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Thesis

Spherically Symmetric Models in Loop Quantum Gravity

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To my grandfather
Erhard Preis (1923 - 2000)

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Contents

Introduction

Loop quantum gravity (LQG) is one of the most promising enterprises towards a quantum theory of gravity. The search for a theory of quantum gravity is motivated by the belief that such a framework may solve most of the inconsistencies of the two today's accepted and prominent models of nature - general relativity (GR) and the standard model of particle physics, which is described by quantized field theories. Both of them suffer from difficulties which can be separated into two kinds. On the one hand, there are the inherent ones: the spacetime singularities of GR and the UV divergences of quantum field theory (QFT). On the other hand both frameworks neglect to incorporate the essential features of each other.
In QFT the dynamical entities can be regarded as actors on a "stage" which is provided by the fixed spacetime background. The Einstein field equations, however tell us that the local metric configuration is determined by the energy content in that very region. This corresponds to Mach's principle, which states that geometry is a dynamical field itself. Both fields, geometry and matter should act on a "stage" represented by a four dimensional differential manifold. As a classical theory general relativity is deterministic whereas the quantum principle in its København interpretation teaches us that a good theory of nature should be intrinsically probabilistic and has to reflect the fact that in measurements the measuring apparatus becomes entangled with the measured system[1]. Nature displays an intrinsic discreteness, which for instance can be seen from the following observation: the values of angular momentum/spin of elementary particles differ by integer multiples of Planck's constant $\hbar$ and take values half integer times Planck's constant. With simple dimensional considerations one can derive a fundamental length starting with the three fundamental constants $\hbar$, G and c

$$\ell_P = \sqrt{\frac{\hbar 8\pi G}{c^3}} \tag{0.0.1}$$

which is called the Planck length. It turns out that this is the minimal interval in which one can locate a massless particle. Due to the momentum uncertainty the dynamical

[1]See [1] for an interesting relational interpretation of quantum mechanics.

mass increases until the measuring uncertainty of location reaches the corresponding Schwarzschild radius. At this point, physical predictability comes to an end. These heuristic considerations suggest that spacetime might be discrete at the Planck level. Such ideas were, for instance, incorporated in Wheeler's spacetime-foams [2] or Penrose's spin-networks [3]. If the theory provides a picture of a discrete space, this could be understood as a natural cut-off preventing UV-divergences.

It is believed that quantum gravity, once a fusion of the quantum principle with general covariance is achieved, will solve the inherent problems of each of the two frameworks, in which we describe large scale and small scale physics respectively. Indeed, LQG seems to support this hope to some extend: First, on a kinematical level, geometrical operators such as length, area and volume do have a discrete spectrum. However, [4] expressed doubts if this holds on the physical Hilbert space, too. Second, in a symmetry reduced model, Ashtekar, Bojowald and Lewandowsky [5] tackled the Big Bang singularity[2] as well as the Schwarzschild singularity [6][3] and found that the spectra of curvature and energy density are bounded. Third LQG passes one of the main "reality checks" any candidate of a quantum theory of gravity has to undergo: loop quantum gravity calculations reproduce the Hawking-Bekenstein formula up to a dimensionless constant, the so-called Barbero-Immirzi parameter, which is of the same numerical value for a variety of black holes. In this way, it explains the celebrated formula and the entropy interpretation of the area of a black hole horizon in terms of state counting. Another reality check, however, – having a well understood semiclassical limit – is still an open issue[4].

According to this list of topics, I will focus on the second one. This thesis aims at working out the classical part of symmetry reduction used in LQG in general and at presenting a spherically symmetric model and its quantization inspired by LQG. Furthermore, it presents the loop quantized version of gravity coupled to Yang Mills fields in general and spherically symmetric LQG coupled to (loop quantum) electrodynamics in particular.

The thesis is structured as follows: In chapter 1 I will derive the formulas needed for the initial value formulation of GR. Chapter 2 discusses the symplectic framework of GR. In chapter 3 I will depart from the metric formulation of GR. Thereby, I am interested in a tetrad formulation of GR, perform a $3+1$ split of spacetime and establish a relation to the ADM formalism. Afterwards, I analyze connections on principal fiber bundles, since they provide the fundamental variables of LQG. Furthermore, this analysis furnishes the tools which are needed for the general formalism of symmetry reduction based on invariant connections on invariant principal fiber bundles presented in chapter 5. Chapter 6 is disigned to provide the necessary notions of the symmetry group, on which the spherical symmetry reduction scheme is based on, as well as to give a more detailed presentation of

[2]Within the model of a homogeneous and isotropic universe.

[3]In the LQG model of the Kantowski-Sachs spacetime, i.e. the interior of a Schwarzschild black hole.

[4]Actually this is, together with finding the true Hamiltonian, the main problem of LQG.

the gauge group of LQG, by discussing the group of rotations $\sim SO(3)$ and its universal cover. Since LQG is formulated in terms of so-called spin connections I devolop a geometrical picture of spinors, the spin group and its relation to $SU(2)$ using the Clifford algebra on three dimensional Euclidean space in chapter 7. This procedure covers the basic tools needed to proceed to the development of a connection dynamics for general relativity and its symmetry reduction on the classical phase space in chapter 8. In the following chapter 9 we work out the connection dynamical formulation of Yang-Mills theory and apply the symmetry reduction framework as well. In chapter 10 the presentation of the Reissner-Nordström solution, i.e. the spherically symmetric metric of a charged static black hole, is derived from the constraint equations of symmetry reduced Einstein-Maxwell theory in terms of connections. Finally I come to the quantization of full Einstein-Yang-Mills theory and its symmetry reduced version in chapter 11. Chapter 12 presents a summary and a discussion of the issues presented in the thesis. Furthermore, it provides an outlook for future research.

Chapter 1

Initial value formulation of general relativity

Loop quantum gravity is a canonically quantized theory, i.e. that the quantization starts from a Hamiltonian formulation of the theory. In order to obtain such a formulation from the Lagrangean formulation via a Legendre transformation one needs to split spacetime into space and time ($3+1$ split). This procedure also allows for setting up an initial value formulation of GR, which will be shown in this chapter.

For the sake of simplicity we assume a four dimensional globally hyperbolic spacetime $\mathcal{M} = \mathbb{R} \times \Sigma$[1] and, for later convenience choose Σ being compact. Such a topology admits a foliation in Cauchy surfaces Σ_t labeled by the so-called timefunction t. Let n^a be the unit normal to the spatial hypersurface. Given a foliation, consider the tangential vector field t^a with affine parameter t. We perform a orthogonal decomposition of t^a into timelike and spatial components

$$t^a = Nn^a + N^a.$$

N, the so-called lapse function and N^a the so-called shift are completely arbitrary since the foliation of $\mathcal{M}$ is not fixed. The flow identifies the points of the spatial slices and its generator can therefore be interpreted as flow of time. Note, that t^a is not necessarily timelike.

On each slice the spacetime metric g_{ab} induces a spatial metric h_{ab}, which gives rise to

$$g_{ab} = -n_a n_b + h_{ab}.$$

[1]See [7] for a short discussion of this somewhat restrictive assumption and and more general approaches. This first step, necessary for the canonical quantization, is also a very problematic one. Clearly it spoils manifest general covariance but we will see that the spatial diffeomorphism constraint together with the relativistic Hamiltonian generate the desired four diffeomorphisms.

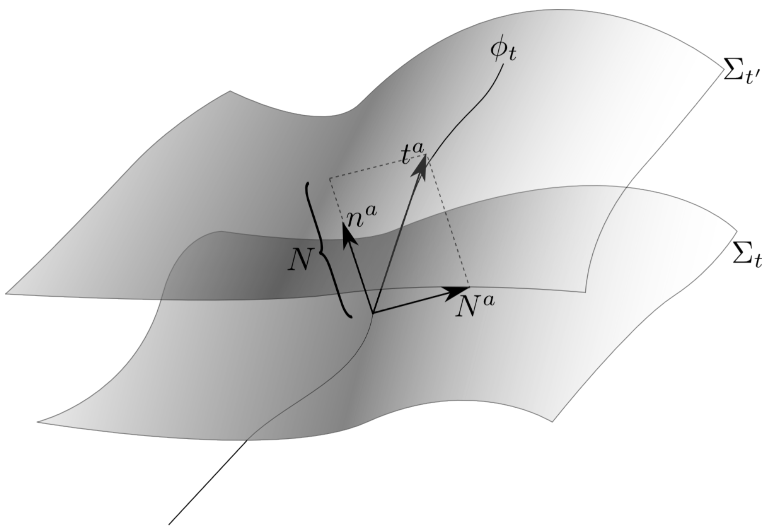


Figure 1.1: *Foliation of spacetime with equal time slices.*

The surface normal of a $t = const.$ slice is given by $n_a = f\nabla_a t$. Therefore, we obtain

$$t^a n_a = -N = f t^a \nabla_a t = f. \tag{1.0.1}$$

Of course a specification of the 3-metric as configuration variableis not enough information to determine its evolution. We also need its variation under evolution, i.e. its spatial velocity moving along integral curves of t^a or the embedding data of the slice, which is equivalent as will be shown immediately. Geometrically the latter is captured by the notion of the extrinsic curvature. Let ξ^a be the unit futurepointing tangent of a geodetic timelike congruence. A congruence is a family of curves, such that through each point $p \in \mathcal{M}$ there passes precisely one curve. Since ξ^a is geodetic we have $(\xi\nabla)\xi^a = 0$. We define

$$\begin{aligned} &B_{ab} := \nabla_b \xi_a \\ &\Rightarrow \ \xi^a B_{ab} = B_{ab}\xi^b = 0 \ \text{ hence is purely spatial.} \end{aligned} \tag{1.0.2}$$

Let η^a be an orthogonal Lie-dragged (Jacobi) vector field, then $(\xi\nabla)\eta^a = B^a{}_b\eta^b$ is the rate of change of a displacement to a nearby geodesic. Via geodesic deviation one would obtain the well known Raychaudhuri equations. We define the spatial metric $h_{ab} = g_{ab} + \xi_a\xi_b$ and split B_{ab} in its trace (expansion), symmetric (shear) and antisymmetric (vorticity) part.

$$B_{ab} = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab}$$

Via the Frobenius' theorem B_{ab} is symmetric, iff ξ_a is hypersurface orthogonal. In this case, we will refer to B_{ab} as extrinsic curvature $K_{ab} = h_a{}^c\nabla_c\xi_b$ of the so determined

hypersurface.
Now let ξ^a and n^a coincide on Σ_t, then $K_{ab} = h_a{}^c \nabla_c n_b$ describes the extrinsic curvature of Σ_t. To firstly show that K_{ab} is a symmetric tensor we refer to the fact $n_a = -N\nabla_a t$, see app. A.1. This allows for showing that the extrinsic curvature is proportional to the Lie-derivative with respect to n^a, see app. A.2.

$$K_{ab} = \frac{1}{2}\pounds_n h_{ab} \tag{1.0.3}$$

Since we aim at interpreting this expression as the time derivative of h_{ab}, $\dot{h}_{ab} := h_a{}^c h_b{}^d \pounds_t h_{cd}$, we use the relation between the Levi-Civita connection D_a associated with h_{ab} and its four dimenional analogue

$$D_a T_{b_1 \ldots b_n} = h_a{}^c h_{b_1}{}^{d_1} \ldots h_{b_n}{}^{d_n} \nabla_c T_{d_1 \ldots d_n},$$

where $T_{b_1 \ldots b_n} \in \prod_{i=1}^n \mathcal{T}_i^* \mathcal{M}$[2]. App. A.3 shows that D_a meets all conditions for being a Levi-Civita connection.
Using

$$n^a = \frac{1}{N}(t^a - N^a)$$

we can rewrite (1.0.3) (see app. A.4)

$$\begin{aligned} K_{ab} &= \frac{1}{2N}(\dot{h}_{ab} - h_a{}^c h_b{}^d \pounds_N h_{cd}) = \\ &= \frac{1}{2N}(\dot{h}_{ab} - D_a N_b - D_b N_a) \end{aligned} \tag{1.0.4}$$

The last ingredients to complete the 3+1 split are the curvature tensors. The Riemannian of the connection D_a is defined analog to the Lorentzian case

$${}^{(3)}R^d{}_{cab} v^c := [D_a, D_b] v^d, \quad v^a \in \mathcal{T}\Sigma, \tag{1.0.5}$$

and the calculation of the commutator (app. A.5) gives us the first Gauß relation between the intrinsic curvature ${}^{(3)}R^d{}_{cab}$ and the projected part of $R^d{}_{cab}$. As expected the relation is coded by products of the extrinsic curvature.

$${}^{(3)}R^d{}_{cab} = h_a{}^e h_b{}^f h_c{}^g h^d{}_h R^h{}_{gef} - K_a{}^d K_{bc} + K_b{}^d K_{ac}$$

[2]Of course this generalizes to tensors of arbitrary valence.

The rest of the Gauß relations is obtained by tracing the Riemannian

$$
\begin{aligned}
{}^{(3)}R_{cb} &= g_d{}^a\, {}^{(3)}R^d{}_{cab} = h_d{}^a\, {}^{(3)}R^d{}_{cab} = \\
&= h_b{}^f h_c{}^g h^e{}_h R^h{}_{gef} - K_a{}^a K_{bc} + K_b{}^a K_{ac} \qquad (1.0.6)\\
{}^{(3)}R &= h^{cb} h_b{}^f h_c{}^g h^e{}_h R^h{}_{gef} + K^{ab}K_{ab} - (K_a{}^a)^2 = \qquad (1.0.7)\\
&= R + 2n^a n^b R_{ab} + K^{ab}K_{ab} - (K_a{}^a)^2. \qquad (1.0.8)
\end{aligned}
$$

In (1.0.5) we only considered vector fields in $\mathcal{T}\Sigma$. The missing data of the behavior of a vector field n^a in $\mathcal{TM}$ along a closed curve in Σ projected to Σ are given by the Codazzi relation. By a slight abuse of notation

$$[D_a, D_b]n^d = D_a K_b{}^d - D_b K_a{}^d = h_a{}^e h_b{}^f h^d{}_h R^h{}_{gef} n^g \qquad (1.0.9)$$

d and a contracted yields

$$D_a K_b{}^a - D_b K_a{}^a = h_b{}^f R_{fg} n^g$$

At this point, it is necessary to spotlight the action of D_a on a general element in $\mathcal{TM}$. This suggestive form is for notational simplicity only, saying: Act with $h_a{}^b \nabla_b$ on an arbitrary tensorfield over $\mathcal{M}$,i.e. $\in \mathcal{XM}$, and project all indices to Σ[3]. Now we can finish the initial value formulation by splitting the Einstein field equations into evolution and initial value constraints. In the next chapter, it will be shown how all this translates into the canonical (ADM) formulation, which provides a better view on the physical significance of the constraints and a quite different interpretation of "evolution".

From $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = \kappa T_{ab}$ we obtain

$$
\begin{aligned}
2G_{ab}n^a n^b &= 2R_{ab}n^a n^b + R = \\
&= {}^{(3)}R - K^{ab}K_{ab} + (K_a{}^a)^2 = 2\kappa T_{ab} n^a n^b =: 2\kappa\rho
\end{aligned}
$$

using (1.0.8) and with (1.0.9)

$$
\begin{aligned}
h_a{}^c G_{cb} n^b &= h_a{}^c R_{cb} n^b = D_b K_a{}^b - D_a K_b{}^b = \\
&= D_b(K_a{}^b - h_a{}^b K_c{}^c) = \kappa T_{cb} h_a{}^c n^b =: -\kappa J_a
\end{aligned}
$$

where ρ denotes the energy density and J_a denotes the momentum density. These equations are the constraints for the initial data: h_{ab} and K_{ab}.

Equation (1.0.4) is one of the evolution equations. To obtain the complete set of evolution

[3] As was shown in (1.0.2) $K_{ab} = h_a{}^c \nabla_c n_b = h_a{}^c h_b{}^d \nabla_c n_d =$"$D_a n_b$" is $\in \mathcal{T}\Sigma$ and therefore the rest of (1.0.9) is well defined.

equations we need the time derivative of K_{ab} (see app. A.6)

$$\dot{h}_{ab} = 2NK_{ab} + 2D_{(a}N_{b)}, \tag{1.0.10}$$

$$\begin{aligned}\dot{K}_{ab} &= 2K_{(a}{}^{e}D_{b)}N_e + N^e D_e K_{ab} - N\big(\,^{(3)}R_{ab} + K_{ab}K - 2K_a{}^c K_{cb}\big) + \\ &+ 8\pi\big(J_{ab} - \frac{1}{2}Jh_{ab} + \frac{1}{2}\rho h_{ab}\big) + D_a D_b N,\end{aligned} \tag{1.0.11}$$

with $J_{ab} := h_a^c h_b^d T_{cd}$ and $J := h^{ab} J_{ab}$

It can be shown that these equations are consistent, i.e. the constraints propagate.

Chapter 2

Canonical general relativity: geometrodynamics

> "There is no spacetime, there is no time, there is no before, there is no after. The question of what happens 'next' is without meaning."[2]

In the last century the action principle was the most often used path towards a quantization of field theories. Quantum mechanics demands that the quantum of the action exchange is a multiple of $\hbar$. Once the classical action is found it is used to quantize the theory via several methods. Two important methods are the path integral formulation and the canonical formulation. The latter passes via a Legendre transformation from the Langrangean to a Hamiltonian formulation. Therefore one defines the configuration space, derives the conjugate momenta (and possibly constraints) and finally arrives at Hamilton's equations. For constraint systems one applies the so-called Dirac formalism. Then in the quantization procedure one has to translate the Poisson bracket structure into a commutator algebra. In the loop quantum gravity context this program is called refined algebraic quantization.

The classical action functional is given by the expression

$$S[g_{ab}] = \int_{\mathcal{R}} \mathcal{L}[g_{ab}],$$

where g_{ab} is the 4-metric and $\mathcal{L}[g_{ab}]$ is the Lagrangean scalar density[1]. Hence, the action is manifestly invariant under 4-diffeomorphisms. $\partial\mathcal{R}$ is assumed to be nowhere null and g_{ab}

[1]It is a 4-form and therefore proportional to the volume form $\omega_g = \sqrt{-\det g} d^4x$

is held fixed there[2]. Furthermore we have to demand $\pounds_t\omega_g = 0$ and define $\omega_g = -Ndt\wedge\omega_h$. After the variation of the action functional the principle of minimal action gives us the Euler-Lagrange equations which, in case of GR are the Einstein equations.

$$\delta S = -\int_{\mathcal{R}} \mathcal{L}(g_{ab}) = \int_{\mathcal{R}} \omega_g \delta g^{ab}\big(\frac{1}{\kappa}G_{ab} - T_{ab}\big) = 0$$

Solving for $\mathcal{L}(g_{ab})$ yields the Einstein-Hilbert-Hawking action (here without matter) (see app. A.7)

$$S[g_{ab}] = -\frac{1}{\kappa}\int_{\mathcal{R}} \omega_g R - \frac{2}{\kappa}\int_{\partial\mathcal{R}} \omega_h K.$$

If matter is to be taken into account, it is more elegant to obtain the energy momentum tensor from a variational principle, too. We write for the complete action

$$S = S_G + \alpha_M S_M.$$

The energy momentum tensor density is given by

$$\omega_g T^{ab} = \alpha_M \frac{\delta S_M}{\delta g_{ab}}.$$

For electromagnetism one chooses $\alpha_M = e^2/2\pi$ such that the equations of motion take the form $\nabla_a F^{ab} = -4\pi j^b$ (j^b denoting the 4-current).

For the sake of simplicity, let us assume that the region $\mathcal{R}$ is bounded by two oriented and compact constant time surfaces Σ_0 and Σ_τ[3]. With this choice of boundary, the trace of the extrinsic curvature of the spatial slices discussed in the preceeding chapter and the trace of the extrinsic curvature of the boundary of $\mathcal{M}$ coincide.

In the following the action will be expressed as a functional of quantities intrinsic to the spatial slices Σ. Therefore we use the Gauß-Codazzi relation (1.0.8) together with

$$n^a n^b R_{ab} = \nabla_d((n\nabla)n^d) - \nabla_b(n^b(\nabla n)) - K_d{}^b K_b{}^d + K^2,$$

to obtain

$$^{(3)}R = R - K^{ab}K_{ab} + K^2 + 2\nabla_d((n\nabla)n^d) - 2\nabla_b(n^b(\nabla n)). \tag{2.0.1}$$

---

[2]See [8] for a discussion about boundary conditions. There Wald argues that it suffices to fix the induced metric, since one can find a gauge transformation that fixes g_{ab}.

[3]See [8] for generalizations to variations that respect asymptotic flat spacetimes. Usually one omits all boundary terms from the beginning and ends up with (2.0.2).

This is inserted in the action functional

$$S[g_{ab}] = -\frac{1}{\kappa}\int dt \int_\Sigma \omega_h N[\,^{(3)}R + K^{ab}K_{ab} - K^2]. \tag{2.0.2}$$

Note, how nicely the boundary terms of the Einstein-Hilbert-Hawking action and the second total divergence in (2.0.1) cancel due to the choice of boundary, while the first one does not contribute since

$$\int_{\mathcal{R}} \omega_g \nabla_d((n\nabla)n^d) = -\int_{\partial\mathcal{R}} \omega_h n_d((n\nabla)n^d) = 0. \tag{2.0.3}$$

Formula (2.0.2) is the point of departure for the ADM approach to canonical general gelativity. As the coordinates of configuration space $\mathcal{C}$ we choose the field variables h_{ab}, N and N^a. Hence, we start with 20 phase space degrees of freedom[4] per spacetime point. $\mathcal{C}$ is a Lagrangean submanifold of the phase space Γ, i.e. in each point p of $\mathcal{C}$, $\mathcal{T}_p\mathcal{C} = \mathcal{T}_p\mathcal{C}^\perp$, where $\mathcal{T}_p\mathcal{C}^\perp$ is the symplectic complement, which is the space of Hamiltonian vector fields annihilating the vector fields of $\mathcal{T}_p\mathcal{C}$ with respect to the symplectic structure at that point. Their respective conjugate momenta are given by the variations of the action with respect to the configuration variables

$$\kappa\delta_h L = \int_\Sigma \Pi^{ab}\delta h_{ab}$$
$$\kappa\delta_N L = \int_\Sigma \Pi\delta N$$
$$\kappa\delta_{\vec{N}} = \int_\Sigma \Pi_a\delta N^a$$

[4] A priori the spacetime metric has 10 degrees of freedom.

Since in case of GR the configuration variables appear algebraically (Π_h denoting $\Pi^{ab}h_{ab}$) the momenta are given by

$$\Pi^{ab} = \frac{\partial}{\partial \dot{h}_{ab}}\omega_h N[\ {}^{(3)}R + K^{ab}\frac{1}{2N}(\dot{h}_{ab} - 2D_{(a}N_{b)}) - Kh^{ab}(\dot{h}_{ab} - 2D_{(a}N_{b)})] =$$

$$= \omega_h(K^{ab} - Kh^{ab})$$

$$\Rightarrow K^{ab} = \omega_h^{-1}[\Pi^{ab} - \frac{1}{2}\Pi_h h^{ab}]$$

$$\dot{h}_{ab} = N\omega_h^{-1}[2\Pi_{ab} - \Pi_h h_{ab}]$$

$$\Pi = \frac{\kappa\partial\mathcal{L}}{\partial\dot{N}} = 0 \tag{2.0.4}$$

$$\Pi_a = \frac{\kappa\partial\mathcal{L}}{\partial\dot{N}^a} = 0 \tag{2.0.5}$$

It can be seen that the conjugate momenta are tensor densities of weight one, i.e. tensor valued 3-forms. The equations (2.0.4) and (2.0.5) show that the matrix of second functional derivatives of S with respect to the generalized positions is singular - we say the Lagrangean and therefore the Legendre transform is singular, i.e. not all velocities can be expressed in terms of conjugated momenta. These equations put us in the realms of a constraint Hamiltonian system. The physical significance of constraints of dynamical systems lies in the fact that the respective variations of the dynamical variables are not independent, i.e. we started with too many degrees of freedom. In the symplectic picture, constraints define a so-called constraint surface $\mathcal{S}$ in the infinite dimensional phase space Γ coordinatized by $(h_{ab}, N, N^a; \Pi^{ab}, \Pi, \Pi_a)$. $\Pi = 0$ and $\Pi_a = 0$ are called the primary constraints, since they come directly from the Langrangean. In order to treat systems with singular Lagrangean using Dirac's procedure [9] which is a self-consistent algorithm that incorporates the constraints via Lagrange multiplier fields, which are (partly) specified upon the demand of the preservation of the constraints under time evolution. Thereby one has to pass from the Hamiltonian to the total Hamiltonian by adding the primary constraints multiplied by Lagrange multipliers, i.e. completely unspecified fields. Then one has to check the consistency conditions, i.e. the evolution of the constraints should vanish weakly. This means that after computing the Poisson brackets of the constraints with the Hamiltonian, the result vanishes as one sets the constraints to be zero. In the geometrical picture, this means that the conditions have to hold on the constraint surface (which is usually dubbed "on shell"). If these conditions are not met, they yield further (secondary) constraints, which then will also be added to the total Hamiltonian and will have to be checked with regard to the vanishing of their evolution. Once all constraints are met, one analyzes the Poisson brackets among them. This is the so-called hypersurface deformation algebra. All brackets having vanished weakly - the algebra closes on shell - is referred to as constraints forming a first class constraint system, as will be the case here. The constraint surface is then said to be a coisotropic submanifold which is defined by $\mathcal{T}_p\mathcal{S}^\perp \subset \mathcal{T}_p\mathcal{S}$. Thus, the induced symplectic form on $\mathcal{S}$ is degenerate[5]. Here, the Hamiltonian vector fields associated with the constraint functions act within the constraint surface, which is why they are regarded as generators of gauge transformations. For this reason these vector-fields sometimes are regarded as the characteristic null directions of the induced symplectic form, and $\mathcal{T}_p\mathcal{S}^\perp$ being referred to as its characteristic distribution completely spanned by the Hamiltonian vector-fields. The Lie brackets of the vector-fields also vanish weakly[6], i.e. they are surface forming according to Frobenius' theorem. Hence, $\mathcal{T}_p\mathcal{S}^\perp$ is called an integrable distribution or foliation then, and the maximal connected integral manifolds of the foliation are being referred to as leaves of K. If the factor space $\mathcal{S}/K$ is Hausdorff one can factor out gauges and end up with the reduced *symplectic* phase space Γ' [10]. Thus, each first class constraint decreases the number of degrees of freedom by 2 per spacetime point. The remainder has to be divided by 2 to give the physical configuration degrees of freedom.

Extending the Hamiltonian gives (see app. A.8)

$$\begin{aligned} H_T &= \int_\Sigma NC + N^aV_a + \lambda\Pi + \lambda^a\Pi_a. \\ C &= \omega_h^{-1}(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi_h^2) - \omega_h \ {}^{(3)}R \qquad (2.0.6) \\ V_a &= -2h_{ac}D_b\Pi^{cb} \qquad (2.0.7) \end{aligned}$$

(2.0.6) and (2.0.7) are exactly the initial value constraints derived in the previous section. Here, the consistency conditions show that C and V_a are secondary constraints

$$\begin{aligned} 0 \approx \dot{\Pi} &= -\frac{\delta H_T}{\delta N} = -C \\ 0 \approx \dot{\Pi}_a &= -\frac{\delta H_T}{\delta N^a} = -V_a \\ \Rightarrow \ H_T &= \int_\Sigma NC + N^aV_a + \lambda\Pi + \lambda^a\Pi_a + \gamma C + \gamma^a V_a. \end{aligned}$$

We define the constraint functionals

$$C[N] = \int_\Sigma CN, \quad V[N^a] = \int_\Sigma V_aN^a \qquad (2.0.8)$$

[5]Lorentzian null surfaces provide an analogous situation, since the induced metric is degenerate because the surface normal is null and therefore, lies tangential to the surface.

[6]They are involutive.

to calculate the so-called hypersurface deformation algebra (see app. A.11) and the new consistency conditions in one step

$$\begin{aligned}
\{V[\vec{N}], V[\vec{N}']\} &= V[\pounds_{\vec{N}} \vec{N}'] \approx 0 \\
\{V[\vec{N}], C[N]\} &= C[\pounds_{\vec{N}} N] \approx 0 \\
\{C[N], C[N']\} &= V[h^{-1}(NdN' - N'dN)] \approx 0,
\end{aligned}$$

showing that all constraints are first class. Hence, we arrive at a Hamiltonian constrained to vanish completely as it is supposed to in a general covariant canonical theory [1]. Calculating the equations of motions for N and N^a we obtain

$$\begin{aligned}
\dot{N} &= \frac{\delta H_T}{\delta \Pi} = \lambda \\
\dot{N}^a &= \frac{\delta H_T}{\delta \Pi_a} = \lambda^a
\end{aligned}$$

which tell us that lapse as well as shift are completely arbitrary, i.e. they are Lagrange multipliers reflecting the freedom in foliating spacetime and should not be regarded as dynamical variables. The (reduced) Hamilton functional reads

$$H_E = \int_\Sigma NC + N^a V_a. \tag{2.0.9}$$

The analysis of the action of the Hamiltonian on arbitrary fields $t_{ab} \in \mathcal{X}\Gamma$, carried out in detail in app. A.10

$$\begin{aligned}
\{V(N), t_{ab}\}_{\mathrm{EOM}} &= \pounds_{N^a} t_{ab} \\
\{C(N), t_{ab}\}_{\mathrm{EOM}} &= \pounds_{Nn^a} t_{ab}.
\end{aligned} \tag{2.0.10}$$

shows that it generates spacetime diffeomorphisms only on shell, i.e. if the Einstein vacuum equations hold $G_{ab} = 0$ and on the constraint surface[7]. V generates spatial diffeomorphisms (tangential to Σ_t), while C generates diffeomorphisms "perpendicular" to Σ_t. Inserting h_{ab} or Π^{ab} in 2.0.10 yields the evolution equations of the previous section. Due to the arbitrariness of the respective parameters N and N^a they have to be regarded as pure gauge - time translation is pure gauge. "Dynamics is not about time evolution, it is about relations between partial observables"[1]. Thus, general covariance is implemented in the canonical formalism of general relativity and its dynamics (the field equations) is covered by the evolution equations together with the constraints. We obtained 8 constraints and and since they are first class we have to fix 8 gauge parameters. Thus, the true dynamical

[7]Otherwise one has terms proportional to C and R_{ab} in 2.0.10 for $t^{ab} = \Pi^{ab}$[11].

degrees of freedom of pure gravity are

$$\frac{20-8-8}{2} = 2 \tag{2.0.11}$$

which is exactly the desired result for a graviton[8].

Finally we want to make some excurse into Hamilton-Jacobi theory of gravity for a moment, since it provides the starting point for a possible quantization[9]. Another reason is to shed some light on spatial diffeomorphism invariance of the Hamilton principal function. The Hamilton principal function, obtained by taking the extremal form of ADM-action and fixing the spatial metric on the initial slice and on the final slice, is always a (special) solution to the Hamilton-Jacobi equations. The principal function is therefore a functional of the elements of boundary states on ∂R. In the relativistic case the Hamilton Jacobi equations read

$$C(h_{ab}, \frac{\delta S_{Hpf}}{\delta h_{ab}}) = 0$$
$$S_{Hpf}(h''_{ab}) = \int_{h'_{ab}}^{h''_{ab}} \Pi^{ab}\dot{h}_{ab}$$

Now varying the final metric gives

$$\delta S_{Hpf} =: \int_{\Sigma''} \Pi^{ab}\delta h''_{ab}$$

Let ξ^a be a spatial vector-field then

$$\tilde{h}_{ab} = h_{ab} + \pounds_\xi h_{ab} = h_{ab} + 2D_{(a}\xi_{b)}.$$

With regard to the variation of S_{Hpf} follows

$$\begin{aligned}\delta S_{Hpf} &= \int_{\Sigma''} \frac{\delta S_{Hpf}}{\delta h''_{ab}}\delta h''_{ab} = \\ &= \int_{\Sigma''} \frac{\delta S_{Hpf}}{\delta h''_{ab}}(2D_{(a}\xi_{b)}) = -2\int_{\Sigma''} D_a\frac{\delta S_{Hpf}}{\delta h''_{ab}}\xi_b,\end{aligned}$$

since Σ is compact and h_{ab} is symmetric. It can be seen immediately that the first term in the integrand is just the diffeomorphism constraint which has to vanish. Thus, the phase function is independent of the specific spatial metric components and depends only on the

[8]This can be seen e.g. from the Weyl curvature in spinor theory, too.

