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DISSERTATION

Investigations on the Renormalizability of a Non-Commutative U(1) Gauge Theory

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To Edith and Mirjam

Kurzfassung

Bei Betrachtung sehr kleiner Skalen in der Nähe der Planck Länge oder - äquivalent dazu - sehr hoher Energien (weit höher als jene, welche durch die heutigen Teilchenbeschleuniger erreicht werden), erwartet man, dass die Raumzeit eine quantisierte Struktur aufweist. Mit Ausnahme der Gravitation können heute alle fundamentalen Naturkräfte durch Quantenfeldtheorien (QFT's), im speziellen durch Eichfeldtheorien, beschrieben werden. Deren Herzstück sind die Renormierungsverfahren, die es erst ermöglichen, die Divergenzen, welche bei der perturbativen Entwicklung der das Modell repräsentierenden Wirkung auftreten und durch hohe interne Impulse verursacht werden, in einer konsistenten Art und Weise zu behandeln. In den letzten Jahren wurden zahlreiche Versuche unternommen, eine konsistente und renormierbare Theorie auch für nicht-kommutative Geometrien zu formulieren. Jedoch ist es v.a. letztgenannter Punkt, welcher eines der bedeutendsten Probleme der Quantenfeldtheorien darstellt: ganz allgemein wird die Nicht-Kommutativität der Raum(zeit) durch das sogenannte Sternprodukt implementiert, im einfachsten Fall durch das Moyal-Weyl-Produkt. Dieses führt zu einer zusätzlichen Phase in den Vertizes, welche vom nicht-kommutativen Parameter θ abhängt, und in Folge zu einer Modifikation des Wechselwirkungsanteils der Theorie. Des Weiteren bewirkt diese Phase ein Mischen von hohen und niedrigen Energien, welche direkt mit dem in Erscheinung treten einer neuen Klasse von Divergenzen für niedrige Energien verknüpft ist. Während es eine Reihe diverser Renormierungsschemata für die im Ultravioletten (UV) auftretenden Divergenzen gibt, stellen deren Gegenspieler im infraroten (IR) Sektor das vielleicht größte Hindernis bei der Formulierung einer konsistenten θ-deformierten Quantenfeldtheorie dar. Ein erster Ausweg aus dieser misslichen Lage konnte für ein skalares Modell durch Grosse und Wulkenhaar aufgezeigt werden [6], und zwar durch die Einführung eines geeigneten Zusatzterms - in diesem Fall eines Oszillatorterm - in der Wirkung, welcher zu einer Entkopplung des UV und IR Sektors führt. Dieselbe Vorgangsweise wurde auch von Gurau *et. al.* umgesetzt [7], welche einen Term der Form $\frac{1}{p^2}$ zu der skalaren Wirkung hinzufügten. Für beide Modelle konnte Renormierbarkeit gezeigt werden. Das letztgenannte Modell führt des Weiteren zu einem translationsinvarianten Eichfeldpropagator, was Impulserhaltung in jedem Raumpunkt impliziert.

Wie bereits erwähnt basiert das Standardmodell auf der Formulierung entsprechender Eichfeldtheorien. Es ist daher essentiell, zu einer Formulierung entsprechender renormierbarer Modelle auch im Nicht-Kommutativen zu gelangen. Damit sind auch Ziel und Inhalt dieser Dissertation umrissen, welche in dem Versuch der Formulierung einer renormierbaren, θ-deformierten $U(1)$ Eichtheorie (kurz $U_\star(1)$) besteht.

In einem ersten Schritt wurde eine lokalisierte Version des in [8] eingeführten Modells einer gründlichen Analyse unterzogen. Dieses Modell stellt eine nicht-kommutative Verallgemeinerung der (im Euklidischen formulierten) üblichen $U(1)$ Eichtheorie dar. Es basiert auf der Einführung eines dem von Gurau *et. al.* analogen Terms, welcher in Folge zu einem für niedrige Energien gedämpften Eichbosonpropagator führt. Im Rah-

men von perturbativen Berechnungen auf 1-Loop Niveau [1] konnten wir zeigen, dass diese Wirkung zu zusätzlichen physikalischen Freiheitsgraden und damit einer Modifikation der Physik führt. Ein Ausweg konnte in [2] aufgezeigt werden. In der dort vorgestellten Wirkung konnte die Modifikation des IR-Sektors durch die Einführung eines sogenannten "soft breaking" Terms erreicht werden, analog zum Gribov-Zwanziger Verfahren, welches uns von der kommutativen Yang-Mills Theorie her bekannt ist. In Folge wurde der Versuch der 1-Loop Renormierung unternommen, welcher jedoch ohne Erfolg war. Der Grund hierfür ist das Fehlen geeigneter Terme in der Wirkung, welche die Absorption der IR-Divergenzen erlauben würde, siehe [3]. Üblicherweise wird man in diesem Fall den Versuch unternehmen, eine sogenannte effektive Wirkung zu konstruieren, welche renormierbar sein soll. Dies wird durch die Anwendung von Renormierungs- schemata wie z.B. der Algebraischen Renormierung erreicht, welche in diesem Fall jedoch nicht angewandt werden kann, aufgrund der inhärenten Nicht-Lokalität des Stern-Produkts. Ideen bezüglich der Anwendbarkeit und möglichen Erweiterung bestehender Renormeriungsschemata auf nicht-kommutative Eichfeldtheorien wurden in [4] diskutiert. Schlussendlich wurde eine neue Wirkung konstruiert [5], das sogenannte BRSW Model (benannt nach den Anfangsbuchstaben von Daniel Blaschke, Rene Sedmik, Michael Wohlgenannt und des Autors), für welches 1-Loop Renormierbarkeit gezeigt werden konnte. Obwohl ein strenger Beweis noch ausständig ist, denken wir, dass es sich hierbei um einen vielversprechenden Kandidaten einer ersten vollständig renormierbaren $U_\star(1)$ Eichtheorie handelt.

Abstract

When considering very small scales near the Planck-length, or equivalently very high energies (far from being reached by today's particle accelerators), space-time is expected to be quantized. Today, all but one forces governing nature (i.e. gravitation) are described via Quantum Field Theories (short QFTs) and more precisely gauge field theories (GFTs). Their heart is the art of renormalization, which allows to handle the divergences for high internal momenta appearing in the course of the perturbative development of the action in a consistent manner. Over the last years numerous attempts have been made to formulate consistent and renormalizable theories also on non-commutative spaces. Yet, it is the latter that represents a major problem for non-commutative QFTs: generally, the non-commutativity is implemented via the so-called star product, which in the simplest case is given by the Moyal-Weyl product, and which leads to a modification of the interaction terms of the theories by introducing additional phase factors depending on the non-commutative parameter θ. Then, this phase leads to a mixing of high and low energies, which is directly linked to the appearance of a new class of divergences for small momenta. While there exist various traditional renormalization schemes in order to handle UV divergences, their counterparts in the IR sector form a major obstacle in formulating consistent θ-deformed QFTs. However, a first way out of this misery could be achieved by Grosse and Wulkenhaar for a scalar model [6]. The idea was to add a suitable term to the action, in their case an oscillator term, leading to a decoupling of the high and low energy sectors. Later, the same philosophy has been followed by Gurau *et. al.* [7] by adding a $\frac{1}{p^2}$-like term to the scalar action. Both models have been shown to be renormalizable, and additionally, the latter model leads to a translation invariant propagator, which implies momentum conservation in all space points.

Now, the standard model is formulated via gauge field theories. It is therefore crucial to find their non-commutative, renormalizable counterparts. Having said this we have already addressed the goal and content of this dissertation, which consists in finding a potentially renormalizable θ-deformed $U(1)$ gauge theory, denoted $U_\star(1)$.

In a first step, we studied in detail a localized version of the model introduced in [8], which represents an extension of ordinary $U(1)$ gauge theory (formulated on Euclidean space) to the non-commutative setting, and is based on adding a term similar to the one of Gurau *et. al.*, leading to an IR-damped gauge boson propagator. In the course of one-loop calculations [1], we have shown that it implements additional degrees of freedom and hence modifies the original physical content of the theory. A way out was found in [2] by implementing the modification of the IR sector through the introduction of a soft breaking term similar to the approach of Gribov and Zwanziger known from commutative Yang Mills theory. However, when trying to show renormalizability at one-loop level, it turned out that the action does not contain the appropriate terms for absorbing the IR divergences, c.f. [3]. Usually, in such cases one constructs an effective renormalizable action via application of renormalization schemes such as Algebraic Renormalization, which in this case fails, due to the inherent non-locality of the star product. Some

ideas regarding the applicability and possible extension of traditional renormalization schemes to non-commutative GFTs have been discussed in [4]. Finally a new action, the BRSW model [5] (named after the initials of Daniel Blaschke, Rene Sedmik, Manfred Schweda, Michael Wohlgenannt and the author) was constructed. It could be shown to be renormalizable to one-loop order. Although a rigorous proof is still missing, we expect it to be a very promising candidate for the first fully renormalizable $U_\star(1)$ gauge field theory.

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Contents

Chapter 1

Introduction

1.1 Motivation for studying NCQFT

The universe we know is made of roughly 61 different scales using powers of ten [9], as depicted in Fig. 1.1. The upper limit is given by the distance of the earth to the edges

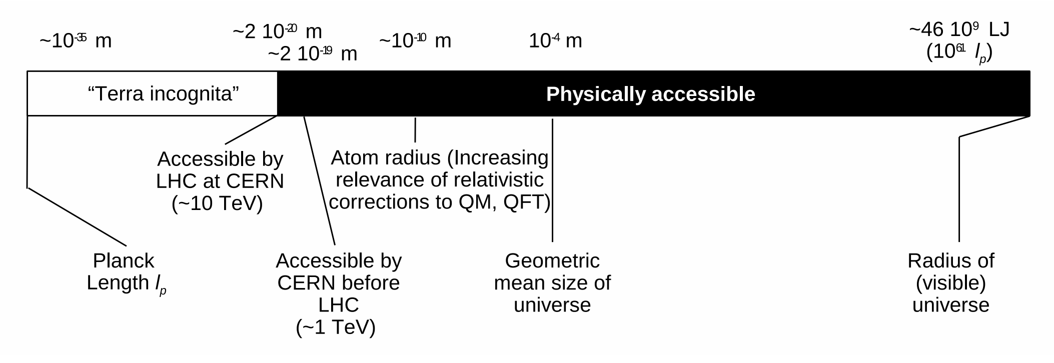


Figure 1.1: The scales of the physical universe

of the observable universe of 46 billion light-years or $4.410^{26}m^1$, whereas the lower limit is given by the Planck-length which one obtains by combining the three fundamental constants of physics[2]:

$$l_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6\ 10^{-35}m. \tag{1.1}$$

As of today, the physically accessible universe ranges from the size of the observable universe down to $10^{-20}m$. For its description we have two basic theoretical building blocks at our disposal (Fig. 1.2): Einstein's theory of general relativity (GRT) on one hand for the description of large scale phenomena, and quantum theory as the appropriate tool for understanding the microscopic world. Especially, quantum field theories or QFTs provide the description of particles and their interactions which is compatible with special relativity, forming the basis of the *standard model of particle physics*. This latter encompasses three of the four basic forces of nature and all its related phenomena. Now the question is: what does a theory look like that describes the terra incognita of

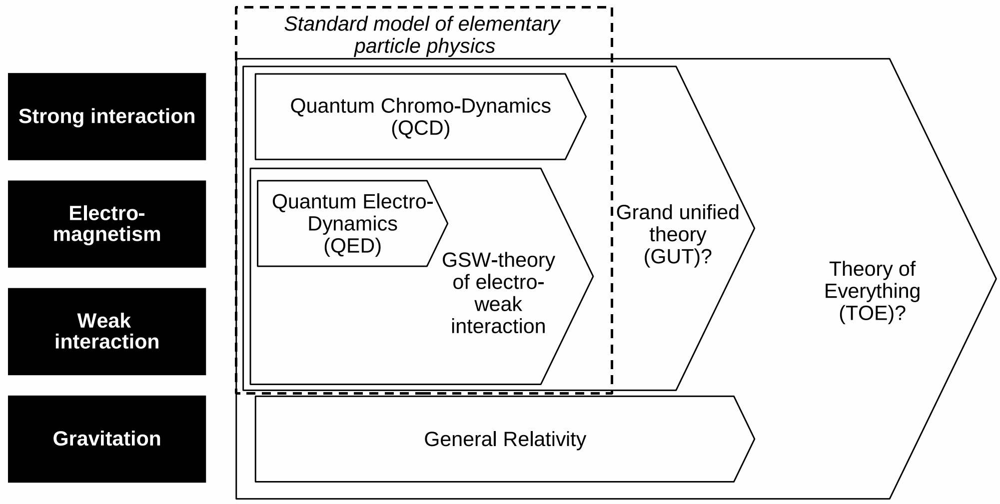


Figure 1.2: The theoretical buildings in theoretical physics

physics, being composed of the remaining scales between the above mentioned $10^{-20}m$ and the Planck length? We definitely expect a final unification of all four forces governing nature, leading to the so called *Theory Of Everything (TOE)*, being the Holy Grail of theoretical physics. Indeed, at very small scales / very high energies both GRT and QFT enter the picture, thus only their combination will give the right physics. This can qualitatively be seen from Fig. 1.3[3], which shows the physical theories suitable for

[1]Due to the expansion of the universe, co-moving coordinates must be used, leading to a size which is bigger than the expected 13.7 billion light years when considering the age of universe [9].

[2]The three fundamental constants are Newton's gravitation constant G, Planck's constant $\hbar$ and the speed of light c.

[3]Source: unofficial talks by Univ.Prof.Dr. Harald Grosse.

describing phenomena for a given scale of energy (c), quantization ($\hbar$) and (space-)time deformation, expressed via the Einstein tensor (symbolically denoted by G) or the parameter θ expressing non-commutativity in *Non-Commutative Quantum Field Theory (NCQFT)*. At this stage, the latter is one very promising *ansatz*, as detailed in what follows.

In ordinary quantum physics, i.e. ordinary quantum mechanics and quantum field the-

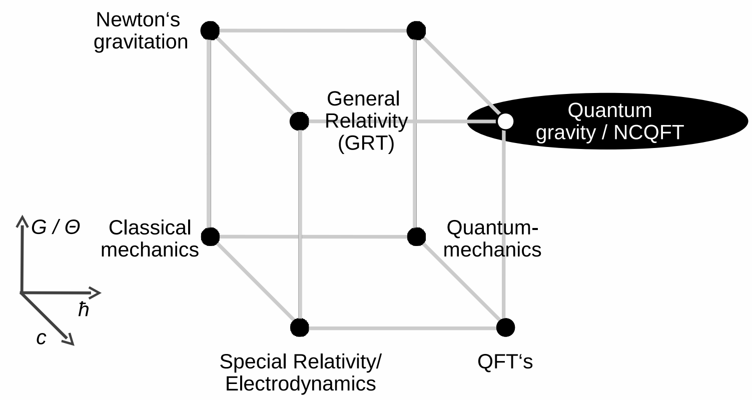


Figure 1.3: Classification of theories

ory, the non-commutativity between conjugated pairs of variables such as the operators corresponding to space coordinates x_i and momentum coordinates p_j is expressed by the famous Heisenberg uncertainty relation $[\hat{x}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{i,j}$. This leads to the quantization of the phase space, where the notion of a point is replaced by that of a Planck cell. Now, generalizing the notion of non-commutativity to ordinary space arises very naturally, leading to

$$[\hat{x}_i, \hat{x}_j] = \mathrm{i}\theta_{ij}. \tag{1.2}$$

Here the space(time) variables have been replaced by operators, and θ_{ij} is a matrix with mass dimension -2 (c.f. (1.7)).

Indeed, there are several physical considerations motivating the above *ansatz*: as is well-known, GRT describes gravitation by a modification of space-time. Thus a quantum field theory of gravity should quantize space-time itself.

Another very strong argument arises from the following *Gedankenexperiment* (Wheeler 1957, [10]; DeWitt 1962, c.f.[11],[12]): following GRT, any concentration of energy in space-time leads to a curvature of space-time:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}, \tag{1.3}$$

with $G_{\mu\nu}$ the Einstein tensor, $g_{\mu\nu}$ the space-time metric, and $R_{\mu\nu}$, R the Ricci-tensor and Ricci-scalar respectively. According to Heisenberg's uncertainty principle the measurement of the position of a particle with accuracy a will lead to an uncertainty and

therefore energy concentration in this region of the order[4] $1/a$. It follows that measuring the position of a particle with accuracy near the Planck length will lead to a curvature of space-time so strong that a black hole will be created, preventing light or any other signal to leave the region under consideration. This problem can be circumvented by "smearing" space coordinates, e.g. non-commutativity between the spatial directions.
Whatsoever, the original motivation for using non-commutative space(time) in QFT[5] originates from the problem of the appearance of ultraviolet or UV-divergences in ordinary QFT's and goes back to Snyder. In fact, this problem already appears in the classical Maxwell-theory and originates from the idealized description of point-like particle interactions. In classical electrodynamics, this leads to a divergence at the interaction space point, whereas in QFT the problem gets worse: an infinite number of divergences appears. In mathematical terms, the problem originates in the multiplication of distributions at the same point and thus causes singularities. Snyder's idea was to smear out point-like interactions by using non-commutative coordinates and thus to regularize the UV divergences.
Against all initial hopes, the use of a non-commutative geometry does not remedy the UV-problem, indeed it gets worse: now a new class of divergences appears, the so called *UV/IR-mixing*[6], which is a general characteristic to all non-commutative theories. So the initial motivation has not been fulfilled, nonetheless the physical motivation introduced at the beginning of this section still remains valid, which is the reason for the intense study of non-commutative theories over the last two decades. The next but one section will give a brief historical overview of the progresses made in this exciting field of research up to now, forming the basis for the present thesis. More precisely, a QFT has to be renormalizable in order to be self-consistent[7]. So far, renormalizability could be shown for some non-commutative scalar models [13, 6, 7]. The next logical step would then be the extension of non-commutative geometry to the simplest theory describing the interaction of *physical* particles, namely the formulation of a non-commutative U(1) gauge theory, the resulting algebra called $U_\star(1)$, representing the gauge group of a non-commutative formulation of Quantum Electrodynamics (NCQED).

1.2 Outline

After a short historical overview of the field of non-commutative geometry, we will introduce the Moyal-Weyl correspondence, which allows to implement the non-commutativity between space variables of (1.2) in field theories while working with ordinary functions

[4] In natural units with $\hbar = c = 1$.

[5] Note that a non-commutative geometry very naturally arises also in condensed matter theory, the most prominent example being the Landau problem, describing the two dimensional motion of a charged particle in an external magnetic field.

[6] Considered in detail in Section 1.5.

[7] A theory is not necessarily renormalizable in order to be physically relevant; however, in that case it is necessarily only a piece of the big picture, c.f. Fermi's (non-renormalizable) theory of weak interaction, which has been replaced the Glashow-Weinberg-Salam theory of electroweak interaction.

instead of operator valued objects. This naturally leads to the problem of UV/IR mixing, which is strongly connected to the appearance of a new type of graphs called non-planar (Section 1.5). In Chapter 2 we will consider two scalar models where this problem could be overcome by the introduction of a new term in the action, leading to a decoupling of the ultraviolet and infrared regimes and finally to renormalizability. In particular, for the model of Gurau *et. al.* of Section 2.2 this has been achieved by introducing a term of the form $\frac{1}{p^2}$. The aim of this thesis being the construction of a non-commutative gauge field theory or NCGFT, various models intending to implement the same damping behaviour for a $U_\star(1)$ or θ-deformed $U(1)$ theory will be studied in Chapter 3. For the case where this could finally be achieved, we will provide one-loop calculations and explicitly show the desired transversality of the vacuum polarization with respect to external momenta. Yet, we will also see that this model contains a major drawback, given by the introduction of new physical degrees of freedom, hence altering the original content of the theory. Motivated by [14], we will study how infrared effects have been implemented for commutative Yang-Mills theories in the framework of the Gribov-Zwanziger approach, c.f. Chapter 4. There we will systemize this method of soft breaking, which then will lead to an improved non-commutative gauge model in Chapter 5. In the latter we will also derive the complete one-loop correction to the gauge boson propagator. Unfortunately, we will see that the attempt for showing one-loop renormalizability will fail. For such a case, one usually tries to construct an effective (and in the end hopefully) renormalizable action, and the intention was to apply Algebraic Renormalization to the model under consideration. However it turned out that for non-commutative QFTs, the application of the latter fails, due to the inherent non-locality of the star product. Based on this finding, problems are to be expected for the application of some prominent traditional renormalization schemes and their possible extension to NCGFTs, which will be discussed in Chapter 6. Instead of following further this path, a new type of non-commutative gauge models still based on the idea of soft breaking will be introduced in Chapter 7, where one-loop renormalizability for the full gauge boson propagator of the starting action, and the derivation of appropriate counterterms for the vertex corrections will be shown. Although a rigorous proof is still missing, this model is expected to be renormalizable to all orders. A short conclusion and outlook will complete the investigations on the renormalizability of $U_\star(1)$ gauge theories.
Useful formulae and detailed calculations have been shifted to the appendix. It also contains a chapter which explains the evaluation of symmetry factors for any Feynman diagram. There we will also give an overview of the programmed routines which have been developed in order to afford the very involving loop calculations for the various models.

1.3 A brief historical overview

The idea of generalizing the notion of non-commutativity from phase-space to ordinary space-time has already been expressed by the founding fathers of quantum theory, especially Heisenberg, the motivation being the introduction of a natural UV-cutoff and thus avoiding the problem of UV-divergences which is characteristic to quantum field theory. It was then Snyder (1947, [15, 16]) who formalized the idea of using a non-commutative geometry in order to smear out the point-like particle interactions, leading to non-local interactions. At the same time, the renormalization program became successful, and non-commutative geometry had not been investigated any further until the 1980's when Connes, Woronowicz and Drinfeld generalized the notion of a differential structure to the non-commutative setting. Especially Connes's reformulation of the standard model based on an (inner) non-commutative geometry attracted much attention [17, 18, 19]. It was discovered in the late 1990's that simple limits of M-theory[8] lead to non-commutative gauge theories, keeping some of its basic characteristics of nonlocality but at the same time being simpler[9] [20, 21, 22], thus leading to a renewed and intense interest for non-commutative geometries amongst high energy physicists. For introductions to the field of non-commutative geometry and its application in quantum field theories, see e.g. [23, 24, 25] and [11, 26] containing also a historical overview of the field.

1.4 The Moyal-Weyl product

Ordinary quantum mechanics is based on the following commutation relations, which lead to a quantization of phase space, replacing thus the notion of a point with that of a Planck cell:

$$\begin{aligned} \left[\hat{x}^i, \hat{p}^j\right] &= \mathrm{i}\hbar\delta^{ij}, \\ \left[\hat{x}^i, \hat{x}^j\right] &= \left[\hat{p}^i, \hat{p}^j\right] = 0, \end{aligned} \tag{1.4}$$

with $i,\ j = 1, 2, 3$, the $\hat{x}^i$, $\hat{p}^j$ being (unbounded) operators acting on a Hilbert space. Here and in the following the hat indicates operator valued objects. A non-vanishing commutator relation always implies an uncertainty relation, in this case the famous Heisenberg uncertainty:

$$\triangle x \triangle y \geq \frac{\hbar}{2}\,. \tag{1.5}$$

In order to implement non-commutativity between space-time variables one generalizes the above concept [27, 28, 29]. The simplest case is a Heisenberg algebra

$$\begin{aligned} [\hat{x}^\mu, \hat{x}^\nu] &= \mathrm{i}\theta^{\mu\nu}, \\ [\hat{x}^\mu, \theta^{\mu\nu}] &= 0, \end{aligned} \tag{1.6}$$

[8] In M-theory developed in the mid 1990's the former rather independently existing different string theories together with supergravity are shown to be limits of a more fundamental theory.

[9] Non-locality is a fundamental property to string theories, as the notion of point like particles is being replaced by that of (open or closed) strings

where $\theta^{\mu\nu}$ is a real[10], constant and antisymmetric deformation matrix, with mass dimension -2, which - after bringing it into its canonical form - is given by:

$$(\theta_{\mu\nu}) = \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta_{d/2} \\ & & & -\theta_{d/2} & 0 \end{pmatrix}, \qquad \text{with } \theta_i \in \mathbb{R}\,. \tag{1.7}$$

Note, that the usual coordinates have been replaced by Hermitian operators $\hat{x}^\mu$ with $\mu = 0, 1, \cdots, (d-1)$, where d is the dimension of the θ-deformed (Minkowskian or Euclidean) space under consideration. The deformation matrix might be of full rank or not, depending whether one wishes to express non-commutativity between all coordinates or only between selected directions. Especially models formulated on the Euclidean $\mathbb{R}^4_\theta$ have been investigated intensely in the literature (e.g. [6, 7]). One can implement noncommutativity also by other deformations (see [34] for a detailed discussion), the Lie-case

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}C^{\mu\nu}_\rho \hat{x}^\rho, \tag{1.8}$$

or the quantum group space

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}R^{\mu\nu}_{\rho\sigma} \hat{x}^\rho \hat{x}^\sigma. \tag{1.9}$$

However, we will limit ourselves to the simplest case (1.6). At this point, it has to be mentioned that this procedure spoils Lorentz-invariance [35, 36], due to the preferred direction of noncommutativity.
In the following we want to use fields instead of operator valued objects. In order to keep the property (1.6) the multiplication law of functional (field) space has to be modified. This is being done through the so called Weyl-Moyal correspondence:

$$\hat{\phi}(\hat{x}) \Longleftrightarrow \phi(x), \tag{1.10}$$

where the l.h.s is a operator valued object and the r.h.s. is an ordinary field depending on (commuting) space-time coordinates $x \equiv x^\mu$. A possible correspondence is given by

$$\begin{aligned} \hat{\phi}(\hat{x}) &= \int \frac{d^dk}{(2\pi)^d} e^{\mathrm{i}k\hat{x}} \phi(k), \\ \phi(k) &= \int d^dx e^{-\mathrm{i}kx} \phi(x)\,. \end{aligned} \tag{1.11}$$

[10] For experimental considerations regarding the size of θ see for example the following publications: [30] contains investigations on the influence of non-commutative space on the Lamb shift; in [31] estimates for the predicted additional energy-loss in stars induced by space-time non-commutativity are derived; relevant considerations in the context of astrophysical observations, relying in particular on γ ray bursts are discussed in [32, 33].

For the product of two operator valued objects it follows

$$\begin{aligned}\hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x}) &= \int \frac{d^dk_1}{(2\pi)^d}\int \frac{d^dk_2}{(2\pi)^d}\; e^{\mathrm{i}k_1\hat{x}}\phi_1(k_1)e^{\mathrm{i}k_2\hat{x}}\phi_2(k_2)\\ &= \int \frac{d^dk_1}{(2\pi)^d}\int \frac{d^dk_2}{(2\pi)^d}\; e^{\mathrm{i}(k_1+k_2)\hat{x}-\frac{1}{2}[\hat{x}^\mu,\hat{x}^\nu]k_{1,\mu}k_{2,\nu}}\phi_1(k_1)\phi_2(k_2), \end{aligned} \tag{1.12}$$

where in the last line of (1.12) the Baker-Campbell-Hausdorff-formula has been used[11]. This leads to the definition of the Groenewold-Moyal-Weyl star product

$$\hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x}) \Longleftrightarrow (\phi_1 \star \phi_2)(x), \tag{1.13}$$

that is given by

$$(\phi_1 \star \phi_2)(x) = e^{\frac{\mathrm{i}}{2}\theta_{\mu\nu}\partial^x_\mu\partial^y_\nu}\phi_1(x)\phi_2(y)|_{x=y}\,. \tag{1.14}$$

By implementing the star product we have achieved that we can work with the ordinary commuting coordinates, whereas the modified product guarantees that (1.6) holds:

$$\begin{aligned}&[\hat{x}^\mu,\hat{x}^\nu] = [x^\mu \stackrel{\star}{,} x^\nu]\,,\\ &[x^\mu \stackrel{\star}{,} x^\nu] = x^\mu \star x^\nu - x^\nu \star x^\mu = \mathrm{i}\theta^{\mu\nu}\,. \end{aligned} \tag{1.15}$$

Following the definition of the star product, we have the following identities:

$$\int d^dx(\phi_1 \star \phi_2)(x) = \int d^dx\phi_1(x)\cdot\phi_2(x)\,, \tag{1.16a}$$

$$\int d^dx(\phi_1 \star \phi_2 \star \cdots \star \phi_n)(x) = \int d^dx(\phi_n \star \phi_1 \cdots \star \phi_{n-1})(x)\,, \tag{1.16b}$$

$$\frac{\delta}{\delta\phi_1}\int d^dx(\phi_1 \star \phi_2 \star \cdots \star \phi_n)(x) = (\phi_2 \star \cdots \star \phi_n)(x)\,. \tag{1.16c}$$

(1.16a) states that under the integral the star product does not affect the product of two fields, which directly translates into the calculation of propagators being derived from the bilinear part of the action. Thus the propagators in the non-commutative case remain unchanged with respect to the commutative counterpart of a given model[12], i.e. when replacing in the action the ordinary product by the star product. (1.16b) states the invariance of a product of integrated fields under cyclic permutations.
Furthermore, associativity holds:

$$[(\phi_1 \star \phi_2) \star \phi_3)] = [\phi_1 \star (\phi_2 \star \phi_3)]\;. \tag{1.17}$$

[11] The Baker-Campbell-Hausdorff-formula gives $e^Ae^B = e^{A+B+\frac{1}{2}[A,B]}$under the condition $[A,[A,B]] = [B,[A,B]]$. In the present case we have $[x_\mu,[x_\nu,x_\rho]] = [x_\mu,\theta_{\nu\rho}] = 0$, from which follows the vanishing of higher order commutators.

[12] Note that this is only true in Euclidean space, and in Minkwoski space if time is commutative.

1.5 UV/IR mixing

By writing down explicitly the star product of n fields (which is being used in order to implement non-commutativity amongst the position coordinates as explained above),

$$\begin{aligned}&\phi_1(x) \star \cdots \star \phi_n(x) = \\ &= \int \frac{d^d k_1}{(2\pi)^d} \cdots \int \frac{d^d k_n}{(2\pi)^d} \phi_1(k_1) \cdots \phi_n(k_n) e^{\mathrm{i} \sum_{i=1}^n k_i^\mu x_\mu} e^{-\frac{\mathrm{i}}{2} \sum_{i<j}^n k_i^\mu \theta_{\mu\nu} k_j^\nu}\end{aligned} \tag{1.18}$$

one can see that with comparison to the ordinary product of fields one observes an additional phase factor. According to (1.16a), this affects only the product of more than two fields, from which it follows, that replacing the ordinary product by the star product in a given Lagrangian will affect the interaction part of the theory. More precisely, this will generate not only the ordinary planar but also the so called non-planar graphs, finally leading to the infamous UV/IR-mixing problem typical for θ-deformed quantum field theories. After a classification of the Feynman graphs, the analysis of a θ-deformed ϕ^4-theory will allow for a better understanding of the problem [37, 38].

1.5.1 Classification of graphs

As stated in (1.16b), the (compared with commutative theories) additional phase factor is only symmetric under cyclic permutations. This property is similar to matrix theories, where we observe a symmetry under cyclic permutations of the indices. Note that in matrix theories the Feynman graphs are ribbon graphs (c.f. e.g. [26]). Therefore the same double line notation may be applied in our case by assigning double lines to propagators and the order of connecting them for a given graph reflects the ordering of the fields. Moreover, this notation allows to distinguish different types of graphs by their topological properties (e.g. [39]), i.e. by their Euler characteristics χ, given by

$$\chi = V - I + F \to 2 - 2g, \tag{1.19}$$

with V, I, F the number of vertices, internal lines and faces. Internal lines are lines wich connect to vertices at both ends. A face is a line which delimits a surface. This is equivalent to say that the number of faces corresponds to the number of single lines in a graph, after having closed the external lines at their ends. For connected diagrams this reduces to the r.h.s of (1.19), where g denotes the genus. Now the Feynman graphs can be classified: graphs with $g = 0$ are called planar, they can be drawn on a plane without crossing lines; graphs with $g \geq 1$ are called non-planar, and can be drawn without crossing lines only on a surface with genus g. Moreover, the planar graphs are further classified with respect to the number of broken faces B, i.e. faces broken by external lines. They are called planar regular for $B = 1$ and planar irregular for $B > 1$.

As an example, let us analyze the graphs depicted in Fig. 1.4. Both graphs have two vertices, two internal lines, and two faces. In order to distinguish them, the two faces of a graph are represented by continuous and dashed lines respectively. From (1.19) it follows,

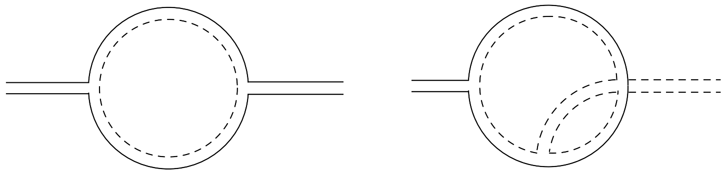


Figure 1.4: Planar (planar regular) and non-planar (planar irregular) one-loop graphs

that both graphs have genus g, and are therefore considered as planar. However, the left graph has only one broken face (the one delimited by the continuous line), whereas the right graph has both faces broken by external legs. It follows that the left graph is planar regular, and the right one planar irregular.
There is a second way to classify graphs, namely by considering whether a given graph has lines crossing each other, i.e. propagators crossing over each other or over external lines [26]. If this is the case, a graph is said to be non-planar, and planar otherwise. The reason is that the additional phase factor for planar graphs will depend on external momenta only, whereas the expression for a non-planar graph contains also a vertex and therefore phase factor for each internal crossing of lines. The behavior of the latter will therefore substantially depend on its inner structure. In particular it can be expected, that the additional factor being a phase will serve as a UV regulator due to rapid oscillations for high internal momenta. In this convention, in Fig. 1.4 the left graph is again planar, but the right graph is non-planar. We will follow this second convention.

1.5.2 One loop self energy in $\phi^4_\star$-theory

Let us start with the formulation of the usual ϕ^4-action on the four-dimensional Moyal space (e.g.[40]). This is done by replacing the ordinary product by the star product, and the usual integration by the non-commutative integration which is a combination of the usual integration and a trace [11]. This results in the simplest non-commutative model with the tree-level action

$$S[\phi] = \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{1}{2}m^2\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi \right) , \tag{1.20}$$

where λ is the coupling constant. Rewriting this action in momentum space we get (for $i,\ j = 1, \ldots, 4$)

$$\begin{aligned} S = &\frac{1}{2}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\phi(-p)\phi(p)(p^2+m^2)+ \\ &+\frac{\lambda}{4!}\int\prod_i \frac{\mathrm{d}^4p_i}{(2\pi)^4}(2\pi)^4\delta(\sum_i p_i)\exp(-\frac{\mathrm{i}}{2}\sum_{i<j}p_i^\mu\theta_{\mu\nu}p_j^\nu)\prod_i\phi(p_i). \end{aligned} \tag{1.21}$$

It can also be written in terms of vertex functions

$$S = \int \mathrm{d}^4 p \left\{ \frac{1}{2}\phi(-p)\phi(p)\Gamma_0^{(2)} + \frac{\lambda}{4!}\prod_i \phi(p_i)\Gamma_0^{(4)} \right\} \tag{1.22}$$

with $\Gamma_j^{(i)}$ denoting the i-point function of loop order j. By comparison of the vertex functions with those of the ordinary theory, that can be retained from (1.21) by considering the limit $\theta_{\mu\nu} \to 0$, it becomes now evident that the propagator, given by the inverse of[13]

$$\Gamma_0^{(2)} = p^2 + m^2 \tag{1.23}$$

remains unchanged. Only the interacting part of the theory will be modified by an additional phase factor.
Calculation of the one loop self energy diagrams leads to

$$\Gamma_1^{(2)} = \frac{\lambda}{6(2\pi)^4} \int \mathrm{d}^4 k \frac{2 + e^{\mathrm{i}k\tilde{p}}}{k^2 + m^2} \equiv \Gamma_{1,pl}^{(2)} + \Gamma_{1,n.p.}^{(2)}. \tag{1.24}$$

In the last equation we have introduced the notation $\tilde{p}_\mu = \theta_{\mu\nu} p^\nu$. It can be seen that the integral splits into a planar and non-planar contribution, the corresponding graphs being drawn in Fig. 1.5. Note that in the limit $\theta \to 0$ we get the commutative result as

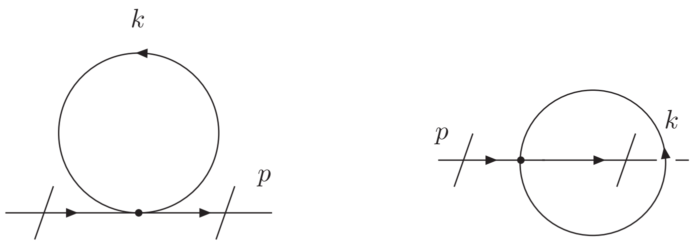


Figure 1.5: Planar and non-planar one-loop graphs

expected. By using Schwinger parametrization (c.f. (A.1)), bringing the exponential in the non-planar expression to a quadratic form and integration over k (c.f. (A.3)) we get the following integrals, depending on the Schwinger parameter α:

$$\Gamma_{1,pl.}^{(2)} = \frac{\lambda}{48\pi^2} \int_0^\infty \mathrm{d}\alpha \frac{e^{-\alpha m^2 - \frac{1}{\Lambda^2 \alpha}}}{\alpha^2}, \tag{1.25a}$$

$$\Gamma_{1,n.p.}^{(2)} = \frac{\lambda}{96\pi^2} \int_0^\infty \mathrm{d}\alpha \frac{e^{-\alpha m^2 - \frac{\tilde{p}^2}{4\alpha}}}{\alpha^2}. \tag{1.25b}$$

[13]Note that for non-commutative time the calculation of the propagator becomes more involved, i.e. it is not simply given by the inverse of (1.23).

In order to regularize the small α divergence in (1.25a) we have multiplied the integrand by $\exp(-\frac{1}{\Lambda^2\alpha})$, where Λ is an UV-cutoff: $\Lambda^2 \to \infty$ creates UV-divergences as one can see in the next formulae. Now (A.5) can be applied for both cases, leading - after approximation of the modified Bessel functions, c.f. (A.10b) - to

$$\Gamma^{(2)}_{1,pl.} = \frac{\lambda}{48\pi^2} 2\sqrt{\Lambda^2 m^2} K_1\Big(2\sqrt{\frac{m^2}{\Lambda^2}}\Big) \approx \frac{\lambda}{48\pi^2}\Big\{\Lambda^2 + m^2 \ln\Big(\frac{m^2}{\Lambda^2}\Big)\Big\}, \tag{1.26a}$$

$$\Gamma^{(2)}_{1,n.p.} = \frac{\lambda}{96\pi^2} 4\sqrt{\frac{m^2}{\tilde{p}^2}} K_1(\sqrt{m^2\tilde{p}^2}) \approx \frac{\lambda}{96\pi^2}\Big\{\frac{4}{\tilde{p}^2} + m^2 \ln\big(m^2\tilde{p}^2\big)\Big\}. \tag{1.26b}$$

From this result, we can observe the following:

- The non planar vertex function is now finite as long as $\tilde{p}^2 \neq 0$, i.e. the momentum acts as UV-regulator with the effective cutoff

$$\Lambda_{eff} = \frac{1}{\sqrt{\tilde{p}^2}}. \tag{1.27}$$

  Therefore, otherwise UV divergent graphs are rendered finite.

- In the limit $\tilde{p}^2 = \tilde{p}_\mu\tilde{p}^\mu \to 0$, i.e. for the commutative limit and/or vanishing external momenta, the original UV divergence reappears. From the latter case we can see, that the former UV divergence has just been replaced by a IR singularity. Indeed, this becomes also clear when considering the regulating exponential for (1.25a) as described above, which is just inverse to the momentum dependent term already present in (1.25b).

- The planar contribution can be absorbed in a mass redefinition, $m_{ren} = m + \Gamma^{(2)}_{1,pl.}$. Thus, the usual UV-renormalization schemes may be applied. Unfortunately this is not the case for the non-planar expression, the reason being its non-locality, i.e. dependence from the integration parameter. Note that IR divergences have already been present in commutative QFTs, in the form of singularities appearing in some ill-defined loop integrals requiring regularization. Contrary to that non-planar graphs lead to divergences in the *external* momenta, and are thus of totally different nature.
  Finally, the one loop 1PI effective action can be written as

$$\begin{aligned} S &= \int \mathrm{d}^4p \left\{\frac{1}{2}\phi(-p)\phi(p)\Big(\Gamma^{(2)}_0 + \Gamma^{(2)}_{1,pl.} + \Gamma^{(2)}_{1,n.p.}(\tilde{p}^2)\Big) + \dots\right\} \\ &= \int \mathrm{d}^4p \left\{\frac{1}{2}\phi(-p)\phi(p)\Big(p^2 + m^2_{ren} + \Gamma^{(2)}_{1,n.p.}(\tilde{p}^2)\Big) + \dots\right\}, \end{aligned} \tag{1.28}$$

  with $m^2_{ren} = m^2 + \Gamma^{(2)}_{1,pl.}$.

- As a result of the above said, inserting the non-planar graph into graphs of higher loop order will lead to IR-divergences of increasing order.

In summary, this mixing of the high and low energy sector is the (in)famous UV/IR mixing problem. In order to make a non-commutative theory renormalizable, one has to handle this obstacle. This project forms a major part of this thesis.

Chapter 2

Renormalizable non-commutative scalar models

In Section 1.5 we have elaborated on the example of the θ-deformed scalar $\phi^4_\star$-model, that the implementation of non-commutativity generally leads to the problem of UV/IR mixing[1]. As is well known, the planar graphs lead to UV-divergences which can be absorbed in the parameters of the theory, hence are renormalizable (under the condition that this is true for the commutative limit of the model). Contrary to that, the non-planar graphs may contain IR-divergences which are not of the renormalizable type, and therefore destroy renormalizability of the model. *Yet, renormalization is the soul of QFT.* Therefore intense work has been done in this direction, finally leading to several models that overcome the UV/IR problem (for the scalar case) and which could be shown to be renormalizable.

In the following we will describe two of those models[2]: the model with oscillator term in Section 2.1 for its historical importance, showing the path for the construction of several other models; the second in Section 2.2 due to its importance for the present work. For both of them we will show their strengths and shortcomings. Whatsoever, both rely on the same philosophy: based on the fact *that noncommutativity relevant at very short distances modifies the physics of the model at very large distances*, the idea is to alter the free theory, as at long distances a theory is a (almost) free one. This is done by adding a properly chosen new term to the action, which is not present in

[1] This is the case at least for models with a constant deformation matrix, c.f. (1.6), formulated on the Euclidean space. There are some hints, that in Minkowskian space-time UV/IR mixing would not occur (e.g. [41]).

[2] After the first model of Section 2.1 has been introduced, several models have been proposed, c.f. e.g. [7, 42].

the naïve model[3]. This term modifies the propagator in such a way that the theory is rendered finite in the IR-sector by mixing short and long distances. At this point it has to be mentioned that both theories are formulated on Groenewald-Moyal deformed (also called θ-deormed) Euclidean space $\mathbb{R}^4_\theta$.

2.1 Model with oscillator term

In the years 2003/04, H. Grosse and R. Wulkenhaar published a series of papers, in which they introduced a non-commutative scalar model on the Euclidean space, first in $\mathbb{R}^2_\theta$ [13] than in $\mathbb{R}^4_\theta$ [6], which where shown to be renormalizable to all orders (c.f. also [43]). This has been achieved by adding to the action a marginal harmonic oscillator-like potential of the form $\tilde{x}\phi^2$, leading to the action

$$\begin{aligned} S[\phi] &= S_{int} + S_{free}\,, \\ S_{free} &= \int \mathrm{d}^4x \Big(\frac{1}{2}\left(\partial_\mu\phi\right)\star\left(\partial^\mu\phi\right) + \frac{\Omega^2}{2}\left(\tilde{x}_\mu\phi\right)\star\left(\tilde{x}^\mu\phi\right) + \frac{\mu_0^2}{2}\phi\star\phi\Big)(x)\,, \\ S_{int} &= \int \mathrm{d}^4x \frac{\lambda}{4!}\left(\phi\star\phi\star\phi\star\phi\right)(x)\,, \end{aligned} \tag{2.1}$$

where $\tilde{x}_\mu = 2(\theta^{-1})_{\mu\nu}x^\nu$ and Ω is a oscillator parameter with $dim(\Omega) = 0$. Retrospectively, the oscillator term can be made plausible by denoting that it is required in order to get an action that is covariant under Langmann-Szabo duality [44] as the naïve action is not. More specifically, the original interaction part of the action is LZ-covariant, while the free part has to be altered before fulfilling the same property. Roughly speaking, LZ-covariance denotes form-invariance when passing from space to momentum coordinates, i.e. under the transformations

$$p_\mu \to \tilde{x}_\mu \text{ and } \hat{\phi}(p) = \int \mathrm{d}^4x \exp\left\{(-1)^a \mathrm{i} p_{a,\mu} x^\mu_a\right\} \phi(x_a)\,. \tag{2.2}$$

The index $a = \{1, \ldots, 4\}$ in the Fourier transformation follows a cyclic ordering at the vertex. By writing down explicitly the action in both momentum and position coordinate space, one can see the following property of LZ-covariance:

$$S\left(\phi, m, \lambda, \Omega\right) = \Omega^2 S\left(\phi, \frac{m}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}\right)\,. \tag{2.3}$$

At $\Omega = 1$, one has complete equivalence of the theory in momentum and coordinate space when looking at the bilinear part of the action. Based on this result, it is intuitive that the most suited base should be found in phase-space. There the basis consists of Gaussian wave packages. Due to the non-commuting space coordinates, there is one best-focused Gaussian, and by defining creation and annihilation operators, one gets

[3]Here and in the following with naïve model we address non-commutative models that result from the commutative theory by simply replacing the ordinary by a star product, e.g. (1.20).

successively all base functions. Applying the star product to those functions reduces to a ordinary matrix product, hence the naming matrix base. In order to show renormalizability, Grosse and Wulkenhaar formulated the propagator in matrix base, imposed maxima on the base-space indices which is equivalent to impose a cutoff, and checked renormalizability by using the Polchinski scheme [45].
In summary, the Grosse-Wulkenhaar or GW model was the first one to overcome the UV/IR mixing problem. The β-function of the model vanishes at all orders, hence is asymptotically safe and therefore free of any Landau ghost, and should be fully consistent at a non-perturbative, i.e. constructive level.
On the other hand, the model also has some shortcomings, the most important being the explicit breaking of translation invariance. This can directly be seen from (2.1) by the presence of space coordinates. Yet, translation invariance is a direct expression of the homogeneity of space and should therefore be fulfilled in order to describe physical phenomena. This implies that the theory will be able to describe local phenomena only. Another disadvantage is the difficulty of generalizing the model to gauge theories, as keeping gauge invariance and LZ-duality simultaneously leads to theories with non trivial vacua, i.e. non vanishing tadpole contributions [46, 47], rendering perturbation theory difficult.

2.2 A translation invariant scalar model

In [7] Gurau *et al.* presented another solution to the *UV/IR* mixing for the ϕ^4-model on Euclidean $\mathbb{R}^4_\theta$ by adding to the action a $1/p^2$-term, resulting in a action, which in momentum space [48] is given by

$$
\begin{aligned}
S[\phi] &= \frac{1}{2}\int d^4p\,\phi(-p)\Big(p^2+m^2+\frac{a}{\theta^2p^2}\Big)\phi(p)+S_{int}[\phi]\\
S_{int}[\phi] &= \frac{\lambda}{4!\,3}\int d^4p_1\ldots d^4p_4\phi(p_1)\ldots\phi(p_4)\,\delta^{(4)}(p_1+p_2+p_3+p_4)\times\\
&\quad\times\big[\cos\frac{p_1\tilde{p}_2}{2}\cos\frac{p_3\tilde{p}_4}{2}+\cos\frac{p_1\tilde{p}_3}{2}\cos\frac{p_2\tilde{p}_4}{2}+\cos\frac{p_1\tilde{p}_4}{2}\cos\frac{p_2\tilde{p}_3}{2}\big]\,. \qquad (2.4)
\end{aligned}
$$

leading to the convention that the model is commonly referred to as $1/p^2$ scalar model. In coordinate space the action is given by

$$
\begin{aligned}
S[\phi] &= \int \mathrm{d}^4x\left[\frac{1}{2}\Big(\partial_\mu\phi\star\partial_\mu\phi+m^2\phi^{\star 2}-\phi\star\frac{a^2}{\widetilde{\widetilde{\square}}}\phi\Big)+\frac{\lambda}{4!}\phi^{\star 4}\right],\\
\widetilde{\widetilde{\square}} &= \tilde{\partial}^\mu\tilde{\partial}_\mu=\theta^2\square,\ \text{where}\ \tilde{\partial}_\mu=\theta_{\mu\nu}\partial^\nu\,. \qquad (2.5)
\end{aligned}
$$

with $a > 0$ a dimensionless parameter. In the last equation, the usual short-hand notation in physics has been used, where $1/\square$ is the symbolical notation for the Green function of the differential operator

$$
\square_x G(x-x')=\delta^{(4)}(x-x'), \qquad G(x)=\frac{1}{|x-x'|^2}\,. \qquad (2.6)
$$

Considering the bilinear part this leads to the propagator

$$G(p) = \frac{1}{p^2 + m^2 + \frac{a^2}{\tilde{p}^2}}\,, \tag{2.7}$$

which mixes long and short scales. Compared to the propagator of the naïve model $(p^2 + m^2)^{-1}$, (2.7) contains an additional term, that leads to a damping behavior for vanishing external momentum,

$$\lim_{p\to 0} G(p) \propto \lim_{p\to 0} \frac{\tilde{p}^2}{a} = 0. \tag{2.8}$$

In order to understand how the above propagator leads to renormalizability, let us investigate the divergence structure in loop calculations [49]. When effecting one-loop calculations, the divergence structure is dominated by large internal momenta, i.e. where the a-dependent part of the propagator is vanishing. As in the naïve model, for the self energy one finds a quadratic UV divergence coming from the planar sector and a quadratically IR divergent term from the non-planar graph of the form $\frac{1}{\tilde{p}^2}$. However, due to the presence of the additional term in the tree-level action (2.5) of the same form, the latter divergence can be absorbed by renormalizing the parameter a [49, 39]. Going over to higher loop insertions of the non-planar tadpole graph, one finds that due to the new propagator the IR-dangerous insertions will be damped. It is this mechanism that leads to the renormalizability of the model, which has been proved generically in the paper of Gurau *et al.* [7] by multiscale analysis.
Compared to the model of Grosse and Wulkenhaar, the main advantage of the $1/p^2$-model is that it does not break translation invariance. Furthermore, a commutative limit mechanism can be written down [50]. A possible drawback is the missing of an analog to the Langmann-Szabo symmetry and hence of asymptotic safety. This may even not be required when extending the model to gauge theories, as at least non abelian gauge theories are asymptotically free even in the commutative limit.
In summary, this represents the reason for the extensive study of a generalization to gauge theories of the $1/p^2$-model.

Chapter 3

A localized translation-invariant gauge model

Several suggestions have been made on how to handle the UV/IR mixing problem in gauge field theories. The one by Slavnov [51, 52] reduces the degrees of freedom of the gauge field through the addition of a constraint, while the extension of the Grosse-Wulkenhaar model (described in Section 2.1) to gauge field theory [47, 46, 53, 54] breaks translation invariance. Motivated by the model presented in Section 2.2 which avoids all of the mentioned drawbacks, in [8] its generalization to a θ-deformed $U_\star(1)$-theory, leading to a damping of the gauge propagator in the IR sector, has been presented. At this point it has to be remarked that a similar term had already been introduced earlier in [55] based on a resummation procedure (however, not leading to the desired damping behavior), and before in [56].
In this chapter we will first discuss the construction of the model, before proceeding to the calculation of the Feynman rules. It follows power counting and one loop calculations, and finally a discussion of the physical implications of the results. Detailed

calculations are given in Section C. Due to the tediousness of the very involving and long loop calculations, algorithms have been developed in order to automatize the work, which are presented in the appendix in Section F.
Here and in the following, the deformation matrix is always given by its canonical form as block-diagonal matrix:

$$\theta_{\mu\nu} = \theta \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \text{where} \quad \dim\theta = -2\,, \tag{3.1}$$

if not mentioned otherwise.

3.1 Construction of the $1/p^2$ $U_\star(1)$ model

As described in Section 2.2, the additional term in the Euclidean momentum space action of the form $a/\tilde{p}^2$ leads to renormalizability of the model while keeping translation invariance. The motivation for this term stems from the fact that the 1-loop-self-energy of the naiv̈e $\phi^{\star 4}$-model is quadratically IR divergent, i.e. is of the same form as the additional term, and thus might be absorbed in the parameter a.
We will now undertake the construction of a translation invariant gauge model along the same line.

3.1.1 The naïve $U_\star(1)$ model and its IR-divergence structure

In order to be able to construct an appropriate counterterm for the IR divergences, let us investigate the IR-divergence structure of the naïve $U_\star(1)$ model, formulated on Euclidian space $\mathbb{R}^4_\theta$. We first have to formulate the model. The gauge invariant action is the same as in the commutative case[1]:

$$S_{\text{inv}} = \int \mathrm{d}^4x \frac{1}{4} F_{\mu\nu} \star F^{\mu\nu}\,, \tag{3.2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g\,[A_\mu \stackrel{\star}{,} A_\nu]\;. \tag{3.3}$$

Due to the star product, the field strength tensor $F_{\mu\nu}$ is now of non-abelian character even in the case of a θ-deformed $U(1)$. The action is invariant under the infinitesimal gauge transformations of the form

$$\delta A_\mu = \partial_\mu \Lambda + \mathrm{i}\,[\Lambda \stackrel{\star}{,} A_\mu] \equiv D_\mu \Lambda(x) \Rightarrow \delta F_{\mu\nu} = \mathrm{i}\,[\Lambda \stackrel{\star}{,} F_{\mu\nu}]\;. \tag{3.4}$$

On the r.h.s. the gauge covariant derivative D_μ has been defined. As is well known, the two-point Green function of (3.2), being the inverse of the operator in the bilinear

[1] In order to exhibit the non-commutative character, let us replace the ordinary product by the star product even in the bilinear case.

part of the action (which we denote by $K_{\mu\nu}$) does not exist. Indeed, it can be written as a projection operator, whereas the only projection operator possessing an inverse is the unity operator $\mathbb{1}_4$ (c.f. Section C.1.1). In order to get an invertible $K_{\mu\nu}$ one adds a gauge fixing term. It can be linear or nonlinear, as well as covariant, e.g.

$$\frac{\delta S_{gf,cov}}{\delta b} = \partial^\mu A_\mu(x) - \alpha b(x) = 0\,, \tag{3.5}$$

or non covariant (i.e. with respect to Lorentz transformations), as in the case of axial gauges (c.f. [57] and references therein):

$$\frac{\delta S_{gf,nc}}{\delta b} = n^\mu A_\mu(x) = 0\,, \tag{3.6}$$

where n_μ is an axial vector, and b denotes the Lagrange multiplier field implementing the gauge fixing; α in (3.5) is a dimensionless parameter, leading to the conventional classification of the gauge fixing into Feynman gauge ($\alpha = 1$), Landau gauge ($\alpha = 0$) and unitary gauge ($\alpha \to \infty$).

Although we will work mostly with covariant gauges, there will be one exception: in Section F we will present an algorithm for the calculation of counterterms in the framework of algebraic renormalization, by the example of an axially gauge fixed action. From now on we refer always to the covariant case, if not mentioned otherwise.

Whereas in ordinary QED or $U(1)$ (its gauge part) we would now be ready for calculations, due to the present Moyality we encounter the same difficulties as in ordinary non-abelian, e.g. Yang-Mills theory. In particular, we observe a coupling between the physical gauge field A_μ and the unphysical field b (c.f. the commutator in (3.8)) which would directly translate into a gauge dependence of physical obersvables. This coupling can be seen on the example of the Ward identity expressing the breaking of gauge invariance applied on the tree level action $\Gamma^{(0)}$. One arrives at the Ward identity by considering the variation of a functional, i.e. $\delta\left(\mathcal{F}[A_\mu]\right) = \int \mathrm{d}^4x \frac{\partial \mathcal{F}}{\partial A_\mu}\delta A_\mu$. Inserting δA_μ given in(3.4), partial integration in $\partial_\mu \Lambda$, and respecting the invariance of the product under cyclic permutations in the integral when considering the commutator[2] allows to write

$$\begin{aligned} \delta\mathcal{F} &= \int \mathrm{d}^4x \mathcal{W}(x)\mathcal{F}[A]\Lambda(x), \qquad \text{with} \\ \mathcal{W}(x) &= -\partial_\mu \frac{\delta}{\delta A_\mu(x)} + \mathrm{i}\left[A_\mu(x), \frac{\delta}{\delta A_\mu(x)}\right]\,. \end{aligned} \tag{3.7}$$

Application to the action

$$\begin{aligned} \Gamma^{(0)} &= S_{inv} + S_{gf}\,, \\ \mathcal{W}(x)\Gamma^{(0)} &= -\Big[\Box b(x) - \mathrm{i}\left[A_\mu(x), \partial^\mu b(x)\right]\Big]. \end{aligned} \tag{3.8}$$

[2]Note that in the case of $U_\star(1)$ cyclic permutativity under the integral is given due to the Weyl-Moyal product, while in commutative $SU(N)$ this is the case due to the matrix character of the fields, hence formally leading to the same WI.

directly exhibits the coupling between the gauge and the auxiliary field. This problem has been overcome by Becchi, Rouet and Stora, and independently by Tyutin who assumed the existence of an enlarged symmetry besides the usual gauge symmetry [58, 59], allowing to decouple the physical from the unphysical sector. We therefore introduce the Faddeev-Popov ($\phi\pi$) ghost and antighost fields c and $\bar{c}$ in the usual way (c.f. e.g. [60]), which finally leads to the following action

$$
\begin{aligned}
\Gamma^{(0)} &= S_{\mathrm{inv}} + S_{\mathrm{gf}} + S_{\phi\pi} \,, \\
S_{\mathrm{inv}} &= \int \mathrm{d}^4x \frac{1}{4} F^{\mu\nu} \star F_{\mu\nu} \,, \\
S_{\mathrm{gf}} + S_{\phi\pi} &= s \int \mathrm{d}^4x \, \bar{c} \star \left[\partial^\mu A_\mu - \frac{\alpha}{2} b \right] = \int \mathrm{d}^4x \left[b \star \partial^\mu A_\mu - \frac{\alpha}{2} b \star b - \bar{c} \star \partial^\mu D_\mu c \right] . \quad (3.9)
\end{aligned}
$$

The action has been constructed in a way to fulfill

$$
s S_{inv} = 0 \qquad \text{and} \qquad s(S_{gf} + S_{\phi\pi}) = 0 \,, \qquad (3.10)
$$

which means invariance with respect to the BRST transformations, which are listed next:

$$
\begin{aligned}
&sA_\mu = D_\mu c \equiv \partial_\mu c - \mathrm{i}g \left[A_\mu \stackrel{\star}{,} c \right] , && s\bar{c} = b, \\
&sc = \mathrm{i}gc \star c, && sb = 0, \\
&s^2 \varphi = 0 \quad \text{for} \quad \varphi \in \{ A_\mu, c, \bar{c}, b \} \,, && \qquad (3.11)
\end{aligned}
$$

their properties being nilpotency, non linearity and increment of the ghost number by one.

