$f \star_M g = fg + \alpha\omega^{MN}\partial_M f(X)\partial_N g(X) + \alpha^2\omega^{MN}\omega^{KL}\partial_M\partial_K f(X)\partial_N\partial_L g(X) + \mathcal{O}(\alpha^3) \tag{6.20}$$

Now we can use the summation convetions in equations (6.11) and (6.12) to write out explicitly the different parts of the star product on the non(anti)commutative superspace. Let us do it order by order in α.

Zeroth Order Moyal-Weyl Star Product:

$$f \star_{M_0} g = f(X) g(X) \tag{6.21}$$

which coincides with the ordinary pointwise product and ω and derivatives of the functions are absent so our Einstein summation convention are not applicable in this trivial case.

First Order Moyal-Weyl Star Product:

$$f \star_{M_1} g = \alpha\omega^{MN}\partial_M f(X)\partial_N g(X) \tag{6.22}$$

Let us split the first order Moyal-Weyl star product (6.22) on non-anti-commutative superspace to get bosonic, fermionic and mixed components

$$\begin{aligned} f \star_{M_1} g =& \alpha\omega^{mn}\partial_{X^m} f(X)\partial_{X^n} g(X) + \alpha\omega^{m\nu}\partial_{X^m} f(X)\partial_{X^\nu} g(X) + \\ &+ \alpha\omega^{\mu n}\partial_{X^\mu} f(X)\partial_{X^n} g(X) + \alpha\omega^{\mu\nu}\partial_{X^\mu} f(X)\partial_{X^\nu} g(X) + \\ &+ \alpha\omega^{m\bar{\nu}}\partial_{X^m} f(X)\partial_{X^\nu} g(X) + \alpha\omega^{\bar{\mu}n}\partial_{X^{\bar{\mu}}} f(X)\partial_{X^n} g(X) + \\ &+ \alpha\omega^{\mu\bar{\nu}}\partial_{X^\mu} f(X)\partial_{X^{\bar{\nu}}} g(X) + \alpha\omega^{\bar{\mu}\nu}\partial_{X^{\bar{\mu}}} f(X)\partial_{X^\nu} g(X) + \\ &+ \alpha\omega^{\bar{\mu}\bar{\nu}}\partial_{X^{\bar{\mu}}} f(X)\partial_{X^{\bar{\nu}}} g(X) \end{aligned} \tag{6.23}$$

We apply now the summation convetions in equations (6.11) to get the explicit star product on superspace

$$\begin{aligned} f \star_{M_1} g \overset{NW}{=} & \sum_{m,n} \alpha\omega^{mn}\partial_{X^m} f(X)\partial_{X^n} g + \sum_{m,\nu} \alpha\omega^{m\nu}\partial_{X^m} f(X)\partial_{X^\nu} g + \\ &+ \sum_{\mu,n} \alpha\omega^{\mu n}\partial_{X^\mu} f(X)\partial_{X^n} g(X) - \sum_{\mu,\nu} \alpha\omega^{\mu\nu}\partial_{X^\mu} f(X)\partial_{X^\nu} g(X) + \\ &+ \sum_{m,\bar{\nu}} \alpha\omega^{m\bar{\nu}}\partial_{X^m} f(X)\partial_{X^{\bar{\nu}}} g(X) + \sum_{\bar{\mu},n} \omega^{\bar{\mu}n}\partial_{X^{\bar{\mu}}} f(X)\partial_{X^n} g\partial_{X^\nu} g(X) - \\ &- \sum_{\mu,\bar{\nu}} \omega^{\mu\bar{\nu}}\partial_{X^\mu} f(X)\partial_{X^{\bar{\nu}}} g(X) - \sum_{\bar{\mu},\nu} \alpha\omega^{\bar{\mu}\nu}\partial_{X^{\bar{\mu}}} f(X)\partial_{X^\nu} g(X) - \\ &- \sum_{\bar{\mu},\bar{\nu}} \alpha\omega^{\bar{\mu}\bar{\nu}}\partial_{X^{\bar{\mu}}} f(X)\partial_{X^{\bar{\nu}}} g(X) \end{aligned} \tag{6.24}$$

6.2.2 Kontsevich Star Product

The Kontsevich star product belongs to a very large equivalence class of star products. The different star products in this class are related by gauge transformations[33]. In the chapter 4 we have calculated the associative star product for spacetime dependent ω. Here we just give its explicit form on the non(anti)commutative superspace.

Zeroth and First Order Star Products: Zeroth order star product just coincides with the Moyal-Weyl star product we have calculated in the last section.i.e.

$$f \star_{K_0} g = f \star_{M_0} g \tag{6.25}$$

and for the first order Kontsevich star product ω depends on the spacetime X but since it is undifferentiated at this order the first order Kontsevich star product is of the same form of the Moyal-Weyl star product.

$$f \star_{K_1} g \simeq f \star_{M_1} g \tag{6.26}$$

Second Order Star Product: The second order Kontsevich Star Product is given in equation (4.32) gives extremely lengthy equation when written down on the non(anti)commutative superspace explicitly. Therefore, we have put the results in the appendix C .

Chapter 7

Non-Associative Star Product

In this chapter we first define the nonassociative star product by generalizing the map $\mathcal{M}^{(O)}$ defined in the section 3.1 to the map $\mathcal{M}^{(O)}_{R/L}$ which embeds the commutative coordinate space into the noncommutative nonassociative coordinates space defined in the section 2.3. We then proceed to calculate the star product using the method elucidated in the chapter 4, but with relaxing condition (4.1) introduced in the chapter 4. It turns out that this star product is also nonassociative . At the end of this chapter we will introduce the so-called cyclicity condition. It is weaker than associativity and of interest in the context of string theory [27]. We check the condition on our representation, in order to obey cyclicity at first order in α.

7.1 Non-Associative Star Product Via Isomorphism

In order to define the noncommutative nonassociative star product we have to define the reference position of the brackets. We have two obvious choices for such ordering. Either to put the brackets starting from the very right or from the very left.

Having a given associator with another formal parameter, means that we can change the bracketing of our products arbitrarily by iterative use of the associator. This is similar to the PBW-property for non-commutativity. This implies that we can again map monomials 1:1 from the commutative and associative algebra to the non-commutative and non-associative algebra. The map $\mathcal{M}^{(O)}$ in equation (3.2) can be modified to:

$$X^{M_1}\dots X^{M_p} \overset{\mathcal{M}_R^{(O)}}{\mapsto} (\hat{X}^{M_1}\dots(\hat{X}^{M_{p-2}}(\hat{X}^{M_{p-1}}\hat{X}^{M_p})))_O \tag{7.1}$$

$$X^{M_1}\dots X^{M_p} \overset{\mathcal{M}_R^{-1(O)}}{\hookleftarrow} (\hat{X}^{M_1}\dots(\hat{X}^{M_{p-2}}(\hat{X}^{M_{p-1}}\hat{X}^{M_p})))_O \tag{7.2}$$

$$1 \underset{\mathcal{M}_R^{-1(O)}}{\overset{\mathcal{M}_R^{(O)}}{\rightleftharpoons}} 1 \tag{7.3}$$

We can then have all three different orderings we defined in chapter 3 for nonassociative monomials as well. For Weyl ordering of monomials we have

$$X^{M_1}\dots X^{M_p} \overset{\mathcal{M}_R^{(W)}}{\mapsto} (\hat{X}^{(M_1}\dots(\hat{X}^{M_{p-2}}(\hat{X}^{M_{p-1}}\hat{X}^{M_p)}))) \tag{7.4}$$

$$X^{M_1}\dots X^{M_p} \overset{\mathcal{M}_R^{-1(W)}}{\hookleftarrow} (\hat{X}^{(M_1}\dots(\hat{X}^{M_{p-2}}(\hat{X}^{M_{p-1}}\hat{X}^{M_p)}))) \tag{7.5}$$

and for index-valued ordering of monomials

$$X^{M_1}\dots X^{M_p} \overset{\mathcal{M}_R^{(S)}}{\longmapsto} (\hat{X}^{:M_1}\dots(\hat{X}^{M_{p-2}}(\hat{X}^{M_{p-1}}\hat{X}^{M_p:}))) \tag{7.6}$$

$$X^{M_1}\dots X^{M_p} \overset{\mathcal{M}_R^{-1(S)}}{\longleftarrow\!\shortmid} (\hat{X}^{:M_1}\dots(\hat{X}^{M_{p-2}}(\hat{X}^{M_{p-1}}\hat{X}^{M_p:}))) \tag{7.7}$$

and also for anti-index-valued ordering of monomials

$$X^{M_1}\dots X^{M_p} \overset{\mathcal{M}_R^{(A_S)}}{\longmapsto} (\hat{X}^{;M_1}\dots(\hat{X}^{M_{p-2}}(\hat{X}^{M_{p-1}}\hat{X}^{M_p;}))) \tag{7.8}$$

$$X^{M_1}\dots X^{M_p} \overset{\mathcal{M}_R^{-1(A_S)}}{\longleftarrow\!\shortmid} (\hat{X}^{;M_1}\dots(\hat{X}^{M_{p-2}}(\hat{X}^{M_{p-1}}\hat{X}^{M_p;}))) \tag{7.9}$$

Let us give a simple example of such a nonassociative Weyl ordered star product for functions $f(X) = \varphi^{M_1}(X) = X^{M_1}$ and $g(X) = \varphi^{M_2M_3}(X) = X^{M_2}X^{M_3}$