[9]See mechanics textbooks for discussions about the relation of Hamilton-Jacobi and the eikonal equation of geometrical optics and its generalization to finite particle wavelength, which is nothing else but the Schrödinger equation. This is why the Hamilton principal function is sometimes referred to as the phase function or dynamical path length. It is the classical analogue of the quantum mechanicle wavefunction

3-geometry[10] of a compact spatial hypersurface. The totality of 3-geometries constitutes the superspace, i.e. the dynamical arena of general relativity. One leaf of history of 3-geometry in superspace is a classical 4-geometry.

The uncertainty principle completely spoils the notion of a classical spacetime (at least in the 3+1 picture): It is not possible to specify all the initial data on the initial slice with arbitrary precession for h_{ab} and Π^{ab} in their operator version do not commute. 3-geometry is the primary concept [2]. From Wheeler's point of view each 3-geometry will be assigned a probability amplitude and its phase is given by the Hamilton principal function. The Hamilton-Jacobi equations become the Wheeler-DeWitt equation

$$C\psi = 0$$

The considerations presented so far appear to be quite natural, and the development of a quantum theory of gravity seems to be straightforward. General solutions of the Wheeler-DeWitt equation, however, could not be found. Only with regard to symmetry reduced situations, so-called minisuperspace models, interesting results could be achieved. The major obstacle to a solution of the Wheeler-DeWitt equation cintinues to be the nonpolynomial form of the Hamiltonian constraint. The program of quantizing gravity via geometrodynamics got stuck in the '70s of the last century.

In the 1980s [11] Abhay Ashtekar gave new life to the canonical approach. He turned attention to connection dynamics instead of geometrodynamics. General relativity was cast in a gauge theoretic (Yang-Mills) form. Ashtekar used the so-called Sen connection as new configuration variable and found that the Hamiltonian could be written in polynomial form. With this, a rigorous solution to the dynamics seemed to be within the reach.

[10] 3-geometry is the equivalence class of 3-metrics modulo active diffeomorphisms.

[11] See [11] for a historical overview of LQG.

Chapter 3

Vierbein formalism

> "[...] the gravitational field can be viewed as the field that determines, at each point of spacetime, the preferred frames in which motion is inertial."[1]

To each point in spacetime one can assign a Lorentz frame[1], i.e. a tetrad e_I^a and its dual/inverse e_a^I, such that

$$g_{ab} = \eta_{IJ} e_a^I e_b^J. \tag{3.0.1}$$

$$e_I^a e_a^J = \delta_I^J \tag{3.0.2}$$

Here I, J are Lorentz algebra ($\mathcal{SO}(1,3)$ or $\mathcal{SL}(2,\mathbb{C})$ respectively) indices. By virtue of (3.0.1) the cotetrad e_a^I provides an isomorphism between the tangent space at each spacetime point and the internal space with the (kinematic) Minkowski metric η_{IJ}. The cotetrad contains the information to determine an inertial frame at any point in the manifold, which is the frame of a freely falling observer. Its matrix of components is the Jacobian matrix of the change of coordinates from arbitrary coordinates to inertial ones. Of course, due to Lorentz invariance (local equivalence principle) no inertial frame is preferred. Hence, the equivalence class of Lorentz frames with respect to (proper) Lorentz transformations describes the gravitational field. In return, each inertial frame at a given point p determines a system of Riemannian normal coordinates in a neighborhood of that point, sometimes also called inertial coodinate system[12] $\{x^I\}$. The doublet (p, e_I^a) serves as initial datum for geodesics through p, which are treated as coordinate lines. In such a coordinate system, the metric at p is flat, and if torsion vanishes, the Christoffel standard symbols vanish, too.

[1] See [1] page 59 ff. for a discussion of why one should regard the tetrad as the gravitational field rather than its metric form.

At this point we again encounter Mach's principle or, in Wheeler's terminology, "Inertia here is determined by energy and matter there" [2]. This argument is enforced by a gedanken-experiment invented by Einstein: the hole argument. It reveals that a generally covariant theory can only remain deterministic, if we remove any physical significance from the spacetime point. Only spacetime coincidences, i.e. intersections of particle worldlines, have a physical meaning. [2]

Cartan's theory of moving frames provides an efficient framework to calculate spin coefficients, i.e. the analogue of Christoffel symbols and curvature quantities. Let d denote the exterior derivative and $\lrcorner$ the interior product (antiderivative). The exterior derivative acting on forms is independent of a specific choice of a torsion free, i.e. symmetric covariant derivative. Its action is then extended to a linear operator ∇ on vector-fields v^a

$$(\nabla v^a)b := \nabla_b v^a \tag{3.0.3}$$

. For this purpose, we regard a vector-field as a vector-valued 0-form. Obviously due to the interplay of the several directional derivatives in tangent space $\nabla^2 v^a \neq 0$ in general. This expression captures the notion curvature. We may write

$$\nabla^2 v^a = R^a{}_b v^b$$

where $R^a{}_b$ is a tensor-valued 2-form. Any tensor quantity can now be decomposed in terms of the tetrad. Therefore, we expand the action of the exterior derivative on a tetrad field in the tetrad

$$\nabla e^a_I = \omega^J{}_I e^a_J$$

$\omega^J{}_I$ is a ($\mathcal{SO}(1,3)$-valued) matrix of 1-forms called the spin connection. From this relation, it follows that

$$\nabla^2 e^a_I = R^a{}_b e^b_I = R^J{}_I e^a_J \tag{3.0.4}$$

and calculated explicitly

$$\begin{aligned} \nabla(\omega^J{}_I e^a_J) &= (d\omega^J{}_I e^a_J + \omega^J{}_K \wedge \omega^K{}_I) e^a_J \\ R^J{}_I &= d\omega^J{}_I + \omega^J{}_K \wedge \omega^K{}_I. \end{aligned} \tag{3.0.5}$$

[2]See [1] for a discussion about the Global Positioning System as a technical manifestation of exactly these ideas.

Equation (3.0.5) is Cartan's second structure equation. In the notation presented here the Kronecker-δ can be regarded as a vector valued 1-form which is annihilated by the exterior derivative (provided the derivative is symmetric) then we have

$$\begin{aligned} \nabla\delta^a = \nabla(e^a_I e^I) = e^a_J \omega^J{}_I \wedge e^I + e^a_J de^J = 0 \\ de^J = -\omega^J{}_I \wedge e^I, \end{aligned} \tag{3.0.6}$$

where the last equation actually determines a so-called gauge covariant exterior derivative D compatible with the cotetrad. The torsion 2-form is defined via the first structure equation

$$T^I = De^I = de^I + \omega^J{}_I \wedge e^I.$$

Hence, here (3.0.6) is the condition for a torsion free spin connection, which has been implemented from the beginning.
Again, due to the above made choice of the metric compatible covariant derivative we obtain another important relationship

$$\begin{aligned} \nabla g^{ab} = 0 = \nabla(\eta^{IJ} e^a_I e^b_J) = \omega^{KJ} e^a_K e^b_J + \omega^{KI} e^a_I e^b_K \\ \omega^{IJ} = -\omega^{JI}, \end{aligned} \tag{3.0.7}$$

which tells us that $\omega^J{}_I$ belong to the Lie algebra $\mathcal{SO}(1,3)$ The vacuum Einstein field equations are then given by

$$G^I = R^I - \frac{1}{2} e^I R = 0.$$

where

$$R^I = e_J \lrcorner R^{IJ}$$

denotes the Ricci 1-form and

$$e_I \lrcorner R^I$$

denotes the curvature scalar.

3.1 The $3+1$ split for vierbeins

As a natural extension of the above procedure we perform the $3+1$ for tetrads[3]. For this it is used the same notation as for the metric formulation before.
To each point in space one can assign an $SO(3)$ frame[4], i.e. a triad e_i^a and its dual e_a^i

$$h_{ab} = \delta_{ij} e_a^i e_b^j.$$

Here i, j are $\mathcal{SO}(3)$ indices. While the metric is obviously invariant under proper rotations[5], the triad is not, i.e. the degrees of freedom are increased. It is convenient to choose the simply connected double cover $SU(2)$ instead of $SO(3)$, which does not alter the Lie algebra, since $\mathcal{SO}(3)$ and $\mathcal{SU}(2)$ are isomorphic. This choice allows us further to incorporate spinors[6] which will be relevant with regard to quantum theory.
Let us calculate dn_a

$$\begin{aligned}(dn)_{ba} &= 2\nabla_{[b}n_{a]} = 2\delta^c_{[b}\nabla_c n_{a]} = 2h_{[b}{}^c\nabla_c n_{a]} - n_{[b}n\nabla n_{a]} = \\ &= 2B_{[ba]} - n_{[b}n\nabla n_{a]} = (B_J \wedge e^J + (n\nabla)n \wedge n)_{ba},\end{aligned} \tag{3.1.1}$$

then perform the pull back to Σ for example denoted by $\downarrow e_a^I = h_a{}^b e_b^I$. Furthermore we define $K :=\downarrow B$, $e_a^0 := n_I e_a^I = -n_a$ and $\Gamma^i{}_j :=\downarrow \omega^i{}_j$

$$\downarrow (de^0) = -K_I \wedge \downarrow e^I = - \downarrow \omega^0{}_J \wedge \downarrow e^J,$$

since with $K_a^I n_I = 0$ and (3.0.7) $\omega^0{}_0 = 0$ all internal indices take effectively the values $1, 2, 3$

$$\downarrow (de^0) = -K_i \wedge e^i = - \downarrow \omega^0{}_j \wedge e^j.$$

Therefore, the pull back of the 4-curvature 2-form is

$$\downarrow {}^{(4)}R^I{}_J =\downarrow d\omega^I{}_J + \downarrow \omega^I{}_k \wedge \downarrow \omega^k{}_J + K^I \wedge K_J. \tag{3.1.2}$$

[3] For a rigorous derivation of the triad representation of general relativity via an action principle see [13] and [14].
[4] Remark: $SO(3)$ is the subgroup of $SO(1,3)$ leaving n_a invariant.
[5] Actually it is invariant under the action of $O(3)$.
[6] In the pure gravity context the internal gauge group of connection dynamics is $SO(3)$, but as one introduces (half integer) spinorial matter fields the group gets enlarged to $SU(2)$ (see chapter 7). Another advantage of using $SU(2)$ is that it is well suited for the spinor formulation of connection dynamics [13].

The (i,j)-th component is given by

$$R^i{}_j + K^i \wedge K_j,$$

where $R^i{}_j = d\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j$ denotes the intrinsic curvature 2-form. The vacuum initial value constraints are given by

$$G^0 = 0$$

A short calculation shows that the scalar constraint is given by $G^0_0 = -1/2 e_j \lrcorner (e_i \lrcorner \downarrow {}^{(4)}R^{ij}) = 0$ which in the $3+1$-split reads

$$-R - K^2 + K^i_j K^j_i = -R - e_j \lrcorner (e_i \lrcorner K^i \wedge K^j). \tag{3.1.3}$$

If (3.1.3) is multiplied by ω_h, we obtain the scalar constraint of the ADM formalism (compare equation (1.0.7)). The volume form is absorbed into the Lie algebra valued vector density

$$\tilde{E}_i := \omega_h e_i,$$

which will serve as the momentum variable canonically conjugated to the configuration variable K^i (This can be motivated from cotangent bundle constructions, since the canonically given one form on the contangent bundle is given by $\Theta = \int \tilde{E} \lrcorner \delta K$.). Using it allows to rewrite (3.1.3)

$$C = -\omega_h R - \omega_h^{-1} \tilde{E}_j \lrcorner (\tilde{E}_i \lrcorner (K^i \wedge K^j)).$$

The diffeomorphism constraint is obtained by calculating the pull back of G^0 to Σ

$$\begin{aligned} \downarrow G^0 &= \downarrow {}^{(4)}R^0 = \downarrow (e_J \lrcorner {}^{(4)}R^{0J})) = \downarrow (e_j \lrcorner {}^{(4)}R^{0j} = \\ &= e_i \lrcorner \downarrow (d\omega^{0i} + \omega^0{}_k \wedge \omega^{ki}) = e_i \lrcorner (dK^i + \Gamma^i{}_k \wedge K^k) = \\ &= e_i \lrcorner (DK^i), \end{aligned} \tag{3.1.4}$$

where D is the gauge covariant exterior derivative in Σ. It now has to be checked if the Poisson algebra of the triad formulation is equivalent to the one of the ADM formulation, i.e. if

$$\{\tilde{E}^a_i(x), K^j_b(y)\} = \frac{\kappa}{2} \delta^a_b \delta^j_i \delta(x,y) \tag{3.1.5}$$

is the only nonvanishing Poisson bracket. Π^{ab} and h_{cd} expressed in the new variables then would have the same Poisson bracket as in the ADM formulation. Obviously this cannot be the case, as there are 18 phase space degrees of freedom and the constraints leave 10. The root of the problem can be found in (3.1.1), where the symmetry condition has not been imposed on the K_{ab}. This condition is known as the Gauß constraint for reasons which will be explained later 8

$$G_{ik} = K_{a[i}\ \tilde{E}^a_{j]}.$$

Its smeared version where $\Lambda^i{}_j$ is an antisymmetric matrix

$$G[\Lambda] = \int_\Sigma \tilde{E}_i \lrcorner K^j \Lambda^i{}_j$$

generates exactly the before mentioned redundant rotations.

$$\begin{aligned} E[\lambda] &= \int_\Sigma \tilde{E}_i \lrcorner \lambda^i \\ \{E[\lambda], G[\Lambda]\} &= \frac{\kappa}{2} \int_\Sigma \Lambda^k{}_i \tilde{E}_k \lrcorner \lambda^i. \end{aligned}$$

So far, no drastic simplifications have been achieved. The previous procedure has just been an intermediate step towards one of the most important conceptual and, in the author's point of view, also phenomenological changes (see ch. 11.4) within canonical and non-perturbative gravity of the last decades.

Furthermore, note that the Poisson structure (3.1.5) is invariant with respect to the following rescaling:

$$\begin{aligned} \tilde{E}^a_i &\to \frac{\tilde{E}^a_i}{\gamma} \\ K^i_a &\to \gamma K^i_a. \end{aligned}$$

This invariance, which seems to be quite trivial, but has important consequences with regard to quantization will be discussed later in chapter 8.

All these notions presented so far will become more clear in the bundle formalism which is also essential to construction of invariant fiber bundles developed by [15] and presented in ch. 5. This should therefore be explained before we continue. For the section about symmetry reduction in LQG developed by Bojowald and Kastrup it actually is essential. This will allow a closer look on the one form e^i_a, the frame, the group structure[7] and the *connection* between points on our "stage" with respect to their respective eqivalence class of inertial frames.

[7]The indices i, I were called Lie algebra indices. As introduced here they were only used as names for distinct orthogonal directions.

Chapter 4

Fiber bundle theory

4.1 A short introduction to fiber bundle theory

A fiber bundle consists of [16]

- three topological spaces, called total space E, base space B and the standard fiber F
- a surjection $\pi : E \to B$, called the projection. F is homeomorphic to the fiber over x $F_x := \pi^{-1}(x)$, $\forall x \in B$ displayed in figure 4.1
- the structure group G of homeomorphisms of the fiber F acting effectively from the left.
- a local trivialization that is a cover $\{\mathcal{U}_\alpha\}$ of B together with homeomorphisms Φ_α $\forall\, \mathcal{U}_\alpha$

  $$\Phi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to \mathcal{U}_\alpha \times F$$

  such that

  $$\pi\Phi_\alpha^{-1}(x, f) = x, \quad x \in \mathcal{U}_\alpha,\ f \in F.$$

  This means that locally the total space is a Cartesian product and that we can coordinatize it by collections (x, f).

The doublet $(\pi^{-1}\mathcal{U}_\alpha, \Phi_\alpha)$ is a chart. To obtain an atlas we need transition functions, i.e. a kind of translation rule in the overlap region of two open sets:
Considering two charts $(\pi^{-1}\mathcal{U}_\alpha, \Phi_\alpha)$, $(\pi^{-1}\mathcal{U}_\beta, \Phi_\beta)$ with non-empty overlap, then the map

$$\Phi_\alpha \circ \Phi_\beta^{-1} : \mathcal{U}_\alpha \times F \to \mathcal{U}_\alpha, \Phi_\alpha \times F$$

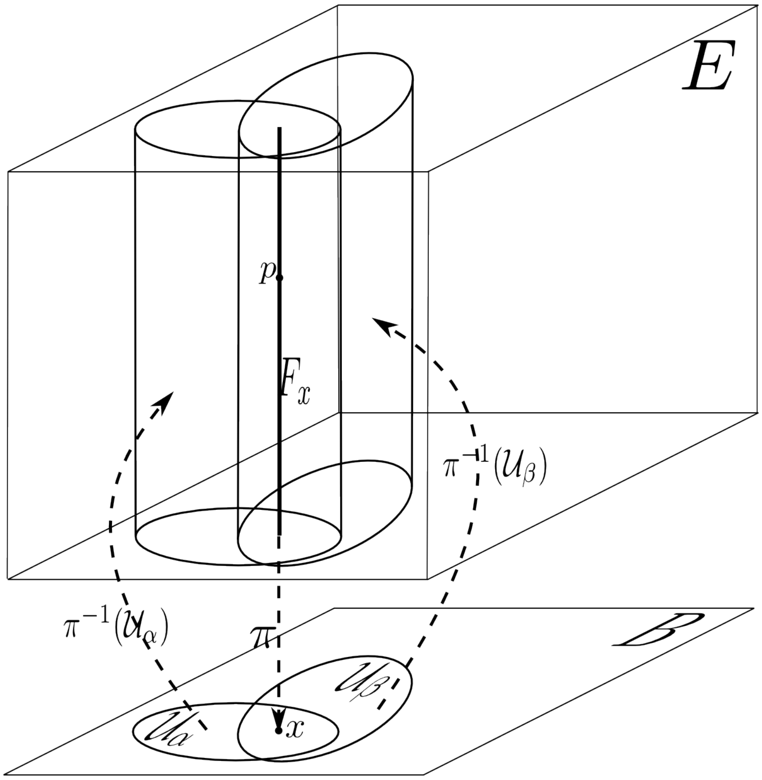


Figure 4.1: *The fiber bundle and the projection to the base space.*

is also a homeomorphism. Keeping a point x fixed, $\Phi_\alpha \circ \Phi_\beta^{-1}$ maps the standard fiber bijectively to itself via the structure group, and we call $\Phi_\alpha \circ \Phi_\beta^{-1}$ the transition function.

$$g_{\alpha\beta}(x) := \Phi_\alpha \circ \Phi_\beta^{-1}|_x : \{x\} \times F \to \{x\} \times F \tag{4.1.1}$$

Since $\{x\} \times F \sim F$ we have

$$g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$$

The transition functions satisfy the consistency conditions

$$\begin{aligned} g_{\alpha\alpha} &= 1 \\ g_{\alpha\beta}^{-1} &= g_{\beta\alpha} \\ g_{\alpha\beta} \circ g_{\beta\gamma} &= g_{\alpha\gamma}. \end{aligned}$$

Actually, the single cocycle condition $g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$, which has to be met by the transition functions, suffices to determine all of these three properties.
A fiber bundle can always be reconstructed given B, F, G and $g_{\alpha\beta}$. To construct another bundle, called the associated principal bundle $P(B, G)$, it is necessary to choose the fiber

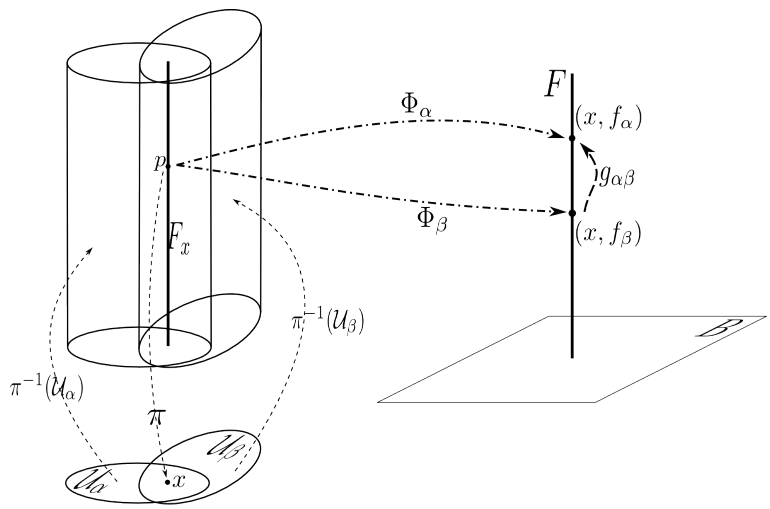


Figure 4.2: *On the trivialization of fiber bundles.*

to be the structure group and additionally define a free right action of the structure group on the resulting fiber bundle. Given the set

$$\tilde{E} = \bigcup_{\alpha} \mathcal{U}_\alpha \times G. \tag{4.1.2}$$

we can define the equivalence relation for two coordinates $(x, g) \in \mathcal{U}_\alpha \times G$ and $(x', g') \in \mathcal{U}_\beta \times G$

$$(x, g) \sim (x', g') \tag{4.1.3}$$

iff $x = x'$ and $g_{\beta\alpha}(x)g = g'$. Then the total space P is given by

$$P = \tilde{E}/\sim$$

which is the set of equivalence classes $[(x, g)]$. The bundle projection is just

$$\pi : P \to B, \ [(x, g)] \to x.$$

At last we describe the homeomorphism ϕ_α by its inverse

$$\Phi_\alpha^{-1}(x, g) = [(x, g)].$$

Let us now introduce the notion of a cross section of a bundle: It is a continuous map $s : B \to E$, which meets $(\pi \circ s)(x) = x$, $\forall x \in B$.

To each point $p \in P$ we assign a tangent space $\mathcal{T}_p P$. The lift of a curve is simply

$$\gamma : \mathbb{R} \to B \quad t \to \gamma(t)$$
$$\hat{\gamma} : \mathbb{R} \to P \quad t \to \hat{\gamma}(t) : \quad \pi \circ \hat{\gamma} = \gamma$$

Consider a flow

$$\Phi_t : B \to B.$$

A flow is generated by a vector-field on B. The flow and the generator can then be lifted to objects on P. This is called the complete lift of a field.

$$\tilde{()} : \mathcal{X}(B) \to \mathcal{X}(P)$$

We can also think of a flow which leaves the base point invariant. Therefore, its lift lies tangent to the fiber in that point.
Now we define the action of G on the manifold P from the right. Given the point $p \in P$, the right action

$$R_g : P \times G \to P, \quad p \mapsto pg \tag{4.1.4}$$

is such that if $\phi_\alpha(p) = (x, h_\alpha)$, then

$$\phi_\alpha(R_g p) = (x, h_\alpha g), \quad \forall \mathcal{U}_\alpha. \tag{4.1.5}$$

Hence, the canonical right action moves a point p on the principle fiber bundle only within the fiber over $\pi(p)$, i.e. vertically.
Consider a vertical flow induced by the action of a one parameter subgroup $g_t = exp(tA)$, where A is an element of the Lie algebra $\mathcal{G}$ isomorphic to $\mathcal{T}_e G$[1]. It induces a vector-field $\tilde{A} \in \mathcal{T}_p P$ called the fundamental vector-field. If G acts freely on P the map $f : \mathcal{G} \to \mathcal{T}_p P$ is a Lie algebra isomorphism and for nonvanishing A, $\tilde{A}$ is nonvanishing, too. This poses the question, of which Lie algebra element corresponds to the fundamental vector-field $R_{g*}\tilde{A}$ with $p = p' g^{-1}$.
$\tilde{A}$ is tangent to the curve $\hat{\gamma}(t)$ induced by R_{g_t} at $\hat{\gamma}(0) = p$. Then for $\hat{\gamma}_g(t)$ with $\hat{\gamma}_g(0) = p'$ we have

$$\hat{\gamma}_g(t) = \hat{\gamma}(t)g = pg_t g = p' g^{-1} g_t g$$

[1]Some detailed comments on Lie algebras and the exponential map can be found in ch. 6

and is therefore induced by $R_{ad_{g^{-1}gt}}$. Thus, $R_{g*}\tilde{A}$ corresponds with the Lie algebra element $Ad_{g^{-1}}A$, i.e. the adjoint representation of G. Such objects pushed forward to $\mathcal{T}B$ are projected to the zero vector. Due to the isomorphism $f : \mathcal{G} \to V_p \subset \mathcal{T}_pP$ V_p has the same dimension as the standard fiber G and is called the vertical subspace of $\mathcal{T}_pP$, which is canonically given on a principal fiber bundle. This leads to the question of what can we find out about the horizontal subspace.

4.2 Connection on a principle fiber bundle

The connection Γ gives a correspondence of points in any two fibers over a curve $\gamma(t)$ in B. As physicists we are, for instance, interested in parallely transporting the frame of reference, i.e. laboratories with three possibly orthonormal directions, the electrical reference potential etc., from one point to the other. The connection is an assignment of the horizontal subspace $H_p \subset \mathcal{T}_pP$ such that

- $\mathcal{T}_pP = V_p \oplus H_p$
- $H_{pg} = (R_g)_*H_p$ with $R_gp = pg$
- H_p depends differentially on p, i.e. given a differentiable vector-field $\tilde{u} \in \mathcal{T}_pP$, its horizontal $\tilde{u}_H$ as well as its vertical $\tilde{u}_V$ component is differentiable.

We define the horizontal lift of $\gamma(t)$ through x with tangent u as the curve $\hat{\gamma}(t)$ with tangent $\tilde{u}$ which is horizontal and projects to $\pi_*\tilde{u} = u$. Since the projection is an isomorphism of H_p onto $\mathcal{T}_{\pi(p)}B$ (same dimension) the lift is unique and H_p is invariant by G.
This notion can be captures via a $\mathcal{G}$-valued 1-form ω on P called the connection 1-form with the properties

- $\omega(\tilde{u}_V = \tilde{A}) = A$
- $\omega(\tilde{u}_H) = 0$

Its pull back, with respect to a group action g, is given by

$$
\begin{aligned}
&R_g^*\omega_p(\tilde{u}) = \omega_{pg}(R_{g*}\tilde{u}) \\
&R_{g*}\tilde{u} = \tilde{u}_H + R_{g*}\tilde{u}_V \\
&R_g^*\omega_p(\tilde{u}) = 0 + Ad_{g^{-1}}A = Ad_{g^{-1}}\omega_p(\tilde{u}).
\end{aligned}
$$

Next we describe the connection in local trivializations: Consider the local (canonical) section $\sigma(x) = \Phi^{-1} \circ Id(x)$, where $x \in \mathcal{U}$ and $Id(x) = (x, e) \in \mathcal{U} \times G$ where e the identity element of G. With the help of σ we can pull back ω_p to $\mathcal{U}_\alpha$ with $\pi(p) = x$, $\Phi(p) =$

$(x, g(x))$. For a vector-field $\tilde{v} \in \mathcal{T}(\pi^{-1}\mathcal{U})$ tangent to the curve $\hat{\gamma}(t)$ with $\pi(\hat{\gamma}(t)) = x(t)$ at p and $\pi_*\tilde{v} = v \in \mathcal{T}_x B$. The corresponding tangent vector in $\mathcal{T}_{(x,g)}\mathcal{U} \times G$ can be identified with an object $(v, w) \in \mathcal{T}_x\mathcal{U} \oplus \mathcal{T}_g G$, which is given by

$$v + g'(x)v.$$

Thus, on $\pi^{-1}\mathcal{U}$ using the right action on P

$$\tilde{v} = R_{g*}\sigma_*(v) + \Phi_*^{-1}g'(x)v$$

and acting with the connection on it yields

$$\omega(\tilde{v}) = Ad_{g^{-1}}\sigma * \omega(v) + \Phi^{-1*}\omega(g'(x)v).$$

Since $\Phi_*^{-1}g'(x)v$ is vertical $\Phi^{-1*}\omega(g'(x)v)$ gives the corresponding Lie algebra element. Therefore, we can write

$$\omega(\tilde{v}) = Ad_{g^{-1}}\mathcal{A}(v) + \Theta_{MC}(g'(x)v).$$

Θ_{MC} is the Maurer-Cartan form on G defined by the mapping $\Theta_{MC} : \mathcal{T}_g G \to \mathcal{T}_e G \simeq \mathcal{G}$[2] and determined $\Theta_{MC}(A) = A$ for $A \in \mathcal{G}$. $\mathcal{A} := \sigma^*\omega \in \Lambda^1 \otimes \mathcal{G}$ is called gauge potential. Accordingly one refers to the local trivialization as the local gauge. In our case $G \subset GL(n)$, one can use the explicit form of $\Theta_{MC} = g^{-1}dg$

$$\omega(\tilde{v}) = g^{-1}\mathcal{A}(v)g + g^{-1}dg(v).$$

In another local gauge $\hat{\mathcal{U}}$ the same bundle point p is given by $\hat{\Phi}(p) = (x, \hat{g})$. The transition function is then defined by $h\hat{g} = \Phi \circ \hat{\Phi}^{-1}\hat{g} = g$ and the action of the connection on $\tilde{v}$ is given by

$$\begin{aligned}
\omega(\tilde{v}) &= Ad_{\hat{g}^{-1}}\hat{\mathcal{A}}(v) + \Phi^{-1*}\omega(\hat{g}'(x)v) = \\
&= Ad_{(h\hat{g})^{-1}}\mathcal{A}(v) + \Theta_{MC}((h\hat{g})'(x)v) = \\
&= Ad_{\hat{g}^{-1}}Ad_{h^{-1}}\mathcal{A}(v) + \Theta_{MC}(\hat{g}'(x)v) + Ad_{g^{-1}}\Theta_{MC}(h'(x)v) \\
\Rightarrow \hat{\mathcal{A}} &= Ad_{h^{-1}}\mathcal{A}(v) + \Theta_{MC}(h'(x)v)
\end{aligned} \tag{4.2.1}$$

In some of the major the LQG references (e.g. [11]) the trivializations are defined as $\Phi_\alpha : \mathcal{U}_\alpha \times G \to \pi^{-1}(\mathcal{U})$. Thus, under local gauge transformations the gauge potential

[2]Sometimes one writes $\Theta_{MC}(\tilde{A}) = L_{g^{-1}*}\tilde{A}$

transforms according to

$$^{g}\mathcal{A} = Ad_g\mathcal{A} + gd(g^{-1}) = Ad_g\mathcal{A} - dgg^{-1}. \tag{4.2.2}$$

Before we proceed, we will discuss briefly a different point of view on gauge transformations. Changing trivializations can be regarded as a change of (fibre-) coordinates, hence as a passive transformation. The active gauge transformations are described via vertical automorphisms of the bundle. Roughly speaking generic automorphisms are diffeomorphisms on P mapping fibres into fibres and inducing diffeomorphisms of the base manifold. In addition automorphisms commute with the right action of the structure group on the principal bundle. Vertical automorphisms leave the base point invariant. For details the reader is referred to [17].

