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DISSERTATION

Optimal Consumption and Pareto Optimal Risk Sharing

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o. Univ.–Prof. Dr. Walter Schachermayer

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Forschungsgruppe Finanz- und Versicherungsmathematik

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von

Dipl.Ing. Mag. Christina Ziehaus

Matrikelnummer: 0025592
Zentagasse 27-29/2/14, 1050 Wien

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Abstract

This thesis consists of two parts, the first one deals with the optimal consumption and investment problem, whereas the second part focuses on the Pareto optimal allocation of a risky position between two agents.
In the first part we start by considering the deterministic optimal consumption problem as a motivation for the stochastic problem. The optimal consumption process is calculated assuming a special form of the utility function. We see that the choice of the utility function plays a major role in the form of the consumption process. Not only do we consider the problem of optimal consumption, but also the problem of optimal terminal wealth, since both of them are linked closely and it is of economical importance to consider both of them at the same time, since it is not always optimal to finance the consumption by a big loan, which leaves us broke at the final time.
Subsequently we turn to the stochastic model and consider the problem of maximizing the utility of consumption and terminal wealth in a geometric Ornstein-Uhlenbeck market, after a quick detour over to the Black-Merton-Scholes model. We calculate the optimal consumption and wealth processes for power, logarithmic and exponential utility as well as their behavior depending e.g. on subjective discounting or the time horizon. We also use a specific example to show the identity of the solutions calculated by the primal and the dual method. In the stochastic case we show explicit results for the optimal processes, which are illustrated by numerical simulations and their limiting behavior depending on the time horizon or the weight on consumption and respectively on the terminal wealth. This first part was inspired by the paper of Walter Schachermayer and Hans Föllmer titled *"Asymptotic Arbitrage and Large Deviations"*.
The second part is joint work with Michael Kupper from the Humbolt University in Berlin and Ranja Reda from the Technical University of Vienna. We consider the utility of a risky position and remember that a utility function always implies a risk measure. The innovation of this work lies in the fact that we do not consider classical utility functions as in the first part of the thesis, but we relax our assumptions on a utility function, so instead of concavity we use quasi-concavity. Using this utility we want to find a Pareto optimal allocation. An allocation is called Pareto optimal if it is impossible to make someone better off without making someone else worse off. When all agents have reached a Pareto optimal distribution, then there would not be any more changes, since at least one of the agents would object.
We prove that Pareto optimal allocations do exist and we can give a characterization of them. We show that this position is closely linked to the optimal distribution of risk in a sense that the sum of the utilities is maximized in such as allocation.
Finally we present some examples of quasiconcave risk measures, which are not concave in order to motivate the relaxation of the definition of utility functions.

Zusammenfassung

Diese Dissertation besteht aus zwei Teilen. Der erste Teil beschäftigt sich mit dem optimalen Konsumproblem. Zu Beginn diskutieren wir den optimalen Konsum und das optimale Endvermögen in einem deterministischen Modell. Diese Fragestellungen kann man nicht voneinander trennen, da unser Konsum nicht durch unendlich hohe Schulden zum Endzeitpunkt finanziert werden soll. Nach diesen Vorbereitungen betrachten wir kurz den optimalen Konsum im Black-Merton-Scholes Markt und widmen uns anschließend dem wichtigsten Abschnitt des ersten Teiles, dem optimalen Konsum im geometrischen Ornstein-Uhlenbeck Markt.
Wir zeigen in diesem stochastischen Modell die optimalen Konsumstrategien für unterschiedliche Nutzenfunktionen, betrachten das Verhalten dieser Strategien abhängig von unterschiedlichen Parametern und illustrieren die Ergebnisse anhand von Simulationen. Schließlich vergleichen wir die primale und die duale Lösungsmethode anhand eines Beispiels. Den Anstoß für diesen ersten Teil gab das paper *"Asymptotic Arbitrage and Large Deviations"* von Walter Schachermayer und Hans Föllmer.
Der zweite Teil beschäftigt sich mit Pareto optimalen Allocationen einer risikobehafteten Position und ist in Zusammenarbeit mit Michael Kupper von der Humboldt Universität zu Berlin und Ranja Reda von der Technischen Universität Wien entstanden. Wir betrachten den Nutzen einer risikobehafteten Position und versuchen diesen zwischen zwei Agenten optimal zu verteilen. Als Pareto optimal bezeichnen wir eine Position dann, wenn es keine Veränderung gibt, die einen Agenten echt besser stellt, die nicht mindestens einen anderen Agenten schlechter stellt. Im Gegensatz zu den Analysen im ersten Teil der Dissertation betrachten wir hier nicht klassische Nutzenfunktionen, sondern betrachten eine größere Klasse, nämlich die Klasse aller quasiconcaven Nutzenfunktionen. Wir zeigen, dass es eine Pareto optimale Alllocation gibt, und beschreiben diese Allocation.

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Part I

Optimal Consumption

Chapter 1

Introduction

The first part of the thesis deals with the optimal consumption problem, which is in turn an optimal investment problem, since we need to invest optimally to ensure optimal consumption.
As a motivation we start by considering the deterministic optimal consumption problem and solve it for the exponential utility, while in the geometric Ornstein-Uhlenbeck market we will also consider power and logarithmic utility. The logarithmic utility represents a limiting case of the power utility, and also the exponential utility presents a different limit of the power utility. Without going into detail, the main difference is that power and logarithmic utility on the one hand and exponential utility on the other hand is that the first two will not allow for negative consumption, but the exponential utility is not restricted to positive consumption and wealth. Both choices have economical sound interpretations depending on the availability of credits. Because of the possibility of negative consumption we take a close look at the exponential utility.
We make the connection with the traditional Black-Merton-Scholes model, but then we turn quickly to the problem of maximizing the utility of consumption and terminal wealth in a geometric Ornstein-Uhlenbeck market, which is the main topic of this part. We calculate the optimal consumption and wealth processes for power, logarithmic and exponential utility as well as their behavior depending e.g. on subjective discounting or the time horizon. Again the exponential utility plays a very important role for us. In the stochastic case we consider two methods for solving this problem. The main emphasis lies on the dual method which corresponds to the Lagrange method presented in the non stochastic part. We also consider the primal approach which goes back to Merton in the 1960s. Finally we use a specific example to show the identity of the solutions calculated by the primal and the dual method and illustrate our findings by some numerical simulations.

Chapter 2

Duality Theory in the Deterministic and Black-Merton-Scholes Setting

Since the 1960s the problem of optimal consumption has been studied in great detail. The finite horizon problem of optimal consumption and terminal wealth in the time period $(0,T)$ consists in maximizing

$$E\Big[\int_0^T e^{-\rho t}U_1(C(t))dt + U_2(X(T),T)\Big], \tag{2.1}$$

where $(X(t))_{0\le t\le T}$ is the wealth and $(C(t))_{0\le t\le T}$ the consumption process and ρ is the constant discounting factor. The function U_1 measures the utility of consumption and the function U_2 the utility of the terminal wealth. In Merton's wording [Merton 1969] U_2 is called "bequest valuation function".
We also consider the infinite horizon problem

$$E\Big[\int_0^\infty e^{-\rho t}U_1(C(t))dt\Big] \tag{2.2}$$

under a suitable transversality condition at $t=\infty$.
The problem was first presented and solved in "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case" [Merton 1969]. In this paper Merton employs the primal approach to solve this problem for a finite time horizon T. The idea of dynamic programming is that if we have a strategy which is optimal on $(0,t_1)$ and $0\le t_1\le T$ and we follow an optimal strategy after t_1 until T then the (combined) strategy is globally optimal. Since then this and similar problems have been treated under additional constraints, as for example under transaction costs in [Øksendal, Sulem].
Besides using the primal approach there is also the dual approach which uses the Lagrangian method. This method involves deriving the first order condition and calculating the Lagrange multiplier, in order to get the optimal consumption process. This approach has also been used in various other settings for example in [Cuoco]. In [Cox, Huang] the problem has been solved for a general market model.
As utility of consumption we consider the exponential utility as done in [Merton 1969]

$$U_1(C) = -\frac{e^{-\eta C}}{\eta} \quad C\in\mathbb{R}. \tag{2.3}$$

Using the exponential utility function to measure utility of consumption and terminal wealth needs careful consideration, since it also assigns a finite utility to negative values of wealth or consumption. For notational convenience, we assume $\eta = 1$, which we may assume w.l.o.g. by choosing the unit by which we measure consumption appropriately. By $\hat{C}(t)$ and $\hat{X}(t)$ we denote the optimal consumption and wealth processes. The initial wealth is denoted by x.
Section 2.1 deals with the problem in the case that there is only a bond and no risky asset, in section 2.2 the problem with infinite time horizon is analyzed and finally in section 2.3 we introduce a risky asset and recall the solution in the Black-Merton-Scholes model.

2.1 The deterministic model for finite time horizon

We assume that there is only a bond yielding interest r, and there will be no risky asset. In Merton's notation [Merton 1969], this corresponds to $\alpha = r$ and $\sigma = 0$, which gives

$$dS_t = rS_t dt. \tag{2.4}$$

We fix the finite horizon $T > 0$, and choose as utility of terminal wealth

$$U_2(X,T) = \begin{cases} 0 & \text{if } X \geq 0 \\ -\infty & \text{if } X < 0 \end{cases}. \tag{2.5}$$

This form of the bequest valuation requires that at time T the wealth $X(T)$ has to be non-negative, in which case the bequest valuation function yields a contribution zero to the target functional (2.1). This is reasonable from an economic point of view and corresponds to the limiting case $\varepsilon = 0$ in Merton's analysis (p.251).
Now we are left with the following deterministic optimization problem: Given $\rho > 0$ we search for the deterministic, $\mathbb{R}$-valued function $(C(t))_{0\leq t\leq T}$ optimizing

$$\max_{C(t)} \Big[-\int_0^T e^{-\rho t} e^{-C(t)} dt \Big] \tag{2.6}$$

under the constraint

$$\int_0^T e^{-rt} C(t) dt \leq x \tag{2.7}$$

for a given initial wealth $x \in \mathbb{R}$. The budget constraint (2.7) ensures that $X(T) \geq 0$ and the wealth process $(X(t))_{0\leq t\leq T}$ satisfies the deterministic ODE

$$X'(t) = rX(t) - C(t).$$

To solve (2.6), we take a closer look at the first order condition

$$e^{-\rho t} e^{-C(t)} = \lambda e^{-rt}$$

which has to be satisfied for $0 \leq t \leq T$ for some Lagrange multiplier $\lambda > 0$. Writing $\lambda = e^{-a}$ we get

$$\begin{aligned} -\rho t - C(t) &= -a - rt \quad \text{or} \\ \hat{C}(t) &= a + (r - \rho)t \end{aligned}$$

for some $a \in \mathbb{R}$ which is determined by the budget constraint. Inserting the optimal consumption process into (2.7) and calculating a gives for a and $(\hat{C}(t))_{0\leq t\leq T}$

$$\begin{aligned} a &= \frac{rx + (r-\rho)[Te^{-rT} - \frac{1}{r}(1-e^{-rT})]}{1-e^{-rT}}, && (2.8) \\ \hat{C}(t) &= \frac{rx + (r-\rho)[Te^{-rT} - \frac{1}{r}(1-e^{-rT})]}{1-e^{-rT}} + (r-\rho)t. && (2.9) \end{aligned}$$

We can also calculate the resulting wealth process $\hat{X}(t)$ from

$$\begin{aligned} \hat{X}(t)e^{-rt} &= x - \int_0^t e^{-rs}\hat{C}(s)ds \\ \hat{X}(t) &= e^{rt}\Big[-\frac{a}{r}(1-e^{-rt}) - \frac{r-\rho}{r}\Big[-te^{-rt} - \frac{1}{r}(e^{-rt}-1)\Big] + x)\Big]. && (2.10) \end{aligned}$$

It follows that at time $t = T$ the wealth is zero, which is no surprise.

Lemma 1. The wealth process is concave for $r > \frac{\rho T}{x+T}$ and convex for $r < \frac{\rho T}{x+T}$.

Proof. To prove this we take a look at the second derivative and verify that it is positive when $r > \frac{\rho T}{x+T}$ and negative otherwise. For later usage we will write down a simplified version of the wealth process and its first derivative as well.

$$\begin{aligned} X(t) &= x\Big(e^{rt} + \frac{1-e^{rt}}{1-e^{-rT}}\Big) + \frac{r-\rho}{r}\Big(\frac{1-e^{rt}}{1-e^{-rT}}Te^{-rT} + t\Big) \\ X'(t) &= -xr\frac{e^{r(t-T)}}{1-e^{-rT}} + \frac{r-\rho}{r}\Big(1 - \frac{rTe^{r(t-T)}}{1-e^{-rT}}\Big) && (2.11) \\ X''(t) &= \frac{e^{r(t-T)}}{1-e^{-rT}}(-r^2x - (r-\rho)rT) \end{aligned}$$

The first factor of the second derivative is clearly always positive, but the sign of the second factor will depend on the choice of ρ and r. It follows that

$$\begin{aligned} -r^2x - (r-\rho)rT &< 0 \\ r &> \frac{\rho T}{x+T}, \end{aligned}$$

which proves the assumption. $\square$

Lemma 2. The minimum of the wealth process is attained at T for $r > \frac{\rho T}{x+T}$ and for $r < \frac{\rho T}{x+T}$ it is attained at $t_m = \frac{1}{r}\log\left(\frac{(1-e^{-rT})(r-\rho)}{(rx+(r-\rho)T)r}\right) + T$.

Proof. Inserting T in the process $X(T)$ gives zero, therefore for $r > \frac{\rho T}{x+T}$ it follows by concavity that the minimal wealth is attained at zero.
But for $r < \frac{\rho T}{x+T}$ we take a look at the first derivative of the wealth process given by equation 2.11 and set it equal to zero. Thus we get the equation

$$\begin{aligned} 0 &= -xr\frac{e^{r(t-T)}}{1-e^{-rT}} + \frac{r-\rho}{r}\Big(1 - \frac{rTe^{r(t-T)}}{1-e^{-rT}}\Big) \\ \frac{r-\rho}{r} &= xr\frac{e^{r(t-T)}}{1-e^{-rT}} + \frac{(r-\rho)Te^{r(t-T)}}{1-e^{-rT}} \end{aligned}$$

which is solved by $t_m = \frac{1}{r}\log\left(\frac{(1-e^{-rT})(r-\rho)}{(rx+(r-\rho)T)r}\right) + T$. $\square$

Lemma 3. Under the assumption $r < \frac{\rho T}{x+T}$ and T big enough the following two inequalities can be derived:

(a) $0 \leq \frac{(1-e^{-rT})(r-\rho)}{(rx+(r-\rho)T)r} \leq 1$

(b) $0 \leq t_m$

Proof. We start by proving the first inequality in (a) $0 \leq \frac{(1-e^{-rT})(r-\rho)}{(rx+(r-\rho)T)r}$. This follows from the fact that $r-\rho$ is negative under the assumption in eq.(2.12) as well as $rx+(r-\rho)T$, thus the ratio is positive.
The second inequality in (a) $\frac{(1-e^{-rT})(r-\rho)}{(rx+(r-\rho)T)r} \leq 1$ can be transformed to

$$1 - e^{-rT} < \frac{r^2 x}{r-\rho} + rT.$$

The limit of the left hand side is 1 for $T \to \infty$ and the corresponding limit for the left hand side is infinity. Since both functions are monotone, at least for $T_1 > \frac{1}{r}\Big(1 - \frac{r^2 x}{r-\rho}\Big)$ this inequality is fulfilled.
Finally for (b) we can rewrite $0 \leq t_m$ in the form

$$\begin{aligned} 0 &\leq \frac{1}{r}\log\Big(\frac{(1-e^{-rT})(r-\rho)}{(rx+(r-\rho)T)r}\Big) + T \\ e^{-rT}(rx+(r-\rho)T)r &\geq (1-e^{-rT})(r-\rho) \end{aligned} \tag{2.12}$$

Since the left hand side is converging to zero for $T \to \infty$ and the right hand side is monotonically decreasing to $1/(\rho - r)$, which is negative, we see that for T bigger than some constant T_2 this inequality is fulfilled. Thus for T bigger than $\max(T_1, T_2)$ both inequalities (a) and (b) are fulfilled. □

Lemma 4. The wealth process is negative at t_m for $r < \frac{\rho T}{x+T}$ and T big enough.

