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DISSERTATION

LIBOR Market Models - An Approximations Approach

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Preface

This dissertation is the result of $3\frac{1}{2}$ years of research at the Christian Doppler laboratory for portfolio risk management (CD-lab PRisMa) at the research group for financial and actuarial mathematics at the Vienna University of Technology.
We found LIBOR market models an especially interesting field concerning both theory and practice. For practitioners they are an essential tool for pricing interest rate derivatives. Concerning theory LIBOR models driven by jump processes have interesting dynamics. We see "approximations" of various types as essential for a better understanding of such models. Approximations to obtain pricing formulae. Approximations to get a better understanding of the dynamics . And finite dimensional approximations of infinite dimensional interest rate market models. Finding better approximations seems to be the key, to better understand those models and is mathematically challenging.
The structure of this work is as follows:

- Chapter 1 deals with the construction of LIBOR market models and provides an insight into several different approaches and resulting models. We are especially interested in the behavior of spot-LIBOR modeling, as we can show as a first original theorem, that this type of model allows an extension of a given tenor structure to larger one in a consistent way.

- Chapter 2 proceeds to deal with the question of how we can interpolate rates starting from a discrete tenor grid and thus build a continuous tenor model. We have extended a known method for the log normal case to our models with jumps but we go beyond even that, as we combine interpolation and extension of a given discrete model to give an existence result for LIBOR term structures. All of this is original work, though the interpolation for a log-normal forward LIBOR market model has been done in [15].

- Chapter 3 shows results of a practical implementation of a LIBOR market model, Laplace pricing methods for a frozen drift variant and possibilities of more sophisticated approximations. We also show some smiles generated by our models.

- Chapter 4 deals with another new idea: a completely discrete LIBOR analogon. We construct such a model. Show its properties and then give approximation results and we give proof of the convergence of a particular approximation from chapter 3 against a exact Lévy LIBOR model.

- Chapter 5 deals with measure change techniques necessary for chapter 4. The representation is original though the knowledge of the possibility of such a procedure is of course not.

- Chapter 6 compares our work in chapter 4 to the work of Glasserman and Zhao [8] on arbitrage free interpolation. We felt there should be some relation and indeed we found, that, after several calculations, one suggested arbitrage free interpolation coincides in the log normal case with the discrete log-normally driven model.

- The appendix provides important notions and theorems which are necessary to understand this work but certainly too specific to be considered general knowledge even in financial mathematics.

Finally I would like to thank all those people who supported me in one way or the other in those $3\frac{1}{2}$ exciting years. Especially I want to thank my advisor Friedrich Hubalek for helping me to find my way in the world of financial mathematics and supporting my various ideas and critically examining of the respective results, the staff of FAM, my family and all the dear people who make life so much more interesting and enjoyable.
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Furthermore the two reviewers of this dissertation Josef Teichmann and again Friedrich Hubalek deserve extra thanks for agreeing to review this work on such short notice.

Contents

Chapter 1

Interest Rate Modeling

1.1 LIBOR-Essentials

In this section, we will review the notions essential to understand LIBOR market models. Most definitions here are taken from the excellent book [4] by Brigo and Mercurio. We also construct a classic log-normal forward LIBOR model.
We then proceed to give the defintion of interest rate market models from the paper [3] and a very general formulation by Eberlein and Özkan ([6]).
Finally we show the existence of a semimartingale driven forward LIBOR model and discuss spot-LIBOR market models.

Our starting point is the bank account process.

Definition 1 (Bank Account Process) *We define $B(t)$ to be the value of a bank account at time $t \geq 0$. We assume $B(0) = 1$. The dynamics are given as*

$$dB(t) = r(t)B(t)dt \qquad B(0) = 1 \tag{1.1}$$

where $r(t)$ is a positive stochastic process.

As a consequence

$$B(t) = \exp\,(\int_0^t r(s)ds). \tag{1.2}$$

We therefore have, that investing 1 unit of currency at time 0 will yield $B(T)$ units of currency at time T.
From this observation we define

Definition 2 (Stochastic Discount Factor) *The stochastic discount factor $D(t,T)$ between two time instants t and T is the amount at time t that is "equivalent" to one unit of currency payable at time T, and is given by*

$$D(t,T) = \frac{B(t)}{B(T)} = \exp\,(-\int_t^T r(s)ds). \tag{1.3}$$

Now the traded objects are the so called zero-coupon bonds

Definition 3 (Zero-Coupon Bond) *A T-maturity zero-coupon bond (pure discount bond) is a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments. The contract value at time $t < T$ is denoted by $B(t,T)$.*

This implies

$$B(T,T) = 1 \qquad \forall T > 0. \tag{1.4}$$

Necessary to understand the conventions in the interest-rate market is the following

Definition 4 (Time To Maturity) *The time to maturity $T - t$ is the amount of time from the present time t to the maturity $T > t$.*

There are now several type of "interest rates" we can define. For instance

- The simple forward rate for $[S,T]$ contracted at t, henceforth referred to as the LIBOR forward rate(which we will be modeling).

$$L(t;S,T) := \frac{B(t,T) - B(t,S)}{(S-T)B(t,T)} \tag{1.5}$$

- The instantaneous forward rate with maturity T, contracted at t

$$f(t,T) = -\frac{\partial \log B(t,T)}{\partial T} \tag{1.6}$$

- The instantaneous forward rate is related to the short rate via

$$r(t) = f(t,t) \tag{1.7}$$

- Then there is the so called simply compounded spot interest rates defined by the formula

$$L_s(t,T) = \frac{1 - B(t,T)}{\tau(t,T)B(t,T)}. \tag{1.8}$$

  Here $\tau(t,T)$ is the number of days between t and T in a given day-count convention (see [4]).

Essential for LIBOR market models in practice is the concept of the tenor structure

Definition 5 (Tenor-Structure) *We choose an ordered set of values $0 = T_0 < T_1 < T_2 < \dots$ (often equidistant). This set of times is called the tenor-structure of our model.*
We speak of discrete tenor models if only LIBOR rates $L(t,T_i)$ with maturities on such a discrete grid are modeled.
We speak of a continuous tenor model if all rates with maturities in an interval $[0, T_{\max}]$ are modeled.

Definition 6 (LIBOR Forward Rates - Discrete Tenor) *We will only use the forward LIBOR-rates. We set*

$$L(t,T_i) := L(t;T_i,T_i+\delta_i) = \frac{1}{\delta_i}\left(\frac{B(t,T_i) - B(t,T_i+\delta_i)}{B(t,T_i+\delta_i)}\right) = \frac{1}{\delta_i}\left(\frac{B(t,T_i)}{B(t,T_i+\delta_i)} - 1\right) \tag{1.9}$$

where we assume a given time-grid $\{T_i\}_{i\in I}$, possibly only finite, for which it holds that $T_i + \delta_i = T_{i+1}$.

Definition 7 (LIBOR Forward Rates - Continuous Tenor) *We assume a fixed (for notational simplicity) $\delta > 0$. We set*

$$L(t,T) := L(t,T,T+\delta) = \frac{1}{\delta}(\frac{B(t,T) - B(t,T+\delta)}{B(t,T+\delta)}) = \frac{1}{\delta}(\frac{B(t,T)}{B(t,T+\delta)} - 1) \tag{1.10}$$

for all $T \in (0, T_{\max})$.

1.2 Measures And No-Arbitrage

Suppose we are given a filtered probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$, where the filtration $(\mathcal{F}_t)_t$ is the $\mathbb{P}$ augmentation of a certain driving semimartingale $(X_t)_t$.
The classic BGM model [3] is constructed under the risk neutral measure, defined by the choice of numeraire $B(t)$.

Definition 8 (Risk Neutral Measure) *The risk-neutral measure $\mathbb{P}^*$ is defined as the measure with numeraire $B(t)$(the Bank account defined above).*

For the development of LIBOR theory (especially for models with discrete tenor) the two concepts of "spot-measure" and "forward measure" are essential. To define them we need conditions on our setting

- For any date T the Bond-price is a positive special semimartingale in t where the process of left-limits is also strictly positive.

- for any fixed $T \leq T^*$ the forward process

  $$F_B(t,T,T^*) := \frac{B(t,T)}{B(t,T^*)}$$

  follows a martingale under $\mathbb{P}^*$, the risk neutral measure, or equivalently

  $$B(t,T) = \mathbb{E}_{\mathbb{P}}(\frac{B(t,T^*)}{B(T,T^*)}|\mathcal{F}_t).$$

Definition 9 (Forward Measure) *Let U be a fixed maturity date. A probability measure $\mathbb{P}_U$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_U)$ is called a forward martingale measure for the date U if for any maturity T the forward process $F_B(t,T,U)$ follows a local martingale(see appendix A for definition of a local martingale) under $\mathbb{P}_U$.*
This is certainly satisfied, if we choose as numeraire $B(t,T_U)$. Therefore, when we speak of "the" forward measure $\mathbb{P}_U$ we will always mean the measure defined by choosing as numeraire $B(t,T_U)$.

In comparison a spot measure is defined as

Definition 10 (Spot Measure) *A spot martingale measure is any probability measure $\mathbb{P}_s$ equivalent to the risk neutral measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F}_T^*)$ for which there exists a process $B \in \mathcal{A}^+$(see appendix C for the definition of $\mathcal{A}^+$) with $B(0) = 1$ and such that for any maturity T the bond price $B(t,T)$ satisfies*

$$B(t,T) = \mathbb{E}_{\mathbb{P}_s}(B(t)/B(T)|\mathcal{F}_t).$$

For instance, the bank account process, will be such a process in the continuous tenor framework of [3]. In that setting, the risk neutral measure is the spot measure.
As it is shown in [15], the spot-LIBOR measure introduced below, is a spot-measure by this definition in the discrete tenor framework.
Referring to the bond-market and the special situation of term-structure-modeling, we have two appropriate no-arbitrage notions.

Definition 11 *A family of bond prices $B(t, T)$ satisfies the weak no-arbitrage condition if and only if there exists a probability measure $\mathbb{Q}$ equivalent to the risk neutral measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F}_{T^*})$ such that for any maturity $T < T^*$ the forward process $F_B(t, T, T^*) = B(t, T)/B(t, T^*)$ belongs to $\mathcal{M}_{loc}(\mathbb{Q})$(see appendix A for definition of $\mathcal{M}_{loc}$).*

Definition 12 *The family $B(t, T)$ satisfies the no-arbitrage condition if in addition the inequality $B(T, U) \leq 1$ holds for any maturities $T, U \in [0, T^*]$ such that $T \leq U$.*

The models we treat will always fulfill the weak no-arbitrage condition. In fact, forward-modeling gives this property automatically.

Now the classic HJM interest rate models are usually specified under the risk neutral martingale measure $\mathbb{P}^*$ with numeraire $B(t)$ (the bank account process).
LIBOR Market Models in practice model the driving process of LIBOR-rate dynamics already under the so called terminal measure, thus assuming (in the classical models) log-normality of the LIBOR-rate-process.
This lognormality is essential if we want a model that prices the classical interest rate derivatives called caps (or more precisely the caplets) via Black's formula.

Definition 13 (Cap) *A cap is a contract on the LIBOR-rate which has the following present value at t*

$$\sum_{\alpha=1}^{\beta} D(t, T_i) N_{T_i} (L(T_{i-1}, T_i) - K)^+. \tag{1.11}$$

Where N_{T_i} is the nominal value. (which is 1 for 1 unit of currency).
As can be seen from this formula, a cap can be decomposed into a sum of European options(called caplets) with payoffs $D(t, T_i) N_{T_i} (L(T_{i-1}, T_i) - K)^+$ over disjoint time-intervals.

Thus if the LIBOR-rates were simultaneously (under a single measure) lognormal, those summands would each be priced by Black's formula and then summed up.

1.3 Log Normal Discrete Tenor Forward LIBOR Modeling

We have to restrict ourselves to a finite number of maturities and are therefore in a discrete tenor framework.

Definition 14 (Terminal Measure) *Given a finite tenor-Structure $0 = T_0 < T_1 < \cdots < T_n < T_{n+1} =: T^*$, we call the forward measure $\mathbb{P}_{T^*}$ associated to the last time $T_{n+1} =: T^*$, the terminal measure with respect to the tenor structure.*

We want to prove the following:

Theorem 1 *Given a volatility structure $\lambda(.,T_1),\dots,\lambda(.,T_n)$ where each $\lambda(.,T_i)$ is assumed to be bounded and the $\lambda(.,T_i)$ strictly increasing in i, a probability measure $\mathbb{P}_{T_{n+1}}$ and a standard $\mathbb{P}_{T_{n+1}}$-Wiener process W_t^{n+1}. Define the processes $L(t,T_1),\dots,L(t,T_n)$ by*

$$dL(t,T_i) = -L(t,T_i)\Big(\sum_{k=i+1}^{N}\frac{\delta_k L(t,T_k)}{1+\delta_k L(t,T_k)}\lambda(t,T_k)\Big)dt + L(t,T_i)\lambda(t,T_i)dW_t^{n+1} \tag{1.12}$$

for $i = 1,\dots,n$.
Then the $\mathbb{P}_{T_{i+1}}$-dynamics are given by

$$dL(t,T_i) = L(t,T_i)\lambda(t,T_i)dW_t^{i+1}. \tag{1.13}$$

Thus there exists a LIBOR model with the given volatility structure.

Proof:
A good derivation is in [15], which we will adapt to give our proof:
Given a finite tenor structure $\{T_i\}_{i=1}^{n+1}$, we model the LIBOR rate $L(t,T)$ under the terminal measure $\mathbb{P}_{T_{n+1}}$ as a log-normal 1-dimensional martingale. Therefore it fulfills

$$dL(t,T_n) = L(t,T_n)\lambda(t,T_n)dW_t^{n+1} \tag{1.14}$$

where W_t^{n+1} denotes standard Brownian Motion under $\mathbb{P}_{T_{n+1}}$. The initial condition is

$$L(0,T_n) = \frac{1}{\delta}\Big(\frac{B(0,T_n)}{B(0,T_{n+1})} - 1\Big). \tag{1.15}$$

We also have the dynamics of the forward rate process $F_B(t,T_n,T_{n+1})$ from the relation $1 + \delta_n L(t,T_n) = F_B(t,T_n,T_{n+1})$

$$dF_B(t,T_n,T_{n+1}) = F_B(t,T_n,T_{n+1})\frac{\delta_n L(t,T_n)}{1+\delta_n L(t,T_n)}\lambda(t,T_n)dW_t^{n+1}. \tag{1.16}$$

From the definition of the forward-measure $\mathbb{P}_{T_n}$ through the numeraire $\frac{B(t,T_n)}{B(0,T_n)}$, we have the property

$$\frac{d\mathbb{P}_{T_n}}{d\mathbb{P}_{T_{n+1}}} = \frac{B(0,T_{n+1})}{B(0,T_n)}\frac{B(t,T_n)}{B(t,T_{n+1})} = \frac{F_B(t,T_n,T_{n+1})}{F_B(0,T_n,T_{n+1})} \tag{1.17}$$

We want the measure change to be described by a stochastic exponential, which yields

$$\frac{F_B(t,T_n,T_{n+1})}{F_B(0,T_n,T_{n+1})} = \mathcal{E}\Big(\int_0^t \lambda(t,T_n)\frac{\delta_n L(s,T_n)}{1+\delta_n L(s,T_n)}dW_s^{n+1}\Big). \tag{1.18}$$

From this we can calculate standard Brownian Motion for $\mathbb{P}_{T_n}$ in terms of $\mathbb{P}_{T_{n+1}}$ and we get

$$W_t^n = -\int_0^t \frac{\delta_n L(s,T_{n+1})}{1+\delta_n L(s,T_{n+1})}\lambda(s,T_{n+1})ds + dW_t^{n+1} \tag{1.19}$$

and plug this into the equation for $L(t,T_{n-1})$ under $\mathbb{P}_{T_n}$(since it has to be a martingale under that measure)

$$dL(t,T_{n-1}) = L(t,T_{n-1})\lambda(t,T_{n-1})dW_t^n \tag{1.20}$$

to get the dynamics of $L(t, T_{n-1})$ under $\mathbb{P}_{T_{n+1}}$:

$$dL(t, T_{n-1}) = -L(t, T_{n-1})\lambda(t, T_{n-1})\frac{\delta_n L(t, T_n)}{1 + \delta_n L(t, T_n)}\lambda(t, T_n)dt + L(t, T_{n-1})\lambda(t, T_{n-1})dW_t^{n+1} \tag{1.21}$$

More generally, if we switch between two consecutive forward measures $\mathbb{P}_{T_i}$ and $\mathbb{P}_{T_{i+1}}$ we get

$$\frac{d\mathbb{P}_{T_i}}{d\mathbb{P}_{T_{i+1}}} = \frac{B(0, T_{i+1})}{B(0, T_i)}\frac{B(t, T_i)}{B(t, T_{i+1})} \tag{1.22}$$

$$= \frac{F_B(t, T_i, T_{i+1})}{F_B(0, T_i, T_{i+1})}$$

which will result in

$$\frac{F_B(t, T_i, T_{i+1})}{F_B(0, T_i, T_{i+1})} = \mathcal{E}\Big(\int_0^t \frac{\delta_i L(s, T_i)}{1 + \delta_i L(s, T_i)}\lambda(t, T_i)dW_s^{i+1}\Big). \tag{1.23}$$

From this we can calculate standard Brownian Motion for $\mathbb{P}_{T_i}$ in terms of $\mathbb{P}_{T_{i+1}}$ and we get

$$W_t^i = -\int_0^t \frac{\delta_i L(s, T_i)}{1 + \delta_i L(s, T_i)}\lambda(s, T_i)ds + dW_t^{i+1}. \tag{1.24}$$

Now for arbitrary i in relation to $n+1$, we have the property

$$\frac{d\mathbb{P}_{T_i}}{d\mathbb{P}_{T_{n+1}}} = \frac{B(0, T_{n+1})}{B(0, T_i)}\frac{B(t, T_i)}{B(t, T_{n+1})} = \frac{B(0, T_{n+1})}{B(0, T_n)}\frac{B(0, T_n)}{B(0, T_{n-1})}\cdots\frac{B(0, T_{i+1})}{B(0, T_i)} \tag{1.25}$$

$$\frac{B(t, T_n)}{B(t, T_{n+1})}\frac{B(t, T_{n-1})}{B(t, T_n)}\cdots\frac{B(t, T_i)}{B(t, T_{i+1})} = \prod_{j=i}^{n}\frac{F_B(t, T_j, T_{j+1})}{F_B(0, T_j, T_{j+1})}$$

We want the measure change to be described by a stochastic exponential, which yields

$$\prod_{j=i}^{n}\frac{F_B(t, T_j, T_{j+1})}{F_B(0, T_j, T_{j+1})} = \mathcal{E}\Big(\int_0^t \frac{\delta_i L(s, T_i)}{1 + \delta_i L(s, T_i)}dW_s^{n+1} - \sum_{j=i}^{n-1}\int_0^t \frac{\delta_i L(s, T_i)}{1 + \delta_i L(s, T_i)}\frac{\delta_{j+1} L(t, T_{j+1})}{1 + \delta_{j+1} L(t, T_{j+1})}\lambda(s, T_{j+1})ds\Big). \tag{1.26}$$

From this we can calculate standard Brownian Motion for $\mathbb{P}_{T_i}$ in terms of $\mathbb{P}_{T_{n+1}}$ and we get

$$dW_t^i = -\sum_{k=i+1}^{n}\frac{\delta_k L(t, T_k)}{1 + \delta_k L(t, T_k)}\lambda(t, T_k)dt + dW^{n+1}(t) \tag{1.27}$$

and plug this into the equation for $L(t, T_{i-1})$ under $\mathbb{P}_{T_i}$

$$dL(t, T_{i-1}) = L(t, T_{i-1})\lambda(t, T_{i-1})dW_t^i \tag{1.28}$$

to get the dynamics of $L(t, T_i)$ under $\mathbb{P}_{T_{n+1}}$ which is just the equation 1.12 from the theorem. □

There are now some standard choices for the form of the volatility functions

- For each $i = 1, \ldots, n$, assume that the corresponding volatility is constant in time, i.e. that

$$\lambda(t, T_i) = \lambda_i \in \mathbb{R}_+$$

for $0 \leq t \leq T_{i-1}$

- For each $i = 1, \dots, n$, assume that $\lambda(., T_i)$ is piece-wise constant, i.e. that

$$\lambda(t, T_i) = \lambda_{ij} \in \mathbb{R}_+, \qquad \text{for} \quad T_{j-1} < t \leq T_j, \quad j = 0, \dots, i-1$$

- As above but with the requirement that the volatility only depends on the number of resettlement dates left to maturity, i.e. that

$$\lambda(t, T_i) = \lambda_{ij} = \beta_{i-j} \in \mathbb{R}_+, \qquad \text{for} \quad T_{j-1} < t \leq T_j, \quad j = 0, \dots, i-1$$

  where β_i are fixed numbers.

- As above, but with the further specification that

$$\lambda(t, T_i) = \lambda_{ij} = \beta_i \gamma_j \qquad \text{for} \quad T_{j-1} < t \leq T_j, \quad j = 0, \dots, i-1$$

  where β_i and γ_j are fixed numbers.

- Assume some simple functional parametrized form of the volatilities such as for example

$$\lambda(t, T_i) = q_i(T_{i-1} - t)e^{\beta_i(T_{i-1}-t)}$$

  where $q_i(.)$is some real polynomial and β_i is a real number.

A problem we see immediately from equation 1.12 however is that under the terminal measure the rates will never be simultaneously lognormal, but(except for the first modeled rate) will always incorporate some stochastic "drift" term we denote by

$$\ell(t, T_i) := \frac{\delta_i L(t, T_i)}{1 + \delta_i L(t, T_i)} \tag{1.29}$$

In order to have the rates simultaneously log-normal under the terminal measure one often does "freeze" the drift, in other words replace $\ell(t, T_i)$ by its deterministic initial value $\ell(0, T_i)$.

1.4 Risk Neutral Modeling - Continuous Tenor Forward Rates

The original LIBOR Market Model constructed in the paper [3] starts from the HJM framework where the dynamics of the term structure of interest rates are modeled by an infinite dimensional SDE for the forward rates $f(t, T)$. We obviously are in a continuous tenor framework.
We assume all processes to be defined on a probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ where the filtration $\{\mathcal{F}_t | t \geq 0\}$ is the $\mathbb{P}$-augmentation of the natural filtration generated by a d-dimensional Brownian Motion $W = \{W(t) | t \geq 0\}$. In the original BGM-Paper the authors use the Musiela parametrization $x = T - t$. We will adopt this for the derivation of the result, but for comparability afterwards give the main SDE in standard parametrization.

1.4.1 Musiela Parametrization - Log-Normal Rates

We look at the process $r(t, x) := f(t, T - t)$ and assume that the process $\{r(t, x) | t, x \geq 0\}$ satisfies

$$dr(t, x) = \frac{\partial}{\partial x}\left(\left(r(t, x) + \frac{1}{2}|\sigma(t, x)|^2\right)dt + \sigma(t, x)dW(t)\right), \tag{1.30}$$

where for all $x \geq 0$ the volatility process $\{\sigma(t,x)|t \geq 0\}$ is $\mathcal{F}_t$ adapted with values in $\mathbb{R}^d$ while $|.|$ stands for the usual Euclidean Norm in $\mathbb{R}^d$. We assume that the function $x \to \sigma(t,x)$ is at least absolutely continuous and the derivative $\tau(t,x) = \frac{\partial}{\partial x}\sigma(t,x)$ is bounded on $\mathbb{R}^2_+ \times \Omega$. It follows then that

$$dD(t,x) = D(t,x)((r(t,0) - r(t,x))dt - \sigma(t,x)dW(t) \tag{1.31}$$

and hence $\sigma(t,x)$ can be interpreted as price volatility. Obviously we have $\sigma(t,0) = 0$.
The spot rate process $\{r(t,0)|t \geq 0\}$ satisfies

$$dr(t,0) = \frac{\partial}{\partial x} r(t,x)|_{x=0}dt + \frac{\partial}{\partial x}\sigma(t,x)|_{x=0}dW(t) \tag{1.32}$$

and hence is not Markov in general. The process

$$B(t) = \exp\Big(\int_0^t r(s,0)ds\Big), t \geq 0 \tag{1.33}$$

represents the amount generated at time $t \geq 0$ by continuously reinvesting 1 amount of currency in the spot rate $r(s,0)$, $0 \leq s \leq t$.
In order to have no-arbitrage between zero-coupon bonds $B(.,T)$ we need the process $B(t,T)/B(t)$ to be a martingale under $\mathbb{P}^*$ for all maturities T.
Given the dynamics of $r(t,x)$ we see that

$$\frac{B(t,T)}{B(t)} = B(0,T)\exp\Big(-\int_0^t \sigma(s,T-s)dW(s) - \frac{1}{2}\int_0^t |\sigma(s,T-s)|^2 ds\Big), \tag{1.34}$$

where the right hand side is a martingale. It also follows that

$$dB(t,T) = B(t,T)(r(t,0)dt - \sigma(t,T-t)dW(t)). \tag{1.35}$$

The LIBOR rate is then

$$1 + \delta L^*(t,x) = \exp\Big(\int_x^{x+\delta} r(t,u)du\Big). \tag{1.36}$$

and is assumed to have a lognormal volatility structure

$$dL^*(t,x) = \ldots dt + L^*(t,x)\gamma(t,x)dW(t) \tag{1.37}$$

where the deterministic function $\gamma : \mathbb{R}^2_+ \to \mathbb{R}^d$ is bounded and at least piecewise continuous.
This LIBOR rate relates to our standard notation $L^*(t,T)$ as

$$L^*(t,x) = L^*(t,t+x) \qquad \forall x \geq 0. \tag{1.38}$$

By applying Ito's formula we get, that the lognormal volatility structure can only hold for all $x \geq 0$ if and only if

$$\sigma(t,x+\delta) - \sigma(t,x) = \frac{\delta L^*(t,x)}{1+\delta L^*(t,x)}\gamma(t,x). \tag{1.39}$$

We have then, as equation for $L^*(t,x)$

$$dL^*(t,x) = \Big(\frac{\partial}{\partial x} L^*(t,x) + L^*(t,x)\gamma(t,x)\sigma(t,x+\delta)\Big)dt + L^*(t,x)\gamma(t,x)dW(t). \tag{1.40}$$

If we have the HJM volatility process σ defined on $0 \leq x < \delta$ then through the recurrence relation above, we have defined σ for all times and x.
Now if we set $\sigma(t,x) = 0$ for all $0 \leq x < \delta$, then we get for $x \geq \delta$

$$\sigma(t,x) = \sum_{k=1}^{[\delta^{-1}x]} \frac{\delta L^*(t,x-k\delta)}{1+\delta L^*(t,x-k\delta)} \gamma(t,x-k\delta). \tag{1.41}$$

Combining those results, our process $\{L^*(t,x)|t,x \geq 0\}$ must satisfy

$$dL^*(t,x) = \Big(\frac{\partial}{\partial x} L^*(t,x) + L^*(t,x)\gamma(t,x)\sigma(t,x) + \frac{\delta(L^*(t,x))^2}{1+\delta L^*(t,x)} |\gamma(t,x)|^2\Big) dt + L^*(t,x)\gamma(t,x) dW(t) \tag{1.42}$$

in Musiela-parametrization.Now BGM in [3] use the following

Lemma 1 *For all $x \geq 0$ let $\{\xi(t,x)|t \geq 0\}$ be an adapted bounded stochastic process with values in $\mathbb{R}$, $a(.,x) : \mathbb{R}_+ \to \mathbb{R}^d$ be a deterministic bounded and piecewise continuous function and let*

$$M(t,x) = \int_0^t a(s,x) dW(s). \tag{1.43}$$

For all $x \geq 0$ the equation

$$dy(t,x) = y(t,x)a(t,x) \times \left(\Big(\frac{\delta y(t,x)}{1+\delta y(t,x)} a(t,x) + \xi(t,x) \Big) dt + dW(t) \right) \qquad y(0,x) > 0 \tag{1.44}$$

(where δ is constant) has a unique strictly positive solution an $\mathbb{R}_+$. Moreover if for some $k \in \{0,1,2,3,\dots\}$, $y(0,.) \in \mathcal{C}(\mathbb{R}_+)$ and for all $t \geq 0$, $a(t,.)$, $M(t,.)$ and $\xi(t,.) \in \mathcal{C}^k(\mathbb{R}_+$ then for all $t \geq 0$, $y(t,.) \in \mathcal{C}^k(\mathbb{R}_+)$.

from which they get their main result

Theorem 2 *Let $\gamma : \mathbb{R}_+^2 \to \mathbb{R}^d$ be a deterministic bounded piecewise continuous function $\delta > 0$ be a constant and let*

$$M(t,x) = \int_0^t \gamma(s,x+t-s) dW(s). \tag{1.45}$$

Then the equation admits a unique nonnegative solution $L(t,x)$ for any $t \geq 0$ and any nonnegative initial condition $L(0,.) = L_0$. If $L_0 > 0$ then $L(t,.) > 0$ for all times and if $L_0, \gamma(t,.), M(t,.), \in \mathcal{C}^k(\mathbb{R}_+)$ and $\frac{\partial^j}{\partial^j x}\gamma(t,x)|_{x=0} = 0 \forall j$ then $L(t,.) \in \mathcal{C}^k(\mathbb{R}_+)$ for all times.

