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DIPLOMARBEIT

On Approaches to Operational Risk and the Application of general formulae from Ruin Theory

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The purpose of this thesis is to give an introductory overview on operational risk, and in particular some applications of extreme value theory and methods using ruin probabilities. The latter approach is studied in detail, based on the article [DIK08].

The thesis consists of 5 chapters.

In chapter 1 we have present some tools that are often used in operational risk and in ruin theory. We recollect some important probability distributions and their properties. Then we review the risk measures, Value-at-risk and expected shortfall. Next we introduce the basics of the theory of copulas and the basic notions of Extreme Value Theory

In chapter 2 is give the framework of operational risk. We describe three approaches: the Basic Indicator Approach (BIA), the Standardized Approach (SA) and the Advanced Measurement Approaches (AMA).

Here we give a typical Advanced Measurement Approach solution for calculating of an operational risk charge for one year, using historical losses.

In chapter 3 we discuss two methods of the Advanced Measurement Approaches: Extreme Value Theory in Operational risk and Ruin theory in operational risk.

In chapter 4 contained the main results of the thesis. In this chapter we have given the Kaishev–Dimitrov formula in two cases: for discrete claim distribution and continuous claim distribution. We have calculated the probability of non-ruin in a finite time x for each of these two cases.

In chapter 5 we have considered three alternative distributions of consecutive losses: Logarithmic, Exponential and Pareto. We have given solutions with graphs. The solutions are presented with Mathematica programs. For calculation purposes we also use a formula from Picard –Lefevre [PL97].

1. Introduction

1.1 Some important probability distributions

In this section we summarize some important probability distributions that are used often in operational risk and ruin theory.(See [Cru02])

Normal (Gauss) Distribution

The normal distribution with mean μ and variance σ^2 has distribution with probability density function (p.d.f)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

and cumulative distribution function (c.d.f)

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} exp\left(-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right) dt = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

on the Interval $x \in (-\infty, +\infty)$.

For $\mu = 0$ and $\sigma = 1$

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} exp\left(-\frac{1}{2}t^2\right) dt = \Phi(x)$$

This formula is known as the standard normal.

Log-normal Distribution

The log-normal distribution is a single-tailed probability distribution of any random variable whose logarithm is normally distributed. The probability density function of the lognormal distribution is given by

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{ln\, x-\mu}{\sigma}\right)^2\right), x > 0$$

where μ and σ are the mean and standard deviation of the variable's natural logarithm.

The Cumulative distribution function is given by

$$F(x) = \Phi\left(\frac{ln\, x-\mu}{\sigma}\right)$$

where Φ denotes the cumulative distribution function of the standard normal distribution.

Exponential Distribution

The exponential distributions are a class of continuous probability distributions. The probability density function of an exponential distribution is

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

where $\lambda > 0$ is the parameter of the distribution (the rate parameter). The distribution is supported on the interval $[0, \infty)$.

The cumulative distribution function is given by

$$F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

The inverse cumulative distribution function (quantile function) is:

$$F^{-1}(x;\lambda) = -\frac{1}{\lambda}\log(1-x)$$

for $0 \leq x < 1$.

Pareto Distribution

The Pareto distribution was named after the Italian economist Vilfredo Pareto, who formulated an economic law (Pareto´s Law) dealing with the distribution of income over a population.

The probability density function is

$$f(x;\alpha,x_m) = \alpha \frac{x_m^{\alpha}}{x^{\alpha+1}}$$

for $x > x_m$.

The classical cumulative distribution functions of a Pareto random variable with parameters k and x_m is

$$F(x) = 1 - \left(\frac{x}{x_m}\right)^{-\alpha}$$

for all $x > x_m$, where x_m is the (necessarily positive) minimum possible value of X, and α is a positive parameter.

The quantile function is:

$$F^{-1}(u) = x_m(1-u)^{-1/\alpha}$$

The family of ***generalized Pareto distributions (GPD)*** has three parameters μ, σ and ξ. The cumulative distribution function is

$$F_{(\xi,\mu,\sigma)}(x) = 1 - \left(1 + \frac{\xi(x-\mu)}{\sigma}\right)^{-1/\xi}$$

for $x \geq \mu$, and $x \leq \mu - \sigma/\xi$ when $\xi < 0$, where $\mu \in R$ is the location parameter, $\sigma > 0$ the scale parameter and $\xi \in R$ the shape parameter. Note that some references give the "shape parameter" as $k = -\xi$.

The probability density function is

$$f_{(\xi,\mu,\sigma)}(x) = \frac{1}{\sigma}\left(1 + \frac{\xi(x-\mu)}{\sigma}\right)^{\left(-\frac{1}{\xi}-1\right)}$$

for $x \geq \mu$, and $x \leq \mu - \sigma/\xi$ when $\xi < 0$.

Beta Distribution

The beta distribution is a family of continuous probability distributions defined on the interval $[0,1]$ parameterized by two positive shape parameters, typically denoted by α and β. It is the special case of the Dirichlet distribution with only two parameters.

The probability density function of the beta distribution is:

$$f(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

where Γ is the gamma function. The beta function, B, appears as a normalization constant to ensure that the total probability integrates to unity.

The cumulative distribution function is

$$F(x;\alpha,\beta) = \frac{B_x(\alpha,\beta)}{B(\alpha,\beta)} = I_x(\alpha,\beta)$$

where $B_x(\alpha,\beta)$ is the incomplete beta function and $I_x(\alpha,\beta)$ is the regularized incomplete beta function.

g-and-h distribution

Let $Z \sim N(0,1)$ be a standard normal random variable. A random variable X is said to have a g-and-h distribution with parameters a, b, g, $h \in \mathbb{R}$, if X satisfies

$$X = a + b\,\frac{exp(gZ)-1}{g}\,exp\left(\frac{hZ^2}{2}\right)$$

for $g \neq 0$. And

$$X = a + bZ\, exp\left(\frac{hZ^2}{2}\right)$$

for $g = 0$.

We write $X \sim$ g-and-h, or when X has distribution function

$$F(x) = \Phi\left(k^{-1}(x)\right),$$

where

$$k(x) = \begin{cases} \dfrac{exp(gx) - 1}{g}\, exp\left(\dfrac{hx^2}{2}\right), & for\ g \neq 0 \\ x\, exp\left(\dfrac{hx^2}{2}\right), & for\ g = 0 \end{cases}$$

and Φ denotes the standard normal distribution function.

For applications of g-and-h in operational risk see [DEL09].

Poisson Distribution

The Poisson distribution is named after the French mathematician and physicist Simeon Denis Poisson. The Poisson distribution has probability mass function:

$$p_k = \frac{e^{-\lambda}\lambda^k}{k!},$$

where $k = 0,1,2, \ldots$.

The cumulative function (a step function) is given by

$$F(x) = e^{-\lambda t} \sum_{i=0}^{\lfloor x \rfloor} \frac{(\lambda t)^i}{i!}.$$

The probability generating function is:

$$P(z) = e^{\lambda(z-1)}.$$

Logarithmic distribution

The logarithmic distribution (also known as the logarithmic series distribution or the log-series distribution) is a discrete probability distribution. The probability mass function of a $Log\ (\alpha)$ is

$$f(i) = -\frac{1}{ln(1-\alpha)}\frac{\alpha^i}{i},$$

for i≥ 1, and where $0 < \alpha < 1$.

The cumulative distribution function is

$$F(i) = 1 + \frac{B(\alpha; i+1,0)}{ln(1-\alpha)},$$

where B is the incomplete beta function.

1.2 Important risk measures

- **Value-at-Risk (VaR) (** See [EFMcN05])

Value-at-Risk is probably the most widely used risk measure in financial institutions and has also made its way into the Basel II capital-adequacy framework-hence it merits an extensive discussion.

"VaR answers the question: how much can I lose with x% probability over a pre-set horizon"

Given some confidence level $\alpha \in (0,1)$. The VaR at the confidence level α is given by the smallest number l such that the probability that the loss L exceeds l is no larger than $(1-\alpha)$.

$$VaR_\alpha = inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\} = inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\}.$$

In probabilistic terms, VaR is thus simply a quantile of the loss distribution.

Typical values for α are $\alpha = 0.95$ or $\alpha = 0.99$; in market risk management the time horizon Δ is usually 1 or 10 days, in credit risk management and operational risk management Δ is usually one year.

Mean-VaR: Denote by μ the mean of the loss distribution. Sometimes the statistic

$$VaR_\alpha^{mean} = VaR_\alpha - \mu$$

is used for capital-adequacy purposes instead of ordinary VaR. If the time horizon Δ equals one day, VaR_α^{mean} is sometimes referred to as daily earnings at risk. The distinction between ordinary VaR and VaR_α^{mean} is of little relevance in market risk management, where the time horizon is short and μ is close to zero. It becomes relevant in credit where the risk-management horizon is longer. In particular, in loan pricing one uses VaR^{mean} to determine the economic capital needed as a buffer against unexpected losses in a loan portfolio.

Given some increasing function $T: \mathbb{R} \to \mathbb{R}$, the generalized inverse of T is defined by

$$T^{\leftarrow}(y) = inf\{x \in \mathbb{R} : T(x) \geq y\},$$

where we use the convention that the infimum of an empty set is ∞.

Given some distribution function F, the generalized inverse $F^{\leftarrow}$is called the quantile function of F. For $\alpha \in (0,1)$ the α-quantile of F is given by

$$q_\alpha(F) = F^{\leftarrow}(\alpha) = inf\{x \in \mathbb{R} : F(x) \geq \alpha\}.$$

For a random variable X with distribution function F we often use the alternative notation

$$q_\alpha(X) = q_\alpha(F).$$

If F is continuous and strictly increasing, than

$$q_\alpha(F) = F^{-1}(\alpha),$$

where F^{-1} is the ordinary inverse of F.

- **Expected shortfall (ES)**

Expected shortfall is closely related to VaR.

For a loss L with $E(|L|) < \infty$ and distribution function F_L the expected shortfall at confidence level $\alpha \in (0,1)$ is defined as

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 q_u(F_L)du,$$

where $q_u(F_L) = F_L^{\leftarrow}(u)$ is the quantile function of F_L.

Expected shortfall is thus related to VaR by

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(F)du.$$

Instead of fixing a particular confidence level α we average VaR over all levels $u \geq \alpha$ and thus “look further into the tail” of the loss distribution. Obviously ES_α depends only on the distribution of L and obviously $ES_\alpha \geq VaR_\alpha$.

For continuous loss distributions an even more intuitive expression can be derived which shows that expected shortfall can be interpreted as the expected loss that is incurred in the event that VaR is exceeded.

1.3 Introduction to copulas

Copulas help in the understanding of dependence at a deeper level. They allow us to see the potential pitfalls of approaches to dependence that focuses only dependence measures. Copulas express dependence on a quantile scale, which is useful for describing the dependence of extreme outcomes and is natural in a risk-management context, where VaR has led us to think of risk in terms of quantiles of loss distributions. **(** See [EFMcN05])

A copula is a multivariate joint distribution defined on the d-dimensional unit cube $[0,1]^d$ such that every marginal distribution is uniform on the interval $[0,1]$.
Specifically,

$$C: [0,1]^d \to [0,1]$$

is an d-dimensional copula (briefly, d-copula) if:

1. $C(u_1, \dots, u_d)$ is increasing in each component u_i
2. $C(1, \dots, 1, u_i, 1, \dots, 1) = u_i$ for all $i \in \{1, \dots, d\}$, $u_i \in [0,1]$
3. For all $(a_1, \dots, a_d)$, $(b_1, \dots, b_d) \in [0,1]^d$ with $a_i \leq b_i$ is

$$\sum_{i_1=1}^{2} \dots \sum_{i_d=1}^{2} (-1)^{i_1+\dots+i_d} C\left(u_{1i_1}, \dots, u_{di_d}\right) \geq 0,$$

where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all $j \in \{1, \dots, d\}$.

The first property is clearly required of any multivariate distribution function and the second property is the requirement of uniform marginal distributions. The third property is less obvious, but the so-called rectangle inequality in

$$\sum_{i_1=1}^{2} \dots \sum_{i_d=1}^{2} (-1)^{i_1+\dots+i_d} C\left(u_{1i_1}, \dots, u_{di_d}\right) \geq 0$$

ensures that if the random vector $(U_1, \dots, U_d)^{'}$ has distribution function C, then

$$P(a_1 \leq U_1 \leq b_1, \dots, a_d \leq U_d \leq b_d)$$

is non-negative.

The importance of copulas is summarized by the following Sklar's theorem, which shows that:

- all multivariate distribution functions correspond to copulas;
- copulas may be used in conjunction with univariate distribution functions to construct multivariate distribution functions.

Sklar's theorem (1959).

Let F be a joint distribution function with margins $F_1, \dots, F_d$.
Then there exists a copula $C: [0,1]^d \to [0,1]$ such that, for all $x_1, \dots, x_d$ in $\overline{\mathbb{R}} = [-\infty, \infty]$

$$F(x_1, \dots, x_d) = C\big(F_1(x_1), \dots, F_d(x_d)\big).$$

If the margins are continuous, then C is unique; otherwise C is uniquely determined on

$$Ran\ F_1 \times Ran\ F_2 \times \dots \times Ran\ F_d,$$

where $Ran\ F_i = F_i(\overline{\mathbb{R}})$ denotes the range of F_i.
Conversely, if C is a copula and $F_1, \dots, F_d$ are univariate distribution functions, then the function F defined in

$$F(x_1, \dots, x_d) = C\big(F_1(x_1), \dots, F_d(x_d)\big)$$

is a joint distribution function with margins $F_1, \dots, F_d$.

Copulas subdivided into three categories:

- fundamental copulas represent a number of important special dependence structures;
- implicit copulas are extracted from well-known multivariate distributions using Sklar's Theorem, but do not necessarily possess simple closed form expressions;
- explicit copulas have simple closed-form expressions and follow general mathematical constructions known to yield copulas.

Fundamental copulas.

Random variables with continuous distributions are independent if and only if their dependence structure is given by

$$\Pi(u_1, \dots, u_d) = \prod_{i=1}^{d} u_i.$$

The copula $\Pi(u_1, \dots, u_d)$ is independence copula.