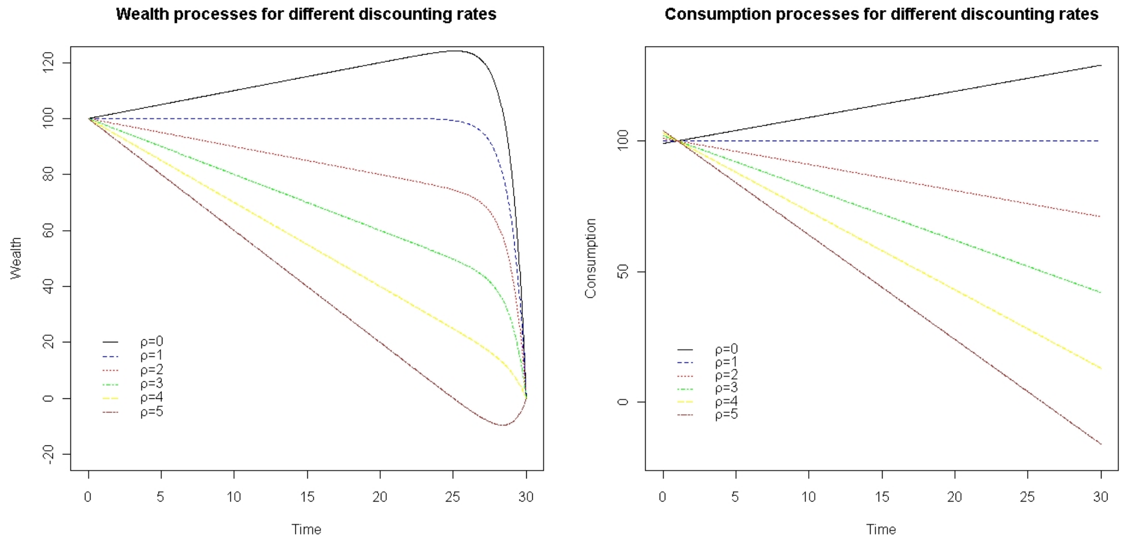
Proof. We start by inserting t_m into the wealth process.

$$\begin{aligned} X(t_m) &= \frac{x}{1-e^{-rT}}\Big(1 - \frac{(1-e^{-rT})(r-\rho)}{(rx+(r-\rho)T)r}\Big) \\ &+ \frac{r-\rho}{r}\Big(\frac{Te^{-rT} - \frac{(1-e^{-rT})(r-\rho)T}{(rx+(r-\rho)T)r}}{1-e^{-rT}} + \frac{1}{r}\log\Big(\frac{(1-e^{-rT})(r-\rho)}{(rx+(r-\rho)T)r}\Big) + T\Big) \end{aligned}$$

The inequality $X(t_m) \leq 0$ corresponds to

$$\begin{aligned} &\frac{x}{1-e^{-rT}}\Big(1 - \frac{(1-e^{-rT})(r-\rho)}{(rx+(r-\rho)T)r}\Big) - \frac{(r-\rho)^2 T}{(rx+(r-\rho)T)r^2} \\ &\leq -\frac{r-\rho}{r}\Big(Te^{-rT} + \frac{1}{r}\log\Big(\frac{(1-e^{-rT})(r-\rho)}{(rx+(r-\rho)T)r}\Big) + T\Big) \end{aligned} \tag{2.13}$$

Taking the limit of the left hand side for $T \to \infty$ we get $x - (r-\rho)/r^2$ whereas the right hand side of eq.(2.13) converges to infinity. Since both sides are monotone for big T it follows that for T big enough this inequality will be fulfilled.
Indeed T has to be chosen in such a way that it is bigger than $\max(T_1, T_2)$, and that the inequality (2.13) is fulfilled. □



(a) The Wealth Processes

(b) The Consumption Processes

Figure 2.1: In (a) we see six examples of wealth processes for different values of ρ. The initial capital is 100 and the interest rate $r = 1$. We optimize over the time interval $(0, 30)$. Note that only for $\rho = 5$ the inequality $r < \frac{\rho T}{x+T}$ is fulfilled and we get negative wealth. In (b) we see the corresponding consumption processes. Only for $\rho = 5$ negative consumption is realized.

Remark 1. The above Lemmas prove that the wealth process can become negative for $r < \frac{\rho T}{x+T}$ and big T. Indeed in Fig. 1 we see different realizations of the wealth process for the same time interval $(0, T)$ but different choices of ρ for fixed interest rate $r = 1$.

In this section we solved the optimal consumption problem in a deterministic market for exponential utility with a finite time horizon. We employed the dual method and noticed that for $r < \frac{\rho T}{x+T}$ and T big enough the wealth process as well as the consumption process will become negative at some point in time. Nevertheless, the wealth process will be exactly zero at final time T, even if it was negative before.

2.2 The deterministic problem for infinite time horizon

We consider the limit $T \to \infty$: In this case the above formula for consumption simplifies to

$$\hat{C}(t) = rx + (r - \rho)\Big(t - \frac{1}{r}\Big). \tag{2.14}$$

Simplifying even further, for $x = 0$ we obtain

$$\hat{C}(t) = (r - \rho)t - \frac{(r - \rho)}{r}. \tag{2.15}$$

Assuming $r > \rho$ we get the following situation: At time $t = 0$ we start with negative consumption until at time $t = 1/r$ we have consumption zero and from then on the consumption process is positive.

For $\rho > r$ on the other hand we start with positive consumption and after time $t = \frac{1}{r}$ we have negative consumption.
Turning again to (2.14) we see that the influence of x only consists in a constant shift of this consumption pattern by rx.
The formula for the wealth becomes

$$\hat{X}(t) = x + \frac{r-\rho}{r}t, \tag{2.16}$$

hence one becomes infinitely rich at time $t = \infty$ (under the assumption $r > \rho$). However, discounted to time zero, the wealth $\hat{X}(t)e^{-rt}$ clearly tends to zero.
If we recall our results from the section above, we can notice that the limit of $r < \frac{\rho T}{x+T}$, which corresponds to the change between convex and concave wealth processes, is $r < \rho$. Thus for the infinite time horizon case we have negative wealth at infinity for any $r < \rho$. In any finite time problem the wealth process will eventually return to zero, in the infinite time horizon case though the wealth will no longer return to zero.

Remark 2. From the previous considerations we argue that the good transversality condition for $t \to \infty$ should be the weaker requirement

$$\lim_{t\to\infty} e^{-rt}X(t) \geq 0. \tag{2.17}$$

We also calculate the indirect utility function $J(X)$

$$\begin{aligned} J(W) &= -\int_0^\infty e^{-\rho t}e^{-a-(r-\rho)t}dt, \\ &= -\frac{1}{r}e^{-rx+(r-\rho)/r} \end{aligned} \tag{2.18}$$

which is exactly the same as in Merton (63).
Finally we can express $\hat{C}(t)$ in terms of $\hat{X}(t)$

$$\hat{C}(t) = r\hat{X}(t) + \frac{r-\rho}{r} \tag{2.19}$$

which is again in line with Merton (64).
We find the same results as in Merton, but there is negative consumption present in some cases which is in contradiction to p.249 of [Merton 1969]. This can be avoided by imposing the constraint $C(t) \geq 0$.
The problem of negative consumption in the exponential utility case has been discussed in [Cox, Huang] for the more general stochastic case. Suppose one can find an optimal solution without the nonnegativity constraint, then the optimal constrained policy would be to buy an insurance package which insures us against the negative consumption, and to invest the rest of the initial capital into the unconstrained policy.

2.3 Exponential utility of consumption in the Black-Merton-Scholes model

In this section we turn back to the general setting where we also have a risky asset characterized by a and σ.

We consider a slight variant of Merton's ("primal") method. We define the value function

$$I(X,t) = \max_{C(s)} E\Big[\int_t^T e^{-\rho s} U(C(s))ds + B(X(T),T)\Big]. \tag{2.20}$$

Our Ansatz for the value function is $I(X,t) = De^{-\rho t}e^{-rX(t)}$, where D is a constant. We choose $I(X,t)$ of this form due to two reasons.
First the factor $e^{-\rho t}$ is a reasonable choice since the utility of consumption (and bequest) are discounted by that rate.
The second argument deals with the term $e^{-rX(t)}$. Suppose we have initial wealth $x + h$ instead of x, then we can finance an additional consumption of rh during the entire period. So $\hat{C}(t)$ would be replaced by $\hat{C}(t) + rh$. Now for $h \searrow 0$, this should be an optimal investment,which leads to an increase of utility from $U(\hat{C}(t))$ to $U(\hat{C}(t) + rh) \approx U(\hat{C}(t))e^{-rh}$, hence the scaling in wealth with $e^{-rX(t)}$.
Now we calculate the constant D. Therefore we calculate the optimal investment in the exponential case. We have

$$\frac{I_{XX}(X,t)}{I_X(X,t)} = r \quad \forall\ X \in \mathbb{R}, t \geq 0, \tag{2.21}$$

hence by the basic arguments of Merton the Euro amount of investment in the stock is $\frac{a-r}{\sigma^2 r}$, in accordance with (65).
To calculate D, we plug our Ansatz into the corresponding HJB equation (17") in [Merton 1969] with $\eta = 1$ as assumed in the beginning

$$0 = -J_X(X) - \rho J(X) + J_X(X)rX + J_X(X)\log(J_X(X)) - \frac{(a-r)^2}{2\sigma^2}\frac{J_X(X)^2}{J_{XX}(X)},$$

where $J(X) = e^{\rho t} I(X,t) = De^{-rX}$ and get

$$I(X,t) = -\frac{1}{r}e^{-\rho t - rX(t) + \frac{r-\rho-(a-r)^2/2\sigma^2}{r}} \tag{2.22}$$

in accordance with eq. 61 in [Merton 1969].

Chapter 3

Optimal Consumption and Investment in a Geometric Ornstein-Uhlenbeck Market[1]

3.1 Introduction

The problem of optimal consumption and investment in a stock market has been studied in great detail since the 1960s. In [Merton 1969] one can find a collection of seminal articles using the primal approach to solve this problem. Though most of the work is concentrated on solving this problem in the Black-Merton-Scholes market model, one can apply the same methods in other specific market settings.
The primal (or dynamic programming) approach involves calculating and solving the Hamilton-Jacobi-Bellman equation to find the optimal consumption and investment processes.
In [Cox, Huang], [Delbaen, Schachermayer] and [Karatzas, Zitkovic], one can review the dual approach to solve this problem. Using this approach we do not need to solve the HJB equation, but instead the density of the martingale measure is used to calculate the optimal consumption process and the optimal terminal wealth.
In [Karatzas, Lehoczky, Shreve, Xu] the dual approach is used for maximizing utility of terminal wealth in an incomplete market. In [He, Pearson] the consumption is optimized under short-sale constraints in a discrete-time, discrete-state-space securities market which is incomplete. This is extended in [Girotto, Ortu]. Also [Elliott, Kopp] and [Karatzas, Shreve] provide a good insight into the dual method.
The Ornstein-Uhlenbeck process was first presented in [Uhlenbeck, Ornstein]. The problem of optimizing the terminal wealth in an Ornstein-Uhlenbeck market when the horizon tends to infinity has been considered in great detail in [Föllmer, Schachermayer]. Here we present an extension of the terminal wealth problem, since we want to optimize the utility of consumption and terminal wealth. In [Bormetti, Cazzola, Montagna, Nicrosini], [Florens-Landais, Pham] and [Barndorff-Nielsen, Shephard] the importance of the Ornstein-Uhlenbeck process for financial modeling is shown.
We consider an exponential Ornstein-Uhlenbeck market with one stock $(S_t)_{0\leq t\leq T}$ and one

[1] This chapter is an extended version of the corresponding paper, which is submitted but not yet published.

bond $(B_t)_{0\leq t\leq T}$ governed by

$$\begin{aligned} S_t &= \exp(Y_t), \\ dY_t &= -\beta Y_t dt + \sigma dW_t, \\ dB_t &= rB_t dt, \end{aligned}$$

with β, σ and r constants and $(W_t)_{0\leq t\leq T}$ is a standard Brownian Motion, which is complete. Using Itô's formula we can also write down the SDE for the stock price S_t

$$dS_t = S_t\sigma\Big[dW_t - \frac{1}{\sigma}\Big(\beta Y_t - \frac{\sigma^2}{2}\Big)dt\Big]. \tag{3.1}$$

We want to optimize the finite time problem

$$V(x) = \sup_{C,w} \mathbb{E}^P\Big[\int_0^T U_1(t,C_t)dt + U_2(T,X_T)\Big],$$

where $x = X_0$ is the initial wealth, $(X_t)_{0\leq t\leq T}$ is the wealth process, $w = (w_t)_{0\leq t\leq T}$ is the investment process and $C = (C_t)_{0\leq t\leq T}$ is the consumption process. The function $V(x)$ is called the value function and the functions $U_1(.)$ and $U_2(.)$ are utility functions. For a utility function we use definition 4.1 from [Karatzas, Shreve], which we repeat here briefly.

Definition 1 (Utility function)**.** A utility function is a concave, nondecreasing, upper semicontinuous function $U : \mathbb{R} \to [-\infty,\infty)$ where $dom(U) = \{x \in \mathbb{R} \mid U(x) > -\infty\}$ is a nonempty subset of $[0,\infty)$ and U' is continuous, positive and strictly decreasing on the interior of $dom(U)$ with $U'(\infty) = 0$.

Remark 3. The following results can also be applied for a more general model

$$S_t = \exp(Y_t + \mu t) \quad \text{with } \mu \text{ constant.} \tag{3.2}$$

This will only result in a change of constants (see [Föllmer, Schachermayer]), so without loss of generality we may assume $\mu = 0$.

In the next sections we do not only solve the optimal consumption and terminal wealth problem under a subjective discounting rate ρ for power, logarithmic and exponential utility, but we also review the limiting behavior of the value function, as well as the dependence on the discounting rate ρ. A positive rate ρ means that we would prefer to consume sooner rather than later, whereas a negative rate will encourage us to save some money for later consumption.
Section 3.3 is dedicated to the comparison of the primal and the dual approach.
In the last section of the first part we present numerical results and compare the results of simulated processes with the analytical solution of the value function.

3.2 Optimal Consumption

In the market described in the introduction of this chapter which is defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ we consider the finite horizon problem

$$V(x) = \sup_{C,w} \mathbb{E}^P\Big[\int_0^T \exp(-\rho t)\theta U_1(C_t)dt + \exp(-\rho T)(1-\theta)U_2(X_T)\Big]. \tag{3.3}$$

The factor ρ is our subjective discounting factor, θ and respectively $1-\theta$ are the weights of the utility of consumption and terminal wealth, and $\theta \in [0,1]$. Choosing θ equal to zero corresponds to the optimal terminal wealth problem.
We will use the dual approach described in [Cox, Huang] to write down the general form of the solution of this problem.
Clearly a positive ρ means that we concentrate on the here and now, so we would consume rather sooner than later. A negative ρ on the other hand represents an individual who is willing to consume less today for the possibility to consume more tomorrow, thus he is focused on the future.
To solve this optimization problem we need the inverse functions I_1 and I_2 of the first derivatives of U_1 and U_2. Then the optimal consumption process $C^* = (C_t^*)_{0\le t\le T}$ and the optimal terminal wealth X_T^* are given by

$$\begin{aligned} C_t^* &= I_1\Big(\frac{\lambda}{\theta}H_t e^{\rho t}\Big) \quad \text{for } 0 \le t \le T \text{ and} \\ X_T^* &= I_2\Big(\frac{\lambda}{1-\theta}H_T e^{\rho T}\Big), \end{aligned}$$

which was shown in [Cox, Huang] Theorem 2.1. Since we are in a geometric Ornstein-Uhlenbeck market, we have a unique martingale measure, under which the discounted stock price is a martingale. Here H_T defines the equivalent martingale measure Q via $dQ/dP = H_T e^{rT}$, and $(H_t)_{0\le t\le T}$ is sometimes called *state price density process* [Karatzas, Shreve]:

$$H_t = \exp\Big(-\int_0^t \phi_s dW_s - \frac{1}{2}\int_0^t \phi_s^2 ds - rt\Big) \text{ for } 0 \le t \le T.$$

The density function H_t is defined using the market price of risk ϕ_t (Sharpe ratio, see [Sharpe]), which is in this model given by

$$\phi_t = \frac{\sigma^2 - 2r}{2\sigma} - \frac{\beta}{\sigma}Y_t.$$

Finally the Lagrange multiplier λ is defined by the budget constraint

$$x = \mathbb{E}^P\Big[\int_0^T H_t I_1\Big(\frac{\lambda}{\theta}H_t \exp(\rho t)\Big)dt + H_T I_2\Big(\frac{\lambda}{1-\theta}H_T \exp(\rho T)\Big)\Big]. \tag{3.4}$$

In order to calculate the investment process $(w_t)_{0\le t\le T}$, which represents the number of shares of stock we hold at time t, we use the first main result of [Cox, Huang] which is given by Theorem 2.1, and provides the solution of the optimal consumption and investment problem. Analogous to [Cox, Huang] we define

$$F((\lambda H_t)^{-1}, S_t, t) = (\lambda H_t)^{-1}\mathbb{E}^P\Big[\int_t^T (\lambda H_s) I_1\Big(\frac{\lambda}{\theta}H_s e^{\rho s}\Big)ds + (\lambda H_T) I_2\Big(\frac{\lambda}{1-\theta}H_T e^{\rho T}\Big)\Big|\mathcal{F}_t\Big]$$

for $0 \le t \le T$ and Theorem 2.1 then allows us to write

$$w_t = F_S((\lambda H_t)^{-1}, S_t, t) + \frac{\phi}{\sigma\lambda H_t S_t}F_{(\lambda H_t)^{-1}}((\lambda H_t)^{-1}, S_t, t) \text{ for } 0 \le t \le T,$$

using the notation of our particular market setting. The development of the wealth process is given by $X_t^* = F((\lambda H_t)^{-1}, S_t, t)$ for $t \in [0,T]$ as stated in [Cox, Huang]. By No-Arbitrage arguments it is clear that for any utility function as in Definition 1, the consumption and the wealth process will always be nonnegative.