See [3] for the proof. $\square$

This theorem applied to 1.42 says effectively that a LIBOR term structure model exists, embedded in an overall HJM framework.

1.4.2 Standard Parametrization With Jumps

A good derivation without the Musiela parametrization is done in [6] based on the work of Jamshidian [11]. Proofs are to be found in [6].

We assume some complete stochastic basis $(\Omega, \mathcal{F}_{T^*}, \mathbb{P}^*, (\mathcal{F}_t)_t)$. In this setting we assume there holds

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)^T dW_t^* + \int_{\mathbb{R}^r} \delta(t,x,T)(\mu - \nu^*)(dt,dx) \tag{1.46}$$

where W^* is a standard Brownian Motion in $\mathbb{R}^d$, μ is the random measure of jumps of a semi-martingale with continuous compensator ν^* for which $\nu^*(dt,dx) = F(t,dx)dt$ is assumed to hold. The coefficients are continuous in the second variable. $\alpha : \Omega \times [0,T^*] \times [0,T^*] \to \mathbb{R}$ and $\sigma : \Omega \times [0,T^*] \times [0,T^*] \to \mathbb{R}_+^d$ are assumed to be $\mathcal{P} \times \mathcal{B}([0,T^*])$ measurable(see Appendix A for the definition of $\mathcal{P}$) and $\delta : \Omega \times [0,T^*] \times \mathbb{R}^r \times [0,T^*] \to \mathbb{R}$ is assumed to be $\mathcal{P} \times \mathcal{B}(\mathbb{R}^r) \times \mathcal{B}([0,T^*])$ measurable. We denote $\Delta := \{(s,u) \in \mathbb{R}_+ \times \mathbb{R}_+ | 0 \le s \le u \le T^*\}$. Then

- If $(t,T) \notin \Delta$ we have $\alpha(t,T) = \delta(t,x,T) = 0$ and $\sigma(t,T) = (0,0,\ldots,0)^T$.
- For all $(t,T) \in \Delta$ there holds

$$\int_0^T \int_t^T |\alpha(s,u)| du ds < \infty \tag{1.47}$$

$$\int_0^T \int_t^T |\sigma(s,u)|^2 du ds < \infty \tag{1.48}$$

$$\int_0^T \int_{\mathbb{R}} \int_t^T |\delta(s,x,u)|^2 du \nu^*(ds,dx). \tag{1.49}$$

- We denote

$$A(t,T) := -\int_t^T \alpha(t,u)du \qquad S(t,T) := -\int_t^T \sigma(t,u)du$$

$$D(t,x,T) = -\int_t^T \delta(t,x,u)du$$

- We get two conditions for $\mathbb{P}^*$ to be a martingale measure. The first

$$\int_0^t \int_{\mathbb{R}^r} e^{D(s,x,T)} - 1 - D(s,x,T)F(s,dx)ds < \infty \qquad \forall (t,T) \in \Delta \tag{1.50}$$

- The second

$$A(t,T) + \frac{1}{2}|S(t,T)|^2 + \int_{\mathbb{R}^r} (e^{D(t,xT)} - 1 - D(t,x,T))F(t,dx) = 0 \qquad [d\mathbb{P}^* \times dt] \tag{1.51}$$

From the forward rate process we calculate the bond price processes through

$$B(t,T) = \exp(-\int_t^T f(t,u)du) \tag{1.52}$$

resulting in

$$B(t,T) = B(0,T) \exp\Big(\int_0^t (f(s,s) + A(s,T))ds + \int_0^t S(s,T)^\top dW^* + \int_0^t \int_{\mathbb{R}^r} D(s,x,T)(\mu - \nu^*)(ds,dx) \Big). \tag{1.53}$$

In this context the bank account is given by

$$B(t) = \exp(\int_0^t f(s,s)ds). \tag{1.54}$$

Calculating the discounted bond prices $Z(t,T)$ under the riskneutral measure ($Z(t,T) := \frac{B(t,T)}{B(t)}$) we get

$$\begin{aligned}Z(t,T) = Z(0,T)\exp\Big(&\int_0^t S(s,T)^T dW_s^* - \frac{1}{2}\int_0^t |S(s,T)|^2 ds + \\ &\int_0^t \int_{\mathbb{R}^r} D(s,x,T)(\mu-\nu^*)(ds,dx) - \int_0^t \int_{\mathbb{R}^r} (e^{D(s,x,T)} - 1 - D(s,x,T))F(s,dx)ds\Big).\end{aligned} \tag{1.55}$$

Using the forward-rate process

$$F_B(t,T,U) := \frac{B(t,T)}{B(t,U)}. \tag{1.56}$$

we derive the dynamics of the LIBOR-rate process under the risk neutral measure

Theorem 3 *If $f(.,T)$ satisfies the conditions stated above including the martingale conditions under $\mathbb{P}^*$, then the dynamics of $L(.,T)$ under $\mathbb{P}^*$ are given by*

$$\begin{aligned}\frac{\delta}{1+\delta L(t_-,T)} dL(t,T) = &(S(t,T+\delta) - S(t,T))^T S(t,T+\delta)dt + \\ &\int_{\mathbb{R}^r} (e^{D(t,x,T)-D(t,x,T+\delta)} - 1 + e^{D(t,x,T+\delta)} - e^{D(t,x,T)})\nu^*(dt,dx) + \\ &(S(t,T) - S(t,T+\delta))^T dW_t^* + \int_{\mathbb{R}^r} (e^{D(t,x,T)-D(t,x,T+\delta)} - 1)(\mu-\nu^*)(dt,dx)\end{aligned} \tag{1.57}$$

We rewrite this, to suit our purposes later on and to be in conformity with the BGM approach

$$S(t,T) =: -c(t,T) \qquad D(t,x,T) := -h(t,x,T)$$

we then get

$$\begin{aligned}\frac{\delta}{1+\delta L(t_-,T)} dL(t,T) = &-(c(t,T) - c(t,T+\delta))^T c(t,T+\delta)dt + \\ &\int_{\mathbb{R}^r} (e^{-h(t,x,T)+h(t,x,T+\delta)} - 1 + e^{-h(t,x,T+\delta)} - e^{-h(t,x,T)})\nu^*(dt,dx) + \\ &(c(t,T+\delta) - c(t,T))^T dW_t^* + \int_{\mathbb{R}^r} (e^{-h(t,x,T)+h(t,x,T+\delta)} - 1)(\mu-\nu^*)(dt,dx).\end{aligned} \tag{1.58}$$

We want our LIBOR-dynamics to be of the form

$$dL(t,T) = \dots dt + L(t_-,T)\Big(\lambda(t,T)c_s^{\frac{1}{2}} dW_t^* + \int_{\mathbb{R}} (e^{(\lambda(t,T)x)} - 1)(\mu - \nu^T)(dt,dx)\Big) \tag{1.59}$$

under the risk neutral-measure and remember that under its proper forward measure it has to be a martingale.
In this form, martingality is equivalent to the disappearance of the drift term.

Comparing to our assumed form in the case $d = 1$(one-dimensional driving process), we get relations between the coefficients:

$$c(t, T + \delta) - c(t, T) = \ell(t_-, T)\lambda(t, T)c_s^{\frac{1}{2}}. \tag{1.60}$$

Concerning the jump-part we derive by comparison

$$\frac{(1 + \delta L(t_-, T))}{\delta}(e^{-h(t,x,T)+h(t,x,T+\delta)} - 1) = L(t_-, T)(e^{\lambda(t,T)x} - 1) \tag{1.61}$$

which gives

$$(e^{-h(t,x,T)+h(t,x,T+\delta)} - 1) = \frac{\delta}{1 + \delta L(t_-, T)}L(t_-, T)(e^{\lambda(t,T)x} - 1) \tag{1.62}$$

we proceed to

$$e^{-h(t,x,T)+h(t,x,T+\delta)} = \ell(t_-, T)(e^{\lambda(t,T)x} - 1) + 1 \tag{1.63}$$

which finally yields

$$-h(t, x, T) + h(t, x, T + \delta) = \log(\ell(t, T)(e^{(\lambda(t,T)x)} - 1) + 1) \tag{1.64}$$

with initial conditions

$$c(t, T) = 0 \qquad \forall T \in [t, t + \delta)$$

and

$$h(t, x, T) = 0 \qquad \forall T \in [t, t + \delta)$$

Furthermore we define a function $i : \mathbb{R}_+ \to \mathbb{N}$ through

$$i(x) := n + 1 \qquad \forall x \in (n, n + 1], n \in \mathbb{N} \tag{1.65}$$

From this we calculate

$$c(t, T) = \sum_{k=i(t)}^{[\delta^{-1}T]} \frac{\delta L(t, T - k\delta)}{1 + \delta L(t, T - k\delta)} c_t^{\frac{1}{2}} \qquad \forall t \leq T \tag{1.66}$$

and

$$e^{h(t,x,T)} = \prod_{k=i(t)}^{[\delta^{-1}T]} \Big(\ell(t_-, T - k\delta)(e^{\lambda(t,T)x} - 1) + 1\Big) \qquad \forall t \leq T \tag{1.67}$$

where $[\delta^{-1}T]$ denotes the Gauss-Bracket of $\delta^{-1}T$.
Thus uniquely defining our model

$$dL(t, T) = (L(t_-, T)c(t, T)c_t^{\frac{1}{2}} + \frac{\delta L^2(t_-, T)}{1 + \delta L(t_-, T)}|c_t^{\frac{1}{2}}|^2 + \frac{L(t_-, T)}{1 + \delta L(t_-, T)}$$
$$\int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(1-e^{-h(t,x,T+\delta)})F(dx, t))dt+L(t_-, T)c_t^{\frac{1}{2}}dW_t^*+L(t_-, T)\int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(\mu-\nu)(dx, dt)$$

for arbitrary semimartingales.
The equation above gives the continuously compounded LIBOR forward rate dynamics driven by an arbitrary semimartingale under the risk neutral measure in standard parametrization.

Existence of such a term structure model (e.g. a solution to this system of SDEs), is more difficult to prove than in the Brownian case.
However, as we will show below, there is a way of using finite dimensional methods, to provide a proof of the existence of such models.

1.4.3 Comparison For Log-Normal Rates

Suppose now, we wish to model the LIBOR-rates to be of the form

$$dL(t,T) = \ldots dt + L(t,T)c_t^{\frac{1}{2}}dW_t^*. \tag{1.68}$$

Then we assume $D(t,x,T) = -h(t,x,T) = 0$ for $T < \delta$. Our equation becomes

$$\frac{\delta}{1+\delta L(t_-,T)}dL(t,T) = -(c(t,T)-c(t,T+\delta))^T c(t,T+\delta)dt + (c(t,T+\delta)-c(t,T))^T dW_t^* \tag{1.69}$$

with $L(t_-,T) = L(t,T)$ due to the pathwise continuity now. We rewrite this equation again

$$dL(t,T) = -\frac{1}{\delta}\Big((1+\delta L(t,T))(c(t,T)-c(t,T+\delta))^T c(t,T+\delta)dt + (1+\delta L(t,T))(c(t,T+\delta)-c(t,T))^T dW_t^*\Big) \tag{1.70}$$

By comparison to our supposed dynamics we get the condition

$$c(t,T+\delta) - c(t,T) = \frac{\delta L(t,T)}{1+\delta L(t,T)}c_t^{\frac{1}{2}}. \tag{1.71}$$

we use this to rewrite

$$\frac{1}{\delta}(1+\delta L(t,T))\Big(-(c(t,T)-c(t,T+\delta))^T c(t,T+\delta)dt + (c(t,T+\delta)-c(t,T))^T dW_t^*\Big) = \\ (L(t,T)c_t^{\frac{1}{2}}c(t,T) + \frac{\delta L^2(t,T)}{1+\delta L(t,T)}|c_t^{\frac{1}{2}}|^2)dt + L(t,T)c_t^{\frac{1}{2}}dW_t^* \tag{1.72}$$

and get

$$dL(t,T) = (L(t,T)c_t^{\frac{1}{2}}c(t,T) + \frac{\delta L^2(t,T)}{1+\delta L(t,T)}|c_t^{\frac{1}{2}}|^2)dt + L(t,T)c_t^{\frac{1}{2}}dW_t^* \tag{1.73}$$

which almost looks like the equation in Musiela Parametrization.

1.5 Semimartingale Discrete Tenor Forward LIBOR Modeling

This section is an adaptation of [6] for our purposes later on.
We start with a theorem summarizing our aims in this section

Theorem 4 (Forward LIBOR Market Model Existence) *Given a discrete finite tenor structure $\{T_i\}_{i\in I}$, positive deterministic functions $\{\lambda(t,T_i)\}_{i\in I}$ and a strictly decreasing strictly positive initial term structure $(B(0,T_i))_{i\in I}$ and a semimartingale X_t which is given as*

$$X_t = \int_0^t b(s,T_n)ds + \int_0^t c_s^{\frac{1}{2}}dW_s^{n+1} + \int_0^t\int_{\mathbb{R}} x(\mu-\nu_t^{n+1})(ds,dx) \tag{1.74}$$

and for which it holds that

$$\int_0^t (|b(s,T_n)| + c_s)ds < \infty \tag{1.75}$$

as well as

$$\int_0^{T_{n+1}}\int_{|x|\geq 1}\exp(ux)F_s(dx)ds < \infty \qquad u < M, M \geq \sum_{i=1}^n |\lambda(.,T_i)| \tag{1.76}$$

and

$$\int_0^{T_{n+1}} \int_{\mathbb{R}} (x^2 \wedge 1) F_s(dx) ds < \infty, \tag{1.77}$$

there is a LIBOR Model $L(t, T_i)_{t \le T_i, i \in I}$ *such that for all* $i \in I$ *under* $\mathbb{P}_{T_{i+1}}$ *there holds*

$$dL(t,T_i) = L(t_-,T_i)\Big(\lambda(t,T_i)c_t^{\frac{1}{2}}dW_t^{i+1} + \int_{\mathbb{R}}(e^{\lambda(t,T_i)x} - 1)(\mu - \nu_t^{i+1})(dt,dx)\Big) \tag{1.78}$$

For notational simplicity we will assume $\{T_i\}_{i\in I}$ to be equally spaced:

$$T_i - T_{i-1} = \delta > 0 \qquad \forall i$$

Note that the LIBOR rate is apriori modeled as driven by X_t and not derived from any HJM context per se.
Proof:
We get initial values

$$L(0,T_i) = \frac{1}{\delta}\Big(\frac{B(0,T_i)}{B(0,T_i+\delta)} - 1)$$

and model the rate $L(t,T_n)$ under $\mathbb{P}_{T_{n+1}}$ to be

$$L(t,T_n) = L(0,T_n)\exp\Big(\int_0^t \lambda(s,T_n)dX_t\Big) \tag{1.79}$$

Since $L(t,T_n)$ has to be a martingale under $\mathbb{P}_{T_{n+1}}$ by definition of the forward measure, we make $L(t,T_n)$ into a martingale by choosing

$$\int_0^t \lambda(s,T_n)b(s,T_n)ds = -\frac{1}{2}\int_0^t c_s\lambda(s,T_n)^2 ds - \int_0^t\int_{\mathbb{R}}(e^{\lambda(s,T_n)x} - 1 - \lambda(s,T_n)x)\nu_t^{n+1}(ds,dx) \tag{1.80}$$

Now $L(t,T_n)$ is a positive martingale. We therefore can write it as a stochastic exponential of the stochastic logarithm $H(t,T_n)$ defined as

$$H(t,T_n) = \int_0^t \lambda(s,T_n)c_s^{\frac{1}{2}}dW_s^{n+1} + \int_0^t\int_{\mathbb{R}}(e^{\lambda(s,T_n)x} - 1)(\mu - \nu_t^{n+1})(ds,dx). \tag{1.81}$$

So

$$L(t,T_n) = L(0,T_n)\mathcal{E}(H(t,T_n)) \tag{1.82}$$

and expressed in a SDE

$$dL(t,T_n) = L(t_-,T_n)\Big(\lambda(t,T_n)c_t^{\frac{1}{2}}dW_t^{n+1} + \int_{\mathbb{R}}(e^{\lambda(t,T_n)x} - 1)(\mu - \nu_t^{n+1})(dt,dx)\Big). \tag{1.83}$$

using $F_B(t,T_n,T_n+\delta) = 1 + \delta L(t,T_n)$ we get

$$dF_B(t,T_n,T_n+\delta) = F_B(t_-,T_n,T_n+\delta)\Big(\ell(t_-,T_n)c_s^{\frac{1}{2}}dW_t^{n+1} + \int_{\mathbb{R}}\ell(t_-,T_n)(e^{\lambda(t,T_n)} - 1)(\mu - \nu_t^{n+1})(dt,dx)\Big). \tag{1.84}$$

Since $F_B(t,T_n,T_{n+1})$ is a measure change between the forward measures $\mathbb{P}_{T_n}$ and $\mathbb{P}_{T_{n+1}}$, we want to represent this as a stochastic exponential

$$\frac{dP_{T_n}}{dP_{T_{n+1}}} = \frac{F_B(t,T_n,T_{n+1})}{F_B(0,T_n,T_{n+1})}\mathcal{E}\Big(M(t,T_n)\Big). \tag{1.85}$$

Our result will be

Lemma 2 (Induction - Induction Start) *Under the conditions of the theorem we get the measure change between $\mathbb{P}_{T_n}$ and $\mathbb{P}_{T_{n+1}}$ to be*

$$\frac{d\mathbb{P}_{T_n}}{d\mathbb{P}_{T_{n+1}}} = \mathcal{E}(M(t,T_n)) \tag{1.86}$$

with

$$M(t,T_n) = \int_0^t \alpha(s,T_n,T_{n+1})c_s^{\frac{1}{2}}dW_s^{n+1} + \int_0^t\int_{\mathbb{R}}(\beta(s,x,T_n,T_{n+1})-1)(\mu-\nu_t^{n+1})(ds,dx). \tag{1.87}$$

and as characteristics of X_t under $\mathbb{P}_{T_n}$ (henceforth denoted as X_t^n)

$$W_t^n = W_t^{n+1} - \int_0^t \alpha(s,T_n,T_{n+1})c_s^{\frac{1}{2}}ds \tag{1.88}$$

and

$$\nu_t^n = \beta(t,x,T_n,T_{n+1})\nu_t^{n+1} \tag{1.89}$$

where

$$\alpha(s,T_n,T_{n+1}) = \ell(t_-,T_n)\lambda(t,T_n)$$

and

$$\beta(s,x,T_n,T_{n+1}) = \ell(t_-,T_n)(e^{\lambda(t,T_n)x}-1)+1.$$

and the drift-condition from the martingality assumption on $L(t,T_{n-1}) = L(0,T_{n-1})\exp(\int_0^t\lambda(s,T_{n-1})dX_s^n)$ as

$$\int_0^t \lambda(s,T_{n-1})b(s,T_n)ds = -\frac{1}{2}\int_0^t c_s\lambda(s,T_{n-1})^2ds - \int_0^t\int_{\mathbb{R}}(e^{\lambda(s,T_{n-1})x}-1-\lambda(s,T_{n-1})x)\nu_t^n(ds,dx). \tag{1.90}$$

Since $F_B(t,T_n,T_{n+1})$ is a positive martingale, it is possible to construct stochastic exponential of the stochastic logarithm again, which yields

$$M(t,T_n) = \int_0^t \alpha(s,T_n,T_{n+1})c_s^{\frac{1}{2}}dW_s^{n+1} + \int_0^t\int_{\mathbb{R}}(\beta(s,x,T_n,T_{n+1})-1)(\mu-\nu_t^{n+1})(ds,dx).$$

This yields as Brownian Motion under $\mathbb{P}_{T_n}$ under $\mathbb{P}_{T_{n+1}}$

$$W_t^n = W_t^{n+1} - \int_0^t \alpha(s,T_n,T_{n+1})c_s^{\frac{1}{2}}ds$$

and as new compensator

$$\nu_t^n = \beta(t,x,T_n,T_{n+1})\nu_t^{n+1}.$$

The only thing we have not determined for our driving process X_t under $\mathbb{P}_{T_n}$ is the drift characteristic $b(t,T_n)$.
We model the next rate $L(t,T_{n-1})$ under $\mathbb{P}_{T_n}$ to be

$$L(t,T_{n-1}) = L(0,T_{n-1})\exp\Big(\int_0^t \lambda(s,T_{n-1})dX_t^n\Big) \tag{1.91}$$

where X_t^n denotes X_t under $\mathbb{P}_{T_{n+1}}$ and there holds

$$X_t^n = \int_0^t b(s,T_{n-1})ds + \int_0^t c_s^{\frac{1}{2}}dW_s^n + \int_0^t\int_{\mathbb{R}}x(\mu-\nu_s^n)(ds,dx). \tag{1.92}$$

Since $L(t,T_{n-1})$ has to be a martingale under $\mathbb{P}_{T_n}$ we get a condition on the drift-characteristic

$$\int_0^t \lambda(s,T_{n-1})b(s,T_{n-1})ds = -\frac{1}{2}\int_0^t c_s\lambda(s,T_{n-1})^2 ds - \int_0^t\int_{\mathbb{R}}(e^{\lambda(s,T_{n-1})x}-1-\lambda(s,T_{n-1})x)\nu_t^n(ds,dx).$$

which concludes the proof of the lemma. □

Proceeding in our construction, we can again represent $L(t,T_{n_1})$ as stochastic exponential of its stochastic logarithm

$$H(t,T_{n-1}) = \int_0^t \lambda(s,T_{n-1})c_s^{\frac{1}{2}}dW_s^n + \int_0^t\int_{\mathbb{R}}(e^{\lambda(s,T_{n-1})x}-1)(\mu-\nu_t^n)(ds,dx) \tag{1.93}$$

and derive the dynamics

$$dL(t,T_{n-1}) = L(t_-,T_{n-1})\Big(\lambda(t,T_{n-1})c_t^{\frac{1}{2}}dW_t^n + \int_{\mathbb{R}}(e^{\lambda(t,T_{n-1})x}-1)(\mu-\nu_t^n)(dt,dx)\Big). \tag{1.94}$$

We proceed to give the general construction principle

Lemma 3 (Induction - Inductive Step) *Under the conditions of the theorem, given a LIBOR-rate process $L(t,T_{i+1})$ through*

$$L(t,T_{i+1}) = L(0,T_i)\exp(\int_0^t \lambda(s,T_{i+1})dX_s^{i+2}) \tag{1.95}$$

we can derive W^{i+1} and ν^{i+1} in terms of W^{i+2} and ν^{i+2} and we model $L(t,T_i)$ it to be

$$L(t,T_i) = L(0,T_i)\exp\Big(\int_0^t \lambda(s,T_i)dX_s^{i+1}\Big) \tag{1.96}$$

and it will be the solution of

$$dL(t,T_i) = L(t_-,T_i)\Big(\lambda(t,T_i)c_t^{\frac{1}{2}}dW_t^{i+1} + \int_{\mathbb{R}}(e^{\lambda(t,T_i)x}-1)(\mu-\nu_s^{i+1})(ds,dx)\Big) \tag{1.97}$$

for all $i\in I$.

Assuming we are given

$$L(t,T_{i+1}) = L(0,T_{i+1})\exp(\int_0^t \lambda(s,T_{i+1})dX_s^{i+2})$$

we get as condition on the drift

$$\int_0^t \lambda(s,T_{i+1})b(s,T_{i+1})ds = -\frac{1}{2}\int_0^t c_s\lambda(s,T_{i+1})^2 ds - \int_0^t\int_{\mathbb{R}}(e^{\lambda(s,T_{i+1})x}-1-\lambda(s,T_{i+1})x)\nu_t^{i+2}(ds,dx). \tag{1.98}$$

We represent $L(t,T_{i+1}) = L(0,T_{i+1})\mathcal{E}(H(t,T_{i+1}))$ with

$$H(t,T_{i+1}) = \int_0^t \lambda(s,T_{i+1})c_s^{\frac{1}{2}}dW_s^{i+2} + \int_0^t\int_{\mathbb{R}}(e^{\lambda(s,T_{i+1})x}-1)(\mu-\nu_t^{i+2})(ds,dx). \tag{1.99}$$

and get as dynamics

$$dL(t,T_{i+1}) = L(t_-,T_{i+1})\Big(\lambda(t,T_{i+1})c_t^{\frac{1}{2}}dW_t^{i+2} + \int_{\mathbb{R}}(e^{\lambda(t,T_{i+1})x}-1)(\mu-\nu_s^{i+2})(ds,dx)\Big). \tag{1.100}$$

Through $1 + \delta L(t, T_{i+1}) = F_B(t, T_{i+1}, T_{i+2})$ we get

$$dF_B(t, T_{i+1}, T_{i+2}) = F_B(t_-, T_{i+1}, T_{i+2})\Big(\ell(t_-, T_{i+1})c_s^{\frac{1}{2}}dW_t^{i+2} + \int_{\mathbb{R}} \ell(t_-, T_{i+1})(e^{\lambda(t,T_{i+1})x} - 1)(\mu - \nu_t^{i+2})(dt, dx)\Big). \tag{1.101}$$

We have

$$\frac{d\mathbb{P}_{T_{i+1}}}{d\mathbb{P}_{T_{i+2}}} = \frac{B(0, T_{i+2})}{B(0, T_{i+1})}\frac{B(t, T_{i+1})}{B(t, T_{i+2})} = \frac{F_B(t, T_{i+1}, T_{i+2})}{F_B(0, T_{i+1}, T_{i+2})} = \mathcal{E}(M(t, T_{i+1})) \tag{1.102}$$

with

$$M(t, T_{i+1}) = \int_0^t \alpha(s, T_{i+1}, T_{i+2})c_s^{\frac{1}{2}}dW_s^{i+2} + \int_0^t \int_{\mathbb{R}} (\beta(s, x, T_{i+1}, T_{i+2}) - 1)(\mu - \nu_t^{i+2})(ds, dx). \tag{1.103}$$

where

$$\alpha(t, T_{i+1}, T_{i+2}) = \ell(t_-, T_{i+1})\lambda(t, T_{i+1}) \tag{1.104}$$

and

$$\beta(t, x, T_{i+1}, T_{i+2}) = \ell(t_-, T_{i+1})(e^{\lambda(t,T_{i+1})x} - 1) + 1. \tag{1.105}$$

We get as Brownian Motion under $\mathbb{P}_{T_{i+1}}$

$$W_t^{i+1} = W_t^{i+2} - \int_0^t \alpha(s, T_{i+1}, T_{i+2})c_s^{\frac{1}{2}}ds \tag{1.106}$$

and as new compensator

$$\nu_t^{i+1} = \beta(t, x, T_{i+1}, T_{i+2})\nu_t^{i+2}. \tag{1.107}$$

Given those components we model the next rate $L(t, T_i)$ under $\mathbb{P}_{T_{i+1}}$ as

$$L(t, T_i) = L(0, T_i)\exp\Big(\int_0^t \lambda(s, T_i)dX_s^{i+1}\Big) \tag{1.108}$$

with

$$X_t^{i+1} = \int_0^t b(s, T_i)ds + \int_0^t c_s^{\frac{1}{2}}dW_s^{i+1} + \int_0^t \int_{\mathbb{R}} x(\mu - \nu_s^{i+1})(ds, dx). \tag{1.109}$$

We choose the drift-characteristic $b(s, T_i)$ under $\mathbb{P}_{T_{i+1}}$ to solve

$$\int_0^t \lambda(s, T_i)b(s, T_i)ds = -\frac{1}{2}\int_0^t c_s\lambda(s, T_i)^2ds - \int_0^t \int_{\mathbb{R}} (e^{\lambda(s,T_i)x} - 1 - \lambda(s, T_i)x)\nu_t^{i+1}(ds, dx) \tag{1.110}$$

since $L(t, T_i)$ has to be a martingale under $\mathbb{P}_{T_{i+1}}$.
Being a positive martingale, we can represent $L(t, T_i) = L(0, T_i)\mathcal{E}(H(t, T_i))$ with

$$H(t, T_i) = \int_0^t \lambda(s, T_i)c_s^{\frac{1}{2}}dW_s^{i+1} + \int_0^t \int_{\mathbb{R}} (e^{\lambda(s,T_i)x} - 1)(\mu - \nu_t^{i+1})(ds, dx). \tag{1.111}$$

That implies for the dynamics of $L(t, T_i)$

$$dL(t, T_i) = L(t_-, T_i)\Big(\lambda(t, T_i)c_t^{\frac{1}{2}}dW_t^{i+1} + \int_{\mathbb{R}} (e^{\lambda(t,T_i)x} - 1)(\mu - \nu_s^{i+1})(ds, dx)\Big). \tag{1.112}$$

That concludes the proof of our lemma. □

It also concludes the proof of our theorem, if combined with the lemma before, which may be seen as the inductions beginning and then the above lemma is the inductive step. □

1.6 Spot-Modeling And Spot-LIBOR Modeling

We use the results of Jamshidian [11] to obtain a new approach to modeling of an unbounded tenor term structure model.