$$\begin{aligned}
f\star g(\hat{X}) &= \varphi^{M_1}\star\varphi^{M_2M_3}(\hat{X}) = \\
&= \hat{X}^{M_1}(\hat{X}^{(M_2}\hat{X}^{M_3)}) = \qquad (7.10)\\
&= \underbrace{\tfrac{1}{3}\Big(\hat{X}^{M_1}(\hat{X}^{(M_2|}\hat{X}^{|M_3)}) + \hat{X}^{(M_2|}(\hat{X}^{M_1}\hat{X}^{|M_3)}) + \hat{X}^{(M_2|}(\hat{X}^{|M_3)}\hat{X}^{M_1})\Big)}_{\hat{X}^{(M_1}(\hat{X}^{M_2}\hat{X}^{M_3))}} + \\
&\quad + \tfrac{1}{3}\underbrace{[\hat{X}^{M_1},\hat{X}^{(M_2|}]}_{2\alpha\omega^{M_1(M_2|}}\hat{X}^{|M_3)} + \tfrac{1}{3}\underbrace{\Big(\hat{X}^{M_1}(\hat{X}^{(M_2}\hat{X}^{M_3)}) - (\hat{X}^{M_1}\hat{X}^{(M_2})\hat{X}^{M_3)}\Big)}_{-\beta\kappa^{M_1(M_2M_3)}} + \\
&\quad + \tfrac{1}{3}\underbrace{\Big((\hat{X}^{(M_2|}\hat{X}^{M_1})\hat{X}^{|M_3)} - \hat{X}^{(M_2|}(\hat{X}^{M_1}\hat{X}^{|M_3)})\Big)}_{\beta\kappa^{(M_2|M_1|M_3)}} + \\
&\quad + \tfrac{1}{3}\Big[\hat{X}^{M_1},\hat{X}^{(M_2|}\hat{X}^{|M_3)}\Big] + \tfrac{1}{3}\underbrace{\Big((\hat{X}^{(M_2|}\hat{X}^{|M_3)})\hat{X}^{M_1} - \hat{X}^{(M_2|}(\hat{X}^{|M_3)}\hat{X}^{M_1})\Big)}_{\beta\kappa^{(M_2M_3)M_1}} = \qquad (7.11)\\
&= \hat{X}^{(M_1}(\hat{X}^{M_2}\hat{X}^{M_3)}) + \\
&\quad + \tfrac{2}{3}\alpha\omega^{M_1(M_2|}\hat{X}^{|M_3)} - \tfrac{1}{3}\beta\kappa^{M_1(M_2M_3)} + \tfrac{1}{3}\beta\kappa^{(M_2|M_1|M_3)} + \tfrac{1}{3}\beta\kappa^{(M_2M_3)M_1} + \\
&\quad + \tfrac{1}{3}\underbrace{\Big[\hat{X}^{M_1},\hat{X}^{(M_2|}\Big]}_{2\alpha\omega^{M_1(M_2|}}\hat{X}^{|M_3)} + \tfrac{1}{3}\hat{X}^{(M_2|}\underbrace{\Big[\hat{X}^{M_1},\hat{X}^{|M_3)}\Big]}_{2\alpha\omega^{M_1|M_3)}} + \\
&\quad + \tfrac{1}{3}\beta\kappa^{(M_2|M_1|M_3)}(\hat{X},\alpha) - \tfrac{1}{3}\beta\kappa^{M_1(M_2M_3)} - \tfrac{1}{3}\beta\kappa^{(M_2M_3)M_1} \qquad (7.12)\\
&= \hat{X}^{(M_1}(\hat{X}^{M_2}\hat{X}^{M_3)}) + \tfrac{4}{3}\alpha\omega^{M_1(M_2|}\hat{X}^{|M_3)} + \tfrac{2}{3}\alpha\hat{X}^{(M_2|}\omega^{M_1|M_3)} + \\
&\quad - \tfrac{2}{3}\beta\kappa^{M_1(M_2M_3)} + \tfrac{2}{3}\beta\kappa^{(M_2|M_1|M_3)} \qquad (7.13)
\end{aligned}$$

Mapped back into the commutative space, this particular non-associative star product takes to lowest orders in the formal parameters the following form:

$$\begin{aligned}
&\varphi^{M_1}\star\varphi^{M_2M_3}(X) = \\
&\quad = X^{M_1}X^{M_2}X^{M_3} + 2\alpha\omega^{M_1(M_2|}X^{|M_3)} + \\
&\qquad - \tfrac{2}{3}\beta\kappa^{M_1(M_2M_3)} + \tfrac{2}{3}\beta\kappa^{(M_2|M_1|M_3)} + \mathcal{O}(\alpha^2,\alpha\beta,\beta^2)
\end{aligned} \tag{7.14}$$

For $\beta \to 0$ this agrees with (3.34). One can avoid to evaluate this star product at a certain point in coordinate space, if one rewrites it as

$$\begin{aligned}
&\varphi^{M_1} \star \varphi^{M_2 M_3} = \\
&= \quad \varphi^{M_1} \cdot \varphi^{M_2 M_3} + \alpha \partial_L \varphi^{M_1} \omega^{LK} \partial_K \varphi^{M_2 M_3} + \\
&\quad -\tfrac{1}{3} \beta \partial_{K_1} \varphi^{M_1} \left(\kappa^{K_1 K_2 K_3} - \kappa^{K_2 K_1 K_3} \right) \partial_{K_2} \partial_{K_3} \varphi^{M_2 M_3} + \mathcal{O}(\alpha^2, \alpha\beta, \beta^2)
\end{aligned} \tag{7.15}$$

Note that the calculation of $\varphi^{M_2 M_3} \star \varphi^{M_1}(X)$ will be more involved, because we would already be starting with an expression that does not have our reference bracketing (brackets starting from the right).

7.2 Non-Associative Star Product Order by Order

As we have seen in the chapter 4 that the star product defined by the equation (4.2)

$$f \star g(X) = f(\hat{X}) g(X) \tag{7.16}$$

gives the associative star product only if

$$\Gamma^{(K_1 \ldots K_k)}(X) = 0 \tag{7.17}$$

and some other coefficient functions like $\Gamma_1^{IJ}(X)$ are excluded due to the consistency condition in equation (2.13). If we relax these conditions the star product becomes non-associative. Let us first check this for relaxing all the constraints on the coefficient functions Γ's in the following.

7.2.1 First Order Star Product

At first order α expansion of the polydifferential operator looks

$$\hat{X}^M = X^M + \alpha \Gamma_0^{ML}(X) \partial_L + \mathcal{O}(\alpha^2) \tag{7.18}$$

Since in principle we would like to calculate the star product of three functions $f(X), g(X)$ and $h(X)$ and would like to see how the results are effected by choosing the different order of the functions for calculating the star product.i.e.

$$(f \star g) \star h - f \star (g \star h) = ? \tag{7.19}$$

Let us first check (7.19) for a simple choice of the functions $f(X) = X^M, g(X) = X^N$ and $h(X) = X^K$. Calculations go as in the following

$$(f \star g)(X) = \hat{X}^M X^N = (X^M + \alpha \Gamma_0^{ML}(X) \partial_L + \mathcal{O}(\alpha^2)) X^N = \tag{7.20}$$

$$= X^M X^N + \alpha \Gamma_0^{ML}(X) \partial_L (X^N) + \mathcal{O}(\alpha^2) = \tag{7.21}$$

$$= X^M X^N + \alpha \Gamma_0^{MN}(X) + \mathcal{O}(\alpha^2) \tag{7.22}$$

and

$$(g \star h)(X) = X^N X^K + \alpha \Gamma_0^{NK}(X) + \mathcal{O}(\alpha^2) \tag{7.23}$$

so, we get

$$(f \star (g \star h))(X) = \hat{X}^M (X^N X^K + \alpha \Gamma_0^{NK} + \mathcal{O}(\alpha^2)) = \tag{7.24}$$

$$= (X^M + \alpha \Gamma_0^{ML}(X) \partial_L + \mathcal{O}(\alpha^2))(X)(X^N X^K + \alpha \Gamma_0^{NK}(X) + \mathcal{O}(\alpha^2)) = \tag{7.25}$$

$$= X^M X^N X^K + \alpha X^M \Gamma_0^{NK}(X) + \alpha \Gamma_0^{ML}(X) \partial_L (X^N X^K) + \mathcal{O}(\alpha^2) = \tag{7.26}$$

$$= X^M X^N X^K + \alpha X^M \Gamma_0^{NK}(X) + 2\alpha \Gamma_0^{M(N} X^{K)}(X) + \mathcal{O}(\alpha^2) \tag{7.27}$$

$$
\begin{aligned}
(f\star g)\star h)(X) =&(f\star g(X))|_{X\to\hat{X}}h(X) && (7.28)\\
=&\Big(\hat{X}^{(M}\hat{X}^{N)}+\alpha\Gamma^{MN}(\hat{X})+\mathcal{O}(\alpha^2)\Big)X^K && (7.29)\\
=&X^MX^NX^K+\alpha X^{(M}\Gamma_0^{N)K}(X)+\alpha\Gamma_0^{(MN)}(X)X^K+\\
&+\alpha\Gamma_0^{(M|K}(X)X^{|N)}+\alpha\Gamma_0^{MN}(X)X^K+\mathcal{O}(\alpha^2) && (7.30)
\end{aligned}
$$

The condition (7.19) gives

$$
(f\star(g\star h))(X)-(f\star g)\star h)(X)=-\alpha\Gamma^{(MN)}(X)X^K+\mathcal{O}(\alpha^2) \tag{7.31}
$$

Now let us repeat the above procedure for slightly more general functions. The operator function $f(\hat{X})$ to this order in α is given by the equation (2.87)

$$
\begin{aligned}
f(\hat{X}) =&f(X)+\frac{\alpha}{2}\Gamma^{KK}(X)\partial_K\partial_Kf(X)+\\
&+\alpha\partial_Kf(X)\Gamma^{KL}(X)\partial_L+\mathcal{O}(\alpha^2) && (7.32)
\end{aligned}
$$

so star product of $f(x)$ and $g(X)$ is

$$
\begin{aligned}
(f\star g)(X) =&f(\hat{X})g(X)= && (7.33)\\
=&f(X)g(X)+\frac{\alpha}{2}\Gamma_0^{KK}(X)\partial_K\partial_Kf(X)g(X)+\\
&+\alpha\partial_Kf(X)\Gamma_0^{KL}(X)\partial_Lg(X)+\mathcal{O}(\alpha^2) && (7.34)
\end{aligned}
$$

and

$$
\begin{aligned}
(g\star h)(X) =&g(\hat{X})h(X)= && (7.35)\\
=&g(X)h(X)+\frac{\alpha}{2}\Gamma_0^{KK}(X)\partial_K\partial_Kg(X)h(X)+\\
&+\alpha\partial_Kg(X)\Gamma_0^{KL}(X)\partial_Lh(X)+\mathcal{O}(\alpha^2) && (7.36)
\end{aligned}
$$

$$
\begin{aligned}
(f\star(g\star h)(X) =&f(\hat{X})\Bigg(g(X)h(X)+\frac{\alpha}{2}\Gamma_0^{KK}(X)\partial_K\partial_Kg(X)h(X)+\\
&+\alpha\partial_Kg(X)\Gamma_0^{KL}(X)\partial_Lh(X)\Bigg)+\mathcal{O}(\alpha^2) && (7.37)\\
=&\Bigg(f(X)+\frac{\alpha}{2}\Gamma^{KK}(X)\partial_K\partial_Kf(X)+\alpha\partial_Kf(X)\Gamma^{KL}(X)\partial_L+\mathcal{O}(\alpha^2)\Bigg)\times\\
&\times\Bigg(g(X)h(X)+\frac{\alpha}{2}\Gamma_0^{KK}(X)\partial_K\partial_Kg(X)h(X)+\\
&+\alpha\partial_Kg(X)\Gamma_0^{KL}(X)\partial_Lh(X)\Bigg)+\mathcal{O}(\alpha^2) && (7.38)\\
=&f(X)g(X)h(X)+\frac{\alpha}{2}\Gamma^{KK}(X)\partial_K\partial_Kf(X)g(X)h(X)+\\
&+\alpha\partial_Kf(X)\Gamma^{KL}(X)\partial_Lg(X)h(X)+ && (7.39)\\
&+\alpha\partial_Kf(X)g(X)\Gamma^{KL}(X)\partial_Lh(X)+\\
&+\frac{\alpha}{2}f(X)\Gamma_0^{KK}(X)\partial_K\partial_Kg(X)h(X)+\alpha f(X)\partial_Kg(X)\Gamma_0^{KL}(X)\partial_Lh(X) && (7.40)
\end{aligned}
$$