To prevent us from negative terminal wealth or consumption in the exponential utility case we can use the method described in [Cox, Huang]. We do not invest all of our initial capital in our strategy but only the part $\tilde{X}_0$, and the rest $x - \tilde{X}_0$ is used to buy insurance in case we have negative wealth or consumption during $[0, T]$. This corresponds to a change of the Lagrange multiplier and thus to a shift of the constant in the consumption and the terminal wealth process.
This insurance against negative consumption and wealth can be represented by European Put options, this continuum of put options on consumption and the put option on the terminal wealth cost $x - \tilde{X}_0$. We consume the positive part of the unconstrained consumption process calculated with the initial capital $\tilde{X}_0$.
in the following subsections we calculate the optimal consumption strategies for Power, Logarithmic and Exponential Utility.

3.2.1 Power Utility

We assume that U_1 and U_2 are of the form

$$U_1(x) = U_2(x) = \frac{x^\alpha}{\alpha},$$

with α a given constant in $(-\infty, 1) \setminus \{0\}$ and thus the functions I_1 and I_2 take the form

$$I_1(x) = I_2(x) = x^{\frac{1}{\alpha-1}}.$$

The consumption process and the terminal wealth are given by

$$\begin{aligned} C_t^* &= \left(\frac{\lambda}{\theta}\right)^{\frac{1}{\alpha-1}} H_t^{\frac{1}{\alpha-1}} e^{\frac{\rho}{\alpha-1}t}, \quad t \in [0, T] &(3.5)\\ X_T^* &= \left(\frac{\lambda}{1-\theta}\right)^{\frac{1}{\alpha-1}} H_T^{\frac{1}{\alpha-1}} e^{\frac{\rho}{\alpha-1}T}. &(3.6)\end{aligned}$$

Since the last two terms in either process are exponential functions, they are positive for sure. It remains to show the positivity of the Lagrange multiplier λ to prove that the optimal consumption process $(C_t^*)_{0 \le t \le T}$ and the optimal terminal wealth X_T^* are well defined. Thus we take a closer look at the definition of the Lagrange multiplier λ.

$$\begin{aligned} x &= \mathbb{E}^P \Big[\int_0^T \left(\frac{\lambda}{\theta}\right)^{\frac{1}{\alpha-1}} H_t^{\frac{\alpha}{\alpha-1}} e^{\frac{\rho}{\alpha-1}t} dt + \left(\frac{\lambda}{1-\theta}\right)^{\frac{1}{\alpha-1}} H_T^{\frac{\alpha}{\alpha-1}} e^{\frac{\rho}{\alpha-1}T} \Big] \\ \lambda^{\frac{1}{\alpha-1}} &= x \Big[\mathbb{E}^P \Big[\int_0^T H_t^{\frac{\alpha}{\alpha-1}} \theta^{-\frac{1}{\alpha-1}} e^{\frac{\rho}{\alpha-1}t} dt + H_T^{\frac{\alpha}{\alpha-1}} (1-\theta)^{-\frac{1}{\alpha-1}} e^{\frac{\rho}{\alpha-1}T} \Big] \Big]^{-1} \\ &= x \Big[\int_0^T \mathbb{E}^P \Big[H_t^{\frac{\alpha}{\alpha-1}} \Big] \theta^{-\frac{1}{\alpha-1}} e^{\frac{\rho}{\alpha-1}t} dt + \mathbb{E}^P \Big[H_T^{\frac{\alpha}{\alpha-1}} \Big] (1-\theta)^{-\frac{1}{\alpha-1}} e^{\frac{\rho}{\alpha-1}T} \Big]^{-1} \end{aligned}$$

All terms inside the expectation are positive as well as the initial capital x so λ is well-defined. Also, λ is finite since the sum of the expectation of terminal wealth and consumption is not equal to zero. All terms, with exception of θ and $(1-\theta)$, are of exponential form, therefore both summands are strictly positive.
Finally we can write down the wealth process $(X_t^*)_{0 \le t \le T}$

$$X_t^* = (\lambda H_t)^{-1} \mathbb{E}^P \Big[\int_t^T \left(\frac{\lambda}{\theta}\right)^{\frac{\alpha}{\alpha-1}} H_s^{\frac{\alpha}{\alpha-1}} e^{\frac{\alpha\rho s}{\alpha-1}} ds + \left(\frac{\lambda}{1-\theta}\right)^{\frac{\alpha}{\alpha-1}} H_T^{\frac{\alpha}{\alpha-1}} e^{\frac{\alpha\rho T}{\alpha-1}} \Big| \mathcal{F}_t \Big], \quad t \in [0, T]$$

which can be used to write down the optimal investment process $(w_t)_{0\leq t\leq T}$ as stated in the beginning of section 2.
If we assume $\theta \to 0$, we see that the solution of the optimal consumption and terminal wealth problem converges to the solution of the optimal terminal wealth problem. Indeed for $\theta = 0$ we have zero consumption and for $r = \rho = 0$ we get the same optimal terminal wealth as in [Föllmer, Schachermayer].

Proposition 1. The expected utility of consumption for power utility at time $t \in [0, T]$ is given by $u_t^{pow} = \mathbb{E}^P[\frac{1}{\alpha}C_t^\alpha]$ and it is of the form

$$\mathbb{E}^P\Big[\frac{1}{\alpha}C_t^\alpha\Big] = \frac{1}{\alpha}\Big(\frac{\lambda}{\theta}\Big)^{\frac{\alpha}{\alpha-1}}(1 - A_2(t))^{-1/2}\exp\Big(A_1(t) + (1 - A_2(t))^{-1}A_3(t) + \frac{\alpha(\rho - r)}{\alpha - 1}t\Big), \tag{3.7}$$

for $t \in [0, T]$ where

$$\begin{aligned}
A_1(t) &= \frac{\sigma^2 - 2r}{2\sigma^2}\frac{\alpha}{\alpha - 1}Y_0 - \frac{(\sigma^2 - 2r)^2}{8\sigma^2}\frac{\alpha}{\alpha - 1}t - \frac{\beta}{2\sigma^2}((\alpha - 1)^{-1} + (1 - \alpha)^{-1/2})(Y_0^2 + \sigma^2)t,\\
A_2(t) &= -\frac{\alpha - 1}{4\beta}(1 - e^{-2\beta(1-\alpha)^{-1/2}t})^2((\alpha - 1)^{-1} + (1 - \alpha)^{-1/2}),\\
A_3(t) &= \frac{\beta}{2\sigma^2}((\alpha - 1)^{-1} + (1 - \alpha)^{-1/2})e^{-2\beta(1-\alpha)^{-1/2}t} - \frac{\sigma^2 - 2r}{2\sigma^2}\frac{\alpha}{\alpha - 1}e^{-2\beta(1-\alpha)^{-1/2}t}\\
&\quad - \frac{(\sigma^2 - 2r)^2}{32\beta}\frac{\alpha^2}{\alpha - 1}(1 - e^{-2\beta(1-\alpha)^{-1/2}t})^2.
\end{aligned}$$

Proof. We can write

$$\mathbb{E}^P\Big[\frac{1}{\alpha}C_t^\alpha\Big] = \frac{1}{\alpha}\Big(\frac{\lambda}{\theta}\Big)^{\frac{\alpha}{\alpha-1}}e^{\frac{\alpha\rho}{\alpha-1}t}\mathbb{E}^P\Big[H_t^{\frac{\alpha}{\alpha-1}}\Big] \quad t \in [0, T].$$

Therefore it remains to calculate $\mathbb{E}^P[H_t^{\frac{\alpha}{\alpha-1}}]$. To achieve this we employ the same procedure as in [Föllmer, Schachermayer]. We define the measure P^δ by

$$\begin{aligned}
\phi_t^\delta &= \exp\Big(\int_0^t \frac{\beta - \delta}{\sigma}Y_s dW_s - \frac{1}{2}\int_0^t \frac{(\beta - \delta)^2}{\sigma^2}Y_s^2 ds\Big)\\
&= \exp\Big(\int_0^t \frac{\beta - \delta}{\sigma}Y_s dY_s + \frac{1}{2}\int_0^t \frac{\beta^2 - \delta^2}{\sigma^2}Y_s^2 ds\Big)
\end{aligned}$$

which is constructed in such a way as to eliminate the term $\int_0^t Y_s^2 ds$ if we set $\delta = \beta\sqrt{1 - \frac{\alpha}{\alpha-1}}$. So we can write

$$\begin{aligned}
\mathbb{E}^P[H_t^{\frac{\alpha}{\alpha-1}}] &= \mathbb{E}^{P^\delta}[H_t^{\frac{\alpha}{\alpha-1}}(\phi^\delta)^{-1}]\\
&= e^{-\frac{\alpha r}{\alpha-1}t}\mathbb{E}^{P^\delta}\Big[\exp\Big(-\frac{\sigma^2 - 2r}{2\sigma^2}\frac{\alpha}{\alpha - 1}(Y_t - Y_0) - \frac{(\sigma^2 - 2r)^2}{8\sigma^2}\frac{\alpha}{\alpha - 1}t\\
&\quad + \frac{\beta}{2\sigma^2}((\alpha - 1)^{-1} + (1 - \alpha)^{-1/2})(Y_t^2 - Y_0^2 - \sigma^2)t\Big)\Big].
\end{aligned}$$

Since Y_t is under P^δ Gaussian with mean $m = e^{-\delta t}$ and variance $v^2 = (1 - e^{-\delta t})\sigma^2/(2\delta)$ we can use

$$\mathbb{E}^P[\exp(\xi Y^2 + \zeta Y)] = (1 - 2\xi v^2)^{-1/2}\exp\Big((1 - 2\xi v^2)^{-1}\Big(\xi m^2 + \zeta m + \frac{1}{2}\zeta^2 v^2\Big)\Big)$$

and thus get the result stated in eq.(3.7). $\square$

3.2.2 Logarithmic Utility

Remark 4. The problem of optimizing logarithmic utility has been studied in great detail in various market settings. In [Merton 1969] it is shown that different assumptions on the price behavior lead to the same optimal consumption process, if one uses logarithmic utility.

Nevertheless we give here the solution of the optimal consumption problem, but then turn immediately to the analysis of the value function itself. We assume that U_1 and U_2 are of the form

$$U_1(x) = U_2(x) = \ln x,$$

and thus the functions I_1 and I_2 take the form

$$I_1(x) = I_2(x) = \frac{1}{x}.$$

The consumption process and the terminal wealth are given by

$$C_t^* = \frac{\theta}{\lambda} H_t^{-1} e^{-\rho t}, \quad t \in [0,T] \tag{3.8}$$

$$X_T^* = \frac{1-\theta}{\lambda} H_T^{-1} e^{-\rho T}. \tag{3.9}$$

In this case the calculation of λ leads to a very nice result, since the H_t terms cancel. We get two different solutions: one for ρ equal to zero and one for $\rho \neq 0$

$$\lambda = \begin{cases} \frac{\theta(T-1)+1}{x} & \text{for } \rho = 0, \\ x^{-1}\Big[\frac{\theta}{\rho} + e^{-\rho T}\Big(1-\theta-\frac{\theta}{\rho}\Big)\Big] & \text{else.} \end{cases}$$

There are no problems for the logarithmic utility in any case. We can write down the optimal invested wealth as well as the optimal investment process for $t \in [0,T]$

$$X_t^* = \begin{cases} \frac{\theta(T-1)+1}{\lambda H_t} & \text{for } \rho = 0 \\ \frac{1}{\lambda H_t}(\frac{\theta}{\rho}e^{-\rho t} - (1-\theta-\frac{\theta}{\rho})e^{-\rho T}) & \text{else} \end{cases},$$

$$w_t^* = \begin{cases} \frac{\phi}{\lambda\sigma S_t H_t}(\theta(T-1)+1) & \text{for } \rho = 0 \\ \frac{\phi}{\lambda\sigma\rho S_t H_t}(\theta e^{-\rho t} + (\rho - 2\theta)e^{-\rho T}) & \text{else} \end{cases}.$$

Proposition 2 (Behavior of the Value Function under Logarithmic Utility)**.** For the logarithmic utility function the following limits can be calculated:

$$\begin{aligned} \lim_{\rho\to\infty} V^{ln}(x) &= 0 \\ \lim_{\rho\to-\infty} V^{ln}(x) &= \infty \quad \text{if } \ln(x) > 0 \text{ and} \\ & \Big(\frac{\beta Y_0^2}{4\sigma^2} - \frac{1}{8}\Big)\Big(e^{-2\beta T}+1\Big) - \frac{(\sigma^2-2r)\theta Y_0}{2\sigma^2}\Big(e^{-\beta T}+1\Big) > -\frac{(\sigma^2+2r)^2+2\beta\sigma^2}{8\sigma^2} \\ \lim_{T\to\infty} V^{ln}(x) &= \frac{\theta}{\rho}\Big(\ln(x\rho) - \frac{9}{8} + \frac{\beta Y_0^2}{4\sigma^2} - \frac{(\sigma^2-2r)Y_0}{2\sigma^2} + \frac{(\sigma^2+2r)^2+2\beta\sigma^2}{8\sigma^2\rho}\Big) \\ & - \frac{\theta}{\rho+2\beta}\Big(\frac{\beta Y_0^2}{4\sigma^2} - \frac{1}{8}\Big) + \frac{(\sigma^2-2r)\theta Y_0}{2\sigma^2(\rho+\beta)} \quad \text{if } \rho > 0. \end{aligned} \tag{3.10}$$

Proof. The value function under logarithmic utility is given by

$$\begin{aligned} V^{ln}(x) &= \mathbb{E}^P\Big[\int_0^T \theta e^{-\rho t}\ln\Big(\frac{x\theta}{H_t e^{\rho t}(\frac{\theta}{\rho}+e^{-\rho T}(1-\theta-\frac{\theta}{\rho}))}\Big)dt \\ &\quad + (1-\theta)e^{-\rho T}\ln\Big(\frac{x(1-\theta)}{H_T e^{\rho T}(\frac{\theta}{\rho}+e^{-\rho T}(1-\theta-\frac{\theta}{\rho}))}\Big)\Big], \end{aligned}$$

and by interchanging integration and expectation and integrating the deterministic elements of the value function it can be rewritten as

$$\begin{aligned} V^{ln}(x) &= \frac{\theta}{\rho}(1-e^{-\rho T})(\ln(\theta x) - \ln\Big(\frac{\theta}{\rho}+e^{-\rho T}\Big(1-\theta-\frac{\theta}{\rho}\Big)\Big) - 1) + \theta T e^{-\rho T} \\ &\quad + (1-\theta)e^{-\rho T}(\ln((1-\theta)x) - \ln\Big(e^{\rho T}\frac{\theta}{\rho}+1-\theta-\frac{\theta}{\rho}\Big) \\ &\quad - \theta\int_0^T e^{-\rho t}\mathbb{E}^P[\ln H_t]dt - (1-\theta)\mathbb{E}^P[\ln H_T]). \end{aligned}$$

We now consider H_t in detail. Since H_t is of exponential form, we can write

$$\begin{aligned} \ln(H_t) &= -\int_0^t \Big(\frac{\sigma^2-2r}{2\sigma} - \frac{\beta}{\sigma}Y_s\Big)dW_s + \int_0^t \frac{\beta(\sigma^2-2r)}{2\sigma^2}Y_s ds \\ &\quad - \int_0^t \frac{\beta^2}{2\sigma^2}Y_s^2 ds - \frac{(\sigma^2+2r)^2}{8\sigma^2}t \\ &= \frac{1}{\sigma}\int_0^t \Big(\frac{\beta}{\sigma}Y_s - \frac{\sigma^2-2r}{2\sigma}\Big)\Big(dY_s + \beta Y_s ds\Big) + \int_0^t \frac{\beta(\sigma^2-2r)}{2\sigma^2}Y_s ds \\ &\quad - \int_0^t \frac{\beta^2}{2\sigma^2}Y_s^2 ds - \frac{(\sigma^2+2r)^2}{8\sigma^2}t \\ &= \frac{1}{\sigma}\int_0^t \Big(\frac{\beta}{\sigma}Y_s - \frac{\sigma^2-2r}{2\sigma}\Big)dY_s + \int_0^t \frac{\beta^2}{2\sigma^2}Y_s^2 ds - \frac{(\sigma^2+2r)^2}{8\sigma^2}t \\ &= \frac{\sigma^2-2r}{2\sigma^2}(Y_0-Y_t) - \frac{\beta}{2\sigma^2}(Y_t^2-Y_0^2-\sigma^2 t) + \frac{\beta^2}{2\sigma^2}\int_0^t Y_s^2 ds - \frac{(\sigma^2+2r)^2}{8\sigma^2}t \\ &= \frac{\sigma^2-2r}{2\sigma^2}(Y_0-Y_t) + \frac{\beta}{4\sigma^2}(Y_t^2-Y_0^2) + \frac{\beta}{2\sigma}\int_0^t Y_s dW_s \\ &\quad - \frac{(\sigma^2+2r)^2+2\beta\sigma^2}{8\sigma^2}t. \end{aligned}$$