1.6.1 The Spot Measure

A spot measure by the definition of [15], which we adopted, in discrete tenor LIBOR Market Model theory is given by the choice of numeraire $B(t,T_1)/B(0,T_1)$ for a given tenor structure. We start from the dynamics under the proper forward-measure $\mathbb{P}_{T_{n+1}}$ for a finite discrete tenor model.(with tenor $\{T_i|i=1,\dots,n,n+1\}$)

$$dL(t,T_n)=L(t_-,T_n)(\lambda(t,T_n)c_t^{\frac{1}{2}}dW_t^{n+1}+\int_{\mathbb{R}}(e^{\lambda(t,T_n)x}-1)(\mu-\nu_t^{n+1})(dt,dx))$$

and build the measure change as follows

$$(\frac{d\mathbb{P}_{T_1}}{d\mathbb{P}_{T_{n+1}}})_t=\frac{B(0,T_{n+1})B(t,T_1)}{B(0,T_1)B(t,T_{n+1})}=\frac{F_B(t,T_1,T_{n+1})}{F_B(0,T_1,T_{n+1})}=\prod_{j=1}^{n}\frac{F_B(t,T_j,T_{j+1})}{F_B(t,T_j,T_{j+1})} \tag{1.113}$$

yielding a Brownian Motion

$$W_t^1=W_t^{n+1}-\int_0^t\sum_{j=1}^{n}\ell(s_-,T_j)\lambda(s,T_j)c_s^{\frac{1}{2}}ds\Rightarrow W_t^{n+1}=W_t^1+\int_0^t\sum_{j=1}^{n}\ell(s_-,T_j)\lambda(s,T_j)c_s^{\frac{1}{2}}ds$$

and a compensator

$$\nu^1=\prod_{j=1}^{n}\beta(t,x,T_j,T_{j+1})\nu_t^{n+1}\Rightarrow\nu^{n+1}=\prod_{j=1}^{n}\frac{1}{\beta(t,x,T_j,T_{j+1})}\nu_t^1.$$

Note that there is no LIBOR-rate to be modeled under $\mathbb{P}_{T_1}$ in the usual forward modeling approach, yet still we can define this forward-measure properly.
The dynamics of $L(t,T_n)$ under that measure $\mathbb{P}_{T_1}$ are then

$$dL(t,T_n)=L(t_-,T_n)\Big(\sum_{j=1}^{n}\ell(t_-,T_j)\lambda(t,T_j)\lambda(t,T_n)c_t dt+$$
$$\int_{\mathbb{R}}(e^{\lambda(t,T_n)x}-1)(1-\prod_{j=1}^{n}\frac{1}{\beta(t,x,T_j,T_{j+1})})\nu_t^1(dt,dx)+\lambda(t,T_n)c_t^{\frac{1}{2}}dW_t^1+\int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(\mu-\nu_t^1)(dt,dx)\Big)$$

More generally starting from an arbitrary rate under its forward measure

$$dL(t,T_i)=L(t_-,T_i)(\lambda(t,T_i)c_t^{\frac{1}{2}}dW_s^{i+1}+\int_{\mathbb{R}}(e^{\lambda(t,T_i)x}-1)(\mu-\nu_t^{i+1})(dx,dt))$$

The measure change becomes

$$(\frac{d\mathbb{P}_{T_1}}{d\mathbb{P}_{T_{i+1}}})_t=\frac{B(t,T_1)}{B(t,T_{i+1})}=\frac{F_B(t,T_1,T_{i+1})}{F_B(0,T_1,T_{i+1})}=\prod_{j=1}^{i}\frac{F_B(t,T_j,T_{j+1})}{F_B(0,T_j,T_{j+1})} \tag{1.114}$$

and the dynamics therefore

$$dL(t,T_i) = L(t_-,T_i)\Big(\sum_{j=1}^{i} \ell(t_-,T_j)\lambda(t,T_j)\lambda(t,T_i)c_t dt+$$

$$\int_{\mathbb{R}}(e^{\lambda(t,T_i)x}-1)(1-\prod_{j=1}^{i}\frac{1}{\beta(t,x,T_j,T_{j+1})})\nu^1(dx,dt)+c_t^{\frac{1}{2}}\lambda(t,T_i)dW_t^1+\int_{\mathbb{R}} e^{\lambda(t,T_i)x}-1(\mu-\nu_t^1)(dt,dx)\Big)$$

by the same arguments concerning W_t^1 and ν_t^1 as above.
Then therefore there exists an equivalent measure $\mathbb{P}_{T_1}$ to the other forward measures and the risk neutral measure, such that LIBOR-dynamics are of the form

$$dL(t,T_i) = L(t_-,T_i)\Big(\sum_{j=1}^{i} \lambda(t,T_j)\ell(t_-,T_j)\lambda(t,T_i)c_t dt+$$

$$\int_{\mathbb{R}}(e^{\lambda(t,T_i)x}-1)(1-\prod_{j=1}^{i}\frac{1}{\beta(t,x,T_j,T_{j+1})})\nu_t^1(dx,ds)+\lambda(t,T_i)c_t^{\frac{1}{2}}dW_t^1+\int_{\mathbb{R}}(e^{\lambda(t,T_i)x}-1)(\mu-\nu_t^1)(dx,dt)\Big)$$

where

$$\nu_t^{i+1} = (\prod_{j=1}^{i}\frac{1}{(\ell(t,T_j)(e^{\lambda(t,T_j)x}-1)+1)})\nu_t^1. \tag{1.115}$$

So the LIBOR-rates in this case depend only on the rates modelled for shorter maturities.
Of course, a problem of this approach is, that $B(t,T_1)$ is essentially only defined on $[0,T_1]$ since afterwards the bond has matured.
Therefore we have to extend $B(t,T_1)$ beyond T_1 in such a way that it stays a semimartingale.
We will not go deeper into diverse possibilities for this, as we are primarily interested in an particular extension, which was introduced explicitly by Jamshidian.

1.6.2 Spot-LIBOR Measure

Introduced by Jamshidian in [11], was the numeraire

$$B^*(t) = \frac{B(t,T_{i(t)})}{B(0,T_1)}\prod_{j=1}^{i(t)-1}\frac{B(T_j,T_j)}{B(T_j,T_{j+1})} \tag{1.116}$$

with $i(t) = \min\{i : t \le T_i\}$.
The idea behind this is the following: We wish to extend a given $B(t,T_i)$ as explained above to time intervals beyond $[0,T_i]$. We do this by making $B(t,T_i)$ proportional to $B(t,T_{i+1})$ on $[T_i,T_{i+1}]$, proportional to $B(t,T_{i+2})$ on $[T_{i+2},T_{i+3}]$ etc. That gives

$$B(t,T_i) = B(t,T_{j+1})\prod_{k=i}^{j}\frac{B(T_k,T_k)}{B(T_k,T_{k+1})}, \qquad t\in[T_j,T_{j+1}), \qquad 1\le i\le j\le n. \tag{1.117}$$

For $B(t,T_1)$ this yields

$$B(t,T_1) = B(t,T_{i(t)})\prod_{j=1}^{i(t)-1}\frac{B(T_j,T_j)}{B(T_j,T_{j+1})} = B(t,T_{i(t)})\prod_{j=1}^{i(t)-1}(1+\delta_j L(T_j,T_j)) \qquad \forall t\le T_{n+1} \tag{1.118}$$

This can be interpreted as as the value of a bond from investing a given amount $B(0,T_1)$ at time 0 at spot LIBOR rate $L(0,T_1)$ and at T_1 reinvesting the principal interest at the prevailing spot LIBOR rate $L(T_1,T_1)$ and so on.
The numeraire $B^*(t)$ is then given through the extended $B(t,T_1)$ as

$$B^*(t) = \frac{B(t,T_1)}{B(0,T_1)}$$

We will show the following

Theorem 5 (Spot-LIBOR Dynamics) *There is a measure denoted by $\mathbb{P}_{Ls}$ given through the numeraire $B^*(t)$, equivalent to the forward-measures and the risk neutral measure such that the dynamics of the LIBOR rates for a given tenor structure are*

$$L(t,T_s) = L(t_-,T_s)(\sum_{j=i(t)}^{s} \ell(t_-,T_j)\lambda(t,T_j)\lambda(t,T_s)c_t dt + \int_{\mathbb{R}}(e^{\lambda(t,T_s)x}-1)$$

$$(1-\prod_{j=i(t)}^{s}\frac{1}{\beta(t,x,T_j,T_{j+1})})\nu_t^{i(t)}(dx,dt) + \lambda(t,T_s)c_t^{\frac{1}{2}}dW_t^{i(t)} + \int_{\mathbb{R}}(e^{\lambda(t,T_s)x}-1)(\mu-\nu_t^{i(t)})(dt,dx))$$

Proof:
To understand the LIBOR-market-model under the measure induced by that numeraire, we first look at the first time interval under consideration $[0,T_1]$ and see our spot-measure as discussed above. How to continue for the other time-intervals? To answer this, we look at the form of the measure change for each time interval

$$\frac{d\mathbb{P}_{Ls}}{d\mathbb{P}_{T_{s+1}}} = \frac{B(t,T_{i(t)-1})}{B(0,T_1)}\prod_{j=1}^{i(t)-1}\frac{B(T_j,T_j)}{B(T_j,T_{j+1})}\frac{B(0,T_{s+1})}{B(t,T_{s+1})} =$$

$$\frac{B(t,T_{i(t)})}{B(t,T_{s+1})}\prod_{j=0}^{i(t)-1}\frac{B(T_j,T_j)}{B(T_j,T_{j+1})} = \frac{F_B(t,T_{i(t)},T_{s+1})}{F_B(0,T_{i(t)},T_{s+1})}\prod_{j=1}^{i(t)-1}F_B(T_j,T_j,T_{j+1}) = \frac{F_B(t,T_{i(t)},T_{s+1})}{F_B(0,T_{i(t)},T_{s+1})}C$$

So for each interval $t \in (T_i,T_{i+1}]$ our numeraire is then $B(t,T_{i+1})C$. We therefore can express the Spot-LIBOR numeraire dynamics by a sequence of forward-measure dynamics.
We compute measure changes accordingly

$$(\frac{d\mathbb{P}_{Ls}}{d\mathbb{P}_{T_{s+1}}})_t = \prod_{j=i(t)}^{s} F_B(t,T_j,T_{j+1})F_B(0,T_j,T_{j+1}) \tag{1.119}$$

Inserting the resulting equation for the Brownian Motion

$$W_t^{s+1} = W_t^{i(t)} + \int_0^t \sum_{j=i(t)}^{s}\lambda(t,T_j)\ell(u_-,T_j)c_u^{\frac{1}{2}}du \tag{1.120}$$

and the compensator

$$\nu_t^{s+1} = \prod_{j=i(t)}^{s}\frac{1}{\beta(t,x,T_j,T_{j+1})}\nu_t^{i(t)} \tag{1.121}$$

into the forward dynamics

$$dL(t,T_s) = L(t_-,T_s)\Big(\lambda(t,T_s)c_t^{\frac{1}{2}}dW_t^{s+1} + \int_{\mathbb{R}}(e^{\lambda(t,T_s)x}-1)(\mu-\nu_t^{s+1})(dt,dx)\Big) \tag{1.122}$$

This yields then the following dynamics

$$L(t,T_s) = L(t_-,T_s)(\sum_{j=i(t)}^{s}\ell(t_-,T_j)\lambda(t,T_j)\lambda(t,T_s)c_t dt + \int_{\mathbb{R}}(e^{\lambda(t,T_s)x}-1)$$
$$(1-\prod_{j=i(t)}^{s}\frac{1}{\beta(t,x,T_j,T_{j+1})})\nu_t^{i(t)}(dx,dt) + \lambda(t,T_s)c_t^{\frac{1}{2}}dW_t^{i(t)} +$$
$$\int_{\mathbb{R}}(e^{\lambda(t,T_s)x}-1)(\mu-\nu_t^{i(t)})(dx,dt)) \qquad \forall t\in[T_{i(t)-1},T_{i(t)}] \qquad \square$$

Those dynamics, where each rate is dependent only on finitely many(already calculated) rates, form the basis for our extension of a given model to an infinite time horizon.
This is especially interesting since in an HJM framework we would assume models to be defined for arbitrary large maturities.
For the time being we can only work on a discrete time-grid, but this problem can be solved by "filling" the gaps as we show in the section on continuous tenors.

1.7 Extending The Tenor

We assume we are given a finite tenor-structure and we are working under the Spot-LIBOR measure.
Say we add another point to the tenor structure $T_{n+2} > T_{n+1}$ and $T_{n+2} - T_{n+1} = \delta_{n+1}$. We have the following relation between Brownian Motion under $\mathbb{P}_{T_{n+2}}$ for an arbitrary forward measure and Brownian Motion for $\mathbb{P}_{Ls}$:

$$W_t^{n+2} = W_t^{i(t)} + \int_0^t\sum_{j=i(s)}^{n+1}\frac{\delta_j L(u_-,T_j)}{1+\delta_j L(u_-,T_j)}\lambda(u,T_j)c_u^{\frac{1}{2}}du \qquad u_-\in[T_{i(s)-1},T_i(s)) \tag{1.123}$$

and the compensator

$$\nu^{n+2} = \prod_{j=i(t)}^{n+1}\frac{1}{\beta(t,x,T_j,T_{j+1})}\nu_t^{i(t)} \qquad \forall t\in[T_{i-1},T_i]. \tag{1.124}$$

We can therefore write down a SDE for $L(t,T_{n+1})$ under $\mathbb{P}_{T_{n+2}}$

$$dL(t,T_{n+1}) = L(t_-,T_{n+1})(\sum_{j=i(t)}^{n+1}\ell(t_-,T_j)\lambda(t,T_j)\lambda(t,T_{n+1})c_t dt + c_t^{\frac{1}{2}}dW^i(t)_t +$$
$$\int_{\mathbb{R}}(e^{\lambda(t,T_{n+1})x}-1)(1-\prod_{j=i(t)}^{n+1}\frac{1}{\beta(t,x,T_j,T_{j+1})})\nu^{i(t)}(dx,dt) + \int_{\mathbb{R}}(e^{\lambda(t,T_{n+1})x}-1)(\mu-\nu_t^{i(t)})(dx,dt))$$

for a a priori unspecified positive bounded function $\lambda(t, T_{n+1})$(to be determined through calibration for instance).
Obviously we can repeat this procedure, choosing a positive function $\lambda(t, T_{n+2})$, a new point in time $T_{n+3} > T_{n+2}$ getting a well defined, solvable(finite-dimensional, with Lipschitz-Coefficients if the $\lambda(., T_i)$ are chosen that way) SDE for $L(t, T_{n+2})$.
Therefore, if we extend our tenor-structure, to an arbitrarily large (even countably infinite) set of time points $\{T_i\}_{i=1}^{\infty}$, we get for any possible rate

$$dL(t,T_s) = L(t_-,T_s)(\sum_{j=i(t)}^{s} \ell(t_-,T_j)\lambda(t,T_j)\lambda(t,T_s)c_s dt + \lambda(t,T_s)c_t^{\frac{1}{2}} dW_t^{i(t)} +$$
$$\int_{\mathbb{R}} (e^{\lambda(t,T_s)x} - 1)(1 - \prod_{j=i(t)}^{s} \frac{1}{\beta(t,x,T_j,T_{j+1})})\nu^{i(t)}(dx,dt) + \int_{\mathbb{R}} (e^{\lambda(t,T_s)x} - 1)(\mu - \nu_t^{i(t)})(dt,dx))$$

which is a finite dimensional SDE, dependent only on already calculated rates and therefore solvable with purely finite-dimensional methods. We sum all of the above up in

Theorem 6 (LIBOR-Extension) *For any given finite tenor-structure $\{T_i\}_{i=1}^{n+1}$, strictly decreasing, positive initial term structure $(B(0,T_i))_{i=1}^{n+1}$ and volatility functions $\{\lambda(., T_i)\}_{i=1}^{n}$ and a corresponding LIBOR-Market-Model $(L(., T_i))_{i=1}^{n}$ we may choose positive functions $\{\lambda(., T_i)\}_{i=n+1}^{\infty}$ and from that obtain a unique extension of our model $\{L(., T_i)\}_{i=1}^{\infty}$ by demanding each of our LIBOR-rate processes fulfills the finite dimensional SDE*

$$dL(t,T_s) = L(t_-,T_s)(\sum_{j=i(t)}^{s} \ell(t_-,T_j)\lambda(t,T_j)\lambda(t,T_s)c_t dt + \lambda(t,T_s)c_t^{\frac{1}{2}} dW_t^{i(t)} +$$
$$\int_{\mathbb{R}} (e^{\lambda(t,T_s)x} - 1)(1 - \prod_{j=i(t)}^{s} \frac{1}{\beta(t,x,T_j,T_{j+1})})\nu^{i(t)}(dx,dt) + \int_{\mathbb{R}} (e^{\lambda(t,T_s)x} - 1)(\mu - \nu_t^{i(t)})(dt,dx))$$

given an initial condition $L(0,T_s)$.

Since we know how to switch between forward and spot-LIBOR measures, it does not matter under which measure we originally specify our LIBOR-Market-Model.
Such an infinite discrete tenor structure may serve as a skeleton for a full continuous tenor term structure model. We would have to fill the gaps between the tenor points. We will address this question in the section on continuous tenors.

Chapter 2

Continuous Tenor

We wish to extend the construction of Musiela and Rutkowski [15] to semimartingale driven LIBOR-Models so as to get a "full" tenor-structure in that case as well.

2.1 Construction Concept

In analogy to the work of Musiela and Rutkowski [15], we wish to "fill the gaps" between the discrete tenor dates $\{T_i\}_{i\in 1,\dots,n+1}$. For that we assume an equidistant tenor-time-grid.
We assume to be working up to a terminal maturity T_{n+1} and wish to specify the dynamics of $L(t,T)$ for all $T \in [0, T_{n+1}]$.
As in [15] we use backward induction for this

1. First, we define a forward LIBOR-market model on a given equidistant discrete grid $T_i = i\delta$.

2. Secondly, numeraires for the interval (T_n, T_{n+1}). We have values for the spot-LIBOR numeraire at T_n and T_{n+1}, in short $B^*(T_n)$ and $B^*(T_{n+1})$. Both $B^*(T_n)$ and $B^*(T_{n+1})$ are $\mathcal{F}_{T_n}$ measurable random variables.
We define a spot martingale measure through $\frac{d\mathbb{P}_{Ls}}{d\mathbb{P}_{T_{n+1}}} = B^*(T_{n+1})B(0,T_{n+1})$.
We attempt to satisfy intial conditions in our model for the interpolated rates via a function $\gamma : [T_n, T_{n+1}] \to [0,1]$ such that $\gamma(T_n) = 0$ and $\gamma(T_{n+1}) = 1$ and the process
$$\log B^*(T) = (1-\gamma(T))\log B^*(T_n) + \gamma(T)\log B^*(T_{n+1}), \qquad \forall T \in [T_n, T_{n+1}],$$
satisfies $B(0,t) = \mathbb{E}_{\mathbb{P}_{Ls}}(1/B^*_t)$ for every $T \in [T_n, T_{n+1}]$. We have that $0 < B^*(T_n) < B^*(T_{n+1})$ and $B(0,t), t \in [T_n, T_{n+1}]$ is assumed to be a strictly decreasing function, so such a γ exists and is unique.

3. Thirdly, given the spot-LIBOR numeraires $B^*(t)$ for all $t \in [T_n, T_{n+1}]$ the forward measure for any date $T \in (T_n, T_{n+1})$ can be defined by the formula
$$\frac{d\mathbb{P}_T}{d\mathbb{P}_{Ls}} = \frac{1}{B^*(T)B(0,T)}.$$
If we use this and the definition of our spot martingale measure, we get
$$\frac{d\mathbb{P}_T}{d\mathbb{P}} = \frac{d\mathbb{P}_T}{d\mathbb{P}_{Ls}}\frac{d\mathbb{P}_{Ls}}{d\mathbb{P}} = \frac{B^*(T_{n+1})B(0,T_{n+1})}{B^*(T)B(0,T)}$$

which gives for every $T \in [T_n, T_{n+1}]$

$$\frac{d\mathbb{P}_T}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathbb{E}_{\mathbb{P}}(\frac{B^*(T_{n+1})B(0,T_{n+1})}{B^*(T)B(0,T)}|\mathcal{F}_t)$$

Using stochastic exponentials to describe this we get

$$\frac{d\mathbb{P}_T}{d\mathbb{P}}|_{\mathcal{F}_t} = \frac{B(0,T_{n+1})}{B(0,T)}\mathcal{E}_t\Big(-\int_0^\cdot \alpha(u,T,T_{n+1})c_u^{\frac{1}{2}}dW_u^{n+1}+\int_0^\cdot\int_{\mathbb{R}}(\beta(u,x,T,T_{n+1})-1)(\mu-\nu_t^{n+1})(dx,du)\Big)$$

which we use to describe the forward volatility $\alpha(t,T,T_{n+1})$ for any maturity $T \in (T_n, T_{n+1})$. We get a $\mathbb{P}_T$ Wiener process W^T and a $\mathbb{P}_T$ compensator for the jump-part. Given those ingredients we define the forward LIBOR rate process $L(t,T-\delta)$ for arbitrary $T \in (T_n, T_{n+1})$ by setting

$$dL(t,T-\delta) = L(t_-,T-\delta)\Big(\lambda(t,T-\delta)c_t^{\frac{1}{2}}dW_t^T + \int_{\mathbb{R}}(e^{\lambda(t,T-\delta)x}-1)(\mu-\nu_t^T)(dt,dx)\Big)$$

with usual initial condition

$$L(0,T-\delta) = \delta^{-1}(\frac{B(0,T-\delta)}{B(0,T)}-1).$$

Finally we know

$$\alpha(t,T_n,T_{n+1}) = \ell(t_-,T)\lambda(t,T_n)$$

and

$$\beta(t,x,T_n,T_{n+1}) = \ell(t_-,T)(e^{(\lambda(t,T_n)x)}-1)+1$$

and thus we are able to define the forward measure for the date T.
To define forward probability measures $\mathbb{P}_U$ and the corresponding driving processes for all maturities $U \in (T_{n-1}, T_n)$ we put

$$\alpha(t,U,T) = \alpha(t,T-\delta,T) = \frac{\delta L(t,T-\delta}{1+\delta L(t,T-\delta)}\lambda(t,T-\delta)$$

and

$$\beta(t,x,U,T) = \beta(t,x,T-\delta,T) = \ell(t_-,T)(e^{(\lambda(t,T-\delta)x)}-1)+1$$

with $U = T-\delta$ such that $T = U+\delta$ belongs to (T_n, T_{n+1}).
The relations between those coefficients are derived from the necessary relations between forward measure changes(see the section on forward modeling).
The coefficient $\alpha(t,U,T_{n+1})$ is calculated through

$$\alpha(t,U,T_{n+1}) = \alpha(t,U,T) - \alpha(t,T,T_{n+1}), \qquad \forall t \in [0,T-\delta].$$

For the jump part, $\beta(t,x,U,T_{n+1})$ is calculated through

$$\beta(t,x,U,T_{n+1}) = \beta(t,x,U,T)\beta(t,x,T,T_{n+1}), \qquad \forall t \in [0,T-\delta].$$

Continuing this Backward construction, we get a continuous tenor LIBOR model.
Since we construct a family of forward measures, we can construct a family of forward processes $F(t,T_{n+1},T)$ which fulfill the SDE

$$dF(t,T,T_{n+1}) = F(t_-,T,T_{n+1})(\alpha(t,T,T_{n+1})c_t^{\frac{1}{2}}dW_t^{n+1}+\int_{\mathbb{R}}(\beta(t,x,T,T_{n+1})-1)(\mu_t-\nu_t^{n+1})(dx,dt)).$$

By construction we have that from those forward processes we get a family of bond prices $B(t,T)$ by $B(t,T) := F(t,T,t)$. The family of bond prices obtained thus always satisfies the weak no-arbitrage condition.