$$
\begin{aligned}
((f\star g)\star h)(X) =&(f\star g(X))|_{X\to\hat{X}}h(X) &(7.41)\\
=&\Big(f(X)g(X)+\frac{\alpha}{2}\Gamma_0^{KK}(X)\partial_K\partial_K f(X)g(X)+\\
&+\alpha\partial_K f(X)\Gamma_0^{KL}(X)\partial_L g(X)\Big)|_{X\to\hat{X}}h(X)+\mathcal{O}(\alpha^2) &(7.42)\\
=&\Big(f(X)g(X)h(X)+\frac{\alpha}{2}\Gamma_0^{KK}(X)\partial_K\partial_K f(X)g(X)h(X)+\\
&+\alpha\partial_K f(X)\Gamma_0^{KL}(X)\partial_L g(X)h(X)+\\
&+\frac{\alpha}{2}\Gamma_0^{PP}(X)\partial_P\partial_P(f(X)g(X))h(X)+\\
&+\alpha\partial_P(f(X)g(X))\Gamma_0^{PQ}(X)\partial_Q h(X)+\mathcal{O}(\alpha^2) &(7.43)\\
=&\Big(f(X)g(X)h(X)+\frac{\alpha}{2}\Gamma_0^{KK}(X)\partial_K\partial_K f(X)g(X)h(X)+\\
&+\alpha\partial_K f(X)\Gamma_0^{KL(X)}\partial_L g(X)h(X)+\frac{\alpha}{2}\Gamma_0^{PP}(X)\partial_P\partial_P f(X)g(X)h(X)+\\
&+\frac{\alpha}{2}\Gamma_0^{PP}(X)\partial_P f(X)\partial_P g(X)h(X)+\frac{\alpha}{2}\Gamma_0^{PP}(X)f(X)\partial_P\partial_P g(X)h(X)+\\
&+\alpha\partial_P f(X)g(X)\Gamma_0^{PQ}(X)\partial_Q h(X)+\alpha f(X)\partial_P g(X)\Gamma_0^{PQ}(X)\partial_Q h(X)+\mathcal{O}(\alpha^2)\\
& &(7.44)
\end{aligned}
$$

The condition (7.19) gives

$$
\begin{aligned}
&(f\star(g\star h))(X)-(f\star g)\star h)(X)= &(7.45)\\
&=-\frac{\alpha}{2}f(X)\Gamma_0^{KK}(X)\partial_K\partial_K g(X)h(X)-\alpha\Gamma_0^{PP}(X)\partial_P f(X)\partial_P g(X)h(X)+\mathcal{O}(\alpha^2) &(7.46)
\end{aligned}
$$

Cyclicity: Cyclicity is a bit weaker condition than associativity but, we can use it to find some further constraints on the coefficient functions $\Gamma(X)$'s. Cyclicity for some arbitrary measure is given by

$$
\begin{aligned}
&\int f\star(g\star h)(X)\mu(X)=\int(f\star g)\star h(X)\mu(X)\qquad\forall f,g,h &(7.47)\\
\Rightarrow&\int\Big(f\star(g\star h)(X)-(f\star g)\star h(X)\Big)\mu(X)=0 &(7.48)
\end{aligned}
$$

At this order of α the cyclicity condition (7.48) becomes

$$
\int\Big(-\frac{\alpha}{2}f(X)\Gamma_0^{KK}(X)\partial_K\partial_K g(X)h(X)-\alpha\Gamma_0^{PP}(X)\partial_P f(X)\partial_P g(X)h(X)+\mathcal{O}(\alpha^2)\Big)\mu(X)=0 \tag{7.49}
$$

We can see that restrictions on the totally symmetrized $\Gamma(X)$'s depend upon the measure of integration $\mu(X)$ and on the class of the functions $f(X), g(X)$ and $h(X)$. For example, for $f(X)=f_M X^M, g(X)=g_N X^N$ and for arbitrary choice of function $h(X)$ (7.49) will give

$$
\begin{aligned}
-\int\alpha\Gamma_0^{PP}(X)\partial_P f(X)\partial_P g(X)h(X)\mu(X)+\mathcal{O}(\alpha^2)=0\qquad\forall f,g,h &\quad(7.50)\\
\Rightarrow\Gamma_0^{PP}(X)\mu(X)=0 &\quad(7.51)\\
\Rightarrow\Gamma_0^{PP}(X)\overset{\mu(X)\neq0}{=}0 &\quad(7.52)
\end{aligned}
$$

As an another example we can also choose $f(X) = X^M, g(X) = X^N$ and $h(X) = X^K$ for which (7.49) will give

$$-\int \alpha\Gamma_0^{PP}(X)X^K\mu(X)+\mathcal{O}(\alpha^2) = 0 \tag{7.53}$$

We can now choose $\mu(X) = e^{\frac{1}{2}X^K C_{KL} X^L}$ with some invertible matrix C_{KL} and thus $\partial_K((C^{-1})^{KL}\mu) = X^K\mu(X)$ to force $\Gamma_0^{PP}(X) = 0$. For the physical interpretation of cyclicity see [27] and for derivation of the non-associative star product from string theory see [34], [35] , [36] [37].

Conclusions

In this thesis we have defined different star products depending on an ordering prescription for the coordinate monomials of the generators of the non-commutative algebra. We obtained along with the Weyl-ordered star product by symmetrically ordering the monomials, two other star products which we call the index-value-ordered star product and the anti-index-value-ordered star product. The index-value-ordered star product comes from ordering the monomials by putting the higher index to the very right and in contrast the anti-index-value-ordered star product comes from ordering the monomials by putting the higher index to the very left. An interesting extension of this approach of defining the star product by embedding the commutative coordinate algebra into a non-commutative algebra is to define the non-associative star product. This is possible by embedding the commutative coordinate algebra into a (non-commutative) non-associative algebra. In future we intend to find a representation for our (non-commutative) non-associative algebra which could be used to effectively calculate the star product up to an arbitrary order.

We were able to do order by order calculations of the Weyl-ordered star product by using the method developed in [10] based on the polydifferential representation of the coordinate operators. We have, however, presented a more detailed explanation of the properties of the expansion coefficients. Relaxing some conditions on these expansion coefficients gave a more general star product which is non-associative . We have discussed at lowest order alternative conditions on the representation which implement the weaker cyclicity condition instead of associativity. Both approaches, embedding and polydiffernetial representation of the coordinate operators, to get the non-associative star product are quite independent of each other.

One of the important results of the thesis is the star product on the non(anti)commutative superspace. We obtained this by avoiding the Fourier transformation performed in [10] and by using the conventions about graded objects developed in [32]. We could also write down the twisted representation for the graded star product using these conventions.

We also obtained corrections to the transformation of the star product under the Lie derivative which we call quantum correction.

Appendix A

Twist Representation of the Star Product on Non(Anti)Commutative Superspace

In this appendix we define the star product on a non(anti)commutative superspace via twisting Procedure. For details see [38],[39],[40] [41] [42][43].

A.1 Universal Envelope of a Lie Algebra

As under Lie bracket multiplication, Lie algebras are non-associative, it is natural and quite useful in some applications to extend these non-associative Lie algebras to unital associative algebras with tensor product multiplication which is associative see for details [44]. Universal enveloping Lie algebra provides one such example of the extension of the non-associative Lie algebra $\mathcal{G}$ over some field F to the associative and unital algebra $\mathcal{U}(\mathcal{G})$ over F through universal property explained below.

A.1.1 Hopf Algebra

We will briefly review the concept of Hopf algebra in this section. We will explain how the usual structure of the algebra is extended by the Hopf algebra. To define Hopf algebra we start with the usual vector space H over a field F and define the following operations on it.

$$m : H \otimes H \rightarrow H \qquad (Multiplication) \tag{A.1}$$

$$i : F \rightarrow H \qquad (Unit) \tag{A.2}$$

The above two maps give a vector space A the usual structure of the algebra over a field F. To extend this to Hopf algebra we define three more operations on A in the following.

$$\Delta : H \rightarrow H \otimes H \qquad (Comultiplication) \tag{A.3}$$

$$\epsilon : H \rightarrow F \qquad (Counit) \tag{A.4}$$

$$\gamma : H \rightarrow H \qquad (Antipode) \tag{A.5}$$

A.1.2 Twisting the Hopf Algebra

A systematic way to transform a Hopf algebra into an another Hopf algebra is known as the twisting of the algebra and was first developed by Drinfel'd. If we have some Hopf algebra $(H, F, m, i, \triangle, \epsilon, \gamma)$ we can twist it by choosing a biproduct element $\mathcal{F} \in H \otimes H$ called a twist element.

A.1.3 Graded Twist

Since we are using ω^{MN} whose indices carry the gradings. We call the twist element constructed out of it a graded twist. In order to deform the superspace we use the following twist which is an element of the Hopf algebra.

$$\mathcal{F}^G = e^{\frac{i}{2}\omega^{MN}\partial_M \otimes \partial_N} \tag{A.6}$$

where the inverse of the above twist is given by

$$(\mathcal{F}^G)^{-1} = e^{-\frac{i}{2}\omega^{MN}\partial_M \otimes \partial_N} \tag{A.7}$$

The graded star product is defined by the following formula

$$\begin{aligned} f \star g =& m_\star(f \otimes g) && \text{(A.8)} \\ =& m((\mathcal{F}^G)^{-1} f \otimes g) && \text{(A.9)} \\ =& m(e^{-\frac{i}{2}\omega^{MN}\partial_M \otimes \partial_N} f \otimes g) && \text{(A.10)} \\ =& f.g - \frac{i}{2}\omega^{MN}\partial_M f \partial_N g - \frac{1}{8}\omega^{MN}\omega^{KL}\partial_M \partial_K f \partial_N \partial_L g + \mathcal{O}(\omega^3) && \text{(A.11)} \end{aligned}$$

This star product gives us a deformation of the superspace for non-degenerate constant ω.

$$[X^M, X^N] = \omega^{MN} \tag{A.12}$$

We can get different components of the above graded twist element by applying (6.11) to the (A.6) and to its inverse (A.7).