The fact that the squared Ornstein-Uhlenbeck process Y_t^2 is a Cox-Ingersoll-Ross process (see [Cox, Ingersoll, Ross]) can be used to calculate the expectation of Y_t^2 using

$$\begin{aligned} Y_t &= Y_0 - \int_0^t \beta Y_s ds + \int_0^t \sigma dW_s \quad \text{and} \\ Y_t^2 &= Y_0^2 + \int_0^t (\sigma^2 - 2\beta Y_s^2)ds + \int_0^t 2\sigma Y_s dW_s. \end{aligned}$$

Thus we get

$$\begin{aligned} \mathbb{E}^P[Y_t] &= Y_0 e^{-\beta t} \quad \text{and} \\ \mathbb{E}^P[Y_t^2] &= Y_0^2 e^{-2\beta t} + \frac{\sigma^2}{2\beta}(1-e^{-2\beta t}), \end{aligned}$$

and we can calculate

$$\begin{aligned}\mathbb{E}^P[\ln(H_t)] &= -\frac{(\sigma^2+2r)^2+2\beta\sigma^2}{8\sigma^2}t+\frac{(\sigma^2-2r)Y_0}{2\sigma^2}(1-e^{-\beta t})\\ &\quad+\Big(\frac{\beta Y_0^2}{4\sigma^2}-\frac{1}{8}\Big)e^{-2\beta t}+\frac{1}{8}-\frac{\beta Y_0^2}{4\sigma^2}.\end{aligned}\tag{3.11}$$

Plugging in our results and integrating yields

$$\begin{aligned}V^{ln}(x) &= \frac{\theta(1-e^{-\rho T})}{\rho}\Big(\ln(\theta x)-\ln\Big(\frac{\theta}{\rho}+e^{-\rho T}\Big(1-\theta-\frac{\theta}{\rho}\Big)\Big)-\frac{9}{8}+\frac{\beta Y_0^2}{4\sigma^2}-\frac{(\sigma^2-2r)Y_0}{2\sigma^2}\\ &\quad+\frac{(\sigma^2+2r)^2+2\beta\sigma^2}{8\sigma^2\rho}\Big)+e^{-\rho T}\Big(T\theta\Big(1-\frac{(\sigma^2+2r)^2+2\beta\sigma^2}{8\sigma^2\rho}\Big)\\ &\quad+\frac{(\sigma^2-2r)\theta Y_0}{2\sigma^2}\Big(1-\theta-\frac{\theta}{(\rho+\beta)}\Big)e^{-\beta T}+(1-\theta)\Big(\ln((1-\theta)x)-\ln\Big(\frac{\theta}{\rho}e^{\rho T}+1-\theta-\frac{\theta}{\rho}\Big)\Big)\\ &\quad+\Big(\frac{\theta}{\rho+2\beta}-1+\theta\Big)\Big(\frac{\beta Y_0^2}{4\sigma^2}-\frac{1}{8}\Big)e^{-2\beta T}-(1-\theta)\Big(\frac{(\sigma^2-2r)Y_0}{2\sigma^2}+\frac{1}{8}-\frac{\beta Y_0^2}{4\sigma^2}\\ &\quad-\frac{(\sigma^2+2r)^2+2\beta\sigma^2}{8\sigma^2}T\Big)\Big)-\frac{\theta}{\rho+2\beta}\Big(\frac{\beta Y_0^2}{4\sigma^2}-\frac{1}{8}\Big)+\frac{(\sigma^2-2r)\theta Y_0}{2\sigma^2(\rho+\beta)}.\end{aligned}$$

Using this representation we can deduce the limits stated in eq.(3.10). The limit $T\to\infty$ is a straightforward calculation, whereas for the limit $\rho\to\infty$ we have to employ l'Hôpital's rule several times to calculate $(1-e^{-\rho T})\ln\Big(\frac{\theta}{\rho}+e^{-\rho T}\Big(1-\theta-\frac{\theta}{\rho}\Big)\Big)$ and $e^{-\rho T}\ln\Big(\frac{\theta}{\rho}e^{\rho T}+1-\theta-\frac{\theta}{\rho}\Big)$ which are both equal to zero in the limit.
For the limit $\rho\to-\infty$ we see that the value function consists of a sum of terms which each converge to infinity under the conditions stated in the theorem. □

Once we let ρ tend to minus infinity, that implies that we would prefer later consumption. Letting ρ tend to plus infinity we model the need for sooner consumption.
If we emphasize the present more than the future, we are eager to consume sooner rather than later. Therefore we will consume most of our capital in the beginning and there is not much money left for investment.
On the other hand, if we value the future more than the present, we can now trade profitable. We can increase our wealth and thus the overall utility of consumption, if we do not feel compelled to consume early due to a high (subjective) discounting rate.

Proposition 3. The expected utility of consumption for logarithmic utility at time $t\in[0,T]$ is given by $u_t^{\ln}=\mathbb{E}^P[\ln(C_t)]$ and it takes the form

$$\begin{aligned}u_t^{\ln} &= \ln\frac{\theta}{\lambda}-\Big(\rho-\frac{(\sigma^2+2r)^2+2\beta\sigma^2}{8\sigma^2}\Big)t-\frac{(\sigma^2-2r)Y_0}{2\sigma^2}(1-e^{-\beta t})\\ &\quad-\Big(\frac{\beta Y_0^2}{4\sigma^2}-\frac{1}{8}\Big)e^{-2\beta t}-\frac{1}{8}+\frac{\beta Y_0^2}{4\sigma^2}.\end{aligned}$$

Proof. We calculate

$$\begin{aligned}\mathbb{E}^P[\ln(C_t)] &= \mathbb{E}^P[\ln\Big(\frac{\theta}{\lambda}H_t^{-1}e^{-\rho t}\Big)]=\ln\frac{\theta}{\lambda}-\mathbb{E}^P[\ln(H_te^{\rho t})]\\ &= \ln\frac{\theta}{\lambda}-\Big(\rho-\frac{(\sigma^2+2r)^2+2\beta\sigma^2}{8\sigma^2}\Big)t-\frac{(\sigma^2-2r)Y_0}{2\sigma^2}(1-e^{-\beta t})\\ &\quad-\Big(\frac{\beta Y_0^2}{4\sigma^2}-\frac{1}{8}\Big)e^{-2\beta t}-\frac{1}{8}+\frac{\beta Y_0^2}{4\sigma^2}.\end{aligned}$$

□

Remark 5. If we set $\theta = 0$ in this problem, we will have zero utility of consumption and thus have the optimal terminal wealth problem as considered in [Föllmer, Schachermayer]. Since the form of the consumption process is very similar to the wealth process we can immediately write down the growth rate of terminal wealth using the same calculations as in the preceding proposition which is

$$\lim_{T\nearrow\infty} \frac{\mathbb{E}^P[\ln(X_T)]}{T} = \frac{(\sigma^2+2r)^2+2\beta\sigma^2}{8\sigma^2}$$

for $\rho = 0$ and exactly the same as in proposition 5.1 in [Föllmer, Schachermayer] for $r = 0$.

Proposition 4. Let $\rho > 0$ and $T \to \infty$, then the $\lim_{t\to\infty} \mathbb{E}^P[\ln C_t]$ is approximately linear with drift $\frac{(\sigma^2+2r)^2+2\beta\sigma^2}{8\sigma^2} - \rho$.

Proof. Letting $T \to \infty$ the Lagrange multiplier converges to $\frac{\theta}{x\rho}$ for $\rho > 0$. Now we can deduce

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}^P[\ln C_t] &= \lim_{t\to\infty} \mathbb{E}^P[\ln x\rho - \ln H_t - \rho t] \\ &= \ln x\rho - \Big(\rho - \frac{(\sigma^2+2r)^2+2\beta\sigma^2}{8\sigma^2}\Big)t - \frac{(\sigma^2-2r)Y_0}{2\sigma^2}(1-e^{-\beta t}) \\ &\quad - \Big(\frac{\beta Y_0^2}{4\sigma^2} - \frac{1}{8}\Big)e^{-2\beta t} - \frac{1}{8} + \frac{\beta Y_0^2}{4\sigma^2},\end{aligned}$$

which concludes the proof. □

3.2.3 Exponential Utility

Finally we assume that U_1 and U_2 are of the form

$$U_1(x) = U_2(x) = -\frac{\exp(-\eta x)}{\eta},$$

with $\eta > 0$ constant, thus the functions I_1 and I_2 take the form

$$I_1(x) = I_2(x) = -\frac{\ln x}{\eta}.$$

The consumption process and the terminal wealth are given by

$$C_t^* = \frac{\ln\theta}{\eta} - \frac{\ln\lambda}{\eta} - \frac{\ln(H_t e^{\rho t})}{\eta}, \quad t \in [0,T] \tag{3.12}$$

$$X_T^* = \frac{\ln(1-\theta)}{\eta} - \frac{\ln\lambda}{\eta} - \frac{\ln(H_T e^{\rho T})}{\eta}, \tag{3.13}$$

and the optimal invested wealth for $t \in [0,T]$ is given by

$$X_t^* = -\frac{1}{\eta H_t}\mathbb{E}^P\Big[\int_t^T H_s(\ln\Big(\frac{\lambda}{\theta}\Big) + \ln H_s + \rho s)ds + H_T(\ln\Big(\frac{\lambda}{1-\theta}\Big) + \ln H_T + \rho T)\Big|\mathcal{F}_t\Big].$$

The Lagrange multiplier λ is well defined by

$$\begin{aligned}\lambda &= \exp\Big(-\frac{\eta x + E_1}{e^{-rT}(1-\frac{1}{r})+\frac{1}{r}}\Big), \quad \text{where} \\ E_1 &= \mathbb{E}^P\Big[\int_0^T H_t(\ln H_t + \rho t - \ln\theta)dt + H_T(\ln H_T + \rho T - \ln(1-\theta))\Big].\end{aligned}$$

We want to calculate the Lagrange multiplier more precisely. To do that we take a closer look at E_1 and get by interchanging the expectation and integration, and changing the measure from P to Q

$$E_1 = \int_0^T (\mathbb{E}^Q(\ln H_t) + \rho t - \ln\theta)e^{-rt}dt + (\mathbb{E}^Q(\ln H_T) + \rho T - \ln(1-\theta))e^{-rT}.$$

We see that it remains to calculate $E^Q[\ln H_t]$ and respectively $E^Q[\ln H_T]$. To do this we rewrite $\ln H_t$ in such a way as to get rid of the Brownian Motion W_t under the measure P and replace it with W_t^Q which is a Brownian Motion under Q. We get for $t \in [0,T]$

$$\ln H_t = \frac{1}{2}\int_0^t \Big(\frac{\sigma^2-2r}{2\sigma} - \frac{\beta}{\sigma}Y_s\Big)^2 ds - rt - \int_0^t \Big(\frac{\sigma^2-2r}{2\sigma} - \frac{\beta}{\sigma}Y_s\Big)dW_s^Q.$$

Since we know that

$$\begin{aligned}\mathbb{E}^Q[Y_t] &= Y_0 - \frac{\sigma^2}{2}t, \\ \mathbb{E}^Q[Y_t^2] &= \sigma^2 t + Y_0^2 - \sigma^2 Y_0 t + \frac{\sigma^4}{4}t^2,\end{aligned}$$

we finally get

$$\mathbb{E}^Q \ln H_t = \Big(\frac{1}{2}\Big(\frac{\beta}{\sigma}Y_0 - \frac{\sigma^2-2r}{2\sigma}\Big)^2 - r\Big)t + \Big(\frac{\beta^2}{4}(1-Y_0) + \frac{\beta}{8}(\sigma^2-2r)\Big)t^2 + \frac{\beta^2\sigma^2}{24}t^3.$$

Integrating with respect to t and collecting similar terms we get

$$\begin{aligned}E_1 &= \Big(\frac{1}{2}\Big(\frac{\beta}{\sigma}Y_0 - \frac{\sigma^2-2r}{2\sigma}\Big)^2 - r + \rho\Big)\Big(e^{-rT}\Big(1 - \frac{T}{r} - \frac{1}{r^2}\Big) + \frac{1}{r^2}\Big) \\ &\quad + \Big(\frac{\beta^2}{4}(1-Y_0) + \frac{\beta}{8}(\sigma^2-2r)\Big)\Big(e^{-rT}\Big(1 - \frac{T^2}{r} - \frac{T}{r^2} - \frac{2}{r^3}\Big) + \frac{2}{r^3}\Big) \\ &\quad + \frac{\beta^2\sigma^2}{24}\Big(e^{-rT}\Big(1 - \frac{T^3}{r} - \frac{3T^2}{r^2} - \frac{6T}{r^3} - \frac{6}{r^4}\Big) + \frac{6}{r^4}\Big) \\ &\quad + \frac{\ln\theta}{r}(e^{-rT}-1) - \ln(1-\theta)e^{-rT}.\end{aligned}$$

Unlike the power and logarithmic utility the solution of the exponential utility consumption and terminal wealth problem does not coincide with the solution of the pure terminal wealth problem if we consider the limit $\theta \to 0$. This is due to the fact that we consider the unconstrained problem, and thus the possibility to consume gives us (by negative consumption) the possibility to increase the terminal wealth.

Proposition 5 (Behavior of the Value Function under Exponential Utility)**.** In the case of exponential utility of consumption and terminal wealth we see that the value function is of the form

$$V^{exp}(x) = -\frac{\lambda}{\eta}\Big(e^{-rt}\Big(1-\frac{1}{r}\Big) + \frac{1}{r}\Big).$$

For $\rho \to -\infty$ we have $V^{exp}(x) \to -\infty$ and for $\rho \to \infty$ we have $V^{exp}(x) \to 0$. Furthermore the value function is monotone in ρ.

Proof. The value function for the exponential utility can be calculated by inserting the optimal consumption process $(C_t^*)_{0\le t\le T}$ and the optimal terminal wealth X_T^* into eq.(3.3), as well as the explicit form of $I_1(.)$ and $I_2(.)$. By using Fubini's theorem we get:

$$\begin{aligned}
V^{exp}(x) &= -\mathbb{E}^P\Big[\int_0^T \frac{\theta}{\eta}\exp(-\rho t)\exp\Big(-\eta\Big(\frac{\ln\theta}{\eta}-\frac{\ln\lambda}{\eta}-\frac{\ln(H_t e^{\rho t})}{\eta}\Big)\Big)dt \\
&\quad + \frac{1-\theta}{\eta}\exp(-\rho T)\exp\Big(-\eta\Big(\frac{\ln(1-\theta)}{\eta}-\frac{\ln\lambda}{\eta}-\frac{\ln(H_T e^{\rho T})}{\eta}\Big)\Big)\Big] \\
&= -\mathbb{E}^P\Big[\int_0^T \frac{\theta}{\eta}\exp\Big(\ln\frac{\lambda H_t}{\theta}+\rho t-\rho t\Big)dt + \frac{1-\theta}{\eta}\exp\Big(\ln\frac{\lambda H_t}{1-\theta}+\rho T-\rho T\Big)\Big] \\
&= -\frac{\lambda}{\eta}\mathbb{E}^P\Big[\int_0^T H_t dt + H_T\Big] \\
&= -\frac{\lambda}{\eta}\Big(e^{-rT}\Big(1-\frac{1}{r}\Big)+\frac{1}{r}\Big).
\end{aligned}$$

As we can see, all terms depending directly on ρ cancel. It remains to look at λ, which still depends on ρ. Since λ depends on ρ only through E_1 which is linear in ρ, this proves that the limits of the value function are $-\infty$ for $\rho \to -\infty$ and zero for $\rho \to \infty$, as well as the monotonicity. □

We see that if ρ tends to minus infinity the value function tends to minus infinity, and for $\rho \to \infty$ the value function tends to zero. We also see that by increasing our subjective discounting rate, we will cause the value function to increase as well.
Because of the fact that in this model the benefit of consumption in the present is so high, the utility gained when consuming early on can never be made up for, even though profitable investments would be realized later on.
Assuming that the conditions in Proposition 2 for $\lim_{\rho\to-\infty} V^{ln}(x) = \infty$ are fulfilled, these two models present two very different individuals due to several reasons. The individual with logarithmic utility has for $\rho \to \infty$ a utility of zero, which is neither the best nor the worst outcome. Whereas the individual with the exponential utility has for $\rho \to \infty$ the utility of zero, which represents the optimum for him.
Since the exponential utility is always negative it is optimal to multiply with a small number in order to increase the supremum. Therefore one would choose a high positive ρ. The logarithmic utility on the other hand can be positive and negative. For $\rho \to -\infty$ the optimal value of the logarithmic utility is attained for the exponential utility on the other hand this gives the worst outcome.

Proposition 6. The expected utility of consumption for exponential utility at time $t \in [0, T]$ is given by $u_t^{\exp} = \mathbb{E}^P[-\frac{1}{\eta}\exp(-\eta C_t)]$ and it takes the form $-\frac{\lambda}{\eta}e^{(\rho-r)t}$.

Proof. We calculate

$$\mathbb{E}^P\Big[-\frac{1}{\eta}\exp(-\eta C_t)\Big] = -\frac{\lambda}{\eta}\mathbb{E}^P[H_t e^{\rho t}] = -\frac{\lambda}{\eta}e^{(\rho-r)t}.$$

□

Lemma 5. In the problem given by eq.(3.3) for exponential utility and $\rho < r$, the consumption process is a submartingale.