2.2 Spot-LIBOR Interpolation

Now we want to carry out interpolation for a model given under the spot-LIBOR measure. We assume a finite equidistant tenor-structure and the spot-LIBOR dynamics

$$dL(t,T_s) = L(t_-,T_s)\Big(\lambda(t,T_s)\sum_{j=i(t)}^{s}\lambda(t,T_j)\ell(t_-,T_j)c_t dt + \lambda(t,T_s)c_t^{\frac{1}{2}}dW_t^{i(t)} + \int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(1-\prod_{j=i(t)}^{s}\frac{1}{\beta(t,x,T_j)})\nu_t^{i(t)}(dx,dt) + \int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(\mu-\nu_t^{i(t)})(dx,dt)\Big)$$

for all $T_s \in \{T_i\}_{i=1}^{n+1}$.
We start in the interval (T_n,T_{n+1}). We define the spot-LIBOR numeraire process $B(T)$ for all $T\in(T_n,T_{n+1})$ just as in the section on forward interpolation above and assume a positive bounded function $\lambda(t,T_s)$.
We can calculate a change of measure from $\mathbb{P}_{T_{n+1}}$ to $\mathbb{P}_{T+\delta}$ as in the section above as well

$$\frac{d\mathbb{P}_{T+\delta}}{d\mathbb{P}_{T_{n+1}}} = \mathcal{E}(\int_0^t \alpha(s,T+\delta,T_{n+1})c_t^{\frac{1}{2}}dW_s^{n+1} + \int_0^t\int_{\mathbb{R}}(\beta(s,x,T+\delta,T_{n+1})-1)(\mu-\nu_s^{n+1})(ds,dx)) \tag{2.1}$$

We know therefore, that a Brownian Motion for the forward measure $\mathbb{P}_{T+\delta}$ is given as

$$W_t^{T+\delta} = W_t^{n+1} + \int_0^t \lambda(s,T)\ell(s,T)c_s^{\frac{1}{2}}ds \qquad \forall t\in[0,T_{n+1}]$$

in terms of $\mathbb{P}_{T_{n+1}}$ with the compensator being

$$\nu_t^{T+\delta} = \nu_t^{n+1}\frac{1}{\beta(t,x,T_{n+1},T+\delta)} = \nu_t^{n+1}\frac{1}{\ell(t_-,T)(e^{\lambda(t,T)x}-1)+1}.$$

A forward LIBOR-rate for $T\in(T_n,T_{n+1}]$ has to fulfill

$$dL(t,T) = L(t_-,T)\Big(\lambda(t,T)c_t^{\frac{1}{2}}dW_t^{T+\delta} + \int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(\mu-\nu_t^{T+\delta})(dt,dx)\Big)$$

under its proper forward measure $\mathbb{P}_{T+\delta}$.
Under $\mathbb{P}_{T_{n+1}}$ we then get $W_t^{T+\delta} = W_t^{n+1} + \int_0^t \lambda(t,T)\ell(s_-,T)c_s^{\frac{1}{2}}ds$ and $\nu_t^T = \nu_t^{n+1}\frac{1}{\beta(t,x,T_{n+1},T+\delta)}$.
From this we get

$$dL(t,T) = L(t_-,T)\Big(\lambda(t,T)c_t^{\frac{1}{2}}dW_t^{n+1} + \lambda(t,T)^2 c_t\ell(t_-,T_{n+1})dt + \int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(1-\frac{1}{\beta(t,x,T_{n+1},T+\delta)})\nu_t^{n+1} + \int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(\mu-\nu_t^{n+1})\Big).$$

in terms of $\mathbb{P}_{T_{n+1}}$.
In terms of the spot-LIBOR measure we get

$$dL(t,T) = L(t_-,T)\Big(\sum_{j=i(t)}^{i(T)-1} \lambda(t,T_j)\lambda(t,T)c_t\ell(t_-,T_j)dt + \ell(t_-,T)c_t\lambda(t,T)^2)dt+$$
$$\int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(1-\prod_{j=i(t)}^{i(T)-1}\frac{1}{\beta(t,x,T_j,T_{j+1})})\nu_t^{i(t)}(dt,dx)+\int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(1-\frac{1}{\beta(t,x,T_{n+1},T+\delta)})\nu_t^{n+1}(dt,dx)+$$
$$\lambda(t,T)c_t^{\frac{1}{2}}dW_t^{n+1} + \int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(\mu-\nu_t^{n+1})(dt,dx)\Big)$$

So the interpolation on the interval $T \in (T_n, T_{n+1}]$ is well defined and we have that

$$W_t^T = W_t^{n+1} + \int_0^t \lambda(s,T)\ell(s_-,T)c_s^{\frac{1}{2}}ds = W_t^{i(T)} + \int_0^t \lambda(s,T)\ell(s_-,T)c_s^{\frac{1}{2}}ds \tag{2.2}$$

and

$$\nu_t^T = \nu_t^{i(T)}\frac{1}{\beta(t,x,T_{i(t)},T+\delta)}. \tag{2.3}$$

Once we have the interpolated model for a whole interval, we use the relation

$$\frac{d\mathbb{P}_{T-k\delta}}{d\mathbb{P}_{T-(k-1)\delta}} = \frac{F_B(t,T-k\delta,T-(k-1)\delta)}{F_B(0,T-k\delta,T-(k-1)\delta)} \qquad \forall k \le i(T)$$

to get Brownian Motions

$$W_t^{T-(k-1)\delta} = W_t^{i(T-k\delta)} + \int_0^t \lambda(t,T-k\delta)\ell(s,T-k\delta)c_t^{\frac{1}{2}}ds \qquad \forall -\infty < k \le i(T)-1$$

and Compensators

$$\nu_t^{T-k\delta} = \nu_t^{i(T-k\delta)}\frac{1}{\beta(t,x,i(T-k\delta),T-(k-1)\delta)} \qquad \forall -\infty < k \le i(T)-1$$

for all remaining maturities $T-k\delta \in [T_n-k\delta, T_{n+1}-k\delta]$ and our interpolated processes become solutions of

$$dL(t,T) = L(t_-,T)\Big(\sum_{j=i(t)}^{i(T)-1} \lambda(t,T)\lambda(t,T_j)c_t\ell(t_-,T_j)dt + \ell(t_-,T)c_t\lambda(t,T)^2)dt+$$
$$\int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(1-\prod_{j=i(t)}^{i(T)-1}\frac{1}{\beta(t,x,T_j,T_{j+1})})\nu^i(t)_t(dt,dx)+$$
$$\int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(1-\frac{1}{\beta(t,x,i(T)-1,T+\delta)})\nu_t^{i(T)}(dt,dx)+\lambda(t,T)c_t^{\frac{1}{2}}dW_t^{n+1}+\int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(\mu-\nu_t^{i(T)})(dt,dx)\Big)$$

For an arbitrary starting interval the procedure works as follows:

1. We look at $T \in (T_k, T_{k+1})$. We want to define the dynamics of $L(t,T)$ in an arbitrage-free way for all $T \in (T_k, T_{k+1})$. For that, we interpolate between two spot-LIBOR numeraires.

$$\log B(T)^* = (1-\gamma(t))\log B^*_{T_k} + \gamma(t)\log B^*_{T_{k+1}}, \qquad \forall T \in [T_k, T_{k+1}],$$

2. We determine the measure change between $\mathbb{P}_{T_{k+1}}$and $\mathbb{P}_{T+\delta}$.

$$\frac{d\mathbb{P}_{T+\delta}}{\mathbb{P}_{T_{k+1}}} = \mathcal{E}(\int_0^t \alpha(s,T,T_{k+1})c_t^{\frac{1}{2}}dW_s^{k+1} + \int_0^t \int_{\mathbb{R}} (e^{\lambda(s,T)} - 1)(\mu - \nu_s^{k+1})(ds,dx)) \tag{2.4}$$

3. We determine the Brownian Motion and the compensator for $\mathbb{P}_{T+\delta}$ in terms of the forward measure $\mathbb{P}_{T_{k+1}}$:

$$W_t^T = W_t^{k+1} + \int_0^t \lambda(s,T)\ell(s_-,T)c_s^{\frac{1}{2}}ds \tag{2.5}$$

and

$$\nu_t^T = \nu_t^{k+1}\frac{1}{\beta(t,x,T_{k+1},T+\delta)}. \tag{2.6}$$

4. From this we can determine the dynamics of $L(t,T)$ under the spot-LIBOR measure:

$$dL(t,T) = L(t_-,T)\Big(\lambda(t,T)(\sum_{j=i(t)}^{k}\lambda(t,T_j)\ell(t_-,T_j)c_t + \ell(t_-,T)c_t\lambda(t,T))dt+$$
$$\int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(1-\prod_{j=i(t)}^{k+1}\frac{1}{\beta(t,x,T_j,T_{j+1})})\nu_t^{i(t)}(dt,dx)+\int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(1-\frac{1}{\beta(t,x,T_{k+1},T+\delta)})\nu_t^{k+1}(dt,dx)+$$
$$\lambda(t,T)c_t^{\frac{1}{2}}dW_t^{i(t)} + \int_{\mathbb{R}}(e^{\lambda(t,T)x} - 1)(\mu - \nu_t^{i(t)})(dt,dx)\Big).$$

From one fully determined interval we can determine the LIBOR-rate process dynamics of any other by consequence of the forward measure changes to be

$$dL(t,T) = L(t_-,T)\Big(\lambda(t,T)(\sum_{j=i(t)}^{i(T)-1}\lambda(t,T_j)c_t\ell(t_-,T_j) + \ell(t_-,T)\lambda(t,T)c_t)dt+$$
$$\int_{\mathbb{R}}(e^{\lambda(t,T)x} - 1)(1 - \prod_{j=i(t)}^{i(T)-1}\frac{1}{\beta(t,x,T_j,T_{j+1})})\nu_t^{i(t)}(dt,dx)+$$
$$\int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(1-\frac{1}{\beta(t,x,i(T),T+\delta)})\nu_t^{i(T)-1}(dt,dx)+\lambda(t,T)c_t^{\frac{1}{2}}dW_t^{i(t)}+\int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(\mu-\nu_t^{i(t)})(dt,dx)\Big).$$

Hereby we have also shown, that our method does not depend on the particular choice of the starting interval(since every interval yields the same SDE). Also the restriction to a finite tenor structure is not necessary.

2.3 Existence Of LIBOR-Term Structure Models

We use the results we have derived so far to prove the following theorem

Theorem 7 *Given a equidistant discrete tenor structure $\{T_i\}_{i\in I}$(I possibly infinite), volatility functions $\{\lambda(t,T_i)\}_{i\in\mathbb{N}}$, an initial strictly positive, strictly decreasing term-structure $\Big(B(0,T)\Big)$ and a driving process*

$$X_t := \int_0^t b(s,T_1)ds + \int_0^t c_s^{\frac{1}{2}}dW_s^1 + \int_0^t \int_{\mathbb{R}} x(\mu - \nu_s^1)(ds,dx)$$

fulfilling

$$\int_0^t (|b(s,T_1)| + c_s)ds < \infty \qquad \forall t \in \mathbb{R}_+ \tag{2.7}$$

as well as

$$\int_0^\infty \int_{|x|\geq 1} \exp(ux)F_s(dx)ds < \infty \qquad u < M, M \geq \sum_{i\in\mathbb{N}} |\lambda(.,T_i)|, M < \infty \tag{2.8}$$

and

$$\int_0^\infty \int_{\mathbb{R}} (x^2 \wedge 1)F_s(dx)ds < \infty \tag{2.9}$$

then there is a LIBOR termstructe $\{L(t,T)\}_{t\leq T, T\in\mathbb{R}_+}$ *fulfilling*

$$dL(t,T) = L(t_-,T)\Big(\lambda(t,T)(\sum_{j=i(t)}^{i(T)-1} \lambda(t,T_j)c_t\ell(t_-,T_j)dt + \ell(t_-,T)c_t\lambda(t,T))dt +$$
$$\int_{\mathbb{R}} (e^{\lambda(t,T)x} - 1)(1 - \prod_{j=i(t)}^{i(T)-1} \frac{1}{\beta(t,x,T_j,T_{j+1})})\nu_t^{i(t)}(dt,dx) + \int_{\mathbb{R}} (e^{\lambda(t,T)x} - 1)(1 - \frac{1}{\beta(t,x,T_{i(T)},T+\delta)})\nu_t^{i(T)}(dt,dx) +$$
$$\lambda(t,T)c_t^{\frac{1}{2}}dW_t^{i(t)} + \int_{\mathbb{R}} (e^{\lambda(t,T)x} - 1)(\mu - \nu_t^{i(t)})(dt,dx)\Big).$$

for all $T \in \mathbb{R}_+$.

Proof:

1. Start with a finite tenor structure $\{T_i\}_{i=1}^{n+1}$.
2. Define a LIBOR-Market-Model under the Spot-LIBOR measure $\mathbb{P}_{Ls}$ as solution to the corresponding SDE's.

   $$dL(t,T_s) = L(t_-,T_s)\Big(\lambda(t,T_s)\sum_{j=i(t)}^{s} \lambda(t,T_j)\ell(t_-,T_j)c_t dt + \lambda(t,T_s)c_t^{\frac{1}{2}}dW_t^{i(t)} +$$
   $$\int_{\mathbb{R}} (e^{\lambda(t,T_s)x} - 1)(1 - \prod_{j=i(t)}^{s} \frac{1}{\beta(t,x,T_j,T_{j+1})})\nu_t^{i(t)}(dt,dx) + \int_{\mathbb{R}} (e^{\lambda(t,T_s)x} - 1)(\mu - \nu_t^{i(t)})(dt,dx)\Big)$$

   Since there are only finitely many factors entering into each equation, we have the usual existence and uniqueness theorems in finite dimension.
3. Interpolate between every two tenor points. This is possible through arbitrage free interpolation under the spot-LIBOR measure.
4. Extend this LIBOR-Market-Model to an infinite tenor structure as in the LIBOR extension theorem. This is well defined, as shown in the theorem.
5. Interpolate/extend for every extension for the tenor-grid, the LIBOR-rate dynamics in between.

6. Interpret the resulting family of LIBOR-rate processes $\{L(t,T)\}_{t,T}$ as a solution of an infinite dimensional problem

$$dL(t,T) = L(t_-,T)\Big(\sum_{j=i(t)}^{i(T)-1} \lambda(t,T_j)c_t\ell(t_-,T_j)dt + \ell(t_-,T)c_t\lambda(t,T)^2)dt+$$

$$\int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(1-\prod_{j=i(t)}^{i(T)-1}\frac{1}{\beta(t,x,T_j,T_{j+1})})\nu_t^{i(t)}(dt,dx)+\int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)$$

$$(1-\frac{1}{\beta(t,x,T_{i(T)},T+\delta)})\nu_t^{i(T)}(dt,dx)+\lambda(t,T)c_t^{\frac{1}{2}}dW_t^{i(t)}+\int_{\mathbb{R}}(e^{\lambda(t,T)x}-1)(\mu-\nu_t)^{i(t)})(dt,dx)\Big). \qquad \forall T\in\mathbb{R}_+\square$$

That way, we get an existence result for a considerably big class of term structure models without using a priori existence of an HJM model giving rise to the continuous (and unbounded) tenor LIBOR-Market-Model.

Naturally we might ask, whether a surrounding HJM model exists and how limiting procedures might work. Questions on the lines of [7] and [10] will be considered in future work.
There is another interpolation method by Schlögl in [17] pages 197-218, which is more flexible, but does not lead to an SDE analogous to the continuously compounded risk neutrally modeled rates of [3].

Chapter 3

A Kou-type Model

Our goal is to implement a tractable Lévy LIBOR model along the lines of Eberlein and Özkan [6]. This is achieved by choosing a (possibly time-inhomogeneous) compound Poisson process as driving process for the LIBOR rates. Such processes are both simple and flexible: They move by jumps only, and we have great freedom to choose the jump intensity and quite arbitrary distributions for the jump sizes constrained only by some integrability conditions.
It follows from the central limit theorem, that such processes include as limiting case the traditional lognormal LIBOR models. Concerning calibration an interesting reference is [2].

3.1 Some notation

We fix a positive integer $n \in \mathbb{N}$, a terminal horizon $T^* > 0$, choose a tenor structure

$$0 = T_{-1} < T_0 < T_1 < \cdots < T_n < T_{n+1} = T^*,$$

and define $\delta_i = T_i - T_{i-1}$ for $i = 0, \dots, n+1$. We set up a model for the LIBOR rates

$$L(t, T_k) = \frac{1}{\delta_{k+1}} \left[\frac{B(t, T_k)}{B(t, T_{k+1})} - 1 \right], \qquad (k = 0, \dots, n) \tag{3.1}$$

with $B(t, T)$ denoting, as usual, zero-coupon bond prices. Thus $L(t, T_k)$ is the simple interest rate for $[T_k, T_{k+1}]$ contracted at time $0 \le t \le T_k$. The caplet mechanism is the following: Fixing takes place at time T_k, and the corresponding caplet with strike K pays $(L(T_k, T_{k+1}) - K)_+$ at time T_{k+1}, where $k = 0, \dots, n$. Finally, the forward measure for time T_k is denoted by P^{T_k}.

3.2 Model ingredients

In setting up the model we use the following ingredients:

- For every tenor date T_k a 'volatility function' is chosen, which is denoted[1] by $\lambda(t, T_k)$. In our first implementation those functions will be piecewise constant,

$$\lambda(t, T_k) = \lambda_{ik}, \qquad (T_{i-1} \le t < T_i), \tag{3.2}$$

[1]This function is denoted by $\lambda(t, T_k)$ to distinguish it in a general Lévy LIBOR model from the 'Brownian volatility' $\sigma(t, T_k)$, though the latter is not used in our pure jump model.

see [4] for more details and other possibilities. The values λ_{ik} are to be determined later by calibrating the model to at-the-money caps and floors.

- We choose a time-dependent jump intensity $\gamma(t)$, governing the frequency of jumps. In our implementation those functions will be piecewise constant,

$$\gamma(t,) = \gamma_i, \qquad (T_{i-1} \leq t < T_i). \tag{3.3}$$

- We choose a time-dependent family $G(t, dx)$ of probability distributions for the jump-sizes. In our implementation it will be piecewise constant and admit probability densities. In principle we are completely free to choose arbitrary distributions, as long they obey sufficient integrability conditions. We choose a two-sided exponential distribution, as in the Kou jump-diffusion model [14]. Namely

$$g(t,x) = p_i \alpha_i e^{-\alpha_i x} I_{(x>0)} + (1-p_i)\beta_i e^{\beta_i x} I_{(x<0)}, \qquad (T_{i-1} \leq t < T_i). \tag{3.4}$$

This means: If a jump occurs in the interval $(T_{i-1}, T_i]$, then it is decided with probability p_i whether the jump is positive or negative. If positive the jump size is drawn from an exponential distribution with parameter α_i. If negative the modulus of the jump size is drawn from an exponential distribution with parameter β_i. This description is to be understood under the terminal measure $P^{T_{n+1}}$. The parameter α_i determines the right tail, the parameter β_i the left tail. In that way we can model semi-heavy tails, excess kurtosis, and skewness. The values of the parameters can be chosen from calibration to the cap and floor smile, or calibration to further instruments, such as swaptions, for example.

3.3 Exact calculations

Let $X(t)$ be the additive process driving the model, $N(t)$ the associated counting process, and S_j and X_j its j-th jump time and jump size. Below we show, that the LIBOR rates evolve according to

$$L(t, T_k) = L(0, T_k) \exp\left[-A(t, T_k) + \sum_{j=1}^{N(t)} \lambda(S_j, T_k) X_j\right]. \tag{3.5}$$

The 'logarithmic returns' of the LIBOR rates move by jumps, compensated with an absolutely continuous drift term. The jump term is fairly simple: If the j-th jump occurs at time $t = S_j$ the 'volatility' function $\lambda(t, T_k)$ is evaluated at $t = S_j$ and multiplied with the jump size X_j to produce the corresponding jump of $L(t, T_k)$ at $t = S_j$. The drift $A(t, T_k)$ must be chosen to make $L(t, T_k)$ a martingale under the forward measure $\mathbb{P}_{T_{k+1}}$. The form of the drift term is computed from subsequent measure changes $\mathbb{P}_{T_{n+1}} \mapsto \mathbb{P}^{T_n}, \ldots, \mathbb{P}^{T_{k+2}} \mapsto \mathbb{P}^{T_{k+1}}$ and the demand of martingality under the proper forward measure. Starting from the terminal measure $\mathbb{P}^{T_{n+1}}$ we change to the forward measure for time T_n, then from T_n to T_{n-1} and so on. Each time we apply the Girsanov theorem. The calculations are given below, let us simply describe the result: Firstly, since we work with piecewise constant quantities, we consider an interval of constancy, say $T_{i-1} \leq t < T_i$. Then $A(t, T_k)$ can be written as a sum of the drift at the beginning of the interval, plus an integral over some cumulant process, cf. [12],

$$A(t, T_k) = A(T_{i-1}, T_k) + \int_{T_{i-1}}^{t} \kappa^{T_{k+1}}(s, \lambda_{ik}) ds \qquad T_{i-1} \leq t < T_i, \qquad (i = 0, \ldots, k) \tag{3.6}$$

The drift at the interval boundaries is obtained by adding up the integrals over the intervals of constancy,

$$A(T_i, T_k) = \sum_{j=0}^{i} \int_{T_{j-1}}^{T_j} \kappa^{T_{k+1}}(s, \lambda_{jk}) ds \qquad (i = 0, \ldots, k) \tag{3.7}$$

The 'cumulant process' can be evaluated exactly. We have

$$\kappa^{T_{k+1}}(t, \theta) := \sum_{j=1}^{2^{n-k}} m_{kj}(t) \left[\kappa^{T_{n+1}}(t, \theta + \psi_{kj}) - \kappa^{T_{n+1}}(t, \psi_{kj})\right] \tag{3.8}$$

where

$$\kappa^{T_{n+1}}(t, \theta) = \gamma_i \left[p_i \frac{\alpha_i}{\alpha_i - \theta} + (1 - p_i) \frac{\beta_i}{\beta_i + \theta} - 1\right] \qquad T_{i-1} \leq t < T_i \tag{3.9}$$

valid for $-\beta_i < \Re(\theta) < \alpha_i$! We use this as definition of κ^{T_k} for $k < n + 1$ while the equation for $\kappa^{T_{n+1}}$ means just

$$\kappa^{T_{n+1}}(t, \theta) := \int_0^t \int_{\mathbb{R}} (e^{\theta x} - 1) \nu_s(ds, dx). \tag{3.10}$$

The coefficients $m_{kj}(t)$ and ψ_{kj} can be evaluated recursively with k going backwards from n down to 1 as follows: Let

$$\ell(t, T_k) = \frac{\delta_{k+1} L(t, T_k)}{1 + \delta_{k+1} L(t, T_k)}. \tag{3.11}$$

We start the recursion with

$$m_{n1}(t) = 1, \qquad \psi_{n1} = 0 \tag{3.12}$$

and proceed by

$$m_{k-1,j}(t) = (1 - \ell(t-, T_k)) m_{kj}(t), \qquad \psi_{k-1,j} = \psi_{kj} \qquad (j = 1, \ldots, 2^{n-k}) \tag{3.13}$$

and

$$m_{k-1,j}(t) = \ell(t-, T_k) m_{k,j-2^{n-k}}(t), \qquad \psi_{k-1,j} = \lambda_{ik} + \psi_{k,j-2^{n-k}} \qquad (j = 2^{n-k} + 1, \ldots, 2^{n-k+1}). \tag{3.14}$$

All calculations so far are elementary and explicit, and they can be implemented directly in a computer program, except for the integrals in (3.6) and (3.7). In principle, they are not difficult: Inbetween two jumps, we have to integrate deterministic functions. Due to the recursive nature, it is not clear, whether we can evaluate the integrals analytically. Even if so, we have to evaluate about 2^n terms, at each jump, which is too costly for an efficient implementation. Instead we take up a suggestion from the literature based on the observation, that the time-dependent and random components $\ell(t, T_k)$ do not vary much, in comparison with all other quantities.

3.4 The Exact Calculations Yield A LIBOR Market Model

Before we come to approximation, it is important, that we see, that the above model constitutes a LIBOR market model in the sense of [6].
We assume a LIBOR market model

$$L(t, T_j) = L(0, T_j) \exp\Big(\int_0^t \lambda(s, T_j) dY_s^{j+1}\Big) \tag{3.15}$$

with the same volatility structure as above, same initial values and Y_s^{n+1} of the form

$$dY_s^{n+1} = b_s^{n+1} ds + \int_{\mathbb{R}} x(\mu - \nu_s^{n+1})(ds, dx) \tag{3.16}$$

where $\int_{\mathbb{R}} x\mu(s, dx) = X_s$ with X_s denoting the same Kou-Process as described above and

$$\int_0^t \lambda(s, T_n) b_s^{n+1} ds = -\int_0^t \int_{\mathbb{R}} (e^{\lambda(s,T_n)x} - 1 - \lambda(s, T_n)x)\nu^{n+1}(ds, dx). \tag{3.17}$$

Now with b_s chosen as in the equation above, we have

$$\int_0^t \lambda(s, T_n) dY_s^{n+1} = -\int_0^t \int_{\mathbb{R}} (e^{\lambda(s,T_n)x} - 1)\nu^{n+1}(ds, dx) + \int_{\mathbb{R}} \lambda(s, T_n)x\mu(ds, dx). \tag{3.18}$$

Comparing with 3.5 for $k = n$ we get

$$A(t, T_n) = \int_0^t \int_{\mathbb{R}} (e^{\lambda(s,T_n)x} - 1)\nu^{n+1}(ds, dx) \tag{3.19}$$

which holds true by the definition of κ.
For the later rates, we have 3.5 in our model and

$$L(t, T_k) = L(0, T_k) \exp(\int_0^t \lambda(s, T_k) dY_s^{k+1}) \tag{3.20}$$

in the Eberlein Özkan approach. We have

$$\int_0^t \lambda(s, T_k) b_s^{k+1} ds = -\int_0^t \int_{\mathbb{R}} (e^{\lambda(s,T_k)x} - 1 - \lambda(s, T_k)x)\nu^{k+1}(ds, dx). \tag{3.21}$$

for the later rates as well as

$$dY_s^{k+1} = b_s^{k+1} ds + \int_{\mathbb{R}} x(\mu - \nu_s^{k+1})(ds, dx). \tag{3.22}$$

That combined gives us again

$$\int_0^t \lambda(s, T_k) dY_s^{k+1} = -\int_0^t \int_{\mathbb{R}} (e^{\lambda(s,T_k)x} - 1)\nu^{k+1}(ds, dx) + \int_{\mathbb{R}} \lambda(s, T_k)x\mu(ds, dx). \tag{3.23}$$

That means we would need

$$A(t, T_k) = \int_0^t \int_{\mathbb{R}} (e^{\lambda(s,T_k)x} - 1)\nu^{k+1}(ds, dx) \tag{3.24}$$

to hold.
Concerning the concrete form of the cumulant process, we have the recursion for our compensators in the Eberlein Özkan representation

$$\nu_t^k = \Big(\ell(t_-, T_k)(e^{\lambda(t,T_k)x} - 1) + 1\Big)\nu_t^{k+1} = \Big(\ell(t_-, T_k)e^{\lambda(t,T_k)x} + (1 - \ell(t_-, T_k))\Big)\nu_t^{k+1}. \tag{3.25}$$

Combining this we get

Lemma 4 *The representation in (3.8) holds for $k = n$*

Proof:
We start with $\kappa^{T_n}(t, \lambda_i n)$

$$\kappa^{T_n}(t, \lambda_i n) = \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} (e^{\lambda_{i(n-1)}x} - 1)\nu_t^n(dt, dx) = \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} (e^{\lambda_{i(n-1)}x} - 1)(\ell(t_-, T_n)(e^{\lambda_{in}x} - 1) + 1)\nu_t^{n+1}(dt, dx). \tag{3.26}$$

Now

$$\int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} (e^{\lambda_{i(n-1)}x} - 1)(\ell(t_-, T_n)(e^{\lambda_{in}x} - 1) + 1)\nu_t^{n+1}(dt, dx) = \tag{3.27}$$

$$\int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} (e^{\lambda_{i(n-1)}x} - 1)(\ell(t_-, T_n)e^{\lambda_{in}x} + (1 - \ell(t_-, T_n)))\nu_t^{n+1}(dt, dx) =$$

$$\ell(t_-, T_n) \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} (e^{(\lambda_{i(n-1)} + \lambda_{in})x} - 1)\nu_t^{n+1}(dt, dx) -$$

$$\ell(t_-, T_n) \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} (e^{\lambda_{in}x} - 1)\nu_t^{n+1}(dt, dx) + (1 - \ell(t_-, T_n)) \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} (e^{\lambda_{i(n-1)}x} - 1)\nu_t^{n+1}(dt, dx).$$

In terms of our assumed cumulant processes this means

$$\kappa^{T_n}(t, \lambda_{i(n-1)}) = \ell(t_-, T_n)\kappa^{T_{n+1}}(t, \lambda_{i(n-1)}) - \ell(t_-, T_n)\kappa^{T_{n+1}}(t, \lambda_{in}) + (1 - \ell(t_-, T_n))\kappa^{T_{n+1}}(t, \lambda_{i(n-1)}). \tag{3.28}$$

Now we look at 3.8 again:

$$\kappa^{T_n}(t, \theta) = \sum_{j=1}^{2} m_{(n-1)j}(t) \left[\kappa^{T_{n+1}}(t, \theta + \psi_{(n-1)j}) - \kappa^{T_{n+1}}(t, \psi_{(n-1)j})\right]$$

and we have as values of m $m((n-1)1) = (1 - \ell(t_-, T_n))$ with corresponding $\psi_{(n-1),1} = 0$ and $m((n-1)2) = \ell(t_-, T_n)$ with $\psi_{(n-1),2} = \lambda_{in}$. Plugging this into 3.8 we get just our equation above. $\square$

We proceed to give an inductive step

Lemma 5 *The equality 3.8 is true for any k provided it held for $k + 1$.*

Proof:
We have

$$\int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} (e^{\lambda_{i(k-1)}x} - 1)(\ell(t_-, T_k)(e^{\lambda_{ik}x} - 1) + 1)\nu_t^{k+1}(dt, dx) = \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} (e^{\lambda_{i(k-1)}x} - 1)(\ell(t_-, T_k)e^{\lambda_{ik}x} +$$

$$(1 - \ell(t_-, T_k)))\nu_t^{k+1}(dt, dx) = \ell(t_-, T_l) \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} (e^{(\lambda_{i(k-1)} + \lambda_{ik})x} - 1)\nu_t^{k+1}(dt, dx) -$$

$$\ell(t_-, T_k) \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} (e^{\lambda_{ik}x} - 1)\nu_t^{k+1}(dt, dx) + (1 - \ell(t_-, T_k)) \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} (e^{\lambda_{i(k-1)}x} - 1)\nu_t^{k+1}(dt, dx)$$

for a given time interval $[t_{i-1}, t_i]$ with $i \leq k$. So it holds that

$$\kappa^{T_k}(t, \lambda_{i(k-1)}) = \ell(t_-, T_k)\kappa^{T_{k+1}}(t, \lambda_{i(k-1)}) - \ell(t_-, T_k)\kappa^{T_{k+1}}(t, \lambda_{ik}) + (1 - \ell(t_-, T_k))\kappa^{T_{k+1}}(t, \lambda_{i(k-1)}). \tag{3.29}$$

Now we use that 3.8 holds for $k+1$:

$$\kappa^{T_k}(t, \lambda_{i(k-1)}) = \ell(t_-, T_k) \sum_{j=1}^{2^{n-k}} m_{(k-1)j}(\kappa(t, \lambda_{i(k-1)} + \lambda_{ik} + \psi_{k(j-1)}) -$$

$$\kappa(t, \lambda_{kj} + \psi_{k(j-1)})) + (1 - \ell(t_-, T_k)) \sum_{j=1}^{2^{n-k}} m_{(k-1)j}(\kappa^{T_{n+1}}(t, \lambda_{i(k-1)} + \psi_{k(j-1)}) - \kappa^{T_{n+1}}(t, \psi_{k(j-1)})).$$

Since $m_{(k-1)j} = (1 - \ell(t_-, T_k))m_{kj}$ and $\psi_{kj} = \psi_{k(j-1)}$ for $k \in \{1, \ldots, 2^{n-k}\}$ and $m_{(k-1)j} = \ell(t_-, T_k)m_{kj}$ and $\psi_{kj} = \lambda_{ik} + \psi_{k(j-1)}$ for $k \in \{1, \ldots, 2^{n-k}\}$, we get that 3.8 holds for k. $\square$.