A.1.4 Bosonic twist

We obtain the bosonic component of the twist (A.6) when both the indices of ω are bosonic

$$\mathcal{F}^b = e^{\frac{i}{2}\omega^{mn}\partial_m \otimes \partial_n} \tag{A.13}$$

and its inverse

$$\mathcal{F}^{-1b} = e^{-\frac{i}{2}\omega^{mn}\partial_m \otimes \partial_n} \tag{A.14}$$

Bosonic component of the graded star product becomes

$$\begin{aligned} f \star_b g =& m_\star(f \otimes g) && \text{(A.15)} \\ =& m((\mathcal{F}^b)^{-1} f \otimes g) && \text{(A.16)} \\ =& m(e^{-\frac{i}{2}\omega^{mn}\partial_m \otimes \partial_n} f \otimes g) && \text{(A.17)} \\ =& f.g - \frac{i}{2}\omega^{mn}\partial_m f \partial_n g - \frac{1}{8}\omega^{mn}\omega^{pq}\partial_m \partial_p f \partial_q \partial_n g + \mathcal{O}(\omega^3) && \text{(A.18)} \end{aligned}$$

This component of the star product gives the deformation of the bosonic coordinates

$$[x^m, x^n] = \omega^{mn}(X) \tag{A.19}$$

A.1.5 Fermionic Twist

Similarly we get fermionic component of the twist (A.6) when both the indices are fermionic

$$\mathcal{F}^f = e^{\frac{i}{2}\omega^{\mu\nu}\partial_\mu\otimes\partial_\nu} \tag{A.20}$$

and its inverse

$$\mathcal{F}^{-1f} = e^{-\frac{i}{2}\omega^{\mu\nu}\partial_\mu\otimes\partial_\nu} \tag{A.21}$$

Fermionic component of the graded star product becomes

$$\begin{aligned}
f \star_f g =& m_\star(f \otimes g) && \text{(A.22)}\\
=& m((\mathcal{F}^f)^{-1} f \otimes g) && \text{(A.23)}\\
=& m(e^{-\frac{i}{2}\omega^{\mu\nu}\partial_\mu\otimes\partial_\nu} f \otimes g) && \text{(A.24)}\\
=& f.g - \frac{i}{2}\omega^{\mu\nu}\partial_\mu f \partial_\nu g - \frac{1}{8}\omega^{\mu\nu}\omega^{\rho\sigma}\partial_\mu\partial_\rho f \partial_\nu\partial_\sigma g + \mathcal{O}(\omega^3) && \text{(A.25)}
\end{aligned}$$

$$\begin{aligned}
f \star_f g =& m_\star(f \otimes g) && \text{(A.26)}\\
=& m((\mathcal{F}^f)^{-1} f \otimes g) && \text{(A.27)}\\
=& m(e^{-\frac{i}{2}\omega^{\bar\mu\bar\nu}\partial_{\bar\mu}\otimes\partial_{\bar\nu}} f \otimes g) && \text{(A.28)}\\
=& f.g - \frac{i}{2}\omega^{\bar\mu\bar\nu}\partial_{\bar\mu} f \partial_{\bar\nu} g - \frac{1}{8}\omega^{\bar\mu\bar\nu}\omega^{\bar\rho\bar\sigma}\partial_{\bar\mu}\partial_{\bar\rho} f \partial_{\bar\nu}\partial_{\bar\sigma} g + \mathcal{O}(\omega^3) && \text{(A.29)}
\end{aligned}$$

$$\begin{aligned}
f \star_f g =& m_\star(f \otimes g) && \text{(A.30)}\\
=& m((\mathcal{F}^f)^{-1} f \otimes g) && \text{(A.31)}\\
=& m(e^{-\frac{i}{2}\omega^{\mu\bar\nu}\partial_\mu\otimes\partial_{\bar\nu}} f \otimes g) && \text{(A.32)}\\
=& f.g - \frac{i}{2}\omega^{\mu\bar\nu}\partial_\mu f \partial_{\bar\nu} g - \frac{1}{8}\omega^{\mu\bar\nu}\omega^{\rho\bar\sigma}\partial_\mu\partial_\rho f \partial_{\bar\nu}\partial_{\bar\sigma} g + \mathcal{O}(\omega^3) && \text{(A.33)}
\end{aligned}$$

This component of the star product gives the deformation of the fermionic coordinates

$$\{\theta^\mu, \theta^\nu\} = \omega^{\mu\nu}(X), \{\bar\theta^{\bar\mu}, \bar\theta^{\bar\nu}\} = \omega^{\bar\mu\bar\nu}(X), \{\theta^\mu, \bar\theta^{\bar\nu}\} = \omega^{\mu\bar\nu}(X) \tag{A.34}$$

A.1.6 Mixed Twist

In mixed case we get mixed component of the twist (A.6) when one of the indices is bosonic and the other one is fermionic

$$\mathcal{F}^m = e^{\frac{i}{2}\omega^{m\nu}\partial_m\otimes\partial_\nu} \tag{A.35}$$

and its inverse

$$\mathcal{F}^{-1m} = e^{-\frac{i}{2}\omega^{m\nu}\partial_m\otimes\partial_\nu} \tag{A.36}$$

mixed component of the graded star product becomes

$$\begin{aligned}
f \star_m g =& m_\star(f \otimes g) && \text{(A.37)}\\
=& m((\mathcal{F}^m)^{-1} f \otimes g) && \text{(A.38)}\\
=& m(e^{-\frac{i}{2}\omega^{m\nu}\partial_m\otimes\partial_\nu} f \otimes g) && \text{(A.39)}\\
=& f.g - \frac{i}{2}\omega^{m\nu}\partial_m f \partial_\nu g - \frac{1}{8}\omega^{m\nu}\omega^{n\sigma}\partial_m\partial_n f \partial_\nu\partial_\sigma g + \mathcal{O}(\omega^3) && \text{(A.40)}
\end{aligned}$$

$$
\begin{aligned}
f \star_m g =& m_\star(f \otimes g) && \text{(A.41)}\\
=& m((\mathcal{F}^m)^{-1} f \otimes g) && \text{(A.42)}\\
=& m(e^{-\frac{i}{2}\omega^{m\bar{\nu}}\partial_m \otimes \partial_{\bar{\nu}}} f \otimes g) && \text{(A.43)}\\
=& f.g - \frac{i}{2}\omega^{m\bar{\nu}}\partial_m f \partial_{\bar{\nu}} g - \frac{1}{8}\omega^{m\bar{\nu}}\omega^{n\bar{\sigma}}\partial_n\partial_n f \partial_{\bar{\nu}}\partial_{\bar{\sigma}} g + \mathcal{O}(\omega^3) && \text{(A.44)}
\end{aligned}
$$

this component of the star product gives the deformation of the mixed coordinates

$$
[x^m, \theta^\nu\} = \omega^{m\nu}(X), [x^m, \bar{\theta}^{\bar{\nu}}\} = \omega^{m\bar{\nu}}(X) \qquad \text{(A.45)}
$$

For twisted represention of star product on ordinary spacetime see [37],[45] [46],[47],[48].

Appendix B

Diffeomorphisms Coefficients

For $n = 2$ the equation (5.43) becomes

$$\sum_{m=0}^{2}\sum_{p=0}^{2m} D_\xi^{(m)}(\partial)^{p+1}(f\star g(X))^{2-m} = \mathcal{L}_\xi(f\star g(X))^2 + \Omega_\xi(f\star g(X))^2 + \\ + ((D_\xi'^2 f)\star g(X))^0 + (f\star(D_\xi'^2 g)(X))^0 \qquad \text{(B.1)}$$

We first calculate the L.H.S of the equation (B.1) and put coefficients infront of every term which are to be determined latter from the corresponding terms on the R.H.S of the equation (B.1).

The (m=0, p=0) part of the L.H.S of the equation (B.1) is

$$A_1^{(2)}\frac{\alpha^2}{2}\xi^P\partial_P\omega^{KL}(X)\omega^{IJ}(X)\partial_I\partial_K f\partial_J\partial_L g + A_2^{(2)}\frac{\alpha^2}{2}\xi^P\omega^{KL}(X)\partial_P\omega^{IJ}(X)\partial_I\partial_K f\partial_J\partial_L g + \\ + A_3^{(2)}\frac{\alpha^2}{2}\xi^P\omega^{KL}(X)\omega^{IJ}(X)\partial_P\partial_I\partial_K f\partial_J\partial_L g + A_4^{(2)}\frac{\alpha^2}{2}\xi^P\omega^{KL}(X)\omega^{IJ}(X)\partial_I\partial_K f\partial_P\partial_J\partial_L g + \\ + A_5^{(2)}\frac{\alpha^2}{3}\xi^P\partial_P\omega^{KL}(X)\partial_L\omega^{IJ}(X)\partial_I\partial_K f\partial_J g + A_6^{(2)}\frac{\alpha^2}{3}\xi^P\omega^{KL}(X)\partial_P\partial_L\omega^{IJ}(X)\partial_I\partial_K f\partial_J g - \\ - A_7^{(2)}\frac{\alpha^2}{3}\xi^P\partial_P\omega^{KL}(X)\partial_K\omega^{IJ}(X)\partial_I f\partial_J\partial_L g - A_8^{(2)}\frac{\alpha^2}{3}\xi^P\omega^{KL}(X)\partial_P\partial_K\omega^{IJ}(X)\partial_I f\partial_J\partial_L g + \qquad \text{(B.2)}$$

The (m=1, p=0) part of the L.H.S of the equation (B.1) is

$$B_1^{(2)}\alpha\partial_R\omega^{PQ}(X)\partial_Q\xi^R\partial_P\omega^{MN}(X)\partial_M f\partial_N g + B_2^{(2)}\alpha\partial_R\omega^{PQ}(X)\partial_Q\xi^R\omega^{MN}(X)\partial_P\partial_M f\partial_N g + \\ + B_3^{(2)}\alpha\partial_R\omega^{PQ}(X)\partial_Q\xi^R\omega^{MN}(X)\partial_M f\partial_P\partial_N g + \qquad \text{(B.3)}$$

The (m=1, p=1) part of the L.H.S of the equation (B.1) is

$$
\begin{aligned}
&C_1^{(2)}\alpha\omega^{PR}(X)\partial_R\xi^Q\partial_P\partial_Q\omega^{MN}(X)\partial_M f\partial_N g+C_2^{(2)}\alpha\omega^{PR}(X)\partial_R\xi^Q\partial_Q\omega^{MN}(X)\partial_P\partial_M f\partial_N g+\\
&+C_3^{(2)}\alpha\omega^{PR}(X)\partial_R\xi^Q\partial_Q\omega^{MN}(X)\partial_M f\partial_P\partial_N g+C_4^{(2)}\alpha\omega^{PR}(X)\partial_R\xi^Q\partial_P\omega^{MN}(X)\partial_Q\partial_M f\partial_N g+\\
&+C_5^{(2)}\alpha\omega^{PR}(X)\partial_R\xi^Q\omega^{MN}(X)\partial_M\partial_P\partial_Q f\partial_N g+C_6^{(2)}\alpha\omega^{PR}(X)\partial_R\xi^Q\omega^{MN}(X)\partial_M\partial_Q f\partial_P\partial_N g+\\
&+C_7^{(2)}\alpha\omega^{PR}(X)\partial_R\xi^Q\partial_P\omega^{MN}(X)\partial_M f\partial_Q\partial_N g+C_8^{(2)}\alpha\omega^{PR}(X)\partial_R\xi^Q\omega^{MN}(X)\partial_P\partial_M f\partial_Q\partial_N g+\\
&+C_9^{(2)}\alpha\omega^{PR}(X)\partial_R\xi^Q\omega^{MN}(X)\partial_M f\partial_P\partial_Q\partial_N g
\end{aligned}
\tag{B.4}
$$