Proof. According to the previous calculations we have

$$C_t = \frac{\ln\theta}{\eta} - \frac{\ln\lambda}{\eta} - \frac{\ln(H_t e^{\rho t})}{\eta}, \quad t \in [0,T].$$

We assume $s < t$ and prove the submartingale property using Jensen's inequality and the martingale property of $H_t e^{rt}$.

$$\begin{aligned}
\mathbb{E}^P[C_t|\mathcal{F}_s] &= \frac{\ln\theta}{\eta} - \frac{\ln\lambda}{\eta} - \mathbb{E}^P\Big[\frac{\ln(H_t e^{\rho t})}{\eta}\Big] \\
&\geq \frac{\ln\theta}{\eta} - \frac{\ln\lambda}{\eta} - \frac{\ln(\mathbb{E}^P[H_t e^{\rho t}])}{\eta} \\
&= \frac{\ln\theta}{\eta} - \frac{\ln\lambda}{\eta} - \frac{\ln(H_s e^{\rho t - r(t-s)})}{\eta} \\
&= \frac{\ln\theta}{\eta} - \frac{\ln\lambda}{\eta} - \frac{\ln(H_s e^{\rho s})}{\eta} - \frac{\ln(e^{\rho(t-s)-r(t-s)})}{\eta} \\
&= C_s - \frac{(\rho - r)(t-s)}{\eta} \\
&\geq C_s.
\end{aligned}$$

□

In the case of the exponential utility there is always the risk of encountering negative consumption and ending up with liabilities at final time T. This depends largely on the initial capital we are endowed with. The main contribution of this Lemma is that it shows that depending on $\rho \gtrless r$ there is either an upward or downward drift of the consumption process.

The critical rate of consumption is exactly the interest rate of the bond which is not very surprising. Assuming that we discount higher than what we can get in return by investing profitably we would rather consume much in the beginning than wait until consumption is not very satisfying for oneself anymore.

We can relate this result to the deterministic model in section 2.1 and 2.2 where we have increasing and decreasing consumption processes depending on $\rho \gtrless r$ as well, reflecting the relation between the deterministic and the stochastic model.

3.3 Comparison of the Primal and Dual Approach

3.3.1 The Primal Approach

In [Merton 1969] one can find the primal solution for consumption under exponential utility. We want to optimize the infinite time problem

$$V(x) = \sup_{C,w} \mathbb{E}^P\Big[- \int_0^\infty \frac{\exp(-\eta C_t)}{\eta} dt \Big],$$

where η is the scaling parameter of the utility function, which has been studied in [Merton 1969] to illustrate the equality of the primal and the dual approach. The constants D_1, D_2 and

D_3 (which will be used later on) are given by

$$\begin{aligned} D_1 &= \frac{\beta}{\sigma^2 r}\Big[\frac{\sigma^2}{2} + \frac{\beta\sigma^2}{2r} - r - \beta\Big], \\ D_2 &= \frac{\beta^2}{2r^2} - 1 + \frac{1}{r\sigma^2}\Big(\frac{1}{2} + \frac{\beta}{r} + \frac{\beta^2}{r^2}\Big)\Big(\frac{\sigma^2}{2} - r\Big)^2 \quad \text{and} \\ D_3 &= \frac{\beta^2}{r^2}\Big(\frac{\sigma^2}{2} - r\Big). \end{aligned}$$

For a detailed description of the stochastic control approach see for example [Øksendal]. We define

$$J(X, Y, t) = \sup_{C,w} \mathbb{E}^P\Big[- \int_t^\infty \frac{\exp(-\eta C_s)}{\eta} ds\Big],$$

and by J_t we denote the derivative with respect to time, and analogous the other derivatives. Using Theorem 11.2.1 in [Øksendal] we obtain the HJB equation

$$\begin{aligned} 0 &= -\frac{\exp(-\eta C^*)}{\eta} + J_t + J_X\Big(w^* X\Big[\frac{\sigma^2}{2} - \beta Y - r\Big] + rX - C^*\Big) \\ &\quad + \frac{1}{2}J_{XX}(w^* X)^2\sigma^2 - J_Y\beta Y + \frac{1}{2}J_{YY}\sigma^2 + J_{YX}w^* X\sigma^2 \end{aligned} \tag{3.14}$$

which coincides with [Merton 1969]. Employing the definitions from above we find the solution of the HJB equation

$$J(X, Y, t) = -\frac{1}{\eta r}\exp\Big(- \eta r X_t - \frac{\beta^2}{2\sigma^2 r}Y_t^2 + D_1 Y_t - D_2\Big),$$

and the corresponding optimal consumption process $(C_t^*)_{0\le t\le T}$ and investment process $(w_t^*)_{0\le t\le T}$ are given by the first order conditions

$$\begin{aligned} C_t^* &= -\frac{\ln(J_X)}{\eta}, \\ w_t^* X_t &= -\frac{J_X(\frac{\sigma^2}{2} - \beta Y - r)}{J_{XX}\sigma^2} - \frac{J_{XY}}{J_{XX}}. \end{aligned}$$

Inserting the solution of the HJB equation we get the explicit form

$$\begin{aligned} C_t^* &= rX_t + \frac{\beta^2}{2\sigma^2\eta r}Y_t^2 - \frac{1}{\eta}D_1 Y_t + \frac{1}{\eta}D_2 \quad \text{and} \\ w_t^* X_t &= \frac{1}{\eta r\sigma^2}\Big[\Big(1 + \frac{\beta}{r}\Big)\Big(\frac{\sigma^2}{2} - \beta Y_t - r\Big) + D_3\Big] \end{aligned}$$

for $t \in [0, T]$ as shown in [Merton 1969]. We now calculate the limit of the wealth process X_t for $t \to \infty$ to show whether our optimal strategy implies amassing infinite debts at infinity.

$$\begin{aligned} X_t &= x + \int_0^t w_t X_t \frac{dS_t}{S_t} + \int_0^t (1 - w_t)X_t \frac{dB_t}{B_t} - \int_0^t C_s^* ds \\ X_t &= x - \int_0^t \frac{1}{\eta r\sigma^2}\Big[\Big(1 + \frac{\beta}{r}\Big)\Big(\frac{\sigma^2}{2} - \beta Y_s - r\Big) + D_3\Big]\Big(\beta Y_s - \frac{\sigma^2}{2} + r\Big)ds \\ &\quad - \int_0^t \frac{\beta^2}{2\sigma^2\eta r}Y_s^2 - \frac{D_1 Y_s}{\eta} + \frac{D_2}{\eta}ds + \int_0^t \frac{1}{\eta r\sigma}\Big[\Big(1 + \frac{\beta}{r}\Big)\Big(\frac{\sigma^2}{2} - \beta Y_s - r\Big) + D_3\Big]dW_s. \end{aligned} \tag{3.15}$$

We insert the expressions we stated in Proposition 2 concerning $\mathbb{E}^P[Y_t]$ and $\mathbb{E}^P[Y_t^2]$ into eq.(3.15) and apply Fubini's theorem to get

$$\begin{aligned}\mathbb{E}^P[X_t] &= x + \frac{\beta^2}{\eta r\sigma^2}\int_0^t \Big(\frac{1}{2}+\frac{\beta}{r}\Big)E^P[Y_s^2]ds \\ &\quad - \frac{\beta}{\eta r\sigma^2}\int_0^t \Big(\frac{\sigma^4}{2} - 2\sigma^2 r + 2r^2 + \frac{\beta^2\sigma^2}{2r^2} - \frac{\beta^2}{r} - \frac{\sigma^2}{2} + \frac{\beta\sigma^2}{2r^2} - r - \beta\Big)E^P[Y_s]ds \\ &\quad + \frac{1}{\eta r\sigma^2}\int_0^t \Big(\frac{\sigma^4}{8} - \frac{3\sigma^2 r}{2} + \frac{r^2}{2} + \frac{\beta\sigma^2}{2r}\Big)ds.\end{aligned}$$

It remains to substitute the formulas for the expectation of Y_s and Y_s^2 and to integrate with respect to t. Most terms of X_t are either constant or tend to zero with exception of some linear terms. To determine the behavior of the wealth process, we take a closer look at these terms and see that

$$\mathbb{E}^P[X_t] \approx \frac{1}{\eta r\sigma^2}\Big[\frac{1}{2}(r^2 - 3\sigma^2 r) + \frac{\beta\sigma^2}{4} + \frac{\beta^2\sigma^2}{r} + \frac{\sigma^2}{8}\Big]t \quad \text{for } t \to \infty.$$

The discounted wealth process $X_t e^{-rt}$ on the other hand tends to zero for $t \to \infty$. Concluding, we can say that this problem and its solution are well-defined and especially the limit of the terminal wealth does not tend to minus infinity for reasonable choices for the constants r, σ and β.
Since the wealth process increases at a slower (linear) rate than the discounting factor, which is exponential, we see that in the future our wealth will not be worth as much as now. This reflects the importance of the near future to the investor.
Another question is for which value of Y_t the consumption is the smallest. To calculate this, we complete the terms in C_t^* to a total square and thus write

$$C_t^* = rX_t + \Big(\frac{\beta}{\sigma\sqrt{2\eta r}}Y_t - \frac{\sigma\sqrt{r}}{\sqrt{2\eta}\beta}D_1\Big)^2 + \frac{1}{\eta}D_2 - \frac{\sigma^2 r D_1^2}{2\eta\beta^2}.$$

We see that the consumption depends linearly on the current wealth X_t and in a quadratic way on the logarithm of the stock price. From this form we see that for a given wealth X_t the consumption C_t^* is minimal if

$$Y_t = \frac{1}{\beta}\Big[\frac{\sigma^2}{2} + \frac{\beta\sigma^2}{2r} - r - \beta\Big] = \tilde{Y}.$$

So we consume much if the stock price, and thus its logarithm, is either very big, or very small. This can be explained by the fact that in these cases we can expect the price to return to its original level. This additional knowledge gives us the possibility to consume more. Moreover, we can ask if the consumption process is positive or not. To answer this question we consider the consumption for the worst case of Y_t. Inserting $\tilde{Y}$ and simplifying gives

$$C_t^*|_{Y_t=\tilde{Y}} = rX_t + \frac{1}{\eta}\Big[\frac{\beta^2}{2r}\Big(\frac{\sigma^2}{4r^2} + \frac{1}{\sigma^2}\Big) - 1\Big]. \tag{3.16}$$

Thus we see that even if we have to consume as little as possible, we will still have positive consumption if our current wealth is *big enough*, where *big enough* depends on the choice of the constants.

3.3.2 The Dual Approach

We can solve the problem given by eq.(3.14) also using the dual method. In [Cox, Huang] and [Karatzas, Zitkovic] the dual procedure for solving this problem can be found. Given the utility U_1 we can calculate the first derivative U_1' and the inverse functions of the first derivative I_1. We get as before in the exponential utility case $I_1(z) = -\frac{\ln z}{\eta}$.
Then the optimal consumption process C_t^* is given by

$$\begin{aligned} C_t^* &= I_1(\lambda H_t) \quad \text{for } t \in [0,\infty] \text{ where } \lambda \text{ is given by} \\ \lambda &= \exp\Big(-\eta r x - r \int_0^\infty e^{-rt}\mathbb{E}^Q[\ln H_t]dt \Big) \end{aligned}$$

and H_t is the same as in section 2. The Lagrange multiplier λ is given by the budget constraint. Thus by inserting the optimal consumption process as well as the optimal terminal wealth into the budget constraint we can calculate λ.
Finally we get the optimal consumption process

$$C_t^* = -rx - \frac{r}{\eta}\int_0^\infty e^{-rt}\mathbb{E}^Q[\ln H_t]dt - \frac{\ln H_t}{\eta} \quad \text{for } 0 \le t \le T. \tag{3.17}$$

Lemma 6. The expectation of the consumption process $\mathbb{E}^P[C_t]$ grows approximately linear in t at the rate

$$\frac{\mathbb{E}^P[C_t]}{t} \approx \frac{\sigma^4 + 6\sigma^2 r + r^2 + 2\beta\sigma^2}{8\sigma^2\eta}. \tag{3.18}$$

Since we assume the positivity of β, r, ρ and η it follows that the drift is positive and so the expected consumption is increasing in t.

Proof. We can use the calculations from Example 2 and thus we can follow that

$$\begin{aligned} \mathbb{E}^P[\ln(H_t)] &= -\frac{\sigma^4 + 6\sigma^2 r + r^2 + 2\beta\sigma^2}{8\sigma^2}t + \frac{(\sigma^2 - r)Y_0}{2\sigma^2}(1 - e^{-\beta t}) \\ &+ \Big(\frac{\beta Y_0}{4\sigma^2} - \frac{1}{8}\Big)e^{-2\beta t} + \frac{1}{8} - \frac{\beta Y_0^2}{4\sigma^2}. \end{aligned}$$

Since C_t^* is given by a constant minus $\frac{1}{\eta}\ln H_t$, this proves the assumption. □

Remark 6. This is in line with Lemma 2, since in this case we have $\rho = 0$ and a positive r, thus the inequality $\rho < r$ is fulfilled.

Finally, we turn to the calculation of the Lagrange multiplier λ. Using once again the expressions we have for $\mathbb{E}^Q[Y_t]$ and $\mathbb{E}^Q[Y_t^2]$, we can calculate $\mathbb{E}^Q[\ln H_t]$ and thus we calculate

$$\ln\lambda = \eta r x + \frac{1}{r^3}\Big(r^2\Big(\frac{1}{2}\Big(\frac{\beta}{\sigma}Y_0 - \frac{\sigma^2 - 2r}{2\sigma}\Big)^2 - 1\Big) + 2r\Big(\frac{\beta^2}{4}(1 - Y_0) + \frac{\beta}{8}(\sigma^2 - 2r)\Big) + \frac{\beta^2\sigma^2}{4}\Big)$$

for the logarithm of the Lagrange multiplier.

3.3.3 Comparison of the two Approaches

We show that the consumption process we calculated using the primal approach is indeed the same as the consumption process calculated by the dual method. This can be done by inserting the following expressions for the wealth X_t and the logarithm of the stock price Y_t into the primal solution of the consumption problem.

$$\begin{aligned}
X_t &= x - \int_0^t \frac{1}{\eta r \sigma^2}\Big[\Big(1+\frac{\beta}{r}\Big)\Big(\frac{\sigma^2}{2} - \beta Y_s - r\Big) + D_3\Big]\Big(\beta Y_s - \frac{\sigma^2}{2} + r\Big)ds \\
&- \int_0^t \Big(\frac{\beta^2}{2\sigma^2 \eta r} Y_s^2 - \frac{D_1 Y_s}{\eta} + \frac{D_2}{\eta}\Big)ds + \int_0^t \frac{1}{\eta r \sigma}\Big[\Big(1+\frac{\beta}{r}\Big)\Big(\frac{\sigma^2}{2} - \beta Y_s - r\Big) + D_3\Big]dW_s \\
Y_t &= Y_0 - \int_0^t \beta Y_s ds + \int_0^t \sigma dW_s \\
Y_t^2 &= Y_0^2 + \int_0^t (\sigma^2 - 2\beta Y_s^2) ds + \int_0^t 2\sigma Y_s dW_s
\end{aligned}$$

Collecting the integrals with respect to s and those with respect to W_s, we get in both approaches

$$\int_0^t \Big(\frac{\beta^2 Y_s^2}{2\eta\sigma^2} + \frac{\beta r Y_s}{\eta \sigma^2} - \frac{\beta Y_s}{2\eta} + \frac{r}{2\eta} + \frac{r^2}{2\eta\sigma^2} + \frac{\sigma^2}{8\eta}\Big)ds \quad \text{and} \quad \int_0^t \Big(-\frac{r}{\eta\sigma} + \frac{\sigma}{2\eta} - \frac{\beta Y_s}{\eta\sigma}\Big)dW_s,$$

proving the equal behavior of the two processes.

3.4 Simulation

To illustrate the results of section 3.2 we want to present some simulated processes and the corresponding optimal consumption strategy. We concentrate on the dependence of the consumption process on the weight θ and the time horizon T. Also we discuss the economical interpretation of our findings. Furthermore we compare the simulated with the analytical results.
For the simulation we used the program R. We considered the time interval $[0, 1]$ and simulated a Brownian motion with stepsize 500. Using this Brownian motion we calculated the processes $(Y_t)_{0 \le t \le T}$, $(S_t)_{0 \le t \le T}$, $(H_t)_{0 \le t \le T}$ and $(C_t)_{0 \le t \le T}$. The analytical computations and graphical representations were computed using Wolfram Mathematica 6.0.

Several papers are concerned with the calibration of the parameters in an Ornstein-Uhlenbeck model. Depending on whether they want to model interest rates, commodities or stock prices, not only the value but also the order of magnitude between the constants vary. Also these models are often combined with other influences. We present here the results for two very different choices of parameters. Models related to the one we consider here can be found for example in [Schwartz, Smith].

In Table 3.1 we see different realizations of the optimal consumption and terminal wealth problem for different price processes and utility functions. We see that the parameter η in the exponential utility case has a huge influence on the magnitude of the value function. Moreover we see that the simulated results are quite accurate compared to the analytical solution.