This we we have shown that if we use 3.8 to define the $\kappa^{T_{k+1}}$, then is holds that

$$\kappa^{T_{k+1}}(t, \lambda_{ik}) = \int_{t_{i-1}}^{t_i} (e^{\lambda_{ik}x} - 1)\nu_t^{k+1}(dt, dx) \tag{3.30}$$

Which is also the final step in showing that our approach conforms to LIBOR market models as constructed in [6]. $\square$

3.5 The Piecewise Frozen Drift Approximation

The basic idea is to replace $\ell(t-, T_k)$ by $\ell(T_{i-1}, T_k)$ for $T_{i-1} \le t < T_i$. This changes slightly the dynamic structure of the model, so we give the complete description for the approximation. The approximate LIBOR rates satisfy

$$\tilde{L}(t, T_k) = L(0, T_k) \exp\left[-\tilde{A}(t, T_k) + \sum_{j=1}^{N(t)} \lambda(S_j, T_k)X_j\right]. \tag{3.31}$$

The approximate drift is

$$\tilde{A}(t, T_k) = \tilde{A}(T_{i-1}, T_k) + \tilde{\kappa}_{i-1}^{T_{k+1}}(\lambda_{ik})(T_{i-1} - t) \qquad T_{i-1} \le t < T_i, \qquad (i = 0, \ldots, k) \tag{3.32}$$

The drift at the interval boundaries is obtained by adding up the integrals over the intervals of constancy,

$$\tilde{A}(T_i, T_k) = \sum_{j=0}^{i} \tilde{\kappa}_{j-1}^{T_{k+1}}(\lambda_{jk})\delta_j \qquad (i = 0, \ldots, k) \tag{3.33}$$

Now

$$\tilde{\kappa}_i^{T_{k+1}}(\theta) = \sum_{j=1}^{2^{n-k}} \tilde{m}_{kji}\left[\kappa^{T_{n+1}}(T_i, \theta + \tilde{\psi}_{kji}) - \kappa^{T_{n+1}}(T_i, \tilde{\psi}_{kji})\right] \tag{3.34}$$

Note, that the terminal cumulant process $\kappa^{T_{n+1}}(t, \theta)$, is as given in (3.9) above, it is not affected by the approximation. The coefficients $\tilde{m}_{kji}$ and $\tilde{\psi}_{kji}$ can be evaluated recursively with k going backwards from n down to 1 as follows: We start the recursion with

$$\tilde{m}_{n1i} = 1, \qquad \tilde{\psi}_{n1i} = 0 \tag{3.35}$$

and proceed by

$$\tilde{m}_{k-1,j,i}(t) = (1 - \ell(T_i-, T_k))\tilde{m}_{kji}, \qquad \tilde{\psi}_{k-1,j,i}(t) = \tilde{\psi}_{kji} \qquad (j = 1, \ldots, 2^{n-k}) \tag{3.36}$$

and

$$\tilde{m}_{k-1,j,i} = \ell(T_i-, T_k)\tilde{m}_{k,j-2^{n-k},i}, \qquad \tilde{\psi}_{k-1,j,i} = \lambda_{ik} + \tilde{\psi}_{k,j-2^{n-k},i} \qquad (j = 2^{n-k}+1, \dots, 2^{n-k+1}). \tag{3.37}$$

3.6 Completely Frozen Drift

In the former model the drift, though piecewise constant, is still a stochastic object in the sense that it depends on $\tilde{L}$. Therefore calculating things like the variance of a given rate under the terminal measure are fairly involved.
The Lévy-Khinchine formula on which most calculations concerning exponential Lévy-models are based for instance would not be valid .
However if we freeze the drift for each rate at the very beginning of our considered time-span then each process decomposes into a deterministic drift function and a (Lévy) jump part.
The formulae are as follows:

$$\tilde{L}(t, T_k) = L(0, T_k) \exp\left[-\tilde{A}(t, T_k) + \sum_{j=1}^{N(t)} \lambda(S_j, T_k) X_j\right]. \tag{3.38}$$

There's no principal difference in the form of the driving process and $\tilde{A}(t, T_k)$ is still given by

$$\tilde{A}(T_i, T_k) = \sum_{j=0}^{i} \tilde{\kappa}_{j-1}^{T_{k+1}}(\lambda_{jk})\delta_j \qquad (i = 0, \dots, k). \tag{3.39}$$

$\tilde{\kappa}$ is given by our recursion and therefore we have that $\tilde{\kappa}$ is deterministic if λ is piecewise constant and $\ell(T_i, T_j)$ is constant for each j.
Now if $\tilde{\kappa}$ is deterministic then of course $\tilde{A}$ is deterministic too.
Now since we have a deterministic drift our rate-process $\tilde{L}$ is a Lévy-process and we can apply the Lévy-Khinchine formula and calculate

$$\mathbb{E}(\tilde{L}(T_s, T_k)) = L(0, T_k) \exp(-\tilde{A}(T_s, T_k)) \exp(\sum_{i=1}^{s} \tilde{\kappa}_{n+1}^{T}(\lambda_{ik})). \tag{3.40}$$

So calculating Moments is well possible in this setting.

3.7 Pricing Caplets In The Frozen Drift Version

In order to calibrate to market data we need to be able to price caps in our model. In fact we need a pricing formula for the caplets making up the concrete cap. We apply bilateral Laplace-Transforms (see also [6]) to obtain this.
The j-th caplet's payoff (for strike K) can be expressed as

$$\delta_j(L(0, T_{j-1})e^{(-\tilde{A}(T_{j-1}, T_{j-1}) + \sum_{z=1}^{N(T_{j-1})} \lambda(S_z, T_{j-1})X_{j-1})} - K)^+ \tag{3.41}$$

From now on $\mathbb{E}_{\mathbb{P}_{T_j}}$ denotes the expectation w.r.t. $\mathbb{P}_{T_j}$. For pricing we need

$$B(0, T_j)\delta_{T_j}\mathbb{E}_{\mathbb{P}_{T_j}}(e^{(-\tilde{A}(T_{j-1}, T_{j-1}) + \sum_{z=1}^{N(T_{j-1})} \lambda(S_z, T_{j-1})X_{j-1}} - K)^+.$$

We do not attempt to calculate this expectation directly but instead calculate the Laplace-transform

$$L[v_K](z) = \int_{\mathbb{R}} e^{-zx}(e^x - K)^+ dx. \qquad z \in \mathbb{C} \tag{3.42}$$

Then we calculate the moment-generating function of $L(T_{j-1}, T_{j-1})$ under $\mathbb{P}_{T_j}$:

$$m(u) = \mathbb{E}_{\mathbb{P}_{T_j}}(e^{(-u\tilde{A}(T_{j-1},T_{j-1})+u\sum_{z=1}^{N(T_{j-1})} \lambda(S_z,T_{j-1})X_{j-1})}). \tag{3.43}$$

Now $\tilde{A}(t, T_k)$ is deterministic for frozen drift and the jump-part $\sum_{z=1}^{N(t)} \lambda(S_z, T_k)X_z$ is time-inhomogeneous Lévy. At least for every time-interval $[T_i, T_{i+1}]$ we may apply the Lévy-Khinchine formula to the jump-part. This yields

$$m(u) = e^{-u\tilde{A}(T_{j-1},T_{j-1})} e^{\sum_{z=1}^{j-1} \int_{\mathbb{R}} (e^{u\lambda(z(j-1))}-1)d\nu^{T_j}}. \tag{3.44}$$

Through Laplace-inversion we get the j-th Caplet-Price $V_j(\zeta_j, K)$ at time 0, where $\zeta_j = -\log L(0, T_{j-1})$. The concrete formula is(see [6])

$$V_j(\zeta_j, K) = \delta_j B(0, T_j) \frac{1}{2\pi} \lim_{M\to\infty} \int_{-M}^{M} L[v_K](R + iu)m(R + iu)du. \tag{3.45}$$

Here $R > 1$ to ensure convergence of the integral.
All this works for frozen drift however we also aim to establish pricing rules for the piecewise-constant version as well. Since ℓ is stochastic in this case, the expectation above gets a lot more complicated.

3.8 Pricing Caplets In The Piece-Wise Model

So far we see only one option in pricing in the piece-wise model without freezing the drift. We must use the driving-process-increments of the first modelled rate as independent variables(since that is always a Lévy-Process) and explicitly write all functions in terms of those variables. In principle this is possible and the expectation of the caplet pay-off then becomes a multidimensional integral w.r.t the increment variables and their densities.
Let $p_1(x), \ldots p_n(x)$ denote the distributions functions of the respective driving process increments $X_1, \ldots, X_n$. Then

$$L(T_i, T_k) = L(0, T_k)f(X_1, \ldots, X_i) \tag{3.46}$$

hence

$$\ell(T_i, T_k) = g(X_1, \ldots, X_n). \tag{3.47}$$

And therefore

$$\mathbb{E}(\ell(T_i, T_k)) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{\delta L(x_1, \ldots, x_i)}{1 + \delta L(x_1, \ldots, x_i)} p_1(x_1)p_2(x_2) \ldots p_i(x_i)dx_1dx_2 \ldots dx_i. \tag{3.48}$$

A bit more explicitly we look at the first ℓ term appearing in the first forward.
First we obviously have

$$L(T_i, T_n) = L(0, T_j)f(X_1, \ldots, X_i). \tag{3.49}$$

Now with

$$\ell(T_i, T_n) = \frac{\delta_i L(T_i, T_n)}{1 + \delta_i L(T_i, T_n)}$$

we get

$$\ell(T_i, T_n) = \frac{\delta L(0,T_n)\exp(\sum_{l=1}^{i} A(X_l) + \sum_{r=1}^{N_{T_i}} \lambda_{r,n} S(r))}{1+\delta L(0,T_n)\exp(\sum_{l=1}^{i} A(X_l) + \sum_{r=1}^{N_{T_i}} \lambda_{r,n} S())} \tag{3.50}$$

so

$$\ell(T_i, T_n) = g(X_1, \dots, X_i). \tag{3.51}$$

While complicated in principle this can be calculated. For pricing we need a moment generating function. We have to calculate the expectation of $\exp(\kappa(.)^{T_i})$(given here for the k-th Rate)

$$\mathbb{E}(\exp(\tilde{\kappa}_i^{T_{k+1}}(\theta))) = \mathbb{E}(\exp(\sum_{j=1}^{2^{n-k}} \tilde{m}_{kji} \left[\kappa^{T_{n+1}}(T_i, \theta + \tilde{\psi}_{kji}) - \kappa^{T_{n+1}}(T_i, \tilde{\psi}_{kji})\right])) =$$
$$\int_{\mathbb{R}} \dots \int_{\mathbb{R}} \exp(\sum_{j=1}^{2^{n-k}} \tilde{m}_{kji}(x_1, \dots, x_i) \left[\kappa^{T_{n+1}}(T_i, \theta + \tilde{\psi}_{kji}) - \kappa^{T_{n+1}}(T_i, \tilde{\psi}_{kji})\right]) p_1(x_1) p_2(x_2) \dots p_i(x_i) dx_1 dx_2 \dots dx_i \tag{3.52}$$

A Caplet-price for T_{n-1} would then look like

$$\mathbb{E}((L(T_{n-1}, T_{n-1}) - K)^+) = \delta B(T_n) \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \dots \int_{\mathbb{R}} L[v_K](\theta)$$
$$\exp(\sum_{j=1}^{2^{n-k}} \tilde{m}_{kji}(x_1, \dots, x_i) \left[\kappa^{T_{n+1}}(T_i, \theta + \tilde{\psi}_{kji}) - \kappa^{T_{n+1}}(T_i, \tilde{\psi}_{kji})\right]) dx_1 dx_2 \dots dx_i du \qquad \theta = R + iu \tag{3.53}$$

The formulae look quite messy when written down explicitly but in theory everything is nice and calculable analytically (though too costly in general).

3.9 Monte-Carlo-Simulation

3.9.1 Frozen Drift

For pricing complex path-dependent options or options in a more complicated model, Monte-Carlo-Simulation is always an option. Assume for instance that we want to compare the results of Monte-Carlo-Pricing to our analytical pricing formula in the frozen drift case.
We have to calculate

$$B(0,T_j)\delta_{T_j} \mathbb{E}_{\mathbb{P}_{T_j}}(e^{(-\tilde{A}(T_{j-1},T_{j-1}) + \sum_{z=1}^{N(T_{j-1})} \lambda(S_z, T_{j-1}) X_{j-1}} - K)^+. \tag{3.54}$$

We do this by calculating paths of the respective rate under the forward measure $\mathbb{P}_{T_j}$. In that case, our jump-size distribution changes. We have the compensator

$$\left[\prod_{k=j+1}^{n} \beta(t, x, T_j, T_{j+1})\right] F^{T_{n+1}}(t, dx). \tag{3.55}$$

This decomposes into a sum

$$\nu^{T_{j+1}} = \sum_{u=1}^{2^{n-j}} A_{u(n-j+1),t_i} B_{u(n-j+1),t_i} \exp(C_{u(n-j+1),t_i} x) F(dx). \tag{3.56}$$

Where

$$A_{0,t_i} = 1 \qquad A_{n,t_i} = c((1-\ell(t_i,T_n))A_{n-1,t_i}, A_{n-1,t_i})$$

$$B_{0,t_i} = 1 \qquad B_{n,t_i} = c(B_{n-1,t_i}, \ell(t_i,T_n)B_{n-1,t_i})$$

$$C_{0,t_i} = 0 \qquad C_{n,t_i} = c(C_{n-1,t_i}, (\psi_{in} + C_{n-1,t_i}).$$

So we see that the distribution of jump-sizes changes considerably with each rate. But its still a mixture of exponential distributions since we assume our drift-frozen. Then $A_{u(n-j+1),t_i}B_{u(n-j+1),t_i}$ is deterministic at all times. $A_{u(n-j+1),t_i}B_{u(n-j+1),t_i}$ is also always positive(even in the stochastic ℓ case). So we can directly simulate from two-sided exponential distributions with exponents modified by $(C_{u(n-j+1),t_i}x)$. The intensity of our Poisson-process changes, for instance to

$$\gamma_{T_{j+1}} = A_{u(n-j+1),t_i}B_{u(n-j+1),t_i}\frac{\gamma\alpha}{(\alpha - C_{u(n-j+1),t_i})} \tag{3.57}$$

and analogously for β.

3.9.2 Piece-Wise Frozen Drift

All the above equations hold, but $A_{u(n-j+1),t_i}B_{u(n-j+1),t_i}$ is a stochastic term now and therefore the distribution need not be a sum of exponential random variables anymore. In fact the distribution is almost certainly more complex.
It is still an open question if we can simulate random variables with the distribution of that object and will be a topic for further research. A "last resort" would of course be the "increment variable" approach presented in the pricing section.

3.9.3 Full Model

In the full model we would have to simulate the jump-times for the last rate and then recalculate all process characteristics for the earlier rates at each jump.

3.10 Numerical Illustration

3.10.1 Caplet-Pricing

In view of calibrating our model to market-data we need to be able to calculate caplet-prices in our models. At the moment our explicit formula relies on the completely frozen drift approximation. Though we hope to ease this restriction in the future for the time being we calculate in the frozen-drift-approximation our caplet prices depending on strike and time to maturity.

We compare our approach through cumulant functions to values taken from the Dissertation of Kluge [13].
Initial conditions are

μ	α	β	δ
0	1.5	0	1.5

and

λ_9	0.2
λ_8	0.19
λ_7	0.18
λ_6	0.17
λ_5	0.16
λ_4	0.15
λ_3	0.14
λ_2	0.13
λ_1	0.12

The prices we obtain are

65.717	42.047	20.617	7.053	2.296	0.867	0.374	0.179	0.097	0.051
93.666	70.364	48.072	28.768	15.041	7.305	3.542	1.781	0.938	0.518
91.664	69.104	47.990	30.192	17.348	9.425	5.035	2.718	1.504	0.857
109.461	87.269	65.968	46.749	30.989	19.420	11.729	6.966	4.134	2.477
106.849	85.272	64.704	46.308	31.261	20.095	12.489	7.623	4.630	2.825
114.821	93.692	73.328	54.658	38.736	26.230	17.142	10.940	6.894	4.329
111.924	91.337	71.519	53.384	37.935	25.783	16.914	10.826	6.831	4.285
117.013	96.876	77.304	59.058	43.074	30.063	20.211	13.207	8.466	5.367
113.944	94.308	75.186	57.318	41.641	28.880	19.243	12.428	7.853	4.897

In implied Black volatilities we get

25.33	21.62	18.94	18.81	20.87	23.28	25.49	27.45	29.36	30.74
22.48	20.56	19.12	18.32	18.25	18.79	19.64	20.58	21.52	22.41
19.89	18.69	17.89	17.51	17.52	17.82	18.30	18.85	19.43	19.99
18.67	17.77	17.14	16.76	16.62	16.67	16.86	17.14	17.47	17.82
17.19	16.52	16.06	15.80	15.71	15.75	15.88	16.08	16.32	16.58
16.08	15.52	15.13	14.89	14.78	14.77	14.84	14.96	15.12	15.30
14.89	14.42	14.10	13.91	13.82	13.81	13.86	13.96	14.09	14.23
13.85	13.44	13.15	12.97	12.87	12.84	12.86	12.92	13.01	13.12
12.76	12.39	12.14	11.98	11.90	11.87	11.88	11.93	12.01	12.10

As the Table above shows our approach via calculation of the cumulant functions yields nearly identical caplet prices and implied black volatilities as the implementation in [13] . Of course our own finite variation models show different smiles.

3.10.2 Data And Models

We are given LIBOR rates and caplet-prices for a particular day.
Our tenor structure is

$$T_n = nY \qquad \forall n \in \{1, 2, \dots, 10\}$$

We are given an initial term structure

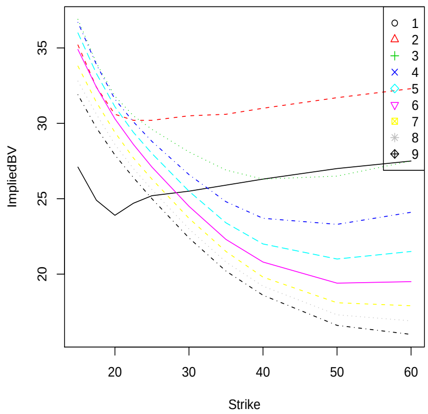
$L(0, T_{10})$	0.0407229153738
$L(0, T_7)$	0.03626087939424
$L(0, T_5)$	0.03189591283065
$L(0, T_3)$	0.02621803947455
$L(0, T_2)$	0.02306848307215
$L(0, T_1)$	0.020769754278

and caplet prices given by implied black volatilities

maturities	strikes								
	15	17.5	20	22.5	25	30	35	40	50	60
1Y	27.1	24.9	23.9	24.7	25.2	25.5	25.9	26.3	27.0	27.5
2Y	35.2	32.4	30.6	30.2	30.2	30.5	30.6	31.0	31.7	32.3
3Y	36.9	34.0	31.8	30.5	29.6	28.1	26.9	26.3	26.5	27.5
4Y	36.7	33.9	31.6	30.1	28.8	26.6	24.8	23.7	23.3	24.1
5Y	36.0	33.3	31.1	29.4	28.0	25.5	23.4	22.0	21.0	21.5
6Y	34.9	32.4	30.3	28.6	27.1	24.5	22.3	20.8	19.4	19.5
7Y	33.8	31.4	29.4	27.7	26.3	23.7	21.5	19.8	18.1	17.9
8Y	32.8	30.6	28.6	27.0	25.6	23.0	20.8	19.2	17.3	16.9
9Y	31.9	29.7	27.9	26.4	25.0	22.4	20.2	18.6	16.6	16.0

We look at the smiles

Maturities $1Y$ (1) to $9Y$ (9), 10 Strikes



From that, we see that the behavior at 2Y and 1Y is significantly different from the behavior of the other maturities.

Through a linear spline interpolation we calculate LIBOR-rates

$L(0, T_{10})$	0.0407229153738
$L(0, T_9)$	0.03946148
$L(0.T_8)$	0.03798937
$L(0, T_7)$	0.03626087939424
$L(0, T_6)$	0.03423581
$L(0, T_5)$	0.03189591283065
$L(0, T_4)$	0.02922813
$L(0, T_3)$	0.02621803947455
$L(0, T_2)$	0.02306848307215
$L(0, T_1)$	0.020769754278

From this, we calculate bond-prices through

$$B(0, T_{i+1}) = \prod_{k=1}^{i} (\frac{1}{(L(0, T_k)} + 1)) \tag{3.58}$$

which yields

$B(0, T_{10})$	0.7597843
$B(0, T_9)$	0.7897666
$B(0.T_8)$	0.8197693
$B(0, T_7)$	0.8494948
$B(0, T_6)$	0.8785780
$B(0, T_5)$	0.9066010
$B(0, T_4)$	0.9330993
$B(0, T_3)$	0.9575633
$B(0, T_2)$	0.9796529

We use yearly tenor structure and try 2 models empirically below. The Kou-type model and the NiG-Lévy model as outlined in Wolfgang Kluge's dissertation [13].
The cumulant of the Kou-type process is

$$\kappa(z) = \mu z + \gamma(p\frac{\eta_1}{\eta_1 - z} + (1-p)\frac{\eta_2}{(\eta_2 + z) - 1}). \tag{3.59}$$

The cumulant of the NiG-Process is

$$\kappa(z) = \mu z + \delta(\sqrt{(\alpha^2 - \beta^2)} - \sqrt{(\alpha^2 - (\beta + z)^2)}) \tag{3.60}$$

however Kluge uses an approximation
Parameters are determined by calibration(see below)

3.11 Calibration

To calibrate our models to market data we use a non-linear minimizing procedure implemented in R. Our targetfunction is

$$O = \sum_{j=1}^{k} \sum_{i=1}^{n} (G[i,j] - V[i,j])^2 \tag{3.61}$$

where G denotes the empirical caplet prices in implied Black volatilities for the maturities $1Y, \ldots, nY$ and k given strikes, V denotes the caplet prices implied by our model given in implied Black volatilites and our input are the process parameter plus some starting values for the volatilities λ_i.
We use the "optim"-routine implemented in R, with box constraints. This method is due to [5] and is implemented in the R "stats" package.
It is not fast, but it works well otherwise, and for existence of the Laplace integral or even the model itself, we require some boundary conditions.
We then proceed by using the result of this step as initial value and using as input in the next step all parameters of the process plus the volatilities for the two longest rates.
We get as parameters for NiG

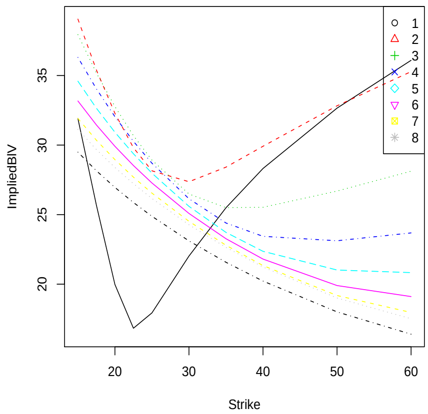
α	β	δ	μ
0.67	-0.58	0.26	0

We get lambda coefficients

λ_{T_9}	0.3946148
λ_{T_8}	0.3798937
λ_{T_7}	0.3626087939424
λ_{T_6}	0.3423581
λ_{T_5}	0.3189591283065
λ_{T_4}	0.2922813
λ_{T_3}	0.2621803947455
λ_{T_2}	0.2306848307215
λ_{T_1}	0.20769754278

and smiles

Maturities $1Y(1)$ to $8Y(8)$, 10 Strikes



The concrete values of the implied Black-volatilities for the calculated prices are

maturities	strikes									
	15	17.5	20	22.5	25	30	35	40	50	60
1 Y	31.87	25.64	19.95	16.82	17.93	22.03	25.5	28.32	32.67	36.11
2Y	39	35.35	32.24	29.78	28.13	27.35	28.41	29.92	32.83	35.29
3Y	37.88	35.13	32.72	30.63	28.88	26.45	25.48	25.52	26.69	28.13
4Y	36.23	33.98	31.99	30.21	28.66	26.1	24.36	23.41	23.11	23.68
5Y	34.52	32.58	30.86	29.31	27.92	25.54	23.68	22.32	20.99	20.82
6Y	33.09	31.37	29.84	28.46	27.21	25.02	23.21	21.76	19.86	19.08
7Y	31.83	30.27	28.88	27.62	26.47	24.45	22.73	21.28	19.14	17.93
8Y	31.03	29.59	28.3	27.14	26.08	24.19	22.57	21.17	18.94	17.47
9Y	29.39	28.05	26.86	25.78	24.79	23.03	21.49	20.15	17.94	16.35

We get the relative error in volatilities in percent

maturities	strikes									
	15	17.5	20	22.5	25	30	35	40	50	60
1Y	17.6	2.97	16.53	31.9	28.8	13.62	1.54	7.69	20.987	31.31
2Y	10.816	9.1	5.366	1.4	6.87	10.32	7.15	3.5	3.56	9.26
3Y	2.66	3.32	2.89	0.44	2.44	5.86	5.27	3	0.72	2.28
4Y	1.26	0.24	1.22	0.38	0.5	1.89	1.76	1.21	0.83	1.75
5Y	4.1	2.2	0.78	0.3	0.28	0.16	1.18	1.47	0.04	3.19
6Y	5.17	3.17	1.52	0.49	0.4	2.13	4.1	4.59	2.39	2.14
7Y	5.83	3.6	1.78	0.3	0.65	3.16	5.72	7.47	5.72	0.16
8Y	5.4	3.31	1.04	0.51	1.86	5.19	8.5	10.2	9.5	3.35
9Y	7.86	5.54	3.73	2.35	0.84	2.8	6.4	8.31	8.07	2.16

And in the Kou-case

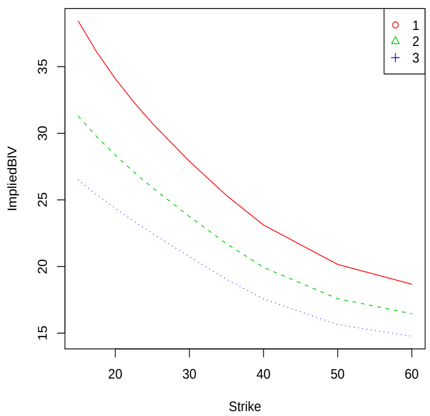
γ	η_1	η_2	p
1	48.2	5.2	0.9

We get λ coefficients

λ_{T_9}	3
λ_{T_8}	1
λ_{T_7}	1

and smiles

Maturities $7Y$ (1) to $9Y$ (3), 10 Strikes



The concrete values of the volatilities are

maturities	strikes									
	15	17.5	20	22.5	25	30	35	40	50	60
7Y	38.41	36.09	34.1	32.34	30.75	27.9	25.33	23.12	20.16	18.67
8Y	31.28	29.74	28.36	27.09	25.92	23.75	21.72	19.93	17.58	16.46
9Y	26.5	25.39	24.36	23.4	22.48	20.73	19.05	17.56	15.64	14.77

We get the relative Error in volatilities in percent

maturities	Strikes									
	15	17.5	20	22.5	25	30	35	40	50	60
7Y	13.65	14.94	15.98	16.74	16.9	17.71	17.82	16.74	11.37	4.32
8Y	4.63	2.82	0.85	0.36	1.26	3.25	4.4	3.82	1.66	2.63
9Y	16.9	14.5	12.67	11.36	10.07	7.44	5.69	5.58	5.76	7.72

For the Kou-model the number of rates to be fitted has been reduced in order to obtain a better result. Even for a very small model however, the NiG-model, at the moment seems to be better than the Kou-Model. The fact that in the Kou model the parameters are close to the allowed box-constraints indicates, that the optimization procedure will have to be improved in the future, as there should be at least one local minimum of the problem besides the values at the boundary.