4.3 Pseudotensorial and tensorial forms

Let ρ be a representation of the structure group on a finite dimensional vector space V, where $\rho(a)$ is a linear transformation of V and we have $\rho(ab) = \rho(a)\rho(b)$. A pseudotensorial r-form φ on P of type (ρ, V) is a V-valued r-form that meets

$$R_a^*\varphi = \rho(a^{-1})\varphi.$$

For instance, let ρ be the adjoint representation of G in $\mathcal{G}$, then $R_a^*\varphi = Ad_{a^{-1}}\varphi$. Thus, the connection 1-form is a pseudotensorial 1-form of type $(Ad, \mathcal{G})$. If such a form additionally satisfies $\varphi(X_1, \cdots, X_r) = 0$, whenever one of the X_i's is vertical, then it is being referred to as a tensorial form of degree r on P of type (ρ, V). Having fixed a connection we can define a special tensorial form: the covariant derivative of a pseudotensorial form. Let h denote the projection of $\mathcal{T}_uP$ to the horizontal subspace H_u, then $D\varphi := (d\varphi)h$ is a tensorial form if φ is pseudotensorial, since the projection as well as the exterior derivative commute with the right action of G and hX_i vanishes if X_i is vertical. The tensorial form $\Omega := D\omega$, which is of type AdG is called curvature form of ω. It satisfies Cartan's second structure equation

$$d\omega(X, Y) = -[\omega(X), \omega(Y)] + \Omega(X, Y)$$

and the Bianchi identity $D\Omega = 0$. Moreover, for any tensorial 1-form ψ on P of type AdG we obtain

$$D\psi(X, Y) = d\psi + [\psi(X), \omega(Y)] + [\omega(X), \psi(Y)].$$

4.4 The frame bundle

The past chapters covered the general framework needed for the classical part of LQG or any kind of Yang-Mills theory. In the case of GR the bundles of interest are already determined by the (differential) basmanifold Σ. We can construct the tangent bundle (a special vector bundle), where the fiber F_x is the set $\mathcal{T}_x\Sigma$ of all tangent vectors of curves through x diffeomorphic to $\mathbb{R}^d$. Thus, locally the tangent bundle has the trivialization $\mathbb{R}^d \times \mathbb{R}^d$. The general linear group $GL(d, \mathbb{R})$ acts on $\mathbb{R}^d$ in a natural manner. It acts on $\mathbb{R}^d$ via linear operators represented on $\mathbb{R}^d$ as regular $d \times d$ matrices as follows way: for $\xi^i \in \mathbb{R}^d$ and for $a \in GL(d, \mathbb{R})$ with $\rho(a) = a^i{}_j$ we have $\tilde{\xi}^i = a^i{}_j \xi^j$. Hence, the transition functions of the tangent bundle are the Jacobians of the vector space transformations. Next we choose the set of all ordered bases (linear frames) of $\mathcal{T}_x\Sigma$ as a fiber of the so-called frame bundle $L(\Sigma)$. A basis we denote by $u = (X_1, \cdots, X_d) \in L(\Sigma)$. Introducing a local coordinate system in an open neighborhood $\mathcal{U}$ of $x \in \Sigma$ allows us to express the basis in the form $X_i = X^j{}_i \frac{\partial}{\partial x^j}$. To ensure completeness of u the $d \times d$ matrix $X^j{}_i$ has to be regular, which shows that the fiber of $L(\Sigma)$ is in $1:1$ correspondence with the group $GL(d, \mathbb{R})$. We regard u as a non-singular linear mapping $u : \mathbb{R}^d \to \mathcal{T}_x\Sigma$: a vector in ξ^i in $\mathbb{R}^d$ is mapped to $\xi = \xi^i X_i \in \mathcal{T}_x\Sigma$. With the natural basis of $\mathbb{R}^d$ [3] we define u by $ub_i = X_i$. Then a linear transformation of $\mathbb{R}^d$ by $a \in GL(d, \mathbb{R})$, $b_i \to a^j{}_i b_j$ is mapped to a transformation of $\mathcal{T}_x\Sigma$ by the composite map ua which is the desired (free) right action on $L(\Sigma)$. Thus, the frame bundle is an example of a principal $GL(d, \mathbb{R})$-bundle associated with $\mathcal{T}\Sigma$. Furthermore, Σ is a Riemannian manifold, i.e. equipped with a metric. Hence, each tangent space is an inner product space. The metric defines which frames are orthonormal. So the structure group can be reduced to $O(d, \mathbb{R})$. If Σ is orientable, i.e. it is equipped with a volume form, we can select the oriented orthonormal frames, and the structure group is reduced to $SO(d)$ represented by orthogonal matrices with unit determinant. The principal $SO(d, \mathbb{R})$-bundle is a subbundle of the principal $GL(d, \mathbb{R})$-bundle.

On $L(\Sigma)$ there exists a canonical $\mathbb{R}^n$-valued 1-form of type $(GL(n, \mathbb{R}), \mathbb{R}^n)$ called the soldering form given by

$$\theta(X) = u^{-1}(\pi(X)), X \in \mathcal{T}_u L(\Sigma)$$

i.e. it is a map from $\mathcal{T}L(\Sigma)$ to $\mathbb{R}^n$ and therefore solders the tangent bundle with the Euclidean vector bundle $P \times_G \mathbb{R}^n$ associated to the frame bundle P. It is clearly horizontal due to the bundle projection in the argument of u^{-1} and behaves under the right action

[3] $b_1 = (1, 0, \cdots, 0)$ etc.

of G as follows:

$$R_a^*\theta(X) = \theta(R_{a*}X) \tag{4.4.1}$$

$R_{a*}X$ is a tangent vector at ua and thus we have

$$(ua)^{-1}(\pi(R_{a*}X)) = a^{-1}u^{-1}(\pi(X)) = a^{-1}\theta(X) \tag{4.4.2}$$

which is the desired property of a tensorial form, where a is the matrix representation of $GL(n, \mathbb{R})$ on $\mathbb{R}^n$.
A connection on the bundle of linear frames is called linear connection. We use the linear connection to define the tensorial form Θ of type $(GL(n, \mathbb{R}), \mathbb{R}^n)$, called torsion, by $\Theta = D\theta$. It satisfies Cartan's first structure equation

$$d\theta = -(\omega(X)\theta(Y) - \omega(Y)\theta(X)) + \Theta(X, Y).$$

We can now decompose the structure equations with respect to the basis (e_i, E_i^j) of $(\mathbb{R}^n, \mathcal{GL}(n, \mathbb{R}))$ to obtain

$$\begin{aligned} d\theta^i &= -\omega^i{}_j \wedge \theta^j + \Theta^i \\ d\omega^i{}_j &= -\omega^i{}_k \wedge \omega^k{}_j + \Omega^i{}_j. \end{aligned}$$

Since we are dealing with a Riemannian manifold, one can show [12] that there exists a unique torsion free metric connection, i.e. a connection that defines parallel displacements which preserves the fiber and has vanishing torsion. Then the two structure equations reduce to the equations (3.0.6) and (3.0.5) in the discussion about the vielbein formalism.

Chapter 5

Invariant connections

According to the aim to construct symmetry reduced models, such as spherical symmetric (Reissner-Nordström) solutions, in this chapter we discuss invariant connections. At this point we will follow closely the discussions in [12], [18] and [15]. The application to LQG was done in [19]. Since this thesis has a particular interest in rotational symmetry, this example will be considered within every intermediate step of the general symmetry reduction formalism.

We consider a compact Lie group K acting on the principal fiber bundle $P(\Sigma, G)$ via automorphisms and via the projection every element of K induces a transformation on Σ in a natural manner[1]. The differentiable manifold Σ as well as P together with the action of K are called a differentiable K-manifolds. $u_0 \in P$ and $x_0 = \pi(u_0) \in \Sigma$ will serve as a reference points. The curve induced by the action of one parameter subgroups of K starting at $u_0 \in P$ will be denoted by u_t, while the projected curve on the base manifold will be denoted by x_t.

Let us first concentrate on the base manifold Σ. J_{x_0} is the so-called isotropy subgroup of K with respect to $x_0 = \pi(u_0)$, i.e. the elements of K leaving x_0 invariant.

$$J_{x_0} = \{k \in K \mid kx_0 = x_0\}$$

E.g. for $K = SU(2)$ we have $J_{x_0} = U(1)$, i.e. the rotations for which x_0 lies on the rotation axis of J_{x_0} but not in the symmetry center that is stabilized by all elements of K. The action of K decomposes the base manifold into orbits. An orbit with reference point x_0 is defined as

$$\mathcal{O}_{x_0} = Kx_0 = \{x \in \Sigma : \exists k \in K, kx_0 = x\}.$$

The so-called orbit space will be denoted by Σ/K. The group K itself is decomposed into orbits with respect to the isotropy subgroup J_{x_0}. The coset space K/J_{x_0} is a homogeneous

[1]Here the action of an element k on the manifold Σ will be denoted by the same letter.

space. If J_{x_0} is not a normal subgroup of K, i.e. an invariant subgroup with respect to the adjoint action of K, then $\mathcal{O}_k\mathcal{O}_{k'} = \mathcal{O}_{kk'}$ does not induce a group structure on the homogeneous space K/J_{x_0}, as is the case for $K = SU(2)$ and $J_{x_0} = U(1)$.
The isotropy groups of different points in the orbit are conjugate to each other, i.e. $y = gx_0 \in \mathcal{O}_{x_0}$ entails

$$\begin{aligned} J_y = J_{gx_0} &= \{k \in K \mid kgx_0 = gx_0\} \\ \to\ g^{-1}kg \in J_{x_0} &\quad \to\ J_y = gJ_{x_0}g^{-1} \end{aligned}$$

The conjugacy class $[J_{x_0}]$ is called isotropy type. Furthermore, we show that the map $q : K \to \Sigma, g \to gx$ with $x \in \Sigma$ and $g \in K$ is constant on the cosets kJ_x and that it induces an injective map $q_x : K/J_x \to \Sigma; [k] \mapsto kx$, where the image is $\mathcal{O}_x$. The last assertion is trivial. For the first one we have

$$\forall g = kj \in kJ_x,\ q(g) = gx = kjx = kx$$

Assume for $gJ_x, \tilde{g}J_x \in K/J_x; g \neq \tilde{g}$ that $q_x(gJ_x) = q_x(\tilde{g}J_x)$, then

$$\begin{aligned} gx &= \tilde{g}x \\ x = g^{-1}\tilde{g}x,\ \to g^{-1}\tilde{g} \in J_x &\ \text{ contradiction.} \end{aligned} \tag{5.0.1}$$

Thus one can construct an x-dependent bijection between the orbit and K/J_x.
Let $J := J_{x_0}$, then the set of points whose isotropy type is $[J]$, which is called the orbit bundle $\Sigma_{[J]}$

$$\Sigma_{[J]} = \{x \in \Sigma \mid [J_x] = [J]\},$$

is a submanifold of Σ. Furthermore, if the orbit space Σ/K is connected, then one can show [20] that there exists a unique isotropy type $[J]$, such that $\Sigma_{[J]}$ is open and dense in Σ and that the space $B := \Sigma_{[J]}/K$ is connected. The space K/J is called principal orbit type of Σ and the submanifold $\Sigma_{[J]}$ is the so-called principal orbit bundle. It is a fiber bundle with base B standard fiber K/J and N_J/J is the structure group, where N_J is the normalizer of J in K. Thus, locally we have $\Sigma_{[J]} \simeq B \times K/J$. Orbits with dimension less than the dimension of the principal orbit are called singular orbits. For example, in case of spherical symmetry, the orbit of the symmetry center is a single point. Orbits which are not of the principal orbit type but have same dimension are called exceptional orbits. Elsewhere, we require that there is only one isotropy type isomorphic to $U(1)$. Hence, the principal orbit type is $SU(2)/U(1)$, which in turn can be shown to be isomorphic to the 2-sphere S^2 [21]. For our purposes here $\Sigma_{[U(1)]}/SU(2)$ we will take to be $\mathbb{R}$ or $\mathbb{R}^+$.
Let us now return to the symmetric principal G-bundle. We restrict P to the bundle $P\mid_B$

over B. For $j \in J$, ju_0 lies in the same fiber as u_0, since $B \simeq B \times \{eJ\}$ is a fixed point set of J. Now we define $\lambda(j) = a$, $a \in G$ by $ju_0 = u_0 a$ in order to construct a homomorphism $\lambda : J \to G$. Let j, $j' \in J$ then

$$u_0\lambda(jj') = (jj')u_0 = j(u_0\lambda(j')) = (ju_0)\lambda(j') = (u_0\lambda(j))\lambda(j') = u_0\lambda(j)\lambda(j')$$

using commutativity of $K \subset Aut(P)$ with the right action of G. Thus, λ preserves the multiplicative structure. It is important to note that the homomorphism is only defined with respect to a certain reference point and therefore will be labeled by the reference point λ_{u_0}. The homomorphisms with respect to different reference points in the same fibre $u_0' = u_0 a$, are related via conjugation

$$\begin{aligned} ju_0' = u_0'\lambda_{u_0'}(j) = u_0 a \lambda_{u_0'} = ju_0 a &= u_0\lambda_{u_0}(j)a \\ \lambda_{u_0 a} &= Ad_{a^{-1}}\lambda_{u_0} \end{aligned}$$

Thus, for points in the same fiber, all corresponding homomorphisms belong to the same conjugacy class $[\lambda]$. Points p, whose corresponding homomorphisms are the same representative, have to be related by elements of the centralizer Z_λ of $\lambda(J) \subset G$

$$Z_\lambda = \{g \in G \mid \lambda(J)g = g\lambda(J)\}.$$

Since G acts transitively in the fiber, each point $u \in G_x$, $x \in B$, can be characterized by $u = pa$, where $\phi_\alpha(p) = z =: \phi_\alpha^Q(p) \in Z_\lambda$. Thus, the transition functions of $P\mid_B$ only take values in Z_λ.

$$\begin{aligned} \phi_\beta \circ \phi_\alpha^{-1} &= \phi_{Q\ \beta} a \circ a^{-1} \phi_{Q\ \alpha}^{-1} = \\ &= \phi_{Q\ \beta} \circ \phi_{Q\ \alpha}^{-1} \end{aligned}$$

Due to proposition 5.3 of chapter I in [12] we can construct the reduced bundle $Q(B, Z_\lambda) = \{p \in P\mid_B : \lambda_p = \lambda\}$ [2]. Therefore, a K-symmetric fiber bundle $P(\Sigma, G)$ can be classified by the conjugacy class $[\lambda]$ and the reduced bundle $Q(B, Z_\lambda)$. Conversely P can be recovered from the classifying pair $(Q, [\lambda])$ as will be done below[15]. The K-invariant connection on P induces a connection $\tilde{\omega}$ on Q.
We define a linear mapping Λ between the Lie algebras $\Lambda : \mathcal{K} \to \mathcal{G}$ by

$$\Lambda_p(X) = \omega_p(\tilde{X}),$$

[2] This is understood in the same sense as the $SO(3)$ bundle is the reduced bundle of the $GL(3)$ bundle, if there exists a fiber metric.

where $\tilde{X} \in \mathcal{T}_p P$ is the vector-field induced by $X \in \mathcal{K}$ and ω is a K-symmetric connection 1-form on P. Clearly for $X \in \mathcal{J}$ the induced vector-field is vertical, i.e.

$$\Lambda_p(X) = d\lambda(X), \quad \forall X \in \mathcal{J} \tag{5.0.2}$$

is a Lie algebra homomorphism. If $X \in \mathcal{K}$ the adjoint representation of J in $\mathcal{K}$ is mapped to

$$\Lambda_p(Ad_j X) = Ad_{\lambda_p(j)}\Lambda_p(X) \forall X \in \mathcal{K} \tag{5.0.3}$$

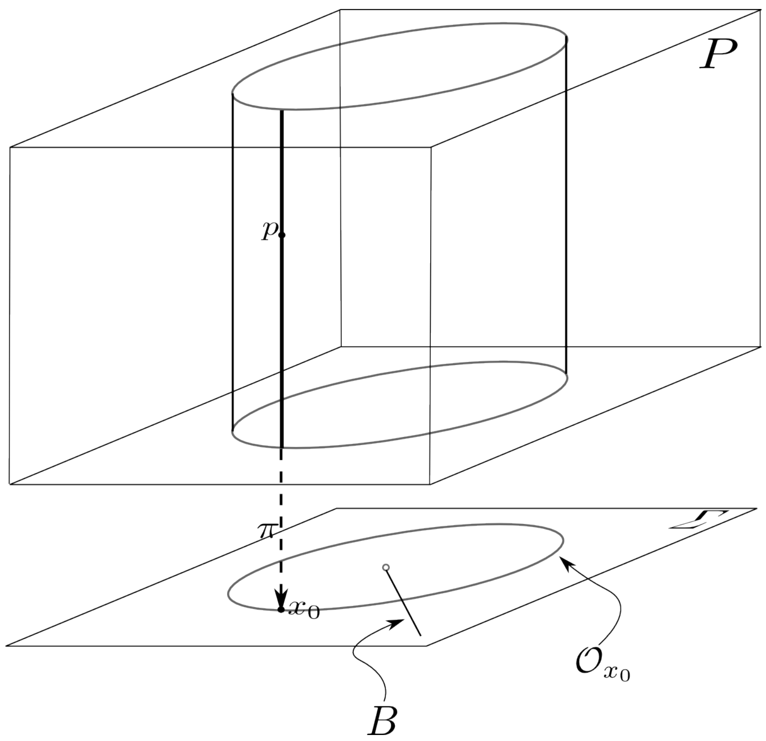
In the following, we assume that the Lie algebra $\mathcal{K}$ can be decomposed into the vector space direct sum $\mathcal{K} = \mathcal{J} \oplus \mathcal{J}^\perp$, such that $ad_J J^\perp \subset J^\perp$. K is then said to be reductive. If we keep λ constant on B[3] the remaining components $\Lambda_p \mid_{\mathcal{J}^\perp}$ together with $\tilde{\omega}$ contain all the information about the invariant connection on ω. More precisely, the invariant connections on P are determined by a scalar field Λ on Q meeting the equations (5.0.2) and (5.0.3) together with a connection $\tilde{\omega}$ in Q. This is the so-called generalized Wang Theorem. It is generalized allowing the action of K on Σ to be intransitive. Otherwise, the set of invariant connections in P is in $1:1$-correspondence with the set of linear maps $\Lambda \mid_{\mathcal{J}^\perp}$ [12]. $\Lambda_p \mid_{\mathcal{J}^\perp} =: \tilde{\Phi}$ we will call the Higgs field.
In order to prove this theorem we will follow the hints given in [15], orienting ourselves by the detailed and transparent calculations in [18] and using the same notation as [12][4]. The first step consists in showing that we can reconstruct the original fiber bundle from the classifying pair $(Q, [\lambda])$. We define the principal G'-bundle by

$$P'(\Sigma, G') = Q(B, Z) \times K(K/J, J) \tag{5.0.4}$$

with structure group $G' = Z \times J$ and base $\Sigma = B \times K/J$. We can extend the homomorphism λ to a homomorphism $\rho : G' \to G$ given by $\rho(g') = z\lambda(j)$ with $z \in Z$. We then construct the trivial principal G-bundle $P' \times G$. Since the second factor in (5.0.4) is K-symmetric so is P'. Using the extended homomorphism, the left action of G' on G is given by $\tilde{\rho} : G' \times G \to G, (g', g) \to \rho(g')g$. The right action of G' on $P' \times G$ is defined by $(p', g) \to (p'g', \rho(g')^{-1}g)$. For points on $P' \times G$ (p', g) we define the equivalence relation $(p', g) \sim (p'g', \rho(g')^{-1}g)$ and obtain the set of the so defined equivalence classes denoted by $P' \times_{G'} G$, which is also a K-invariant principal G-bundle associated with P'. The projection $\Psi : P' \times G \to P' \times_{G'} G$ is K-equivariant. The bundle $P' \times_{G'}$ G is equivalent with the original K-invariant G-bundle via an K- and G-equivariant bundle isomorphism.

[3]See [18] for a discussion on those properties B has to fulfill in order to keep λ constant.

[4]A geometric interpretation of this scheme, in particular an interpretation of the Higgs, field can be found in [22].



We use Ψ to prove the assertion given above. For a K-invariant connection 1-form ω on $P' \times_{G'} G$, $\Psi^*\omega$ is a K-invariant connection on $P' \times G$. This is determined by a K-invariant $\mathcal{G}$-valued 1-form on the base $P' = Q \times K$, which can be proven analog to the proof of existence and uniqueness for connections in e.g. [12]. A $\mathcal{G}$-valued 1-form given in a patch $\mathcal{U}_\alpha$ suffices to determine the connection. Using triviality of $P' \times G$ $\Psi^*\omega_\alpha$ can be extended to $\Psi^*\omega$. A K-invariant 1-form on Q is just the connection $\tilde{\omega}$. A K-invariant 1-form on K is the canonical $\mathcal{K}$-valued 1-form (the Maurer Cartan form) on K. We use the linear map Λ to map this form to a $\mathcal{G}$-valued form.
Thus, we can decompose the invariant connection 1-form in the following way

$$\omega = \tilde{\omega} + \Lambda \circ \iota^*_{K/J} \Theta^K_{MC}$$

where σ_Q is a section in Q and Θ^K_{MC} is the Maurer-Cartan form on K.

Chapter 6

Rotations and the Peter-Weyl theorem

In order to obtain a spherically symmetric gauge potential we will concentrate in the following on the group $K = SU(2)$ and its relation to rotations. For the Ashtekar variables this is the gauge as well as the symmetry group.

Rotations are operations on an Euclidean vector space preserving angles and the Euclidean norm. General rotations can be parametrized by three numbers (ϕ, θ, ψ) called Euler angles, which give a chart of $SO(3)$, with the range $0 < \theta < \pi$, $0 < \phi < 2\pi$ and $0 < \psi < 2\pi$. In order to rotate a vector we write in accordance with [17]

$$V' = R(\phi, \theta, \psi)V,$$
$$R(\phi, \theta, \psi) = r_3 \circ r_2 \circ r_1.$$

r_1 describes the rotation about the z-axis by the angle ϕ, r_2 rotates about the x'-axis (the r_1 rotated x-axis) by θ and r_3 is the rotation about the z'-axis by ψ, the situation is visualized in figure 6.1. These fundamental rotations are important in classical mechanics, describing the angular momentum in a principal axis coordinate system and are known as precession, nutation and proper rotation. Two parameter sets are called equivalent if $R(\phi, \theta, \psi) = R(\bar{\phi}, \bar{\theta}, \bar{\psi})$. The operator group of rotations acting on $\mathbb{R}^3$ is then the quotient with respect to this equivalence relation $\mathbb{R}^3/\sim$. In order to represent $R(\phi, \theta, \psi)$ explicitly as a matrix the well known fundamental rotation matrices for the rotation about the (fixed) z- and x-axis, $\tilde{r}_z$ and $\tilde{r}_x$ respectively have to be introduced. Then we have $r_1 = \tilde{r}_z(\phi)$, $r_2 = \tilde{r}_z(\phi)\tilde{r}_x(\theta)\tilde{r}_z(\phi)^{-1}$ and $r_3 = \tilde{r}_z(\phi)\tilde{r}_x(\theta)\tilde{r}_z(\psi)\tilde{r}_x(\theta)^{-1}\tilde{r}_z(\phi)^{-1}$, hence, $\tilde{R}(\phi, \theta, \psi) = \tilde{r}_z(\phi)\tilde{r}_x(\theta)\tilde{r}_z(\psi)$. The set of matrices $\{\tilde{R}(\phi, \theta, \psi)\}$ form the group $SO(3)$.

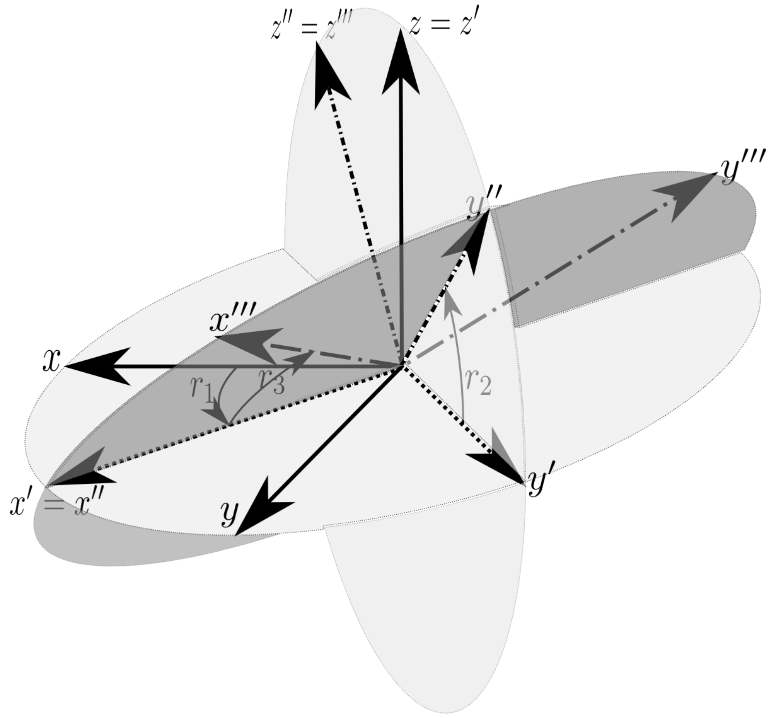


Figure 6.1: *The action of the rotation group on an orthonormal basis.*

6.1 The group of unimodular unitary 2x2 matrices

Now consider the group of unitary matrices with unit determinant, $SU(2)$. A general element $g \in SU(2)$ is of the form

$$g = \begin{pmatrix} a - ib & -c - id \\ c + id & a + ib \end{pmatrix}$$

with the condition

$$a^2 + b^2 + c^2 + d^2 = 1.$$

Thus, we establish a mapping $f : SU(2) \to S^3 \subset \mathbb{R}^4$.
The Lie algebra $\mathcal{G}$ of a Lie group G is the Lie algbra of left invariant vector-fields on G. We denote a tangent vector in $\mathcal{T}_g G$ by v_g then given a left translation L_h together with its differential $L'_h : \mathcal{T}_g G \to \mathcal{T}_{hg} G$ a left invariant vector-field v satisfies

$$L'_h v_g = v_{hg}. \tag{6.1.1}$$

In particular we have $L'_h v_e = v_h$. Therefore, we have established a bijective correspondence between the left invariant vector-fields and the vectors of tangents to G at e [17].
The exponential map maps the line tv_e, $t \in \mathbb{R}$ onto the one parameter subgroup $g_{v_e}(t)$,

i.e. a curve tangent to v_e and is defined by

$$\exp : \mathcal{T}_e G \to G, \quad v_e \mapsto \exp(v_e) = g_{v_e}(1) \tag{6.1.2}$$

The name "exponential mapping" results from the property $g_{v_e}(t)g_{v_e}(s) = g_{v_e}(s+t)$ (composition of curves).
The integral curve $\gamma(t)$ of vector-fields w is given by the unique solution to the ordinary differential equation

$$\frac{d}{dt}\gamma(t) = w\gamma(t) \tag{6.1.3}$$

more generally for a function $f : G \to \mathbb{K}$

$$\frac{d}{dt}f(\gamma(t)) = wf(\gamma(t)). \tag{6.1.4}$$

Accordingly, the vector-field w is the generator of the one parameter group $\gamma(t)$ uniquely defined by the equation

$$w(\gamma(0)) = \frac{d}{dt}\gamma(t)\Big|_{t=0} \tag{6.1.5}$$

Now we can show that a left invariant vector-fields v are generators of right translations $R_{\gamma(t)}$, since

$$\frac{d}{dt}R_{\gamma(t)}h\Big|_{t=0} = \frac{d}{dt}L_h\gamma(t)\Big|_{t=0} = L'_h v_e \tag{6.1.6}$$

Hence, we can define a left invariant vector-field on G from now on denoted by L^v via the differential equation

$$L^v f(g) = \frac{d}{dt}f(g\exp(tv))\big|_{t=0}. \tag{6.1.7}$$

The very same can be done with right invariant vector-fields

$$R^v f(g) = \frac{d}{dt}f(\exp(tv)g)\big|_{t=0}. \tag{6.1.8}$$

If G is a subgroup of $GL(n)$ the exponential map is explicitly given by the series expansion

$$\exp(v) = \sum_{n=0}^{\infty}\frac{1}{n!}v^n \tag{6.1.9}$$

Using the exponential map and the identity $\det(\exp(A)) = \exp(\operatorname{tr} A)$ we can analyze the Lie algebra $\mathcal{SU}(2)$ of $SU(2)$. The matrix A has to be antihermitean, $A^\dagger = -A$, traceless and hold the form

$$A = \begin{pmatrix} -\frac{i}{2}z & \frac{1}{2}y - \frac{i}{2}x \\ -\frac{1}{2}y - \frac{i}{2}x & \frac{i}{2}z \end{pmatrix}. \tag{6.1.10}$$

$\mathcal{SU}(2)$ is a 3 dimensional vectorspace. We cast A in a form that simplifies identifying the basis. A expanded in this basis is

$$A = x\tau_1 + y\tau_2 + z\tau_3 = V^i\tau_i, \; V \in \mathbb{R}^3.$$

That this is really a basis of $\mathcal{SU}(2)$ can be shown by calculating the commutator among the basis elements. Indeed,

$$\begin{aligned} [\tau_i, \tau_j] &= \varepsilon_{ijk}\tau_k \\ \tau_i^2 &= -\frac{1}{4}\mathbb{1} \\ \{\tau_i, \tau_j\} &= -\frac{1}{2}\delta_{ij}\mathbb{1} \\ \tau_i\tau_j\tau_k &= -\frac{1}{4}(\frac{1}{2}\varepsilon_{ijk}\mathbb{1} + \delta_{ij}\tau_k - \delta_{ij}^{km}\tau_m) \end{aligned}$$

with the structure constants $c_{ij}^k = \varepsilon_{ijk}$. The last three identities are needed for following calculations. Exponentiation yields a general element of $SU(2)$ (in a neighborhood of the identity). As already mentioned, in case of a matrix group one can Taylor-expand the exponential function, to find ($n^in_i = 1, \; \omega \in \mathbb{R}$)

$$\begin{aligned} &(n^i\tau_i)^2 = -\frac{1}{4}\mathbb{1} \quad \Rightarrow \quad (n^i\tau_i)^{2n} = \frac{(-1)^n}{2^{2n}}\mathbb{1}, \\ &(n^i\tau_i)^{2n+1} = 2\frac{(-1)^n}{2^{2n+1}}n^i\tau_i \\ &g = \exp(\omega n^i\tau_i) = \sum_{n=0}^{\infty}\frac{1}{n!}(\omega n^i\tau_i)^n = \\ &= \sum_{n=0}^{\infty}[\frac{(-1)^n}{(2n)!}(\frac{\omega}{2})^{2n}\mathbb{1} + 2\frac{(-1)^n}{(2n+1)!}(\frac{\omega}{2})^{2n+1}n^i\tau_i] = \\ &= \mathbb{1}\cos(\frac{1}{2}\omega) + 2n^i\tau_i\sin(\frac{1}{2}\omega). \end{aligned}$$

Consider (6.1.10) as a map $h : \mathbb{R}^3 \to \mathcal{SU}(2), V \mapsto A$, with $\det(A) =\| V \|$. Then one can show that there is a 2:1 correspondence $F : SU(2) \to SO(3)$ between rotations in $\mathbb{R}^3$ and the adjoint representation of $SU(2)$ in $\mathcal{SU}(2)$, i.e. between $V' = R(g)V$ and $h(V') = Mh(V)M^{-1}$. It can be easily checked that $h(V')$ is traceless and antihermitean and, furthermore that the determinant is invariant under that operation, which corresponds with the invariance of the Euclidean norm. Thus, any M determines a rotation. Obviously,

M and $-M$ determine the same $R(g)$. What remains to be shown is that $F^{-1}(Id) = \{-1, 1\}$ [17]: In order to meet the condition $h(V') = h(V)$, M has to commute with all $\mathcal{SU}(2)$-matrices which are traceless and antihermitian. M clearly commutes with all imaginary multiples of 1. Hence M commutes with all imaginary matrices, hence also with all real matrices, hence with all complex matrices and is therefore ± 1. This homomorphism restricted to a neighborhood of any point on the manifold of $SU(2)$ is a homeomorphism, i.e. a bicontinuous bijection. This determines $SU(2)$ as a double cover of $SO(3)$. Since the group manifold of $SU(2)$ is isomorphic to the 3-sphere which is simply connected[1], $SU(2)$ is the universal cover of $SO(3)$.

In order to present the concrete relation of rotations with the adjoint representation of $SU(2)$ explicitly, we show that if we calculate $h(V') = Mh(V)M^{-1}$, we can identify τ_1, τ_2 and τ_3 as the generators of rotations about the x-, y- and z-axis respectively.

$$h(V') = V^i \tau_i \cos\omega + \sin\omega n^i V^j \varepsilon_{ijk} \tau_k + 2\sin^2\frac{\omega}{2} V^i n_i n^j \tau_j \tag{6.1.11}$$

For $\omega = \phi$, $V = (x, y, z)$, $n^i = e_z^i$ and $V' = (x', y', z')$ this yields $x \to \cos\phi x - \sin\phi y$, $y \to \sin\phi x + \cos\phi y$ and $z \to z$. Hence, we can use the Euler angles to parametrize elements in $SU(2)$. For a general element $g \in SU(2)$ we write, according to the order of fundamental rotations,

$$\begin{aligned} g &= \exp(\phi\tau_3)\exp(\theta\tau_1)\exp(\psi\tau_3) = \\ &= \cos\frac{\theta}{2}\cos\frac{1}{2}(\psi+\phi)\mathbb{1} + 2\sin\frac{\theta}{2}\cos\frac{1}{2}(\psi-\phi)\tau_1 - \\ &\quad - 2\sin\frac{\theta}{2}\sin\frac{1}{2}(\psi-\phi)\tau_2 + 2\cos\frac{\theta}{2}\sin\frac{1}{2}(\psi+\phi)\tau_3 \end{aligned} \tag{6.1.12}$$

We can now compare this expression with the one above for $SU(2)$ elements. This allows us to use the Euler chart of the S^3 imbedded in $\mathbb{R}^4$ with coordinates

$$\begin{aligned} a &= \cos\frac{\theta}{2}\cos\frac{1}{2}(\psi+\phi) =: x_0 \\ b &= \cos\frac{\theta}{2}\sin\frac{1}{2}(\psi+\phi) =: x_3 \\ c &= -\sin\frac{\theta}{2}\sin\frac{1}{2}(\psi-\phi) =: x_2 \\ d &= \sin\frac{\theta}{2}\cos\frac{1}{2}(\psi-\phi) =: x_1 \end{aligned}$$

The inverse g^{-1} is given by $(\psi, \theta, \phi) \to (-\phi, -\theta, -\psi)$. Fixing the value of ψ amounts to restricting oneself to the 2-sphere, i.e. a point on the free z'-axis has the isotropy group $J = U(1)$ and the ϕ and θ rotations generate the S^2.