Now we want to study the divergence structure of the action (3.9), which has already been done extensively in the literature ([61] for the formulation on Minkowskian space). Due to the fact that the star product does not modify the bilinear part of the action, hence only the interacting part, the propagators remain the same as in the commutative case, while the vertices will carry additional phases. The graphs are the same as for the gauge part of ordinary QCD (including their non-planar counterparts). For the IR sector, this finally leads to a contribution for the vacuum polarization of the form ($\tilde{k}$ are the external momenta)

$$
\Pi^{IR}_{\mu\nu} \propto \frac{\tilde{k}_\mu \tilde{k}_\nu}{(\tilde{k}^2)^2}, \qquad (3.12)
$$

which is independent of the chosen gauge fixing.

We will now try to formulate an additional term for the action which should allow to cure the UV/IR mixing by a damping of the photon propagator. The conditions imposed on such a term are:

- Damping of the gauge propagator for vanishing momenta, which is equivalent to finiteness in the IR sector

- Invariance with respect to the extended gauge transformations, i.e. BRST invariance
- Dimensional consistency

3.1.2 A gauge analogon to the scalar $1/p^2$-model

In analogy to the new term of the model (2.4), we construct the counterterm by multiplying (3.12) by gauge fields in a bilinear way. This leads to

$$\Gamma_{new} = a \int d^4k A_\mu(-k) \frac{\tilde{k}_\mu \tilde{k}_\nu}{(\tilde{k}^2)^2} A_\nu(k). \tag{3.13}$$

Note that the negative argument of the first gauge field propagator stems from the Fourier transform from coordinate to momentum space, i.e.

$$a \int \mathrm{d}^4x A_\mu(x) \frac{\tilde{\partial}_\mu \tilde{\partial}_\nu}{(\tilde{\partial}^2)^2} A_\nu(x) \Longrightarrow -a \int \frac{d^4k_{1,2}}{(2\pi)^2} \delta(k_1 + k_2) A_\mu(k_1) \frac{\tilde{k}_{2\mu} \tilde{k}_{2\nu}}{(\tilde{k}_2^2)^2} A_\nu(k_2)\,. \tag{3.14}$$

As observed in [61], this term is invariant with respect to the gauge transformation $\delta A_\mu(x) = \partial_\mu \Lambda(x)$, which in momentum space yields to $\delta A_\mu = \mathrm{i}k_\mu \Lambda$. This can immediately be seen when taking into consideration the relation $\tilde{k}_\mu k_\mu = \theta_{\mu\nu} k_\nu k_\mu = 0$. However, it is not invariant under the full gauge transformation $\delta A_\mu = D_\mu c$.

3.1.3 A BRST invariant action

A BRST-invariant generalization can be obtained by the following considerations:

- Similar to the gauge invariant part of the original action, the gauge fields should be replaced by the field strength tensors
- Partial derivatives should be replaced by their covariant counterparts
- In order to respect dimensionality, $\theta_{\mu\nu}$ should be inserted where appropriate

This leads to the following additional term in the action, which was already introduced in [55]:

$$\Gamma_{new} = \frac{a}{4} \int \mathrm{d}^4x \widetilde{F} \star \frac{1}{(\widetilde{D}^2)^2} \widetilde{F}. \tag{3.15}$$

where

$$\begin{aligned}
\widetilde{F} &= \theta^{\mu\nu} \widetilde{F}_{\mu\nu}, && \text{with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g \left[A_\mu \stackrel{\star}{,} A_\nu\right], \\
\widetilde{D}^2 &= \widetilde{D}^\mu \star \widetilde{D}_\mu, && \text{with } \widetilde{D}_\mu = \theta_{\mu\nu} D^\nu
\end{aligned} \tag{3.16}$$

from which follows

$$\frac{1}{\widetilde{D}^2}\star\widetilde{F}=\frac{1}{\theta^2 D^2}\star\widetilde{F}\equiv\frac{1}{\theta^2}Y. \tag{3.17}$$

On the r.h.s. we have introduced the abbreviation Y. Although in the above action there appears the inverse of the fourth power of $\widetilde{D}_\mu$, it is sufficient to consider (3.17), because partial integration leads to

$$\Gamma_{new}=\frac{a}{4}\int d^4x\frac{1}{\widetilde{D}^2}\widetilde{F}\star\frac{1}{\widetilde{D}^2}\widetilde{F}\,, \tag{3.18}$$

as shown in Section C.1.2. The expression Y has to be understood as formal power series in A_μ, which has to be determined recursively. First we can write

$$\begin{aligned}
D^2Y&=\partial^\mu(D_\mu Y)-ig\,[A_\mu \stackrel{\star}{,} D_\mu Y]\\
&=\Box Y-\mathrm{i}g\partial^\mu\,[A_\mu \stackrel{\star}{,} Y]-\mathrm{i}g\,[A_\mu \stackrel{\star}{,} \delta_\mu Y]+(\mathrm{i}g)^2\,[A^\mu \stackrel{\star}{,} [A_\mu \stackrel{\star}{,} Y]]\\
\Rightarrow Y&=\frac{1}{\Box}\Big(\widetilde{F}+\mathrm{i}g\partial^\mu\,[A_\mu \stackrel{\star}{,} Y]+\mathrm{i}g\,[A_\mu \stackrel{\star}{,} \delta_\mu Y]-(\mathrm{i}g)^2\,[A^\mu \stackrel{\star}{,} [A_\mu \stackrel{\star}{,} Y]]\Big)\,,
\end{aligned} \tag{3.19}$$

where the last line follows by applying the Green function of the operator $\Box$ to the preceding line. Indeed, it represents a recursive relation for Y:

$$\begin{aligned}
Y^{(0)}&=\frac{1}{\Box}\widetilde{F}\,,\\
Y^{(1)}&=Y^{(0)}+\frac{1}{\Box}\Big(\mathrm{i}g\partial^\mu\left[A_\mu \stackrel{\star}{,} Y^{(0)}\right]+\mathrm{i}g\left[A_\mu \stackrel{\star}{,} \delta_\mu Y^{(0)}\right]-(\mathrm{i}g)^2\left[A^\mu \stackrel{\star}{,}\left[A_\mu \stackrel{\star}{,} Y^{(0)}\right]\right]\Big)\,,
\end{aligned} \tag{3.20}$$

which may be continued up to arbitrary order. Together with the BRST-transformations (3.11) and the relation $s\left(\frac{1}{\widetilde{D}^2}\star F_{\mu\nu}\right)=\mathrm{i}g\left[c \stackrel{\star}{,} \frac{1}{\widetilde{D}^2}F_{\mu\nu}\right]$, as was previously shown in Ref. [8] we can now directly show the BRST invariance of (3.18):

$$\begin{aligned}
s\Gamma_{new}&\propto s\int d^4x\frac{1}{\widetilde{D}^2}\widetilde{F}\star\frac{1}{\widetilde{D}^2}\widetilde{F}=\\
&\int d^4x\left\{\mathrm{i}g\left[c \stackrel{\star}{,} \frac{1}{\widetilde{D}^2}F_{\mu\nu}\right]\star(\frac{1}{\widetilde{D}^2}F_{\mu\nu})+\mathrm{i}g(\frac{1}{\widetilde{D}^2}F_{\mu\nu})\left[c \stackrel{\star}{,} \frac{1}{\widetilde{D}^2}F_{\mu\nu}\right]\right\}\\
&=\int d^4x\left\{c\star\left(\frac{1}{\widetilde{D}^2}F_{\mu\nu}\right)^2-\left(\frac{1}{\widetilde{D}^2}F_{\mu\nu}\right)^2\star c\right\}=0\,.
\end{aligned} \tag{3.21}$$

In the last step the invariance of the star product under cyclic permutations under the integral has been used. However, calculation of the gauge propagator leads to (similar as in [55])

$$G^A_{\mu\nu}(k)=\frac{1}{k^2}\Big(\delta_{\mu\nu}-a\frac{\tilde{k}_\mu\tilde{k}_\nu}{(\tilde{k}^2)^2k^2}\Big), \tag{3.22}$$

which is not damped for vanishing momentum k.

3.1.4 Action with IR-damping for the gauge propagator

In [8], the authors introduced a new model formulated on Euclidean space $\mathbb{R}_4$ which reads

$$\begin{aligned}
\Gamma &= S_{\text{inv}} + S_{\text{gf}}\,,\\
S_{\text{inv}} &= \int \mathrm{d}^4x \left[\frac{1}{4}F^{\mu\nu} \star F_{\mu\nu} + \frac{1}{4}F^{\mu\nu} \star \frac{1}{D^2\widetilde{D}^2} \star F_{\mu\nu}\right]\,,\\
S_{\text{gf}} &= s\int \mathrm{d}^4x\,\bar{c}\star\left[\left(1+\frac{1}{\square\widetilde{\square}}\right)\partial^\mu A_\mu - \frac{\alpha}{2}b\right]\\
&= \int \mathrm{d}^4x\left[b\star\left(1+\frac{1}{\square\widetilde{\square}}\right)\partial^\mu A_\mu - \frac{\alpha}{2}b\star b - \bar{c}\star\left(1+\frac{1}{\square\widetilde{\square}}\right)\partial^\mu D_\mu c\right]. \qquad (3.23)
\end{aligned}$$

Here again, $\frac{1}{D^2\widetilde{D}^2}\star F_{\mu\nu}$ in the gauge invariant part of the action S_{inv} is to be understood as a formal power series in the gauge field A_μ (c.f. previous section). The field strength tensor $F_{\mu\nu}$ has already been defined in (3.3). As introduced above, b is the Lagrange multiplier field implementing the gauge fixing, c and $\bar{c}$ are the ghost and antighost, and α is a dimensionless parameter which determines the kind of gauge fixing. Note that $\left(1+\frac{1}{\square\widetilde{\square}}\right)$ improves the IR behavior in the ghost sector, a term that has already been introduced in [55].
The new expression in the gauge invariant action is indeed BRST invariant, fulfils dimensional consistency conditions, and leads to the gauge field propagator

$$G^{AA}_{\mu\nu}(k) = \frac{1}{\left(k^2+\frac{1}{\tilde{k}^2}\right)}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \alpha\frac{k_\mu k_\nu}{\left(k^2+\frac{1}{\tilde{k}^2}\right)}\right)\,, \qquad (3.24)$$

which shows the desired damping behavior for $k \to 0$. We almost reached the target that we imposed to ourselves at the beginning of the chapter. However, this model incorporates a drawback arising from the fact that the series expansion of the $\frac{1}{D^2\widetilde{D}^2}\star F_{\mu\nu}$ term introduces an infinite number of gauge boson vertices. This can be seen by looking at the recursive relation in (3.20): a term up to order n will contain up to $n+1$ fields, and as consequence a corresponding vertex with the same number of legs. Hence it is not possible to compute a finite sum of contributions in a given loop order. Furthermore, any approximation to the infinite series $\frac{1}{D^2\widetilde{D}^2}\star F_{\mu\nu}$ with just a finite number of terms cannot be gauge invariant, and therefore any explicitly calculated (physical) quantities using such an approximation will fail to be gauge invariant as well.
Fortunately, there is a way out, which will be described in the next section.

3.2 Localization of the translation invariant gauge model

In [1], we succeeded in localizing the model (3.23) by the introduction of a new (dynamic) field $B_{\mu\nu}$. As in the corresponding scalar model a new parameter a' has been introduced,

which is expected to play an essential role in the renormalization procedure, being the reason it has been introduced in the action.
Let us start by writing down the Euclidean action (3.23) with the additional parameter a':

$$
\begin{aligned}
\Gamma^{(0)} &= S_{\text{inv}} + S_{\text{gf}}\,,\\
S_{\text{inv}} &= \int \mathrm{d}^4x \left[\frac{1}{4}F^{\mu\nu}\star F_{\mu\nu} + \frac{1}{4}F^{\mu\nu}\star\frac{a'^2}{D^2\widetilde{D}^2}\star F_{\mu\nu}\right],\\
S_{\text{gf}} &= s\int \mathrm{d}^4x\,\bar{c}\star\left[\left(1+\frac{a'^2}{\Box\widetilde{\Box}}\right)\partial^\mu A_\mu - \frac{\alpha}{2}b\right]\\
&= \int \mathrm{d}^4x\left[b\star\left(1+\frac{a'^2}{\Box\widetilde{\Box}}\right)\partial^\mu A_\mu - \frac{\alpha}{2}b\star b - \bar{c}\star\left(1+\frac{a'^2}{\Box\widetilde{\Box}}\right)\partial^\mu D_\mu c\right].
\end{aligned}
\tag{3.25}
$$

The fields are defined as in section Section 3.1.4. The gauge fixed action (3.25) is invariant under the BRST transformations (3.11). As can be seen in (3.25), the gauge fixing part S_{gf} is BRST exact. Moreover, it follows that $s\left(\frac{1}{\widetilde{D}^2}\star F_{\mu\nu}\right) = \mathrm{i}g\left[c \stackrel{\star}{,} \frac{1}{\widetilde{D}^2}F_{\mu\nu}\right]$, as was previously shown in Ref. [8].
The issue of the appearance of an infinite number of vertices still remains. However, this can be circumvented by the introduction of a new antisymmetric field $B_{\mu\nu}$ of mass dimension two, as will be shown in what follows. Let us start with

Lemma 1 *Auxiliary fields allow to linearize terms of the action which are quadratic in the dynamic fields.*

Consider e.g. the gauge fixing term $(\partial A)^2/2\alpha$. Introduction of the auxiliary field b leads to the equivalent linearized expression of our model. This can be summarized as follows:

$$
\mathcal{L}_{aux} = \frac{(\partial A)^2}{2\alpha} \equiv b(\partial A) - \frac{\alpha}{2}b^2, \qquad \text{with } b = \frac{\partial A}{\alpha}.
$$

In a similar way, we obtain for the action term under consideration, after partial integration and extracting θ, the expression

$$
S^{(2)}_{\text{inv}} = \int \mathrm{d}^4x \frac{1}{2}\left(\frac{2\theta^2}{a'^2}\right)^{-1}\left(\frac{1}{D^2}F_{\mu\nu}\right)^2, \tag{3.26}
$$

which has been written in analogy to our foregoing example. In the same way we now introduce the new field

$$
B_{\mu\nu} = \left(\frac{2\theta^2}{a'^2}\right)^{-1}\left(\frac{1}{D^2}F_{\mu\nu}\right)
$$

and obtain for the action term

$$
S^{(2)}_{\text{inv}} = \int \mathrm{d}^4x B_{\mu\nu}\left(\frac{1}{D^2}F_{\mu\nu}\right) - \frac{\theta^2}{a'^2}B^2_{\mu\nu}\,.
$$

The redefinition $B_{\mu\nu} \Rightarrow D^2 B_{\mu\nu}$ allows to finally eliminate the inverse of covariant derivatives:

$$S^{(2)}_{\text{inv}} = \int \mathrm{d}^4x B_{\mu\nu} \star F_{\mu\nu} - B_{\mu\nu} \star \frac{\theta^2 (D^2)^2}{a'^2} B_{\mu\nu} \,.$$

In a last step we again conduct a redefinition by rescaling $B_{\mu\nu} \Rightarrow a' B_{\mu\nu}$ in order to pull the to-be-renormalized constant a' to the numerator. With $\theta^2 D^2 = \widetilde{D}^2$ the respective term in the action replacing the infinite power series finally reads

$$S^{(2)}_{\text{inv}} = \int \mathrm{d}^4x \left[a' B_{\mu\nu} \star F_{\mu\nu} - B_{\mu\nu} \star \widetilde{D}^2 D^2 \star B_{\mu\nu} \right] \,, \tag{3.27}$$

and gauge invariance is given if $B_{\mu\nu}$ transforms covariantly, i.e.

$$s B_{\mu\nu} = \mathrm{i}g \left[c \stackrel{\star}{,} B_{\mu\nu} \right] . \tag{3.28}$$

The equivalence between the two formulations of the new term in the action can be seen by reinserting its equation of motion

$$\frac{\delta S^{(2)}_{\text{inv}}}{\delta B_{\rho\sigma}} = a' F_{\rho\sigma} - 2 \widetilde{D}^2 D^2 \star B_{\rho\sigma} = 0 \tag{3.29}$$

into (3.27). This leads to

$$\begin{aligned} S^{(2)}_{\text{inv}} &= \int \mathrm{d}^4x \left[\left(\frac{a'^2}{2 D^2 \widetilde{D}^2} \star F_{\mu\nu} \right) \star F_{\mu\nu} - \left(\frac{a'^2}{2 D^2 \widetilde{D}^2} \star F_{\mu\nu} \right) \star \frac{1}{2} F_{\mu\nu} \right] \\ &= \int \mathrm{d}^4x \left[\frac{1}{4} F_{\mu\nu} \star \frac{a'^2}{D^2 \widetilde{D}^2} \star F_{\mu\nu} \right] . \end{aligned} \tag{3.30}$$

Note that the second line of (3.30) vanishes in the limit $a' \to 0$ while in the "improved" formulation given in Eqn. (3.27) the second term does not. Hence, the newly introduced $B_{\mu\nu}$ is a dynamic field whose existence is independent from the value of the parameter a', in contrast to the auxiliary gauge fixing field b, which by definition does not propagate. We will come back to this point in Section D.

3.3 Feynman rules

Without the Lagrange multiplier field b, the model gives rise to four propagators and eight vertices. The main results will be given in this section, while detailed calculations can be found in the appendix in Section C.2.

Equations of motion

We follow the usual approach for the calculation of the equations of motion by defining first the generating functional for the connected Green functions:

$$Z^c[j^\phi] = S\left[\phi[j^\phi]\right] + \int \mathrm{d}^4x j^\phi \phi[j^\phi] \tag{3.31}$$

with $\phi = \{A, B, b, c, \bar{c}\}$ and the external sources $j^\phi = \{j^A_\mu, j^B_{\mu\nu}, j^b, j^c, j^{\bar{c}}\}$. This leads to the following equations of motion

$$\frac{\delta S_{\mathrm{bil}}}{\delta A^\nu} = -\left(\Box\delta_{\nu\mu} - \partial_\nu\partial_\mu\right)A^\mu - 2a'\partial^\mu B_{\mu\nu} - \left(1 + \frac{a'^2}{\Box\widetilde{\Box}}\right)\partial_\nu b = -j^A_\nu\,, \tag{3.32a}$$

$$\frac{\delta S_{\mathrm{bil}}}{\delta B^{\mu\nu}} = a'\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - 2\widetilde{\Box}\Box B_{\mu\nu} = -j^B_{\mu\nu}\,, \tag{3.32b}$$

$$\frac{\delta S_{\mathrm{bil}}}{\delta b} = \left(1 + \frac{a'^2}{\Box\widetilde{\Box}}\right)\partial^\mu A_\mu - \alpha b = -j^b\,, \tag{3.32c}$$

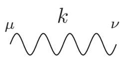
$$\frac{\delta S_{\mathrm{bil}}}{\delta \bar{c}} = -\left(\Box + \frac{a'^2}{\widetilde{\Box}}\right)c = j^{\bar{c}}\,. \tag{3.32d}$$

Propagators

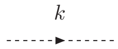
One arrives at the tree level propagators by expressing in (3.32) each field as functional of external sources and variation with respect to the latter, i.e.

$$G^{ba} = \langle 0|\phi_a(x_\mu)\phi_b(x_\nu)|0\rangle_{(0)} = -\frac{\delta\phi_a(x_\mu)}{\delta j^b(x_\nu)}\,. \tag{3.33}$$

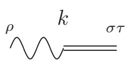
The generalized index a denotes the species of the fields as well as Lorentz indices, and the positions in Euclidean space are denoted by x_μ. For obvious reasons, with respect to the corresponding expression in Minkowskian space there is no time ordering in the expectation value of the product of fields. With the definitions $\tilde{k}^2 = \theta^2 k^2$ and $a'^2 = \theta^2 a^2 \theta^2$, the propagators are given by the following expressions:



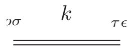
$$= G^{AA}_{\mu\nu}(k) = \frac{1}{\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \alpha\frac{k_\mu k_\nu}{\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)}\right)\,, \tag{3.34a}$$



$$= G^{\bar{c}c}(k) = \frac{-1}{\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)}\,, \tag{3.34b}$$



$$= G^{AB}_{\rho,\sigma\tau}(k) = \frac{-\mathrm{i}\,a'}{2}\frac{\left(k_\tau\delta_{\rho\sigma} - k_\sigma\delta_{\rho\tau}\right)}{k^2\tilde{k}^2\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)}, \tag{3.34c}$$



$$= G^{BB}_{\rho\sigma,\tau\epsilon}(k)$$
$$= \frac{-1}{4k^2\tilde{k}^2}\left[\delta_{\rho\tau}\delta_{\sigma\epsilon} - \delta_{\rho\epsilon}\delta_{\sigma\tau} + a'^2\frac{k_\sigma k_\tau \delta_{\rho\epsilon} + k_\rho k_\epsilon \delta_{\sigma\tau} - k_\sigma k_\epsilon \delta_{\rho\tau} - k_\rho k_\tau \delta_{\sigma\epsilon}}{k^2\tilde{k}^2\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)}\right]. \tag{3.34d}$$

The latter two are antisymmetric in the index pairs corresponding to the $B_{\mu\nu}$ fields, i.e.

$$G^{AB}_{\rho,\sigma\tau}(k) = -G^{AB}_{\rho,\tau\sigma}(k) = -G^{BA}_{\sigma\tau,\rho}(k)\,,$$
$$G^{BB}_{\rho\sigma,\tau\epsilon}(k) = -G^{BB}_{\sigma\rho,\tau\epsilon}(k) = -G^{BB}_{\rho\sigma,\epsilon\tau}(k)\,. \tag{3.35}$$

As for the non local model, the gauge and ghost propagators go to zero for vanishing momentum and non-vanishing a'. This can also be seen from Fig. 3.1 (where $a'^2/\tilde{k}^2$ has been replaced by a^2/k^2), which shows the qualitative behavior of the damping term with the maximum of $1/2a$ at positions $\pm\sqrt{a}$.

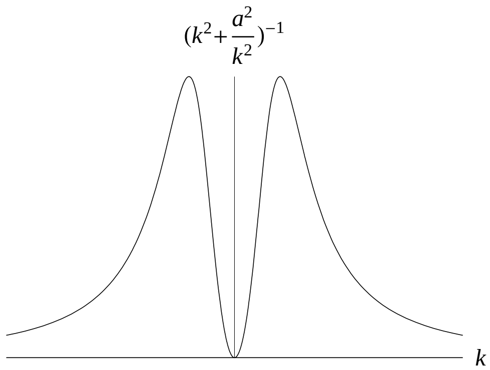


Figure 3.1: Qualitative representation of the damping for the gauge propagator.

Notice furthermore the relations

$$2k^2\tilde{k}^2 G^{AB}_{\rho,\mu\nu}(k) = \mathrm{i}a'k_\mu G^{AA}_{\rho\nu}(k) - \mathrm{i}a'k_\nu G^{AA}_{\rho\mu}(k),$$
$$2k^2\tilde{k}^2 G^{BB}_{\mu\nu,\rho\sigma}(k) = \frac{1}{2}\left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right) + \mathrm{i}a'k_\mu G^{BA}_{\rho\sigma,\nu}(k) - \mathrm{i}a'k_\nu G^{BA}_{\rho\sigma,\mu}(k), \tag{3.36}$$

which follow from the equations of motion for $B_{\mu\nu}$ Eqn. (3.32b), c.f. Section C.2.1 for a detailed calculation.

Vertices

The interaction part of the action is given by

$$S_{inv,int} = S^{(1)}_{inv,int} + S^{(2)}_{inv,int} + S_{gf,int}\,,$$

where

$$
\begin{aligned}
S^{(1)}_{inv,int} &= \frac{1}{4}\int \mathrm{d}^4x\Big\{ -2\mathrm{i}g\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\star\left[A_\mu \stackrel{\star}{,} A_\nu\right] + (\mathrm{i}g)^2\left[A_\mu \stackrel{\star}{,} A_\nu\right]^2\Big\}\,,\\
S^{(2)}_{inv,int} &= \int \mathrm{d}^4x\Big\{\mathrm{i}a'gB_{\mu\nu}\left[A_\mu \stackrel{\star}{,} A_\nu\right]\\
&\quad - 2\mathrm{i}g\theta^2\Big(\Box B_{\mu\nu}\star\partial_\mu\left[A_\mu \stackrel{\star}{,} B_{\mu\nu}\right] + \Box B_{\mu\nu}\star\left[A_\mu \stackrel{\star}{,} \partial_\mu B_{\mu\nu}\right]\Big)\\
&\quad + (\mathrm{i}g)^2\theta^2\Big(\Box B_{\mu\nu}\left[A_\mu \stackrel{\star}{,}\left[A_\mu \stackrel{\star}{,} B_{\mu\nu}\right]\right] + \left(\partial_\mu\left[A_\mu \stackrel{\star}{,} B_{\mu\nu}\right]\right)^2\\
&\qquad\quad + 2\partial_\mu\left[A_\mu \stackrel{\star}{,} B_{\mu\nu}\right]\left[A_\mu \stackrel{\star}{,} \partial_\mu B_{\mu\nu}\right] + \left[A_\mu \stackrel{\star}{,} \partial_\mu B_{\mu\nu}\right]^2\Big)\\
&\quad - (\mathrm{i}g)^3\theta^2\Big(\partial_\mu\left[A_\mu \stackrel{\star}{,} B_{\mu\nu}\right]\left[A_\mu \stackrel{\star}{,}\left[A_\mu \stackrel{\star}{,} B_{\mu\nu}\right]\right] + \left[A_\mu \stackrel{\star}{,} \partial_\mu B_{\mu\nu}\right]\left[A_\mu \stackrel{\star}{,}\left[A_\mu \stackrel{\star}{,} B_{\mu\nu}\right]\right]\Big)\\
&\quad + (\mathrm{i}g)^4\theta^2\Big(\left[A_\mu \stackrel{\star}{,}\left[A_\mu \stackrel{\star}{,} B_{\mu\nu}\right]\right]^2\Big)\Big\}\\
S_{gf,int} &= \mathrm{i}g\int \mathrm{d}^4x\,\bar{c}\Big(1 + \frac{a'^2}{\Box\widetilde{\Box}}\Big)\partial_\mu\left[A_\mu \stackrel{\star}{,} c\right]
\end{aligned}
\tag{3.37}
$$

where $S_{gf,int}$ is the gauge fixing interaction part of (3.25), and the gauge invariant interaction has been split into the $B_{\mu\nu}$-independent part $S^{(1)}_{inv,int}$ and $B_{\mu\nu}$-dependent part $S^{(2)}_{inv,int}$ (c.f. (3.27)). Writing the action with respect to the interacting fields one gets, with obvious notation,

$$
S_{int} = S^{3A} + S^{4A} + S^{BAA} + S^{BBA} + S^{2B2A} + S^{2B3A} + S^{2B4A} + S^{\bar{c}Ac}\,,
\tag{3.38}
$$

which leads to eight different vertices. Note that in addition to the usual ones describing interactions between (anti)ghost and gauge bosons, one now also has interactions of gauge bosons with $B_{\mu\nu}$ fields giving rise to numerous new Feynman diagrams, c.f. Section 3.5.

Their explicit calculation is done by first performing a Fourier transform of the interaction under consideration:

$$
\begin{aligned}
S^{\phi_1,\dots,\phi_n}_{int}(x) &\to S^{\phi_1,\dots,\phi_n}_{int}(k)\,,\\
S^{\phi_1,\dots,\phi_n}_{int}(k) &= \int \mathrm{d}^4x \int \frac{d^nq}{(2\pi)^n} e^{\mathrm{i}\sum_{i=1}^n k_i^\mu x_\mu}\mathcal{L}\left[\tilde{\phi}_1(k_1),\dots,\tilde{\phi}_n(k_n)\right] e^{-\frac{\mathrm{i}}{2}\sum_{i<j}k_i\times k_j}\\
&= \int \frac{d^nq}{(2\pi)^n}\delta\left(\sum_{i=1}^n k_i\right)(2\pi)^4\mathcal{L}\left[\tilde{\phi}_1(k_1),\dots,\tilde{\phi}_n(k_n)\right] e^{-\frac{\mathrm{i}}{2}\sum_{i<j}k_i\times k_j}\,,
\end{aligned}
\tag{3.39}
$$

with $\phi_i \in \{A_\mu, B_{\mu\nu}, c, \bar{c}\}$[3]. In the last equation we observe the additional phase factor due to non-commutativity. Now functional variation can be performed, leading to

$$
\tilde{V}^{\phi_1,\dots,\phi_n}(k_1,\dots,k_n) = -(2\pi)^{4n}\frac{\delta}{\delta\phi_1(-k_1)}\cdots\frac{\delta}{\delta\phi_n(-k_n)}S^{\phi_1,\dots,\phi_n}_{int}(k_1,\dots k_n)\,.
\tag{3.40}
$$

The vertices give rather lengthy expressions; thus, they are listed in Section C.2.2. There we will also give one explicit example for the application of (3.40).

[3]Note that in (3.39) the non-commutative phase factor has been written explicitly, and that the interaction Lagrangian is given via the usual product between the fields.

3.4 Power Counting

In order to determine the superficial degree of (ultraviolet) divergence of an arbitrary Feynman graph of the present model, we take into account the powers of internal momenta k each Feynman rule contributes and also that each loop integral over 4-dimensional space increases the degree by 4. For example, the gauge boson propagator behaves like $1/k^2$ for large k and therefore reduces the degree of divergence by 2, whereas each ghost vertex (cf. Eqn. (C.25c) in Section C.2.2) contributes one power of k to the numerator of a graph, hence increasing the degree by one. Continuing these considerations for all other Feynman rules we arrive at

$$d_\gamma = 4L - 2I_A - 2I_c - 5I_{AB} - 4I_{BB} + V_c + V_{3A} + 3V_{BBA} + 2V_{2B2A} + V_{2B3A}\,, \tag{3.41}$$

where the I and V denote the number of the various types of internal lines and vertices, respectively (see Eqn. (3.34) and Appendix C.2.2). The number of loop integrals L is given by

$$\begin{aligned} L = {} & I_A + I_c + I_{AB} + I_{BB} - \\ & - (V_c + V_{3A} + V_{4A} + V_{BAA} + V_{BBA} + V_{2B2A} + V_{2B3A} + V_{2B4A} - 1)\,. \end{aligned}$$

Furthermore, we take into account the relations

$$\begin{aligned} E_{c/\bar{c}} + 2I_c &= 2V_c\,, && \text{(3.42a)}\\ E_A + 2I_A + I_{AB} &= V_c + 3V_{3A} + 4V_{4A} + 2V_{BAA} + V_{BBA} \\ &\quad + 2V_{2B2A} + 3V_{2B3A} + 4V_{2B4A}\,, && \text{(3.42b)}\\ E_B + 2I_{BB} + I_{AB} &= V_{BAA} + 2V_{BBA} + 2V_{2B2A} + 2V_{2B3A} + 2V_{2B4A}\,, && \text{(3.42c)}\\ E_\theta &= 2I_{AB} + 2I_{BB} - 2V_{BBA} - 2V_{2B2A} - 2V_{2B3A} - 2V_{2B4A}\,, && \text{(3.42d)}\\ E_{a'} &= I_{AB} + V_{BAA}\,, && \text{(3.42e)} \end{aligned}$$

between the various Feynman rules describing how they (and how many) can be connected to one another. The $E_{c/\bar{c}}$, E_A and E_B denote the number of external lines of the respective fields. Their only possible numerical prefactor is one, as they will connect only on one side to other fields. For the internal lines and vertices, the prefactor equals their number of possibilities to connect to the field under consideration. E.g. (3.42b) considers the possible connections for the A_μ field. The E_θ and $E_{a'}$ count the negative powers of θ and positive powers of a' in a graph, respectively. Using these relations one can eliminate all internal lines and vertices from the power counting formula. From the last three lines of Eqn. (3.42) it follows that $E_B + E_\theta = E_{a'}$, and therefore we find two alternative expressions for the power counting, reading

$$d_\gamma = 4 - E_A - E_{c/\bar{c}} - 2E_B - 2E_\theta\,, \tag{3.43a}$$

$$d_\gamma = 4 - E_A - E_{c/\bar{c}} - 2E_{a'}\,. \tag{3.43b}$$

In Eqn. (3.43a) the superficial degree of divergence is reduced by the number of external legs weighted by the dimension of the respective fields (and parameters). However, this formula can be misleading, since E_θ may also become negative in some graphs. Therefore, in practice, it is more convenient to use Eqn. (3.43b).

3.5 One loop calculations

3.5.1 A remark on UV/IR mixing in the present model

Before analyzing the one loop graphs in detail, let us consider the integrals to be expected for the one- and two-point functions (graphs with one respectively two external legs). The one-point functions will be composed by a 3-vertex and one propagator. A look at the vertices in Section C.2.2 reveals that each of them contains one sine, i.e. we encounter a purely non-planar structure, and using the decomposition integrals for the sine given in Section A.1.4 the integrals are of the form

$$\int d^4k \frac{e^{\mathrm{i}k\tilde{p}}}{\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)} \,. \tag{3.44}$$

We observe the following:

- Superficially, i.e. by naïve power counting, one would expect this integral to diverge quadratically necessitating the introduction of a UV cutoff. However, it is regularized by the phase and no cutoff is needed. This is the usual mechanism leading to UV/IR mixing.
- For dimensional reasons the above integral is expected to be $\propto \frac{1}{\tilde{p}^2}$, which is an IR divergent expression. The explicit calculation of the integral can be found in Section A.2.2, and confirms the expected bevahiour, which has already been found in scalar theories [49, 7].
- However, insertions of such an IR divergent expression into higher loop graphs are regularized by the IR damping behavior of additional propagators (see also ref. [49]). This is the reason why the present gauge model is expected to remedy the UV/IR mixing problem.

Considering the two point functions, we observe that the involved vertices (one 4-vertex or two 3-vertices) lead to a squared sine in the momenta. Applying the corresponding decomposition formulae this leads to

$$\sin^2\left(\frac{k\tilde{p}}{2}\right) = \frac{1}{2}\left(1 - \cos\left(k\tilde{p}\right)\right) = \frac{1}{2} - \frac{1}{4}\sum_{\eta=\pm 1} e^{\mathrm{i}\eta k\tilde{p}} \,. \tag{3.45}$$

Here one directly can see that one gets planar (phase independent)[4] and non-planar (phase dependent) contributions, which reflects the UV/IR mixing of non-commutative quantum field theories.

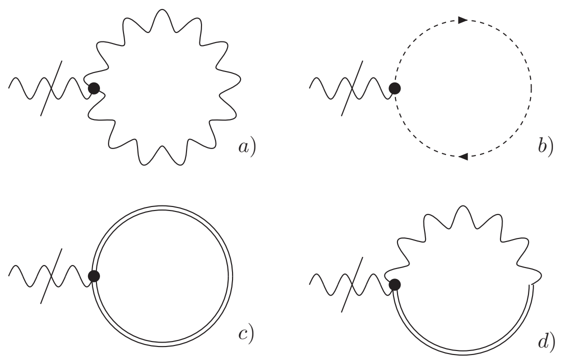


Figure 3.2: One-loop tadpole graphs

3.5.2 Vanishing Tadpole Graphs

From the Feynman rules Eqn. (3.34) and Eqn. (C.25) arise four possible one-loop tadpole graphs with one external gauge boson line. These are depicted in Figure 3.2. A short look at the relevant vertices in Section C.2.2 shows that for each of them the integrand incorporates a factor of the form

$$\delta^4(p+k-k)\sin\left(\frac{k\tilde{p}}{2}\right), \tag{3.46}$$

p and k being the external and internal momenta, respectively. Note that due to the oscillating phase one only encounters non-planar one-loop tadpole graphs. From momentum conservation at the vertices (expressed by the δ-functional) follows $p=0$ leading to a vanishing sine-function. Hence, all four graphs vanish.

3.5.3 Bosonic Vacuum Polarization

The present model gives rise to twelve 1PI one-loop graphs with external boson lines. These are collected in Fig. 3.3 where the first three graphs are already known to appear in theories like QCD. Due to the corresponding lengthy expressions they are given in Section C.3. Computing the respective integrals according to the Feynman rules Eqn. (3.34) and Eqn. (C.25) one encounters expressions of the form

$$\Pi_{\mu\nu} = \int d^4k\,\mathcal{I}_{\mu\nu}(k,p)\sin^2\left(\frac{k\tilde{p}}{2}\right),$$

where $\mathcal{I}_{\mu\nu}$ is a function in the external and internal momenta p and k, respectively. Details are given in Appendix C.3.

[4]Note that also planar expressions can be accompanied by a phase, however it would depend on external momenta only.

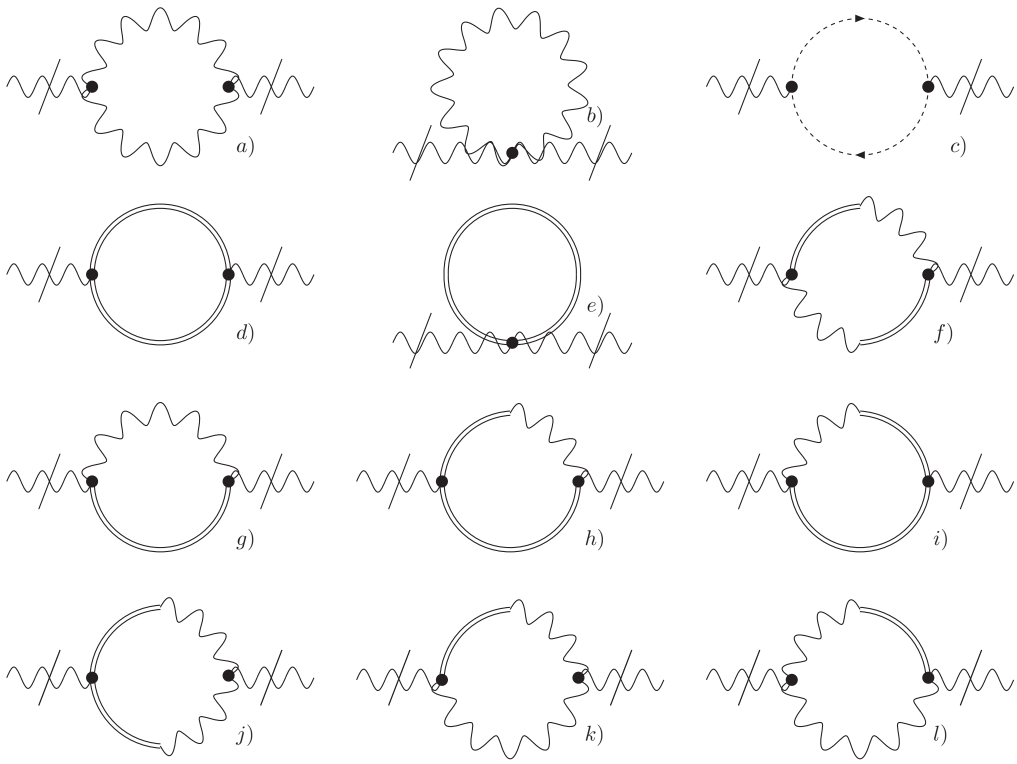


Figure 3.3: Summary of contributions to the one-loop boson vacuum polarization

Taylor expansion

Before proceeding to the evaluation of the integrals let us first note, that we are mainly interested in the IR behavior of the theory, i.e. for vanishing external momenta. Hence it is natural to consider the expansion of the single-graph results around $p \to 0$ according to the rule[5]

$$\begin{aligned}\Pi_{\mu\nu} = \int d^4k\, \mathcal{I}_{\mu\nu}(p,k) \sin^2\left(\tfrac{k\tilde{p}}{2}\right) \approx \int d^4k \sin^2\left(\tfrac{k\tilde{p}}{2}\right) \Bigg\{ &\mathcal{I}_{\mu\nu}(0,k) + p_\rho \left[\partial_{p_\rho}\mathcal{I}_{\mu\nu}(p,k)\right]_{p=0} \\ &+ \frac{p_\rho p_\sigma}{2}\left[\partial_{p_\rho}\partial_{p_\sigma}\mathcal{I}_{\mu\nu}(p,k)\right]_{p=0} + \mathcal{O}\left(p^3\right) \Bigg\} = \\ &= \sum_{j=0,1,2} \Pi^{(j)}_{\mu\nu} + \mathcal{O}\left(p^3\right). \end{aligned} \tag{3.47}$$

Here the integrand $\mathcal{I}_{\mu\nu}(p,k)$ has been separated from the phase factor in order to keep the regularizing effects in the non-planar parts due to rapid oscillations for large k. Note

[5]Note, that the phase is not expanded in order to benefit from the regularizing effects in the non-planar parts due to rapid oscillations for large k.

that the expansion requires $p^2 \ll k^2$. Yet, this condition is not fulfilled for the integration range about $k \approx 0$. The resulting error can be ignored if it is finite. This is always the case for the non-planar graphs due to the regulating effect of the phase factor. For the planar graphs this is the case as long as the power counting in the *internal* momenta shows a mass dimension > -4. A closer look to the expression in Section C.3 now shows the following:

- The first term in the expression for the graph e) shows a logarithmic behavior for vanishing internal momenta, i.e. mass dimension $= -4$ in k.
- As will be shown explicitly below, the second order expansion of the divergent graphs will lead to logarithmic divergences for both large and small k. Indeed this can be understood also in an intuitive way by considering the fact that the maximum UV divergence of the unexpanded expressions is of second order. Hence, differentiating twice will lead to at most logarithmic divergences.

In summary, for the expanded as well as unexpanded case we will encounter divergences, demanding the introduction of a regulator mass. In addition, the evaluation of the unexpanded expressions has the disadvantage of being rather complicated. We will hence apply the expansion (3.47) and introduce a regulator mass μ for the expressions in second order.

Evaluation to lowest order

To lowest order in the expansion (3.47) only five of the graphs depicted in Fig. 3.3, namely graphs $a-e$, are found to diverge superficially and read

$$\Pi^{(0,\mathrm{a})}_{\mu\nu} \approx s_a \frac{8g^2}{(2\pi)^4} \int d^4k \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right)}{\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)^2} \left\{ 6k_\mu k_\nu + \alpha\, k^2 \frac{\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right)}{\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)} \right\}, \tag{3.48a}$$

$$\Pi^{(0,\mathrm{b})}_{\mu\nu} \approx -s_b \frac{8g^2}{(2\pi)^4} \int d^4k \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{1}{k^2 + \frac{a'^2}{\tilde{k}^2}} \left[2\delta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} + \alpha \frac{\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right)}{\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)} \right], \tag{3.48b}$$

$$\Pi^{(0,\mathrm{c})}_{\mu\nu} \approx -s_c \frac{4g^2}{(2\pi)^4} \int d^4k \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{k_\mu k_\nu}{k^4}, \tag{3.48c}$$

$$\Pi^{(0,\mathrm{d})}_{\mu\nu} \approx s_d \frac{192g^2}{(2\pi)^4} \int d^4k \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{k_\mu k_\nu}{k^4} \left(2 + \frac{a'^2\left(\frac{a'^2}{\tilde{k}^2} - 2\right)}{\tilde{k}^2\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)} \right), \tag{3.48d}$$

$$\Pi^{(0,\mathrm{e})}_{\mu\nu} \approx -s_e \frac{24g^2}{(2\pi)^4} \int \mathrm{d}^4k \frac{\sin^2(\frac{k\tilde{p}}{2})}{k^4} \left[4k_\mu k_\nu + 2k^2\delta_{\mu\nu}\right] \left(2 - \frac{a'^2}{\tilde{k}^2\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)} \right), \tag{3.48e}$$

Table 3.1: Symmetry factors for the graphs in Fig. 3.3

$s_a=\frac{1}{2}$	$s_b=\frac{1}{2}$	$s_c=1$
$s_d=\frac{1}{2}$	$s_e=\frac{1}{2}$	$s_f=1$
$s_g=1$	$s_h=1$	$s_i=1$
$s_j=\frac{1}{2}$	$s_k=1$	$s_l=1$

where the symmetry factors are listed in Table 3.1. For a detailed explanation of the calculation of the symmetry factors see Section B. The other graphs (graphs $f)-l)$ of Fig. 3.3) are found to be finite. This observation is consistent with the power counting formula (3.43), since graphs $g)-l)$ come with two overall powers of a', i.e. $E_a = 2$, and graph f even has 4 powers of a', i.e. $E_a = 4$. After summing up all these contributions (and neglecting the finite terms) one arrives at the following expression for the quadratic IR divergence:

$$
\begin{aligned}
\Pi^{(0)}_{\mu\nu} &= \sum_j \Pi^{(0,\mathrm{j})}_{\mu\nu} \\
&= \frac{4g^2}{(2\pi)^4} \int d^4k \sin^2\left(\frac{k\tilde{p}}{2}\right) \left[\frac{-1}{\left(k^2+\frac{a'^2}{\tilde{k}^2}\right)} \left(2\delta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} + \alpha \frac{(k^2\delta_{\mu\nu}-k_\mu k_\nu)}{\left(k^2+\frac{a'^2}{\tilde{k}^2}\right)} \right) - \frac{k_\mu k_\nu}{(k^2)^2} \right. \\
&\qquad + \frac{1}{\left(\left(k^2+\frac{a'^2}{\tilde{k}^2}\right)\right)^2} \left(6k_\mu k_\nu + \alpha k^2 \frac{(k^2\delta_{\mu\nu}-k_\mu k_\nu)}{\left(k^2+\frac{a'^2}{\tilde{k}^2}\right)} \right) \\
&\qquad \left. + \frac{12}{k^2}\left(2\frac{k_\mu k_\nu}{k^2} - \delta_{\mu\nu}\right) \right] \\
&= 14\frac{g^2}{\pi^2}\frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2} + \text{finite terms}. \qquad (3.49)
\end{aligned}
$$

From this result, we observe the following:

- The term is quadratically divergent in the IR sector. Hence the zero order terms in Eqn. (3.49) *do not* produce logarithmic IR divergences.
- The expected transversality with respect to p_μ is given. Furthermore, the contribution from graphs $d)$ and $e)$ in Fig. 3.3 is transversal by itself (as is the contribution from the other three graphs). This was expected, as the graphs $a)$ to $c)$ are known to be transveral from QCD.
- The result is independent from the gauge parameter α. In the limit $a' \to 0$, the α-dependent terms even drop out before integrating over k.
- a' does not actually play a role in one-loop UV/IR mixing since that effect is

dominated by large k for which

$$\frac{1}{\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)} \approx \frac{1}{k^2}.$$

Therefore it is not surprising that the result for the quadratic IR divergence of (3.49) does not depend on a'.

Second order evaluation

Proceeding to the next order in the expansion (3.47), it has to be noted that the first order terms vanish due to the symmetric integration of an odd power in k. Thus it remains to calculate the explicit expressions for the second order terms of the graphs depicted in Fig. 3.3. With gauge fixing parameter $\alpha = 1$ they read:

$$\left(\frac{\partial^2}{\partial p_\rho p_\sigma} \mathcal{I}^{(\mathrm{a})}_{\mu\nu}\right)_{p=0} \frac{p_\rho p_\sigma}{2} = s_a \frac{4g^2}{(2\pi)^4} \frac{1}{k^4} \left\{ 10 \frac{k_\mu k_\nu}{k^2} \left(4 \frac{(kp)^2}{k^2} - p^2\right) + \delta_{\mu\nu} \left(4 \frac{(kp)^2}{k^2} + 3p^2\right) \right. \\ \left. - 2p_\mu p_\nu - 10 \frac{(kp)}{k^2} (k_\mu p_\nu + p_\mu k_\nu) + \text{finite terms} \right\}, \tag{3.50a}$$

$$\left(\frac{\partial^2}{\partial p_\rho p_\sigma} \mathcal{I}^{(\mathrm{b})}_{\mu\nu}\right)_{p=0} \frac{p_\rho p_\sigma}{2} = 0\,, \tag{3.50b}$$

$$\left(\frac{\partial^2}{\partial p_\rho p_\sigma} \mathcal{I}^{(\mathrm{c})}_{\mu\nu}\right)_{p=0} \frac{p_\rho p_\sigma}{2} = s_c \frac{4g^2}{(2\pi)^4} \left\{ k_\mu k_\nu \left(\frac{p^2}{k^6} - 4 \frac{(kp)^2}{k^8}\right) + 2p_\mu k_\nu \frac{kp}{k^6} \right\}, \tag{3.50c}$$

$$\left(\frac{\partial^2}{\partial p_\rho p_\sigma} \mathcal{I}^{(\mathrm{d})}_{\mu\nu}\right)_{p=0} \frac{p_\rho p_\sigma}{2} = s_d \frac{96g^2}{(2\pi)^4} \frac{1}{k^4} \left[p_\mu p_\nu - 4 \frac{(kp)}{k^2} (k_\mu p_\nu + p_\mu k_\nu) \right. \\ \left. + 4 \frac{k_\mu k_\nu}{k^4} \left(5(kp)^2 - k^2 p^2\right) + \text{finite terms} \right], \tag{3.50d}$$

$$\left(\frac{\partial^2}{\partial p_\rho p_\sigma} \mathcal{I}^{(\mathrm{e})}_{\mu\nu}\right)_{p=0} \frac{p_\rho p_\sigma}{2} = -s_e \frac{24g^2}{(2\pi)^4} \frac{p_\mu p_\nu}{k^4} \left(2 - \frac{a'^2}{\tilde{k}^2 \left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)}\right). \tag{3.50e}$$

As discussed at the beginning of this section, we encounter logarithmic divergences for large *and* for small k. As expected, the sine-squared (cf. Eqn. (3.47)) cures this divergence for the non-planar sector, i.e. one has the integral

$$\int d^4k \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right)}{k^4} = \frac{1}{2} \int d^4k \frac{(1 - \cos(k\tilde{p}))}{k^4}\,. \tag{3.51}$$

In order to calculate the planar, i.e. phase independent, part one has to introduce a regulator mass μ, and in the end those contributions depending on this cutoff coming from the non-planar sector will exactly cancel the ones from the planar sector.

The regulator mass may be introduced in the following way in order to arrive at integrable expressions (namely integrals leading to modified Bessel functions) after the initial Gauss integration:

$$\int d^4k \frac{e^{ik\tilde{p}}}{(k^2)^2} \longrightarrow \int d^4k \frac{e^{ik\tilde{p}}}{(k^2+\mu^2)^2} \,. \tag{3.52}$$

This integral leads to Bessel-integrals of the type (for details of this and the following integral c.f. Section A.2)

$$I_\alpha \equiv \int_0^\infty d\alpha \alpha^{-1} e^{-\frac{\tilde{p}^2}{4\alpha}-\alpha\mu^2} = 2K_0\left(\sqrt{\tilde{p}^2\mu^2}\right) \,, \tag{3.53}$$

which for vanishing regulator reduces to

$$\lim_{\mu\to 0} I_\alpha = -2\gamma_E - \ln\left(\frac{\tilde{p}^2}{4}\right) - \lim_{\mu\to 0}\ln(\mu^2) \,. \tag{3.54}$$

For the planar sector, the parameter integral denoted I'_α is given by

$$I'_\alpha \equiv \int_0^\infty d\alpha \alpha^{-1} e^{-\frac{1}{4\Lambda^2\alpha}-\alpha\mu^2} = 2K_0\left(\sqrt{\frac{\mu^2}{\Lambda^2}}\right) \,, \tag{3.55}$$

where Λ is an ultraviolet cutoff, leading to

$$\lim_{\mu\to 0} I'_\alpha = -2\gamma_E - \ln\left(\frac{1}{4\Lambda^2}\right) - \lim_{\mu\to 0}\ln(\mu^2) \,. \tag{3.56}$$

Returning to Eqn. (3.50), we see that we need to solve three types of integrals (and their planar counterparts, i.e. with $\tilde{p} = 0$):

$$\int d^4k \frac{e^{ik\tilde{p}}}{(k^2+\mu^2)^2} = \pi^2 I_\alpha \,, \tag{3.57a}$$

$$\int d^4k \frac{k_\mu k_\nu e^{ik\tilde{p}}}{(k^2+\mu^2)^3} = \frac{\pi^2}{4}\delta_{\mu\nu} I_\alpha + \text{finite terms} \,, \tag{3.57b}$$

$$\int d^4k \frac{k_\mu k_\nu k_\rho k_\sigma e^{ik\tilde{p}}}{(k^2+\mu^2)^4} = \frac{\pi^2}{24}\left(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}\right) I_\alpha + \text{finite terms} \,, \tag{3.57c}$$

where by "finite terms" we mean terms which converge for both $\mu \to 0$ and $p \to 0$ (the full expressions are given in Section A.2). Plugging these formulae into Eqn. (3.50) we find

$$\left(\frac{\partial^2}{\partial p_\rho p_\sigma}\left(\Pi^{(a)}_{\mu\nu} + \Pi^{(c)}_{\mu\nu}\right)\right)_{p=0} \frac{p_\rho p_\sigma}{2} + \left(\frac{\partial^2}{\partial p_\rho p_\sigma}\left(\Pi^{(d)}_{\mu\nu} + \Pi^{(e)}_{\mu\nu}\right)\right)_{p=0} \frac{p_\rho p_\sigma}{2}$$

$$= \frac{2g^2\pi^2}{(2\pi)^4} \left[\frac{5}{3} \left(p^2\delta_{\mu\nu} - p_\mu p_\nu \right) - 2 \left(p^2\delta_{\mu\nu} - p_\mu p_\nu \right) \right] \left(I'_\alpha - I_\alpha + \text{finite terms} \right) . \tag{3.58}$$

As expected, the result is transversal with respect to p^μ, i.e. $p^\mu \Pi^{\log}_{\mu\nu} = 0$. Furthermore, as already in first order the sum of B-dependent graphs (i.e. graphs d and e in Fig. 3.3) is also transversal by itself.
Hence we are led to the final expression where the infrared cutoff μ drops out in the sum of planar and non-planar parts, as expected:

$$\begin{aligned} \Pi^{(2)}_{\mu\nu}(p) &\equiv \left(\frac{\partial^2}{\partial p_\rho p_\sigma} \left(\Pi^{(\mathrm{a})}_{\mu\nu} + \Pi^{(\mathrm{c})}_{\mu\nu} + \Pi^{(\mathrm{d})}_{\mu\nu} + \Pi^{(\mathrm{e})}_{\mu\nu} \right) \right)_{p=0} \frac{p_\rho p_\sigma}{2} = \\ &= \frac{g^2}{24\pi^2} \left(p_\mu p_\nu - p^2 \delta_{\mu\nu} \right) \left(\ln \left(\Lambda^2 \tilde{p}^2 \right) + \text{finite terms} \right) . \end{aligned} \tag{3.59}$$

Notice, that we only have a logarithmic divergence in the UV cutoff Λ coming from the planar parts and that the result (3.59) is well-behaved for $p \to 0$, i.e. there is no logarithmic infrared divergence in the external momentum.

3.6 Discussion

Based on the translation invariant scalar model introduced in Section 2.2, in the present chapter we have discussed the construction of a translation invariant non-commutative gauge model in Euclidean space based on earlier work [8, 7]. The behavior regarding divergences of the theory is improved mainly by inserting operators of the form $\left(1 + \frac{a'^2}{\square\widetilde{\square}}\right)$ in the action, where the parameter a' (introduced in [1]) is a free parameter of the theory. As has already been worked out in great detail in [49] for a similar scalar model these kinds of additional factors lead to a damping in all propagators, thereby taming the divergences arising from the UV/IR mixing present in non-commutative models.
The action (3.25) is invariant with respect to the BRST transformations Eqns. (3.11) and (3.28). Furthermore, a power counting formula (3.43) allowing to estimate the upper bound degree of UV divergence of the model has been introduced in Section 3.4. It appears that powers of the newly introduced parameter a', which are closely tied to the number of external $B_{\mu\nu}$-legs and powers of θ, play an essential role.
One-loop calculations show that, due to the phase factors associated with vertices and momentum conservation, tadpole graphs with one external line vanish. Examination of one-loop corrections to the photon propagator show the well-known $\frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2}$ divergence [61, 62, 63]. Detailed computation of the divergent part, the results being given in Eqns. (3.49) and (3.59), shows transversality and gauge independence of the quadratic divergence as expected. The next logical step would be to conduct the entire one-loop program, i.e. calculate the one-loop corrections to the other propagators and the vertices before going over to higher loops. Then it would remain to be shown that the damping mechanism caused by the inserted operators $\frac{1}{D^2\widetilde{D}^2}$ and $\frac{1}{\widetilde{\square}}$ mentioned earlier suffices to

render the theory renormalizable at higher loop orders. Before rolling up one's sleeves and starting to work off the program, let us recapitulate the localization of the damping term in the action. In order to realize the desired damping for the gauge propagator, the gauge invariant equivalent of the $\frac{1}{\widetilde{\Box}}$ operator, namely $\frac{1}{D^2\widetilde{D}^2}$, acting on $F_{\mu\nu}$ as been introduced. However this leads to an infinite number of gauge boson vertices, and hence avoids any practical calculations. As has been shown in Section 3.2 this problem can be circumvented by the introduction of a new real antisymmetric field $B_{\mu\nu}$. Initially thought to be a simple Lagrange multiplier, $B_{\mu\nu}$ appears to have its own dynamic properties (c.f. the e.o.m Eqn. (3.29)), and it remains in the action even for $a' = 0$. Even if it is related to the gauge field by the Ward-like identities (3.36) suggesting a close relation between the new field and the field strength $F_{\mu\nu}$ of the gauge boson A_μ, it represents an additional degree of freedom and has to be considered as being physical. This latter statement directly translates into the statement that the theory does not any more represent a pure generalization of $U(1)$-gauge theory to the non-commutative setting, but a deep modification of its physics. This was not the original objective of the model. Yet, a way out restoring the original physical content of the theory while maintaining a finite number of tree-level interactions has been found. Before presenting the corresponding model in Section 5, we will first introduce some notions which will be key in understanding the ideas leading to its construction in the following chapter.