3.12 Remarks

It follows from (3.5) that all LIBOR rates jump simultaneously. This is clearly an oversimplification. As [6] note, it is not difficult to extend the model to several driving processes, see also [2]. Moreover, lognormal LIBOR models used in practice use two or more factors. The extension to several driving processes will be addressed in future work.

Furthermore we are interested in extensions of pricing-formulae to more sophisticated ℓ-approximations. The main aim in that case being an analytically solvable model at all stages and a good fit in both the pricing as well as the statistical properties of the model.

Multi-currency settings serve as a long time goal of those extensions at the single currency, single factor level.

Chapter 4

The discrete timegrid LIBOR-version

The aim of this Section is to develop a discrete LIBOR analogon(discrete in time and space). The piece-wise frozen drift model is then to be situated in between a discrete time model and the exact model. When the former converges to the later, so will the piece-wise drift approximation as we shall see below.

4.1 The Model

Definition 15 (Discrete Forward LIBOR Model) *Assume a given timegrid, say a finite tenor structure like for continuous time LIBOR models:*

$$0 = T_{-1} < T_0 < T_1 < \cdots < T_n < T_{n+1} = T^*.$$

and a process build from positive discrete random variables $(Y_s)_s$

$$\hat{L}(T_i, T_n) = L_0 \prod_0^i Y_s \tag{4.1}$$

in discrete time with

$$Y_s = \exp\left(\lambda_{sn}(X_s + b_s)\right). \tag{4.2}$$

Here $(X_s)_s$ is an i.i.d family of random variables (for notational simplicity. We will derive formulae for the general case below), $(b_s)_s$ and $(\lambda_{ij})_{i,j}$ are sequences of real numbers. $(t_i)_i$ is a sequence of times with $\{t_i | i \in I\} \supset \{T_i | i = -1, \ldots, n+1\}$ However for notational simplicity we assume $\{t_i | i \in I\} = \{T_i | i = -1, \ldots, n+1\}$ and $\Delta t_i = 1$ for all i.

In order for $\hat{L}(t_i, T_n)$ to be a martingale we need that

$$\mathbb{E}(\hat{L}(t_i, T_n) | \mathcal{F}_s) = \hat{L}(t_s, T_n) \quad \forall s < i. \tag{4.3}$$

Now written as a product there holds

$$\mathbb{E}(\hat{L}(0, T_n) \prod_0^i Y_k | \mathcal{F}_s) = \mathbb{E}(\hat{L}(0, T_n) \prod_0^s Y_k \prod_{s+1}^i Y_k | \mathcal{F}_s). \tag{4.4}$$

Here $\hat{L}(0,T_n)\prod_0^s Y_k$ is $\mathcal{F}_s$-measurable and since the Y_k are independent (which only holds for the terminal rate), the conditional expectation becomes

$$\hat{L}(0,T_n)\prod_0^s Y_k \mathbb{E}(\prod_{s+1}^i Y_k|\mathcal{F}_s) = \hat{L}(0,T_n)\prod_0^s Y_k \mathbb{E}(\prod_{s+1}^i Y_k). \tag{4.5}$$

Things will be considerably more difficult for the earlier rates, since the random variables will no longer be i.i.d.
By the tower law or projection property of the conditional expectation, we can look at the former formula

$$\hat{L}(0,T_n)\prod_0^s Y_k \mathbb{E}(\prod_{s+1}^i Y_k|\mathcal{F}_s) = \hat{L}(0,T_n)\prod_0^s Y_k \mathbb{E}(\prod_{s+1}^i Y_k|\mathcal{F}_{i-1}|\mathcal{F}_s). \tag{4.6}$$

We see that it would be sufficient if

$$\mathbb{E}(\hat{L}(0,T_n)\prod_0^i Y_k|\mathcal{F}_{i-1}) = \hat{L}(0,T_{n+1})\prod_0^{i-1} Y_k \qquad \forall i \leq n \tag{4.7}$$

held. We can write

$$\hat{L}(0,T_n)\prod_0^i Y_k = \hat{L}(0,T_n)(\prod_0^{i-1} Y_k)Y_i \tag{4.8}$$

which (by independence which only holds for the terminal rate) yields a martingale-condition on the "increments"

$$\mathbb{E}(\hat{L}(0,T_n)\prod_0^i Y_k|\mathcal{F}_{i-1}) = \hat{L}(0,T_n)(\prod_0^{i-1} Y_k)\mathbb{E}(Y_i) = \hat{L}(0,T_n)\prod_0^{i-1} Y_k. \tag{4.9}$$

Therefore our condition becomes

$$\mathbb{E}(\exp(\lambda_{sn}(X_s+b_s))) = 1 \qquad \forall s < n \tag{4.10}$$

or

$$\mathbb{E}(\exp(\lambda_{sn}(X_s))) = \mathbb{E}(\exp(-\lambda_{sn}b_s)) = \exp-(\lambda_{sn}b_s). \tag{4.11}$$

4.1.1 Explicit Drift

Going on along the lines of the analogy to the continuous time model, we want to have some sort of "stochastic difference equation" for $\hat{L}(t_i,T_n)$. Directly from the model assumptions we derive

$$\Delta\hat{L}(t_i,T_n) = \hat{L}(t_i,T_n) - \hat{L}(t_{i-1},T_n) = \hat{L}(t_{i-1},T_n)(\exp(\lambda_{sn}(X_i+b_i))) - 1). \tag{4.12}$$

This has almost the form of the stochastic differential equation in the continuous time model, except that the drift b_i does enter in the equation. However, accepting that difference we may write

$$\Delta\hat{L}(t_i,T_n) = \hat{L}(t_{i-1},T_n)\int_{\mathbb{R}}(\exp(\lambda_{in}(x+b_i)) - 1)d(\mu_i - \nu_i^{n+1}). \tag{4.13}$$

Where

$$\int_{\mathbb{R}} f(x)d\mu_i = f(X_i) \tag{4.14}$$

(see the next chapter for the definition of μ and ν).
And $\nu_i^{T_n}$ is the compensator of the discrete process $\sum_i X_i$ as defined for general semimartingales. This also gives

$$\int_{\mathbb{R}} (\exp(\lambda_{in} x) d(\nu_i^{n+1}) = \exp(\lambda_{in} b_i)$$

The last equation holds since

$$\int_{\mathbb{R}} (\exp(\lambda_{in}(x+b_i)) - 1) d(\mu_i) = \exp(\lambda_{in}(X_i + b_i)) - 1$$

and

$$-\int_{\mathbb{R}} (\exp(\lambda_{in}(x+b_i)) - 1) d(\nu_i^{n+1}) = -\exp(\lambda_{in}(-b_i + b_i)) + 1 = 0.$$

We then proceed to look at the forward-rate process $F(T_i, T_n, T_{n+1}) = 1 + \delta_i L(T_i, T_{n+1})$. The dynamics are derived as

$$\Delta \hat{F}(T_i, T_n, T_{n+1}) = \frac{\delta \hat{L}(T_i, T_n)}{1 + \delta \hat{L}(T_i, T_n)} \Delta \hat{L}(T_i, T_n).$$

We have described $\Delta \hat{L}(t_i, T_n)$ above. Therefore we get an analogy to the SDE of the forward rate in continuous time

$$\Delta \hat{F}(t_i, T_n, T_{n+1}) = \hat{F}(t_{i-1}, T_n, T_{n+1}) \int_{\mathbb{R}} \ell(t_{i-1}, T_n)(\exp(\lambda_{in}(x+b_i)) - 1) d(\mu_i - \nu_i^{n+1}). \quad (4.15)$$

Here ℓ is completely analogous to continuous time:

$$\ell(t_{i-1}, T_n) = \frac{\delta L(t_{i-1}, T_n)}{1 + \delta L(t_{i-1}, T_n)}.$$

Now we need a appropriate forward measure, fulfilling the same conditions as in the continuous LIBOR-models. We choose as measure change the stochastic exponential derived from the difference equation representation of the forward. This gives us

$$\nu_i^n = (\ell(t_{i-1}, T_n)(\exp(\lambda_{in}(x+b_i)) - 1) + 1) d\nu_i^{n+1} \quad (4.16)$$

See the next chapter on discrete grisanov procedures.

That way we guaranty that the martingality condition on the forward-rate-process is fulfilled. Due to our difference-equation-representation, we can write the measure-change down as a change in the compensator, as in the continuous time case. The difference however being, that the drift b_i appears explicitly in the measure change. This is due to the predictability of jump-times in the time-discrete case, which does not allow for a complete analogy to Lévy-LIBOR-models.(the reason being that the drift can not adapt continuously but has to compensate the coming jump at t_i already at t_{i-1})

For the next rate we have the dynamics

$$\hat{L}(t_i, T_{n-1}) = \hat{L}(0, T_{n-1}) \exp \sum_{s=1}^{i} \lambda_{s(n-1)}(X_s^n + b_s^{n-1}). \quad (4.17)$$

This looks completely analogous to before. However as in continuous time there is a serious change in dependencies.
Concerning the distribution of X_s^n we know that the jump-sizes will remain unchanged. However the jump-probabilities have to change to fulfill

$$\mathbb{E}_{\mathbb{P}^{T_n}}(X_s^n|\mathcal{F}_{t_{s-1}}) = \int_{\mathbb{R}} x d\nu_s^n. \tag{4.18}$$

Now $d\nu_s^n$ is path dependent due to the ℓ term as described above. Therefore the random variables $(X_s^n)_s$ are no longer i.i.d . This also has consequences for determining b_s^n of course.
If we look at the martingale condition now

$$\mathbb{E}_{\mathbb{P}_{T_n}}(\hat{L}(t_i,T_{n-1})|\mathcal{F}_s) = \mathbb{E}_{\mathbb{P}^{T_n}}(\hat{L}(t_s,T_{n-1})) \tag{4.19}$$

we can use the tower-law to get

$$\hat{L}(t_s,T_{n-1})\mathbb{E}_{\mathbb{P}_{T_n}}(\exp\sum_{u=s+1}^{i}\lambda_{u(n-1)}(X_u^n+b_u^n)|\mathcal{F}_u) = \hat{L}(t_s,T_{n-1})\mathbb{E}_{\mathbb{P}_{T_n}}(\exp\sum_{u=s+1}^{i}\lambda_{u(n-1)}(X_u^n+b_u^n)|\mathcal{F}_{i-1}|\mathcal{F}_s). \tag{4.20}$$

Now the drift $b_{t_i}^n$ may always be chosen $\mathcal{F}_{t_i-1}$-measurable. Therefore one condition we get if we choose $s=i-1$ is

$$\exp(\lambda_{i(n-1)}b_i^n) = \mathbb{E}_{\mathbb{P}_{T_n}}(\exp(\lambda_{i(n-1)}X_i^n)|\mathcal{F}_{i-1}). \tag{4.21}$$

We know that $\mathbb{E}_{\mathbb{P}_{T_n}}(\exp(\lambda_{i(n-1)}X_i^n)|\mathcal{F}_{i-1})$ is pathdependent, so b_i^n has to be as well.
Since we assume that condition to be fulfilled for every b_u^n with $u\in\{1,\ldots,(n-1)\}$ and b_u^n is $\mathcal{F}_{u-1}$ measurable always by construction, we get that our martingale condition is fulfilled as long as eq (4.21) holds for any u.
So the drift-characteristic is now a path-dependent function and conditional expectation does not trivialise to simple expectation. We may still calculate everything explicitly(e.g. pathwise) but calculations may easily get fairly complex.

Working with this stochastic drift now, everything else works analogously, so we can write down formulae for an arbitrary rates and dependent, non-equally distributed random variables.

Theorem 8 (Discrete LIBOR Equations - Explicit Drift) *We have derived the following equations:*

- *The general dynamics*

$$\hat{L}(t_i,T_j) = \hat{L}(0,T_j)\exp(\sum_{s=i}^{i}\lambda_{sj}(X_s^{j+1}+b_s^{j+1})). \tag{4.22}$$

- *The general drift condition*

$$\exp(\lambda_{ij}b_i^{j+1}) = \mathbb{E}_{\mathbb{P}_{T_{j+1}}}(\exp-(\lambda_{ij}X_i^{j+1})|\mathcal{F}_{i-1}). \qquad \forall i \tag{4.23}$$

- *The general forward*

$$\Delta\hat{F}(t_i,T_j,T_{j+1}) = \hat{F}(t_{i-1},T_j,T_{j+1})\int_{\mathbb{R}}\ell(t_{i-1},T_j)(\exp(\lambda_{ij}x)-1)d(\mu_i-\nu_i^{j+1}). \tag{4.24}$$

- *The general measure change*

$$\nu_i^j = (\ell(t_{i-1},T_j)(\exp\lambda_{ij}(x+b_i^{T_j})-1)+1)d\nu_i^{j+1}. \tag{4.25}$$

4.1.2 Implicit Drift

Another way to treat the discrete case is to include the drift in the random-jump-measure μ which has the effect, the that the jump-measure changes for each rate - unlike the continuous model - but in exchange the discrete girsanov (see the next chapter) looks exactly as in the continuous time case.
We set

$$X_i + b_i =: \tilde{X}_i \tag{4.26}$$

with b_i determined from the martingale condition.
Then - for the terminal rate - $\tilde{\nu}^{n+1}$ is calculated through

$$\mathbb{E}(\tilde{X}_i) = \int_{\mathbb{R}} (x) d\tilde{\nu}^{n+1} = \mathbb{E}(X_i) + b_i. \tag{4.27}$$

With this notation our difference equation for $\Delta L(t_i, T_n)$ becomes

$$\Delta \hat{L}(t_i, T_n) = \hat{L}(t_{i-1}, T_n) \int_{\mathbb{R}} (\exp(\lambda_{in} x) - 1) d(\tilde{\mu_i} - \tilde{\nu}^{n+1}). \tag{4.28}$$

Thus we get our formula for the forward rate process

$$\Delta \hat{F}(t_i, T_n, T_{n+1}) = \hat{F}(t_{i-1}, T_n, T_{n+1}) \int_{\mathbb{R}} \ell(t_{i-1}, T_n)(\exp(\lambda_{in} x) - 1) d(\tilde{\mu_i} - \tilde{\nu}^{n+1}) \tag{4.29}$$

The measure change now is completely analogous to the continuous time case. Accordingly our new ν_i^n is given as

$$\nu_i^n = (\ell(t_{i-1}, T_{n-1})(\exp(\lambda_{in} x) - 1) + 1) d\tilde{\nu}_i^{n+1} \tag{4.30}$$

The next rate is then given through

$$\hat{L}(t_i, T_{n-1}) = \hat{L}(0, T_{n-1}) \exp(\sum_{s=1}^{i} \lambda_{s(n-1)} \tilde{X}_s^n) \tag{4.31}$$

where

$$\tilde{X}_s^n = b_s^n + \int_{\mathbb{R}} x d(\mu_i^n - \nu_i^n) \tag{4.32}$$

with $\tilde{\mu}_i^n$ given as

$$\int_{\mathbb{R}} x d\tilde{\mu}_i^n = X_i^n + b_i^n \tag{4.33}$$

and the jump-probabilities of X_i^n calculated from ν_i^n given as

$$\mathbb{E}_{\mathbb{P}_{T_n}}(X_i^n) = \int_{\mathbb{R}} (x(\ell(t_{i-1}, T_n)(\exp(\lambda_{in} x) - 1) + 1)) d\tilde{\nu}_i^n. \tag{4.34}$$

One should however remember, that concerning jump-sizes

$$\int_{\mathbb{R}} x d\mu_i^n = X_i + b_i. \qquad (\text{under} \quad \mathbb{P}_{T_n}) \tag{4.35}$$

Of course we might exclude the drift from the jump measure again by

$$\int_{\mathbb{R}} (x(\ell(t_{i-1}, T_n)(\exp(\lambda_{i(n+1)} x) - 1) + 1)) d\tilde{\nu}_i^n = \int_{\mathbb{R}} (x(\ell(t_{i-1}, T_n)(\exp(\lambda_{in} x + b_i) - 1) + 1)) d\nu_i^n \tag{4.36}$$

Then we manage to obtain the explicit drift jump-measure again. In the end both approaches yield the same model after all but representations and interpretations are quite different(changing jump-measure vs fixed jump-measure).

Theorem 9 (Discrete LIBOR Equations - Implicit Drift) *We have obtained the following formulae*

- $$\hat{L}(t_i, T_j) = \hat{L}(0, T_j) \exp(\sum_{s=1}^{i} \lambda_{ij} \tilde{X}_i^{j+1}). \tag{4.37}$$

- $$\tilde{X}_i^{j+1} = X_i^{j+1} + b_i^{j+1} =: \int_{\mathbb{R}} x d\tilde{\mu}_i^{j+1}. \tag{4.38}$$

- $$b_i^{j+1} = -\frac{1}{\lambda_{ij}} \log (\hat{L}(0, T_j) \mathbb{E}_{\mathbb{P}_{T_{j+1}}} (\exp (\lambda_{ij} X_i^{j+1}) | \mathcal{F}_{i-1})). \tag{4.39}$$

- $$\Delta \hat{F}(t_i, T_j, T_j) = \hat{F}(t_{i-1}, T_j, T_j) \int_{\mathbb{R}} (\ell(t_{i-1}, T_j)(\exp \lambda_{ij} x - 1)) d\tilde{\nu}_i^{j+1} \tag{4.40}$$

- $$\tilde{\nu}_i^j = (\ell(t_{i-1}, T_j)(\exp \lambda_{ij} x - 1) + 1) d\tilde{\nu}_i^{j+1} \tag{4.41}$$

4.1.3 Conclusions and Comparison

Generally the inclusion of the Drift in the jump-measure seems more natural if we wish to stress analogies between the continuous time and the discrete model. From the point of view of simply considering discrete models, the explicit drift seems the more natural choice because it does not involve changing the jump-sizes.
Compared to the continuous time models, measure change and LIBOR-dynamics change significantly in the sense that either the drift characteristic of the driving process enters in the measure change or the jump-measure changes(incorporating the drift characteristic then). This is due to predictability of jump-times.

4.2 Approximation Property

We have restricted ourselves twofold in the above section. We assumed our random Variables to be discrete in space and the process to be discrete in time.
In order to relate our piece-wise-frozen-drift approximation to the discrete model, we have to consider a LIBOR-model which will be discrete in time, but not in space. However formulae hold just the same, as we never used the actual form of the distribution. All we need is a certain integrability to make sure everything exist properly(especially the exponentials). We demand

$$\mathbb{E}(\exp (\sum_i \lambda_i X_i)) < \infty \quad \forall \lambda_i \in \mathbb{R} \tag{4.42}$$

for the terminal rate.
It is then also clear now, that the piece-wise frozen drift may be approximated again as a discrete-time model, with a certain increment distribution.
The key to showing that our piece-wise-frozen-drift model converges to a LIBOR model is then the following continuous mapping theorem

Theorem 10 (Continuous Mapping Theorem) *Assume (E_1, d_1) and (E_2, d_2) to be metric spaces and $\phi : E_1 \to E_2$ Borel-measurable and U_ϕ be the set of discontinuouities of ϕ. Then*

- *If $\mu_1, \mu_2, \dots$ is sequence of probability measures on E_1 and there holds $(\mu_i) \to \mu$ weakly with μ again a probability measure and $\mu(U_\phi)) = 0$. Then $\mu_i \circ \phi^{-1} \to \mu \circ \phi^{-1}$ weakly.*
- *If $X, X_1, X_2, \dots$ are E_1-valued rv's with $\mathbb{P}[X \in U_\phi] = 0$ and $X_n \to X$ weakly then $\phi(X_n) \to \phi(X)$ weakly.*

Proof:
Standard text books on probability and/or stochastic processes. For instance Bauer [1]
Using this we get the main theorem of this chapter

Theorem 11 (Approximation Property) *By the continuous mapping theorem if the terminal rate of a discrete model converges weakly to a continuous time model, then the other rates will converge weakly as well to the corresponding processes in the continuous time model.*

Proof:
Assume as given an exact Lévy-LIBOR model $L(t, T_n)$ with increments $X_i^{T_n}$ as in [6] and discrete Models $(\hat{L}(t_i, T_n)^{(k)})_{k\in\mathbb{N}}$ with increments $X_i^{(k),n+1}$.
By assumption weakly $X_i^{(k),n+1} \to X_i^{n+1}$ so - by the continuous mapping theorem - $\exp X_i^{(k),n+1} \to \exp X_i^{n+1}$ in the sense of weak convergence.
The drift parts converge as continuous functions of the jump-part (expectations of the exponential)
If we have the stochastic Difference Equation

$$\Delta\hat{L}^{(k)}(t_i, T_n) = \hat{L}^{(k)}(t_{i-1}, T_n) \int_{\mathbb{R}} (\exp(\lambda_{in}(x + b_i^{n+1})) - 1) d(\mu_i - \nu_i^{n+1,(k)}). \tag{4.43}$$

we see that we can transform it into the corresponding stochastic integral equation

$$\hat{L}^{(k)}(t_i, T_n) = \hat{L}^{(k)}(0, T_n) + \sum_{j=1}^{i} \hat{L}^{(k)}(t_{j-1}, T_n) \int_{\mathbb{R}} (\exp(\lambda_{jn}(x + b_j^{n+1})) - 1) d(\mu_j - \nu_j^{n+1}) \tag{4.44}$$

To see that this converges to the stochastic integral representation of $L(t, T_n)$ we need conditions on the convergence of stochastic Integrals in a general semimartingale setting. Such conditions can be found for instance in Jacod,Shiryaev [9].
First we need certain existence and uniqueness results:
We use theorem 16 from the appendix B.
Now more specifically we have the following result from Jacod, Shiryaev [9]: Assume the following setting:

- We are given an equation
  $$Y = Z + f(Y_-) \cdot X$$
  as above
- For each n we have a stochastic basis $\mathcal{B}^n = (\Omega^n, \mathcal{F}^n, \mathbf{F}^n, \mathbb{P}^n)$ and an equation
  $$Y^n = Z^n + f_n(Y_-^n) \cdot X^n. \tag{4.45}$$
- X^n is a d-dimensional semimartingale on that basis.

- Z^n is a q-dimensional cadlag adapted process.
- f_n are functions $\mathbb{R}^q \to \mathbb{R}^q \times \mathbb{R}^d$ such that each equation above admits a unique solution.

Then there holds

Theorem 12 *Assume the functions f_n fulfill Lipschitz and linear boundedness with constants not dependent on n and $f_n \to f$ at least pointwise. Assume further that the sequence X^n is $P - UT$. Then if Y^n denotes the unique solution of the sequence of equations there holds*
If $(X^n, Z^n) \to (X, Z)$ weakly, then $(X^n, Z^n, Y^n) \to (X, Z, Y)$ weakly.

(see also 17 in appendix B for a source for the proof)

Now we need the P-UT condition which is defined in 24 in appendix B.
Sufficient conditions on the sequence of processes are 18 in appendix B.

In our concrete case and using the notation from the appendix, we have that:

$$\begin{aligned}
Y_t^{(k)} &= \hat{L}^{(k)}(t, T_n) \qquad \forall k \in \mathbb{N} \\
Z^{(k)} &= \hat{L}^{(k)}(0, T_n) \qquad \forall k \in \mathbb{N} \\
f^{(k)}(x) &= x \qquad \forall k \in \mathbb{N} \\
X_t^{(k)} &= (\exp(\lambda_{in}(x + b_i^{n+1})) - 1)d(\mu_i^{(k)} - \nu_i^{n+1,(k)}) \qquad \forall k \in \mathbb{N}.
\end{aligned}$$

and

- Our $(X_t^{(k)})_t$ is weakly convergent by assumption and the continuous mapping theorem. Therefore certainly tight. In absence of a drift component, we automatically have that $Var(B^{n,i})_t$ is tight. Therefore $(X_t^k)_k$ is $P - UT$.
- We have that $f(x)^k = x$ and therefore f is Lipschitz and linearly bounded.
- We have assumed $X_t^k \to X_t$ weakly and know that $Z^k = L(0, T_n) \to L(0, T_n)$. Therefore we have $(X^n, Z^n) \to (X, Z)$ weakly and by the theorem on weak convergence of solutions of SDEs we have $(X^n, Z^n, Y^n) \to (X, Z, Y)$ weakly.