We use this coordinates to caclulate the adjoint representation of a general $SU(2)$ element

[1]Actually any n-sphere, $n > 1$, is simply connected

given by (6.1.12).

$$\begin{aligned}
&g(\phi,\theta,\psi)\tau_j g^{-1}(\phi,\theta,\psi) = \\
&= x_0^2\tau_j + 2x_0x_i\varepsilon_{ijk}\tau_k + 2x_jx_k\tau_k - x_kx_k\tau_j = \\
&= (x_0^2 - \sum_{k=1}^{3} x_kx_k + 2x_jx_j)\tau_j + \sum_{k\neq j}(2x_0x_i\varepsilon_{ijk} + 2x_jx_k)\tau_k \\
&= \begin{cases}
(\cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta)\tau_1 + (\sin\psi\cos\phi\cos\theta + \cos\psi\sin\phi)\tau_2 + \\
\quad +(\sin\theta\sin\psi)\tau_3 \\
(\cos\psi\cos\phi\cos\theta - \sin\psi\sin\phi)\tau_2 + (-\sin\psi\cos\phi - \cos\psi\sin\phi\cos\theta)\tau_1 + \\
\quad +(\sin\theta\cos\psi)\tau_3 \\
(\cos\theta)\tau_3 + (\sin\theta\sin\phi)\tau_1 + (-\sin\theta\cos\phi)\tau_2
\end{cases}
\end{aligned} \tag{6.1.13}$$

This expression allows for reading off the general $SO(3)$ matrix

$$\begin{pmatrix}
\cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta & -\sin\psi\cos\phi - \cos\psi\sin\phi\cos\theta & \sin\theta\sin\phi \\
\sin\psi\cos\phi\cos\theta + \cos\psi\sin\phi & \cos\psi\cos\phi\cos\theta - \sin\psi\sin\phi & -\sin\theta\cos\phi \\
\sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta
\end{pmatrix}. \tag{6.1.14}$$

Finally we also establish the isomorphism $\mathcal{F} : \mathcal{SU}(2) \to \mathcal{SO}(3)$ which is the differential of the adjoint representation of $SU(2)$, i.e. it is the adjoint representation of $\mathcal{SU}(2)$. $\mathcal{SO}(3)$ is the algebra of traceless antisymmetric real 3×3 matrices. From the commutation relations of $\mathcal{SU}(2)$ we have $F : \tau_i \mapsto c_{ij}^k\tau_k$. Thus, we obtain a matrix representation of the $\mathcal{SU}(2)$ generators $\tau_i \mapsto c_{ij}^k = \varepsilon^k{}_{ij} =: (\gamma_i)^k{}_j$. We defined the (k,j)-th component of the $\mathcal{SO}(3)$-generator γ_i. Due to the properties of the permutation symbol ε_{kij} it follows immideately that the γ's also satisfy the commutation relations $[\gamma_i, \gamma_j] = \varepsilon^k{}_{ij}\gamma_k$.

6.2 The Maurer Cartan form on $SU(2)$

For the sake of completeness, we first caclulate the full Maurer-Cartan form on $SU(2)$, $\Theta_{MC}^{SU(2)} = g^{-1}dg$, using the parametrization introduced above. A tedious but straight forward calculation[2] gives

$$\begin{aligned}
\Theta_{MC}^{SU(2)} &= (\tau_1\sin\theta\sin\psi + \tau_2\sin\theta\cos\psi + \tau_3\cos\theta)d\phi + \\
&+ (\tau_1\cos\psi - \tau_2\sin\psi)d\theta + \tau_3 d\psi.
\end{aligned}$$

[2]At this point it is convenient to choose different coordinates for the intermediate steps of the calculation: $u := \psi + \phi$, $v = \psi - \phi$.

Based on this expression we calculate the Maurer-Cartan form restricted to $S^2 \simeq SU(2)/U(1)$, $\iota^*_{S^2} g^{-1}dg$, which is the pull back to a $\psi = const.$-surface.

$$\iota^*_{S^2}\Theta^{SU(2)}_{MC} = (\tau_1 \sin\theta \sin\psi + \tau_2 \sin\theta \cos\psi + \tau_3 \cos\theta)d\phi + \\ + (\tau_1 \cos\psi - \tau_2 \sin\psi)d\theta.$$

Then we choose $\psi = 0$ which yields

$$(\tau_2 \sin\theta + \tau_3 \cos\theta)d\phi + \tau_1 d\theta. \tag{6.2.1}$$

6.3 Haar measure, representations and Peter Weyl theorem

In order to provide the necessary tools for the loop quantization of GR this thesis strives for demands an excurse to representation theory. Therefore we will state some very important theorems and properties of representations of compact finite dimensional Lie groups, such as $SU(2)$. We follow closely the tremendously useful introductory chapter on mathematical physics in [11] and also [17]. We will use the following theorem:
On a compact, finite-dimensional Lie group, G, there exists a measure, invariant under left and right translations, which is unique if fixed to be a probability measure. Furthermore it is invariant under inversions. The measure, denoted by μ_H, is called Haar measure. By left invariance we mean

$$\int_G d\mu_H(g) f(hg) = \int_G \mu_H(g) f(g)$$

A proof can be found in [11]. At this point, we give the concrete forms of the measures on the compact Lie groups we are concerned with; $SU(2)$ and $U(1)$.
Given the left invariant Maurer-Cartan form Θ_{MC}, we can build a metric on the group manifold given by

$$g = -\frac{1}{N}\,\mathrm{Tr}(\Theta_{MC} \otimes \Theta_{MC}), \tag{6.3.1}$$

where N is the normalization of the Lie algebra generators. In case of $SU(2)$ we use $N = 1/2$. Due to the trace the metric is invariant under inversion, left and right translations. Consequently, we find a volume form given by

$$d\mu_H = \frac{1}{G}\sqrt{\det g}\; d^3x \tag{6.3.2}$$

In case of $SU(2)$ this procedure yields

$$g_{ab} = d\phi_a d\phi_b + 2\cos\theta d\phi_{[a} d\psi_{b]} + d\psi_a d\psi_b + d\theta_a d\theta_b$$
$$d\mu_H = \frac{1}{(4\pi)^2}\sin\theta d\psi d\theta d\phi, \int_{SU(2)} d\mu_H = 1.$$

Using the pull back ι_{S^2} we can also obtain the well known volume form on the homogeneous space, i.e. the 2-sphere (omitting the normalization constant):

$$d\Omega = \sin\theta d\theta d\phi$$

In case of $U(1)$ all this becomes trivial. The metric and normalized Haar measure read[3]

$$g = d\psi_a d\psi_b$$
$$d\mu_H = \frac{1}{4\pi} d\psi \tag{6.3.3}$$

The existence of such a Haar measure allows for equipping a finite linear representation space V with an inner product such that the representation is unitary. Let the representation be denoted by $\rho : G \to \mathscr{B}(V)$. By definition, the carrier space is a Hilbert space, i.e. a complete inner product space. $\mathscr{B}(V)$ denotes the bounded linear operators on V. The map ρ is a homomorphism. An inner product satisfies the property $\langle u, \rho(g)v\rangle = \langle \rho^\dagger(g)u, v\rangle$, where $\dagger$ denotes the adjoint w.r.t. the inner product. A representation is called unitary if $\rho^\dagger(g) = \rho^{-1}(g) = \rho(g^{-1})$. Then define

$$\langle u, v\rangle' := \int d\mu_H(g)\langle \rho(g)u, \rho(g)v\rangle. \tag{6.3.4}$$

If we calculate $\langle \rho(h)u, \rho(h)v\rangle'$. We find

$$\langle \rho(h)u, \rho(h)v\rangle' = \int d\mu_H(g)\langle \rho(gh)u, \rho(gh)v\rangle = \langle u, v\rangle' \tag{6.3.5}$$

due to right invariance of the Haar measure, and we conclude that the representation is unitary.

Furthermore, we show that a finite dimensional unitary representation is either irreducible or completely reducible, i.e. the representation decomposes into a direct sum of irreducible representations. A representation is called irreducible, if there does not exist any non-trivial invariant subspace. By an invariant subspace W we mean $\rho(g)W \subset W \;\forall g \in G$. To prove the assertion above suppose that there exists an invariant subspace $V_1 \subset V$,

[3]The factor 4π is due to the fact that we considered $U(1)$ as a subgroup of $SU(2)$, where the range of ψ is $[0, 4\pi)$.

which is invariant and choose one element u. Then choose any $v \in V_1^\perp$. We find that

$$0 = \langle \rho(g^{-1})u, v\rangle = \langle u, \rho(g)v\rangle \tag{6.3.6}$$

Hence, $V_1^\perp$ is invariant too. Now upon iteration this process terminates, since V is finite dimensional.

Proving the theorem, which is to be presented in the following requires Schur's Lemma: If an intertwiner $A : V_1 \to V_2$ between two finite dimensional irreducible representations, ρ_1 and ρ_2, commutes with the representations, $A\rho_1(g) = \rho_2(g)A\ \forall g \in G$, then either $A = 0$ or the representations are equivalent.

First, from the commutation property it follows that the kernel and the image of A are an invariant. From the irreducibilty, it follows that either $Ker(A) = V_1$, from which we conclude that $A = 0$, or $Ker(A) = \{0\}$ and $Im(A) = V_2$, and we conclude that A is invertible. If an intertwiner is invertible the representations are called equivalent. We could now choose an other intertwiner B, then the combination $C = A - zB$, $z \in \mathbb{C}$ is again an intertwiner. We may choose z such that C is not invertible, but then $C = 0$. If $\rho_1 = \rho_2$ we may choose $A = \lambda\mathbb{1}$.

In the following we will prove parts one of the most important theorems used in the development of loop quantum gravity - the Peter Weyl theorem. It providesa decomposition of the representation space of infinte dimensional unitary representations of compact Lie groups into a direct sum of finite dimensional irreducible representations. Furthermore it equips us with an orthonormal and complete basis for the Hilbert space $L_2(G, d\mu_H)$, namely the suitably normalized coefficients of the representation matrices. Choose one representative ρ_j out of each of the equivalence classes of $\infty > d_j$-dimensional unitary irreducible representations. Denote the normalized martix coefficients of the m'th row and n'th column

$$g \mapsto b^{(j)}_{mn}(g) := \sqrt{d_j}\rho_j(g)_{mn}$$

and consider the (k, l) coefficient of the matrix $A^{jj',(n_0n_0')}$ defined by

$$\begin{aligned} A^{jj',(n_0n_0')}_{kl} &= \int_G d\mu_H(g) b^{(j)}_{kn_0}(g) b^{(j')}_{n_0'l}(g^{-1}) = \\ &= \int_G d\mu_H(g) b^{(j)}_{kn_0}(g) \bar{b}^{(j')}_{ln_0'}(g) = \\ &= \langle b^{(j')}_{ln_0'} b^{(j)}_{kn_0}\rangle \end{aligned} \tag{6.3.7}$$

Obviously, $A^{jj',(n_0n_0')}$ is an intertwiner between the j'th and j''th representation. By Schur's lemma we can conclude that it either vanishes or $j = j'$ and $A^{jj',(n_0n_0')}_{kl} = \lambda^{j,(n_0n_0')}\delta_{kl}$.

We rewrite equation (6.3.7) in the form

$$\lambda^{j,(n_0 n_0')}\delta_{kl} = \int_G d\mu_H(g) d_j \rho^{-1}_{n_0' l}(g)\rho_{kn_0}(g).$$

Multiplying both sides with δ^{kl} (tracing) yields

$$\lambda^{j,(n_0 n_0')} d_j = d_j \delta_{n_0' n_0}. \tag{6.3.8}$$

Hence, we can summarize the results in the orthonormality condition

$$\langle b^{(j')}_{m'n'} b^{(j)}_{mn} \rangle = \delta_{jj'}\delta_{mm'}\delta_{nn'}. \tag{6.3.9}$$

Showing completeness involves more effort. We will only sketch the proof. One has to utilze utilize the generalization of Weierstraß' result that on an compact intervall any continuous function can be approximated by linear combinations of polynomials, which is known as the Stone-Weierstraß theorem. It generalizes the result to compact Haussdorff spaces and more general algebras [4].

For our purpose to construct a spherically symmetry reduced sector of loop quantum gravity, we will need the unitary irreducible representations of $U(1)$. The basis of the Abelian Lie algebra $\mathcal{U}(1)$ is given by the imaginary unit. An arbitrary element of $U(1) \sim S^1$ is given using the exponential map

$$g(\phi) = \exp(i\phi) \tag{6.3.10}$$

On the n-dimensional complex vector space V g is represented as an unitary element of $GL(n, \mathbb{C})$ given by

$$\rho(g) = \exp(T\phi),$$

where T is a diagonal and antihermitean. In order to ensure periodicity $g(2\pi) = 1$ we conclude that the diagonal entries of T are given by im, $m \in \mathbb{Z}$[5], and further that $\rho(g)$ is diagonal with entries given by $\exp(im\phi)$. This representation is clearly reduible and decomposes into a direct sum of one dimensional unitary irreducible representations. The orthogonal basis is given by the functions

$$\rho_{rs} = \exp im\phi$$

$$\int \frac{1}{2\pi} \exp -im'\phi \exp im\phi = \delta_{mm'}. \tag{6.3.11}$$

[4] See [11] for detials.

[5] Alternatively we can have $m \in \frac{1}{2}\mathbb{Z}$ if we are interested in representations up to phase.

Chapter 7

Spin structure

As already mentioned in the beginning it is necessary to incorporate spinors in a complete quantum theory of gravity. Spinors turn out to be a (geo-)metrical concept as can be seen from $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$ which involves the metric. This leads to the question of how to generalize the concept to arbitrary manifolds and if there are any mathematically founded *obstructions* with regard to this concern. In particular it has to be clarified, given a tangent bundle over an n-dimensional paracompact differential manifold and its associated $O(n)$-principal bundle, which additional structure is needed in order to construct a spinor bundle and its associated $Spin(n)$-principal bundle from the given data. The spin group $Spin(n)$ is defined as the double cover of $SO(n)$. For $n > 2$ it is the universal cover. In the cases which are of interest for LQG and symmetry reduction we have $Spin(3) = SU(2)$ and $Spin(2) = U(1)$.

The most intuitive and beautiful approach to spinor formalism is via Clifford algebras [23]. Given the Euclidean vectorspace $F = \mathbb{R}^3$, isomorphic to the fibers of our tangent bundle over the 3-dimensional paracompact differential, hence Riemannian, manifold Σ. As discussed in ch 4, the fiber is an inner product space, i.e. equipped with a metric defined by

$$(x|y) = x^i y^i g_{ij},$$
$$x, y \in F, \quad g_{ij} = \delta_{ij}$$

The components of x and y are given with respect to an orthonormal frame e_i.

Now we will introduce an associative product on F which is distributive with respect to addition. The new product shall incorporate both, the notion of the inner as well as the exterior product. Given a vector x the square with respect to the new multiplication equals to its length squared,

$$x^2 = x^i e_i x^j e_j = \frac{1}{2} x^i x^j (e_i e_j + e_j e_i) = (x|x)$$

from this equation we deduce the defining equation for the Clifford algebra $\mathcal{C}(F)$ associated with the metric on F

$$(e_i e_j + e_j e_i) = 2\delta_{ij}. \tag{7.0.1}$$

In particular, this implies

$$\begin{aligned}(e_i)^2 &= 1 \\ e_i e_j &= -e_j e_i, \quad i \neq j.\end{aligned} \tag{7.0.2}$$

The Clifford algebra over the reals is then the linear span of dimension 2^3 of the basis elements

$$\mathcal{B} = \{1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3\}, \tag{7.0.3}$$

referred to as Pauli algebra. The elements of the 1-dimensional subspace spanned by $\iota = e_1e_2e_3$ are called pseudoscalars and share the following property with purely imaginary numbers $(\alpha\iota)^2 = -a^2$, $\alpha \in \mathbb{R}$, since in particular $\iota^2 = -1$. The Pauli-number ι is called unit right handed pseudoscalar. Therefore, the Clifford numbers are sometimes called hypercomplex numbers. Note that they already apear in Clifford algebras over the reals [24]. We denote the linear subspaces spanned by products of $p \leq 3$ e_i's with $\mathcal{C}_p$. The elements ι and 1 commute with all Pauli-numbers. Thus, in general, the center of the Clifford algebra associated to an n odd dimensional Euclidean vector space is $\mathcal{C}_0 \oplus \mathcal{C}_n$. Note that this subalgebra is isomorphic to the field of complex numbers $\mathbb{C}$.
Elements of $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_2$ are called scalars, vectors and pseudovectors respectively, due to their behavior under involution defined below. There are two important linear transformations of $\mathcal{C}(F)$ to itself and a duality rotation:

- the main automorphism called involution denoted by $*$: for $u \in \mathcal{C}_p$, we have $u^* = (-1)u$. It can be interpreted as space reflection. The even elements are invariant under this operation.
- The main antiautomorphism called reversion denoted by $\dagger$: $(e_{i_1} \cdots e_{i_{p-1}} e_{i_p})^\dagger = e_{i_p} e_{i_{p-1}} \cdots e_{i_1}$. The invariant elements in this case are scalars and vectors.
- The two main (anti-) automorphisms are sometimes combined to an antiautomorphism, called Clifford conjugation, denoted by an overbar $\bar{\ }$. In the Pauli algebra only scalars and pseudoscalars are invariant.
- duality rotation: for $u \in \mathcal{C}_p$, we have $\iota u \in \mathcal{C}_{n-p}$. For the example presented here

this implies

$$\begin{aligned}\iota e_1 &= e_2 e_3\\ \iota e_2 &= e_3 e_1\\ \iota e_3 &= e_1 e_2\end{aligned}$$

The reversion operation is used to define a scalar product for Clifford numbers $u,\ v \in \mathcal{C}(F)$ by

$$\langle u, v\rangle := (u^\dagger v)_S, \tag{7.0.4}$$

where the subscript S denotes the restriction to the scalar part of the product $u^\dagger v$. Obviously this coincides with the usual inner product for vectors. It is symmetric and positive definite in the Euclidean case [24].
The even subalgebra $\mathcal{C}_+ = \oplus_{p\ \mathrm{even}} \mathcal{C}_p$ of a Clifford algebra is again a Clifford algebra isomorphic to a Clifford algebra over a vectorspace with dimension lowered by 1. In case of the Pauli algebra $\mathcal{C}_+$ is spanned by

$$\{1, e_1e_2, e_1e_3, e_2e_3\}, \tag{7.0.5}$$

with the following beautiful properties:

$$\begin{aligned} j := -e_2e_3,\quad k := -e_3e_1,\quad l &:= -e_1e_2\\ j^2 = k^2 = l^2 &= -1\\ jkl &= -1\\ jk = l,\quad kl = j,\quad lj &= k \end{aligned} \tag{7.0.6}$$

Thus, we found the algebra of quaternions $\mathcal{H}_2$. Furthermore, we recover the $\mathcal{SU}(2)$-like algebra

$$\begin{aligned}\tau_i &:= -\frac{1}{2}\iota e_i\\ [\tau_i, \tau_j] &= \varepsilon_{ij}{}^k \tau_k.\end{aligned}$$

Finally, we take a different and very fruitful look upon Pauli-numbers. The relations we presented above show that any Pauli number can be decomposed in a sum of a formally complex scalar and a complex vector

$$u = \alpha + \iota\beta + a + \iota b \tag{7.0.7}$$

where $\alpha, \beta \in \mathbb{R}$ and $a, b \in \mathcal{C}_1$. Complex conjugation is realized by the reversion operation $\dagger$. Later we will introduce the notion of spinors, which is usually done via a complexification of the vectorspace F. At this point this becomes completely unnecessary, because the complex structure is already built in.

7.1 Reflections and rotations

The considerations presented in the following trace back to Hamilton's idea, namely to represent isometries of a vector space as successive actions of reflections. Every (non-isotropic) vector in $\mathcal{C}_1$ determines a reflection. First we define the inverse of vectors by

$$u^{-1} := \frac{u}{(u|u)}.$$

Any vector v can be decomposed into a part parallel $v_\parallel$ and perpendicular $v_\perp$ to u, where the first commutes and the latter anticommutes with u. We then have

$$v' = -uvu^{-1} = -v_\parallel + v_\perp,$$

hence we observe that v is reflected with respect to the (hyper)-plane perpendicular to u. Furthermore, this determines an isometry, since

$$(v'|w') = \frac{1}{2}(v'w' + w'v') = \frac{1}{2}(uvu^{-1}uwu^{-1} + uwu^{-1}uvu^{-1}) = u\frac{1}{2}(vw + wv)u^{-1} = (v|w).$$

We generalize this result to the theorem [23]: Any isometry of $\mathcal{C}_1$ can be written in the equivalent form

$$v \mapsto (-1)^r u_1 \cdots u_r v u_r^{-1} \cdots u_1^{-1} = u_1 \cdots u_r v[(u_1 \cdots u_r)^{-1}]^*, \tag{7.1.1}$$

with $r \leqslant n$. The elements in $\mathcal{C}(F)$ leaving $\mathcal{C}_1$ invariant, when acting as above, constitute the so-called Clifford group. The subgroup of elements Λ in $\mathcal{C}(F)$ with the properties

$$\begin{aligned} (-1)^r \Lambda_r v \Lambda_r^{-1} &\in \mathcal{C}_1 \\ \Lambda\Lambda^\dagger &= 1 \end{aligned}$$

is called $Pin(n)$. From these properties it follows that any Λ_r *can* be written in the form

$$\begin{aligned} \Lambda_r &= u_1 \cdots u_r, \\ u_i^2 &= 1. \end{aligned} \tag{7.1.2}$$

In the following, we restrict ourselves to this form of the operators Λ, by which the transformation formula (7.1.1) gets the remarkable simple and beautiful form using the Clifford conjugation

$$v \mapsto \Lambda v \bar{\Lambda}$$

The subgroup of even elements is called $Spin(n)$ and their respective actions are called rotations. Here the (anti-) automorphisms reversion and Clifford conjugation coincide. The action of odd elements is called reflections.

In our particular case $n = 3$, the e_i determine reflections with respect to planes perpendicular to e_i, while the elements $-\iota e_i$ determine rotations in planes perpendicular to e_i and ι determines a space reflection.

Let us put more emphasis on the rotations in 3 dimensions[1]. Choose any two unit vectors b and a, and denote the enclosed angle by α. Denote the vector perpendicular to a_1 in the plane S spanned by a and b by $a_\perp$. Then we decompose b w.r.t. a and $a_\perp$: $b = \cos\alpha a + \sin\alpha a_\perp$. Now choose an arbitrary vector v in $\mathcal{C}_1 \simeq F$ which can be decomposed into a part perpendicular with respect to S and a part parallel to S, $v = v_0 + v_t$. The component v_0 is invariant with respect to the action of $\Lambda = ba = \cos\alpha - \sin\alpha a a_\perp$, since it anticommutes with a as well as b. Hence, we can restrict the discussion of rotations to the plane S [2]. Further we decompose v_t with respect to a, $v_t = v_1 a + v_2 a_\perp$. Then

$$\begin{aligned} \Lambda v_t \Lambda^\dagger &= (\cos\alpha - a a_\perp \sin\alpha) v_t (\cos\alpha + a a_\perp \sin\alpha) = \\ = \cos(2\alpha)(v_1 a + v_2 a_\perp) + \sin(2\alpha)(v_1 a_\perp - v_2 a) &= \cos(2\alpha) v_t + \sin(2\alpha) v_t^\perp . \end{aligned}$$

Therefore, the the successive reflection determined by two unit vectors rotates any vector by the *doubled* enclosed angle and about an axis (right oriented) perpendicular to these two unit vectors. Using the expansions of the cosinus and sinus together with $I := -a a_\perp$, $I^2 = -1$ gives the remarkable formula

$$v_{\text{rot}} = e^{\frac{\alpha}{2} I} \ v \ e^{-\frac{\alpha}{2} I}.$$

The Pauli-bivector $1/2I = R$, i.e. the generator of the rotation, is called rotor. Hence, for example, if $a = e_1$, $R = \tau_3$. Note that in three dimensions any bivector is necessarily a *simple* bivector [23], i.e. it can be decomposed in an antisymmetric product of two vectors $I = 1/2(ab - ba) =: a \wedge b$[3].

[1] In higher dimension it can be shown [23] that any "rotation" can be decomposed in rotations with respect to mutually orthogonal 2 dimensional planes.

[2] We choose the order of the reflections such that we the first reflection is with respect to a, and the second with respect to b. This also corresponds to the sign of the elementary generators τ_i. Acting on vectors this choice is without any effect, but is important in case of spinors.

[3] Remark: Here we also made contact with the Grassmann-algebra.

As a side result we obtained a matrix representation of $Spin(3)$ corresponding to the defining representation of $SO(3)$ and, therefore, also established a 2:1 homomorphism $F : Spin(3) \to SO(3)$[4]. Hence, it is a univeral cover of $SO(3)$ and since all universal covers are isomorphic, we obtain the isomorphism $Spin(3) \simeq SU(2)$. Furthermore, the induced map $F' : \mathcal{S}pin(3) \to \mathcal{SO}(3)$ is an isomorphism. One might wonder, what the considerations have to do with spinors. Regarding the isomorphism of $Spin(3) \simeq SU(2)$, we know, of course that the natural representation space is $\mathbb{C}_2$, whose elements are called spinors. But Clifford algebras can do more.

7.2 Spinors

We now aim representing the Clifford algebra on one of its subagebras. In doing so, we will encounter an entirely geometric definition of spinors in three dimensions. Therefore we need to find the (one-sided) proper minimal ideals of the Clifford algebra[5]. As we have already seen in eq. (7.0.7), a general element of the Pauli algebra can be written in the compact form

$$\Phi = \Phi^0 + \Phi^i e_i,$$

where Φ^0 and Φ^i are elements of the center. We can immideatly find an nilpotent element of the Pauli algebra as follows: Choose two orthogonal elements u, $v \in \mathcal{C}_1$, with $u^2 = v^2$ and define the element $w = u + \iota v$. Then we have

$$w^2 = (u + \iota v)(u + \iota v) = u^2 - v^2 + \iota(u|v) = 0.$$

In the above described complexified vectorspace terminology we found a so-called isotropic vector. For convenience we choose $u = e_1$ and $v = e_2$ and rescale by a factor $1/2$, i.e. $w = 1/2(e_1 + \iota e_2)$. Furthermore we define a complementary isotropic vector w' by $ww' = 1$, which yields

$$w' = \frac{1}{2}(e_1 - \iota e_2) = w^\dagger$$

These two independent isotropic vectors span the two isotropic (complex) vectorspaces W and W'. Finally, we find a unit vector u orthogonal to $W \oplus W'$ spanning the one dimensional vectorspace U. Clearly, with our choice $u = e_3$. From the two nilpotent Clifford numbers, we construct idempotents, i.e. projection operators $P_3^+ = we_1$ and $P_3^- = w'e_1$.

[4]Of course, for the same reason, we see that $Pin(3)$ is a double cover of $O(3)$

[5]Remark: The Pauli algebra does not admit a two-sided proper ideal.

Now the set of all Pauli numbers of the form $\Phi_+ = \Phi P_3^+$ and $\Phi_- = \Phi P_3^-$, called (left)[6] spinors, form two independent minimal left ideals $\mathscr{I}_\pm$ as we will show now[7]. We find

$$\Phi_+ = \frac{1}{\sqrt{2}}(\Phi^0 + \Phi^3)\frac{1}{\sqrt{2}}(1+e_3) + \frac{1}{\sqrt{2}}(\Phi^1 + \iota\Phi^2)\frac{1}{\sqrt{2}}(e_1 - \iota e_2)$$
$$\Phi_- = \frac{1}{\sqrt{2}}(\Phi^1 - \iota\Phi^2)\frac{1}{\sqrt{2}}(e_1 + \iota e_2) + \frac{1}{\sqrt{2}}(\Phi^0 - \Phi^3)\frac{1}{\sqrt{2}}(1-e_3)$$

and define the two basis spinors $u_{1+} = 1/\sqrt{2}(1+e_3)$ and $u_{2-} = 1/\sqrt{2}(e_1 - \iota e_2)$ in $\mathscr{I}_+$ and $u_{1-} = 1/\sqrt{2}(e_1 + \iota e_2)$ $u_{2-} = 1/\sqrt{2}(1-e_3)$ in $\mathscr{I}_-$, such that they obey the completeness and orthonormality conditions with respect to the above defined scalar product for arbitrary Clifford numbers (7.0.4):

$$\langle u_{a\pm}, u_{b\pm}\rangle = \delta_{ab}$$
$$\sum_{a=1}^{2} u_{a\pm}u_{a\pm}^\dagger = 2. \tag{7.2.1}$$

Furthermore, we have

$$\langle u_{a\mp}, u_{b\pm}\rangle = 0.$$

Their behavior under multiplication by vectors from the left is

$$\begin{aligned} e_1u_{1\pm} &= u_{2\pm}, & e_1u_{2\pm} &= u_{1\pm}, \\ e_2u_{1\pm} &= \iota u_{2\pm}, & e_2u_{2\pm} &= -\iota u_{1\pm}, \\ e_3u_{1\pm} &= u_{1\pm}, & e_3u_{2\pm} &= -u_{2\pm}, \end{aligned} \quad .$$

Since all other basis objects of Pauli algebra are generated by multiplication of vectors and since vectors multiplication from the left leaves $\mathscr{I}_\pm$ invariant, we have indeed, constructed two minimal ideals, which serve as equivalent representations of our Pauli algebra. $\mathscr{I}_\pm$ is isomorphic to $\mathbb{C}^2$. The last line in eq. (8.0.1 is an a posteriori justification for the enumeration chosen for the basis spinors, according to the splitting of $\mathscr{I}_\pm$ into eigenspaces with respect to the left action of e_3 with positiv and negative eigenvalues, i.e. up and down spinors in the usual sense. The above table also allows us to read off the matrix representations of the Clifford numbers $\Phi_{ab} = \langle u_b, \Phi u_a\rangle$, which is independent of the choice of the minimal ideal. For example, we find that the basis vectors e_i are represented by the Pauli matrices. The corresponding minimal right ideals $\mathscr{I}_\pm^\dagger$ are simply found by

[6] The notion of left spinors is not to be confused with the notion of left handed spinors, introduced below.

[7] Here we could have used the two nilpotent elements to generate the minimal ideals. The advantage of using the pojection operators is that they allow to write any Pauli number as a sum $\Phi = \Phi_+ + \Phi_-$.

reversion. Their elements are called right or conjugate spinors.

7.3 Obstructions

In the following we will extend the notion of spin-soace to spinor fields. For this purpose, we first have to construct a spinor bundle $Spin(\Sigma)$ together with a 2:1-bundle morphism $F : Spin(\Sigma) \rightarrow SO(\Sigma)$ out of the $O(\Sigma)$ bundle associated to the tangent bundle $\mathcal{T}\Sigma$ over a differential paracompact 3-dimensional manifold Σ. The pair $[Spin(\Sigma), F]$ is called spin structure. Completely analogous to the soldering form on frame bundles, we will define a spinor field as a tensorial scalar on $Spin(\Sigma)$ of type $(\rho, \mathbb{C}^2)$[8]. Parallel transport of spinors is given by the so-called spin connection on the spin bundle.

Certainly, given a spin structure allows for reconstructing an $SO(3)$ bundle, but the reverse, starting from a tangent bundle, is possible only under certain restrictions. We saw that the notion of an oriented frame in the vector space isomorphic to the tangent space over a point in Σ is mandatory for obtaining spinors. Hence additionally to requiring that Σ is a paracompact manifold, which allows for obtaining an $O(3)$ bundle, Σ must be orientable, i.e. the $O(3)$ bundle must be reducible to a $SO(3)$ bundle. Furthermore, we know from the discussion on the group $SO(3)$ (ch 6) that the group manifold is not simply but doubly connected. A rotation about 2π is not homotopic to the trivial path, i.e. no rotation. One could imagine that if the bundle $SO(\Sigma)$ is simply connected a 2π rotation within a fiber over a point x can be undone by transporting the triad along a one parameter sequence of closed curves through x [8]. In order to ensure that this is impossible, the fundumanetal group of $SO(\Sigma)$ has to be isomorphic to $\mathbb{Z}_2$, i.e. $SO(\Sigma)$ has to be doubly connected. Only then we can consistently assign a change of sign of a spinor under a 2π rotation. The bundle $Spin(\Sigma)$ is then the double (universal) cover of $SO(\Sigma)$.