Chapter 4

Excursion: The Gribov-Zwanziger approach

In this chapter we will undertake an excursion to commutative non-abelian Yang Mills theory[1], being formally similar to a $U_\star(1)$ field model, due of the non-commutativity of the gauge fields based on the star product. In particular we will see that restricting the integration range of the functional space for the generating functional to the first Gribov horizon, will lead to a propagator with the same IR-damping behaviour as in our gauge model introduced in Section 3.2. First a introduction to the Gribov problem will be given, followed by the derivation of the Gribov-Zwanziger action, both leading to an IR damped gauge propagator. We will than show how this latter action can be obtained by the method of softly breaking the BRST invariance of the Faddeev-Popov action without changing its physical content in the UV sector. Indeed, that is exactly what we need: constructing an action which leads to the IR damping as in the aforementioned model while avoiding new degrees of freedom. This goal has been achieved in [2] and will be detailed in Section 5.

[1]Note that a Yang-Mills theory is defined via the coupling of fields by covariant derivatives. It can be abelian as in the case of ordinary QED, or non abelian as in QCD and non-commutative gauge theory.

4.1 Gribov ambiguities

In [64] (c.f. also [65] for an excellent pedagogic introduction to the topic) Gribov states that fixing the divergence of the gauge field in non-abelian theories does not uniquely fix its gauge and that this problem can be circumvented by restricting the integration range in the path integral to the first Gribov horizon.

Considering the commutative Euclidean Yang-Mills action in four dimensions

$$S_{YM} = \frac{1}{4g^2}\int \mathrm{d}^4x F^a_{\mu\nu}F^a_{\mu\nu} \text{ with } F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu \tag{4.1}$$

where a, b, c denote the color indices of the adjoint representation of a semi-simple Lie group G with its structure constants f^{abc}. It will not be affected by the gauge transformation

$$A_\mu \to \tilde{A}_m = S^\dagger\partial_\mu S + S^\dagger A_\mu S = A_\mu + S^\dagger(\partial_\mu S + [A_\mu,\, S]) \text{ with } S = e^{\alpha^a\lambda^a} = e^\alpha$$
$$\Rightarrow \tilde{F}_{\mu\nu} = S^\dagger F_{\mu\nu} S\,,$$

infinitesimal case:

$$\delta A_\mu = \partial_\mu\alpha + [A_\mu,\, \alpha] = D_\mu\alpha \text{ with } S = 1+\alpha \Rightarrow \delta F_{\mu\nu} = [F_{\mu\nu},\, \alpha]\,. \tag{4.2}$$

Here α denotes the gauge parameter, λ^a are the generators of the group and $A_\mu = A^a_\mu\lambda^a$. When calculating the generating functional this leads to an overcounting and in consequence to its divergence, because field configurations which are just connected by the gauge transformation are considered individually, although being equivalent with respect to the action. This can be seen schematically by

$$\mathcal{Z} = \int dA e^{\mathrm{i}S_{YM}} \approx \int d\bar{A} e^{\mathrm{i}S_{YM}} \int d\Lambda\,, \tag{4.3}$$

with $\bar{A}$ belonging to different gauge classes defined by the gauge parameter Λ. In order to take into account only inequivalent field configurations when quantizing the action, in a first step one fixes the divergence by $\partial_\mu A_\mu = f$ (i.e. $f = 0$ in the Landau gauge expressing transversality), leading to the partition function

$$\mathcal{Z} = \mathcal{N}\int DA_\mu \delta(\partial A)\det(\mathcal{M}^{ab})e^{-S_{YM}} \text{ where } \mathcal{M}^{ab}(A) = -\partial_\mu\left(\partial_\mu\delta^{ab} - f^{abc}A^c_\mu\right).$$

$\mathcal{M}$, $\mathcal{N}$ denote the Faddeev-Popov operator and a normalization factor respectively (see e.g. [66], p. 245 for its derivation). Now consider two fields being connected by a gauge transformation, and impose the condition that both are of vanishing divergence, i.e.

$$\tilde{A} = A + \delta A,\, \partial A = \partial\tilde{A} \Leftrightarrow \partial_\mu\left(\partial_\mu\alpha + [A_\mu,\, \alpha]\right) = 0\,. \tag{4.4}$$

Note that on the r.h.s. we find the Faddeev-Popov operator, whose determinant enters the Faddeev-Popov quantization formula. From the last equation we can see that if the condition is fulfilled, there exist different field configurations which are equivalent in the above sense, even after imposing a gauge condition, which are called Gribov copies. Let us study when this is the case. For this we write down its eigenvalue equation, which is a kind of Schrödinger equation, A_μ being the potential.

$$-\partial_\mu \left(\partial_\mu \psi + [A_\mu,\, \psi]\right) = \epsilon(A)\psi \tag{4.5}$$

For small potential, there will be only positive solutions, $\epsilon > 0$. When it becomes larger, one of the eigenvalues will first vanish, than become negative, indicating the existence of bound states. For ever increasing potential, a second zero eigenvalue will appear etc. According to Gribov, the functional space might be divided into regions C_n where the Fadeev-Popov operator has n negative eigenvalues, being encompassed by the lines l_{n+1} of n zero eigenvalues. In particular, C_0 called the first Gribov region, contains only positive eigenvalues, i.e. the $\phi\pi$-operator is strictly positive, and on its surrounding line l_1 (the first Gribov horizon) appears the first vanishing eigenvalue. Now, it can be proved that for any field on C_n there exists an equivalent field on C_0, i.e. each gauge orbit passes through the first Gribov region (in particular also $A_\mu = 0$, which allows for the usual perturbation theory). It is therefore sufficient to restrict the domain of integration in the path integral to the first Gribov horizon[2]

$$C_0 = \{A_\mu,\, \partial A = 0, -\partial_\mu \left(\partial_\mu \bullet + [A_\mu,\, \bullet]\right) = -\partial_\mu D_\mu \bullet > 0\}\ . \tag{4.6}$$

Next we will calculate the gauge propagator on C_0. The generating functional now contains an additional factor $\mathcal{V}_0$ which implements the restriction of the integration range:

$$\begin{aligned}\mathcal{Z} &= \mathcal{N} \int DA_\mu \delta(\partial A) \det(\mathcal{M}^{ab}) e^{-S_{YM}} \mathcal{V}(C_0) \\ &= \mathcal{N} \int DA_\mu Dc D\bar{c} \delta(\partial A) e^{-\left(S_{YM} + \int \mathrm{d}^4 x \bar{c}^a \partial_\mu D^{ab}_\mu c^b\right)} \mathcal{V}(C_0).\end{aligned} \tag{4.7}$$

According to Gribov, we can write

$$\mathcal{V}(C_0) = \theta(1 - \sigma(0, A)) \tag{4.8}$$

with $\sigma(0, A)$ being given in the Landau gauge by (N being the dimension of the adjoint representation of the semi-simple Lie group)

$$\sigma(0, A) = \frac{N}{4(N^2 - 1)} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} A^a_\mu(q) A^a_\mu(-q)\,. \tag{4.9}$$

Actually the argument of the step function stems from the ghost propagator in the external A^a_μ-field with ghost momentum k,

$$(G^{c\bar{c}})^{ab}(k, A) = \frac{\delta^{ab}}{k^2} \frac{1}{1 - \sigma(k, A)} = (\mathcal{M}^{-1})^{ab}(k, A)\,. \tag{4.10}$$

[2]Actually even within the first Gribov horizon there exist Gribov copies, and one should restrict the integration range to the first modular region (FMR). Yet there are indications that the expectation values in either regions are the same, [67].

The first vanishing of the $\phi\pi$-operator at low momenta therefore coincides with $\sigma(0,A)$ approximating unity. Applying the step function (A.1.5) we obtain for the quadratic part in A_μ of (4.7), after transformation to momentum space, and evaluating the integral over β at its saddle point,

$$\mathcal{Z}_{quadr} = \mathcal{N} \int \frac{d\beta}{2\pi \mathrm{i}\beta} e^{\beta} DA_\mu e^{-\frac{1}{2g^2}\int \frac{d^4q}{(2\pi)^4} A^a_\mu(q) \mathcal{Q}^{ab}_{\mu\nu} A^b_\nu(-q)}, \quad \text{where}$$
$$\mathcal{Q}^{ab}_{\mu\nu} = \left[\left(q^2 + \frac{\gamma^4}{q^2}\right)\delta_{\mu\nu} + \left(\frac{1}{\alpha} - 1\right) q_\mu q_\nu\right]\delta^{ab}\,. \tag{4.11}$$

The parameter γ is called Gribov parameter and is determined by the gap equation

$$\gamma^2 \approx \Lambda^2 e^{-\frac{1}{g^2}}, \tag{4.12}$$

which stems from a diverging integral that has been regularized by the UV-cutoff Λ. The gluon propagator now can be easily be obtained:

$$G^{ab}_{\mu\nu}(k) = \langle A^a_\mu(k) A^b_\nu(-k)\rangle = \delta^{ab} g^2 \frac{1}{k^2 + \frac{\gamma^4}{k^2}} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right). \tag{4.13}$$

In the *ultraviolet*, the propagator shows the well-known $\approx \frac{1}{k^2}$-behaviour, whereas in the *infrared* region it vanishes, due to the Gribov parameter γ. This is the same as in our non-commutative gauge model, where a' implements the damping. Whereas in the latter this has been achieved by introducing suitable additional terms in the action, in the present case it arises in a very natural way due to the restriction of the functional space to the first Gribov horizon.
Now, referring to the before mentioned divergence of the gap-equation and as a consequence of γ, the corresponding action should be enhanced in order to obtain a renormalized version of the gap equation, based on the introduction of suitable counterterms.

4.2 The Gribov-Zwanziger action

The construction of a local, renormalizable effective action which implements the restriction of the functional space to the first Gribov horizon has been obtained by Zwanziger in [68]. As will be shown, this has been done by introducing a nonlocal horizon function in the measure, which in a second step will be localized by introducing suitable additional fields.
We start with the generating functional in (4.7) with the replacement (4.8). It can be shown (c.f. the cited papers), that in the thermodynamical limit the step function can be replaced by a δ-distribution, leading to

$$\begin{aligned}\mathcal{Z} &= \mathcal{N}\int DA_\mu \delta(\partial A)\det(\mathcal{M}^{ab}) e^{-S_{YM}} \delta(1-\sigma(0,A))\\ &= \mathcal{N}\int DA_\mu Dc D\bar{c} Db e^{-(S_{YM}+S_{gf})}\delta(1-\sigma(0,A))\,,\end{aligned} \tag{4.14}$$

where the action terms are given by

$$S_{YM} = \frac{1}{4}\int \mathrm{d}^4x F^a_{\mu\nu}F^a_{\mu\nu}, \text{ and } S_{gf} = \int \mathrm{d}^4x b^a \partial A^a + \bar{c}^a \partial_\mu D^{ab}_\mu c^b \,. \tag{4.15}$$

This can be understood in an intuitive way by considering that the ratio between the volumes of a ball with radius R and its surface is given by $V_{sphere}/V_{surface} \sim R/N$. When the dimension N goes to infinity (which holds in the present case, remember that the A-space is infinite dimensional), the volume of the ball gets concentrated on its surface, i.e. the sphere of radius R.
Now let us consider for a moment the microcanonical ensemble and its equivalence with the canonical Boltzmann ensemble in the thermodynamic limit in statistical mechanics [69]. More specifically, given a Hamiltonian H the averages of the microcanonical ensemble can be constructed via

$$Z(E) = \int d\mu\delta(E-H) \sim \int d\beta f(\beta)\,, \quad \text{with } f(\beta) = \int d\mu e^{\beta(E-H)}, \tag{4.16}$$

with $d\mu$ the measure and E the energy of the system. In the thermodynamical limit, it is exact to evaluate the integral on the r.h.s. at its saddle point $\bar{\beta}$ given by $f'(\bar{\beta}) = 0$, leading to the following gap equation:

$$E = \langle H \rangle_{\bar{\beta}} = \frac{\int d\mu H e^{-\bar{\beta}H}}{\int d\mu e^{-\bar{\beta}H}} \tag{4.17}$$

In other words, the gap equation determines the parameter $\bar{\beta}$.
Assuming the same equivalence between the microcanonical and the canonical ensemble we introduce the Boltzmann factor also in our model:

$$\begin{aligned} &\delta(1-\sigma(0,A)) \Longrightarrow e^{-\gamma^4 H(x)}, \\ &\qquad H = (A, \mathcal{M}^{-1}A) \equiv \int \mathrm{d}^4x h(x) = \int \mathrm{d}^4x f^{abc} A^b_\mu (\mathcal{M}^{-1})^{ad} f^{dec} A^e_\mu. \end{aligned} \tag{4.18}$$

In the last line, the horizon function has been defined. The parameter γ is the Gribov parameter introduced above and has mass dimension one. It will turn out later that this is an important property when implementing infrared effects in a renormalizable way. In this context the gap equation or horizon condition which determines γ reads

$$g^2\langle h \rangle = f \equiv 4(N^2-1)\,, \tag{4.19}$$

where the expectation value on the left is calculated with (4.14). On the right hand side we have the dimension of the adjoint representation of $SU(N)$ multiplied with the dimension of the underlying (here Euclidean) space. It equals therefore the number of components or degrees of freedom f of A^a_μ in four dimensions, and is obtained by calculating the lowest eigenvalue of the $\phi\pi$-operator.
Now the horizon term is nonlocal: it contains the inverse of $\mathcal{M}$, i.e. of differential operators. In order to localize it, by gaussian quadrature for the Boltzmann factor one

introduces pairs of complex conjugate commuting bosonic fields $(\phi, \bar{\phi}) \equiv (\phi_\mu^{ac}, \bar{\phi}_\mu^{ac})$ of dimension one and ghost number zero, and anticommuting Grassmann fields $(\omega, \bar{\omega}) \equiv (\omega_\mu^{ac}, \bar{\omega}_\mu^{ac})$ of ghost numbers $(1, -1)$, and with the abbreviation $\mathcal{M} \equiv \mathcal{M}^{ab}$ this leads to

$$\begin{aligned} e^{-\gamma^4(A, \frac{1}{\mathcal{M}} A)} &\Rightarrow \int D\omega D\bar{\omega} D\phi D\bar{\phi} e^{(\bar{\omega}, \mathcal{M}\omega) - (\bar{\phi}, \mathcal{M}\phi) - \gamma^2(A, \phi - \bar{\phi})} \\ S_{loc} &= S_{YM} + S_{gf} + (\bar{\phi}, \mathcal{M}\phi) - (\bar{\omega}, \mathcal{M}\omega) + \gamma^2(A, \phi - \bar{\phi}). \end{aligned} \tag{4.20}$$

The equivalence between the original and localized version can be seen by the following variable definition (indices are suppressed),

$$\begin{aligned} &\underbrace{\left(\bar{\phi} + \frac{\gamma^2}{\mathcal{M}} A\right)}_{\bar{\phi}'} \mathcal{M} \underbrace{\left(\phi - \frac{\gamma^2}{\mathcal{M}} A\right)}_{\phi'} = \bar{\phi}\mathcal{M}\phi + \gamma^2 A(\phi - \bar{\phi}) - A\frac{\gamma^4}{\mathcal{M}} A \Rightarrow \\ &\bar{\phi}'\mathcal{M}\phi' + A\frac{\gamma^4}{\mathcal{M}} A = \bar{\phi}\mathcal{M}\phi + \gamma^2 A(\phi - \bar{\phi}) \,. \end{aligned} \tag{4.21}$$

Now the A field is decoupled from the auxiliary fields. Therefore the functional integration over the latter can be performed, which gives unity and finally leads to the above equivalence.
The action is now local. The next step now consists of formulating the theory in a BRST invariant way. Terms that can be written as exact BRST transformations do not change the physical content of a theory, as they will not modify the expectation value for a given (single of composite) gauge invariant operator. For this purpose, consider the following action:

$$S_0 = S_{YM} + s \int \mathrm{d}^4 x \left(\bar{c}^a \partial A^a + \omega_\mu^{ac} \mathcal{M}^{ab} \phi_\mu^{bc} \right) . \tag{4.22}$$

With the following BRST-transformations

$$\begin{aligned} sA_\mu^a &= -D_\mu^{ab} c^b & s\bar{c}^a &= b^a & s\phi_\mu^{ab} &= \omega_\mu^{ab} & s\bar{\omega}_\mu^{ab} &= \bar{\phi}_\mu^{ab}, \\ sc^a &= \frac{g}{2} f^{abc} c^b c^c & sb^a &= 0 & s\omega_\mu^{ab} &= 0 & s\bar{\phi}_\mu^{ab} &= 0, \\ s^2 &= 0, \end{aligned} \tag{4.23}$$

this leads to

$$S_0 = S_{loc}(\gamma = 0) + \int \mathrm{d}^4 x \bar{\omega}_\mu^{ac} \partial_\nu \left(g f^{abd} \phi_\mu^{bc} D_\nu^{de} c^e \right) . \tag{4.24}$$

By a variable shift in S_{loc} [70] given by $\omega_\mu^{ac} \to \omega_\mu^{ac} - (\mathcal{M}^{-1})^{ab} \partial_\nu \left(g f^{bed} \phi_\mu^{ec} D_\nu^{dn} c^n \right)$ it transforms into S_0 and the partition function results in

$$\mathcal{Z} = \int D\Phi e^{-S_0 - \gamma^2(A, \phi - \bar{\phi})} \equiv e^{-\Gamma_{GZ}}, \tag{4.25}$$

where $\Phi = \{A, c, bc, b, \phi, \bar{\phi}, \omega, \bar{\omega}\}$ denotes all fields and the action is $BRST$-exact expect in the last term. Γ_{GZ} denotes the effective action. It can be shown (e.g. [71]) that introduction of local sources for the latter that take appropriate values in the *ultraviolet* yield to $BRST$-covariance and renormalizability while in the *infrared* they take the values leading to (4.13).
The gap equation now reads

$$\frac{\partial \Gamma_{GZ}}{\partial \gamma^2} = 0. \tag{4.26}$$

Notice that the gluon propagator becomes

$$G^{ab}_{\mu\nu}(k) = \langle A^a_\mu(k) A^b_\nu(-k) \rangle = \delta^{ab} \frac{1}{k^2 + \frac{\gamma^4}{k^2}} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right), \tag{4.27}$$

which up to g^2 is the same as in Section 4.1, as expected.

4.3 Soft breaking of BRST invariance

We have seen in the last section that the Gribov-Zwanziger action consists of a BRST-exact term plus a soft breaking. To emphasize it again, this finally leads to a damped gauge boson propagator. Based on [72] it will now be shown how to introduce infrared effects in a *systematic* way by introducing BRST-doublets, without changing the physical content of the theory and allowing renormalizability. In the present case this will reproduce the same action. Lessons learned, this will allow for constructing a non-commutative gauge model in the same spirit.

4.3.1 The methodology

The goal is the introduction of non-perturbative, i.e. infrared effects while keeping locality and renormalizability, supposed this is the case for the original action $S_{inv}(\phi)$. Furthermore, the space of local observables which coincides with the cohomology classes of the BRST operator in the space of local field polynomials should remain unchanged. This is the same as saying that the correlation functions $\langle O_1(x_1) \dots O_n(x_n) \rangle$ of the theory should stay the same, while their infrared behaviour will be modified in a desired way. From another point of view this means that the physical content of the theory will remain unchanged in the UV. This is done in three steps:

1. Introduction of fields forming BRST-doublets, e.g. of auxiliary fields and corresponding ghosts in a way that BRST-doublets are formed:

$$\begin{aligned} s\alpha &= \beta\,, \\ s\beta &= 0. \end{aligned} \tag{4.28}$$

They allow to construct invariant terms that can be written as exact BRST transforms, i.e. $S_{inv}(\alpha,\beta) = s\tilde{S}(\alpha,\beta)$: From the nilpotency of s it follows that they are of vanishing cohomology, and no new observables are introduced.

2. The new fields are coupled (linearly) to the original fields ϕ of the theory whose IR behaviour shall be modified, and to a parameter γ with mass dimension > 0. This leads to a *soft breaking* of the BRST invariance:

$$s\left(S_{inv}(\phi) + S_{inv}(\alpha,\beta) + S_{sb}(\gamma,\alpha,\beta,\phi)\right) = sS_{sb} = \gamma\Delta(\alpha,\beta,\phi)\,. \tag{4.29}$$

Soft breaking means a breaking term with a field polynomial Δ of lower mass dimension than the invariant Lagrangian, which is possible due to the massive new parameter (obviously the whole action term S_{sb} is of the same dimension as the remaining action, i.e. zero). Due to its super-renormalizability, this term will not affect the UV region of the theory. Indeed, it has already been shown by Zwanziger in [73] that terms of this type do not spoil renormalizability. However, the Gribov like soft breaking parameter γ is necessarily a physical one, as $s\frac{\partial S_{sb}}{\partial\gamma} = \Delta \neq 0$.

3. In order to restore BRST invariance in the UV region (which is necessary to allow e.g. the application of algebraic renormalization), sources are introduced for the breaking term. They are constructed such that BRST exactness is achieved in the UV, while taking their physical values in the IR, i.e. restoring the original soft breaking term. We therefore have three kind of sources: first, the usual sources for any elementary field introduced in the generating functional (c.f. (3.31)). Second, those for the composite, i.e. non-linear BRST-transfoms (i.e. of the gauge and ghost fields), c.f [57] and [60] for a detailed discussion. Finally those introduced above.

4. Gap actions have to be imposed which determine the new parameter as a function of the original parameters of the theory, i.e. the coupling constant, $\gamma = \gamma(g)$.

Although being obvious, it should be mentioned that the constructed polynomials have to be consistent with respect to the dimensionality and quantum numbers (i.e. ghost numbers).

4.3.2 The Gribov-Zwanziger action in the framework of soft breaking

Consider the Gribov-Zwanziger action derived in the last section: it can be written as

$$S_{GZ} = S_{inv}^{(1)}(A_\mu,\, c,\, \bar{c},\, b) + S_{inv}^{(2)}(\phi,\bar{\phi},\omega,\bar{\omega}) + S_{sb}(\gamma, A_\mu, \phi, \bar{\phi}) \tag{4.30a}$$

$$S_{inv}^{(1)}(A_\mu,\, c,\, \bar{c},\, b) = S_{YM} + S_{gf} \tag{4.30b}$$

$$S_{inv}^{(2)}(\phi,\, \bar{\phi},\, \omega,\, \bar{\omega}) = s\tilde{S}(\bar{\omega},\phi) = s\int \mathrm{d}^4x\, \omega_\mu^{ac}\partial_\nu D_\nu \phi_\mu^{ac} \tag{4.30c}$$

$$S_{sb} = -\gamma^2 g\int \mathrm{d}^4x \left(f^{abc}A_\mu^a(\phi_\mu^{bc} + \bar{\phi}_\mu^{bc})\right) = -\gamma^2\int \mathrm{d}^4x D_\mu\phi\,. \tag{4.30d}$$

The fields and their BRST transformations as well as S_{YM} and S_{gf} are given in the previous section.

- According to *(1)* of the previous discussion we observe that the BRST doubletts $(\phi,\, \omega)$ and $(\bar{\phi},\, \bar{\omega})$ (c.f. (4.23) for their transformations) lead to a BRST invariant term in the action given by (4.30c).
- According to *(2)*, the new fields are coupled to the gauge field, as we intend to modify its IR behaviour, c.f. (4.30d). Furthermore we observe the Gribov parameter γ of dimension 1. It follows from the fact that the soft breaking can be written as

$$sS_{sb} = \gamma^2 \Delta, \tag{4.31}$$

  thus γ is a physical parameter. Indeed this leads to a modified gauge propagator, as has been elaborated in detail above and can bee seen from (4.13).
- According to *(3)*, external sources are introduced in the action (all indices, e.g. denoting colors, components or of the Lorentz type are suppressed):

$$\begin{aligned} S =& S^{(1)}_{inv} + S^{(2)}_{inv} - (M_\mu,\, D\phi) - (U, sD\phi) \\ & - (sD\bar{\omega}, V) - (D\bar{\omega},\, N) - (K, sA) - (L, sc)\,. \end{aligned} \tag{4.32}$$

  Based on this action, with additional sources for the elementary fields, renormalizability can be shown. The original action is obtained by the physical values of the sources, given by

$$M_{ph} = -V_{ph} \propto \gamma^2 \times \text{(Index structure)} \quad \text{and} \quad K = L = N = U = 0. \tag{4.33}$$

  For details refer to [73], Section 4 ff. This program will be explicitly applied to non-commutative gauge theory in Section 5.
- Finally, the gap equation is given by (4.26) according to *(4)*.

Chapter 5

Localization via BRST-doublets

In this chapter, we will apply the method of introducing infrared effects to the propagators of a θ-deformed gauge theory via introducing auxiliary fields [2, 3]. Contrary to Section 3, they will be introduced as BRST doublets, therefore avoiding the introduction of new degrees of freedom. In order to achieve this, the method introduced in Section 4 will be followed.

First, the model of Vilar *et al.* given in [14] will be discussed in short in Section 5.1, which follows the same philosophy, and was published in response to [2]. It will then be discussed how the same objective can be reached in a simpler way, as we could show in [3]. In particular, the construction of the model will be given in Section 5.2, followed by its symmetry content (Section 5.3). They form the basis for the possible application of Algebraic Renormalization. The Feynman rules will be derived in Section 5.4, followed by power counting formulae (Section 5.5), and one loop calculations are given in

Section 5.6. There, we will also undertake the attempt of renormalizing the full gauge boson propagator. Together with Section 5.7 dealing with higher loop calculations, we will finally see the problems appearing in the renormalization program. Conclusions will be given in Section 5.8.

5.1 The model of Vilar et. al.

In response to the model presented in the last chapter Section 3.2, Vilar *et al.* [14] presented a model which avoids the introduction of new degrees of freedom. This has been achieved by localizing the term under consideration $S^{(2)}_{inv}$ (c.f. second line in (3.30)), i.e.

$$S^{(2)}_{inv} \equiv S_{nloc} = \int \mathrm{d}^4x \left[\frac{1}{4} F_{\mu\nu} \star \frac{a'^2}{D^2 \widetilde{D}^2} \star F_{\mu\nu} \right] , \tag{5.1}$$

in a way compatible with the methodology in Section 4.3.1. Instead of (3.27), their localized term reads:

$$\begin{aligned} S_{\text{nloc}} \to S_{\text{loc}} &= S_{\text{loc},0} + S_{\text{break}} \, , \\ S_{\text{loc},0} &= \int \mathrm{d}^4x \left(\bar{\chi}_{\mu\nu} \star D^2 B_{\mu\nu} + \bar{B}_{\mu\nu} \star D^2 \chi_{\mu\nu} + \gamma^2 \bar{\chi}_{\mu\nu} \star \chi_{\mu\nu} \right) , &(5.2) \\ S_{\text{break}} &= \int \mathrm{d}^4x \left[-\mathrm{i}\frac{\gamma}{2} B_{\mu\nu} \star F^{\mu\nu} + \mathrm{i}\frac{\gamma}{2} \bar{B}_{\mu\nu} \star F^{\mu\nu} \right] , &(5.3) \end{aligned}$$

with $(B_{\mu\nu}, \bar{B}_{\mu\nu})$, and $(\chi_{\mu\nu}, \bar{\chi}_{\mu\nu})$ being two pairs of auxiliary complex conjugated anti-symmetric tensorial bosonic fields of mass dimension one, and γ a parameter of mass dimension one corresponding to a Gribov-like parameter according to (*2*), Section 4.3.1. The term S_{nloc} is now split into a BRST invariant part $S_{\text{loc},0}$, and a breaking term S_{break} as can be seen by explicit calculation with the definitions in Ref. [14].
According to (*1*), Section 4.3.1, ghosts are furthermore added for each of the bosonic fields by the action term S_G, given by $\{\psi_{\mu\nu}, \bar{\psi}_{\mu\nu}, \xi_{\mu\nu}, \bar{\xi}_{\mu\nu}\}$. This is being done in such a way that BRST doublet structures are formed. This results in a trivial BRST cohomology for $S_{\text{loc},0}$ from which follows [72] that

$$sS_{\text{loc},0} = 0 \quad \Rightarrow \quad S_{\text{loc},0} = s\hat{S}_{\text{loc},0} \, , \tag{5.4}$$

i.e. the part of the action depending on the auxiliary fields and their associated ghosts can be written as an exact expression with respect to the nilpotent BRST operator s. The breaking term does not join this nice property due to a non-trivial cohomology. However, it is constructed such that its mass dimension is smaller than four, the dimension of the underlying Euclidean space. As mentioned in (*1*), Section 4.3.1 such a breaking is referred to as "soft" and does not spoil renormalizability. In fact, S_{break} is the actual origin of the avoidance of UV/IR mixing featured by this theory as it alters the IR sector while not affecting the UV part.

With the usual gauge fixing term S_{gf} the tree level action of Ref. [14] is finally given by[1]:

$$
\begin{aligned}
S &= S_0 + S_{\text{break}} + S_{\text{G}} + S_{\text{gf}}, \\
S_0 &= \int \mathrm{d}^4x \left[\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \bar{\chi}_{\mu\nu} \star D^2 B^{\mu\nu} + \bar{B}_{\mu\nu} \star D^2 \chi^{\mu\nu} + \gamma^2 \bar{\chi}_{\mu\nu} \star \chi^{\mu\nu} \right] , \\
S_{\text{G}} &= \int \mathrm{d}^4x \left[-\bar{\psi}_{\mu\nu} \star D^2 \star \xi^{\mu\nu} - \bar{\xi}_{\mu\nu} \star D^2 \psi^{\mu\nu} - \gamma^2 \bar{\psi}_{\mu\nu} \star \psi^{\mu\nu} \right] , \\
S_{\text{gf}} &= \int \mathrm{d}^4x \left[\mathrm{i}b \star \partial^\mu A_\mu + \bar{c} \star \partial^\mu D_\mu c \right] ,
\end{aligned} \tag{5.5}
$$

with S_{break} already defined in (5.3). The total action leads to 19 tree level propagators, which are given in Section D.1.

5.2 Construction of the action

The starting point is the localized gauge invariant part $S^{(2)}_{\text{inv}}$, in the following denoted S_{loc} (c.f. Eqn. (3.27)),

$$
S_{\text{loc}} = \int \mathrm{d}^4x \left[a' \mathcal{B}_{\mu\nu} \star F_{\mu\nu} - \mathcal{B}_{\mu\nu} \star \widetilde{D}^2 D^2 \star \mathcal{B}_{\mu\nu} \right] . \tag{5.6}
$$

Note that the former notation $B_{\mu\nu}$ for the real auxiliary field has been changed to $\mathcal{B}_{\mu\nu}$, in order to avoid any confusion with respect to the auxiliary fields introduced below. The latter action term is equivalent to (5.1).
From now on we will omit the star when writing the product of fields, and the product in x-space has to be understood in the sense of the star product, if not mentioned otherwise.

Introduction of fields forming BRST-doublets

According to Section 4.3.1, (*1*), Eqn. (4.28) we introduce fields forming BRST doublets[2]. They are given by a pair of ghost and antighost fields $(\psi_{\mu\nu},\ \bar{\psi}_{\mu\nu})$, and a pair of fields $(B_{\mu\nu},\ \bar{B}_{\mu\nu})$. The latter result from turning $\mathcal{B}_{\mu\nu}$ into a pair of complex conjugated fields. The doublet structure can be seen by considering the BRST-transformations of the new fields,

$$
\begin{aligned}
s\bar{\psi}_{\mu\nu} &= \bar{B}_{\mu\nu} + \mathrm{i}g \left\{ c, \bar{\psi}_{\mu\nu} \right\} , \qquad & s\bar{B}_{\mu\nu} &= \mathrm{i}g \left[c, \bar{B}_{\mu\nu} \right] , \\
sB_{\mu\nu} &= \psi_{\mu\nu} + \mathrm{i}g \left[c, B_{\mu\nu} \right] , \qquad & s\psi_{\mu\nu} &= \mathrm{i}g \left\{ c, \psi_{\mu\nu} \right\} .
\end{aligned} \tag{5.7}
$$

[1] The total action including external sources according to (*1*), Section 4.3.1, can be found in [14].
[2] The same has been done in [14], however leading to a more complex action than derived in what follows, c.f. Section 5.1.

The transformation laws for the other fields are the same as before, i.e.

$$\begin{aligned} sA_\mu &= D_\mu c\,, & sc &= \mathrm{i}gcc\,, \\ s\bar{c} &= b\,, & sb &= 0\,, \\ sF_{\mu\nu} &= \mathrm{i}g\left[c, F_{\mu\nu}\right]\,, \end{aligned} \tag{5.8}$$

which are valid when considering Landau gauge fixing, given by

$$S_{\phi\pi} = \int \mathrm{d}^4x \left(b\partial^\mu A_\mu - \bar{c}\partial^\mu D_\mu c\right)\,. \tag{5.9}$$

Note that contrary to the general approach defined in Section 4.3.1, we have now introduced two BRST-doublets. Indeed, one doublet would suffice for eliminating the additional degree of freedom of the previous model, but only with complex conjugated fields we will restore hermiticity of the action[3].

Invariant and soft breaking terms in the action

Eqn. (5.6) is now being replaced by [2, 3]

$$S_{\mathrm{loc}} = \int \mathrm{d}^4x \left[\frac{\lambda}{2}\left(B_{\mu\nu} + \bar{B}_{\mu\nu}\right)F^{\mu\nu} - \mu^2 \bar{B}_{\mu\nu} D^2 \widetilde{D}^2 B^{\mu\nu} + \mu^2 \bar{\psi}_{\mu\nu} D^2 \widetilde{D}^2 \psi^{\mu\nu}\right]\,, \tag{5.10}$$

where (as in the remainder of this section) all field products are considered to be star products. The parameters λ and μ both have mass dimension 1 and replace the former dimensionless parameter a'. They are connected via the relation $a' \equiv \lambda/\mu$. The equivalence of this localized action and the original non-local version can be shown by employing the path integral formalism:

$$\begin{aligned} Z &= \int \mathcal{D}(\bar{\psi}\psi\bar{B}BA)\, \exp\left\{-\left(\int \mathrm{d}^4x \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + S_{\mathrm{loc}}\right)\right\} \\ &= \int \mathcal{D}(\bar{B}BA)\, \det{}^4\left(\mu^2 D^2 \widetilde{D}^2\right) \exp\Bigg\{ - \int \mathrm{d}^4x \Bigg[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\lambda}{2}\left(B_{\mu\nu} + \bar{B}_{\mu\nu}\right)F^{\mu\nu} \\ &\qquad\qquad - \mu^2 \bar{B}_{\mu\nu} D^2 \widetilde{D}^2 B^{\mu\nu}\Bigg]\Bigg\} \\ &= \int \mathcal{D}(\bar{B}BA)\, \det{}^4\left(\mu^2 D^2 \widetilde{D}^2\right) \exp\Bigg\{- \int \mathrm{d}^4x \Bigg[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\lambda^2}{4\mu^2} F_{\mu\nu} \frac{1}{\widetilde{D}^2 D^2} F^{\mu\nu} - \\ &\qquad - \left(\bar{B}_{\mu\nu} - \frac{\lambda}{2\mu^2}\frac{1}{\widetilde{D}^2 D^2}F_{\mu\nu}\right)\mu^2 D^2 \widetilde{D}^2 \left(B^{\mu\nu} - \frac{\lambda}{2\mu^2}\frac{1}{\widetilde{D}^2 D^2}F^{\mu\nu}\right)\Bigg]\Bigg\} \\ &= \int \mathcal{D}A\, \det{}^4\left(D^2 \widetilde{D}^2\right) \det{}^{-4}\left(D^2 \widetilde{D}^2\right) \exp\left\{- \int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\lambda^2}{4\mu^2} F_{\mu\nu} \frac{1}{\widetilde{D}^2 D^2} F^{\mu\nu}\right]\right\}. \end{aligned} \tag{5.11}$$

[3] Note that also for the Gribov-Zwanziger action two BRST-doublets have been introduced, c.f. Section 4.3.2.

With the BRST doublet structure of Eqn. (5.7) one can write

$$S_{\mathrm{loc}} = \int \mathrm{d}^4x \left[\frac{\lambda}{2} B_{\mu\nu} F^{\mu\nu} + s\left(\frac{\lambda}{2}\bar{\psi}_{\mu\nu} F^{\mu\nu} - \mu^2 \bar{\psi}_{\mu\nu} D^2 \widetilde{D}^2 B^{\mu\nu}\right)\right], \tag{5.12}$$

Following the discussion Section 4.3.1, (*2*), we observe the following: first the localized action is split into a BRST-invariant part and a breaking term (c.f. Eqn. (4.29)), as

$$sS_{\mathrm{break}} = \int \mathrm{d}^4x \frac{\lambda}{2} \psi_{\mu\nu} F^{\mu\nu}, \quad \text{with} \quad S_{\mathrm{break}} = \int \mathrm{d}^4x \frac{\lambda}{2} B_{\mu\nu} F^{\mu\nu} . \tag{5.13}$$

Second, the mass dimension d_m of the field dependent part of S_{break} fulfills the condition $d_m\left(\psi_{\mu\nu}F^{\mu\nu}\right) = 3 < D = 4$, the breaking is therefore considered to be "soft", and does not spoil renormalizability of the action. Here λ plays the role of the Gribov parameter. Next, the new fields are coupled linearly to the gauge field as required in order to modify its IR behaviour.
Finally, according to the same discussion it should be noticed that the mass μ is a physical parameter despite the fact that the variation of the action $\frac{\partial S}{\partial \mu^2} = s\left(\bar{\psi}_{\mu\nu} D^2 \widetilde{D}^2 B^{\mu\nu}\right)$ yields an exact BRST form. Following the argumentation in Ref. [72] this is a consequence of the introduction of the soft breaking term. For vanishing Gribov-like parameter λ the contributions to the path integral of the μ dependent sectors of S_{loc} (and below in S_{new} in (5.19)) cancel each other. If $\lambda \neq 0$ one has to consider the additional breaking term which couples the gauge field A_μ to the auxiliary field $B_{\mu\nu}$ and the associated ghost $\psi_{\mu\nu}$. This mixing is reflected by the appearance of $a' = \lambda/\mu$ in the damping factor $\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)$ featured by all field propagators (5.32c)–(5.32f).

Restoring BRST invariance in the UV sector

In order to restore BRST invariance in the UV region (as is a prerequisite for a possible application of algebraic renormalization) an additional set of sources, given by

$$\begin{aligned}
s\bar{Q}_{\mu\nu\alpha\beta} &= \bar{J}_{\mu\nu\alpha\beta} + \mathrm{i}g\left\{c, \bar{Q}_{\mu\nu\alpha\beta}\right\}, & s\bar{J}_{\mu\nu\alpha\beta} &= \mathrm{i}g\left[c, \bar{J}_{\mu\nu\alpha\beta}\right], \\
sQ_{\mu\nu\alpha\beta} &= J_{\mu\nu\alpha\beta} + \mathrm{i}g\left\{c, Q_{\mu\nu\alpha\beta}\right\}, & sJ_{\mu\nu\alpha\beta} &= \mathrm{i}g\left[c, J_{\mu\nu\alpha\beta}\right],
\end{aligned} \tag{5.14}$$

is introduced, and coupled to the breaking term which then takes the form

$$\begin{aligned}
S_{\mathrm{break}} &= \int \mathrm{d}^4x\, s\left(\bar{Q}_{\mu\nu\alpha\beta} B^{\mu\nu} F^{\alpha\beta}\right) \\
&= \int \mathrm{d}^4x \left(\bar{J}_{\mu\nu\alpha\beta} B^{\mu\nu} F^{\alpha\beta} - \bar{Q}_{\mu\nu\alpha\beta} \psi^{\mu\nu} F^{\alpha\beta}\right) .
\end{aligned} \tag{5.15}$$

Eqn. (5.13) is reobtained if the sources $\bar{Q}$ and $\bar{J}$ take their "physical values" given by

$$\bar{Q}_{\mu\nu\alpha\beta}\big|_{\text{phys}} = 0\,, \qquad \bar{J}_{\mu\nu\alpha\beta}\big|_{\text{phys}} = \frac{\lambda}{4}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right)\,,$$
$$Q_{\mu\nu\alpha\beta}\big|_{\text{phys}} = 0\,, \qquad J_{\mu\nu\alpha\beta}\big|_{\text{phys}} = \frac{\lambda}{4}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right)\,. \tag{5.16}$$

Note that the Hermitian conjugate of the counterterm S_{break} in Eqn. (5.10), (i.e. the term $\int \mathrm{d}^4x \bar{B}_{\mu\nu}F^{\mu\nu}$) may also be coupled to external sources which, however, is not required for BRST invariance but restores Hermiticity of the action,

$$\frac{\lambda}{2}\int \mathrm{d}^4x \bar{B}_{\mu\nu}F^{\mu\nu} \;\longrightarrow\; \int \mathrm{d}^4x\, s\left(J_{\mu\nu\alpha\beta}\bar{\psi}^{\mu\nu}F^{\alpha\beta}\right) = \int \mathrm{d}^4x\, J_{\mu\nu\alpha\beta}\bar{B}^{\mu\nu}F^{\alpha\beta}. \tag{5.17}$$

Including external sources Ω^{ϕ}, $\phi \in \{A, c, B, \bar{B}, \psi, \bar{\psi}, J, \bar{J}, Q, \bar{Q}\}$ for the non-linear BRST transformations the complete action with Landau gauge $\partial^{\mu}A_{\mu} = 0$ and general $Q/\bar{Q}$ and $J/\bar{J}$ reads[4]:

$$S = S_{\text{inv}} + S_{\phi\pi} + S_{\text{new}} + S_{\text{break}} + S_{\text{ext}}\,, \tag{5.18}$$

with

$$\begin{aligned}
S_{\text{inv}} &= \int \mathrm{d}^4x \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,\\
S_{\phi\pi} &= \int \mathrm{d}^4x\, s\left(\bar{c}\,\partial^{\mu}A_{\mu}\right) = \int \mathrm{d}^4x \left(b\,\partial^{\mu}A_{\mu} - \bar{c}\,\partial^{\mu}D_{\mu}c\right)\,,\\
S_{\text{new}} &= \int \mathrm{d}^4x\, s\left(J_{\mu\nu\alpha\beta}\bar{\psi}^{\mu\nu}F^{\alpha\beta} - \mu^2\bar{\psi}_{\mu\nu}D^2\widetilde{D}^2B^{\mu\nu}\right)\\
&= \int \mathrm{d}^4x \left(J_{\mu\nu\alpha\beta}\bar{B}^{\mu\nu}F^{\alpha\beta} - \mu^2\bar{B}_{\mu\nu}D^2\widetilde{D}^2B^{\mu\nu} + \mu^2\bar{\psi}_{\mu\nu}D^2\widetilde{D}^2\psi^{\mu\nu}\right)\,,\\
S_{\text{break}} &= \int \mathrm{d}^4x\, s\left(\bar{Q}_{\mu\nu\alpha\beta}B^{\mu\nu}F^{\alpha\beta}\right) = \int \mathrm{d}^4x \left(\bar{J}_{\mu\nu\alpha\beta}B^{\mu\nu}F^{\alpha\beta} - \bar{Q}_{\mu\nu\alpha\beta}\psi^{\mu\nu}F^{\alpha\beta}\right)\,,\\
S_{\text{ext}} &= \int \mathrm{d}^4x \Big(\Omega^{A}_{\mu}D^{\mu}c + \mathrm{i}g\,\Omega^{c}cc + \Omega^{B}_{\mu\nu}\left(\psi^{\mu\nu} + \mathrm{i}g\,[c, B^{\mu\nu}]\right) + \mathrm{i}g\,\Omega^{\bar{B}}_{\mu\nu}\left[c, \bar{B}^{\mu\nu}\right]\\
&\quad + \mathrm{i}g\,\Omega^{\psi}_{\mu\nu}\left\{c, \psi^{\mu\nu}\right\} + \Omega^{\bar{\psi}}_{\mu\nu}\left(\bar{B}^{\mu\nu} + \mathrm{i}g\left\{c, \bar{\psi}^{\mu\nu}\right\}\right) + \Omega^{Q}_{\mu\nu\alpha\beta}\left(J^{\mu\nu\alpha\beta} + \mathrm{i}g\left\{c, Q^{\mu\nu\alpha\beta}\right\}\right)\\
&\quad + \mathrm{i}g\,\Omega^{J}_{\mu\nu\alpha\beta}\left[c, J^{\mu\nu\alpha\beta}\right] + \Omega^{\bar{Q}}_{\mu\nu\alpha\beta}\left(\bar{J}^{\mu\nu\alpha\beta} + \mathrm{i}g\left\{c, \bar{Q}^{\mu\nu\alpha\beta}\right\}\right) + \mathrm{i}g\,\Omega^{\bar{J}}_{\mu\nu\alpha\beta}\left[c, \bar{J}^{\mu\nu\alpha\beta}\right]\Big)\,.
\end{aligned} \tag{5.19}$$

Tab. 5.1 summarizes properties of the fields and sources contained in the model (5.19).

[4] Note that compared to (3.25), in $S_{\phi\pi}$ the damping term $\left(1 + \frac{1}{\Box\widetilde{\Box}}\right)$ has been omitted, and the gauge parameter has been set to 0.

5.3 Symmetry content

Following the objective of applying the method of algebraic renormalization we explore the symmetry content of the theory given in Eqn. (5.18). The Slavnov-Taylor identity is given by

$$\begin{aligned}\mathcal{B}(S) = \int \mathrm{d}^4x \Big[&\frac{\delta S}{\delta\Omega^A_\mu}\frac{\delta S}{\delta A^\mu} + \frac{\delta S}{\delta\Omega^c}\frac{\delta S}{\delta c} + b\frac{\delta S}{\delta\bar{c}} + \frac{\delta S}{\delta\Omega^B_{\mu\nu}}\frac{\delta S}{\delta B^{\mu\nu}} + \frac{\delta S}{\delta\Omega^{\bar{B}}_{\mu\nu}}\frac{\delta S}{\delta\bar{B}^{\mu\nu}} \\ &+ \frac{\delta S}{\delta\Omega^\psi_{\mu\nu}}\frac{\delta S}{\delta\psi^{\mu\nu}} + \frac{\delta S}{\delta\Omega^{\bar{\psi}}_{\mu\nu}}\frac{\delta S}{\delta\bar{\psi}^{\mu\nu}} + \frac{\delta S}{\delta\Omega^Q_{\mu\nu\alpha\beta}}\frac{\delta S}{\delta Q^{\mu\nu\alpha\beta}} + \frac{\delta S}{\delta\Omega^J_{\mu\nu\alpha\beta}}\frac{\delta S}{\delta J^{\mu\nu\alpha\beta}} \\ &+ \frac{\delta S}{\delta\Omega^{\bar{Q}}_{\mu\nu\alpha\beta}}\frac{\delta S}{\delta\bar{Q}^{\mu\nu\alpha\beta}} + \frac{\delta S}{\delta\Omega^{\bar{J}}_{\mu\nu\alpha\beta}}\frac{\delta S}{\delta\bar{J}^{\mu\nu\alpha\beta}}\Big] = 0\,. \end{aligned} \tag{5.20}$$

It expresses the invariance of the action under BRST transformations on a functional level, i.e. generalizes $s\phi$ from the fields to the total action, with ϕ an arbitrary field of the action. It can be written very generally as

$$\mathcal{B}(S) = \int \mathrm{d}^4x \sum_\phi (s\,\phi)\frac{\delta S}{\delta\phi}\,,$$

which shows that only fields with non-vanishing BRST-transformations enter the symmetry.
Furthermore we have the gauge fixing condition

$$\frac{\delta S}{\delta b} = \partial^\mu A_\mu = 0\,, \tag{5.21}$$

the ghost equation

$$\mathcal{G}(S) = \partial_\mu \frac{\delta S}{\delta\Omega^A_\mu} + \frac{\delta S}{\delta\bar{c}} = 0\,, \tag{5.22}$$

Table 5.1: Properties of fields and sources.

Field	A_μ	c	$\bar{c}$	$B_{\mu\nu}$	$\bar{B}_{\mu\nu}$	$\psi_{\mu\nu}$	$\bar{\psi}_{\mu\nu}$	$J_{\alpha\beta\mu\nu}$	$\bar{J}_{\alpha\beta\mu\nu}$	$Q_{\alpha\beta\mu\nu}$	$\bar{Q}_{\alpha\beta\mu\nu}$
$g_\sharp$	0	1	-1	0	0	1	-1	0	0	-1	-1
Mass dim.	1	0	2	1	1	1	1	1	1	1	1
Statistics	b	f	f	b	b	f	f	b	b	f	f
Source	Ω^A_μ	Ω^c	b	$\Omega^B_{\mu\nu}$	$\Omega^{\bar{B}}_{\mu\nu}$	$\Omega^\psi_{\mu\nu}$	$\Omega^{\bar{\psi}}_{\mu\nu}$	$\Omega^J_{\alpha\beta\mu\nu}$	$\Omega^{\bar{J}}_{\alpha\beta\mu\nu}$	$\Omega^Q_{\alpha\beta\mu\nu}$	$\Omega^{\bar{Q}}_{\alpha\beta\mu\nu}$
$g_\sharp$	-1	-2	0	-1	-1	-2	0	-1	-1	0	0
Mass dim.	3	4	2	3	3	3	3	3	3	3	3
Statistics	f	b	b	f	f	b	b	f	f	b	b

and the antighost equation

$$\bar{\mathcal{G}}(S) = \int \mathrm{d}^4x \frac{\delta S}{\delta c} = 0\,. \tag{5.23}$$

The latter two are given by the equations of motion for the ghost and antighost on a functional level.
Following the notation of Ref. [14] the identity associated to the BRST doublet structure is given by

$$\begin{aligned} U^{(1)}_{\alpha\beta\mu\nu}(S) = \int \mathrm{d}^4x \Bigg(& \bar{B}_{\alpha\beta}\frac{\delta S}{\delta\bar{\psi}^{\mu\nu}} + \Omega^{\bar{\psi}}_{\mu\nu}\frac{\delta S}{\delta\Omega^{\bar{B}}_{\alpha\beta}} + \psi_{\alpha\beta}\frac{\delta S}{\delta B^{\mu\nu}} - \Omega^{B}_{\mu\nu}\frac{\delta S}{\delta\Omega^{\psi}_{\alpha\beta}} \\ & + J_{\mu\nu\rho\sigma}\frac{\delta S}{\delta Q^{\alpha\beta}{}_{\rho\sigma}} + \Omega^{Q}_{\alpha\beta\rho\sigma}\frac{\delta S}{\delta\Omega^{J}_{\mu\nu\rho\sigma}} + \bar{J}_{\mu\nu\rho\sigma}\frac{\delta S}{\delta\bar{Q}^{\alpha\beta}{}_{\rho\sigma}} + \Omega^{\bar{Q}}_{\alpha\beta\rho\sigma}\frac{\delta S}{\delta\Omega^{\bar{J}}_{\mu\nu\rho\sigma}} \Bigg) = 0\,, \end{aligned} \tag{5.24}$$

in short:

$$U^{(1)}(S) = \int \mathrm{d}^4x \sum_{\phi} \left\{ (s\,\phi)\frac{\delta S}{\delta\phi} + \Omega^{\phi}\frac{\delta S}{\delta\Omega^{s\phi}} \right\}\,, \quad \forall\phi:\ s\phi = ig\,[c,\phi]_{\pm}$$

with $[\,,\,]_+ \equiv \{\,,\,\}$ the anticommutator for fermionic fields, and $[\,,\,]_-$ the commutator in the case of bosonic fields.
It is interesting to mention that the first two terms of the second line,

$$\int \mathrm{d}^4x \left(J_{\mu\nu\rho\sigma}\frac{\delta S}{\delta Q^{\alpha\beta}{}_{\rho\sigma}} + \Omega^{Q}_{\alpha\beta\rho\sigma}\frac{\delta S}{\delta\Omega^{J}_{\mu\nu\rho\sigma}} \right) = 0\,,$$

constitute a symmetry by themselves. These terms stem from the insertion of conjugated field partners J and Q for $\bar{J}$ and $\bar{Q}$, respectively, which are not necessarily required as discussed above in Section 5.2.

Furthermore, we have the linearly broken symmetries $U^{(0)}$ and $\tilde{U}^{(0)}$:

$$U^{(0)}_{\alpha\beta\mu\nu}(S) = -\Theta^{(0)}_{\alpha\beta\mu\nu} = -\tilde{U}^{(0)}_{\alpha\beta\mu\nu}(S)\,, \tag{5.25}$$

with

$$\begin{aligned} U^{(0)}_{\alpha\beta\mu\nu}(S) = \int \mathrm{d}^4x \Bigg[& B_{\alpha\beta}\frac{\delta S}{\delta B_{\mu\nu}} - \bar{B}_{\mu\nu}\frac{\delta S}{\delta\bar{B}_{\alpha\beta}} - \Omega^{B}_{\mu\nu}\frac{\delta S}{\delta\Omega^{B}_{\alpha\beta}} + \Omega^{\bar{B}}_{\alpha\beta}\frac{\delta S}{\delta\Omega^{\bar{B}}_{\mu\nu}} \\ & + J_{\alpha\beta\rho\sigma}\frac{\delta S}{\delta J_{\mu\nu\rho\sigma}} - \bar{J}_{\mu\nu\rho\sigma}\frac{\delta S}{\delta\bar{J}_{\alpha\beta\rho\sigma}} - \Omega^{J}_{\mu\nu\rho\sigma}\frac{\delta S}{\delta\Omega^{J}_{\alpha\beta\rho\sigma}} + \Omega^{\bar{J}}_{\alpha\beta\rho\sigma}\frac{\delta S}{\delta\Omega^{\bar{J}}_{\mu\nu\rho\sigma}} \Bigg]\,, \end{aligned}$$

$$\tilde{U}^{(0)}_{\alpha\beta\mu\nu}(S) = \int d^4x \left[\psi_{\alpha\beta}\frac{\delta S}{\delta\psi_{\mu\nu}} - \bar{\psi}_{\mu\nu}\frac{\delta S}{\delta\bar{\psi}_{\alpha\beta}} - \Omega^{\psi}_{\mu\nu}\frac{\delta S}{\delta\Omega^{\psi}_{\alpha\beta}} + \Omega^{\bar{\psi}}_{\alpha\beta}\frac{\delta S}{\delta\Omega^{\bar{\psi}}_{\mu\nu}}\right.$$
$$\left. + Q_{\alpha\beta\rho\sigma}\frac{\delta S}{\delta Q_{\mu\nu\rho\sigma}} - \bar{Q}_{\mu\nu\rho\sigma}\frac{\delta S}{\delta\bar{Q}_{\alpha\beta\rho\sigma}} - \Omega^{Q}_{\mu\nu\rho\sigma}\frac{\delta S}{\delta\Omega^{Q}_{\alpha\beta\rho\sigma}} + \Omega^{\bar{Q}}_{\alpha\beta\rho\sigma}\frac{\delta S}{\delta\Omega^{\bar{Q}}_{\mu\nu\rho\sigma}}\right], \tag{5.26}$$

$$\Theta^{(0)}_{\alpha\beta\mu\nu} = \int d^4x \left[\bar{B}_{\mu\nu}\Omega^{\bar{\psi}}_{\alpha\beta} - \psi_{\alpha\beta}\Omega^{B}_{\mu\nu} + \bar{J}_{\mu\nu\rho\sigma}\Omega^{\bar{Q}}_{\alpha\beta\rho\sigma} - J_{\alpha\beta\rho\sigma}\Omega^{Q}_{\mu\nu\rho\sigma}\right]. \tag{5.27}$$

In short the linearly broken symmetries can be written as

$$\int d^4x \sum_{\phi}(-1)^n \left\{\phi\frac{\delta S}{\delta\phi} - \Omega^{\phi}\frac{\delta S}{\delta\Omega^{\phi}}\right\}, \tag{5.28}$$

where $U^{(0)}(S)$ / $\tilde{U}^{(0)}(S)$ is obtained by summing over bosonic / fermionic fields, only. In the last sum, $n=1$ for the fields denoted with an overbar, and $n=0$ otherwise. The above relations form the starting point for the algebraic renormalization procedure[5]. In order to assure the completeness of the set of symmetries it has to be assured that the algebra generated by them closes. From the Slavnov-Taylor identity (5.20) one derives the linearized Slavnov operator

$$\mathcal{B}_S = \int d^4x \left[\frac{\delta S}{\delta\omega^A_\mu}\frac{\delta}{\delta A_\mu} + \frac{\delta S}{\delta A_\mu}\frac{\delta}{\delta\omega^A_\mu} + \frac{\delta S}{\delta c}\frac{\delta}{\delta\omega^c} + \frac{\delta S}{\delta\omega^c}\frac{\delta}{\delta c} + b\frac{\delta S}{\delta\bar{c}} + \frac{\delta S}{\delta\omega^B_{\mu\nu}}\frac{\delta}{\delta B_{\mu\nu}} + \frac{\delta S}{\delta B_{\mu\nu}}\frac{\delta}{\delta\omega^B_{\mu\nu}}\right.$$
$$+ \frac{\delta S}{\delta\omega^{\bar{B}}_{\mu\nu}}\frac{\delta}{\delta\bar{B}_{\mu\nu}} + \frac{\delta S}{\delta\bar{B}_{\mu\nu}}\frac{\delta}{\delta\omega^{\bar{B}}_{\mu\nu}} + \frac{\delta S}{\delta\omega^{\psi}_{\mu\nu}}\frac{\delta}{\delta\psi_{\mu\nu}} + \frac{\delta S}{\delta\psi_{\mu\nu}}\frac{\delta}{\delta\omega^{\psi}_{\mu\nu}} + \frac{\delta S}{\delta\omega^{\bar{\psi}}_{\mu\nu}}\frac{\delta}{\delta\bar{\psi}_{\mu\nu}} + \frac{\delta S}{\delta\bar{\psi}_{\mu\nu}}\frac{\delta}{\delta\omega^{\bar{\psi}}_{\mu\nu}}$$
$$+ \frac{\delta S}{\delta\omega^{Q}_{\mu\nu\alpha\beta}}\frac{\delta}{\delta Q_{\mu\nu\alpha\beta}} + \frac{\delta S}{\delta Q_{\mu\nu\alpha\beta}}\frac{\delta}{\delta\omega^{Q}_{\mu\nu\alpha\beta}} + \frac{\delta S}{\delta\omega^{J}_{\mu\nu\alpha\beta}}\frac{\delta}{\delta J_{\mu\nu\alpha\beta}} + \frac{\delta S}{\delta J_{\mu\nu\alpha\beta}}\frac{\delta}{\delta\omega^{J}_{\mu\nu\alpha\beta}}$$
$$\left. + \frac{\delta S}{\delta\omega^{\bar{Q}}_{\mu\nu\alpha\beta}}\frac{\delta}{\delta\bar{Q}_{\mu\nu\alpha\beta}} + \frac{\delta S}{\delta\bar{Q}_{\mu\nu\alpha\beta}}\frac{\delta}{\delta\omega^{\bar{Q}}_{\mu\nu\alpha\beta}} + \frac{\delta S}{\delta\omega^{\bar{J}}_{\mu\nu\alpha\beta}}\frac{\delta}{\delta\bar{J}_{\mu\nu\alpha\beta}} + \frac{\delta S}{\delta\bar{J}_{\mu\nu\alpha\beta}}\frac{\delta}{\delta\omega^{\bar{J}}_{\mu\nu\alpha\beta}}\right]. \tag{5.29}$$

Furthermore, the $\mathcal{U}^{(0)}$ and $\tilde{\mathcal{U}}^{(0)}$ symmetries are combined to define the operator $\mathcal{Q}$ as

$$\mathcal{Q} \equiv \delta_{\alpha\mu}\delta_{\beta\nu}\left(\mathcal{U}^{(0)}_{\alpha\beta\mu\nu} + \tilde{\mathcal{U}}^{(0)}_{\alpha\beta\mu\nu}\right). \tag{5.30}$$

Notice that the action is invariant under $\mathcal{Q}$, i.e. $\mathcal{Q}(S) = 0$ because of $\mathcal{U}^{(0)}_{\alpha\beta\mu\nu}(S) = -\tilde{\mathcal{U}}^{(0)}_{\alpha\beta\mu\nu}(S)$.