The (m=2, p=0) part of the L.H.S of the equation (B.1) is

$$
\begin{aligned}
&\partial_R\omega^{MN}\partial_M\omega^{RQ}\partial_N\partial_Q\xi^P(E_1\partial_P fg+E_2 f\partial_P g)+\\
&+\partial_Q\partial_R\omega^{MN}\partial_N\omega^{PQ}\partial_M\xi^R(E_3\partial_P fg+E_4 f\partial_P g)+\\
&+\partial_Q\omega^{MN}\partial_R\partial_N\omega^{PQ}\partial_M\xi^R(E_5\partial_P fg+E_6 f\partial_P g)+\\
&+\omega^{MN}\partial_N\partial_R\omega^{PQ}\partial_M\partial_Q\xi^R(E_7\partial_P fg+E_8 f\partial_P g)+\\
&+\partial_R\omega^{MN}\partial_M\omega^{PQ}\partial_N\partial_Q\xi^R(E_9\partial_P fg+E_{10} f\partial_P g)+\\
&+\partial_R\omega^{PN}\partial_M\omega^{RQ}\partial_N\partial_Q\xi^M(E_{11}\partial_P fg+E_{12} f\partial_P g)
\end{aligned}
\tag{B.5}
$$

The (m=2, p=1) part of the L.H.S of the equation (B.1) is

$$
\begin{aligned}
&\omega^{MN}\partial_M\omega^{RP}\partial_N\partial_R\xi^Q\Big(F_1^{(2)}\partial_P f\partial_Q g+F_2^{(2)}\partial_Q f\partial_P g+F_3^{(2)}\partial_P\partial_Q fg+F_4^{(2)} f\partial_P\partial_Q g\Big)+\\
&+\partial_R\omega^{MN}\partial_N\omega^{RP}\partial_M\xi^Q\Big(F_5^{(2)}\partial_P f\partial_Q g+F_6^{(2)}\partial_Q f\partial_P g+F_7^{(2)}\partial_P\partial_Q fg+F_8^{(2)} f\partial_P\partial_Q g\Big)+\\
&+\partial_M\omega^{PN}\partial_R\omega^{MQ}\partial_N\xi^R\Big(F_9^{(2)}\partial_P f\partial_Q g+F_{10}^{(2)}\partial_Q f\partial_P g+F_{11}^{(2)}\partial_P\partial_Q fg+F_{12}^{(2)} f\partial_P\partial_Q g\Big)+\\
&+\partial_N\partial_R\omega^{MP}\partial_M\omega^{NQ}\xi^R\Big(F_{13}^{(2)}\partial_\alpha f\partial_Q g+F_{14}^{(2)}\partial_Q f\partial_P g+F_{15}^{(2)}\partial_P\partial_Q fg+F_{16}^{(2)} f\partial_P\partial_Q g\Big)+\\
&+\omega^{MP}\partial_N\omega^{RQ}\partial_M\partial_R\xi^N\Big(F_{17}^{(2)}\partial_P f\partial_Q g+F_{18}^{(2)}\partial_Q f\partial_P g+F_{19}^{(2)}\partial_P\partial_Q fg+F_{20}^{(2)} f\partial_P\partial_Q g\Big)+\\
&+\omega^{MP}\partial_M\partial_R\omega^{NQ}\partial_N\xi^R\Big(F_{21}^{(2)}\partial_P f\partial_Q g+F_{22}^{(2)}\partial_Q f\partial_P g+F_{23}^{(2)}\partial_P\partial_Q fg+F_{24}^{(2)} f\partial_P\partial_Q g\Big)
\end{aligned}
\tag{B.6}
$$

The (m=2, p=2) part of the L.H.S of the equation (B.1) is

$$
\begin{aligned}
&\omega^{PM}\partial_M\omega^{QN}\partial_N\xi^R\Big(G_1^{(2)}\partial_R\partial_P f\partial_Q g+G_2^{(2)}\partial_P f\partial_R\partial_Q g+G_3^{(2)}\partial_R\partial_Q f\partial_P g+G_4^{(2)}\partial_Q f\partial_R\partial_P g+\\
&+G_5^{(2)}\partial_R\partial_P\partial_Q fg+G_6^{(2)}\partial_P\partial_Q f\partial_R g+G_7^{(2)}\partial_R f\partial_P\partial_Q g+G_8^{(2)} f\partial_R\partial_P\partial_Q g\Big)+\\
&+\partial_N\omega^{PM}\partial_M\omega^{QN}\xi^R\Big(G_9^{(2)}\partial_R\partial_P f\partial_Q g+G_{10}^{(2)}\partial_P f\partial_R\partial_Q g+G_{11}^{(2)}\partial_R\partial_Q f\partial_P g+G_{12}^{(2)}\partial_Q f\partial_R\partial_P g+\\
&+G_{13}^{(2)}\partial_R\partial_P\partial_Q fg+G_{14}^{(2)}\partial_P\partial_Q f\partial_R g+G_{15}^{(2)}\partial_R f\partial_P\partial_Q g+G_{16}^{(2)} f\partial_R\partial_P\partial_Q g\Big)+
\end{aligned}
$$

$$+\,\omega^{PM}\omega^{QN}\partial_M\partial_N\xi^R\Big(G^{(2)}_{17}\partial_R\partial_P f\partial_Q g+G^{(2)}_{18}\partial_P f\partial_R\partial_Q g+G^{(2)}_{19}\partial_R\partial_Q f\partial_P g+G^{(2)}_{20}\partial_Q f\partial_R\partial_P g+$$

$$+\,G^{(2)}_{21}\partial_R\partial_P\partial_Q fg+G^{(2)}_{22}\partial_P\partial_Q f\partial_R g+G^{(2)}_{23}\partial_R f\partial_P\partial_Q g+G^{(2)}_{24}f\partial_R\partial_P\partial_Q g\Big) \tag{B.7}$$

Coefficients: We can now use (B.1) to determine the coefficients $A^{(2)}, B^{(2)}, C^{(2)}, E^{(2)}, F^{(2)}$ and $G^{(2)}$ in equations (B.2),(B.3) ,(B.4) , (B.5), (B.6) and (B.7).

$$A^{(2)}_1,\ldots,A^{(2)}_8=1 \tag{B.8}$$

$$B^{(2)}_1,\ldots,B^{(2)}_3=0 \tag{B.9}$$

$$C^{(2)}_1,\ldots,C^{(2)}_9=0 \tag{B.10}$$

$$E^{(2)}_1,\ldots,E^{(2)}_{12}=0 \tag{B.11}$$

$$F^{(2)}_1,\ldots,F^{(2)}_4=\frac{\alpha^2}{3} \tag{B.12}$$

$$G^{(2)}_{22},G^{(2)}_{23}=\frac{2\alpha^2}{3} \tag{B.13}$$

$$F^{(2)}_5,\ldots,F^{(2)}_{24}=0 \tag{B.14}$$

$$G^{(2)}_1,\ldots,G^{(2)}_{16}=0 \tag{B.15}$$

$$G^{(2)}_{17}=G^{(2)}_{18},G^{(2)}_{19}=G^{(2)}_{20}=\frac{\alpha^2}{6} \tag{B.16}$$

$$G^{(2)}_{21}=G^{(2)}_{24}=\frac{7\alpha^2}{6} \tag{B.17}$$

Appendix C

Second Order Star Product on Non(Anti)Commutative Superspace

Applying the graded summation convention (6.11) to the second order star product in equation (4.32) gives the following extremely lengthy equation. Where for the purpose of brevity we have not shown the derivatives acting on functions.

$$
\begin{aligned}
(f\star g)^{(2)} =& \frac{\alpha^2}{2}\Bigg(\sum_{m,n,k,l}\omega^{mn}\omega^{kl}+\sum_{m,n,\kappa,l}\omega^{mn}\omega^{\kappa l}+\sum_{m,n,k,\lambda}\omega^{mn}\omega^{k\lambda}-\sum_{m,n,\kappa,\lambda}\omega^{mn}\omega^{\kappa\lambda}+\sum_{m,n,k,\bar\lambda}\omega^{mn}\omega^{k\bar\lambda}- \\
&-\sum_{m,n,\bar\kappa,\lambda}\omega^{mn}\omega^{\bar\kappa\lambda}-\sum_{m,n,\bar\kappa,\bar\lambda}\omega^{mn}\omega^{\bar\kappa\bar\lambda}-\sum_{m,n,\kappa,\bar\lambda}\omega^{mn}\omega^{\kappa\bar\lambda}+\sum_{m,n,\bar\kappa,l}\omega^{mn}\omega^{\bar\kappa l}+\sum_{m,\nu,k,l}\omega^{m\nu}\omega^{kl}+ \\
&+\sum_{m,\nu,\kappa,l}\omega^{m\nu}\omega^{\kappa l}+\sum_{m,\nu,k,\lambda}\omega^{m\nu}\omega^{k\lambda}-\sum_{m,\nu,\kappa,\lambda}\omega^{m\nu}\omega^{\kappa\lambda}+\sum_{m,\nu,k,\bar\lambda}\omega^{m\nu}\omega^{k\bar\lambda}-\sum_{m,\nu,\bar\kappa,\lambda}\omega^{m\nu}\omega^{\bar\kappa\lambda}- \\
&-\sum_{m,\nu,\bar\kappa,\bar\lambda}\omega^{m\nu}\omega^{\bar\kappa\bar\lambda}-\sum_{m,\nu,\kappa,\bar\lambda}\omega^{m\nu}\omega^{\kappa\bar\lambda}+\sum_{m,\nu,\bar\kappa,l}\omega^{m\nu}\omega^{\bar\kappa l}+\sum_{\mu,n,k,l}\omega^{\mu n}\omega^{kl}+\sum_{\mu,n,\kappa,l}\omega^{\mu n}\omega^{\kappa l}+ \\
&+\sum_{\mu,n,k,\lambda}\omega^{\mu n}\omega^{k\lambda}-\sum_{\mu,n,\kappa,\lambda}\omega^{\mu n}\omega^{\kappa\lambda}+\sum_{\mu,n,k,\bar\lambda}\omega^{\mu n}\omega^{k\bar\lambda}-\sum_{\mu,n,\bar\kappa,\lambda}\omega^{\mu n}\omega^{\bar\kappa\lambda}-\sum_{\mu,n,\bar\kappa,\bar\lambda}\omega^{\mu n}\omega^{\bar\kappa\bar\lambda}+ \\
&+\sum_{i,j,\kappa,\bar\lambda}\omega^{\mu n}\omega^{\kappa\bar\lambda}+\sum_{\mu,n,\bar\kappa,l}\omega^{\mu n}\omega^{\bar\kappa l}-\sum_{\mu,\nu,k,l}\omega^{\mu\nu}\omega^{kl}-\sum_{\mu,\nu,\kappa,l}\omega^{\mu\nu}\omega^{\kappa l}-\sum_{\mu,\nu,k,\lambda}\omega^{\mu\nu}\omega^{k\lambda}+ \\
&+\sum_{\mu,\nu,\kappa,\lambda}\omega^{\mu\nu}\omega^{\kappa\lambda}-\sum_{\mu,\nu,k,\bar\lambda}\omega^{\mu\nu}\omega^{k\bar\lambda}+\sum_{\mu,\nu,\bar\kappa,\lambda}\omega^{\mu\nu}\omega^{\bar\kappa\lambda}+\sum_{\mu,\nu,\bar\kappa,\bar\lambda}\omega^{\mu\nu}\omega^{\bar\kappa\bar\lambda}+\sum_{\mu,\nu,\kappa,\bar\lambda}\omega^{\mu\nu}\omega^{\kappa\bar\lambda}- \\
&-\sum_{\mu,\nu,\bar\kappa,l}\omega^{\mu\nu}\omega^{\bar\kappa l}+\sum_{m,\bar\nu,\kappa,l}\omega^{m\bar\nu}\omega^{\kappa l}+\sum_{m,\bar\nu,k,\lambda}\omega^{m\bar\nu}\omega^{k\lambda}-\sum_{m,\bar\nu,\kappa,\lambda}\omega^{m\bar\nu}\omega^{\kappa\lambda}+\sum_{m,\bar\nu,k,\bar\lambda}\omega^{m\bar\nu}\omega^{k\bar\lambda}- \\
&-\sum_{m,\bar\nu,\bar\kappa,\lambda}\omega^{m\bar\nu}\omega^{\bar\kappa\lambda}-\sum_{m,\bar\nu,\bar\kappa,\bar\lambda}\omega^{m\bar\nu}\omega^{\bar\kappa\bar\lambda}-\sum_{m,\bar\nu,\kappa,\bar\lambda}\omega^{m\bar\nu}\omega^{\kappa\bar\lambda}+\sum_{m,\bar\nu,\bar\kappa,l}\omega^{m\bar\nu}\omega^{\bar\kappa l}+\sum_{m,\bar\nu,k,l}\omega^{m\bar\nu}\omega^{kl}+ \\
&+\sum_{\bar\mu,n,\kappa,l}\omega^{\bar\mu n}\omega^{\kappa l}+\sum_{\bar\mu,n,k,\lambda}\omega^{\bar\mu n}\omega^{k\lambda}-\sum_{\bar\mu,n,\kappa,\lambda}\omega^{\bar\mu n}\omega^{\kappa\lambda}+\sum_{\bar\mu,n,k,\bar\mu}\omega^{\bar\mu n}\omega^{k\bar\mu}-\sum_{\bar\mu,n,\bar\kappa,\lambda}\omega^{\bar\mu n}\omega^{\bar\kappa\lambda}-
\end{aligned}
$$