The comonotonicity copula is the Fréchet upper bound copula from

$$max\left\{\sum_{i=1}^{d} u_i + 1 - d, 0\right\} \le C(u) \le min\{u_1, \dots, u_d\}:$$

$$M(u_1, \dots, u_d) = min\{u_1, \dots, u_d\}$$

Observe that this special copula is the joint distribution function of the random vector $(U, \dots, U)$, where $U \sim U(0,1)$. Suppose that the random variables $X_1, \dots, X_d$ have continuous distribution functions and are perfectly positively dependent in the sense that they are almost surely strictly increasing functions

The countermonotonicity copula is the two-dimensional Fréchet lower bound copula from

$$max\left\{\sum_{i=1}^{d} u_i + 1 - d, 0\right\} \le C(u) \le min\{u_1, \dots, u_d\}:$$

$$W(u_1, u_2) = max\{u_1 + u_2 - 1, 0\}$$

This copula is the joint distribution function of the random vector $(U, 1 - U)$, where $U \sim U(0,1)$. If X_1 and X_2 have continuous distribution functions and are perfectly negatively dependent in the sense that X_2 is almost surely a strictly decreasing function of X_1.

Implicit copulas:

- the Gauss copula
- the t copula

The Gaussian and t copulas are copulas implied by well-known multivariate distribution functions and do not themselves have simple closed forms.

Explicit copulas:

- the bivariate Gumbel copula

$$C_\theta^{Gu}(u_1, u_2) = exp\left\{-\left((-\ln u_1)^\theta + (-\ln u_2)^\theta\right)^{1/\theta}\right\}, \qquad 1 \le \theta < \infty.$$

If $\theta = 1$ we obtain the independence copula as a special case, and the limit of C_θ^{Gu} as $\theta \to \infty$ is the two-dimensional comonotonicity copula. Thus the Gumbel copula interpolates between independence and perfect dependence and the parameter θ represents the strength of dependence.

- the bivariate Clayton copula

$$C_\theta^{Cl}(u_1, u_2) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}, 0 < \theta < \infty$$

In the limit as $\theta \to 0$ we approach the independence copula, and as $\theta \to \infty$ we approach the two-dimensional comonotonicity copula.

The bivariate Gumbel copula and bivariate Clayton copula have simple closed forms.

The Gumbel copula and the Clayton copula belong to the family of so called Archimedean copulas.

A d-dimensional Clayton copula is

$$C_{\theta,\delta}^{Cl}(u) = \left(u_1^{-\theta} + \cdots + u_d^{-\theta} - d + 1\right)^{-1/\theta} =$$
$$= \left(\sum_{i=1}^{d} u_i^{-\theta} - d + 1\right)^{-1/\theta}, \qquad for \quad \theta \geq 0,$$

where $0 \leq u_i \leq 1$ and $\theta \in (0, \infty)$ is a parameter.

The limiting case $\theta = 0$ should be interpreted as the d-dimensional independence copula.

The density is given by

$$c^{Cl}(u_1, \dots, u_d; \theta) = \theta^d \frac{\Gamma\left(1/_\theta + d\right)}{\Gamma\left(1/_\theta\right)} \left(\prod_{i=1}^{d} u_i^{-\theta-1}\right) \left(\sum_{i=1}^{d} u_i^{-\theta} - d + 1\right)^{-1/\theta - d}.$$

As $\theta \to 0$, the Clayton copula converges to the product copula with density $c(u_1, \dots, u_d) = 1$.

The Clayton copula has lower tail dependence with coefficient $\lambda_L = 2^{-1/\theta}$, which makes it convenient for modeling dependence in the left tails of the marginal distributions, i.e. between very small claims.

Based on the Clayton copula, one can model upper tail dependence using the multivariate Rotated Clayton copula, defined as

$$C^{RCl}(u_1, \dots, u_d; \theta) = 1 + \sum_{l=1}^{d} (-1)^l \sum_{1 \leq i_1 < \cdots < i_l = d} C^{Cl}\left(1 - u_{i_1}, \dots, 1 - u_{i_l}\right),$$

with density $c^{RCl}(u_1, \dots, u_d; \theta) = c^{Cl}(1 - u_1, \dots, 1 - u_d; \theta)$ and $\theta \in (0, \infty)$. The value $\theta = 0$ corresponds to independence as for C^{Cl}

The Rotated Clayton copula has upper tail dependence with coefficient $\lambda_U = 2^{-1/\theta}$ and is suitable for modeling dependence between extreme insurance losses.

1.4 Basics of extreme value theory

The reason for use of the EVT lies in the fact that EVT has solid foundations in the mathematical theory of the behavior of extremes and, moreover, many applications have indicated that EVT appears to be a satisfactory scientific approach in treating rare, large losses. (See [EKM97])

EVT is applied to real data in two related ways.

The first approach (see Reiss and Thomas, 2001, p. 14 ff) deals with the maximum (or minimum) values the variable takes in successive periods, for example months or years. These observations constitute the extreme events, also called block (or per-period) maxima.

Classical EVT is concerned with limiting distributions for normalized maxima $M_n = max(X_1, \dots, X_n)$ of i.i.d. random variables. The only possible non-degenerate limiting distributions for normalized block maxima are in the GEV family.

At the heart of this approach is the "Limit laws for Maxima" (Fisher – Tippet Theorem), which states that there are only three types of distributions which can arise as limiting distributions of extreme values in random samples: the Weibull type, the Gumbel type and the Fréchet type.

Fisher – Tippet Theorem

Let (X_n) be a sequence of i.i.d. random variables. If there exist norming constants $c_n > 0$, $d_n \in \mathbb{R}$ and some non degenerate distribution function H such that

$$c_n^{-1}(M_n - d_n) \xrightarrow{d} H$$

where $M_n = max(X_1, \dots, X_n)$,
then H belong to the type of one of the following three distribution functions:

Fréchet $\quad \Phi_\alpha(x) = \begin{cases} 0, & x \le 0 \\ exp(-x^{-\alpha}), & x > 0 \end{cases} \quad \alpha > 0,$

Weibull $\quad \psi_\alpha(x) = \begin{cases} exp(-(-x)^{-\alpha}), & x \le 0 \\ 1, & x > 0 \end{cases} \quad \alpha > 0,$

Gumbel $\quad \Lambda(x) = exp(-e^{-x}), x \in \mathbb{R}.$

This result is very important, since the asymptotic distribution of the maxima always belongs to one of these three distributions, regardless of the original one. Therefore the majority of the distributions used in finance and actuarial sciences can be divided into these three classes, according to their tail-heaviness:

- light-tail distributions with finite moments and tails, converging to the Weibull curve (Beta, Weibull);
- medium-tail distributions for which all moments are finite and whose cumulative distribution functions decline exponentially in the tails, like the Gumbel curve (Normal, Gamma, Lognormal);

- heavy-tail distributions, whose cumulative distribution functions decline with a power in the tails, like the Fréchet curve (T-Student, Pareto, Log-Gamma, Cauchy).

The Weibull, Gumbel and Fréchet distributions can be represented in a single three parameter model, known as the **Generalized Extreme Value distribution**:

$$H_\xi(x) = \begin{cases} exp\left(-(1+\xi x)^{-1/\xi}\right), & \xi \neq 0 \\ exp(-e^{-x}), & \xi = 0 \end{cases} ,$$

where: $1 + \xi x > 0$.

A three-parameter family is obtained by defining $H_{\xi,\mu,\sigma}(x) = H_\xi((x-\mu)/\sigma)$ for a location parameter $\mu \in \mathbb{R}$ and a scale parameter $\sigma > 0$. The parameter ξ is known as the shape parameter of the generalized extreme value distribution and H_ξ defines a type of distribution, meaning a family of distributions specified up to location and scaling.

When $\xi > 0$, the distribution is a Fréchet distribution; when $\xi = 0$, it is a Gumbel distribution and when $\xi < 0$ it is a Weibull distribution.

The role of the generalized extreme value distribution in the theory of extremes is analogous to that of the normal distribution (and more generally the stable laws) in the central limit theory for sums of random variables. More generally, the generalized extreme value distribution given by $H_\xi(x)$ describes the limit distribution of suitably normalized maxima. Observe that the Weibull distribution is a short-tailed distribution with a so-called finite right endpoint. The right endpoint of a distribution will be denoted by

$$x_F = sup\{x \in \mathbb{R}: F(x) < 1\}.$$

The Gumbel and Fréchet distributions have infinite right endpoints, but the decay of the tail of the Fréchet distribution is much slower than that of the Gumbel distribution. Suppose that block maxima M_n of i.i.d. random variables converge in distribution under an appropriate normalization. Recalling that

$$P(M_n \leq x) = F^n(x),$$

we observe that this convergence means that there exist sequences of real constants (d_n) and (c_n), where $(d_n) > 0$ for all n, such that

$$\lim_{n\to\infty} P\left(\frac{M_n - d_n}{c_n} \leq x\right) = \lim_{x\to\infty} F^n(c_n x + d_n) = H(x),$$

for some non-degenerate distribution function $H(x)$.

The distributions lead to limits for maxima.

The Fréchet case: The distributions that lead to the Fréchet limit $H_\xi(x)$ for $\xi > 0$ have a particularly elegant characterization involving slowly varying or regularly varying functions. Slowly varying functions are functions which, in comparison with power functions, change relatively slowly for large x, an example being the logarithm $L(x) = ln(x)$. Regularly varying functions are functions which can be represented by power functions multiplied by slowly varying functions, i.e. $h(x) = x^\rho L(x)$.

Distributions giving rise to the Fréchet case are distributions with tails that are regularly varying functions with a negative index of variation. Their tails decay essentially like a power function and the rate of decay $\alpha = 1/\xi$ is often referred to as the tail index of the distribution. These distributions are of particular interest in financial applications because they are heavy-tailed distributions with infinite higher moments.

The Gumbel case: The characterization of distributions in this class is more complicated than in the Fréchet class. The distributions in this class have tails that have an essentially exponential decay. A positive-valued random variable with a distribution function in MDA(H_0) has finite moments of any positive order, i.e. $E(X^k) < \infty$ for every $k > 0$ ([EKM97], p. 148).

If

$$\lim_{n\to\infty} P\left(\frac{M_n - d_n}{c_n} \le x\right) = \lim_{x\to\infty} F^n(c_n x + d_n) = H(x)$$

holds for some non-degenerate distribution function H, then F is said to be in the maximum domain of attraction of H, written $F \in MDA(H)$

However, there is a great deal of variety in the tails of distributions in this class, so, for example, both the normal and the lognormal distributions belong to the Gumbel class (EKM, pp. 145–147). The normal distribution is thin tailed, but the lognormal distribution has much heavier tails and we would need to collect a lot of data from the lognormal distribution before we could distinguish its tail behavior from that of a distribution in the Fréchet class. The Gumbel class is also interesting because it contains many distributions with much heavier tails than the normal, even if these are not regularly varying power tails. Examples are hyperbolic and generalized hyperbolic distributions (with the exception of the special boundary case that is Student *t*).

The Weibull case. This is perhaps the least important case for financial modeling, at least in the area of market risk, since the distributions in this class all have finite right endpoints.

The second approach to EVT is the Peaks Over Threshold (POT) method, tailored for the analysis of data bigger than preset high thresholds.
The severity component of the POT method is based on a Generalized Pareto Distribution.

The so-called POT model makes the following assumptions:

- exceedences occur according to a homogeneous Poisson process in time;
- excess amount above the threshold are i.i.d. and independent of exceedance times;
- the distribution of the excess amounts is generalized Pareto.

The interpretation of ξ in the generalized Pareto distribution

$$F_{(\xi,\mu,\sigma)}(x) = \begin{cases} 1 - \left(1 + \dfrac{\xi(x-\mu)}{\sigma}\right)^{-1/\xi}, & \xi \neq 0 \\ 1 - exp\left(\dfrac{-x}{\sigma}\right), & \xi = 0 \end{cases},$$

where $\sigma > 0$, and $x \geq 0$ when $\xi \geq 0$ and $0 \leq x \leq -\sigma/\xi$ when $\xi < 0$, is the same as in the Generalized Extreme Value distribution:

when $\xi > 0$ than the generalized Pareto distribution is known as the Pareto "Type II" distribution;

when $\xi = 0$ the generalized Pareto distribution corresponds to the Exponential distribution;

when $\xi < 0$ is probably the most important for operational risk data, because the generalized Pareto distribution takes the form of the ordinary Pareto distribution with tail index $\alpha = 1/\xi$ and indicates the presence of heavy-tail data.

The role of the GPD in EVT is as a natural model for the excess distribution over a high threshold. We define this concept along with the mean excess function, which will also play an important role in the theory.

There are various alternative ways of describing this model. It might also be called a marked Poisson point process, where the exceedance times constitute the points and the GPD-distributed excesses are the marks. It can also be described as a (non-homogeneous) two-dimensional Poisson point process, where points (t, x) in two-dimensional space record times and magnitudes of exceedances

The basic result underlying the POT method is that the marked point process of excesses over a high threshold u, under fairly general (though very precise) conditions, can be well approximated by a compound Poisson process:

$$\sum_{k=1}^{N(u)} Y_k \delta_{T_k},$$

where (Y_k) iid have a generalized Pareto distribution and $N(u)$ denotes the number of exceedances of u by (X_k). The exceedances of u form (in the limit) a homogeneous Poisson process and both are independent.

2 Operational risk

This Chapter is based on [DIK08], [ECB04], [EKM97], [EFMcN05], [Mos04].

The actuarial community working on Solvency 2 so far defied a precise definition, and as a consequence a detailed quantitative capital measurement for operational risk. The situation in the banking world is very different indeed, not only did Basel II settle on a precise definition.

“Operational risk is defined as the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events. This definition includes legal risk, but excludes strategic and reputational risk.” (BCBS128)

In consultative document on the New Basel Capital Accord (also referred to as Basel II or the Accord), the Basel Committee for Banking Supervision continues its drive to increase market stability in the realms of market risk, credit risk and, most recently, operational risk. The approach is based on a three pillar concept where Pillar 1 corresponds to a Minimal Capital Requirement, Pillar 2 stands for a Supervisory Review Process and finally Pillar 3 concerns Market Discipline. Applied to credit and operational risk, within Pillar 1, quantitative modeling techniques play a fundamental role, especially for those banks opting for an advanced, internal measurement approach.