Those conditions fulfilled entail that the sequence of discrete processes converges weakly to a weak solution of the proper SDE in continuous time. From that we get that $\Delta\hat{F}(t_i, T_n, T_{n+1})$ also converges to the SDE of the forward in continuous time.
Therefore also the measure change converges to the proper one in continuous time by the continuous mapping theorem since we can represent the exponential-change as stochastic exponential of the integral

$$M_{t_i}^{(k)} = \sum_{j=1}^{i} \int_{\mathbb{R}} \ell(t_{j-1}, T_n)(\exp(\lambda_{jn}(x + b_j)) - 1)d(\mu_j^{(k)} - \nu_j^{n+1,(k)}). \tag{4.46}$$

which means

$$\mathcal{E}(M_{t_i}^{(k)}) = 1 + \sum_{j=1}^{i} \mathcal{E}(M_{t_{j-1}}^{(k)})\Delta(M_{t_j}^{(k)}) \tag{4.47}$$

Our situation with respect to our convergence theorems is

$$Y^{(k)} = \mathcal{E}(M_{t_i}^{(k)})$$

$$Z^{(k)} = 1$$
$$f(x)^{(k)} = x$$
$$X_t^{(k)} = M_{t_i}^{(k)}$$

We have that $M_{t_i}^{(k)}$ is weakly convergent to $M_{t_i}^{(k)}$ and a martingale and driftless in its SDE representation, hence $P-UT$. With $Z^{(k)}$ independent of k and f clearly being linearly bounded and smooth, we have all conditions for weak convergence of $Y^{(k)}$, hence the measure changes converge properly.
So we have now that the forward measures $\mathbb{P}_{T_n}^{(k)}$ converge to the proper forward measure in continuous time.
So, for modeling the next rate we have the compensator $\nu_j^{T_n,(k)}$, of which we know that

$$\nu_j^{T_n,(k)} \to \nu_t^{T_n}$$

The next rate is modelled as

$$\hat{L}^{(k)}(t_i, T_{n-1})^{(k)} = \hat{L}^{(k)}(0, T_{n-1}) \exp\left(\sum_{j=1}^{i} \lambda_{j(n-1)} X_j^{n,(k)}\right) = \hat{L}^{(k)}(0, T_{n-1}) \exp\left(\int_{\mathbb{R}} \sum_{j=1}^{i} \lambda_{j(n-1)} x(\mu_j - \nu_j^{n,(k)}) + b_j^{n,(k)}\right) \tag{4.48}$$

and we derive the integral equation

$$\hat{L}^{(k)}(t_i, T_{n-1})^{(k)} = \hat{L}^{(k)}(0, T_{n-1}) + \sum_{j=1}^{i} \hat{L}^{(k)}(t_{j-1}, T_{n-1}) \int_{\mathbb{R}} (\exp(\lambda_{j(n-1)}(x + b_j^{n,(k)})) - 1) d(\mu_j - \nu_j^{n,(k)}) \tag{4.49}$$

With

$$Y_t^{(k)} = L^{(k)}(t, T_{n-1}) \qquad \forall k \in \mathbb{N}$$
$$Z^{(k)} = L(0, T_{n-1}) \qquad \forall k \in \mathbb{N}$$
$$f^{(k)}(x) = x \qquad \forall k \in \mathbb{N}$$
$$X_t^{(k)} = (\exp(\lambda_{i(n-1)}(x + b_i^n)) - 1) d(\mu_i^{(k)} - \nu_i^{n,(k)}) \qquad \forall k \in \mathbb{N}.$$

we can apply our convergence results from above and get again weak convergence. $\hat{L}^{(k)}(t_j, T_{n-1}) \to L(t, T_{n-1})$ as well as the convergence of measure changes (through the forward) and compensator $\nu_j^{n-1,k} \to \nu_{t_j}^{n-1}$.
We continue this procedure inductively until we get the convergence of every rate down to $L(t, T_1)$. $\square$
NOTE: This theorem only proves the convergence of an exact discrete model to an exact time continuous model. All approximations affecting the measure change (especially freezing the drift) need separate proofs. The case of the piece-wise constant approximation will be treated below.
In the above proof we have not actually needed the RV's to be discrete in space. With some adaptations the proof also holds for discrete models which are discrete only in time.

Corollary 1 *The former theorem holds even if $(X_i^k)_k$ are not discretely distributed, but fulfill the integrability conditions*

$$\mathbb{E}(\exp(\sum_i \lambda_i X_i)) < \infty \quad \forall \lambda_i \in \mathbb{R}. \tag{4.50}$$

Proof:
The integrability is clearly necessary, to ensure everything, especially measure changes are well-defined and our rates stay integrable.
We again have an exact Lévy-LIBOR model $L(t, T_n)$ with increments $X_i^{n+1,(k)}$ and discrete Models $(\hat{L}(t_i, T_n)^{(k)})_{k\in\mathbb{N}}$ with increments $X_i^{(k),n+1}$.
$\exp(X_i^{(k),n+1})$ is well-defined due to our integrability condition.
By assumption weakly $X_i^{(k),n+1} \to X_i^{n+1}$ so - by the continuous mapping theorem - $\exp X_i^{(k),n+1} \to \exp X_i^{n+1}$ in the sense of weak convergence.
The drift parts converge as continuous functions of the jump-part(expectations of the exponential).
If we have the stochastic Difference Equation

$$\Delta\hat{L}^{(k)}(t_i, T_n) = \hat{L}^{(k)}(t_{i-1}, T_n) \int_{\mathbb{R}} (\exp(\lambda_{in}(x + b_i)) - 1) d(\mu_i - \nu_i^{n+1}). \tag{4.51}$$

The $P - UT$ condition and weak convergence follow by directly applying the same general conditions used in the approximation theorem to our respective situation.
The model formulae and all theorems we used hold completely analogously and everything is carried out as for space-discrete variables. □

Theorem 13 *Given a Lévy-LIBOR-model as defined in [6] with driving process X_t fulfilling the integrability condition and chosen to make $L(t, T_n)$ a martingale already and subdivisions of $[0, n]$ into nk equidistant parts. We have a discretization of the driving process*

$$Y_i^{(k)} := X_{\frac{i}{k}i} - X_{\frac{(i-1)}{k}} \qquad \forall 1 \leq i \leq nk \tag{4.52}$$

as well as

$$\lambda_{ij}^{(k)} = \lambda(T_i, T_j) \qquad \forall i, \frac{i}{k} \leq j - 1 \tag{4.53}$$

in the terminal rate.
Then the conditions above are fulfilled and the model converges.

Proof:
We need to show

$$\hat{L}^{(k)}(t_i, T_n) = \hat{L}^{(k)}(0, T_n) \exp\Big(\sum_{j=1}^{i} \lambda_{(\frac{j}{k}),n} Y_{\frac{j}{k}}\Big) \to L(t, T_n) = L(0, T_n) \exp \int_0^{\frac{i}{k}} \lambda(s, T_n) dX_s^{n+1}. \tag{4.54}$$

For an additive driving process, it is clear, that X_i^k will converge weakly to X_t. For the whole $\int_0^t \lambda(s, T_n) dX_s^{n+1}$ we have in this discretization essentially the definition of the stochastic integral as limit of sums, since our $Y_i^{(k)}$ are process increments of X_s^{n+1}. Therefore we also get convergence of $\sum_{j=1}^{i} \lambda_{(\frac{j}{k}),n} Y_{\frac{j}{k}} \to \int_0^{\frac{i}{k}} \lambda(s, T_n) dX_s^{n+1}$.
For weak convergence we need that the grid gets infinitely fine so that for the probability spaces the approximations are defined on we have $(\Omega^k, \mathcal{F}^k, \mathbb{P}^{(k)}_{T_{n+1}}) \to (\Omega, \mathcal{F}, \mathbb{P}_{T_{n+1}})$. Convergence here is ensured so long as the mesh of our grid tends to 0. With weak convergence in the terminal rate, we have weak convergence for all rates by the approximation theorem 11 and its corollary. □

4.3 Piece-Wise Frozen-Drift Model Convergence

Our piece-wise frozen drift model is certainly finer than a corresponding time discrete model.We can show that if the time-discretized model converges weakly to a fully continuous model, then the piece-wise frozen-drift model in continuous time will also converge to the same model.

Theorem 14 *Assume piece-wise constant volatilities*

$$\lambda(t,T_j) = \lambda(t_i,T_j) \qquad \forall t \in [t_i, t_{i+1}). \tag{4.55}$$

Then the piece-wise-frozen drift approximation $\{\tilde{L}(t,T_j)\}_j$ converges to the exact model $\{L(t,T_j)\}_j$ as gridsize gets smaller.

Proof:
Given a Lévy-driven-exact model $\{L(t,T_j)\}_j$. We move on to the piece-wise frozen drift approximation $\{\tilde{L}(t,T_j)\}_j^{(k)}$ for a certain equidistant time grid $\{\frac{i}{k}|i \in 0,\dots,kn\}$.
Now we take a time discrete model $\{\hat{L}(t_i,T_j)\}_j^{(k)}$, which may be seen as a piece-wise approximation with everything fixed between the grid-points, on the same grid, build from the exact model. We look at the defining quantities of the piece-wise frozen drift model(seen as a LIBOR market model)

$$\tilde{L}(t,T_j)^{(k)} = \tilde{L}^{(k)}(0,T_j)\exp(\int_0^t \lambda(s,T_j)dX_s^{j+1,(k)})$$

$$X^{j+1,(k)}(t) = \int_0^t b^{j+1,(k)}(s) + \int_0^t \int_{\mathbb{R}} x(\mu - \nu_s^{j+1,(k)})(dx,ds)$$

$$\ell^{(k)}(t_-,T_j) = \ell^{(k)}(t_{\frac{i}{k}},T_j) \qquad \forall t \in [t_i,t_{i+1}) \quad \forall 1 \le i \le nk, \forall j \in 1,\dots,n$$

and the discretization

$$\hat{L}(k)(t,T_j) = \hat{L}^{(k)}(0,T_j)\exp(\sum_{s=1}^{i} \lambda_{sj}\Delta X_s^{j+1,(k)}) \tag{4.56}$$

as well as

$$\lambda_{ij}^{(k)} = \lambda(T_i,T_j) \qquad \forall i, \frac{i}{k} \le j-1. \tag{4.57}$$

We already know that the discrete exact model converges to the continuous time exact model. Let $(X_s^{n+1})_s$ denote the random-variables driving the longest modelled LIBOR in the discrete model.
We assume a sequence of equidistant time grids with mesh going to 0 as $k \to \infty$. From this we get a sequence of models $\{\hat{L}^{(k)}(t_i,T_j)\}_j$ and $\{\tilde{L}^{(k)}(t,T_j)\}_j$ derived from discretizing and building the piece.wise frozen drift approximation the exact model along the corresponding grid.
The drift in the piece-wise-frozen-drift approximation is a continuous compensation, given through the same equation as in the exact LIBOR-market model

$$\begin{aligned}\tilde{L}^{(k)}(u,T_n) = \mathbb{E}(\tilde{L}^{(k)}(t,T_n)|\mathcal{F}_u) = \tilde{L}^{(k)}(0,T_n)(\exp(\int_0^u \lambda(r,T_n)dX_r^{n+1})\mathbb{E}(\exp(\int_u^t \lambda(r,T_n)dX_r^{n+1})))\\ = \tilde{L}^{(k)}(0,T_n)\exp(\int_0^u \lambda(r,T_n)dX_r^{n+1})\end{aligned}$$

which results in

$$\mathbb{E}(\exp(\int_u^t \lambda(r,T_n)dX_r^{n+1})) = 0 = \mathbb{E}(\exp(\int_u^t \lambda(r,T_n)b_r^{n+1})dr + \int_u^t \int_{\mathbb{R}} \lambda(r,T_n)x(\mu - \nu_r^{n+1})(dr,dx)) \tag{4.58}$$

or

$$\int_0^t \lambda(s,T_n)b_r^{n+1}dr = -\int_0^t \int_{\mathbb{R}} (e^{\lambda(r,T_n)x} - 1 - \lambda(r,T_n)x)\nu^{n+1}.(dr,dx) \tag{4.59}$$

The drift in the discrete model jumps at each grid point and is given through

$$\mathbb{E}(\exp(\lambda_{sn}\Delta X_s^{n+1})) = \mathbb{E}(\exp(-\lambda_{sn}b_s^{n+1})) = \exp -(\lambda_{sn}b_s^{n+1}) \tag{4.60}$$

Now assume $\lambda(s,T)$ chosen piece-wise constant and $(t-u) = 1$. We integrate in the piece-wise model to get

$$\mathbb{E}(\exp(\lambda_{tn}b_r^{n+1})dr + \lambda_{tn}\Delta X_t^{n+1}) \tag{4.61}$$

with ΔX_t^{n+1} a process increment of length 1.
Now if s in the equation for the discrete model is the index of the gridpoint coinciding with time t, we see that the drift for the terminal rate is equal at the gridpoints in both models.
As a consequence for all refinements, the two models are perfectly equal in distribution at the gridpoints for the terminal rate. This also implies equality of the forward rates distributionally

$$F_B(t,T_n,T_{n+1}) = 1 + (T_{n+1} - T_n)L(t,T_n)$$

and therefore of the measure changes at the gridpoints and the compensators for the next rate.
We proceed to the later rates:
Assume we have equal compensators at the grid points $\tilde{\nu}_{t_i}^{j+1,(k)} = \hat{\nu}_{t_i}^{j+1,(k)}$ and piece-wise constant volatilities $\lambda(t,T_j) = \lambda_{[t]j} \in \mathbb{R}_+$ for rate $L(t,T_j)$ and $\hat{L}(t,T_j)$.
Now since in the very first interval we have a de facto frozen drift model in the piece-wise frozen drift approximation, we get

$$\mathbb{E}(\exp(\int_0^{t_1} \lambda(r,T_j)b_r^{j+1})dr + \int_0^{t_1} \int_{\mathbb{R}} \lambda(r,T_j)x(\mu - \nu_r^{j+1})(dr,dx)) = 1 = \mathbb{E}(\exp(\lambda_{1j}(X_1^{j+1}) + \lambda_{1j}b_1^{j+1})) \tag{4.62}$$

and thus coincidence of the drifts $b_{t_1}^{j+1}$ and b_1^{j+1} for the first intervals.
Then inductively

$$\mathbb{E}(\exp(\int_{t_i}^{t_{i+1}} \lambda(r,T_j)b_r^{j+1})dr + \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} \lambda(r,T_j)x(\mu - \nu_r^{j+1})(dr,dx)) = 1 = \mathbb{E}(\exp(\lambda_{ij}\Delta(X_{i+1}^{j+1}) + \lambda_{ij}b_i^{j+1})) \tag{4.63}$$

since then

$$\begin{aligned}\mathbb{E}(\exp(\int_0^{t_s} \lambda(t_r,T_j)dX_r^{j+1})|\mathcal{F}_{t_{s-1}}) &= \exp(\int_0^{t_{s-1}} \lambda(t_r,T_j)dX_r^{j+1})\mathbb{E}(\exp(\int_{t_{s-1}}^{t_s} \lambda(t_{s-1},T_j)dX_r^{j+1})) \\ &= \exp(\sum_{i=1}^{s-1} \lambda_{ij}\Delta X_i^{j+1})\mathbb{E}(\lambda_{(s-1,j)}\Delta X_s^{j+1}).\end{aligned}$$

So at the grid points the 2 drifts coincide.
Having this for all rates, we get that distributionally we have total equality between the piece-wise frozen drift with piece-wise constant(at least for some grid refinement) volatilities $\lambda(t,T_j) = \lambda_{[t]j}$ and the discrete exact models:
Since the later model converges weakly, we may now conclude by convergence of distribution functions, for refining the grid, that the former model also converges weakly. □

Corollary 2 *The (weak) convergence of the piece-wise frozen drift model holds, even if the volatilities* $\{\lambda(t,T_i)\}_{t,i}$ *are not piece-wise constant.*

Proof:
The convergence of the discrete model is not affected. For the piece-wise frozen drift models: choose a discretization of the volatility functions:

$$\lambda(t,T_i) = \lambda(\frac{s}{k},T_i) \qquad \forall t \in [\frac{s}{k},\frac{s+1}{k}), \quad s \in 0,1,\dots,ik. \tag{4.64}$$

Then, if we build piece-wise frozen drift models $\tilde{L}^{(k)}(t,T_i)$ with those volatilities, we have distributional equality at the gridpoints $\{\frac{s}{k}|s \in 0,1,\dots,nk\}$ to the discrete models $\hat{L}^{(k)}(\frac{s}{k},T_i)$ on the same grid and therefore weak convergence. $\square$

Chapter 5

Discrete Measure Change

5.1 Implicit Drift

Suppose we are given a probability space $(\Omega, \mathcal{F}, P)$ with a sequence of random variables $(X_1, \ldots, X_n)$ and another probability measure $P' \ll P$. To describe the properties of $(X_1, \ldots, X_n)$ under P' we introduce the following quantities: For $k = 0, 1, \ldots, n$ let $\mathcal{F}_k = \sigma(X_1, \ldots, X_k)$ and P_k resp. P'_k denote the restrictions of P resp. P' to $\mathcal{F}_k$.

For $k = 1, \ldots, n$ let $g_k(dx_1, \ldots, dx_k)$ and $g'_k(dx_1, \ldots, dx_k)$ denote the distributions of $(X_1, \ldots, X_k)$ under P and P', respectively. Let $f_k(x_1, \ldots, x_{k-1}; x_k)$ and $f'_k(x_1, \ldots, x_{k-1}; x_k)$ denote the factorized conditional distributions of X_k given $X_1, \ldots, X_{k-1}$ under P and P', respectively. Obviously we have

$$g_k(dx_1, \ldots, dx_k) = \prod_{i=1}^{k} f_k(x_1, \ldots, x_{i-1}; dx_i) \tag{5.1}$$

and

$$g'_k(dx_1, \ldots, dx_k) = \prod_{i=1}^{k} f'_k(x_1, \ldots, x_{i-1}; dx_i). \tag{5.2}$$

Let

$$Z_k = \frac{d\mathbb{P}'_k}{d\mathbb{P}_k}, \tag{5.3}$$

then $(Z_0, \ldots, Z_n)$ is the density process for the measure change $P \mapsto P'$. Let

$$z_k(x_1, \ldots, x_k) = \frac{dg'_k}{dg_k}(x_1, \ldots, x_k), \tag{5.4}$$

then we have

$$Z_k(\omega) = z_k(X_1(\omega), \ldots, X_k(\omega)) \tag{5.5}$$

for $\omega \in \Omega$. Next we define

$$y_k(x_1, \ldots, x_{k-1}; x_k) = \frac{df'_k}{df_k}(x_1, \ldots, x_{k-1}, x_k), \tag{5.6}$$

i.e., the derivative of the measure f'_k with respect to f_k when the conditioning variables $x_1, \ldots, x_{k-1}$ are treated as parameters. We obtain

$$z_k(x_1, \ldots, x_k) = \prod_{i=1}^{k} y_i(x_1, \ldots, x_{i-1}, x_i). \tag{5.7}$$

We can introduce the predictable functions

$$Y_k(\omega, x) = y_k(X_1(\omega), \ldots, X_{k-1}(\omega), x), \tag{5.8}$$

then

$$Z_k = \prod_{i=1}^{k} Y_i(X_i). \tag{5.9}$$

Let

$$M_k = \sum_{i=1}^{k} (Y_i(X_i) - 1), \tag{5.10}$$

then $(M_0, \ldots, M_n)$ is a martingale under P, and we have

$$Z_k = \mathcal{E}(M)_k, \tag{5.11}$$

in complete analogy with the continuous time notation, when considering the Girsanov Theorem for a purely discontinuous process.

For applications we have the following: If we can write the density process in the form of the stochastic difference equation for the stochastic exponential in discrete time,

$$\Delta Z_k = Z_{k-1} \Delta M_k, \tag{5.12}$$

where

$$\Delta M_k = Y_k(X_k) - 1 \tag{5.13}$$

and Y_k is of the form (5.8), we understand, in view of the above derivations, how the change of measure $P \mapsto P'$ is reflected in a change of distributions $g_k \mapsto g'_k$, respectively in a change of conditional distributions $f_k \mapsto f'_k$.
We will see in the next section that the conditional distributions f_k and f'_k correspond in discrete time in a simple way to the compensated jump measures ν and ν'. Thus we can describe the change $\nu \mapsto \nu'$ induced by $P \mapsto P'$. This is exactly the content of the Girsanov Theorem for discrete time, when the drift is included in the jump part.

5.2 Measure Change And Characteristics - Implicit Drift

We would like to formulate this in terms of the characteristics of a discrete process.
We assume a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$, with Ω discrete.
We assume as given a discrete process $(X_i)_{i\in\mathbb{N}}$. We construct a decomposition of our process into

$$X_i = \sum_{j=1}^{i} (b_j + V_j) \tag{5.14}$$

where $(V_i)_i$ is then a martingale with $\mathbb{E}(V_i) = 0$ for all i. We get this by setting

$$b_j := \mathbb{E}(X_j | \mathcal{F}_{j-1}) \tag{5.15}$$

and

$$V_i := X_i - b_i. \tag{5.16}$$

In order to express this in terms of semimartingale characteristics we define:

Definition 16 (Random Measure Of Jumps) *We call*

$$\int_{\mathbb{R}} f(x)\mu_i(dx) := f(X_i) \qquad \mu := (\mu_i)_i \tag{5.17}$$

our random measure of jumps.

and

Definition 17 (Compensator) *We call our compensator the sequence ν_i for which it holds that*

$$\int_{\mathbb{R}} f(x)\nu_i(dx) := \mathbb{E}(f(X_i)|\mathcal{F}_{i-1}). \tag{5.18}$$

This also implies

$$b_j = \int_{\mathbb{R}} x\nu_j(dx). \tag{5.19}$$

Therefore $\int_0^t \int_{\mathbb{R}} f(x)(\mu_s - \nu_s)(ds, dx)$ will always yield a martingale, which is what the compensated jump-measure should fulfill.
That way we have found a first characteristic $(\mu - \nu)$. We assume that we do not have a diffusion component. That determines a second characteristic $D = 0$.
So, $X_i = \sum_{j=1}^i b_j + \sum_{j=1}^i \int_{\mathbb{R}} x(\mu_j - \nu_j)$.
Now concerning the measure change: Our X_i have the above representation. Under the new measure $\mathbb{P}'$ we know that we can also find a representation

$$X_i = \sum_{j=1}^i b_j^{\mathbb{P}'} + \sum_{j=1}^i \int_{\mathbb{R}} x(\mu_j - \nu_j^{\mathbb{P}'})(dx). \tag{5.20}$$

with

$$b_j^{\mathbb{P}'} = \int_{\mathbb{R}} x\nu_j^{\mathbb{P}'}(dx). \tag{5.21}$$

So what we are looking for is $\nu_j^{\mathbb{P}'}$.

We have that the $f_k(x_1, \ldots, x_{k-1}; x_k)$ are the factorized conditional distributions of X_k given $X_1, \ldots, X_{k-1}$ or in other words

$$f_k(x_1, \ldots, x_{k-1}; x_k) = \mathbb{E}(b_k|\mathcal{F}_{k-1}) + \mathbb{E}(b_k| \int_{\mathbb{R}} x(\mu_j - \nu_j)(dx)|\mathcal{F}_{k-1}) = b_k = \int_{\mathbb{R}} x\nu_k(dx) \tag{5.22}$$

and

$$f'_k(x_1, \ldots, x_{k-1}; x_k) = \mathbb{E}^{\mathbb{P}'}(b_k^{\mathbb{P}'}|\mathcal{F}_{k-1}) + = \mathbb{E}^{\mathbb{P}'}(\int_{\mathbb{R}} x(\mu_j - \nu_j^{\mathbb{P}'})|\mathcal{F}_{k-1})(dx) = b^{\mathbb{P}'_k} = \int_{\mathbb{R}} x\nu^{\mathbb{P}'}(dx) \tag{5.23}$$

so

$$df_k(x_1, \ldots, x_{k-1}; x_k) = x\nu_k \tag{5.24}$$

and

$$df'_k(x_1, \ldots, x_{k-1}; x_k) = x\nu_k^{\mathbb{P}'}. \tag{5.25}$$

This gives for our function y_k defined above

$$y_k(x_1, \ldots, x_{k-1}; x_k) = \frac{df'_k}{df_k}(x_1, \ldots, x_{k-1}, x_k) = \frac{\nu_k^{\mathbb{P}'}}{\nu_k}. \tag{5.26}$$

In order to calculate this in practice we use the formulae from above:
If we can write the density process Z_k as

$$\Delta Z_k = Z_{k-1}\Delta M_k, \tag{5.27}$$

where

$$\Delta M_k = Y_k(X_k) - 1 \tag{5.28}$$

and understand Y_k to be of the form (5.8), we understand, in view of the above derivations, how the change of measure $P \mapsto P'$ is reflected in a change of distributions $g_k \mapsto g'_k$, respectively in a change of conditional distributions $f_k \mapsto f'_k$.

$$\Delta M_k = Y_k(\sum_j b_j + \sum_j \int_{\mathbb{R}} x(\mu - \nu)(dx)) - 1$$

Below we give a (slight) reformulation, where the drift is treated explicitly.

5.3 Measure Change And Characteristics - Explicit Drift

We mimic the calculations above:
Suppose we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a discrete martingale $(X_i)_i$, and a sequence of variables $(b_i)_i$ such that b_i is $\mathcal{F}_{i-1}$ measurable for all i, and another probability measure $\mathbb{P}' \ll \mathbb{P}$. We also assume a representation

$$X_i = \sum_{j=1}^{i} \int_{\mathbb{R}} x(\mu_j - \nu_j)(dx). \tag{5.29}$$

We introduce sum variables $Y_i = X_i + b_i$, and immediately proceed to decompose $Y_i = \tilde{Y}_i + \tilde{b}_i$ such that we have a sequence of "drift" variables $(\tilde{b}_1, \dots, \tilde{b}_n)$ defined as

$$\tilde{b}_i := \mathbb{E}(Y_i|\mathcal{F}_{i-1}) \tag{5.30}$$

and therefore

$$\tilde{Y}_i := Y_i - \mathbb{E}(Y_i|\mathcal{F}_{i-1}). \tag{5.31}$$

and the $\tilde{Y}_i$ define a martingale.
Now for the Y_i our measure change behaves just as above. In terms of characteristics we have

$$\int_{\mathbb{R}} x\tilde{\mu}_i = Y_i \tag{5.32}$$

and

$$\int_{\mathbb{R}} x\tilde{\nu}_i = \mathbb{E}(Y_i|\mathcal{F}_{i-1}) = \tilde{b}_i. \tag{5.33}$$

This definition yields

$$\int_{\mathbb{R}} x\tilde{\mu}_i = Y_i = X_i + b_i = \int_{\mathbb{R}} x\nu_i(dx) + b \tag{5.34}$$

We get our measure change through

$$\tilde{y}_k(x_1, \dots, x_{k-1}; x_k) = \frac{d\tilde{f}'_k}{d\tilde{f}_k}(x_1, \dots, x_{k-1}, x_k) = \frac{\tilde{\nu}_k^{\mathbb{P}'}}{\tilde{\nu}_k}. \tag{5.35}$$

Since

$$\int_{\mathbb{R}} f(x)\tilde{\nu}_k = \int_{\mathbb{R}} f(x+b_i)\nu_k \tag{5.36}$$

we know how to calculate the measure change for $X_k + b_k$:

$$\tilde{z}_k(x_1,\ldots,x_k) = z_k(x_1+b,\ldots,x_k+b). \tag{5.37}$$

5.4 Explicit Drift

We use the same notation as in the section on implicit drift. Suppose we are given a probability space $(\Omega,\mathcal{F},P)$ with a sequence of random variables $(X_1,\ldots,X_n)$ and a sequence of drift-variables $(b_1,\ldots,b_n)$ with b_i being $\mathcal{F}_{i-1}$ measurable for all i and another probability measure $P' \ll P$.

Without restriction of generality we may assume $\mathbb{E}(X_i|\mathcal{F}_{i-1}) = X_{i-1}$ since we could include the difference $X_i - \mathbb{E}(X_i|\mathcal{F}_{i-1})$ in the b_i.