$$
-\sum_{\bar{\mu},n,\bar{\kappa},\bar{\lambda}}\omega^{\bar{\mu}n}\omega^{\bar{\kappa}\bar{\lambda}}-\sum_{\bar{\mu},n,\kappa,\bar{\lambda}}\omega^{\bar{\mu}n}\omega^{\kappa\bar{\lambda}}+\sum_{\bar{\mu},n,\bar{\kappa},l}\omega^{\bar{\mu}n}\omega^{\bar{\kappa}l}+\sum_{\bar{\mu},n,k,l}\omega^{\bar{\mu}n}\omega^{kl}-\sum_{\mu,\bar{\nu},\kappa,l}\omega^{\mu\bar{\nu}}\omega^{\kappa l}-
$$

$$
-\sum_{\mu,\bar{\nu},k,\lambda}\omega^{\mu\bar{\nu}}\omega^{k\lambda}+\sum_{\mu,\bar{\nu},\kappa,\lambda}\omega^{\mu\bar{\nu}}\omega^{\kappa\lambda}-\sum_{\mu,\bar{\nu},k,\bar{\lambda}}\omega^{\mu\bar{\nu}}\omega^{k\bar{\lambda}}+\sum_{\mu,\bar{\nu},\bar{\kappa},\lambda}\omega^{\mu\bar{\nu}}\omega^{\bar{\kappa}\lambda}+\sum_{\mu,\bar{\nu},\bar{\kappa},\bar{\lambda}}\omega^{\mu\bar{\nu}}\omega^{\bar{\kappa}\bar{\lambda}}+
$$

$$
+\sum_{\mu,\bar{\nu},\kappa,\bar{\lambda}}\omega^{\mu\bar{\nu}}\omega^{\kappa\bar{\lambda}}-\sum_{\mu,\bar{\nu},\bar{\kappa},l}\omega^{\mu\bar{\nu}}\omega^{\bar{\kappa}l}+\sum_{\mu,\bar{\nu},k,l}\omega^{\mu\bar{\nu}}\omega^{kl}-\sum_{\bar{\mu},\nu,\kappa,l}\omega^{\bar{\mu}\nu}\omega^{\kappa l}-\sum_{\bar{\mu},\nu,k,\lambda}\omega^{\bar{\mu}\nu}\omega^{k\lambda}+
$$

$$
+\sum_{\bar{\mu},\nu,\kappa,\lambda}\omega^{\bar{\mu}\nu}\omega^{\kappa\lambda}-\sum_{\bar{\mu},\nu,k,\bar{\lambda}}\omega^{\bar{\mu}\nu}\omega^{k\bar{\lambda}}+\sum_{\bar{\mu},\nu,\bar{\kappa},\lambda}\omega^{\bar{\mu}\nu}\omega^{\bar{\kappa}\lambda}+\sum_{\bar{\mu},\nu,\bar{\kappa},\bar{\lambda}}\omega^{\bar{\mu}\nu}\omega^{\bar{\kappa}\bar{\lambda}}+\sum_{\bar{\mu},\nu,\kappa,\bar{\lambda}}\omega^{\bar{\mu}\nu}\omega^{\kappa\bar{\lambda}}-
$$

$$
-\sum_{\bar{\mu},\nu,\bar{\kappa},l}\omega^{\bar{\mu}\nu}\omega^{\bar{\kappa}l}-\sum_{\bar{\mu},\nu,k,l}\omega^{\bar{\mu}\nu}\omega^{kl}-\sum_{\bar{\mu},\bar{\nu},\kappa,l}\omega^{\bar{\mu}\bar{\nu}}\omega^{\kappa l}-\sum_{\bar{\mu},\bar{\nu},k,\lambda}\omega^{\bar{\mu}\bar{\nu}}\omega^{k\lambda}+\sum_{\bar{\mu},\bar{\nu},\kappa,\lambda}\omega^{\bar{\mu}\bar{\nu}}\omega^{\kappa\lambda}-
$$

$$
-\sum_{\bar{\mu},\bar{\nu},k,\bar{\lambda}}\omega^{\bar{\mu}\bar{\nu}}\omega^{k\bar{\lambda}}+\sum_{\bar{\mu},\bar{\nu},\bar{\kappa},\lambda}\omega^{\bar{\mu}\bar{\nu}}\omega^{\bar{\kappa}\lambda}+\sum_{\bar{\mu},\bar{\nu},\bar{\kappa},\bar{\lambda}}\omega^{\bar{\mu}\bar{\nu}}\omega^{\bar{\kappa}\bar{\lambda}}+\sum_{\bar{\mu},\bar{\nu},\kappa,\bar{\lambda}}\omega^{\bar{\mu}\bar{\nu}}\omega^{\kappa\bar{\lambda}}-\sum_{\mu,\bar{\nu},\bar{\kappa},l}\omega^{\mu\bar{\nu}}\omega^{\bar{\kappa}l}-
$$

$$
-\sum_{\bar{\mu},\bar{\nu},k,l}\omega^{\bar{\mu}\bar{\nu}}\omega^{kl}\Bigg)+\frac{\alpha^2}{3}\Bigg(\sum_{m,n,k,l}\omega^{mn}\partial_n\omega^{kl}+\sum_{m,n,\kappa,l}\omega^{mn}\partial_n\omega^{\kappa l}+\sum_{m,n,k,\lambda}\omega^{mn}\partial_n\omega^{k\lambda}-
$$

$$
-\sum_{m,n,\kappa,\lambda}\omega^{mn}\partial_n\omega^{\kappa\lambda}+\sum_{m,n,k,\bar{\lambda}}\omega^{mn}\partial_n\omega^{k\bar{\lambda}}-\sum_{m,n,\bar{\kappa},\lambda}\omega^{mn}\partial_n\omega^{\bar{\kappa}\lambda}-\sum_{m,n,\bar{\kappa},\bar{\lambda}}\omega^{mn}\partial_n\omega^{\bar{\kappa}\bar{\lambda}}-
$$

$$
-\sum_{m,n,\kappa,\bar{\lambda}}\omega^{mn}\partial_n\omega^{\kappa\bar{\lambda}}+\sum_{m,n,\bar{\kappa},l}\omega^{mn}\partial_n\omega^{\bar{\kappa}l}+\sum_{m,\nu,k,l}\omega^{m\nu}\partial_\nu\omega^{kl}+\sum_{m,\nu,\kappa,l}\omega^{m\nu}\partial_\nu\omega^{\kappa l}+
$$

$$
+\sum_{m,\nu,k,\lambda}\omega^{m\nu}\partial_\nu\omega^{k\lambda}-\sum_{m,\nu,\kappa,\lambda}\omega^{m\nu}\partial_\nu\omega^{\kappa\lambda}+\sum_{m,\nu,k,\bar{\lambda}}\omega^{m\nu}\partial_\nu\omega^{k\bar{\lambda}}-\sum_{m,\nu,\bar{\kappa},\lambda}\omega^{m\nu}\partial_\nu\omega^{\bar{\kappa}\lambda}-
$$

$$
-\sum_{m,\nu,\bar{\kappa},\lambda}\omega^{m\nu}\partial_\nu\omega^{\bar{\kappa}\bar{\lambda}}-\sum_{m,\nu,\kappa,\bar{\lambda}}\omega^{m\nu}\partial_\nu\omega^{\kappa\bar{\lambda}}+\sum_{m,\nu,\bar{\kappa},l}\omega^{m\nu}\partial_\nu\omega^{\bar{\kappa}l}+\sum_{\mu,n,k,l}\omega^{\mu n}\partial_n\omega^{kl}+
$$

$$
+\sum_{\mu,n,\kappa,l}\omega^{\mu n}\partial_n\omega^{\kappa l}+\sum_{\mu,n,k,\lambda}\omega^{\mu n}\partial_n\omega^{k\lambda}-\sum_{\mu,n,\kappa,\lambda,}\omega^{\mu n}\partial_n\omega^{\kappa\lambda}+\sum_{\mu,n,k,\bar{\lambda}}\omega^{\mu n}\partial_n\omega^{k\bar{\lambda}}-
$$

