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Arakelov Geometry with a View towards Integral Points

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unter Anleitung von
Univ.Prof. Dipl.-Ing. Dr. techn. Michael Drmota

durch
Claudia I. Scheimbauer
Hartäckerstr. 32
1190 Wien

Datum

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L'algèbre n'est qu'une géométrie écrite – la géométrie n'est qu'une algèbre figurée.

-Sophie Germain (1776–1831)

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Contents

Introduction

In this work, we deal with a number of issues related to Arakelov theory. Arakelov theory originally was developed as a tool for solving problems in diophantine geometry. It combines Grothendieck's theory of schemes with Hermitian complex geometry: schemes give a geometric interpretation of diophantine problems and Hermitian complex geometry serves as a tool for controlling the height of points on such schemes.

In 1974, Suren Y. Arakelov introduced an intersection theory on so-called arithmetic surfaces, [Ara74], [Ara75]. His ground-breaking idea was to "complete" an arithmetic surface over a ring of integers $\mathcal{O}_K$ of a number field K by taking the places at infinity, i.e. the embeddings $\sigma : K \hookrightarrow \mathbb{C}$, into account. He "added" the complex manifolds $X_\sigma(\mathbb{C})$, where $\sigma : K \hookrightarrow \mathbb{C}$ is an embedding and $X_\sigma = X \times_{K,\sigma} \mathbb{C}$. Instead of considering divisors on X as a formal linear combination of points in X as in the classical way, he adds an infinite part to the sum, i.e. $D = D_{fin} + D_{inf} = \sum_{i=0}^{k} n_i P_i + \sum_{\sigma:K\hookrightarrow\mathbb{C}} \lambda_\sigma X_\sigma$, where X_σ is seen as a formal symbol and $\lambda_\sigma \in \mathbb{R}$. This has an interpretation as line bundles on X: D_{fin} defines a line bundle $\mathcal{L} = \mathcal{O}(D_{fin})$ on X, and the addition of D_{inf} corresponds to defining a metric on $\mathcal{L}$. Along these lines, the main objects of study in Arakelov geometry, and in particular in this thesis, are *Hermitian vector bundles on arithmetic varieties.*

Arakelov's approach was taken up and extended by Gerd Faltings in his seminal paper [Fal84]. In this paper, he proved analogs of the Riemann-Roch theorem, the Hodge index theorem, and Noether's formula for arithmetic varieties. Furthermore, the work on this paper eventually led Faltings to his proof of the Mordell conjecture, now called Faltings' theorem.

Finding a generalization of Arakelov's intersection product for higher dimensional arithmetic varieties is not straightforward and was given by Henri Gillet and Christoph Soulé in a series of papers, among them [GS90]. In a joint paper with Jean-Benoît Bost, [BGS94], they further introduced the notion of a *height of an arithmetic variety* as a special case of the intersection product.

Throughout the last three decades, the methods of Arakelov theory have been used as a tool to solve a number of hitherto unsolved problems in arithmetic geometry, e.g. the Bogomolov conjecture by Emmanuel Ullmo in 1998, [Ull98], and Serge Lang's generalization of Mordell's conjecture by Faltings in 1991, [Fal91]. They were also used by Paul Vojta for a new proof of Mordell's conjecture, [Voj91].

One of the most-studied problems in arithmetic is to determine the number of integral or rational points on a curve. Faltings' proof of the Mordell conjecture gave solutions to these problems for curves of genus greater than one. For curves of genus one, i.e. elliptic curves, this already had been achieved in the 1920s by the Mordell-Weil theorem for rational points, and by Siegel's theorem for integral points. However, these results are not effective in the sense that the Mordell-Weil theorem does not give an explicit value for the rank of the finitely generated Abelian group of rational points and Siegel's theorem does not give a bound for the height of the finitely many integral points. While the effectivity of Mordell-Weil's theorem remains unsolved, for Siegel's theorem the effectivity was established in 1970 by Alan Baker and John Coates in [BC70].

In this context, Arakelov theory seems to provide a viable, new ansatz to finding an effective bound for the heights of the finite number of integral points on a given elliptic curve. While solving this problem is by far beyond the scope of this thesis, we, in this work, give an introduction to the necessary subset of tools drawn from Arakelov theory that we hope can be used to tackle this issue.

This work is structured as follows: in part I, i.e. chapters 1 to 3, we introduce the basic notions from algebraic geometry needed throughout this thesis. Part II, i.e. chapters 4 to 9, deals with Arakelov geometry. Part III, i.e. chapter 10, gives an overview of the problem of finding integral points on elliptic curves and contains an outlook on possible future work using Arakelov geometry.

To be more precise, chapter 1 deals with sheaves, a tool for keeping track of local data on a topological space; in chapter 2, we introduce schemes, the main objects of study in modern algebraic geometry. They are generalizations of classical varieties and form the means to use algebraic geometry in the study of problems in arithmetic. In chapter 3, vector bundles are defined both on a complex manifold and on a variety. We illustrate the connections between vector bundles, projective modules, sheaves, and divisors. Subsequent to chapters 1 to 3, we give a summary of the most important notions and connections.

Chapter 4 first deals with Hermitian complex geometry, i.e. endowing a vector bundle on a complex manifold with a Hermitian metric; in particular, we impose the Fubini-Study metric on the so-called twisting sheaf $\mathcal{O}(1)$ on projective space to obtain the Hermitian vector bundle $\overline{\mathcal{O}(1)}$. Then, we extend these techniques to vector bundles on arithmetic varieties and impose metrics on them, thus defining the main objects of study in Arakelov geometry.

Chapter 5 is on the arithmetic degree of a Hermitian vector bundle on an arithmetic variety. We define the degree and give some properties. Furthermore, as an example, we calculate the arithmetic degree of $\overline{\mathcal{O}(1)}$ on the projective n-space over $\mathcal{O}_K$, $\mathbb{P}^n_{\mathcal{O}_K}$. In chapter 6, we introduce heights of Jean-Benoît Bost, Henri Gillet, and Christoph Soulé, [BGS94]; more precisely, the height of a point in $\mathbb{P}^n_{\mathcal{O}_K}$ and, recursively, the height of $\mathbb{P}^n_{\mathcal{O}_K}$, both with respect to $\overline{\mathcal{O}(1)}$. We follow [BGS94] in calculating the height of $\mathbb{P}^n_{\mathcal{O}_K}$. We refrain from introducing the arithmetic intersection product on arithmetic varieties in this thesis, as this would require too much background material; but we give remarks on how it is related to the degree and the height in section 6.3, and thus give a connection between our results on the arithmetic degree of $\overline{\mathcal{O}(1)}$ on projective n-space over $\mathcal{O}_K$ and the height of $\mathbb{P}^n_{\mathcal{O}_K}$ with respect to $\overline{\mathcal{O}(1)}$.

We discuss canonical polygons in chapter 7, and thus draw a link to geometry of numbers. In this context, we present recent results of Thomas Borek [Bor05] and slope inequalities, following [Via05].

In chapter 8, we explain a more geometric interpretation of Hermitian vector bundles, as used in [Sou92]. We first introduce geometric Chow groups as in classical intersection theory and then extend them to arithmetic Chow groups using the Poincaré-Lelong formula. We give a sketch of proof of the correspondence between the first arithmetic Chow group and the arithmetic Picard group.

Chapter 9 first deals with the problem of attaching an arithmetic surface to an elliptic curve and general properties of arithmetic surfaces such as divisors and integral points on them. Secondly, we introduce theta functions and, using them, impose a metric on the line bundle $\mathcal{O}(O_E)$ on an elliptic curve, where O_E is the origin. Thirdly, we briefly present a result of Jürg Kramer on the degree of the resulting Hermitian line bundle.

Finally, the last chapter, chapter 10, serves as an introduction to the problem of (effectively) determining the integral points on an elliptic curve. We give a historical overview of the existing results and briefly discuss the idea of applying the techniques in Arakelov theory to this problem. This chapter is intended to be an outlook on future work.

We also provide two appendices for reference, one on algebraic number theory – basic definitions, the product formula, and the height of a point – and the other on elliptic curves – Weierstrass equations, the Weierstrass $\wp$-function, and curves of genus one.

Part I

Basics

Chapter 1

Sheaves

Schemes allow us to translate arithmetic problems into geometric problems and thus look at them from a different perspective. Therefore, we first need some basic definitions from algebraic geometry. We present the most important definitions and theorems in the first three sections. We omit many of the proofs here, as the intention of this chapter is to recall the most important notions needed. Nevertheless, we give references as to where to find more details, further explanations, and illustrative examples.

We first need to introduce the notion of a *sheaf*. Sheaves provide an important technical tool for the study of algebraic geometry. They form a tool to keep track of local (algebraic) data on a topological space and to pass from local information to global information.

1.1 Definition

Definition 1.1.1. A *presheaf* $\mathcal{F}$ over a topological space X consists of the following data:

(i) For every open set $U \subset X$ a set $\mathcal{F}(U)$,

(ii) For every pair U, V of open subsets of X such that $V \subset U$, a *restriction homomorphism*

$$\rho_{U,V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V),$$

satisfying the following conditions:

(a) $\rho_{U,U} = id_U$, and

(b) given open subsets $U \supset V \supset W$ of X, restriction is compatible, i.e.

$$\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}.$$

Remark 1.1.2. We usually additionally require that our presheaf has some algebraic structure (e.g. *presheaf of Abelian groups, of rings*). By that, we mean that the $\mathcal{F}(U)$ have the given structure and that the restriction homomorphisms preserve the structure (e.g. group homomorphism, ring homomorphism).

Remark 1.1.3. Sometimes one also requires that $\mathcal{F}(\varnothing) = 0$, where 0 is the trivial group, ring, etc. (e.g. [Har77]), but, as is pointed out in [Sha94c], p. 16, a presheaf (up to isomorphism) does not depend on the choice of the element $\mathcal{F}(\varnothing)$.

Definition 1.1.4. Let $\mathcal{F}, \mathcal{G}$ be two presheaves over X. Then a *morphism (of presheaves)*

$$h : \mathcal{F} \longrightarrow \mathcal{G}$$

is a collection of maps

$$h_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$

for each open set U in X such that the h_U commute with the restriction maps. If a morphism has a two-sided inverse, we call it an *isomorphism*.

If the maps h_U are inclusions, we say that $\mathcal{F}$ is a *subpresheaf* of $\mathcal{G}$.

Example 1.1.5. If $\mathcal{F}$ is a presheaf on X and $U \subset X$ open, then we define the *restriction of the presheaf* $\mathcal{F}$ to be the presheaf defined by

$$\forall V \subset U \text{ open}, \quad V \mapsto \mathcal{F}(V).$$

We denote it by $\mathcal{F}|_U$.

Definition 1.1.6. Let $\mathcal{F}$ be a presheaf on X, and let $x \in X$ be a point. Then the *stalk* $\mathcal{F}_x$ *of* $\mathcal{F}$ *at* x is defined as the direct limit of the groups $\mathcal{F}(U)$ taken over all open sets U containing x, with respect to the restriction maps $\rho_{U,V}$ for $V \subset U$, i.e.

$$\mathcal{F}_x = \varinjlim \mathcal{F}(U).$$

Definition 1.1.7. A presheaf is called a *sheaf*, if for every open covering $\{U_i\}_i$ of an open subset $U \subset X$ such that $\forall i : U_i \subset U$, $\mathcal{F}$ satisfies the following:

1. If $s, t \in \mathcal{F}(U)$ and $\forall i : \rho_{U,U_i}(s) = \rho_{U,U_i}(t)$, then $s = t$.
2. If $s_i \in \mathcal{F}(U_i)$ and if for all i, j such that $U_i \cap U_j \neq \varnothing$,
$$\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j),$$
then there exists an $s \in \mathcal{F}(U)$ such that $\rho_{U,U_i}(s) = s_i$.

Morphisms of sheaves are simply morphisms of the underlying presheaves. A *subsheaf* is a subpresheaf of a sheaf which itself is a sheaf.

An *isomorphism of sheaves* is a morphism which has a two-sided inverse.

Definition 1.1.8. Given a presheaf or a sheaf $\mathcal{F}$, we call the elements $s \in \mathcal{F}(U)$ *sections of $\mathcal{F}(U)$ over U*. A *global section* is an element of $\mathcal{F}(X)$. The set of global sections is often denoted by $\Gamma(X, \mathcal{F})$.

Example 1.1.9. Let X be a complex manifold. Let U be an open subset of X. Define $\mathcal{O}_X(U)$ to be the set of holomorphic maps on U. This clearly defines a sheaf on X. This sheaf is called the *structure sheaf of the manifold* X and is denoted by $\mathcal{O}_X$.

1.2 Construction of sheaves

Given a presheaf $\mathcal{F}$, it is possible to uniquely (up to isomorphism) associate a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ to it, satisfying a certain universal property. This allows us to construct sheaves out of given sheaves.

Theorem 1.2.1. *Given a presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ and a morphism $\theta : \mathcal{F} \to \mathcal{F}^+$, with the property that for any sheaf $\mathcal{G}$, and any morphism $\varphi : \mathcal{F} \to \mathcal{G}$, there is a unique morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \psi \circ \theta$. Furthermore, the pair $(\mathcal{F}^+, \theta)$ is unique up to isomorphism.*

Proof. See [Har77], Proposition-Definition II.1.2. □

Definition 1.2.2. $\mathcal{F}^+$ is called the *sheaf associated to the presheaf* $\mathcal{F}$, or *sheafification of* $\mathcal{F}$.

Example 1.2.3. Given a morphism of sheaves, $\varphi : \mathcal{F} \to \mathcal{G}$, we can use the construction above to define the *image* $\operatorname{im} \varphi$ *of the morphism* φ to be the sheaf associated to the image presheaf of φ. By the universal property of the sheafification, there is a morphism $\operatorname{im} \varphi \to \mathcal{G}$. The *kernel* $\ker \varphi$ *of the morphism* φ already is a sheaf, so we do not need sheafification in this case. A morphism is called *injective* if $\ker \varphi = 0$. Then the morphism $\operatorname{im} \varphi \to \mathcal{G}$ from above is injective.

Example 1.2.4. Let $f : X \to Y$ be a continuous map of topological spaces and $\mathcal{F}$ be a sheaf on X. The *direct image sheaf* $f_*\mathcal{F}$ *on* Y is defined by mapping an open set $V \subset Y$ to

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V)).$$

This sheaf (or rather the vector bundle associated with this sheaf, see section 3.4), is often called the *pushforward of* $\mathcal{F}$.

Definition 1.2.5. Let $\mathcal{R}$ be a presheaf of commutative rings and let $\mathcal{F}$ be a presheaf of Abelian groups over a topological space X such that for every open subset $U \subset X$, $\mathcal{F}(U)$ is an $\mathcal{R}(U)$ module. Furthermore, let the module structure be compatible with the restriction homomorphisms ρ of $\mathcal{F}$ via the restriction ring homomorphisms σ of $\mathcal{R}$, i.e. for all open subsets U, V of X such that $V \subset U$,

$$\rho_{U,V}(\alpha s) = \sigma_{U,V}(\alpha)\rho_{U,V}(s) \quad \forall \alpha \in \mathcal{R}(U),\ s \in \mathcal{F}(U).$$

Then $\mathcal{F}$ is called a *presheaf of $\mathcal{R}$-modules.* If $\mathcal{F}$ is a sheaf, it is called a *sheaf of $\mathcal{R}$-modules.*

We can use now use algebraic constructions of modules for constructing new sheaves of modules, e.g. given a sheaf $\mathcal{R}$ of commutative rings over a topological space X and two sheaves of $\mathcal{R}$-modules $\mathcal{F}$ and $\mathcal{G}$, we can define the *direct sum of* $\mathcal{F}$ *and* $\mathcal{G}$ to be the sheaf associated to the presheaf $(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$.

Definition 1.2.6. Given two sheaves $\mathcal{F}$ and $\mathcal{G}$ and using the corresponding algebraic construction and sheafification, we get the following:

- *the dual sheaf* $\mathcal{F}^\vee$,
- *the direct sum of sheaves* $\mathcal{F} \oplus \mathcal{G}$,
- *the tensor product of sheaves* $\mathcal{F} \otimes_{\mathcal{R}} \mathcal{G}$,
- *the* m*-th exterior product of a sheaf* $\bigwedge^k \mathcal{F}$,
- *the tensor, symmetric, and antisymmetric algebra of a sheaf* $\mathcal{F}$*, etc.*

See [Har77], chapter II.5 for details.

Definition 1.2.7. A sheaf of $\mathcal{R}$-modules $\mathcal{F}$ over X is *free* if it is isomorphic to a direct sum of copies of $\mathcal{R}$. It is *locally free* if there is a covering of X by open sets U such that for every U, $\mathcal{F}|_U$ is a free $\mathcal{R}|_U$-module. In particular, a locally free sheaf is a sheaf of $\mathcal{R}$-modules. A locally free sheaf of rank 1 is called an *invertible sheaf.*

Locally free sheaves and, in particular, invertible sheaves will be our main objects of study in the following chapters. We will study locally free sheaves and invertible sheaves from another point of view in chapter 3.

Chapter 2

Schemes

In algebraic geometry, the definition of the basic object of study, *varieties*, changed throughout the different stages of development. Yet, the intuition should always be that one can think of a variety as the zero set of a system of polynomial equations. The simplest and thus best studied case, of course, is that of a zero set of just one polynomial equation. However, the setting can be quite different, for example a variety can be a subset of affine or projective space (affine and projective varieties). A good introduction to varieties as zero sets of equations is [Sha94b]. In this thesis, we consider a more general notion, the notion of a *scheme*. Schemes turn out to be interesting for us because, contrary to the "classical" definition of a variety, they can be defined over the integers $\mathbb{Z}$, or, more generally, over the ring of integers $\mathcal{O}_K$ of a number field K. Therefore, schemes are very useful for applications in number theory and allow introducing geometric intuitions to problems in number theory.

2.1 The spectrum of a ring

Definition 2.1.1. Let R be a ring, commutative with one. As a set, we define the *spectrum* $\operatorname{Spec} R$ *of the ring* R to be the set of all prime ideals of R. The prime ideals of R are called *points of* $\operatorname{Spec} R$.

If $\mathfrak{a} \subset R$ is any ideal of R, let

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{a} \subset \mathfrak{p}\}.$$

It is easy to see that the $V(\mathfrak{a})$ form the closed sets of a topology on $\operatorname{Spec} R$, the *Zariski topology*, see [Har77] p. 70, Lemma II.2.1. The topological space $\operatorname{Spec} R$ is compact. In general, it is not Hausdorff, but it always is T_0. The closure of a point $\mathfrak{p}$ is homeomorphic to $\operatorname{Spec} R/\mathfrak{p}$, so a point is closed if and only if $\mathfrak{p}$ is maximal. In particular, $\operatorname{Spec} R$ may contain non-closed points.

Definition 2.1.2. A point is called a *generic point of a topological space* if it is dense (as a set).

Remark 2.1.3. We will sometimes denote a point in $X = \operatorname{Spec} R$ by x, if we want to stress the fact that it is a point of the topological space X, and sometimes by $\mathfrak{p}$, if we consider it as a prime ideal in R.

Example 2.1.4. Let $R = \mathbb{Z}$. Then $\operatorname{Spec} R = \{(0)\} \cup \{(p) : p \text{ prime}\}$. The closed sets of $\operatorname{Spec} R$ are the finite sets of prime ideals not containing the zero ideal. In particular, the point (0) is not closed; in fact, it is a generic point.

Remark 2.1.5. Note that every homomorphism of rings, $\varphi : A \to B$, induces a continuous map ${}^a\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$, see [Sha94c], p. 6 and 10. As every ring R allows a natural map $\mathbb{Z} \to R$, we always get a map $\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}$.

Definition 2.1.6. The *residue field at a point* $x \in \operatorname{Spec} R$, where x is the point given by a prime ideal $\mathfrak{p}$, is the field of fractions of the quotient ring $R/\mathfrak{p}$ and is denoted by $k(x)$. Thus, given a point $x \in \operatorname{Spec} R$ which corresponds to the prime ideal $\mathfrak{p}$, there is a homomorphism

$$R \longrightarrow k(x)$$

with kernel $\mathfrak{p}$. The image of an element $f \in A$ is denoted by $f(x)$.

Next we define a sheaf of rings $\mathcal{O}$ on $\operatorname{Spec} R$, the *structure sheaf of* $\operatorname{Spec} R$.

Definition 2.1.7. Let $R_{\mathfrak{p}}$ be the localization of R in the prime ideal $\mathfrak{p}$. Then define a sheaf of rings on $\operatorname{Spec} R$, denoted by $\mathcal{O}$, and called the *structure sheaf of* $\operatorname{Spec} R$, by sending an open set $U \subset \operatorname{Spec} R$ to the ring $\mathcal{O}(U)$ consisting of functions

$$s : U \longrightarrow \bigsqcup_{\mathfrak{p} \in \operatorname{Spec} R} R_{\mathfrak{p}},$$

satisfying

1. $s(\mathfrak{p}) \in R_{\mathfrak{p}}$, $\forall\, \mathfrak{p} \in U$, and
2. s locally is a quotient of elements in R.

This means that for each $\mathfrak{p} \in U$, there are a neighborhood V of $\mathfrak{p}$ in U and elements $r, f \in R, f(\mathfrak{p}) \neq 0$, such that for every $\mathfrak{q} \in V$, $s(\mathfrak{q}) = r/f \in R_{\mathfrak{q}}$.

The ring $\mathcal{O}(U)$ is commutative and the element 1 which gives 1 in each $R_{\mathfrak{p}}$ is an identity element. The restriction map is the obvious restriction, which is a ring homomorphism. Therefore, $\mathcal{O}$ is a presheaf of rings, and, since it is defined locally, it even is a sheaf of rings.

Remark 2.1.8. Sometimes, e.g. in [Har77], the spectrum of a ring is defined as the pair $(\operatorname{Spec} R, \mathcal{O})$.

The global sections of this sheaf form a ring which is isomorphic to R, i.e.

$$\Gamma(\operatorname{Spec} R, \mathcal{O}) \cong R.$$

Furthermore, the stalk of the structure sheaf at a point $\mathfrak{p}$ is isomorphic to the local ring $R_{\mathfrak{p}}$, i.e.

$$\mathcal{O}_{\mathfrak{p}} \cong R_{\mathfrak{p}},$$

see [Har77], p. 71, Proposition II.2.2.

Remark 2.1.9. The construction above is similar to the construction of the sheafification of a presheaf, namely constructing a sheaf out of its stalks, which in this case are the localizations $R_{\mathfrak{p}}$. We will see several constructions like this one.

2.2 Ringed spaces and schemes

Definition 2.2.1. A *ringed space* is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. X is called the *underlying topological space* and $\mathcal{O}_X$ is called the *structure sheaf.*

A *morphism of ringed spaces from* $(X, \mathcal{O}_X)$ *to* $(Y, \mathcal{O}_Y)$ is a pair $(f, f^{\#})$, where $f : X \to Y$ is a continuous map and $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of rings on Y.

A ringed space $(X, \mathcal{O}_X)$ is a *locally ringed space*, if for every point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring. A *morphism of locally ringed spaces* is a morphism $(f, f^{\#})$ of ringed spaces, such that at every point, the induced map between the stalks is a *local homomorphism* of local rings. That is, a homomorphism such that the preimage of the maximal ideal of the codomain is the maximal ideal of the domain (see [Har77], p. 72, for an explanation of the induced map on the stalks).

An *isomorphism* is a morphism with a two-sided inverse.

Proposition 2.2.2. *(Proposition II.2.3 in [Har77])*

1. *If R is a ring, then* $(\operatorname{Spec} R, \mathcal{O})$ *is a locally ringed space.*
2. *If $\varphi : A \to B$ is a homomorphism of rings, then φ induces a natural morphism of locally free ringed spaces*

$$(f, f^{\#}) : (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \longrightarrow (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}).$$

3. *If A and B are rings, then any morphism of locally ringed spaces from* $\operatorname{Spec} B$ *to* $\operatorname{Spec} A$ *is induced by a homomorphism of rings $\varphi : A \to B$ as in 2.*

Definition 2.2.3. A locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic (as a locally ringed space) to the spectrum of a ring together with its structure sheaf is called an *affine scheme.*

A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ in which every point x has an open neighborhood U, called an *affine neighborhood of* x, such that the pair $(U, \mathcal{O}_X|_U)$ is an affine scheme. X is called the *underlying topological space* of the scheme $(X, \mathcal{O}_X)$ and $\mathcal{O}_X$ is called the *structure sheaf.*

A *morphism of schemes* is a morphism as locally ringed spaces. An *isomorphism* is a morphism with a two-sided inverse.

Let $x \in X$ be a point. Then the *local ring in* x is the stalk $\mathcal{O}_{X,x}$. This indeed is a local ring. Its maximal ideal usually is denoted by $\mathfrak{m}_{X,x}$, and the *residue field in* $x \in X$, denoted by $k(x)$, is $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$. In case X is affine, i.e. $X = \operatorname{Spec} R$, this definition coincides with that in definition 2.1.6.

Remark 2.2.4. Let X be a scheme and $x \in X$. Let U be an affine neighborhood of x. The residue field of x with respect to the affine scheme U was defined in definition 2.1.6. In fact, it is independent of U, and therefore well-defined on the whole scheme X. Moreover, this coincides with the residue field in x defined above.

Example 2.2.5. Let R be a ring. Then $\mathbb{A}^n_R := \operatorname{Spec} R[x_1, \dots x_n]$ is called the *affine space over the ring* R, see [EH00], p. 33, chapter I.2.4.

Let k be a field. Then the *affine line over* k, $\mathbb{A}^1_k$, is $\operatorname{Spec} k[x]$. It has one generic point, namely the zero ideal. The other points correspond to the maximal ideals of $k[x]$ and therefore are closed points. They are in one-to-one correspondence with the nonconstant irreducible monic polynomials. Furthermore, if k is algebraically closed, they are of the form $(x - a)$ for some $a \in k$, so the closed points of $\mathbb{A}^1_k$ are in one-to-one correspondence to the elements of k.

The *affine plane over* k, $\mathbb{A}^2_k$, is $\operatorname{Spec} k[x, y]$. Let k be algebraically closed. Then the closed points correspond to the ordered pairs of elements of k. There is a generic point which corresponds to the zero ideal. Furthermore, every irreducible polynomial $f(x, y)$ defines a prime ideal of $\operatorname{Spec} k[x, y]$, and so gives a point in $\mathbb{A}^2_k$. Its closure consists of the point together with all closed points (a, b) such that $f(a, b) = 0$.

This definition of affine space over an algebraically closed field gives us a generalization of the "classical" affine space. The closed points correspond to the points in the classical case, but here we get additional, non-closed points.

Remark 2.2.6. Schemes and classical algebraic geometry

We now explain how the notion of a scheme fits to the intuition of a variety being the zero set of polynomial equations, as stated in the beginning of this chapter. For simplicity, we just consider one equation. See [Har77], Proposition II.2.6 for more details.

Let R be a ring and assume that the polynomial $f \in R[x_1, \dots, x_n]$ is irreducible. We consider the *zero set of* f, i.e. the set of all n-tupels $(a_1, \dots, a_n) \in \mathbb{A}^n$ such that $f(a_1, \dots, a_n) = 0$. This usually is denoted by $Z(f)$.

The *affine coordinate ring* of $Z(f)$ is defined to be $R[x_1, \dots, x_n]/(f)$. The elements of this ring define functions on $Z(f)$, as $f(x) = 0$ on $Z(f)$. Note that the affine coordinate ring is the set of global sections of the structure sheaf $\mathcal{O}_X$ on $X = \operatorname{Spec} R[x_1, \dots, x_n]/(f)$.

Given a point $P = (a_1, \dots, a_n) \in Z(f)$, define $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$, a point in $\operatorname{Spec} R[x_1, \dots, x_n]$. Then by Hilbert's Nullstellensatz, $f \in \mathfrak{m}$, so $(f) \subset \mathfrak{m}$. This in turn means that $\mathfrak{m}$ defines a prime ideal in $\operatorname{Spec} R[x_1, \dots, x_n]/(f)$. Thus, every point $P \in Z(f)$ determines a point in the affine scheme $\operatorname{Spec} R[x_1, \dots, x_n]/(f)$.

On the other hand, if $R = k$ is an algebraically closed field, every maximal ideal $\bar{\mathfrak{m}}$ of $k[x_1, \dots, x_n]/(f)$ determines a maximal ideal $\mathfrak{m}$ of $k[x_1, \dots, x_n]$ which is of the form $\mathfrak{m} = (x_1 - a_1, \dots, x_1 - a_n)$ (see [Eis95] Corollary 1.6) and contains (f). Therefore, $f(a_1, \dots, a_n) = 0$.

Thus, for an algebraically closed field k we get a one-to-one correspondence between the points of the zero set of the polynomial f and the closed points of the affine scheme $\operatorname{Spec} k[x_1, \dots, x_n]/(f)$.

Definition 2.2.7. The *dimension of a scheme* is the dimension of the underlying topological space. If Z is an irreducible closed subset of X, then the *codimension of Z in X* is the supremum of integers n such that there exists a chain of distinct irreducible closed subsets of X above Z, i.e.

$$Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n.$$

If Y is a closed subset of X, the codimension of Y in X is the infimum of the codimensions of all irreducible closed subsets of Y in X.

Remark 2.2.8. The dimension of the spectrum of a ring R is equal to the Krull dimension of R.

Definition 2.2.9. Let S be a fixed scheme. Then a *scheme over S* is a scheme X together with a morphism $X \to S$. If R is a ring and $S = \operatorname{Spec} R$, we say the X is a *scheme over R* or an *R-scheme*. If X and Y both are schemes over S, a *morphism of X to Y as schemes over S* is a morphism of schemes which is compatible with the given morphisms to S.

Example 2.2.10. Every spectrum of a ring R, $\operatorname{Spec} R$, is a scheme over $S = \operatorname{Spec} \mathbb{Z}$, see remark 2.1.5.

Definition 2.2.11. Let R be a ring and X an R-scheme. The set of *R-valued points of X* is defined as the set

$$X(R) = \{R\text{-morphisms} \operatorname{Spec} R \to X\}.$$

Note that for an affine scheme $X = \operatorname{Spec} B$,

$$\begin{aligned} X(R) &= \{R\text{-morphisms} \operatorname{Spec} R \to X\} \\ &\cong \{\text{ring homomorphisms } B \to R\}, \end{aligned}$$

by proposition 2.2.2.

Example 2.2.12. This notion agrees with our intuition, see also remark 2.2.6: If X is given by the equation $f = 0$, where $f \in R[x_1, \ldots, x_n]$, i.e. $X = \operatorname{Spec} R[x_1, \ldots, x_n]/(f)$, then

$$\begin{aligned} X(R) &\cong \{R\text{-algebra homomorphisms } R[x_1, \ldots, x_n]/(f) \to R\} \\ &= \{P \in R^n : f(P) = 0\}. \end{aligned}$$

Remark 2.2.13. We sometimes call $X(R)$ the set of *sections of the R-scheme X*.

Definition 2.2.14. A *geometric point* of a scheme X over a field K is a morphism $\operatorname{Spec} \bar{K} \to X$, where $\bar{K}$ is the algebraic closure of K.

Remark 2.2.15. The addendum "geometric" usually means that the object somehow is considered over the algebraic closure, e.g. the geometric fiber of a morphism or geometrically irreducible [Har77], [EH00].

Next we need to define a projective construction to define the generalization of projective space. We first need some basic definitions, see [Har77], chapter I.2:

A *graded ring* is a ring R, together with a decomposition $R = \bigoplus_{d \geqslant 0} R_d$ of R, where R_d are Abelian groups, such that for any $d, e \geqslant 0$, $R_d \cdot R_e \subset R_{d+e}$. This decomposition is called a *grading* of the ring R. An element of R_d is called a *homogeneous element of degree d*. An ideal $\mathfrak{a} \subset R$ is called a *homogeneous ideal* if $\mathfrak{a} = \bigoplus_{d \geqslant 0}(\mathfrak{a} \cap R_d)$. Denote by R_+ the ideal $\bigoplus_{d>0} R_d$.

Definition 2.2.16. Let R be a ring, commutative with one. Let the set $\operatorname{Proj} R$ be the set of all homogeneous prime ideals $\mathfrak{p}$, which do not contain all of R_+.

If $\mathfrak{a}$ is a homogeneous ideal of R, we define the subset

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj} R : \mathfrak{a} \subset \mathfrak{p}\}.$$

These sets satisfy the axioms of closed sets of a topology, which is not difficult to prove.

A sheaf of rings is given on $\operatorname{Proj} R$ by the following construction: for every homogeneous prime ideal $\mathfrak{p}$, we consider the ring $R_{(\mathfrak{p})}$ which is the ring of elements of degree 0 in the localization $T^{-1}R$, where T is the multiplicative system consisting of all homogeneous elements of R which are not in $\mathfrak{p}$. Then we proceed as in the case of the structure sheaf of the spectrum of a ring. Let $\mathcal{O}(U)$ be the set of functions

$$s : U \longrightarrow \bigsqcup_{\mathfrak{p} \in \operatorname{Proj} R} R_{(\mathfrak{p})},$$

satisfying

1. $s(\mathfrak{p}) \in R_{(\mathfrak{p})}$, $\forall\, \mathfrak{p} \in U$, and
2. s locally is a quotient of elements in R.

This means that for every $\mathfrak{p} \in U$, there is a neighborhood V of $\mathfrak{p}$ in U and homogeneous elements $r, f \in R$ of the same degree, such that for all $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = r/f \in R_{(\mathfrak{p})}$. This clearly defines a presheaf of rings, with the natural restrictions, and, since it is defined locally, it even is a sheaf of rings.

For any graded ring R, the pair $(\operatorname{Proj} R, \mathcal{O})$ is a scheme and, similar to the case of the spectrum of a ring, the stalk of $\mathcal{O}$ in a point $\mathfrak{p} \in \operatorname{Proj} R$ is isomorphic to the local ring $R_{(\mathfrak{p})}$ ([Har77], p. 76, Proposition II.2.5).

Example 2.2.17. Let R be a ring. The *projective n-space* over R is defined to be $\mathbb{P}^n_R := \operatorname{Proj} R[x_0, \ldots, x_n]$. The *arithmetic projective n-space* is $\mathbb{P}^n_{\mathcal{O}_K}$, where K is a number field and $\mathcal{O}_K$ its ring of integers.