In order to adress these issues we make use of sheaf cohomology. For giving an exhaustive introduction to cohomology theory would go beyond the scope of this thesis, we will introduce the necessary notions only exemplarily[9]. Roughly speaking, in our case the sheaves are cartesian products of the multiplicative Abelian group $\mathbb{Z}_2$ and the base manifold Σ. Choose a locally finite cover $\mathcal{U}$ of Σ. A n-simplex σ is a collection of $n+1$ $\mathcal{U}_\alpha$ with nonvanishing intersection, denoted by $|\sigma|$, which is called support of σ. The β'th partial boundary $\partial_\beta\sigma$ of σ is the same collection as before, but with $\mathcal{U}_\beta$ being omitted. A n-cochain with values in $\mathbb{Z}_2$ assigns to each n-simplex an element $\mathcal{Z}_2(|\sigma|)$. The set of n-cochains forms an Abelian group, denoted by $C^n(\Sigma, \mathbb{Z}_2)$ with respect to pointwise multiplication. We define

[8] Alternatively we could have defined a spinor field as a section of the associated spinor bundle $Spin(\Sigma) \times_{Spin(3)} \mathbb{C}^2$.

[9] A comprehensive introduction to sheaf cohomology can be found in [25].

a nilpotent map, called the differential, $d : C^n(\Sigma, \mathbb{Z}_2) \rightarrow C^{n+1}(\Sigma, \mathbb{Z}_2)$, by

$$(d\tau)(|\sigma|) = \prod_{\alpha=0}^{n} [\tau(|\partial_\alpha \sigma|)]^{(-1)^n}.$$

The obtained $(n+1)$-chain is called $n+1$-coboundary $\in B^{n+1}(\Sigma, \mathbb{Z}_2)$. The kernel of d is called n-cocycle $\in Z^n(\Sigma, \mathbb{Z}_2)$. Of course, a coboundary is always a cocycle. Finally, we define the n'th cohomology group as the quotient $H^n(\Sigma, \mathbb{Z}_2) = Z^n(\Sigma, \mathbb{Z}_2)/B^{n+1}(\Sigma, \mathbb{Z}_2)$. We will motivate now that questions about orientability and the possibilty of constructing $Spin(\Sigma)$ can be answered affirmatively if the so-called first and second Stiefel-Whitney classes of $H^1(\Sigma, \mathbb{Z}_2)$ and $H^2(\Sigma, \mathbb{Z}_2)$ are trivial.
Let us discuss orientability: Choose an orthonormal frame e_α over each $\mathcal{U}_\alpha \in \mathcal{U}$. Changing the orientation of $e_\alpha \mapsto \omega_\alpha e_\alpha$[10] gives rise to the 0-chain ω. By using the differential d we obtain the 1-coboundary $d\omega$. For $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$, with $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \oslash$, we write according to the local definition for the differential

$$(d\omega)(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) = \omega_\beta \omega_\alpha,$$

which indicates wether we changed the orientation in $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ in in both neighborhoods or only in one of them.
In the overlaps $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ the frames change via the transition functions $g_{\alpha\beta}$ in $O(3)$, therefore $\tau_{\alpha\beta} := \det(g_{\alpha\beta}) = \pm 1$. The 1-chains $\tau_{\alpha\beta}$ tell us if the orientation changes in the transition from one to another trivialization. Since the $g_{\alpha\beta}$ satisfy the compatibility condition $g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\beta} = 1$ in the triple overlap $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$, τ turns out to be a 1-cocycle:

$$(d\tau)(\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma) = \tau_{\beta\gamma} \tau_{\alpha\gamma}^{-1} \tau_{\alpha\beta} = 1.$$

If we change orientation in $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ seperately $\tau_{\alpha\beta}$ maps to $\omega_\alpha \tau_{\alpha\beta} \omega_\beta$ and therefore changes by a coboundary. The set of equivalence classes $[\tau]$ under this action is the cohomology group. If this class is trivial we have $\tau_{\alpha\beta} = \omega_\alpha \omega_\beta$. Consequently, we could choose the 0-chains such that all $\tau_{\alpha\beta} = 1$. This corresponds to a global choice of orientation and consequently Σ is orientable and, therefore, we reduce the $O(3)$ bundle to an $SO(3)$ bundle.
From the bundle $SO(\Sigma)$ we construct the bundle $Spin(\Sigma)$ by choosing one of the two preimages of $g_{\alpha\beta} \in SO(3)$ denoted by $\Lambda_{\alpha\beta} \in Spin(3)$. These should also – but will not do so in general – satisfy the compatibility condition, which actually can be seen as a cocycle condition, $\Lambda_{\beta\gamma} \Lambda_{\alpha\gamma}^{-1} \Lambda_{\alpha\beta} = \zeta_{\alpha\beta\gamma} 1$. This defines a 2-cochain $\zeta_{\alpha\beta\gamma} = \pm 1$. Again, ζ is actually

[10]This is can be done via a $\omega_\alpha = \pm id \in O(3)$.

a 2-cocycle:

$$\begin{aligned}
&\Lambda_{\gamma\delta}\Lambda_{\delta\alpha}^{-1}\Lambda_{\alpha\gamma} = \zeta_{\alpha\gamma\delta}1 \\
&\quad \rightarrow \quad \Lambda_{\alpha\gamma}^{-1} = \zeta_{\alpha\gamma\delta}^{-1}\Lambda_{\gamma\delta}\Lambda_{\delta\alpha} \\
&\Lambda_{\gamma\delta}\Lambda_{\delta\beta}\Lambda_{\beta\gamma} = \zeta_{\beta\gamma\delta}1 \\
&\quad \rightarrow \quad \Lambda_{\beta\gamma}\Lambda_{\gamma\delta} = \zeta_{\beta\gamma\delta}\Lambda_{\beta\delta} \\
&\Lambda_{\beta\delta}\Lambda_{\alpha\delta}^{-1}\Lambda_{\alpha\beta} = \zeta_{\alpha\beta\delta}1 \\
&\Lambda_{\beta\gamma}\Lambda_{\alpha\gamma}^{-1}\Lambda_{\alpha\beta} = \\
&= \Lambda_{\beta\gamma}\zeta_{\alpha\gamma\delta}^{-1}\Lambda_{\gamma\delta}\Lambda_{\alpha\delta}^{-1}\Lambda_{\alpha\beta} = \\
&= \zeta_{\beta\gamma\delta}\zeta_{\alpha\gamma\delta}^{-1}\Lambda_{\beta\delta}\Lambda_{\alpha\delta}^{-1}\Lambda_{\alpha\beta} = \\
&= \zeta_{\beta\gamma\delta}\zeta_{\alpha\gamma\delta}^{-1}\zeta_{\alpha\beta\delta}1 = \zeta_{\alpha\beta\gamma}1
\end{aligned}$$

This leads to the conclusion that

$$\begin{aligned}
&(d\zeta)(\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma \cap \mathcal{U}_\delta) = \\
&\qquad \zeta_{\beta\gamma\delta}\zeta_{\alpha\gamma\delta}^{-1}\zeta_{\alpha\beta\delta}\zeta_{\alpha\beta\gamma}^{-1} = 1.
\end{aligned}$$

If, instead of $\Lambda_{\alpha\beta}$, we choose its negative in the overlap $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, this gives rise to a 1-cochain ω, from which we build a 2-coboundary. The 2-cocycles ζ change by that coboundary. If the equivalence class $[\zeta]$ is trivial we can choose the 1-cochain such that we obtain $\zeta = 1$ globally.

Now, having established

$$\Lambda_{\beta\gamma}\Lambda_{\alpha\gamma}^{-1}\Lambda_{\alpha\beta} = 1,$$

we can deduce the relations

$$\Lambda_{\gamma\gamma} = 1$$

and

$$\Lambda_{\beta\alpha} = \Lambda_{\alpha\beta}^{-1},$$

by setting $\beta = \gamma$ and $\gamma = \alpha$, respectively. Now we can construct the $Spin(\Sigma)$ bundle from a given Σ with vanishing first and second Stiefel Whitney class, $Spin(3)$ and $\Lambda_{\alpha\beta}$ as was shown in section 4.1.

Chapter 8

Gravitational connection dynamics

The main ingredient in loop quantum gravity is the passage from geometrodynamics to spin-connection dynamics. Instead of describing the dynamical gravitational field by means of metrics, one passes to a formulation of canonical GR using connections. Thereby also the "fourth" fundamental interaction is cast in a gauge theory. In order to do so we now define a new canonical pair of variables: $(A^i_a, \tilde{E}^a_i)$. The configuration variable is a $Spin(3)$-valued gauge potential coming from a connection 1-form on a $Spin(3)$-bundle over Σ

$$A := \Gamma + \gamma K. \tag{8.0.1}$$

$\Gamma = \Gamma^i \tau_i$ is a $Spin(3)$-gauge potential defined by

$$\Gamma^i{}_j = \varepsilon^i{}_{kj} \Gamma^k$$

where $\omega^i{}_j$ is the unique, torsion free and metric compatible $\mathcal{SO}(3)$-valued gauge potential. In this step the dimension of Σ was crucial. Only in three dimensions is it possible to include the spacetime information into a spatial connection. This uses the duality of the generator of rotations and the axis of rotation. We encounter the same duality in the context of Pauli algebra. The subspace of the generators of rotations, the rotors, is spanned by τ_i, generating rotations about the e_i. The rotors could be obtained via the duality rotation $\tau_i = -1/2\iota e_i$. This allows to incorporate the $\mathcal{C}_1$ valued field $K = K^i e_i$ into a $Spin(3)$-connection, as a difference tensor, which can be identified with the exterior curvature after imposing the Gauß constraint. From that point of view it could be assumed that $\gamma \in \mathcal{C}_3$, e.g. $\gamma = -1/2\iota$, since then

$$A^i \tau_i = \Gamma^i \tau_i + K^i \Big(-\frac{1}{2} \iota e_i \Big)$$

If we instead take $\gamma \in \mathcal{C}_0$, A becomes a so-called paravector valued 1-form. In geometric algebra, paravectors are used to generate proper Lorentz transformations, the vector part generating boosts, while, as discussed before, the pseudovectors generate the rotations. Hence, we obtain a $Spin(1,3)$-connection.
The discussion presented here differs from the conventions in the LQG literature so far as one uses $K = K^i\tau_i$ in the definition of A. In the following we will refrain from taking this more geometrically motivated approach for reasons of better compatibility with the literature and beyond use the LQG conventions. We will proceed with a $Spin(3)$-connection. The factor γ is the already mentioned Barbero-Immirzi parameter.
The geometrical operators of the quantized theory contain the Barbero-Immirzi parameter, as will be shown later, which therefore has a nontrivial effect on their spectra. In symplectic geometry the rescaling with γ is a symplectomorphism, i.e. a canonical transformation. The crucial point is that it "cannot be implemented unitarily in the quantum theory" as a consequence of the choice of configuration space, which is an affine space [26]. It can be interpreted as an quantization ambiguity similar to the so-called Θ-angle in QCD. In the case of LQG the parameter enters via a topological term called the Holst modification of the Palatini action [14]. In general it takes values in $\mathbb{C} \setminus \{0\}$. At the beginning of LQG the theory was formulated in terms of selfdual connections, i.e. γ was chosen to be the positive or the negative imaginary unit, in the signature convention used here. It can then be interpreted as the pull back of the selfdual $SL(2,\mathbb{C})$ connection of complexified general relativity to Σ. The drawback of this choice is, firstly that the gauge group is noncompact, which is crucial for the quantization scheme favored nowadays. Secondly, the reality conditions are extremely difficult to control. The advantage, however, is the simple polynomial form of the constraints.
The Poisson bracket now reads

$$\{E^a_i(x), A^j_b(y)\} = \frac{\kappa}{2}\delta^j_i\delta^a_b\delta^{(3)}(x,y) \tag{8.0.2}$$

Here the Gauß constraint takes the form to which its name traces back[1]

$$G_i = \mathcal{D}_a\tilde{E}^a_i = \partial_a\tilde{E}^a_i - \omega^j{}_i\tilde{E}^a_j + \gamma\varepsilon^j{}_{ik}K^k_a\tilde{E}^a_j. \tag{8.0.3}$$

The curvature of A is denoted by $F = dA + 1/2[A,A]$.
The corresponding "curvature scalar" is

$$\frac{\beta^2\tilde{E}_j\lrcorner(\tilde{E}_i\lrcorner F^i{}_j)}{\omega_h} = \omega_h^{(3)}R + \gamma^2\omega_h(K^{ab}K_{ab} - K^2) + \mathcal{P}(G_i)$$

[1]Note that $\tilde{E}$ is a vector-valued density thus, $\nabla_a\tilde{E}^a_i = \partial_a\tilde{E}^a_i$.

For $\gamma = \pm i$ and on the Gauß constraint surface this is exactly the Hamiltonian constraint. For an arbitrary value of γ we therefore have the following form of the (on G-shell) Hamiltonian constraint

$$C = \frac{1}{\omega_h} \tilde{E}_j \lrcorner (\tilde{E}_i \lrcorner (\varepsilon^{ij}{}_k F^k - (1+\gamma^2)(K^i \wedge K^j))) \tag{8.0.4}$$

Finally, the (on G-shell) diffeomorphism constraint reads

$$V = -2\tilde{E}_i \lrcorner F^i \tag{8.0.5}$$

At least at the moment only a partial spacetime interpretation of the Ashtekar connection in the case of a real valued parameter can be found using the so-called Holst approach. As shown in the discussion of the vielbein formalism, the $\mathcal{SO}(3)$-valued 1-form K^i can be regarded as the projection of a partially (temporal) gauge fixed connection component.
In that discussion we stated that the Barbero-Immirzi parameter would not alter the equations of motion of the classical theory. This is only true if there is no matter that couples to the connection in a first order action principle, as it is the case for fermions [27] in Einstein-Cartan theory. Then the spin-density gives rise to a connection component with nonvanishing torsion, which in turn implies that the Holst term in the action does not vanish, because the Bianchi identity is modified. This has physical effects: a weak four fermion interaction, well known from Einstein-Cartan theory, comes into play with a coupling constant depending on the Barbero-Immirzi parameter that spoils the interpretation of the (inverse) γ-parameter in analogy with the Θ-angle. This effect has been the starting point for [28], who got rid of this spurious four fermion interaction, using a modification of the fermion part of Palatini-Holst action coupled to fermions. The Holst term can be seen as part of a term called the Nieh-Yan invariant. The other part modifies the fermionic action after using the equantions of motion of the connection. This idea has then been further developed for any kind of matter coupled to (Einstein-Cartan) gravity, yielding the most general first order covariant approach to LQG. These approaches to an formulation of an action principle for LQG support my point of view that the notions of spinors and gravitation are linked.
We have now reached the realm of a gravitational Yang-Mills theory.

8.1 Spherically symmetric gravitational connection dynamics

Let us again turn to the main focus of this thesis: tracing the formulation of spherically symmetric models in LQG.
Before we make use of the linear map Λ to map $\iota^*_{S^2}\Theta^{SU(2)}_{MC}$ to values in $\mathcal{G}$ we rewrite

equation (5.0.3) in its infinitesimal version and choose $J = U(1) = \exp\langle\tau_3\rangle$

$$j = exp(t\tau_3)$$
$$\frac{d}{dt}\,|_{t=0}\; Ad_j X = [\tau_3, X] = ad_{\tau_3} X$$
$$\Lambda \circ ad_{\tau_3} = ad_{d\lambda(\tau_3)} \circ \Lambda$$

For an arbitrary $X = a_0\tau_1 + b_0\tau_2 \;\in \mathcal{J}^\perp$ we have

$$ad_{\tau_3} X = a_0\tau_2 - b_0\tau_3$$
$$\Lambda(a_0\tau_2 - b_0\tau_3) = ad_{d\lambda(\tau_3)}\Lambda(a_0\tau_1 + b_0\tau_2). \tag{8.1.1}$$

Now the question arises if and how we can find all possible conjugacy classes of the homomorphisms $\lambda : J \to G$. This boils down to the question if and how we can determine all conjugacy classes of G, if we identify the homomorphisms with their images. Indeed, this is possible due to the notion of the torus. At this point, we will only motivate the useful relation

$$\hom(J, G)/Ad_G \cong \hom(J, T(G))/W, \tag{8.1.2}$$

where $T(G)$ is the maximal torus of G and W is the Weyl group with respect to the torus[2]. The maximal torus of a compact connected Lie group (which is the case for the gauge group of LQG) is a Lie group $T \subset G$ which is isomorphic to $\mathbb{R}^k/\mathbb{Z}^k$. It is the same thing as the maximal Abelian subgroup of G which in our case is just $U(1)$. Let $S\Delta(2) \subset SU(2)$ be the subgroup of diagonal matrices

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$
$$z \in U(1)$$

The normalizer N of $T(G)$ consists of those elements out of G which do not lead out of the torus when acting on it via conjugation. The Weyl group of $T(G)$ is N/T, which can be shown to be compact and discrete, and therefore finite. For $SU(2)$ the Weyl group is the symmetric group $S(2)$ which is the group of permutations of two elements, hence acting on T gives

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \to \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}$$

[2]For detailed proofs of the involved theorems and lemmas see [29] chapter IV.

The argument for equation (8.1.2) is based on three facts: The maximal tori cover G, the maximal tori are all conjugate and if two elements of a maximal torus are conjugate in G, they lie on the same orbit under the action of the Weyl group. Hence, there is a canonical homeomorphism[3]

$$\kappa : T/W \to \mathrm{Con}(G).$$

The homomorphisms $\mathrm{hom}(U(1), T(SU(2))$ are then given by

$$\tilde{\lambda}_k : z \to diag(z^k, z^{-k}),$$

with $k \in \mathbb{Z}$, since $z = exp(2\pi i\phi)$, $\phi \in \mathbb{R}/\mathbb{Z}$. Due to the equivalence with respect to the Weyl group $diag(z^k, z^{-k}) \sim diag(z^{-k}, z^k)$ we have $k \in \mathbb{N}_0$.
For the sake of simplicity we make a partial gauge choice called the τ_3-gauge, i.e. $\lambda_k(exp(t\tau_3)) = \exp(kt\tau_3)$, which in the group theoretical terminology amounts to choose a specific maximal torus. It follows that $d\lambda : \tau_3 \to k\tau_3$, and we can continue with calculating (8.1.1).

$$a_0\Lambda(\tau_2) - b_0\lambda(\tau_3) = k(a_0[\tau_3, \lambda(\tau_1)] + b_0[\tau_3, \Lambda(\tau_2)]) \tag{8.1.3}$$

We formulate the ansatz

$$\Lambda(\tau_i) = a_i\tau_1 + b_i\tau_2 + c_i\tau_3$$

and find the equations

$$\begin{aligned} c_1 = c_2 = 0, \quad a_1 = kb_2, \quad b_1 = -ka_2 \\ a_2 = -kb_1, \quad b_2 = ka_1 \end{aligned}$$

which have non-trivial solutions - hence non-vanishing Higgs field components - iff $k = 1$,

$$\begin{aligned} a_1 =: A_1, \quad b_1 =: A_2 \\ a_2 = -A_2, \quad b_2 = A_1 \end{aligned}$$

The centralizer of λ_k is $U(1)$.
Therefore we finally arrive at a connection which in the τ_3-gauge and with $k = 1$ can be gauged to

$$\begin{aligned} A &= \tilde{\omega} + \Lambda \circ \iota^*_{S^2}\Theta^{SU(2)}_{MC} = \\ &= A_x(x)dx\tau_3 + (A_1(x)\tau_1 + A_2(x)\tau_2)d\theta + (A_1(x)\tau_2 - A_2(x)\tau_1)\sin\theta d\phi + \tau_3\cos\theta d\phi. \end{aligned}$$

[3] proposition IV,(2.6) in [29]

The invariant densitized triad can be found by imposing the symplectic structure of the full theory

$$\frac{2}{\kappa\gamma}\int_\Sigma \mathbf{d}A^i_a \wedge \mathbf{d}\tilde{E}^a_i \tag{8.1.4}$$

From the tangent bundle construction we find

$$\tilde{E}^a = \tilde{E}^x(x)\partial^a_x\tau_3 + (\tilde{E}^1(x)\tau_1 + \tilde{E}^2(x)\tau_2)\partial^a_\theta + (\tilde{E}^1(x)\tau_2 - \tilde{E}^2\tau_1)\frac{1}{\sin\theta}\partial^a_\phi$$

which gives the symplectic structure

$$\frac{2}{\kappa\gamma}\int_\Sigma \mathbf{d}A_x \wedge \mathbf{d}\tilde{E}^x + 2\mathbf{d}A_1 \wedge \mathbf{d}\tilde{E}^1 + 2\mathbf{d}A_2 \wedge \mathbf{d}\tilde{E}^2.$$

We can write the densitized scalars $\tilde{E}^I$ as

$$\tilde{E}^I = \bar{E}^I \wedge (d\theta \wedge \sin\theta d\phi) = \bar{E}^I \wedge d^2\Omega \tag{8.1.5}$$

which yields the following reduced symplectic structure

$$\frac{8\pi}{\kappa\gamma}\int_B (\mathbf{d}A_x \wedge \mathbf{d}\bar{E}^x + 2\mathbf{d}A_1 \wedge \mathbf{d}\bar{E}^1 + 2\mathbf{d}A_2 \wedge \mathbf{d}\bar{E}^2).$$

Thus, the Poisson structure of the configuration variables (A_x, A_1, A_2) and their respective conjugate momenta $(\bar{E}^x, \bar{E}^1, \bar{E}^2)$ is[4]

$$\{A_x, \bar{E}^x(y)\} = \frac{\kappa\gamma}{8\pi}\bar{\delta}(x,y), \quad \{A_\alpha(x), \bar{E}^\beta(y)\} = \frac{\kappa\gamma}{16\pi}\bar{\delta}(x,y)\delta^\alpha_\beta \quad \alpha,\beta = 1,2$$

For the reduced quantized theory it will be crucial to undo the partial gauge fixing we performed before by choosing an arbitrary element out of the conjugacy class $[\lambda_1]$, i.e. $d\lambda_1(\tau_3) = g\tau_3 g^{-1}$ with $g \in SU(2)$. Again we parametrize $SU(2)$ with the Euler angles $g = exp(-\phi\tau_3)exp(-\theta\tau_1)exp(-\psi\tau_3)$.

$$d\lambda(\tau_3) = \sin\theta\sin\phi\tau_1 + \sin\theta\cos\phi\tau_2 + \cos\theta\tau_3 =: n^i\tau_i \tag{8.1.6}$$

In the λ_1-gauge the $U(1)$-connection on B is $A^i := A_x n^i dx$ and the conjugate momentum is $\bar{E}^i = \bar{E}^x n_i \partial_x$ with the Poisson structure

$$\{A_i, \bar{E}^j\} = \frac{\kappa\gamma}{8\pi}\bar{\delta}(x,y)\delta^j_i$$

[4]Note that the delta function is a density of weight one with respect to the one dimensional base manifold B. The same is true for the triad, i.e. we could write $\bar{E}^I(x) = E^I(x)dx$.

8.1.1 Spherically symmetric geometric objects

Let us now present several geometric objects of the symmetry reduced model: volume, area, triad, extrinsic curvature and curvature.
The volume form is given by the formula[5]

$$\omega_h = \sqrt{|*\tilde{E}_1 \wedge \tilde{E}_2 \wedge \tilde{E}_3|} \tag{8.1.7}$$

In the spherically reduced context this gives

$$\begin{aligned} \omega_h &= \{|E^x|[(E^1)^2 + (E^2)^2](\partial_x \wedge \partial_\theta \wedge \frac{1}{\sin\theta}\partial_\phi)\lrcorner(dx \wedge d\theta \wedge \sin\theta d\phi)^3\}^{\frac{1}{2}} = \\ &= \sqrt{|E^x|(E^t)^2}dx \wedge \omega_{S^2} \end{aligned}$$

which can be reduced upon integration over S^2 to a 1-form on B

$$\omega_B = 4\pi\sqrt{|E^x|}E^t dx. \tag{8.1.8}$$

The volume form now allows us to read off the $SU(2)$ triad

$$\begin{aligned} E_1^a &= \frac{1}{\sqrt{|E^x|}E^t}(E^1\partial_\theta^a - E^2\frac{1}{\sin\theta}\partial_\phi^a), \\ E_2^a &= \frac{1}{\sqrt{|E^x|}E^t}(E^2\partial_\theta^a + E^1\frac{1}{\sin\theta}\partial_\phi^a), \\ E_3^a &= \mathrm{sgn}(E^x)\frac{\sqrt{|E^x|}}{E^t}\partial_x^a. \end{aligned} \tag{8.1.9}$$

which by being orthonormal parametrizes the metric

$$ds^2 = \frac{(E^t)^2}{|E^x|}dx^2 + |E^x|(d\theta^2 + \sin^2\theta d\phi^2). \tag{8.1.10}$$

The area of an arbitrary 2-surface S is given by the formula

$$Ar(S) = \int_S \sqrt{*\tilde{E} \cdot *\tilde{E}}$$

The area of the 2-sphere generated by the orbits of the symmetry group therefore is

$$Ar(S^2, x) = 4\pi|E^x(x)|$$

[5] See app. A.14 for the definition of the $*$ operation for densitized tensorfields.

The calculations can be drastically simplified by using polar like phase space coordinates. To that end, we define the angle coordinates η and β

$$\begin{aligned}\sin\eta &= \frac{E^1}{E^t}, \quad \cos\eta = -\frac{E^2}{E^t}\\ \sin\beta &= \frac{A_1}{A_t}, \quad \cos\beta = -\frac{A_2}{A_t}\end{aligned}$$

In the new coordinates the co-triad with respect to (8.1.9) read

$$\begin{aligned}e^1 &= \sqrt{|E^x|}(\sin\eta d\theta + \cos\eta\sin\theta d\phi)\\ e^2 &= \sqrt{|E^x|}(-\cos\eta d\theta + \sin\eta\sin\theta d\phi)\\ e^3 &= \mathrm{sgn}(E^x)\frac{E^t}{\sqrt{|E^x|}}dx.\end{aligned} \tag{8.1.11}$$

Starting from this expression we obtain the spin connection using the relation $\Gamma^i{}_j = \varepsilon^i{}_{kj}\Gamma^k$, where the prime denotes the derivative w.r.t. x

$$\begin{aligned}\Gamma^2{}_1 &= -\eta' dx + cos\theta d\phi = \Gamma^3\\ \Gamma^\alpha{}_3 &= \frac{E^{x\prime}}{2\sqrt{|E^x|E^t}}e^\alpha\end{aligned} \tag{8.1.12}$$

In addition, we introduce an adapted, rotated internal $\mathcal{SU}(2)$ basis, which allows us to write A, E and Γ in a compact form.

$$\begin{aligned}\Lambda^1_\eta &= \tau^1\sin\eta - \tau^2\cos\eta,\\ \Lambda^2_\eta &= \tau^1\cos\eta + \tau^2\sin\eta,\\ \Lambda^3_\eta &\equiv \Lambda^3 = \tau^3,\end{aligned} \tag{8.1.13}$$

with

$$\begin{aligned}[\Lambda^i_\eta, \Lambda^j_\eta] &= \varepsilon^{ij}{}_k\Lambda^k_\eta\\ \Lambda^1_\eta \cdot \Lambda^\beta_1 &= \cos(\eta-\beta) =: \cos\alpha.\end{aligned}$$

We end up with the following results:

$$\begin{aligned}\Gamma = -\eta'\Lambda_3 dx - \frac{E^{x\prime}}{2E^t}(-\Lambda_2^\eta d\theta + \Lambda_1^\eta \sin\theta d\phi) + \cos\theta\Lambda_3^\eta d\phi\\ \tilde{E} = \tilde{E}^x\Lambda^3\partial_x + \tilde{E}^t(\Lambda_\eta^1\partial_\theta + \Lambda_\eta^2\frac{1}{\sin\theta}\partial_\phi)\\ A = A_x\Lambda_3 dx + A_t(\Lambda_1^\beta d\theta + \Lambda_2^\beta \sin\theta d\phi) + \cos\theta\Lambda_3 d\phi =\\ = A_x\Lambda_3 dx + \cos\theta\Lambda_3 d\phi + A_t\cos\alpha(\Lambda_1^\eta d\theta + \Lambda_2^\eta\sin\theta d\phi) +\\ + A_t\sin\alpha(-\Lambda_2^\eta d\theta + \Lambda_1^\eta \sin\theta d\phi).\end{aligned}$$

Looking at the internal directions of A and E we see that E_t is not the conjugate momentum to A_t. This results from the internal directions of Γ being perpendicular to the ones of E, while those of the extrinsic curvature K are parallel, as is shown by the following calculation. We start with equation (1.0.4) and use (8.1.10).

$$\frac{\dot{h}_{ab}}{2N} = N^{-1}\big[\big(\operatorname{sgn}(E^x)\frac{E^t}{\sqrt{|E^x|}}\big)^{\cdot}\frac{E^t}{\sqrt{|E^x|}}dx_a dx_b + (\sqrt{|E^x|})^{\cdot}\sqrt{|E^x|}(d\Omega)^2_{ab}\big] \tag{8.1.14}$$

The reduced shift is tangential to B (see (8.1.18)), $N^a = N^x\partial_x^a$.

$$\begin{aligned}D_a N_b =& D_a(N^x \operatorname{sgn}(E^x)\frac{E^t}{\sqrt{|E^x|}}e_b^3) = (N^x \operatorname{sgn}(E^x)\frac{E^t}{\sqrt{|E^x|}})' dx_a e_b^3 -\\ &- (N^x \operatorname{sgn}(E^x)\frac{E^t}{\sqrt{|E^x|}})\Gamma^3_{ai}e_b^i =\\ =& \big(N^x \operatorname{sgn}(E^x)\frac{E^t}{\sqrt{|E^x|}}\big)' dx_a e_b^3 + N^x(\sqrt{|E^x|})'(e_a^1 e_b^1 + e_a^2 e_b^2)\end{aligned}$$

Then we find for $K_a^i e_b^i = K_{ab}$

$$\begin{aligned}K =& N^{-1}\big\{\big[\big(\operatorname{sgn}(E^x)\frac{E^t}{\sqrt{|E^x|}}\big)^{\cdot} - \big(N^x \operatorname{sgn}(E^x)\frac{E^t}{\sqrt{|E^x|}}\big)'\big]\Lambda^3 dx +\\ &+ [(\sqrt{|E^x|})^{\cdot} - N^x(\sqrt{|E^x|})'](\Lambda_\eta^1 d\theta + \Lambda_\eta^2\sin\theta d\phi)\big\}\end{aligned}$$

Since $A = \Gamma + \gamma K$ we find the following relations between the spin connection, the Ashtekar connection and the extrinsic curvature:

$$\begin{aligned}A_x &= -\eta' + \gamma K_x \\ A_t\cos\alpha &= \gamma K_t\\ A_t\sin\alpha &= \Gamma_t\end{aligned} \tag{8.1.15}$$

The reduced symplectic form now reads

$$\Omega = \frac{1}{G\gamma}\int_B \delta A_x \wedge \delta\bar{E}^x + \delta A_t \wedge \delta\bar{P}^t + \delta\beta \wedge \delta\bar{P}^\beta,$$
$$\bar{P}^t = 2\bar{E}^t \cos\alpha, \quad \bar{P}^\beta = 2A_t\bar{E}^t \sin\alpha$$

or

$$\Omega = \frac{1}{G\gamma}\int_B \delta A_x \wedge \delta\bar{E}^x + 2\gamma\delta K_t \wedge \delta\bar{E}^t + \delta\eta \wedge \delta\bar{P}^\eta,$$
$$\bar{P}^\eta = \bar{P}^\beta$$

depending on which of the following two options we choose:

- Connection representation: Configuration variables (A_x, A_t, β).
- Triad representation: Momentum variables $(\bar{E}_x, \bar{E}_t, \bar{P}^\eta)$.

Clearly, the first choice would be much closer to the full theory. The advantage of the second, however, is that geometric operators and the constraints are more compact and easier to quantize in this representation. This can be seen for for instance from the form of the volume element, which plays an important role in the Hamiltonian, One might object that - from the full theory's perspective - holonomies are built using connections instead of the extrinsic curvature. From the reduced theory's viewpoint the connection components A^1 and A^2 are Higgs scalars. There are no edges but those in the reduced basis manifold B.