$$
-\sum_{\mu,n,\bar{\kappa},\lambda}\omega^{\mu n}\partial_n\omega^{\bar{\kappa}\lambda}-\sum_{\mu,n,\bar{\kappa},\bar{\lambda}}\omega^{\mu n}\partial_n\omega^{\bar{\kappa}\bar{\lambda}}-\sum_{\mu,n,\kappa,\bar{\lambda}}\omega^{\mu n}\partial_n\omega^{\kappa\bar{\lambda}}+\sum_{\mu,n,\bar{\kappa},l}\omega^{\mu n}\partial_n\omega^{\bar{\kappa}l}-
$$

$$
-\sum_{\mu,\nu,k,l}\omega^{\mu\nu}\partial_\nu\omega^{kl}-\sum_{\mu,\nu,\kappa,l}\omega^{\mu\nu}\partial_\nu\omega^{\kappa l}-\sum_{\mu,\nu,k,\lambda}\omega^{\mu\nu}\partial_\nu\omega^{k\lambda}+\sum_{\mu,\nu,\kappa,\lambda}\omega^{\mu\nu}\partial_\nu\omega^{\kappa\lambda}-
$$

$$
-\sum_{\mu,\nu,k,\bar{\lambda}}\omega^{\mu\nu}\partial_\nu\omega^{k\bar{\lambda}}+\sum_{\mu,\nu,\bar{\kappa},\lambda}\omega^{\mu\nu}\partial_\nu\omega^{\bar{\kappa}\lambda}+\sum_{\mu,\nu,\bar{\kappa},\bar{\lambda}}\omega^{\mu\nu}\partial_\nu\omega^{\bar{\kappa}\bar{\lambda}}+\sum_{\mu,\nu,\kappa,\bar{\lambda}}\omega^{\mu\nu}\partial_\nu\omega^{\kappa\bar{\lambda}}-
$$

$$
-\sum_{\mu,\nu,\bar{\kappa},l}\omega^{\mu\nu}\partial_\nu\omega^{\bar{\kappa}l}+\sum_{m,\bar{\nu},\kappa,l}\omega^{m\bar{\nu}}\partial_{\bar{\nu}}\omega^{\kappa l}+\sum_{m,\bar{\nu},k,\lambda}\omega^{m\bar{\nu}}\partial_{\bar{\nu}}\omega^{k\lambda}-\sum_{m,\bar{\nu},\kappa,\lambda}\omega^{m\bar{\nu}}\partial_{\bar{\nu}}\omega^{\kappa\lambda}+
$$

$$
+\sum_{m,\bar{\nu},k,\bar{\lambda}}\omega^{m\bar{\nu}}\partial_{\bar{\nu}}\omega^{k\bar{\lambda}}-\sum_{m,\bar{\nu},\bar{\kappa},\lambda}\omega^{m\bar{\nu}}\partial_{\bar{\nu}}\omega^{\bar{\kappa}\lambda}-\sum_{m,\bar{\nu},\bar{\kappa},\bar{\lambda}}\omega^{m\bar{\nu}}\partial_{\bar{\nu}}\omega^{\bar{\kappa}\bar{\lambda}}-\sum_{m,\bar{\nu},\kappa,\bar{\lambda}}\omega^{m\bar{\nu}}\partial_{\bar{\nu}}\omega^{\kappa\bar{\lambda}}+
$$

$$
+\sum_{m\bar{\nu},\bar{\kappa},l}\omega^{m\bar{\nu}}\partial_{\bar{\nu}}\omega^{\bar{\kappa}l}+\sum_{m\bar{\nu},k,l}\omega^{m\bar{\nu}}\partial_{\bar{\nu}}\omega^{kl}+\sum_{\bar{\mu},n,\kappa,l}\omega^{\bar{\mu}n}\partial_n\omega^{\kappa l}+\sum_{\bar{\mu},n,k,\lambda}\omega^{\bar{\mu}n}\partial_n\omega^{k\lambda}-
$$

$$
+\sum_{\bar{\mu},n,k,\bar{\lambda}}\omega^{\bar{\mu}n}\partial_n\omega^{k\bar{\lambda}}-\sum_{\bar{\mu},n,\bar{\kappa},\lambda}\omega^{\bar{\mu}n}\partial_n\omega^{\bar{\kappa}\lambda}-\sum_{\bar{\mu},n,\bar{\kappa},\bar{\lambda}}\omega^{\bar{\mu}n}\partial_n\omega^{\bar{\kappa}\bar{\lambda}}-\sum_{\bar{\mu},n,\kappa,\bar{\lambda}}\omega^{\bar{\mu}n}\partial_n\omega^{\kappa\bar{\lambda}}+
$$

$$
+\sum_{\bar{\mu},n,\bar{\kappa},l}\omega^{\bar{\mu}n}\partial_n\omega^{\bar{\kappa}l}+\sum_{\bar{\mu},n,k,l}\omega^{\bar{\mu}n}\partial_n\omega^{kl}-\sum_{\mu,\bar{\nu},\kappa,l}\omega^{\mu\bar{\nu}}\partial_{\bar{\nu}}\omega^{\kappa l}-\sum_{\mu,\bar{\nu},k,\lambda}\omega^{\mu\bar{\nu}}\partial_{\bar{\nu}}\omega^{k\lambda}+
$$