2.3 The projective space associated to a projective module

Let R be a ring (e.g. $\mathcal{O}_K$) and M a finitely generated projective R-module. Denote by $\mathcal{S}(M)$ the symmetric algebra on M,

$$\mathcal{S}(M) = \bigoplus_{k \geqslant 0} \mathcal{S}^k(M), \qquad \mathcal{S}^k(M) = \mathcal{T}^k(M)/\langle\{m \otimes n - n \otimes m\}\rangle,$$

where $\mathcal{T}^k(M) = M^{\otimes k}$, and $\langle\{m \otimes n - n \otimes m\}\rangle$ is the ideal generated by the elements $m \otimes n - n \otimes m$, for $m, n \in M$. $\mathcal{S}(M)$ is a finitely generated commutative algebra and therefore, in particular, a ring. If M is a free R-module of finite rank r, $\mathcal{S}(M)$ is the polynomial ring in r variables, see [Eis95], Appendix A.2.3.

We define the associated *projective space* to be $\mathbb{P}(M) := \operatorname{Proj} \mathcal{S}(M)$.

In fact, $\mathbb{P}(M)$ is a scheme over $\operatorname{Spec} R$ ([Har77], p. 162 in chapter II.7).

Remark 2.3.1. In the literature, sometimes $\mathbb{P}(M)$ is defined to be $\operatorname{Proj} \mathcal{S}(M^\vee)$, where $M^\vee = \operatorname{Hom}(M, R)$, e.g. in [BGS94]. We will stay with above as is used in e.g. [Gro61], 4.1 and [Har77].

An R-valued point in $\mathbb{P}(M)$ is an injection $\xi : \operatorname{Spec}(R) \longrightarrow \mathbb{P}(M)$. Equivalently, a point is given by a ring homomorphism $\xi^\# : \mathcal{S}(M) \longrightarrow R$. Consider the elements of M as elements of $\mathcal{S}(M)$. Then $\xi^\#$ is induced by $\xi^\# : M \longrightarrow R$, i.e. $\xi^\# \in M^\vee = \operatorname{Hom}(M, R)$. This, in turn, is equivalent to fixing the projective module $M_\xi = \operatorname{Ker} \xi^\# \subseteq M$ of corank one, see [EH00], p.103.
Summing up, a point in $\mathbb{P}(M)$ can be identified with a projective submodule of M of corank 1.

Example 2.3.2. Let R be a ring and $M = R^{n+1}$. Then $\mathcal{S}(M) = R[x_0, \ldots, x_n]$, and the projective n-space over R is $\mathbb{P}^n_R = \mathbb{P}(M)$.

2.4 Properties of schemes

In the following sections, we need some properties of schemes. We now define the most important ones in this and the next section. We refer to [Har77], chapters II.3 and II.4, [Sha94c], and [EH00].

Some properties are based on those of the underlying topological space:

Definition 2.4.1. A scheme is *connected* or *irreducible* if the underlying topological space has this property.

Other properties come from the covering of the scheme by affine sets:

Definition 2.4.2. A scheme X over a ring R is *of finite type over* R if X has a finite covering by open affine sets U_i, i.e. $U_i = \operatorname{Spec} A_i$ for some rings A_i, such that the A_i are algebras of finite type over R. An *algebraic scheme* is a scheme X of finite type over a field k.

Definition 2.4.3. A scheme X is *Noetherian* if X has a finite covering by open affine sets which are the spectra of Noetherian rings.

Some properties come from the analogous property of the rings $\mathcal{O}_X(U)$ for open sets $U \subset X$ or of the stalks $\mathcal{O}_{X,x}$:

Definition 2.4.4. A scheme X is *reduced* if for every open set U, the ring $\mathcal{O}_X(U)$ is reduced, i.e. has no non-trivial nilpotent elements.

Definition 2.4.5. A scheme X is *integral* if for every open set U, the ring $\mathcal{O}_X(U)$ is an integral domain. This is equivalent to being both reduced and irreducible.

Definition 2.4.6. A scheme X is called *normal* if for every point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is an integrally closed domain.

Definition 2.4.7. Let X be a Noetherian scheme. Then X is *regular* (or *nonsingular*) at a point $x \in X$, if the local ring $\mathcal{O}_{X,x}$ at x is a regular local ring. X is *regular* (or *nonsingular*), if it is regular at every point. The scheme is called *singular* if it is not regular.

Definition 2.4.8. Let X be an integral scheme and η a generic point of X. Then the local ring $\mathcal{O}_{X,\eta}$ is a field, the *function field of* X, and is denoted by $k(X)$. If $U = \operatorname{Spec} A$ is an open affine set of X, then $k(X)$ is isomorphic to the quotient field of A. The elements of $k(X)$ are called *rational functions on* X.

Definition 2.4.9. An *open subscheme* of a scheme X is a scheme whose underlying topological space U is an open subset of X and whose structure sheaf $\mathcal{O}_U$ is isomorphic to the restriction sheaf $\mathcal{O}_X|_U$.

Definition 2.4.10. A *closed immersion* is a morphism $f : Y \to X$ of schemes such that f induces a homeomorphism of the underlying topological space of Y onto a closed subset of the underlying topological space of X, and furthermore, the induced map $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ of sheaves on X is surjective. A *closed subscheme* of a scheme X is an equivalence class of closed immersions, where two closed immersions $f : Y \to X$ and $f' : Y' \to X$ are equivalent if there is an isomorphism $i : Y' \to Y$ such that $f' = f \circ i$. A morphism $i : Y \to X$ is an *immersion* if it gives an isomorphism of Y to an open subscheme of a closed subscheme of X.

2.5 Fiber product

As a set, the fiber product of two sets over a third set is defined in an analogous way to the construction of the pullback of a vector bundle which we will see in example 3.2.5 in the next section (see also [Sha94c] or [EH00]). This construction satisfies a universal property, which we take as the definition for the fiber product of schemes. In fact, the construction is a categorical construction, called the (categorial) pullback. The fiber product is the pullback in the category of schemes and the pullback bundle is the pullback in the category of vector bundles.

Definition 2.5.1. Let S be a scheme, and let X and Y be two schemes over S. The *fiber product of X and Y over S*, $X \times_S Y$, is a scheme together with morphisms $p_X : X \times_S Y \to X$ and $p_Y : X \times_S Y \to Y$ such that the compositions of these morphisms with the given maps $X \to S$ and $Y \to S$, respectively, coincide, i.e. p_X and p_Y make the diagram below commutative. Furthermore, given any scheme Z over S and given morphisms $f : Z \to X$ and $g : Z \to Y$, which make the following diagram commute, there exists a unique morphism $\theta : Z \to X \times_S Y$ such that $f = p_X \circ \theta$ and $g = p_Y \circ \theta$.

The definition is much easier to see in the commutative diagram:

$$
\begin{tikzcd}
Z \arrow[drr, bend left, "f"] \arrow[dr, dotted, "\theta"] \arrow[ddr, bend right, "g"'] & & \\
& X \times_S Y \arrow[r, "p_X"'] \arrow[d, "p_Y"] & X \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}
$$

If X and Y are any schemes without reference to a base scheme S, we take $S = \operatorname{Spec}\mathbb{Z}$ and define the *product of X and Y* to be $X \times Y := X \times_S Y$.

Since we defined the fiber product by a universal property, the uniqueness of the fiber product up to isomorphism is clear. However, it is a priori not clear that such a scheme exists. A proof for the existence can be found in e.g. [Har77], Theorem II.3.3.

A few interesting applications of fiber products lead us to definitions we need in the next chapters.

Definition 2.5.2. Let $f : X \to Y$ be a morphism of schemes, and let $y \in Y$ be a point with residue field $k(y)$. Let $\operatorname{Spec} k(y) \to Y$ be the natural morphism, which, as a map between sets, sends the only point of $\operatorname{Spec} k(y)$ to $y \in Y$. Then the *fiber of the morphism* f *over the point* y is the scheme

$$X_y = X \times_Y \operatorname{Spec} k(y).$$

This is a scheme over $k(y)$ and furthermore, the underlying topological space is homeomorphic to the inverse image $f^{-1}(y) \subset X$.

If η is a generic point of Y, the scheme X_η is called the *generic fiber* of the scheme X.

Example 2.5.3. Let X be a scheme over $\mathbb{Z}$. Then the fiber over the generic point (0), $X_{\mathbb{Q}} := X_{(0)}$, is a scheme over $\mathbb{Q}$, and the fiber over a closed point corresponding to a prime $p \in \mathbb{Z}$ is a scheme X_p over the finite field $\mathbb{F}_p$. X_p is called the *reduction* $\operatorname{mod} p$ *of the scheme* X.

Definition 2.5.4. Let S be a fixed scheme, the *base scheme*. Let X be a scheme over S. If $S' \to S$ is another scheme over S, let $X' = X \times_S S'$. This is a scheme over S' and is said to be obtained from X by making a *base extension*.

Next we define a property which, in a certain sense, is analogous to that of the Hausdorff separation axiom. However, the underlying topological space of a scheme is usually not Hausdorff, since it is endowed with the Zariski topology.

Definition 2.5.5. Let $f : X \to Y$ be a morphism of schemes. The *diagonal morphism* is the unique morphism $\Delta : X \to X \times_Y X$ such that the composition with the projection maps p_X, p_Y is the identity map on X. The morphism f is called *separated* if the diagonal morphism Δ is a closed immersion. One also says that X is *separated over* Y. A scheme is called *separated* if it is separated over $\operatorname{Spec} \mathbb{Z}$.

We now define the notion of *flatness* of a morphism of schemes. This property intuitively means that the fibers vary "smoothly". We will not elaborate on this, but one should keep the intuitive idea in mind; see [Eis95], chapter 6, for some illustrative examples and [Har77] for an extensive study of flat morphisms. First we define *flat modules*, an important property of modules in commutative algebra and algebraic geometry.

Definition 2.5.6. Let R be a ring and M an R-module. The module M is *flat over* R if the functor $N \mapsto M \otimes_R N$ is an exact functor for R-modules N. That means that, given an exact sequence of R-modules

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0,$$

the following sequence also is exact:

$$0 \longrightarrow M \otimes_R N_1 \longrightarrow M \otimes_R N_2 \longrightarrow M \otimes_R N_3 \longrightarrow 0.$$

The analogon for schemes is the following:

Definition 2.5.7. Let $\pi : X \to Y$ be a morphism of schemes. We say X *is flat over* Y, or π *is a flat morphism*, if for every $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,y}$-module,[1] where $y = \pi(x)$.

2.6 Varieties

We can now formulate the definition of a variety in the language of schemes. This is an extension of the classical definition of a variety, as is shown in [Har77], Propositions II.2.6 and II.4.10 or in [Sha94c], Example 5 in chapter V.3.1.

Definition 2.6.1. A *variety* over an algebraically closed field k is a reduced separated scheme of finite type over k.

Remark 2.6.2. This definition is the one used in e.g. [Sha94c]. [Har77] additionally requires the scheme to be irreducible, and therefore integral. Furthermore, note that varieties are, by definition, algebraic schemes.

Definition 2.6.3. A *morphism of varieties* is a morphism of schemes over k. A variety X that is an affine scheme is called an *affine variety*.

Definition 2.6.4. If Y is a scheme and n a non-negative integer, we define *projective* n*-space over* Y to be $\mathbb{P}^n_Y := \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} Y$. A morphism $f : X \to Y$ of schemes is called *projective*, if it factors into a closed immersion $i : X \to \mathbb{P}^n_Y$, for some n, and the projection p_Y from the fiber product, $p_Y : \mathbb{P}^n_Y \to Y$. A morphism $f : X \to Y$ is *quasi-projective* if it factors into an open immersion $j : X \to X'$ and a projective morphism $g : X' \to Y$. One also says that X is *(quasi-)projective over* Y. A scheme is called *(quasi-)projective* if it is (quasi-)projective over $\mathbb{Z}$.

Remark 2.6.5. Note that a projective scheme is a scheme which allows a closed immersion into $\mathbb{P}^n_{\mathbb{Z}}$, i.e. $f : Y \to \mathbb{P}^n_{\mathbb{Z}}$. It therefore is homeomorphic to a closed subset of $\mathbb{P}^n_{\mathbb{Z}}$. In fact, if Y is a projective, integral and irreducible scheme over a field k, it corresponds to a projective variety in the sense of classical algebraic geometry, i.e. the zero set of homogeneous elements of the polynomial ring over k, see [Har77], Proposition II.4.10.

Interestingly, schemes which are projective over an affine scheme arise from the Proj construction.

Proposition 2.6.6. *(Corollary II.5.16 b in [Har77]) Let A be a ring. A scheme Y over* $\operatorname{Spec} A$ *is projective if and only if it is isomorphic to* $\operatorname{Proj} S$ *for some graded ring S, where $S_0 = A$, and S is finitely generated by S_1 as an S_0-algebra.*

Definition 2.6.7. An irreducible variety of dimension one is called a *curve*. *Surfaces* are irreducible varieties of dimension two.

[1] Note that the stalk $\mathcal{O}_{X,x}$ always can be given an $\mathcal{O}_{Y,y}$-module structure using the natural map $\pi^{\#} : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$.

2.7 Čech cohomology and the genus

Cohomology is a standard tool in topology, geometry and, in particular, algebraic geometry. There are quite a lot of cohomology theories in different contexts, e.g. De Rham cohomology, sheaf cohomology, Étale cohomology, crystalline cohomology, and many more. In this section, we define Čech cohomology, which is an important tool for calculating global sections of a given sheaf. We will need it for the definition of the genus of a projective algebraic curve. There are several notions of "genus", which in certain cases are equal. We use [Har77], chapter III.4 and the course notes [Wüs08].

Let X be a topological space, and let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of X. We choose a well-ordering of I. For any integer $n \geqslant 0$ consider the set I_n of $n+1$-tupels of elements in I such that $i_0 < i_1 < \ldots < i_n$. Then, for every $0 \leqslant k \leqslant n$, there is a natural projection $\pi_k : I_n \to I_{n-1}$ which maps the $n+1$-tupel $\mathbf{i} = (i_0, \ldots, i_n)$ to the n-tuple $\pi_k(\mathbf{i}) = (i_0, \ldots, \widehat{i_k}, \ldots, i_n)$, where the element i_k is omitted. We denote the intersection $U_{i_0} \cap \ldots \cap U_{i_n}$ by $U_{i_0,\ldots,i_n}$.

Let $\mathcal{F}$ be a sheaf of Abelian groups on X with restriction maps $\rho_{U,V}$. We define a complex $C^{\cdot}(\mathfrak{U}, \mathcal{F})$:

$$C^n(\mathfrak{U}, \mathcal{F}) = \prod_{(i_0,\ldots,i_n) \in I_n} \mathcal{F}(U_{i_0,\ldots,i_n}),$$

i.e. an element $s \in C^n(\mathfrak{U}, \mathcal{F})$ is a family $s = (s_{\mathbf{i}})$ such that $\mathbf{i} = (i_0, \ldots, i_n) \in I_n$. It is called a *cochain*.

We define the *coboundary maps* $d : C^n(\mathfrak{U}, \mathcal{F}) \to C^{n+1}(\mathfrak{U}, \mathcal{F})$ by setting

$$(ds)_{\mathbf{i}} := \sum_{k=0}^{n+1} (-1)^k \rho_{U_{\pi_k(\mathbf{i})}, U_{\mathbf{i}}}(s_{\pi_k(\mathbf{i})})$$

for every $\mathbf{i} \in I_{n+1}$.

To understand all the indices, we describe this briefly. $\rho_{U_{\pi_k(\mathbf{i})}, U_{\mathbf{i}}}(s_{\pi_k(\mathbf{i})})$ means that for an $\mathbf{i} \in I_{n+1}$ and a $0 \leqslant k \leqslant n+1$ we take the image of $s_{\pi_k(\mathbf{i})} \in \mathcal{F}(U_{\pi_k(\mathbf{i})})$ under the restriction map, an element in $\mathcal{F}(U_{\mathbf{i}})$. For fixed $\mathbf{i}$ and every k this is in $\mathcal{F}(U_{\mathbf{i}})$, which is an Abelian group, so we can add them.

Actually, the map d depends on n, but by abuse of notation we omit the specification and simply write d. One can check that $d \circ d = 0$, and therefore the coboundary maps define a complex of Abelian groups:

$$C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \cdots \xrightarrow{d} C^n \xrightarrow{d} \cdots,$$

where $C^n := C^n(\mathfrak{U}, \mathcal{F})$.

The cohomology groups of this complex are called *Čech-cohomology groups* and are defined as follows:

Again, we denote $C^n(\mathfrak{U},\mathcal{F})$ by C^n. Then the image of C^{n-1} under d in C^n is a group, denoted by $\check{B}^n(\mathfrak{U},\mathcal{F})$, whose elements are called *coboundaries*. The kernel of d in C^n also is a group, denoted by $\check{Z}^n(\mathfrak{U},\mathcal{F})$, whose elements are called *cocycles*. Furthermore, for negative n, we set $\check{B}^n(\mathfrak{U},\mathcal{F}) = \check{Z}^n(\mathfrak{U},\mathcal{F}) = 0$.

Since $d \circ d = 0$, $\check{B}^n(\mathfrak{U},\mathcal{F}) \subset \check{Z}^n(\mathfrak{U},\mathcal{F})$ and we define the *n-th Čech-cohomology group with respect to the covering* $\mathfrak{U}$ as the quotient group

$$\check{H}^n(\mathfrak{U},\mathcal{F}) = \check{Z}^n(\mathfrak{U},\mathcal{F})/\check{B}^n(\mathfrak{U},\mathcal{F}).$$

Lemma 2.7.1. *(e.g. Lemma III.4.1 in [Har77]) Let X be a topological space, $\mathcal{F}$ a sheaf of Abelian groups on X, and $\mathfrak{U}$ an open overing of X. Then,*

$$\check{H}^0(\mathfrak{U},\mathcal{F}) \cong \Gamma(X,\mathcal{F}).$$

Proof. Since $\check{H}^0(\mathfrak{U},\mathcal{F}) = \check{Z}^0(\mathfrak{U},\mathcal{F})$, we need to study the kernel of the map $d : C^0(\mathfrak{U},\mathcal{F}) \to C^1(\mathfrak{U},\mathcal{F})$. If $\alpha \in C^0(\mathfrak{U},\mathcal{F})$ is in the kernel, and if $\alpha = (\alpha_i)_i$, where $\alpha_i \in \mathcal{F}(U_i)$, then for each $i < j$, $(d\alpha)_{(i,j)} = \rho_{U_i,U_i\cap U_j}(\alpha_i) - \rho_{U_j,U_i\cap U_j}(\alpha_j) =$ "$(\alpha_i - \alpha_j)|_{U_i\cap U_j}$" $= 0$. Thus, α_i and α_j coincide on $U_i \cap U_j$. By the glueing axiom of the sheaf $\mathcal{F}$, they define a global section of $\mathcal{F}$. □

There is an ordering of open coverings with respect to refinements and one can show that this defines maps of the respective cohomology groups. Furthermore, if the covering $\mathfrak{U}$ is fine enough, the cohomology group is independent of the chosen covering and we set

$$\check{H}^n(X,\mathcal{F}) := \check{H}^n(\mathfrak{U},\mathcal{F}).$$

This is the same as defining

$$\check{H}^n(X,\mathcal{F}) := \varinjlim_{\mathfrak{U}} \check{H}^n(\mathfrak{U},\mathcal{F})$$

as the direct limit of the cohomology groups with respect to the ordering of open coverings.

This purely abstract concept gives us the definition of the genus of a projective algebraic curve.

Definition 2.7.2. Let X be a projective algebraic curve. Then the *genus of* X is defined as

$$g = \dim \check{H}^1(X,\mathcal{O}_X).$$

Example 2.7.3. This rather abstract approach coincides with an easy definition if the curve is given as the zero set of an irreducible polynomial $f(x,y) = 0$ of degree d. Then $g = \frac{1}{2}(d-1)(d-2)$. In this case, if $g = 0$, then $d = 1$ or $d = 2$. Thus curves of genus 0 are lines and conics. The degree of a curve of genus one satisfies the following quadratic equation: $2 = d^2 - 3d + 2$, i.e. $0 = d(d-3)$. Thus curves given by a polynomial and of genus one are given by a polynomial of degree 3. They are called *elliptic curves*, see also appendix B.

Remark 2.7.4. We will not discuss this rather abstract notion of the genus in more detail, as it is more useful for the following to keep in mind the above example. One should just remember that one can define it for general varieties; we will need this in the last section, but usually the intuition given above will suffice.

Chapter 3

Vector bundles

In this chapter, we give the definition and properties of *vector bundles*, both on a complex manifold and on a variety. Several connections between vector bundles and other objects, and the connection between vector bundles on a variety and on a complex manifold are of particular importance for us in the following chapters.

3.1 Definition

We use [Wel07] and [Sha94c] for the following definitions.

Definition 3.1.1. Let k be the field[1] $\mathbb{C}$. A continuous map $\pi : E \to X$ of Hausdorff spaces is a *vector bundle* if

1. the *fiber* over every $p \in X$, $E_p := \pi^{-1}(p)$, is a k-vector space, and
2. for every $p \in X$ there is a neighborhood U of p and a homeomorphism

$$h : \pi^{-1}(U) \to U \times k^r$$

such that

$$\forall q \in U : h(E_q) \subset \{q\} \times k^r$$

and such that (U, h) is a *local trivialization*, i.e. the composition

$$h^p : E_p \xrightarrow{h} \{p\} \times k^r \xrightarrow{p_2} k^r$$

is a k-vector space isomorphism, where p_2 is the projection on the second coordinate.

[1] In general, one can also consider $k = \mathbb{R}$. However, for our purposes, $k = \mathbb{C}$ suffices.

E is called the *total space* and X is called the *base space*. One often says that E is a vector bundle over X.

We call a vector bundle *holomorphic* if

- the underlying spaces E and X are holomorphic complex manifolds,
- π is a holomorphic morphism,
- and the local trivializations are holomorphic.

In the following, if not specified otherwise, we usually mean holomorphic vector bundles over complex manifolds, but omit this specification.

Let $(U_i, h_i), (U_j, h_j)$ be two trivializations of a given vector bundle $\pi : E \to X$. Consider the map

$$h_i \circ h_j^{-1} : (U_i \cap U_j) \times k^r \longrightarrow (U_i \cap U_j) \times k^r.$$

It induces a map

$$g_{ij} : U_i \cap U_j \longrightarrow GL(r, k),$$

where

$$g_{ij}(p) = h_i^p \circ (h_j^p)^{-1} : k^r \longrightarrow k^r.$$

The functions g_{ij} are called the *transition functions* of the vector bundle with respect to the local trivializations $(U_i, h_i), (U_j, h_j)$.

The transition functions satisfy the *compatibility conditions*

(a) $g_{ij} \cdot g_{jk} \cdot g_{ki} = 1 \quad$ on $U_i \cap U_j \cap U_k$,

(b) $g_{ii} = 1 \quad$ on U_i.

In fact, suppose we are given an open covering $\{U_i\}$ of X, and we have maps $g_{ij} : U_i \cap U_j \longrightarrow GL(r, k)$ for every non-empty intersection $U_i \cap U_j$ which satisfy the compatibility conditions, then we can define a vector bundle $\pi : E \to X$ having these transition functions. This process is often called "glueing".

In the following definition we introduce the analogon of a vector bundle over a (complex) manifold in the context of algebraic geometry, vector bundles over varieties.

Definition 3.1.2. A *family of vector spaces* over a variety X is a morphism $\pi : E \to X$ such that every fiber over a point $x \in X$, $E_x = \pi^{-1}(x)$, is a vector space over $k(x)$, and the algebraic variety-structure of E_x as a vector space coincides with that of $E_x \subset E$ as the inverse image of x under π.

A family of vector spaces is *trivial*, if $E = X \times V$, where V is a vector space over a field k. (In this case, $k = k(x), \ \forall x \in X$.)

A family of vector spaces $\pi : E \to X$ is a *vector bundle* if every point $x \in X$ has a neighborhood U such that the restriction $E|_U$, defined as $\pi^{-1}(U) \to U$, is trivial.

Remark 3.1.3. Note that condition 1 in the definition of a vector bundle over a Hausdorff space, 3.1.1, corresponds to the condition in a family of vector spaces and condition 2 in definition 3.1.1 corresponds to the additional condition for a vector bundle over a variety to be locally trivial. Therefore we do not, in general, distinguish between vector bundles on complex manifolds or on varieties if we say vector bundle, unless explicitly specified. We sometimes call vector bundles on varieties *algebraic vector bundles* to distinguish them from holomorphic vector bundles on complex manifolds.

Remark 3.1.4. *Transition functions* for a vector bundle over a variety are defined in an analogous way to the above. Furthermore, also in this case, we can obtain a vector bundle by glueing together trivial bundles by the data of transition functions over an open covering of the variety.

Example 3.1.5. Let X be a complex manifold and V a k-vector space. Then $X \times V \xrightarrow{p_1} X$, where p_1 denotes the projection onto the first coordinate, is a vector bundle. In fact, if $X = \mathbb{C}^n$, every vector bundle is trivial, see [GH78], p. 307.

Definition 3.1.6. The dimension of the fiber E_x over a point $x \in X$ of a vector bundle is a locally constant function on X. In particular, if X is connected, the dimension of the fibers is constant. In this case, it is called the *rank of the vector bundle E*.

Definition 3.1.7. A *morphism* of a vector bundle $\pi_E : E \to X$ to a vector bundle $\pi_F : F \to X$ is a morphism $f : E \to F$ such that the diagram

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & X & \end{tikzcd}$$

commutes and for every $p \in X$, $f_p : E_p \to F_p$ is k-linear in the case of holomorphic vector bundles and $k(x)$-linear in the case of vector bundles over a variety.

An *isomorphism* of vector bundles is a morphism of vector bundles which is an isomorphism on the total spaces and a vector space isomorphism on the fibers. Being isomorphic defines an equivalence relation on the set of vector bundles over a given base space X.

Using the notion of isomorphic bundles, the condition that a holomorphic vector bundle is locally trivial becomes the following:

2′. For every $x \in X$ there is an open neighborhood U of x and a bundle isomorphism

$$h : E|_U \xrightarrow{\sim} U \times k^r,$$

respectively the same condition with $k = k(x)$ if the vector bundle is over a variety.

Remark 3.1.8. Let X be a non-singular n-dimensional variety over the field of complex numbers $\mathbb{C}$. Then, the topological space $X(\mathbb{C})$ of complex points on X is, in fact, an orientable complex manifold, [Sha94c], p. 117ff, chapters VII.1 and VII.2. There are some basic relationships between properties of X and those of $X(\mathbb{C})$, e.g. if X is irreducible, $X(\mathbb{C})$ is connected. The correspondence between complex analytic spaces and schemes was partially established by Serre in his famous paper GAGA, [Ser56]. Furthermore, if X is a variety as above, and $E \to X$ is a vector bundle, then $E(\mathbb{C}) \to X(\mathbb{C})$ is a topological vector bundle. This will be important in the next chapter.

3.2 Construction of holomorphic vector bundles

Let X be a complex manifold and $\pi : E \to X$ be a surjective map such that

1. E_p is a k-vector space,
2. for each $p \in X$ there is a neighborhood U of p and a bijective map
$$h : \pi^{-1}(U) \longrightarrow U \times k^r \quad \text{such that } h(E_p) \subset \{p\} \times k^r, \text{ and}$$
3. $h^p : E_p \xrightarrow{h} \{p\} \times k^r \xrightarrow{p_2} k^r$ is a k-vector space isomorphism.

If for every $(U_i, h_i), (U_j, h_j)$ as in 2., $h_i \circ h_j^{-1}$ is an isomorphism, we can make E into a vector bundle over X by giving it the topology such that the h_i are homeomorphisms, see the remark in [Wel07], p. 16.

Definition 3.2.1. Given two k-vector spaces A and B, we can form new vector spaces such as the direct sum, the tensor product, the vector space of linear maps from A to B, the dual space, and the symmetric and antisymmetric tensor products of a certain degree. By the construction above, we can extend these constructions to vector bundles over X and define the *dual, tensor product, direct sum, exterior product, etc. of vector bundles.*

As an example, we give the construction for the *direct sum of vector bundles E and F*; the other constructions are similar.

Example 3.2.2. Given two vector bundles E and F over a complex manifold X of ranks r and s, respectively, we define the total space of the direct sum of vector bundles to be
$$E \oplus F = \bigsqcup_{p \in X} E_p \oplus F_p.$$

We then get a canonical surjective map $\pi : E \oplus F \to X$. We now need to show that the properties required for the construction from above are satisfied.

We can assume that the open coverings coincide, so denote the local trivializations of E and F as (U, h_E) and (U, h_F), respectively. We define

$$h_{E\oplus F} : \pi^{-1}(U) \to U \times k^{r+s}.$$

For $q \in U$, $e \in E_q$, and $f \in F_q$, we define

$$h_{E\oplus F}(e + f) = (q, h_E(e) + h_F(f)) \in U \times (k^r \oplus k^s) \cong U \times k^{r+s}.$$

This map satisfies the conditions for the construction of a vector bundle given above.

Remark 3.2.3. We define analogous constructions for vector bundles on varieties in the next section in remark 3.4.2, using the concept of locally free sheaves. One can also define these construction without using locally free sheaves, straight by the help of transition functions.

Example 3.2.4. We now define the natural bundle on projective space, the *tautological (or canonical) bundle* T. It is both a holomorphic vector bundle over the complex space $\mathbb{P}^n_{\mathbb{C}}$ and a vector bundle over the variety $\mathbb{P}^n_{\mathbb{C}}$. In this example we define T as a holomorphic vector bundle and later – in the examples 3.4.6, 3.4.7, and in remark 3.4.12 – we give the algebraic interpretation and a generalization of this concept.

Let T be the disjoint union of the lines, i.e. one-dimensional subspaces, in $\mathbb{C}^{n+1}$. Consider the natural projection

$$\pi : T \longrightarrow \mathbb{P}^n_{\mathbb{C}},$$

given by $\pi(v) = p$, if v is a vector in the line in $\mathbb{C}^{n+1}$ determined by the point $p = (x_0 : \ldots : x_n) \in \mathbb{P}^n_{\mathbb{C}}$, i.e. v and $(x_0, \ldots, x_n)$ are linearly dependent. So, we define the fiber $T_p = p \subset \mathbb{C}^{n+1}$. We may identify T with the subset of $\mathbb{P}^n_{\mathbb{C}} \times \mathbb{C}^{n+1}$ containing all pairs (p, v) such that $v \in p$, where we consider p as a line in $\mathbb{C}^{n+1}$.

To define a vector bundle $T \to \mathbb{P}^n_{\mathbb{C}}$, we need to define local trivialization maps. Let U_i be the standard open sets in $\mathbb{P}^n_{\mathbb{C}}$, i.e. $U_i = \{(x_0 : \ldots : x_n) \in \mathbb{P}^n_{\mathbb{C}} : x_i \neq 0\}$. Since

$$\pi^{-1}(U_i) = \{v = t(x_0, \ldots, x_n) \in \mathbb{C}^{n+1} : t \in \mathbb{C}, x_i \neq 0\},$$

we can uniquely write any $v \in \pi^{-1}(U_i)$ in the form

$$v = t_i \left(\frac{x_0}{x_i}, \ldots, 1, \ldots, \frac{x_n}{x_i} \right).$$

Let the local trivialization

$$h_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{C}$$

be defined as

$$h_i(v) = ((x_0 : \ldots : x_n), t_i).$$

These maps are bijective and $\mathbb{C}$-linear.

To apply the construction above to define a vector bundle, we need to show that $h_i \circ h_j^{-1}$ is an isomorphism. Suppose that $v = t(x_0, \dots, x_n) \in \pi^{-1}(U_i \cap U_j)$. Then,

$$h_i(v) = ((x_0 : \ldots : x_n), t_i),$$
$$h_j(v) = ((x_0 : \ldots : x_n), t_j),$$

with $t_i = tx_i$ and $t_j = tx_j$; and therefore $t_i = \frac{x_i}{x_j} t_j$. Setting $g_{ij} : U_i \cap U_j \to GL(1, \mathbb{C}) = \mathbb{C}^* = \mathbb{C}\backslash\{0\}$,

$$g_{ij}((x_0 : \ldots x_n)) = \frac{x_i}{x_j},$$

we get well-defined transition functions and $h_i \circ h_j^{-1}$ is an isomorphism as required.

Example 3.2.5. Let $f : X \to Y$ be a morphism and $\pi : E \to Y$ a bundle. The *pullback of the vector bundle* E, denoted by $f^*E \to X$, is defined as follows: we define the space

$$E' = \{(x, e) \in X \times E : f(x) = \pi(e)\}.$$

Let $x \in X$ and give $E'_x = \{x\} \times E_{f(x)}$ the structure of a k-vector space induced by $E_{f(x)}$. Set

$$f^*\pi : E' \to X, \quad (x, e) \mapsto x.$$

Then E' is a fibered family of vector spaces. The local trivializations arise from those for E, i.e. if (U, h) is a local trivialization for E, $E|_U \xrightarrow{\sim} U \times k^n$, then

$$E'|_{f^{-1}(U)} \xrightarrow{\sim} f^{-1}(U) \times k^n$$

is a local trivialization of E'. Then, setting $g : E' \to E$, $(x, e) \mapsto e$, the following diagram commutes:

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d, "f^*\pi"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

In fact, the pullback of a vector bundle is unique up to isomorphism.

More generally, the pullback is defined by a universal property using the fact that the pullback makes the above diagram commutative. This construction is a categorical construction, e.g. the pullback in the category of schemes over a given scheme is the fiber product, see section 2.5. The dual construction in the sense of category theory is that of the pushforward, see example 1.2.4.

3.3 Global sections

Definition 3.3.1. A *global section* of a vector bundle $\pi : E \to X$ is a morphism $s : X \to E$ such that $\pi \circ s = \mathrm{id}_X$. We denote by $\Gamma(X, E)$ the set of global sections of E over X. Furthermore, elements of $\Gamma(U, E) := \Gamma(U, E|_U)$ are called *sections of E*.

Remark 3.3.2. We often identify a section $s : X \to E$ with its image $s(X) \subset E$. An important example is that of the *zero section* $0 : X \to E$ which is given by $s(x) = 0_x \in E_x$, where 0_x is the zero element in the vector space E_x. The image of the zero section, in fact, is isomorphic to the base space X.