[5]We will see below, why the application of the algebraic procedure finally fails.

Having defined the operators $\mathcal{B}_S$, $\bar{\mathcal{G}}$, $\mathcal{Q}$ and $\mathcal{U}^{(1)}$ we may derive the following set of graded commutators:

$$\begin{aligned}
&\{\bar{\mathcal{G}},\bar{\mathcal{G}}\} = 0\,, && \{\mathcal{B}_S,\mathcal{B}_S\} = 0\,, && \{\bar{\mathcal{G}},\mathcal{B}_S\} = 0\,,\\
&[\bar{\mathcal{G}},\mathcal{Q}] = 0\,, && [\mathcal{Q},\mathcal{Q}] = 0\,, && \left\{\bar{\mathcal{G}},\mathcal{U}^{(1)}_{\mu\nu\alpha\beta}\right\} = 0\,,\\
&\left\{\mathcal{B}_S,\mathcal{U}^{(1)}_{\mu\nu\alpha\beta}\right\} = 0\,, && \left\{\mathcal{U}^{(1)}_{\mu\nu\alpha\beta},\mathcal{U}^{(1)}_{\mu'\nu'\alpha'\beta'}\right\} = 0\,, && \left[\mathcal{U}^{(1)}_{\mu\nu\alpha\beta},\mathcal{Q}\right] = 0\,,\\
&[\mathcal{B}_S,\mathcal{Q}] = 0\,,
\end{aligned} \tag{5.31}$$

which shows that the algebra of symmetries closes.

5.4 Feynman rules

5.4.1 Propagators

From the action (5.19) with $J/\bar{J}$ and $Q/\bar{Q}$ set to their physical values given by (5.16) one finds the propagators

$$G^{\bar{c}c}(k) = -\frac{1}{k^2}\,, \tag{5.32a}$$

$$G^{\bar{\psi}\psi}_{\mu\nu,\rho\sigma}(k) = \frac{(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho})}{2\mu^2k^2\tilde{k}^2}\,, \tag{5.32b}$$

$$G^{AA}_{\mu\nu}(k) = \frac{1}{\left(k^2+\frac{a'^2}{\tilde{k}^2}\right)}\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\,, \tag{5.32c}$$

$$G^{AB}_{\mu,\rho\sigma}(k) = \frac{\mathrm{i}a'}{2\mu}\frac{(k_\rho\delta_{\mu\sigma}-k_\sigma\delta_{\mu\rho})}{k^2\tilde{k}^2\left(k^2+\frac{a'^2}{\tilde{k}^2}\right)} = G^{A\bar{B}}_{\mu,\rho\sigma}(k) = -G^{\bar{B}A}_{\rho\sigma,\mu}(k)\,, \tag{5.32d}$$

$$G^{\bar{B}B}_{\mu\nu,\rho\sigma}(k) = \frac{-1}{2\mu^2k^2\tilde{k}^2}\left[\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}-a'^2\frac{k_\mu k_\rho\delta_{\nu\sigma}+k_\nu k_\sigma\delta_{\mu\rho}-k_\mu k_\sigma\delta_{\nu\rho}-k_\nu k_\rho\delta_{\mu\sigma}}{2k^2\tilde{k}^2\left(k^2+\frac{a'^2}{\tilde{k}^2}\right)}\right]\,, \tag{5.32e}$$

$$G^{BB}_{\mu\nu,\rho\sigma}(k) = \frac{a'^2}{2\mu^2k^2\tilde{k}^2}\left[\frac{k_\mu k_\rho\delta_{\nu\sigma}+k_\nu k_\sigma\delta_{\mu\rho}-k_\mu k_\sigma\delta_{\nu\rho}-k_\nu k_\rho\delta_{\mu\sigma}}{2k^2\tilde{k}^2\left(k^2+\frac{a'^2}{\tilde{k}^2}\right)}\right] = G^{\bar{B}\bar{B}}_{\mu\nu,\rho\sigma}(k)\,, \tag{5.32f}$$

where the abbreviation $a' \equiv \lambda/\mu$ is used. Notice, that they obey the following symmetries and relations:

$$G^{AB}_{\mu,\rho\sigma}(k) = G^{A\bar{B}}_{\mu,\rho\sigma}(k) = -G^{BA}_{\rho\sigma,\mu}(k) = -G^{\bar{B}A}_{\rho\sigma,\mu}(k), \tag{5.33a}$$

$$G^{\phi}_{\mu\nu,\rho\sigma}(k) = -G^{\phi}_{\nu\mu,\rho\sigma} = -G^{\phi}_{\mu\nu,\sigma\rho}(k) = G^{\phi}_{\nu\mu,\sigma\rho}(k), \tag{5.33b}$$

$$\text{for } \phi \in \{\bar{\psi}\psi, \bar{B}B, BB, \bar{B}\bar{B}\},$$

$$2k^2\tilde{k}^2 G^{AB}_{\rho,\mu\nu}(k) = \mathrm{i}\frac{a'}{\mu}\left(k_\mu G^{AA}_{\rho\nu}(k) - k_\nu G^{AA}_{\rho\mu}(k)\right), \tag{5.33c}$$

$$\frac{1}{\mu^2}\left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right) = \mathrm{i}\frac{a'}{\mu}\left(k_\mu G^{BA}_{\rho\sigma,\nu}(k) - k_\nu G^{BA}_{\rho\sigma,\mu}(k)\right) - 2k^2\tilde{k}^2 G^{B\bar{B}}_{\mu\nu,\rho\sigma}(k), \tag{5.33d}$$

$$0 = \mathrm{i}\frac{a'}{\mu}\left(k_\mu G^{BA}_{\rho\sigma,\nu}(k) - k_\nu G^{BA}_{\rho\sigma,\mu}(k)\right) - 2k^2\tilde{k}^2 G^{BB}_{\mu\nu,\rho\sigma}(k), \tag{5.33e}$$

$$G^{B\bar{B}}_{\mu\nu,\rho\sigma}(k) = G^{\bar{\psi}\psi}_{\mu\nu,\rho\sigma}(k) + G^{BB}_{\mu\nu,\rho\sigma}(k). \tag{5.33f}$$

Note that the relations (5.33c) to (5.33e) directly follow from the equations of motion for the fields $B_{\mu\nu}$ and $\bar{B}_{\mu\nu}$.

5.4.2 Vertices

The action (5.19) leads to 13 tree level vertices whose rather lengthy expressions are listed in Appendix D.3. One immediately finds the following vertex relation:

$$\widetilde{V}^{\bar{\psi}\psi(n\times A)}_{\mu\nu,\rho\sigma,\xi_1\dots\xi_n}(q_1,q_2,k_{\xi_1},\dots k_{\xi_n}) = -\widetilde{V}^{\bar{B}B(n\times A)}_{\mu\nu,\rho\sigma,\xi_1\dots\xi_n}(q_1,q_2,k_{\xi_1},\dots k_{\xi_n}), \tag{5.34}$$

i.e. all vertices with one B, one $\bar{B}$ and an arbitrary number of A legs have exactly the same form as the ones with one ψ, one $\bar{\psi}$ and an arbitrary number of A legs. This is due to the fact that the $\bar{\psi}\psi nA$ and $\bar{B}BnA$ vertices stem from terms in the action which are of the same structure, and are thus equal in their form. We therefore expect all divergent contributions to the vacuum polarization coming from the ψ sector to exactly cancel those coming from the B sector. We will come back to this point in Section 5.6.

Finally, the vertices obey the following additional relations:

$$\begin{aligned}\widetilde{V}^{\bar{\psi}\psi(n\times A)}_{\mu\nu,\rho\sigma,\xi_1\dots\xi_n}(q_1,q_2,k_{\xi_1},\dots k_{\xi_n}) &= -\widetilde{V}^{\psi\bar{\psi}(n\times A)}_{\rho\sigma,\mu\nu,\xi_1\dots\xi_n}(q_2,q_1,k_{\xi_1},\dots k_{\xi_n})\\ &= -\widetilde{V}^{\bar{\psi}\psi(n\times A)}_{\nu\mu,\rho\sigma,\xi_1\dots\xi_n}(q_1,q_2,k_{\xi_1},\dots k_{\xi_n})\\ &= -\widetilde{V}^{\bar{\psi}\psi(n\times A)}_{\mu\nu,\sigma\rho,\xi_1\dots\xi_n}(q_1,q_2,k_{\xi_1},\dots k_{\xi_n}),\end{aligned} \tag{5.35}$$

and

$$\begin{aligned}\widetilde{V}^{\bar{B}B(n\times A)}_{\mu\nu,\rho\sigma,\xi_1\dots\xi_n}(q_1,q_2,k_{\xi_1},\dots k_{\xi_n}) &= +\widetilde{V}^{B\bar{B}(n\times A)}_{\rho\sigma,\mu\nu,\xi_1\dots\xi_n}(q_2,q_1,k_{\xi_1},\dots k_{\xi_n})\\ &= -\widetilde{V}^{\bar{B}B(n\times A)}_{\nu\mu,\rho\sigma,\xi_1\dots\xi_n}(q_1,q_2,k_{\xi_1},\dots k_{\xi_n})\\ &= -\widetilde{V}^{\bar{B}B(n\times A)}_{\mu\nu,\sigma\rho,\xi_1\dots\xi_n}(q_1,q_2,k_{\xi_1},\dots k_{\xi_n}),\\ &\text{for } n\in 1,2,3,4\,.\end{aligned} \tag{5.36}$$

In the first line on the r.h.s. of (5.35) we observe a minus sign, contrary to (5.36). This is due to the reversed order of variation with respect to the fermionic fields ψ and $\bar{\psi}$. More specifically, this can be understood as follows (indices are suppressed):

- The vertex $\widetilde{V}^{\bar{\psi}\psi(n\times A)}$ stems from a term in the action with the same fields in a given order, e.g. it is proportional to the integrated product $\bar{\psi}\psi(n \times A)$.
- According to (3.40) the vertex under consideration will be obtained by functional variation of this action term, i.e.

$$\begin{aligned}\widetilde{V}^{\bar{\psi}\psi(n\times A)} &\propto +\frac{\delta}{\delta\bar{\psi}}\frac{\delta}{\delta\psi}\left(\frac{\delta}{\delta A}\right)^n \int \mathrm{d}^4x\bar{\psi}\psi A_1 \dots A_n \\ &= -\frac{\delta}{\delta\psi}\frac{\delta}{\delta\bar{\psi}}\left(\frac{\delta}{\delta A}\right)^n \int \mathrm{d}^4x\bar{\psi}\psi A_1 \dots A_n \,. \end{aligned} \tag{5.37}$$

  In the first line we have to pull the derivation with respect to ψ through the field $\bar{\psi}$. Due to their fermionic character, this leads to a overall minus.

5.5 Power counting

The superficial degree of UV divergence is determined by the number of external legs of the various fields denoted by E. Its explicit form is given by:

$$d_\gamma = 4 - E_A - E_{c/\bar{c}} - 2E_B - 2E_{\bar{B}} - 2E_{\psi\bar{\psi}} - 2E_\theta \,, \tag{5.38a}$$

$$d_\gamma = 4 - E_A - E_{c/\bar{c}} - 2E_\lambda \,, \tag{5.38b}$$

where

$$E_\lambda = E_B + E_{\bar{B}} + E_{\psi/\bar{\psi}} + E_\theta \,, \tag{5.39}$$

and E_θ counts negative powers of θ. It is possible that E_θ becomes negative. Hence, the first version (counting E_λ, i.e. the overall powers of λ in a graph) is probably more useful, as $E_\lambda \geq 0$.

5.6 One-loop calculations

As the main objective of this section we will investigate the possibility of renormalizing the IR divergences at one-loop level. Given the high number of vertices and propagators compared to ordinary (i.e. commutative) Yang-Mills theory, this directly translates into a very high number of one loop graphs. A short overview of the number of Feynman graphs for each type is given in Tab. 5.3. However, the main results can already be deduced by considering the vacuum expectation value of the gauge field, c.f. Section 5.6.3. In order to get there, all two point functions allowing to connect both of their external (amputated) legs to gauge propagators have been evaluated and calculated. In order to master the complexity and high effort regarding calculations also for this limited task, algorithms have been developed with *Mathematica*®, which are explained in Section F. Due to the lenghty results, details have been collected in Section D.4. In the following we will limit ourselves to the discussion of the main observations and results.

5.6.1 Vanishing tadpoles

As has been the case for the previous model with dynamical $B_{\mu\nu}$, the tadpole graphs (defined by having only one external field) all vanish due to momentum conservation. Remember that, since all of these graphs have only one vertex, there exists a delta function $\delta^4(p+k-k) = \delta^4(p)$, i.e. $p \to 0$ where p is the external momentum and k is the internal momentum of the loop. In consequence, $\int d^4k \sin\frac{k\theta p}{2}\delta^4(p) = 0$. This is the case for all tadpoles, independent of the external leg, because all 3-vertices contain a sine in their analytical expressions.

5.6.2 Two point functions with amputated external legs

For each type of two-point functions, being characterized by the kind of (amputated) external legs[6], the result is obtained based on the following method:

- Evaluation of all possible graphs.
- Evaluation of their superficial degree of divergence, based on power counting of Section 5.5. Due to our interest in the divergence behaviour of theory, only the divergent graphs will be considered further.
- Evaluation of the analytic expressions of the remaining graphs and expansion up to second order of the integrands according to (3.47). Note that we expand up to second order because of the appearance of utmost quadratic divergences.
- Summing up all contributions, order by order, and considering planar/nonplanar expressions individually. Evaluation of the divergent part of the integrals[7].
- Finally, the result is obtained by summing up the planar and non planar integrated results of each order.

While the final results and any relevant observations are given in what follows, for more detailed information please refer to the appendix, Section D.4.

Vacuum polarization

The action (5.19) gives rise to eleven divergent graphs for the one-loop correction to the vacuum polarization. The final result is given by

$$\Pi_{\mu\nu}(p) = \frac{2g^2}{\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\tilde{p}^2)^2} - \frac{5g^2}{12\pi^2}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\ln\left(\Lambda^2\right) + \text{finite terms}\,. \tag{5.40}$$

It exhibits a quadratic IR divergence in $\tilde{p}^2$ and a logarithmic divergence in the cutoff Λ. Furthermore, the transversality condition $p_\mu\Pi_{\mu\nu}(p) = 0$ is fulfilled.

[6]For the conventions regarding the graphical representation of the fields refer to Section D.2

[7]It should be obvious that also the divergent integrals may still contain a finite part.

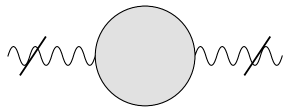


Figure 5.1: Total two point function for the $A_\mu A_\nu$ propagator with amputated external legs.

Note, that explicit calculations reveal that the graphs coming from the $\psi/\bar\psi$ sector cancel with those from the $B/\bar{B}$ sector[8]. This was expected, as mentioned above. What remains, are the three "classical" graphs known from commutative Yang-Mills theory, e.g. QCD given in Fig. 5.2, which are transversal on their own.

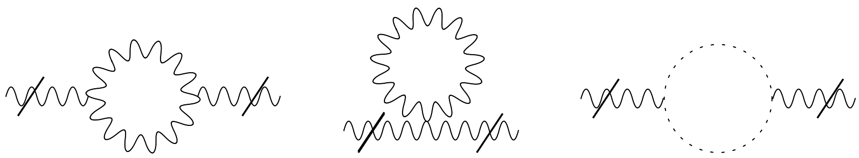


Figure 5.2: The three "classical" one loop Feynman graphs for the vacuum polarization.

Corrections to the AB and $A\bar{B}$ propagator

The action (5.19) gives rise to eight divergent graphs with one external A_μ and one $B_{\mu\nu}$. The sum of all divergent contributions (c.f. Fig. 5.3) is given by

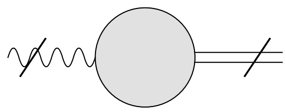


Figure 5.3: Total two point function for the $A_\mu B_{\nu_1\nu_2}$ propagator with amputated external legs.

$$\Sigma^{\mathrm{AB}}_{\mu 1,\nu 1\nu 2}(p) = \frac{3\mathrm{i}g^2}{32\pi^2}\lambda\left(p_{\nu 1}\delta_{\mu 1\nu 2} - p_{\nu 2}\delta_{\mu 1\nu 1}\right)\left(\ln\Lambda + \ln|\tilde{p}|\right) + \text{finite}\,, \tag{5.41}$$

which shows a logarithmic divergence for $\Lambda \to \infty$.
From the symmetry between B and $\bar{B}$ in the sense that both have identical interactions with the gauge field, it follows

$$\Sigma^{\mathrm{AB}}_{\mu 1,\nu 1\nu 2} \equiv \Sigma^{\mathrm{A}\bar{B}}_{\mu 1,\nu 1\nu 2}\,, \tag{5.42}$$

and as implied by Eqn. (5.33a) it follows furthermore

$$\Sigma^{\mathrm{BA}}_{\mu 1\mu 2,\nu 1} \equiv -\Sigma^{\mathrm{AB}}_{\nu 1,\mu 1\mu 2}. \tag{5.43}$$

[8] This can be seen by inserting the symmetry factors of Tab. D.1 into the Taylor expanded expressions given in (D.5b) and (D.6b).

Corrections to the BB and $\bar{B}\bar{B}$ propagator

The action (5.19) gives rise to nine divergent graphs with two external $B_{\mu\nu}$. The sum

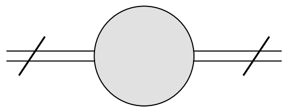


Figure 5.4: Total two point function for the $B_{\mu_1\mu_2}B_{\nu_1\nu_2}$ propagator with amputated external legs.

of all divergent contributions (c.f. Fig. 5.4) is given by

$$\Sigma^{\mathrm{BB}}_{\mu1\mu2,\nu1\nu2}(p) = \frac{g^2\lambda^2}{64\pi^2}\left(\delta_{\mu1\nu1}\delta_{\mu2\nu2} - \delta_{\mu2\nu1}\delta_{\mu1\nu2}\right)\left(\ln\Lambda^2 + \ln\tilde{p}^2\right) + \text{finite}\,, \tag{5.44}$$

leading a logarithmic divergence for both the planar and the non-planar part.
Due to symmetry reasons this result is also equal to the according correction to the $\bar{B}\bar{B}$ propagator, i.e.

$$\Sigma^{\bar{\mathrm{B}}\bar{\mathrm{B}}}_{\mu1\mu2,\nu1\nu2}(p) = \Sigma^{\mathrm{BB}}_{\mu1\mu2,\nu1\nu2}(p)\,. \tag{5.45}$$

Corrections to the $B\bar{B}$ propagator

The action (5.19) gives rise to ten divergent graphs with one external $B_{\mu\nu}$ and one external $\bar{B}_{\mu\nu}$. The sum of all divergent contributions (c.f. Fig. 5.5) is given by

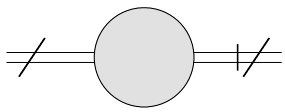


Figure 5.5: Total two point function for the $B_{\mu_1\mu_2}\bar{B}_{\nu_1\nu_2}$ propagator with amputated external legs.

$$\begin{aligned}\Sigma^{\mathrm{B}\bar{B}}_{\mu1\mu2,\nu1\nu2}(p) = {} & \frac{g^2}{2\pi^2}\Lambda^2\mu^2\tilde{p}^2\left(\delta_{\mu2\nu1}\delta_{\mu1\nu2} - \delta_{\mu1\nu1}\delta_{\mu2\nu2}\right) \\ & + \frac{g^2\lambda^2}{64\pi^2}\left(\delta_{\mu1\nu1}\delta_{\mu2\nu2} - \delta_{\mu2\nu1}\delta_{\mu1\nu2}\right)\left(\ln\Lambda^2 + \ln\tilde{p}^2\right) + \text{finite}\,,\end{aligned} \tag{5.46}$$

which is logarithmically divergent in $\tilde{p}^2$ and quadratically in Λ. Furthermore the relation

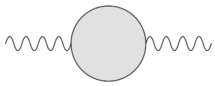
$$\Sigma^{\mathrm{B}\bar{\mathrm{B}}}_{\mu1\mu2,\nu1\nu2} \equiv \Sigma^{\bar{\mathrm{B}}\mathrm{B}}_{\nu1\nu2,\mu1\mu2} \tag{5.47}$$

holds.

5.6.3 The complete gauge propagator and attempt for renormalization

The complete gauge boson propagator up to one loop

In the standard renormalization procedure, the complete (also called dressed) propagator up to the first loop level, is given by



$$\equiv \Delta'(p) = \frac{1}{\mathcal{A}} + \frac{1}{\mathcal{A}}\,\Sigma(\Lambda,p)\,\frac{1}{\mathcal{A}}\,, \tag{5.48}$$

where

$$\frac{1}{\mathcal{A}} \equiv G^{\mathrm{AA}}_{\mu\nu}(p)\,,$$
$$\Delta'(p) \equiv G^{\mathrm{AA,1l-complete}}_{\mu\nu}(p) \qquad\qquad \Sigma(\Lambda,p) \equiv \left(\Pi^{\mathrm{plan}}\right)_{\mathrm{regul.}}(\Lambda,p) + \Pi^{\mathrm{n\text{-}pl}}(p)\,,$$

where "regul." indicates that the planar contribution has been regularized by the UV cutoff Λ, and "complete" stands for the complete or dressed one loop propagator. For $\mathcal{A} \neq 0$, one can apply the formula

$$\frac{1}{\mathcal{A}+\mathcal{B}} = \frac{1}{\mathcal{A}} - \frac{1}{\mathcal{A}}\,\mathcal{B}\,\frac{1}{\mathcal{A}+\mathcal{B}} = \frac{1}{\mathcal{A}} - \frac{1}{\mathcal{A}}\,\mathcal{B}\,\frac{1}{\mathcal{A}} + \mathcal{O}(\mathcal{B}^2)\,, \tag{5.49}$$

which allows one to rewrite expression (5.48) to order Σ as

$$\Delta'(p) = \frac{1}{\mathcal{A} - \Sigma(\Lambda,p)}\,, \tag{5.50}$$

and thus (in the case of renormalizability) to absorb any divergences in the appropriate parameters of the theory present in $\mathcal{A}$ (see [49] for an example).

However, in our case (5.49) cannot be applied directly, as the complete one loop correction to the gauge boson propagator is given by the sum of all the results of section 5.6.2 after multiplication with *appropriate, i.e. different* external legs:

$$\begin{aligned}
G^{\mathrm{AA,1l-complete}}_{\mu\nu}(p) = G^{\mathrm{AA}}_{\mu\nu}(p) &+ G^{\mathrm{AA}}_{\mu\rho}(p)\Pi_{\rho\sigma}(p)G^{\mathrm{AA}}_{\sigma\nu}(p) \\
&+ G^{\mathrm{AA}}_{\mu\rho}(p)2\Sigma^{\mathrm{AB}}_{\rho,\sigma1\sigma2}(p)G^{\mathrm{BA}}_{\sigma1\sigma2,\nu}(p) \\
&+ G^{\mathrm{AA}}_{\mu\rho}(p)2\Sigma^{\mathrm{A\bar{B}}}_{\rho,\sigma1\sigma2}(p)G^{\mathrm{\bar{B}A}}_{\sigma1\sigma2,\nu}(p) \\
&+ G^{\mathrm{AB}}_{\mu,\rho1\rho2}(p)\Sigma^{\mathrm{BB}}_{\rho1\rho2,\sigma1\sigma2}(p)G^{\mathrm{BA}}_{\sigma1\sigma2,\nu}(p) \\
&+ G^{\mathrm{AB}}_{\mu,\rho1\rho2}(p)2\Sigma^{\mathrm{B\bar{B}}}_{\rho1\rho2,\sigma1\sigma2}(p)G^{\mathrm{\bar{B}A}}_{\sigma1\sigma2,\nu}(p) \\
&+ G^{\mathrm{A\bar{B}}}_{\mu,\rho1\rho2}(p)\Sigma^{\mathrm{\bar{B}\bar{B}}}_{\rho1\rho2,\sigma1\sigma2}(p)G^{\mathrm{\bar{B}A}}_{\sigma1\sigma2,\nu}(p) + \mathcal{O}\left(g^4\right)\,.
\end{aligned} \tag{5.51}$$

Note, that the factors 2 stem from the (not explicitly written) mirrored contributions $AB \leftrightarrow BA$, $A\bar{B} \leftrightarrow \bar{B}A$, and $B\bar{B} \leftrightarrow \bar{B}B$. Since the factor $\mathcal{A}$ must be the same for all summands we have to use the Ward Identities (5.33a) and (5.33c), i.e.

$$
\begin{aligned}
G^{AB}_{\mu,\rho\sigma}(k) &= G^{A\bar{B}}_{\mu,\rho\sigma}(k) = -G^{BA}_{\rho\sigma,\mu}(k) = -G^{\bar{B}A}_{\rho\sigma,\mu}(k) \\
2k^2\tilde{k}^2 G^{AB}_{\rho,\mu\nu}(k) &= \mathrm{i}\frac{a'}{\mu}\left(k_\mu G^{AA}_{\rho\nu}(k) - k_\nu G^{AA}_{\rho\mu}(k)\right),
\end{aligned}
\tag{5.52}
$$

which allow us to express the (tree level) AB and $A\bar{B}$ propagators uniquely in terms of AA-propagators. This leads (in analogy to (5.49)) to the following representation for the dressed one-loop gauge boson propagator:

$$
G^{\mathrm{AA,1l-complete}}_{\mu\nu}(p) = \frac{1}{\mathcal{A}} - \frac{1}{\mathcal{A}}\Big(\sum \mathcal{B}_i\Big)\frac{1}{\mathcal{A}}\,,
\tag{5.53}
$$

where $1/\mathcal{A}$ once more stands for the tree level gauge boson propagator. The $\mathcal{B}_i$'s are given by the one-loop corrections (with amputated external legs) of the two-point functions relevant for the dressed gauge boson propagator, multiplied by any prefactors coming from (5.52) and the factor 2 where needed (c.f. (5.51)). Thus, the full propagator is given by

$$
\begin{aligned}
&G^{\mathrm{AA,1l-complete}}_{\mu\nu}(p) = G^{\mathrm{AA}}_{\mu\nu}(p) + G^{\mathrm{AA}}_{\mu\rho}(p)\Pi_{\rho\sigma}(p)G^{\mathrm{AA}}_{\sigma\nu}(p) \\
&+\left(\frac{\mathrm{i}a'}{\mu p^2\tilde{p}^2}\right)\Bigg\{2G^{\mathrm{AA}}_{\mu\rho}(p)\left(\Sigma^{\mathrm{AB}}_{\rho,\sigma1\sigma2}(p) + \Sigma^{\mathrm{A\bar{B}}}_{\rho,\sigma1\sigma2}(p)\right)p_{\sigma2}G^{\mathrm{AA}}_{\nu\sigma1}(p) \\
&\quad+\left(\tfrac{\mathrm{i}a'}{\mu p^2\tilde{p}^2}\right)p_{\rho1}G^{\mathrm{AA}}_{\mu\rho2}(p)\left(\Sigma^{\mathrm{BB}}_{\rho1\rho2,\sigma1\sigma2}(p) + 2\Sigma^{\mathrm{B\bar{B}}}_{\rho1\rho2,\sigma1\sigma2}(p) + \Sigma^{\mathrm{\bar{B}\bar{B}}}_{\rho1\rho2,\sigma1\sigma2}(p)\right)p_{\sigma2}G^{\mathrm{AA}}_{\nu\sigma1}(p)\Bigg\}.
\end{aligned}
\tag{5.54}
$$

The expression $\mathcal{B} = \sum\limits_i \mathcal{B}_i$ can now be extracted from (5.54) and is explicitly given by

$$
\begin{aligned}
\mathcal{B} =& \frac{g^2}{8\pi^2\mu^4}\Bigg\{\tilde{p}_\mu\tilde{p}_\nu\left(\frac{16\mu^4}{(\tilde{p}^2)^2} + \frac{\theta^4\lambda^4}{2(\tilde{p}^2)^4}\right) - 7\lambda^2\mu^2\frac{\theta^4}{(\tilde{p}^2)^4}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\left(4 - \tilde{p}^2\Lambda^2\right) \\
&\qquad + \left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\left[\ln 2 - \ln\tilde{p}^2 - \ln\Lambda\right]\left(\frac{5}{3}\mu^4 + \frac{3\lambda^2\mu^2\theta^2}{(\tilde{p}^2)^2} + \frac{\lambda^4\theta^4}{(\tilde{p}^2)^4}\right)\Bigg\} \\
&+ \text{finite}\,.
\end{aligned}
\tag{5.55}
$$

Next we need to derive the inverse of the tree level gauge boson propagator (5.32c)[9] $\mathcal{A} \equiv \Gamma^{AA}_{\mu\nu}(p)$. According to (D.4.1) we get, after writing the gauge fixing term of the action (5.19) with general α,

$$
\mathcal{A}_{\mu\nu} \equiv \Gamma^{AA}_{\mu\nu}(p) = \left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)\left\{\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \frac{1}{\alpha}\frac{k_\mu k_\nu}{\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)}\right\}
$$

[9] A remark regarding the symbolic notation: at the beginning of this section the propagator $G^{AA}_{\mu\nu}$ has been identified with $\frac{1}{\mathcal{A}}$. Now one could naively think of $\mathcal{A}$ to be simply its inverse. However, this would imply a index structure in the denominator. Therefore note that in a more stringent (less lazy) way we would have to write $\frac{1}{\mathcal{A}} \equiv (\frac{1}{\mathcal{A}})_{\mu\nu}$ and $\mathcal{A} \equiv (\mathcal{A})_{\mu\nu}$, where the second is defined via requiring $(\frac{1}{\mathcal{A}})_{\mu\nu}(\mathcal{A})_{\nu\rho} = \delta_{\mu\rho}$.

$$= \left(k^2 + \frac{a'^2}{\tilde{k}^2}\right) \left\{ \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \left(1 - \frac{1}{\alpha} \frac{k^4}{k^4 + a^2}\right) \right\}. \tag{5.56}$$

With the last two equations we can now apply (5.50), i.e. in the present case

$$G^{\mathrm{AA,1l-complete}}_{\mu\nu}(p) = \frac{1}{\mathcal{A} - \mathcal{B}} \ . \tag{5.57}$$

Attempt for renormalization and discussion

Before proceeding to the renormalization of the gauge boson propagator, let us briefly recapitulate the notion of renormalization (c.f. for example [66] chapter 9, section 6.).

- The bare Lagrangian $\mathcal{L}_B$ up to the first loop order of the action is obtained by adding to the original tree level Lagrangian $\mathcal{L}_0$ (here 0 denotes zero loop order) the expression $\mathcal{L}_1$. It is constructed in a way that the resulting propagator is $G^{\mathrm{AA,1l-complete}}$, i.e.

$$\int \mathrm{d}^4x \mathcal{L}_0(g, m, \dots) \Rightarrow G^{\mathrm{AA}}_{\mu\nu}(p) \,,$$
$$\int \mathrm{d}^4x \mathcal{L}_B(g, m, \Lambda, \dots) \Rightarrow G^{\mathrm{AA,1l-complete}}_{\mu\nu}(p), \ .$$

  In this symbolic expressions, we have made explicit the dependence of the action on the parameters of the theory (g is the coupling constant, m the mass, Λ a UV cutoff introduced by regularization of divergent expressions. Whereas the first two are physical parameters, the latter is called unphysical).

- If the terms present in $\mathcal{L}_1$ are of the same form as those of $\mathcal{L}_0$, than they can be combined by defining the bare or renormalized parameters of the theory, leading to a action which is of the same form as $\mathcal{L}_0$, but where the original parameters have been replaced by the renormalized parameters:

$$(g, m, \Lambda, \dots) \Rightarrow (g_R, m_R, \dots) \,, \Leftrightarrow$$
$$\int \mathrm{d}^4x \mathcal{L}_B(g, m, \Lambda, \dots) \Rightarrow \int \mathrm{d}^4x \mathcal{L}_B(g_R, m_R, \dots)$$

  The form invariance can be expressed via renormalization conditions, for example by stating for the two-point function of the scalar ϕ^4-theory at the tree level obeys $\Gamma^{(2)}_{(0)}(p)\big|_{p^2=0} = m^2$. Then, if they are still valid after loop calculations, i.e. if $\Gamma^{(2)}_{(1)}(p)\big|_{p^2=0} = m_R^2$ in the present example, this is equivalent with the form invariance or stability of the Langrangian.
  Note that the UV cutoff Λ is not present any more in the action, because it has been

absorbed in the physical parameters. In this approach, the bare or renormalized quantities are the physical quantities of the theory[10].

- Equivalently we can write down the one loop propagator (or its inverse, the vertex function), examine the possibility of absorbing the divergences in the parameters of the tree level propagator, leading to renormalized parameters, and introduce them in the action. The two equivalent possibilities are summarized as follows, applied to the case under consideration, i.e. the renormalization of the gauge boson propagator, which depends on the parameter[11] $a' = \lambda/\mu$.

$$\begin{array}{cc} \mathcal{L}_0(a',g) & \mathcal{L}_0(a',g) \\ \downarrow & \downarrow \\ G^{\mathrm{AA}}_{\mu\nu}(a') + \text{Feynman rules} & G^{\mathrm{AA}}_{\mu\nu}(a') + \text{Feynman rules} \\ \downarrow & \downarrow \\ \text{1 loop calculations} & \text{1 loop calculations} \\ \downarrow & \downarrow \\ G^{\mathrm{AA}}_{\mu\nu,1l-complete}(a') & \mathcal{L}_B(a',g,\Lambda) = \mathcal{L}_0(a',g) + \mathcal{L}_1(a',g,\Lambda) \\ \downarrow & \downarrow \\ a' \to a'_R,\ G^{\mathrm{1l-ren,AA}}_{\mu\nu}(a'), & a' \to a'_R,\ \mathcal{L}_{ren}(a'_R,g) \\ \downarrow & \downarrow \\ \mathcal{L}_{ren}(a'_R,g) & G^{\mathrm{1l-ren,AA}}_{\mu\nu}(a') \end{array}$$

  We will follow the approach on the l.h.s.

- If the loop calculations lead to terms which can not be absorbed in the parameters of the theory, the form of the action will be modified, and the *original* theory $\mathcal{L}_0$ is not renormalizable. Symbolically,

$$\text{if} \quad \mathcal{L}_B(a',g,\Lambda) \equiv \mathcal{L}_{ren}(a'_R,g,\Lambda) + \mathcal{L}'_1(a',g,\Lambda) \Longrightarrow \text{non renormalizable}\,,$$

  where $\mathcal{L}_{ren} = \mathcal{L}_0 + \mathcal{L}''_1$ is the sum of the tree level action and the one loop part that can be absorbed in the parameters of the first. $\mathcal{L}'_1$ denotes the part of $\mathcal{L}_1$ that leads to a modification of the form of the action, which is equivalent to a breaking of the renormalization conditions.

- Given the case that by loop calculations we encounter divergent terms that cannot be absorbed in the parameters of the theory. If their number is bounded, we could introduce new terms in the action, so called counterterms, which compensate them. This would then lead to a new theory given by the so called *effective action*, which is renormalizable. However, if the number of divergent loops increases with the order, no renormalizable action can be defined, as the effective action will never become stable.

[10]Note that other authors use different conventions. E.g. in [66], the original parameters are the physical ones and therefore finite, and the bare quantities are divergent. In the approach followed here, we assume the original parameters to be divergent; then, adding the divergent expressions from one loop calculations lead to a compensation of the infinite expressions.

[11]Note that wave function renormalization is not being made explicit for reasons of simplicity.

Let us apply the above said to the case under consideration. Making explicit the qualitative divergence structure of $\mathcal{A}$ and $\mathcal{B}$, we can write for the inverse of (5.57) (with $a'^2/\tilde{p}^2 = a^2/p^2$):

$$\Gamma^{AA,1l-complete}_{\mu\nu}(p) = \mathcal{A}\left(p^2, \frac{a'^2}{\tilde{p}^2}\right) + \mathcal{B}\left(p^2, \ln(p^2), \frac{1}{p^2}, \frac{1}{p^4}, \frac{1}{p^6}, \ln(\Lambda), \Lambda^2\right). \tag{5.58}$$

We observe the following: in contrast to commutative gauge models and even though the vacuum polarization tensor $\Pi_{\mu\nu}$ only had a logarithmic UV divergence, the full $\mathcal{B}$ diverges quadratically in the UV cutoff Λ. Secondly, despite the fact that $\Pi_{\mu\nu}$ exhibited the usual quadratic IR divergence, $\mathcal{B}$ behaves like $\frac{1}{(\tilde{p}^2)^3}$ in the IR limit. Both properties arise due to the existence (and the form) of the mixed AB and $A\bar{B}$ propagators. Whereas the first point represents no problem, the second does: from the above discussion follows that terms of the form p^2 and $\frac{1}{p^2}$ can be absorbed by wave function and a' renormalization, while the more divergent terms cannot be absorbed, i.e. the form of the propagator is modified implying new counterterms in the effective action. Therefore, *the original theory is not renormalizable.*
Next it should be examined whether a finite number of counterterms could be introduced, leading to a renormalizable effective action. This is done in the next section, where higher loop orders are examined.

5.7 Higher loop calculations

In the light of attempting to construct a renormalizable effective action, it should be understood whether the number of possible counterterms is finite, or if it increases with the loop order. Motivated by the previous section, which showed that renormalizability of the model (5.18) is impeded by the IR divergences not absorbable in the parameters of the action, we will now investigate the qualitative IR behaviour of higher loop expressions. In other words, the aim is to identify possible poles at $\tilde{p} \to 0$ for loops of higher order. More specifically, we will focus on the integrands of those expressions, which result from inserting the one-loop corrections with amputated external legs of Section 5.6.2.
Due to the high number of possibilities of constructing such graphs, we will solely focus on the IR behaviour of chains of n non-planar insertions denoted by $\Xi^{\phi_1\phi_2}(p,n)$, with the external fields ϕ_1 and ϕ_2. These expressions may be part of a higher loop graph. For example, connecting them with the legs of a vertex leads to a 1PI graph of order $n+1$. More generally it could be inserted in any loop graph of order i, which leads to a graph of order $i+n$. Being interested in the IR divergence structure, we will examine only a few exemplary configurations in this section - especially those for which one expects the worst IR behaviour.
To start with, let us state that amongst all types of two point functions, the vacuum polarization shows the highest, namely a quadratic divergence. Amongst the propagators

those with two external double-indexed legs (B, $\bar{B}$, ψ and $\bar{\psi}$) feature the highest (quartic) divergence in the limit of vanishing external momenta. This directly leads to the most divergent chains, as they are composed by one loop corrections and propagators in between, as investigated in the following.

5.7.1 Chain of vacuum polarizations

A chain of n vacuum polarizations $\Pi^{\mathrm{np}}_{\mu\nu}(p)$ (see Eqns. (D.6a) and (D.6b)) with $(n+1)$ AA-propagators ($(n-1)$ between the individual vacuum polarization graphs, and one at each end) leads to the following expression (for a graphical representation, see Fig. 5.6):

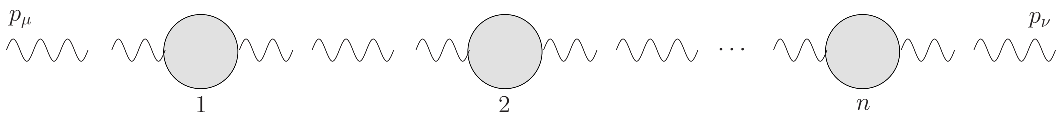


Figure 5.6: A chain of n non-planar insertions, concatenated by gauge field propagators.

$$
\begin{aligned}
\Xi^{AA}_{\mu\nu}(p,n) &= \left(G^{AA}(p)\Pi^{\mathrm{np}}(p)\right)^n_{\mu\rho} G^{AA}_{\rho\nu}(p) \\
&= \left(\frac{2g^2}{\pi^2}\right)^n \frac{1}{\left(p^2+\frac{a'^2}{\tilde{p}^2}\right)^{n+1}} \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^{n+1}} \,.
\end{aligned}
\tag{5.59}
$$

Note that due to transversality, from the propagator (5.32c) only the term with the Kronecker delta enters the calculation. For vanishing momenta, i.e. in the limit $\tilde{p}^2 \to 0$ the expression reduces to

$$
\lim_{\tilde{p}^2 \to 0} \Xi^{AA}_{\mu\nu}(p,n) = \left(\frac{2g^2}{\pi^2}\right)^n \frac{\tilde{p}_\mu \tilde{p}_\nu}{a'^{2(n+1)}} \,,
\tag{5.60}
$$

exhibiting IR finiteness which is independent from the number of inserted loops.

5.7.2 Chain of truncated AB / $A\bar{B}$ one loop corrections

Another representative is the chain

$$
\Xi^{A\phi}(p,n) \equiv G^{A\phi}(p) \left(\Sigma^{\mathrm{np},\phi\mathrm{A}}(p) G^{A\phi}(p)\right)^n \,, \qquad \text{where} \quad \phi \in \{B, \bar{B}\},
$$

which could replace any single G^{AB} (or $G^{A\bar{B}}$) line. Obviously, one has

$$
\Xi^{A\phi}_{\mu,\nu 1\nu 2}(p,n) = \frac{\mathrm{i}a'}{2\mu} \left(-\frac{3g^2}{32\pi^2} a'^2\right)^n \frac{(p_{\nu 1}\delta_{\mu\,\nu 2} - p_{\nu 2}\delta_{\mu\,\nu 1})}{p^2 \left[\tilde{p}^2 \left(p^2 + \frac{a'^2}{\tilde{p}^2}\right)\right]^{n+1}} n \ln \tilde{p}^2 \,,
\tag{5.61}
$$

which for $\tilde{p}^2 \ll 1$ (and neglecting dimensionless prefactors) behaves like

$$\Xi^{A\phi}_{\mu,\nu 1\nu 2}(p,n) \approx n \frac{(p_{\nu 1}\delta_{\mu\,\nu 2} - p_{\nu 2}\delta_{\mu\,\nu 1})}{\mu p^2} \ln \tilde{p}^2 \,. \tag{5.62}$$

The latter insertion can be regularized since the pole at $p = 0$ is independent of n.

5.7.3 Chain including double-indexed propagators

In contrast to the previous examples, higher divergences are expected for chain graphs being concatenated by propagators with four indices, i.e. $G^{\bar{B}B}_{\mu\nu,\rho\sigma}$, $G^{BB}_{\mu\nu,\rho\sigma}$, $G^{\bar{\psi}\psi}_{\mu\nu,\rho\sigma}$, due to the inherent quartic IR singularities. Let us first consider the combination $\Xi^{\bar{B}B}(p,n) \equiv \left(G^{\bar{B}B}(p)\Sigma^{\mathrm{p,B\bar{B}}}(p)\right)^n G^{\bar{B}B}(p)$. As before, we can approximate for $\tilde{p}^2 \ll 1$ and, omitting dimensionless prefactors and indices, and find

$$\Xi^{A\phi}(p,n) \underset{\tilde{p}^2 \ll 1}{\propto} \frac{n}{\mu^2} \frac{\ln \tilde{p}^2}{(p^2\tilde{p}^2)^n} \,, \tag{5.63}$$

which represents a singularity $\forall\, n > 1$ (since in any graph, at $n = 0$, the divergence is regularized by the phase factor being a sine function which behaves like p for small momenta). Regarding the index structures, no cancellations can be expected since the product of an arbitrary number of contracted, completely antisymmetric tensors is again an antisymmetric tensor with the outermost indices of the chain being free.
Exactly the same result is obtained for $\Xi^{BB}(p) \equiv \left(G^{BB}(p)\Sigma^{\mathrm{p,BB}}(p)\right)^n G^{BB}(p)$. From this it is clear that the damping mechanism seen in $\Xi^{AA}(p,n)$ fails for higher insertions of $B/\bar{B}$ (and also $\psi/\bar{\psi}$) fields). Furthermore, the divergence increases with the loop order.

5.7.4 Summary

The considered examples show a damping for the chain of the vacuum polarization. The chain including both gauge and auxiliary fields contains poles, however independent of the order and hence can be regularized. A problem regarding renormalizability is expected from chains involving only the (doubled indexed) auxiliary fields, as the degree of divergence increases with the order, and hence one could expect that this leads to an infinite number of counterterms in the action. It should be noted, however, that this is nothing more than a hint. One could e.g. expect to cancel the divergences coming from the sector of auxiliary bosonic fields $B, \bar{B}$ with those from the fermionic sector $\psi, \bar{\psi}$[12]. As in the one loop case, this assumption would have to be confirmed by explicit loop calculations.

[12] Note that we encounter a similar mechanism in supersymmetric or SUSY theories, where quadratic divergences coming from the bosonic and fermionic sectors cancel each other.

5.8 Discussion

In the spirit of the Gribov-Zwanziger approach, and due to suggestions of Vilar *et al.* [14], we constructed a non-commutative gauge model [2, 3] wich avoids the introduction of new degrees of freedom. After deriving the Feynman rules, explicit loop-calculations were presented, and our hope was to show renormalizability - at least at the one-loop level. In this respect, unexpected difficulties appeared. The soft breaking term, being required to implement the IR damping behaviour of the $1/p^2$ model in a way being compatible with the Quantum Action Principle or QAP of Algebraic Renormalization (AR), gives rise to mixed propagators G^{AB} and $G^{A\bar{B}}$. These, in turn, allow the insertion of one-loop corrections with external B-fields into the dressed AA propagator (see Section 5.6.3) and, therefore, enter the renormalization. Despite all corrections featuring the expected $\frac{1}{p^2}$ IR behaviour, the dressed propagators with external AB or $A\bar{B}$ legs multiplicatively receive higher poles due to the inherent quadratic divergences in $G^{AB}(p)$ (and $G^{A\bar{B}}(p)$) for $p \to 0$. As a consequence, the resulting corrections cannot be absorbed in a straightforward manner.
In order to investigate the possibility of the construction of an effective action (i.e. with a finite number of counterterms to be added to the tree level action), we have also investigated the structure of singularities in higher-loop integrands by studying chain graphs consisting of tree-level propagators, and one-loop corrections of various types. It turned out that chains containing gauge fields benefit from the damping of the propagator (5.32c) while those consisting (solely) of concatenated B and $\bar{B}$ fields and insertions do (expectedly) not. Hence, at first sight, there exist divergences which increase order by order, which would indicate non-renormalizability. However, as we pointed out in Section 5.7, due to the symmetry between the $B/\bar{B}$ and $\psi/\bar{\psi}$ sectors, cancellations can be expected. These already appear in our one-loop calculations, and there is strong evidence that they appear to all orders. An intuitive argument can be given when considering the action (5.10) for $\lambda \to 0$, i.e. vanishing damping. In this case, the $B/\bar{B}$ and $\psi/\bar{\psi}$ fields may simply be integrated out in the path integral formalism (see Ref. [2]), and the contributions cancel exactly.
In order to prove renormalizability of the model, one could explicitly conduct the calculation of all divergent loops to arbitrary order and hence directly determine any counterterms. Yet, tables Tab. 5.2 and Tab. 5.3 will persuade the reader of the impossibility of realizing this task. The first table lists the number of possible configurations leading to one loop graphs, depending on the number of involved vertices. E.g. the first line states that there are four possible configurations when involving only one loop, which can easily be verified, as there are four different types of vertices. In this context the type is characterized only by the number of external legs. Table Tab. 5.3 contains a listing of exemplary configurations for one loop graphs and the corresponding number of divergent expressions, where each configuration is characterized by the kind and respective number of vertices entering the loops. E.g. the second line indicates that when combining two vertices with three legs ($2 \times 3V$) in all possible ways leading to one loop

# of involved vertices	# of configurations
1	4
2	10
3	19
4	28
5	37

Table 5.2: Number of possible (one loop) configurations for a given number of involved vertices.

Configuration	# of divergent graphs
$1 \times 3V$	12
$2 \times 3V$	76
$1 \times 4V$	12
$1 \times 3V$, $1 \times 6V$	200

Table 5.3: Number of divergent graphs for exemplary one-loop configurations.

graphs, this will lead to 76 divergent expressions[13]. Fortunately, different renormalization schemes exist, allowing to prove renormalizability in a more efficient way, one of them being Algebraic Renormalization (AR), which has been used also by Vilar *et al.* [14], where they claimed to have proven the renormalizability of their action, which differs from our model only on how it is localized. In the light of that renormalization scheme it is most important to maximize the symmetry content of the theory which is the basis for the generation of constraints to potential counterterms. Therefore, the symmetries and their resulting algebra has been studied in Section 5.3.

However, as will be discussed in the next chapter (see also [4]) the foundations of AR are only proved to be valid in *local* QFTs so far, and hence may not be applicable in non-commutative field theories, as the deformation inherently implies non-locality. Nonetheless, the authors claim to have shown renormalizability using Algebraic Renormalization. Indeed, an explicit argument which shows that the proof of renormalizability is questionable comes from the following considerations: it has to be noted that renormalizability of the non-local model (3.25) cannot depend on how it is localized. The reason is the equivalence of the respective path integrals (see [2]). Therefore, we expect the same problems of IR divergences to appear in *all* localized versions of (3.25) (and in particular of the term under consideration (5.1)), including the one of Vilar *et al.* [14].

[13]The number of possible divergent graphs has been evaluated via the *Mathematica*® routine described in Section F, where power counting of Section 5.5 is being applied. This means more precisely that we count the number of graphs with positive superficial divergence. Note, however, that due to internal cancellations or symmetries, the effective degree of divergence can be smaller, which implies that some of the expressions may even become convergent. An example is the vanishing of all tadpoles, i.e. superficially divergent $1 \times 3V$ graphs.

In fact, from the discussion in Appendix D.1, one notices that the propagators (D.1f)-(D.1h) and (D.1s) of their action all exhibit the same quartic IR divergences as those of our present model (5.19), even though the operator D_μ appears at most quadratically as D^2 in the according action (5.5).
In this respect it has to be noted that in commutative space the model of Vilar *et al.* [14] should indeed be renormalizable, since the action, apart from the star product, is completely local and provides the necessary symmetries for the Quantum Action Principle. Since the propagators are the same in both spaces, and hence show the same quartic IR divergences, one may expect related IR problems to cancel when considering the sum of bosonic and fermionic sectors (i.e. B/χ and ψ/ξ). These cancellations should also take place in non-commutative space (in both models), but the problem of proving renormalization remains (cf. Section 5.6.3).
Based on this findings regarding the problem of applying AR to non-commutative gauge theories, the next chapter Section 6 will be dedicated to a general review of existing non-commutative field theories. In particular we will investigate on the possibility of generalizing renormalization schemes which have been succesful for non-commutative scalar theories and / or commutative gauge theories to non-commutative gauge models. An alternative approach which avoids the above-mentioned uncertainties will then be presented in Chapter 7.

Chapter 6

Renormalizability of Non-Commutative Gauge Field Models

The last chapter has revealed an unexpected complexity in our attempt to prove renormalizability for the respective gauge model: on the one hand, although the model has been constructed on the basis of Gurau's renormalizable $\frac{1}{p^2}$ scalar model, one loop renormalization seems far more complex for the gauge version (c.f. Section 5.6.3). On the other hand, standard renormalization schemes might not be applicable in the case of θ-deformed gauge theories, a discussion already initiated in Section 5.7.4. In this chapter, which heavily relies on our paper [4], we will therefore try to better understand which are the problems specific to non-commutative gauge models. We will point out obstacles for renormalization on a very general basis, and not particular to any specific model. This will be done by first taking a step backwards and reviewing the history of non-commutative renormalization. In order to allow for a self-contained discussion, we will repeat some of the findings of the previous chapters (especially Section 1.5 and

Chapter 2), where it is required. In doing so, this will allow us to see the big picture, revealing where non-commutative renormalization succeeded, and for which reasons it failed whenever this was the case. In the successful cases, renormalization schemes have been applied. We will therefore investigate the possibility of extending them to non-commutative gauge theories in Section 6.2. A final discussion of non-commutative renormalization will conclude this chapter.
As in the previous work, in this chapter we presume θ-deformed $\mathbb{R}^4_\theta$ endowed with the non local Groenewold-Moyal star product [28, 29]

$$(\Phi_1 \star \Phi_2)(x) \equiv \Phi_1(x)\, \mathrm{e}^{\frac{\mathrm{i}}{2}\overleftarrow{\partial}_\mu \theta_{\mu\nu} \overrightarrow{\partial}_\nu} \Phi_2(x)\,, \qquad \text{where} \qquad \mathrm{i}\theta_{\mu\nu} \equiv [x_\mu \stackrel{\star}{,} x_\nu]\,. \tag{6.1}$$

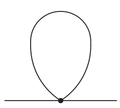
In the simplest case considered here, the antisymmetric real matrix $\theta_{\mu\nu}$ is constant and has mass dimension -2. Again we will use the notation $\tilde{p}_\mu \equiv \theta_{\mu\nu} p_\nu$.

6.1 Non-commutative renormalization: a review

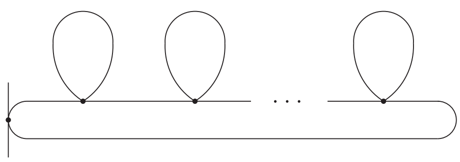
Ever since the first non-commutative quantum field theory models were constructed, the biggest obstacle has been the infamous so-called UV/IR mixing problem [37], where certain types of Feynman graphs, the non-planar graphs, exhibit new unrenormalizable IR singularities in exceptional momenta (see [11, 26, 9] for a review).
Historically, (an incomplete list of references is given by [74, 75, 76, 77, 78]) the IR divergences have been neglected in the discussion of renormalization. Instead, direct correspondences between known commutative results and the outcome of planar part calculations of the non-commutative counterparts have been sought. Soon afterwards, there appeared a series of publications [37, 38, 26, 79, 80] describing the finally discovered UV/IR mixing in all detail.

6.1.1 Implications of the UV/IR mixing

After discovering the UV/IR mixing, it was not clear at this point how to apply renormalization in the presence of this new effect. More precisely, when computing a simple tadpole graph with n non-planar insertions for ϕ^4-theory naïvely generalized to the non-commutative case, one gets [37, 49, 48]



$$\propto \frac{1}{\tilde{p}^2} \quad \Rightarrow$$



$$\propto \int d^4p \frac{1}{(\tilde{p}^2)^n} \frac{1}{(p^2+m^2)^{n+1}}\,. \tag{6.2}$$

Hence, the IR singularity grows order by order. The case is very similar for a $U_\star(1)$-field theory (the star denoting the θ-deformation of the product), where one finds the same quadratic IR behaviour in the non-planar part of the self-energy (which is independent of the gauge fixing [61, 62, 81, 63]). Due to BRST invariance, it takes the form

$$\Pi^{\mathrm{IR}}_{\mu\nu} \propto g^2 \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2}\,, \tag{6.3}$$

i.e. $p_\mu \Pi^{\mathrm{IR}}_{\mu\nu} = 0$ due to $p_\mu \tilde{p}_\mu = p_\mu \theta_{\mu\nu} p_\nu = 0$, and no term exists in the tree level action to absorb this divergence. Note, that contrary to the scalar case now the divergent term is endowed with an index structure.

As discussed in Section 1.5, the $\frac{1}{\tilde{p}^2}$ singularities are intimately tied to the Groenewold-Moyal product. More specifically, the deformation gives rise to phase factors, e.g. for one loop order of the type $\mathrm{e}^{\mathrm{i}k_\mu \theta_{\mu\nu} p_\nu}$, with k_μ being an internal momentum to be integrated out, and p_μ being an external momentum. In the UV limit the rapid oscillation effectively eliminates UV divergences in (non-planar) loop integrals. However, this damping behaviour vanishes in the limit $p_\mu \to 0\ \forall\mu$ or $\tilde{p}_\mu \to 0\ \forall\mu$, where the phase becomes unity. Naturally, in this limit the original divergence has to reappear, and in the case of a quadratic divergence is represented in the form $\frac{1}{\tilde{p}^2}$.

As a consequence of the UV/IR mixing, the following new problems appeared:

- The IR divergences are no ‘classical’ singularities appearing in some ill-defined loop integrals requiring regularization, but divergences in the *external* momentum. Therefore, the well known renormalization schemes from commutative quantum field theory cannot be applied straightforwardly.
- The standard choice for the renormalization conditions [57, 60] cannot be taken due to the appearance of the $1/\tilde{p}^2$ term in loop corrections[1].

From the short discussion we conclude the following:

- The naïve approach of starting from a renormalizable commutative model and replacing all products with Groenewold-Moyal products does not lead to a renormalizable non-commutative theory.
- Introduction of a θ-deformed product into quantum field theories results in inherent non-locality (i.e. divergences for vanishing external momenta). Therefore, the renormalization schemes known from commutative theory that require locality, cannot be used directly, c.f. Section 6.2.