Finally, in order to be able to treat the constraints we need the curvature 2-form.

$$\begin{aligned}
F =& dA + \frac{1}{2}[A, A] = \\
=& [A_t'\Lambda_1^\beta + A_t(\beta' + A_x)\Lambda_2^\beta]dx \wedge d\theta + [A_t'\Lambda_2^\beta + A_t(\beta' + A_x)\Lambda_1^\beta]dx \wedge \sin\theta d\phi + \\
&+ (A_t^2 - 1)\Lambda_3 d\theta \wedge \sin\theta d\phi = \\
=& \{[\gamma K_t' + \Gamma_t(\eta' + A_x)]\Lambda_1^\eta + [-\Gamma_t' + \gamma K_t(\eta' + A_x)]\Lambda_2^\eta\}dx \wedge d\theta \\
&+ \{[\gamma K_t' + \Gamma_t(\eta' + A_x)]\Lambda_2^\eta - [-\Gamma_t' + \gamma K_t(\eta' + A_x)]\Lambda_1^\eta\}dx \wedge \sin\theta d\phi \\
&+ (\Gamma_t^2 + \gamma^2 K_t^2 - 1)\Lambda_3 d\theta \wedge \sin\theta d\phi
\end{aligned} \tag{8.1.16}$$

8.1.2 The spherically symmetric constraints

Let us begin with the simplest of the three first class constraints of connection dynamics - the Gauß constraint:

$$\begin{aligned} G[\lambda] &= \frac{1}{\kappa\gamma}\int_\Sigma \lambda\cdot\mathcal{G} \\ \mathcal{G} = \mathcal{D} * \tilde{E} &= (\tilde{E}^{x\prime} + \tilde{P}^\eta)\Lambda^3 + \tilde{E}^t\cot\theta\Lambda^1_\eta \\ G[\lambda] &= \frac{1}{2G\gamma}\int_B \lambda^3(\bar{E}^{x\prime} + \bar{P}^\eta) \end{aligned} \tag{8.1.17}$$

Next, let us turn to the diffeomorphism constraint.

$$\begin{aligned} D[S] &= \frac{1}{\kappa\gamma}\int_\Sigma S\lrcorner\mathcal{D} \\ \mathcal{D} &= -2(\tilde{E}\cdot\lrcorner F) = \\ &= 2(\eta'\tilde{P}^\eta - A_x\tilde{E}^{x\prime} + 2\gamma\tilde{E}^t K_t')dx \\ D[S] &= \frac{1}{G\gamma}\int_B S^x[2\gamma\bar{E}^t K_t' - (\eta' + A_x)\bar{E}^{x\prime} + \eta'\mathcal{G}_3] \end{aligned} \tag{8.1.18}$$

Finally, we have to compute the scalar constraint.

$$\begin{aligned} C[N] &= \frac{1}{\kappa}\int_\Sigma N\mathcal{H} \\ \mathcal{H} &= \frac{\omega_h^{-1}}{2}[\tilde{E},\tilde{E}]\cdot\lrcorner(F - \frac{1+\gamma^2}{2}[K,K]) = \\ &= -2|E^x|^{-\frac{1}{2}}[\tilde{E}^t K_t^2 + \frac{2}{\gamma}\tilde{E}^x K_t(\eta' + A_x) + 2\tilde{E}^x\Gamma_t' - \tilde{E}^t(\Gamma_t^2 - 1)] \\ C[N] &= -\frac{1}{G}\int_B N|E^x|^{-\frac{1}{2}}[\tilde{E}^t K_t^2 + \frac{2}{\gamma}\tilde{E}^x K_t(\eta' + A_x) + 2\tilde{E}^x\Gamma_t' - \tilde{E}^t(\Gamma_t^2 - 1)] \end{aligned} \tag{8.1.19}$$

Chapter 9

Yang-Mills connection dynamics

Now we will start to incorporate matter fields in our model. Before turning to spherically symmetric Maxwell theory we will discuss the general framework of gauge field theories; Yang-Mills theory. The action for Yang-Mills fields is defined by

$$\begin{aligned} S[\mathcal{A}] = -\frac{1}{2g^2}\int_{\mathcal{M}} \mathcal{F} \star \mathcal{F}, \\ \mathcal{F} = d\mathcal{A} + \mathcal{A}^2 \end{aligned}$$

where the star denotes the hodge dual as well as contraction of the internal indices[1] and g denotes the coupling constant. The constants are chosen such that the Hamiltonian density will coincide with the energy density in the case of Maxwell theory. We choose the normalization of the Yang-Mills gauge-generators in the same fashion as for the gravitational sector

$$\begin{aligned} T^{a\dagger} = -T^a \quad \text{antihermitean}, [T^a, T^b] = f^{ab}{}_c T^c, \\ \mathrm{tr}(T^a T^b) = -\frac{1}{2}\delta^{ab}, \end{aligned}$$

We will now perform the 3+1-split. To that aim we first split-off terms containing $n \propto dt$.

$$\begin{aligned} \mathcal{A} &= -\Phi dt + A \\ -\Phi &= N n \lrcorner \mathcal{A} \\ \mathcal{F} &= F - n \wedge n \lrcorner \mathcal{F} \end{aligned} \tag{9.0.1}$$

On a $t = const$ folium the pullback F of $\mathcal{F}$ is the curvature of the pullback A of $\mathcal{A}$, i.e.

$$F = {}^{(3)}dA + A^2$$

[1]The definition can be found in app. A.14.

Now let us take a closer look at on the second term in (9.0.1),

$$\begin{aligned} n\lrcorner\mathcal{F} = \frac{1}{N}d\Phi + n\lrcorner dA + \frac{1}{N}[A,\Phi] = \frac{1}{N}(d\Phi + [A,\Phi]) + \pounds_n A \\ \pounds_n A = \frac{1}{N}\pounds_{Nn} A = \frac{1}{N}(\pounds_t A - \pounds_V A) \\ \Rightarrow \quad n\wedge n\lrcorner\mathcal{F} = n\wedge \frac{1}{N}(D\Phi + \dot{A} - {}^{(3)}\pounds_V A), \end{aligned}$$

where in the last step we used the definitions

$$\begin{aligned} n\wedge \pounds_t A := n\wedge \dot{A} \\ \forall\ \psi\in\Lambda^p\mathcal{M},\ \text{with}\ \ n\lrcorner\psi = 0,\ p\leqslant \dim(\mathcal{M})-2;\quad n\wedge d\psi := {}^{(3)}d\psi \\ D := {}^{(3)}d\,\cdot + [A,\cdot] \end{aligned}$$

In order to be able to vary with respect to the shift V in our expressions we rearrange some terms:

$$D\Phi = -D(t\lrcorner\mathcal{A}) + D(V\lrcorner A) = -D(t\lrcorner\mathcal{A}) + {}^{(3)}d(V\lrcorner A) - V\lrcorner A^2.$$

Thus, we obtain

$$D\Phi - {}^{(3)}\pounds_V A = -D(t\lrcorner\mathcal{A}) - V\lrcorner F. \tag{9.0.2}$$

For the $3+1$-split of the field strenght we obtain

$$\mathcal{F} = dt(\dot{A} - D(t\lrcorner\mathcal{A}) - V\lrcorner F) + F \tag{9.0.3}$$

and this result we insert into the action functional.

$$\begin{aligned} S[A, t\lrcorner\mathcal{A}, N, V] = -\frac{1}{2g^2}\int_{\mathcal{M}}[dt(\dot{A} - D(t\lrcorner\mathcal{A}) - V\lrcorner F) + F]\star\mathcal{F} = \\ = \frac{1}{2g^2}\int_{\mathcal{I}} dt\int_{\Sigma} N^{-1}[(\dot{A} - D(t\lrcorner\mathcal{A}) - V\lrcorner F) * (\dot{A} - D(t\lrcorner\mathcal{A}) - V\lrcorner F)] - NF * F = \\ = \int dt L[A, t\lrcorner\mathcal{A}, N, V; \dot{A}, (t\lrcorner\mathcal{A})^{\cdot}, \dot{N}, \dot{V}], \end{aligned}$$

where in the second line the new star operation with respect to $\Lambda^p\Sigma$ is understood and we used the $3+1$-split of the Hodge star (see. app. A.14). The momenta conjugate to the configuration variables are given by

$$\frac{g^2\delta L}{\delta\dot{A}} = \tilde{\Pi}$$
$$\frac{g^2\delta L}{\delta(t\lrcorner\mathcal{A})^\cdot} = 0$$
$$\frac{g^2\delta L}{\delta\dot{N}} = 0$$
$$\frac{g^2\delta L}{\delta\dot{V}} = 0.$$

Hence we have 3 primary constraints. We define the momentum conjugate to A as the Hodge dual of a vector and Lie algebra valued density

$$*\tilde{\Pi} =: \Sigma = *N^{-1}(\dot{A} - D(t\lrcorner\mathcal{A}) - V\lrcorner F) \equiv - * \tilde{\mathcal{E}},$$

where $\tilde{\mathcal{E}}$ is the Yang-Mills electric field. In analogy with the connection formulation of general relativity we call Σ the Yang-Mills-Plebanski 2-form [1]. The momentum conjugate to A is then defined as the 1-current $\tilde{\Pi}$ associated to Σ via the star operation extended to densitized vector-fields (see A.14).

Then we rearrange the action functional in the form $S = \int dt \int_\Sigma \dot{q}p - \mathcal{H}$ and add the primary constraints with some auxilliary multipliers

$$S = \frac{1}{g^2}\int_{\mathcal{I}} dt \int_\Sigma \{\dot{A}\Sigma + (t\lrcorner\mathcal{A})^\cdot\tilde{P}_\phi + \dot{N}\tilde{P}_N + \dot{V}\lrcorner\tilde{P}_V -$$
$$-[(t\lrcorner\mathcal{A})D\Sigma - V\lrcorner(\tilde{\Pi}\lrcorner F) + N\frac{1}{2}(\Sigma * \Sigma + F * F) + \lambda\tilde{P}_N + \gamma\lrcorner\tilde{P}_V + \chi\tilde{P}_\phi]\} \tag{9.0.4}$$

where we assumed suitable fall off conditions for the fields in order to shift the covariant derivative via partial integration and used the identity $V\lrcorner F\Sigma = -V\lrcorner(\tilde{\Pi}\lrcorner F)$.

The consistency conditions for the time evolution of the primary constraints yield the secondary constraints

$$\mathcal{C} = \frac{1}{2g^2}(\Sigma * \Sigma + F * F) \approx 0, \tag{9.0.5}$$
$$\mathcal{G} = \frac{1}{g^2}D\Sigma \approx 0,$$
$$\mathcal{V} = \frac{1}{g^2}\tilde{\Pi}\lrcorner F \approx 0.$$

These terms are again added to the Hamiltonian with Lagrange multipliers. The equations of motion for V, N and $t\lrcorner\mathcal{A}$ tell us that their velocities are proportional to Lagrange multipliers and are hence Lagrange multipliers themselves. Their respective consistency conditions do not yield any secondary constraints and the constraint algorithm stops.

In order to facilitate compareability with the main LQG references, we formulate the constraints in a way known from Yang-Mills theory.

$$C[N] = \int_\Sigma N \frac{h_{ab}}{2g^2\omega_h}(\tilde{\mathcal{E}}_i^a\tilde{\mathcal{E}}_i^b + \tilde{\mathcal{B}}_i^a\tilde{\mathcal{B}}_i^b) \tag{9.0.6}$$

$$D[V] = -\frac{1}{g^2}\int_\Sigma V^b\tilde{\mathcal{E}}_i^a F_{ab}^i \tag{9.0.7}$$

$$G[\phi] = -\frac{1}{g^2}\int_\Sigma \phi^i D_a\tilde{\mathcal{E}}_i^a. \tag{9.0.8}$$

The Yang-Mills magnetic field, $\tilde{\mathcal{B}}$, is the 1-current associated to the field strength F via the star operation. Note that we used the electric field in these equations, which is the negative of the momentum conjugate to A.
The vector constraint $D[V]$ in that form is often being reffered to as the diffeomorphism constraint, but in fact the generator of diffeomorphisms is the combination $D'[V] = D[V] - G[V \lrcorner A]$ [13] and we obtain

$$D'[V] = -\frac{1}{g^2}\int_\Sigma \pounds_V A\Sigma. \tag{9.0.9}$$

This, however, is exactly the expression we would have directly obtained, if we did not use the r.h.s. of (9.0.2). Therefore, the diffeomorphism constraint arises completely naturally in our $3+1$ split of the action. So while isolating V in the above procedure was necessary to identify it as a Lagrange multiplier, the constraint which acts via Poisson brackets is the smeared version anyway. Furthermore, the Lagrange multiplier of the Gauß law now becomes the component of the connection $\mathcal{A}$ proportional to the hypersurface 1-form n and no terms proportional to A appear as is implicitly the case in $t\mathcal{A}$. Omitting the redundant constraint terms we end up with the following form of the action

$$S = \int dt \int_\Sigma \dot{A}\Sigma - [C[N] + G[\Phi] + D'[V]]. \tag{9.0.10}$$

One can check that the diffeomorphism and Gauss constraints are first class.

9.1 Spherically symmetric Maxwell theory

Maxwell's theory of electromagnetism can be cast into Yang-Mills form with (Abelian) structure group $U(1)$. The associated bundle is the complex line bundle, the sections of which are the wave functions describing a charged quantum particle, subject to internal $U(1)$ rotations. According to the symmetry reduction scheme we first have to look for all possible group homomorphisms $\lambda : J \to U(1)$ up to conjugation, which of course is trivial in $U(1)$ and $J = U(1)$ in our case. Thus, the homomorphisms are classified by an integer

$n \in \mathbb{Z}$, $\lambda_n : z \mapsto z^n$ with $z \in \mathbb{C}$, $|z| = 1$. The centralizer for any value of n is $U(1)$ again. Now, the remaining information about the reduced connection can be found in the linear map $\Lambda : \mathcal{SU}(2) \to \mathcal{U}(1)$, $\Lambda(X) = \omega(\tilde{X})$. $\Lambda|_{\mathcal{J}}$ is already determined by the differential of the group homomorphism (5.0.2) $\Lambda(c\tau_3) = in/2c$, where the $1/2$ comes from the spinorial origin of the isotropy subgroup, such that for $n = 1$ the homomorphism is id. Since $U(1)$ is Abelian the right hand side of equation (5.0.3) always gives $\Lambda(X)$. Making an ansatz for $X \in \mathcal{J}^{\perp}$, $X = a\tau_1 + b\tau_2$ and for $j = \exp(c\tau_3)$ (5.0.3) yields an equation independent of n

$$\begin{aligned}(a\cos^2(\frac{c}{2}) - 2b\sin(\frac{c}{2})\cos(\frac{c}{2}) + 4a\sin^2(\frac{c}{2}))\Lambda(\tau_1) + \\ +(b\cos^2(\frac{c}{2}) + 2a\sin(\frac{c}{2})\cos(\frac{c}{2}) + 4b\sin^2(\frac{c}{2}))\Lambda(\tau_2) = a\Lambda(\tau_1) + b\Lambda(\tau_2),\end{aligned}$$

which has to be met for all values of a and b. This can only be the case if $\mathcal{J}^{\perp}$ lies in the kernel of Λ. Hence we find for the symmetric magnetic potential

$$A(x) = i\varphi(x)dx + i\frac{n}{2}\cos\theta d\phi, \tag{9.1.1}$$

where the second term stems from (6.2.1). The conjugate momentum is given by

$$\tilde{\mathcal{E}} = -ip(x)\partial_x\omega_h =: -i\tilde{q}(x)\partial_x, \tag{9.1.2}$$

where $(-i)$ denotes the generator dual to i such that the Cartan Killing metric is $k = 1$. The symplectic form reads (with e denoting the Maxwell coupling constant, i.e. the elementary charge)

$$\frac{1}{e^2}\int_{\sigma} \mathbf{d}A(x) \wedge *\mathbf{d}\mathcal{E} = \frac{4\pi}{e^2}\int_B \mathbf{d}\phi \wedge \mathbf{d}q$$

Note that the connection is not well defined on whole of the z-axis. This second term is often being referred to as the monopole connection for the following reason. If we calculate the field strength, we find

$$F = -i\frac{n}{2}\sin\theta d\theta d\phi, \tag{9.1.3}$$

which is closed but not exact, since (9.1.1) is not defined everywhere.

Let us disregard gravity for a moment and discuss spherically symmetric electro magnetism in Euclidean space. Therefore the volume element in symmetry adapted coordinates reads $\omega_h = x^2 \sin\theta dx d\theta d\phi$. From (9.1.3) we calculate the (dual) densitized magnetic vector-field

$$\tilde{\mathcal{B}} = -i\frac{n}{2x^2}\partial_x\omega_h. \tag{9.1.4}$$

The magnetic field is the radially symmetric field of a Dirac monopole of (quantized) charge $n/2$, as can be seen from calculating the divergence of the field.

$$\begin{aligned}\langle i\psi, d * \tilde{\mathcal{B}}\rangle = -i\langle d\psi, *\tilde{\mathcal{B}}\rangle = -i\int_{\mathbb{R}^3} d\psi F = \\ = -i\lim_{\varepsilon\to\infty}\int_{\mathbb{R}^3\backslash B_\varepsilon} d\psi F = -i\lim_{\varepsilon\to\infty}\int_{\mathbb{R}^3\backslash B_\varepsilon}[d(\psi F) - 0] = \\ = i\lim_{\varepsilon\to\infty}\int_{\partial\backslash B_\varepsilon}(\psi(0) + \varepsilon\partial_x\psi(0) + \mathcal{O}(\varepsilon^2))F = i\psi(0)\int_{S^2} F \\ = 4\pi\frac{n}{2}\langle\psi(x), \tilde{\delta}^{(3)}(x)\rangle\end{aligned} \tag{9.1.5}$$

where we introduced the (dual) $\mathcal{U}(1)$-valued test scalar field $i\psi$ and B_ε is a ball of radius ε around the origin. Outside B_ε the divergence of magnetic field vanishes. The additional sign in the third line stems from the orientation of the boundary $\partial(\mathbb{R}^3\backslash B_\varepsilon)$ compared to the opposite orientation of ∂B_ε. From the third line, we can see that we calculated a quantity which is proportional to the integrated first Chern class. Chern classes are topological invariants, i.e. a gauge invariant polynomials, sometimes called characteristic polynomials in F, used to classify the fiber bundle. Of course, in our $U(1)$-case the simplest gauge invariant polynomial is the curvature itself, which is also the highest Chern class in three dimensions.

In the discussions on the Dirac monopole [16] the quantization of the charge is found by defining two new gauge potentials, singular only on the upper or lower half z-axis respectively, by transforming the original potential given in (9.1.1) with $\exp(\mp in/2\phi)$ and a priori $n \in \mathbb{R}$. The gauge transformation relating these two gauge potentials is given by $h = \exp(in\phi)$. This gauge transformation is only well defined if it is single valued, i.e. $\exp(\mp in2\pi) = 1$, from which follows that n is integer. The quantization found here is a remarkable result, since the topological charge is quantized from the outset due to our reduction scheme and justifies the identification of different classes of homomorphisms λ with different topological charges.

Finally, we compute the matter part of the constraints for canonical "vacuum" Einstein-

Maxwell theory from equations (9.0.6) to (9.0.8)

$$\mathcal{G}^M = -\frac{i}{e^2}\tilde{q}' = 0, \tag{9.1.6}$$

$$\mathcal{V}^M \equiv 0, \tag{9.1.7}$$

$$\mathcal{C}^M = \frac{\omega_h}{2e^2}\Big(p^2\frac{(E^t)^2}{|E^x|} + \frac{n^2}{4|E^x|^2}\Big). \tag{9.1.8}$$

If we again neglect the gravity-matter coupling, equation (9.1.6) allows the calculation of the electric field of a point charge.

$$(px^2)' = p'x^2 + 2xp = 0$$

$$\frac{p'}{p} = \frac{-2}{x} \rightarrow \quad p = qx^{-2} \rightarrow \quad \mathcal{E} = \frac{q}{x^2}\partial_x$$

$$\langle\psi, d * \mathcal{E}\rangle = 4\pi q\langle\psi, \tilde{\delta}^{(3)}(x)\rangle. \tag{9.1.9}$$

We generalize this result to Einstein-Maxwell theory: Apart from some possibly isolated singular points on the reduced base manifold B, for example a center of symmetry that was excluded from the outset, the quantity $p\sqrt{|E^x|}E^t$ is a constant which we call the electric charge q (density). Using the charge in the expression for the elctric field, we rewrite the Maxwell Hamiltonian (9.1.8)

$$\mathcal{C}^M = \frac{E^t}{2\sqrt{|E^x|}e^2}\Big(\frac{q^2}{|E^x|} + \frac{n^2}{4|E^x|}\Big). \tag{9.1.10}$$

Chapter 10

The Reissner-Nordström solution

In this chapter the knowledge of the sections above will be used to obtain the classical spherically symmetric solution to the coupled initial value formulation of Einstein-Maxwell theory. The initial data have to meet the following constraints:

- Gauß constraints:

$$\begin{aligned}G^G[\lambda] &= \frac{1}{2G\gamma}\int_B \lambda^3(\bar{E}^{x\prime} + \bar{P}^\eta) = 0\\ G^M[\Lambda = i\alpha] &= 4\pi\int_B \alpha\tilde{q}' = 0\end{aligned}$$

- Vector constraints:

$$\begin{aligned}V^G[S] &= \frac{1}{2G\gamma}\int_B S^x[2\gamma\bar{E}^t K_t' - (\eta' + A_x)\bar{E}^{x\prime} + \eta'\mathcal{G}_3] = 0\\ V^M[S] &\equiv 0\end{aligned}$$

- Scalar constraints:

$$\begin{aligned}C[N] :=& C^G[N] + \frac{e^2}{2\pi}C^M[N] =\\ &-\frac{1}{G}\int_B N|E^x|^{-\frac{1}{2}}[\bar{E}^t K_t^2 + \frac{2}{\gamma}\bar{E}^x K_t(\eta' + A_x) + 2\bar{E}^x\Gamma_t' -\\ &-\bar{E}^t(\Gamma_t^2 - 1) - \bar{E}^t\frac{G}{|E^x|}\big(q^2 + \frac{n^2}{4}\big)] = 0\end{aligned}$$

We choose the Lagrange multipliers conveniently to simplify the equations of motion. Let us set S^x, λ^3 and α to zero. The matter part of the scalar constraint suggests the choice

$$N = \frac{|E^x|^{\frac{1}{2}}}{E^t},$$

which, of course, forces us to exclude all points in the phase space where $E^x = 0$ and $E^t = 0$, which in turn will have a certain physical interpretation.
By doing so we find the equations of motion for the momentum coefficients, where the dot denotes the derivative with respect to "time" T, i.e. the parameter of the Hamiltonian flow in phase space,

$$\dot{P}^\eta = -G\gamma \frac{\delta C[N]}{\delta \eta} = -\Big(\frac{2K_t E^x}{E^t}\Big)', \tag{10.0.1}$$

$$\dot{E}^x = -G\gamma \frac{\delta C[N]}{\delta A_x} = \frac{2K_t E^x}{E^t}, \tag{10.0.2}$$

$$\dot{E}^t = -\frac{G}{2}\frac{\delta C[N]}{\delta K_t} = K_t + \frac{(\eta' + A_x)E^x}{\gamma E^t}, \tag{10.0.3}$$

$$\dot{e}^t = -\frac{\delta C[N]}{\delta \phi} = 0, \tag{10.0.4}$$

and for the connection coefficients

$$\dot{\eta} = G\gamma \frac{\delta C[N]}{\delta P^\eta} = 0 \tag{10.0.5}$$

$$\begin{aligned}\dot{A}_x = G\gamma \frac{\delta C[N]}{\delta E^x} &= \\ &= -\frac{2K_t(\eta' + A_x)}{E^t} - \gamma \frac{E^x (E^t)''}{(E^t)^3} + \gamma \frac{3(E^x)''}{2(E^t)^2} - \gamma \frac{3(E^x)'(E^t)'}{(E^t)^3} - \\ &\quad + \gamma \frac{3E^x[(E^t)']^2}{(E^t)^4} - \gamma \frac{G}{|E^x|^2}(\theta(E^x) - \theta(-E^x) + 2E^x\delta(E^x))\big(q^2 + \frac{n^2}{4}\big)\end{aligned} \tag{10.0.6}$$

$$\dot{K}_t = \frac{G}{2}\frac{\delta C[N]}{\delta E^t} = \frac{K_t(\eta' + A_x)E^x}{\gamma (E^t)^2} - \frac{E^x(E^x)''}{2(E^t)^3} + \frac{((E^x)')^2}{4(E^t)^3} \tag{10.0.7}$$

$$\dot{\varphi} = \frac{\delta C[N]}{\delta q} = \frac{2q}{|E^x|}. \tag{10.0.8}$$

Now, we use the Gauß constraint to find $P^\eta = -(E^x)'$, which makes (10.0.1) redundant. Due to equation (10.0.5) we can replace $\dot{A}_x$ by $\gamma \dot{K}_x$ and with equation (8.1.15) we can replace $\eta' + A_x$ with γK_x everywhere. The delta function in (10.0.6) has support at the point which we excluded and we can drop this term. The combination of the Heaviside functions in (10.0.6) is simply the signum function $\mathrm{sgn}(E^x)$ which drops out of the equation. This is because – by the same argument as before – the derivative with respect to x of the signum function in the other terms in this equations has only support on the excluded points.
If we require the triads to be stationary $\dot{E}^i \equiv 0$ we find from (10.0.2) that $K_t = 0$ and then from (10.0.3) $K_x = 0$. The remaining equation, obtained from (10.0.7), reads

$$-E^x(E^x)'' + \frac{((E^x)')^2}{2} = 0 \tag{10.0.9}$$

which yields the solution $E^x = cx^2$, $c \in \mathbb{R}$. The constant c now carries the signum of E^x. Next, by the substitution $E^t = \frac{x}{\sqrt{F(x)}}$ we obtain from (10.0.6) the equation

$$-\frac{c}{2}F'' - \frac{cF'}{x} + \frac{Q^2 G \,\mathrm{sgn}(c)}{c^2 x^4} = 0.$$
$$Q^2 = q^2 + \frac{n^2}{4} \tag{10.0.10}$$

Replacing $F'(x)$ by $H(x)$ gives

$$-\frac{1}{2}H'x^4 - Hx^3 + \frac{Q^2 G}{|c|^3} = 0,$$

where we solve the homogeneous equation via a the separation of variables

$$\frac{1}{2}\frac{H'}{H} + \frac{1}{x} = 0$$

We find

$$H(x) = \frac{k}{x^2}$$

Now, we use the method of variation of constants to obtain the full solution

$$H(x) = \frac{\tilde{k}}{x^2} - \frac{2Q^2 G}{|c|^3 x^3},$$

or

$$F(x) = \left(b - \frac{\tilde{k}}{x} + \frac{Q^2 G}{|c|^3 x^2}\right) \tag{10.0.11}$$

respectively. Finally, we can write the spatial metric as

$$\begin{aligned} h_{ab} &= \frac{1}{|c|\left(b - \frac{\tilde{k}}{x} + \frac{Q^2 G}{|c|^3 x^2}\right)} dx_a dx_b + |c| x^2 (d\Omega^2_{S^2})_{ab} = \\ &= \frac{|c|}{b|c|^2 - \frac{\tilde{k}|c|^2}{x} + \frac{Q^2 G}{|c| x^2}} dx_a dx_b + |c| x^2 (d\Omega^2_{S^2})_{ab} \end{aligned} \tag{10.0.12}$$

We define $r := \sqrt{|c|}x$ and $2MG = \tilde{k}|c|^{5/2}$. Furthermore, we require the metric to be asymptotically flat, i.e. $\lim_{r\to\infty} h_{ab} = \delta_{ab}$.

$$h_{ab} = \frac{1}{1 - \frac{2MG}{r} + \frac{Q^2 G}{r^2}} dr_a dr_b + r^2 (d\Omega^2_{S^2})_{ab}. \tag{10.0.13}$$

We also rescale the evolution parameter T by $T/\sqrt{|c|} = t$ to obtain the full Lorentzian metric

$$g_{ab} = -\Big(1 - \frac{2MG}{r} + \frac{\big(q^2 + \frac{n^2}{4}\big)G}{r^2}\Big)dt_a dt_b + \frac{1}{1 - \frac{2MG}{r} + \frac{\big(q^2+\frac{n^2}{4}\big)G}{r^2}} dr_a dr_b + r^2 d^2\Omega_{S^2}, \tag{10.0.14}$$

which is exactly the well known Reissner-Nordström solution. Note that we could have also used the Newtonian limit to determine $\tilde{k}|c|^2$. The classical singulatrity occurs at $r = 0$, while there exist two horizons at $r_\pm = GM \pm G\sqrt{M^2 - q^2}$.
For densitized triad

$$E^x = \mathrm{sgn}(r) r^2 \tag{10.0.15}$$

$$E^t = \sqrt{|c|}\sqrt{\frac{r^4}{r^2 - 2MGr + G(q^2 + \frac{n^2}{4})}} \tag{10.0.16}$$

$$\tilde{E} = \Big(\mathrm{sgn}(r) r^2 \partial_r \Lambda^3_\eta + \sqrt{\frac{r^4}{r^2 - 2MGr + G(q^2 + \frac{n^2}{4})}}(\Lambda^1_\eta \partial_\theta + \Lambda^2_\eta \frac{1}{\sin\theta}\partial_\phi)\Big) dr d\theta \sin\theta d\phi \tag{10.0.17}$$

Note that the local frame is smooth everywhere and does not diverge at the singularity, it only degenerates at this location and changes orientation.

Chapter 11

Quantum theory

Finally, let us pass to the quantization of the classical theories presented in the previous chapters. First, we introduce further notions known from fiber bundle theory – parallel transport – and discuss its physical significance by means of the Aharonov effect. Furthermore we present the quantization scheme of LQG and carry out the quantization of several classical quantities.

11.1 Parallel transport

Let $x(t)$, $0 \leq t \leq 1$, be a *piecewise differentiable* curve of class C^1 in B and $u(t)$ its horizontal lift and let us denote the tangent vectors to curves by a dot, e.g. $\dot{u}(t)$. Now consider a general lift $w(t)$ of the curve $x(t)$ with $\pi w(t) = x(t)$ and $w(0) = u(0)$, which must be of the form $u(t) = w(t)a(t)$, where $a(0) = e$. We look for a condition for $a(t)$ to make $u(t)$ a horizontal curve. Due to the Leibniz formula we have

$$\begin{aligned} \dot{u}(t) &= \dot{w}(t)a(t) + w(t)\dot{a}(t) \\ \omega(\dot{u}(t)) &= Ad_{a(t)^{-1}}\omega(\dot{w}(t)) + a(t)^{-1}\dot{a}(t) = 0 \end{aligned}$$

where the last term of the second equation is a curve $Y(t)$ in $\mathcal{G}$. The condition for $a(t)$ making $u(t)$ a horizontal curve then is

$$\dot{a}(t) = -\omega(\dot{w}(t))a(t) \tag{11.1.1}$$

for every t.

Now we define the parallel displacement along the curve $x(t)$ as the map $\tau : \pi^{-1}(x(0)) \to \pi^{-1}(x(1))$. The horizontal lift starts at $u(0) \in \pi^{-1}(x(0))$ and ends at $u(1) \in \pi^{-1}(x(1))$. The mapping is an isomorphism since it commutes with the right action on P which leaves horizontal vectors invariant. In a local trivialization the parallel transport along x_t with the tangent vector-field v_t is given defined by the initial value problem

$$Ad_{h_t}\mathcal{A}(v_t) - \dot{h}_t h_t^{-1} = 0,$$

$$h_t\mathcal{A}(v_t) = \dot{h}_t. \tag{11.1.2}$$

$$h_0 = e \tag{11.1.3}$$

by virtue of (4.2.2). The G element h_t is called the parallel transport along x_t. We denote the parallel transport along the full curve c with respect to the gauge potential A by $h_c(A)$.

Finally we can define a equivalence relation for points on P, which can be joined by a horizontal curve, $u \sim v$.

11.1.1 Solution of the parallel transport equation

In the following we will show that there exists a unique solution of equation (11.1.2) and demonstrate how we can construct it. For this purpose, we require that h_t is are continuous on $[0,1] \subset \mathbb{R}$, hence it is bounded on that intervall with respect to some suitable norm. Furthermore, we restrict ourselves to subgroups of $GL(n,\mathbb{R})$, so that we can choose any submultiplicative matrix norm, e.g. the norm $\|\cdot\|_M$ induced by the Hilbert-Schmidt inner product $\langle A, B\rangle = \mathrm{tr}(A^\dagger B)$. The set of continuous bounded functions $h_t \in \mathcal{C}([0,1], GL(n,\mathbb{R}), \|\cdot\|_\infty = \sup_t \|\cdot\|_M)$ is a complete metric space. Choose once and for all one gauge potential A on P.