Example 3.3.3. Let X be an algebraic variety and consider the trivial bundle $X \times k$ over X. A section of this bundle, $s : X \to X \times k$, corresponds to a regular function on X. Therefore, $\Gamma(X, X \times k) = \mathcal{O}_X(X)$. In particular, $\Gamma(\mathbb{P}^n \times k) = k$. Similarly, $\Gamma(\mathbb{P}^n \times V) = V$, where V is a k-vector space. As every vector bundle is locally trivial, we can consider sections of a vector bundle over $\mathbb{P}^n$ as "twisted" vector-valued functions. This will be the intuition for the definition of Serre's twisting sheaf, see examples 3.4.6, 3.4.7, and remark 3.4.12.

Example 3.3.4. The global sections of the dual of the tautological bundle on $\mathbb{P}^n_{\mathbb{C}}$ correspond to the homogeneous polynomials in $n+1$ variables of degree 1, i.e. $\Gamma(\mathbb{P}^n_{\mathbb{C}}, E^\vee) \cong \mathbb{C}x_0 + \cdots + \mathbb{C}x_n$. For an explicit calculation of the global sections of the powers of the tautological bundle on $\mathbb{P}^n_{\mathbb{C}}$, see [Wel07], p. 22, Example 2.13.

Module of global sections

The set of global sections can be given an algebraic structure: if s_1 and s_2 are sections of a vector bundle $\pi : E \to X$, then

$$(s_1 + s_2)(x) := s_1(x) + s_2(x) \in E_x$$

defines a section of E and thus gives $\Gamma(X, E)$ an additive structure.

Moreover, setting

$$(fs)(x) := f(x)s(x),$$

where $f \in \mathcal{O}_X(X)$ and $s \in \Gamma(X, E)$, gives $\Gamma(X, E)$ an $\mathcal{O}_X(X)$-module structure.

The module of global sections is very interesting because it, in fact, determines the vector bundle on a given variety.

Proposition 3.3.5. *(Corollary A.3.3 in [Eis95]) Given a vector bundle E on a variety X, its module of global sections $\Gamma(X, E)$ forms a finitely generated projective module. Furthermore, any finitely generated projective module arises uniquely from a vector bundle in this way.*

Thus, one can identify vector bundles with projective modules. In the next chapters, we will identify them and we will not distinguish between the two, except where necessary.

Definition 3.3.6. Let $E \to X$ be a vector bundle over a complex manifold or over a variety. For every open $U \subset X$ let $\mathcal{L}_E(U) = \Gamma(U, E)$. Then this, together with the natural restriction maps, defines a presheaf on X. This even is a sheaf, called the *sheaf of sections of the vector bundle E*. In fact, since $\Gamma(U, E)$ is an $\mathcal{O}_X(U)$-module, $\mathcal{L}_E$ is a sheaf of modules. Moreover, since vector bundles are locally trivial, it is locally free. (See [Wel07], p. 40 for a proof of the last assertion.)

3.4 Locally free sheaves and vector bundles

Theorem 3.4.1. *Let X be a complex manifold or a variety. Then there is a one-to-one correspondence between (isomorphism classes of) vector bundles over X and (isomorphism classes of) locally free sheaves over X.*

Proof. We follow [Wel07] respectively [Sha94c]. Given a locally free sheaf $\mathcal{F}$, we construct a vector bundle. This construction turns out to be the inverse of the map $E \mapsto \mathcal{L}_E$. We can assume that X is connected, otherwise consider the connected components separately.

Take an open covering $(U_\alpha)_\alpha$ of X such that the $\mathcal{F}|_{U_\alpha}$ are free sheaves of rank r_α. Since X is connected, the rank is independent of the choice of α, and we set $r = r_\alpha$. Let $\varphi_\alpha : \mathcal{F}_{U_\alpha} \xrightarrow{\sim} \mathcal{O}^r_{U_\alpha}$ be the corresponding isomorphisms. Define $g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ on $\mathcal{O}^r_{U_\alpha \cap U_\beta}$. Then,

$$g_{\alpha\beta} : \mathcal{O}^r_{U_\alpha \cap U_\beta} \longrightarrow \mathcal{O}^r_{U_\alpha \cap U_\beta}.$$

Furthermore, $g_{\alpha\beta}$ determines an invertible mapping of vector-valued functions $(g_{\alpha\beta})_{U_\alpha \cap U_\beta}$, which we can write as

$$g_{\alpha\beta} : \mathcal{O}_X(U_\alpha \cap U_\beta)^r \longrightarrow \mathcal{O}_X(U_\alpha \cap U_\beta)^r,$$

which is given by a nonsingular $r \times r$ matrix of functions in $\mathcal{O}_X(U_\alpha \cap U_\beta)$. The matrices of the functions $g_{\alpha\beta}$ satisfy the compatibility conditions (see section 3.1). Hence one obtains a vector bundle $E \to X$ by "glueing". This construction gives $\mathcal{F} = \mathcal{L}_E$ and furthermore preserves isomorphism classes. □

Remark 3.4.2. Constructions of the dual, sum, tensor product, etc. of vector bundles and of locally free sheaves on complex manifolds are preserved under this correspondence. Furthermore, we define these constructions on vector bundles on varieties by the corresponding construction on the associated locally free sheaf.

Example 3.4.3. Let E be a vector bundle over X and let $f : X \to Y$ be a continuous map of topological spaces. We call the vector bundle corresponding to the pushforward of the sheaf corresponding to E the *pushforward of the vector bundle* E, and denote it by f_*E. Compare also examples 1.2.4 and 3.2.5.

Note that, as a special case of the above theorem, invertible sheaves correspond to line bundles. Therefore invertible sheaves are sometimes called *line sheaves*. They play a special role because of the following proposition:

Proposition 3.4.4. *(Propostion II.6.12 in [Har77]) If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a ringed space X, so is $\mathcal{L} \otimes \mathcal{M}$. If $\mathcal{L}$ is any invertible sheaf on X, then there exists an invertible sheaf $\mathcal{L}^{-1}$ on X such that $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$. In particular, the invertible sheaves on X form a group.*

Definition 3.4.5. The group of locally free sheaves of rank one on a ringed space X is called the *Picard group of* X. It is denoted by $\operatorname{Pic} X$.

Example 3.4.6. In example 3.2.4, we saw an important line bundle on $\mathbb{P}^n_{\mathbb{C}}$, the tautological bundle. The line sheaf associated to the dual of this vector bundle is called the *twisting sheaf of Serre* and denoted by $\mathcal{O}(1)$. This can also be defined in another approach, in a more general way, which is more suitable in the context of sheaves, see the next example and [Har77], p. 117 in chapter II.5. Tensor products $\mathcal{O}(1)^{\otimes m}$, denoted by $\mathcal{O}(m)$, for $m \in \mathbb{Z}$, also are line bundles on $\mathbb{P}^n_{\mathbb{C}}$ by the above proposition.

Example 3.4.7. Let M be a finitely generated projective module. To define the twisting sheaf of Serre on $\mathbb{P}(M)$, consider the trivial bundle

$$\begin{tikzcd} M \times \mathbb{P}(M) \arrow[d] \\ \mathbb{P}(M) \end{tikzcd}$$

and define the subbundle $\mathcal{H}(M)$ as follows: for $\xi \in \mathbb{P}(M)$, let M_ξ be the associated projective submodule of corank 1 (see section 2.3). We define the fiber $\mathcal{H}(M)_\xi = M_\xi$, and $\mathcal{H}(M) = \bigsqcup(M_\xi, \xi)$. Then the *twisting sheaf of Serre* $\mathcal{O}(1)$ is the dual of the quotient bundle Q in the following exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H}(M) \arrow[r] & M \times \mathbb{P}(M) \arrow[r] \arrow[d] & Q \arrow[r] & 0 \\ & & \mathbb{P}(M) & & \end{tikzcd}$$

i.e. $\mathcal{O}(1) = Q^\vee$.

Remark 3.4.8. For $M = \mathbb{C}^{n+1}$, in fact, $\mathbb{C}^{n+1}/\mathcal{H}(\mathbb{C}^{n+1})_\xi \cong L_\xi$, where L_ξ is the line represented by $\xi \in \mathbb{P}^n_{\mathbb{C}}$. So we can also describe the twisting sheaf as the dual of the line bundle $\mathcal{L}$ which is a subbundle of the trivial bundle $\mathbb{C}^{n+1} \times \mathbb{P}^n_{\mathbb{C}}$, where the fiber over a point $\xi \in \mathbb{P}^n_{\mathbb{C}}$ is given by the line represented by ξ in $\mathbb{C}^{n+1}$, i.e. $\mathcal{L}_\xi = L_\xi$. We then have an exact sequence of vector bundles,

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L} \arrow[r] & \mathbb{C}^{n+1} \times \mathbb{P}^n_{\mathbb{C}} \arrow[r] \arrow[d] & \tilde{Q} \arrow[r] & 0 \\ & & \mathbb{P}^n_{\mathbb{C}} & & \end{tikzcd}$$

and $\mathcal{O}(1) = \mathcal{L}^\vee$.

Remark 3.4.9. For $M = R^n$, $\mathbb{P}(M) = \mathbb{P}^n_R$. In this case, the global sections of $\mathcal{O}(1)$ can be identified with R^n, i.e. $\Gamma(\mathbb{P}^n_R, \mathcal{O}(1)) \cong Rx_0 + \cdots Rx_n$. Note that this is analogous to the complex case, see example 3.3.4.

In particular, we just defined the twisting sheaf on $\mathbb{P}_k^n$ for some field k. This is an important invertible sheaf on $\mathbb{P}_k^n$ since it in fact generates the Picard group:

Proposition 3.4.10. *(Corollary II.6.17 in [Har77]) If $X = \mathbb{P}_k^n$ for some field k, then every invertible sheaf of X is isomorphic to $\mathcal{O}(m) := \mathcal{O}(1)^{\otimes m}$ for some $m \in \mathbb{Z}$.*

As vector bundles correspond to projective modules, the projective bundle of a vector bundle is defined as follows:

Definition 3.4.11. Let $\mathcal{E} \to X$ be a vector bundle over a scheme X or a holomorphic vector bundle over a complex space X. Let E be the finitely generated projective module of global sections of $\mathcal{E}$. Then the associated *projective bundle* is defined as

$$\mathbb{P}(\mathcal{E}) := \mathbb{P}(E).$$

This is a bundle $\pi : \mathbb{P}(\mathcal{E}) \to X$ and the fiber over a point $x \in X$ is the projective space $\mathbb{P}(E_x)$ (see [GH78], p. 515, [Har77], p. 162ff in section II.7, and [Laz04], Appendix A).

Remark 3.4.12. In this case, the tautological bundle $T \to \mathbb{P}(E)$, in fact, is the subbundle of the pullback bundle $\pi^*E \to \mathbb{P}(E)$ whose fiber at a point $(x, v) \in \mathbb{P}(E)$, where $x \in X$ and $v \in \mathbb{P}(E_x)$, is the line in E_x represented by v, see [GH78], p.605. Also in this case, the twisting sheaf is defined to be the dual of the tautological bundle.

Using the twisting sheaf, we can define an important property of an invertible sheaf.

Definition 3.4.13. Let X be a scheme over another scheme S. Furthermore, let $\mathcal{L}$ be an invertible sheaf over X. Then $\mathcal{L}$ is *very ample relative to* S, if there is an immersion $i : X \to \mathbb{P}_S^n$ for some n such that

$$i^*(\mathcal{O}(1)) \cong \mathcal{L}.$$

Let X be a scheme of finite type over a Noetherian ring A, and let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is *ample* if $\mathcal{L}^{\otimes m}$ is very ample over $\operatorname{Spec} A$ for some $m > 0$.

Remark 3.4.14. Let the corresponding immersion of a very ample invertible sheaf $\mathcal{L}$ over a scheme X over $\operatorname{Spec} A$ be $i : X \to \mathbb{P}_A^n$. Then $\mathcal{L}$ admits global sections $s_0, \dots, s_n$ which define i. Furthermore, these sections *generate* $\mathcal{L}$, i.e. for each $x \in X$, the stalk $\mathcal{L}_x$ is generated by the images of the s_i in $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module. Furthermore, $s_i = i^*x_i$, where the x_i are the homogeneous coordinates on $\mathbb{P}_A^n$, which give rise to global sections.

There is a more general definition of ampleness which is independent of the base scheme, see [Har77], p. 153, section II.7.

3.5 Divisors

We now introduce the notion of a divisor on a scheme X. For this, we need the scheme to satisfy some properties. Throughout this section, we assume that the schemes we consider all have these properties. However, first we need a definition.

Definition 3.5.1. A scheme X is *regular in codimension one* if every local ring $\mathcal{O}_{X,x}$ of X of dimension one is regular.

Assumption 3.5.2. X is a Noetherian integral separated scheme which is regular in codimension one.

Definition 3.5.3. Let X be a scheme satisfying assumption 3.5.2. A *prime divisor* on X is a closed integral subscheme Y of codimension one. A *Weil divisor* D is an element of the free Abelian group $\operatorname{Div} X$ generated by prime divisors. We can write D as a finite sum $D = \sum_i n_i Y_i$, where the Y_i are prime divisors, and the n_i are integers. A divisor $D = \sum_i n_i Y_i$ is called *effective* if all $n_i \geqslant 0$.

If Y is a prime divisor on X, let η be its generic point (it exists by [Har77], Exercise II.2.9). Then its local ring $\mathcal{O}_{X,\eta}$ has Krull dimension one ([Liu02], Exercise 2.5.2, compare also Example 7.2.6) and therefore is regular by the assumption about X. Thus, $\mathcal{O}_{X,\eta}$ is a discrete valuation ring ([Eis95], Proposition 11.1). Furthermore, the quotient field of $\mathcal{O}_{X,\eta}$ is the function field $k(X)$ of X. Denote the corresponding discrete valuation by v_Y. If $f \in k(X)^*$, $v_Y(f) \in \mathbb{Z}$. If it is positive, we say f has a *zero along* Y of order $v_Y(f)$, and if it is negative, we say f has a *pole along* Y of order $-v_Y(f)$. An alternative notation for $v_Y(f)$ is $\operatorname{ord}_Y(f)$, the *order of vanishing of* f, see also definition 8.1.3.

Using this, we can define the divisor of a function.

Definition 3.5.4. Let X satisfy the assumption and let $f \in k(X)^*$. Then let the *divisor of* f, denoted by $\operatorname{div} f$, be

$$\operatorname{div} f = \sum \operatorname{ord}_Y(f) \cdot Y,$$

where the sum is taken over all prime divisors of X. One can show that this sum indeed is finite ([Har77], Lemma II.6.1) and therefore $\operatorname{div} f$ is well-defined. Any divisor which is equal to the divisor of a rational function is called *principal divisor*. Often one denotes it by $(f) = \operatorname{div} f$.

Note that if $f, g \in k(X)^*$, $\operatorname{div}(f/g) = \operatorname{div} f - \operatorname{div} g$, and therefore sending a function to its divisor is a homomorphism from the multiplicative group $k(X)^*$ to the additive group $\operatorname{Div} X$. Its image $P(X)$ is a subgroup of $\operatorname{Div} X$. The quotient group

$$\operatorname{Cl} X = \operatorname{Div} X / P(X)$$

is called the *divisor class group of* X. Two divisors D, D' are called *linearly equivalent* if there exists a nonzero rational function f such that $D - D' = \operatorname{div} f$.

Divisors are interesting for us because, in fact, by the following theorem, they correspond to invertible sheaves and therefore, by theorem 3.4.1, to line bundles. So, we can identify line bundles, line sheaves, and divisors. Sometimes it is useful to consider our object of study as a vector bundle or sheaf, sometimes it is more useful to regard it as a divisor.

Theorem 3.5.5. *(Corollary II.6.16 in [Har77]) If X is a Noetherian, integral, separated scheme whose local rings are unique factorization domains, then there is a natural isomorphism* $\operatorname{Cl} X \cong \operatorname{Pic} X$.

Remark 3.5.6. The proof of this theorem requires the theory of *Cartier divisors*, which, if the scheme satisfies certain properties, correspond to Weil divisors. The proof then is quite straightforward. For a different proof of this theorem, see [Sha94c], VI.1.4 or [Wel07], III.4, p. 107.

Example 3.5.7. The prime divisor in $\mathbb{P}^n_k$ which is given by a hyperplane corresponds to the invertible sheaf $\mathcal{O}(1)$, compare proposition 3.4.10.

Remark 3.5.8. For projective curves we give an explicit construction of the sheaf corresponding to a divisor D. We follow [Wüs08].

Let $\bar{X}$ be a projective non-singular curve. In particular, $\bar{X}$ is integral. The prime divisors of $\bar{X}$ are points of $\bar{X}$. Let $D = \sum_{P \in \bar{X}} n_P P$. Every point of $\bar{X}$ is contained in an affine open set in $\bar{X}$. We define

$$\mathcal{O}(D)_P = \{\text{functions } f \in k(X) \text{ such that } \operatorname{ord}_P(f) \geqslant -n_P\}.$$

This is an $\mathcal{O}_{\bar{X},P}$-module and, in fact, finitely generated. $\mathcal{O}_{\bar{X},P}$ is a regular local ring since $\bar{X}$ is projective, and of dimension one. Therefore, the maximal ideal of $\mathcal{O}_{\bar{X},P}$ is generated by one element, π, called a *uniformizing element in* P. Therefore $\mathcal{O}(D)_P$ is a fractional ideal of the form

$$\mathcal{O}(D)_P = (\pi^{-n_P}).$$

This defines a sheaf $\mathcal{O}(D)$ similar to the construction in definition 2.2.16:

For any open subset of $\bar{X}$ we define $\mathcal{O}(D)(U)$ as the set of functions

$$s : U \longrightarrow \bigsqcup_{P \in U} \mathcal{O}(D)_P$$

such that

1. $s(P) \in \mathcal{O}(D)_P$ for every $P \in U$, and
2. for every $P \in U$ there is an affine open set $U = \operatorname{Spec} R$ such that $P \in U$ and an open neighborhood V of P in X and elements $g, h \in R$ such that $\frac{g}{h} \in \mathcal{O}(D)_Q = (\pi_Q^{-n_Q})$ and $s(Q) = \frac{g}{h}$ for every $Q \in V$. Here π_Q is a uniformizing element for Q.

Summary

We briefly summarize the most important concepts from part I for reference. First, we recall the connections, especially those used for identifications in the following. Then, we give a résumé of the definitions of projective space and its twisting sheaf.

Connections

Vector bundles on varieties – vector bundles on complex manifolds

In remark 3.1.8, we discussed the correspondence between complex analytic spaces and schemes established in Serre's paper GAGA, [Ser56]. Furthermore, we noted that, if E is a vector bundle on a variety X, then $E(\mathbb{C}) \to X(\mathbb{C})$ is a topological vector bundle.

Vector bundles – projective modules

Given a (fixed) variety X, proposition 3.3.5 establishes the connection between vector bundles on X and their finitely generated projective modules of global sections:

Proposition. *(Corollary A.3.3 in [Eis95]) Given a vector bundle E on a variety X, its module of global sections $\Gamma(X, E)$ forms a finitely generated projective module. Furthermore, any finitely generated projective module arises uniquely from a vector bundle in this way.*

Thus, throughout the thesis, we identify vector bundles and finitely generated projective modules. We only explicitly differentiate between the two when necessary.

Vector bundles – locally free sheaves

Theorem 3.4.1 asserts the one-to-one correspondence between isomorphism classes of vector bundles on a fixed complex manifold or variety and isomorphism classes of locally free sheaves on that complex manifold or variety.

Theorem. *Let X be a complex manifold or a variety. Then there is a one-to-one correspondence between (isomorphism classes of) vector bundles over X and (isomorphism classes of) locally free sheaves over X.*

Therefore, we will talk about vector bundles and locally free sheaves interchangeably.

Line bundles – invertible sheaves – divisors

Subject to certain assumptions on the base space, theorem 3.5.5 relates isomorphism classes of line bundles to the class group, i.e. equivalence classes of divisors.

Theorem. *(Corollary II.6.16 in [Har77]) If X is a Noetherian, integral, separated scheme whose local rings are unique factorization domains, then there is a natural isomorphism* $\operatorname{Cl} X \cong \operatorname{Pic} X$.

Later on, we particularly need the line bundle $\mathcal{O}(D)$ associated to a divisor D from remark 3.5.8.

Projective space and the twisting sheaf of Serre

We briefly recall the construction of the projective space associated to a finitely generated projective module and its twisting sheaf.

In section 2.3 we defined the projective space $\mathbb{P}(M)$ associated to a finitely generated projective module M:

Definition. Let R be a ring and M a finitely generated projective R-module. Denote by $\mathcal{S}(M)$ the symmetric algebra on M,

$$\mathcal{S}(M) = \bigoplus_{k \geqslant 0} \mathcal{S}^k(M), \qquad \mathcal{S}^k(M) = \mathcal{T}^k(M)/\langle\{m \otimes n - n \otimes m\}\rangle,$$

where $\mathcal{T}^k(M) = M^{\otimes k}$, and $\langle\{m \otimes n - n \otimes m\}\rangle$ is the ideal generated by the elements $m \otimes n - n \otimes m$, for $m, n \in M$. Then the associated *projective space* is $\mathbb{P}(M) := \operatorname{Proj} \mathcal{S}(M)$. If n is a non-negative integer, then the *projective n-space over R* is $\mathbb{P}^n_R = \mathbb{P}(R^{n+1})$. If Y is a scheme and n a non-negative integer, then the *projective n-space over Y* is $\mathbb{P}^n_Y := \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} Y$.

In example 3.2.4, we introduced the *tautological bundle.* This is a line bundle on complex projective space:

As a topological space, the total space of the tautological bundle $T \to \mathbb{P}^n_{\mathbb{C}}$ is a disjoint union of lines. Furthermore, using the open cover $(U_i)_i$ of $\mathbb{P}^n_{\mathbb{C}}$ by the standard open sets $U_i = \{(x_0 : \ldots : x_n) \in \mathbb{P}^n_{\mathbb{C}} : x_i \neq 0\}$, the tautological bundle is determined by the transition functions $g_{ij} : U_i \cap U_j \to GL(1, \mathbb{C}) = \mathbb{C}^*$, $g_{ij}((x_0 : \ldots : x_n)) = \frac{x_i}{x_j}$.

The line bundle which is dual to the tautological bundle corresponds to a locally free sheaf, the *twisting sheaf* $\mathcal{O}(1)$. More generally, we defined the twisting sheaf for the projective space associated to a finitely generated projective module M in example 3.4.7 as follows:

Definition. Let $\mathcal{H}(M)$ be the subbundle of the trivial bundle over $\mathbb{P}(M)$ such that for $\xi \in \mathbb{P}(M)$, the fiber $\mathcal{H}(M)_\xi = M_\xi$, where M_ξ is the projective submodule of corank 1 associated to ξ (see section 2.3). Then $\mathcal{H}(M) = \bigsqcup(M_\xi, \xi)$. The *twisting sheaf of Serre* $\mathcal{O}(1)$ is the dual of the quotient bundle Q, i.e. $\mathcal{O}(1) = Q^\vee$, in the following exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H}(M) \arrow[r] & M \times \mathbb{P}(M) \arrow[r] \arrow[d] & Q \arrow[r] & 0 \\ & & \mathbb{P}(M) & & \end{tikzcd}$$

In example 3.3.4 and remark 3.4.9 we saw that for $\mathbb{P}^n_R$, the projective n-space over a ring R, the corresponding finitely generated projective module of *global sections* is

$$\Gamma(\mathbb{P}^n_R, \mathcal{O}(1)) \cong Rx_0 + \cdots Rx_n.$$

Part II

Arakelov geometry

Chapter 4

Hermitian vector bundles

In this chapter, we introduce *Hermitian vector bundles* on an arithmetic variety over $S = \operatorname{Spec} \mathcal{O}_K$. First, however, we need to define the notion of a Hermitian vector bundle on a complex manifold. We construct metrics on the constructions of holomorphic vector bundles of section 3.2 and define an important example of a Hermitian metric on a vector bundle, the Fubini-Study metric on $\mathcal{O}(1)$ on complex projective space.

4.1 Hermitian vector bundles on complex manifolds

Definition 4.1.1. A *Hermitian metric* on a complex manifold M of dimension n is given by a positive definite Hermitian inner product

$$h_z(\cdot,\cdot) : T'_z(M) \otimes \overline{T'_z(M)} \longrightarrow \mathbb{C}$$

for every $z \in M$, which depends smoothly on z, i.e. given local coordinates z_i on M, the functions

$$h_{ij}(z) = h_z\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)$$

are smooth. Writing $z_j = x_j + iy_j$, $T'_z(M) = \{\frac{\partial}{\partial x_j}\}\mathbb{C}$ is the holomorphic tangent space at z. Given a basis $\{dz_i \otimes d\bar{z}_j\}$ for $(T'_z(M) \otimes \overline{T'_z(M)})^*$, the Hermitian metric is given by

$$ds^2 = \sum_{i,j} h_{ij}(z)\, dz_i \otimes d\bar{z}_j.$$

The *(1,1)-form associated to the metric* is

$$\omega(z) = \frac{i}{2}\sum_{j,k} h_{jk}(z)\, dz_j \wedge d\bar{z}_k.$$

(See [GH78], p. 27ff.)

Remark 4.1.2. The associated differential form determines the metric; in fact, any positive differential form ω of type $(1,1)$ on a manifold M gives a Hermitian metric on M, see [GH78].

Remark 4.1.3. The real part of a Hermitian inner product gives a Riemannian metric, the *induced Riemannian metric.*

Example 4.1.4. The *standard or Euclidean Hermitian metric* on $\mathbb{C}^n$ is given by

$$ds^2 = \sum_{i=1}^{n} dz_i \otimes d\bar{z}_i.$$

We will see another example in section 4.1.2.

Definition 4.1.5. Let X be a complex variety and $\mathcal{E}$ a holomorphic vector bundle on X. A *Hermitian metric* h on $\mathcal{E}$ is a Hermitian inner product on each fiber $\mathcal{E}_z$ of $\mathcal{E}$ such that the functions representing h locally are C^∞.
A *Hermitian vector bundle* $\bar{\mathcal{E}}$ *on a complex variety* X is a pair $(\mathcal{E}, h)$, where $\mathcal{E}$ is a locally free sheaf of finite rank on X and h is a Hermitian metric on $\mathcal{E}$.

Example 4.1.6. Let M be a complex manifold. Consider the trivial bundle $\pi : E = \mathbb{C}^n \times M \to M$ where π is the projection onto the second coordinate. Then every fiber over a point $z \in M$ is $E_z = \mathbb{C}^n$, and thus can be endowed with the standard metric. This turns E into a Hermitian vector bundle.

4.1.1 Construction of metrics

Given a Hermitian vector bundle on a complex variety X, we construct a metric on the dual, the tensor product, the direct sum, the pullback, and the (m-th) exterior product of Hermitian vector bundles on X. We then, given a Hermitian bundle and a subbundle, induce a metric on the subbundle and on the quotient. Using these constructions and the definition in example 3.4.7, a Hermitian metric on $\mathcal{O}(1)$ is defined by the standard metric on $\mathbb{C}$.

We follow the rather abstract approach in [Bos99], [Via05].

Notation. Let $\mathcal{E}^c$ be the complex conjugate vector bundle of $\mathcal{E}$, i.e. let its $\mathbb{C}$-structure be given by the one of $\mathcal{E}$ composed with complex conjugation. Let $\mathcal{E}^\vee$ be the dual bundle, i.e. the bundle of homomorphisms from $\mathcal{E}$ to the trival bundle. Let $\Gamma(X, \mathcal{E})$ be the space of global smooth sections of $\mathcal{E}$.

Remark 4.1.7. A Hermitian metric h is an element of $\Gamma(X, \mathcal{E}^\vee \otimes \mathcal{E}^{c\vee})$: The metric h is sesquilinear on each fiber, so we can consider it as a linear form from $\mathcal{E}_z \otimes \mathcal{E}_z^c$ to $\mathbb{C}$, i.e. an element of the dual space $(\mathcal{E}_z \otimes \mathcal{E}_z^c)^\vee$. Since the fiber $\mathcal{E}_z$ is finite-dimensional, $(\mathcal{E}_z \otimes \mathcal{E}_z^c)^\vee = \mathcal{E}_z^\vee \otimes (\mathcal{E}_z^c)^\vee$. Furthermore, h varies smoothly in $z \in X$, so h is a smooth global section of $\mathcal{E}^\vee \otimes (\mathcal{E}^c)^\vee$.
Furthermore, a Hermitian metric is *positive*, i.e. the induced quadratic form h_z is positive definite for all z.

The dual $\overline{\mathcal{E}}^\vee$

We first define the canonical metric $h^\vee$ on the dual $\mathcal{E}^\vee$ of a given Hermitian vector bundle $(\mathcal{E}, h)$, i.e. a positive element $h^\vee \in \Gamma(X, \mathcal{E} \otimes \mathcal{E}^c)$.

For every $z \in X$, h induces an isomorphism $\Phi_z : \mathcal{E}_z^c \to \mathcal{E}_z^\vee, a \mapsto h_z(\cdot, a)$, where

$$\begin{array}{rcccl} h_z(\cdot, a) & : & \mathcal{E}_z & \longrightarrow & \mathbb{C} \\ & & b & \longmapsto & h_z(b, a). \end{array}$$

Now define

$$\begin{array}{rcccl} h_z^\vee & : & \mathcal{E}_z^\vee \otimes (\mathcal{E}^{\vee c})_z & \longrightarrow & \mathbb{C} \\ & & (v, v') & \longmapsto & h_z(\Phi_z^{-1}(v'), \Phi_z^{-1}(v)). \end{array}$$

To check that $h^\vee$ is a Hermitian metric on $\mathcal{E}^\vee$, we need to check that $h^\vee$ is smooth and that it is positive. Since h is smooth, all maps defined above are smooth, and therefore $h^\vee$ also is smooth, i.e. $h^\vee \in \Gamma(X, \mathcal{E} \otimes \mathcal{E}^c)$. Now, to show that $h^\vee$ is positive, let us fix a $z \in X$. We choose an orthogonal basis of $\mathcal{E}_z$. By the definition of $h^\vee$, one sees that the dual basis is orthogonal in $\mathcal{E}_z^\vee$. Therefore, $h^\vee$ is positive.

The tensor product $\bar{\mathcal{E}} \otimes \bar{\mathcal{E}}'$

Let $\bar{\mathcal{E}} = (E, h)$ and $\bar{\mathcal{E}}' = (E', h')$ be Hermitian vector bundles on X. We define a Hermitian metric $h \otimes h'$ on $\mathcal{E} \otimes \mathcal{E}'$, i.e. a positive element in $\Gamma\left(X, (\mathcal{E} \otimes \mathcal{E}')^\vee \otimes (\mathcal{E} \otimes \mathcal{E}')^{c\vee}\right)$, canonically depending on h and h'.

Since $(\mathcal{E}^\vee \otimes \mathcal{E}^{c\vee}) \otimes (\mathcal{E}'^\vee \otimes (\mathcal{E}'^c)^\vee) \cong (\mathcal{E} \otimes \mathcal{E}')^\vee \otimes (\mathcal{E} \otimes \mathcal{E}')^{c\vee}$, we can consider the natural embedding

$$\Phi : \Gamma\left(X, \mathcal{E}^\vee \otimes (\mathcal{E}^c)^\vee\right) \otimes \Gamma\left(X, \mathcal{E}'^\vee \otimes (\mathcal{E}'^c)^\vee\right) \hookrightarrow \Gamma\left(X, (\mathcal{E} \otimes \mathcal{E}')^\vee \otimes (\mathcal{E} \otimes \mathcal{E}')^{c\vee}\right).$$

Define the metric $h \otimes h'$ (by abuse of notation) as the image of

$$h \otimes h' \in \Gamma\left(X, \mathcal{E}^\vee \otimes (\mathcal{E}^c)^\vee\right) \otimes \Gamma\left(X, \mathcal{E}'^\vee \otimes (\mathcal{E}'^c)^\vee\right)$$

under Φ. We give $(h \otimes h')_z$ explicitly for fixed $z \in X$:

$$\begin{array}{rcccl} (h \otimes h')_z & : & (\mathcal{E} \otimes \mathcal{E}')_z \times (\mathcal{E} \otimes \mathcal{E}')_z^c & \longrightarrow & \mathbb{C} \\ & & \left(\sum_i e_i \otimes e_i', \sum_i f_i \otimes f_i'\right) & \longmapsto & \sum_{i,j} h_z(e_i, f_j) \cdot h_z'(e_i', f_j'), \end{array}$$

which is exactly what one expects the metric of the tensor product to be. By this we see the positivity of $h \otimes h'$ and therefore we get a canonical metric on the tensor product $\mathcal{E} \otimes \mathcal{E}'$.

The direct sum $\bar{\mathcal{E}} \oplus \bar{\mathcal{E}}'$

Again, let $\bar{\mathcal{E}} = (\mathcal{E}, h)$ and $\bar{\mathcal{E}}' = (\mathcal{E}', h')$ be Hermitian vector bundles on X. To canonically induce a metric on the direct sum, we need to find a positive element in $\Gamma(X, (\mathcal{E} \oplus \mathcal{E}')^\vee \otimes (\mathcal{E} \oplus \mathcal{E}')^{c\vee})$. We will denote the Hermitian metric on the direct sum as $h \oplus h'$.

By commuting the dual and the direct sum as well as using the distributive law, we get

$$(\mathcal{E} \oplus \mathcal{E}')^\vee \otimes (\mathcal{E} \oplus \mathcal{E}')^{c\vee} = (\mathcal{E}^\vee \otimes \mathcal{E}^{c\vee}) \oplus (\mathcal{E}^\vee \otimes (\mathcal{E}'^c)^\vee) \oplus (\mathcal{E}'^\vee \otimes \mathcal{E}^{c\vee}) \oplus (\mathcal{E}'^\vee \otimes (\mathcal{E}'^c)^\vee).$$

Therefore, we get a canonical embedding of global sections:

$$\Gamma(X, \mathcal{E}^\vee \otimes \mathcal{E}^{c\vee}) \oplus \Gamma(X, \mathcal{E}'^\vee \otimes (\mathcal{E}'^c)^\vee) \hookrightarrow \Gamma(X, (\mathcal{E} \oplus \mathcal{E}')^\vee \otimes (\mathcal{E} \oplus \mathcal{E}')^{c\vee}).$$

Like in the previous section, we define the metric $h \oplus h'$ (by abuse of notation) as the image of

$$h \oplus h' \in \Gamma(X, \mathcal{E}^\vee \otimes \mathcal{E}^{c\vee}) \oplus \Gamma(X, \mathcal{E}'^\vee \otimes \mathcal{E}'^{c\vee}).$$

We can give this metric explicitly as

$$\begin{array}{cccll} (h \oplus h')_z & : & (\mathcal{E} \oplus \mathcal{E}')_z \times (\mathcal{E} \oplus \mathcal{E}')^c_z & \longrightarrow & \mathbb{C} \\ & & (e, e', f, f') & \longmapsto & h_z(e, f) + h'_z(e', f'), \end{array}$$

which, again, is exactly what one expects.