6.1.2 Successful mechanisms in renormalizable scalar models

As has been shown in Chapter 2, the problems could be overcome at least in the scalar case, first by the Grosse & Wulkenhaar breakthrough [13, 6, 43], and later by the model of Gurau *et al.* [7]. Their approach was the following:

[1]One should also mention, that supersymmetry can in principle improve the situation by reducing the degrees of divergences, and hence the UV/IR mixing (An incomplete list of references is given by [82, 83, 84, 85, 86]).

- Introduction of an additional counterterm in the action in order to be able to absorb and damp the non-local IR divergences which are generated in loop calculations.
- Application of a suitable renormalization scheme in order to prove renormalizability to all orders of perturbation theory. For the GW-model, this was achieved by using flow equations [45, 87], i.e. via application of the Polchinski approach and Multiscale Analysis in a matrix base formulation [13, 6]. For the $\frac{1}{p^2}$-model, Multiscale Analysis (MSA) in x-space [7, 43] has been used.

Both models have furthermore in common that they are formulated on Groenewold-Moyal deformed [28, 29] Euclidean space $\mathbb{R}^4_\theta$ (rather than Minkowski).
The models differ in some points, which however do not affect renormalizability: while translation invariance is broken explicitly in the Grosse-Wulkenhaar model by adding an oscillator-like term to the action, the scalar $1/p^2$ model avoids this problem through a non-local bilinear term of the form $\phi \star \frac{a}{\square}\phi$ for the quadratic one-loop IR divergence inherently generated by the phase factors of the non-planar part at one-loop level. On the other hand, the Grosse-Wulkenhaar model implements the so-called Langmann-Szabo duality [44] and kills the infamous Landau ghost [88, 89], whereas the scalar $1/p^2$ model does not.

6.1.3 State-of-the-art

Before proceeding, let us briefly summarize the open problems of non-commutative quantum field theory. Despite of almost a decade of work in this field, two important steps have not been achieved yet:

- A good handling and efficient computation of Feynman diagrams on non-commutative Minkowski space-time and the construction of a candidate for a renormalizable scalar model, such as a Minkowskian version of the Grosse-Wulkenhaar or the scalar $1/p^2$ model, although for the latter a promising candidate has recently been put forward [90];
- The construction of a renormalizable gauge model, or more precisely, a rigorous proof of renormalizability of one of the promising candidates [53, 46, 47, 91, 8, 1, 2, 3].

Regarding the first point, we only would like to mention that it has been claimed that the UV/IR mixing might not be present in a Minkowskian non-commutative QFT if one considers proper Feynman rules taking into account a generalized notion of time ordering [92, 93, 94, 95, 96, 97]. However, these conjectures still lack a rigorous proof. The present chapter focuses on the second point.

6.1.4 Generalization to gauge models

Along the lines of the above approach and following the same ideas, we have constructed our gauge models in Section 3.1 and Section 5.2. Now the question is: if renormalizability was so straightforward in the scalar case, what causes difficulties when generalizing the same ideas to gauge models?
Let us recapitulate our findings of the work done so far in this thesis, and review them on a very general basis. Remember that we started with $U(1)$ gauge fields on $\mathbb{R}^4_\theta$. We have seen that in the naïve ansatz (i.e. by simply replacing the ordinary product by a star product), the gauge invariant Yang-Mills action is given by

$$\begin{aligned} S_{\mathrm{YM}\star} &= \int \mathrm{d}^4x \, \frac{1}{4} F_{\mu\nu} \star F_{\mu\nu} \,, \qquad \text{with} \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g \left[A_\mu \stackrel{\star}{,} A_\nu\right] \,. \end{aligned} \tag{6.4}$$

The star product (6.1) modifies the initial $U(1)$ algebra in a way that it becomes non-Abelian[2]. We have called the resulting algebra $U_\star(1)$ (c.f. Section 1.1)[3].
As indicated above, the action (6.4) does not lead to a renormalizable model no matter how the gauge fixing and Faddeev-Popov terms are chosen. The reason is, as in the naïve scalar case, that one finds a quadratic IR divergence in the non-planar part of the self-energy according to (6.3). Therefore one may suggest that the obstacles are in principle the same for scalar and gauge models, and that the solutions that worked for the scalar versions [7, 43], i.e. of the method described in Section 6.1.2, should easily be generalized to gauge models. Obviously, a higher degree of complexity is expected, which can also be seen when remembering that the work done so far in this thesis respects mainly the first point of the above mentioned method. Whatsoever, a straightforward generalization to gauge models does not directly lead to a renormalizable theory due to the following:

- In contrast to the scalar theory where renormalizability can be restored by adding a simple non-local term (see Chapter 2), gauge theories contain an additional requirement for counterterms regarding the tensor structure as observed in Section 6.1.1. Indeed, the form of Eqn. (6.3) cannot simply be generated by contracting $F_{\mu\nu}$ with $\theta_{\mu\nu}$ (c.f. [8] and Section 3.1.3), but requires 'fine tuning' of the action. This has been done in [1, 2, 3, 46, 47, 53, 8], as discussed in detail in the foregoing chapters. However, as we have seen in Section 5.6.3, contrary to the scalar case, the additional term does not allow to absorb all IR-divergences coming from one loop calculations.

[2] In θ-deformed $U(N)$ algebras all defining commutator relations are replaced by star-commutators $\left[X^a(x)T^a \stackrel{\star}{,} Y^b(x)T^b\right] \equiv \left(X^a(x)T^a \star Y^b(x)T^b - Y^b(x)T^b \star X^a(x)T^a\right)$ where X, Y are arbitrary functions on $\mathbb{R}^4_\theta$, and T are the generators of $U(N)$. Hence, even in the special case $N \to 1$ these commutators do not vanish in contrast to the commutative case. Hence, any Moyal-deformed gauge theory is of the non-Abelian type.

[3] It has been shown [98, 99, 100, 101, 102] that only enveloping algebras, such as $U(N)$ or $O(N)$ and $USp(2N)$, survive the introduction of a deformed product (in the sense that commutators of algebra elements are again algebra elements), while e.g. $SU(N)$ does not.

- Furthermore, there is an additional demand for gauge invariance. In commutative theories, several techniques have been established over the years to conduct renormalization in the presence of symmetries [60, 57] but application of these is prevented by an inherent property introduced by the deformation (6.1): non-locality.

From the first observation follows the need of constructing an effective action, allowing to absorb all divergences. This directly leads to the necessity of generalizing the renormalization schemes to non-commutative gauge models. We will discuss this issue in detail in Section 6.2.

6.2 Renormalization schemes in the light of non-commutative gauge theories

In this section we will review some representative renormalization procedures. Far from being exhaustive, we will only discuss some of the possible techniques, being selected due to their functioning for non-commutative scalar (Polchinski, MSA) and/or commutative gauge theories. For each of them, we address issues and possible solutions regarding their applicability for θ-deformed gauge theories.

6.2.1 The BPHZ approach

In the BPHZ subtraction procedure[4] ([103], see [57], [60] for a introduction to the field), the divergences are absorbed in the parameters of the theory leading to renormalized quantities.
Lets consider a loop integral $J(p)$ with degree of (UV) divergence n according to naïve power counting. By subtracting from its integrand $I(p)$ the first n terms of its Taylor expansion in the external momentum p, we get an integral which is convergent denoted by $\hat{J}(p)$. By considering the equivalence

$$\frac{\partial}{\partial p_{i_1}^{\mu_1}} \cdots \frac{\partial}{\partial p_{i_n}^{\mu_n}} \hat{J}(p) = \frac{\partial}{\partial p_{i_1}^{\mu_1}} \cdots \frac{\partial}{\partial p_{i_n}^{\mu_n}} J(p) \,, \tag{6.5}$$

we get the general solution:

$$J(p) = \hat{J}(p) + c + \sum_{l=1}^{n} \frac{1}{l!} c_{\mu_1 \ldots \mu_l}^{i_1 \ldots i_l} p_{i_l}^{\mu_1} \ldots p_{i_l}^{\mu_l} \,. \tag{6.6}$$

If the starting action is the most general one (i.e. includes all possible tree level terms) and suppose it is power-counting renormalizable, than the coefficients lead to a finite renormalization of the parameters of the theory. If it is renormalizable but not the most

[4]BPHZ stands for Bogoliubov, Parasiuk, Hepp and Zimmerman.

general one, this leads to the new counterterms in the action, which are given by the product of the coefficients $c^{i_1 \dots i_l}_{\mu_1 \dots \mu_l}$ with l differentiated fields.
Obviously, this procedure is only well-defined if the convergence radius of the series does not vanish. In the case of non-commutative field theories, we definitely expect it to work for the planar graphs, being (up to a factor) the same as in the commutative case. However, consider a typical non-planar integral at one loop level:

$$J^{n.p.}(k,p) \propto \int d^4k \mathcal{I}(k,p) \exp\left(\mathrm{i}kp\right) = \int d^4k \mathcal{I}^{n.p.}(k,p)\,, \tag{6.7}$$

where k, p are internal and external momenta, respectively, and $\mathcal{I}(k,p)$ is essentially the integrand of the planar expression, i.e. a polynomial where BPHZ might be applied. Note that the phase factor present in this (momentum space) integral is a direct expression of the non-locality of the star product in coordinate space. Now let us look at the Taylor expansion of the whole integrand, which is given by

$$\begin{aligned}
\mathcal{I}^{n.p.}(k,p) =& \mathcal{I}(k,0) + \sum_{l=0}^{\infty} \frac{1}{l!} p_{\mu_1} \dots p_{\mu_l} \left[\frac{\partial}{\partial p_{\mu_1}} \dots \frac{\partial}{\partial p_{\mu_l}} \mathcal{I}^{n.p.}(k,p) \right] \Bigg|_{p=0} \\
=& \mathcal{I}(k,0) + \sum_{l=0}^{\infty} \frac{1}{l!} p_{\mu_1} \dots p_{\mu_l} \left[\frac{\partial}{\partial p_{\mu_1}} \dots \frac{\partial}{\partial p_{\mu_l}} \exp\left(\mathrm{i}kp\right) \right] \mathcal{I}(k,p) \Bigg|_{p=0} \\
& + \text{other term with derivations acting also on } \mathcal{I}(k,p) \\
=& \mathcal{I}(k,0) + \sum_{l=0}^{\infty} \frac{1}{l!} (kp)^l \mathcal{I}(k,0) + \dots\,.
\end{aligned} \tag{6.8}$$

We see that at each order there appear additional powers in k, steming from the derivation of the phase factor, which is equivalent to a increase of the power counting degree of divergence by one at each order. Hence, the BPHZ subtraction scheme is not directly applicable to non-planar expressions, without previous modifications.

6.2.2 Algebraic Renormalization

The Algebraic Renormalization procedure (in short AR, c.f. [60] for a introduction to the topic) enables one to determine the most general counterterms for a given action which are allowed by its symmetry content. Hence, all possible terms resulting from the *brute force ansatz* of explicit loop calculations can be retained algebraically[5].

Methodology

The application of the Algebraic Renormalization procedure can be summarized by the following algorithm:

[5]However, this does not mean that explicit loop calculations would become dispensable, as the prefactors still have to be evaluated explicitly whenever required.

Given a particular action, i.e. a theory at tree level. Determine all symmetries of the theory. Examples are the gauge invariance (broken after gauge fixing) for ordinary QED or YM, the broken Ward Identity $\mathcal{WI}$ for the gauge fixed $U(1)$ action, and the Slavnov Taylor identity for Yang-Mills theories. Based on them, we also construct their algebras, the so called consistency conditions. For our gauge model with soft breaking, this has been done in Section 5.3.

Now construct the set of all solutions of the consistency conditions Δ_i, and let us define $\Delta = \sum r_i \Delta_i$, where r_i are numerical coefficients. They are determined by the so called Quantum Action Principle or QAP. It tells us that for a given tree level (generally order $n-1$) action, a given symmetry is only true as long as one ignores results from next loop order calculations, and that the breakings of the symmetries are field polynomials, which have to be local (i.e. polynomials of fields defined at the same point in spacetime[6]). This can be understood when considering that indeed loop calculations might lead to new terms in the action, disturbing the nice symmetry. By denoting a symmetry generally with Υ and a possible breaking already present at order $(n-1)$ with Ξ we can write:

$$\Upsilon(\Gamma_{(n-1)}) = \Xi + \hbar^n \Delta_{(n)} \,. \tag{6.9}$$

For the solutions we have the following possibilities[7]:

1. Those that can be written as variations of the symmetry under consideration, i.e. $\Delta = \Upsilon(\hat{\Delta})$. By redefining the action, i.e. $\Gamma_1 = \Gamma_0 - \hat{\Delta}_i$, we can restore the original symmetry content. Again we have two cases:
   (a) Polynomials $\hat{\Delta}_i$ of the same form as those already present in the action, so called invariant counterterms, leading only to a change in the prefactor. For the above example of the gauge condition, e.g. $\Upsilon \equiv \frac{\delta}{\delta b}$ we would have $\Delta = \partial A$, $\hat{\Delta} = \int d^4x b \partial A$. Than $\hat{\Delta}$ is a invariant counterterm.
   (b) Polynomials $\hat{\Delta}_i$ not of the same form, i.e. noninvariant counterterms. They will be subtracted from the action $\Gamma_{(n-1)}$, leading to a new effective action Γ_n being invariant under the transformation under consideration. For the example of the Slavnov-Taylor identity $\mathcal{S}$, i.e. the generalization of the BRST symmetry s on a functional level, $\Delta = s(\hat{\Delta})$ and $\Gamma_{(1)} = \Gamma_{(0)} - \hbar\hat{\Delta}$. Due to $s^2 = 0$ we have: $s\Gamma_{(1)} = 0$, hence the symmetry is again valid at first loop order. Note that $\hat{\Delta}$ has ghost number zero.

[6]This is a obvious consequence of the fact that the original action is made of local polynomials. Now, a symmetry or $\mathcal{WI}$ always expresses the invariance of the whole action under the variation of "some part", e.g. one of the local fields. It follows then that a possible breaking has to be defined at the same spacetime point, i.e. be local as well.

[7]In the following, the term "(non)invariant" is intended to be used in the context of form invariance, whereas in the standard literature it is usually referred to as "BRST-invariance". However, note that counterterms may result from *all* symmetries (not only BRST symmetry), and that our notation seems to distinguish more precisely the various cases.

2. Polynomials that cannot be written as variations of a given symmetry $\Delta_i \neq \Upsilon(\hat{\Delta}_i)$. Hence the symmetry cannot be restored, Δ_i is a so called anomaly. It is of ghost number one.

Another way of constructing the non-invariant counterterms which are not anomalies, i.e. those given under (1b) relies on first constructing all Δ_i with ghost number zero and that respect any other conditions (e.g. PC symmetry). Evaluate if they vanish under application of symmetry, e.g. s. If not, they have to be included in the original action.

In a more formal language, in order to apply the AR, first the validity of the so-called Quantum Action Principle (QAP), which is based on the assumption of locality, [104, 105, 106] has to be proved. Secondly, one has to show that the symmetry content of the theory at tree level is stable under quantum corrections, i.e. that the theory is free of anomalies, i.e. the set of polynomials under 2 has to be empty. This latter point involves the computation of the cohomology [107, 108, 109] $H(s) = \mathrm{Ker}\, s/\,\mathrm{Im}\, s$ (see also [110] for an exemplary application of this concept in commutative theories and further references) of the nilpotent BRST operator s, or its generalized nilpotent operator δ which collects all Ward identities in the presence of further symmetries, e.g. supersymmetry. There, one has to show triviality of the respective cohomology group for ghost number one local functionals (i.e. anomalies). This again requires locality in all steps [109].

Excursion: additional counterterm for axially gauged U(1) theory

According to the second method presented in the last section, a short algorithm has been developed in *Mathematica*® programming language and applied to the axially gauged $U(1)$ action presented in [57]. It could be shown that the set of noninvariant counterterms given in [57], Eqn.(4.99) is not complete. One missing term, given by

$$\hat{\Delta}_{2,new} = (nA)(nd)(dA) \tag{6.10}$$

could be identified, where n is the axial gauge vector, which is fixed and of mass dimension zero, and following the notation of [57], the index “2” refers to the counterterms depending on n. For details c.f. Section F.2.

Applicability to non-commutative quantum field theories

Unfortunately, from the short discussion above it follows that the AR cannot directly be applied: the QAP does not exist in its usual form, due to the inherent non-locality of the star product. The proof of the triviality of the cohomology group $H^{(1)}$, or equivalently ghost number one local functionals corresponding to anomalies, fails for the same reason. Furthermore, the computation of the cohomology class has to be worked out rigorously for the ghost number 0 functionals $\mathcal{F}$, representing the most general quantum level action, to fulfill $s\mathcal{F} = 0$.

In this respect, some efforts for a generalization to non-commutative spaces have been made. For example, the notion of BRST cohomology and the Chern character has been introduced in [111] using Connes' notation of spectral triples [112, 113]. Another contribution has been the generalization of the descent equations describing Yang-Mills anomalies to non-commutative spaces [114]. It has also been shown [74] that the symmetry content compatible with the QAP can be established for non-commutative $U_\star(N)$ theories and is invariant under an explicit one-loop UV renormalization. Furthermore, the classification of anomalies by computation of the cohomology class $H^{(1)}$ of the BRST operator for general functionals (i.e. counterterms) with ghost number one has already been achieved [111].

However, in order to fully restore the foundations of AR in the non-local case, still some work has to be done:

- The computation of the cohomology class has to be worked out rigorously for the ghost number 0 functionals $\mathcal{F}$, representing the most general quantum level action, to fulfill $s\mathcal{F} = 0$, as mentioned above.
- It has to be assured in a rigorous way that trivial cohomology alone is sufficient to prove the absence of anomalies also in the non-commutative case. If this turns out not to be the case, one has to find out which additional requirements are necessary.
- Whereas the absence of anomalies would imply (power counting) renormalizability[8] in the commutative case, this is not true for the non-commutative case. Therefore a proof that the triviality of the cohomology is sufficient to guarantee renormalizability in the presence of non-locality is missing as well.

The last point is of very general nature, as it applies to *all* non-commutative theories: It concerns the appearance of dimensionless operator insertions in the action. A parameter of non-commutativity θ with mass dimension -2 allows to freely add composite field operators[9] of zero mass dimension, such as $D^2\widetilde{D}^2$ or $\widetilde{F}^2$, to the action, where $\widetilde{D}_\mu = D_\nu\theta_{\mu\nu}$ is a contracted covariant derivative and $\widetilde{F} = F_{\mu\nu}\theta_{\mu\nu}$ is a field strength. Being invariant under all symmetries appearing in the QAP (and gauge transformations in general), there is no constraint or theorem preventing insertions of arbitrary powers of these operators both at tree level or as quantum corrections. This is the reason why the sufficiency of a trivial cohomology class for renormalizability has been questioned above. We have seen that the AR starts from the most general set of all possible counterterms, and restricts them by applying constraints (we will call this a *top-down* approach). Since this set is *a priori infinite* in the presence of invariant dimensionless insertions, the attempt to achieve a finite number of counterterms will fail, independent of the cohomology.

We have to conclude, that it is essential to identify constraints that limit the appearance of insertions of massless operators into the action. In this context also the issue of field redefinitions might be important and maybe some classes of insertions can be

[8] Note that a power counting renormalizable theory can also be non-renormalizable, c.f. [57].

[9] Note that this also occurs in scalar field theories. For example, the non-local term $\Box^{-1}\phi^2$ could be inserted into the tree level action to arbitrary power.

rewritten as such redefinitions (cf. [115] in the context of non-commutative $U_\star(1)$ gauge theory with Seiberg-Witten maps).

Most general $U_\star(1)$-action with IR damping

In the context of the foregoing discussion regarding the insertion of an arbitrary number of dimensionless operators, we have to review our action (3.23) which has been constructed in order to get an IR-damped gauge propagator. A generalized form including damping terms is then given by

$$\begin{aligned}\Gamma = \int \mathrm{d}^4x \Bigg\{ \Bigg[&\frac{1}{4} F_{\mu\nu} \left(1 + (D^2 \widetilde{D}^2)^m\right)^n F_{\mu\nu} + \frac{1}{4} (\widetilde{F})^p \frac{1}{(\widetilde{D}^2)^2} (\widetilde{F})^q \\ &+ \frac{1}{4} F_{\mu\nu} \theta_{\nu\rho} \theta_{\rho\sigma} \frac{1}{(\widetilde{D}^2)^2} F_{\sigma\mu} + \dots \Bigg] \\ &+ b \left(1 + (\partial^2 \widetilde{\partial}^2)^r\right)^s \partial_\mu A_\mu - \bar{c} \left(1 + (\partial^2 \widetilde{\partial}^2)^u\right)^v \partial_\mu s A_\mu \Bigg\}, \end{aligned} \tag{6.11}$$

where $\theta_{\mu\nu}$ is given by (3.1) and of dim $\theta = -2$, and $\widetilde{D}_\mu \equiv \theta_{\mu\nu} D_\nu$, from which it follows that $\widetilde{D}^2 D^2$ and $\widetilde{F} \equiv \theta_{\mu\nu} F_{\mu\nu}$ are both dimensionless. We see that the models (3.25) and (5.18) represent minimal versions with the choice $m = -1, n = 1$ and $r = u = -1 = -s = -v$ in (3.25), and $s = v = 0$ in (5.18). However, in the light of AR nothing prevents the other terms of (6.11) to appear in the course of loop calculations.

6.2.3 Multiscale Analysis

Multiscale Analysis or MSA is inspired by the ideas of Wilson concerning the Renormalization Group, i.e. the behaviour of a theory under change of scales. To start with, let us first state that in this approach, the Schwinger-parametrized propagator gets sliced with respect to the integration variable. E.g. for ϕ^4-theory:

$$\begin{aligned} \hat{\Delta}(p) &= \frac{1}{p^2 + m^2} = \int_0^\infty d\alpha e^{-\alpha(p^2+m^2)} \Longleftrightarrow \Delta(x) = \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-\alpha m^2 - \frac{(x_\mu - x_\nu)^2}{4\alpha}}, \\ \Delta(x) &= \sum_{i=0}^\infty C^i(x_\mu, x_\nu), \quad \text{with } C^i \propto \int_{M^{-2i}}^{M^{-2(i-1)}} \frac{d\alpha}{\alpha^2} e^{-\alpha m^2 - \frac{(x_\mu - x_\nu)^2}{4\alpha}}, \text{ for } i \geq 0, \\ &\qquad\qquad \text{and } C^0 \propto \int_1^\infty \frac{d\alpha}{\alpha^2} e^{-\alpha m^2 - \frac{(x_\mu - x_\nu)^2}{4\alpha}}. \end{aligned} \tag{6.12}$$

Here M is a fixed number. C_i can intuitively be imagined to be the piece of field oscillating with Fourier momenta mainly of size M^i. The integration result for each slice can be estimated by an upper bound, which shows that the slicing indeed corresponds to a scale or energy decomposition of the propagator:

$$C^i(x_\mu, x_\nu) \leq K M^{2i} e^{-kM^i |x_\mu - x_\nu|}, \tag{6.13}$$

with K a positive constant. From this result one directly can see, that under the integral, i.e. in the course of loop calculations, the sum in (6.12) will diverge, and in order to regularize it, one replaces the upper summation limit by a (large) integer ρ, serving as UV cut-off. Furthermore we introduce the decomposition of the fields by defining $\Phi_i = \sum_{j=0}^{i} \phi_j$.
Now, after having introduced the ideas of slicing and scale decomposition we shortly review the philosophy behind MSA. It relies on the idea that the original or bare couplings of the action are the divergent quantities, and the bare action $S_\rho(\Phi_\rho)$ itself represents the physics at high energy scales. But one is interested in a effective renormalized action $S_0(\Phi_0) = S_0(\phi_0)$ in the last slowly varying field. This is done as follows: By performing $\rho - i$ steps, starting from the highest scale ρ, one gets to an effective action for the remaining field $\Phi_i = \Phi_{i-1} + \phi_i$, which has been split into a background field and a (high energy) fluctuating field. Now in a first step we perform functional integration, which is performed over ϕ_i only (The action obviously depends on Φ_i, i.e. is given by $S_i(\Phi_i)$). By computation of the logarithm one gets $S_{i-1}(\Phi_{i-1})$. By recursion one finally gets S_0. The renormalization proof and evaluation of the infinite counterterms is performed according the following algorithm:

- Slicing of the propagator, including the evaluation of the upper bounds for the slices according to (6.13).
- General definition of the graphs of the theory. For a N-point function (i.e. N external legs) of loop order $(n-1)$ (i.e. with n internal vertices) one gets:

$$A_G(z_1, \dots, z_N) = \int \mathrm{d}^4x \prod_{i=1}^{n} dx_i \prod_{l \in G} C_l(x_{\mu,l}, x_{\nu,l}) \,. \tag{6.14}$$

  Here, l stands for all lines of a graph G. In perturbation theory, going over to higher loop orders is equivalent to consider further, i.e. new "inner" loops, which is equivalent to consider higher energies. Therefore each propagator in a given graph is characterized by one index (i.e. one slice instead of the full propagator), where the inner most propagators carry the highest indices.
- Estimating the upper bounds for the most general graph expression leads to power counting, and as a consequence allows to determine the superficially divergent graphs. Note that only those (sub)graphs will be superficial, where the upper estimate will not decrease with growing difference for the inner and outer slice indices.
- The divergent graphs will then lead (as usual in perturbative renormalization) to a redefinition of the parameters of the theory.
- In the usual way, an action is called renormalizable and stable, if the divergent graphs will only lead to a redefinition of parameters already present. This is the case, if for each divergent loop having N external legs (to arbitrary loop order), a N-point tree level function is already present in the original action. Otherwise

this leads to a finite number of counterterms, i.e. to a new renormalizable effective action. If the number of primitively divergent graphs grows with the loop order, the theory is not renormalizable.

The above procedure has been applied to the Grosse-Wulkenhaar oscillator model in the matrix base [43] and the a/p^2-model of Gurau *et al.* in x-space [7]. Consider that by extending the same procedure to a gauge theory, in a first step breaks gauge symmetry due to the slicing of the propagator. Therefore, in the end one has to show that gauge invariance has been restored for the effective action.
In Ref. [116], attempts to treat pure Yang-Mills models within this renormalization approach are described. The authors tried to construct Schwinger functions (i.e. the Euclidean counterpart of the Green functions) of the field in Feynman or Landau gauge with an IR cut-off. However, their approach was unsuccessful: On the one hand the functional integrals they obtained lacked sufficient positivity, and on the other hand the related Gribov problem [64] was not solved.

Therefore it may be suggested to handle the Gribov problem using a soft breaking mechanism [2, 72] similar to the one present in the Gribov-Zwanziger action ([68, 73, 71], c.f. also Chapter 4) when using the MSA for gauge theories, and see if renormalizability can in principle be achieved.

6.2.4 Polchinski approach

Let us further mention the Polchinski approach which has successfully been applied to the Grosse-Wulkenhaar model [13, 6]. Similar to MSA it relies on the Renormalization Group approach of Wilson. In contrast to MSA where different scales are treated simultaneously, the Polchinski apporach goes through the scales in an inductive way.
The case of (commutative) spontaneously broken $SU(2)$ Yang-Mills theory has been discussed by C. Kopper and V. F. Müller [117]. Their starting point was the classical BRST invariant action including all (i.e. a finite set of) counterterms satisfying certain symmetry constraints. Since the regularization (which is required in the Polchinski approach) breaks the local gauge symmetry explicitly, the counterterms are only required to be invariant under a global $SO(3)$ isosymmetry. The authors showed that this *ansatz* solves the flow equations to all orders by induction. In the case of non-commutative gauge theories the set of all possible counterterms is infinite, but one could choose a restricted, finite set of counterterms instead. Renormalizability would then be established, if it could be shown that this finite set solves the flow equations.

6.3 Discussion

We have reviewed the current status of renormalization of non-commutative quantum field models. The effect of UV/IR mixing gives rise to the well-known quadratic IR divergence in the external momentum p. In gauge theories this singularity is endowed with a new type of (transversal) tensor structure $p_\rho\theta_{\rho\mu}p_\sigma\theta_{\sigma\nu} \equiv \tilde{p}_\mu\tilde{p}_\nu$ which cannot be absorbed in a straightforward manner. For scalar models it has been shown that this can be done by adding a corresponding non-local term into the tree level action. This leads to renormalizability because the insertion alters the propagator in a way that it 'damps' in the IR limit. In this respect, mainly two approaches have been followed: The Grosse-Wulkenhaar model featuring an oscillator-like term [13, 6], and the $1/p^2$ model by Gurau *et al.* [7]. Both have been proven to be renormalizable up to all orders using 'bottom up' schemes, such as the so-called Multiscale Analysis and the Polchinski approach. From the point of view of renormalization the inherent non-locality introduced by the θ-deformed Groenewold-Moyal product turns out to be a great obstacle since almost all 'classical' procedures rely on the presumption of locality[10] . It is well known that omitting the latter requirement generally leads to non-renormalizability since it allows for the insertion of arbitrarily high powers of massless operators into the action. This is exactly the problem one is facing in non-commutative theories. In addition, in gauge models, the mentioned insertions are completely invariant under any symmetry compatible with the well known Quantum Action Principle, BRST, or gauge transformations. At present, there is no criterion to rule out these terms, and the set of all possible counterterms, which is the starting point for 'classical' renormalization schemes such as the Algebraic Renormalization program, is *a priori* infinite.

In order to find a way out of this misery, and towards renormalizability of non-commutative gauge models, we have made several suggestions. The first one is to use schemes, such as the Polchinski approach or Multiscale Analysis, which both should in principle work out for gauge theories. However, there are indications that, in addition to the standard gauge fixing, a soft breaking mechanism is required in order to avoid the Gribov problem violating positivity of functional integrals. This point will have to be studied more thoroughly before the mist clears.

Another approach is to rigorously prove that trivial cohomology automatically induces absence of anomalies, and renormalizability, even in the presence of a deformed product. Since the latter point is rather questionable one will have to find an additional criterion to restore validity of the Algebraic Renormalization procedure, or find some proper modification. Finally, it will be of great importance to investigate possible conditions to restrict the appearance of arbitrary powers of massless operator insertions, which affect *all* non-commutative models on the market, including the model of Vilar *et al.* [14]. It follows that - contrary to what has been stated by the authors - their renormalization proof cannot be regarded as being complete.

Finally, let us summarize the findings of this chapter in the light of our aim to construct an effective renormalizable action for the model presented in Chapter 5. We may state that at this point, a generalized multiscale approach is the appropriate tool at our hands. However, instead of applying this approach to the rather complicated action (5.18), in the next Chapter we will break new ground and present an alternative and

[10]In the context of renormalization schemes requiring locality also other procedures than BPHZ or AR have to be mentioned, e.g. the bi-algebra based approach of Epstein and Glaser [118].

more simple model, leading finally to a promising candidate for a renormalizable action remedying the UV/IR mixing problem of non-commutative gauge theory.

Chapter 7

The BRSW model

In this chapter which heavily relies on [5] we will present a new type of $U_\star(1)$ gauge model, and deliver strong arguments for the conjecture that it is renormalizable to all orders. A careful analysis of the problems encountered for the previous models and lessons learned will lead to the construction of the new model in Section 7.1. After presenting its symmetry content (Section 7.2) the Feynman rules and power counting will be derived in Section 7.3. The fact that one-loop corrections to the gauge boson propagator will reduce to the three graphs (and the same divergence structure) which already appear for the naïve $U_\star(1)$-model will reveal the simplicity of the new action with respect to the previous models, as shown in Section 7.4. There we will also see that the gauge boson propagator allows for the absorption of all divergences of the vacuum polarization, i.e. one-loop renormalizability of the model will be shown. A short analysis of the IR-behaviour at higher loop order will lead to the strong conjecture of renormalizability to all orders of the BRSW model. After deriving the one-loop corrections to the vertices and the β-function, we will conclude by a short summary of the findings and address the next step of a rigorous renormalization proof still to be done.

7.1 Construction of the BRSW model

7.1.1 Preliminary considerations

Before proceeding to the construction of the new model, let us briefly summarize the problems discussed in Section 5.8 for the previous model, the ideas behind its construction, which will very naturally lead to the more simple action of this Chapter. The drawbacks associated with the model (5.18) are the following:

- The tree level gauge propagator does not allow for the absorption of the IR divergences appearing in the full one loop expression. The reason is the appearance of IR divergences of new type due to the new Feynman rules, involving the gauge field A_μ and the auxiliary fields. Yet, even if we would only have encountered the IR divergence of the vacuum polarization already present in the case of the naïve gauge model, there would be no term in the propagator capable of absorbing it. This latter is given by

$$\Pi^{\text{IR}}_{\mu\nu}(k) \propto \frac{\tilde{k}_\mu \tilde{k}_\nu}{(\tilde{k}^2)^2}\,, \tag{7.1}$$

and is the same found in the previous models of this thesis.

- The mass dimension of -2 of the matrix $\theta_{\mu\nu}$ implementing non-commutativity between the space variables leads to the possibility of arbitrary insertions of the dimensionless operator $D^2\widetilde{D}^2$ and $\widetilde{F} = \theta_{\mu\nu}F_{\mu\nu}$ in the effective action. Even when considering a minimal tree level action, they may and do appear as counter terms in the effective n-loop action.
- The action is rather complicated, and concrete calculations are very involving, due to the appearance of new vertices and mixed propagators with respect to the naïve action involving the auxiliary fields.

Let us recapitulate the ideas behind the construction of the last model:

- The starting point was the model of Gurau *et. al.* remedying the UV/IR mixing for the scalar $\phi^4_\star$-theory by implementing a damping behaviour for the propagator.
- The $\frac{1}{p^2}$-term was then implemented for gauge models in a *gauge invariant* way, leading to the desired damping also for the gauge field propagator at tree level. Yet, this latter is non-local[1], i.e. an infinite number of vertices appeared. In order to localize it and according the Gribov-Zwanziger approach, new fields forming BRST-doublets have been introduced. This finally led to a finite number of vertices while keeping the mentioned damping behaviour.
- As a side-effect, also mixed propagators and vertices appeared, leading to a rather complicated IR-divergence structure for the full gauge boson propagator, impeding a direct IR-renormalization as mentioned before.

Now let us make a step backwards, recapitulate our goal and the assumptions our work was based on so far, leading to a constructive criticism of the latter and as its consequence to a new model avoiding all mentioned drawbacks.

> *Our goal is the construction of a renormalizable $U_\star(1)$ gauge theory, remedying the UV/IR mixing problem present in the naïve model obtained by simply replacing the ordinary product by the star product.*

Our work was then based on the following assumptions:

[1] Let us mention again that we encountered two types of non-locality: a inherent one due to the star product, and a second one (addressed in the context above) due to the inverse of a covariant derivative.

1. *This goal may be achieved via a damping of the gauge propagator.*
2. *The damping has to be implemented in a gauge invariant way.*
3. *The antisymmetric matrix implementing non-commutativity is of mass dimension -2.*

While the first assumption must be regarded as essential keeping in mind that IR problems are expressed via the behaviour of the gauge propagator, assumption 2 grew historically. Indeed, from the Gribov Zwanziger approach (see Chapter 4) we know that the damping is implemented via the soft breaking. This can nicely be seen when looking at (5.11), showing the equivalence with the undamped action for vanishing Gribov-like parameter λ. Here the question arises: *Why not implement the damping directly via the soft breaking, avoiding the detour via a gauge invariant action term?* A further simplification arises when considering that the propagators stem from the bilinear part of the action. The appearance of the additional Feynman rules spoiling renormalizability (in the IR sector) was a side effect for achieving our goal of implementing the damping, due to the coupling of the full field strength tensor $F_{\mu\nu}$ to the auxiliary fields. This leads to the next question: *can't we avoid additional Feynman rules involving A_μ while implementing the desired damping, by simply modifying the bilinear part only?* Let us therefore split $F_{\mu\nu}$ into a non-interacting and a interacting part:

$$f_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\mu A_\nu \,, \qquad F_{\mu\nu} = f_{\mu\nu} + \mathrm{i}g\,[A_\mu, A_\nu] \;. \tag{7.2}$$

Regarding the last assumption it should be noted that it is the property of antisymmetry which is essential for constructing the non-commutative counterparts for fields and variables. We are therefore free to split $\theta_{\mu\nu}$ into a parameter ε carrying the dimensions and a dimensionless matrix $\theta_{\mu\nu}$,

$$\theta^{old}_{\mu\nu} \to \varepsilon\theta_{\mu\nu}, \tag{7.3}$$

with

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} . \tag{7.4}$$

The Moyal deformed product now gets

$$[x_\mu \stackrel{\star}{,} x_\nu] \equiv x_\mu \star x_\nu - x_\nu \star x_\mu = \mathrm{i}\varepsilon\theta_{\mu\nu} \,. \tag{7.5}$$

We will from now on use the abbreviations $\tilde{v}^\mu \equiv \theta^{\mu\nu} v_\nu$ for vectors v and $\tilde{M} \equiv \theta^{\mu\nu} M_{\mu\nu}$ for matrices M.

7.1.2 The model

Based on the previous considerations and equipped with the new conventions, the following action S formulated on Euclidean $\mathbb{R}^4$ is put forward:

$$
\begin{aligned}
S &= S_{\text{inv}} + S_{\text{gf}} + S_{\text{aux}} + S_{\text{break}} + S_{\text{ext}}\,,\\
S_{\text{inv}} &= \int \mathrm{d}^4x \tfrac{1}{4} F_{\mu\nu}F_{\mu\nu}\,,\\
S_{\text{gf}} &= \int \mathrm{d}^4x\, s\,(\bar{c}\,\partial_\mu A_\mu) = \int \mathrm{d}^4x\,(b\,\partial_\mu A_\mu - \bar{c}\,\partial_\mu D_\mu c)\,,\\
S_{\text{aux}} &= -\int \mathrm{d}^4x\, s\,\left(\bar{\psi}_{\mu\nu}B_{\mu\nu}\right) = \int \mathrm{d}^4x\,\left(-\bar{B}_{\mu\nu}B_{\mu\nu} + \bar{\psi}_{\mu\nu}\psi_{\mu\nu}\right)\,,\\
S_{\text{break}} &= \int \mathrm{d}^4x\, s\left[\left(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu}\right)\frac{1}{\tilde{\Box}}\left(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f}\right)\right] =\\
&= \int \mathrm{d}^4x\Bigg[\left(\bar{J}_{\mu\nu\alpha\beta}B_{\mu\nu} + J_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu}\right)\frac{1}{\tilde{\Box}}\left(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f}\right)\\
&\qquad - \bar{Q}_{\mu\nu\alpha\beta}\psi_{\mu\nu}\frac{1}{\tilde{\Box}}\left(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f}\right)\\
&\qquad - \left(\bar{Q}_{\mu\nu\alpha\beta}B_{\mu\nu} + Q_{\mu\nu\alpha\beta}\bar{B}_{\mu\nu}\right)\frac{1}{\tilde{\Box}}\,s\left(f_{\alpha\beta} + \sigma\frac{\theta_{\alpha\beta}}{2}\tilde{f}\right)\Bigg]\,,\\
S_{\text{ext}} &= \int \mathrm{d}^4x\,\left(\omega^A_\mu sA_\mu + \omega^c sc\right)\,,
\end{aligned}
\tag{7.6}
$$

where all products are implicitly assumed to be the deformed Moyal product. This will also apply to the rest of this chapter. As usual, A_μ denotes the gauge field, $\bar{c}$ and c are the (anti-)ghosts and the multiplier field b implements the gauge fixing, in the present case of the Landau type $\partial_\mu A_\mu = 0$. ω^A_μ and ω^c are external sources introduced for the non-linear BRST transformations sA_μ and sc (c.f. (7.8)). Next, the complex field $B_{\mu\nu}$ and its conjugate $\bar{B}_{\mu\nu}$ together with their associated ghosts $\psi_{\mu\nu}$ and $\bar{\psi}_{\mu\nu}$ have been introduced into the bilinear part of the action in order to implement the IR damping. Note that these new unphysical fields interact with the gauge field A_μ utmost bilinearly, hence the appearance of new vertices for A_μ has been avoided.
The additional sources $\bar{Q}, Q, \bar{J}, J$ allow to restore BRST invariance of S_{break} in the ultraviolet. In the infrared these sources take their "physical values"

$$
\begin{aligned}
\bar{Q}_{\mu\nu\alpha\beta}\big|_{\text{phys}} &= 0\,, & \bar{J}_{\mu\nu\alpha\beta}\big|_{\text{phys}} &= \frac{\gamma^2}{4}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right)\,,\\
Q_{\mu\nu\alpha\beta}\big|_{\text{phys}} &= 0\,, & J_{\mu\nu\alpha\beta}\big|_{\text{phys}} &= \frac{\gamma^2}{4}\left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right)\,,
\end{aligned}
\tag{7.7}
$$

implementing a "soft breaking" of the BRST symmetry, where γ is a new parameter of the theory of mass dimension 1. Finally, also note the introduction of a new dimensionless parameter σ.
With general $\bar{Q}, Q, \bar{J}, J$, the action (7.6) is invariant under the following BRST transformations:

$$
\begin{aligned}
&sA_\mu = D_\mu c\,, && sc = igcc\,, \\
&s\bar{c} = b\,, && sb = 0\,, \\
&s\bar{\psi}_{\mu\nu} = \bar{B}_{\mu\nu}\,, && s\bar{B}_{\mu\nu} = 0\,, \\
&sB_{\mu\nu} = \psi_{\mu\nu}\,, && s\psi_{\mu\nu} = 0\,, \\
&s\bar{Q}_{\mu\nu\alpha\beta} = \bar{J}_{\mu\nu\alpha\beta}, && s\bar{J}_{\mu\nu\alpha\beta} = 0\,, \\
&sQ_{\mu\nu\alpha\beta} = J_{\mu\nu\alpha\beta}, && sJ_{\mu\nu\alpha\beta} = 0\,.
\end{aligned}
\tag{7.8}
$$

Dimensions and ghost numbers of all fields appearing in the action are listed in Table 7.1.

Table 7.1: Properties of fields and sources.

Field	A_μ	c	$\bar{c}$	$B_{\mu\nu}$	$\bar{B}_{\mu\nu}$	$\psi_{\mu\nu}$	$\bar{\psi}_{\mu\nu}$	$J_{\alpha\beta\mu\nu}$	$\bar{J}_{\alpha\beta\mu\nu}$	$Q_{\alpha\beta\mu\nu}$	$\bar{Q}_{\alpha\beta\mu\nu}$	Ω^A_μ	Ω^c	b
$g_\sharp$	0	1	-1	0	0	1	-1	0	0	-1	-1	-1	-2	0
Mass dim.	1	0	2	2	2	2	2	2	2	2	2	3	4	2
Statistics	b	f	f	b	b	f	f	b	b	f	f	f	b	b

7.2 Symmery content

As in the foregoing chapters, let us for completness reasons also list the symmetry content which would be necessary when applying the Algebraic Renormalization procedure. As detailed in Section 6.2.2, the symmetries allow to exclude possible counter terms in the effective action. The Slavnov-Taylor identity describing the BRST symmetry content of the model is given by

$$
\mathcal{B}(S) = \int \mathrm{d}^4x \left(\frac{\delta S}{\delta \omega^A_\mu}\frac{\delta S}{\delta A_\mu} + \frac{\delta S}{\delta \omega^c}\frac{\delta S}{\delta c} + b\frac{\delta S}{\delta \bar{c}} \right) = 0\,, \tag{7.9}
$$

from which one derives the linearized Slavnov-Taylor operator

$$
\mathcal{B}_S = \int \mathrm{d}^4x \left(\frac{\delta S}{\delta \omega^A_\mu}\frac{\delta}{\delta A_\mu} + \frac{\delta S}{\delta A_\mu}\frac{\delta}{\delta \omega^A_\mu} + \frac{\delta S}{\delta \omega^c}\frac{\delta}{\delta c} + \frac{\delta S}{\delta c}\frac{\delta}{\delta \omega^c} + b\frac{\delta}{\delta \bar{c}} \right)\,. \tag{7.10}
$$

Furthermore we have the gauge fixing condition

$$
\frac{\delta S}{\delta b} = \partial^\mu A_\mu = 0\,, \tag{7.11}
$$

the ghost equation

$$
\mathcal{G}(S) = \partial_\mu \frac{\delta S}{\delta \omega^A_\mu} + \frac{\delta S}{\delta \bar{c}} = 0\,, \tag{7.12}
$$

and the antighost equation

$$\bar{\mathcal{G}}(S) = \int \mathrm{d}^4x \frac{\delta S}{\delta c} = 0 \,. \tag{7.13}$$

The identity associated to the BRST doublet structure of the auxiliary fields is given by

$$\mathcal{U}^{(1)}_{\alpha\beta\mu\nu}(S) = \int \mathrm{d}^4x \left(\bar{B}_{\alpha\beta} \frac{\delta S}{\delta \bar{\psi}_{\mu\nu}} + \psi_{\mu\nu} \frac{\delta S}{\delta B_{\alpha\beta}} + J_{\mu\nu\rho\sigma} \frac{\delta S}{\delta Q_{\alpha\beta\rho\sigma}} + \bar{J}_{\alpha\beta\rho\sigma} \frac{\delta S}{\delta \bar{Q}_{\mu\nu\rho\sigma}} \right) = 0 \,, \tag{7.14}$$

and we finally also have the symmetries $\mathcal{U}^{(0)}$ and $\tilde{\mathcal{U}}^{(0)}$:

$$\mathcal{U}^{(0)}_{\alpha\beta\mu\nu}(S) = \int \mathrm{d}^4x \left[B_{\alpha\beta} \frac{\delta S}{\delta B_{\mu\nu}} - \bar{B}_{\mu\nu} \frac{\delta S}{\delta \bar{B}_{\alpha\beta}} + J_{\alpha\beta\rho\sigma} \frac{\delta S}{\delta J_{\mu\nu\rho\sigma}} - \bar{J}_{\mu\nu\rho\sigma} \frac{\delta S}{\delta \bar{J}_{\alpha\beta\rho\sigma}} \right] = 0 \,, \tag{7.15a}$$

and

$$\tilde{\mathcal{U}}^{(0)}_{\alpha\beta\mu\nu}(S) = \int \mathrm{d}^4x \left[\psi_{\alpha\beta} \frac{\delta S}{\delta \psi_{\mu\nu}} - \bar{\psi}_{\mu\nu} \frac{\delta S}{\delta \bar{\psi}_{\alpha\beta}} + Q_{\alpha\beta\rho\sigma} \frac{\delta S}{\delta Q_{\mu\nu\rho\sigma}} - \bar{Q}_{\mu\nu\rho\sigma} \frac{\delta S}{\delta \bar{Q}_{\alpha\beta\rho\sigma}} \right] = 0 \,. \tag{7.15b}$$

The symmetry operators $\mathcal{U}^{(0)}$ and $\tilde{\mathcal{U}}^{(0)}$ may be combined to the operator $\mathcal{Q}$ describing the reality of the action [14] as

$$\mathcal{Q} \equiv \delta_{\alpha\mu}\delta_{\beta\nu} \left(\mathcal{U}^{(0)}_{\alpha\beta\mu\nu} + \tilde{\mathcal{U}}^{(0)}_{\alpha\beta\mu\nu} \right) \,, \tag{7.16}$$

which obviously also generates a symmetry of the action (7.6). With the definitions of the operators $\mathcal{B}_S$, $\bar{\mathcal{G}}$, $\mathcal{Q}$ and $\mathcal{U}^{(1)}$ we get the following graded commutators:

$$\begin{aligned}
&\left\{\bar{\mathcal{G}}, \bar{\mathcal{G}}\right\} = 0 \,, && \left\{\mathcal{B}_S, \mathcal{B}_S\right\} = 0 \,, && \left\{\bar{\mathcal{G}}, \mathcal{B}_S\right\} = 0 \,, \\
&\left[\bar{\mathcal{G}}, \mathcal{Q}\right] = 0 \,, && \left[\mathcal{Q}, \mathcal{Q}\right] = 0 \,, && \left\{\bar{\mathcal{G}}, \mathcal{U}^{(1)}_{\mu\nu\alpha\beta}\right\} = 0 \,, \\
&\left\{\mathcal{B}_S, \mathcal{U}^{(1)}_{\mu\nu\alpha\beta}\right\} = 0 \,, && \left\{\mathcal{U}^{(1)}_{\mu\nu\alpha\beta}, \mathcal{U}^{(1)}_{\mu'\nu'\alpha'\beta'}\right\} = 0 \,, && \left[\mathcal{U}^{(1)}_{\mu\nu\alpha\beta}, \mathcal{Q}\right] = 0 \,, \\
&\left[\mathcal{B}_S, \mathcal{Q}\right] = 0 \,.
\end{aligned} \tag{7.17}$$

It can be seen that these symmetry operators form a closed algebra.

7.3 Feynman rules and power counting

7.3.1 Propagators

In order to calculate the gauge boson propagator we first integrate over the auxiliary fields $B, \bar{B}, \psi, \bar{\psi}$ in the path integral, and assign to $J, \bar{J}, Q, \bar{Q}$ their physical values given in Eqn. (7.7). The action becomes

$$S_{\mathrm{nl}} = \int \mathrm{d}^4x \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{\gamma^4}{4} \left[\frac{1}{\tilde{\Box}} f_{\mu\nu} \frac{1}{\tilde{\Box}} f_{\mu\nu} + \left(\sigma + \sigma^2 \frac{\theta_{\mu\nu}\theta_{\mu\nu}}{4} \right) \frac{1}{\tilde{\Box}} \tilde{f} \frac{1}{\tilde{\Box}} \tilde{f} \right] + s \left(\bar{c} \partial_\mu A_\mu \right) \right) , \tag{7.18}$$

which by using the abbreviation $\theta^2 = \theta_{\mu\nu}\theta_{\mu\nu}$ and with the definition of $\tilde{f}$ reduces to

$$S_{\mathrm{nl}} = \int \mathrm{d}^4x \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \gamma^4 \left[\partial_\mu A_\nu \frac{1}{2\tilde{\Box}^2} f_{\mu\nu} + \left(\sigma + \tfrac{\theta^2}{4} \sigma^2 \right) (\tilde{\partial} A) \frac{1}{\tilde{\Box}^2} (\tilde{\partial} A) \right] + s \left(\bar{c} \partial_\mu A_\mu \right) \right) . \tag{7.19}$$

With the abbreviation

$$\bar{\sigma}^4 \equiv 2 \left(\sigma + \frac{\theta^2}{4} \sigma^2 \right) \gamma^4 \, , \tag{7.20}$$

and considering the case where $\theta_{\mu\nu}$ has the simple block diagonal form given in (7.4) so that $\tilde{k}^2 = k^2$, we get for the gauge field propagator:

$$\begin{aligned} G^{AA}_{\mu\nu}(k) &= \frac{1}{k^2 \left(1 + \frac{\gamma^4}{(\tilde{k}^2)^2} \right)} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - \frac{\bar{\sigma}^4}{\left[\bar{\sigma}^4 + k^2 \left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2} \right) \right]} \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2} \right) \\ &= \left[k^2 + \frac{\gamma^4}{\tilde{k}^2} \right]^{-1} \left[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - \frac{\bar{\sigma}^4}{\left(k^2 + (\bar{\sigma}^4 + \gamma^4) \frac{1}{\tilde{k}^2} \right)} \frac{\tilde{k}_\mu \tilde{k}_\nu}{(\tilde{k}^2)^2} \right] . \end{aligned} \tag{7.21}$$

For the general action, we encounter furthermore the following propagators:

$$G^{\bar{c}c}(k) = \frac{-1}{k^2} \, , \tag{7.22a}$$

$$\begin{aligned} G^{BA}_{\mu\nu,\rho}(k) &= \frac{\mathrm{i}\gamma^2 \left(k_\mu \delta_{\sigma\nu} - k_\nu \delta_{\sigma\mu} - \sigma \tilde{k}_\sigma \theta_{\mu\nu} \right)}{2k^2 \left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2} \right)} \left[\delta_{\rho\sigma} - \frac{\bar{\sigma}^4}{\left[\bar{\sigma}^4 + k^2 \left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2} \right) \right]} \frac{\tilde{k}_\rho \tilde{k}_\sigma}{\tilde{k}^2} \right] \\ &= G^{\bar{B}A}_{\mu\nu,\rho}(k) \, , \end{aligned} \tag{7.22b}$$

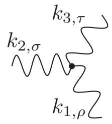
$$\begin{aligned} G^{BB}_{\mu\nu,\rho\sigma}(k) &= -\gamma^4 \frac{\left(k_\mu k_\rho \delta_{\nu\sigma} + k_\nu k_\sigma \delta_{\mu\rho} - k_\mu k_\sigma \delta_{\nu\rho} - k_\nu k_\rho \delta_{\mu\sigma} \right)}{2k^2 \tilde{k}^2 \left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2} \right)} \\ &\quad + \frac{\gamma^4}{4\tilde{k}^2 \left[k^2 \left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2} \right) + \bar{\sigma}^4 \right]} \left[\sigma\theta_{\mu\nu} \left(k_\rho \tilde{k}_\sigma - k_\sigma \tilde{k}_\rho \right) + \sigma\theta_{\rho\sigma} \left(k_\mu \tilde{k}_\nu - k_\nu \tilde{k}_\mu \right) \right. \\ &\qquad \left. - \sigma^2 \tilde{k}^2 \theta_{\mu\nu} \theta_{\rho\sigma} - \bar{\sigma}^4 \frac{\left(k_\mu \tilde{k}_\nu \tilde{k}_\rho k_\sigma + k_\rho \tilde{k}_\sigma \tilde{k}_\mu k_\nu - k_\mu \tilde{k}_\nu \tilde{k}_\sigma k_\rho + k_\sigma \tilde{k}_\rho \tilde{k}_\mu k_\nu \right)}{k^2 \tilde{k}^2 \left(\tilde{k}^2 + \frac{\gamma^4}{\tilde{k}^2} \right)} \right] \\ &= G^{\bar{B}\bar{B}}_{\mu\nu,\rho\sigma}(k), \end{aligned} \tag{7.22c}$$

$$G^{B\bar{B}}_{\mu\nu,\rho\sigma}(k) = -\frac{1}{2}\left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right) + G^{BB}_{\mu\nu,\rho\sigma}(k)\,, \tag{7.22d}$$

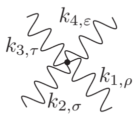
$$G^{\bar{\psi}\psi}_{\mu\nu\rho\sigma}(k) = -\frac{1}{2}\left(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right)\,. \tag{7.22e}$$

7.3.2 Vertices

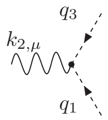
The model (7.6) leads to three vertices which equal those of the “naïve” implementation of QED on non-commutative spaces found, for example, in Ref. [81]:



$$= \widetilde{V}^{3A}_{\rho\sigma\tau}(k_1,k_2,k_3) = 2\mathrm{i}g(2\pi)^4\delta^4(k_1+k_2+k_3)\sin\left(\tfrac{\varepsilon}{2}k_1\tilde{k}_2\right)\times \\ \times\left[(k_3-k_2)_\rho\delta_{\sigma\tau} + (k_1-k_3)_\sigma\delta_{\rho\tau} + (k_2-k_1)_\tau\delta_{\rho\sigma}\right], \tag{7.23a}$$



$$= \widetilde{V}^{4A}_{\rho\sigma\tau\epsilon}(k_1,k_2,k_3,k_4) = -4g^2(2\pi)^4\delta^4(k_1+k_2+k_3+k_4)\times \\ \times\Big[(\delta_{\rho\tau}\delta_{\sigma\epsilon} - \delta_{\rho\epsilon}\delta_{\sigma\tau})\sin\left(\tfrac{\varepsilon}{2}k_1\tilde{k}_2\right)\sin\left(\tfrac{\varepsilon}{2}k_3\tilde{k}_4\right) \\ + (\delta_{\rho\sigma}\delta_{\tau\epsilon} - \delta_{\rho\epsilon}\delta_{\sigma\tau})\sin\left(\tfrac{\varepsilon}{2}k_1\tilde{k}_3\right)\sin\left(\tfrac{\varepsilon}{2}k_2\tilde{k}_4\right) \\ + (\delta_{\rho\sigma}\delta_{\tau\epsilon} - \delta_{\rho\tau}\delta_{\sigma\epsilon})\sin\left(\tfrac{\varepsilon}{2}k_2\tilde{k}_3\right)\sin\left(\tfrac{\varepsilon}{2}k_1\tilde{k}_4\right)\Big], \tag{7.23b}$$



$$= \widetilde{V}^{\bar{c}Ac}_{\mu}(q_1,k_2,q_3) = -2\mathrm{i}g(2\pi)^4\delta^4(q_1+k_2+q_3)q_{3\mu}\sin\left(\tfrac{\varepsilon}{2}q_1\tilde{q}_3\right)\,. \tag{7.23c}$$

7.3.3 Power counting

Regarding the superficial degree of UV divergence d_γ one gets the following relations for the number of loops L, external lines E_ϕ, internal lines I_ϕ and vertices V_ϕ for fields $\phi \in \{A, c, \bar{c}\}$ (c.f. Section 3.4 for an explanation of the method):

$$\begin{aligned} L &= I_A + I_{c\bar{c}} - \left(V_{\bar{c}Ac} + V_{3A} + V_{4A} - 1\right)\,, \\ E_{c\bar{c}} + 2I_{c\bar{c}} &= 2V_{\bar{c}Ac}\,, \\ E_A + 2I_A &= 3V_{3A} + 4V_{4A} + V_{\bar{c}Ac}\,. \end{aligned} \tag{7.24}$$

By simply counting the UV powers of the Feynman rules one gets:

$$d_\gamma = 4L - 2I_A - 2I_{c\bar{c}} + V_{3A} + V_{4A} + V_{\bar{c}Ac}\,. \tag{7.25}$$

This system of equations can be resolved by eliminating the I_ϕ and V_ϕ which leads to

$$d_\gamma = 4 - E_A - E_{c\bar{c}}\,. \tag{7.26}$$

This is again in agreement with the respective relations for the “naïve” implementation of non-commutative $U_\star(1)$.

7.3.4 Analysis

Let us start by investigating the behaviour of the gauge boson propagator in the IR ($k^2 \to 0$) and UV limit ($k^2 \to \infty$). We get

$$G^{AA}_{\mu\nu}(k) \approx \begin{cases} \frac{\tilde{k}^2}{\gamma^4}\left[\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - \frac{\bar{\sigma}^4}{(\bar{\sigma}^4+\gamma^4)}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right], & \text{for } \tilde{k}^2 \to 0 \\ \frac{1}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right), & \text{for } k^2 \to \infty \end{cases}. \tag{7.27}$$

In the IR limit we encounter (by construction) a term of the same form as the one-loop vacuum polarization (7.1). Contrary to the previous models, we have no vertices involving the auxiliary fields. Hence the only contribution for the propagator at one-loop level will stem from the vacuum polarization. As will be explicitly shown in Section 7.4, this will allow for the absorption of the divergence and hence lead to renormalization. Another advantageous property of the gauge propagator is that the UV limit (from which divergences originate), admits to neglect the term proportional to γ which reduces the number of terms in Feynman integrals considerably.