Assume we wish to know the distribution of $Y_i = X_i + b_i$ under the new measure:
We use the variables with implicit drift Y_i to compute a measure change $\tilde{Z}_k$ as outlined in the section on implicit drift variables above. Then we use:

$$\mathbb{E}(Y_i|\mathcal{F}_{i-1}) = b_i \tag{5.38}$$

$$\mathbb{E}_{\mathbb{P}'}(Y_i|\mathcal{F}_{i-1}) = \mathbb{E}(Y_i\frac{d\mathbb{P}'}{d\mathbb{P}}|\mathcal{F}_{i-1}) = \mathbb{E}(Y_i\tilde{Z}_i|\mathcal{F}_{i-1}) = b_i^{\mathbb{P}'} \tag{5.39}$$

and then from

$$Y_i - b_i^{\mathbb{P}'} = X_i \qquad \text{under} \quad \mathbb{P}' \tag{5.40}$$

we get the distribution of X_i.
Defining an explicit drift is essentially only interesting, when we wish to work with the semi-martingale analogy.

Chapter 6

Comparison To Arbitrage-free Euler-Discretization

6.1 Introduction

We operate in the following framework:
Our tenor structure is

$$0 = T_0 < T_1 < \cdots < T_N < T_{N+1}$$

Define a right-continuous function $\eta : [0, T_{N+1}) \to \{1, \dots, N+1\}$ by taking $\eta(t)$ to be the integer satisfying

$$T_{\eta(t)-1} \leq t < T_{\eta(t)}.$$

The forward LIBOR-rate for the accrual period $[T_i, T_{i+1}]$, $t \leq T_i$ is

$$L(t, T_i) = \frac{1}{\delta}\left(\frac{B(t, T_i)}{B(t, T_{i+1})} - 1\right), \qquad i = 1, \dots, N.$$

From this follows, that the price of any bond $B(t, T_n)$ that has not yet matured at time T_i is given by

$$B(T_i, T_n) = \prod_{j=1}^{n-1} \frac{1}{1 + \delta L(T_i, T_j)}, \qquad i < n$$

more generally, at an arbitrary time

$$B(t, T_n) = B(t, T_{\eta(t)}) \prod_{j=\eta(t)}^{n-1} \frac{1}{1 + \delta L(t, T_j)}.$$

Choosing as numeraire $B(t, T_{N+1})$ we get as equations for our LIBOR rates

$$\frac{dL(t, T_n)}{L(t, T_n)} = -\sum_{i=n+1}^{N} \frac{\delta\lambda(t, T_n)\lambda(t, T_i)' L(t, T_i)}{1 + \delta L(t, T_i)} dt + \lambda(t, T_n) dW_t, \qquad n = 1, \dots, N. \tag{6.1}$$

6.2 Comparison to our Work

6.2.1 A Log-Normal Time-Discretization

Glassermann and Zhao in [8] present a straightforward Euler Discretization of a forward LIBOR market model directly

$$\hat{L}((j+1)h, T_N) = \hat{L}(jh, T_N)\mu_N(jh)h + \hat{L}(jh, T_N)\lambda(jh, T_N)[W_{(j+1)h} - W_{jh}] \tag{6.2}$$

or for $\log L(., T_N)$

$$\hat{L}((j+1)h, T_N) = \hat{L}(jh, T_N)\times\exp\Big([\mu_N(jh) - \frac{1}{2}\lambda(jh, T_N)\top\lambda(jh, T_N)]h + \lambda(jh, T_N)[W_{(j+1)h} - W_{jh}]\Big). \tag{6.3}$$

By comparison our approximation in the first interval gives

$$\hat{L}(t_1, T_N) = L(t_0)\Big(e^{\lambda(0,T_N)Y + \lambda(0,T_N)b_0^{N+1}} - 1\Big). \tag{6.4}$$

In order to get a martingale we need some drift adjustment in the direct approach.
Suggestions for the approach taken in [8] lead to

$$\mathbb{E}\Big[\frac{1}{(1+\delta L(0,T_1))(1+\hat{\mu}_1(0)h + \sqrt{h}\lambda(0,T_1)\xi_1)}\Big] = \frac{1}{1+\delta L(0,T_1)} \qquad \xi_1\ N(0,1) \tag{6.5}$$

which is never finite or, for the log-discretization to quantities of the form

$$\mathbb{E}[\frac{1}{1+\exp X}] \qquad X\ N(a,b)$$

which are complicated to calculate.
By comparison in our model, we have

$$b_1^{N+1} = -\frac{1}{\lambda(0,T_N)}\log\mathbb{E}(\exp(\lambda(0,T_N)Y_1)) \qquad Y_1\ N(0,1) \tag{6.6}$$

which is well-defined and very well calculable.
However, a priori our log-normal discrete time model differs from the Euler-Discretization only by a drift adjustment, just as Glassermann and Zhao try in [8] . Why is our drift-condition so much nicer then?
Essentially the difference is , that in equation 6.3 one would first have to summarize

$$\mu_N(jh) - \frac{1}{2}\lambda(jh, T_N)\top\lambda(jh, T_N) = \hat{\mu}_N jh \tag{6.7}$$

and then determine the condition for martingality of $L(., T_N)$ for $\hat{\mu}_N jh$. Then the drift condition becomes equal to ours. So our method is quite different from straightforward Euler discretization (Weak Convergence therefore had to be proved separately and would not have followed from convergence of the Eulerscheme).
How is the relation of our model then, to the suggested alternative discretization in [8]? We wish to prove the following main result of this section

Theorem 15 *The discretization of the logarithms of differences of discounted bond-prices under the terminal measure*

$$X_n(t) := \frac{1}{\delta}(D(t,T_n) - D(t,T_{n+1})) = L_n(t)\prod_{i=n+1}^{N}(1+\delta L_i(t)). \tag{6.8}$$

implies the same dynamics for the implied discretized LIBOR rates $(L_n(t_i))_{i\leq n}$ *as our discrete LIBOR models* $(\hat{L}(t_i,T_n))_{i\leq n}$ *for standard normal driving-process variables*

$$\hat{L}(t_i,T_N)=\hat{L}(t_{i-1},T_N)e^{\lambda(t_i,N)Y_{i-1}^{N+1}+b_i^{N+1}} \qquad Y_s\ N(0,1)\forall s \tag{6.9}$$

Proof:
How do differences of deflated bond-prices behave in our discrete model?
Here we have a priori the problem that $(B(t_j,T_i))_{j\leq i}$ were not defined originally for our discrete LIBOR analog. However if we look closely at 6.8, we get a much clearer representation

$$\frac{1}{\delta}(D_n(t)-D_{n+1}(t))=\frac{1}{\delta}\Big(\frac{B(t,T_n)-B(t,T_{n+1})}{B(t,T_{N+1})}\Big)=\frac{1}{\delta}\Big(\frac{B(t,T_n)-B(t,T_{n+1})}{B(t,T_{n+1})}\Big)\frac{B(t,T_{n+1})}{B(t,T_{N+1})}= \tag{6.10}$$

$$L(t,T_n)F_B(t,T_{n+1},T_{N+1})=L(t,T_n)\prod_{j=n+1}^{N}F_B(t,T_j,T_{j+1}).$$

$F_B(t,T_j,T_{j+1})$ is well defined in our discrete setting, as $\hat{F}_B(t_i,T_j,T_{j+1})$.
We have equations for those processes

$$L(t,T_n)=\frac{X_n(t)}{1+\delta X_{n+1}(t)+\cdots+\delta X_N(t)} \tag{6.11}$$

which also hold for the discretization by purely algebraic calculations from 6.8

$$\hat{L}(t,T_n)=\frac{\hat{X}_n(t)}{1+\delta\hat{X}_{n+1}(t)+\cdots+\delta\hat{X}_N(t)}. \tag{6.12}$$

Now we want to derive the dynamics of the $(\hat{X}_n)_n$ in our discrete LIBOR model. In [8] the continuous time dynamics are derived, and the solution to their log-discretization is taken as definition of the $(\hat{X}_n)_n$

$$\frac{dX_n}{X_n}(t_i)=\Big(\lambda(t_i,T_n)+\sum_{j=n+1}^{N}\frac{\delta X_j\lambda(t_i,T_j)}{1+\delta X_j+\cdots+\delta X_N}\Big)dW \tag{6.13}$$

and

$$\frac{\Delta\hat{X}_n}{X_n}(t_i)=\Big(\lambda(t_i,T_n)+\sum_{j=n+1}^{N}\frac{\delta\hat{X}_j\lambda(t_i,T_j)}{1+\delta\hat{X}_j+\cdots+\delta\hat{X}_N}\Big)dW. \tag{6.14}$$

Lemma 6 *The dynamics derived for* $\hat{X}_N$ *from our discrete LIBOR dynamics with standard normal driving variables coincide with the dynamics derived in [8] equation* (24) *for* $n=N$.

Proof:
Suppose now, we have our discrete LIBOR analog $(\hat{L}(t_i,T_j))_{ij}$. How would the dynamics of the $(\hat{X}_n)_n$ defined through 6.8 look? We have $X_N=L_N$ and therefore

$$\Delta\hat{X}_N(t_s)=\Delta\hat{L}(t_s,T_N) \qquad \forall s \tag{6.15}$$

which means

$$\Delta\hat{X}_N(t_s)=\hat{X}(t_{s-1},T_N)\int_{\mathbb{R}}(e^{\lambda(s,T_N)x+b_s^{N+1}}-1)(\mu-\nu_s^{N+1})(ds,dt) \tag{6.16}$$

The discretization of $\log X_N(t)$ in [8] gives(with $h = 1$)

$$\hat{X}_N((i+1)) = \hat{X}_N(i) \exp\Big(-\frac{1}{2}\lambda(i,T_)N^2 + \lambda(i,T_N)\xi_{i+1}\Big) = \hat{X}_N(i)\exp(-\log m(\lambda(i,T_N)))\exp\lambda(i,T_N)\xi_{i+1} \tag{6.17}$$

where $m(.)$ denotes the moment generating function of ξ_{i+1}. Thus $\hat{X}_N$ derived by discretizing the $\log X_N(t)$ is exactly the same as deriving $\hat{X}_N$ from our $\hat{L}(t_i, T_n)$ in our discrete LIBOR market model. $\square$

Lemma 7 *The dynamics derived for $\hat{X}_n$ from our discrete LIBOR dynamics with standard normal driving variables coincide with the dynamics derived in [8] equation* (24) *for all n.*

Proof:
For the higher rates first of all we observe

$$\lambda(i,T_n) + \sum_{j=n+1}^{N} \frac{\delta X_j(i)\lambda(i,T_j)}{1+\delta X_j(i)+\cdots+\delta X_N(i)} = \lambda(i,T_n) + \sum_{j=n+1}^{N} \ell(t_i,T_j)\lambda(i,T_j). \tag{6.18}$$

Then we remember

$$\hat{X}_n(i) = \hat{L}(t_i,T_n) \prod_{j=n+1}^{N} F_B(t_i,T_j,T_{j+1}).$$

We proceed to use our discrete LIBOR model $\hat{L}(t_i,T_n)$.
We know from the stochastic difference equation that

$$F_B(t_i,T_j,T_{j+1}) = F_B(t_{i-1},T_j,T_{j+1})\mathcal{E}\Big(\lambda(i,T_n)\ell(t_{i-1},T_j)Y_i^{j+1}\Big). \tag{6.19}$$

Now we derive

$$\hat{X}_n(i) = \hat{X}_n(i-1) + \Delta\hat{X}_n(i) = \hat{X}_n(i-1)\exp\Big(\lambda(i,T_n)Y_i^{n+1} + b_i^{n+1}\Big) \prod_{j=n+1}^{N} \mathcal{E}\Big(\lambda(i,T_j)\ell(t_{i-1},T_j)Y_i^{j+1}\Big) \tag{6.20}$$

So

$$\hat{X}_n(i) = \hat{X}_n(i-1)\exp\Big(\lambda(i,T_n)Y_i^{n+1} + b_i^{n+1}\Big)\mathcal{E}\Big(\sum_{j=n+1}^{N}\lambda(i,T_j)\ell(t_{i-1},T_j)Y_i^{N+1}\Big). \tag{6.21}$$

Now we know in our case of Y_i^{N+1} being standard normally distributed

$$\mathcal{E}\Big(\sum_{j=n+1}^{N}\lambda(i,T_j)\ell(t_{i-1},T_j)Y_i^{N+1}\Big) = \exp\Big(\sum_{j=n+1}^{N}\lambda(i,T_j)\ell(t_{i-1},T_j)Y_i^{N+1} - \frac{1}{2}(\sum_{j=n+1}^{N}\lambda(i,T_j)\ell(t_{i-1},T_j))^2\Big). \tag{6.22}$$

Furthermore we consider everything under the terminal measure. Therefore

$$\lambda(i,T_n)Y_i^{n+1} = \lambda(i,T_n)Y_i^{N+1} + \lambda(i,T_n)\sum_{j=n+1}^{N}\lambda_j\ell(t_{i-1},) \tag{6.23}$$

and our drift under the terminal measure

$$b_i^{n+1} = \lambda(i,T_n)^2. \qquad \text{under} \quad \mathbb{P}_{N+1} \tag{6.24}$$

Thus we can summarize

$$\hat{X}_n(i) = \hat{X}_n(i-1)\exp\Bigg(-\frac{1}{2}\sigma_n^2(i-1) + \sigma_n(i-1)Y_i^{n+1}\Bigg) \tag{6.25}$$

with

$$\sigma_n(i-1) = \lambda(i, T_n) + \sum_{j=n+1}^{N} \lambda(i, T_j)\ell(t_{i-1}, T_j) \tag{6.26}$$

and with 6.18 those are just equations (24) and (25) in [8]. □.

This lemma also concludes the proof of our theorem. □

6.2.2 Advantages Of Our Approach

Our Approach has several advantages

- The derivation of our drift-condition for $(L(t_i, T_j))_{i \le j}$ is a straightforward adaptation of the corresponding situation in Lévy LIBOR models or in fact general semimartingale driven LIBOR market models, instead of a not so obvious consequence of a discretization of a different quantity(which only holds in the special case of log-normal model).
- Having only the exponential moment condition on our variables in the discrete analogon, we can model very different driving processes as well, without the need to derive a new drift correction.
- We can develop all interesting notions for pricing, measures etc in a purely discrete context, without the need to laboriously translate from the continuous time context.

Appendix A

Essentials

We assume basic measure theoretic notions such as measure, σ-Algebra and elementary knowledge of stochastic processes, such as filtrations and stopping times, and the stochastic integral as known.
There are some notions however, we would like to state explicitly.

Definition 18 (Local Martingale) *Assume as given a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *satisfying the usual hypotheses.*
An adapted cádlág process X *is a local martingale is there exists a sequence of increasing stopping times* T_n *with* $\lim_{n\to\infty} T_n = \infty$ *a.s., such that* $X_{t\hat{T}_n} 1_{\{T_n>0\}}$ *is a uniformly integrable martingale for each* n*. Such a sequence* (T_n) *of stopping times is called a fundamental sequence.*
The space of all local martingales on $(\Omega, \mathcal{F})$ *for a given measure* $\mathbb{Q}$ *is called* $\mathcal{M}_{loc}(\mathbb{Q})$*.*

We also need the property of predictable measurability.

Definition 19 (Predictably Measurable) *Assume as given a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *satisfying the usual hypotheses.*
Let $\mathbb{L}$ *denote the space of all adapted processes with caglad paths. The predictable sigma-algebra* $\mathcal{P}$ *on* $\mathbb{R}_+ \times \Omega$ *is the smallest* σ*-algebra making all processes in* $\mathbb{L}$ *measurable. We also let* $\mathcal{P}$ *denote all processes that are predictably measurable.*

Appendix B

Weak convergence of processes

Definition 20 (Pathspace) *Let*

$$\mathbb{D}(\mathbb{R}) := \{f \quad \text{function on } \mathbb{R} \mid \quad f \text{ is cádlág}\}$$

be the vector space of cádlág-functions on $\mathbb{R}$.

We then have a σ-Field generated by the evaluation functionals

Definition 21 (Measurability On Pathspace) *The mappings* $t : \mathbb{D}(\mathbb{R}) \to \mathbb{R}$ *defined as* $t(f) := f(t)$ *generate a* σ*-Field for* $t \leq s$*, which we denote by* $\mathcal{D}_t^0(\mathbb{R})$*. We further denote by* $\mathcal{D}(\mathbb{R}) := \cup_{t \geq 0} \mathcal{D}_t^0(\mathbb{R})$ *the union of those* σ*-Fields.*
We get a filtration on $\mathbb{D}(\mathbb{R})$ *by taking* $\mathbf{D}(\mathbb{R}) := (\mathcal{D}_t(\mathbb{R}))_{t \geq 0}$ *as said filtration.*

We want $\mathcal{D}(\mathbb{R})$ to be the Borel-Field for our topology on $\mathbb{D}(\mathbb{R})$. To that end we define the Skorokhod Topology via

Definition 22 (Skorohod Topology) *A sequence* $(\alpha_n)_n \in \mathbb{D}(\mathbb{R})$ *converges to* α *iff there is a sequence* λ_n *of strictly increasing continuous functions with* $\lambda_n(0) = 0$ *and* $\lambda \to \infty$ *such that*

$$\sup_s |\lambda_n(s) - s| \to 0$$

$$\sup_{s \leq N} |\alpha \circ \lambda_n(s) - \alpha(s)| \to 0 \qquad \forall N \in \mathbb{N}$$

both hold.

This characterization of convergence of sequences suffices because there is a metrizable topology bearing just this characterization of convergence on $\mathbb{D}(\mathbb{R})$.
The proof consist of defining the proper metric and calculating the properties.(See [9])
Now we can define weak convergence of processes by a sort of convergence in a "weak topology on the pathspace".

Definition 23 (Weak Convergence Of Processes) *We say that a sequence of processes* $(X_t^n)_{t \in I}$ *converges weakly to* Y *iff the paths of the* X_t^n *converge in* $D[I]$ *to* Y *in the Skhorohod Topology.*

A related concept is that of P-UT(predictable uniform tightness) which we will need latter for the weak-convergence result on SDE's.

Definition 24 (Predictably Uniformly Tight) *For each integer n let $\mathcal{B}^n = (\Omega^n, \mathcal{F}^n, \mathbf{F}^n, \mathbb{P}^n)$ be a stochastic basis. We denote by $\mathcal{H}$ the set of all predictable processes H^n on $\mathcal{B}^n$ having the form*

$$H_t^n = Y_0 1_{(0)} + \sum_{i=1}^{k} Y_i 1_{(s_i, s_{i+1}]}(t)$$

with $k \in \mathbb{N}$, $0 = s_0 < s_1 < ... < s_k < s_{k+1}$ and Y_i is $\mathcal{F}_{s_i}^n$-measurable with $|Y_i| \leq 1$.
We can define an elementary stochastic integral by

$$H^n \cdot X_t^n = \sum_{i=1}^{k} Y_i (X_{\inf\{t, s_{i+1}\}}^n - X_{\inf\{t, s_i\}}^n).$$

Now a sequence (X^n) of adapted(on their respective Bases) cadlag d-dimensional processes is P-UT if for every $t > 0$ the family of random variables $(\sum_{1 \leq i \leq d} H^{n,i} \cdot X_t^{n,i} : n \in \mathbb{N}, H^{n,i} \in \mathcal{H}^n)$ is tight in $\mathbb{R}$ meaning

$$\lim_{a \to \infty} \sup_{H^{n,i} \in \mathcal{H}} \mathbb{P}(|\sum_{i=1}^{d} H^{n,i} \cdot X_t^{n,i}| > a) = 0.$$

Before we get to the 2 main results, a short reminder on Existence and Uniqueness in SDE's

Theorem 16 (Existence And Uniqueness) *Given an equation*

$$Y = Z + f(Y_-) \cdot X \tag{B.1}$$

where

- *The driving term is a d-dimensional semi-martingale X, given on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$.*
- *The solution Y is a q-dimensional process.*
- *There is an "initial condition" which is a q-dimensional cadlag adapted process Z*
- *There is a "coefficient" function $f : \mathbb{R}^r \to \mathbb{R}^q \times \mathbb{R}^d$ which is locally Lipschitz und linearly bounded.*

then we know that we have locally a unique strong solution to the SDE.
If our f is timedependent, then we simply include time as an additional dimension in the Lipschitz and linear-growth conditions.

Proof:
All proofs can be found in [9]. □

Now for the two main results.
Assume the following setting:

- We are given an equation
$$Y = Z + f(Y_-) \cdot X$$
as above

- For each n we have a stochastic basis $\mathcal{B}^n = (\Omega^n, \mathcal{F}^n, \mathbf{F}^n, \mathbb{P}^n)$ and an equation

$$Y^n = Z^n + f_n(Y_-^n) \cdot X^n. \tag{B.2}$$

- X^n is a d-dimensional-semimartingale on that basis.
- Z^n is a q-dimensional cadlag adapted process.
- f_n are functions $\mathbb{R}^q \to \mathbb{R}^q \times \mathbb{R}^d$ such that each equation above admits a unique solution.

Then there holds

Theorem 17 (Weak Approximation) *Assume the functions f_n fulfill Lipschitz and linear boundedness with constants not dependent on n and $f_n \to f$ at least pointwise. Assume further that the sequence X^n is P-UT. Then if Y^n denotes the unique solution of the sequence of equations there holds*
If $(X^n, Z^n) \to (X, Z)$ weakly, then $(X^n, Z^n, Y^n) \to (X, Z, Y)$ weakly.

Proof:
All proofs can be found in [9]. □

In the case of our theorem in the mainwork on discrete approximation we need to show especially $P - UT$ of the driving process of the SDE. However we have weak convergence of the sequence of driving processes already and in that case there holds

Theorem 18 (Tightness And P-UT) *Let $(X^n)_n$ be a sequence of d-dimensional semimartingales with characteristics and second modified characteristics (B^n, C^n, ν^n) and $\tilde{C}^n$.*
If the sequence $(X^n)_n$ is tight, then the sequence is $P - UT$ iff the sequence $Var(B^{n,i})_t$ is tight.

Proof:
All proofs can be found in [9]. □

Appendix C

Variation and Increasing Process

Assume a fixed stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$.

Definition 25 (Bounded Variation Processes) *We denote by $\mathcal{V}^+$(resp. $\mathcal{V}$) the set of all real-valued processes A that are cadlag, adapted with $A_0 = 0$ and whose each path $t \to A_t(\omega)$ is non-decreasing(resp. has a finite variation over each finite interval $[0,t]$)*

We abbreviate calling a process $A \in \mathcal{V}^+$(resp. $\mathcal{V}$) an adapted, increasing process(resp an adapted process with finite variation).

Furthermore, we call $\mathcal{A}$ the set of processes which are both, of finite variation and predictably measurable. In other words

$$\mathcal{A} = \mathcal{V} \cap \mathcal{P}. \tag{C.1}$$

Definition 26 (Variation Process) *We denote by $Var(A)$ the variation process of A, that is the process such that $Var(A)_t(\omega)$ is the total variation of the function $s \to A_s(\omega)$ on the interval $[0,t]$. Of course $Var(A) = A$ if $A \in \mathcal{V}^+$.*

Theorem 19 (Decomposition) *Let $A \in \mathcal{V}$. There exists a unique pair (B, C) of adapted increasing processes such that $A = B - C$ and $Var(A) = B + C$(hence $Var(A) \in \mathcal{V}^+$). Moreover if A is predictable, then B, C and $Var(A)$ are also predictable.*

Proof:
All proofs can be found in [9]. $\square$

Appendix D

Doob-Meyer Decomposition and Compensators

In this section, we wish to give the basic facts about Doob-Meyer decompositions.

D.1 Doob-Meyer-Decomposition

We know that processes X of the form $X = M + A$ with M a local martingale and A an FV process are semimartingales. Those processes are called special semimartingales.
The question now is, which semimartingales are special martingales?
There's a chain of results(found for instance in [16] chap. 3) of increasing generality. The results culminate in the Bichteler-Dellacherie Theorem.

Theorem 20 (Bichteler-Dellacherie Theorem) *An adapted, cadlag process X is a semimartingale if and only if it is a classical semimartingale.*
That is: X is a semimartingale if and only if it can be written $X = M + A$, where M is a local martingale and A is an FV-process.

Proof:
See [16] p.146 □

D.2 Compensator

What is a compensator?
Given a process of locally integrable variation X, we have that $X = M + A$ with some local martingale M and an FV process A in a unique way. In other words there is an unique FV process A such that $X - A$ is a local martingale. This is the defining property of the compensator and very important for modeling purposes.

Definition 27 (Compensator) *Given A an FV process with $A_0 = 0$ with locally integrable total variation. The unique FV process $\tilde{A}$ such that $A - \tilde{A}$ is a local martingale is called the compensator of A.*

The application of this concept lies mainly in studying jump-measures:
As will be seen in the next section on characteristics, the "compensated jump-measure" is an essential tool to represent a semimartingale or for that purpose a Lévy-process.

Appendix E

Characteristics of Semimartingales

What are the Characteristics?
The main idea stems from an observation on Lévy-processes:
Let X be a real valued process with independent increments with $X_0 = 0$ and without fixed times of discontinuity. It is well known, that X_t has a distribution that is infinitely divisible and its characteristic function is of the form $\mathbb{E}(\exp(iuX_t)) = \exp(\psi_t(u))$, with

$$\psi_t(u) = iub_t - \frac{u^2}{2}c_t + \int (e^{iux} - 1 - iuh(x))F_t(dx)$$

where $b_t \in \mathbb{R}, c_t \in \mathbb{R}_+$ and F_t is a positive measure which integrates $\max\{1, x^2\}$ and h is any bounded Borel function with compact support which "behaves like x" near the origin. Moreover the property of independent increments immediately yields:

$$\exp(iuX_t)/\exp(\psi_t(u)) \qquad \text{is a martingale.}$$

Definition 28 (Truncation Function) *We call $\mathcal{C}_t^d$(for truncation function) the class of all functions $h : \mathbb{R}^d \to R^d$ which are bounded and satisfy $h(x) = x$ in a neighbourhood U of 0.*

Now we chose $h \in \mathcal{C}_t^d$. Since there is a symmetric open subset $V \subset U$ where $h(x) = x$ holds, there exists a bound $b \in \mathbb{R}_+$ such that $\Delta X_s - h(\Delta X_s) \neq 0$ if and only if $|\Delta X_s| > b$. In other words we truncate the small jumps, which could possible constrain integrability of the RV's of the process by defining a new process

$$\hat{X}(h)_t = \sum_{s \leq t} [\Delta X_s - h(\Delta X_s)]$$

$$X(h) = X - \hat{X}(h)$$

This Process is a special semimartingale since $\Delta X(h) = h(\Delta X)$ is bounded. Therefore we get the canonical decomposition

$$X(h) = X_0 + M(h) + B(h).$$

Now we can define our semimartingale characteristics

Definition 29 (Characteristics) *Let $h \in \mathcal{C}_t^d$ be fixed. We call characteristic of X the triplet (B, C, ν) consisting in:*

- $B = (B^i)_{i \leq d}$ *is a predictable process in* $\mathcal{V}^d$*, namely the process* $B = B(h)$ *appearing in the canonical decomposition.*
- $C = (C^{ij})_{i,j \leq d}$ *is a continuous process in* $\mathcal{V}^d \times \mathcal{V}^d$*, namely*

$$C^{ij} =< X^{i,c}, X^{j,c} >$$

- ν *is a predictable random measure on* $\mathbb{R}_+ \times \mathbb{R}^d$*, namely the compensator of the random measure* μ^X *associated to the jumps of* X*.*

Essential is the following result on Characteristics:

Theorem 21 *(Canonical Representation For Semimartingales] Let* X *be a* d*-dimensional semimartingale, with characteristics* (B, C, ν) *relative to a truncation function* $h \in \mathcal{C}_t^d$*, and with the measure* μ^X *associated to its jumps. Then* $W^i(\omega, t, x) = h^i(x)$ *belongs to* $G_{loc}(\mu^X)$ *for all* $i \leq d$ *and the following representation holds:*

$$X = X_0 + X^c + h * (\mu^X - \nu) + (x - h(x)) * \mu^X + B.$$

where "$*$" denotes the integral to the respective measure.

Proof:
All proofs can be found in [9]. $\square$

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