By this, we see the positivity of $h \oplus h'$ and therefore get a well-defined metric on the direct sum.

The pullback $f^*\bar{\mathcal{E}}$

Let $f : Y \to X$ be a morphism of complex manifolds and $\bar{\mathcal{E}}$ a Hermitian vector bundle on X. Since

$$(f^*\mathcal{E} \otimes f^*\mathcal{E}^c)^\vee = (f^*\mathcal{E})^\vee \otimes (f^*\mathcal{E})^{c\vee} = f^*\mathcal{E}^\vee \otimes (f^*\mathcal{E}^c)^\vee = f^*\left(\mathcal{E}^\vee \otimes (\mathcal{E}^c)^\vee\right),$$

f^*h defines an element of $\Gamma\left(X, (f^*\mathcal{E})^\vee \otimes (f^*\mathcal{E})^{c\vee}\right)$ which is positive by the definition of the pullback of a morphism, and therefore defines a Hermitian metric on $f^*\bar{\mathcal{E}}$.

Exact metric sequences

Let $\bar{\mathcal{E}} = (\mathcal{E}, h)$ be a Hermitian vector bundle. Given an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{E}' \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{E}'' \longrightarrow 0,$$

we canonically induce Hermitian metrics h', h'' on $\mathcal{E}', \mathcal{E}''$; respectively. By the exact sequence, we get induced maps

$$\alpha \otimes \alpha^c : \mathcal{E}' \otimes \mathcal{E}'^c \longrightarrow \mathcal{E} \otimes \mathcal{E}^c$$

and

$$\beta \otimes \beta^c : \mathcal{E} \otimes \mathcal{E}^c \longrightarrow \mathcal{E}'' \otimes \mathcal{E}''^c.$$

Dualizing the first map, we get the surjective map

$$(\alpha \otimes \alpha^c)^\vee : (\mathcal{E} \otimes \mathcal{E}^c)^\vee \longrightarrow (\mathcal{E}' \otimes \mathcal{E}'^c)^\vee.$$

Now we define h' to be the image of h under the map $(\alpha \otimes \alpha^c)^\vee$.

In section 4.1.1, we canonically constructed a metric $h^\vee \in \Gamma(X, \mathcal{E} \otimes \mathcal{E}^c)$ on the dual. Define h'' to be the dual of $h''^\vee := (\beta \otimes \beta^c)(h^\vee)$. Then, $h'' = (h''^\vee)^\vee \in \Gamma(X, (\mathcal{E}'' \otimes \mathcal{E}''^c)^\vee)$ is the desired Hermitian metric on the quotient bundle.

Naturally, we can think of h' as the restriction norm. Furthermore, h'' is the restriction norm on the orthogonal complement of $\mathcal{E}'$ in $\mathcal{E}$, which is canonically isomorphic to $\mathcal{E}/\mathcal{E}'$. Therefore, h' and h'' are positive.

Definition 4.1.8. Given Hermitian vector bundles $\bar{\mathcal{E}} = (\mathcal{E}, h)$, $\overline{\mathcal{E}'} = (\mathcal{E}', h_1)$, and $\overline{\mathcal{E}''} = (\mathcal{E}'', h_2)$, we define the sequence

$$0 \longrightarrow \overline{\mathcal{E}'} \xrightarrow{\alpha} \bar{\mathcal{E}} \xrightarrow{\beta} \overline{\mathcal{E}''} \longrightarrow 0, \tag{4.1}$$

to be *metric exact*, if h_1 and h_2 are the metrics induced on $\mathcal{E}'$ and $\mathcal{E}''$ by the metric h on $\mathcal{E}$ as described above, i.e. in the above notation, $h_1 = h'$ and $h_2 = h''$.
Metric exact sequences have interesting additivity properties (e.g. proposition 5.1.8) which, in general, do not hold when the sequence is just exact.

Remark 4.1.9. Not all exact sequences of Hermitian vector bundles are metric exact. In the above notation, if $h_1 \neq h'$ or $h_2 \neq h''$, then (4.1) is not metric exact. It is an interesting question to study $h_1 - h'$ and $h_2 - h''$. This leads to the study of the *Bott-Chern secondary characteristic class*, see e.g. [Sou92].

The exterior product $\bigwedge^m \bar{\mathcal{E}}$

Since

$$\bigwedge^m \bar{\mathcal{E}} = \mathcal{T}^m(\bar{\mathcal{E}})/\langle\{e \otimes e\}\rangle,$$

we get a metric on $\bigwedge^m \bar{\mathcal{E}}$ by inducing a metric on $\mathcal{T}^m(\bar{\mathcal{E}})$ and then on the quotient by the construction in the previous sections. Thus, we obtain a metric on the m-th exterior product.

By this and the construction of a metric on the direct sum of Hermitian vector bundles, we get a metric on the *exterior product*

$$\bigwedge \bar{\mathcal{E}} = \bigoplus_{m=1}^{\infty} \bigwedge^m \bar{\mathcal{E}},$$

and on the *determinant bundle*

$$\det \bar{\mathcal{E}} = \bigwedge^{\operatorname{rk}\mathcal{E}} \bar{\mathcal{E}}.$$

Furthermore, the metric on the determinant bundle is, for $z \in X$, explicitly given by

$$\begin{array}{rccc} (\bigwedge^r h)_z : & \bigwedge^r \mathcal{E}_z \times \bigwedge^r \mathcal{E}_z & \longrightarrow & \mathbb{C} \\ & (e_1 \wedge \ldots \wedge e_r, f_1 \wedge \ldots \wedge f_r) & \longmapsto & \det(h_z(e_i, f_j)_{i,j}), \end{array}$$

where $r = \operatorname{rk}\mathcal{E}$.

4.1.2 The Fubini-Study metric

In this section, we construct a metric on $\mathbb{P}^n_{\mathbb{C}}$ and on the twisting sheaf $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)$, which we deduce from the standard Hermitian metric on $\mathbb{C}^n$. We use [GH78], [Huy05], [Voi02], and [Laz04].

The Fubini-Study metric on $\mathbb{P}^n_{\mathbb{C}}$

We start by inducing a metric on $\mathbb{P}^n_{\mathbb{C}}$.

Let $x_0, \ldots, x_n$ be coordinates on $\mathbb{C}^{n+1}$ and let $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n_{\mathbb{C}}$ be the natural projection. Given an open set $U \subset \mathbb{P}^n_{\mathbb{C}}$, let $Z : U \to \mathbb{C}^{n+1}\backslash\{0\}$ be a lift of U, i.e. a holomorphic map with $\pi \circ Z = id|_U$, and define the differential form

$$\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log\|Z\|^2.$$

Here $\|\cdot\|$ is the norm on $\mathbb{C}^{n+1}$.

This differential form is, in fact, independent of the lifting: if $Z' : U \to \mathbb{C}^{n+1}\backslash\{0\}$ is another lifting, then there is a nonzero holomorphic map $f : \mathbb{C} \to \mathbb{C}$ such that $Z' = f \cdot Z$. Then,

$$\begin{aligned} \frac{i}{2\pi}\partial\bar{\partial}\log\|Z'\|^2 &= \frac{i}{2\pi}\partial\bar{\partial}\log\|f\cdot Z\|^2 \\ &= \frac{i}{2\pi}\left(\partial\bar{\partial}\log\|Z\|^2 + \partial\bar{\partial}\log|f|^2\right) \\ &= \omega_{FS}, \end{aligned} \tag{4.2}$$

since $\partial\bar{\partial}\log f\bar{f} = \partial\bar{\partial}f - \bar{\partial}\partial\bar{f} = 0 - 0 = 0$ ([GH78], p. 30)).

Liftings always exist locally, so this defines a global differential form ω_{FS} on $\mathbb{P}^n_{\mathbb{C}}$ of type $(1,1)$. In fact, ω_{FS} is positive ([Huy05], Example 3.1.9).

We now derive a local representation of ω_{FS}: choosing coordinates, let

$$\mathbb{P}^n_{\mathbb{C}} = \bigcup_{k=0}^{n} U_k$$

be the standard open covering with the sets $U_k = \{[x] = (x_0 : \ldots : x_n) \in \mathbb{P}^n_{\mathbb{C}} : x_k \neq 0\}$. Clearly, $U_k \cong \mathbb{C}^n$ by the map $\alpha_k : (x_0 : \ldots : x_n) \mapsto (\frac{x_0}{x_k}, \ldots, \widehat{\frac{x_k}{x_k}}, \ldots, \frac{x_n}{x_k})$, where $\widehat{\cdot}$ means that this coordinate is omitted.

We define the differential forms ω_k on U_k, which are local representations of ω_{FS}:

$$\omega_k = \frac{i}{2\pi}\partial\bar{\partial}\log\left(\sum_{l=0}^{n}\left|\frac{x_l}{x_k}\right|^2\right).$$

Under α_k, if z_l are the coordinates in $\mathbb{C}^n$, this corresponds to the form

$$\omega_k = \frac{i}{2\pi}\partial\bar{\partial}\log\left(1 + \sum_{l=1}^{n}|z_l|^2\right).$$

It is immediate that the ω_k glue together to a well-defined differential form on $\mathbb{P}^n_{\mathbb{C}}$, i.e. $\omega_k|_{U_k \cap U_j} = \omega_j|_{U_k \cap U_j}$: since $\partial\bar{\partial}\log|z|^2 = \partial\left(\frac{1}{z\bar{z}}\bar{\partial}(z\bar{z})\right) = \partial\left(\frac{1}{z\bar{z}}z\,d\bar{z}\right) = 0$,

$$\log\left(\sum_{l=0}^{n}\left|\frac{x_l}{x_k}\right|^2\right) = \log\left(\left|\frac{x_j}{x_k}\right|^2\right) + \log\left(\sum_{l=0}^{n}\left|\frac{x_l}{x_j}\right|^2\right).$$

Using this, we get the local representation of ω_{FS}:

$$\begin{aligned}
\omega_{FS} &= \frac{i}{2\pi}\partial\bar{\partial}\log(1 + \sum_j z_j\bar{z}_j) \\
&= \frac{i}{2\pi}\partial\left(\sum_j \frac{z_j}{1 + \sum|z_j|^2}d\bar{z}_j\right) \\
&= \frac{i}{2\pi}\left(\frac{\sum dz_j \wedge d\bar{z}_j}{1 + \sum|z_j|^2} - \frac{(\sum \bar{z}_j dz_j) \wedge (\sum z_j d\bar{z}_j)}{(1 + \sum|z_j|^2)^2}\right). \qquad (4.3)
\end{aligned}$$

(See [GH78] p. 30, [Voi02] p.76, and [Laz04] p.43.)

In fact, the differential form ω_{FS} is positive, and thus defines a Hermitian metric on $\mathbb{P}^n_{\mathbb{C}}$. It is called the *Fubini-Study metric*, see [GH78].

The Fubini-Study metric on $\overline{\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)}$

By the construction given in section 4.1.1 applied to $M = \mathbb{C}^{n+1}$ endowed with the standard metric, we define a Hermitian metric on $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)$, the *Fubini-Study metric on* $\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)$.

We give the norm explicitly:

Consider the point $[x] = (x_0 : \ldots : x_n) \in \mathbb{P}^n_{\mathbb{C}}$ represented by $x = (x_0, \ldots, x_n) \in \mathbb{C}^{n+1}\backslash\{0\}$, and a section $s \in \Gamma(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(1))$. In example 3.3.4 we saw that the global sections of $\mathcal{O}(1)$

are the homogeneous polynomials of degree one, i.e. $\Gamma(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(1)) = \mathbb{C}x_0 + \cdots + \mathbb{C}x_n$. By the construction of the metric on $\mathcal{O}(1)$, the norm h is determined by

$$h_{[x]}(s([x]), s([x])) = \frac{|s(x)|^2}{\|x\|^2}, \tag{4.4}$$

where $s(x)$ denotes the evaluation of the corresponding linear functional in $\mathbb{C}x_0 + \cdots + \mathbb{C}x_n$, see [Laz04], p. 43.

To derive a local representation of the metric, again take U_i to be the standard open sets $U_i = \{[x] \in \mathbb{P}^n_{\mathbb{C}} : x_i \neq 0\}$. Consider the commutative diagram

$$\begin{tikzcd} \varphi_i^*\mathcal{O}(1) \arrow[r] \arrow[d] & \mathcal{O}(1) \arrow[d] \\ U_i \arrow[r, "\varphi_i"'] \arrow[u, bend left, "\varphi_i^* s"] & \mathbb{P}^n_{\mathbb{C}} \arrow[u, bend right, "s"'] \end{tikzcd}$$

where φ_i denotes the inclusion. Since $U_i \cong \mathbb{C}^n$, and using the remark in example 3.1.5, we obtain that $\varphi_i^*\mathcal{O}(1)$ is trivial. Note that $\varphi_i^*\mathcal{O}(1) = \mathbb{C}^{n+1} \times U_i$.

Identifying $U_i \cong \mathbb{C}^n$, let $s_i : U_i \to \mathbb{C}^{n+1}, \quad s_i(z_1, \ldots, z_n) := (z_1, \ldots, z_{i-1}, 1, z_i, \ldots, z_n)$. Since $(\varphi_i^* s)(z) \in \mathbb{C}^{n+1} \times \{z\}$, we can regard $\varphi_i^* s : U_i \to \mathbb{C}^{n+1} \times U_i$ as a function $\varphi_i^* s : U_i \to \mathbb{C}^{n+1}$. Furthermore, setting $\varphi_i^* s := s_i$ defines a global section $s : \mathbb{P}^n_{\mathbb{C}} \to \mathcal{O}(1)$.

The dual of $\mathcal{O}(1)$, the tautological bundle, inherits the standard metric from $\mathbb{C}^{n+1}$ since it is a subbundle of the trivial bundle, so the norm $h^\vee$ on $\mathcal{O}(-1)$ is $h_z^\vee(s_i^\vee(z), s_i^\vee(z)) = 1 + \sum_j |z_j|^2$, where $s_i^\vee$ is the section dual to s_i. Therefore, since $\mathcal{O}(1)^\vee = \mathcal{O}(-1)$,

$$\forall i : \quad h_z(s_i(z), s_i(z)) = \frac{1}{1 + \sum_j |z_j|^2}.$$

Note that this is independent of the choice of i. (See also e.g. [Voi02], p. 76, chapter 3.2.2.)

One obtains the same result by using the section $s = x_0$ (under the identification $\Gamma(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(1)) = \mathbb{C}x_0 + \cdots + \mathbb{C}x_n$) in (4.4) and local coordinates on U_i.

4.2 Hermitian vector bundles on arithmetic varieties

Now we can turn to arithmetic and define the arithmetic analogon of complex varieties and Hermitian vector bundles. Throughout this section, let K be a number field of degree $[K : \mathbb{Q}]$, $\mathcal{O}_K$ its ring of integers and $S = \operatorname{Spec} \mathcal{O}_K$ the associated scheme.

Let X be an S-scheme. We denote by $X_{\mathbb{C}}$ the scheme $X_{\mathbb{C}} = \coprod_{\sigma : K \hookrightarrow \mathbb{C}} X_\sigma$, where $X_\sigma = X \times_{S,\sigma} \operatorname{Spec} \mathbb{C}$ is the fiber product of X and $\operatorname{Spec} \mathbb{C}$ over S using the map $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} K$ which is induced by an embedding $\sigma : K \hookrightarrow \mathbb{C}$. Denote this map, by abuse of notation, also by $\sigma : \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} K$.

Example 4.2.1. Let $X = S = \operatorname{Spec} \mathcal{O}_K$. We show that $S_{\mathbb{C}} = \operatorname{Spec} K \otimes_{\mathbb{Q}} \mathbb{C}$, see [Via01], p. 73. Clearly $S_\sigma = \operatorname{Spec} \mathbb{C}$, so $S_{\mathbb{C}} = \coprod_{\sigma: K \hookrightarrow \mathbb{C}} \operatorname{Spec} \mathbb{C}$. Since K is an algebraic number field, there is an $f(x) \in \mathbb{Q}[x]$ such that $K \cong \mathbb{Q}[x]/(f(x))$. Tensorizing with $\mathbb{C}$, we get that $K \otimes_{\mathbb{Q}} \mathbb{C} \cong (\mathbb{Q}[x]/(f(x))) \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}[x]/(f(x))$. The polynomial $f(x)$ splits over $\mathbb{C}$ and its roots correspond to the embeddings of K into $\mathbb{C}$. Therefore $f(x)$ is of the form $f(x) = \prod_{\sigma: K \hookrightarrow \mathbb{C}} (x - \alpha_\sigma)$. We deduce that

$$K \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}[x]/(f(x)) \cong \prod_{\sigma: K \hookrightarrow \mathbb{C}} \mathbb{C}[x]/(x - \alpha_\sigma) \cong \prod_{\sigma: K \hookrightarrow \mathbb{C}} \mathbb{C}.$$

Thus, $S_{\mathbb{C}} = \coprod_{\sigma: K \hookrightarrow \mathbb{C}} \operatorname{Spec} \mathbb{C} = \operatorname{Spec} (K \otimes_{\mathbb{Q}} \mathbb{C})$.

Definition 4.2.2. The set of *complex points of the scheme* X is $X(\mathbb{C}) := X_{\mathbb{C}}(\mathbb{C}) = \coprod_{\sigma: K \hookrightarrow \mathbb{C}} X_\sigma(\mathbb{C})$. Note that X_σ is a complex scheme, so this is well-defined. Moreover, the complex conjugation on the coordinates of complex points in X induces an antiholomorphic involution $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$.

In the following, we use the definitions[1] of [BGS94]:

Definition 4.2.3. An *arithmetic variety* X is a scheme over S, i.e. there is a map $\pi : X \to S$, such that π is flat and quasiprojective, and whose generic fiber $X_K = X \times_S \operatorname{Spec} K$ is regular.

Example 4.2.4. ([Sou92], p. 3, chapter 0.2.2) Let $f_1, \ldots, f_k \in \mathbb{Z}[x_0, \ldots, x_n]$ be homogeneous polynomials with integer coefficients. Consider the system of polynomial equations

$$f_1(x_0, \ldots, x_n) = \cdots = f_k(x_0, \ldots, x_n) = 0.$$

Similar to the construction in remark 2.2.6, we define the projective scheme

$$X = \operatorname{Proj} \mathbb{Z}[x_0, \ldots, x_n]/(f_1, \ldots, f_k).$$

Then, under certain conditions on $f_1, \ldots, f_k$, e.g. $\mathbb{Z}[x_0, \ldots, x_n]/(f_1, \ldots, f_k)$ is torsion-free, this is an arithmetic variety.

For more examples of spectra of rings finitely generated over $\mathbb{Z}$ and their properties, see [EH00].

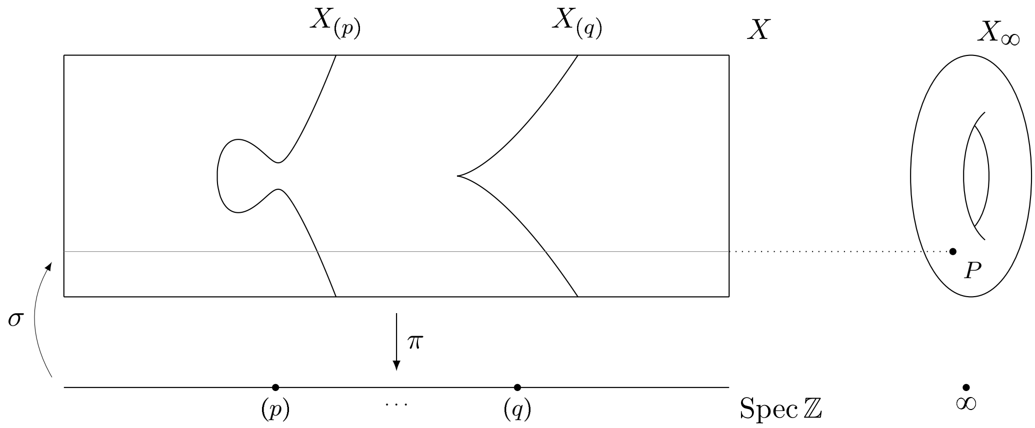
Definition 4.2.5. A *Hermitian vector bundle* $\bar{E}$ on an arithmetic variety X over S is a pair (E, h), where E is a locally free sheaf of finite rank on X and h is a C^∞ Hermitian scalar product on the holomorphic vector bundle $E(\mathbb{C})$ on $X(\mathbb{C})$ (see remark 3.1.8) which is invariant under F_∞.

Remark 4.2.6. A Hermitian vector bundle $\bar{E}$ on S is the same as a finitely generated projective $\mathcal{O}_K$-module E together with Hermitian scalar products on the $[K : \mathbb{C}]$ complex vector spaces $E_\sigma = E \otimes_{\mathcal{O}_K, \sigma} \mathbb{C}$ associated to the embeddings $\sigma : K \hookrightarrow \mathbb{C}$, which are invariant under F_∞.

[1] The notion of an arithmetic variety is not consistent in the literature. In e.g. [Sou92], an arithmetic variety is a regular scheme which is projective and flat over $\mathbb{Z}$.

Definition 4.2.7. Since the dual, the tensor product, the direct sum, the pullback, and the exterior product of locally free sheaves is locally free ([Har77], Propositions II.5.5, II.5.7, II.5.8, and Example II.5.16), we can apply the constructions of the previous section to the setting on an arithmetic variety and obtain the *dual*, the *tensor product*, the *direct sum*, the *pullback*, and the *m-th exterior product* of vector bundles on arithmetic varieties.

One should keep the following picture in mind:



We "add" the complex variety $X_\infty := X(\mathbb{C})$ to an arithmetic variety X. Points on X are sections of π and determine a point on X_∞ (definition 2.2.11). Thus, to control the "size" of points on an arithmetic variety as suggested in the introduction, we need to introduce metrics. In Arakelov geometry, given an algebraic vector bundle $E \to X$, this is done by endowing the vector bundle $E_\infty \to X_\infty$ with a Hermitian metric.

Example 4.2.8. Let $R = \mathcal{O}_K$ and M be a finitely generated projective $\mathcal{O}_K$-module. Let $M_\mathbb{C} = M \otimes_\mathbb{Z} \mathbb{C}$. Given a Hermitian metric on $M_\mathbb{C}$, we construct a metric on the twisting sheaf $\mathcal{O}(1)$ of the projective space $\mathbb{P}(M)$ associated to M. First of all, we see that the metric on $M_\mathbb{C}$ induces a Hermitian metric on the trivial bundle $M \times \mathbb{P}(M) \to \mathbb{P}(M)$: every fiber $(M \times \mathbb{P}(M))_P = M$, and so we have a Hermitian metric on $(M \times \mathbb{P}(M))_P \otimes_\sigma \mathbb{C}$. Then we apply the constructions of metrics from the previous sections and deduce a metric on $\mathcal{O}(1)$.

Example 4.2.9. The Fubini-Study metric induces a metric on the tautological bundle $\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)$ of $\mathbb{P}^n_{\mathcal{O}_K}$, i.e. on every fiber at infinity.

The Hermitian line bundles on a fixed arithmetic variety with the tensor product form a group, the arithmetic analogon of the Picard group of invertible sheaves:

Definition 4.2.10. The set of Hermitian line bundles on a fixed arithmetic variety forms a group under the operation of the tensor product of Hermitian line bundles with neutral element the trivial bundle $\mathcal{O}_X$ endowed with the metric induced by $\|1\|_\sigma = 1$ and inverse element the dual bundle. This group is called the *arithmetic Picard group of X* and is denoted by $\widehat{\operatorname{Pic}}(X)$.

Remark 4.2.11. Note that this is an extension of the classical Picard group on an arithmetic variety. The underlying vector bundles on the arithmetic variety X form the Picard group of X.

Chapter 5

The Arakelov degree

This chapter deals with the concept of the *Arakelov degree* of a Hermitian vector bundle over an arithmetic variety. The degree is an important notion in Arakelov geometry. We first consider Hermitian vector bundles over affine schemes and then extend this concept to arbitrary schemes. As an example, we compute the degree of the twisting sheaf over $\mathbb{P}^n_{\mathcal{O}_K}$, the projective n-space over a ring of integers of a number field K, endowed with the Fubini-Study metric.

5.1 The Arakelov degree of a Hermitian vector bundle over $\operatorname{Spec}\mathcal{O}_K$

Definition 5.1.1. Let $\bar{L} = (L, h)$ be a Hermitian line bundle over S, i.e. a Hermitian vector bundle of rank 1. Then the *Arakelov degree of* $\bar{L}$ is defined as

$$\widehat{\deg}\ \bar{L} := \log \#(L/s\mathcal{O}_K) - \sum_{\sigma: K \hookrightarrow \mathbb{C}} \log \|s\|_\sigma,$$

where s is any non-zero global section of L. Let $\bar{E} = (E, h)$ be a Hermitian vector bundle of rank r over S, then the *Arakelov degree of* $\bar{E}$ is defined to be

$$\widehat{\deg}\ \bar{E} := \widehat{\deg}\,(\det \bar{E}),$$

which can be expressed as

$$\widehat{\deg}\ \bar{E} = \log \#\left(E/(s_1\mathcal{O}_K + \cdots + s_r\mathcal{O}_K)\right) - \frac{1}{2}\sum_{\sigma: K \hookrightarrow \mathbb{C}} \log\det(\langle s_i, s_j\rangle_\sigma), \tag{5.1}$$

where $(s_1, \ldots, s_r) \in E^r$ is a base of E_K over K.

Remark 5.1.2. The independence of this definition of the choice of s, respectively $s_1, \ldots, s_r$, essentially follows from the product formula (theorem A.2.14) and the following lemma (e.g. [Via05]).

Lemma 5.1.3. *Let $\bar{L}$ be a Hermitian line bundle. Then*

$$\widehat{\deg}\ \bar{L} = -\sum_{\mathfrak{p}\nmid\infty} \log \|s\|_{\mathfrak{p}} - \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log \|s\|_{\sigma},$$

where $\mathfrak{p}$ runs over the prime ideals of $\mathcal{O}_K$ and $\|\cdot\|_{\mathfrak{p}}$ is the norm corresponding to the non-archimedean valuation $v_{\mathfrak{p}}$.

Proof. We need to prove that

$$\log \#(L/s\mathcal{O}_K) = -\sum_{\mathfrak{p}\nmid\infty} \log \|s\|_{\mathfrak{p}}.$$

First of all,

$$L/s\mathcal{O}_K \cong \prod_{\mathfrak{p}} (L/s\mathcal{O}_K)_{\mathfrak{p}} \cong \prod_{\mathfrak{p}} L_{\mathfrak{p}}/s\mathcal{O}_{K\mathfrak{p}}.$$

Then, since L has rank one, there exists an isomorphism $i_{\mathfrak{p}} : L_{\mathfrak{p}} \to \mathcal{O}_{K\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$. Therefore,

$$L_{\mathfrak{p}}/s\mathcal{O}_{K\mathfrak{p}} \cong \mathcal{O}_{K\mathfrak{p}}/i_{\mathfrak{p}}(s)\mathcal{O}_{K\mathfrak{p}} \cong (\mathcal{O}_K/(i_{\mathfrak{p}}(s)))^{\operatorname{ord}_{\mathfrak{p}}(i_{\mathfrak{p}}(s))}.$$

Since $\operatorname{ord}_{\mathfrak{p}}(i_{\mathfrak{p}}(s)) = \operatorname{ord}_{\mathfrak{p}}(s) = v_{\mathfrak{p}}(s)$, we get that

$$\#(L/s\mathcal{O}_K) = \prod_{\mathfrak{p}} \mathfrak{N}\mathfrak{p}^{v_{\mathfrak{p}}(s)}.$$

By definition, $\|s\|_{\mathfrak{p}} = \mathfrak{N}\mathfrak{p}^{-v_{\mathfrak{p}}(s)}$, see definitions A.1.5 and A.2.12. □

In the next propositions we follow [Via05]:

Proposition 5.1.4. *Let $\bar{E}$ and $\bar{F}$ be Hermitian vector bundles and $\bar{L}$ a Hermitian line bundle over S. Furthermore, $\mathcal{O}_K$ is endowed with the norms $\|1\|_{\sigma} = 1$, $\forall \sigma : K \hookrightarrow \mathbb{C}$. Then,*

(i) $\widehat{\deg}\,(\bar{E}\otimes\bar{F}) = m\cdot\widehat{\deg}\ \bar{E} + n\cdot\widehat{\deg}\ \bar{F}$, *for* $n = \operatorname{rk}\bar{E}$ *and* $m = \operatorname{rk}\bar{F}$

(ii) $\widehat{\deg}\,(\bar{E}\oplus\bar{F}) = \widehat{\deg}\ \bar{E} + \widehat{\deg}\ \bar{F}$

(iii) $\widehat{\deg}\ \bar{L}^{\vee} = -\widehat{\deg}\,\bar{L}$, *for $\bar{L}^{\vee}$ the dual of $\bar{L}$.*

Proof. See [Bos99], [Via05].

(i) Let s, t be non-zero global sections of $\bar{E}, \bar{F}$, respectively. Then $s\otimes t$ is a non-zero global section of $\bar{E}\otimes\bar{F}$.

Since

$$\bigwedge^{nm}(\bar{E}\otimes\bar{F}) \cong (\bigwedge^{n}\bar{E})^{\otimes m}\otimes(\bigwedge^{m}\bar{F})^{\otimes n},$$

(see e.g. [Neu99], p. 233, Exercise III.4.3), we can reduce the proof to the case in which $\bar{E}, \bar{F}$ are line bundles.

The induced metric on the tensor product (4.1.1) is

$$\|s \otimes t\|_\sigma = \|s\|_\sigma \cdot \|t\|_\sigma,$$

so

$$\log \|s \otimes t\|_\sigma = \log \|s\|_\sigma + \log \|t\|_\sigma.$$

Lemma 5.1.3 and the definition of the norm, $\|s\|_\mathfrak{p} = \mathfrak{N}\mathfrak{p}^{-v_\mathfrak{p}(s)}$, reduce the claim to

$$v_\mathfrak{p}(s \otimes t) = v_\mathfrak{p}(s) + v_\mathfrak{p}(t).$$

By the construction of the tensor product of two line bundles, a local trivialization of $E \otimes F$ is given by the product of the trivializations of E and F. So, if $i_\mathfrak{p} : E_\mathfrak{p} \to \mathcal{O}_{K\mathfrak{p}}$ is a trivialization of E in $\mathfrak{p}$ and $j_\mathfrak{p} : F_\mathfrak{p} \to \mathcal{O}_{K\mathfrak{p}}$ is a trivialization of F in $\mathfrak{p}$, then the trivialization of $E \otimes F$ in $\mathfrak{p}$ is $k_\mathfrak{p}(s \otimes t) = i_\mathfrak{p}(s) \cdot j_\mathfrak{p}(t)$. This yields the desired result.

(ii) From [Bou70], Proposition A.III §7.10, we know that

$$\bigwedge(E \oplus F) \cong \bigwedge E \otimes \bigwedge F,$$

and therefore,

$$\bigwedge^{n+m}(E \oplus F) \cong \left(\bigoplus_{i=1}^{n+m} \bigwedge^{i} E\right) \otimes \left(\bigoplus_{j=1}^{n+m} \bigwedge^{n+m-j} F\right) = \bigwedge^{n} E \otimes \bigwedge^{m} F. \tag{5.2}$$

So,

$$\widehat{\deg}\,(\bar{E} \oplus \bar{F}) = \widehat{\deg}\,(\bigwedge^{n} \bar{E} \otimes \bigwedge^{m} \bar{F}) \overset{\text{(i)}}{=} \widehat{\deg}\ \bar{E} + \widehat{\deg}\ \bar{F}.$$

(iii) This follows from (i) since the isomorphism $\bar{L} \otimes \overline{L^\vee} \cong \mathcal{O}_K$ is an isomorphism by the choice of the norm on $\mathcal{O}_K$.

□

Definition 5.1.5. A submodule F of an $\mathcal{O}_K$-module E is said to be *saturated* if $F = (F \otimes_{\mathcal{O}_K} K) \cap E$. Otherwise, we define its *saturation* $F_s = (F \otimes_{\mathcal{O}_K} K) \cap E$.

Remark 5.1.6. Let E be a finitely generated projective module over $\mathcal{O}_K$ and F a submodule. Then $F \otimes_{\mathcal{O}_K} K$ is torsion-free and thus its saturation is torsion-free. Moreover, it is finitely generated, and therefore it even is projective ([Neu99], Proposition III.4.3). Thus, if $\bar{E}$ is a Hermitian vector bundle over Spec $\mathcal{O}_K$ and F is a subbundle of $\bar{E}$, F_s also is a subbundle of $\bar{E}$ and inherits a metric. Furthermore, for $r = \operatorname{rk} F_s = \operatorname{rk} F$, since $\bigwedge^r F \subseteq \bigwedge^r F$, we get the inequality $\widehat{\deg}\,\bar{F} \leqslant \widehat{\deg}\,\bar{F}_s$.

Proposition 5.1.7. *Let E be a finitely generated projective module over a ring of integers $\mathcal{O}_K$ of a number field K and let F be a saturated submodule of E. Then F and E/F are torsion-free, and the exact sequence*

$$0 \longrightarrow F \longrightarrow E \xrightarrow{p} E/F \longrightarrow 0,$$

splits.