Looking at the ghost propagator (7.22a) we see that it is quadratically IR divergent, as usual in Landau gauge. Obviously, we could add a damping factor to the gauge fixing term $b(\partial A)$ and the ghost sector $\bar{c}\partial_\mu D^\mu c$ as has been done in (3.25) Ref. [1]. However, such dampings appear in vertex expressions with an inverse power relative to the respective propagators and, thus, cancel each other. Moreover, these factors contribute to UV divergences at higher loop orders, and are omitted, hence.

Regarding the other propagators (7.22) involving the auxiliary fields $(B, \bar{B}, \psi, \bar{\psi})$ it may be observed that all of them tend towards a constant in the infrared as well as in the ultraviolet. However, eventually they will not contribute to the physical results, because respective vertices are missing. This simplifies explicit calculations considerably.

7.4 Vacuum polarization and renormalization

Vacuum polarization

The Feynman rules of Section 7.3 lead to three graphs for the vacuum polarization $\Pi_{\mu\nu}(p)$, depicted in Fig. 7.1. Note that they are the same as encountered in theories like

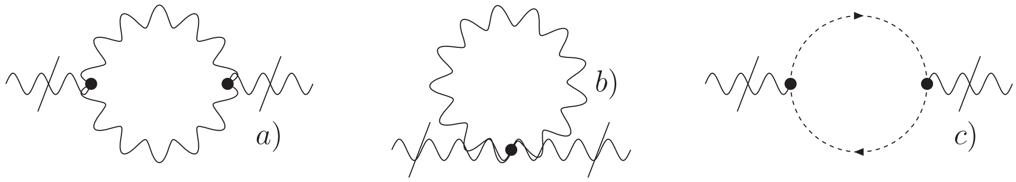


Figure 7.1: One loop corrections to the gauge boson propagator.

commutative QCD and the naïve extension of QED to the non-commutative setting. In order to get the expressions for the three graphs, we expand the integrands according to (3.47) around small external momenta (which is allowed since we are interested in the behaviour for $p^2 \to 0$). Note that the phase factor is excluded from the expansion, in order to keep its regulating effect for high internal momenta. By noting furthermore that all divergences arise from large momenta k of the integrand, the tree level gauge propagator may be replaced by its values at high momenta (i.e. the expression for $k \to \infty$ in (7.27)). Finally the sum of the integrated results is given by

$$\Pi_{\mu\nu}(p) = \frac{2g^2}{\pi^2\varepsilon^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\tilde{p}^2)^2} + \frac{13g^2}{3(4\pi)^2}\left(p^2\delta_{\mu\nu} - p_\mu p_\nu\right)\ln\left(\Lambda^2\right) + \text{finite terms}\,. \tag{7.28}$$

As usual, Λ denotes a ultraviolet cutoff, and ‘finite terms’ collects contributions being finite in the limits $\Lambda \to \infty$ and $\tilde{p}^2 \to 0$, respectively. As expected, in the IR we observe a quadratic divergence with the tensor structure Eqn. (7.1).

Renormalization of the one-loop gauge propagator

Along the same line as in Section 5.6.3 and according to (5.50), we can write for the complete gauge boson propagator at one-loop level

$$G^{\mathrm{AA,1l-complete}}_{\mu\nu}(k) = \left[\Gamma^{AA,tree}_{\mu\nu}(k) - \Gamma^{AA,corr.}_{\mu\nu}(k)\right]^{-1}, \tag{7.29}$$

where the tree level vertex function $\Gamma^{\mathrm{AA,1l-complete}}_{\mu\nu}(k)$ has been identified with $\mathcal{A}$, and $\Sigma(\Lambda, p) \to \Gamma^{AA,corr.}_{\mu\nu}(k) = \Pi_{\mu\nu}(k)$ is given by the vacuum polarization. Renormalizability (at one loop level) is given, if all divergences present in $\Pi_{\mu\nu}(k)$ may be absorbed in the parameters and the wave function of the tree level action.

First, we calculate the tree level vertex function. It is given by the inverse of the gauge boson propagator, which due to Landau gauge fixing does not exist. We therefore have to consider the propagator derived from the action (7.6) with general gauge fixing parameter α^2 (c.f (D.12)), leading to

$$G^{AA}_{\mu\nu}(k) = \frac{1}{k^2\mathcal{D}}\left(\delta_{\mu\nu} - (1-\alpha\mathcal{D})\frac{k_\mu k_\nu}{k^2} - \mathcal{F}\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right), \text{ with}$$

[2]Note, that the quadratic IR divergence is independent of the gauge fixing [63, 62, 81].

$$\mathcal{D}(k) \equiv \left(1 + \frac{\gamma^4}{(\tilde{k}^2)^2}\right) \quad \text{and} \quad \mathcal{F}(k) \equiv \frac{1}{\tilde{k}^2} \frac{\bar{\sigma}^4}{\left(k^2 + (\bar{\sigma}^4 + \gamma^4)\frac{1}{\tilde{k}^2}\right)} . \tag{7.30}$$

This leads to the vertex function

$$\Gamma^{AA,\text{tree}}_{\mu\nu}(k) = \left(G^{-1}_{AA}\right)_{\mu\nu}(k) = k^2 \mathcal{D} \left(\delta_{\mu\nu} + \left(\frac{1}{\alpha\mathcal{D}} - 1\right)\frac{k_\mu k_\nu}{k^2} + \frac{\bar{\sigma}^4}{k^2 \tilde{k}^2 \mathcal{D}} \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2}\right) . \tag{7.31}$$

With the abbreviations

$$\begin{aligned} \Gamma^{AA,\text{corr.}}_{\mu\nu}(k) =& \mathit{\Pi}_1 \frac{\tilde{k}_\mu \tilde{k}_\nu}{(\tilde{k}^2)^2} + \mathit{\Pi}_2 \left(k^2 \delta_{\mu\nu} - k_\mu k_\nu\right) , \quad \text{with} \\ \mathit{\Pi}_1 =& \frac{2g^2}{\pi^2 \varepsilon^2} , \quad \text{and} \quad \mathit{\Pi}_2 = \frac{13 g^2}{3(4\pi)^2} \ln \Lambda^2 , \end{aligned} \tag{7.32}$$

splitting the vacuum polarization into momentum dependent/independent parts, we get for the complete one-loop vertex function:

$$\begin{aligned} \Gamma^{AA,\text{1l-compl.}}_{\mu\nu}(k) &= \Gamma^{AA,\text{tree}}_{\mu\nu}(k) - \Gamma^{AA,\text{corr.}}_{\mu\nu}(k) \\ &= k^2(\mathcal{D} - \mathit{\Pi}_2)\left\{\delta_{\mu\nu} + \left(\frac{1}{\alpha(\mathcal{D} - \mathit{\Pi}_2)} - 1\right)\frac{k_\mu k_\nu}{k^2} + \frac{\bar{\sigma}^4 - \mathit{\Pi}_1}{k^2 \tilde{k}^2 (\mathcal{D} - \mathit{\Pi}_2)} \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2}\right\} . \end{aligned} \tag{7.33}$$

By introduction of the wave-function renormalization Z_A and the renormalized parameters γ_r and $\bar{\sigma}_r$,

$$\begin{aligned} Z_A &= \frac{1}{\sqrt{1 - \mathit{\Pi}_2}} , \\ \gamma_r^4 &= \gamma^4 Z_A^2 , \\ \bar{\sigma}_r^4 &= \left(\bar{\sigma}^4 - \mathit{\Pi}_1\right) Z_A^2 , \end{aligned} \tag{7.34}$$

the one-loop two-point vertex function is cast into the same form as its tree-level counter part, i.e.

$$\begin{aligned} \Gamma^{AA,\text{ren}}_{\mu\nu}(k) &= \frac{k^2 \mathcal{D}_r}{Z_A^2}\left\{\delta_{\mu\nu} + \left(\frac{Z_A^2}{\alpha\mathcal{D}_r} - 1\right)\frac{k_\mu k_\nu}{k^2} + \frac{\bar{\sigma}_r^4}{k^2 \tilde{k}^2 \mathcal{D}_r} \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2}\right\} , \\ \mathcal{D}_r(k) &\equiv \left(1 + \frac{\gamma_r^4}{(\tilde{k}^2)^2}\right) . \end{aligned} \tag{7.35}$$

Finally, we may also write $\bar{\sigma}_r$ in terms of the renormalized σ_r:

$$\begin{aligned} \bar{\sigma}_r^4 &= 2\left(\sigma_r + \frac{\theta^2}{4}\sigma_r^2\right)\gamma^4 Z_A^2 , \\ \sigma_r &= -\frac{2}{\theta^2} \pm 2\sqrt{\left(1 + \frac{\theta^2}{2}\sigma\right)^2 - \frac{g^2\theta^2}{\pi^2\gamma^4\varepsilon^2}} . \end{aligned} \tag{7.36}$$

The renormalized propagator in Landau gauge fixing (i.e. $\alpha \to 0$) finally becomes

$$\begin{aligned} G^{AA,\mathrm{ren}}_{\mu\nu}(k) &= \frac{Z_A^2}{k^2\mathcal{D}_r}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - \mathcal{F}_r\frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}\right), \\ \mathcal{F}_r &\equiv \frac{1}{\tilde{k}^2}\frac{\bar{\sigma}_r^4}{\left(k^2 + (\bar{\sigma}_r^4 + \gamma_r^4)\,\frac{1}{\tilde{k}^2}\right)}\,. \end{aligned} \tag{7.37}$$

Now that the two-point vertex function for the gauge boson can be written as sum of a longitudinal, transversal and non-commutative part according to

$$\begin{aligned} \Gamma^{AA}_{\mu\rho} =& \Gamma^{AA,T}(\delta_{\mu\rho} - \frac{k_\mu k_\rho}{k^2}) + (\Gamma^{AA,NC})\,\frac{\tilde{k}_\mu\tilde{k}_\rho}{\tilde{k}^2} + (\Gamma^{AA,L})\,\frac{k_\mu k_\rho}{k^2}\,, \quad \text{with} \\ \Gamma^{AA,T} &= k^2\mathcal{D}\,, \quad \Gamma^{AA,NC} = \frac{\bar{\sigma}^4}{\tilde{k}^2} \quad \text{and} \quad (\Gamma^{AA,L}) = \frac{k^2}{\alpha}\,. \end{aligned} \tag{7.38}$$

The renormalization conditions are then given by

$$\frac{(\tilde{k}^2)^2}{k^2}\Gamma^{AA,T}\Big|_{k^2=0} = \gamma^4\,, \tag{7.39a}$$

$$\frac{1}{2k^2}\frac{\partial(k^2\Gamma^{AA,T})}{\partial k^2}\Big|_{k^2=0} = 1\,, \tag{7.39b}$$

$$\tilde{k}^2\Gamma^{AA,NC}\Big|_{k^2=0} = \bar{\sigma}^4\,, \tag{7.39c}$$

$$\Gamma^{AA,L}\Big|_{k^2=0} = 0\,, \tag{7.39d}$$

$$\frac{\partial\Gamma^{AA,L}}{\partial k^2}\Big|_{k^2=0} = \frac{1}{\alpha}\,. \tag{7.39e}$$

7.5 IR damping at higher loop orders

In the previous section we have shown renormalizability of the gauge propagator at one loop level. In order to investigate the IR behaviour at higher loop order, let us analyze the behaviour of a chain of n vacuum polarizations connected by gauge boson propagators, which might be part of another loop graph:

$$\begin{aligned} \Xi^{AA}_{\mu\nu}(p,n) &= \left(G^{AA}(p)\Pi^{\mathrm{np}}(p)\right)^n_{\mu\rho} G^{AA}_{\rho\nu}(p)\Big|_{\mathrm{IR}} \\ &\approx \frac{g^{2n}}{\left[\varepsilon^2\left(\bar{\sigma}^4+\gamma^4\right)\right]^{n+1}}\tilde{p}_\mu\tilde{p}_\nu\,. \end{aligned} \tag{7.40}$$

We observe a vanishing, i.e. IR-finite behaviour, which is independent of the number of graphs. We therefore conclude the absence of IR problems at higher loop order. Obviously, for vanishing soft breaking parameter γ (implying $\bar{\sigma} \to 0$) the divergent behaviour of the "naïve" model without soft breaking is recovered.

7.6 Vertex corrections and β-function

In this section we will first discuss the one-loop corrections for the three vertices. The planar result for the vertex with three external A-legs will then lead to the β-function.

7.6.1 Methodology for the calculation of the vertex expressions

The calculation of the respective expressions for the vertex corrections is basically done along the same line as for the propagator corrections. However, some additional difficulties appear.
First, when considering vertices with n external legs, we get integrands which contain products of the same number of *sine* functions, each of them depending on the internal momentum. Such integrals are difficult to evaluate. Yet, a product of *sine*-functions can be reduced to a sum of expressions, each of them containing only one trigonometric function depending on the integration variable, as explained in Section E.1. The resulting integrals can then easily be calculated, the only practical difficulty being given by the high number of summands.
A more basic difficulty is given by the fact that a Taylor expansion of the type (3.47) can not be applied for $n > 2$ external momenta. This can be understood by considering the example $p_{1,\mu}p_{2,\nu}/(p_1^2+p_2^2)\big|_{\{|p_1|,|p_2|\}\to 0}$. Here the limit is not well defined for its derivatives, i.e. the limit depends on the order of its application. In order to calculate the expressions, there exist several possibilities:

- *Taylor expansion of the integrand:* Momentum conservation is applied, which leads to the elimination of one of the momenta. The Taylor expansion of the integrand without the phase, denoted by $\mathcal{I}(k, p_1, \ldots, p_n)$ is then (for vertices with 3 external A-legs) given by

$$\mathcal{I}(k, p_1, p_2, p_3) \approx \mathcal{I}(k, 0, 0, 0) + \sum_{i=1,2} \left[\frac{\partial}{\partial p_i}\mathcal{I}(k, p_1, p_2, -p_1 - p_2)\right]\Big|_{p_2 \approx p_1} p_i \,. \tag{7.41}$$

  Being interested in the divergence structure only, it follows from power counting that it is sufficient to consider the expansion only up to first order for the case $n = 3$ and zero order for $n = 4$. In the next step, the limit $p \to 0$ can be taken. A possible criticism is given by considering that the above Taylor expansion leads to integrals with one external momentum only, similar to the case of one-loop propagator corrections.

- *Approximation of the denominator:* When considering expressions of the form

$$\int d^4k \frac{k_{1,\mu}(k+p)_{2,\nu}(k+p)_{3,\sigma}}{(k_1+p_2)^2} \sin\left(k(\tilde{p}_1 - 2\tilde{p}_2)\right) \tag{7.42}$$

  one may first approximate the denominator by setting all external momenta to zero. This is justified by the fact that *all* divergences (in the UV and IR sector)

Graph	# of expressions	$Symmetry factors$
3A $a)$	1/2	3
3A $b)$	1	6
3A $c)$	2	6
4A $a)$	1/2	6
4A $b)$	1	12
4A $c)$	1	24
4A $d)$	2	24
$\bar{c}Ac\ a)$	1	2
$\bar{c}Ac\ b)$	1	2

Table 7.2: Symmetry factors and number of different expressions for the vertex corrections

result from integration over high internal momenta. With the integration formulae given in the appendix it is then easy to evaluate such integrals. In the cases below, the second method has been applied.

Before proceeding to the explicit results, note that for each graph there exist different expressions resulting from permuting the external legs. For the example of the graph Fig. 7.2 $a)$ we have $3! = 6$ different possibilities of permuting the external legs. However, we have to divide by 2, because interchanging the external legs from the $4A$-vertex leads to identical expressions. The resulting number of graphs is then 3. Furthermore, each of those expressions has to be multiplied by the correct symmetry factor steming from internal symmetries. The corresponding information is listed in Tab. 7.2.

7.6.2 Corrections to the $3A$-vertex

The graphs are displayed in Fig. 7.2. Although power counting allows linear divergences,

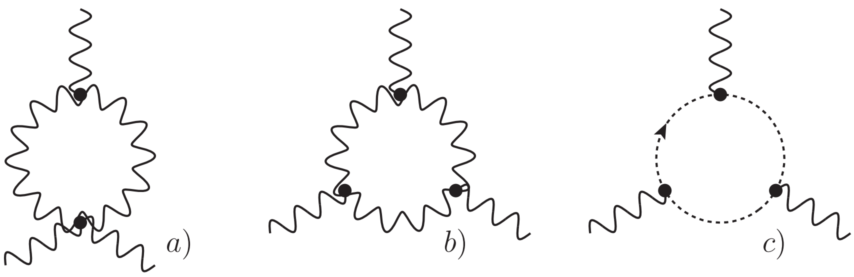


Figure 7.2: One loop corrections to the 3A-vertex.

in the planar part we observe only a logarithmic UV divergence:

$$\begin{aligned}\Gamma^{3A,\mathrm{UV}}_{\mu\nu\rho}(p_1,p_2,p_3) =& \frac{\mathrm{i}g^3\ln\Lambda}{\pi^2}(2\pi)^4\delta^4(p_1+p_2+p_3)\times \\ & \sin\left(\varepsilon\frac{p_1\tilde{p}_2}{2}\right)\left[(p_1-p_2)_\mu\delta_{\nu\rho}+(p_2-p_3)_\nu\delta_{\rho\mu}+(p_3-p_1)_\rho\delta_{\mu\nu}\right].\end{aligned} \tag{7.43}$$

The non-planar parts $\Gamma^{3A,\mathrm{IR}}_{\mu\nu\rho}(p_1,p_2,p_3)$ result in linear IR divergences of the form [78, 77, 62]

$$\left\{g^3\sum_{i=1,2,3}\frac{\tilde{p}_{i,\mu}\tilde{p}_{i,\nu}\tilde{p}_{i,\rho}}{\varepsilon(\tilde{p}_i^2)^2}\,,\quad g^3\sum_{i=1,2,3}\frac{\delta_{\mu_1\mu_2}\tilde{p}_{i,\mu_3}}{\tilde{p}_i^2}\quad\text{with}\quad\{\mu_1,\mu_2,\mu_3\}\in\{\mu,\nu,\rho\}\right\}, \tag{7.44}$$

and the corresponding counterterms (up to numeric prefactors) which have to be added to the Langrangian are then given by

$$\left\{\frac{(\tilde{\partial}A)^3}{\tilde{\square}^2},\quad\frac{(\tilde{\partial}A)}{\tilde{\square}}A^2\right\}. \tag{7.45}$$

7.6.3 Corrections to the $4A$-vertex

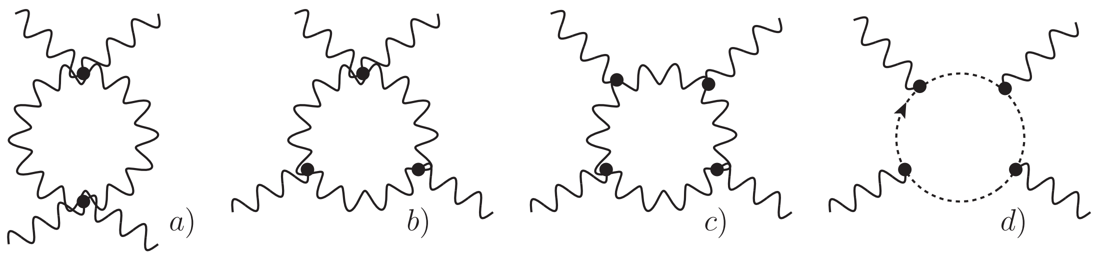


Figure 7.3: One loop corrections to the 4A-vertex.

The planar part $\Gamma^{4A,\mathrm{UV}}_{\mu\nu\rho}(p_1,p_2,p_3)$ is given by the tree level vertex multiplied by a correction term, as follows from the following considerations: Renormalization of the $3A$ vertex leads to a expression of the form

$$\begin{aligned}\Gamma^{(1)}_{3A,complete} &= \Gamma^{(0)}_{3A}-\Gamma^{(1)}_{3A} = g\left(1-\frac{\mathrm{i}g^2\ln\Lambda^2}{(2\mathrm{i})\pi^2}\right)\left(\frac{\Gamma^{(0)}_{3A}}{g}\right) \\ &\Rightarrow g'_r = g\left(1-\frac{\mathrm{i}g^2\ln\Lambda^2}{(2\mathrm{i})\pi^2}\right) = gZ_g^{3A}\equiv g(1-f_g^{3A})\,.\end{aligned} \tag{7.46}$$

Here, g'_r denotes the renormalized coupling by considering the corrections stemming from the one-loop vertices only, while the full renormalized coupling has to include also the wave function renormalization, c.f. Section 7.6.5. Furthermore, Z_g^{3A} denotes the renormalization due to the $3A$ vertex, and $\Gamma^{(i)}_{3A}$ denotes the $3A$ vertex at order i. Now, the same consideration applies to the correction of the $4A$-vertex:

$$\begin{aligned}\Gamma^{(1)}_{4A,complete} &= \Gamma^{(0)}_{4A} - \Gamma^{(1)}_{4A} = g^2 Z_g^{4A} = g^2(1-f_g^{4A}) \equiv (g')^2_r \\ &\Rightarrow (1-f_g^{4A}) = (1-f_g^{3A})^2 \Leftrightarrow f_g^{4A} = 2f_g^{3A} - (f_g^{3A})^2\,. \end{aligned} \tag{7.47}$$

Hence, the planar part of the one-loop correction to the $4A$-vertex is given by the tree level expression (7.23b), multiplied by

$$\frac{g^2 \ln \Lambda^2}{2\pi^2}\left\{2 - \frac{g^2 \ln \Lambda^2}{2\pi^2}\right\}. \tag{7.48}$$

For the non-planar part, we get expressions of the form

$$\begin{aligned}&\left\{g^4 \sum_{i=1\dots4} \frac{\tilde{p}_{i,\mu}\tilde{p}_{i,\nu}\tilde{p}_{i,\rho}\tilde{p}_{i,\sigma}}{\tilde{p}_i^4},\, g^4 \sum_{i=1\dots4} \frac{\tilde{p}_{i,\mu_1}\tilde{p}_{i,\mu_2}}{\tilde{p}_i^2}\delta_{\mu_3\mu_4}\,,\, g^4 \sum_{i=1\dots4} K_0\left(\sqrt{\tilde{p}_i^2\eta^2}\right)\delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4}\right\} \\ &\quad \text{with} \quad \{\mu_1,\mu_2,\mu_3,\mu_4\} \in \{\mu,\nu,\rho,\sigma\}\end{aligned} \tag{7.49}$$

Only the terms with a Bessel function represent a problem, the others being finite. They are divergent in both of their arguments, the external momentum as well as the regulator mass η. At this point it is not clear how to construct appropriate counterterms. Yet, it is strongly expected that they will vanish when summing up all corresponding contributions (c.f. [119]). Yet, this has to be clarified by explicit calculations.

7.6.4 Corrections to the $\bar{c}Ac$-vertex

The two graphs which enter the one-loop correction of the $\bar{c}Ac$ vertex are given in Fig. 7.4. Explicit calculation show that they are both finite, hence do not have to be considered further.

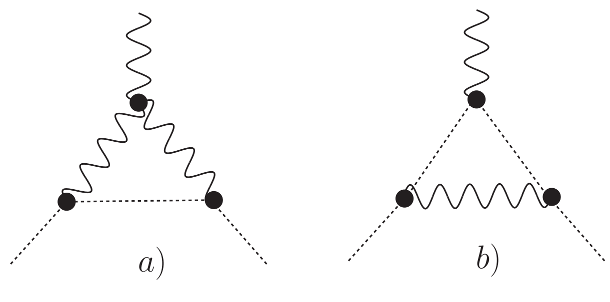


Figure 7.4: One loop corrections to the $\bar{c}Ac$-vertex.

7.6.5 The β-function

As can be seen from (7.34), we have employed the following convention for the renormalization of the gauge field:

$$A^r = Z_A^{-1} A\,. \tag{7.50}$$

Therefore, we obtain for the renormalized coupling

$$g_r = g'_r Z_A^3 = g Z_g Z_A^3\,, \tag{7.51}$$

where Z_g denotes the vertex correction to the three-photon vertex, i.e. is identical to Z_g^{3A} considered before. The wave function renormalization has to be included, because the vertex correction has been calculated using the unrenormalized fields A, and therefore would yield a counter term of the form

$$g(Z_g - 1)[A_\mu, A_\nu]\partial_\mu A_\nu\,, \tag{7.52}$$

where $(Z_g - 1) = -f_{3A}$ according to the definition in (7.46). This expression has to be compared with the expression in the renormalized action, namely

$$g_r[A^r_\mu, A^r_\nu]\partial_\mu A^r_\nu\,. \tag{7.53}$$

In other words, when renormalizing the gauge fields, this has to be compensated by a corresponding term to be included in the coupling. How to obtain the expression for g'_r has been discussed above. Obviously, the coupling is also renormalized by 1-loop corrections from the four-point functions. But these corrections just reproduce the above result for g'_r due to gauge invariance, c.f. Section 7.6.3.
Let us proceed to the calculation of the β-function. It is given by the logarithmic derivative of the bare coupling with respect to the cut-off, for fixed g_r:

$$\beta(g,\Lambda) = \Lambda \frac{\partial g}{\partial \Lambda}\Big|_{g_r\, fixed}\,, \tag{7.54}$$

$$\beta(g) = \lim_{\Lambda\to\infty} \beta(g,\Lambda)\,. \tag{7.55}$$

In the present case, we have

$$Z_A = \left(1 - \frac{26g^2}{3(4\pi)^2}\ln\Lambda\right)^{-1/2}\,, \tag{7.56}$$

$$Z_g = 1 - \frac{g^2}{2\pi^2}\ln\Lambda\,, \tag{7.57}$$

Note that Z_A stems from (7.28), where we have used $\ln\Lambda^2 = 2\ln\Lambda$. By applying to $Z_A^{3/2}$ the Taylor expansion $(1-x)^{-\frac{3}{2}} = 1 + \frac{3x}{2} + \mathcal{O}(x^2)$ we obtain

$$g_r = g\left(1 + \frac{5}{16\pi^2} g^2 \ln\Lambda\right) + \mathcal{O}(g^5)\,. \tag{7.58}$$

Differentiation with respect to Λ, considering the relation

$$\frac{\partial g_r}{\partial \Lambda} = 0\,,$$

and subsequent multiplication with Λ leads to

$$0 = \beta(g,\Lambda) + 3\,vg^2\beta(g,\Lambda)\ln\Lambda + vg^3\,, \tag{7.59}$$

where the we have used the definition

$$g_r = g(1 + g^2 v\ln\Lambda) + \mathcal{O}(g^5).$$

For $\beta(g,\Lambda)$, this leads to the solution

$$\beta(g,\Lambda) = -g^3 v + \mathcal{O}(g^5)\,. \tag{7.60}$$

Finally we obtain for the β-function

$$\beta(g) = \lim_{\Lambda\to\infty}\beta(g,\Lambda) = -g^3 v = -\frac{5g^3}{16\pi^2} < 0\,. \tag{7.61}$$

Hence, it is negative and will decrease with g, which exhibits asymptotic freedom, contrary to commutative $U(1)$ gauge theory. This confirms the result obtained in [120], where the β-function has been calculated for the naïve $U(1)$ gauge theory.

7.7 Conclusion and outlook

We have first analyzed the problems encountered in the renormalization attempt for the previous model according to Section 5.6.3, which can be summarized by a lack of terms in the action (or equivalently in the gauge boson propagator) capable of absorbing the one-loop divergences, and the problems empeding the application of schemes like Algebraic Renormalization. This is due to the possibility of constructing an arbitrary number of counterterms which are invariant with respect to all symmetries, which again stems from the mass dimension of the deformation matrix with $dim(\theta^{old}_{\mu\nu}) = -2$. Based on these findings, we have split the latter into two parts and shifted the tensor structure to a constant matrix with $dim(\theta_{\mu\nu}) = 0$. By this, contracting operators with the deformation matrix does not lead to dimensionless operator insertions in the tree level action, although they might still appear at higher order. Based on the Gribov-Zwanziger approach already applied for the previous model, we have introduced auxiliary fields forming BRST-doublets. The fact that it is always the soft breaking term which implements the damping finally led to an action avoiding the detour via a gauge invariant damping term. A splitting of the field strength tensor $F_{\mu\nu}$ into a linear and interacting part allowed for affecting only the bilinear part of the action, avoiding the introduction of new vertices.

In summary, this led to a non-commutative action which - with respect to the Feynman rules - is as simple as the naïve $U_\star(1)$ gauge model while remedying the UV/IR mixing problem. Furthermore, the possibility of absorbing all divergences present at one-loop level for the propagator corrections, and the missing of IR divergent terms in higher loop orders does not only show one-loop renormalizability of the action but also suggests renormalizability to all orders. At this point, a rigorous proof is still missing. This would be the next logical step, which however requires to establish the foundations for schemes like Multiscale Analysis to θ-deformed gauge theories.

Summary and outlook

In our attempt of finding a renormalizable non-commutative $U(1)$ gauge theory, short $U_\star(1)$, three models defined on Euclidean θ-deformed space ($\mathbb{E}^4_\theta$) were discussed. All of them are based on the idea of Gurau *et. al.* [7] of adding additional terms to the action, which allow to absorb the *IR*-divergences appearing in the course of one-loop calculations. While in the model of Chapter 3 [1] a new dynamical field appeared, this drawback could be avoided in Chapter 5 (c.f. [2, 3]) by the introduction of a soft breaking term known from the Gribov-Zwanziger approach, c.f. Chapter 4. However, it turned out not to contain the appropriate terms to absorb all divergences appearing at one-loop level, i.e. to be non-renormalizable without adding new terms. With the target of arriving at a renormalizable effective action, the possible extension of standard renormalization schemes known from commutative QFTs was discussed, c.f. Chapter 6 and [4]. There it was found, that their direct application without prior modifications is impeded by the inherent non-locality of θ-deformed gauge field theories (GFTs). However, a new type of model was developed (Chapter 7, [5]), which implements the *IR* damping behaviour directly via the soft breaking term. For the latter, one-loop renormalizability could be shown, and there are strong hints that it is renormalizable to all orders.

The next step would consist in delivering a rigorous proof for the renormalizability of the model, e.g. by application of Multiscale Analysis. The next logical step is then the transition to non-commutative Minkowski space-time[3] $\mathbb{M}^4_\theta$. In the long run, it will be necessary to consider dynamic deformation matrices. Possibly, the formulation of consistent and renormalizable gauge field theories with a deformation matrix depending on the space-time metric will one day lead to a description of gravity[4] via non-commutative QFT and to a unified description of all forces by the resulting *Theory of Everything.*

[3]For a discussion of problems and possible solutions concerning non-commutative time see e.g. [95].

[4]Note that there exist models under the notion of *Emergent gravity*, where gravity arises naturally within the framework of NCQFT, c.f. [121] for a review.

Appendix A

Useful formulae and integrals

A.1 Useful formulae

A.1.1 Schwinger parametrization

With $\Gamma(N)$ the Gamma function, $\Gamma(N) = (n-1)!$ for $n > 0$, $n \in \mathbb{N}$ one gets:

$$\frac{1}{k^{2N}} = \frac{1}{\Gamma(N)} \int_0^\infty \mathrm{d}\alpha\, \alpha^{N-1} e^{-\alpha k^2}, \quad \forall N \in \mathbb{N},\, Re(k^2) > 0. \tag{A.1}$$

The more general form reads:

$$\frac{1}{(k^2+\beta)^N} = \frac{1}{\Gamma(N)} \int_0^\infty \mathrm{d}\alpha\, \alpha^{N-1} e^{-\alpha(k^2+\beta)}, \quad \forall N \in \mathbb{N},\, Re(k^2+\beta) > 0. \tag{A.2}$$

A.1.2 Integration formulae

Integration of quadratic forms ([66], p. 179 (5A.3)):

$$\int \mathrm{d}x \exp\left(-\alpha x^2 + \beta x + \gamma\right) = \exp\left(\frac{\beta^2}{4\alpha} + \gamma\right)\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}. \tag{A.3}$$

A special case is the Gaussian integral with $\beta = \gamma = 0$:

$$\int \mathrm{d}x \exp\left(-\alpha x^2\right) = \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}. \tag{A.4}$$

Parameter integrals ([122] 3.471; K_ν are the modified Bessel functions of the second kind):

$$\int_0^\infty \mathrm{d}x x^{\nu-1} e^{-\frac{\beta}{x} - \gamma x} = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\beta\gamma}), \tag{A.5}$$

$$\int_0^\infty \frac{dx}{x^\nu} e^{-\frac{\alpha^2}{x}} = \Gamma(\nu-1)\left(\alpha^2\right)^{(1-\nu)}, \quad \text{for} \quad \nu > 1,\, \alpha^2 > 0. \tag{A.6}$$

A.1.3 Modified Bessel functions

The modified Bessel functions represent the solutions to the modified Bessel's differential equation, which is given by

$$x^2 \frac{d^2 y(x)}{dx^2} + x \frac{d^y(x)}{dx} - (x^2 + \alpha^2) y(x) = 0 \,, \qquad \{\alpha,\, x\} \in \mathbb{C} \,. \tag{A.7}$$

where α defines the order of the equation. The two linearly independent solutions are given by the modified Bessel functions of the first kind $I_\alpha(x)$ and second kind $K_\alpha(x)$:

$$I_\alpha(x) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha} \,, \tag{A.8a}$$

$$K_\alpha(x) = \frac{\pi}{2} \frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)} . \tag{A.8b}$$

In this thesis only the solutions (A.8b) with α being integers will be of interest, i.e. $K_n(x)$, $n \in \mathbb{Z}$. They are exponentially decaying and singular at the origin (c.f. Fig. A.1). From the definitions (A.8) follows the property:

$$K_n(x) = K_{-n}(x) \,. \tag{A.9}$$

Series expansion of the modified Bessel functions

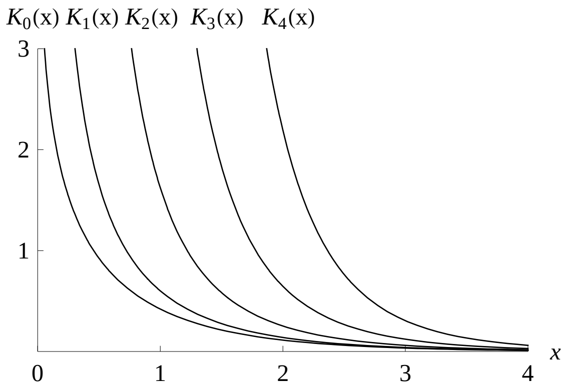


Figure A.1: The modified Bessel functions of the second kind

With $\gamma_E \approx 0.577$ the Euler-Mascheroni constant we get:

$$K_0(x) \approx \ln\frac{2}{x} - \gamma_E - \frac{x^2}{4}\left(\gamma_E - 1 + \ln\frac{x}{2}\right) + O(x^4) \,, \tag{A.10a}$$

$$K_1(x) \approx \frac{1}{x} + \frac{x}{2}\left(\gamma_E - \frac{1}{2} + \ln\frac{x}{2}\right) + \frac{x^3}{16}\left(\gamma_E - \frac{5}{4} + \ln\frac{x}{2}\right) + O(x^4) \,, \tag{A.10b}$$

$$K_2(x) \approx \frac{2}{x^2} - \frac{1}{2} - \frac{x^2}{8}\left(\gamma_E - \frac{3}{4} + \ln\frac{x}{2}\right) + O(x^4)\,, \tag{A.10c}$$

$$K_3(x) \approx \frac{8}{x^3} - \frac{1}{x} + \frac{x}{8} + \frac{x^3}{48}\left(\gamma_E - \frac{11}{12} + \ln\frac{x}{2}\right) + O(x^4)\,, \tag{A.10d}$$

$$K_4(x) \approx \frac{48}{x^4} - \frac{4}{x^2} + \frac{1}{4} - \frac{x^2}{48} + O(x^4)\,. \tag{A.10e}$$

A.1.4 Decomposition formulae for trigonometric functions

$$\exp(\pm \mathrm{i}x) = \cos(x) \pm \mathrm{i}\sin(x) \tag{A.11a}$$

$$\sin(x) = \frac{e^{\mathrm{i}x} - e^{-\mathrm{i}x}}{2\mathrm{i}} \tag{A.11b}$$

$$\cos(x) = \frac{e^{\mathrm{i}x} + e^{-\mathrm{i}x}}{2} \tag{A.11c}$$

$$\sin^2(x) = \frac{1}{2}\Big(1 - \cos(2x)\Big) = \frac{1}{2}\left(1 - \frac{e^{2\mathrm{i}x} + e^{-2\mathrm{i}x}}{2}\right) \tag{A.11d}$$

$$\sin(x \pm x) = \sin(x)\,\cos(y)\,\pm\,\cos(x)\,\sin(y) \tag{A.11e}$$

A.1.5 Various

Integral representation of the heavyside step function

$$\theta(x) = \int_{-\mathrm{i}\infty+e}^{\mathrm{i}\infty+e} \frac{d\beta}{2\pi\mathrm{i}\beta} e^{\beta x}\,. \tag{A.12}$$

A.2 Integrals for one loop corrections to the propagators

In this chapter we solve the integrals appearing in the expressions for the various one-loop corrections of the propagators. The integrals we encounter are of the form

$$I(m,n,a') \equiv \int d^4k \frac{k_{\mu_1}\dots k_{\mu_m}}{\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)^n} \sin^2\left(\frac{k\tilde{p}}{2}\right)\,, \tag{A.13}$$

where k, p are the internal and external momenta, $m \in \{0,2,4\}$ and $n \in \{1,\dots,4\}$. In Section A.2.2 it will be shown that a' in the denominator can be put to zero before calculating the integrals. Hence, we may write $I(m,n,a') \equiv I(m,n)$ for the integrals. In Section A.2.1 the method for their calculation will be given generically.

A.2.1 Methodology for calculating the integrals

After setting $a' = 0$ in the denominator of (A.13), we get

$$I(m,n) \equiv \int d^4k \frac{k_{\mu_1}\dots k_{\mu_m}}{(k^2)^n}\sin^2\left(\frac{k\tilde{p}}{2}\right). \tag{A.14}$$

The integrals can be evaluated by following the methodology given below.

1. Simplification of the trigonometric function: by applying the decomposition formula for the sine squared given in (A.11) we get:
$$\begin{aligned}&\int d^4k \frac{k_{\mu_1}\dots k_{\mu_m}}{(k^2)^n}\frac{1}{2}\Big(1-\cos\left(k\tilde{p}\right)\Big)\\ &= \int d^4k \frac{k_{\mu_1}\dots k_{\mu_m}}{(k^2)^n}\frac{1}{2}\Big(1-e^{\mathrm{i}(k\tilde{p})}\Big)\\ &\equiv I^{pl.}(m,n) + I^{n.p.}(m,n)\,,\end{aligned} \tag{A.15}$$
splitting the integral explicitly into a planar and non-planar part, which have to be evaluated individually. Contrary to what the definition (A.11c) might suggest, it is sufficient to evaluate the exponential with positive argument as indicated in the second line: due to the even integration range, only the cosine will survive, leading to the desired result.

2. For the momenta in the nominator we can write:
$$k_\mu = \Big[-\mathrm{i}\,\partial^z_\mu\, e^{\mathrm{i}kz}\Big]_{z=0} \Rightarrow k_{\mu_1}\dots k_{\mu_m} = (-\mathrm{i})^m \partial^z_{\mu_1}\dots\partial^z_{\mu_m}\, e^{\mathrm{i}kz}\big|_{z=0}. \tag{A.16}$$
Together with Schwinger parametrization of the denominator (c.f. (A.1)) we get:
$$\frac{1}{2}\frac{(-\mathrm{i})^m}{\Gamma(n)}\int_0^\infty \mathrm{d}\alpha\alpha^{(n-1)}\partial^z_{\mu_1}\dots\partial^z_{\mu_m}\int d^4k\left(e^{-\alpha k^2+\mathrm{i}kz} - e^{-\alpha k^2+\mathrm{i}k(\tilde{p}+z)}\right)\Big|_{z=0}. \tag{A.17}$$

3. Next we bring the exponential to a quadratic form:
$$-\alpha k^2 + \mathrm{i}k(\tilde{p}+z) = \alpha\left(\mathrm{i}k + \frac{\tilde{p}+z}{2\alpha}\right)^2 - \frac{(\tilde{p}+z)^2}{4\alpha}. \tag{A.18}$$
Application of (A.3) than leads to a factor π^2/α^2 (Note that the square follows from the integration over 4 dimensions, which leads to a power of 4 compared to (A.3)):
$$\frac{1}{2}\frac{(-\mathrm{i})^m\pi^2}{\Gamma(n)}\int_0^\infty \mathrm{d}\alpha\alpha^{(n-3)}\partial^z_{\mu_1}\dots\partial^z_{\mu_m}\left(e^{-\frac{z^2}{4\alpha}} - e^{-\frac{(\tilde{p}+z)^2}{4\alpha}}\right)\Big|_{z=0}. \tag{A.19}$$
The derivations are given by
$$\partial_\mu e^{-\frac{(\tilde{p}+z)^2}{4\alpha}} = -\frac{\tilde{p}+z}{2\alpha}e^{-\frac{(\tilde{p}+z)^2}{4\alpha}}\,, \tag{A.20a}$$

$$\partial_\mu\partial_\nu e^{-\frac{(\tilde{p}+z)^2}{4\alpha}} = \left\{-\frac{\delta_{\mu\nu}}{2\alpha} + \frac{(\tilde{p}+z)_\mu(\tilde{p}+z)_\nu}{(2\alpha)^2}\right\}, \tag{A.20b}$$

$$\begin{aligned}\partial_\mu\partial_\nu\partial_\rho e^{-\frac{(\tilde{p}+z)^2}{4\alpha}} &= \left\{\frac{\delta_{\mu\nu}(\tilde{p}+z)_\rho + \delta_{\mu\rho}(\tilde{p}+z)_\nu \delta_{\rho\nu}(\tilde{p}+z)_\mu}{(2\alpha)^2}\right. \\ &\quad \left. - \frac{(\tilde{p}+z)_\mu(\tilde{p}+z)_\nu(\tilde{p}+z)_\rho}{(2\alpha)^3}\right\} e^{-\frac{(\tilde{p}+z)^2}{4\alpha}}, \end{aligned} \tag{A.20c}$$

$$\begin{aligned}\partial_\mu\partial_\nu\partial_\rho\partial_\sigma e^{-\frac{(\tilde{p}+z)^2}{4\alpha}} &= \left\{\frac{\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\nu\rho}\delta_{\mu\sigma}}{(2\alpha)^3}\right. \\ &\quad - \frac{\delta_{\mu\nu}(\tilde{p}+z)_\rho(\tilde{p}+z)_\sigma + \delta_{\mu\rho}(\tilde{p}+z)_\nu(\tilde{p}+z)_\sigma + \delta_{\nu\rho}(\tilde{p}+z)_\mu(\tilde{p}+z)_\sigma}{(2\alpha)^3} \\ &\quad - \frac{\delta_{\mu\sigma}(\tilde{p}+z)_\nu(\tilde{p}+z)_\rho + \delta_{\nu\sigma}(\tilde{p}+z)_\mu(\tilde{p}+z)_\rho + \delta_{\rho\sigma}(\tilde{p}+z)_\mu(\tilde{p}+z)_\nu}{(2\alpha)^3} \\ &\quad \left. + \frac{(\tilde{p}+z)_\mu(\tilde{p}+z)_\nu(\tilde{p}+z)_\rho(\tilde{p}+z)_\sigma}{(2\alpha)^4}\right\} e^{-\frac{(\tilde{p}+z)^2}{4\alpha}}. \end{aligned} \tag{A.20d}$$

This leads to integrals of the form

$$\begin{aligned} &I(m,n) = I(m,n)^{pl.} + I(m,n)^{n.p.} \\ &\frac{1}{2}\frac{(-\mathrm{i})^m\pi^2}{\Gamma(n)}\left\{A_{m-1}\int_0^\infty \mathrm{d}\alpha\alpha^{(n-3+(m-1))} - \sum_{i=m-1,m} A_i \int_0^\infty \mathrm{d}\alpha\alpha^{(n-3+i)}\left(e^{-\frac{\tilde{p}^2}{4\alpha}}\right)\right\}, \end{aligned} \tag{A.20e}$$

where the prefactors A_i can be deduced from (A.20) in an obvious way.

4. Evaluation of the integrals: in order to evaluate the planar integrals we introduce a UV-cut off Λ in the exponential. Furthermore, if $n-3+(m-1)=-1$, a regulator mass for the IR sector μ is required. In this case the integrals become

$$I(m,n)^{pl.} \sim \int_0^\infty \mathrm{d}\alpha \frac{1}{\alpha} e^{-\frac{1}{\Lambda^2\alpha}+\mu^2\alpha}, \tag{A.21a}$$

$$I(m,n)^{n.p.} \sim \int_0^\infty \mathrm{d}\alpha \frac{1}{\alpha} e^{-\frac{\tilde{p}^2}{4\alpha}+\mu^2\alpha}, \tag{A.21b}$$

which can be evaluated by application of (A.5), leading to the modified Bessel function of zero order. For $n-3+(m-1) < -1$ one has two possibilities: direct evaluation by applying (A.6), or again introduction of a regulator mass, leading to Bessel functions of higher order by application of (A.5).

A.2.2 Neglecting the a' term in the denominator

In this thesis the integrals are evaluated for $a' \to 0$, due to the following

Lemma 2 *Setting $a'=0$ in the integrand of the integrals* (A.14)*, i.e. approximating*

$$\frac{1}{\left(k^2+\frac{a'^2}{\tilde{k}^2}\right)} \approx \frac{1}{k^2}$$

will not affect the divergence behaviour of the result.

This can be understood intuitionally as the integration over high internal momenta will not be affected by the approximation. In the low-energy sector, for the non-planar graphs the oscillating phase will regulate the divergence, and for the planar graphs, a regulator mass will be introduced. Hence, the approximation is valid and the integrals will be denoted by $I(m,n,a') \to I(m,n)$.
In order to see the equivalence, we will give a explicit proof for the simplest integral $I(0,1,a') \longleftrightarrow I(0,1,0) \equiv I(0,1)$,

$$\frac{1}{2}\int d^4k \frac{1-e^{\mathrm{i}k\tilde{p}}}{\left(k^2+\frac{a'^2}{\tilde{k}^2}\right)} \quad \Longleftrightarrow \frac{1}{2} \quad \int d^4k \frac{1-e^{\mathrm{i}k\tilde{p}}}{k^2}. \tag{A.22}$$

Integral with $a' \neq 0$

In order to do the explicit calculation of the integral

$$I(0,1,a') \equiv \frac{1}{2}\int d^4k \frac{1-e^{\mathrm{i}k\tilde{p}}}{\left(k^2+\frac{a'^2}{\tilde{k}^2}\right)}. \tag{A.23}$$

the denominator can be written as

$$\frac{1}{\left(k^2+\frac{a'^2}{\tilde{k}^2}\right)} = \frac{k^2}{(k^2+\mathrm{i}a)(k^2-\mathrm{i}a)} = \frac{1}{2}\left[\frac{1}{(k^2+\mathrm{i}a)}+\frac{1}{(k^2-\mathrm{i}a)}\right],$$

where $a \equiv a'/\theta$. (Note that the rescaled parameter a has mass dimension two.) Using Schwinger parametrization and introduction of the UV-cutoff Λ for the planar sector one arrives at

$$\begin{aligned}
&\frac{1}{4}\sum_{\xi=\pm1}\int d^4k \int_0^\infty d\alpha \Big\{ \exp\left[-\alpha\left(k^2+\mathrm{i}\xi a\right)\right] - \exp\left[-\alpha\left(k^2+\mathrm{i}\xi a\right)+\mathrm{i}k\tilde{p}\right] \Big\} \\
&= \frac{\pi^2}{4}\sum_{\xi=\pm1}\int_0^\infty d\alpha \frac{1}{\alpha^2}\Big\{ \exp\left[-\alpha\left(\mathrm{i}\xi a\right)-\frac{1}{\Lambda^2\alpha}\right] - \exp\left[-\alpha\left(\mathrm{i}\xi a\right)-\frac{\tilde{p}^2}{4\alpha}\right] \Big\} \\
&= \frac{\pi^2}{2}\sum_{\xi=\pm1}\Big\{ \left(\Lambda^2(\mathrm{i}\xi a)\right)^{\frac{1}{2}} K_{-1}\left(2\sqrt{\frac{\mathrm{i}\xi a}{\Lambda^2}}\right) - 2\left(\frac{\mathrm{i}\xi a}{\tilde{p}^2}\right)^{\frac{1}{2}} K_{-1}\left(\tilde{p}^2(\mathrm{i}\xi a)\right) \Big\}.
\end{aligned} \tag{A.24}$$

For small arguments $x \to 0$ the modified Bessel function of first type $K_{-1} = K_1$ admits the expansion (A.10b) (c.f. Section A.1.3). Being interested in the IR behaviour of the theory, this expansion can be applied to the last equation for the limit of small external momenta p, yielding,

$$\frac{\pi^2}{2} \sum_{\xi=\pm 1} \left\{ \left(\Lambda^2 (\mathrm{i}\xi a)\right)^{\frac{1}{2}} \frac{1}{2} \sqrt{\frac{\Lambda^2}{\mathrm{i}\xi a}} - 2 \left(\frac{\mathrm{i}\xi a}{\tilde{p}^2}\right)^{\frac{1}{2}} \frac{1}{\sqrt{\tilde{p}^2 (\mathrm{i}\xi a)}} \right\} + \mathcal{O}(\tilde{p}^2) \approx \pi^2 \left\{ \frac{\Lambda^2}{2} - \frac{2}{\tilde{p}^2} \right\} . \tag{A.25}$$

Integral with $a' = 0$

For $a' = 0$ the integral reads

$$I(0,1,0) \to I(0,1) \equiv \frac{1}{2} \int d^4k \frac{1 - e^{\mathrm{i}k\tilde{p}}}{k^2} . \tag{A.26}$$

Following the method detailed in Section A.2.1, we get the result (A.27), which is the same as (A.25), up to finite contributions.

This completes the proof of Lemma 2. □

A.2.3 Integrals

In the following formulae, (b) and (c) give the planar and non-planar solutions respectively, for the integrals given in (a). Only those results are given, which appear in the calculations for the presented models. Regarding the notation, it shall be repeated that $I(m, n)$ stands for an integral with the argument sine square in $(k\tilde{p})/2$ multiplied by a fraction with $m\,(n)$ powers of k in the (de)nominator. Furthermore, Λ denotes the UV-cutoff, while μ is a IR-regulator mass.

- $I(0, 1)$

$$\int d^4k \frac{1}{k^2} \sin^2\left(\frac{k\tilde{p}}{2}\right) = \tag{A.27a}$$

$$+ \frac{1}{2}\pi^2\Lambda^2 \tag{A.27b}$$

$$- 2\frac{\pi^2}{\tilde{p}^2} . \tag{A.27c}$$

- $I(0, 2)$

$$\int d^4k \frac{1}{k^4} \sin^2\left(\frac{k\tilde{p}}{2}\right) = \tag{A.28a}$$

$$+\pi^2 K_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right) \tag{A.28b}$$
$$-\pi^2 K_0\left(\sqrt{\mu^2\tilde{p}^2}\right). \tag{A.28c}$$

- $I(2,2)$

$$\int d^4k \frac{k_\mu k_\nu}{k^4}\sin^2\left(\frac{k\tilde{p}}{2}\right) = \tag{A.29a}$$
$$+\frac{1}{4}\pi^2\delta_{\mu\nu}\Lambda^2 \tag{A.29b}$$
$$-\pi^2\left\{\frac{\delta_{\mu\nu}}{\tilde{p}^2} - 2\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\tilde{p}^2)^2}\right\}. \tag{A.29c}$$

- $I(2,3)$

$$\int d^4k \frac{k_\mu k_\nu}{k^6}\sin^2\left(\frac{k\tilde{p}}{2}\right) = \tag{A.30a}$$
$$+\frac{\pi^2}{4}\delta_{\mu\nu}K_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right) \tag{A.30b}$$
$$-\frac{\pi^2}{4}\left\{\delta_{\mu\nu}K_0\left(\sqrt{\mu^2\tilde{p}^2}\right) - \frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}\right\}. \tag{A.30c}$$

- $I(4,2)$

$$\int d^4k \frac{k_\mu k_\nu k_\rho k_\sigma}{k^4}\sin^2\left(\frac{k\tilde{p}}{2}\right) = \tag{A.31a}$$
$$+\frac{\pi^2}{8}\Lambda^4\left(\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\nu\rho}\delta_{\mu\sigma}\right) \tag{A.31b}$$
$$\begin{aligned}&-2\pi^2\left\{\frac{1}{\tilde{p}^4}\left(\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\nu\rho}\delta_{\mu\sigma}\right) - \frac{4}{\tilde{p}^6}\big(\delta_{\mu\nu}\tilde{p}_\rho\tilde{p}_\sigma\right.\\ &+\delta_{\mu\rho}\tilde{p}_\nu\tilde{p}_\sigma+\delta_{\nu\rho}\tilde{p}_\mu\tilde{p}_\sigma+\delta_{\mu\sigma}\tilde{p}_\nu\tilde{p}_\rho+\delta_{\nu\sigma}\tilde{p}_\mu\tilde{p}_\rho+\delta_{\rho\sigma}\tilde{p}_\mu\tilde{p}_\nu\big)\\ &\left.+24\frac{\tilde{p}_\mu\tilde{p}_\nu\tilde{p}_\rho\tilde{p}_\sigma}{\tilde{p}^8}\right\}.\end{aligned} \tag{A.31c}$$

- $I(4,4)$

$$\int d^4k \frac{k_\mu k_\nu k_\rho k_\sigma}{k^8}\sin^2\left(\frac{k\tilde{p}}{2}\right) = \tag{A.32a}$$
$$+\frac{\pi^2}{24}\left(\delta_{\mu\nu}\delta_{\rho\sigma}+\delta_{\mu\rho}\delta_{\nu\sigma}+\delta_{\nu\rho}\delta_{\mu\sigma}\right)K_0\left(2\sqrt{\frac{\mu^2}{\Lambda^2}}\right) \tag{A.32b}$$

$$
\begin{aligned}
&- \frac{\pi^2}{24}\Bigg\{ \left(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\nu\rho}\delta_{\mu\sigma}\right) K_0\left(\sqrt{\mu^2\tilde{p}^2}\right) \\
&- \Big(\delta_{\mu\nu}\tilde{p}_\rho\tilde{p}_\sigma + \delta_{\mu\rho}\tilde{p}_\nu\tilde{p}_\sigma + \delta_{\nu\rho}\tilde{p}_\mu\tilde{p}_\sigma \\
&+ \delta_{\mu\sigma}\tilde{p}_\nu\tilde{p}_\rho + \delta_{\nu\sigma}\tilde{p}_\mu\tilde{p}_\rho + \delta_{\rho\sigma}\tilde{p}_\mu\tilde{p}_\nu\Big)\frac{1}{\tilde{p}^2} \\
&+ \frac{2}{\tilde{p}^4}\tilde{p}_\mu\tilde{p}_\nu\tilde{p}_\rho\tilde{p}_\sigma \Bigg\}.
\end{aligned}
\tag{A.32c}
$$

Appendix B

Symmetry factors

The algorithm derived here for the calculation of the combinatorial factors heavily relies on [123]. A proof relying on the Wick expansions leading to a given graph can be found in [124]. For a general formula being applicable to scalar QFT as well as QED (spinor and scalar) and QCD c.f [125]. A note regarding the correct wording: what in the main part of this thesis are called symmetry factors in the literature are commonly denoted as *combinatoric factors*, while *symmetry factor* is assigned to the internal symmetries of a graph. In order to avoid confusion, the following explanations will rely on the usual.

B.1 Origin of combinatoric factors

The necessity to multiply the analytical expression for a given Feynman graph with a symmetry factor relies on the fact that, given its ingredients (the Feynman rules), there might be several possibilities to connect them in order to get the same result.
The problem can be best understood in terms of Wick contractions (c.f. for example [126], Chapter 4). Consider ϕ^4-theory on four dimensional Minkowski space, where all n-point functions result from the expansion

$$\begin{aligned} \langle 0|T\left[\phi(x_1)\dots\phi(x_n)\right]|0\rangle &\approx \langle 0|T\left[\phi(x_1)\dots\phi(x_n)\exp\left\{\mathrm{i}\int d^4z\mathcal{L}_{int}(z)\right\}\right]|0\rangle\,, \\ \mathcal{L}_{int} &= \frac{g}{4!}\phi(z)\phi(z)\phi(z)\phi(z), \end{aligned} \tag{B.1}$$

with $\mathcal{L}_{int}$ denoting the interaction Lagrangian. The r.h.s. of the first line leads to the vacuum expectation value of the product of i (time ordered[1]) fields, where $i = n + j\times m$ depends on the order m, and the number of fields in the interaction terms j (in the present example $j = 4$). For example the expression in second order for the 2-point

[1]Note that for obvious reasons there is no time ordering in the non-commutative theories considered in this thesis, due to the underlying Euclidean manifold.

function reads

$$\frac{1}{2!}\langle 0|T\left[\phi(x_1)\phi(x_2)\frac{\mathrm{i}g}{4!}\int d^4z'\phi(z')\phi(z')\phi(z')\phi(z')\frac{\mathrm{i}g}{4!}\int d^4z''\phi(z'')\phi(z'')\phi(z'')\phi(z'')\right]|0\rangle\,. \tag{B.2}$$

For such an expression, Wick's theorem tells us that it is equivalent to evaluate the sum of all possible contractions, and due to the expectation value only the fully contracted expressions will survive. Finally, each of them can graphically be represented by a Feynman graph. However, consider the following two Wick contractions:

$$\langle 0|\phi(x)\phi(y)\int dz'\,\phi(z')\phi(z')\phi(z')\phi(z')\int dz''\,\phi(z'')\phi(z'')\phi(z'')\phi(z'')|0\rangle\,,$$

$$\langle 0|\phi(x)\phi(y)\int dz'\,\phi(z')\phi(z')\phi(z')\phi(z')\int dz''\,\phi(z'')\phi(z'')\phi(z'')\phi(z'')|0\rangle\,. \tag{B.3}$$

Both of them lead to the same analytical expression, and the same graphical representation, i.e. the Feynman graph depicted in Fig. B.1. When performing all Wick

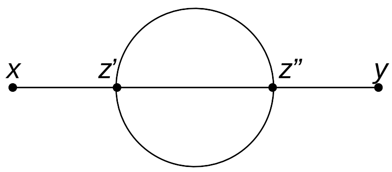


Figure B.1: Two point graph in scalar ϕ^4-theory

contractions for the example under considerations, one will encounter in total 192 ones leading to identical analytical expressions. Therefore, one draws the graph just once, and multiplies it by its multiplicity $M = 192$, i.e. the number of expectation values leading to a topologically equivalent graph. Furthermore, by pulling the factors $1/4!$ from the vertices before the expectation value, and considering the term $1/n!$ from the expansion to order n (in the present case $n = 2$), one ends up with the following expression for the graph in Fig. B.1:

$$\begin{aligned}&\frac{192}{2!\,4!\,4!}\times\text{One of the expectation values in (B.3)}\Rightarrow\\ &\frac{192}{2!\,4!\,4!}\times(\mathrm{i}g)^2\int dz'\int dz'' D_F(x-z')D_F(y-z'')D_F(z'-z'')D_F(z'-z'')D_F(z'-z''),\end{aligned} \tag{B.4}$$

with $D_F(x-y)$ the Feynman propagator between the space-time points x and y.