$$
\begin{aligned}
&+\sum_{\mu,\bar\nu,\kappa,\lambda}\omega^{\mu\bar\nu}\partial_{\bar\nu}\omega^{\kappa\lambda}-\sum_{\mu,\bar\nu,k,\bar\lambda}\omega^{\mu\bar\nu}\partial_{\bar\nu}\omega^{k\bar\lambda}+\sum_{\mu,\bar\nu,\bar\kappa,\lambda}\omega^{\mu\bar\nu}\partial_{\bar\nu}\omega^{\bar\kappa\lambda}+\sum_{\mu,\bar\nu,\bar\kappa,\bar\lambda}\omega^{\mu\bar\nu}\partial_{\bar\nu}\omega^{\bar\kappa\bar\lambda}+\\
&+\sum_{\mu,\bar\nu,\kappa,\bar\lambda}\omega^{\mu\bar\nu}\partial_{\bar\nu}\omega^{\kappa\bar\lambda}-\sum_{\mu,\bar\nu,\bar\kappa,l}\omega^{\mu\bar\nu}\partial_{\bar\nu}\omega^{\bar\kappa l}-\sum_{\mu,\bar\nu,k,l}\omega^{\mu\bar\nu}\partial_{\bar\nu}\omega^{kl}-\sum_{\bar\mu,\nu,\kappa,l}\omega^{\bar\mu\nu}\partial_{\nu}\omega^{\kappa l}-\\
&-\sum_{\bar\mu,\nu,k,\lambda}\omega^{\bar\mu\nu}\partial_{\nu}\omega^{k\lambda}+\sum_{\bar\mu,\nu,\kappa,\lambda}\omega^{\bar\mu\nu}\partial_{\nu}\omega^{\kappa\lambda}-\sum_{\bar\mu,\nu,k,\bar\lambda}\omega^{\bar\mu\nu}\partial_{\nu}\omega^{k\bar\lambda}+\sum_{\bar\mu,\nu,\bar\kappa,\lambda}\omega^{\bar\mu\nu}\partial_{\nu}\omega^{\bar\kappa\lambda}+\\
&+\sum_{\bar\mu,\nu,\bar\kappa,\bar\lambda}\omega^{\bar\mu\nu}\partial_{\nu}\omega^{\bar\kappa\bar\lambda}+\sum_{\bar\mu,\nu,\kappa,\bar\lambda}\omega^{\bar\mu\nu}\partial_{\nu}\omega^{\kappa\bar\lambda}-\sum_{\bar\mu,\nu,\bar\kappa,l}\omega^{\bar\mu\nu}\partial_{\nu}\omega^{\bar\kappa l}-\sum_{\bar\mu,\nu,k,l}\omega^{\bar\mu\nu}\partial_{\nu}\omega^{kl}-\\
&-\sum_{\bar\mu,\bar\nu,\kappa,l}\omega^{\bar\mu\bar\nu}\partial_{\bar\nu}\omega^{\kappa l}-\sum_{\bar\mu,\bar\nu,k,\lambda}\omega^{\bar\mu\bar\nu}\partial_{\bar\nu}\omega^{k\lambda}+\sum_{\bar\mu,\bar\nu,\kappa,\lambda}\omega^{\bar\mu\bar\nu}\partial_{\bar\nu}\omega^{\kappa\lambda}-\sum_{\bar\mu,\bar\nu,k,\bar\lambda}\omega^{\bar\mu\bar\nu}\partial_{\bar\nu}\omega^{k\bar\lambda}+\\
&+\sum_{\bar\mu,\bar\nu,\bar\kappa,\lambda}\omega^{\bar\mu\bar\nu}\partial_{\bar\nu}\omega^{\bar\kappa\lambda}+\sum_{\mu,\bar\nu,\bar\kappa,\bar\lambda}\omega^{\bar\mu\bar\nu}\partial_{\bar\nu}\omega^{\bar\kappa\bar\lambda}+\sum_{\mu,\bar\nu,\bar\kappa,\bar\lambda}\omega^{\bar\mu\bar\nu}\partial_{\bar\nu}\omega^{\kappa\bar\lambda}-\sum_{\mu,\bar\nu,\bar\kappa,l}\omega^{\mu\bar\nu}\partial_{\bar\nu}\omega^{\bar\kappa l}-\\
&-\sum_{\bar\mu,\bar\nu,k,l}\omega^{\bar\mu\bar\nu}\partial_{\bar\nu}\omega^{kl}-\sum_{m,n,k,l}\omega^{mn}\partial_{m}\omega^{kl}-\sum_{m,n,\kappa,l}\omega^{mn}\partial_{m}\omega^{\kappa l}-\sum_{m,n,k,\lambda}\omega^{mn}\partial_{m}\omega^{k\lambda}+\\
&+\sum_{m,n,\kappa,\lambda}\omega^{mn}\partial_{m}\omega^{\kappa\lambda}-\sum_{m,n,k,\bar\lambda}\omega^{mn}\partial_{m}\omega^{k\bar\lambda}+\sum_{m,n,\bar\kappa,\lambda}\omega^{mn}\partial_{m}\omega^{\bar\kappa\lambda}+\sum_{m,n,\bar\kappa,\bar\lambda}\omega^{mn}\partial_{m}\omega^{\bar\kappa\bar\lambda}+\\
&+\sum_{m,n,\kappa,\bar\lambda}\omega^{mn}\partial_{m}\omega^{\kappa\bar\lambda}-\sum_{m,n,\bar\kappa,l}\omega^{mn}\partial_{m}\omega^{\bar\kappa l}-\sum_{m,\nu,k,l}\omega^{m\nu}\partial_{m}\omega^{kl}-\sum_{m,\nu,\kappa,l}\omega^{m\nu}\partial_{m}\omega^{\kappa l}-\\
&-\sum_{m,\nu,k,\lambda}\omega^{m\nu}\partial_{m}\omega^{k\lambda}+\sum_{m,\nu,\kappa,\lambda}\omega^{m\nu}\partial_{m}\omega^{\kappa\lambda}-\sum_{m,\nu,k,\bar\lambda}\omega^{m\nu}\partial_{m}\omega^{k\bar\lambda}+\sum_{m,\nu,\bar\kappa,\lambda}\omega^{m\nu}\partial_{m}\omega^{\bar\kappa\lambda}+\\
&+\sum_{m,\nu,\bar\kappa,\bar\lambda}\omega^{m\nu}\partial_{m}\omega^{\bar\kappa\bar\lambda}+\sum_{m,\nu,\kappa,\bar\lambda}\omega^{m\nu}\partial_{m}\omega^{\kappa\bar\lambda}-\sum_{m,\nu,\bar\kappa,l}\omega^{m\nu}\partial_{m}\omega^{\bar\kappa l}-\sum_{\mu,n,k,l}\omega^{\mu n}\partial_{\mu}\omega^{kl}-\\
&-\sum_{\mu,n,\kappa,l}\omega^{\mu n}\partial_{\mu}\omega^{\kappa l}-\sum_{\mu,n,k,\lambda}\omega^{\mu n}\partial_{\mu}\omega^{k\lambda}+\sum_{\mu,n,\kappa,\lambda}\omega^{\mu n}\partial_{\mu}\omega^{\kappa\lambda}-\sum_{\mu,n,k,\bar\lambda}\omega^{\mu n}\partial_{\mu}\omega^{k\bar\lambda}+\\
&+\sum_{\mu,n,\bar\kappa,\lambda}\omega^{\mu n}\partial_{\mu}\omega^{\bar\kappa\lambda}+\sum_{\mu,n,\bar\kappa,\bar\lambda}\omega^{\mu n}\partial_{\mu}\omega^{\bar\kappa\bar\lambda}+\sum_{\mu,n,\kappa,\bar\lambda}\omega^{\mu n}\partial_{\mu}\omega^{\kappa\bar\lambda}-\sum_{\mu,n,\bar\kappa,l}\omega^{\mu n}\partial_{\mu}\omega^{\bar\kappa l}+\\
&+\sum_{\mu,\nu,k,l}\omega^{\mu\nu}\partial_{\mu}\omega^{kl}-\sum_{\mu,\nu,\kappa,l}\omega^{\mu\nu}\partial_{\mu}\omega^{\kappa l}+\sum_{\mu,\nu,k,\lambda}\omega^{\mu\nu}\partial_{\mu}\omega^{k\lambda}-\sum_{\mu,\nu,\kappa,\lambda}\omega^{\mu\nu}\partial_{\mu}\omega^{\kappa\lambda}+\\
&+\sum_{\mu,\nu,k,\bar\lambda}\omega^{\mu\nu}\partial_{\mu}\omega^{k\bar\lambda}-\sum_{\mu,\nu,\bar\kappa,\lambda}\omega^{\mu\nu}\partial_{\mu}\omega^{\bar\kappa\lambda}-\sum_{\mu,\nu,\bar\kappa,\bar\lambda}\omega^{\mu\nu}\partial_{\mu}\omega^{\bar\kappa\bar\lambda}-\sum_{\mu,\nu,\kappa,\bar\lambda}\omega^{\mu\nu}\partial_{\mu}\omega^{\kappa\bar\lambda}+\\
&+\sum_{\mu,\nu,\bar\kappa,l}\omega^{\mu\nu}\partial_{\mu}\omega^{\bar\kappa l}-\sum_{m,\bar\nu,\kappa,l}\omega^{m\bar\nu}\partial_{m}\omega^{\kappa l}-\sum_{m,\bar\nu,k,\lambda}\omega^{m\bar\nu}\partial_{m}\omega^{k\lambda}+\sum_{m,\bar\nu,\kappa,\lambda}\omega^{m\bar\nu}\partial_{m}\omega^{\kappa\lambda}-\\
&-\sum_{m,\bar\nu,k,\bar\lambda}\omega^{m\bar\nu}\partial_{m}\omega^{k\bar\lambda}+\sum_{m,\bar\nu,\bar\kappa\lambda}\omega^{m\bar\nu}\partial_{m}\omega^{\bar\kappa\lambda}+\sum_{m,\bar\nu,\bar\kappa,\bar\lambda}\omega^{m\bar\nu}\partial_{m}\omega^{\bar\kappa\bar\lambda}+\sum_{m,\bar\nu,\kappa,\bar\lambda}\omega^{m\bar\nu}\partial_{m}\omega^{\kappa\bar\lambda}-\\
&-\sum_{m,\bar\nu,\bar\kappa,l}\omega^{m\bar\nu}\partial_{m}\omega^{\bar\kappa l}+\sum_{\bar\mu,\bar\nu,k,l}\omega^{\bar\mu\bar\nu}\partial_{\bar\mu}\omega^{kl}-\sum_{\bar\mu,n,\kappa,l}\omega^{\bar\mu n}\partial_{\bar\mu}\omega^{\kappa l}-\sum_{\bar\mu,n,k,\lambda}\omega^{\bar\mu n}\partial_{\bar\mu}\omega^{k\lambda}+\\
&+\sum_{\bar\mu,n,\kappa,\lambda}\omega^{\bar\mu n}\partial_{\bar\mu}\omega^{\kappa\lambda}-\sum_{\bar\mu,n,k,\bar\lambda}\omega^{\bar\mu n}\partial_{\bar\mu}\omega^{k\bar\lambda}+\sum_{\bar\mu,n,\bar\kappa,\lambda}\omega^{\bar\mu n}\partial_{\bar\mu}\omega^{\bar\kappa\lambda}+\sum_{\bar\mu,n,\bar\kappa,\bar\lambda}\omega^{\bar\mu n}\partial_{\bar\mu}\omega^{\bar\kappa\bar\lambda}+\\
&+\sum_{\bar\mu,n,\kappa,\bar\lambda}\omega^{\bar\mu n}\partial_{\bar\mu}\omega^{\kappa\bar\lambda}-\sum_{\bar\mu,n,\bar\kappa,l}\omega^{\bar\mu n}\partial_{\bar\mu}\omega^{\bar\kappa l}-\sum_{\bar\mu,n,,k,l}\omega^{\bar\mu n}\partial_{\bar\mu}\omega^{kl}+\sum_{\mu,\bar\nu,\kappa,l}\omega^{\mu\bar\nu}\partial_{\mu}\omega^{\kappa l}-
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\mu,\bar{\nu},k,\lambda}\omega^{\mu\bar{\nu}}\partial_{\mu}\omega^{k\lambda}-\sum_{\mu,\bar{\nu},\kappa,\lambda}\omega^{\mu\bar{\nu}}\partial_{\mu}\omega^{\kappa\lambda}+\sum_{\mu,\bar{\nu},k,\bar{\lambda}}\omega^{\mu\bar{\nu}}\partial_{\mu}\omega^{k\bar{\lambda}}-\sum_{\mu,\bar{\nu},\bar{\kappa},\lambda}\omega^{\mu\bar{\nu}}\partial_{\mu}\omega^{\bar{\kappa}\lambda}-\\
&-\sum_{\mu,\bar{\nu},\bar{\kappa},\bar{\lambda}}\omega^{\mu\bar{\nu}}\partial_{\mu}\omega^{\bar{\kappa}\bar{\lambda}}-\sum_{\mu,\bar{\nu},\kappa,\bar{\lambda}}\omega^{\mu\bar{\nu}}\partial_{\mu}\omega^{\kappa\bar{\lambda}}+\sum_{\mu,\bar{\nu},\bar{\kappa},l}\omega^{\mu\bar{\nu}}\partial_{\mu}\omega^{\bar{\kappa}l}+\sum_{\bar{\mu},\bar{\nu},k,l}\omega^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}\omega^{kl}+\\
&+\sum_{\bar{\mu},\nu,\kappa,l}\omega^{\bar{\mu}\nu}\partial_{\bar{\mu}}\omega^{\kappa l}+\sum_{\bar{\mu},\nu,k,l}\omega^{\bar{\mu}\nu}\partial_{\bar{\mu}}\omega^{k\lambda}-\sum_{\bar{\mu},\nu,\kappa,\lambda}\omega^{\bar{\mu}\nu}\partial_{\bar{\mu}}\omega^{\kappa\lambda}+\sum_{\bar{\mu},\nu,k,\bar{\lambda}}\omega^{\bar{\mu}\nu}\partial_{\bar{\mu}}\omega^{k\bar{\lambda}}-\\
&-\sum_{\bar{\mu},\nu,\bar{\kappa},\lambda}\omega^{\bar{\mu}\nu}\partial_{\bar{\mu}}\omega^{\bar{\kappa}\lambda}-\sum_{\bar{\mu},\nu,\bar{\kappa},\bar{\lambda}}\omega^{\bar{\mu}\nu}\partial_{\bar{\mu}}\omega^{\bar{\kappa}\bar{\lambda}}-\sum_{\bar{\mu},\nu,\kappa,\bar{\lambda}}\omega^{\bar{\mu}\nu}\partial_{\bar{\mu}}\omega^{\kappa\bar{\lambda}}+\sum_{\bar{\mu},\nu,\bar{\kappa},l}\omega^{\bar{\mu}\nu}\partial_{\bar{\mu}}\omega^{\bar{\kappa}l}+\\
&+\sum_{\bar{\mu},\nu,k,l}\omega^{\bar{\mu}\nu}\partial_{\bar{\mu}}\omega^{kl}-\sum_{\bar{\mu},\bar{\nu},\kappa,l}\omega^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}\omega^{\kappa l}-\sum_{\bar{\mu},\bar{\nu},k,\lambda}\omega^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}\omega^{k\lambda}-\sum_{\bar{\mu},\bar{\nu},\kappa,\lambda}\omega^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}\omega^{\kappa\lambda}-\\
&+\sum_{\bar{\mu},\bar{\nu},k,\bar{\lambda}}\omega^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}\omega^{k\bar{\lambda}}-\sum_{\bar{\mu},\bar{\nu},\bar{\kappa},\lambda}\omega^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}\omega^{\bar{\kappa}\lambda}-\sum_{\bar{\mu},\bar{\nu},\bar{\kappa},\bar{\lambda}}\omega^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}\omega^{\bar{\kappa}\bar{\lambda}}-\sum_{\bar{\mu},\bar{\nu},\kappa,\bar{\lambda}}\omega^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}\omega^{\kappa\bar{\lambda}}-\\
&-\sum_{\mu,\bar{\nu},\bar{\kappa},\lambda}\omega^{\mu\bar{\nu}}\partial_{\bar{\nu}}\omega^{\bar{\kappa}\lambda}+\sum_{\bar{\mu},\bar{\nu},k,l}\omega^{\bar{\mu}\bar{\nu}}\partial_{\bar{\mu}}\omega^{kl}\Bigg)
\end{aligned}
\tag{C.1}
$$

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