Proof. By remark 5.1.6, F is torsion-free and moreover projective. Assume that E/F has non-trivial torsion T. Then $F \subset p^{-1}(T)$ and the following sequence is exact:

$$0 \longrightarrow F \longrightarrow p^{-1}(T) \xrightarrow{p} T \longrightarrow 0.$$

Since K is a flat $\mathcal{O}_K$-module ([Har77], Example III.9.1.1) and T is torsion, it follows that $T \otimes_{\mathcal{O}_K} K = 0$. Therefore, we obtain

$$F \otimes_{\mathcal{O}_K} K = p^{-1}(T) \otimes_{\mathcal{O}_K} K.$$

However, F is saturated, so

$$p^{-1}(T) \supset F = (p^{-1}(T) \otimes_{\mathcal{O}_K} K) \cap E,$$

which is a contradiction. Recall that over $\mathcal{O}_K$, torsion-free implies projective. Thus, the exact sequence splits. □

Proposition 5.1.8. *Let $\bar{E}$ be a Hermitian vector bundle on* $\operatorname{Spec} \mathcal{O}_K$ *and F a saturated submodule of $\bar{E}$. Endow F and E/F with the metrics induced by $\bar{E}$. Then,*

$$\widehat{\deg}\, \bar{E} = \widehat{\deg}\, \bar{F} + \widehat{\deg}\, \overline{E/F}.$$

Proof. From the previous proposition, we know that

$$E \cong F \oplus (E/F).$$

Let $m = \operatorname{rk} F$ and $n = \operatorname{rk}(E/F)$. By (5.2), we get an isomorphism

$$\bigwedge^{m} F \otimes \bigwedge^{n} E/F \cong \bigwedge^{m+n} E,$$

given by

$$\begin{array}{ccc} \bigwedge^m F \otimes \bigwedge^n E/F & \longrightarrow & \bigwedge^{m+n} E \\ (f_1 \wedge \ldots \wedge f_m) \otimes (\overline{e_1} \wedge \ldots \wedge \overline{e_n}) & \longmapsto & (f_1 \wedge \ldots \wedge f_m \wedge e_1 \wedge \ldots \wedge e_n). \end{array}$$

By the definitions of the metrics induced on F and E/F, this isomorphism is an isometry. Furthermore, it is canonical, because it does not depend on the choice of representatives of $\overline{e_1}, \ldots, \overline{e_n}$. □

5.2 The Arakelov degree of a Hermitian vector bundle over an arithmetic variety

Let $\bar{E} = (E, h)$ be a Hermitian vector bundle on an arithmetic variety X over $S = \operatorname{Spec} \mathcal{O}_K$.

We first define a metric on the pushforward of the vector bundle $\bar{E}$ under the map $\pi : X \to S$. We will then define the Arakelov degree of $\bar{E}$ as the Arakelov degree of the pushforward.

Moret-Bailly showed in [MB85] that the pushforward $\pi_* E$ of a locally free sheaf on an arithmetic variety to $\operatorname{Spec} \mathcal{O}_K$ also is locally free. The vector bundle $\pi_* E$ on S is given as the projective module $\Gamma(S, \pi_* E) = \Gamma(X, E)$. We now induce a metric on $\pi_* E$. Let $s \in \Gamma(S, \pi_* E) = \Gamma(X, E)$ and define

$$\|s\|_\sigma^2 = \int_{X_\sigma(\mathbb{C})} \|s_x\|_{E_\sigma}^2 \, d\mu_\sigma(x),$$

where $d\mu_\sigma$ is a measure on $X_\sigma(\mathbb{C})$ and $\|\cdot\|_{E_\sigma}^2 = h_{\sigma,x}(\cdot,\cdot)$. In the case of a projective space X we use the Fubini-Study metric as the measure $d\mu_\sigma$.

Definition 5.2.1. The *Arakelov degree of a Hermitian vector bundle $\bar{E}$ over an arithmetic variety X* is

$$\widehat{\deg}\ \bar{E} := \widehat{\deg}\,(\pi_* \bar{E}),$$

where $\pi : X \to \operatorname{Spec} \mathcal{O}_K$.

5.3 An example: the Arakelov degree of $\overline{\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)}$

Consider $\mathbb{P}^n_{\mathcal{O}_K} = \operatorname{Proj} \mathcal{O}_K[x_0, \dots, x_n]$ from example 2.2.17 and the metric on $\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)$ which is induced by the Fubini-Study metric from section 4.1.2. In this section, we compute the Arakelov degree of $\overline{\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)}$. For simpler notation, we omit the index $\mathbb{P}^n_{\mathcal{O}_K}$ and simply write $\mathcal{O}(1)$ for $\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)$.

Theorem 5.3.1. *The Arakelov degree of $\overline{\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)}$ is*

$$\widehat{\deg}\ \overline{\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)} = \frac{[K : \mathbb{Q}]}{2} \cdot (n+1) \log(n+1).$$

Proof. Recall that the global sections of $\mathcal{O}(1)$ are the homogeneous polynomials of degree one, i.e. $\Gamma(\mathbb{P}^n_{\mathcal{O}_K}, \mathcal{O}(1)) = \mathcal{O}_K x_0 + \cdots + \mathcal{O}_K x_n$ (see e.g. [Wel07], p. 22, Example 2.13).

To compute the Arakelov degree of $\overline{\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)}$, by the definition above we need to choose a global section $s \in \mathcal{O}_K x_0 + \cdots + \mathcal{O}_K x_n$. Then,

$$\|s\|_{\mathcal{O}(1),\sigma}^2 = \int_{\mathbb{P}^n_{\mathbb{C},\sigma}} \|s(p)\|_{FS}^2 \, \omega(p), \tag{5.3}$$

where $\|\cdot\|_{FS}$ is the Fubini-Study metric and ω is the differential form associated to the Fubini-Study metric (section 4.1.2).

Recall that the matrix representation of the Fubini-Study metric is given by $h_z = \frac{1}{1+\sum_i |z_i|^2}$ on $\mathcal{O}(1)_z \otimes \mathcal{O}(1)_z^c$.

First, we compute the first part of formula (5.1) in the definition of the Arakelov degree (section 5.1). Choosing $s_i = x_i$, we get

$$\begin{aligned}
\log \# \big(\Gamma\big(\mathbb{P}^n_{\mathcal{O}_K}, \mathcal{O}(1)\big) / (s_1\mathcal{O}_K + \cdots + s_r\mathcal{O}_K)\big) = \\
&= \log \# \left((x_0\mathcal{O}_K + \cdots x_n\mathcal{O}_K)/(x_0\mathcal{O}_K + \cdots + x_n\mathcal{O}_K)\right) \\
&= 0.
\end{aligned}$$

For the second part of formula (5.1) in the definition of the Arakelov degree, we need to construct the matrix $\langle s_i, s_j\rangle_\sigma$ to compute its determinant. The computation is similar for every σ, so we omit the index in the following.

Since the standard open set $U_j = \{p \in \mathbb{P}^n_{\mathbb{C},\sigma} : x_j \neq 0\}$ is dense in $\mathbb{P}^n_{\mathbb{C},\sigma}$, we can calculate the integral in (5.3) as follows:

$$\begin{aligned}
\|x_j\|^2_{\mathcal{O}(1)} &= \int_{U_j} \|x_j\|^2_{FS}\, \omega \\
&= \int_{\mathbb{C}} \frac{1}{1+\sum_j |z_j|^2}\, \omega,
\end{aligned}$$

and, using the explicit formula for the differential form ω in (4.3) from section 4.1.2, we get

$$\|x_j\|^2_{\mathcal{O}(1)} = \frac{i}{2\pi} \int_{\mathbb{C}} \frac{1}{1+\sum |z_k|^2} \left(\frac{\sum dz_k \wedge d\bar{z}_k}{1+\sum |z_k|^2} - \frac{(\sum \bar{z}_k dz_k) \wedge (\sum z_k d\bar{z}_k)}{(1+\sum |z_k|^2)^2} \right). \quad (5.4)$$

Similarly, one can compute the inner product of two sections:

$$\langle x_i, x_j\rangle_{\mathcal{O}(1),\sigma} = \langle x_i, x_j\rangle_{\mathcal{O}(1)} = \int_{\mathbb{P}^n_{\mathbb{C}}} \mathrm{Re}\left(\langle x_i(p), x_j(p)\rangle_{FS}\right) \omega(p). \quad (5.5)$$

We compute the norms and the inner products in the following lemma and using this, we obtain the desired result,

$$\widehat{\deg}\ \overline{\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)} = -\frac{1}{2}[K:\mathbb{Q}] \cdot \log \left(\frac{1}{n+1}\right)^{n+1} = \frac{[K:\mathbb{Q}]}{2} \cdot (n+1)\log(n+1).$$

□

Lemma 5.3.2. *Consider the sections x_i of $\mathcal{O}(1)$. Then, for every $\sigma : K \hookrightarrow$,*

$$\begin{aligned} \forall j: \quad \|x_j\|^2_{\mathcal{O}(1),\sigma} &= \frac{1}{n+1}, \\ \forall i \neq j: \quad \langle x_i, x_j \rangle_{\mathcal{O}(1),\sigma} &= 0. \end{aligned}$$

Proof. We calculate the integrals in (5.4) and (5.5) using the rotation invariant measure on $\mathbb{S}^{2n+1}$.

First, we calculate

$$\|x_0\|^2_{\mathcal{O}(1),\sigma} = \int_{\mathbb{P}^n_{\mathbb{C}}} \|x_0\|^2_{FS}\, \omega^n.$$

By the definition of the Fubini-Study metric,

$$\int_{\mathbb{P}^n_{\mathbb{C}}} \|x_0\|^2_{FS}\, \omega^n = \int_{\mathbb{S}^{2n+1}} |x_0|^2\, dv, \tag{5.6}$$

where dv is the unique $U(n+1)$-invariant probability measure on the unit sphere $\mathbb{S}^{2n+1}$ in $\mathbb{C}^{n+1}$, see [Laz04], p. 42.

We compute the integral on the right hand side as follows: let $u = (u_1, \ldots, u_m) \in R^m$ and let $|u| = r$ and $\varphi_1, \ldots \varphi_{m-1}$ be the angles defining u. Then the polar coordinates of $u = p(r, \varphi_1, \ldots, \varphi_{m-1})$ are given by

$$p(r, \varphi_1, \ldots, \varphi_{m-1}) = \begin{pmatrix} r\cos\varphi_1 \\ r\sin\varphi_1\cos\varphi_2 \\ \vdots \\ r\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{m-2}\cos\varphi_{m-1} \\ r\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{m-2}\sin\varphi_{m-1} \end{pmatrix}.$$

Moreover, the Jacobian of this transformation is

$$J_p(r, \varphi_1, \ldots, \varphi_{m-1}) = r^{m-1}\sin^{m-2}\varphi_1\sin^{m-3}\varphi_2\cdots\sin^2\varphi_{m-3}\sin\varphi_{m-2}.$$

Therefore the rotation invariant measure σ_m on $S^{m-1} \subset \mathbb{R}^m$ is given by

$$\begin{aligned} &\int_{\mathbb{S}^{m-1}} g(u)\, d\sigma_m(u) = \\ &\qquad = \int_{[0,\pi)^{m-2}\times[0,2\pi)} g(p(1, \varphi_1, \ldots, \varphi_{m-1}))J_p(1, \varphi_1, \ldots, \varphi_{m-1})\, d(\varphi_1, \ldots, \varphi_{m-1}). \end{aligned}$$

This measure is uniquely defined up to a constant. We normalize it, i.e. we consider the unique probability measure. Recall that the volume of the unit sphere is

$$V_m = \sigma_m(\mathbb{S}^{m-1}) = \frac{m\pi^{m/2}}{\Gamma(\frac{m}{2}+1)}.$$

Therefore, since $U(n+1) \subset SO(2(n+1))$ and the $U(n+1)$-invariant probability measure is unique, the measure $v = \frac{\sigma_m}{V_m}$, where $m-1 = 2n+1$, i.e. $m = 2(n+1)$, is the measure we are looking for. Thus,

$$
\begin{aligned}
&\int_{\mathbb{S}^{m-1}} |x_0|^2 \, dv = \\
&\quad = \frac{1}{V_m} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \left((\cos \varphi_1)^2 + (\sin \varphi_1 \cos \varphi_2)^2 \right) \cdot \\
&\qquad\qquad\qquad\qquad \cdot \sin^{m-2} \varphi_1 \, \sin^{m-3} \varphi_2 \cdots \sin \varphi_{m-2} \, d\varphi_1 \ldots d\varphi_{m-1} \\
&\quad = \frac{1}{V_m} \int_0^{\pi} \cos^2 \varphi_1 \sin^{m-2} \varphi_1 \, d\varphi_1 \int_0^{\pi} \sin^{m-3} \varphi_2 \, d\varphi_2 \cdots \int_0^{\pi} \sin \varphi_{m-2} \, d\varphi_{m-2} \underbrace{\int_0^{2\pi} 1 \, d\varphi_{m-1}}_{=2\pi} + \\
&\qquad + \frac{1}{V_m} \int_0^{\pi} \sin^m \varphi_1 \, d\varphi_1 \int_0^{\pi} \cos^2 \varphi_2 \sin^{m-3} \varphi_2 \, d\varphi_2 \int_0^{\pi} \sin^{m-3} \varphi_3 \, d\varphi_3 \cdots \underbrace{\int_0^{2\pi} 1 \, d\varphi_{m-1}}_{=2\pi} \\
&\quad =: I(m) + J(m),
\end{aligned}
$$

where $I(m)$ denotes the first summand and $J(m)$ the second one.

Claim:

$$I(m) = J(m) = \frac{1}{2} \cdot \frac{1}{n+1}.$$

We start by computing the inner integrals. Set $I_k := \int_0^{\pi} \sin^k \varphi \, d\varphi$.
By partial integration,

$$I_k = (k-1) \int_0^{\pi} \cos^2 \varphi \sin^{k-2} \varphi \, d\varphi, \tag{5.7}$$

and, using $\cos^2 \varphi = 1 - \sin^2 \varphi$, we get

$$I_k = \frac{k-1}{k} I_{k-2}, \tag{5.8}$$

and therefore

$$I_k = \begin{cases} \frac{(k-1)(k-3)\cdots 1}{k(k-2)\cdots 2} \pi, & k \text{ even} \\ \frac{(k-1)(k-3)\cdots 2}{k(k-2)\cdots 3 \cdot 1} 2, & k \text{ odd.} \end{cases}$$

For $k \geqslant 2$, we obtain

$$I_k \cdot I_{k-1} = \frac{2\pi}{k}. \tag{5.9}$$

Since $m = 2(n+1)$, m is even. Using (5.7),

$$I(m) = \frac{2\pi}{V_m} \cdot \frac{1}{m-1} I_m \cdot (I_{m-3} \cdot I_{m-4}) \cdots (I_3 \cdot I_2) \cdot I_1.$$

and

$$J(m) = \frac{2\pi}{V_m} I_m \cdot \frac{1}{m-2} I_{m-1} \cdot (I_{m-4} \cdot I_{m-5}) \cdots (I_2 \cdot I_1).$$

By (5.8), $\frac{1}{m-2} I_{m-1} = \frac{1}{m-1} I_{m-3}$, and comparing the two lines, we see that

$$I(m) = J(m).$$

We now compute $I(m) = J(m)$. By (5.9) and using the fact that m is even, we get

$$\begin{aligned}
I(m) &= \frac{2\pi}{(m-1)\, V_m} I_m \cdot \frac{2\pi}{m-3} \cdot \frac{2\pi}{m-5} \cdots \frac{2\pi}{3} \cdot 2 \\
&= \frac{4\pi}{V_m} I_m \frac{(2\pi)^{\frac{m-4}{2}}}{(m-1)(m-3)\cdots 3} \\
&= \frac{4\pi}{V_m} \frac{(m-1)(m-3)\cdots 1}{m(m-2)\cdots 2} \pi \cdot \frac{(2\pi)^{\frac{m-4}{2}}}{(m-1)(m-3)\cdots 3} \\
&= \frac{1}{V_m} \cdot \frac{(2\pi)^{\frac{m}{2}}}{m(m-2)\cdots 2}.
\end{aligned}$$

Recall that $m = 2(n+1)$, and since $V_{2(n+1)} = \frac{2\pi^{n+1}}{n!}$,

$$I(m) = \frac{n!}{2\pi^{n+1}} \cdot \frac{(2\pi)^{n+1}}{2^{n+1}(n+1)!} = \frac{1}{2} \cdot \frac{1}{n+1}.$$

Therefore we obtain the first assertion,

$$\|x_0\|^2_{\mathcal{O}(1)} = I(m) + J(m) = \frac{1}{n+1}.$$

The norms of the other x_i are computed similarly, and, using the same notation, we get

$$\|x_i\|^2_{\mathcal{O}(1)} = \frac{1}{V_m} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \left(u_i^2 + u_{i+1}^2\right) \sin^{m-2}\varphi_1 \, \sin^{m-3}\varphi_2 \cdots \sin\varphi_{m-2} \, d\varphi_1 \ldots d\varphi_{m-1},$$

and the calculation of this integral also results in

$$\|x_i\|^2_{\mathcal{O}(1)} = \frac{1}{n+1}.$$

By another similar computation, since $\forall k : \int_0^{\pi} \cos\varphi \sin^k\varphi \, d\varphi = 0$, we get

$$\langle x_i, x_j \rangle_{\mathcal{O}(1)} = \int_{\mathbb{S}^{m-1}} \mathrm{Re}\langle x_i, x_j \rangle \, dv = 0, \quad \forall i \neq j.$$

□

Example 5.3.3. The special case of $n = 1$ yields $m = 4$ and $\omega_4 = 2\pi^2$. Therefore,

$$\|x_i\|^2_{\mathcal{O}(1)} = \int_{\mathbb{P}^1_{\mathbb{C}}} \|x_i\|^2_{FS}\, \omega = \frac{1}{2},$$

and

$$\widehat{\deg}\left(\overline{\mathcal{O}_{\mathbb{P}^1_{\mathcal{O}_K}}(1)}\right) = \log 2 \cdot [K : \mathbb{Q}].$$

In the following example, we calculate the degree of $\overline{\mathcal{O}_{\mathbb{P}^1_{\mathcal{O}_K}}(1)}$ by hand with the differential form ω without using the rotation invariant measure on $\mathbb{S}^{2n+1}$. Note that the result is the same as in the previous example.

Example 5.3.4. For $n = 1$ we can also explicitely calculate (5.4) and (5.5): expression (4.3) from section 4.1.2 for ω simplifies to

$$\omega = \frac{i}{2\pi} \frac{1}{(1 + |z|^2)^2} dz \wedge d\bar{z}.$$

Passing to polar coordinates $(\rho, \vartheta), dz \wedge d\bar{z} = -2\rho i\, d\rho \wedge d\vartheta$, we get

$$\begin{aligned}
\|x_0\|^2_{\mathcal{O}(1),\sigma} &= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{1}{(1 + |z|^2)^3}\, dz \wedge d\bar{z} \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} \frac{2\rho}{(1 + \rho^2)^3}\, d\rho \wedge d\vartheta \\
&= \int_1^{\infty} \frac{1}{u^3}\, du \\
&= \frac{1}{2}.
\end{aligned}$$

Similarly,

$$\|x_1\|^2_{\mathcal{O}(1),\sigma} = \frac{1}{2}.$$

Using the same coordinate change,

$$\begin{aligned}
\langle x_0, x_1 \rangle_{\mathcal{O}(1),\sigma} &= \int_{U_0 \cap U_1} \frac{\mathrm{Re}\left(x_0(p) \cdot \overline{x_1(p)}\right)}{\sum |x_j(p)|^2}\, \omega \\
&= \frac{i}{2\pi} \int_{\mathbb{C} \setminus \{0\}} \frac{\mathrm{Re}\,(\bar{z})}{(1 + |z|^2)^3}\, dz \wedge d\bar{z} \\
&= -\frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} \frac{2\rho^2}{(1 + \rho^2)^3} \mathrm{Re}\,(\cos\vartheta - i \sin\vartheta)\, d\rho \wedge d\vartheta \\
&= \frac{1}{2\pi} \int_0^{\infty} \frac{2\rho^2}{(1 + \rho^2)^3} \underbrace{\int_0^{2\pi} \cos\vartheta\, d\vartheta}_{=0}\, d\rho \\
&= 0.
\end{aligned}$$

So,

$$\begin{aligned}
\widehat{\deg}\left(\overline{\mathcal{O}_{\mathbb{P}^1_{\mathcal{O}_K}}(1)}\right) &= 0 - \frac{1}{2}\sum_{\sigma} \log\det(\langle x_i, x_j\rangle_{\sigma}) \\
&= -\frac{1}{2}[K:\mathbb{Q}]\cdot\log\left(\frac{1}{2}\right)^2 \\
&= \log 2\cdot[K:\mathbb{Q}].
\end{aligned}$$

Chapter 6

The height of $\mathbb{P}^n_{\mathcal{O}_K}$

Height is an important concept in diophantine geometry. It allows using descent arguments as used in the proof of the Mordell-Weil theorem. In this chapter, we discuss the notion of the height of projective space as a special case of the height of an arithmetic variety as defined by Jean-Benoît Bost, Henri Gillet, and Christophe Soulé in [BGS94].

6.1 The height of a point in $\mathbb{P}^n_{\mathbb{Z}}$

Definition 6.1.1. Let $\bar{L}$ be a Hermitian line bundle on $\operatorname{Spec}\mathcal{O}_K$. Then the *height of* $\operatorname{Spec}\mathcal{O}_K$ with respect to $\bar{L}$ is

$$h_{\bar{L}}(\operatorname{Spec}\mathcal{O}_K) := \widehat{\deg}\,(\bar{L}).$$

Let X be an arithmetic variety, $\bar{L}$ a Hermitian line bundle on X, and P a point on X. Then P corresponds to a section $\varepsilon_P : \operatorname{Spec}\mathcal{O}_K \to X$.

Definition 6.1.2. Let $P \in X$ be a point corresponding to the section $\varepsilon_P : \operatorname{Spec}\mathcal{O}_K \to X$. We define the *height of* P with respect to the line bundle $\bar{L}$ as

$$h_{\bar{L}}(P) := \widehat{\deg}\,\left(\varepsilon_P^* \bar{L}\right).$$

Consider $X = \mathbb{P}^n_{\mathbb{Z}}$ and its universal bundle $\overline{\mathcal{O}(1)}$. In section 4.1.2 we defined the Fubini-Study metric and gave an explicit formula on $U_i = \{(x_0 : \ldots : x_n) : x_0 \neq 0\}$.

Let $P \in X(\mathbb{C})$ be a point. Recall that the metric on $\mathcal{O}(1)_P$ is the dual of the restriction of the standard metric on $\mathbb{C}^n$ to the line corresponding to P. For an arbitrary section s of $\mathcal{O}(1)$ associated to the homogeneous polynomial $\sum \lambda_j X_j$ we then get

$$\left\| \sum \lambda_j X_j \right\|_P = \frac{|\sum \lambda_j z_j|}{(\sum |z_i|^2)^{\frac{1}{2}}}.$$

Example 6.1.3. Let $(x_0, \dots, x_n)$ be a $(n+1)$-tuple of relatively prime integers and let P be the associated point in $\mathbb{P}^n_{\mathbb{Z}}$. Then,

$$h_{\overline{\mathcal{O}(1)}}(P) = \log \sqrt{\sum x_i^2}.$$

Proof. (See [Bos99], section 3.3.2.) The morphism associated to P, $\varepsilon_P : \operatorname{Spec}\mathbb{Z} \to \mathbb{P}^n_{\mathbb{Z}}$, corresponds to a morphism of graded rings,

$$\begin{aligned} \varepsilon_P^{\#} : \quad \mathbb{Z}[X_0, \dots, X_n] &\longrightarrow \mathbb{Z}[Y] \\ X_i &\longmapsto x_i Y. \end{aligned}$$

Furthermore, we have a canonical isomorphism of line bundles $\varepsilon_P^* \mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}$ associating $\varepsilon_P^* X_i$ to x_i. Under this map, a global section $s = \sum \lambda_j X_j$ is sent to $\sum \lambda_j x_j$. By scaling of the λ_j (the section is independent of the scaling) we get $\sum \lambda_j x_j = \pm 1$. Therefore the norm at infinity of $\varepsilon_P^* s$ is

$$\left\| \sum \lambda_j X_j \right\|_P = \frac{|\sum \lambda_j x_j|}{\sqrt{\sum x_i^2}} = \frac{1}{\sqrt{\sum x_i^2}}.$$

Now the product formula and the definition of the Arakelov degree complete the proof. □

Remark 6.1.4. For an arbitrary point $P = (x_0 : \dots : x_n) \in \mathbb{P}^n(K)$, the height defined above coincides with the classical height in Arakelov theory (definition A.3.6)

$$h(P) = \sum_{\mathfrak{p} \nmid \infty} \log(\max_i |x_i|_{\mathfrak{p}}) + \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log \left(\sum_i |\sigma(x_i)|^2 \right)^{\frac{1}{2}},$$

where $\mathfrak{p}$ runs over the prime ideals of $\mathcal{O}_K$ and $|\cdot|_{\mathfrak{p}}$ is the absolute value corresponding to the non-archimedean valuation $v_{\mathfrak{p}}$. (See [BGS94], p. 947.)

In the case $K = \mathbb{Q}$, this coincides with our result in the lemma above because $\mathbb{Z}$ is a principal ideal domain and therefore the finite primes do not contribute to the sum.

Example 6.1.5. Consider the point $(0 : 1) \in \mathbb{P}^1_{\mathcal{O}_K}$:

$$h((0:1)) = \log 1 = 0.$$

6.2 The height of $\mathbb{P}^n_{\mathcal{O}_K}$

In this section, for simpler notation, we often omit the index $\mathbb{P}^n_{\mathcal{O}_K}$ and simply write $\overline{\mathcal{O}(1)}$ for $\overline{\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)}$ as in section 5.3

Definition 6.2.1. The *height of* $\mathbb{P}^n_{\mathcal{O}_K}$ *with respect to* $\overline{\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)}$ *with the metric induced by the Fubini-Study metric* is

$$h_{\overline{\mathcal{O}(1)}}(\mathbb{P}^n_{\mathcal{O}_K}) = h_{\overline{\mathcal{O}(1)}|_{\operatorname{div} s}}(\operatorname{div} s) - \int_{(\mathbb{P}^n_{\mathcal{O}_K})(\mathbb{C})} \log \|s\| \, \omega^n, \tag{6.1}$$

where s is a non-zero section of $\overline{\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)}$ and ω is the differential form associated to the Fubini-Study metric (section 4.1.2).

Remark 6.2.2. For $s = x_0$, $\overline{\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)}|_{\operatorname{div} s} = \overline{\mathcal{O}_{\mathbb{P}^{n-1}_{\mathcal{O}_K}}(1)}$

Lemma 6.2.3. *In the above notation,*

$$h_{\overline{\mathcal{O}(1)}}(\mathbb{P}^1_{\mathcal{O}_K}) = \frac{1}{2} \cdot [K : \mathbb{C}].$$

Proof. We choose the section $s = x_0$.

1. The divisor $\operatorname{div} x_0 = (0 : 1)$, and by example 6.1.5, $h_{\overline{\mathcal{O}(1)}}(\operatorname{div} x_0) = 0$.
2. We now need to compute the integral in formula (6.1). Recall that

$$\mathbb{P}^1_{\mathcal{O}_K}(\mathbb{C}) = \bigsqcup_{\sigma : K \hookrightarrow \mathbb{C}} (\mathbb{P}^1_{\mathcal{O}_K})_\sigma(\mathbb{C}),$$

and that for each σ the computation is similar. Thus,

$$h(\mathbb{P}^1_{\mathcal{O}_K}) = -[K : \mathbb{Q}] \cdot \int_{\mathbb{P}^1_{\mathbb{C}}} \log \|s\| \, \omega, \tag{6.2}$$

where ω is the differential form (4.3) from section 4.1.2.

This integral can be taken over the standard open set $U_0 = \{(x_0 : x_1) : x_0 \neq 0\} \cong \mathbb{C}$. By setting $z = \frac{x_1}{x_0}$,

$$\begin{aligned}
-\int_{\mathbb{P}^1_{\mathbb{C}}} \log \|s\| \, \omega &= -\int_{\mathbb{C}} \log((1 + |z|^2)^{-\frac{1}{2}}) \left(\frac{i}{2\pi} \frac{1}{(1 + |z|^2)^2} \right) dz \wedge d\bar{z} \\
&= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\log(1 + |z|^2)}{2(1 + |z|^2)^2} \, dz \wedge d\bar{z}.
\end{aligned}$$

Passing to polar coordinates (ρ, ϑ) and $dz \wedge d\bar{z} = -2\rho i \, d\rho \wedge d\vartheta$,

$$= -\frac{i}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{\log(1 + \rho^2)}{2(1 + \rho^2)^2} 2\rho i \, d\rho \wedge d\vartheta,$$

and changing the coordinates to $\mu = \rho^2$, we obtain $2\rho\, d\rho \wedge d\vartheta = d\mu \wedge d\vartheta$ and

$$\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\frac{\log(1+\mu)}{2(1+\mu)^2}\, d\mu \wedge d\vartheta \\
&= \int_0^{\infty}\frac{\log(1+\mu)}{2(1+\mu)^2}\, d\mu \\
&= \left(-\frac{\log(1+\mu)}{2(1+\mu)} - \frac{1}{2(1+\mu)}\right)\Bigg|_0^{\infty} \\
&= \frac{1}{2}.
\end{aligned}$$

So,

$$h_{\overline{\mathcal{O}(1)}}(\mathbb{P}^1_{\mathcal{O}_K}) = [K:\mathbb{Q}]\cdot\frac{1}{2}.$$

□

Theorem 6.2.4. *(Bost, Gillet, Soulé, [BGS94]) The height of* $\mathbb{P}^n_{\mathcal{O}_K}$ *is*

$$h_{\overline{\mathcal{O}(1)}}(\mathbb{P}^n_{\mathcal{O}_K}) = [K:\mathbb{Q}]\cdot\sigma_n,$$

where σ_n *is the Stoll number, i.e.* $\sigma_n = \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{k}\frac{1}{l}$.

Proof. Choose the section $s = x_0$. Then, $\operatorname{div} s = \mathbb{P}^{n-1}_{\mathcal{O}_K}$.

Let dv be the unique $U(n+1)$-invariant probability measure on the unit sphere $\mathbb{S}^{2n+1}$ in $\mathbb{C}^{n+1}$. Then, by [BGS94], p. 924,

$$\int_{\mathbb{P}^n_{\mathbb{C}}} \log\|x_0\|\,\omega^n = \int_{\mathbb{S}^{2n+1}} \log|x_0|\, dv.$$

One can compute the integral on the right-hand side as in section 5.3 to be

$$\int_{\mathbb{S}^{2n+1}} \log|x_0|\, dv = -\frac{1}{2}\left(\sum_{l=1}^{n}\frac{1}{l}\right).$$

By definition 6.2.1,

$$\begin{aligned}
h(\mathbb{P}^n_{\mathcal{O}_K}) &= h(\mathbb{P}^{n-1}_{\mathcal{O}_K}) - [K:\mathbb{Q}]\int_{\mathbb{S}^{2n+1}} \log|x_0|\, dv \\
&= h(\mathbb{P}^{n-1}_{\mathcal{O}_K}) + [K:\mathbb{Q}]\,\frac{1}{2}\left(\sum_{l=1}^{n}\frac{1}{l}\right).
\end{aligned} \tag{6.3}$$

The desired result is obtained by induction on n:
In example 6.1.5, we explicitly computed the base case, namely for $n = 1$,

$$\operatorname{div} x_0 = (0:1) \in \mathbb{P}^1.$$

The induction step follows from equation (6.3). □

6.3 Arakelov degree and height – intersection theory

In the light of the historic development of Arakelov Theory, we now discuss the result of theorem 6.2.4 above in detail.

Intersection theory is a branch of algebraic geometry, which studies linear combinations of subvarieties of algebraic varieties and their intersections. Their intersections, on the one hand, are motivated by the set-theoretic intersection, but on the other hand, this is extended such that one can e.g. compute the intersection product of two lines. We will not discuss intersection theory here, as this would go beyond the scope of this thesis, but we will touch it now and then (see e.g. remark 8.1.7 and section 10.4).

In the paper that led to the theory named after him, [Ara74], Arakelov introduced an intersection theory on arithmetic surfaces and thus extended the theory on varieties over fields to a theory on arithmetic surfaces. Gerd Faltings, in his seminal paper [Fal84], pushed this theory further and proved a Riemann-Roch theorem, a Hodge index theorem, and a Noether's formula for arithmetic surfaces. Moreover, this eventually led him to his proof of the Mordell conjecture (now called Faltings' theorem), the Shafarevich conjecture, and a conjecture of Tate.

Henri Gillet and Christophe Soulé later generalized this to a product on arithmetic varieties in a series of papers, among them [GS90], and proved a version of the Riemann-Roch theorem in this context in [GS89] and [GS92].

In the context of this arithmetic intersection product on arithmetic varieties, the height of an arithmetic variety with respect to a Hermitian vector bundle was defined to be the degree of a certain intersection product in [BGS94], and therefore the height is a special case of the degree.

In fact, for $\mathbb{P}^n_{\mathcal{O}_K}$ respectively $\overline{\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)}$, the degree and the height should actually coincide. Now, our results in theorem 5.3.1 and theorem 6.2.4 are not the same, and we will now elaborate on what leads to this difference.

In [BGS94], Proposition 3.3.2, it is shown that, when changing the metric, the height essentially only changes up to a constant. We furthermore observe that, indeed, our results are asymptotically equivalent as n tends to infinity:

Recall that

$$\widehat{\deg}\ \overline{\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)} = \frac{[K:\mathbb{Q}]}{2} \cdot (n+1)\log(n+1),$$

and

$$h_{\overline{\mathcal{O}(1)}}(\mathbb{P}^n_{\mathcal{O}_K}) = \frac{[K:\mathbb{Q}]}{2} \cdot \sum_{k=1}^{n} \sum_{l=1}^{k} \frac{1}{l}.$$

Since

$$\sum_{k=1}^{n} \sum_{l=1}^{k} \frac{1}{l} \sim \sum_{k=1}^{n} \int_1^k \frac{1}{l}\, dl = \sum_{k=1}^{n} \log k \sim \int_1^n \log k \, dk \sim n \log n \sim (n+1)\log(n+1),$$

the degree and the height indeed are asymptotically equivalent, i.e.

$$\widehat{\deg}\ \overline{\mathcal{O}_{\mathbb{P}^n_{\mathcal{O}_K}}(1)} \sim h_{\overline{\mathcal{O}(1)}}(\mathbb{P}^n_{\mathcal{O}_K}).$$

This puts close the assumption that the metrics used in the two calculations were not the same.

Indeed, we used the Fubini-Study metric on the fibers in both cases, but in section 5.2 we defined the norm on the pushforward of a vector bundle to be the L^2-norm, i.e.

$$\|s\|^2 = \int_{X(\mathbb{C})} \|s(x)\|^2 d\mu(x).$$

However in [BGS94], apparently another norm is used. On the line bundle $\mathcal{O}(1)$ on $X = \mathbb{P}(V)$ for an n-dimensional vector space V, they define the norm

$$\|s\|_0 = \exp\left(\int_{X(\mathbb{C})} \log \|s(x)\| \, dv(x) \right),$$

where v is the $U(n)$-invariant probability measure.