Integration of (11.1.2) yields the fixed point problem

$$h_t = e + \int_0^t dt' h_{t'} A(v_{t'}) =: Th_t. \tag{11.1.4}$$

The integrand in (11.1.4) is Lipschitz continuous, i.e. $\exists\, L > 0 \in \mathbb{R}$ s.t.:

$$\begin{aligned}\|(h_t - h'_t)A(v_t)\|_M &\leqslant \|(h_t - h'_t)\|_M \cdot \|A(v_t)\|_M \\ \leqslant \sup_t \left\{\|A(v_t)\|_M\right\} &\cdot \|(h_t - h'_t)\|_M = L\|(h_t - h'_t)\|_M.\end{aligned} \tag{11.1.5}$$

In particular, it is continuous and hence is also bounded, i.e. $\|h_t A(v_t)\|_M \leqslant \Lambda$, for some fixed number Λ, $\forall\, t \in [0,1]$. Therefore, we can show that the operator is indeed a well defined map $T : \mathcal{C}([0,1], GL(n,\mathbb{R})) \to \mathcal{C}([0,1], GL(n,\mathbb{R}))$, i.e. it is continuous in t

$$\begin{aligned}\|Th_t - Th_{\tilde{t}}\|_M &= \|\int_0^t dt' h_{t'} A(v_{t'}) - \int_0^{\tilde{t}} dt' h_{t'} A(v_{t'})\|_M \\ &= \|\int_{\tilde{t}}^t dt' h_{t'} A(v_{t'})\|_M \leqslant \Lambda|t - \tilde{t}|\end{aligned} \tag{11.1.6}$$

and hence also bounded. The map T is also Lipschitz continuous with respect to the uniform norm

$$\begin{aligned}
\|Th - Th'\|_\infty &= \| \int_0^t dt'(h_{t'} - h'_{t'})A(v_{t'})\|_\infty \\
&\leqslant \sup_t \big\{ \int_0^t dt' \|(h_{t'} - h'_{t'})A(v_{t'})\|_M \big\} \\
\leqslant \sup_t \big\{ \int_0^t dt' L \|h_{t'} - h'_{t'}\|_M \big\} &\leqslant \sup_t \big\{ Lt \|h_{t'} - h'_{t'}\|_M \big\} \\
&= L \cdot 1 \|h - h'\|_\infty
\end{aligned}$$

Then we prove by induction that $\|T^n h - T^n h'\|_\infty = L^n/n! \|h - h'\|_\infty$. We show that it is true for $n = 2$[1]

$$\begin{aligned}
\|T^2 h - T^2 h'\|_\infty &\leqslant \| \int_0^t dt'(Th_{t'} - Th'_{t'})A(v_{t'})\|_\infty \\
&\leqslant \sup_t \big\{ \int_0^t dt' \|(Th_{t'} - Th'_{t'})A(v_{t'})\|_M \big\} \\
&\leqslant \sup_t \big\{ \int_0^t dt' L \|Th_{t'} - Th'_{t'}\|_M \big\} \\
&\leqslant \sup_t \big\{ \int_0^t dt' L^2 t' \|h_{t'} - h'_{t'}\|_M \big\} \\
&\leqslant \sup_t \big\{ \frac{(Lt)^2}{2} \|h_{t'} - h'_{t'}\|_M \big\} \\
&= \frac{L^2}{2} \|h - h'\|_\infty
\end{aligned}$$

and the case $n + 1$ is shown similarily using the induction hypothesis. This qualifies T^N being a contraction for a sufficiently high value of N, since[2]

$$\Big| \frac{L^{n+1}/(n+1)!}{L^n/n!} \Big| = \Big| \frac{L}{n+1} \Big| < 1, \quad \text{for} \quad n + 1 > L \tag{11.1.7}$$

Next we prove an extension of Banach's fixed point theorem. Consider the sequence $\{h^{(n)}\}_{n \in \mathbb{N}}$ with $h^{(n+1)} = Th^{(n)}$ and $h^{(0)} = e$. We use

$$\|h^{(n+1)} - h^{(n)}\|_\infty = \|T^n h^{(1)} - T^n h^{(0)}\|_\infty \leqslant \frac{L^n}{n!} \|h^{(1)} - h^{(0)}\|_\infty,$$

[1] We chose the case $n = 2$ also to motivate that formula.

[2] This can also be seen from the fact that $L^n/n!$ is the n-th term in the Taylor series of the exponential function $\exp(L)$.

to show that this sequence is Cauchy:

$$\begin{aligned}\|h^{(m)} - h^{(n)}\|_\infty &\leqslant \|h^{(m)} - h^{(m-1)}\|_\infty + \cdots + \|h^{(n+1)} - h^{(n)}\|_\infty \\ &\leqslant \|h^{(1)} - h^{(0)}\|_\infty \sum_{k=n}^{m-1} \frac{L^k}{k!} = \|h^{(1)} - h^{(0)}\|_\infty \frac{L^n}{n!} \sum_{k=0}^{m-n-1} \frac{L^k}{k!} \\ &\leqslant \|h^{(1)} - h^{(0)}\|_\infty \frac{L^n}{n!} \sum_{i=0}^{\infty} \frac{L^k}{k!} =\leqslant \|h^{(1)} - h^{(0)}\|_\infty \frac{L^n}{n!} e^L, \end{aligned} \tag{11.1.8}$$

hence for any $\varepsilon > 0 \in \mathbb{R}\ \exists\ M \in \mathbb{N}$ given by the equation $\|h^{(1)} - h^{(0)}\|_\infty \frac{L^M}{M!} e^L < \varepsilon$, such that for $\forall m, n > M$ $\|h^{(m)} - h^{(n)}\|_\infty < \varepsilon$ holds.
Since $\mathcal{C}([0,1], GL(n,\mathbb{R}), \|\cdot\|_W)$ is complete, the sequence cornverges and we denote the limit by h_t^*. We show that h_t^* is the unique fixed point of T:

- Fixed point:

$$\lim_{n\to\infty} \|h^{(n+1)} - Th^*\|_\infty = \lim_{n\to\infty} \|Th^{(n)} - Th^*\|_\infty \leqslant \lim_{n\to\infty} L\|h^{(n)} - h^*\|_\infty = 0$$

- Uniqueness: Suppose $\exists\ \tilde{h}^* \neq h^*$, s.t. $T\tilde{h}^* = \tilde{h}^*$, then we obtain the contradiction

$$\begin{aligned} &\|\tilde{h}^* - h^*\|_\infty = \|T\tilde{h}^* - Th^*\|_\infty \\ &= \|T^N\tilde{h}^* - T^N h^*\|_\infty \leqslant \frac{L^N}{N!}\|\tilde{h}^* - h^*\|_\infty \to_{N\to\infty} 0 \end{aligned}$$

Finally, we construct the solution employing the iteration utilized for the proof above.

$$h_t = e + \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n A(v_{t_n}) \cdots A(v_{t_2}) A(v_{t_1}) \tag{11.1.9}$$

Based on the following observation with respect to double integrals in the sum above we will rewrite (11.1.9) as a so-called path ordered exponential. Using the Heaviside step function $\Theta(t - t')$ we rewrite the double integral

$$\begin{aligned} &\int_0^t dt_1 \int_0^{t_1} dt_2 A(v_{t_2}) A(v_{t_1}) \\ &= \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 [A(v_{t_2})A(v_{t_1})\Theta(t_1 - t_2) + A(v_{t_1})A(v_{t_2})\Theta(t_2 - t_1)] \\ &= \frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 \mathcal{P}[A(v_{t_2})A(v_{t_1})] \end{aligned} \tag{11.1.10}$$

and define the path ordering, i.e. we order the connection with the lowest value of the curve parameter to the left. This we generalize to all multi integrals and obtain

$$h_t = \mathcal{P}e^{\int_c A} := \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^t dt_1 \int_0^t dt_2 \cdots \int_0^t dt_n \mathcal{P}[A(v_{t_n}) \cdots A(v_{t_2})A(v_{t_1})] \tag{11.1.11}$$

11.1.2 Paths and graphs

We define the beginning point, final point and range of a (continuous) curve $c : [0,1] \hookrightarrow B$ (where c is an embedding) respectively by $b(c) := c(t)$, $f(c) := c(1)$ and $r(c) := c([0,1])$ [11]. We can now introduce the composition and inversion of curves.

$$\begin{aligned} c_1 \circ c_2 &:= \begin{cases} c_1(2t), & t \in [0,1/2] \\ c_2(2t-1), & t \in [1/2,1] \end{cases} \quad \text{if} \;\; f(c_1) = b(c_2) \\ c^{-1}(t) &:= c(1-t) \end{aligned} \tag{11.1.12}$$

The composition $c(t) \circ c^{-1}(t)$ is called a retracing, but one has to keep in mind that this is not a simply single point.
As can be seen from the defining equation of the parallel transport or equivalently from (11.1.11), we have

$$h_{c_1 \circ c_2}(A) = h_{c_1}(A) h_{c_2}(A), \tag{11.1.13}$$

$$h_{c^{-1}}(A) = h_c(A)^{-1}. \tag{11.1.14}$$

Furthermore, it will be shown later that the parallel transport is invariant under reparametrizations of curves.
This suggests to introduce an equivalence relation of curves c (with the same initial and final points) modulo a finite number of retracings and reparametrizations. The equivalence classes will be called paths $[c] = p_c \in \mathcal{P}$, where the operations defined above introduce a so-called groupoid structure on $\mathcal{P}$, therefore it is natural to consider category theory. A groupoid structure differs from a group structure by not having a natural identity element and not defining composition for all elements. The category under consideration here is the base manifold, which is a collection (in our case it is even a set) of objects, namely the points together with morphisms, the paths, for each ordered pair of points, which we denote by $\hom(x,y)$. The compostion defined for the morphisms has to be associative, which is obviously the case. Beyond identities $id_x \in \hom(x,x)$ need to exist, which in our case are the trivial paths $p \circ p^{-1}$. Since we also defined inversion, all the morphisms are automatically isomorphisms. Note that the closed paths with respect to a certain base point form a group called the hoop group.
As we saw from the definition of the parallel transport we need curves which are piecewise

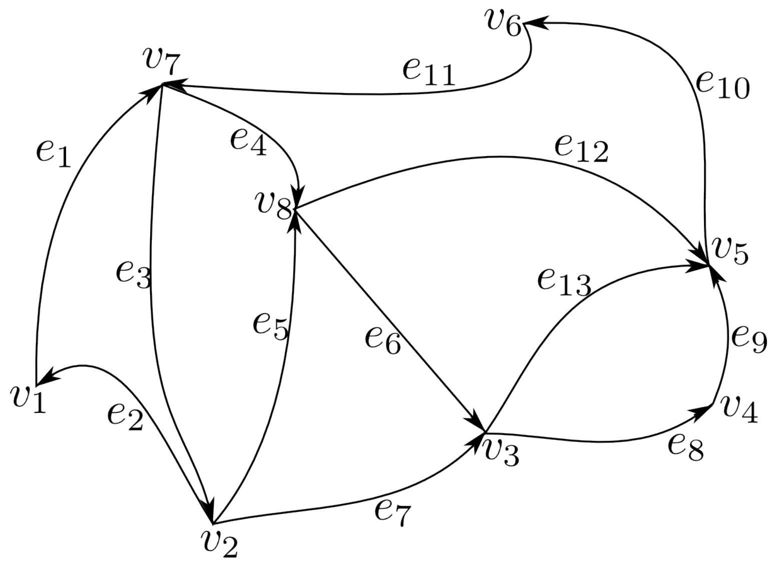


Figure 11.1: *An example of a graph.*

differentiable. For later convenience, we will restrict the set of curves from class C^1 to the *semianalytic* class in the following. Roughly speaking, these are curves which are piecewice analytic, i.e. a compostion of analytic curves for which the intersection points of the analytic pieces they are at least C^1. Compostion of such curves could lead out of these class since the composite object is a priori only C^0 at the intersection point. Therefore, we introduce the notion of edges, which are paths with at least one entirely semianalytic representative.

The union of ranges of finitely many edges $\{e_1, \ldots, e_n\}$, which intersect at most at their endpoints [3] is called a graph $\gamma = \cup_i = 1^n r(e_i)$ with the sets $E(\gamma) = \{e_1, \ldots, e_n\}$ and $V(\gamma) = \{b(e), f(e) | e \in E(\gamma)\}$, where the latter is called the vertex set. The set of graphs is denoted by Γ.

We equip Γ with an order relation denoted by $\prec$ which makes Γ a directed set. We say that $\gamma \prec \gamma'$ iff every edge $e \in E(\gamma)$ is a finite composition of the $e' \in E(\gamma')$ and their inverses. For any pair γ, γ' one can find a γ'' such that $\gamma, \gamma' \prec \gamma''$. If γ and γ' are disjoint then simply $\gamma'' = \gamma \cup \gamma'$. The same is true if the two graphs intersect. The piecewise analycity requirement is crucial here, since the edges can either intersect in finitely many points or the analytic pieces overlap. In such a case we break the edges of the union graph into smaller pieces.

[3]The set of such edges is said to be independent.

11.1.3 Gauge transformations and variations

Under a local gauge transformation $\mathcal{A} \to {}^{g_t}\mathcal{A}$ we have

$$\begin{aligned}
{}^{g_t}\dot{h}_t &= {}^{g_t}h_t\ {}^{g_t}\mathcal{A} = {}^{g_t}h_t(g_t\mathcal{A}g_t^{-1} + g_t(g_t^{-1})^{\cdot}) = \\
&= {}^{g_t}h_t(g_t h_t^{-1} h_t \mathcal{A} g_t^{-1} + g_t(g_t^{-1})^{\cdot}) = \\
&= {}^{g_t}h_t(g_t h_t^{-1} \dot{h}_t g_t^{-1} + g_t(g_t^{-1})^{\cdot}) \\
{}^{g_t}\ h_t^{-1\,g_t}\dot{h}_t &= (h_t g_t^{-1})^{-1}(h_t g_t^{-1})^{\cdot} - g_t(g_t^{-1})^{\cdot} + g_t(g_t^{-1})^{\cdot} \\
{}^{g_t}h_t^{-1}\ {}^{g_t}\dot{h}_t &= (h_t g_t^{-1})^{-1}(h_t g_t^{-1})^{\cdot}.
\end{aligned} \tag{11.1.15}$$

Since ${}^{g_0}h_0 = h_0 = \mathbb{1}$ the transformation property of the parallel transport is

$${}^{g_t}h_t = g_0 h_t g_t^{-1}. \tag{11.1.16}$$

Obviously, for matrix groups the so-called Wilson line $\mathrm{tr}(h_c(A))$, where c is a loop, is invariant under gauge transformations.
The variation of the parallel propagator can be calculated via its defining equation (11.1.2)

$$\delta\dot{h}_t - \delta h_t \mathcal{A}(v_t) = h_t \delta\mathcal{A}(v_t).$$

This equation can be seen as an inhomogeneous first order differential equation in the variable δh since the derivation with respect to the curve parameter and functional variation commute. The initial value is $\delta h_0 = 0$. The homogeneous part of the equation

$$(\delta h_t)^{-1}(\delta h_t)^{\cdot} = h_t^{-1}\dot{h}_t.$$

is solved by $\delta h = Ch$. Now, we find the following solution of the inhomogeneous equation via the method of variations of the constant

$$\begin{aligned}
\dot{C} &= h_t \delta\mathcal{A}(v_t) h_t^{-1} \\
\delta h_t &= \int_0^t h_{t'}\delta\mathcal{A}(v_{t'})h_{t'}^{-1} dt' h_t + C_0 \\
\delta h_0 &= 0 \quad \to \quad C_0 = 0
\end{aligned}$$

Using the composition and inversion properties of the parallel transport we find

$$\delta h_t = \int_0^t h_{t'}\delta\mathcal{A}(v_{t'})h_{t-t'}dt'$$

For the variation w.r.t. the connection the variation of the parallel transport along the full curve $h_1 := h(c)$ reads

$$\frac{\delta h(c)}{\delta \mathcal{A}_a^i(y)} = \int_0^t dt' h_{t'} \tau_i v_{t'}^a h_{1-t'} \delta^{(3)}(x(t'), y) \tag{11.1.17}$$

The variation of the path is only slightly more difficult. Here we use the notation $v_t = \dot{x}(t)$ and $h_{t-t'} = h(x(t'), x(t))$ denoting the tangent vector-field to the curve and the parallel transport from the point $x(t')$ to the point $x(t)$. Furthermore, we write $\mathcal{A}(v_t) = \dot{x} \lrcorner \mathcal{A}(x(t))$, and $(\dot{x} \lrcorner d)\mathcal{A}(x(t)) =: \dot{\mathcal{A}}(x(t))$

$$\delta_x h(x(t)) = \int_0^t h(x_0, x(t'))[\delta\dot{x} \lrcorner \mathcal{A}(x(t')) + \dot{x} \lrcorner (\delta x \lrcorner d)\mathcal{A}(x(t'))]h(x(t'), x(t))dt' \tag{11.1.18}$$

Next we focus on the term in parenthesis

$$\begin{aligned}
&\delta\dot{x} \lrcorner \mathcal{A}(x(t')) + \dot{x} \lrcorner (\delta x \lrcorner d)A(x(t')) = \\
&= (\delta x \lrcorner \mathcal{A}(x(t')))^{\cdot} - \delta x \lrcorner \dot{\mathcal{A}}(x(t')) + \dot{x} \lrcorner (\delta x \lrcorner d)A(x(t') = \\
&= (\delta x \lrcorner \mathcal{A}(x(t')))^{\cdot} + \dot{x} \lrcorner (\delta x \lrcorner d\mathcal{A}).
\end{aligned}$$

The first term inserted in (11.1.18) gives

$$\begin{aligned}
&\int_0^t h(x_0, x(t'))(\delta x \lrcorner \mathcal{A}(x(t')))^{\cdot} h(x(t'), x(t))dt' = \\
&= \int_0^t \{[h(x_0, x(t'))\delta x \lrcorner \mathcal{A}(x(t'))h(x(t'), x(t))]^{\cdot} + h(x_0, x(t'))\dot{x} \lrcorner \delta x \lrcorner [\mathcal{A} \wedge \mathcal{A}]h(x(t'), x(t))\}dt'
\end{aligned} \tag{11.1.19}$$

using the defining equation (11.1.2) several times. In order to obtain the $\mathcal{A}^2$ part, we used the following trick: From $h(x(t'), x(t)) = h^{-1}(x(t), x(t'))$ we find $\dot{h}(x(t'), x(t)) = -\dot{x} \lrcorner \mathcal{A}(x(t'))h(x(t'), x(t))$. Collecting all pieces of the calculation yields

$$\begin{aligned}
\delta_x h(c) = &\int_0^1 h(x_0, x(t'))\dot{x} \lrcorner (\delta x \lrcorner \mathcal{F})(x(t'))h(x(t'), x_1) + \\
&+ h(c)\delta x \lrcorner \mathcal{A}(x_1) - \delta x \lrcorner \mathcal{A}(x_0)h(c),
\end{aligned} \tag{11.1.20}$$

where $\mathcal{F}$ is the field strength (local curvature, see chapter 4.3) of the gauge potential $\mathcal{A}$.

11.2 Holonomy

Next, we analyze special curves: the loop space, i.e. C^k curves in B where the starting and end point are the same. Then the parallel displacement provides an isomorphism of the fiber to itself. The set of these isomorphisms is called the holonomy group $\Phi(x)$ with reference point x and the subgroup $\Phi^0(x)$ corresponding to loops homotopic to zero is called restricted holonomy group. We understand them as subgroups of the structure group G. For a point u in the fiber $\pi^{-1}(x)$ we can write $\tau(u) = ua$, $a \in G$. Say another loop μ determines the element $b \in G$, then $\mu \circ \tau(u) = \mu(ua)$. Since parallel displacement and right action of G commutes, we have

$$\mu \circ \tau(u) = uba$$

Since the inverse and multiplication of parallel displacement is defined, the elements of G corresponding to the parallel displacements form a subgroup of G called the holonomy group $\Phi(u)$ of Γ with reference point u. The same is true in the restricted case. For two points u and $v = ua$, $a \in G$, in a fiber the holonomy groups are conjugate in G. Say $b \in \Phi(u)$, then $u \sim ub$ and $u \sim va^{-1}b$, hence, $v \sim va^{-1}ba$ and it follows that $\Phi(v) = ad_{a^{-1}}\Phi(u)$. For two points u and v in different fibers joined by a horizontal curve we have $u \sim v$, which implies $ub \sim vb$. Since the equivalence relation is transitive, we conclude $u \sim ub$ iff $v \sim vb$ and therefore $\Phi(u) = \Phi(v)$. If B is connected, i.e. all pairs of points in B can be joined by a differentiable curve, all holonomy groups are isomorphic to each other, since for all pairs of lifts u and v in P there exists an element a in G such that $v \sim ua$. Furthermore, if B is paracompact[4] one can show [12] that $\Phi(u)$ is a Lie subgroup of G with identity component $\Phi^0(u)$.

11.3 Variations of the curve and the Ambrose Singer theorem

We can consider a reparametrization of the curve as a variation with δx and $\dot{x}$ collinear and $\delta x(x_0) = \delta x(x_1) = 0$. Then we can see that the parallel transport is invariant with respect to the reparametrization of the curve. If we choose an arbitrary variation of the path and keep the initial and final points fixed, then as expected curvature comes into play.

Let us now consider the expansion of the parallel transport and a holonomy with respect to a "small" curve or loop respectively. For this purpose we consider a one parameter family of curves and loops (c^ε and α^ε). Furthermore, we require $x^0(t) = x_0$ for all t, i.e. for vanishing ε our curves shrink to a single point and then also the tangent vector shall vanish $\dot{x}^0(t) = 0$. For the variation of the curve we can write

$$\delta x = \frac{\partial x^\varepsilon(t)}{\partial \varepsilon}\varepsilon$$

[4]as it is the case for every metrizable space.

Accordingly we trade the formula (11.1.20) for

$$\begin{aligned}\frac{\partial h(c^\varepsilon)}{\partial\varepsilon} &= \int_0^1 h(x_0^\varepsilon, x^\varepsilon(t'))\dot{x^\varepsilon}\lrcorner(\frac{\partial x^\varepsilon(t)}{\partial\varepsilon}\lrcorner\mathcal{F})(x^\varepsilon(t'))h(x^\varepsilon(t'), x_1^\varepsilon)+ \\ &+ h(c)\frac{\partial x^\varepsilon(t)}{\partial\varepsilon}\lrcorner\mathcal{A}(x_1^\varepsilon) - \frac{\partial x^\varepsilon(t)}{\partial\varepsilon}\lrcorner\mathcal{A}(x_0^\varepsilon)h(c)\end{aligned} \tag{11.3.1}$$

Next, we expand the parallel transport such that we keep the initial point of the curve fixed, i.e. $b(x^\varepsilon(t)) = x_0$ for all ε, and vary the length of the curve, i.e. we shift the end point of the curve along the curves tangent. Then we obtain

$$\begin{aligned}h(c^\varepsilon) &= 1 + \varepsilon\frac{\partial h(c^\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0} + \mathcal{O}(\varepsilon^2) = \\ &= 1 + \varepsilon\frac{\partial x^\varepsilon(t)}{\partial\varepsilon}\lrcorner\mathcal{A}(x_1^\varepsilon)\Big|_{\varepsilon=0} + \mathcal{O}(\varepsilon^2),\end{aligned} \tag{11.3.2}$$

In the calculation the second boundary term in (11.3.1) vanishes, since the variation of the curve vanishes in the limit.

The variation of the holonomy reads

$$\begin{aligned}\frac{\partial h(\alpha^\varepsilon)}{\partial\varepsilon} &= \int_0^1 h(x_0^\varepsilon, x^\varepsilon(t'))\dot{x^\varepsilon}\lrcorner(\frac{\partial x^\varepsilon(t)}{\partial\varepsilon}\lrcorner\mathcal{F})(x^\varepsilon(t'))h(x^\varepsilon(t'), x_0^\varepsilon)+ \\ &+ [h(\alpha^\varepsilon), \frac{\partial x^\varepsilon(t)}{\partial\varepsilon}\lrcorner\mathcal{A}(x_0^\varepsilon)]\end{aligned}$$

Expanding a holonomy

$$h(\alpha^\varepsilon) = 1 + \varepsilon\frac{\partial h(\alpha^\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0} + \frac{\varepsilon^2}{2}\frac{\partial^2 h(\alpha^\varepsilon)}{\partial\varepsilon^2}\Big|_{\varepsilon=0} + \mathcal{O}(\varepsilon^3) \tag{11.3.3}$$

all terms to first order in ε vanish, since in the curvature term the tangent vector is the zero vector, while the boundary commutator vanishes up to third order. Finally, we obtain

$$h(\alpha^\varepsilon) = 1 + \frac{\varepsilon^2}{2}\Big(\frac{\partial\dot{x}^\varepsilon}{\partial\varepsilon}\Big|_{\varepsilon=0}\lrcorner\frac{\partial x^\varepsilon}{\partial\varepsilon}\Big|_{\varepsilon=0}\lrcorner\mathcal{F}\Big)(x_0) + \mathcal{O}(\varepsilon^3) \tag{11.3.4}$$

This is essentially the core of the Ambrose-Singer theorem: Roughly speaking, the Lie algebra of the holonomy group $\Phi(u)$, where $u \in P(M,G)$ is some refence point, is spanned by the elements $\Omega_v(X,Y)$ where Ω is the curvature 2-form and X, Y are arbitrary horizontal vectors at $v \in P$, $v \sim u$.

In the following these considerations are illustrated with some examples.

11.3.1 Examples for variations of the holonomy along special paths

First consider a loop which in a coordinate neighborhood of a point M can be parametrized as

$$x^\varepsilon(t) = \varepsilon(\cos(2\pi t)u + \sin(2\pi t)v)$$

where u, v, are some arbitrary orthogonal unit vectors at x_0. Then (11.3.4) yields up to second order

$$h(\alpha^\varepsilon) \approx 1 + Ar^\varepsilon v \lrcorner u \lrcorner \mathcal{F}(x_0),$$

where we used the abbreviations $Ar^\varepsilon = \varepsilon^2 \pi$ (the coordinate area bounded the loop).
Finally, we present a loop most frequently used in loop quantum gravity: the triangle. In a coordiante neighborhood we parametrize the triangle with

$$x^\varepsilon(t) = x_0 + \varepsilon \begin{cases} 3tu, & t \in [0, 1/3] \\ u + (3t-1)(v-u), & t \in [1/3, 2/3] \\ (3-3t)v, & t \in [2/3, 1] \end{cases}$$

The only nonvanishing contributions are coming from the second egde, where the variation vector of the curve and the variation of the tangent vector are not colinear.

$$\begin{aligned} \left.\frac{\partial \dot{x}^\varepsilon(t)}{\partial \varepsilon}\right|_{\varepsilon=0} &= 3(v-u) \\ \left.\frac{\partial x^\varepsilon(t)}{\partial \varepsilon}\right|_{\varepsilon=0} &= u + (3t-1)(v-u) \end{aligned} \tag{11.3.5}$$

Again we find

$$h(\alpha^\varepsilon) \approx 1 + Ar^\varepsilon v \lrcorner u \lrcorner \mathcal{F}(x_0), \tag{11.3.6}$$

with $Ar^\varepsilon = \varepsilon^2/2$ in this case.

11.4 The Aharonov-Bohm effect

This subsection will show that fiber bundle theory provides the most complete picture in which the physics of Yang-Mills theories can be described and understood. The Aharonov-Bohm effect will serve as a guidance.
Consider the following gedankenexperiment: We perform the double slit experiment with (positively) charged particles in non-relativistic quantum mechanics and observe the typ-

ical interference fringes due to the superposition of the coherent wave function ψ_1 and ψ_2 originateing in slit 1 and slit 2, respectively. In the next step we introduce a infinitesimal thin and infinitely long solenoid along the z-axis behind the wall between the two slits. In this idealized situation the magnetic field will be completely confined within the solenoid. The stationary magnetic flux through an arbitrary surface transversal to the solenoid will be denoted by φ_M, then the magnetic field strength is $F = dA = \varphi_M \delta^{(2)}(x,y) dx \wedge dy$ which implies a singular gauge potential.

Denote by $\tau = \tau_z dz$ a test 1-form of compact support and by Z_ϵ a cylindrical region around the z-axis, then

$$\int_{\mathbb{R}^3} \varphi_M \tau_z(0,0,z) dz =< F, \tau >= \int_{\mathbb{R}^3} \varphi_M \delta^{(2)}(x,y) dx \wedge dy \wedge \tau_z(x,y,z) dz =< A, d\tau >= \\ = \lim_{\epsilon \longrightarrow 0} \int_{\mathbb{R}^3 \setminus Z_\epsilon} A \wedge d\tau = \lim_{\epsilon \longrightarrow 0} \int_{\mathbb{R}^3 \setminus Z_\epsilon} -d(A \wedge \tau) + dA \wedge \tau = \lim_{\epsilon \longrightarrow 0} - \int_{\partial Z_\epsilon} -A \wedge \tau. \tag{11.4.1}$$

Using cylindrical symmetry and symmetry adapted coordinates and the Taylor expansion of the test 1-form $\tau(\epsilon e_\rho, z) = \tau(0,z) + \mathcal{O}(\epsilon)$, we find $A = \frac{\varphi_M}{2\pi} d\phi$, which in Cartesian coordinates reads $A = \frac{\varphi_M}{2\pi(x^2+y^2)}(xdy - ydx)$. We can regard the space that our experiment takes place in as the doubly connected manifold $\Sigma \cong \mathbb{R}^2 \times S^1$ with first homotopy group $\pi_1(\Sigma) = \mathbb{Z}$, i.e. the set of winding numbers of loops around the solenoid. Outside the solenoid the gauge potential is closed, and locally we can write $A = d\Lambda$, where $\Lambda = \frac{\varphi_M}{2\pi}\phi$, which is a multi-valued function. Thus, A is a gauge transformed of $A = 0$ with the gauge group element $g = \exp[-i\Lambda]$.

Now, let us consider the principal $U(1)$-bundle over Σ and its associated complex line bundle. A section ψ of this line bundle is interpreted as the wave function describing a charged particle, which is parallel transported along a path p from a to b in the base space according to the rules derived above (In the caclulation we use $-i$ for the generator of $\mathcal{U}(1)$ with the minus sign for the negatively charged electrons, $\alpha = e/(\hbar c)$ and ψ denotes the wave function in case of a vanishing magnetic flux)

$$\psi^{\text{trans}}(b) = \exp\big[\int_p -i\alpha A\big]\psi(b) \tag{11.4.2}$$

The wave function transported along a loop γ in the base space, winding around the solenoid n-times, is given with respect to the wave function for vanishing magnetic flux by the expression

$$\psi^{\text{trans}}(x_0) = \exp\big[\oint_\gamma -i\alpha A\big]\psi(x_0) = \exp\big[-i\alpha\Lambda\big|_{\phi=0}^{\phi=2\pi n}\big]\psi(x_0) = \\ = \exp\big[-in\alpha\varphi_M\big]\psi(x_0). \tag{11.4.3}$$

Thus, a loop around the solenoid lifted to the bundle does not close and the holonomy group is given by the set of elements $\Phi(p) = \{\exp\big[-in\Phi_M\big]\}$, with $x_0 = \pi(p)$ being the base point of the loops.
In our experiment the wavefunction at the point x_0 at the detector is given by the superposition of the wavefunctions parallel transported along a curve through slit 1 and one through slit 2:

$$\begin{aligned} \psi_1^{\text{trans}}(x_0) + \psi_2^{\text{trans}}(x_0) = \\ = \exp\big[\int_1 -i\alpha A\big]\psi_1(x_0) + \exp\big[\int_2 -i\alpha A\big]\psi_2(x_0) = \\ = \exp\big[\int_2 -i\alpha A\big]\big(\exp\big[\oint_\gamma -i\alpha A\big]\psi_1(x_0) + \psi_2(x_0)\big) \end{aligned} \tag{11.4.4}$$

Hence, the interference pattern will be influenced by the phase factor [30]

$$\exp\big[-in\alpha\varphi_M\big]. \tag{11.4.5}$$

Note that the parallel transport, for which the name non-integrable phase factor was coined in [30], is not observable – only the holonomy, i.e. the *relative* phase factor, is. Classically, only the field strength could have an measureable effect governed by the Lorentz force. For a quantum mechanical particle there is a measureable effect, although the particles never enter the region where the field strength is nonvanishing.
The main conclusion of [30] is, "[w]hat provides a complete description [of electromagnetism] that is neither too much nor too little is the phase factor", while the field strength underdescribes electromagnetism and the gauge potential overdescribes it.
The Aharonov-Bohm effect has been confirmed several times. A broad overview on the experiments and also on the long standing debates about the theoretical explanation can be found in [31]. A. Tonomura performed experiments of high precesion which are in principle designed like our Gedanken experiment above. He used the at that time newly developed technique called electron holography.