The framework outlined below presents three methods for calculating operational risk capital charges in a continuum of increasing sophistication and risk sensitivity:

- the Basic Indicator Approach (BIA);
- the Standardized Approach (SA);
- Advanced Measurement Approaches (AMA).

2.1 The Basic Indicator Approach

The Basic Indicator Approach and the Standardized Approach are two elementary approaches to operational risk measurement. Under the Basic Indicator Approach, banks must hold capital for operational risk equal to the average over the previous three years of a fixed percentage (denoted by α) of positive annual Gross Income (GI). Figures for any year in which annual gross income is negative or zero, should be excluded from both the numerator and denominator when calculating the average. Hence the risk capital under the Basic Indicator Approach for operational risk in year t is given by

$$RC_{BI}^{t}(OR) = \frac{1}{Z_t}\sum_{i=1}^{3} \alpha \, max\left(GI^{t-i}, 0\right),$$

where

$$Z_t = \sum_{i=1}^{3} I_{\{GI^{t-i}>0\}}$$

and GI^{t-i} stands for Gross Income year $t-i$ (See [EFMcN05]).
An operational risk-capital charge is calculated on a yearly basis. The Basic Indicator Approach gives a fairly straightforward, volume-based, one-size-fits-all capital charge. Based on the various Quantitative Impact Studies, the Basel Committee suggests that α = 15%.

2.2 The Standardized Approach

Under the Standardized Approach, banks' activities are divided into eight business lines:

1. corporate finance;
2. trading & sales;
3. retail banking;
4. commercial banking;
5. payment & settlement;
6. agency services;
7. asset management;
8. retail brokerage.

Precise definitions of these business lines are to be found in the Basel Committee's final document (Basel Committee on Banking Supervision 2004).Within each business line, gross income is a broad indicator that serves as a proxy for the scale of business operations and thus the likely scale of operational risk exposure. The capital charge for each business line is calculated by multiplying gross income by a factor (denoted by β) assigned to that business line. The total capital charge is calculated as a three-year average over positive gross incomes, resulting in the following capital charge formula:

$$RC_S^t(OR) = \frac{1}{Z_t}\sum_{i=1}^{3} max\left[\sum_{j=1}^{8} \beta_j GI_j^{t-i}, 0\right].$$

Here in any given year $t-i$, negative capital charges (resulting from negative gross income) in some business line j may offset positive capital charges in other business lines (albeit at the discretion of the national supervisor). This kind of "netting" should induce banks to go from the basic indicator to the standardized approach.

2.3 Advanced Measurement Approach

Under an Advanced Measurement Approach, the role of insurance in mitigating operational risk is recognized and the regulatory capital is determined by a bank's own internal risk-measurement system according to a number of quantitative and qualitative criteria set forth in documentation produced by the Basel Committee (Basel Committee on Banking Supervision 2004). The Advanced Measurement Approach lays down general guidelines. In the words of the Basel Committee (Basel Committee on Banking Supervision 2004):

> "Given the continuing evolution of analytical approaches for operational risk, the Committee is not specifying the approach or distributional assumptions used to generate the operational risk measure for regulatory capital purposes. However, a bank must be able to demonstrate that its approach captures potentially severe "tail" loss events. Whatever approach is used, a bank must demonstrate that its operational risk measure meets a soundness standard comparable to that of the internal ratings-based approach for credit risk (comparable to a one year holding period and the 99.9 percent confidence interval)."

In an Advanced Measurement Approach, operational losses should be categorized according to the eight business lines mentioned in Standardized Approach as well as the following seven loss-event types:

- internal fraud;
- external fraud;
- employment practices & workplace safety;
- clients, products & business practices;
- damage to physical assets;
- business disruption & system failures;
- execution, delivery & process management.

Banks are expected to gather internal data on repetitive, high-frequency losses (three to five years of data), as well as relevant external data on non-repetitive low-frequency losses. Moreover, they must add stress scenarios both at the level of loss severity (parameter shocks to model parameters) and correlation between loss types. In the absence of detailed joint models for different loss types, risk measures for the aggregate loss should be calculated by summing across the different loss categories. In general, both so-called expected and unexpected losses should be taken into account (i.e. risk-measure estimates cannot be reduced by subtraction of an expected loss amount).

Skeletal version of a typical Advanced Measurement solution for the calculation of an operational risk charge for year t:

Assume that historical loss data from previous years have been collected in a data warehouse with the structure

$$\{X_k^{t-i,b,l}: i = 1, \dots, T; b = 1, \dots, 8; l = 1, \dots, 7; k = 1, \dots, N^{t-i,b,l}\},$$

where $X_k^{t-i,b,l}$ stands for the k-th loss of type l for business line b in year $t-i$; $N^{t-i,b,l}$ is the number of such losses (See [FEMcN05]). Thresholds may be imposed for each (i, b, l) category and small losses less than the threshold may be neglected; a threshold is typically of the order of €10 000.

The total historical loss amount for business line b in year $t-i$ is obviously

$$L^{t-i,b} = \sum_{l=1}^{7} \sum_{k=1}^{N^{t-i,b,l}} X_k^{t-i,b,l},$$

and the total loss amount for year $t-i$ is

$$L^{t-i} = \sum_{b=1}^{8} L^{t-i,b}.$$

The problem in the Advanced Measurement Approach is to use the loss data to estimate the distribution of L_t for year t and to calculate risk measures such as VaR or expected shortfall for the estimated distribution. The joint distributional structure of the losses for any given year is generally unknown, we would typically resort to simple aggregation of risk measures across loss categories to obtain a formula of the form

$$RC_{AM}^{t}(OR) = \sum_{b=1}^{8} \rho_\alpha(L^{t,b}),$$

where RC is the regulatory capital and ρ_α the risk measure at a confidence level $\alpha \in (0,1)$.

When $\rho_\alpha = VaR_\alpha$ and $\alpha = 0{,}999$, then a capital charge under the Advanced Measurement Approach requires the calculation of a quantity of the type:

$$VaR_{0,999}\left(\sum_{k=1}^{N} X_k\right),$$

where X_k is some sequence of loss severities and N is an random variable describing the frequency with which operational losses occur.

Exploratory data analysis reveals the following stylized facts (confirmed in several other studies):

- loss severities have a heavy-tailed distribution;
- losses occur randomly in time;
- loss frequency may vary substantially over time.

The several classes of loss may have a considerable cyclical component and/or may depend on changing economic co-variables.

The Advanced Measurement Approach focuses on using internal and external loss data, among other techniques, and is often referred to as the Loss Distribution Approach (LDA). There are several examples of works under the Loss Distribution Approach.

Loss Distribution Approach methods are becoming important for internal risk modeling purposes and at Basel-defined business line and event type level modeling in order to improve the stability of the financial services industry.
Loss Distribution Approach methods are flexible and could be used within the whole financial industry sector, by central and commercial banks, insurance companies and supervisory bodies. No doubt, a great potential for developing such methods lies within the paradigm of ruin theory as has already been noted by Embrechts et al. (2004).

Reference and works under the Loss Distribution Approach:

the common Poisson shock models:
- [EVMNAF01]
- [EVMNAF02]
- [B04]

the ruin probability based models:
- [EKS04]
- [ES03]

A more recent paper, considering the effect of insurance on setting the capital charge for operational risk is that of [BCPS06].

The LDA approach has recently been used by Dutta and Perry, who have considered fitting appropriate loss distributions to operational loss data under the 2004 Loss Data Collection Exercise (LDCE) and the Quantitative Impact Study 4 [DP06].

Methods for Advanced Measurement Approach, which we discuss in the next chapter, are:

- Ruin theory in operational risk
- EVT in operational risk

3 Methods for Advanced Management Approach

3.1 Extrem Value Theory in Operational risk

The key attraction of EVT is that it offers a set of ready-made approaches to the most difficult problem in operational risk analysis: how can risks that are both extreme, and extremely rare, be modeled appropriately? But applying EVT to financial institution operational risk raises some difficult issues. Some of these arise from the nature of the data that is available to analysts. Others relate to the purpose of any operational risk analysis, the definition of an "extreme" event, and the meaning of the term operational risk. [M01]

In fact, operational risk data appear to be characterized by two "souls": the first one, driven by high-frequency low impact events, constitutes the body of the distribution and refers to expected losses; the second one, driven by low-frequency high-impact events, constitutes the tail of the distribution and refers to unexpected losses. In practice, the body and the tail of data do not necessarily belong to the same, underlying, distribution or even to distributions belonging to the same family. More often their behavior is so different that it is hard to identify a unique traditional model that can at the same time describe, in an accurate way, the two "souls" of data. [Mos04]

The need for clarification of the new Basel alternative operational risk quantification is obvious. The complexity of the originating causes of operational risk, the "rare event" nature of significant losses and the desire to integrate operational risk capital provision with that for market and credit risks all lead us to capital allocation rules based on results from extreme value theory. Application of EVT to operational risk modeling serves as the principal objective of regulation:

"A real concern of supervisors is the low-probability, high-severity event that can produce losses large enough to threaten a financial institution's health". [M01]

Consequently, in all the cases in which the tail tends "to speaks for itself", EVT appears to be an useful inferential instrument with which to investigate the large losses, owing to its double property of focusing the analysis only on the tail area (hence reducing the disturbance effect of the small/medium-sized data) and treating the large losses by an approach as scientific as the one driven by the Central Limit Theorem for the analysis of the high-frequency low-impact losses. Unlike traditional methods, EVT does not require particular assumptions on the nature of the original underlying distribution of all the observations, which is generally unknown. EVT is applied to real data in two related ways. The first approach deals with the maximum (or minimum) values the variable takes in successive periods, for example months or years. The second approach to EVT is the Peaks Over Threshold (POT) method, tailored for the analysis of data bigger than preset high thresholds. [Mos04]

On the basis of the POT tail severity and frequency estimates, an aggregate figure for each business line and for the eight business lines as a whole is computed by means a semi parametric approach. The POT approach appears to be a viable solution to reduce the estimate error and the computational costs related to the not analytical techniques, like the Monte Carlo simulation, usually implemented in the financial industry to reproduce the highest percentiles of the aggregate loss distribution. The findings clearly indicate that operational losses represent a significant source of risk for banks, given a 1-year period capital charge against expected plus unexpected losses at the 99.9th percentile.

The advantage of the POT approach in the estimate of the tail of the aggregated losses therefore appears directly connected to the two following properties:

1. the POT method takes into consideration the (unknown) relationship between the frequency and the severity of large losses up to the end of the distribution;

2. the POT method makes it possible to employ a semi parametric approach to compute the highest percentiles of the aggregated losses, hence reducing the computational cost and the estimate error related to a not analytical representation of the aggregated losses themselves. In the POT model, it suffices to select a suitable (high) threshold, on which basis the model can be built and the relevant parameters estimated. Once the model is correctly calibrated, the total losses (and their percentiles) are easily obtainable by proper analytical expressions. [Mos04]

The estimation of the parameters of the POT model is usually based on the maximum likelihood method, which requires a relatively large number of observations above the threshold (e.g., more than 100). But operational risk data sets are not homogeneous and are often classified into several subsamples, each associated with a different risk factor or business unit. It might be more realistic to think in terms of 20 or 30 excesses. [MK01]

The conventional maximum likelihood (ML) estimation method performs unstably when it is applied to small or even moderate sample sizes, i.e. less than fifty observations. Bayesian simulation methods for parameter estimates allow one to overcome problems associated with lack of data through intensive computation. [MK01]

To justify the modeling of operational risk using EVT, many obstacles must be overcome. But not all the obstacles are technical in nature. Many are caused by the fact that operational risk continues to be ill-defined for the purpose of calculating risk capital. [Med00]

For example, one might ask how any approaches to operational risk using extreme value theory relate to definitions of "normality" and the problem of internal bank controls and external supervision? More topically, how does EVT relate to the Basel Committee on Banking Supervision's present proposals for controlling operational risk? The Committee has attempted to clarify the complex issues of risk management by adopting a "three – pillared" approach. The first pillar concerns capital allocation, the second supervision and controls, and the third transparency and consistency of risk management procedures. What is the relation of EVT to these three pillars – most problematically, the second and third pillars? Another problem is that, while risk capital is generally understood as a way of protecting a bank against "unexpected" losses – expected losses are covered by business-level reserves – it is not clear to what degree it is used to cover the most extreme risks. Some practitioners and regulators have made it clear that they do not intend to include the risk of the most extreme losses in their calculations of either economic risk capital or regulatory risk capital. So in what way is extreme value theory useful in measuring operational risk? Lastly, how can an analyst deal with market and credit risk management without double counting? Some framework that identifies the roles of credit, market, and other risks must be constructed.

The first step in operational risk management should be a careful analysis of all available data to identify the statistical patterns of losses related to identifiable primal and secondary risk factors.

Ideally, this analysis would form part of the financial surveillance system for the bank. In the future, perhaps such an analysis might also form part of the duties of bank supervisors. In other words, at a conceptual level, it relates to the second of the Basel Committee's three pillars. Here, the important point for analysts is that this surveillance is concerned with the identification of the "normality" of business processes. The identification of suitable of market and credit risk models also forms a natural part of this operational risk assessment.

Such an analysis should allow an analyst to classify banks losses into two categories:

1. significant in value but rare, corresponding to extreme loss events distributions;
2. low value but frequently occurring corresponding to "normal" loss event distributions.

Next, we might take the view that controls procedures will be developed for the reduction of the low value/frequent losses, and for their illumination and disclosure (the third pillar of the Basel approach).

These control procedures, and any continuing expected level of loss, should be accounted for in the operational budget. This allows us to assume that only losses of large magnitude need be considered for operational risk economic capital provision. [Med00]

3.2 Ruin theory in operational risk

A new methodology for modeling operational risk based on risk and ruin theory.

Ruin theory	Operational risk
Claims	Operational losses
Claim frequency	Operational frequency
Claim size	Loss severity
Risk capital accumulation	Premium income

Ruin theory may be viewed as the theoretical foundation of insolvency risk modeling. Under the classical ruin theory model, the (premium) income to an (insurance) company is modeled by a straight line

$$h(t) = u + ct,$$

where u is the company's initial risk capital at time $t = 0$ and c is the premium income per unit of time, received by the company.