	$\beta = 0.03$, $\sigma = 0.2$, $\rho = 1$, $r = 0$, $T = 1$		$\beta = 1.5$, $\sigma = 0.3$, $\rho = 1$, $r = 0$, $T = 1$	
V^{log}	3.9739		6.9176	
Sim	AM: 3.9752	V: 0.0011	AM: 7.0080	V: 2.7607
$V^{exp}_{\eta=0.1}$	-0.0635		$-9.2422 \cdot 10^{-6}$	
Sim	AM: -0.0633	V: $6.3739 \cdot 10^{-6}$	AM: $-8.5648 \cdot 10^{-6}$	V: $1.1332 \cdot 10^{-8}$
$V^{exp}_{\eta=1}$	$-1.820 \cdot 10^{-22}$		$-2.6456 \cdot 10^{-26}$	
Sim	AM: $-1.817 \cdot 10^{-22}$	V: $6.831 \cdot 10^{-47}$	AM: $-2.5328 \cdot 10^{-26}$	V: $2.9832 \cdot 10^{-50}$

Table 3.1: Value function for logarithmic and exponential utility: These simulations were made with stepsize $n = 500$. We compare the arithmetic mean (AM) and variance (V) over 100 simulations with the analytical solution of the problem.

	$\rho = -10$	$\rho = -1$	$\rho = 1$	$\rho = 10$
V^{log}	199834	33.0909	6.91756	0.681379
V^{exp}	$-1.01264 \cdot 10^{-22}$	$-1.18568 \cdot 10^{-25}$	$-2.6456 \cdot 10^{-26}$	$-3.09768 \cdot 10^{-29}$

Table 3.2: The result of the value function under optimal consumption and investment for different values of ρ. The market is given by $\beta = 1.5$, $\sigma = 0.3$, and $T = 1$. The parameter for the exponential utility is given by $\eta = 1$ and the interest rate $r = 0$.

Furthermore we consider the behavior of the value function under logarithmic and exponential utility for different values of ρ. The results are presented in Table 3.2. We see that these analytic results are in accordance with our theoretical results.

Finally Table 3.3 represents the dependence of the value function on θ. For θ equal to zero this solution corresponds to the pure terminal wealth problem and the consumption process is equal to zero. We see that the dependence of the value function on θ is itself depending on ρ since for small ρ the value function decreases in θ, and for big ρ the value function increases in θ. This relation between ρ and θ is presented in Figure 3.1.

Also the impact of θ depends on the choice of ρ. A small change in θ will result in a small change of $V^{\ln}(x)$ for $\rho = 2$, but for $\rho = -2$ on the other hand, even a small change in θ might cause a big change in $V^{\ln}(x)$.

In Figure 3.2 (a) we see an example of two stock price processes governed by a geometric Ornstein-Uhlenbeck process. On the right hand side we see three different optimal

V^{log}	$\theta = 0.1$	$\theta = 0.35$	$\theta = 0.65$	$\theta = 0.9$
$\rho = 0.1$	11.9769	11.1898	10.6069	10.4179
$\rho = 1.5$	3.16772	3.61914	4.29961	4.95703

Table 3.3: The result of the value function under optimal consumption and investment for different values of θ. The market is given by $\beta = 1.5$, $\sigma = 0.2$, $r = 0.02$ and $T = 1$. The initial capital x is 100.

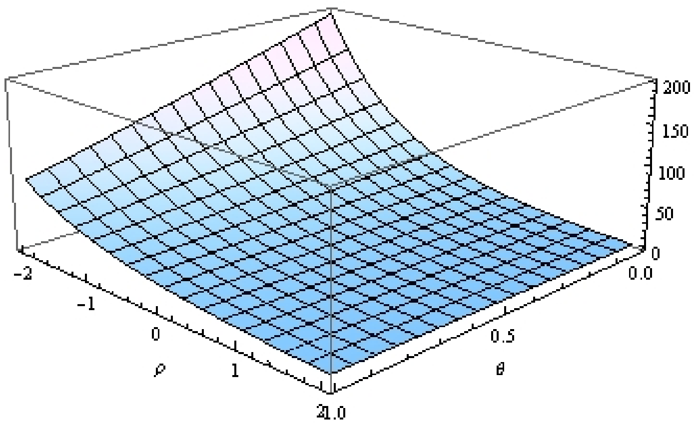
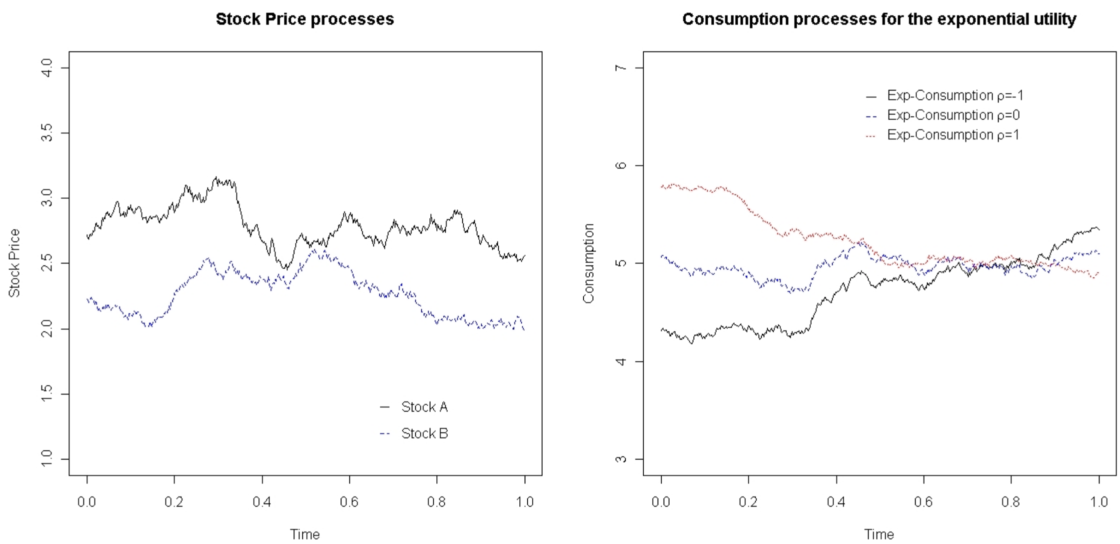


Figure 3.1: The value function for logarithmic utility for different values of ρ and θ, under the assumptions $\beta = 3.5$, $\sigma = 0.2$, $r = 0.02$ and $T = 1$. The initial capital x is 100.



(a) The Stock Price Processes

(b) The Consumption Processes

Figure 3.2: In (a) we see two examples of different stock price processes. Stock A and B are governed by $\sigma = 0.2$ and $\beta = 0.1$.
In (b) we see the corresponding consumption processes for the stock A in (a) for exponential utility under three different assumptions on ρ. We set $r = 0$ and consider the cases ρ equal to -1, 0 and 1.

consumption strategies for different values of ρ for the exponential utility function. All three processes correspond to stock A. As we would expect from the economical interpretation of ρ we see that the consumption process is increasing in average for negative ρ and decreasing for positive ρ. For ρ equal to zero we see that the consumption process is approximately

constant over time. Of course the actual behavior of the consumption processes depends crucially on the stock price itself.
This graphic is also a good example for the submartingale property of the consumption for $\rho < r$ as stated in Lemma 5.

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Part II

Pareto Optimal Risk Sharing

Chapter 4

Introduction[1]

Mainly motivated by insurance problems, optimal risk sharing between multiple agents transferring risks from one to another has been studied extensively, see [15] and references therein. Throughout this part, we consider two economic agents with initial risky endowments (or loss exposures) searching for an optimal re-allocation of their risks. The agents assess their risk using utility functions.

Optimal Risk sharing is closely linked to *Pareto optimality* (also referred to as Pareto efficiency). The term is named after Vilfredo Pareto, an Italian economist who used the concept in his studies of economic efficiency and income distribution [20]. The risk sharing problem consists in finding an *optimal allocation* namely an allocation such that it is *Pareto optimal*. Pareto optimality implies that no agent can be made strictly better off without another agent being made strictly worse off. An additional constraint on Pareto optimal allocations is that the allocation satisfies a rationality constraint, that is, all agents are at least as well off under the new allocation as under the initial exposures [18]. The latter rationality constraint is motivated by the assumption that only an irrational agent would enter into a contract that made the agent (strictly) worse off.
Note, however, that Pareto optimality does not necessarily result in a socially desirable distribution of resources. Pareto optimality makes no statement about either equality or the overall well-being of a society. It follows from the above interpretation of Pareto optimality, that if an economic allocation in any system is not Pareto optimal, there is potential for a Pareto improvement or equivalently for an increase in Pareto efficiency. In this case, by reallocating risk, goods or services, one can improve at least one participant's well-being without reducing any other participant's well-being.

Utility is a measure of the relative satisfaction from, or desirability of, consumption of various goods and services. Economists analyze increasing and decreasing utility, and explain economic behavior in terms of attempts to increase one's utility. Preference relations can often be represented by utility functions satisfying useful properties such as, e.g., monotonicity. In recent years, research focused on risk preferences given in terms of monetary utility functions and risk measures. See [9] for a brief introduction to the history and recent developments of utility functions and risk measures, respectively. Up to the sign, convex risk measures are identical to monetary utility functions. As in [9], in the present paper we concentrate on optimal risk sharing in the context of *quasiconcave* utility

[1] This and the following chapters are based on joint work with Michael Kupper and Ranja Reda.

functions. The quasiconcavity of the utility function has profound implications in decision making processes: it represents the idea that diversification should not decrease utility. Cash sub-additivity corresponds to the fact that adding a certain amount of cash to your risky position increases your utility by at most the amount of cash added in the first place. We refer to [4] and [16] for a motivation for the usage of quasiconvex risk measures and to [8] and [9] for more details on quasiconvex risk measures.

In [15] Jouini et al. consider the problem of optimal risk sharing of some given total risk between two economic agents characterized by law-invariant monetary utility functions or equivalently, law-invariant risk measures. They prove existence of an optimal risk sharing allocation which is in addition increasing in terms of the total risk. Ludkovski and Rüschendorf [18] further show that Pareto optimal allocations are comonotone if the risk measures preserve the convex order. They establish various extensions of the comonotone improvement result of Landsberger and Meilijson [17] which are of interest for the risk sharing problem.

The presented work was inspired by [15] and is a generalization of [11] to quasiconcave utility functions. The main contribution is the result on the existence of a Pareto optimal allocation of a given risk between two agents who assess their risk using quasiconcave and strictly cash sub-additive utility functions. Furthermore we show that the Pareto optimal allocation can be characterized using the sup-convolution of the corresponding utility functions.

This part is structured as follows: Section 5.1 introduces the necessary preliminaries and definitions as well as some examples. Section 5.2 presents the first main result, the characterization of the Pareto optimal allocation. In section 5.3 we prepare for the existence theorem, which is formulated and proved in section 5.4.

Chapter 5

Pareto Optimal Risk Sharing

5.1 Preliminaries

The setup throughout this part is defined as follows: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space and $\mathbb{L}^\infty := \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be the space of bounded random variables. Its topological dual $(\mathbb{L}^\infty)^* := \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})^*$ is isometrically isomorphic to the space of all bounded finitely additive set functions on $\mathcal{F}$ that are absolutely continuous with respect to $\mathbb{P}$. By $\mathcal{M}_{1,f} := \mathcal{M}_{1,f}(\Omega, \mathcal{F}, \mathbb{P})$ we denote the positive unit ball of $(\mathbb{L}^\infty)^*$, it coincides with the set of finitely additive probabilities that are absolutely continuous with respect to $\mathbb{P}$. In particular, $\mathcal{M}_1 := \mathcal{M}_1(\Omega, \mathcal{F}, \mathbb{P})$ is the subset of $\mathcal{M}_{1,f}$ consisting of all its countably additive elements.

We first give the definitions of utility functions, the corresponding acceptance sets and minimal penalty functions. Subsequently, we consider two agents and define both aggregated acceptance sets and attainable allocations.

Definition 2 (Utility Function)**.** A mapping $U : \mathbb{L}^\infty \to \mathbb{R}$ is a *utility function* if it is $\mathbb{L}^\infty$-monotone in the sense that

$$U(X) \geq U(Y) \quad \Leftrightarrow \quad \mathbb{P}(X \geq Y) = 1 \tag{5.1}$$

for any $X, Y \in \mathbb{L}^\infty$. Furthermore, a utility function is *quasiconcave* if it satisfies

$$U(\lambda X + (1-\lambda)Y) \geq \min(U(X), U(Y)) \tag{5.2}$$

for any $X, Y \in \mathbb{L}^\infty$ and $\lambda \in (0,1)$. A utility function is *cash sub-additive* if it satisfies[1]

$$U(X - m) \geq U(X) - m \tag{5.3}$$

for all $X \in \mathbb{L}^\infty$ and $m \in \mathbb{R}_+$. We call a utility function *strictly cash sub-additive*, if it satisfies

$$\limsup_{m\to\infty} \frac{U(m)}{m} \quad < \quad 1 \quad \text{and} \tag{5.4}$$

$$\liminf_{m\to\infty} \frac{U(-m)}{m} \quad = \quad -1. \tag{5.5}$$

[1]We use the notation of [4] but this definition coincides with [16], since $U(X) = U(X+m-m) \geq U(X+m) - m$. Note that for risk measures, with $\rho(X) = -U(X)$, cash sub-additivity is equivalent to $\rho(X - m) \leq \rho(X) + m$.

In other words, a function $f(x)$ is quasiconcave if it is a function whose negative is quasiconvex. A function $f(x)$ is quasiconvex if its lower contour sets are convex sets. That is, if the set $\{x : f(x) \leq K\}$ is a convex set for any constant K.
The definition of *strict cash sub-additivity* represents the fact that the agent values the asymptotic decrease in his asset more drastic than the asymptotic increase. This definition is of particular interest for the existence of a Pareto optimal allocation. Indeed it could be relaxed in such a way, that no agent is valuing an asymptotic increase with a rate higher than the rate of an other agent for asymptotic decrease, since such a situation would lead to the outcome that one agent would consume an infinite amount leaving the other with infinite liabilities.

Example 1 (A quasiconcave, cash sub-additive utility function, which is neither concave nor cash additive)**.** We use example 2 in [4] to construct a utility function which is neither concave nor cash additive. As a basis we use the entropic risk measure with some positive parameter η and apply the arctan to transform it into a quasiconvex risk measure. Taking the negative we have constructed a utility function. We define

$$U(X) = -\arctan\Big(\frac{1}{\eta}\ln \mathbb{E}[e^{-\eta X}]\Big).$$

This utility function is quasiconcave but not concave. An immediate result of this definition is that the utility of any position is now assessed by a number in the interval $[-\pi, \pi]$. In terms of risk measures, an economical interpretation of this risk assessment is that once a risk is higher than a certain threshold, any risk even higher has virtually the same impact on the agent. The same holds for very high returns.

The following definitions 3-6 hold for general utility functions and can be found for example in [13] and in [15], respectively.

Definition 3. A utility function U induces the class

$$A := \{X \in \mathbb{L}^\infty : U(X) \geq 0\}$$

of positions which are acceptable. The set $A_m := \{X \in \mathbb{L}^\infty : U(X) \geq m\}$, $m \in \mathbb{R}$, denotes the *acceptance set* for the level m. Consequently, for an agent i with utility U_i, the acceptance set A^i_m of level m is defined by

$$A^i_m \quad = \quad \{X \in \mathbb{L}^\infty : U_i(X) \geq m\}.$$

Definition 4. We consider the *minimal penalty function*

$$\alpha_{min}(Q, m) = \inf_{X \in A_m} \mathbb{E}_Q[X], \tag{5.6}$$

where A_m is the acceptance set of level m and $Q \in \mathcal{M}_1$. The right continuous version of α_{min} is denoted by α^+_{min} and defined by

$$\begin{aligned}\alpha^+_{min}(Q, m) \quad &= \quad \inf_{X \in A^-_m} \mathbb{E}_Q[X] \quad \text{ where } A^-_m = \bigcup_{m' > m} A_{m'} \\ &= \quad \inf_{m' > m} \alpha_{min}(Q, m').\end{aligned}$$

For U continuous from above, we can rewrite the utility function U, using this penalty function, analogous to the robust representation of risk measures in [8]

$$\begin{aligned} U(X) &= \inf_{Q \in \mathcal{M}_{1,f}} \sup_{m} \{m \,|\, \mathbb{E}_Q[X] \geq \alpha_{min}(Q,m)\} \\ &= \sup_{m} \{m \,|\, \mathbb{E}_Q[X] \geq \alpha_{min}(Q,m) \,\forall\, Q \in \mathcal{M}_{1,f}\}\,. \end{aligned}$$

For this representation we can use any $\alpha_{min}(Q,m)$ in the interval $[\alpha^-_{min}(Q,m), \alpha^+_{min}(Q,m)]$, where $\alpha^-_{min}(Q,m)$ is the left continuous version defined analogously to the right continuous version. All of these penalty functions will give the same utility to any position X.

On one hand, this representation is motivated by the robust representation of risk measures (see for example [8]). On the other hand, one can find a similar representation in [21], which links this representation to the biconjugate. This gives a different motivation, outside the context of risk measures.