Chapter 7

The canonical polygon

Canonical polygons form a tool to bridge between Arakelov theory and geometry of numbers. We first give the definition and, as a continuation of the example in section 5.3, we consider the canonical polygon for the twisting sheaf. We then look at successive minima and discuss morphisms between vector bundles.

In this chapter, if we do not state otherwise, we always use the induced metric on subbundles of a given Hermitian vector bundle $\bar{E}$.

7.1 Definition

Definition 7.1.1. The *normalized Arakelov degree* is

$$\widehat{\mathit{deg}}\,\bar{E} = \frac{1}{[K:\mathbb{Q}]}\widehat{\deg}\,\bar{E},$$

and the *normalized slope* is

$$\widehat{\mu}(\bar{E}) = \frac{1}{\operatorname{rk} E}\,\widehat{\mathit{deg}}\,\bar{E}.$$

Remark 7.1.2. It is sometimes more natural to consider the normalized Arakelov degree or the normalized slope instead of the Arakelov degree since they are invariant under the pullback under the morphism f associated to an extension of number fields [Bos96]. In the literature, the normalized Arakelov degree often is denoted by $\widehat{\deg}_n(\bar{E})$.

Definition 7.1.3. Let $\bar{E}$ be a Hermitian vector bundle over $\operatorname{Spec}\mathcal{O}_K$. Consider the set of points $\{(\operatorname{rk}\bar{F}, \widehat{\mathit{deg}}\bar{F}), F \text{ is a subbundle of } \bar{E}\} \subset [0, \operatorname{rk}\bar{E}] \times \mathbb{R}$. The convex hull of this set is bounded from above ([Bos96], A.3). Its upper boundary defines a piecewise linear function $P_E : [0, \operatorname{rk} E] \to \mathbb{R}$, which is called the *canonical polygon*. If P_E is linear, we say that $\bar{E}$ is *semi-stable*.

Furthermore, for each $i \in \{1, \ldots, \operatorname{rk} \bar{E}\}$, we define the *i-th slope of* $\bar{E}$ to be

$$\widehat{\mu}_i(\bar{E}) = P_{\bar{E}}(i) - P_{\bar{E}}(i-1),$$

so $\widehat{\mu}_i(\bar{E})$ is the slope of $P_{\bar{E}}$ in the interval $(i-1, i)$.

The *maximal and minimal slopes* are denoted by $\widehat{\mu}_{\max}(\bar{E})$, $\widehat{\mu}_{\min}(\bar{E})$, respectively.

Remark 7.1.4. Note that $P_{\bar{E}}(0) = 0$, $P_{\bar{E}}(\operatorname{rk} \bar{E}) = \widehat{deg}(\bar{E})$, and $\sum_i \widehat{\mu}_i(\bar{E}) = \widehat{deg}(\bar{E})$. Moreover, since $P_{\bar{E}}$ is a concave function,

$$\forall i \in \{1, \ldots, \operatorname{rk} \bar{E}\} : \widehat{\mu}_{i-1}(\bar{E}) \geqslant \widehat{\mu}_i(\bar{E}).$$

In particular,

$$\begin{aligned}\widehat{\mu}_{\max}(\bar{E}) &= \widehat{\mu}_1(\bar{E})\\ \widehat{\mu}_{\min}(\bar{E}) &= \widehat{\mu}_{\operatorname{rk} E}(\bar{E}).\end{aligned}$$

Example 7.1.5. We want to calculate the canonical polygon of $\bar{E} = (\pi_* \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1), h_{FS})$, where h_{FS} is the metric induced by the Fubini-Study metric.

Recall that $\Gamma(\operatorname{Spec} \mathbb{Z}, \pi_* \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1)) = \Gamma(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1)) = \mathbb{Z}x_0 + \mathbb{Z}x_1$. In example 5.3.4 we calculated

$$\begin{aligned}\|x_0\|^2_{\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}} = \|x_1\|^2_{\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}} &= \frac{1}{2},\\ \langle x_0, x_1 \rangle_{\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}} &= 0,\end{aligned}$$

and

$$\widehat{\deg}\, \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1) = \log 2.$$

Since $[\mathbb{Q} : \mathbb{Q}] = 1$, $\widehat{deg} \bar{E} = \widehat{\deg}\, \bar{E}$; so $P_E(2) = \log 2$.

Subbundles of $\pi_* \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1)$ correspond to submodules of $\mathbb{Z}x_0 + \mathbb{Z}x_1$. We now consider the submodules of rank 1:
All submodules of $\mathbb{Z}x_0 + \mathbb{Z}x_1$ of rank 1 are of the form $N = \mathbb{Z}(ax_0 + bx_1)$, where $a, b \in \mathbb{Z}$, not both 0. We calculate the degree using the section $s = ax_0 + bx_1$ in definition 5.1.1.

1. Clearly, $\log \#(\mathbb{Z}(ax_0 + bx_1)/\mathbb{Z}(ax_0 + bx_1)) = 0$.
2. Since $\langle x_0, x_1 \rangle_{\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}} = 0$,

$$\|ax_0 + bx_1\|^2 = |a|^2 \|x_0\|^2 + |b|^2 \|x_1\|^2 = \frac{1}{2}(|a|^2 + |b|^2).$$

Therefore,

$$\widehat{\deg}\,(\mathbb{Z}(ax_0 + bx_1)) = -\frac{1}{2} \log \left(\frac{1}{2}(|a|^2 + |b|^2) \right) = \frac{1}{2} \log 2 + \frac{1}{2} \log \left(\frac{1}{|a|^2 + |b|^2} \right).$$

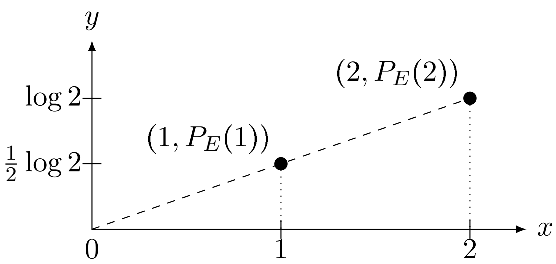
Since $a, b \in \mathbb{Z}$, $\log\left(\frac{1}{|a|^2+|b|^2}\right) \leqslant 0$, and so

$$\widehat{deg}(\mathbb{Z}(ax_0 + bx_1)) = \widehat{\deg}\left(\mathbb{Z}(ax_0 + bx_1)\right) = \frac{1}{2}\log 2 + \frac{1}{2}\log\left(\frac{1}{|a|^2+|b|^2}\right) \leqslant \frac{1}{2}\log 2.$$

This bound is attained at e.g. $a = 1, b = 0$: $\widehat{deg}(\mathbb{Z}x_0) = \frac{1}{2}\log 2$.

The point $(1, \frac{1}{2}\log 2)$ already lies on the line between $(0, P_E(0)) = (0,0)$ and $(2, P_E(2)) = (2, \log 2)$, so taking the convex hull does not change the upper boundary of the set $\{(\operatorname{rk}\bar{F}, \widehat{\deg}\,\bar{F}), F \text{ is a subbundle of } \bar{E}\}$. So $P_E(1) = \frac{1}{2}\log 2$.

In particular, $\widehat{\mu}_1(\bar{E}) = \widehat{\mu}_2(\bar{E}) = \frac{1}{2}\log 2$, so P_E is linear. Thus, $\pi_*\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(1)$ is semi-stable.



7.2 Slopes and successive minima

We start this section by looking at an example.

Example 7.2.1. Consider the lattice $\Lambda = \mathbb{Z}\cdot(a,b) + \mathbb{Z}\cdot(c,d) \subset \mathbb{R}^2$, where $a,b,c,d \in \mathbb{R}$. Λ is a $\mathbb{Z}$-module and we endow $\Lambda \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^2$ with the standard Hermitian metric. We compute the degree of Λ and the degree of a given submodule.

Let $e_1 = (a,b)$ and $e_2 = (c,d)$. Clearly,

$$\|e_1\|^2 = a^2 + b^2, \qquad \|e_2\|^2 = c^2 + d^2 \qquad \text{and } \langle e_1, e_2\rangle = ac + bd.$$

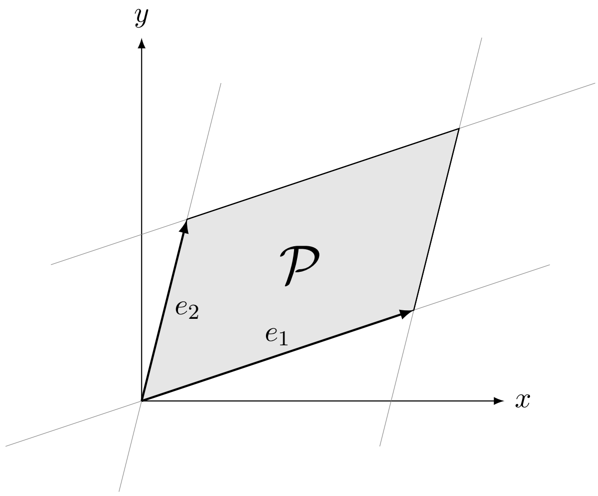
We compute the determinant $\det(e_i, e_j)$:

$$\det(\langle e_i, e_j\rangle) = (a^2+b^2)(c^2+d^2) - (ac+bd)^2 = (ad-bc)^2 = (\det(e_i,e_j))^2.$$

So the Arakelov degree of Λ is

$$\widehat{deg}\Lambda = -\log\left(\operatorname{Vol}(\mathcal{P})\right),$$

where $\mathcal{P}$ is the fundamental parallelogram of the lattice Λ.



Let Λ_1 be a sublattice of Λ. Then, $\Lambda_1 = (\alpha e_1 + \beta e_2)\mathbb{Z}$ for some $\alpha, \beta \in \mathbb{Z}$, and

$$\widehat{deg}((\alpha e_1 + \beta e_2)\mathbb{Z}) = -\log \|\alpha e_1 + \beta e_2\|.$$

So maximizing the degree of a submodule of Λ is equivalent to finding a shortest vector in the lattice Λ. This leeds to the problem of successive minima.

Definition 7.2.2. Let C be a convex body which is symmetric with respect to the origin and Λ a lattice in $\mathbb{R}^n$. The *successive minima* $\lambda_i = \lambda_i(C, L)$, $i = 1, \dots n$, of C with respect to Λ are defined by

$$\lambda_i = \min\{\lambda > 0 : \lambda C \text{ contains } i \text{ linearly independent points of } L\}.$$

See [Gru07] for more on successive minima.

In the example above, we considered a Hermitian vector bundle E over $\operatorname{Spec}\mathbb{Z}$, i.e. a $\mathbb{Z}$-module E endowed with a metric $\|\cdot\|$. We take $C = B_n$, where B_n is the closed unit ball in $\mathbb{R}^n$ endowed with the standard metric, and, since we were dealing with a $\mathbb{Z}$-module, we get the following equality:

Let $\lambda = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n \subset \mathbb{R}^n$. Then

$$\begin{aligned}
\widehat{\mu}_1 = P_\Lambda(1) &= \sup_{\operatorname{rk} N = 1} \widehat{deg} N \\
&= \sup_{(a_1,\dots,a_n)\in\mathbb{Z}^n} \{-\log \|a_1 e_1 + \cdots a_n e_n\|\} \\
&= -\log \min_{(a_1,\dots,a_n)\in\mathbb{Z}^n} \{\|a_1 e_1 + \cdots a_n e_n\|\} \\
&= -\log \lambda_1.
\end{aligned}$$

For arbitrary K, we just get an inequality:

Lemma 7.2.3. *Let $\bar{E}$ be a Hermitian vector bundle on* $\operatorname{Spec}\mathcal{O}_K$*, i.e. an $\mathcal{O}_K$-module endowed with a Hermitian metric. Then*

$$\log \lambda_1(\bar{E}) \geqslant -\widehat{\mu}_1(\bar{E}).$$

Proof. See [Bor05]. Let $s \in E$ be an element such that $\max_{\sigma:K\hookrightarrow\mathbb{C}} \|s\|_\sigma = \lambda_1(\bar{E})$. Then, using the section s of $\overline{s\mathcal{O}_K}$,

$$\widehat{\deg}\,(\overline{s\mathcal{O}_K}) = \underbrace{\log \#(s\mathcal{O}_K/s\mathcal{O}_K)}_{=0} - \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log \|s\|_\sigma \geqslant -\log \lambda_1(\bar{E}) \cdot [K:\mathbb{Q}].$$

Since $\overline{s\mathcal{O}_K}$ is a subbundle of $\bar{E}$, we get the desired result by the definition of $\widehat{\mu}_1(\bar{E})$. $\square$

One can show even more: in fact, the inequality holds for all i.

Proposition 7.2.4. *(Borek, [Bor05]) Let $\bar{E}$ be a Hermitian vector bundle on* $\operatorname{Spec}\mathcal{O}_K$*. Then,*

$$\forall i: \quad \log \lambda_i(\bar{E}) \geqslant -\widehat{\mu}_i(\bar{E}).$$

Example 7.2.5. We take $(a,b) = (2,0)$ and $(c,d) = (3,4)$ in the previous example. Then

$$\widehat{deg}\Lambda = \log(\det \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}) = \log \frac{1}{8} = 3\log 12.$$

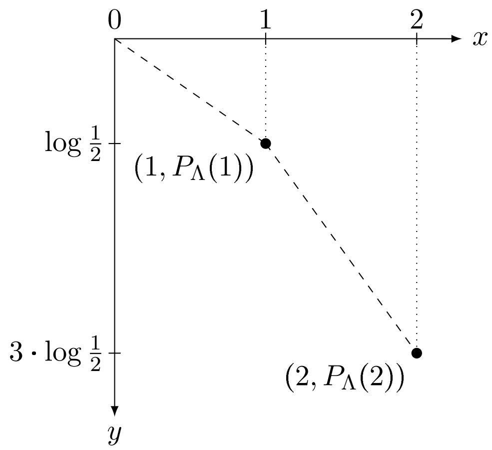
The shortest vector in the lattice is $(2,0)$, and the degree of the generated sublattice is

$$\widehat{deg}((2,0)\cdot\mathbb{Z}) = \log \frac{1}{2}.$$

So we get

$$P_\Lambda(1) = \log \frac{1}{2}, \qquad P_\Lambda(2) = 3\log \frac{1}{2}.$$

Note that Λ is not semi-stable.



7.3 The Arakelov degree and morphisms

In this section, we want relate the Arakelov degrees of Hermitian vector bundles $\bar{E}$ and $\bar{F}$ which are connected via a morphism ϕ. We follow the appendix of [Via05].

Definition 7.3.1. Let $\bar{E}, \bar{F}$ be two Hermitian vector bundles over $\operatorname{Spec} \mathcal{O}_K$ and $\phi : \bar{E} \to \bar{F}$ a morphism. Then, the *norm of* ϕ is the operator norm of ϕ, i.e.

$$\|\phi\|_\sigma := \sup_{0 \neq s \in E} \frac{\|\phi(s)\|_\sigma}{\|s\|_\sigma}.$$

Proposition 7.3.2. *Let $\phi : \bar{E} \to \bar{F}$ be a non-trivial injective morphism. Then*

$$\widehat{deg} \bar{E} \leqslant \sum_{i=1}^{\operatorname{rk} E} \widehat{\mu}_i(\bar{F}) + \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log \| \wedge^{\operatorname{rk} E} \phi \|_\sigma.$$

Proof. We start by proving the statement for a line bundle $\bar{L} = \bar{E}$. Let s be a non-zero section of L. By the injectivity of ϕ, $\phi(s)$ is a non-zero section of $\phi(L)$, and

$$\#(\phi(L)/\phi(s)\mathcal{O}_K) = \#(L/s\mathcal{O}_K).$$

Therefore,

$$\begin{aligned}
\widehat{deg}(\bar{L}) &= \frac{1}{[K:\mathbb{Q}]} \left(\log \#(L/s\mathcal{O}_K) - \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log \|s\|_\sigma \right) \\
&= \frac{1}{[K:\mathbb{Q}]} \left(\log \# \left(\phi(L)/\phi(s)\mathcal{O}_K\right) - \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log \|s\|_\sigma \right) \\
&= \widehat{deg}\, \overline{\phi(L)} + \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log \frac{\|\phi(s)\|_\sigma}{\|s\|_\sigma} \\
&\leqslant \widehat{deg}\, \overline{\phi(L)} + \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log \|\phi\|_\sigma.
\end{aligned}$$

Now for a Hermitian vector bundle $\bar{E}$ with $r := \operatorname{rk} E = \operatorname{rk} \phi(E) > 1$, consider the injective map $\wedge^r \phi : \wedge^r E \to \wedge^r \phi(E)$. $\wedge^r E$ is a line bundle, so by the above we get

$$\widehat{deg}(\bigwedge^r E) \leqslant \widehat{deg}\,(\bigwedge^r \overline{\phi(E)}) + \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log \| \wedge^r \phi \|_\sigma.$$

As $\phi(E) \subseteq F$, $\forall i : \widehat{\mu}_i(\overline{\phi(E)}) \leqslant \widehat{\mu}_i(\bar{F})$, and we get

$$\widehat{deg}\, \overline{\phi(E)} = \sum_{i=1}^{\operatorname{rk} E} \widehat{\mu}_i(\overline{\phi(E)}) \leqslant \sum_{i=1}^{\operatorname{rk} E} \widehat{\mu}_i(\bar{F}),$$

which concludes the proof. □

Corollary 7.3.3. *Under the assumptions above,*

$$\widehat{deg}\bar{E} \leqslant \sum_{i=1}^{\operatorname{rk} E} \widehat{\mu}_i(\bar{F}) + \frac{\operatorname{rk} E}{[K:\mathbb{Q}]} \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log \|\phi\|_\sigma.$$

Proof. By e.g. [Bos99], $\| \wedge^r \phi\| \leqslant \|\phi\|^r$. □

Proposition 7.3.4. *Let $\phi : \bar{E} \to \bar{F}$ be a non-trivial injective morphism. Then*

$$\widehat{\mu}_{\max}(\bar{E}) \leqslant \widehat{\mu}_{\max}(\bar{F}) + \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log \|\phi\|_\sigma.$$

Proof. Let $(r, P_E(r))$ be the point of discontinuity of P_E such that $r > 0$ is minimal. Let E_r be a submodule of E of rank r such that $P_E(\bar{E}_r) = \widehat{deg}\,\overline{E_r} = r \cdot \widehat{\mu}_{\max}(\bar{E})$. By applying corollary 7.3.3 to the morphism $\phi|_{E_r} : E_r \to \phi(E_r)$,

$$\widehat{\mu}_{\max}(\bar{E}) = \frac{\widehat{deg}\,\overline{E_r}}{r} \leqslant \frac{1}{r} \cdot \left(\sum_{i=1}^{r} \widehat{\mu}_i(\overline{\phi(E_r)}) + \frac{r}{[K:\mathbb{Q}]} \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log \|\phi|_{E_r}\|_\sigma \right),$$

and since $\widehat{\mu}_i(\overline{\phi(E_r)}) \leqslant \widehat{\mu}_{\max}(\overline{\phi(E_r)}) \leqslant \widehat{\mu}_{\max}(\bar{F})$ and $\|\phi|_{E_r}\|_\sigma \leqslant \|\phi\|_\sigma$, we get the desired result. □

Chapter 8

Chow groups

In this chapter, we give an outlook on an alternative, more geometric, interpretation of isomorphism classes of Hermitian vector bundles as used e.g. in [BGS94]. We start by considering the classical case of algebraic cycles on algebraic varieties over a field k, e.g. a number field. Then, to bring in arithmetic, we extend this notion to arithmetic varieties, i.e. varieties over the ring of integers of a number field with some additional properties. In 8.2, we introduce arithmetic cycles and Chow groups. We then sketch the proof showing that the first *arithmetic Chow group* and the arithmetic Picard group are isomorphic and thus, this approach indeed represents a new interpretation.

We will be brief on technical details, as the intention of this section is to give an introduction to this setting. However, the technicalities can be found in [Ful98], [Sou92], [BGS94], and [Lan88].

8.1 Geometric Chow groups

Chow rings are a geometric analogon of cohomology rings of a variety in topology. They are a generalization of the divisor class group, which has a long history in the study of algebraic geometry. The notion of rational equivalence was first introduced by Severi. For an overview of the historic development of Chow rings, see [Ful98].

Definition 8.1.1. Let X be an algebraic scheme. An *algebraic cycle* on X is a formal linear combination of irreducible closed subvarieties on X with integer coefficients, i.e. element of the group

$$Z(X) = \bigoplus_{V \subseteq X} \mathbb{Z} \cdot V,$$

where V runs over the irreducible closed subvarieties of X. An *algebraic cycle of dimension p or p-cycle* on X is an algebraic cycle on X such that all subvarieties of X which have non-zero coefficient have dimension p. The group of algebraic cycles of dimension p is denoted by $Z_p(X)$.

A cycle $Z = \sum n_i \cdot V_i$ is called *positive* if it is not zero and each of its coefficients n_i is a non-negative integer.

Remark 8.1.2. The divisors of X are the algebraic cycles of codimension 1, i.e.

$$\mathrm{Div}(X) = Z_{\dim X - 1}(X).$$

We introduce the order of vanishing of r along V for a subvariety V of X of codimension one to define the cycle associated to a rational function. The construction is similar to that of principal divisors in definition 3.5.4.

Let X be a variety which is non-singular in codimension one and V a subvariety of X of codimension one. Then the local ring of X along Y, $\mathcal{O}_{X,V}$, is a discrete valuation domain. Furthermore, $k(X) = \mathrm{Quot}\, \mathcal{O}_{X,V}$. Let $r \in k(X)^*$ be a nonzero rational function on X. This can be written as $r = a/b$, where $a, b \in \mathcal{O}_{X,V}$.

Definition 8.1.3. The *order of vanishing of r along V* is defined to be

$$\mathrm{ord}_V(r) = \mathrm{ord}_V(a) - \mathrm{ord}_V(b),$$

where the orders on the right are those with respect to the valuation of the discrete valuation ring $\mathcal{O}_{X,V}$.

Remark 8.1.4. In case that X possibly is singular, one can define

$$\mathrm{ord}_V(r) = l_{\mathcal{O}_{X,Y}}(\mathcal{O}_{X,Y}/(r)),$$

where $l_{\mathcal{O}_{X,Y}}$ denotes the length of the $\mathcal{O}_{X,Y}$-module $\mathcal{O}_{X,Y}/(r)$.

Definition 8.1.5. For any $p+1$-dimensional subvariety W of X, and any $r \in k(W)^*$, i.e. a non-zero rational function on W, let $\mathrm{div}\, r$ be the p-cycle

$$\mathrm{div}\, r = \sum \mathrm{ord}_V(r) \cdot V,$$

where the sum is taken over all subvarieties of codimension one of W. A p-cycle Z is called *rationally equivalent to zero* if there exist $p+1$-dimensional subvarieties W_i of X and $r_i \in k(W_i)^*$ such that

$$Z = \sum \mathrm{div}\, r_i.$$

Note that the set of p-cycles rationally equivalent to zero form a subgroup of $Z_p(X)$ since $\mathrm{div}\, r^{-1} = -\mathrm{div}\, r$, denoted by $R_p(X)$.

Definition 8.1.6. The group of p-cycles modulo rational equivalence is called the *(geometric) Chow group of dimension p*,

$$CH_p(X) = Z_p(X)/R_p(X).$$

The *(geometric) Chow group* is the direct sum

$$CH(X) = \bigoplus_{p=0}^{\dim X} CH_p(X).$$

Remark 8.1.7. One can equip $CH(X)$ with a product to make it into a ring, the Chow ring. This product is called the *intersection product.* In section 6.3, we briefly mentioned intersection theory and its generalization to arithmetic varieties. Even though it will be mentioned several times in this thesis, we will not go into the details of intersection theory, as this would be beyond the scope. For a thorough reference on intersection theory, see [Ful98].

8.2 Arithmetic Chow groups

Throughout this section, let X be an arithmetic variety and $\mathcal{X}$ be a smooth projective complex equidimensional variety of dimension d.

To define the arithmetic counterpart of algebraic cycles, we first need some terminology of complex geometry. In particular, we need the notions of Green currents on smooth projective complex varieties.

Notation. Let $\mathcal{X}$ be a smooth projective complex equidimensional variety of dimension d. We denote by $A^{p,q}(\mathcal{X})$ the vector space of complex valued differential forms of type (p,q). A current δ is a smooth linear functional on $A^{p,q}(\mathcal{X})$, i.e. a differential form with distribution coefficients. We denote by $D^{p,q}(\mathcal{X})$ the set of all currents. Note that there is an inclusion $A^{p,q}(\mathcal{X}) \hookrightarrow D^{p,q}(\mathcal{X})$ given by $\omega \mapsto \left(\alpha \mapsto \int_{\mathcal{X}} \omega \wedge \alpha\right)$. We remark that, more generally, just like in the theory of distributions, any locally L^1 form of type (p,q) on $\mathcal{X}$ defines a current on $\mathcal{X}$ (see [Sou92], p. 39, chapter II.1 and [BGS94], p. 908 for details).

For any irreducible analytic subvariety $\mathcal{Y}$ of $\mathcal{X}$, we can define a current $\delta_{\mathcal{Y}}$ by setting

$$\delta_{\mathcal{Y}}(\alpha) := \int_{\mathcal{Y}_{ns}} i^*\alpha \quad \forall \alpha \in \mathrm{A}^{d-p,d-q},$$

where $i : \mathcal{Y} \hookrightarrow \mathcal{X}$ and $\mathcal{Y}_{ns}$ is the smooth part of $\mathcal{Y}$. We extend this definition by linearity to any analytic cycle of $\mathcal{X}$.

Definition 8.2.1. A *Green current for an analytic cycle $\mathcal{Z}$ of codimension p on $\mathcal{X}$* is an element $g \in D^{p-1,p-1}(\mathcal{X})$ such that

$$dd^c g + \delta_{\mathcal{Z}} \in \mathrm{A}^{pp}(\mathcal{X}),$$

where $d = \partial + \bar{\partial}$ and $d^c = \frac{i}{4\pi}(\partial - \bar{\partial})$.

Now we can define the arithmetic analogs. For an arithmetic variety X, we denote by $A^{pp}(X_{\mathbb{R}})$ the vector space of real differential forms in $A^{pp}(X(\mathbb{C}))$ and by $D^{pp}(X)$ the set of real currents in $D^{pp}(X(\mathbb{C}))$. We can associate a current δ_Z to a cycle $Z = \sum_i n_i Z_i$ by setting $\delta_Z = \sum_i n_i \delta_{Z_i(\mathbb{C})}$.

Definition 8.2.2. A *Green current for a cycle Z of codimension p on X* is a current $g \in D^{p-1,p-1}(X_{\mathbb{R}})$ such that $dd^c g + \delta_Z$ is smooth, i.e. a Green current for $Z(\mathbb{C})$ which lies in $D^{p-1,p-1}(X_{\mathbb{R}})$.

Definition 8.2.3. An *arithmetic cycle of dimension p on X* is a pair (Z, g_Z), where Z is a cycle of codimension p and g_Z is a Green current for Z. Let $\widehat{Z}^p(X)$ be the group of arithmetic cycles of dimension p, where addition is defined componentwise.

Giving examples for arithmetic cycles is not easy, and we only sketch the idea of an example for an arithmetic cycle. For this, we need the following theorem, the Poincaré-Lelong formula. For more details on this example and a proof of the theorem, see [Sou92], p. 41, Theorem II.2 and p. 54ff, Chapter III.

Theorem 8.2.4. *(The Poincaré-Lelong formula) Let $\bar{L} = (L, \|\cdot\|)$ be a holomorphic line bundle on $\mathcal{X}$ and s a meromorphic section of L. Then $-\log \|s\|^2 \in L^1(\mathcal{X})$ and hence induces a distribution in $D^{00}(\mathcal{X})$ which furthermore is a Green current for* $\operatorname{div}(s)$.

Example 8.2.5. Let y be a point on an arithmetic variety X such that $Y = \overline{\{y\}}$ is a closed integral subscheme of X of codimension $p-1$. Let $f \in k(y)^*$. Then $(\operatorname{div}(f), -[\log |f|^2])$ is an arithmetic cycle on X, where $[\log |f|^2]$ is a certain Green current on X associated to $\log |f|^2$.

Definition 8.2.6. We denote by $\widehat{R}^p(X) \subset \widehat{Z}^p(X)$ the subgroup generated by the pairs $(\operatorname{div} f, -[\log |f|^2])$ from the example above and by pairs $(0, \partial(u) + \bar{\partial}(v))$, where u and v are currents of type $(p-2, p-1)$ and $(p-1, p-2)$, respectively.
Then the *arithmetic Chow group of codimension p of X* is defined as the quotient

$$\widehat{CH}^p(X) := \widehat{Z}^p(X)/\widehat{R}^p(X).$$

Remark 8.2.7. Arithmetic Chow groups clearly are a generalization of geometric Chow groups in classical algebraic geometry as defined in the previous section. To the classical, "geometric", part, an additional, "analytic", part is added.

We now come to the main proposition of this section, namely we discuss the correspondence between the first arithmetic Chow group and the arithmetic Picard group of an arithmetic variety.

Proposition 8.2.8. *Using the notation of the example above, there is an isomorphism*

$$\widehat{c}_1 : \widehat{\operatorname{Pic}}(X) \longrightarrow \widehat{CH}^1(X),$$

mapping the class of $(L, \|\cdot\|)$ to the class of $(\operatorname{div} s, -[\log \|s\|^2])$ for any rational section s of L.

Sketch of proof. We first show the well-definedness of this map. Since L is a line bundle, the set of global sections of L has rank 1. Therefore, any other rational section of L can be written as $s' = fs$, where f is a rational function on X. So, $\operatorname{div} s' = \operatorname{div} s$. Furthermore, a calculation similar to the one in (4.2) in section 4.1.2 shows that $\widehat{c}_1(L, \|\cdot\|)$ does not depend on the choice of the section.

We consider the inverse map. This is given by sending a cycle (Z, g_Z) to the isomorphism class of $(O_X(Z), \|\cdot\|)$, where the metric is locally given by $\|f\|^2 = |f|^2 e^{-g_Z}$. Since g_Z is a Green current for Z, this defines a smooth metric and hence this defines a smooth metric. $\square$

Remark 8.2.9. $\widehat{c}_1(L, \|\cdot\|)$ is called the *first (arithmetic) Chern class of* $(L, \|\cdot\|)$. The concept of Chern classes comes from classical differential geometry and is a topological invariant of Hermitian vector bundles; e.g. given a Hermitian line bundle $(L, \|\cdot\|)$, the associated *first Chern form* is the differential form given by $\widehat{c}_1(L, \|\cdot\|) = -dd^c \log \|s(z)\|^2$. Note that in section 4.1.2, we calculated the first Chern form of the Fubini-Study metric on $\mathcal{O}(1)$. The Chern class then is the class of the first Chern form in the second de Rham cohomology group, [GH78, Lan88, Deb05].

Chapter 9

Arithmetic surfaces and the degree of an elliptic curve

In this chapter, we introduce the notions needed to apply Arakelov theory to elliptic curves. We show how to attach a scheme over $\mathbb{Z}$ – or, more precisely, an *arithmetic surface* – to an elliptic curve. We define theta functions and examine how they correspond to global sections of line bundles on complex tori, i.e. elliptic curves. Using *theta functions*, we define a norm on the line sheaf $\mathcal{O}(P)$ for $P = O_{\mathcal{E}}$. Finally, we state a result of Jürg Kramer on the Arakelov degree of a special line bundle on an elliptic curve.

9.1 Arithmetic surface attached to an elliptic curve

Definition 9.1.1. Let R be a Dedekind domain and K its field of fractions. Then, an *arithmetic surface over* R is a scheme $\mathscr{C}$ over R whose generic fiber is a non-singular connected projective curve C/K and whose special fibers are unions of curves over the appropriate residue fields. Furthermore, we require some technical conditions: $\mathscr{C}$ is integral, normal, excellent, and is flat and of finite type over R.

Remark 9.1.2. One can consider an arithmetic surface as a curve over $\operatorname{Spec} R$ since the relative dimension is one, i.e. the fibers are one-dimensional. Note that the fibers are not necessarily regular, they can also be reducible. Nevertheless, an arithmetic surface is regular in codimension one and therefore has a theory of Weil divisors. See [Sil94] for details.

Now let $\mathscr{C} \subset \mathbb{P}^2_{\mathbb{Z}}$ be the $\mathbb{Z}$-scheme defined by the equation

$$Y^2 = X^3 + aX + b,$$

where $a, b \in \mathbb{Z}$ and $\Delta = -16(4a^3 + 27b^2) \neq 0$. Then the generic fiber of $\mathscr{C}$ is the elliptic curve over $\mathbb{Q}$ defined by the equation above.

The fibers over closed points $\mathfrak{p}$ are given by prime ideals, and the fiber then is

$$\mathscr{C}_{\mathfrak{p}} = \mathscr{C} \times_{\mathbb{Z}} \mathfrak{p},$$

the reduction modulo $\mathfrak{p}$.

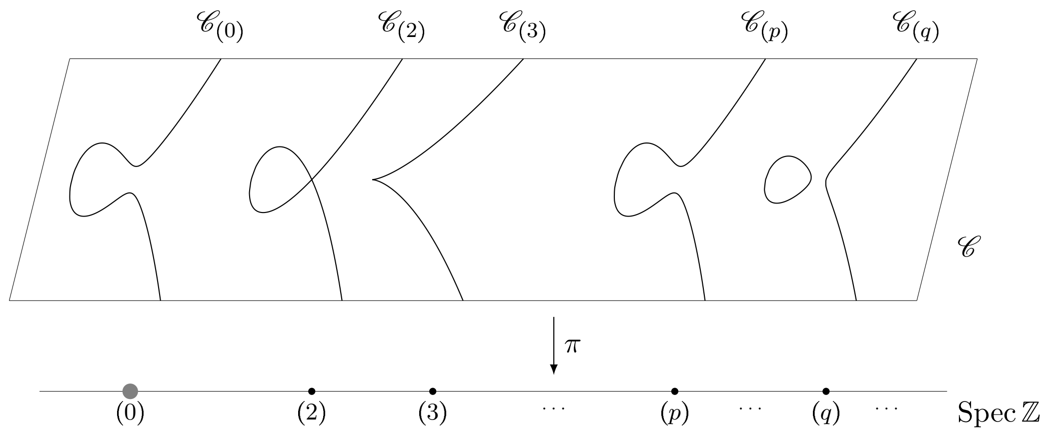
If $\mathfrak{p} = p\mathbb{Z}$, for a prime $p \in \mathbb{Z}$, then $\mathscr{C}_{\mathfrak{p}}$ is exactly what "reduction mod $\mathfrak{p}$" should be, namely the scheme over $\mathbb{Z}/\mathfrak{p} = \mathbb{F}_p$ given by $Y^2 = X^3 + \bar{a}X + \bar{b}$, where $\bar{a} \equiv a \mod p$ and $\bar{b} \equiv b \mod p$.