Embrechts et al. (2004) proposed to take an actuarial point of view and directly apply the (classical) ruin probability model to the context of operational risk, under the LDA (Loss Distribution Approach) approach. Thus, the random variables W_i, $i = 1, 2, \ldots$ in model

$$S_t = \sum_{i=1}^{N_t} W_i$$

are viewed as representing operational risk losses and the aggregate loss amount, $S(t)$, due to different types of operational risk, is expressed as a superposition of the risk processes, corresponding to each type of risk. The rate c is seen "as a premium rate paid to an external insurer for taking (part of) the operational risk losses or as a rate paid to (or accounted for by) a bank internal office". In order to reserve against operational risk, it is proposed to set the initial capital u and the income rate c in such a way that it satisfies the equation

$$P(T \leq x) = P\left(\inf_{0 \leq t \leq x}\big(u + ct - S(t)\big) < 0\right) = \epsilon,$$

where the probability of ruin, $P(T \leq x)$, over a finite time interval, $[0, \infty]$,$0 < x < \infty$ is set to a pre-assigned appropriate (small) value $\epsilon > 0$. As noted, if the time interval is of length x and $c = 0$, the risk capital u is equal to the operational value at risk at significance level of α i.e.,

$$u = OR - VaR_{1-\alpha}^{x},$$

which is another popular risk measure considered in defining the capital charge for operational risk. Although Embrechts et al. (2004) extend the applicability of the classical ruin probability model; the following major limitations may still be outlined:

- the function $h(t)$ is represented by a straight line, which is a simple but not a realistic assumption for the premium income;

- the losses, W_i, $i = 1, 2, \ldots$ are assumed independent and identically distributed which is also a restrictive assumption, not expected to hold for operational risk losses;
- the ruin probability estimates quoted are asymptotic approximations, i.e., for ruin on infinity, and as mentioned by the authors, "are not fine enough for accurate numerical approximations" and their numerical properties are "far less satisfactory", since these estimates are in an integral form.

A more general ruin probability model [IK00] assumes

- any non-decreasing (premium) income function $h(t)$ as an alternative to the classical straight line case;
- any joint distribution of the losses W_i, $i = 1, 2, \ldots$, allowing dependency between the loss amounts, as an alternative to the i.i.d. classical assumption;
- finite time ruin probabilities, as an alternative to the asymptotic approximations of infinite time ruin probabilities.

4 Ruin probability for finite horizon

4.1 Kaishev Dimitrov formula

4.1.1 Discrete claim distribution

Let as consider the counting process $N_t = \#\{i: \tau_1 + \cdots \tau_i \le t\}$, where # in the right-hand side denotes the number of elements in the set $\{\cdot\}$, and $\tau_1, \dots \tau_i, \dots$ are independent, exponentially distributed random variables with mean $E\{\tau_i\} = \frac{1}{\lambda_i}$, $\lambda_i > 0$, i.e., $P(\tau_i > y) = e^{-\lambda_i y}$ for $y > 0$ and $P(\tau_i > y) = 1$ for $y \le 0, i = 1,2, \dots$.

Consider the integer valued random variables $W_1, W_2, W_3, \dots$ are independent of N. The joint distribution of $W_1, \dots, W_i$ denoted by $P(W_1 = w_1, \dots, W_i = w_i) = P_{w_1,\dots,w_i}$, where $w_1 \ge 1, w_2 \ge 1, \dots, w_i \ge 1, i = 1,2, \dots$. Then the risk reserve process of an insurance company is

$$R_t = h(t) - S_t ,$$

where $h(t)$ is a function, representing the premium income and $S(t)$ is the aggregate loss amount at time *t*, defined as

$$S_t = \sum_{i=1}^{N_t} W_i ,$$

We will assume, that $h(t)$ is a non-negative, increasing, real function, defined on $\mathbb{R}_+$ and such that

$$\lim_{t\to\infty} h(t) = +\infty.$$

The function $h(t)$ may be continuous or not. If the $h(t)$ is discontinuous we will assume that

$$h^{-1}(y) = inf\{z \ge 0: h(z) \ge y\} \qquad (\mathbf{1})$$

We will denote $h^{-1}(i) = v_i$, for $i = 0,1, \dots$. We observe $0 = v_0 \le v_1 \le v_2 \le \cdots$.

Under the classical ruin theory model, the (premium) income of an (insurance) company is modeled by a straight line

$$h(t) = u + ct, \qquad (\mathbf{2})$$

where $u \ge 0$ is the company's initial risk capital at time $t = 0$ and $c \ge 0$ is the premium income per unit of time.

The time of ruin T is defined as

$$T := inf\{t > 0, \qquad R_t \le 0\} \qquad (3)$$

In other words the instant of ruin T is the time in which the trajectory $t \to S_t$ of first crosses the boundary $t \to h(t)$ (disregarding the origin, when $h(0) = 0$).

The probability $P(T < \infty)$ called the infinite-time probability of ruin of the company. In other word, this is the probability that the risk process R_t will become negative in some future moment, within an infinite time horizon.

$$P(T < x) = P\left(\min_{0 \le t \le x}\big(u + ct - S(t)\big) < 0\right) = \epsilon \qquad (\mathbf{4})$$

where value $\epsilon > 0$ and the probability of ruin $P(T < x)$ in a finite time interval $[0, x]$, $0 < x < \infty$

We will be interested in the probability of non-ruin (survival), i.e., $P(T > x)$ in a finite time interval $[0, x]$, $x > 0$. [IKK01] have shown, that if W_i – discrete, than the probability of non-ruin in a finite time x is

$$P(T > x) = e^{-x\lambda} \sum_{k=1}^{n} \left(\sum_{\substack{w_1 \ge 1, \dots, w_{k-1} \ge 1 \\ w_1 + \dots + w_{k-1} \le n-1}} P\,(W_1 = w_1, \dots, W_{k-1} = w_{k-1}; \right.$$

$$\left. W_k \ge n - w_1 - \dots - w_{k-1}) \sum_{j=0}^{k-1} (-1)^j\, b_j\big(z_1, \dots, z_j\big)\, \lambda^j \sum_{m=1}^{k-j-1} \frac{(x\lambda)^m}{m!} \right) \qquad (\mathbf{5})$$

where $n = [h(x)] + 1$, $[h(x)]$ is the integer part of $h(x)$, $v_{n-1} \le x < v_n$, where k is such that

$$w_1 + \dots + w_{k-1} \le n - 1,\ w_1 + \dots + w_k \ge n,\ (1 \le k \le n),$$

$$z_l = v_{w_1 + \dots + w_l},\ l = 0,1, \dots, k-1 \text{ and}$$

$$b_j\big(z_1, \dots, z_j\big) = (-1)^{j+1} \frac{z_j^j}{j!} + (-1)^{j+2} \frac{z_j^{j-1}}{(j-1)!} b_1(z_1) + \cdots$$

$$+ (-1)^{j+j} \frac{z_j^1}{1!} b_{j-1}\big(z_1, \dots, z_{j-1}\big), \qquad (\mathbf{6})$$

with $b_0 \equiv 1$, $b_1(z_1) = z_1$, $j = 0,1, \dots, k-1$.

The non-ruin probability on the left-hand side can be rewritten as:

$$P(T > x) = 1 - P\left(\inf_{0 \le t \le x}\big(u + ct - S(t)\big) < 0\right) =$$

$$= 1 - P\left(\inf_{0 \le t \le x}\big(h(t) - S(t)\big) < 0\right) = 1 - \epsilon\,. \qquad (\mathbf{7})$$

4.1.2 Continuous claim distribution

Let the random variables $Y_1, Y_2, \ldots$ have a joint distribution for which

$$P(Y_1 \le Y_2 \le \cdots \le Y_k \le \cdots) = 1,$$

i.e. let the joint density of $Y_1, Y_2, \ldots, Y_k$ have the form

$$f(y_1, \ldots, y_k) = \begin{cases} \varphi(y_1, \ldots, y_k) \; if \; 0 \le y_1 \le \cdots \le y_k, \\ 0 \qquad\qquad\qquad otherwise, \end{cases}$$

where $\varphi(y_1, \ldots, y_k) \ge 0$ when $0 \le y_1 \le \cdots \le y_k$
and where

$$\int_{0 \le y_1 \le \cdots} \int_{\le y_a} \varphi(y_1, \ldots, y_k)\, dy_1 \ldots dy_k = 1.$$

Introduce that the sequence of random variables $T_1 = \tau_1$, $T_2 = \tau_1 + \tau_2$, … which represent the consecutive moments of arrival of claims to an insurance company. The Severities of the claims will correspondingly be represented by the ransom variables $W_1, W_2, \ldots$. Assume that the claim severities are related to the random variables $Y_1, Y_2, \ldots$ through the equalities $W_1 = Y_1$, $W_2 = Y_2 - Y_1$, $W_3 = Y_3 - Y_2$, … , in other words, $Y_1, Y_2, \ldots$ can be viewed as the partial sums of the consecutive claim amounts. The joint density $\psi(w_1, \ldots, w_k)$ of the random variables $W_1, W_2, \ldots W_k$ can be expressed as

$$\psi(w_1, w_2 \ldots, w_k) = f(w_1, w_1 + w_2, \ldots, w_1 + \cdots + w_k)$$

or as

$$f(y_1, y_2 \ldots, y_k) = \psi(y_1, y_2 - y_1, \ldots, y_k - y_{k-1}) \qquad \textbf{(8)}$$

[IK04] have shown, when claims have any continuous joint distribution, than the probability of non-ruin within a finite time x has representation:

$$P(T > x) = e^{-\lambda x}\left(1 + \sum_{k=1}^{\infty} \lambda^k \int_{y_0}^{h(x)} dy_1 \int_{y_1}^{h(x)} dy_2 \ldots \right.$$
$$\left. \times \int_{y_{k-1}}^{h(x)} A_k\left(x; h^{-1}(y_1), \ldots, h^{-1}(y_k)\right) f(y_1, \ldots, y_k) dy_k \right) =$$

$$\textbf{(9)}$$

$$= e^{-\lambda x}\left(1+\sum_{k=1}^{\infty}\lambda^k \int_{y_0}^{h(x)}\int_{y_1}^{h(x)} \dots \int_{y_{k-1}}^{h(x)} A_k\big(x; h^{-1}(y_1), \dots, h^{-1}(y_k)\big)\right.$$
$$\left. \times f(y_1, \dots, y_k) dy_k \dots dy_2 dy_1\right),$$

where $y_0 \equiv 0$ and $A_k\big(x; h^{-1}(y_1), \dots, h^{-1}(y_k)\big)$ are the classical Appell polynomials $A_k(x)$ of degree k with a coefficient in front of x^k equal to $\frac{1}{k!}$. These polynomials are defined by

$$\begin{aligned} &A_0(x) = 1,\\ &A'_k(x) = A_{k-1}(x),\\ &A_k(v_k) = 0. \end{aligned}$$

Proof:

The risk process at time t is given by

$$R_t = h(t) - S(t),$$

with

$$S_t = \sum_{k=1}^{N_t} W_k.$$

The partial sums of consecutive claims is

$$Y_k = \sum_{i=1}^{k} W_i.$$

Cause on the number of claims the probability of non-ruin represent with

$$P[T > x] = \sum_{k\geq 0} P[T > x | N(x) = k] P[N(x) = k].$$

We know that:

$$N(x) \sim Poi(\lambda x) \;\; \Rightarrow$$

$$\Rightarrow \; P[N(x) = k] = e^{-\lambda x}\frac{(\lambda x)^k}{k!}.$$

If up to time x occur exactly k claims, when $T > x$ is valid exactly then when $R(T_1) > 0, \dots, R(T_k) > 0$

This equivalent

$$h(T_1) > S(T_1), \dots, h(T_k) > S(T_k).$$

Obviously

$$S(T_1) = Y_1, \dots, S(T_k) = Y_k,$$

hence equivalent

$$h(T_1) > Y_1, \dots, h(T_k) > Y_k,$$

and almost surely

$$T_1 > h^{-1}(Y_1), \dots, T_k > h^{-1}(Y_k).$$

And so

$$P[T > x | N(x) = k] = P[T_1 > h^{-1}(Y_1), \dots, T_k > h^{-1}(Y_k) | N(x) = k].$$

The $Y_1, \dots, Y_k$ are assumed to be independent of $T_1, \dots, T_k$ and $N(x)$ and thus it follows:

$$P[T > x | N(x) = k] = \tag{10}$$

$$= \int_{0 \le y_1 \le \cdots} \int_{\le y_k \le h(x)} P\,[T_1 > h^{-1}(y_1), \dots, T_k > h^{-1}(y_k) | N(x) = k] f(y_1, \dots, y_k)\, dy_1 \dots dy_k.$$

The joint density of random variables $T_1, \dots, T_k$ is

$$g(t_1, \dots, t_k) = \frac{k!}{x^k} \mathbb{1}_{\{0 \le t_1 \le \cdots \le t_k \le x\}}\,.$$

Therefore

$$P[T > x | N(x) = k] =$$

$$= \int_{0 \le y_1 \le \cdots} \int_{\le y_k \le h(x)} \int_{0 \le t_1 \le \cdots} \int_{\le t_k \le x} \frac{k!}{x^k} \mathbb{1}_{\{t_1 > h^{-1}(y_1), \dots, t_k > h^{-1}(y_k)\}}\, dt_1 \dots dt_k f(y_1, \dots, y_k) dy_1 \dots dy_k$$

$$= \int_0^{h(x)} \int_{y_1}^{h(x)} \dots \int_{y_{k-1}}^{h(x)} \int_{h^{-1}(y_k)}^{x} \int_{h^{-1}(y_{k-1})}^{t_{k-1}} \dots \int_{h^{-1}(y_1)}^{t_2} \frac{k!}{x^k}\, dt_1 \dots dt_k f(y_1, \dots, y_k) dy_1 \dots dy_k\,. \tag{11}$$

We can write:

$P[T > x] =$

$$= \sum_{k=0}^{\infty} e^{-\lambda x} \frac{(\lambda x)^k}{k!} \int_{0}^{h(x)} \int_{y_1}^{h(x)} \dots \int_{y_{k-1}}^{h(x)} \int_{h^{-1}(y_k)}^{x} \int_{h^{-1}(y_{k-1})}^{t_{k-1}} \dots \int_{h^{-1}(y_1)}^{t_2} \frac{k!}{x^k} f(y_1, \dots, y_k) dt_1 \dots dt_k dy_1 \dots dy_k$$

$$= e^{-\lambda x} \lambda^k \sum_{k=0}^{\infty} \int_{0}^{h(x)} \int_{y_1}^{h(x)} \dots \int_{y_{k-1}}^{h(x)} \left[\int_{h^{-1}(y_k)}^{x} \int_{h^{-1}(y_{k-1})}^{t_{k-1}} \dots \int_{h^{-1}(y_1)}^{t_2} dt_1 \dots dt_k \right] f(y_1, \dots, y_k) dy_1 \dots dy_k. \quad \mathbf{(12)}$$

Let

$$A_k(x; v_1, \dots, v_k) = \int_{v_k}^{x} \int_{v_{k-1}}^{t_{k-1}} \dots \int_{v_1}^{t_2} dt_1 \dots dt_k,$$

then

$$A_k(x; v_1, \dots, v_k) = \int_{v_k}^{x} A_{k-1}(t; v_1, \dots, v_{k-1}) \, dt.$$

And it follows that

$$A_k(v_k; v_1, \dots, v_k) = 0 \,,$$

$$A_k'(x; v_1, \dots, v_k) = A_{k-1}(x; v_1, \dots, v_{k-1}).$$

Also we can rewrite the probability of non-ruin as:

$P[T > x] =$

$$= e^{-\lambda x} \lambda^k \sum_{k=0}^{\infty} \int_{0}^{h(x)} \int_{y_1}^{h(x)} \dots \int_{y_{k-1}}^{h(x)} A_k(x; h^{-1}(y_1), \dots, h^{-1}(y_k)) \, f(y_1, \dots, y_k) dy_1 \dots dy_k. \quad \mathbf{(13)}$$

- **Appell Polynomials**

A polynomial sequence $\{p_n(z)\}$ has a generalized Appell representation if the generating function for the polynomials takes on a certain form:

$$K(z,w) = A(w)\psi\big(zg(w)\big) = \sum_{n=0}^{\infty} p_n(z)\, w^n,$$

where the generating function or kernel $K(z,w)$ is composed of the series

$$A(w) = \sum_{n=0}^{\infty} a_n w^n\,, a_0 \neq 0,$$

$$\psi(t) = \sum_{n=0}^{\infty} \psi_n\, t^n, \qquad for\ \forall\, \psi_n \neq 0,$$

$$g(w) = \sum_{n=0}^{\infty} g_n\, w^n,\ \ g_{n1} \neq 0$$

and $p_n(z)$ is a polynomial of degree n.

The generalized Appell polynomials have the explicit representation

$$p_n(z) = \sum_{k=0}^{n} z^k \psi_k\, h_k.$$

The constant is

$$h_k = \sum_{P} a_{j0} g_{j1} g_{j2} \dots g_{jk}\ ,$$

where this sum extends over all partitions of n into $k\ +\ 1$ parts; that is, the sum extends over all $\{j\}:\ j_0 + j_1 + \cdots + j_k = 0$.

For the Appell polynomials, this becomes the formula

$$p_n(z) = \sum_{k=0}^{n} \frac{a_{n-k} z^k}{k!} = \sum_{k=0}^{n} \frac{1}{k!} a_{n-k} z^k.$$

4.2 Picard – Lefévre Formula

(See [PK97])

The family of polynomials $\{e_n, n \epsilon \mathbb{N}\}$ is defined by formal generating function:

$$\sum_{n=0}^{\infty} e_n(x)s^n = e^{xg(s)} ,$$

where

$$g(s) = \sum_{j=1}^{x} \lambda q_j s^j \,.$$

Then we define the operator Δ by

$$\begin{cases} \Delta e_{n-1} = e_n, & n > 0 \\ \Delta e_0 = 0 \end{cases}.$$

The powers of Δ being built recursively from $\Delta^{r+1}= \Delta(\Delta^r)$, $r \geq 0$, with Δ^0 as the identity operator.

Clearly e_n is a polynomial of degree n, $e_0 = 1$, $e_1(x) = \lambda q_1 x$, $e_n(0) = \delta_{n0}$ and $\{e_n, n \epsilon \mathbb{N}\}$, respectively to the operator Δ, is e family of generalized Appell polynomials.

General

$$P(T > x) = e^{-\lambda x} \sum_{n=0}^{\infty} A_n\,(x)\mathbb{1}_{\{x \geq \upsilon_n\}}. \qquad \mathbf{(14)}$$

When $\upsilon_j < x \leq \upsilon_{j+1}$:

$$P(T > x) = e^{-\lambda x} \sum_{n=0}^{j} A_n\,(x), \qquad \mathbf{(15)}$$

where polynomial's A_n are given by

$$A_n(x) = \int_{\upsilon_n}^{x} \sum_{j=1}^{n} \lambda\, q_j A_{n-j}(t)dt,$$

when $x \geq \upsilon_n,\; n > 0$.

But this can be also calculated by the formula

$$A_n(x) = \sum_{i=0}^{n} b_i\, e_{n-i} \qquad \mathbf{(16)}$$

for $n \geq 0$ and the b_i 's are built recursively from

$$\sum_{i=0}^{n} b_i\, e_{n-i}(\upsilon_n) = \delta_{n0}. \qquad \mathbf{(17)}$$

A_n being a polynomial of degree n defined for any x, and we shall keep the two conditions

$$A_n(v_n) = 0$$

and

$$A_n'(x) = \sum_{j=1}^{n} \lambda\, q_j A_{n-j}(x), \qquad n > 0,$$

which together determine the A_n's uniquely.

The linear boundary case

Let us examine the special case of a constant premium rate c. Here thus

$$h(t) = u + ct\ ,$$

and

$$v_n = Max(0, (n-u)/c), n \geq 0. \tag{18}$$

In the linear case

$$A_n(x) = \begin{cases} e_n(x), \qquad when\ 0 \leq n \leq u \\ \displaystyle\sum_{j=0}^{u} e_j\left(\frac{j-u}{c}\right) f_{n-j}\left(x + \frac{u-n}{c}\right) \\ \qquad = \displaystyle\sum_{j=0}^{u} e_j\left(\frac{j-u}{c}\right)\frac{cx-n+u}{cx-j+u} e_{n-j}\left(x + \frac{u-j}{c}\right), when\ n > u \end{cases}. \tag{19}$$

The two simple cases of this formula are worth noticing:

when $u = 0$

$$A_n(x) = \frac{cx-n}{cx} e_n(x), \qquad n \geq 0,$$

when $u = 1$

$$A_n(x) = \begin{cases} e_n(x), & for\ n = 0{,}1 \\ \dfrac{cx-n+1}{cx+1} e_n\left(x + \dfrac{1}{c}\right), & for\ n > 1 \end{cases}$$

and

$P(T > x | R_0 = u) =$

$$= e^{-\lambda x} \sum_{j=0}^{u} \left\{ e_j(x) + \sum_{n = u + 1}^{[cx+u]} e_j\left(\frac{j-u}{c}\right) \frac{cx - n + u}{cx - j + u} e_{n-j}\left(x + \frac{u-j}{c}\right) \right\}, \quad (\mathbf{20})$$

where $[cx + u]$ is the integer part of $c + ux$.

When $u = 0$,

$$P(T > x | R_0 = 0) = e^{-\lambda x} \sum_{n = 0}^{[cx]} \frac{cx - n}{cx} e_n(x).$$

When $T = +\infty$

$$P(T = +\infty | R_0 = 0) = \left(1 - \frac{\lambda m}{c}\right) \sum_{j = 0}^{u} e^{\lambda(u-j)/c} e_j\left(\frac{j-u}{c}\right).$$

5 Numerical Illustration

In this section we consider three alternative distributions of the consecutive losses.

We discuss interarrival times $(\tau_i)_{i\geq 1}$, which are i.i.d. $Exp(\lambda)$ for all illustrations. Thus the number of losses $[0, x]$ in is $N_x \sim Poi(\lambda x)$.

Then the Expected value and Variance of N_x are given by

$$E[N_x] = \lambda x,$$

$$Var[N_x] = \lambda x.$$

For $\lambda = 20$ und $x = 2$, we have:

$E[N_2] = 40, \qquad Var[N_2] = 40$

and plus/minus three standard derivatives

$$E[N_2] - 3\sqrt{Var[N_2]} \approx 21{,}03$$

$$E[N_2] + 3\sqrt{Var[N_2]} \approx 58{,}97.$$

Typically $20 - 30$ losses.

.

5.1 Logarithmic i.i.d. losses.

Assume, that operational risk losses have logarithmic distribution.

If losses are $W \sim Log(\alpha)$ and $\alpha = 0{,}73$, then

$E[W] \approx 2{,}06,$ $Var[W] \approx 3{,}38$

and

$$E[W] - 3\sqrt{Var[W]} \approx -3{,}45$$

$$E[W] + 3\sqrt{Var[W]} \approx 7{,}58\,.$$

Thus typically losses of size 1,2,..,8.

A set of operational losses arriving in the interval $[0{,}2]$ with inter-arrival times distributed as $Exp(\lambda)$ and with severities simulated from the $Log(\alpha)$ distribution are presented in the **Fig. 1*a***. $[i.e.\ \lambda = 20$ and $\alpha = 0.73]$

We have two Methods to simulate losses.

In Method I we have simulated operational loss data, where the number of losses had Poisson distribution and in Method II we have simulated with exponential distribution for interarrival time.

We use the same simulation methods for all examples and change only the claim distribution.

Method I with Poisson Distribution

fn="log.nb"; Remove["Global`*"];
(* Figure 1a *)
(* SeedRandom[1234]; *)
la=20.; al=0.73; x=2.;
nx=RandomInteger[PoissonDistribution[la*x]]
t=Sort[RandomReal[{0,x}, nx]]
w=RandomInteger[LogSeriesDistribution[al], nx]
ListPlot[Table[{t[[k]], w[[k]]}, {k,1,nx}],Filling→Axis, PlotRange→ {{0,2}, {0,11}}]

55

{0.00839501, 0.0646646, 0.101015, 0.116138, 0.11776, 0.177927, 0.207163,
0.232146, 0.235277, 0.345728, 0.346841, 0.35419, 0.368867, 0.370268,
0.435546, 0.441798, 0.446491, 0.499636, 0.526316, 0.571584, 0.638295,
0.796619, 0.820702, 0.845062, 0.849002, 0.850851, 0.893697, 0.904163,
0.962739, 0.973639, 1.01526, 1.15179, 1.16659, 1.20285, 1.21865, 1.2348,
1.24104, 1.30498, 1.36861, 1.48231, 1.52011, 1.52192, 1.5516, 1.60997,
1.61405, 1.64038, 1.6572, 1.72499, 1.80092, 1.87666, 1.87803, 1.93877, 1.94443,
1.95511, 1.99264}

{1,7,1,1,1,1,1,1,7,1,8,1,1,1,4,1,
1,1,1,1,1,1,1,3,2,2,1,2,1,1,1,1,1,3,1,
3,1,7,1,5,2,1,3,4,1,1,4,1,1,1,1,4,1,1,2}

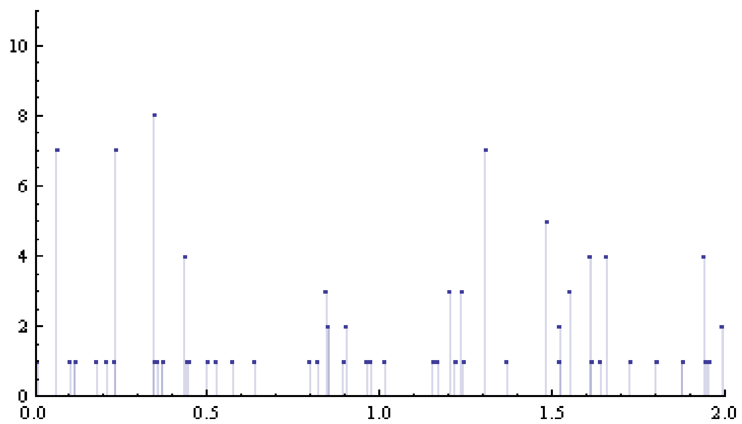

Fig.1a: simulated operational $Log(\alpha)$ losses

Method II with Exponential Distribution

```
fn="log.nb";Remove["Global`*"];
(* Figure 1a *)
(* SeedRandom {1234]; *)
la=20.; al=0.73;
For[n=0; t[0]=0, t[n]≤2, n++;
  t[n]=t[n-1]+RandomReal[ExponentialDistribution[la]];
  w[n]=RandomInteger[LogSeriesDistribution[al]]
  ];
ListPlot[Table[{t[k], w[k]}, {k, 1, n – 1}], Filling→Axis, PlotRange→{{0, 2},{0, 11}}]
```