Example 2 (Certainty Equivalent)**.** In general, the amount of payoff (e.g. money or utility) that an agent would have to receive to be indifferent between that payoff and a given gamble is called that gamble's *certainty equivalent*. We refer to [23] for a detailed discussion of the certainty equivalent in the framework of macroeconomic theory. For a risk averse agent the certainty equivalent is less than the expected value of the gamble because the agent prefers to reduce uncertainty. We have the following definition for U and the minimal penalty function α_{min}

$$\begin{aligned} U(X) &= f^{-1}\mathbb{E}[f(X)] \quad \text{and} \\ \alpha_{min}(Q,m) &= \inf_{X \in A_m} \mathbb{E}_Q[X]. \end{aligned}$$

In [2] (Theorem 5.1) one can find that the certainty equivalent defines a concave utility function if and only if $1/r(s)$ is a concave function, where $r(s)$ is the Arrow-Pratt index of risk aversion given by

$$r(s) = -\frac{f''(s)}{f'(s)}.$$

The cash additivity and cash super-additivity, respectively, for the discrete case is discussed in [14] (Proposition B) based on a result by [19]. The certainty equivalent is cash additive for $f(s) = -e^{-s}$. Choosing $f(s) = -e^{-s}$ this utility would introduce the entropic risk measure by $\rho = -U$, and as utility we get

$$U(X) \quad = \quad -\ln \mathbb{E}[e^{-X}].$$

Also for other functions, e.g. $f(s) = \ln(s)$ or $f(s) = s^\alpha/\alpha$, for $\alpha \in (-\infty, 1)\backslash\{0\}$, the Certainty Equivalent is a concave utility function.

Definition 5. We consider two agents $i = 1, 2$. Agent 1 has the utility U_1 and thus the acceptance set A^1_m of level $m \in \mathbb{R}$ and agent 2 has A^2_m respectively. Their *aggregated acceptance set* is given by

$$A^{1,2}_m \quad = \quad \{X \in \mathbb{L}^\infty : \exists X_1 \in A^1_{m_1}, \exists X_2 \in A^2_{m_2}, X_1 + X_2 = X, m_1 + m_2 \geq m\},$$

and the set of *attainable allocations* is given by

$$\mathcal{A}(X) \quad = \quad \{(X_1, X_2) \in \mathbb{L}^\infty \times \mathbb{L}^\infty : X_1 + X_2 = X\}.$$

In the following paragraph we discuss properties of the aggregate acceptance set $A_m^{1,2}$ defined by two quasiconcave utility functions U_1 and U_2.

Remark 7. The aggregated acceptance set $A_m^{1,2}$ is defined by two quasiconcave utility functions U_1 and U_2 has the following properties:

1. **Monotone:** We have that $A_m^{1,2} \subseteq A_n^{1,2}$ for $m \leq n$. Also if $X \in A_m^{1,2}$ and $\mathbb{P}(Y \geq X) = 1$ then $Y \in A_m^{1,2}$.

2. **Right Continuous:** For any $m \in \mathbb{R}$ we have that $A_m^{1,2} = \bigcup_{n>m} A_n^{1,2}$.

All of these properties follow directly from the definition of $A_m^{1,2}$.
Since every risk measure introduces a utility function by $\rho(X) = -U(X)$ and vice versa we now present here an example for a risk measure without turning it artificially into a utility function.

Example 3 (Economic Index of Riskiness)**.** The economic index of riskiness is defined by[2]

$$\rho(X) = \begin{cases} 1/\lambda(X) & \text{if } \mathbb{E}[X] \geq_{ssd} 0, \\ +\infty & \text{else,} \end{cases} \qquad X \in \mathbb{L}^\infty.$$

where for fixed $c_0 \in \mathbb{R}$ and l a loss function

$$\lambda(X) = \sup\{\lambda > 0 | \mathbb{E}[l(-\lambda X)] \leq c_0\} \quad \text{for } \mathbb{E}[X] \geq 0.$$

As stated in [8] the economic index of riskiness measures whether gambles are rejected depending on the level of wealth. The acceptance set[3] for one and two agents and $m \in \mathbb{R}_-$ is given by

$$\begin{aligned} A_m^i &= \{X \in \mathbb{L}^\infty | \lambda(X) \geq -1/m\} \\ A_m^{1,2} &= \{X \in \mathbb{L}^\infty : \exists (X_1, X_1) \in \mathbb{L}^\infty \times \mathbb{L}^\infty, X_1 + X_2 = X, \\ &\qquad \lambda_i(X_i) \geq -1/m_i, m_1 + m_2 \geq m\}. \end{aligned}$$

For more details we refer to [8] (Example 1.11).

5.2 Optimal Risk Sharing

This section discusses Pareto optimal risk sharing between two agents. As discussed in the introduction, informally speaking, Pareto optimal situations are those in which any change that makes one person better off must necessarily make someone else worse off. Consider a set of alternative allocations, e.g. payoff functions for a set of individuals. A change from one allocation to another that can make at least one individual better off without making any other individual worse off is referred to as a *Pareto improvement.* Consequently, an allocation is defined as Pareto optimal when no further Pareto improvements can be made.

We start with the definition of the sup-convolution and the set of extrema.

[2]For the definition of $\geq_{ssd}$ see definition 12 in section 5.3.
[3]In our understanding, which is up to the sign the same as in [8].

Definition 6 (Sup-Convolution)**.** We define the *sup-convolution* of utility functions

$$\square_{i=1}^{n} U_i(X) = \sup_{\substack{(X_i)_{i=1}^n \in \mathbb{L}^\infty \\ \sum_{i=1}^n X_i = X}} \sum_{i=1}^{n} U_i(X_i), \quad X \in \mathbb{L}^\infty \tag{5.7}$$

and $\alpha_{min}^{\square}(Q,m) = \inf_{m_1+m_2 \geq m}(\alpha_{min}^1(Q,m_1) + \alpha_{min}^2(Q,m_2))$.

Remark 8. The definition of $\alpha_{min}^{\square}(Q,m)$ follows from

$$\begin{aligned} \alpha_{min}^{\square}(Q,m) &= \inf_{X \in A_m^{1,2}} \mathbb{E}_Q[X] \\ &= \inf_{m_1+m_2 \geq m} \inf_{X_1 \in A_{m_1}^1} \mathbb{E}_Q[X_1] + \inf_{X_2 \in A_{m_2}^2} \mathbb{E}_Q[X_2] \\ &= \inf_{m_1+m_2 \geq m} \left(\alpha_{min}^1(Q,m_1) + \alpha_{min}^2(Q,m_2)\right). \end{aligned}$$

For the characterization of the Pareto optimal allocation we will make use of the set of extrema.

Definition 7 (Set of Extrema)**.** We say that Q lies in the set of extrema $\partial U(X)$, and, X lies in the set of extrema $\partial \alpha_{min}^{+}(Q,.)$, respectively, if

$$\mathbb{E}_Q[X] = \alpha_{min}^{+}\left(Q, U(X)\right).$$

Definition 8. Let (Z_1, Z_2) be an attainable allocation, then it is *Pareto optimal* if for any other attainable allocation (Y_1, Y_2) it holds that $U_1(Y_1) \geq U_1(Z_1)$ and $U_2(Y_2) \geq U_2(Z_2)$ implies $U_1(Y_1) = U_1(Z_1)$ and $U_2(Y_2) = U_2(Z_2)$.

Theorem 1. Assume two agents 1 and 2 with quasiconcave utility functions U_1 and U_2 which are normalized[4] by $U(m) = m$ for all m in $\mathbb{R}$. For an aggregate risk $X \in \mathbb{L}^\infty$ and risk sharing given by the attainable position (X_1, X_2), the following statements are equivalent:

(i) (X_1, X_2) is a Pareto optimal allocation.

(ii) $U_1 \square U_2(X) = U_1(X_1) + U_2(X_2)$.

If the convolution is again quasiconcave, then the above are equivalent to

iii) There exists a $Q \in \mathcal{M}_{1,f}$ such that $X_i \in \partial \alpha_{min}^{+}(Q,.)$ for i=1,2, and $Q \in \partial U(X)$.

Proof. *ii)* $\Longrightarrow$ *i)* follows directly from the definition of Pareto optimality.
i) $\Longrightarrow$ *ii)*: The first part of this proof is completely in line with [15], we repeat it here for completeness. Let (X_1, X_2) be the Pareto optimal allocation. Consider the sets $\widetilde{B} = \{(U_1(Y_1), U_2(Y_2)) : Y_1 + Y_2 = X\}$, $B = \widetilde{B} - \mathbb{R}^2$ and $C = \{(U_1(X_1), U_2(X_2))\} + \mathbb{R}_+^2 \setminus \{0\}$. The sets B and C are convex subsets of $\mathbb{R}^2$ which are non-empty with non-empty interior and $B \cap C = \emptyset$ by the Pareto optimality. By Hahn-Banach separation Theorem, there exists a $\lambda \in \mathbb{R}^2$ such that $\lambda y \leq \lambda z$ for all $(y,z) \in B \times C$. Since $(U_1(X_1), U_2(X_2)) \in B$ and $(U_1(X_1)+1, U_2(X_2)) \in C$ and $(U_1(X_1), U_2(X_2)+1) \in C$ we see that $\lambda_1, \lambda_2 \geq 0$. It follows

[4] Instead of this normalization we can also use that $U(m)$ is linear for $m \in \mathbb{R}$, but not constant.

from the separation inequality, that $(U_1(X_1), U_2(X_2))$ is a maximizer of $\lambda_1 U_1(Y_1) + \lambda_2 U_2(Y_2)$. Since $(U_1(X_1), U_2(X_2))$ is the maximizer, we can write the following inequality

$$\begin{aligned}
\sup_{\substack{X_1, X_2 \in \mathbb{L}^\infty \\ X_1+X_2=X}} \lambda_1 U_1(X_1) + \lambda_2 U_2(X_2) &\geq \lambda_1 U_1(X+c) + \lambda_2 U_2(-c) \\
&\geq \lambda_1 U_1(-\|X\|_\infty + c) + \lambda_2 U_2(-c) \quad \forall c \in \mathbb{R} \\
&= \lambda_1(-\|X\|_\infty + c) - \lambda_2 c \\
&= -\lambda_1 \|X\|_\infty + (\lambda_1 - \lambda_2)c.
\end{aligned}$$

For $c \to \infty$ the RHS grows to (plus or minus) infinity, and thus $\lambda_2 = \lambda_1$ is the only way to ensure that the inequality holds for all $c \in \mathbb{R}$. Thus, it follows that (X_1, X_2) is the optimizer of the sup-convolution.
i) $\Longrightarrow$ *iii)* First we show that a $Q \in \mathcal{M}_{1,f}$ exists which is in the set of extrema $\partial U(X)$. We use the fact that $\mathcal{M}_{1,f}$ is compact in the $\sigma(ba(\mathbb{P}), \mathbb{L}^\infty)$-sense. Then for all $m \in \mathbb{R}$, the mapping $Q \to \alpha^+_{min}(Q, m)$ is $\sigma(ba(\mathbb{P}), \mathbb{L}^\infty)$ upper semi-continuous. Here $ba(P)$ denotes the bounded finitely additive signed measures on $\mathcal{F}$ absolutely continuous with respect to $\mathbb{P}$. We use the representation

$$U(X) = \sup_m \{m \,|\, \mathbb{E}_Q[X] \geq \alpha^+_{min}(Q, m) \forall Q \in \mathcal{M}_{1,f}\}.$$

From this representation we get the following two inequalities:

$$\begin{aligned}
&\forall m' > U(X): \quad \exists Q \in \mathcal{M}_{1,f} \quad \mathbb{E}_Q[X] < \alpha^+_{min}(Q, m'), \\
&\forall m' < U(X): \quad \forall Q \in \mathcal{M}_{1,f} \quad \mathbb{E}_Q[X] \geq \alpha^+_{min}(Q, m').
\end{aligned}$$

We consider a sequence $(m_k)_{k\geq 0} \downarrow m = U(X)$ and a corresponding sequence of measures $(Q_k)_{k\geq 0} \to Q^*$. By right continuity and upper semi-continuity of α^+_{min} it follows that

$$\begin{aligned}
\mathbb{E}^*_Q[X] &\leq \alpha^+_{min}(Q^*, m') \\
\mathbb{E}^*_Q[X] &\geq \alpha^+_{min}(Q^*, m'),
\end{aligned}$$

which implies equality.
From the existence of such a Q it follows that $\mathbb{E}_Q[X] = \alpha^+_{min}(Q, U(X))$ and by the Pareto optimality the RHS is smaller than $\alpha^{1,+}_{min}(Q, U_1(X_1)) + \alpha^{2,+}_{min}(Q, U_2(X_2))$. By the definition of U_i we have that $\mathbb{E}_Q[X_i] \geq \alpha^{i,+}_{min}(Q, U_i(X_i))$ for $i = 1, 2$ and for all $Q \in \mathcal{M}_{1,f}$. Thus it follows that $X_i \in \partial\alpha^+_{min}(Q, .)$ for i=1,2.
iii) $\Longrightarrow$ *i)*: Statement *iii)* translates into
$\mathbb{E}_Q[X] = \alpha^+_{min}(Q, U(X))$ and $\mathbb{E}_Q[X_i] = \alpha^+_{min}(Q, U(X_i))$ for $i = 1, 2$. This translates into $\alpha^{\square,+}_{min}(Q, U(X)) = \alpha^{1,+}_{min}(Q, U(X_1)) + \alpha^{2,+}_{min}(Q, U(X_2))$ and therefore $U(X) = U_1(X_1) + U_2(X_2)$. $\square$

Remark 9. Any monetary utility function introduces a risk measure by $\rho = -U$.
The statements above are equivalent to $\rho_1 \square \rho_2(X) = \rho_1(X_1) + \rho_2(X_2)$, where the inf-convolution of the risk is defined analogously to the sup-convolution of utility.

5.3 On Comonotonicity, Concave Order and Second Order Stochastic Dominance

The study of Pareto optimal risk allocations is closely linked to the comonotonicity property, which is defined as follows (see [17, 18] for definition and discussion).

Definition 9 (Comonotonicity)**.** Two random variables Y and $Z \in \mathbb{L}^0(\mathbb{P})$ are said to be comonotone if

$$(Y(\omega_1) - Y(\omega_2))\,(Z(\omega_1) - Z(\omega_2)) \geq 0, \tag{5.8}$$

$\mathbb{P}(d\omega_1) \times \mathbb{P}(d\omega_2)$-almost surely. In other words, Y and Z move together.

Intuitively, comonotonicity is a very strong positive dependency structure. For diffuse random variables Y and Z this is equivalent to the following: Y increases with Z, and we denote $Y \nearrow Z$, if $Y = f(Z)$ for some non-decreasing map f.

Definition 10 (Comonotone Allocation)**.** An allocation $(Y_1, Y_2) \in \mathcal{A}(X)$ is called comonotone if Y_1, Y_2 are comonotone.

If (Y_1, Y_2) is a comonotone allocation then Y_i and X are comonotone for $i = 1, 2$ [15]. Equivalently, an allocation (Y_1, Y_2) of $X \in \mathbb{L}^\infty$ is called comonotone if there exist increasing functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ such that $f_1 + f_2 = id_{\mathbb{R}}$ and $Y_i = f_i(X)$ for $i = 1, 2$.

Lemma 7. Let f_1 and f_2 be nondecreasing with $f_1 + f_2 = id_{\mathbb{R}}$, then f_i is Lipschitz continuous with Lipschitz constant 1.

Proof. We assume $m \geq 0$, then we get

$$\begin{aligned} f_1(x+m) - f_1(x) &= x + m - f_2(x+m) - f_1(x) \\ &\leq x + m - f_2(x) - f_1(x) = m, \end{aligned}$$

and the same holds for f_2. $\square$

Especially in actuarial science and finance, comonotonicity plays an important role. Dhaene et al. [7] discuss the following example. Consider an insurance contract, i.e. an agreement between a person that faces a certain risk (the insured), and an insurer that promises to cover part of the claim amount. Let X be a non-negative random variable denoting the risk the insured faces during the insurance period, and denote by $\Psi(X)$ the amount the insurer promises to pay in case the claim amount equals X. The amount $X - \Psi(X)$ is then retained by the insured. As discussed in [7], it is reasonable to require that both $\Psi(x)$ and $x - \Psi(x)$ are non-decreasing functions on the set of all possible outcomes of X. Equivalently, one can require that both risk sharing partners have to bear more (or at least as much) if the actual claim x increases. One finds that the risk sharing scheme $(\Psi(X), X - \Psi(X))$ is indeed comonotone.

We next recall the concepts of concave order and second-order stochastic dominance.

Definition 11 (Concave Order)**.** A random variable $Y \in \mathbb{L}^\infty$ is said to precede (or be preferred to) $Z \in \mathbb{L}^\infty$ in *concave order* if $\mathbb{E}[f(Y)] \geq \mathbb{E}[f(Z)]$ for all concave functions f for which the expectation exists. We write $Y \geq_c Z$. A function U is *concave monotone* increasing if $Y \geq_c Z$ implies $U(Y) \geq U(Z)$.

The concept of second-order stochastic dominance is widely used in economics.