Furthermore, the scheme $\mathscr{C}$ is proper over $\mathbb{Z}$, since $\mathscr{C}$ is a closed subscheme of projective space over $\operatorname{Spec}\mathbb{Z}$ ([Sil94], Theorem IV.2.8.). This is a rather technical condition, but we need it below to extend rational points on the elliptic curve which is the generic fiber of an arithmetic surface.

Example 9.1.3. Consider the elliptic curve $Y^2 = X^3 + 3X + 6$ over $\mathbb{Z}$. The fibers over primes p are given by reduction modulo p, i.e. by the equation $Y^2 = X^3 + \bar{3}X + \bar{6}$ over $\mathbb{F}_p$, where $\bar{3} \equiv 3 \mod p$, and $\bar{6} \equiv 6 \mod p$. The discriminant is $\Delta = -2^7 \cdot 3^3 \cdot 5$, so for every prime $p \neq 2, 3, 5$, $\mathscr{C}_{(p)}$ is an elliptic curve over $\mathbb{F}_p$; in particular, the fiber is non-singular.

- For $p = 2$, we get the curve $\mathscr{C}_{(2)} : Y^2 = X(X+1)^2$, which has a double point,
- for $p = 3$, we get $\mathscr{C}_{(3)} : Y^2 = X^3$, which has a cusp,
- and for $p = 5$ we get $\mathscr{C}_{(5)} : Y^2 = (X-1)(X-2)^2$, which also has a double point.

We can illustrate this in the following figure:



The fibers over (2), (3), and (5) are singular. The generic fiber is the fiber over (0). The fibers over other primes $(p), (q)$ are regular. See [Sil94], p.300, Examples IV.2.2.1-2.2.3 and p. 311ff for more details and similar examples.

As stated in the remark above, there is a theory of Weil divisors on arithmetic surfaces. In fact, irreducible divisors only have two possible forms: horizontal or fibral divisors. For this, we need the following result of algebraic geometry.

Proposition 9.1.4. *Let $\phi : C_1 \to C_2$ be a morphism of curves. Then ϕ is either constant or surjective.*

Proof. [Sil86], p. 24, Theorem II.2.3 or [Har77], Proposition II.6.8. □

Proposition 9.1.5. *Every irreducible divisor on an arithmetic surface $\mathscr{C}/R$ is either*

1. *contained in a special fiber* (fibral divisor) *or,*
2. *maps surjectively onto* $\operatorname{Spec} R$ (horizontal divisor).

Proof. If D is an irreducible divisor which is not contained in a special fiber, consider the projection map $\pi : D \to \operatorname{Spec} R$ induced by the projection map of the scheme $\mathscr{C}$. This is not constant, since D does not lie in a special fiber, so, by the previous proposition, it is surjective. □

We now consider rational points on the generic fiber of an arithmetic surface. If R is a ring, recall definition 2.2.11 of R-valued points of an R-scheme X:

If R is a ring and X an R-scheme, the set of *R-valued points of X* is the set

$$X(R) = \{R\text{-morphisms} \operatorname{Spec} R \to X\}.$$

Remark 9.1.6. Note that one can identify the image of a section $\sigma : \operatorname{Spec} R \to \mathscr{C}$ with a horizontal divisor on $\mathscr{C}$.

We can identify rational points on the generic fiber of an arithmetic surface, i.e. points in $C(K)$, with R-valued points of $\mathscr{C}$:

Theorem 9.1.7. *([Sil94], Corollary IV.4.4 (a)) Let R be a Dedekind domain and K its field of fractions, let $\mathscr{C}/R$ be an arithmetic surface and C/K its generic fiber. If $\mathscr{C}$ is proper over R,*

$$C(K) = \mathscr{C}(R).$$

This theorem tells us that rational points on an elliptic curve "extend" to a section of the associated arithmetic surface, i.e. define a horizontal divisor on $\mathscr{C}$.

The following theorem/definition is a bit technical, but we need the result for the example in the next section. The first assertion was proven by Abhyankar and Lipman, the second is due to Lichtenbaum and Shafarevich.

Theorem 9.1.8. *([Sil94] p. 317, Theorem IV.4.5) Let R be a Dedekind domain and K its field of fractions, and let C/K be a non-singular projective curve of genus g. Then*

1. (Resolution of Singularities for Arithmetic Surfaces) *There exists a regular arithmetic surface $\mathscr{C}/R$, proper over R, whose generic fiber is isomorphic to C/K. We call $\mathscr{C}/R$ a* proper regular model *for C/K.*

2. (Minimal Models Theorem) *Assume that $g \geqslant 1$. Then there exists a proper regular model $\mathscr{C}^{\min}/R$ for C/K with the following minimality property:*

   Let $\mathscr{C}/R$ be another proper regular model for C/K. Fix an isomorphism from the generic fiber of $\mathscr{C}$ to the generic fiber of $\mathscr{C}^{\min}$. This induces an R-birational map $\mathscr{C} \to \mathscr{C}^{\min}$, which is an R-isomophism. $\mathscr{C}^{\min}$ is called the minimal proper model *for C/K. It is unique up to isomorphism.*

9.2 Theta functions

Theta functions play an important role in the theory of elliptic curves, or – more generally – in the theory of Abelian varieties. They are related to the Weierstrass $\wp$-function (see [Sha94a], p. 145, 152) and give a projective embedding of tori $\mathbb{C}/\Lambda_\tau$, where $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ (Lefschitz Theorem, see e.g. [Mur93], p. 51).

Remark 9.2.1. For higher dimensional tori, they do not aways give such an embedding. In this case, this holds if there is a positive line bundle on the torus, i.e. a line bundle with positive first Chern class, see e.g. [Mur93]. This is a special case of the Kodaira theorem [GH78], p. 181, which gives a projective embedding for a compact complex manifold with a positive line bundle.

9.2.1 Jacobi theta functions

Definition 9.2.2. Let $\mathcal{H}$ denote the *complex upper half plane*, i.e.

$$\mathcal{H} = \{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}.$$

Definition 9.2.3. The standard *theta function* on $\mathbb{C} \times \mathcal{H}$ is defined as

$$\vartheta(z, \tau) = \sum_{m \in \mathbb{Z}} e^{\pi i m^2 \tau + 2\pi i m z}.$$

Proposition 9.2.4. *For every $k \in \mathbb{Z}$, the theta function satisfies the following:*

1. $\vartheta(z + k, \tau) = \vartheta(z, \tau)$ *and*
2. $\vartheta(z + k\tau, \tau) = \vartheta(z, \tau) e^{-\pi i k^2 \tau - 2\pi i k z}$.

Proof. The first assertion is clear since $e^{2\pi i k} = 1$ for $k \in \mathbb{Z}$.

Since

$$\begin{aligned}\vartheta(z+k\tau,\tau) &= \sum_{m\in\mathbb{Z}} e^{\pi i m^2\tau+2\pi i m(z+k\tau)}\\ &= \sum_{m\in\mathbb{Z}} e^{\pi i(m+k)^2\tau+2\pi i(m+k)z}\cdot e^{-\pi i k^2\tau-2\pi i k z}\\ &= \vartheta(z,\tau)e^{-\pi i k^2\tau-2\pi i k z},\end{aligned}$$

the second property follows. □

Often it is useful to also consider the related theta functions with characteristics:

Definition 9.2.5. The *theta functions with characteristics* (α,β) are given on $\mathbb{C}\times\mathcal{H}$ by

$$\vartheta_{\alpha,\beta}(z,\tau) = \sum_{m\in\mathbb{Z}} e^{\pi i(m+\alpha)^2\tau+2\pi i(m+\alpha)(z+\beta)},$$

for $\alpha,\beta\in\mathbb{C}$. If $l\in\mathbb{Z}$ such that $\alpha,\beta\in\frac{1}{l}\mathbb{Z}$, we say that $\vartheta_{\alpha,\beta}$ has *level* l. A special case are the *Jacobi theta functions*, which have level 2. They are usually written as

$$\begin{aligned}\vartheta_{00}(z,\tau) &= \vartheta_{0,0}(z,\tau)\\ \vartheta_{01}(z,\tau) &= \vartheta_{0,\frac{1}{2}}(z,\tau)\\ \vartheta_{10}(z,\tau) &= \vartheta_{\frac{1}{2},0}(z,\tau)\\ \vartheta_{11}(z,\tau) &= \vartheta_{\frac{1}{2},\frac{1}{2}}(z,\tau).\end{aligned}$$

Remark 9.2.6. Note that $\vartheta_{\alpha,\beta}(z,\tau) = \vartheta(z+\alpha\tau+\beta,\tau)e^{\pi i\alpha^2\tau+2\pi i\alpha(z+\beta)}$, in particular, $\vartheta_{0,0}(z,\tau)=\vartheta_{00}(z,\tau)=\vartheta(z,\tau)$. Often one uses the more general Riemann theta function, which is defined on $\mathbb{C}^n\times\mathbb{H}_n$, where $\mathbb{H}_n$ is the Siegel upper half space. Even, more general, one defines a theta function by means of a functional equation, as we will see later.

We use [Mur93] for the following propositions.

Proposition 9.2.7. *Let $l\in\mathbb{Z}$, $l\geqslant 2$, and let $\alpha,\beta\in\frac{1}{l}\mathbb{Z}$. The theta functions with characteristics (α,β) satisfy*

1. $\vartheta_{\alpha,\beta}(z+l,\tau)=\vartheta_{\alpha,\beta}(z,\tau)$ *and*
2. $\vartheta_{\alpha,\beta}(z+l\tau,\tau)=\vartheta_{\alpha,\beta}(z,\tau)e^{-\pi i l^2\tau-2\pi i l z}$.

Proof. Both properties follow from proposition 9.2.4 and remark 9.2.6:

$$\begin{aligned}\vartheta_{\alpha,\beta}(z+l,\tau) &= \vartheta(z+l+\alpha\tau+\beta,\tau)e^{\pi i\alpha^2\tau+2\pi i\alpha(z+l+\beta)}\\ &= \vartheta(z+\alpha\tau+\beta,\tau)e^{\pi i\alpha^2\tau+2\pi i\alpha(z+\beta)}\cdot e^{2\pi i\alpha l}\\ &= \vartheta_{\alpha,\beta}(z,\tau)e^{2\pi i\alpha l}\\ &= \vartheta_{\alpha,\beta}(z,\tau),\end{aligned}$$

since $\alpha \cdot l \in \mathbb{Z}$. The second property is shown similary:

$$\begin{aligned}
\vartheta_{\alpha,\beta}(z+l\tau,\tau) &= \vartheta(z+l\tau+\alpha\tau+b,\tau)e^{\pi i\alpha^2\tau+2\pi i\alpha(z+l\tau+\beta)} \\
&= \vartheta(z+\alpha\tau+\beta,\tau)e^{-\pi il^2\tau-2\pi il(z+\alpha\tau+\beta)}\cdot e^{\pi i\alpha^2\tau+2\pi i\alpha(z+l\tau+\beta)} \\
&= \vartheta(z+\alpha\tau+\beta,\tau)e^{\pi i\alpha^2\tau+2\pi i\alpha(z+\beta)}\cdot e^{-\pi il^2\tau-2\pi il(z+\beta)} \\
&= \vartheta_{\alpha,\beta}(z,\tau)e^{-\pi il^2\tau-2\pi ilz},
\end{aligned}$$

since $\beta \cdot l \in \mathbb{Z}$. □

Proposition 9.2.8. *The Jacobi theta functions satisfy the following properties for every $k \in \mathbb{Z}$:*

1. $\vartheta_{ij}(z+k,\tau) = \begin{cases} \vartheta_{ij}(z,\tau) & \text{if } i=0 \\ -\vartheta_{ij}(z,\tau) & \text{if } i=1, \end{cases}$ *and*

2. $\vartheta_{ij}(z+k\tau,\tau) = \begin{cases} \vartheta_{ij}(z,\tau)e^{-\pi ik^2\tau-2\pi ikz} & \text{if } j=0 \\ -\vartheta_{ij}(z,\tau)e^{-\pi ik^2\tau-2\pi ikz} & \text{if } j=1. \end{cases}$

Proof. The proof is analogous to the one above. □

Theorem 9.2.9. *Let $\tau \in \mathcal{H}$ be fixed, and, for simpler notation, denote $\vartheta_{\alpha,\beta}(z,\tau)$ simply by $\vartheta_{\alpha,\beta}(z)$. Then the holomorphic map $\varphi : \mathbb{C}/\Lambda_\tau \to \mathbb{P}^2_{\mathbb{C}}$ defined by*

$$\varphi(z) = (\vartheta_{0,0}(z)\vartheta_{1,1}(z)^2 : \vartheta_{1,0}(z)\vartheta_{0,1}(z)\vartheta_{1,1}(z) : \vartheta_{0,0}(z)^3)$$

induces an isomorphism from $\mathbb{C}/\Lambda_\tau$ onto the smooth cubic with homogeneous equation

$$Y^2Z = X(\alpha X - \beta Z)(\beta X + \alpha Z),$$

where $\alpha = \frac{\vartheta_{1,0}(0)^2}{\vartheta_{0,0}(0)^2}$, and $\beta = \frac{\vartheta_{0,1}(0)^2}{\vartheta_{0,0}(0)^2}$.

Proof. See [Deb05], p. 12. The proof is similar to the one for the embedding of an elliptic curve using the Weierstrass $\wp$-function, see B.1.4. □

Note that this gives us a projective embedding of a torus onto an elliptic curve in $\mathbb{P}^2_{\mathbb{C}}$ using theta functions.

9.2.2 Theta functions

The definition of Jacobi theta functions can be extended to a more general notion of theta functions. The transformation properties of the Jacobi theta functions in propositions 9.2.8 and 9.2.4 are taken as a model for a more general transformation behaviour. We follow [Deb05], chapter 4, and [Mur93], chapter 5. These more general theta functions are used in the next section to examine global sections of line bundles on complex tori, i.e. on elliptic curves.

Definition 9.2.10. Let V be a complex vector space, and let Λ be a lattice in V. A *theta function of type (T,J)* associated to Λ is an entire function ϑ on V, not identically zero, such that there is a function $T : V \times \Lambda \to \mathbb{C}$ which is $\mathbb{C}$-linear in the first variable and a function $J : \Lambda \to \mathbb{C}$ such that for all $z \in V$ and $\lambda \in \Lambda$,

$$\vartheta(z+\lambda) = e^{2\pi i(T(z,\lambda)+J(\lambda))}\vartheta(z).$$

Example 9.2.11. For fixed $\tau \in \mathcal{H}$, the standard theta function $\vartheta(z,\tau)$ is a theta function for $V = \mathbb{C}$ and $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$. In this case, if $\lambda = k + l\tau$, $T(z,\lambda) = -lz = -z\mathrm{Im}\,\lambda \cdot \frac{1}{\mathrm{Im}\,\tau}$ and $J(\lambda) = -\frac{1}{2}(\mathrm{Im}\,\lambda/\mathrm{Im}\,\tau)^2\tau$.

By evaluating $\vartheta(z+\lambda_1+\lambda_2)$ in two different ways, one obtains the following identities for the function T:

$$T(\lambda_1,\lambda_2) \equiv T(\lambda_2,\lambda_1) \mod \mathbb{Z},$$

and

$$T(z,\lambda_1+\lambda_2) \equiv T(z,\lambda_1) + T(z,\lambda_2) \mod \mathbb{Z}.$$

We now use the identity above to extend T. Since Λ is a lattice in V, we get an isomorphism $\Lambda \otimes \mathbb{R} \cong \mathbb{C}^n$, and, identifying the two, we can extend T to a form

$$T : V \times V \longrightarrow \mathbb{C}$$

which is $\mathbb{C}$-linear in the first variable and $\mathbb{R}$-linear in the second variable.

Setting $\omega(x,y) = T(x,y) - T(y,x)$ gives an $\mathbb{R}$-bilinear form on V which is alternating, real, takes on integral values on $\Lambda \times \Lambda$, and satisfies $\omega(ix,iy) = \omega(x,y)$, see [Mur93], Proposition 5.1.

Furthermore, a Hermitian form H is defined on V by

$$H(x,y) = \omega(x,iy) + i\omega(x,y).$$

Definition 9.2.12. Let V be a vector space, and let Λ be a lattice in V. A *normalized theta function of type* (H,α) associated to Λ is a theta function ϑ on V such that for every $\lambda \in \Lambda$

$$T = \frac{1}{2i}H \quad \text{and} \quad \mathrm{Im}\,J = -\frac{1}{4}H(\lambda,\lambda).$$

A normalized theta function satisfies

$$\vartheta(z+\lambda) = \alpha(\lambda)e^{\pi H(\lambda,z)+\frac{\pi}{2}H(\lambda,\lambda)}\vartheta(z)$$

for every $\lambda \in \Lambda$, where $\alpha : \Lambda \to \{z \in \mathbb{C} : |z| = 1\}$ is a function such that

$$\alpha(\lambda_1+\lambda_2) = \alpha(\lambda_1)\alpha(\lambda_2)(-1)^{\omega(\lambda_1,\lambda_2)}$$

for every $\lambda_1, \lambda_2 \in \Lambda$. Note that here it is important that ω takes on integral values on $\Lambda \times \Lambda$.

9.2.3 Sections of line bundles over tori

As already suggested in the previous section, theta functions are used to explicitly describe line bundles on a torus $\mathbb{C}/\Lambda$, in particular sections of line bundles. In this section, we follow [GH78], p. 307ff, and explain this correspondence.

Let $\mathcal{L} \to V/\Lambda$ be a line bundle over a torus given by a one-dimensional $\mathbb{C}$-vector space[1] V and a lattice Λ. Denote by $\pi : V \to V/\Lambda$ the projection map. Then $\pi^*\mathcal{L} \to \mathbb{C}$ is a trivial line bundle by example 3.1.5. Thus, we can find a global trivialization

$$\varphi : \pi^*\mathcal{L} \to V \times \mathbb{C}.$$

For $z \in V$, $\lambda \in \Lambda$, we can identify the fibers $(\pi^*\mathcal{L})_z \cong (\pi^*\mathcal{L})_{z+\lambda} \cong \mathcal{L}_{\pi(z)}$. Then the trivialization φ, considered at z and $z+\lambda$, $\varphi_z : (\pi^*\mathcal{L})_z \xrightarrow{\cong} \mathbb{C}$ and $\varphi_{z+\lambda} : (\pi^*\mathcal{L})_{z+\lambda} \xrightarrow{\cong} \mathbb{C}$, determines an automorphism of $\mathbb{C}$:

$$\mathbb{C} \xleftarrow{\varphi_z} (\pi^*\mathcal{L})_z = \mathcal{L}_{\pi(z)} = (\pi^*\mathcal{L})_{z+\lambda} \xrightarrow{\varphi_{z+\lambda}} \mathbb{C}.$$

Automorphisms of $\mathbb{C}$ are given as multiplications by nonzero complex numbers, and we denote the complex number determining the automorphism above by $e_\lambda(z)$. This gives a collection of functions

$$\{e_\lambda \in \mathcal{O}^*(V)\}_{\lambda\in\Lambda}.$$

These functions are called *multipliers for $\mathcal{L}$*.

By definition, they satisfy the compatibility condition

$$e_{\lambda'}(z+\lambda)e_\lambda(z) = e_\lambda(z+\lambda')e_{\lambda'}(z) = e_{\lambda+\lambda'}(z)$$

for all $\lambda, \lambda' \in \Lambda$.

On the other hand, given a set of nonzero entire functions $\{e_\lambda\}_{\lambda\in\Lambda}$ satisying the compatibility condition above, we define a vector bundle over V/Λ as follows: the lattice Λ acts on $V \times \mathbb{C}$ by sending a $\lambda \in \Lambda$ to the map

$$V \times \mathbb{C} \longrightarrow V \times \mathbb{C}, \quad (v,t) \longmapsto (v+\lambda, e_\lambda(v)\cdot t).$$

This indeed is an action by the compatibility condition.

In fact, it suffices to specify the multipliers for a basis of the lattice (of course satisfying the compatibility conditions). Then the compatibility conditions determine the other multipliers e_λ.

Note that the function $\vartheta(z+\lambda)/\vartheta(z)$, where ϑ is a normalized theta function of type (H,α) on $V=\mathbb{C}$, satisfies the compatibility conditions. Thus, for every type (H,α), we get multipliers

$$e_\lambda(z) = \frac{\vartheta(z+\lambda)}{\vartheta(z)} = \alpha(\lambda)e^{\pi H(\lambda,z)+\frac{\pi}{2}H(\lambda,\lambda)}.$$

[1] Clearly, $V \cong \mathbb{C}$, but we use this notation as this can be generalized to higher-dimensional vector spaces.

By the construction above, we get a line bundle $\mathcal{L}(H, \alpha)$. In fact, by the theorem of Apell-Humbert ([Deb05], Theorem 5.17), any line bundle on a complex torus is obtained in this way.

Now, given any line bundle $\mathcal{L}$ on the complex torus V/Λ, consider a section $\sigma : V/\Lambda \to \mathcal{L}$. Then, we get a section $\pi^*\sigma : V \to \pi^*\mathcal{L} \cong V \times \mathbb{C}$ and, by the trivialization of $\pi^*\mathcal{L}$, a map $\bar{\sigma} : V \to V \times \mathbb{C}$:

$$\begin{tikzcd}
V \times \mathbb{C} \arrow[r, "\simeq"] & \pi^*\mathcal{L} \arrow[r] \arrow[d] & \mathcal{L} \arrow[d] \\
& V \arrow[r, "\pi"'] \arrow[u, bend left, "\pi^*\sigma"] \arrow[ul, bend left, "\bar{\sigma}"] & V/\Lambda \arrow[u, bend right, "\sigma"']
\end{tikzcd}$$

Then, since $\mathcal{L} = \mathcal{L}(H, \alpha) = V \times \mathbb{C}/\sim$, where $\sim$ is the relation $(v, t) \sim (v + \lambda, e_\lambda(v) \cdot t)$, we get an entire function ϑ_σ such that $\bar{\sigma}(v) = (v, \vartheta_\sigma(v))$, and

$$\vartheta_\sigma(z + \lambda) = e_\lambda(z) \cdot \vartheta_\sigma(z) = \alpha(\lambda) e^{\pi H(\lambda, z) + \frac{\pi}{2} H(\lambda, \lambda)} \vartheta_\sigma(z).$$

Thus, we establish a correspondence between normalized theta functions and global sections of line bundles on a complex torus.

Remark 9.2.13. Note that the the theta functions $\vartheta_{\alpha,\beta}(z)$ also are normalized theta functions. In fact, if $V/\Lambda = E$ is considered as an elliptic curve with origin O_E, ϑ_{00} has just the transformation properties to correspond to a global section of $\mathcal{O}(O_E)$, the line bundle associated to the prime divisor given by the origin O_E.

9.3 An example: the arithmetic degree of an elliptic curve

To apply the methods in Arakelov geometry, we need to define a metric on line bundles over complex tori, i.e elliptic curves. We first define such a metric on the bundle $\mathcal{O}(O_E)$ and then, as an example, we present a result of Jürg Kramer, [Kra92].

Let E be an elliptic curve over $\mathbb{Q}$ with origin O_E having semistable reduction[2]. Semistable reduction just means that the reductions $\bmod p$ behave nicely. There might be primes p where E_p is singular, but it allows only double points, i.e. it excludes cusps. Furthermore, let $p : \mathscr{E} \to \operatorname{Spec} \mathbb{Z}$ denote the minimal regular model of $E/\mathbb{Q}$, and let $\mathcal{E}(\mathbb{C}) = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$.

Denote by $\mathcal{L}_{\mathscr{E}}$ the line bundle $\mathcal{O}(2O_{\mathscr{E}})$, see 3.5.8. The line bundle $\mathcal{O}(O_{\mathscr{E}})$, in fact, is the line bundle on E giving E *principal polarization*. This is an important concept in the theory of Abelian varieties, see e.g. [Deb05], [Mur93].

Using the construction of global sections of line bundles on tori given in the previous section, we now endow $\mathcal{L}_{\mathscr{E}}$ with a Hermitian metric. Let σ be a section of $\mathcal{L}_{\mathscr{E}}$, and let $y = \operatorname{Im} z$. We equip $\mathcal{L}_{\mathscr{E}}$ with the Hermitian metric given by

$$\|\sigma\|_{\mathscr{E}}(z) = |\sigma(z)| \cdot e^{-\pi y^2/\operatorname{Im}\tau} \cdot (\operatorname{Im}\tau)^{\frac{1}{2}}.$$

[2]This condition is needed for the example of Jürg Kramer from [Kra92].

In this case, $\vartheta^2 = \vartheta_{00}^2$ gives a global section of $\mathcal{L}_{\mathscr{E}}$. For $\vartheta = \vartheta_{00}$, we can show explicitely that the norm indeed is invariant under translation in the lattice and thus is well-defined:

$$\begin{aligned}
\|\vartheta(z+\tau,\tau)\| &= |\vartheta(z+\tau,\tau)| \cdot e^{-\pi(y+\operatorname{Im}\tau)^2/\operatorname{Im}\tau} \cdot (\operatorname{Im}\tau)^{\frac{1}{2}} \\
&= |\vartheta(z,\tau)e^{-\pi i\tau-2\pi iz}| \cdot e^{-\pi y^2/\operatorname{Im}\tau} \cdot e^{-2\pi y-\pi\operatorname{Im}\tau} \cdot (\operatorname{Im}\tau)^{\frac{1}{2}} \\
&= |\vartheta(z,\tau)| \cdot e^{2\pi y+\pi\operatorname{Im}\tau} \cdot e^{-2\pi y-\pi\operatorname{Im}\tau} \cdot e^{-\pi y^2/\operatorname{Im}\tau} \cdot (\operatorname{Im}\tau)^{\frac{1}{2}} \\
&= |\vartheta(z,\tau)| \cdot e^{-\pi y^2/\operatorname{Im}\tau} \cdot (\operatorname{Im}\tau)^{\frac{1}{2}} \\
&= \|\vartheta(z,\tau)\|.
\end{aligned}$$

The first arithmetic Chern class of this Hermitian line bundle can be represented by $(\operatorname{div}\sigma, g(z))$, where σ is any section, and g(z), by abuse of notation, is the Green current associated to

$$g(z) = -\log\|\sigma\|_{\mathscr{E}}^2(z) = -\log|\sigma(z)|^2 + \frac{2\pi y^2}{\operatorname{Im}\tau} - \log\operatorname{Im}\tau.$$

In his habilitation, Kramer explicitely calculated the "analytic part" of the degree of an elliptic curve equipped with this Hermitian line bundle. The analytic part of the degree is the part coming from the points at infinity.

Theorem 9.3.1. *(Kramer) Let $\mathscr{E}/\mathbb{Z}$ be the minimal regular model of an elliptic curve $E/\mathbb{Q}$ having semistable reduction. If $4|m$, the arithmetic degree $\widehat{\deg}\,(\mathcal{L}_{\mathscr{E}}^{\otimes m}, \|\cdot\|_{\mathscr{E}}^m)$ is given by*

$$\widehat{\deg}\,(\mathcal{L}_{\mathscr{E}}^{\otimes m}, \|\cdot\|_{\mathscr{E}}^m) = \frac{4m^2}{3}(\Sigma_{geo} + \Sigma_{ana}).$$

Here,

$$\Sigma_{geo} = ((D_{0,0}\cdot D_{1,1})_2 + (D_{0,1}\cdot D_{1,1})_2 + (D_{1,0}\cdot D_{1,1})_2)\cdot\log 2,$$

where $(D_{j,k}\cdot D_{1,1})_2$ means the intersection number of $D_{j,k}$ and $D_{1,1}$ in the fiber over $2\in\operatorname{Spec}\mathbb{Z}$, and

$$\Sigma_{ana} = -\log\left(|\eta(\tau)|^6\cdot(\operatorname{Im}\tau)^{\frac{3}{2}}\right) - \log 2.$$

Here $\eta(\tau)$ is Dedekind's eta function, i.e. $\eta(\tau) = q^{\frac{1}{24}}\prod_{n\geqslant 1}^{\infty}(1-q^n)$, where $q = e^{2\pi i\tau}$.

Thus, the difficulty is to calculate the geometric part of the degree, the intersection product in the formula above. This corresponds to the "classical" part of the degree. The analytic part, which corresponds to the part at infinity, can be calculated explicitely.

This was extended by Jay Jorgenson and Jürg Kramer to the case of Abelian varieties, see [JK98]. Also in this case, their result is only on the analytic part of the degree.

Part III

A view towards integral points

Chapter 10

An application to integral points on elliptic curves

In this chapter, we give an outlook on how Arakelov theory could potentially be used in the context of *integral points* on elliptic curves. More precisely, the idea is to apply the methods of Arakelov geometry to the problem of finding an effective proof for the finiteness of integral points on elliptic curves and thus giving a bound for their height.

We first give an overview of the existing finiteness results in the context of the number of integral points on elliptic curves. Subsequently, we define integral points on a curve and then extend this definition. We discuss Siegel's theorem and Baker's method of using a bound for linear forms in logarithms. To turn to modern methods from Arakelov geometry, we consider the modern notion of an integral point on an arithmetic surface. Finally, we briefly sketch the ideas of how to apply the presented theory to this problem.

10.1 Integral points

We first discuss the concept of an integral point on an elliptic curve. This notion has changed throughout history – it was extended slowly to fit in with modern mathematics. We start with the "classical" definition and work up to the one we need in our work. We mostly use [Ser89] for the definitions and statements.

Definition 10.1.1. Let $f(x, y) = 0$ be an irreducible plane curve X over $\mathbb{Q}$. Then an *integral point on* X is a pair $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that $f(a, b) = 0$.

This definition can be extended to the following:

Definition 10.1.2. Let K be a number field and S a finite set of places $\mathfrak{p}$ of K containing the infinite places. A rational point $P = (x_1, \dots, x_n)$ of a variety X given by an equation with integral coefficients is *S-integral* if all coordinates x_i satisfy

$$v_{\mathfrak{p}}(x_i) \geqslant 0, \quad \forall\, \mathfrak{p} \notin S.$$

Remark 10.1.3. Note that if $K = \mathbb{Q}$ and $S = \{p_1, \ldots, p_k\} \cup S_\infty$, where S_∞ is the set of infinite places, this means that the denominators of coefficients of the point P are at most divisible by p_i.

Lemma 10.1.4. *If $K = \mathbb{Q}$, X is an irreducible plane curve over $\mathbb{Q}$, and $S = \varnothing$, then the definitions 10.1.1 and 10.1.2 coincide.*

Proof. Let $x \in \mathbb{Q}$, $x = \frac{a}{b}$, where $a, b \in \mathbb{Z}$. Then

$$\forall \text{ primes } p : v_p(x) \geqslant 0 \Leftrightarrow \forall \text{ primes } p : p \mid b \Leftrightarrow b = 1 \Leftrightarrow x \in \mathbb{Z}. \quad \square$$

Definition 10.1.5. Let L be a field with a family of absolute values satisfying the product formula. Let K/L be a finite field extension. Denote by S_∞ the set of all infinite valuations of K. Let S be a finite set of absolute values of K containing S_∞. Then the *ring of S-integers, R_S,* is the set of elements x of K such that

$$v(x) \geqslant 0, \quad \forall\, v \notin S.$$

We can go even further and extend this definition even further as done in [Ser89]:

Definition 10.1.6. Let K be a number field and S a finite set of places $\mathfrak{p}$ of K containing the infinite places. Let X be an affine variety over K. Then, the set of global sections $\Lambda = \Gamma(X, \mathcal{O}_X)$ is a finitely generated algebra over K. Furthermore, let M be a set of rational points of X, i.e. $M \subset X(K)$. We call M *quasi-integral relative to the ring R_S*, if for alle $f \in \Lambda$ there is an $a \in K^*$ such that $f(M) \subset aR_S$. In other words, the denominators of $f(x)$, where $x \in M$, are bounded.

Note that since Λ is finitely generated, one can check this condition for a set of generators of Λ. Furthermore, if we choose an embedding of X into $\mathbb{A}^n$ (this is possible since X is an affine variety) and fix coordinates $z_1, \ldots z_n$, then the condition that a set M is quasi-integral is equivalent to saying that the coordinates (in $\mathbb{A}^n$) of the points of M have a common denominator.

This relates to the concept of R-valued points from definition 2.2.11:

Lemma 10.1.7. *([Ser89], section 7.1) The following properties are equivalent:*

1. *The set M is quasi-integral relative to R_S.*
2. *There is an R_S-scheme X' of finite type such that*
   (a) *$X = X' \times_{R_S} K$,*

(b) every point x of M extends to an R_S-valued point of X'.

3. There is an affine R_S-scheme X' of finite type satisfying (a) and (b).

Proof. 1. $\Rightarrow$ 3. If M is quasi-integral, first choose an immersion of X into $\mathbb{A}^n$. Multiply the coordinate functions in $\mathbb{A}$ by a common denominator, which is possible since M is quasi-integral. Then take the R_S-subalgebra generated by these, and take the corresponding affine R_S-scheme. This satisfies the asserted properties.

3. $\Rightarrow$ 2. is trivial

2. $\Rightarrow$ 1. Let $f \in \Lambda$. Then there is an $a \in R_S, a \neq 0$ such that af extends to X'. The values of af at R_S-integral points of X' (morphisms $x : \operatorname{Spec} R_S \to X'$) are S-integers. □

10.2 Siegel's theorem

Siegel used his earlier work on diophantine approximation to give a first proof of the finiteness of integral points on affine curves which are not exceptional.

The original statement of the theorem in [Sie29] was the following:

> "Die algebraische Gleichung $f(x, y) = 0$ sei nicht dadurch identisch in einem Parameter t lösbar, daß man entweder $x = A/L^n$, $y = B/L^n$ oder $x = C/Q^n$, $y = D/Q^n$ setzt, wo A, B, C, D ganzzahlige Polynome in t, L ein lineares, Q ein indefinites quadratisches Polynom in t bedeuten. Dann hat sie nur endlich viele Lösungen in ganzen rationalen Zahlen."