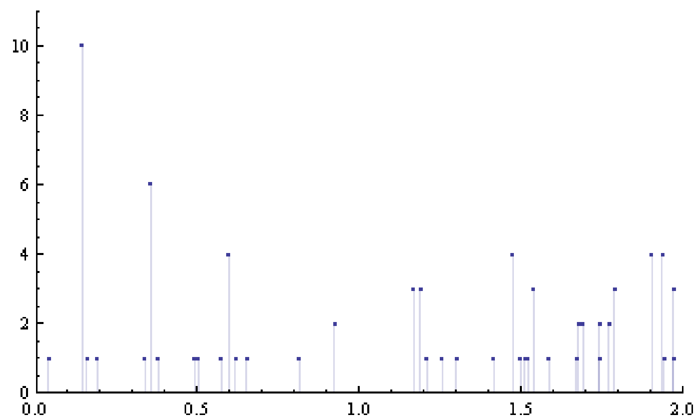

Fig 1a: simulated operational $Log(\alpha)$ losses

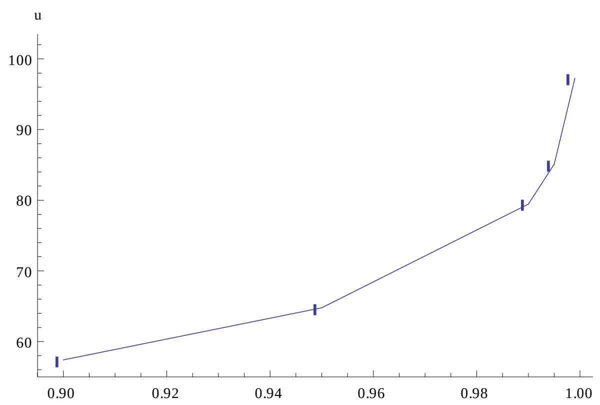

Fig 1b: initial capital u for choices $P(T > 2)$ of equal to 90%,95%, 99%, 99,5% and 99,9%

```
fn = "fig1b.nb"; Remove["Global`*"];
(*Picard and Lefévre*)
c = 25;
x = 2;
al = 73/100;
la = 20;
rh = −1 / Log[1 – al];
e[ n_, z_ ] = Binomial[−la * rh * z, n]*( −al ) ^ n;
pr[ u_ ] := Exp[−la * x] *
        Sum[e[j, x] + Sum[e[j, (j – u) / c]*(u + c*x–n) / (u + c*x – j)*e[n − j, x + (u–j) / c],
                n, Floor[u] + 1, Floor[u + c *x]}],{j, 0, Floor[u]}];
ep = {0.1, 0.05, 0.01, 0.005, 0.001};
(*preassigned small epsilon*)
tmp = Table[{N[N[pr[u], 40]], u}, {u, 50, 100}]

{{0.815784, 50}, {0.829467, 51}, {0.842396, 52}, {0.854587, 53}, {0.866055, 54},
 {0.876819, 55},{0.8869, 56}, {0.896322, 57}, {0.905108, 58}, {0.913285, 59}, {0.920881, 60},
 {0.927921, 61}, {0.934434, 62}, {0.940447, 63}, {0.94599, 64}, {0.951088, 65}, {0.955769, 66},
 {0.960059, 67},{0.963984, 68}, {0.96757, 69}, {0.970838, 70}, {0.973813, 71}, {0.976517, 72},
 {0.978969, 73}, {0.981191, 74}, {0.983199, 75}, {0.985013, 76}, {0.986648, 77}, {0.98812, 78},
 {0.989443, 79}, {0.99063, 80}, {0.991693, 81}, {0.992645, 82}, {0.993496, 83}, {0.994255, 84},
 {0.994931, 85}, {0.995533, 86}, {0.996068, 87}, {0.996543, 88}, {0.996964, 89}, {0.997337, 90},
 {0.997666, 91}, {0.997957, 92}, {0.998214, 93}, {0.99844, 94}, {0.998638, 95}, {0.998813, 96},
 {0.998966, 97}, {0.999101, 98}, {0.999219, 99}, {0.999322, 100}}

ip = Interpolation[tmp] (*interpolation*)

InterpolatingFunction[{{0.815784, 0.999322}}, < >]

res = Table[{1 – ep[[i]], ip[1 – ep[[i]]]}, {i, 1, 5}]

{{0.9, 57.41}, {0.95, 64.7793}, {0.99, 79.4559}, {0.995, 85.108}, {0.999, 97.2371}}

(*Fig 1b*)

ListLinePlot[res, PlotRange → {{0.895, 1.0025}, {55, 103.5}},
        AxesOrigin → {0.895, 55}, PlotMarkers → {"■", Medium}, AxesLabel → "u"]
```

In the **Fig 1b**, for

$$h(t) = u + ct$$

we have presented values of the initial capital u for different choices of the probability of survival $P(T > 2)$, for fixed value of the rate $c = 25$.

In particular, $P(T > 2) = 0.99$ is achieved for $h(t) = 79.4 + 25t$. The same probability 0.99 can be achieved by alternative choices of the capital accumulation function $h(t)$. Assumed that $h(t)$ belongs to the subclass of all piecewise linear functions on $[0, x]$, with one jump of size J , at some instant $t_J \in [0, x]$, that is:

$$h(t) = \begin{cases} u + c_1 t, & 0 \leq t \leq t_J \\ u + c_1 t_J + J + s_1\left(t - t_J\right), & t_J \leq t \leq x \end{cases} \Rightarrow t_j \in [0, \mathrm{t}] .$$

We have two choices of $h(t)$:

- Without jump

$$h_1(t) = 79.4 + 25t$$

and

- With jump

$$h_2(t) = \begin{cases} 59.4 + 27t, & 0 \leq t < 1 \\ 59.4 + 27 + 20 + 23(t - 1), & 1 \leq t \leq 2 \end{cases} .$$

(See **Fig 2a**)

Now, move the location t_J of the jump $J = 20$ from $t_J = 0$ to $t_J = 2$ and keep the rest of the parameters fixed. (**Fig 2b**). A maximum of $P(T > 2) = 0.99$ is achieved for $t_J = 1$. Indeed, both functions $h_1(t)$ and $h_2(t)$ provide equal chances of survival, 99% and also, accumulate equal risk capital at the end of the time interval $x = 2$, that is $h_1(t) = h_2(t) = 129.4$. But the choice $h_2(t)$ is obviously preferable since it requires less capital, $u = 59.4$, to be put aside initially, compared to $u = 79.4$ for the choice $h_1(t)$.

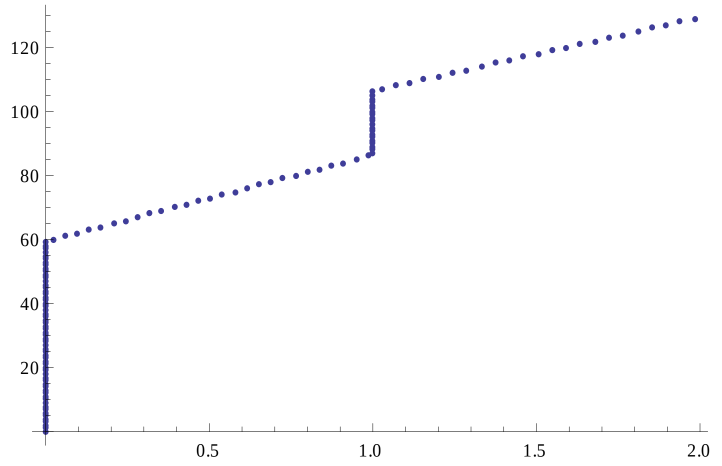

Fig.2a: The capital accumulation function $h_2(t)$

For the calculation of $P(T > 2)$ with capital accumulation function $h_2(t)$ we need

1. υ_n.

We know from (**1**) what

$$\upsilon_n = h_2^{-1}(n) = inf\{t \geq 0: h_2(t) \geq n\}\text{d}$$

$$\upsilon_n = \frac{n-u}{c_1}.$$

Let $t = 0$, then $h_2(t) = 59{,}4$.

Also

$\upsilon_0 = 0, \quad \upsilon_1 = 0, . . . \;, \upsilon_{59} = 0$, for $0 < n \leq 59$.

Let $0 < t < 1$, then

$$h_2(t_j -) = u + c_1 t_j \qquad \Rightarrow \qquad h_2(1 -) = 59{,}4 + 27 * 1$$

$$n_1 = \lfloor h_2(t_j -) \rfloor \qquad \Rightarrow \qquad n_1 = 86 .$$

If $n = 60$, then

$$\upsilon_{60} = \frac{60 - 59{,}4}{27} = 0{,}0222 .$$

(See Table 1.)

Also, $60 \leq n \leq n_1$, where $n_1 = 86$.

Let $t = 1$. Here we have a jump

$$h_2(t_j) = u + c_1 t_j + J \qquad \Rightarrow \qquad h_2(1) = 59{,}4 + 27 * 1 + 20$$
$$n_2 = \lfloor h_2(t_j) \rfloor \qquad \Rightarrow \qquad n_2(1) = 106 .$$

Also $n_1 + 1 \leq n \leq n_2 \;\; \Rightarrow \;\; v_n = t_j$
$61 \leq n \leq 106 \;\; \Rightarrow \;\; v_{106} = 1$.

Let $1 < t < 2$, then

$$h_2(t_j +) = u + c_1 t_j + J + s_1(v_n - t_j) \Rightarrow h_2(1 +) = 59{,}4 + 27 * 1 + 20 + 23(v_n - 1)$$
$$n_3 = \lfloor h_2(x) \rfloor \Rightarrow n_3 = 129 .$$

Also $n_2 + 1 \leq n \leq n_3 \qquad \Rightarrow \qquad v_n = \frac{n-u-c_1 t_j - J}{s_1} + t_j$.
$106 \leq n \leq 129$

2. (b_n) calculation according $(\mathbf{17})$ for $1 \leq n \leq 129$ is given as:

$$\sum_{i=0}^{n} b_i e_{n-i}(v_n) = 0 ,$$

$$\sum_{i=0}^{n-1} b_i e_{n-i}(v_n) + b_n e_0(v_1) = 0 \;\; \Rightarrow b_n = \sum_{i=0}^{n-i} b_i e_{n-i}(v_n),$$

where $b_0 = 1$, $e_0 = 1$ and

$$e_n(x) = \binom{-\lambda\rho x}{n} (-\alpha)^n, \quad n \geq 0 ,$$

$$\rho = -\frac{1}{ln(1-\alpha)} , for\ 0 < \alpha < 1.$$

For $\lambda = 20$ and $\alpha = 0{,}73$, we have

$$e_n(x) = (-0{,}73)^n Binomial[-15{,}2749x, n]$$

and

$$\rho = 0{,}763747.$$

3. A_n calculation according $(\mathbf{16})$
4. $P(T > x)$ calculation according $(\mathbf{17})$

The calculation of A_n and $P(T > x)$ are given in text.

Here, we give the solution with Mathematica for calculating of Floors up to time 0, 1, and 2 for **Fig 2a**. For **Fig 2b** we have used Picard-Lefévre Formula.

```
fn="fig2a.nb"; Remove["Global`*"];
u=59.4;
x=2.0;
J=20.0;
c1=27.0;
s1=23.0;
tj=1.0;

(* Computation of v[n] *)
n0=Floor[u]
59

Table [v[n]=0, {n,0,n0}]
{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
 0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
 0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}

n1=Floor[u+c1*tj]
86

Table [v[n]=(n-u)/c1, {n,n0+1,n1}]

{0.0222222, 0.0592593, 0.0962963, 0.133333, 0.17037, 0.207407, 0.244444,
  0.281481, 0.318519, 0.355556, 0.392593, 0.42963, 0.466667, 0.503704,
  0.540741, 0.577778, 0.614815, 0.651852, 0.688889, 0.725926, 0.762963,
  0.8, 0.837037, 0.874074, 0.911111, 0.948148, 0.985185}

n2=Floor[u+c1*tj+J]
106

Table [v[n]=tj, {n,n1+1,n2}]
{1.,1.,1.,1.,1.,1.,1.,1.,1.,1.,1.,1.,1.,1.,1.,1.,1.,1.,1.,1.}

n3=Floor[u+c1*tj+J+s1*(x-tj)]
129

Table [v[n]=(n-u-c1*tj-J)/s1+tj, {n,n2+1,n3}]

{1.02609, 1.06957, 1.11304, 1.15652, 1.2, 1.24348, 1.28696, 1.33043,
  1.37391, 1.41739, 1.46087, 1.50435, 1.54783, 1.5913, 1.63478, 1.6782,
  1.72174, 1.76522, 1.8087, 1.85217, 1.89565, 1.93913, 1.98261}

ListPlot[Table [{v[n],n}, {n,0,n3}]]
```

```
fn = " fig2b.nb"; Remove["Global`*"];
(* Picard and Lefévre *)
u = SetPrecision[594/10, 80]
c1 = 27;
jj = 20;
s1 = 23;
x = 2;

59.4000000000000000000000000000000000000000000000000000000000000000000000000000
00000

al = SetPrecision[73/100, 80];
la = 20;

rh = −1/Log[1 – al];
e[n_, z_] = Binomial[−la * rh * z, n] * (−al)^n;

h2[t_, tj_] := If[t < tj, u + c1 * t, u + c1 * tj + jj + s1 * (t – tj)]

pr[tj_] := Module[
        {n0, n1, n2, n3, v, b, a},
        n0 = Floor[u];
        n1 = Floor[u + c1 * tj];
        n2 = Floor[u + c1 * tj + jj];
        n3 = Floor[u + c1 * tj + jj + s1 * (x – tj)];
        ;
        Table[v[n] = 0, {n, 0, n0}];
        Table[v[n] = (n – u)/c1, {n, n0 + 1, n1}];
        Table[v[n] = tj, {n, n1 + 1, n2}];
        Table[v[n] = tj + (n − u − c1 * tj – jj) / s1, {n, n2 + 1, n3 + 1}];
        ;
        b[n_] := b[n] = If[n == 0, 1, −Sum[b[i] * e[n − i, v[n]], {i, 0, n – 1}]];
        ;
        Table[a[n] = Sum[b[i] * e[n − i, x], {i, 0, n}], {n, 0, n3 + 1}];
        ;
        Exp[−la _ x] * Sum[a[n], {n, 0, n3}]
        ];

ll = Table[SetPrecision[k / 10, 80], {k, 0, 20}];

pr[ll[[1]]]
0.98445307785074882959902622467450405796183123662717

tmp = Table[{ll[[i]], pr[ll[i]]]}, {i, 1, 21}];

N[tmp]
{{0., 0.984453}, {0.1, 0.985095}, {0.2, 0.985775}, {0.3, 0.986453}, {0.4, 0.986944},
  {0.5, 0.987654}, {0.6, 0.988158}, {0.7, 0.988667}, {0.8, 0.989117}, {0.9, 0.98932},
  {1., 0.989522}, {1.1, 0.989263}, {1.2, 0.988602}, {1.3, 0.987314}, {1.4, 0.98502},
  {1.5, 0.981691}, {1.6, 0.977023}, {1.7, 0.970701}, {1.8, 0.962476}, {1.9, 0.952905},
  {2., 0.941531}}
```

```
plt1 = ListPlot[tmp];

plt2 = ListLinePlot[{{1.0, 0}, {tmp[[11, 1]], tmp[[11, 2]]}}];

Show[plt1, plt2]
```

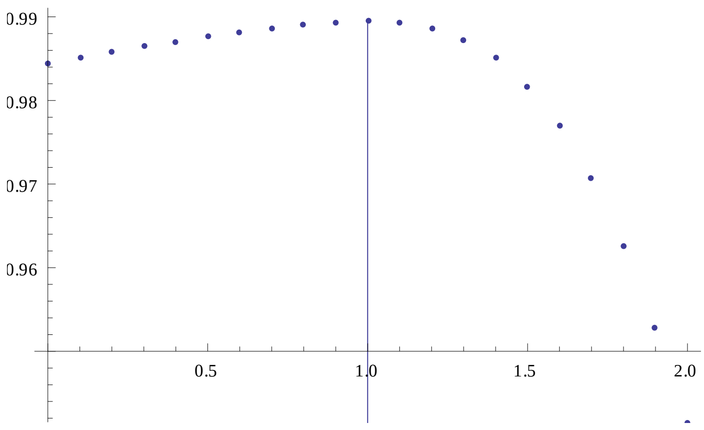

Fig.2b: $P(T > 2)$ as a function of jump in $h_2(t)$

5.2 Exponential i.i.d. losses.

The severities of the consecutive risk losses W_i, $i = 1, 2, \ldots$ are assumed i.i.d. following $Exp(0.5)$, so that their mean mathes the mean of the 2002 LDCE data.