Definition 12 (Second-Order Stochastic Dominance)**.** Given two real-valued bounded random variables Y and Z, i.e. $Y, Z \in \mathbb{L}^\infty$, Y is said to dominate Z for second-order stochastic dominance (notation $Y \geq_{ssd} Z$) whenever $\mathbb{E}[g(Y)] \geq \mathbb{E}[g(Z)]$ for every concave and *nondecreasing* function $g : \mathbb{R} \to \mathbb{R}$. A function U is *ssd monotone* increasing if $Y \geq_{ssd} Z$ implies $U(Y) \geq U(Z)$.

Note that concave order is equivalent to ordering with respect to second stochastic dominance with equal means, see [22] and [24]. Since $id_{\mathbb{R}}$ and $id_{-\mathbb{R}}$ are concave functions, $Y \geq_c Z$ implies $\mathbb{E}[Y] = \mathbb{E}[Z]$. Therefore, $Y \geq_c Z$ if and only if $Y \geq_{ssd} Z$ and $\mathbb{E}[Y] = \mathbb{E}[Z]$ [3]. Consequently, concave order $\geq_c$ implies ordering with respect to second order stochastic dominance $\geq_{ssd}$. On the other hand, if a function is ssd monotone, then it is also concave monotone.

Moreover, one can find that

$$X \geq_c Y \Leftrightarrow \mathbb{E}[X] = \mathbb{E}[Y] \quad \text{and} \quad \mathbb{E}\left[(X-c)^+\right] \leq \mathbb{E}\left[(Y-c)^+\right] \forall c \in \mathbb{R}. \tag{5.9}$$

For proof and discussion of (5.9) we refer to Corollary 2.62 in [13] and [11], respectively.

It is well known that if an allocation $(Y_1, Y_2, ..., Y_n)$ of a random endowment X among n agents with prespecified nondecreasing and concave utility functions is Pareto optimal, then this allocation is comonotone. Indeed, each Y_i is a nondecreasing function of $X = \sum_{i=1}^n Y_i$ [1, 25]. The link between Pareto optimal allocations and the comonotonicity property was originally obtained in [17], who developed an algorithm to construct a concave order improvement of any non-comonotone allocation. However, [17] only proved this result for random variables X supported by a finite set. For sake of completeness, [11] give the full proof and state that every allocation is dominated in concave order by a comonotone allocation. Furthermore, [11] establish that any lower semi-continuous law-invariant concave function is concave monotone. It follows that a Pareto optimal risk allocation is necessarily comonotone.

Remark 10. If $X \geq_{ssd} Y$ then $X + m \geq_{ssd} Y + m$ for all $m \in \mathbb{R}$.

We proceed with a proof for the existence of an optimal risk sharing allocation.

5.4 Existence of Pareto Optimal Allocations

In [15], Jouini et al. discuss the problem of optimal risk sharing of some given total risk between two economic agents characterized by law-invariant monetary (concave, monotone, cash additive and normalized) utility functions. There, a proof for the existence of an optimal risk sharing allocation, which is in addition increasing in terms of the total risk, is given. Furthermore, [11] provide the complete solution to the existence and characterization problem of optimal capital and risk allocations for *not necessarily monotone*, law-invariant concave utility functions on L^p, $p \in [1, \infty]$. Discussing agents, or business units, who redistribute the aggregate risk among themselves in order to maximize total and individual utility, they state the following: As often the case in practice, this redistribution procedure may be subject to frictions (e.g. limited fungibility of capital) in the sense that not every allocation of X is admissible. This can be formalized by restricting the utility functions U_i accordingly, see, e.g. [10]. As the restricted U_i are typically not monotone, contrary to [15], monotonicity is not required in [11].
We adapt the work of [15] and [11] and develop a proof for the existence of Pareto optimal risk allocations for $\mathbb{L}^\infty$*-monotone,* $\mathbb{L}^\infty$*-continuous, quasiconcave, law-invariant, strictly cash sub-additive* utility functions.

Let $\mathcal{A}\nearrow(X) := \{(X_1, X_2) \in \mathcal{A}(X) : X_1 \nearrow X \text{ and } X_2 \nearrow X\}$ be the subset of admissible allocations which increase with the corresponding aggregate risk. Here, $X_i \nearrow X$ denotes that X_i increases with the aggregate risk X, where $i = 1, 2$.

By Denneberg's lemma [6], we observe that $\mathcal{A}\nearrow(X)$ is the subset of $\mathcal{A}(X)$ consisting of all comonotone allocations. Then, $(X_1, X_2) \in \mathcal{A}\nearrow(X)$ if and only if there are nondecreasing functions $f_i : [a, b] \to \mathbb{R}$, with $a = \operatorname{ess\,inf} X$ and $b = \operatorname{ess\,sup} X$, with $f_1 + f_2 = id_{\mathbb{R}}$ such that $X_i = f_i(X)$ for $i = 1, 2$. Then, the functions f_i are all 1-Lipschitz, and the allocations in $\mathcal{A}\nearrow(X)$ are 1-Lipschitz functions of X [3, 15].

Theorem 2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space and let U_1 and U_2 be two $\mathbb{L}^\infty$-monotone, $\mathbb{L}^\infty$-continuous, quasiconcave, law-invariant, strictly cash sub-additive utility functions. Then for every bounded $X \in \mathbb{L}^\infty$,

$$U_1 \Box U_2(X) = \sup_{(X_1, X_2) \in \mathcal{A}\nearrow(X)} U_1(X_1) + U_2(X_2), \tag{5.10}$$

and the set of Pareto optimal allocations in $\mathcal{A}\nearrow(X)$ is non empty.

Example 4. Before we start with the proof we present an example which shows that without the strict cash sub-additivity the supremum can be infinity. Consider two agents one with utility $\mathbb{E}_P[X]$ and the other with $\frac{1}{2}\mathbb{E}_P[X]$. For simplicity we set $X = 0$ and consider only allocations of the form $X_1 = m_1$ and $X_2 = m_2$ with $m_1 + m_2 = 0$. Then

$$U_1 \Box U_2(0) \geq \sup_{m_1 \in \mathbb{R}} \Big(m_1 + \frac{1}{2}(-m_1) \Big) = \infty.$$

First, we prove that the maximization problem in the definition of the sup-convolution $U_1 \Box U_2$ can be restricted to pairs $(X_1, X_2) \in \mathcal{A}\nearrow(X)$. Then, we prove that this set is non empty.

Lemma 8. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space and let U_1 and U_2 be two law-invariant quasiconcave utility functions[5]. Then

$$U_1 \Box U_2(X) = \sup_{(X_1, X_2) \in \mathcal{A}\nearrow(X)} U_1(X_1) + U_2(X_2). \tag{5.11}$$

Proof. For the first step of the proof, we recall that in the concave case, any attainable allocation $(X_1, X_2) \in \mathcal{A}(X)$ is dominated by some comonotone attainable allocation $(\hat{X}_1, \hat{X}_2) \in \mathbb{L}^\infty$ in the sense of second order stochastic dominance. This result was proved by [17] in the context of a finite probability space, and further extended to $\mathbb{L}^\infty$ allocations in a general probability space by [5]. As discussed in the previous section, according to [11], every allocation is dominated in concave order by a comonotone allocation.
It was shown in [4] that for any quasiconcave continuous from above utility function U, law invariance of U is equivalent to preserving second-order stochastic dominance. The corresponding proof is based on the theory of rearrangement invariant Banach spaces. Any quasiconcave continuous from above law-invariant utility function U is second-order stochastic dominance monotone. This provides

$$U_i(\hat{X}_i) \geq U_i(X_i). \tag{5.12}$$

[5] As given in definition 1.

Hence, the maximization problem in the definition of $U_1 \Box U_2$ can be restricted to a subset of $\mathcal{A}(X)$ consisting of comonotone pairs (X_1, X_2). We conclude that this subset is precisely $\mathcal{A}\nearrow(X)$. $\square$

The following Lemma is essential to show that the supremum is actually attained without letting either X_i tend to infinity.

Lemma 9. For every $c \in \mathbb{R}_+$ and $X \in \mathbb{L}^\infty$ for a $\mathbb{L}^\infty$-monotone, strictly cash sub-additive utility function U the following holds

$$\limsup_{m\to\infty} \sup_{X:\|X\|_\infty\le c} \frac{U(X+m)}{m} < 1 \quad \text{and} \tag{5.13}$$

$$\liminf_{m\to\infty} \sup_{X:\|X\|_\infty\le c} \frac{U(X-m)}{m} = -1. \tag{5.14}$$

Proof. Due to the $\mathbb{L}^\infty$-monotonicity of U we have that

$$\sup_{X:\|X\|_\infty\le c} \frac{U(X+m)}{m} = \frac{U(c+m)}{m} \tag{5.15}$$

and therefore

$$\limsup_{m\to\infty} \sup_{X:\|X\|_\infty\le c} \frac{U(X+m)}{m} = \limsup_{m\to\infty} \frac{U(c+m)}{m} = \limsup_{\widetilde{m}\to\infty} \frac{U(\widetilde{m})}{\widetilde{m}} \frac{\widetilde{m}}{\widetilde{m}-c} < 1$$

for $\widetilde{m} = m + c$, which proves the inequality in Eq.(5.13). Again we can use the $\mathbb{L}^\infty$-monotonicity, see Eq.(5.15). Let $\widetilde{m} = m - c$,

$$\liminf_{m\to\infty} \sup_{X:\|X\|_\infty\le c} \frac{U(X-m)}{m} = \liminf_{m\to\infty} \frac{U(c-m)}{m} = \liminf_{\widetilde{m}\to\infty} \frac{U(-\widetilde{m})}{\widetilde{m}} \frac{\widetilde{m}}{\widetilde{m}+c} = -1.$$

which proves Eq.(5.14). $\square$

We are now ready for the

Proof of Theorem 2. By Lemma 8 we know that the supremum is attained in the set of comonotone allocations, therefore we can write

$$U_1 \Box U_2(X) = \sup_{m\in\mathbb{R}} \sup_{\substack{f_1,f_2\nearrow \\ f_1+f_2=id_\mathbb{R} \\ f_i(0)=0}} U_1(f_1(X)+m) + U_2(f_2(X)-m).$$

We know that for $m \to \infty$ the term $U_1(f_1(X) + m)$ increases with rate smaller than one to infinity and at the same time $U_2(f_2(X) - m)$ decreases with rate one to minus infinity, therefore there exists an $m_2 \in \mathbb{R}$ such that $U_1(f_1(X) + m) + U_2(f_2(X) - m)$ is decreasing in m for all m bigger than m_2 for every pair (f_1, f_2).
On the other hand $m \to -\infty$ implies that $U_1(f_1(X)+m)$ decreases with rate one to minus infinity and $U_2(f_2(X)-m)$ increases with a smaller rate. Therefore we also have an $m_1 \in \mathbb{R}$ such that $U_1(f_1(X) + m) + U_2(f_2(X) - m)$ is decreasing in $-m$ for all m smaller than m_1 for every pair (f_1, f_2). Thus we can restrict the first supremum to

$$U_1 \Box U_2(X) = \sup_{m\in[m_1,m_2]} \sup_{\substack{f_1,f_2\nearrow \\ f_1+f_2=id_\mathbb{R} \\ f_i(0)=0}} U_1(f_1(X)+m) + U_2(f_2(X)-m).$$

We may now apply the Ascoli Theorem, which states the following: Consider a sequence of real-valued continuous functions (f_n), $n \in \mathbb{N}$, defined on a closed and bounded interval $[a, b]$ of the real line. If this sequence is uniformly bounded and equicontinuous, then there exists a subsequence (f_{n_k}) that converges uniformly.
Let (X_1^i, X_2^i) be a sequence maximizing Eq.(5.10), then for every i there exist functions $\widetilde{f_1^i}$ and $\widetilde{f_2^i}$ such that (X_1^i, X_2^i) is dominated by $(\widetilde{f_1^i}(X), \widetilde{f_2^i}(X))$. Since X is bounded these functions are defined on the closed and bounded interval given by $[-\|X\|_\infty, \|X\|_\infty]$. By shifting these functions to zero by m we get the functions f_1^i and f_2^i and since $f_1^i(0) = f_2^i(0) = 0$ they are bounded due to the Lipschitz continuity.
The Lipschitz continuity of f_j^i for $j = 1, 2$ and $i \in \mathbb{N}$ with common Lipschitz constant one is sufficient for the equicontinuity of the functions f_j. Therefore, there exist subsequences $f_1^{i_k}$ and $f_2^{i_k}$ such that these subsequences converge uniformly to f_1^∞ and f_2^∞. By the $\mathbb{L}^\infty$-continuity of U_1 and U_2 we have that

$$U_1 \square U_2(X) = \sup_{m \in [m_1, m_2]} U_1\left(f_1^\infty(X) + m\right) + U_2\left(f_2^\infty(X) - m\right)$$

and have thus found the required maximizer, which is not necessarily unique. This maximizer represents a Pareto optimal allocation due to the definition of Pareto optimality. $\square$

Appendix A

Appendix

In [12], Filipovic et al. state the following: Let $f : \mathbb{L}^p \to [-\infty, \infty]$ be a closed convex function. Then, law-invariance of f and concave monotonicity of f are equivalent. Motivated by this, we prove the following result.
We will make use of the definition

Definition 13. We say that U satisfies the Fatou property, if

$$U(X) \geq \limsup_{n\to\infty} U(X_n)$$

whenever $\sup_n \|X_n\|_\infty < \infty$ and $X_n \stackrel{\mathbb{P}}{\to} X$ where $X_n \stackrel{\mathbb{P}}{\to} X$ denotes convergence in probability.

Lemma 10. Let $U : \mathbb{L}^\infty \to \mathbb{R}$ be a quasiconcave utility function, which is continuous from above and has the Fatou property. If U is law-invariant, then U is also concave monotone increasing.

Proof. Let $X \leq_c Y$. We can write

$$U(X) \quad = \quad \sup_m \{m \,|\, \mathbb{E}_Q[X] \geq \alpha_{min}(Q, m) \,\forall\, Q \in \mathcal{M}_{1,f}\}.$$

Denote $U(X) = m^*$, then we know that for all Q

$$\mathbb{E}_Q[X] \quad \leq \quad \alpha_{min}(Q, m^*) \tag{A.1}$$

holds. The above inequality holds for any $\widetilde{X}$ which has the same law as X due to the law-invariance, and thus also for the supremum over all these $\widetilde{X}$. The left hand side of Eq.(A.1) therefore can be rewritten using Lem 4.55 in [13] and fact 3, p.14, of [12].

$$\begin{aligned}
\sup_{\widetilde{X}\sim X} \mathbb{E}[\widetilde{X}\varphi_Q] &= \int_0^1 q_X(t) q_{\varphi_Q}(t) dt \\
&\geq \int_0^1 q_Y(t) q_{\varphi_Q}(t) dt \quad \text{since } X \leq_c Y. \\
&= \sup_{\widetilde{Y}\sim Y} \mathbb{E}[\widetilde{Y}\varphi_Q],
\end{aligned}$$

where φ_Q denotes the Radon-Nikodym derivative. Since the inequality holds for m^* and the supremum over all m is at least m^*, we have that $U(Y) \geq U(X)$. $\square$

Having proved that any attainable allocation $(X_1, X_2) \in \mathcal{A}(X)$ is dominated by some comonotone attainable allocation $(\hat{X}_1, \hat{X}_2) \in \mathbb{L}^\infty$ in the sense of second order stochastic dominance we can give an upper bound to $\mathbb{E}[|\hat{X}_i - X_i|]$.
The following alternative representation from [13] will be useful. We find that if $X \geq_{ssd} Y$ is equivalent to

$$\mathbb{E}[(c - X)_+] \leq \mathbb{E}[(c - Y)_+] \quad \forall c \in \mathbb{R}.$$

Lemma 11. Let (X_1, X_2) and $(\hat{X}_1, \hat{X}_2)$ be different allocations of X with $\hat{X}_i \geq_{ssd} X_i$ for $i = 1, 2$, then $\mathbb{E}[|\hat{X}_i - X_i|] \leq \mathbb{E}[|X_1|] + \mathbb{E}[|X_2|]$.

Proof. We prove the result for $i = 1$, the same holds for $i = 2$ and start by rewriting the absolute value.

$$\mathbb{E}[|X_1 - \hat{X}_1|] = \underbrace{\mathbb{E}[(X_1 - \hat{X}_1)_+]}_{A} \underbrace{-\mathbb{E}[(X_1 - \hat{X}_1)_-]}_{B}$$

We consider the two terms A and B in detail, starting with A

$$\begin{aligned} A = \mathbb{E}[(X_1 - \hat{X}_1)_+] &\leq \mathbb{E}[(X_1)_+ + (-\hat{X}_1)_+] \\ &\leq \mathbb{E}[(X_1)_+ + (-X_1)_+] = \mathbb{E}[|X_1|] \end{aligned}$$

and also for B

$$\begin{aligned} B = \mathbb{E}[-(X_1 - \hat{X}_1)_-] &= \mathbb{E}[(\hat{X}_1 - X_1)_+] = \mathbb{E}[(X_2 - \hat{X}_2)_+] \\ &\leq \mathbb{E}[|X_2|] \end{aligned}$$

which concludes the proof. $\square$

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