In this paper, he also formulates the result for algebraic numbers:

> "Damit $f(x, y) = 0$ in einem algebraischen Zahlkörper unendlich viele ganzartige Lösungen besitzt, ist notwendig und hinreichend, daß sich die Gleichung $f = 0$ entweder in $u = 0$ oder in $ut = 1$ überführen läßt, und zwar durch eine birationale Transformation, welche alle ganzartigen Paare x, y und u, t miteinander verknüpft."

Here, "ganzartig" means that x and y are elements of the algebraic number field such that cx, cy are integral, where c ist a fixed natural number.

A more modern formulation of the statement can be found e.g. in [Ser89]:

Theorem 10.2.1. *(Siegel's theorem) Let K be a number filed. If the smooth affine curve X is not isomorphic to $\mathbb{P}^1\backslash\{0\}$ or $\mathbb{P}^1\backslash\{0, 1\}$, then every subset of $X(K)$ which is quasi-integral relative to R_S is finite.*

Note that the asymptotic behavior for the exceptional cases $(\mathbb{P}^1\backslash\{0\})(K) \cong \mathbb{G}_a(K) = K$ and $(\mathbb{P}^1\backslash\{0,1\})(K) \cong \mathbb{G}_m(K) = K^*$ is fundamentally different. We illustrate this for $\mathbb{Z}$:

$$\#\{x \in \mathbb{G}_a(\mathbb{Z}) : H(x) = |x| \leqslant N\} = 2N + 1$$

but

$$\#\{x \in \mathbb{G}_m(\mathbb{Z}) : H(x) = |x| \leqslant N\} = 2.$$

For arbitrary $\mathcal{O}_K$, the asymptotic of $\#\{x \in \mathbb{G}_m(\mathbb{Z}) : H(x) \leqslant N\}$ depends on the structure of the unit group of $\mathcal{O}_K$. By Dirichlet's unit theorem it is a finitely generated Abelian group, and can be either finite or infinite. In the infinite case, the asymptotic is logarithmic, as we see in the following example.

Example 10.2.2. We make this explicit for $K = \mathbb{Q}(\sqrt{m})$, where $m > 0$. Then, by [Wüs04], the group of units $U_m := \mathbb{G}_m(\mathcal{O}_K) \cong \mu_2 \times \mathbb{Z}$, where μ_2 denotes the group of second roots of unity.

Let $u \in U_m$ such that $\langle u \rangle \cong \mathbb{Z}$. Since $u \cdot u^{-1} = 1$ and, since u is a fundamental unit and $|u| \neq 1$, we can assume that $|u| > 1$. Then, an element $v \in U_m$ can be written uniquely as $v = \xi u^n$, where $n \in \mathbb{N}$. Furthermore, $|v| = |u|^n$. Since $H(v^{-1}) = H(v)$, we can choose $n \geqslant 0$. Then,

$$H(v) = \left(\max\{|u|^n, 1\} \cdot \max\{|u|^{-n}, 1\}\right)^{\frac{1}{2}}.$$

So, $H(v) = (|u|^n)^{\frac{1}{2}}$, and the condition $H(v) < N$ is equivalent to

$$\frac{n}{2} \log |u| < \log N.$$

Therefore,

$$\#\{x \in \mathbb{G}_m(\mathcal{O}_K) : H(x) \leqslant N\} \sim \log N.$$

The proof of Siegel's theorem uses the Thue-Siegel-Roth theorem on the approximation of irrational numbers and the approximation of rational points on Abelian varieties and on curves of genus at least one. In 1929, Siegel originally had a weaker form of the approximation theorem which made the proof more complicated. Furthermore, he proved the theorem only for the usual integers, with $S = S_\infty$. In 1933, Mahler extended it to S-integral points, but only for genus equal to one and over the rational numbers $\mathbb{Q}$. In 1955, Roth proved his theorem which also led to a further extension of Siegel's theorem. Lang extended Roth's theorem, under reasonable hypotheses, to any field equipped with a product formula. Siegel's theorem then holds for all rings of characteristic 0 of finite type over $\mathbb{Z}$. (See the end of section 7.5 of [Ser89] for the historical overview.)

10.3 Baker's method

From a computational point of view, the weakness of Siegel's theorem is that even though it ensures the finiteness of the number of integral points, it does not give a bound for this number. In other words, the result is not effective.

Baker's method gives an answer to the problem of effectivity of Siegel's theorem in all cases in which it applies. However, it does not apply in all cases.

The first step is to reduce solutions of the given equation to solutions of a so-called *S-unit equation*.

Definition 10.3.1. Let K be a number field and S a finite set of places containing all infinite ones. An element $u \in K$ is called an *S-unit of K* if

$$v_{\mathfrak{p}}(x) = 0, \quad \forall \mathfrak{p} \notin S.$$

The S-units form a finitely generated multiplicative group U_S. If $S = S_\infty$, U_S is the set of units in $\mathcal{O}_K$.

The *inhomogeneous S-unit equation* in two variables is the equation

$$\alpha x + \beta y = 1,$$

where α, β are non-zero elements of K.

The main result on S-unit equations is the following:

Theorem 10.3.2. *(Baker, Theorem 3.1 in [BW07]) There are only finitely many solutions of the equation in S-units x and y and all of these can be effectively determined.*

This result is established using a lower bound for non-zero linear forms of logarithms. This bound was obtained by Baker. A more recent refinement of this result on the bound was given by Baker and Wüstholz in [BW93]:

Theorem 10.3.3. *(Baker-Wüstholz) Let the linear form of logarithms $\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n \neq 0$, where $b_1, \cdots, b_n$ are integers and $\alpha_1, \cdots, \alpha_n$ are algebraic numbers with heights at most $A_1, \cdots, A_n$ (all $\geqslant e$), respectively, and we assume that the logarithms have their principal values. Furthermore, let $b_1, \cdots, b_n$ have absolute values at most B ($\geqslant e$). Then,*

$$\log |\Lambda| > -(16nd)^{2(n+2)} \log A_1 \cdots \log A_n \log B,$$

where d denotes the degree of $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$.

This result is best possible with respect to each of $A_1, \ldots, A_n$ and B. Moreover, the function in n and d is quite sharp. Baker and Wüstholz gave an even stronger result, see [BW07], section 7.2. For a historical discussion of these results, see [BW07].

The main idea of this approach to the number of integral points on a curve is to reduce the problem to solving an S-unit equation. Then, by theorem 10.3.2 the unit equation has finitely many solutions and thus, the original curve has only finitely many solutions. The problem therefore is to reduce the integral points on the given curve to solutions of the S-unit equation.

The first success in finding an effective version of Siegel's theorem was accomplished by Alan Baker in [Bak69] for curves of the type

$$y^2 = f(x),$$

where $f(x)$ is a polynomial with at least three distinct roots.

For a curve of genus one given by a polynomial with coefficients over $\mathbb{Z}$ which is irreducible over $\mathbb{C}$, Alan Baker and John Coates gave a proof of the effectivity of Siegel's theorem in 1970, [BC70]. Their result was the following:

Theorem 10.3.4. *(Baker, Coates, [BC70]) Let $f(x, y) = 0$ be an absolutely irreducible polynomial with degree n and with integer coefficients having absolute values at most H such that the curve $f(x, y) = 0$ has genus 1. Then all integer solutions x, y of $f(x, y) = 0$ satisfy*

$$\max(|x|, |y|) < \exp\exp\exp\left((2H)^{10^{n^{10}}}\right).$$

They give an algorithm to transform the given equation to Weierstrass form in such a way that integral solutions of the original equation become integral solutions of the Weierstrass equation and that this process is effective. This involves an effective version of the Riemann-Roch theorem for function fields, see [Coa70], which uses so-called Puiseux expansions for the construction of rational functions on the curve, and is rather technical. Then, the result is obtained by applying the work in [Bak69] to get a unit equation.

Some further work has been done for curves other than curves of genus one, as for the Thue equation, see e.g. [BW07], chapter 3.3 or the work of Schmidt [Sch92]. For more details, see [BW07], chapter 3.

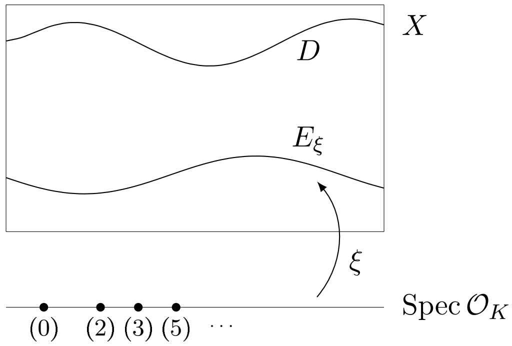
10.4 A modern definition of integral points

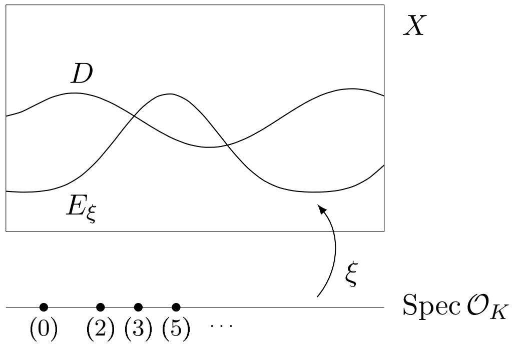
In section 10.1, we already saw several definitions of integral points. We now extend this to arithmetic surfaces.

Definition 10.4.1. Let K be a number field with ring of integers $\mathcal{O}_K$. Let $X \to \operatorname{Spec} \mathcal{O}_K$ be a curve, i.e. X is an arithmetic surface over $\mathcal{O}_K$. Let $D \to \operatorname{Spec} \mathcal{O}_K$ be an effective ample divisor. An $\mathcal{O}_K$-valued point $\xi : \operatorname{Spec} \mathcal{O}_K \to X$ is called *integral*, if the arithmetic intersection product ([BGS94]) (E_ξ, D) of $E_\xi = \xi(\operatorname{Spec} \mathcal{O}_K)$ and D is zero.

Intuitively, one should think of the intersection product as the formal sum of the disjoint irreducible components of the intersection (as sets) with multiplicities. So if the intersection product is zero, the divisor D and E_ξ should not intersect. Recall also the remarks in section 6.3 and remark 8.1.7 on the intersection product.

One can picture the setting as follows:





The left picture shows an integral point ξ, as the divisors D and E_ξ do not intersect. In the right picture, ξ is not an integral point, as D and E_ξ intersect.

Example 10.4.2. One should think of D as the "divisor at infinity". Then, thinking of the fibers over a point $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ as the reduction $\operatorname{mod} \mathfrak{p}$ of e.g. a given equation, it is clear that if we want a point to be integral, we want it not to go to infinity in any of the fibers, as that would translate to the point having a "denominator divisible by $\mathfrak{p}$". We illustrate this intuitively by the example

$$E : y^2 = x^3 + x - 1,$$

considered as an arithmetic surface over $\operatorname{Spec} \mathbb{Z}$. Then $P = (2,3) \in E(\mathbb{Z})$, and the point $\sigma : \operatorname{Spec} \mathbb{Z} \to E$ associates a prime $p \in \operatorname{Spec} \mathbb{Z}$ to the point with coordinates $(\bar{2}, \bar{3})$, where $\bar{2}, \bar{3}$ are the reductions of $2, 3 \operatorname{mod} p$. So e.g. $\sigma(2) = (0,1) \in E_{(2)}(\mathbb{F}_2)$ and $\sigma(3) = (2,0) \in E_{(3)}(\mathbb{F}_3)$. Now consider $2P \in E(\mathbb{Q})$. This point has coordinates $2P = (\frac{145}{6}, -\frac{1825}{36})$. Thus, in the fiber over e.g. 2, this point "goes to infinity". Thus, if e.g. the x-coordinate of a point has a denominator divisible by a prime p, then this point "goes to infinity" when reducing $\operatorname{mod} p$. This is exactly what we want to exclude by the definition above.

10.5 Outlook: from Arakelov theory to integral points

While an answer to the problem of giving an effective upper bound for the height of the finite number of integral points on elliptic curves was given by Alan Baker and John Coates in [BC70] (see section 10.3), the main "limitation" of the existing results is that they depend on the choice of coordinates. In the elliptic curve case, this means that the bound depends on the chosen equation for the curve. In the case of the curves as in the result in [Bak69], the bound depends on the chosen Weierstrass equation of the curve. However, the equation of a curve is not uniquely determined. Therefore, an interesting question for future work is to find a result which is independent of the choice of coordinates. In the following, we briefly discuss an idea how it may be possible to succeed to find such a result using Arakelov theory.

Arakelov geometry allows us to examine the projective module of global sections of a line bundle $\mathcal{L}$ on an arithmetic surface over a ring of integers in a number field more closely. It can be seen as a lattice in the sense of geometry of numbers as was used by Enrico Bombieri and Jeffrey D. Vaaler in their fundamental article [BV83]. Arakelov geometry gives a metric which allows to define convex bodies in the usual way. This allows the application of the techniques of geometry of numbers; in particular, successive minima are defined and one may use Minkowski's first and second theorem, see also chapter 7. The successive minima depend, of course, on the given line bundle $\mathcal{L}$. Geometrical considerations make it necessary to consider the t-th powers $\mathcal{L}^{\otimes t}$ of $\mathcal{L}$, and one has to determine the successive minima of these line bundles in terms of the successive minima of $\mathcal{L}$ – this dependence will be in terms of the first arithmetic Chern class $\widehat{c}_1(\mathcal{L})$ of $\mathcal{L}$ (remark 8.2.9). For this, one needs to apply the arithmetic Riemann-Roch theorem of Henri Gillet and Christoph Soulé, [GS89], [GS92].

This appears to give a promising approach to replacing the rather technical and tedious method of studying Puiseux series as in [Coa70] and [Sch92] to obtain an effective version of the Riemann-Roch theorem for function fields. We suggest to use the modern, new, arithmetic-geometric techniques in Arakelov geometry discussed above instead of making use of Puiseux series. With such a substitute for the old methods, we hope to succeed in modifying the proof of Alan Baker and John Coates in [BC70] to give a coordinate-free result.

However, this is not only a *l'art pour l'art* approach to the problem. It could be a starting point for extending the proposed approach to more complicated geometric situations such as higher dimensional varieties. The first natural, practically unexplored, case would be the study of points on algebraic surfaces, which would correspond to arithmetic threefolds. Although things become much more difficult in these cases, very interesting questions and problems certainly would arise.

Appendix

Appendix A

Algebraic number theory

Throughout this thesis, we rely on several notions from *algebraic number theory.* In this chapter, we give the needed definitions and briefly discuss some results. We mostly use [Neu99] for this chapter.

A.1 Number fields and rings of integers

Definition A.1.1. A *number field* is a finite field extension over $\mathbb{Q}$.

Definition A.1.2. Let $A \subset B$ be a ring extension. An element $x \in B$ is called *integral over* A, if it satisfies a monic equation over A, i.e. $\exists\, a_0, \ldots, a_{n-1} \in A$ such that

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

The set of integral elements of B is, in fact, a ring, the *ring of integers of* B *over* A. The ring of integers of a number field K over $\mathbb{Q}$ usually is denoted by $\mathcal{O}_K$.

Throughout the rest of this section, let K be a number field and $\mathcal{O}_K$ its ring of integers.

Theorem A.1.3. *(Theorem I.3.3 in [Neu99]) Every non-trivial ideal* $\mathfrak{a}$ *of* $\mathcal{O}_K$ *(*$\mathfrak{a} \neq (0), (1)$*) has a factorization in prime ideals which is unique up to ordering.*

Remark A.1.4. The connection between the prime ideals of the respective rings of integers when passing from a field to its extension is very important. In particular, given a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, it is not hard to see that $\mathfrak{p} \cap \mathbb{Z}$ again is a non-zero prime ideal, i.e. there is a prime $p \in \mathbb{Z}$ such that $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$.

Definition A.1.5. Let $\mathfrak{a} \neq (0)$ be an ideal of $\mathcal{O}_K$. Then, the *norm of* $\mathfrak{a}$ is defined as

$$\mathfrak{N}(\mathfrak{a}) = (\mathcal{O}_K : \mathfrak{a}),$$

where $(\mathcal{O}_K : \mathfrak{a})$ denotes the index of $\mathfrak{a}$ in $\mathcal{O}_K$. This indeed is finite by [Neu99], theorem I.2.12. Furthermore, by the Chinese remainder theorem it is multiplicative ([Neu99], theorem II.6.2), i.e. for a non-trivial ideal $\mathfrak{a} = \mathfrak{p}_1^{v_1} \cdots \mathfrak{p}_n^{v_n}$,

$$\mathfrak{N}(\mathfrak{a}) = \mathfrak{N}(\mathfrak{p}_1)^{v_1} \cdots \mathfrak{N}(\mathfrak{p}_n)^{v_n}.$$

A.2 Absolute values and places

Definition A.2.1. Let k be a field. An *absolute value* on k is a function

$$|\cdot| : k \longrightarrow \mathbb{R}$$

with the following properties:

1. $|x| \geqslant 0$, and $|x| = 0 \Leftrightarrow x = 0$,
2. $|xy| = |x||y|$, and
3. $|x + y| \leqslant |x| + |y|$ (triangle inequality).

In the following, we will exclude the trivial absolute value defined by $|0| = 0$ and $|x| = 1$ if $x \neq 0$.

Note that using $|\cdot|$ we can define a topology defined by a metric on k by setting

$$d(x, y) = |x - y|.$$

Definition A.2.2. Two absolute values $|\cdot|_1$ and $|\cdot|_2$ are called *equivalent* if they define the same topology on k. By [Neu99], proposition II.3.3, this is the case if and only if there is a real number $s > 0$ such that

$$|x|_1 = |x|_2^s, \quad \forall\, x \in k.$$

Definition A.2.3. An absolute value $|\cdot|$ is called *non-archimedean*, if the set $\{|n| : n \in \mathbb{N}\}$ is bounded; otherwise it is called *archimedean*.

Non-archimedean absolute values satisfy a stronger version of the triangle inequality, the *ultra-metric inequality* (see theorem II.3.6 in [Neu99]):

$$|x + y| \leqslant \max\{|x|, |y|\}.$$

Definition A.2.4. A *valuation on a field k* is a function

$$v : k \longrightarrow \mathbb{R} \cup \{\infty\}$$

with the following properties:

1. $v(x) = \infty \Leftrightarrow x = 0$,
2. $v(xy) = v(x) + v(y)$, and
3. $v(x + y) \geqslant \min\{v(x), v(y)\}$.

We will exclude the trivial valuation defined by $v(0) = \infty$ and $v(x) = 0$ if $x \neq 0$ in the following.

Remark A.2.5. The denotation of 'absolute value' and 'valuation' is not consistent throughout the literature. Sometimes, one calls an absolute value a valuation and a valuation an exponential valuation. The reason for this is the following:

Let $|\cdot|$ be an absolute value on F. Setting

$$v(x) = -\log|x| \text{ for } x \neq 0, \quad \text{and } v(0) = \infty$$

defines a valuation on F.

On the other hand, given a valuation v on F, we can define an absolute value by fixing a real number $q > 1$ and setting

$$|x| = q^{-v(x)}.$$

Definition A.2.6. A *valuation ring* R is an integral domain with fraction field F such that for every $x \in F$, either $x \in R$, $x^{-1} \in R$, or both. A *discrete valuation ring* is a valuation ring with a value group isomorphic to the integers under addition. A valuation v is called *discrete*, if it has a smallest value $s \in \mathbb{R}$.

There are several definitions of a discrete valuation ring equivalent to the one we gave above, see e.g. [Eis95] for other equivalent definitions[1]. Moreover, if a valuation v is discrete, $v(F^*) = s\mathbb{Z}$. We can normalize the valuation by dividing by s; the new valuation is equivalent to v.

Given any valuation v on a field F, we get a ring $\mathcal{O} = \{x \in F : v(x) \geqslant 0\}$. Its units are the elements $\mathcal{O}^* = \{x \in F : v(x) = 0\}$ and it is a local ring with maximal ideal $\mathfrak{p} = \{x \in F : v(x) > 0\}$ ([Neu99], proposition II.3.8). $\mathcal{O}$ is an integral domain with quotient field F and is a valuation ring. The field $k(\mathfrak{p}) = \mathcal{O}/\mathfrak{p}$ is called the *residue field* (see also 2.2.3).

The statement of the following theorem actually is a classification of so-called Dedekind domains. In particular, $\mathcal{O}_K$ is a Dedekind domain.

Theorem A.2.7. *(Theorem I.11.5 in [Neu99]) The localizations of $\mathcal{O}_K$ in prime ideals $\mathfrak{p}$ are discrete valuation rings.*

[1] A very good overview of equivalent definitions of a discrete valuation ring can be found on the corresponding Wikipedia entry, `http://en.wikipedia.org/wiki/Discrete_valuation_ring` (last retrieved on August 15, 2009).

Example A.2.8. The valuation on a number field K corresponding to a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ is defined by

$$v_{\mathfrak{p}}(a) = \nu_{\mathfrak{p}},$$

if $(a) = \prod_{\mathfrak{p}} \mathfrak{p}^{\nu_{\mathfrak{p}}}$. Thus, we get a non-archimedean valuation for each prime $\mathfrak{p}$ which furthermore are not equivalent. The associated valuation ring is the localization $(\mathcal{O}_K)_{\mathfrak{p}}$.

Definition A.2.9. A *prime or place* $\mathfrak{p}$ of an algebraic number field K is an equivalence class of absolute values on K. The non-archimedean equivalence classes are called *finite primes or finite places*, denoted by $\mathfrak{p} \nmid \infty$, and the archimedean ones *infinite primes or infinite places*, denoted by $\mathfrak{p} \mid \infty$.

Theorem A.2.10. *(Theorem 7.14 in [Mil08]) Let K be an algebraic number field. There exists exactly one prime of K*

1. *for each prime ideal $\mathfrak{p}$,*
2. *for each real embedding, and*
3. *for each conjugate pair of complex embeddings.*

The primes in 2. are called *real primes* and the primes in 3. are called *complex primes.*

Remark A.2.11. Compare these notions to the setting of an affine scheme in algebraic geometry (definitions 2.1.1 and 2.2.3). The points of an affine scheme are the prime ideals of the underlying ring. Then again, recall the observation in example A.2.8 that every prime ideal of $\mathcal{O}_K$ induces a non-archimedean valuation.

Definition A.2.12. The $\mathfrak{p}$-*adic absolute values* $|\cdot|_{\mathfrak{p}}$ for a prime $\mathfrak{p}$ are defined as follows: for a finite prime, let

$$v_{\mathfrak{p}} : K^* \to \mathbb{R}$$

be the normalized valuation induced by the valuation defined as in A.2.8, and for an infinite prime corresponding to an embedding $\sigma : K \hookrightarrow \mathbb{C}$, set

$$v_{\mathfrak{p}}(a) = -\log |\sigma(a)|.$$

Moreover we define $|\cdot|_{\mathfrak{p}}$ separately for the different types of primes:

1. For a finite prime, let
$$|a|_{\mathfrak{p}} = \mathfrak{N}(\mathfrak{p})^{-v_{\mathfrak{p}}(a)}.$$
2. For a real embedding $\sigma : K \hookrightarrow \mathbb{C}$ respectively the corresponding prime we let
$$|a|_{\mathfrak{p}} = |\sigma(a)|.$$
3. For a non-real complex embedding, we define
$$|a|_{\mathfrak{p}} = |\sigma(a)|^2.$$

For $K = \mathbb{Q}$, $\mathcal{O}_K = \mathbb{Z}$, the finite absolute values are simply the p-adic absolute values and the (only) infinite absolute value is the normal real absolute value.

In this context, an important tool is the product formula: it gives a relation between the absolute values of an element. We first consider it in $\mathbb{Q}$ and later extend this to an arbitrary number field.

Theorem A.2.13. *(Theorem II.2.1 in [Neu99]) Let $a \neq 0$ be a rational number. Then*

$$\prod_p |a|_p = 1.$$

Here p runs over all primes in $\mathbb{Z}$ and the symbol ∞.

Proof. Consider the prime factorization of a,

$$a = \pm \prod_{p \neq \infty} p^{v_p(a)}.$$

Since $|a|_p = p^{-v_p(a)}$,

$$a = \pm \prod_{p \neq \infty} \frac{1}{|a|_p}.$$

Moreover, the sign of a is $a/|a|_\infty$, so the equation above gives

$$a = \frac{a}{|a|_\infty} \prod_{p \neq \infty} \frac{1}{|a|_p},$$

which yields the desired result. □

The product formula for arbitrary number fields K follows from the product formula for $\mathbb{Q}$ given in the theorem above.

Theorem A.2.14. *(Product formula) For any nonzero $a \in K$,*

$$\prod_{\mathfrak{p}} |a|_{\mathfrak{p}} = 1,$$

where the product is taken over all primes (finite and infinite) of K.

For a proof, see e.g. [Neu99], Proposition III.1.3 or [Mil08], chapter 8.

A.3 The height of a point

Heights play an important role in diophantine geometry. They make it possible to "count" rational or integral points; they measure the arithmetic complexity of a point on a variety.

We start with the definition of the height of an algebraic number.

Definition A.3.1. Let α be an algebraic number and let $p(x) = a_d x^d + \cdots + a_1 x + a_0$ be the minimal polynomial of α, so $a_i \in \mathbb{Z}$ and $a_0, \cdots, a_d$ are relatively prime. Then the *(absolute) height of* α is

$$H(\alpha) = |a_d| \prod_{i=1}^{d} \max\{|\alpha_i|, 1\},$$

where $|\cdot|$ is the complex absolute value and $\alpha_1, \ldots, \alpha_d$ are the distinct conjugates of $\alpha \in \mathbb{C}$. The *logarithmic height of* α is

$$h(\alpha) = \log H(\alpha).$$

Remark A.3.2. For a rational number $\alpha = \frac{a}{b}$, where a, b are relatively prime integers, the height is $h(\alpha) = \max\{|a|, |b|\}$.

Definition A.3.3. Let P be a point in $\mathbb{P}^n(\mathbb{Q})$ given by coordinates $P = (x_0 : \ldots : x_n)$ such that all $x_i \in \mathbb{Z}$ and $x_0, \ldots, x_n$ are relatively prime. Then the *height of* P is defined to be

$$H(P) = \max\{|x_0|, \ldots, |x_n|\}.$$

The *logarithmic height of* P is

$$h(P) = \log H(P).$$

Note that for any $C \in \mathbb{R}$, the set

$$\{P \in \mathbb{P}^n(\mathbb{Q}) : h(P) \leqslant C\}$$

is finite.

For an arbitrary number field K, this can be generalized to the following:

Definition A.3.4. Let K be a number field and let $P = (x_0 : \ldots : x_n) \in \mathbb{P}^n(K)$ such that all $x_i \in K$. Then the *height of* P is

$$H(P) = \prod_{\mathfrak{p}} \max\{|x_0|_{\mathfrak{p}}, \ldots, |x_n|_{\mathfrak{p}}\},$$

and

$$h(P) = \log H(P) = \sum_{\mathfrak{p}} \log \max_i \{|x_i|_{\mathfrak{p}}\}.$$

Here, $\mathfrak{p}$ runs over all primes of K and the well-definedness follows from the product formula (see [HS00], lemma B.2.1).

Note that this height depends on the number field K; therefore one often denotes it by H_K and h_K, respectively. Sometimes one also uses a normalized version similar to the normalized Arakelov degree as in definition 7.1.1. In case $K = \mathbb{Q}$, this definition coincides with the one above, since the finite places do not contribute to the product or sum, respectively.

Remark A.3.5. The notion of the height of an algebraic number is a special case of the one above. This definition is equivalent to the one given above.

In Arakelov theory, it turns out to be more natural to consider the following height, which uses the ℓ^2-norm instead of the maximum norm at infinity.

Definition A.3.6. Let K be a number field and let $P = (x_0, \ldots, x_n) \in \mathbb{P}^n(K)$ such that all $x_i \in K$. Then

$$h(P) = \sum_{\mathfrak{p} \nmid \infty} \log \max_i \{|x_i|_{\mathfrak{p}}\} + \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log \left(\sum_i |\sigma(x_i)|^2 \right)^{\frac{1}{2}},$$

where $\mathfrak{p}$ runs over all primes of K.

Sometimes, yet other versions are used ([BG06], 2.8), depending on the context. For details on height functions and the relations between them, see [Lan83].

Appendix B

Elliptic Curves

Elliptic curves have been long studied in number theory and algebraic geometry. In this chapter, we give a set of basic definitions related to elliptic curves. For a more detailed reference, see [Sil86] and [Sil94].

B.1 Weierstrass equations

Definition B.1.1. An *elliptic curve over a field K with char$(K) \neq 2, 3$* is the set in $\mathbb{P}^2(K)$ determined by an equation

$$Y^2Z = X^3Z + aXZ^2 + bZ^3, \tag{B.1}$$

with discriminant $\Delta = -16(4a^3 + 27b^2) \neq 0$.

Remark B.1.2.

1. We defined an elliptic curve by a Weierstrass equation (B.1). We will see later that we can also define an elliptic curve over K as a pair (E, O_E), where E is a smooth projective algebraic curve of genus 1 over K and $O_E \in E(K)$. By the theorem of Riemann-Roch, one can deduce a Weierstrass equation from this definition. However, this Weierstrass equation is not unique.

2. One often equivalently defines an elliptic curve by an equation of the form

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3, \tag{B.2}$$

with non-zero discriminant $g_2^3 - 27g_3^3 \neq 0$.

In this case, g_2 and g_3 are multiples of certain values of Eisenstein series. One obtains one equation type from the other by a linear transformation. In fact, given any cubic equation, if char $K \neq 2, 3$, we can obtain an equation of the form (B.1) or (B.2) by a linear transformation.

Definition B.1.3. Let Λ be a lattice in $\mathbb{C}$, of rank 2 over $\mathbb{R}$, i.e. $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, see [Deb05]. Then the Weierstrass $\wp$-function is defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \frac{1}{(z-w)^2} - \frac{1}{w^2}.$$

It is meromorphic on $\mathbb{C}$ (e.g. [Kna92]) and its derivative is computed term by term:

$$\wp'(z) = -2 \sum_{w \in \Lambda} \frac{1}{(z-w)^3}.$$

The complex structure of an elliptic curve is just given by the structure of a complex torus, as we see in the following proposition.

Proposition B.1.4. *([Kna92], Theorems 6.14, 6.15, 6.16) Let Λ be a lattice in $\mathbb{C}$. The map of $\mathbb{C}/\Lambda$ into $\mathbb{P}^2(\mathbb{C})$ given by*

$$z \longmapsto \begin{cases} (\wp(z) : \wp'(z) : 1), & z \notin \Lambda, \\ (0:1:0), & z \in \Lambda \end{cases}$$

and its inverse map are holomorphic. They bijectively map $\mathbb{C}/\Lambda$ onto the elliptic curve $E(\mathbb{C})$, where E is given by a Weierstrass equation of type (B.2).

By this map, the elliptic curve inherits a group structure with neutral element $(0:1:0)$ corresponding to O_E from the remark above. One can geometrically describe the group law by the Chord-Tangent Construction ([Kna92], p. 10).

Proposition B.1.5. *Let $\Lambda = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\Lambda' = \mathbb{C}/\mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$. Then $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are isomorphic if and only if there is a $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ such that*

$$\frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2} = \frac{\omega_1'}{\omega_2'}.$$

Let $\mathcal{H}$ denote the *complex upper half plane*, i.e. $\mathcal{H} = \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$. Setting $\tau = \frac{w_1}{w_2}$, by the above proposition, the elliptic curve over $\mathbb{C}$ corresponds to a lattice $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$, where we can take $\tau \in \mathcal{H}$.

B.2 Curves of genus one

In a more abstract approach, an elliptic curve is defined as follows:

Definition B.2.1. An *elliptic curve* is a nonsingular projective curve of genus one together with a distinguished point P.

We will see how this definition fits to the one in definition B.1.1. For this we need a corollary of the Riemann-Roch theorem.

Theorem B.2.2. *([Sha94b], III.6.5, corollary 2) Let E be an elliptic curve and P a point on E, e.g. the distinguished point. Furthermore, let $n \in \mathbb{Z}$, $n > 0$. Then*

$$\dim \Gamma(E, \mathcal{O}(nP)) = n,$$

where $\mathcal{O}(nP)$ is the sheaf defined in remark 3.5.8.

By this, we can derive a Weierstrass equation for any elliptic curve over a field k, provided that $\operatorname{char} k \neq 2$.

Theorem B.2.3. *([Har77], proposition IV.4.6) Let E be an elliptic curve over a field k, with $\operatorname{char} k \neq 2$, and let $P \in E$ be a given point. Then there is a closed immersion $E \to \mathbb{P}^2$ such that the image is the curve*

$$y^2 = x(x-1)(x-\lambda)$$

for some $\lambda \in k$, and the point P goes to the point at infinity, more precisely, to the projective point $(0:1:0)$ on the y-axis.

Proof. By [Har77], theorem II.5.19, $\Gamma(E, \mathcal{O}(nP))$ is a vector space. Think of the vector spaces $\Gamma(E, \mathcal{O}(nP))$ as contained in each other, i.e.

$$\Gamma(E, \mathcal{O}(P)) \subset \Gamma(E, \mathcal{O}(2P)) \subset \cdots$$

Now choose an $x \in \Gamma(E, \mathcal{O}(2P))$ such that $1, x$ form a basis of $\Gamma(E, \mathcal{O}(2P))$. Furthermore, choose a $y \in \Gamma(E, \mathcal{O}(3P))$ such that $1, x, y$ form a basis of $\Gamma(E, \mathcal{O}(3P))$. Then the seven elements

$$1, x, y, x^2, xy, y^2, x^3$$

are all in $\Gamma(E, \mathcal{O}(6P))$ and therefore satisfy a linear relation. Moreover, y^2, x^3 are both only in $\Gamma(E, \mathcal{O}(6P))$, so their coefficients in the equation are both non-zero. By, if necessary, replacing x and y by scalar multiples, we may assume that their coefficients are equal to 1. Then we have a relation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

for certain $a_i \in k$. By a suitable linear transformation (here we need that $\operatorname{char} k \neq 2$) we get the required form

$$y^2 = x(x-1)(x-\lambda).$$

Since x and y both have a pole at P, P goes to the unique point at infinity $(0:1:0)$. $\square$

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