Also the Expected Value and Variance of an exponentially distributed random variable W with rate parameter β are given by

$$E[W] = \frac{1}{\beta} = \frac{1}{0,5} = 2$$

$$Var[W] = \frac{1}{(0,5)^2} = 4$$

And

$$E[W] - 3\sqrt{Var[W]} \approx -4$$

$$E[W] + 3\sqrt{Var[W]} \approx 8\,.$$

Now, we performed the simulation of exponential operational losses, where the number of losses had Poisson distribution.

```
fn = "fig3.nb"; Remove["Global`*"];
la = 20.0;
be = 0.5;
x = 2.0;
nx = RandomInteger[PoissonDistribution[la*x]]
```

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```
t = Sort[RandomReal[{0, x}, nx]]
```

{0.251113, 0.272001, 0.346121, 0.429404, 0.430883, 0.447037, 0.495517, 0.53443, 0.562677, 0.607895, 0.656842, 0.678083, 0.680966, 0.691327, 0.693063, 0.739811, 0.838745, 0.878367, 0.886294, 0.888972, 1.03823, 1.04278, 1.11817, 1.2166, 1.25536, 1.3035, 1.32231, 1.32929, 1.35001, 1.47228, 1.62822, 1.74475, 1.83042, 1.85391, 1.86735, 1.87441, 1.96648}

```
w = RandomReal[ExponentialDistribution[be], nx]
```

{1.15821, 0.824682, 0.0109363, 0.0419599, 2.41516, 1.07307, 6.90254, 1.57801, 0.811676, 1.46206, 1.46109, 2.64482, 0.609949, 0.649267, 0.155679, 0.134245, 0.233637, 5.2117, 0.484376, 0.946768, 1.19508, 1.99953, 3.36536, 9.95494, 1.00209, 1.26196, 4.12289, 2.29586, 0.566376, 0.0723781, 0.17224, 5.45774, 1.95689, 0.709713, 1.02519, 1.71098, 0.0492167}

```
ListPlot[Table[{t[[k]], w[[k]]}, {k, 1, nx}], Filling → Axis, PlotRange → {{0, 2}, {0, 12}}]
```

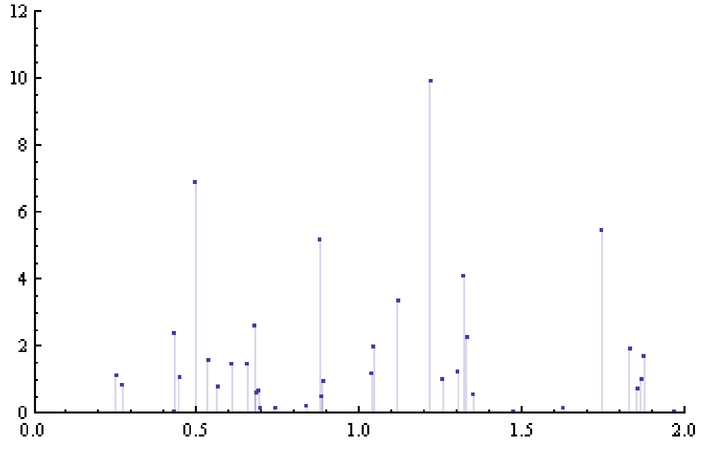

Fig 3: simulated operational $Exp(\beta)$ losses

We can evaluate the example with i.i.d. $Exp(\beta)$ losses with the formulae of Seal [G79].

$$P[T > t] = U(x, t),$$

where $R_0 = x$ the initial surplus.

The losses are $Exp(\beta)$ with distribution function

$$P(y) = 1 - e^{-\beta y},$$

with density

$$p(y) = \beta e^{-\beta y},$$

for $y > 0$.

Distribution function of the risk process is given by

$$F(y, t) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} p^{*k}(y).$$

We know that $Exp(\beta)^{*k} = \Gamma(k, \beta)$ for $k \geq 1$ and $y > 0$

$$F(y, t) = e^{-\lambda t} + \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{k!} \int_0^y \frac{\beta^k}{k!} w^{k-1} e^{-\beta w} dw,$$

where

$$\int_0^y \frac{\beta^k}{k!} w^{k-1} e^{-\beta w} dw$$

is a distribution function of $\Gamma(k, \beta)$.

When $k = 0$, then we have not losses.

So

$$f(y, t) = \frac{\partial F}{\partial y}(y, t)$$

is

$$f(y, k) = \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{k!} \frac{\beta^k}{k!} y^{k-1} e^{-\beta y}.$$

$$U(0,t) = \frac{1}{ct}\int_0^{ct} F(z,t)dz = \frac{1}{ct}\int_0^{ct}\left(e^{-\lambda t} + \sum_{k=1}^{\infty}\frac{(\lambda t)^k}{k!}\int_0^y \frac{\beta^k}{k!}w^{k-1}e^{-\beta w}dw\right)dz$$

If F is replaces by $1-(1-F)$, then

$$U(0,t) = \frac{1}{ct}\int_0^{ct} F(z,t)dz = \frac{1}{ct}\left(ct - \int_0^{ct}(1-F)\,dz\right)$$

$$= \frac{1}{ct}\left(ct - \int_0^{\infty}(1-F)\,dz + \int_{ct}^{\infty}(1-F)dz\right) =$$

$$= \frac{1}{ct}\left(ct - \lambda t p_1 + \int_{ct}^{\infty}(1-F)dz\right).$$

Now consider the case of a positive initial surplus of x. If the claim is absolutely continuous, this means that

$$U(x,t) = F(x+ct,t) - c\int_0^t U(0,t-w)f(x+cw,w)dw =$$

$$= e^{-\lambda t} + \sum_{k=1}^{\infty}\frac{(\lambda t)^k}{k!}\int_0^{x+ct}\frac{\beta^k}{k!}w^{k-1}e^{-\beta w}dw -$$

$$-c\int_0^t\left(e^{-\lambda(t-w)} + \frac{1}{c(t-w)}\int_0^{c(t-w)}(c(t-w)-y)\,f(y,t)dy\right)f(x+cw,w)dw =$$

$$= e^{-\lambda t} + \sum_{k=1}^{\infty}\frac{(\lambda t)^k}{k!}\int_0^{x+ct}\frac{\beta^k}{k!}w^{k-1}e^{-\beta w}dw -$$

$$-c\int_0^t e^{-\lambda(t-w)}\,f(x+cw,w)dw -$$

$$-c\int_0^t\int_0^{c(t-w)}\left(1-\frac{y}{c(t-w)}\right)f(y,t-w)dyf(x+cw,w)dw$$

For $t = 2, c = 25, \lambda = 20$ and $\beta = 0{,}5$ we have inverse function $U(x,2)$ (See **Table 2**)

Instead of Newton Algorithm we use interpolation for

$$1-\varepsilon = 0{,}90; \ldots; 0{,}999$$

Fig 4 illustrate the value of the initial capital charge u, given $h(t) = u + 25t, x = 2$ and Poisson inter-arrival times $\tau_i \sim Exp(20)$. As can be seen , in order to achieve survival probability $P(T > 2) = 0,90$ the capital charge $u = 55,7$ in the case of i.i.d. $W_i \sim Exp(0,5)$ assuming dependence. Furthermore, if a probability of $P(T > 2) = 0,999$ is to be achieved the corresponding values are $u = 98,3$ for i.i.d. losses

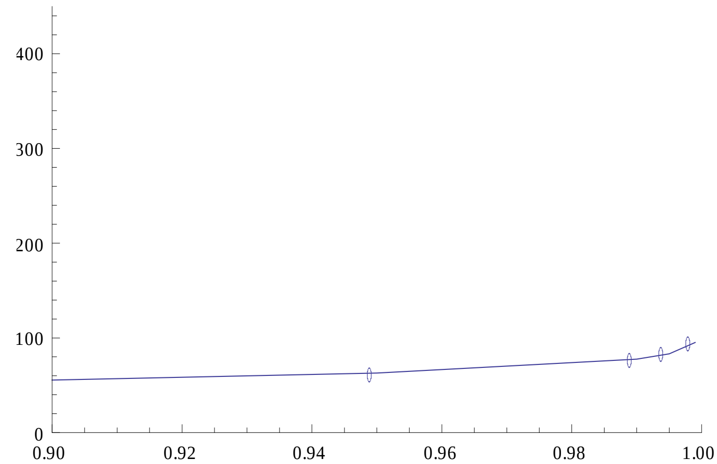

Fig 4: initial capital u for choices $P(T > 2)$ of equal to 90%,95%, 99%, 99,5% and 99,9% in the case of i.i.d. $Exp(0,5)$ losses

Here we present the solution, where we have used Bessel formula and for calculating of survival probabilities $U(0, t)$ and $U(x, t)$ we use the Seal formula and interpolation instead to Newton-Algorithm.

```
fn="exp.nb"; Remove["Global`*"];
c=25; (* premium rate *)
la=20; (* jump intensity lambda *)
be=0.5; (* parameter beta for exponential losses *)
ep={ 0.1, 0.05, 0.01, 0.005, 0.001 }; (* preassigned small epsilon *)

(* random sum density f *)
f[y_, t_ ] = Exp[-la*t–be*y] *Sqrt[be*la*t/y]*BesselI[1, 2 *Sqrt[be*la*t*y]]
```

$3.16228\, e^{-20t-0.5y}\sqrt{\frac{t}{y}}$BesselI[1, 6.32456$\sqrt{ty}$]

```
(*random sum distribution function F*)
F[y_, t_ ]: = Exp[-la*t]+NIntegrate[f[z, t], {z, 0, y}]
```

(*Survival probability $U(0,t)$ according to Seal's first formula *)

```
U0[t_ ]:=
  Exp[-la*t]+1/(c*t)*NIntegrate[(c*t-y)*f[y, t], {y, 0, c*t}]
```

(* Survival probability $U\ (x,t)$ according to Seal's second formula *)

```
U[x_,t_]:=
  F[x+c*t, t]-
    c*NIntegrate[Exp[-la*(t-w)]*f[x+c*w, w], {w, 0, t}]-
    c*NIntegrate[(c*(t-w)-y)/(c*(t-w))*f[y, t-w]*
f[x+c*w, w], {w, 0, t}, {y, 0, c*(t-w)}]

tmp=Table[{U[x, 2], x}, {x, 50, 100}] (* inverse function table *)

{{0.84068,50},{0.852882,51},{0.864375,52},{0.875176,53},{0.885304,54},{0.894781,55},
  {0.90363,56},{0.911875,57},{0.919542,58},{0.926657,59},{0.933247,60},{0.939339,61},
  {0.944959,62},{0.950134,63},{0.95489,64},{0.959253,65},{0.963249,66},{0.966901,67},
  {0.970233,68},{0.973269,69},{0.976029,70},{0.978534,71},{0.980805,72},{0.982859,73},
  {0.984714,74},{0.986387,75},{0.987894,76},{0.989248,77},{0.990463,78},{0.991552,79},
  {0.992526,80},{0.993397,81},{0.994173,82},{0.994865,83},{0.99548,84},{0.996027,85},
  {0.996512,86},{0.996941,87},{0.997321,88},{0.997656,89},{0.997952,90},{0.998213,91},
  {0.998442,92},{0.998643,93},{0.99882,94},{0.998975,95},{0.999111,96},{0.999229,97},
  {0.999333,98},{0.999423,99},{0.999501,100}}

ip=Interpolation[tmp] (* interpolation *)
InterpolatingFunction[{{0.84068, 0.999501}}, < >]

res=Table[{1-ep[[i]], ip[1-ep[[i]]]}, {i, 1, 5}]
(* table with inital capital for given ruin probability *)

{{0.9, 55.5813}, {0.95, 62.9731}, {0.99, 77.606}, {0.995, 83.2093}, {0.999, 95.1737}}

(*Fig 4, thick line graph*)
ListLinePlot[res, PlotRange → {{0.9, 1}, {0, 450}}, PlotMarkers → {"○", Medium}]