Rules leading to the combinatoric factor in scalar and gauge theories

Now we start from the other side, i.e. with the Feynman graph. In what follows we derive a general set of rules leading to the combinatoric factor for a given graph. Simultaneously, we will apply the rules to the two graphs of Fig. B.3. The graph a) is the one from scalar theory considered above (which allows us to verify the result). The graph b) corresponds to k) of Fig. 3.3 of our non-commutative gauge model with real $B_{\mu\nu}$ in Section 3.5.3. Actually it is rotated by an angle of $180\,^\circ$. However, rotating or twisting a graph does not modify its topology, and leads to the identity operation.
Given the Feynman rules, one will get the corresponding analytical expression by translating each graphical element (lines, vertices, etc.) into the corresponding rule. The expression than must be multiplied by the correct numerical or combinatorial factor. From the above considerations, it follows that the multiplicity has to be included. Without performing explicitly Wick contractions, it can be evaluated as follows:

1. Draw the pre-diagram. It consists of the elements of the graph drawn separately, but in the right relative positions with respect to the final graph (for our examples, c.f. Fig. B.2). The elements are the graphical counterparts of the Feynman rules.
2. Count the number of ways the first vertex (e.g. the left one) can be connected to the external legs.
   a) This leads to 2×4, because there are 4 possibilities to connect it with either external leg.
   b) 2×3: the 3A vertex may be connected in 3 ways on the left and right external leg, respectively.
3. Count the number of ways to connect the remaining vertices to the remaining external legs. In our examples, there is just one vertex left, that can be connected *a)* in 4 different ways, *b)* in 2 different ways to the remaining external leg.
4. Take one of the unconnected ends of a vertex and count the ways to connect it to the other vertex. Repeat this, until all internal legs are connected. For *a)* we have first 3 ways, than 2 ways and finally just 1 possibility of connection, leading to $3 \times 2 \times 1 = 6$ possibilities; for *b)* there is just one way to connect the two lines on the left to those one on the right, leading to[2] 1.
5. The product of all factors gives the Multiplicity M: $M_a = 192$, $M_b = 12$.

Furthermore, there are factors contributing to the denominator that are collected in the symmetry factor S:

1. For n identical vertices, we get a factor $1/n!$. Actually, each type of vertex corresponds to an interaction term in the Lagrangian, and a graph with n identical vertices stems from the expansion of the exponential (c.f. (B.1)) to the same order,

[2] One could argue that there are actually two possibilities of interconnecting the internal lines; however, this results in the two graphs k and l of Fig. 3.3.

leading to the factor mentioned above. For the examples this delivers a factor $1/2!$ for *a)*, 1 for *b)*.

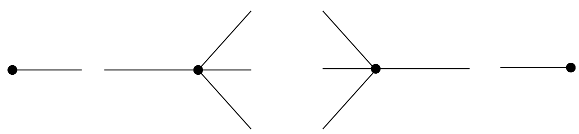
2. Finally, for a general theory, a term of the action involving $n_{\phi_1}, n_{\phi_2}, \ldots, n_{\phi_k}$ fields of type $\phi_1, \phi_2, \ldots, \phi_k$ contains a prefactor $1/(n_{\phi_1}!\, n_{\phi_2}! \ldots n_{\phi_k}!)$. In the case *a)* this reduces to $n_\phi = 4$ identical fields of type ϕ, and from (B.2) we see indeed that each vertex delivers a factor[3] $1/4!$. In the case *b)* this leads to $1/(3!2!)$.

To summarize, this leads to the combinatorial factors

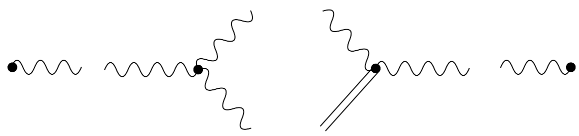
$$C = \frac{M}{S} \Rightarrow C_a = \frac{1}{6},\ C_b = 1\,, \tag{B.5}$$

where C_a is the same as in (B.4), as expected.

Application of this procedure to arbitrary graphs delivers e.g. the table Tab. 3.1.

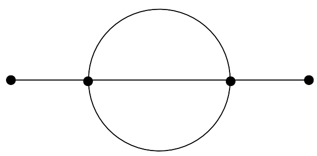


(a) Scalar 2-point graph.

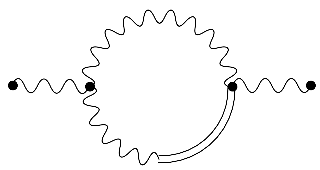


(b) Two-point graph in gauge model with real $B_{\mu\nu}$.

Figure B.2: Prediagrams.



(a) Scalar 2-point graph.



(b) Two-point graph in gauge model with real $B_{\mu\nu}$.

Figure B.3: Examples for the evaluation of the combinatoric factor.

[3]Note that when performing Wick contractions, each vertex will be accompanied by the right numerical factor. However, when calculating the expression for a given graph, the Feynman rule for a vertex is just ig, so the corresponding numerical factor has to be introduced by hand.

Appendix C

Calculations for the gauge model with real $B_{\mu\nu}$

C.1 Calculations for the preliminary models

C.1.1 Gauge propagator for the undamped gauge invariant action

The bilinear part of the action (3.2) reads as

$$\begin{aligned} S_{\mathrm{inv,bil}} &= \int \mathrm{d}^4x \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu) \star (\partial_\mu A_\nu - \partial_\nu A_\mu) = \\ &= -\int \mathrm{d}^4x \frac{1}{2} A_\mu \star \{(\Box A_\mu - \partial_\mu(\partial A)\} = \\ &= -\int \mathrm{d}^4x \frac{1}{2} A_\mu \star \left\{\Box \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right) A_\nu\right\} . \end{aligned} \tag{C.1}$$

By denoting the operator of the bilinear part by $K_{\mu\nu}$ and the projection operator to the orthogonal plane by $P_{\mu\nu}$, we can write

$$K_{\mu\nu} = \Box \left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right) = \Box P_{\mu\nu} \,, \tag{C.2}$$

which shows that the bilinear part is proportional to the projection operator. Now for any projection operator we can write $P^2 = P$ and $PP^{-1} = 1$, from which follows

$$P^2P^{-1} = 1 \quad \Rightarrow \quad P = 1 \,. \tag{C.3}$$

The unity is the only projection operator which has an inverse.

C.1.2 Partial integration of the inverse of covariant derivatives

Let us consider the integral of two fields F_1 and F_2 with the square of the inverse of a covariant derivative to power n:

$$I = \int \mathrm{d}^4x F_1 \star \frac{1}{(D^2)^n} \star F_2 \,. \tag{C.4}$$

This can also be written as

$$I = \int \mathrm{d}^4x \, \mathbb{1} F_1 \star \frac{1}{(D^2)^n} \star F_2 = \int \mathrm{d}^4x \frac{1}{(D^2)^k} (D^2)^k F_1 \star \frac{1}{(D^2)^n} \star F_2 \,. \tag{C.5}$$

Now one may conduct the usual partial integration, leading to

$$I = \int \mathrm{d}^4x \frac{1}{(D^2)^k} F_1 \star (D^2)^k \frac{1}{(D^2)^n} \star F_2 = \int \mathrm{d}^4x \frac{1}{(D^2)^k} F_1 \star \frac{1}{(D^2)^{n-k}} \star F_2. \tag{C.6}$$

For the special case of $k = 1, n = 2$ one obtains

$$\int \mathrm{d}^4x F_1 \star \frac{1}{(D^2)^2} \star F_2 = \int \mathrm{d}^4x \frac{1}{D^2} F_1 \star \frac{1}{D^2} \star F_2 \,, \tag{C.7}$$

which leads to the desired result.

C.2 Feynman rules for the localized gauge model

C.2.1 Propagators

In this section we will derive the propagators (3.34). From the equations of motion (3.32) and with the abbreviation

$$\chi = \left(1 + \frac{a'^2}{\Box\tilde{\Box}}\right) \tag{C.8}$$

it directly follows

$$\frac{\delta S_{\text{bil}}}{\delta B^{\mu\nu}} \Rightarrow B^{\mu\nu} = \frac{1}{2\tilde{\Box}\Box}\left[j^B_{\mu\nu} + a'\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\right] \,, \tag{C.9a}$$

$$\frac{\delta S_{\text{bil}}}{\delta b} \Rightarrow b = \frac{1}{\alpha}\left[j^b + \chi \partial^\mu A_\mu\right] \,, \tag{C.9b}$$

$$\frac{\delta S_{\text{bil}}}{\delta \bar{c}} \Rightarrow c = -\frac{j^{\bar{c}}}{\chi\Box} \,. \tag{C.9c}$$

We will also need the following derivations:

$$\partial_\mu B_{\mu\nu} = \frac{1}{2\tilde{\Box}\Box}\left[\partial_\mu j^B_{\mu\nu} + a'\left(\Box\delta_{\mu\nu} - \partial_\nu\partial_\mu\right) A_\mu\right] \,, \tag{C.10a}$$

$$\partial_\nu \partial_\mu B_{\mu\nu} = \frac{1}{2\widetilde{\Box}\Box}\partial_\nu\partial_\mu j^B_{\mu\nu} = 0\,, \tag{C.10b}$$

$$\partial_\mu b = \frac{1}{\alpha}\left[\partial_\mu j^b + \chi\partial_\mu\partial_\mu A_\mu\right]\,. \tag{C.10c}$$

The vanishing of the second derivative (C.10b) follows from the anti-symmetry of the $B_{\mu\nu}$ field. Inserting the previous expressions for the fields into the equation of motion for the gauge field (3.32a) yields

$$\begin{aligned} j^A_\nu &= \left(\Box\delta_{\nu\mu} - \partial_\nu\partial_\mu\right)A^\mu + 2a'\partial^\mu B_{\mu\nu} + \chi\partial_\nu b \\ &= \Box\chi A_\mu + \left(\frac{\chi^2}{\alpha} - \chi\right)\partial_\mu\partial_\nu A_\mu + \frac{a'}{\Box\widetilde{\Box}}\partial_\mu j^B_{\mu\nu} + \frac{\chi}{\alpha}\partial_\nu j^b\,. \end{aligned} \tag{C.11}$$

Derivation of the previous equation and taking into consideration the vanishing of the second derivative of $j^B_{\mu\nu}$ allows one to express A_μ as

$$\partial_\mu A_\mu = \frac{\alpha}{\chi^2\Box}\left[\partial_\nu j^A_\nu - \frac{\chi}{\alpha}\Box j^b\right]. \tag{C.12}$$

This finally leads to an expression of A_ν as functional of external sources only:

$$A_\nu = \frac{a}{\chi\Box}\left[j^A_\nu - \left(1 - \frac{\alpha}{\chi}\right)\frac{1}{\Box}(\partial j^A) - \frac{a'}{\Box\widetilde{\Box}}\partial_\mu j^B_{\mu\nu} - \partial_\nu j^b\right]\,. \tag{C.13}$$

Inserting the last equation into the expression for $B_{\tau\epsilon}$ (C.9a) after choosing the index names according to the propagators (3.34) results in

$$B_{\tau\epsilon} = \frac{1}{2\Box\widetilde{\Box}}\left[j^B_{\tau\epsilon} + \frac{a'}{\Box\chi}\left(\partial_\tau j^A_\epsilon - \partial_\epsilon j^A_\tau\right) - \frac{a'^2}{\widetilde{\Box}\Box^2\chi}\left(\partial_\tau\partial_\mu j_{\mu\epsilon} - \partial_\epsilon\partial_\mu j_{\mu\tau}\right)\right]\,. \tag{C.14}$$

Fourier transformation of the two expressions ($\partial_\mu \to \mathrm{i}k_\mu$) and (C.9c) gives the results in momentum space:

$$A_\nu(k) = -\frac{1}{k^2 + \frac{a'}{\tilde{k}^2}}\left(j^A_\nu - \frac{k_\mu k_\nu}{k^2}j^A_\mu + \alpha\frac{k_\mu k_\nu}{k^2 + \frac{a'^2}{\tilde{k}^2}}j^A_\mu + \mathrm{i}\frac{a' k_\mu}{\tilde{k}^2 k^2}j^B_{\mu\nu} + \mathrm{i}k_\nu j^b\right)\,, \tag{C.15}$$

$$\begin{aligned} &B_{\tau\epsilon}(k) = \\ &= \frac{1}{2k^2\tilde{k}^2}\left[j^B_{\tau\epsilon} - \frac{\mathrm{i}a'}{k^2 + \frac{a'^2}{\tilde{k}^2}}(k_\tau j^A_\epsilon - k_\epsilon j^A_\tau) - \frac{a'^2}{k^2\tilde{k}^2\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)}\left(k_\tau k_\mu j_{\mu\epsilon} - k_\epsilon k_\mu j_{\mu\tau}\right)\right]\,, \end{aligned} \tag{C.16}$$

$$c(k) = \frac{\bar{c}}{k^2 + \frac{a'^2}{\tilde{k}^2}}\,. \tag{C.17}$$

The final propagators are obtained by

$$G^{AA}_{\mu\nu}(k) = -\frac{\delta A_\nu(k)}{\delta j^A_\mu(k)}\,,$$

$$
\begin{aligned}
G^{AB}_{\rho,\sigma\tau}(k) &= -\frac{\delta B_{\sigma\tau}(k)}{\delta j^A_\rho(k)} = -G^{BA}_{\sigma\tau,\rho}(k) = \frac{\delta A_\rho(k)}{\delta j^B_{\sigma\tau}(k)}\,, \\
G^{BB}_{\rho\sigma,\tau\epsilon}(k) &= -\frac{\delta B_{\tau\epsilon}(k)}{\delta j^B_{\rho\sigma}(k)}\,, \\
G^{\bar{c}c}(k) &= -\frac{\delta c(k)}{\bar{c}(k)}\,.
\end{aligned} \tag{C.18}
$$

Interdependence of the propagators

In order to derive the interdependences between the propagators including A_μ and $B_{\mu\nu}$, one starts with the bilinear part of (3.29), with the r.h.s replaced by the source $-j^B_{\mu\nu} \neq 0$:

$$
\frac{\delta S^{(2)}_{\text{inv, bil}}}{\delta B_{\mu\nu}} = a'\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right) - 2\widetilde{\square}\square \star B_{\mu\nu} = -j^B_{\mu\nu}\,. \tag{C.19}
$$

Fourier transformation and rearranging the terms leads to

$$
2\tilde{k}^2 k^2 \star B_{\mu\nu}(k) = \mathrm{i}a'\left(k_\mu A_\nu(k) - \tilde{k}_\nu A_\mu(k)\right) + j^B_{\mu\nu}(k)\,. \tag{C.20}
$$

In accordance with (3.33) we functionally derivate with respect to j^A_ρ and $j^B_{\rho\sigma}$, leading to the result (3.36).

C.2.2 Vertices

Explicit calculation of the $cA\bar{c}$-vertex

Here we want to apply explicitly and in detail the procedure given in Section 3.3 by the example of the $\bar{c}Ac$ vertex. Application of (3.39) to $S^{\bar{c}Ac}$ in (3.38) leads to

$$
\begin{aligned}
S^{\bar{c}Ac} &= S_{gf,int} = \\
&= -\mathrm{i}g\int \mathrm{d}^4x\left(1+\frac{a'^2}{\square\widetilde{\square}}\right)\partial_\mu\bar{c}(x)\left[A_\mu(x) \overset{\star}{,} c(x)\right] \\
&= -\mathrm{i}g\int \mathrm{d}^4x\int\frac{d^4q_{1,2,3}}{(2\pi)^{12}}\left(1+\frac{a'^2}{q_1^2\tilde{q}_1^2}\right)(\mathrm{i}q_{1\mu})e^{\mathrm{i}\sum q_i^\mu x_\mu} \\
&\qquad \tilde{\bar{c}}(q_1)\left[\tilde{A}_\mu(q_2)\tilde{c}(q_3) - \tilde{c}(q_2)\tilde{A}_\mu(q_3)\right]e^{-\frac{\mathrm{i}}{2}(q_1\times q_2+q_1\times q_3+q_2\times q_3)} \\
&= g\int\frac{d^4q_{1,2,3}}{(2\pi)^8}\delta(q_1+q_2+q_3)\left(1+\frac{a'^2}{q_1^2\tilde{q}_1^2}\right)q_{1\mu} \\
&\qquad \tilde{\bar{c}}(q_1)\tilde{c}(q_2)\tilde{A}_\mu(q_3)e^{-\frac{\mathrm{i}}{2}\left(q_1\times(q_2+q_3)\right)}\left(e^{\frac{\mathrm{i}}{2}(q_2\times q_3)} - e^{-\frac{\mathrm{i}}{2}(q_2\times q_3)}\right)\,.
\end{aligned} \tag{C.21}
$$

With respect to (3.38), in the first line we have partially integrated in order to have the derivatives acting on $\bar{c}$ only. Due to momentum conservation by the δ-function we have $q_3 = -q_1 - q_2$, and by application of the relation (A.11b) we obtain

$$
\begin{aligned}
S^{\bar{c}Ac} =& \\
= 2\mathrm{i}g \int \frac{d^4 q_{1,2,3}}{(2\pi)^8} \delta(q_1 + q_2 + q_3) \Big(1 + \frac{a'^2}{q_1^2 \tilde{q}_1^2}\Big)(q_{1\mu}) \tilde{\bar{c}}(q_1) \tilde{c}(q_2) \tilde{A}_\mu(q_3) \sin\left(\frac{q_1 \times q_2}{2}\right) . \qquad \text{(C.22)}
\end{aligned}
$$

Now one can apply the functional derivation with respect to the fields (3.40), which in our case reads as

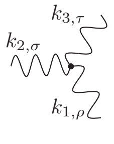
$$
\tilde{V}^{\bar{c}A_\mu c}(k_1, k_2, k_3) = -(2\pi)^{12} \frac{\delta}{\delta\tilde{\bar{c}}(-k_1)} \frac{\delta}{\delta\tilde{A}_\mu(-k_2)} \frac{\delta}{\delta\tilde{c}(-k_3)} S^{\bar{c}Ac} , \qquad \text{(C.23)}
$$

and one gets:

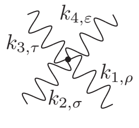
$$
\begin{aligned}
&\tilde{V}^{\bar{c}A_\mu c}(k_1, k_2, k_3) = \\
&= -2\mathrm{i}g(2\pi)^4 \int d^4 q_{1,2,3} \delta(q_1 + q_2 + q_3) \Big(1 + \frac{a'^2}{q_1^2 \tilde{q}_1^2}\Big)(q_{1\mu}) \\
&\qquad\qquad \delta(q_1 - k_1)\delta(q_3 - k_2)\delta(q_2 - k_3) \sin\left(\frac{q_1 \times q_2}{2}\right) = \\
&= -2\mathrm{i}g(2\pi)^4 \delta(q_1 + q_2 + q_3) \Big(1 + \frac{a'^2}{k_1^2 \tilde{k}_1^2}\Big)(k_{1\mu}) \sin\left(\frac{k_1 \times k_3}{2}\right) . \qquad \text{(C.24)}
\end{aligned}
$$

By renaming the momenta of the ghost and antighost, $k_i \to q_i$ one gets the expression (C.25c).

Vertices

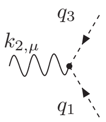


$$
\begin{aligned}
&= \widetilde{V}^{3A}_{\rho\sigma\tau}(k_1, k_2, k_3) \\
&= 2\mathrm{i}g(2\pi)^4 \delta^4(k_1 + k_2 + k_3) \sin\left(\tfrac{k_1 \tilde{k}_2}{2}\right) \times \\
&\quad \times \left[(k_3 - k_2)_\rho \delta_{\sigma\tau} + (k_1 - k_3)_\sigma \delta_{\rho\tau} + (k_2 - k_1)_\tau \delta_{\rho\sigma}\right] , \qquad \text{(C.25a)}
\end{aligned}
$$

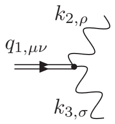


$$
= \widetilde{V}^{4A}_{\rho\sigma\tau\epsilon}(k_1, k_2, k_3, k_4)
$$

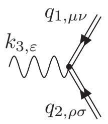
$$
\begin{aligned}
&= -4g^2(2\pi)^4\delta^4(k_1+k_2+k_3+k_4)\times \\
&\quad \times \Big[(\delta_{\rho\tau}\delta_{\sigma\epsilon}-\delta_{\rho\epsilon}\delta_{\sigma\tau})\sin\left(\tfrac{k_1\tilde{k}_2}{2}\right)\sin\left(\tfrac{k_3\tilde{k}_4}{2}\right) \\
&\quad +(\delta_{\rho\sigma}\delta_{\tau\epsilon}-\delta_{\rho\epsilon}\delta_{\sigma\tau})\sin\left(\tfrac{k_1\tilde{k}_3}{2}\right)\sin\left(\tfrac{k_2\tilde{k}_4}{2}\right) \\
&\quad +(\delta_{\rho\sigma}\delta_{\tau\epsilon}-\delta_{\rho\tau}\delta_{\sigma\epsilon})\sin\left(\tfrac{k_2\tilde{k}_3}{2}\right)\sin\left(\tfrac{k_1\tilde{k}_4}{2}\right)\Big],
\end{aligned}
\tag{C.25b}
$$



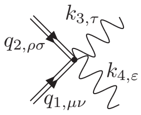
$$
\begin{aligned}
&= \widetilde{V}^c_\mu(q_1,k_2,q_3) \\
&= -2\mathrm{i}g(2\pi)^4\delta^4(q_1+k_2+q_3)q_{1\mu}\left(1+\frac{a'^2}{(q_1)^2(\tilde{q}_1)^2}\right)\sin\left(\tfrac{q_1\tilde{q}_3}{2}\right),
\end{aligned}
\tag{C.25c}
$$



$$
\begin{aligned}
&= \widetilde{V}^{BAA}_{\mu\nu,\rho\sigma}(q_1,k_2,k_3) \\
&= 2ga'(2\pi)^4\delta^4(q_1+k_2+k_3)\left(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}\right)\sin\left(\tfrac{k_2\tilde{k}_3}{2}\right),
\end{aligned}
\tag{C.25d}
$$

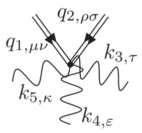


$$
\begin{aligned}
&= \widetilde{V}^{BBA}_{\mu\nu,\rho\sigma,\epsilon}(q_1,q_2,k_3) \\
&= -2\mathrm{i}g\theta^2(2\pi)^4\delta^4(q_1+q_2+k_3)\left(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}\right)\times \\
&\quad \times \left((q_1)^2+(q_2)^2\right)(q_1-q_2)_\epsilon\sin\left(\tfrac{q_1\tilde{q}_2}{2}\right),
\end{aligned}
\tag{C.25e}
$$



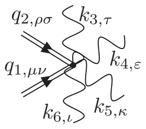
$$
= \widetilde{V}^{2B2A}_{\mu\nu,\rho\sigma,\tau\epsilon}(q_1,q_2,k_3,k_4)
$$

$$
\begin{aligned}
&= 4g^2\theta^2(2\pi)^4\delta^4(q_1+q_2+k_3+k_4)\left(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}\right)\times \\
&\times\Big\{\left[k_{3,\tau}k_{4,\varepsilon}+2\left(q_{1,\tau}k_{4,\varepsilon}+q_{2,\varepsilon}k_{3,\tau}\right)+4q_{1,\tau}q_{2,\varepsilon}-\delta_{\varepsilon\tau}\left({q_1}^2+{q_2}^2\right)\right]\sin\left(\tfrac{q_1\tilde{k}_3}{2}\right)\sin\left(\tfrac{q_2\tilde{k}_4}{2}\right) \\
&\quad+\left[k_{3,\tau}k_{4,\varepsilon}+2\left(q_{2,\tau}k_{4,\varepsilon}+q_{1,\varepsilon}k_{3,\tau}\right)+4q_{1,\varepsilon}q_{2,\tau}-\delta_{\varepsilon\tau}\left({q_1}^2+{q_2}^2\right)\right]\sin\left(\tfrac{q_1\tilde{k}_4}{2}\right)\sin\left(\tfrac{q_2\tilde{k}_3}{2}\right)\Big\},
\end{aligned}
\tag{C.25f}
$$



$$= \widetilde{V}^{2B3A}_{\mu\nu,\rho\sigma,\tau\epsilon\kappa}\,(q_1, q_2, k_3, k_4, k_5)$$

$$\begin{aligned}
&= -8\mathrm{i}g^3\theta^2(2\pi)^4\delta^4(q_1+q_2+k_3+k_4+k_5)\,(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho})\times\\
&\quad\times\Bigg\{[k_3+2q_1]_\tau\,\delta_{\epsilon\kappa}\sin\!\left(\tfrac{k_3\tilde{q}_1}{2}\right)\left[\sin\!\left(\tfrac{k_5\tilde{q}_2}{2}\right)\sin\!\left(\tfrac{k_4(\tilde{k}_5+\tilde{q}_2)}{2}\right)+(k_4\leftrightarrow k_5)\right]\\
&\qquad+[k_4+2q_1]_\epsilon\,\delta_{\tau\kappa}\sin\!\left(\tfrac{k_4\tilde{q}_1}{2}\right)\left[\sin\!\left(\tfrac{k_5\tilde{q}_2}{2}\right)\sin\!\left(\tfrac{k_3(\tilde{k}_5+\tilde{q}_2)}{2}\right)+(k_5\leftrightarrow k_3)\right]\\
&\qquad+[k_5+2q_1]_\kappa\,\delta_{\tau\epsilon}\sin\!\left(\tfrac{k_5\tilde{q}_1}{2}\right)\left[\sin\!\left(\tfrac{k_3\tilde{q}_2}{2}\right)\sin\!\left(\tfrac{k_4(\tilde{k}_3+\tilde{q}_2)}{2}\right)+(k_3\leftrightarrow k_4)\right]\\
&\qquad+(q_1\leftrightarrow q_2)\Bigg\},
\end{aligned}\tag{C.25g}$$



$$= \widetilde{V}^{2B4A}_{\mu\nu,\rho\sigma,\tau\epsilon\kappa\iota}\,(q_1, q_2, k_3, k_4, k_5, k_6)$$

$$\begin{aligned}
&= 4g^4\theta^2(2\pi)^4\delta^4(q_1+q_2+k_3+k_4+k_5+k_6)\,(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho})\times\\
&\quad\times\Bigg\{2\delta_{\tau\epsilon}\delta_{\kappa\iota}\left[\sin\!\left(\tfrac{k_4\tilde{q}_1}{2}\right)\sin\!\left(\tfrac{k_3(\tilde{k}_4+\tilde{q}_1)}{2}\right)\sin\!\left(\tfrac{k_6\tilde{q}_2}{2}\right)\sin\!\left(\tfrac{k_5(\tilde{k}_6+\tilde{q}_2)}{2}\right)+(k_3\leftrightarrow k_4)+(k_5\leftrightarrow k_6)\right]\\
&\qquad+\delta_{\tau\kappa}\delta_{\epsilon\iota}\left[\sin\!\left(\tfrac{k_5\tilde{q}_1}{2}\right)\sin\!\left(\tfrac{k_3(\tilde{k}_5+\tilde{q}_1)}{2}\right)\sin\!\left(\tfrac{k_6\tilde{q}_2}{2}\right)\sin\!\left(\tfrac{k_4(\tilde{k}_6+\tilde{q}_2)}{2}\right)+(k_3\leftrightarrow k_5)+(k_4\leftrightarrow k_6)\right]\\
&\qquad+\delta_{\tau\iota}\delta_{\kappa\epsilon}\left[\sin\!\left(\tfrac{k_6\tilde{q}_1}{2}\right)\sin\!\left(\tfrac{k_3(\tilde{k}_6+\tilde{q}_1)}{2}\right)\sin\!\left(\tfrac{k_4\tilde{q}_2}{2}\right)\sin\!\left(\tfrac{k_5(\tilde{k}_4+\tilde{q}_2)}{2}\right)+(k_3\leftrightarrow k_6)+(k_5\leftrightarrow k_4)\right]\\
&\qquad+(q_1\leftrightarrow q_2)\Bigg\}.
\end{aligned}\tag{C.25h}$$

C.3 One-loop graphs

Using the abbreviation

$$\mathcal{G}(k)\equiv\left(k^2+\frac{a'^2}{\tilde{k}^2}\right),\tag{C.26}$$

the full expressions for the graphs depicted in Fig. 3.3 including the factors listed in Table 3.1 are given by:

$$\begin{aligned}
\Pi^{(\mathrm{a})}_{\mu\nu} &= s_a \frac{4g^2}{(2\pi)^4} \int d^4k \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right)}{\mathcal{G}(k)\mathcal{G}(k+p)} \Bigg\{ \frac{k_\mu k_\nu}{k^2} \left(11k^2 - p^2 + 2kp + \frac{(p^2-k^2)^2}{(k+p)^2} \right) \\
&+ \delta_{\mu\nu} \left[\alpha \left(\frac{(k^2+2kp)^2}{\mathcal{G}(k)} + \frac{(k^2-p^2)^2}{\mathcal{G}(k+p)} \right) + k^2 + 5p^2 - 2kp - \frac{(k^2-p^2)^2}{(k+p)^2} - 4\frac{(kp)^2}{k^2} \right] \\
&+ \alpha k_\mu k_\nu \left(\frac{p^2-k^2-2kp}{\mathcal{G}(k)} - \frac{\frac{p^4}{(k+p)^2}\left(\frac{a'^2}{(\tilde{k}+\tilde{p})^2} - (k+p)^2(\alpha-1)\right)}{\mathcal{G}(k)\mathcal{G}(k+p)} - \frac{(k^2-p^2)^2}{k^2\mathcal{G}(k+p)} \right) \\
&+ p_\mu p_\nu \frac{\left(\frac{a'^2}{(\tilde{k}+\tilde{p})^2} - (k+p)^2(\alpha-1)\right)}{(\mathcal{G}(k+p))\,(k+p)^2} \left(\frac{k^2p^2 + (kp)^2 - 2k^4}{k^2} - \alpha\frac{(kp)^2}{\mathcal{G}(k)} \right) \\
&+ p_\mu p_\nu \left(-3 + \alpha\frac{k^2}{\mathcal{G}(k)} \right) + (k_\mu p_\nu + p_\mu k_\nu) \left(3\frac{kp+2k^2}{k^2} - \alpha\frac{3kp+k^2}{\mathcal{G}(k)} \right) \\
&+ (k_\mu p_\nu + p_\mu k_\nu)(kp) \frac{\left(\frac{a'^2}{(\tilde{k}+\tilde{p})^2} - (k+p)^2(\alpha-1)\right)}{\mathcal{G}(k+p)(k+p)^2} \left(\frac{k^2-p^2}{k^2} + \alpha\frac{p^2}{\mathcal{G}(k)} \right) \Bigg\},
\end{aligned} \tag{C.27a}$$

$$\Pi^{(\mathrm{b})}_{\mu\nu} = -s_b \frac{8g^2}{(2\pi)^4} \int d^4k \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{1}{\mathcal{G}(k)} \left[2\delta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} + \alpha\frac{(k^2\delta_{\mu\nu} - k_\mu k_\nu)}{\mathcal{G}(k)} \right], \tag{C.27b}$$

$$\Pi^{(\mathrm{c})}_{\mu\nu} = -s_c \frac{4g^2}{(2\pi)^4} \int d^4k \frac{k_\mu (k+p)_\nu}{k^2(k+p)^2} \sin^2\left(\frac{k\tilde{p}}{2}\right), \tag{C.27c}$$

$$\begin{aligned}
\Pi^{(\mathrm{d})}_{\mu\nu} &= s_d \frac{4g^2\theta^4}{(2\pi)^4} \int d^4k \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{\left[(k+p)^2 + k^2\right]^2 (2k+p)_\mu (2k+p)_\nu}{k^2\tilde{k}^2(k+p)^2(\tilde{k}+\tilde{p})^2} \times \\
&\times \left[6 - \frac{3a'^2}{\tilde{k}^2\mathcal{G}(k)} - \frac{3a'^2}{(\tilde{k}+\tilde{p})^2\mathcal{G}(k+p)} + \frac{a'^4\left(k^2(k+p)^2 + 2[k(k+p)]^2\right)}{k^2\tilde{k}^2(k+p)^2(\tilde{k}+\tilde{p})^2\mathcal{G}(k)\mathcal{G}(k+p)} \right],
\end{aligned} \tag{C.27d}$$

$$\Pi^{(\mathrm{e})}_{\mu\nu} = -s_e \frac{24g^2}{(2\pi)^4} \int \mathrm{d}^4k \frac{\sin^2(\frac{k\tilde{p}}{2})}{k^4} [p_\mu p_\nu + 4k_\mu k_\nu + 2k^2\delta_{\mu\nu}] \left[2 - \frac{a'^2}{\tilde{k}^2\mathcal{G}(k)} \right], \tag{C.27e}$$

$$\Pi^{(\mathrm{f})}_{\mu\nu} = s_f \frac{4\,a'^4 g^2}{(2\pi)^4} \int d^4k \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{3k_\mu k_\nu + 2k_\mu p_\nu + k_\nu p_\mu}{k^2\tilde{k}^2(k+p)^2(\tilde{k}+\tilde{p})^2\mathcal{G}(k)\mathcal{G}(k+p)}, \tag{C.27f}$$

$$
\begin{aligned}
\Pi^{(\mathrm{g})}_{\mu\nu} &= s_g \frac{4\,a'^2 g^2}{(2\pi)^4} \int d^4k \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right)}{k^2\tilde{k}^2\mathcal{G}(k+p)} \Bigg\{ 2\delta_{\mu\nu} + \frac{(k+p)_\mu (k+p)_\nu}{(k+p)^2} \\
&+ \frac{a'^2}{k^2\tilde{k}^2\mathcal{G}(k)} \Big[\delta_{\mu\nu} \left(\tfrac{[k(k+p)]^2}{(k+p)^2} - k^2 \right) - k_\mu k_\nu - \tfrac{k(k+p)}{(k+p)^2} \left(2k_\mu k_\nu + k_\mu p_\nu + p_\mu k_\nu\right) \Big] \\
&+ \frac{\alpha}{\mathcal{G}(k+p)\mathcal{G}(k)} \Bigg(\delta_{\mu\nu} \left[k^2(k+p)^2 - a'^2 \frac{(kp)^2 - k^2p^2}{k^2\tilde{k}^2} \right] - k^2(k_\mu + p_\mu)(k_\nu + p_\nu) \\
&\qquad - \frac{a'^2}{k^2\tilde{k}^2} \left(k^2 p_\mu p_\nu + p^2 k_\mu k_\nu - (kp)(k_\mu p_\nu + p_\mu k_\nu) \right) \Bigg) \Bigg\},
\end{aligned} \tag{C.27g}
$$

$$
\begin{aligned}
\Pi^{(\mathrm{h+i})}_{\mu\nu} &= s_{\mathrm{h}} \frac{4\,a'^2 g^2}{(2\pi)^4} \int d^4k \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right)}{\tilde{k}^2(k+p)^2\mathcal{G}(k)} \left(\frac{1}{k^2} + \frac{1}{(k+p)^2} \right) (2k_\mu + p_\mu) \times \\
&\times \left[3k_\nu - a'^2 \frac{k_\nu[(k+p)^2 + 2k(k+p)] + 2p_\nu[k(k+p)]}{(k+p)^2(\tilde{k}+\tilde{p})^2\mathcal{G}(k+p)} \right] + \mu \leftrightarrow \nu,
\end{aligned} \tag{C.27h}
$$

$$
\begin{aligned}
\Pi^{(\mathrm{j})}_{\mu\nu} &= s_j \frac{4a'^2 g^2}{(2\pi)^4} \int d^4k \sin^2\left(\frac{k\tilde{p}}{2}\right) \left(\frac{1}{(k+p)^2} + \frac{1}{k^2} \right) \times \\
&\times \frac{(2k+p)_\mu \left[(6k^2 + 6kp + 2p^2)k_\nu + (3k^2 + kp)p_\nu \right]}{\tilde{k}^2(k+p)^2\mathcal{G}(k)\mathcal{G}(k+p)},
\end{aligned} \tag{C.27i}
$$

$$
\begin{aligned}
\Pi^{(\mathrm{k+l})}_{\mu\nu} &= s_{\mathrm{k}} \frac{4\,a'^2 g^2}{(2\pi)^4} \int d^4k \frac{\sin^2\left(\frac{k\tilde{p}}{2}\right)}{k^2\tilde{k}^2\mathcal{G}(k)\mathcal{G}(k+p)} \Bigg\{ 3k_\mu k_\nu + 2p_\mu k_\nu + k_\mu p_\nu \\
&+ \delta_{\mu\nu} \left[k(k-p) + k(k+p)\frac{(p^2 - k^2)}{(k+p)^2} - \alpha \frac{k(k+p)(p^2-k^2)}{\mathcal{G}(k+p)} \right] \\
&+ \frac{1}{(k+p)^2} \Big(k(k+p)(k_\mu k_\nu - p_\mu p_\nu) + (p^2 + 2k^2 + 3(kp))(k_\mu k_\nu + k_\mu p_\nu) \Big) \\
&- \frac{\alpha}{\mathcal{G}(k+p)} \Big(k(k+p)(k_\mu k_\nu - p_\mu p_\nu) + (p^2 + 2k^2 + 3(kp))(k_\mu k_\nu + k_\mu p_\nu) \\
&\qquad - (k+p)^2(2k_\mu k_\nu + k_\mu p_\nu) \Big) + \mu \leftrightarrow \nu \Bigg\}.
\end{aligned} \tag{C.27j}
$$

Appendix D

Calculations for gauge model with soft breaking

D.1 Propagators for the model of Vilar *et al.*

The action (5.5) leads to 19 propagator:

$$G^{A}_{\mu\nu}(k) = \frac{-1}{\left(k^2 + \frac{\gamma^4}{k^2}\right)} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right), \tag{D.1a}$$

$$G^{BA}_{\rho,\sigma\tau}(k) = \frac{-\gamma^3}{\left(k^2 + \frac{\gamma^4}{k^2}\right)} \frac{(k_\sigma \delta_{\rho\tau} - k_\tau \delta_{\rho\sigma})}{2(k^2)^2}, \tag{D.1b}$$

$$G^{\bar{B}A}_{\rho,\sigma\tau}(k) = -G^{BA}_{\rho,\sigma\tau}(k), \tag{D.1c}$$

$$G^{\chi A}_{\rho,\sigma\tau}(k) = \frac{\mathrm{i}\gamma}{\left(k^2 + \frac{\gamma^4}{k^2}\right)} \frac{(k_\sigma \delta_{\rho\tau} - k_\tau \delta_{\rho\sigma})}{2k^2}, \tag{D.1d}$$

$$G^{\bar{\chi}A}_{\rho,\sigma\tau}(k) = -G^{\chi A}_{\rho,\sigma\tau}(k), \tag{D.1e}$$

$$G^{\bar{B}\bar{B}}_{\rho\sigma,\tau\epsilon}(k) = \frac{\gamma^4}{(k^2)^2} \frac{(k_\rho k_\tau \delta_{\sigma\epsilon} + k_\sigma k_\epsilon \delta_{\rho\tau} - k_\rho k_\epsilon \delta_{\sigma\tau} - k_\sigma k_\tau \delta_{\rho\epsilon})}{4(k^2)^2 \left(k^2 + \frac{\gamma^4}{k^2}\right)}, \tag{D.1f}$$

$$G^{BB}_{\rho\sigma,\tau\epsilon}(k) = G^{\bar{B}\bar{B}}_{\rho\sigma,\tau\epsilon}(k), \tag{D.1g}$$

$$G^{B\bar{B}}_{\rho\sigma,\tau\epsilon}(k) = \gamma^2 \frac{(\delta_{\rho\tau}\delta_{\sigma\epsilon} - \delta_{\rho\epsilon}\delta_{\sigma\tau})}{2(k^2)^2} - G^{\bar{B}\bar{B}}_{\rho\sigma,\tau\epsilon}(k), \tag{D.1h}$$

$$G^{\bar{\chi}\bar{\chi}}_{\rho\sigma,\tau\epsilon}(k) = -\gamma^2 \frac{(k_\rho k_\tau \delta_{\sigma\epsilon} + k_\sigma k_\epsilon \delta_{\rho\tau} - k_\rho k_\epsilon \delta_{\sigma\tau} - k_\sigma k_\tau \delta_{\rho\epsilon})}{4(k^2)^2 \left(k^2 + \frac{\gamma^4}{k^2}\right)}, \tag{D.1i}$$

$$G^{\chi\chi}_{\rho\sigma,\tau\epsilon}(k) = G^{\bar{\chi}\bar{\chi}}_{\rho\sigma,\tau\epsilon}(k), \tag{D.1j}$$

$$G^{\chi\bar{\chi}}_{\rho\sigma,\tau\epsilon}(k) = -G^{\bar{\chi}\bar{\chi}}_{\rho\sigma,\tau\epsilon}(k), \tag{D.1k}$$

$$G^{\chi B}_{\rho\sigma,\tau\epsilon}(k) = \frac{\gamma^4}{k^2}\frac{(k_\rho k_\tau \delta_{\sigma\epsilon} + k_\sigma k_\epsilon \delta_{\rho\tau} - k_\rho k_\epsilon \delta_{\sigma\tau} - k_\sigma k_\tau \delta_{\rho\epsilon})}{4(k^2)^2\left(k^2 + \frac{\gamma^4}{k^2}\right)}\,, \tag{D.1l}$$

$$G^{\bar\chi \bar B}_{\rho\sigma,\tau\epsilon}(k) = G^{\chi B}_{\rho\sigma,\tau\epsilon}(k)\,, \tag{D.1m}$$

$$G^{\chi \bar B}_{\rho\sigma,\tau\epsilon}(k) = \frac{(\delta_{\rho\tau}\delta_{\sigma\epsilon} - \delta_{\rho\epsilon}\delta_{\sigma\tau})}{2k^2} - G^{\chi B}_{\rho\sigma,\tau\epsilon}(k)\,, \tag{D.1n}$$

$$G^{\bar\chi B}_{\rho\sigma,\tau\epsilon}(k) = G^{\chi \bar B}_{\rho\sigma,\tau\epsilon}(k)\,, \tag{D.1o}$$

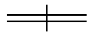
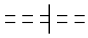
$$G^{\bar c c}(k) = -\frac{1}{k^2}\,, \tag{D.1p}$$

$$G^{\xi,\bar\psi}_{\mu\nu,\rho\sigma}(k) = \frac{(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})}{2k^2}\,, \tag{D.1q}$$

$$G^{\bar\xi,\psi}_{\mu\nu,\rho\sigma}(k) = -G^{\xi,\bar\psi}_{\mu\nu,\rho\sigma}\,, \tag{D.1r}$$

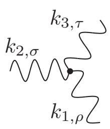
$$G^{\bar\xi\xi}_{\mu\nu,\rho\sigma}(k) = -\gamma^2\frac{(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})}{2(k^2)^2}\,. \tag{D.1s}$$

D.2 Graphical conventions

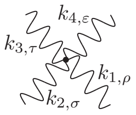
	$= A_\mu$
	$= c$ and $\bar c$
	$= B_{\mu\nu}$
	$= \bar B_{\mu\nu}$
	$= \psi_{\mu\nu}$
	$= \bar\psi_{\mu\nu}$

(D.2)

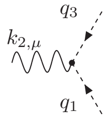
D.3 Vertices



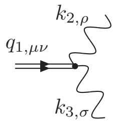
$$\begin{aligned} &= \widetilde{V}^{3A}_{\rho\sigma\tau}(k_1,k_2,k_3) \\ &= 2\mathrm{i}g(2\pi)^4\delta^4(k_1+k_2+k_3)\sin\left(\tfrac{k_1\tilde{k}_2}{2}\right)\times \\ &\quad\times\left[(k_3-k_2)_\rho\delta_{\sigma\tau} + (k_1-k_3)_\sigma\delta_{\rho\tau} + (k_2-k_1)_\tau\delta_{\rho\sigma}\right], \end{aligned} \tag{D.3a}$$



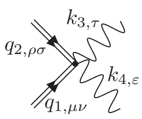
$$
\begin{aligned}
&= \widetilde{V}^{4A}_{\rho\sigma\tau\epsilon}(k_1,k_2,k_3,k_4)\\
&= -4g^2(2\pi)^4\delta^4(k_1+k_2+k_3+k_4)\times\\
&\quad\times\Big[(\delta_{\rho\tau}\delta_{\sigma\epsilon}-\delta_{\rho\epsilon}\delta_{\sigma\tau})\sin\left(\tfrac{k_1\tilde{k}_2}{2}\right)\sin\left(\tfrac{k_3\tilde{k}_4}{2}\right)\\
&\quad+(\delta_{\rho\sigma}\delta_{\tau\epsilon}-\delta_{\rho\epsilon}\delta_{\sigma\tau})\sin\left(\tfrac{k_1\tilde{k}_3}{2}\right)\sin\left(\tfrac{k_2\tilde{k}_4}{2}\right)\\
&\quad+(\delta_{\rho\sigma}\delta_{\tau\epsilon}-\delta_{\rho\tau}\delta_{\sigma\epsilon})\sin\left(\tfrac{k_2\tilde{k}_3}{2}\right)\sin\left(\tfrac{k_1\tilde{k}_4}{2}\right)\Big],
\end{aligned}
\tag{D.3b}
$$



$$
\begin{aligned}
&= \widetilde{V}^{\bar{c}Ac}_{\mu}(q_1,k_2,q_3)\\
&= -2\mathrm{i}g(2\pi)^4\delta^4(q_1+q_2+k_3)q_{1\mu}\sin\left(\tfrac{q_1\tilde{q}_3}{2}\right)
\end{aligned}
\tag{D.3c}
$$

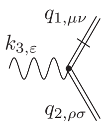


$= \widetilde{V}^{BAA}_{\mu\nu,\rho\sigma}(q_1,k_2,k_3) =$

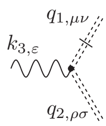


$\widetilde{V}^{\bar{B}AA}_{\mu\nu,\rho\sigma}(q_1,k_2,k_3)$

$$
= \lambda g(2\pi)^4\delta^4(q_1+k_2+k_3)\left(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}\right)\sin\left(\tfrac{k_2\tilde{k}_3}{2}\right),
\tag{D.3d}
$$

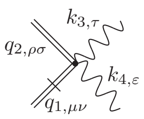


$= \widetilde{V}^{\bar{B}BA}_{\mu\nu,\rho\sigma\epsilon}(q_1,q_2,k_3) = -$

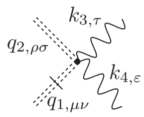


$= -\widetilde{V}^{\bar{\psi}\psi A}_{\mu\nu,\rho\sigma\epsilon}(q_1,q_2,k_3)$

$$
\begin{aligned}
&= -\mathrm{i}\mu^2g(2\pi)^4\delta^4(q_1+q_2+k_3)\left(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}\right)\times\\
&\quad\times\left((\tilde{q}_1)^2+(\tilde{q}_2)^2\right)(q_1-q_2)_\epsilon\sin\left(\tfrac{q_1\tilde{q}_2}{2}\right),
\end{aligned}
\tag{D.3e}
$$

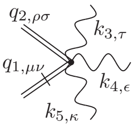


$= \widetilde{V}^{\bar{B}B2A}_{\mu\nu,\rho\sigma,\tau\epsilon}(q_1,q_2,k_3,k_4) =$

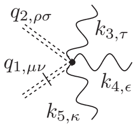


$= -\widetilde{V}^{\bar{\psi}\psi 2A}_{\mu\nu,\rho\sigma,\tau\epsilon}(q_1,q_2,k_3,k_4)$

$$
\begin{aligned}
&= 2\mu^2g^2\theta^2(2\pi)^4\delta^4(q_1+q_2+k_3+k_4)\left(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}\right)\times\\
&\times\Big\{\left[k_{3,\tau}k_{4,\epsilon}+2(q_{1,\tau}k_{4,\epsilon}+q_{2,\epsilon}k_{3,\tau})+4q_{1,\tau}q_{2,\epsilon}-\delta_{\epsilon\tau}\left({q_1}^2+{q_2}^2\right)\right]\sin\left(\tfrac{q_1\tilde{k}_3}{2}\right)\sin\left(\tfrac{q_2\tilde{k}_4}{2}\right)\\
&+\left[k_{3,\tau}k_{4,\epsilon}+2(q_{2,\tau}k_{4,\epsilon}+q_{1,\epsilon}k_{3,\tau})+4q_{1,\epsilon}q_{2,\tau}-\delta_{\epsilon\tau}\left({q_1}^2+{q_2}^2\right)\right]\sin\left(\tfrac{q_1\tilde{k}_4}{2}\right)\sin\left(\tfrac{q_2\tilde{k}_3}{2}\right)\Big\},
\end{aligned}
\tag{D.3f}
$$

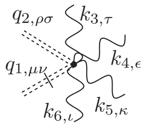


$= \widetilde{V}^{\bar{B}B3A}_{\mu\nu,\rho\sigma,\tau\epsilon\kappa}(q_1,q_2,k_3,k_4,k_5) = -$

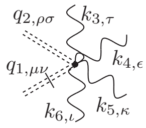


$= -\widetilde{V}^{\bar{\psi}\psi3A}_{\mu\nu,\rho\sigma,\tau\epsilon\kappa}(q_1,q_2,k_3,k_4,k_5)$

$$
\begin{aligned}
&= -4\mathrm{i}g^3\mu^2\theta^2(2\pi)^4\delta^4(q_1+q_2+k_3+k_4+k_5)\left(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}\right)\times\\
&\quad\times\Bigg\{\left[k_3+2q_1\right]_\tau\delta_{\epsilon\kappa}\sin\!\left(\tfrac{k_3\tilde{q}_1}{2}\right)\left[\sin\!\left(\tfrac{k_5\tilde{q}_2}{2}\right)\sin\!\left(\tfrac{k_4(\tilde{k}_5+\tilde{q}_2)}{2}\right)+(k_4\leftrightarrow k_5)\right]\\
&\qquad+\left[k_4+2q_1\right]_\epsilon\delta_{\tau\kappa}\sin\!\left(\tfrac{k_4\tilde{q}_1}{2}\right)\left[\sin\!\left(\tfrac{k_5\tilde{q}_2}{2}\right)\sin\!\left(\tfrac{k_3(\tilde{k}_5+\tilde{q}_2)}{2}\right)+(k_5\leftrightarrow k_3)\right]\\
&\qquad+\left[k_5+2q_1\right]_\kappa\delta_{\tau\epsilon}\sin\!\left(\tfrac{k_5\tilde{q}_1}{2}\right)\left[\sin\!\left(\tfrac{k_3\tilde{q}_2}{2}\right)\sin\!\left(\tfrac{k_4(\tilde{k}_3+\tilde{q}_2)}{2}\right)+(k_3\leftrightarrow k_4)\right]\\
&\qquad+(q_1\leftrightarrow q_2)\Bigg\},
\end{aligned}
\tag{D.4a}
$$



$= \widetilde{V}^{\bar{B}B4A}_{\mu\nu,\rho\sigma,\tau\epsilon\kappa\iota}(q_1,q_2,k_3,k_4,k_5,k_6) = -$



$= -\widetilde{V}^{\bar{\psi}\psi4A}_{\mu\nu,\rho\sigma,\tau\epsilon\kappa\iota}(q_1,q_2,k_{3-6})$

$$
\begin{aligned}
&= 2g^4\mu^2\theta^2(2\pi)^4\delta^4(q_1+q_2+k_3+k_4+k_5+k_6)\left(\delta_{\mu\rho}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}\right)\times\\
&\quad\times\Bigg\{2\delta_{\tau\epsilon}\delta_{\kappa\iota}\Big[\sin\!\left(\tfrac{k_4\tilde{q}_1}{2}\right)\sin\!\left(\tfrac{k_3(\tilde{k}_4+\tilde{q}_1)}{2}\right)\sin\!\left(\tfrac{k_6\tilde{q}_2}{2}\right)\sin\!\left(\tfrac{k_5(\tilde{k}_6+\tilde{q}_2)}{2}\right)+(k_3\leftrightarrow k_4)+(k_5\leftrightarrow k_6)\Big]\\
&\qquad+\delta_{\tau\kappa}\delta_{\epsilon\iota}\Big[\sin\!\left(\tfrac{k_5\tilde{q}_1}{2}\right)\sin\!\left(\tfrac{k_3(\tilde{k}_5+\tilde{q}_1)}{2}\right)\sin\!\left(\tfrac{k_6\tilde{q}_2}{2}\right)\sin\!\left(\tfrac{k_4(\tilde{k}_6+\tilde{q}_2)}{2}\right)+(k_3\leftrightarrow k_5)+(k_4\leftrightarrow k_6)\Big]\\
&\qquad+\delta_{\tau\iota}\delta_{\kappa\epsilon}\Big[\sin\!\left(\tfrac{k_6\tilde{q}_1}{2}\right)\sin\!\left(\tfrac{k_3(\tilde{k}_6+\tilde{q}_1)}{2}\right)\sin\!\left(\tfrac{k_4\tilde{q}_2}{2}\right)\sin\!\left(\tfrac{k_5(\tilde{k}_4+\tilde{q}_2)}{2}\right)+(k_3\leftrightarrow k_6)+(k_5\leftrightarrow k_4)\Big]\\
&\qquad+(q_1\leftrightarrow q_2)\Bigg\}.
\end{aligned}
\tag{D.4b}
$$

D.4 Two point functions

According to Section 5.6.2 we will list the following information for the divergent two point functions (with amputated external legs):

- Feynman graphs
- Symmetry factors
- For the sum of graphs: the integrated results, given individually for each order and planar/non-planar results

Each type of two point functions, characterized by its external legs, is considered individually.

Vacuum polarization

The model (5.19) gives rise to 23 graphs contributing to the vacuum polarization. Omitting convergent expressions, there are 11 graphs left depicted in Fig. D.1. The symmetry

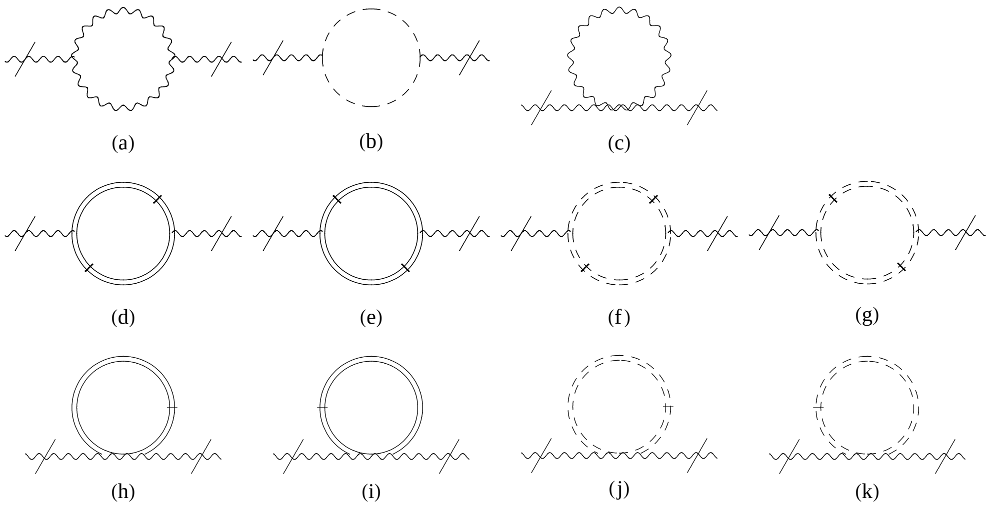


Figure D.1: One loop corrections for the gauge boson propagator

factors are listed in Tab. D.1. Being interested in the divergent contributions one can

Table D.1: Symmetry factors for the one loop vacuum polarization (where the factor (-1) for fermionic loops has been included).

s_{a}	$\frac{1}{2}$	s_{e}	1	s_{i}	1
s_{b}	-1	s_{f}	-1	s_{j}	-1
s_{c}	$\frac{1}{2}$	s_{g}	-1	s_{k}	-1
s_{d}	1	s_{h}	1		

apply the expansion in Eqn. (3.47). Summing up the contributions of the graphs in Fig. D.1 and denoting the result at order i for the planar (p) part by $\Pi^{(i),\mathrm{p}}_{\mu\nu}$, one is left with

$$\begin{aligned}\Pi^{(0),\mathrm{p}}_{\mu\nu}(p) =&\frac{g^2}{16\pi^2}\Lambda^2\delta_{\mu\nu}\left(-10s_{\mathrm{c}}-96s_{\mathrm{h}}-96s_{\mathrm{j}}+12s_{\mathrm{a}}+s_{\mathrm{b}}+96s_{\mathrm{d}}+96s_{\mathrm{f}}\right)\\ =&0\,, \end{aligned}\tag{D.5a}$$

$$
\begin{aligned}
\Pi^{(2),\mathrm{p}}_{\mu\nu}(p) =& -\frac{1}{3}\frac{g^2}{16\pi^2}\Big[\delta_{\mu\nu}p^2\left(22s_\mathrm{a}+s_\mathrm{b}+48(s_\mathrm{d}+s_\mathrm{f})\right)\\
&\qquad + 2p_\mu p_\nu\left(72(s_\mathrm{h}+s_\mathrm{j})-8s_\mathrm{a}+s_\mathrm{b}-96(s_\mathrm{d}+s_\mathrm{f})\right)\Big]\mathrm{K}_0\left(2\sqrt{\frac{M^2}{\Lambda^2}}\right)\\
=& -\frac{5g^2}{12\pi^2}\left(p^2\delta_{\mu\nu}-p_\mu p_\nu\right)K_0\left(2\sqrt{\frac{M^2}{\Lambda^2}}\right)\\
\approx& -\frac{5g^2}{12\pi^2}\left(p^2\delta_{\mu\nu}-p_\mu p_\nu\right)\ln\left(\frac{\Lambda^2}{M^2}\right)+\text{finite}\,.
\end{aligned}
\tag{D.5b}
$$

According to Eqn. (A.10a) we have approximated the modified Bessel function K_0 by

$$
\mathrm{K}_0(x) \underset{x\ll 1}{\approx} \ln\tfrac{2}{x}-\gamma_E+\mathcal{O}\left(x^2\right)\,,
$$

where γ_E denotes the Euler-Mascheroni constant. Note that this is valid for small arguments, i.e. vanishing regulator cutoffs[1] $\Lambda\to\infty$ and $M\to 0$. Note that the first order vanishes identically due to an odd power of k in the integrand which leads to a cancellation under the symmetric integration over the momenta.