(*Fig4*)
```

5.3 Dependent exponential losses.

Now the W_i, $i = 1, 2, \ldots$ are assumed to be dependent, with joint distribution function given by the Rotated Clayton copula, $C^{RCl}(u_1, \ldots, u_k; \theta)$ and the marginal are assumed to be $Exp(0.5)$ or Pareto $(2.41; 1.17)$ distributed.

The Rotated Clayton copula is defined as

$$C^{RCl}(u_1, \ldots, u_k; \theta) = 1 + \sum_{i=1}^{k} (-1)^i \sum_{1 \le i_1 < \cdots < i_i = k} C^{Cl}\left(1 - u_{i_1}, \ldots, 1 - u_{i_i}\right),$$

with density $c^{RCl}(u_1, \ldots, u_k; \theta) = c^{Cl}(1 - u_1, \ldots, 1 - u_k; \theta)$ and parameter $\theta \in (0, \infty)$. The value $\theta = 0$ corresponds to independence. The Rotated Clayton copula has upper tail dependence with coefficient $\lambda_U = 2^{-\frac{1}{\theta}}$ and is suitable for modeling dependence between extreme operational losses.

We (see [DIK08]) report the heavy impact of dependence between loss severities on the value of the initial capital charge u, given $h(t) = u + 25\,t$, $x = 2$ and Poisson inter-arrival times $\tau_i \sim Exp(20)$. In order to achieve survival probability $P(T > 2) = 0{,}90$ the capital charge $u = 55.7$ in the case of i.i.d. $W_i \sim Exp(0.5)$ and $u = 112$ assuming dependence. Furthermore, if a probability of $P(T > 2) = 0{,}999$ is to be achieved, the corresponding values are $u = 98.3$ for i.i.d. losses and $u = 446$, for dependent losses, which is 4.54 higher. The values of the capital charge u have been calculated solving

$$P(T > x) = 1 - P\left(\inf_{0 \le t \le x} \left(h(t) - S(t)\right) < 0\right) = 1 - \epsilon\,,$$

for $x = 2$, with $P(T > x)$ given by

$$P(T > x) = e^{-\lambda x}\left(1 + \sum_{k=1}^{\infty} \lambda^k \int_{y_0}^{h(x)} dy_1 \int_{y_1}^{h(x)} dy_2 \ldots \int_{y_{k-1}}^{h(x)} dy_2\, A_k\left(x; h^{-1}(y_1), \ldots, h^{-1}(y_k)\right) \times f(y_1, \ldots, y_k) dy_k \ldots dy_2 dy_1\right).$$

5.4 Pareto i.i.d. losses

The severities of the consecutive risk losses W_i, $i = 1, 2, ...$ are assumed i.i.d. following Pareto $(2.41; 1.17)$, so that their mean matches the mean of the 2002 LDCE data.
Also the Expected Value of a random variable following a Pareto distribution with $\alpha > 1$ is

$$E[W] = \frac{\alpha x_m}{\alpha - 1} = 1{,}999979$$

$$Var[W] = \left(\frac{x_m}{\alpha - 1}\right)^2 \frac{\alpha}{\alpha - 2} = 4{,}04731$$

And

$$E[W] - 3\sqrt{Var[W]} \approx -4{,}03559$$

$$E[W] + 3\sqrt{Var[W]} \approx 8{,}03517.$$

Now, we performed the simulation of Pareto operational losses, where the number of losses had Poisson distribution.

```
fn = "fig5.nb"; Remive["Global`*"];
la = 20.0;
al = 2.41;
m = 1.17;
x = 2.0;
nx = RandomInteger[PoissonDistribution[la*x]]
```

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```
t = Sort[RandomReal[{0, x}, nx]]
```

{0.180763, 0.184665, 0.210628, 0.241765, 0.264662, 0.325136, 0.725517, 0.83033,
0.869769, 0.906698, 0.947523, 0.955314, 0.971087, 1.02506, 1.06194, 1.06436,
1.11558, 1.16973, 1.3499, 1.35464, 1.45061, 1.46714, 1.47632, 1.49235, 1.50408,
1.51717, 1.56326, 1.62723, 1.64433, 1.69646, 1.74403, 1.77731, 1.80965, 1.86779}

```
w = RandomReal[ParetoDistribution[k, m], nx]
```

{4.01421, 5.11758, 3.90763, 3.65179, 60.0096, 3.25332, 2.6235, 6.52475,
2.95423, 7.33319, 2.41153, 2.71996, 22.5071, 5.01423, 3.37687, 3.36069,
3.2747, 4.08698, 8.90636, 5.99028, 2.9879, 16.29, 6.24734, 4.15028, 3.86768,
2.48354, 3.0344, 3.47868, 21.6089, 2.98171, 2.55807, 6.42044, 14.4792, 4.99283}

```
ListPlot[Table[{t[[a]], w[[a]]}, {a, 1, nx}], Filling → Axis, PlotRange → {{0, 2}, {0, 10}}]
```

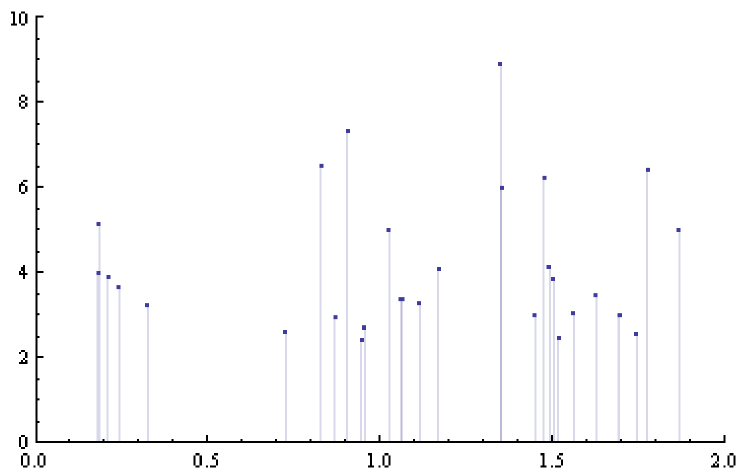

Fig 5: simulated operational $Pareto(\alpha, m)$ losses

5.5 Dependent Pareto

The data for this case we have taken from paper of Kaishev and Dimitrova.

Comparing it with the exponential case, one can see that a lower level of capital is required for probabilities 0.90 and 0.95 and similar or greater u is needed for higher probabilities (between 0.99 and 0.999) for the independent and dependent case, owing to the effect of the heavy-tailedness of the Pareto distribution and its left truncation, which rules out losses smaller than 1.17.

The results can be extreme, if the choice of the Rotated Clayton copula with parameter $\theta = 1$, leads to Kendall's $\tau = 0.33$ and upper tail dependence $\lambda_U = 0.5$, which tailors a reasonably strong dependence.

Table 1. for n and v_n

$\boldsymbol{n}$	$\boldsymbol{v_n}$
0	0
1	0
2	0
3	0
4	0
5	0
6	0
7	0
8	0
9	0
10	0
11	0
12	0
13	0
14	0
15	0
16	0
17	0
18	0
19	0
20	0
21	0
22	0
23	0
24	0
25	0
26	0
27	0
28	0
29	0
30	0
31	0
32	0
33	0
34	0
35	0
36	0
37	0
38	0
39	0
40	0
41	0

42	0
43	0
44	0
45	0
46	0
47	0
48	0
49	0
50	0
51	0
52	0
53	0
54	0
55	0
56	0
57	0
58	0
59	0
60	0,0222222
61	0,0592593
62	0,0962963
63	0,133333
64	0,17037
65	0,207407
66	0,244444
67	0,281481
68	0,318519
69	0,355556
70	0,392593
71	0,42963
72	0,466667
73	0,503704
74	0,540741
75	0,577778
76	0,614815
77	0,651852
78	0,688889
79	0,725926
80	0,762963
81	0,8
82	0,837037
83	0,874074
84	0,911111
85	0,948148
86	0,985185
87	1
88	1
89	1
90	1

91	1
92	1
93	1
94	1
95	1
96	1
97	1
98	1
99	1
100	1
101	1
102	1
103	1
104	1
105	1
106	1
107	1,02609
108	1,06957
109	1,11304
110	1,15652
111	1,2
112	1,24348
113	1,28696
114	1,33043
115	1,37391
116	1,41739
117	1,46087
118	1,50435
119	1,54783
120	1,5913
121	1,63478
122	1,67826
123	1,72174
124	1,76522
125	1,8087
126	1,85217
127	1,89565
128	1,93913
129	1,98261

Table 2: Inverse function $U(x,2)$

$U(x,2)$	$\boldsymbol{x}$
0.901450	55
0.909622	56
0.917242	57
0.924333	58
0.930917	59
0.937021	60
0.942667	61
0.947881	62
0.952686	63
0.957107	64
0.961168	65
0.96489	66
0.968297	67
0.971409	68
0.974248	69
0.976833	70
0.979184	71
0.981318	72
0.983252	73
0.985002	74
0.986583	75
0.988011	76
0.989297	77
0.990454	78
0.991494	79
0.992428	80
0.993265	81
0.994014	82
0.994684	83
0.995283	84
0.995817	85
0.996293	86
0.996717	87
0.997094	88
0.99743	89
0.997728	90
0.997992	91
0.998226	92
0.998434	93
0.998618	94
0.99878	95
0.998924	96
0.999051	97
0.999164	98

0.999263	99
0.999351	100
$U(x,2)$	$\boldsymbol{x}$
0.851557	50
0.862803	51
0.873397	52
0.883358	53
0.892702	54
0.901450	55
0.909622	56
0.917242	57
0.924333	58
0.930917	59
0.937021	60
0.942667	61
0.947881	62
0.952686	63
0.957107	64
0.961168	65
0.96489	66
0.968297	67
0.971409	68
0.974248	69
0.976833	70
0.979184	71
0.981318	72
0.983252	73
0.985002	74
0.986583	75
0.988011	76
0.989297	77
0.990454	78
0.991494	79
0.992428	80
0.993265	81
0.994014	82
0.994684	83
0.995283	84
0.995817	85
0.996293	86
0.996717	87
0.997094	88
0.99743	89
0.997728	90
0.997992	91
0.998226	92
0.998434	93
0.998618	94
0.99878	95

0.998924	96
0.999051	97
0.999164	98
0.999263	99
0.999351	100

References

[B04] Brandts, S. (2004). Operational Risk and Insurance: Quantitative and Qualitative Aspects. Working Paper, EFMA 2004 Basel Meetings Paper

[BCPS06] Bazzarello, D., Crielaard, B., Piacenza, F. and Soprano, A. (2006). Modeling insurance mitigation on operational risk capital. *Journal of Operational Risk*, 1(1), pp. 57-65.

[Cru02] Marcelo G. Cruz. Modeling, measuring and hedging operational risk. Wiley, 2002.

[DEL09] Matthias Degen, Paul Embrechts, Dominik D. Lambrigger. The Quantitative Modeling of Operational Risk: Between g-and-h and EVT. Department of Mathematics, ETH Zurich, CH-8092 Zurich, Switzerland, April 25, 2007.

[DIK08] Dimitrina S. Dimitrova, Zvetang G. Ignatov, and Vladimir K. Kaishev. Operational risk and insurance: a ruin-probabilistic reserving approach. The Journal of Operational Risk, 3(3):39–60, 2008. Preprint www.actuaires.org/ASTIN/Colloquia/Orlando/Papers/Kaishev.pdf.

[DP 06] Dutta, K. and Perry, J. (2006). A Tale of Tails: An Empirical Analysis of Loss Distribution Models for Estimating Operational Risk Capital. Working Paper, Federal Reserve Bank of Boston, No 06-13.

[ECB 04] European Central Bank, Carlos Bernadell, Pierre Cardon, Joachim Coche, Francis X. Diebold and Simone Manganelli. Risk management for central bank foreign reserves., May 2004.

[EFMcN05] Paul Embrechts, Rüdiger Frey, Alexander J. McNeil. Quantitative risk management : concepts, techniques, and tools. 2005 Princeton University Press.

[EKM97] Embrechts, P., C. Klüppelberg and T. Mikosch. Modeling extremal events for insurance and finance. Springer-1997.

[EKS04] Embrechts, P., Kaufmann, R. and Samorodnitsky, G. (2004). Ruin Theory Revisited: Stochastic Models for Operational Risk. *Risk Management for Central Bank Foreign Reserves* (Eds.C. Bernadell et al.) European Central Bank, Frankfurt a.M., pp. 243-261.

[ES03] Embrechts, P. and Samorodnitsky, G. (2003). Ruin Problem, Operational Risk and How Fast Stochastic Processes Mix. *The Annals of Applied Probability*, 13., 1-36.

[EVMNAF01] Ebnöther, S., Vanini, P., McNeil, A. and Antolonez-Fehr, P. (2001). Modeling operational risk. ETH Zürich Working paper.

[EVMNAF02] Ebnöther, S., Vanini, P., McNeil, A. and Antolonez-Fehr, P. (2002). Operational risk: A practitioner's view. ETH Zürich Working paper.

[G97] Hans U. Gerber, An introduction to mathematical risk theory, S. S. Huebner Foundation for Insurance Education, University of Pennsylvania , 1979

[IK00] Zvetan G. Ignatov and V. K. Kaishev. Two-sided bounds for the finite time probability of ruin. Scandinavian Actuarial Journal, (1):46–62, 2000.

[IK04] Zvetan G. Ignatov and Vladimir K. Kaishev. A finite-time ruin probability formula for continuous claim severities. Journal of Applied Probability, 41(2):570–578, 2004.

[IKK01] Zvetan G. Ignatov, Vladimir K. Kaishev, and Rossen S. Krachunov. An improved finite-time ruin probability formula and its Mathematica implementation. Insurance: Mathematics & Economics, 29(3):375–386, 2001. 4th IME Conference (Barcelona, 2000).

[KK06] Christian Kleiber, Samuel Kotz. Statistical Size Distributions in Economics und Actuarial Sciences. Wiley Series in Probability and Statistics. Wiley-Interscience, 2006.

[M01] Medova Elena "Operational risk capital allocation and integration of risks." Judge Institute of Management Working Papers, No.10/2001. Cambridge: University of Cambridge.

[Med00] Medova E., Extreme Value Theory – Extreme values and the measurement of operational risk, Operational Risk (Jily).2000, 11-15

[MK01] E. A. Medova and M. N. Kyriacou. Extremes in operational risk management, 2001.

[Mos04] Marco Moscadelli. The modelling of operational risk: experience with the analysis of the data collected by the Basel Committee, Bank of Italy, Banking Supervision Department July 2004.

[Pan06] Harry H. Panjer. Operational risk. Wiley Series in Probability and Statistics. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, 2006. Modeling analytics.

[PL97] Picard, P., Lefévre, C.,. The probability of ruin in finite time with discrete claim size distribution. Scandinavian Actuarial Journal 1 (58-69), 1997