Of particular interest is the non-planar part (np) which for small p results to:

$$
\begin{aligned}
\Pi^{(0),\mathrm{np}}_{\mu\nu}(p) =& \frac{g^2}{4\pi^2\tilde{p}^2}\Big[\delta_{\mu\nu}\left(96(s_\mathrm{h}+s_\mathrm{j}-s_\mathrm{d}-s_\mathrm{f})-12s_\mathrm{a}-s_\mathrm{b}+10s_\mathrm{c}\right)\\
&\qquad -2\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}\left(48(s_\mathrm{h}+s_\mathrm{j})-96(s_\mathrm{d}+s_\mathrm{f})-12s_\mathrm{a}-s_\mathrm{b}+2s_\mathrm{c}\right)\Big]\\
=& \frac{2g^2}{\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\tilde{p}^2)^2}\,,
\end{aligned}
\tag{D.6a}
$$

$$
\begin{aligned}
\Pi^{(2),\mathrm{np}}_{\mu\nu}(p) =& \frac{g^2}{48\pi^2\tilde{p}^2}\Bigg\{2\theta^2 p_\mu p_\nu p^2\left(72(s_\mathrm{h}+s_\mathrm{j})-8s_\mathrm{a}+s_\mathrm{b}-96(s_\mathrm{d}+s_\mathrm{f})\right)\mathrm{K}_0\left(\sqrt{M^2\tilde{p}^2}\right)\\
&\qquad +\sqrt{\frac{\tilde{p}^2}{M^2}}p^2\Bigg[\sqrt{\frac{\tilde{p}^2}{M^2}}\left(22s_\mathrm{a}+s_\mathrm{b}+48(s_\mathrm{d}+s_\mathrm{f})\right)M^2\delta_{\mu\nu}\,\mathrm{K}_0\left(\sqrt{M^2\tilde{p}^2}\right)\\
&\qquad\qquad +2M^2\left(13s_\mathrm{a}+s_\mathrm{b}+120(s_\mathrm{d}+s_\mathrm{f})\right)\tilde{p}_\mu\tilde{p}_\nu\,\mathrm{K}_1\left(\sqrt{M^2\tilde{p}^2}\right)\\
&\qquad\qquad -3\sqrt{\frac{M^2}{\tilde{p}^2}}\left(16s_\mathrm{a}+s_\mathrm{b}+96(s_\mathrm{d}+s_\mathrm{f})\right)\tilde{p}_\mu\tilde{p}_\nu\Bigg]\Bigg\}\\
=& -\frac{g^2}{48\pi^2}\Bigg[\tilde{p}_\mu\tilde{p}_\nu\left(\frac{21}{\theta^2}-11p^2\sqrt{\frac{M^2}{\tilde{p}^2}}\,\mathrm{K}_1\left(\sqrt{M^2\tilde{p}^2}\right)\right)\\
&\qquad -10\,\mathrm{K}_0\left(\sqrt{M^2\tilde{p}^2}\right)\left(p^2\delta_{\mu\nu}-p_\mu p_\nu\right)\Bigg]\,.
\end{aligned}
\tag{D.6b}
$$

[1]The cutoffs are introduced via a factor $\exp\left[-M^2\alpha-\frac{1}{\Lambda^2\alpha}\right]$ to regularize parameter integrals $\int_0^\infty \mathrm{d}\alpha$. See Section A.2 for a more extensive description of the mathematical details. Note that here the regulator mass is written as M instead of μ, as the latter represents a parameter of the theory under consideration.

Considering the limit $\tilde{p}^2 \to 0$ allows the application of the approximation Eqn. (A.10b) for K

$$\mathrm{K}_1(x) \underset{x \ll 1}{\approx} \tfrac{1}{x} + \tfrac{x}{2}\left(\gamma_E - \tfrac{1}{2} + \ln \tfrac{x}{2}\right) + \mathcal{O}\left(x^2\right) ,$$

which reveals that the second order is IR finite (which is immediately clear from the fact that the terms of lowest order in p are $\mathcal{O}\left(p^2\right)$). We are left with a UV divergence given by the $\ln(M^2)$-term, however which cancels in the sum of planar and non-planar contributions. Hence, collecting all divergent terms one is left with the final result given in Eqn. (5.40), after performing the limit $M \to 0$ and $\Lambda \to \infty$. It is independent of the IR-cutoff M.

Corrections to the AB and $A\bar{B}$ propagator

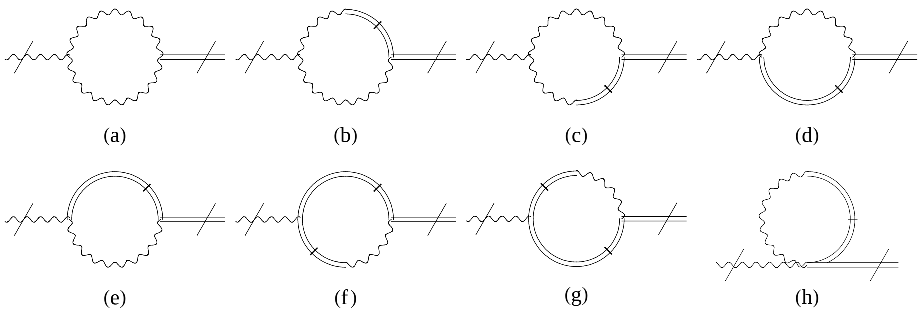


Figure D.2: One loop corrections for $\langle A_\mu B_{\nu_1\nu_2}\rangle$ (with amputated external legs).

The action (5.19) gives rise to eight divergent graphs with one external A_μ and one $B_{\mu\nu}$ which are depicted in Fig. D.2. The symmetry factors are given by Tab. D.2.

Table D.2: Symmetry factors for the graphs depicted in Fig. D.2

(a)	1/2	(e)	1
(b)	1	(f)	1
(c)	1	(g)	1
(d)	1	(h)	1

Applying an expansion of type (3.47) for small external momenta p and summing up the divergent contributions of all graphs one ends up with,

$$\begin{aligned}
\Sigma^{(1),\mathrm{p,AB}}_{\mu1,\nu1\nu2}(p) &= -\frac{3\mathrm{i}g^2}{32\pi^2}\lambda\left(p_{\nu1}\delta_{\mu1\nu2} - p_{\nu2}\delta_{\mu1\nu1}\right)\mathrm{K}_0\left(2\sqrt{\frac{M^2}{\Lambda^2}}\right) + \text{finite} \\
\Sigma^{(1),\mathrm{np,AB}}_{\mu1,\nu1\nu2}(p) &= \frac{3\mathrm{i}g^2}{32\pi^2}\lambda\,\mathrm{K}_0\left(\sqrt{M^2\tilde{p}^2}\right)\left(p_{\nu1}\delta_{\mu1\nu2} - p_{\nu2}\delta_{\mu1\nu1}\right) + \text{finite}\,.
\end{aligned} \tag{D.7}$$

Note that the expansions in zero and second order ($\Sigma^{(0),\mathrm{p+np,AB}}_{\mu1,\nu1\nu2}(p), \Sigma^{(2),\mathrm{p+np,AB}}_{\mu1,\nu1\nu2}(p)$) vanishes identically, due to the odd power of k in the integrand.
Approximating the Bessel functions as in Section 5.6.2 and summing up planar and non-planar parts one finds the final result Eqn. (5.41) where one can see that the IR cutoff M has cancelled.

Corrections to the BB and $\bar{B}\bar{B}$ propagator

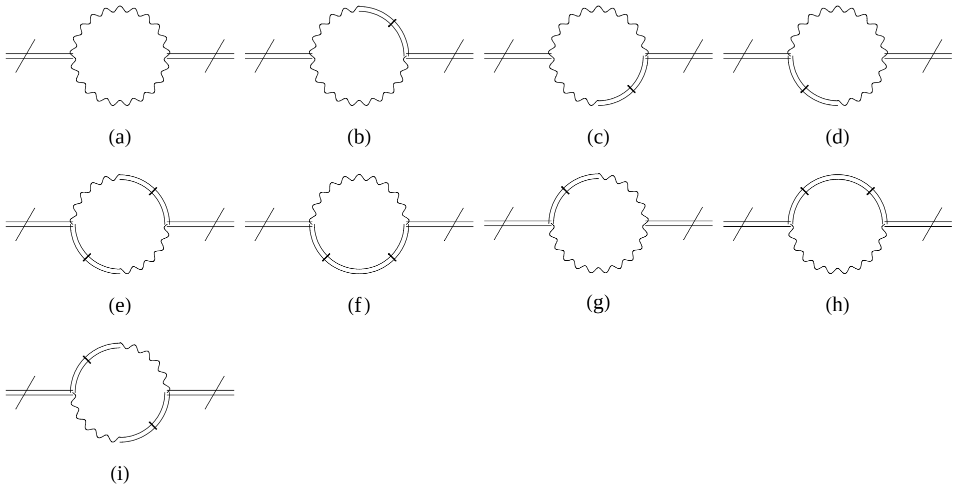


Figure D.3: One loop corrections for $\langle B_{\mu_1\mu_2} B_{\nu_1\nu_2} \rangle$ (with amputated external legs).

The set of divergent graphs contributing to the $B_{\mu1\mu2}B_{\nu1\nu2}$-propagator consists of those depicted in Fig. D.3. The symmetry factors are given by Tab. D.3. Making

Table D.3: Symmetry factors for the graphs depicted in Fig. D.3

(a)	$1/2$	(d)	1	(g)	1
(b)	1	(e)	1	(h)	1
(c)	1	(f)	1	(i)	1

an expansion of the type (3.47) for small external momenta p and summing up the contributions of all nine graphs yields

$$\Sigma^{(0),\mathrm{p,BB}}_{\mu1\mu2,\nu1\nu2}(p) = \frac{g^2\lambda^2}{32\pi^2}\left(\delta_{\mu1\nu1}\delta_{\mu2\nu2} - \delta_{\mu2\nu1}\delta_{\mu1\nu2}\right)\mathrm{K}_0\left(2\sqrt{\frac{M^2}{\Lambda^2}}\right) + \text{finite}\,,$$

$$\Sigma^{(0),\mathrm{np,BB}}_{\mu 1\mu 2,\nu 1\nu 2}(p) = \frac{g^2\lambda^2}{64\pi^2}\Bigg(\frac{\delta_{\mu 1\nu 2}\tilde{p}_{\mu 2}\tilde{p}_{\nu 1} - \delta_{\mu 1\nu 1}\tilde{p}_{\mu 2}\tilde{p}_{\nu 2} - \delta_{\mu 2\nu 2}\tilde{p}_{\mu 1}\tilde{p}_{\nu 1} + \delta_{\mu 2\nu 1}\tilde{p}_{\mu 1}\tilde{p}_{\nu 2}}{\tilde{p}^2}$$
$$+ 2\,\mathrm{K}_0\left(\sqrt{M^2\tilde{p}^2}\right)(\delta_{\mu 1\nu 2}\delta_{\mu 2\nu 1} - \delta_{\mu 1\nu 1}\delta_{\mu 2\nu 2})\Bigg) + \text{finite}\,, \tag{D.8}$$

for the planar/non-planar part, respectively. Note that the expansions in first order vanishes due to the even power of k in the integrand and that also the divergent part of the second order $(\Sigma^{(0),\mathrm{p+np,AB}}_{\mu 1,\nu 1\nu 2}(p), \Sigma^{(2),\mathrm{p+np,AB}}_{\mu 1,\nu 1\nu 2}(p))$ is zero, due to only logarithmic divergence.
Approximating the Bessel functions as in Section 5.6.2 reveals cancellations of contributions depending on M in the final sum, which is given in Eqn. (5.44).

Corrections to the $B\bar{B}$ propagator

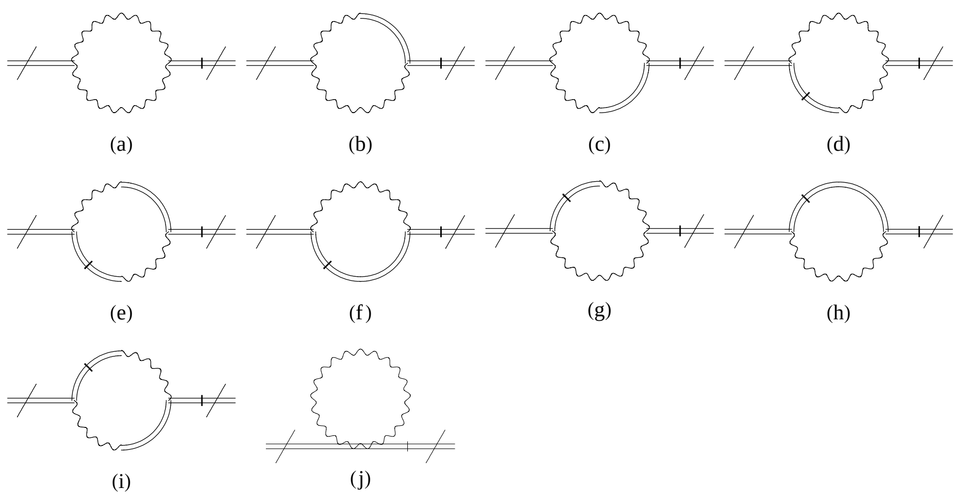


Figure D.4: One loop corrections for $\langle B_{\mu_1\mu_2}\bar{B}_{\nu_1\nu_2}\rangle$ (with amputated external legs).

For the one loop correction to the $B_{\mu 1\mu 2}\bar{B}_{\nu 1\nu 2}$ propagator one finds the ten divergent graphs depicted in Fig. D.4. The respectie symmetry factors are given in Tab. D.4.

Expansion for small external momenta p and summation of the divergent results after integration yields

$$\Sigma^{(0),\mathrm{p,B\bar{B}}}_{\mu 1\mu 2,\nu 1\nu 2}(p) = \frac{g^2\lambda^2}{32\pi^2}(\delta_{\mu 1\nu 1}\delta_{\mu 2\nu 2} - \delta_{\mu 2\nu 1}\delta_{\mu 1\nu 2})\,\mathrm{K}_0\left(2\sqrt{\frac{M^2}{\Lambda^2}}\right) + \text{finite}\,,$$

Table D.4: Symmetry factors for the graphs depicted in Fig. D.4

(a)	$1/2$	(e)	1	(i)	1
(b)	1	(f)	1	(j)	$1/2$
(c)	1	(g)	1		
(d)	1	(h)	1		

$$\begin{aligned}
\Sigma^{(0),\mathrm{np},\mathrm{B}\bar{\mathrm{B}}}_{\mu1\mu2,\nu1\nu2}(p) =& \frac{g^2\lambda^2}{64\pi^2}\Bigg(\frac{\delta_{\mu1\nu2}\tilde{p}_{\mu2}\tilde{p}_{\nu1} - \delta_{\mu1\nu1}\tilde{p}_{\mu2}\tilde{p}_{\nu2} - \delta_{\mu2\nu2}\tilde{p}_{\mu1}\tilde{p}_{\nu1} + \delta_{\mu2\nu1}\tilde{p}_{\mu1}\tilde{p}_{\nu2}}{\tilde{p}^2} \\
&+ 2\,\mathrm{K}_0\left(\sqrt{M^2\tilde{p}^2}\right)\left(\delta_{\mu1\nu2}\delta_{\mu2\nu1} - \delta_{\mu1\nu1}\delta_{\mu2\nu2}\right)\Bigg) + \text{finite}\,, \\
\Sigma^{(2),\mathrm{p},\mathrm{B}\bar{\mathrm{B}}}_{\mu1\mu2,\nu1\nu2}(p) =& \frac{g^2}{2\pi^2}\Lambda^2\mu^2\tilde{p}^2\left(\delta_{\mu2\nu1}\delta_{\mu1\nu2} - \delta_{\mu1\nu1}\delta_{\mu2\nu2}\right)
\end{aligned} \tag{D.9}$$

The final result given in Eqn. (5.46) is the sum of all contributions in Eqn. (D.9). It can be seen that it is logarithmically divergent in $\tilde{p}^2$ and quadratically in Λ. Once more, μ has dropped out in the sum of planar and non-planar contributions.

D.4.1 Calculation of the inverse propagator

The derivation of the inverse of the gauge propagator (5.32c) is based on the relation

$$G^{AA}_{\mu\nu}(p)\Gamma^{AA}_{\nu\rho}(p) = \delta_{\mu\rho}. \tag{D.10}$$

Remember however that so far we worked in the Landau gauge (i.e. $\alpha = 0$), and that the resulting gauge propagator is proportional to a projection operator, which following Section C.1.1 has no inverse. We therefore have to generalize the gauge fixing term in the action (5.19) to an arbitrary gauge, i.e.

$$S_{\phi\pi} \to \int \mathrm{d}^4x \left[b\partial^\mu A_\mu - \frac{\alpha}{2}b^2\right] + S_{ghost}\,, \tag{D.11}$$

leading to the more general propagator

$$G^{AA}_{\mu\nu}(k) = \frac{1}{\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \alpha\frac{k_\mu k_\nu}{k^4}\,. \tag{D.12}$$

$$\tag{D.13}$$

Now we can calculate the inverse via the *ansatz*

$$\left\{\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \alpha\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)\frac{k_\mu k_\nu}{k^4}\right\}\left\{a\delta_{\nu\rho} + b\frac{k_\nu k_\rho}{k^2} + \frac{c}{\alpha}\frac{1}{\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)}\frac{k_\nu k_\rho}{k^4}\right\} \stackrel{!}{=} \delta_{\mu\rho}$$

$$\Longrightarrow a = 1,\, b = -1,\, c = k^4\,, \tag{D.14}$$

leading to the two point vertex function

$$\mathcal{A}_{\mu\nu} \equiv \Gamma^{AA}_{\mu\nu}(p) = \left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)\left\{\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \frac{1}{\alpha}\frac{k_\mu k_\nu}{\left(k^2 + \frac{a'^2}{\tilde{k}^2}\right)}\right\}. \tag{D.15}$$

In the last line the relation $a'^2/\tilde{k}^2 = a^2/k^2$ has been used.

Appendix E

Calculations for the BRSW model

While the one-loop corrections for the gauge propagator can be calculated in the same way as for the previous models, new types of integrals appear in the course of one-loop corrections to the vertices. In particular it follows from the Feynman rules, that each corresponding integral will contain a product of three (four) sine-functions, for the corrections to the vertices with three (four) external legs. We will first show how to simplify such products, followed by a list of the resulting new integrals.

E.1 Simplification of integrals for one-loop corrections to the vertices

The integrals appearing in the course of one-loop corrections to the vertices are of the form

$$\int d^4k \frac{k_{\mu_1} \dots k_{\mu_m}}{(k^2)^n} \mathcal{F}\Big\{ \sin^r(k, p, \varepsilon\theta_{\mu\nu}) \Big\} , \tag{E.1}$$

where $\mathcal{F}\,(\sin^r)$ symbolically denotes the product of r sine-functions, which depend on internal (k) and external (p) momenta, and $r = \{3, 4\}$ equals the number of external legs. In order to solve such integrals, $\mathcal{F}\,(\sin^r)$ can be reduced to expressions containing only one (or even none for the planar part) trigonometric function depending on both internal and external momenta, multiplied by trigonometric functions depending only on external momenta and hence not affecting the integration. This leads to the rather simple integrals given in Section A.2.3 and the next section Section E.2.

A first example is given by the phase factor appearing in the one-loop graph composed by three $\widetilde{V}^{3A}$ vertices (c.f. (7.23a))

$$
\begin{aligned}
&4\sin\left(\varepsilon\frac{p_1}{2}(\tilde{p}_2-\tilde{k})\right)\sin\left(\varepsilon\frac{p_2\tilde{k}}{2}\right)\sin\left(\varepsilon\frac{(p_1+p_2)\tilde{k}}{2}\right)=\\
&\quad\sin\left(\varepsilon\frac{p_1\tilde{p}_2}{2}\right)+\sin\left(\varepsilon\frac{p_1\tilde{p}_2}{2}\right)\cos\left(\varepsilon p_1\tilde{k}\right)-\cos\left(\varepsilon\frac{p_1\tilde{p}_2}{2}\right)\sin\left(\varepsilon p_1\tilde{k}\right)\\
&-\sin\left(\varepsilon\frac{p_1\tilde{p}_2}{2}\right)\cos\left(\varepsilon p_2\tilde{k}\right)-\cos\left(\varepsilon\frac{p_1\tilde{p}_2}{2}\right)\sin\left(\varepsilon p_2\tilde{k}\right)+\cos\left(\varepsilon\frac{p_1\tilde{p}_2}{2}\right)\sin\left(\varepsilon(p_1+p_2)\tilde{k}\right)\\
&-\sin\left(\varepsilon\frac{p_1\tilde{p}_2}{2}\right)\cos\left(\varepsilon(p_1+p_2)\tilde{k}\right),
\end{aligned}
\tag{E.2}
$$

where the sum of momenta in the original expression have been separated by using (A.11c).

In the case of the four-point functions, we encounter products of four sine functions, which is given by

$$
\begin{aligned}
&\sin\left(\varepsilon\frac{p_1\tilde{k}}{2}\right)\sin\left(\varepsilon\frac{p_2\tilde{k}}{2}\right)\sin\left(\varepsilon\frac{p_3(\tilde{k}+\tilde{p}_2)}{2}\right)\sin\left(\varepsilon\frac{p_4(\tilde{k}+\tilde{p}_2)}{2}\right)\\
=&\frac{1}{8}\Big[\cos\left(\varepsilon\frac{p_3\tilde{p}_2}{2}\right)\cos\left(\varepsilon\frac{p_4\tilde{p}_2}{2}\right)\Big(\cos(\varepsilon(p_1+p_2)\tilde{k})+\cos(\varepsilon(p_1+p_3)\tilde{k})+\cos(\varepsilon(p_1+p_4)\tilde{k})\\
&\qquad-\cos(\varepsilon p_1\tilde{k})+\cos(\varepsilon p_2\tilde{k})+\cos(\varepsilon p_3\tilde{k})+\cos(\varepsilon p_4\tilde{k})+1\Big)\\
&-\sin\left(\varepsilon\frac{p_3\tilde{p}_2}{2}\right)\cos\left(\varepsilon\frac{p_4\tilde{p}_2}{2}\right)\Big(-\sin(\varepsilon(p_1+p_2)\tilde{k})+\sin(\varepsilon(p_1+p_3)\tilde{k})\\
&\qquad-\sin(\varepsilon(p_1+p_4)\tilde{k})+\sin(\varepsilon p_1\tilde{k})+\sin(\varepsilon p_2\tilde{k})-\sin(\varepsilon p_3\tilde{k})+\sin(\varepsilon p_4\tilde{k})\Big)\\
&-\cos\left(\varepsilon\frac{p_3\tilde{p}_2}{2}\right)\sin\left(\varepsilon\frac{p_4\tilde{p}_2}{2}\right)\Big(-\sin(\varepsilon(p_1+p_2)\tilde{k})-\sin(\varepsilon(p_1+p_3)\tilde{k})\\
&\qquad+\sin(\varepsilon(p_1+p_4)\tilde{k})+\sin(\varepsilon p_1\tilde{k})+\sin(\varepsilon p_2\tilde{k})+\sin(\varepsilon p_3\tilde{k})-\sin(\varepsilon p_4\tilde{k})\Big)\\
&+\sin\left(\varepsilon\frac{p_3\tilde{p}_2}{2}\right)\sin\left(\varepsilon\frac{p_4\tilde{p}_2}{2}\right)\Big(-\cos(\varepsilon(p_1+p_2)\tilde{k})+\cos(\varepsilon(p_1+p_3)\tilde{k})\\
&\qquad+\cos(\varepsilon(p_1+p_4)\tilde{k})+\cos(\varepsilon p_1\tilde{k})+\cos(\varepsilon p_2\tilde{k})-\cos(\varepsilon p_3\tilde{k})\\
&\qquad-\cos(\varepsilon p_4\tilde{k})-1\Big)\Big].
\end{aligned}
\tag{E.3}
$$

More generally, the simplification of such products of trigonometric functions may be achieved by using the software *Mathematica*®. Without entering into technical details, the overall way to get there is the following:

- Application of the built-in function `TrigReduce[...]` to the entire expression allows to rewrite a product of trigonometric functions as a sum of single trigonometric functions, which depend on sums of the previous arguments.
- By application of `TrigExpand[...]` we again get products of trigonometric functions, yet this time only depending on two momenta (Note that the original expression may contain arguments of the form $(p_1\tilde{p}_2 + p_3\tilde{k})$).
- In the expression evaluated before, we apply the following procedure to each summand: first, separation of the functions depending on external momenta only, from those depending also on internal momenta. Then, application of `TrigReduce[...]` to the part depending on k. Finally, multiplication with the external part separated before.

By this simple algorithm, one is able to simplify the calculation of integrals depending on a product of trigonometric functions.

E.2 Integrals

In addition to the integrals of Section A.2.3, in the course of one-loop corrections of the vertices we encounter also integrands with an odd power in k. Hence, by generalizing the definition (A.14) of $I(m,n)$ to arbitrary trigonometric functions we get:

- $I(1,2)$
$$\int d^4k \frac{k_\mu}{k^4} \sin(k\tilde{p}) = \frac{2\pi^2 \tilde{p}_\mu}{\tilde{p}^2} \tag{E.4}$$
- $I(3,3)$
$$\int d^4k \frac{k_\mu k_\nu k_\rho}{k^6} \sin(k\tilde{p}) = \pi^2 \frac{\tilde{p}_\mu \tilde{p}_\nu \tilde{p}_\rho}{\tilde{p}^4} - \frac{\pi^2}{2} \frac{\delta_{\mu\nu}\tilde{p}_\rho + \delta_{\mu\rho}\tilde{p}_\nu + \delta_{\nu\rho}\tilde{p}_\mu}{\tilde{p}^2} \tag{E.5}$$
- $I(5,3)$
$$\int d^4k \frac{k_\mu k_\nu k_\rho k_\sigma k_\epsilon}{k^8} \sin(k\tilde{p}) = \frac{\pi^2}{12} \left\{ - \Big((1) \Big) \frac{1}{\tilde{k}^2} + 2 \Big((2) \Big) \frac{1}{\tilde{k}^4} - 8 \frac{\tilde{k}_\mu \tilde{k}_\nu \tilde{k}_\rho \tilde{k}_\sigma \tilde{k}_\epsilon}{\tilde{k}^6} \right\}, \tag{E.6a}$$
$$\begin{aligned}(1) \equiv \Big(& \delta_{\mu\nu}\delta_{\rho\sigma}\tilde{k}_\epsilon + \delta_{\mu\rho}\delta_{\nu\sigma}\tilde{k}_\epsilon + \delta_{\rho\nu}\delta_{\mu\sigma}\tilde{k}_\epsilon + \delta_{\mu\nu}\delta_{\rho\epsilon}\tilde{k}_\sigma + \delta_{\mu\nu}\delta_{\sigma\epsilon}\tilde{k}_\rho + \delta_{\mu\rho}\delta_{\nu\epsilon}\tilde{k}_\sigma \\ & + \delta_{\mu\rho}\delta_{\epsilon\sigma}\tilde{k}_\nu + \delta_{\rho\nu}\delta_{\mu\epsilon}\tilde{k}_\sigma + \delta_{\rho\nu}\delta_{\sigma\epsilon}\tilde{k}_\mu + \delta_{\mu\sigma}\delta_{\nu\epsilon}\tilde{k}_\rho + \delta_{\mu\sigma}\delta_{\rho\epsilon}\tilde{k}_\nu + \delta_{\nu\sigma}\delta_{\mu\epsilon}\tilde{k}_\rho \\ & + \delta_{\nu\sigma}\delta_{\rho\epsilon}\tilde{k}_\mu + \delta_{\rho\sigma}\delta_{\mu\epsilon}\tilde{k}_\nu + \delta_{\rho\sigma}\delta_{\nu\epsilon}\tilde{k}_\mu \Big),\end{aligned} \tag{E.6b}$$

$$
\begin{aligned}
(2) \equiv \Big(& \delta_{\mu\nu}\tilde{k}_\rho\tilde{k}_\sigma\tilde{k}_\epsilon + \delta_{\mu\rho}\tilde{k}_\nu\tilde{k}_\sigma\tilde{k}_\epsilon + \delta_{\rho\nu}\tilde{k}_\mu\tilde{k}_\sigma\tilde{k}_\epsilon + \delta_{\mu\sigma}\tilde{k}_\nu\tilde{k}_\rho\tilde{k}_\epsilon + \\
& \delta_{\nu\sigma}\tilde{k}_\mu\tilde{k}_\rho\tilde{k}_\epsilon + \delta_{\rho\sigma}\tilde{k}_\mu\tilde{k}_\nu\tilde{k}_\epsilon + \delta_{\mu\epsilon}\tilde{k}_\nu\tilde{k}_\rho\tilde{k}_\sigma + \delta_{\nu\epsilon}\tilde{k}_\mu\tilde{k}_\rho\tilde{k}_\sigma \\
& + \delta_{\rho\epsilon}\tilde{k}_\mu\tilde{k}_\nu\tilde{k}_\sigma + \delta_{\sigma\epsilon}\tilde{k}_\mu\tilde{k}_\nu\tilde{k}_\rho \Big) \,.
\end{aligned}
\tag{E.6c}
$$

Appendix F

Mathematica algorithms

The main part of the results presented in this thesis, especially the very involved calculations for the explicit evaluation of the Feynman graphs, has been obtained with $Mathematica^{®}$ routines, which have been developed and continuously refined for this purpose. The intention of this chapter is not to provide a full description of the (rather lengthy) source code, nor to present a self-contained manual for its use. Instead, an overview of the developed packages with focus on the main ideas leading to the algorithms, which finally allowed to handle a given problem by the $Mathematica^{®}$ software tool will be presented, followed by the programmed functions based on them. For this purpose, also built in $Mathematica^{®}$ functions and source code will be presented where useful.
This chapter contains two sections. In Section F.1 we will present as an example the algorithms for the calculation of Feynman diagrams for the model of Chapter 5, their adaption to the other models being obvious. In Section F.2 we will then briefly discuss the algorithm for the evaluation of counterterms in the framework of Algebraic Renormalization, which eventually led to the detection of an additional counterterm for an axially gauged commutative QED, which was missing in the respective publication [57], as mentioned in Section 6.2.2.

F.1 One loop calculations

In order to obtain the explicit results for the various loop calculations of this thesis, several packages have been developed. From Fig. F.1 one can nicely see the connection between the (high level) algorithmic steps (displayed on the left with respect to the vertical dashed line), and the respective packages allowing to conduct the calculations in an automated way, shown on the right hand side. The package `VariationalCalc.m` allows to perform functional variation, partial integration and partial differentiation, for simple expressions as well as for a given action. `All1LoopGraphs.m` allows to identify all one-loop graphs with an arbitrary number of external legs and number of involved

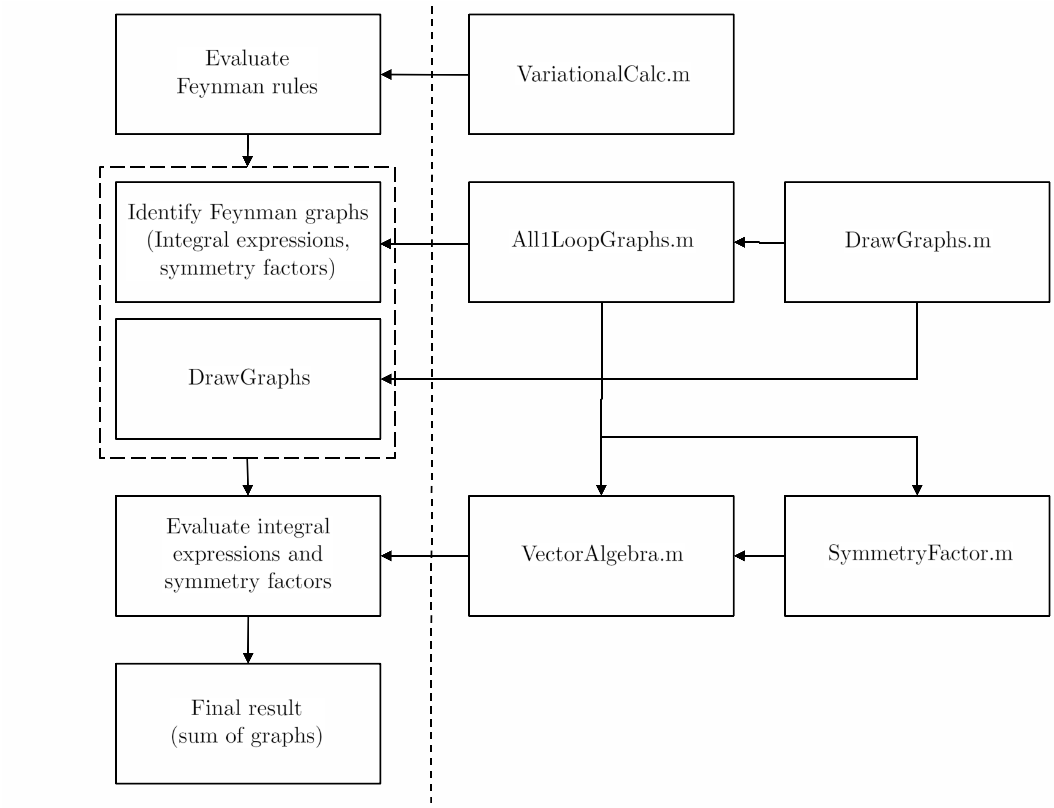


Figure F.1: Interplay between the single steps for loop calculations (displayed on the left) and corresponding *Mathematica*® packages (on the r.h.s.).

vertices, their power counting degree of divergence and to define the corresponding unevaluated integrals. Furthermore, the argument for the evaluation of the combinatoric factor is evaluated. The latter serves then as input for a function performing the explicit calculation of the combinatoric factors based on Wick contractions, which is defined in the package `SymmetryFactor.m`, whereas the explicit calculation of the Feynman integrals and their sum is done via the package `VectorAlgebra.m`. Finally, `DrawGraphs.m` allows to draw and label any of the formerly identified graphs.

In the present section we will focus explicitly on the packages of the second line in Fig. F.1 (`All1LoopGraphs.m` and `DrawGraphs.m`). We will first provide a short overview of the programmed functions, which have been applied in our daily work. Then we will go over to a deep dive into the basic ideas of the algorithms and provide some source code, where necessary for the understanding. For the other packages mentioned before this information can be found in [127]. First of all, in order to allow a straightforward discussion, let us first provide some basic definitions:

- *Configuration of a graph:* It is defined by the number of involved vertices of each kind, which by itself is uniquely characterized by the number of legs. It is displayed

as {`No.of 3V, No.of 4V, No.of 5V, No.of 6V`}.
Example: The configuration {1,1,0,0} denotes all graphs involving one vertex with three legs and one with four legs.

- *Representation of Feynman graphs as a nested list:* In order to operate with Feynman rules and the resulting graphs, to each field of the theory a numeric and a alphanumeric value have been assigned, c.f. Tab. F.1. Based on the theses definitions one can now define the propagators and vertices of the theory, i.e. the vertex $\widetilde{V}^{\bar{B}BA}_{\mu\nu,\rho\sigma\epsilon}$ of Section D.3 is defined by {`-2,2,0`}. Note that the fields are listed in clockwise order. Now it is easy to construct a graph by collecting the appropriate vertices, which gives a nested list. For example, the graph of Fig. B.3b would be given by {{`0,0,0`},{`0,0,2`}} or equivalently {{`A,A,A`},{`A,A,B`}}. Consider that for each vertex, the first and last elements represent the inner legs, i.e. they will connect to inner propagators, while all other legs are amputated and external. Furthermore, they follow a clockwise order.

- *Feynman rules*: they are written as functions of momenta and indices,

$$< \mathtt{type} >< \mathtt{field1} > ... < \mathtt{fieldn} > [\mathtt{p1}, ..., \mathtt{pn}, \mathtt{ind1}, ..., \mathtt{indn}], \tag{F.1}$$

  where `<type>` is given by `g` for propagators and `v` for vertices. By the definition of such a function, i.e. for the gauge boson propagator (5.32c)

$$\mathtt{gAA}[\mathtt{k\_}, \mu\_, \nu\_] := \mathtt{1}/(\mathtt{k}^2 + \mathtt{a}'/(\theta^2\mathtt{k}^2))(\delta_{\mu,\nu} - \mathtt{k}_\mu\mathtt{k}_\nu/\mathtt{k}^2); \tag{F.2}$$

  one deduces the Feynman integrals, which is explained next.

- *Feynman expressions:* For each Feynman graph, the integrand is obtained by multiplying the respective Feynman rules, i.e. vertices and propagators, with the correct arguments of momenta and indices, which express which legs are connected and how they are connected. In general, the below defined functions will provide the integrand as a product of expressions of the l.h.s. form of (F.2). Loading the full definitions given by e.g. (F.2) will then lead to the explicit integrand. Note that each expression is accompanied by a label allowing its unique identification by encoding the configuration of the graph combined with an arbitrary letter.

- *Combinatoric factors:* For each graph, the appropriate combinatoric prefactor has to be evaluated. The package `All1LoopGraphs.m` contains a function allowing to generate the correct syntax, i.e. expressions of the form `SymmetryFactor[...]`. Then, by loading the package `SymmetryFactor.m` and evaluation of this expression one finally gets the numeric factor. Note that in this chapter we consider the first package only. Hence, in this context "combinatoric factor" refers to the input form and not to the final numeric factor.

F.1.1 Programmed functions

Note again that all functions are explicitly designed for one-loop calculations. The following descriptions mainly reproduce the content of the help files, which have been written for each function.

GetAll1LoopGraphs[...]

The syntax is given by

```
GetAll1LoopGraphs[{# of 3V, # of 4V, # of 5V, # of 6V},
       [PrintOptionTable], [PrintOptionGraphs],[OptionDivergence]]
```

The first argument specifies the graph configuration to be considered. Then the following information will be provided:

1. If `PrintOptionTable` is set to `1`, a table will be printed with 3 columns and $n+1$ lines (the number of graphs plus the header):
   - The first column contains the graph represented as nested list, c.f. the definitions above.
   - The second column contains the degree of divergence for each graph.
   - The third column contains the integrand for the respective Feynman graph, or more precisely, the product of the Feynman rules with the correct arguments (indices, momenta). Furthermore, each expression is multiplied by a prefactor allowing the unique identification of the graph.

   For `PrintOptionTable=0` it will be obmitted. Default is `1`.
2. The option `PrintOptionGraphs` allows to print the Feynman diagrams for `1`, or to ommit their printing by setting it to `0`.
3. Finally, `OptionDivergence` specifies whether all graphs (for `1`) or only the divergent graphs (for `0`) will be considered.

AllConfigurations[...]

The function is given by

```
AllConfigurations[#V(min), #V(max)]
```

where `#V(min)`, `#V(max)` indicate the minimum respectively maximum number of vertices to be considered. The function returns a list with two columns. The first column contains all possible graph configurations which are constructed by combining at least `V(min)` and at most `V(max)` vertices, and which are displayed in the form of the first

argument of the function `GetAll1LoopGraphs`. The second column contains the number of divergent graphs for each configuration.
This function allows to determine explicitly, which configurations of vertices lead to divergent 1 loop graphs, and to know their number. It has been used e.g. for the construction of Tab. 5.3, which contains a few of the lines which will be returned by the choice `{#V(min), #V(max)}={1,2}`. By determining the length of the output of `AllConfigurations[i,i]` with $i = 1 \dots 5$ one arrives to Tab. 5.2.
Example: AllConfiguration[1,4] returns a list with e.g. the following configurations: Four 3-vertices {4,0,0,0}, two 3-vertices and two 4-vertices {2,2,0,0}, one 3-vertex and two 4-vertices {1,2,0,0}, ...

GetAllPropGraphs[...] and GetAllGaugePropGraphs[]

The function

```
GetAllPropGraphs[{field1,field2},[PrintOptionGraphs], [OptionDivergence]]
```

returns (for `OptionDivergence=1` all, if set to `0` only the divergent) one loop corrections to the propagator specified by the two (amputated) external legs `{field1,field2}`. It returns a list with 3 columns with the graph represented as nested list, the degree of divergence and the Feynman integrand, as for `GetAll1LoopGraphs`. Furthermore, a list with two columns will be displayed, containing the identification label of the graph and the input form for the function `SymmetryFactor[...]` (to be evaluated by loading the package `SymmetryFactor.m`). Finally the argument `PrintOptionGraphs` allows to print the graphs ("`1`") or to suppress it ("`0`").
Due to the frequently calculation of the vacuum polarization, the function (without arguments) `GetAllGaugePropGraphs[]:=GetAllPropGraphs[A,A,1,1]` has furthermore been defined.

SymFacExpr[...]

```
SymFacExpr[<nested list encoding graph>]
```

The input is given by the Feynman graph written as nested list, c.f. definitions above. It delivers as output the expression `SymmetryFactor[<arguments>]`, where `<arguments>` denotes the arguments of the function following the right syntax, which is required by the package `SymmetryFactor.m` (c.f. explanations below). The final symmetry factor is obtained after loading the respective package and evaluating this expression.

DrawFeynGraph[. . .]

```
DrawFeynGraph[<nested list encoding graph>,[Label]]
```

allows to get the Feynman diagram for a given integral, which optionally can be tagged with a label.

ExportGraphArray[. . .]

The function

```
ExportGraphArray[GraphsList, [iMaxRow], [Path], [bDisplay]]
```

exports the list of graphics elements `GraphsList` to the file given by the optional argument `Path`. If `Path` is not given it is automatically set to the variable `DefaultPath` which has to be set externally. The optional parameter `iMaxRow` defines the number of elements of `GraphsList` aligned in each row, the default value is 5. The optional boolean (True or False) `bDisplay` defines if the resulting table will also be printed to screen (`1`=print+export, `0`=export only).

F.1.2 Algorithms

All functions described in the foregoing section are based on a given combination of the following algorithms. Note that they are presented in a thematical order, which does not reflect how they are arranged in the source code of the packages. Furthermore, all but the last one (defined in `DrawGraphs.m`) are contained in `All1LoopGraphs.m`.

Evaluation of all graphs with a given number of vertices for each type

This is being done by a internal function `GetAllGraphsInternal[{No.of 3V, No.of 4V, No.of 5V, No.of 6V}]`, where the input is a graph configuration. First, for each

Field	A_μ	$c, \bar{c}$	$B_{\mu\nu}$	$\bar{B}_{\mu\nu}$	$\psi_{\mu\nu}$	$\bar{\psi}_{\mu\nu}$
Unicode	`A`	`c`	`B`	`bB`	`P`	`pP`
Assigned $g_\sharp$	0	1	2	-2	3	-3

Table F.1: Designations for the fields.

of the fields, a abbreviation (in *Unicode*) and a respective integer have been assigned (c.f. Tab. F.1), followed by the definition of the propagators and vertices, which have been classified according the number of legs. This latter allows housekeeping of all vertices when adapting the package to a new model:

```
Props={{0,0},{1,1},{0,2},{2,0},{0,-2},{-2,0},{-3,3},{3,-3},{-2,2},
{2,-2},{2,2},{-2,-2}};
ThreeVertBase={{0,0,0},{1,0,1},{2,0,0},{-2,0,0},{-2,2,0},{-3,3,0}};
FourVertBase={{0,0,0,0},{-2,2,0,0},{-3,3,0,0}};
FiveVertBase={{-2,2,0,0,0},{-3,3,0,0,0}};
SixVertBase={{-2,2,0,0,0,0},{-3,3,0,0,0,0}};
```

Next, these lists are extended by adding all cyclic permutations of all of its elements. Based on the latter, all *formally* possible graphs are obtained by simply constructing all combinations of vertices leading to the configuration specified by the input, and may be collected in the list called e.g. `AllGraphs`. For example, the input `{2,0,0,0}` defines all one-loop graphs involving two vertices with 3 legs, i.e. the one loop corrections to the various propagators. Then the corresponding set of all possible graphs is obtained by combining two elements of the extended list of `ThreeVertBase` in all possible ways. This procedure leads to a list of Feynman graphs defined as nested lists. A more general example is given by `{0,2,0,0}`, leading to the graphs defined by

$$\{\{\mathtt{V1}[1], \mathtt{V1}[2], \mathtt{V1}[3], \mathtt{V1}[4]\}, \{\mathtt{V2}[1], \mathtt{V2}[2], \mathtt{V2}[3], \mathtt{V2}[4]\}\}; \tag{F.3}$$

The corresponding vertices are displayed in Fig. F.2. There one can also see the conventions which are valid for all graphs: for all vertices of a graph, the last field (e.g. `V1[4]`) combines with the first field of the following vertex (e.g. `V2[1]`) to a inner propagator (i.e. one belonging to the loop). For the last vertex (in the example of Fig. F.2 `V[2]`), the following vertex is given by the first one (`V1`). It follows that for each vertex, the first and last elements (`V1[1],V1[4],V2[1],V2[4]`) represent the inner legs, while all other legs are external. Next, we have to consider that `AllGraphs` might contain graphs

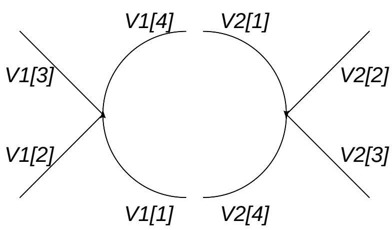


Figure F.2: Graphical representation of a nested list representing a Feynman graph

with with inner propagators not belonging to the list `Props`, i.e. which do not exist for the given model. Those graphs have to be excluded. This can be implemented by the following code line:

$$\begin{aligned}\mathtt{If}[\mathtt{MemberQ}[\mathtt{Props}, \{\mathtt{AllGraphs}[[\mathtt{i}, \mathtt{j}, \mathtt{Length}[\mathtt{AllGraphs}[[\mathtt{i}, \mathtt{j}]]]]],\\ \mathtt{AllGraphs}[[\mathtt{i}, \mathtt{j}+1, 1]]\}]];\end{aligned} \tag{F.4}$$

Here `i` runs over all graphs and `j` over all vertices (If `j==Length[AllGraphs[[i]]]`, than `j+1` has to be replaced by `1`). Finally, one gets all *physically* possible graphs by identifying all graphs being identically under cyclic permutations of the vertices, and by keeping just one of them.

Determination of the degree of divergence

In order to determine the superficial degree of divergence for a given graph contained in the set `AllGraphs`, one performs simple power counting. In the model under consideration, the degree of divergence for the graph `i` is then given by

$$\texttt{DivergenceList}[[\texttt{i}]] = 4 - 2 * \texttt{ccCounter} + ... + \texttt{AAACounter} + 2 * \texttt{bBBAACounter}; \tag{F.5}$$

after having counted the number of involved Feynman rules of each type. For the triple-A vertex this is achieved by e.g.

$$\texttt{If}[\texttt{MemberQ}[\{\{0,0,0\}\}, \texttt{AllGraphs}[[\texttt{i},\texttt{j}]]], \texttt{AAACounter}++]; \tag{F.6}$$

where `i,j` are loop variables running over all graphs and vertices. Remember that `AAACounter` counts the number of vertices with three gauge boson lines, which are encoded by {`0,0,0`}.

Determination of Feynman expressions

As discussed at the beginning of this section, the Feynman expressions are obtained by the following steps:

1. Definition of the functions for the Feynman rules of the form `gAA[<momenta, indices>]` for all vertices and propagators of the graph, with the appropriate fields, momenta and indices.
2. Multiplication of the Feynman rules.
3. Loading the definitions of the functions, which replaces the functions by their corresponding values, i.e. the explicit expressions for the Feynman rules.
4. Performing index contractions.
5. Evaluation of the integral.

Due to the technicality of this task, at this place we will not enter into details of the source code. However, the basic ideas can be grasped by looking at the simple example of the graph {{`A,B,A`},{`A,A,A`}}. First let us give some conventions: external momenta are denoted by p, internal momenta by k; momenta oriented towards the vertex are counted positive, otherwise negative; momentum conservation at each vertex has to be considered. This leads to Fig. F.3.

When programming the routine for the definition of the Feynman rules, one has to consider the following:

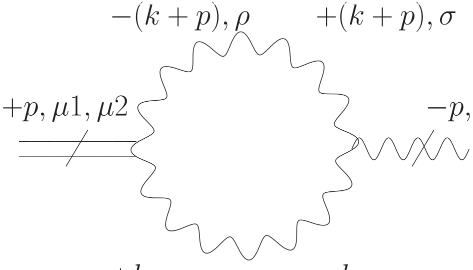


Figure F.3: Momenta and indices for a one loop graph with two external legs

- Fields requiring one index (e.g. A_μ) have to be distinguished from those with two or more indices (e.g. $B_{\mu\nu}$).
- If the functions for the Feynman rules have been defined for the elements of the "base" lists (`ThreeVertBase` etc.) then the presence of vertices obtained by cyclic permutations must be brought to the appropriate form. In the present example this would apply to `A,B,A` where we have to write `vBAA[p,-(p+k),k,m,n,r,t]` instead of `vABA[k,p,-(p+k),t,m,n,r]`.

The final product of Feynman function is given by

$$\mathtt{vBAA}[\mathtt{p}, -(\mathtt{p}+\mathtt{k}), \mathtt{k}, \mathtt{m}, \mathtt{n}, \mathtt{r}, \mathtt{t}] * \mathtt{gAA}[\mathtt{p}+\mathtt{k}, \rho, \sigma] * \mathtt{vAAA}[\mathtt{k}+\mathtt{p}, -\mathtt{p}, -\mathtt{k}), \sigma, \nu, \epsilon] * \mathtt{gAA}[\mathtt{k}, \epsilon, \tau]; \tag{F.7}$$

Loading the corresponding definitions leads to the integrand of the expression for the diagram in Fig. F.3. In order to get the integrated results, the next operations consist in performing the index contractions and solving the integral, which are defined in the package `VectorAlgebra.m`.

Symmetry factors

The purpose of the algorithm is to generate expressions of the form

$$\begin{aligned}&\mathtt{SymmetryFactor}[\{\mathtt{E1...En}\}, \{\{\mathtt{V11}, \mathtt{V12}, \ldots, \mathtt{V1m1}\}, ..., \{\mathtt{Vk1}, \mathtt{Vk2}, ...\mathtt{Vkmk}\}\},\\ &\qquad\{\{\mathtt{F11}, \mathtt{F12}, \mathtt{V1s}, \mathtt{V1e}\}, ..., \{\mathtt{Fj1}, \mathtt{Fj2}, \mathtt{Vjs}, \mathtt{Vje}\}\}];\end{aligned} \tag{F.8}$$

which, after loading the package `SymmetryFactors.m` allows to compute the symmetry factor for the specified graph topology based on Wick contractions as described in Appendix B. In order to specify the topology, an integer is assigned to each vertex, the last numbers being assigned to the external vertices. Then the first argument specifies the external fields, the second is the graph written as nested list. The third argument contains the various propagators and the numbering of the vertices which they connect. For our example the corresponding expression is given by

$$\begin{aligned}\texttt{SymmetryFactor}[\{\texttt{B},\texttt{A}\},\{\{\texttt{A},\texttt{B},\texttt{A}\},\{\texttt{A},\texttt{A},\texttt{A}\}\},\{\{\texttt{B},\texttt{B},3,1\},\\ \{\texttt{A},\texttt{A},4,2\},\{\texttt{A},\texttt{A},1,2\},\{\texttt{A},\texttt{A},2,1\}\}]; \end{aligned} \tag{F.9}$$

The numbering of the vertices can be seen in Fig. F.4. In *Mathematica*® terms, the

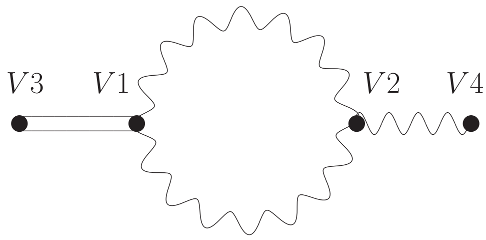


Figure F.4: Numbering of the vertices for the definition of the function `SymmetryFactor[...]`.

correct expressions are obtained by combining the information encoded in the nested list for a graph in the appropriate way. Once again, due to the pure technicality we will not enter in detail into the source code.

Drawing Feynman diagrams

The routine allowing for the graphical representation of the Feynman graphs is implemented in the package `DrawGraphs.m`. Its core definitions are contained in the internal function `FeynGraphInternal[g_,label_]`, its arguments being a given graph `g` given as nested list and a optional label. For all configurations and all fields of the theory it contains a corresponding set of lines that may be printed graphically. In other words, in order to allow the graphical representation of a graph belonging to a given configuration, for each of its elements and all possible values, a graphical element has to be defined. For the example of the configuration `{0,2,0,0}` the set of lines is given by

```
If[GraphConfig=={0,2,0,0},
   ALines={ALines[[1]],...,ALines[[6]]};
   cLines={...};
   ...
];
```

and e.g. the line `ALines[[1]]` is given by

```
{Translate[PolarPlot[{1+0.06 Sin[21t]},{t,Pi,3/2Pi},
PlotStyle->Directive[Thickness[.003],Black],Axes->None][[1]],{1,0}]};
```

Elements	Mass dimension	# indices	Charge conjugation
A_μ	1	1	$-A_\mu$
n_μ	0	1	n_μ
∂_μ	1	1	∂_μ
b	3	0	-b

Table F.2: Elements entering the counterterms generated by Algebraic Renormlaization.

Now, for a given graph, the routine first checks its configuration, and collects all relevant lines to form a list, which then will be accessed and printed by any of the functions making use of it.

F.2 Algebraic Renormalization

This section contains a brief description of the algorithm, which was implemented in *Mathematica*® in order to evaluate the non-invariant counterterms (no anomalies) of an axially gauged commutative $U(1)$ theory. Originally thought to be a finger exercise before applying the algorithm to the non-commutative model of Chapter 5 ([57] contains a listing of the counterterms and hence allows to check the results), the latter motivation disappeared due to the problems mentioned in Section 6.2.2. As a side-effect, the additional term (6.10) which is missing in [57] has been detected.

The gauge part of the Langrangian for the commutative axially gauged fixed QED on Minkowski space $\mathbb{M}^4$ is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + b n^\mu A_\mu \,. \tag{F.10}$$

According to the discussion in Section 6.2.2, the non-invariant counterterms will be obtained by first constructing all monomials allowed by considering the mass dimension and index structure, followed by application of the symmetries: in the present case the gauge Ward identity $\mathcal{WI}$, the gauge condition and charge conjugation. In other words, those monomials which survive the application of the Ward identities are non-invariant countertems to be included in the action.

It follows the cooking recipe, which has been turned into *Mathematica*® source code, leading to all non-invariant counterterms:

1. *Construction of all monomials with mass dimension 4*: the elements allowing to construct the monomials together with their mass dimension and number of indices as well as their behaviour under charge conjugation are listed in Tab. F.2. In a first step, only monomials without the massless axial gauge vector are constructed. This is done by

$$\texttt{Monomials} = \texttt{Tuples}[\texttt{AllFields}, \texttt{i}]; \text{ where } \texttt{AllFields} = \{\texttt{A}, \texttt{d}, \texttt{b}\}\,, \tag{F.11}$$

and `i={2,3,4}`. It suffices to consider only monomials with at least two and at most four elements due to considerations regarding mass dimension. Notice furthermore, that the index structure of the fields has not yet been considered. Then, from this set of monomials all those with mass dimension unequal to four are excluded. Furthermore are excluded those monomials with a derivation as last element. It remains the set of all possible combinations (inlcuding permutations) of `AllFields`.

2. *Insertion of the axial gauge vector* n_μ: For a given monomial with n fields carrying an index, the gauge vector will be inserted $i = n, n-2, n-4, \ldots$ times, as long as i is positive (for n odd) or zero (for n even). More precisely, the insertion follows three rules: first, the n_μ have to be separated by an element of `AllFields`. Second, the inserted axial vector is contracted with the consecutive element of the monomial. Third, for a given i, all possible positions (combinations of positions for $i > 0$) respecting the previously defined rules have to be realized.
Finally, the remaining fields will be combined to form scalars, as required by the fact that the action is a scalar. As an example, let us consider the term $A\partial^2 A$. This will lead to

$$\{A\partial^2 A,\, (nA)(n\partial)(\partial A),\, \partial^2(nA)^2,\, (n\partial)^2 A^2, (n\partial)^2(nA)^2\} + \text{permutations}\,. \quad \text{(F.12)}$$

3. *Exclusion of multiples and non-allowed terms* Now, after having saturated all indices, multiples must be excluded. Monomials can be equivalent with respect to permutations, but also with respect to integration (i.e. $(\partial A)^2 \equiv -A_\mu \partial^2 A^\mu$), remembering that they are integrands. Furthermore, all monomials violating the symmetry under charge conjugation must be eliminated.

4. *Application of Ward identities* First one defines the $\mathcal{WI}$ and the gauge condition as functions, i.e.

$$\begin{aligned}&\texttt{WI[Counterterm\_]} := -\texttt{PD[VarD[Counterterm,A,}\mu\texttt{],}\mu\texttt{]};\\ &\texttt{GC[Counterterm\_]} := \texttt{VarD[Counterterm,b]};\end{aligned} \quad \text{(F.13)}$$

with `PD[argument,index]` a partial differentiation and `VarD[argument, field]` the variation of the argument with respect to `field`. Both functions are defined in the package `VariationalCalc.m`. Having done so, they are applied to the set of monomials.

Those monomials which “break” the symmetry expressed by the identities (F.13), i.e. which give a non-vanishing result remain to form the set of non-invariant counterterms and hence shall be - accompanied by numerical prefactors - included in the action. In the above example this considers all explicitly written monomials in (F.12). All but the second one, i.e. $(nA)(n\partial)(\partial A)$ can be found (up to partial integration and cyclic permutations) in [57], Eqns. (4.98) and (4.99).

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