11.5 Elementary variables - the Poisson *-algebra

If we take a look at the commutation relation (8.0.2), we can see that it is distributional. In order to circumvent this difficulty one usually smears the coordinates of phase space with test fields of compact support. In the sections discussing parallel transport (ch. 11.1) we showed that the connection couples to a curve in a natural way without using a metric and/or smearing fields, which might have awkward transformation properties under gauge transformation, yielding a well defined operator. Of course, this smearing is distributional. Since we are interested in gauge invariant information, i.e. physical

information, the transformation rule (11.1.16) behaves nicely in contrast to the inhomogeneous transformation (4.2.2) of the connection itself. From a physically motivated point of view, the parallel transport along a closed curve, the holonomy carries exactly the relevant physical information as the Aharonov-Bohm effect reveals 11.4. In the procedure of quantizing a classical theory, we will be interested in square integrable function on the configuration space. In our case this is the space of smooth connections, which is infinite dimensional. At this point the question arises of how to build an infinite dimensional integration theory. Following the ideas of Kolmogorov, one builds such a theory from a finite dimensional one [32]. The edges and graphs will serve as probes, extracting a small amount of information of the connection.
An analogous *natural* smearing has to be done with the densitized triad.

$$\tilde{E}_i = \omega_h e_i.$$

We define the operation $*$ as

$$*\tilde{E}_i = e_i \lrcorner \omega_h$$

which is a Lie algebra valued 2-form that can be naturally integrated over a surface $\mathcal{S}$

$$E_i(\mathcal{S}) = \int_{\mathcal{S}} *(\tilde{E}_i^a). \tag{11.5.1}$$

It is known as the gravitational electric flux through S. This particular smearing has important consequences: It allows for calculating a well defined Poisson bracket

$$-\{E_i(\mathcal{S}), h(c)\} \tag{11.5.2}$$

using equation (11.1.17).

$$-\{E_i(\mathcal{S}), h(c)\} = \frac{\kappa\gamma}{2} \int_c \int_{\mathcal{S}} h(x_0, x(t))\tau_i(x(t))h(x(t), x_1)dt \; \dot{x} \lrcorner \tilde{\delta}^3(x, x(t)) \tag{11.5.3}$$

Here, the $\tilde{\delta}^3$ denotes a scalar density that has support at intersection points of the (oriented, analytic, open) surface (with compact support) $\mathcal{S}$ with the (oriented, piecewise analytic) curve c. Analycity of $\mathcal{S}$ and piecewise analycity of c ensures that there are only finitely many intersection points or some analytic pieces of c lie entirely in $\mathcal{S}$. Subdivide c and $\mathcal{S}$ if necessary such that the intersection is of definite type. In case of no intersection points the integral vanishes. For the sake of simplicity we assume that there is a *single* intersection point, which is the starting point x_0 of the curve. As long as there exists a nonvanishing derivative of the curve along a direction transversal to the surface in its Taylor expansion around x_0, the integral has a nonvanishing value (see app. A.13),

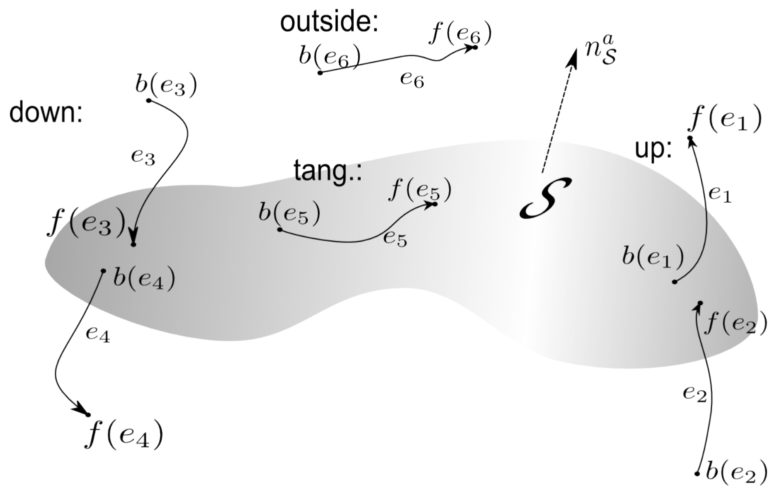


Figure 11.2: *The possible types of intersections of edges with surfaces.*

namely

$$-\{E_i(\mathcal{S}), h(c)\} = \frac{\kappa\gamma}{4}\tau_i h(c). \tag{11.5.4}$$

The factor 1/2 has its origin in the regularization used for delta function, since it is evaluated at the boundary of the integral. In order to show this one can apply Colombeau theory (see [33] for a short introduction and a generalization to paracompact manifolds).

$$\begin{aligned}\int_0^1 \delta(t)dt = \int_{-\infty}^1 \Theta(t)\Theta'(t)dt = \frac{1}{2}\int_{-\infty}^1 (\Theta^2)'(t) \approx \\ \approx \frac{1}{2}\Theta(1) = \frac{1}{2}.\end{aligned}$$

A further motivation for that choice of the numerical prefactor is a physical one. Only if this integral gives 1/2 the area operator of LQG is invariant under the flip of orientation of the surface [11].

We will now distinguish between different cases of the possible relations of c and $\mathcal{S}$: The case we already discussed we will be called "outgoing/above", which means the egde lies transversal above $\mathcal{S}$ and punctures the surface in $b(c)$, i.e. it is oriented in the direction of the surface normal. According to that, we generalize the result from above

$$-\{E_i(\mathcal{S}), h(c)\} = \left\{\begin{array}{ll} \frac{\sigma(\mathcal{S},c)\kappa\gamma}{4}\tau_i h(c) & \text{outgoing: } c\cap\mathcal{S} = b(c) \\ -\frac{\sigma(\mathcal{S},c)\kappa\gamma}{4}h(c)\tau_i & \text{incoming } c\cap\mathcal{S} = f(c) \end{array}\right. ,$$

$$\sigma(\mathcal{S}, c) = \left\{\begin{array}{cc} +1 & \text{above} \\ 0 & \text{tangential or nonintersecting} \\ -1 & \text{below} \end{array}\right. .$$

The possible types of intersections can be seen from figure 11.2. Usually the types "outgoing/above" and "incoming/below" are subsumed in the type "up" and "incoming/above" and "outgoing/below" in the type "down". Hence

$$-\{E_i(\mathcal{S}), h(c)\} = \frac{\sigma(\mathcal{S},c)\kappa\gamma}{4}(\delta_{c\cap\mathcal{S}=b(c)}\tau_i h(c) + \delta_{c\cap\mathcal{S}=f(c)}h(c)\tau_i)$$

$$\sigma(\mathcal{S},c) = \begin{cases} +1 & \text{up} \\ 0 & \text{tangential or nonintersecting} \\ -1 & \text{down} \end{cases}$$

In the following considerations we adopt the convention, used by the loop quantum gravity community, employing the terminus holonomy when actually speaking of the parallel transport.
Remark: All what was said here applies to any Yang Mills field theory.

11.5.1 Spherically symmetric holonomies and fluxes

In the symmetry reduction scheme developed in the preceding chapter we arrived at connection dynamics with a $U(1)$ fiber bundle over a one dimensional manifold, which we choose to be $\mathbb{R}$ or $\mathbb{R}^+$. For the $U(1)$ gauge potential we can proceed as shown above and construct a $U(1) \subset SU(2)$ the parallel transport

$$\begin{aligned} h_3(e) &= \exp\Big(\Lambda_3 \int_e A\Big) = \\ &= \cos\big(\frac{1}{2}\int_{\mathcal{I}} A_x(x)dx\Big)\mathbf{1} + 2\Lambda_3 \sin\big(\frac{1}{2}\int_{\mathcal{I}} A_x(x)dx\Big), \end{aligned}$$

where $\mathcal{I}$ denotes a finite interval of $\mathbb{R}$ and e denotes an edge in the one dimensional base manifold B. For the intrinsic $U(1)$ parallel transport we write

$$h_3(e) = \exp\Big(\frac{i}{2}\int_{\mathcal{I}} A_x(x)dx\Big).$$

Additionally, we obtained scalar fields called the Higgs fields, originating from the gauge potential components in the homogeneous directions of the orbits ($S/J \sim S^2$). These Higgs fields will be treated like matter scalar fields in the full LQG theory, using point holonomies [34], i.e. via exponentiation. In the reduced picture there are no edges which could couple to the connection components in homogeneous direction, the spheres are represented by points/vertices in the radial manifold, in the following denoted by v. The neighboring vertices will be denoted by v_+ and v_- respectively. Nevertheless we will entertain the point of view that the field K_t is integrated over edges, which are parts of great circles on S^2 having parameter length δ. In order to provide for the edges which meet in v to be loops, we could choose $\delta = 2\pi$. We choose the circles to be the equator for the integration of the ϕ coordinate and the meridians for the θ coordinate. Thus, we confine ourselves to a situation, where the radial line cuts the spheres in the equatorial plane, with coordinates $(v, \pi/2, 0)$ with respect to the background structure. For the configuration variable η such interpratation is not available, but also not necessary, since it is only a $U(1)$ scalar even in the three dimensional picture. Therefore, we have use a point holonomy, too.

$$\begin{aligned}
h_1(v) &= \exp\left(\gamma\delta K_t(v)\Lambda^1\right) = \\
&= \cos(\gamma\delta/2K_t(v))\mathbb{1} + 2\Lambda^1 \sin(\gamma\delta/2K_t(v)) \\
h_2(v) &= \exp\left(\gamma\delta K_t(v)\Lambda^2\right) = \\
&= \cos(\gamma\frac{\delta}{2}K_t(v))\mathbb{1} + 2\Lambda^2 \sin(\gamma\frac{\delta}{2}K_t(v)) \\
h_\eta(v) &= \exp(i\eta(v))
\end{aligned}$$

Analogous to the full LQG theory we smear the dualized momenta $*\tilde{E}$ over two dimensional surfaces. We will choose a plaquette in the symmetry orbit S^2_ϵ bounded by four great circles. Furthermore choose a plaquette in the equatorial plane bounded by two radial edges and two congruent equatorial great circles, denoted by S_θ,[5]. Analogously define a plaquette bounded by meridian lines, denoted by S_ϕ. Each boundary curve in the homogeneous directions has parameter length ϵ. All the plaquettes are oriented in the directions ∂_x, ∂_θ and ∂_ϕ, respectively according to the subscript. The situation is displayed in figure 11.3. The momentum conjugate to $\eta(x)$ has a special role and we smear it over a box $\mathcal{V}_\epsilon$ bounded by pairs of the plaquettes S^2_ϵ, S_θ, S_ϕ. We can consider $P^\eta(x) = 2\sin\alpha A_t(x)E^t(x)$ as a 3-form $\tilde{P}^\eta = 2\sin\alpha A_t E^t dx d\Omega_{S^2}$. For the symmetric fluxes we then obtain[6]

$$\begin{aligned}
E_3[S^2_\epsilon] = \int_{S^2_\epsilon} *\tilde{E}_3 = \mathrm{Ar}(S^2_\epsilon)E^x \\
E_1[S_\theta] = \epsilon\int_{\mathcal{I}} E^t dx \\
E_2[S_\phi] = \epsilon\int_{\mathcal{I}} E^t dx \\
E_\eta[\mathcal{V}_\epsilon] = \int_{\mathcal{V}_\epsilon} \tilde{P}^\eta = \mathrm{Ar}(S^2_\epsilon)\int_{\mathcal{I}} P^\eta dx
\end{aligned} \tag{11.5.5}$$

[5] A special case would be the ring bounded by two such great circles. In general we could, at least in the three dimensional picture, allow the boundary curves to be lines of latitude connected by two radial edges. In such a case we would obtain $E_\theta[S_\theta] = \int_{S_\theta} *\tilde{E}_1 = \int_{S_\theta} -E_t \sin\theta dx d\phi = \epsilon\sin\theta\int_{\mathcal{I}} E_t dx$.

[6] The minus sign in the expression of $E_\theta[S_\theta]$ was absorbed such that the plaquette is oriented in the direction of ∂_θ.

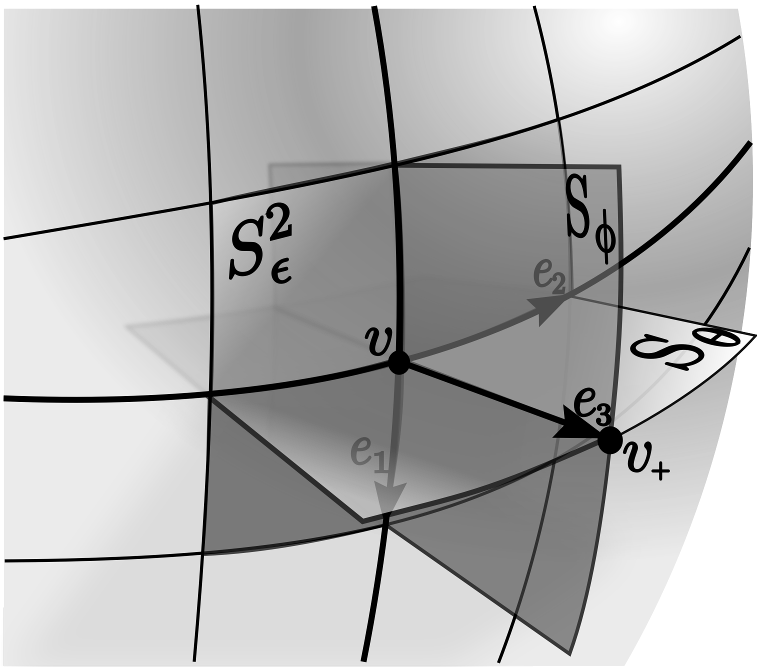


Figure 11.3: *Triangulation for spherically symmetric LQG*

For the calculation of the flux holonomy algebra we choose a setting where all edges tangential to S^2 are centered at the vertex v. A radial edge starts at v and ends at the vertex v^+. Each of the regularizing surfaces is centered at v. In the following calculation we synchronize $\delta = \epsilon$

$$\begin{aligned} -\{E_3[S^2_v], h_3(e_v^{v^+})\} =& G\gamma \operatorname{Ar}(S^2_\epsilon) \int_{\mathbb{R}} dz \delta(v,z) h_3(e_v^{v^+}) \int_v^{v^+} dx \Lambda^3 \delta(x,z) = \\ =& \frac{\kappa^{S^2_\epsilon}\gamma}{4} \Lambda^3 h_3(e_v^{v^+}), \end{aligned} \tag{11.5.6}$$

where we abreviated $\kappa^{S^2_\epsilon} = 2\operatorname{Ar}(S^2_\epsilon)G$ and Λ^3 is either $i/2$ or the full $\mathcal{SU}(2)$ generator.

$$\begin{aligned} -\{E_{1/2}[S_{\theta/\phi}], h_{1/2}(v)\} =& \frac{G\epsilon}{2} \int_{\mathbb{R}} dz \int_{\mathcal{I}} dx \delta(x,z) \Lambda^{1/2} \gamma\epsilon\delta(v,z) h_{1/2}(v) = \\ =& \frac{\kappa^\epsilon \gamma}{4} \Lambda^{1/2} h_{1/2}(v), \end{aligned} \tag{11.5.7}$$

with $\kappa^\epsilon = 2\epsilon^2 G$.
Finally we find

$$\begin{aligned} -\{E_\eta[\mathcal{V}_\epsilon], h_\eta(v)\} =& G \operatorname{Ar}(S^2_\epsilon) \int_{\mathbb{R}} dz \int_{\mathcal{I}} dx \delta(x,z) i \delta(v,z) = \\ =& \frac{\kappa^{S^2_\epsilon}\gamma}{2} i h_\eta(v) \end{aligned} \tag{11.5.8}$$

Note that because of the particular choices of the parametrization and the imbedding of the edges and surfaces we obtain the holonomy flux algebra of full loop quantum gravity in the limit $\mathrm{Ar}(S^2_\epsilon) \to 4\pi$.

For the spherically symmetric Maxwell connection dynamics developed in section 9.1 we proceed analogously. We introduce the holonomies along a radial edge e and along an arbitrary edge in the symmetry orbit S^2 denoted by e_t.

$$g_x(e) = e^{i\int_{\mathcal{I}} \phi dx}$$
$$g_\phi(v) = e^{\frac{i}{2}n\int_{e_t}\cos\theta(\phi)d\phi} = \beta^n_{e_t}.$$

We see that the holonomy corresponding to the monopole connection part simply gives a $U(1)$-phase, depending only on the geometry of e_t, which for example for an eqautorial edge gives 1. This phase is analogous to the Aharanov Bohm phase discussed in section 11.4. The densitized electric vector-field can again be smeared over a 2-surface

$$\mathcal{E}[S^2_\epsilon] = \int_{S^2_\epsilon} *\tilde{\mathcal{E}} = \mathrm{Ar}(S^2_\epsilon)q, \tag{11.5.9}$$

which we recognize as the litaral electric flux through the surface $\mathrm{Ar}(S^2_\epsilon)$. Clearly, for $\mathrm{Ar}(S^2_\epsilon) = 4\pi$ one recovers the classical result for the electric flux of a point charge in the center of a 2-sphere. The Maxwell holonomy flux algebra reads

$$-\{\mathcal{E}[S^2_\epsilon], g_x(e)\} = \frac{q^2\,\mathrm{Ar}(S^2_\epsilon)}{8\pi} i g_x(e). \tag{11.5.10}$$

11.5.2 Cylindrical functions and vector-fields

In order to define Cylindrical functions first notice, that each graph defines a map from the space of smooth connections into the Cartesian product of G. We define the map with respect to some graph γ

$$p_\gamma : \mathcal{A} \to G^{|E(\gamma)|};\ A \mapsto \{h(e)_{e\in E(\gamma)}\},$$

where $\mathcal{A}$ denotes the set of smooth connections and $|E(\gamma)|$ the number of edges in $E(\gamma)$. Given a $\mathcal{C}^\infty$ function $\psi : G^{|E(\gamma)|} \to \mathbb{C}$ we can define function $\Psi_\gamma : \mathcal{A} \to \mathbb{C}$ said to be cylindrical over γ such that $\Psi_\gamma = \psi \circ p_\gamma$. We will sometimes use the notation

$$\Psi_\gamma(A) = \psi(h(e_1), \ldots, h(e_n))$$

The set of functions cylindrical over γ is denoted by $\mathrm{Cyl}^\infty_\gamma$. The space of all infinitly differentiable cylindrical functions (i.e. cylindrical with respect to some graph) will be denoted by $\mathrm{Cyl}^\infty = \cup_{\gamma\in\Gamma}\mathrm{Cyl}^\infty_\gamma$. They form an Abelian algebra with respect to the Poisson

bracket.
The action of fluxes on cylindrical functions Ψ_γ cylindrical over the graph γ is given by

$$
\begin{aligned}
&-\{E_i(\mathcal{S}), \Psi_\gamma\}(A) = \\
&= \sum_{e\in E(\gamma)} \Big[\frac{\partial\psi(\{h(e')\})}{\partial h(e)_{AB}} \frac{\sigma(\mathcal{S},e)\kappa\gamma}{4}(\delta_{e\cap\mathcal{S}=b(e)}\tau_i h_{AB}(e) + \delta_{e\cap\mathcal{S}=f(e)} h_{AB}(e)\tau_i)\Big].
\end{aligned}
$$

Now recall formulas the (6.1.7) and (6.1.8), where we defined left and right invariant vector-fields on a Lie group. According to them we can rewrite the above formula in the compact form

$$
-\{E_i(\mathcal{S}), \Psi_\gamma\}(A) = \sum_{e\in E(\gamma)} \frac{\sigma(\mathcal{S},e)\kappa\gamma}{4}\Big\{\Big[\delta_{e\cap\mathcal{S}=b(e)} R^i_e + \delta_{e\cap\mathcal{S}=f(e)} L^i_e\Big]\psi(\{h(e')\})\Big\},
$$

where R^i_e is the right invariant vector-field on the copy of G^e.
Based on this expression, we can define vector-fields on Cyl^∞

$$
Y^i_S[f] := -\{E_i(\mathcal{S}), \Psi\}(A). \tag{11.5.11}
$$

This completes our search for the elementary variables. The so-called Poisson $*$-algebra $\mathcal{B} \subseteq V^\infty \times \mathrm{Cyl}^\infty$ a subset of smooth cylindrical functions and smooth vector-fields thereon will serve as the algebra to be quantized. The algebra is defined by the Lie algebra relations

$$
[(\Psi, Y^i_{\mathcal{S}}), (\Psi', Y'^j_{\mathcal{S}'})] = (Y^i_{\mathcal{S}}[\Psi'] - Y'^j_{\mathcal{S}'}[\Psi], [Y^i_{\mathcal{S}}, Y'^j_{\mathcal{S}'}])
$$

and the $(*)$ involution operation is given by complex conjugation. Note that the "momentum" variables no longer commute, as one would expect from the Poisson brackets we started with. This results from the distributional smearing we used[7].
Now recall our discussion on Haar measures. We can define an inner product on the space $\mathrm{Cyl}^\infty_\gamma$.

$$
\langle \Psi_\gamma, \Phi_\gamma\rangle := \int_{G^{|E(\gamma)|}} d\mu_H^{|E(\gamma)|} \bar{\psi} \circ \phi(\{h(e_i)\}), \tag{11.5.12}
$$

where $\mu^{|E(\gamma)|}$ denotes the Haar measure on $G^{|E(\gamma)|}$.

11.6 Quantum configuration spaces

We aim at the Cauchy-completion of the space $(\mathrm{Cyl}^\infty_\gamma, d\mu_H^{|E(\gamma)|})$ in order to obtain a Hilbert space. In infinite dimensional theories, i.e. field theories, in performing the Cauchy

[7]For a detailed discussion on this new feature see [35].

completion one obtains states that cannot be realized as function on the configuration space, which in our case is $\mathcal{A}$. For example, in scalar field theory on Minkowski space the quantum states are tempered distributions, living in the topological dual to the space of probes (Schwartz space) [32].

First, let us confine ourselves to one single graph and later extend the discussion to all graphs. The crucial point is the huge redundancy of gauge transformations acting on connections. The parallel transport is only sensitive to gauge transformations G/G^0 which are nontrivial at the end points of the edge. We therefore arrive at the configuration space $\bar{\mathcal{A}}_\gamma$ and the residual quantum gauge group $\bar{G}_\gamma$ given by

$$\begin{aligned}\bar{\mathcal{A}}_\gamma &:= \mathcal{A}_\gamma / G^0_\gamma \\ \bar{G}_\gamma &:= G_\gamma / G^0_\gamma.\end{aligned} \tag{11.6.1}$$

The advantage of this construction is that the elements of $\bar{\mathcal{A}}_\gamma$ can now be identified with G elements given by the parallel transport.

Recall now what has been shown on paths and graphs in the terminology of category theory (ch. 11.1.2): Since the group G is, of course, also a groupoid we consider the connection groupoid homomorphism, which maps a path to an element of G via the parallel transport. The equations (11.1.13) and (11.1.14) qualifies this map homomorphism. Hence, due to the identifications discussed before, we can consider the space of generalized connection over the graph γ as the set $\hom(\gamma, G^{|E(\gamma)|})$.

The Hermitian inner product is invariant with respect to the change of orientation of edges (inversion) and to generalized gauge transformations (left and right translations), due to the properties of the Haar measure. The space of quantum states is denoted by $\mathcal{H}_\gamma = L^2(\bar{\mathcal{A}}_\gamma, d\mu_H^{|E(\gamma)|})$, which has the product structure $L^2(\bar{\mathcal{A}}_\gamma, d\mu_H^{|E(\gamma)|}) \sim L^2(G, d\mu_H)^{\otimes |E(\gamma)|}$. Utilizing the Peter-Weyl theorem we can decompose this Hilbert space into the direct sum

$$\mathcal{H}_\gamma = \oplus_{\vec{j},\vec{l}} \mathcal{H}_{\gamma,\vec{j},\vec{l}}, \tag{11.6.2}$$

where $\vec{j} = (j_{e_1}, \ldots, j_{e_{|E(\gamma)|}})$ labels the irreducible representations of G on the edges and $\vec{l} = (l_{v_1}, \ldots, l_{v_{|V(\gamma)|}})$ labels the irreducible representations of residual gauge transformations at the vertices. The gauge invariant subspace of $\mathcal{H}_\gamma$ is given by $\oplus_{\vec{j}} \mathcal{H}_{\gamma,\vec{j},\vec{l}\equiv 0}$. A basis of the space $\mathcal{H}_\gamma$ is provided by the so-called spin-network function defined by

$$T_s = \prod_{e \in E(\gamma)} b^{(j_e)}_{mn}(h(e)) \tag{11.6.3}$$

Consider now different graphs γ and γ' and a cylindrical functions $\mathit{\Psi}_\gamma$ and $\mathit{\Phi}_{\gamma'}$. Since Γ is a directed set, we can find a graph γ'' which contains the other two graphs. Automatically

Ψ_γ and $\Phi_{\gamma'}$ are also cylindrical with respect to the bigger graph, they are simply constant on the additional edges. In order to obtain a well defined inner product the value of the inner product of the two functions need to be independent of the larger graph chosen. This is, indeed, the case due to the properties of the *normalized* Haar measure. In particular, the constant function is cylindrical over all graphs, and we see that the family of measures $\mu_{H,\gamma}^{|E(\gamma)|}$ is consistent, i.e.

$$\int_{\bar{\mathcal{A}}_\gamma} d\mu_{H,\gamma}^{|E(\gamma)|}\psi = \int_{\bar{\mathcal{A}}_{\gamma''}} d\mu_{H,\gamma''}^{|E(\gamma)|}\psi. \tag{11.6.4}$$

Let us now turn to the space $\bar{\mathcal{A}}$. It is defined as the set $\bar{\mathcal{A}} := \hom(\mathcal{P}, G)$ of all algebraic, arbitrarily non-continuous groupoid morphisms, which we can consider, analogously to the case of scalar field theory on Minkowski space, as a distributional extension of $\mathcal{A}$. The elements of $\bar{\mathcal{A}}$ satisfy relations similar to (11.1.13) and (11.1.14). For regularizations used later it is important to note, that for any generalized connection $\bar{A}$ restricted to the graph γ, there exists a smooth connection A on γ such that $h_A(e) = h_{\bar{A}}(e)$. Now the set $\bar{\mathcal{A}}$ has to be equipped with a topology. This is done in detail in [11] here we only sketch the procedure.
Basically, one uses Tychnov's theorem to equip the set

$$X_\infty := \prod_{\gamma\in\Gamma} \bar{\mathcal{A}}_\gamma \tag{11.6.5}$$

with a compact Hausdorff topology. Then one identifies $\bar{\mathcal{A}}$ with the so-called projective limit $\bar{X}$ of the $\bar{\mathcal{A}}_\gamma$. Finally, one makes use of the fact that $\bar{X}$ is a closed subset of X_∞, which therefore is also a compact Hausdorff space. Having equipped $\bar{\mathcal{A}}$ with a compact Hausdorff topology one can make use of a very important theorem, which is called the Riesz representation theorem. It states that there is a 1:1 correspondence between the positive linear functionals on such spaces and regular Borel probability measures thereon via the formula

$$\Lambda(f) := \int_{\bar{\mathcal{A}}} f d\mu. \tag{11.6.6}$$

Thus, we define the so-called Ashtekar-Lewandowski measure μ^0 on $\bar{\mathcal{A}}$ by the positive linear functional via the spin network basis

$$\Lambda(T_s) := \begin{cases} 1 & s = (\gamma = \emptyset, \vec{j} = 0) \\ 0 & \text{otherwise.} \end{cases} \tag{11.6.7}$$

Note that this is consistent with the consistent family of Haar measures introduced before.
Remark: The Hilbert space $\mathcal{H} = L^2(\bar{\mathcal{A}}, d\mu^{|E(\gamma)|})$ is non-separable, since we have uncount-

ably many graphs and therefore the basis is also uncountable.
It is convenient to introduce a decomposition of $\mathcal{H}$ into mutually orthorgonal Hilbert subspaces $\mathcal{H}'_\gamma$. The Hilbert spaces $\mathcal{H}_\gamma$ cannot be mutually orthogonal, since a function over γ is also cylindrical over every larger graph. Therefore one introduces the Hilbert spaces $\mathcal{H}'_\gamma$, which are subspaces of $\mathcal{H}_\gamma$, and are orthogonal w.r.t. $\mathcal{H}_{\tilde{\gamma}} \subset \mathcal{H}_\gamma$, where $\tilde{\gamma}$ is *strictly* contained in γ. This can be achieved by assigning a nontrivial labeling to all the edges of γ and for any two valent vertex v^2 the label l_{v^2} must be nontrivial. Hence, in a spin net work decomposition of a function in $\mathcal{H}'_\gamma$, only basis elements in which none of the factors $b^{(j_e)}_{mn}(h(e))$ equals 1 $\forall e \in E(\gamma)$ appear. The nontrivial labeling of the two valent vertices distinguishes functions over γ from functions over γ', where γ' is obtained from γ' by simply splitting edges.

11.6.1 Gauge transformations and diffeomorphisms

We have already examined the behavior of the parallel transport under local gauge transformations. This simply extends to the generalized connection. We define the quantum gauge group as the set of local gauge transformations without any continuity requirement $\bar{G} := \mathrm{Fun}(\Sigma, G)$. For a cylindrical function the action of $\bar{G}$ is represented by

$$[\hat{U}(g)\Psi_\gamma](\bar{A}) = \psi(\{g(b(e_i))\bar{h}(e_i)g^{-1}(f(e_i))\}) \tag{11.6.8}$$

We choose the group of diffeomorphisms to be a subgroup of the differentiable diffeomorphism $\mathrm{Diff}^n(\Sigma)$, which maps a graph to a so-called permissible graph, i.e. again a collection of piecwise analytic edges. Loosely speaking, we have something like piecewise analytic diffeomorphisms in mind. Its action on $\mathcal{H}$ is given by

$$[\hat{V}(\varphi)\Psi_\gamma](\bar{A}) = \psi(\{\bar{h}(\varphi(e_i))\}) \tag{11.6.9}$$

Again the inner product on $\mathcal{H}$ is left invariant by the action of diffeomorphisms. It is important to point out that due to this fact the so-called kinematic symmetry group is unitarily represented on $\mathcal{H}$.

11.6.2 Solving the kinematical constraints

The solution space to the Gauß constraint is given by the gauge invariant subspace $\mathcal{H}^G = \oplus_{\vec{j}}\mathcal{H}'_{\gamma,\vec{j},\vec{l}\equiv 0}$.
Finding the solution space to the diffeomorophism constraint requires more effort. We only can implement finite diffeomorphisms because the infintesimal operators $\hat{\phi}_t$ representing the one parameter subgroup ϕ_t fail to exist. This is due to the fact that they are not weakly continuous, since the spaces Cyl_γ and $\mathrm{Cyl}_{\phi_t(\gamma)}$ are orthogonal for any nonvanishing

value of the parameter t.

The solutions to the constraint can be constructed via group averaging, and will in general not belong to Cyl but to a subspace of the algebraic dual Cyl*, which is the space of linear functionals on Cyl with the topology of pointwise convergence [7].

For our discussion, consider some graph γ and the associated Hilbert space $\mathcal{H}'_\gamma$. First, we average over the diffeomorphisms which map γ to itself, but are not trivial on γ. They can exchange edges and their orientation. Denote the subgroup leaving γ invariant by Diff_γ, the sub group with trivial action by TDiff_γ, the coset space by GS_γ and its volume by N_γ. We define the projection operator $\hat{P}_{\mathrm{GS}_\gamma}$, which maps $\mathcal{H}'_\gamma$ to the sub space invariant under the induced action of GS_γ by

$$\hat{P}_{\mathrm{GS}_\gamma}\Psi'_\gamma := \frac{1}{N_\gamma}\sum_{\phi\in\mathrm{GS}_\gamma}\hat{V}(\phi)\Psi'_\gamma.$$

Now we can associate to any element Ψ'_γ an element $\eta(\Psi'_\gamma)$ of Cyl* defined by its linear action on *arbitrary* elements Φ_β of Cyl

$$(\eta(\Psi'_\gamma)|\Phi_\beta\rangle = \sum_{\varphi\in\mathrm{Diff}\,/\,\mathrm{Diff}_\gamma}\langle\hat{V}(\varphi)\hat{P}_{\mathrm{GS}_\gamma}\Psi'_\gamma,\Phi_\beta\rangle.$$

Since the inner product on the right hand side is invariant under diffeomorphisms, $\eta(\Psi'_\gamma)$ is invariant under the action of $\mathrm{Diff}(\Sigma)$. It was crucial here that we used elements of $\mathcal{H}'_\gamma$, since only then the right hand side consists of a finite number of terms. Consider the spin network decomposition of Φ_β. The right hand side of this equation is only nonvanishing for terms in this decomposition which belong to a decomposition of an element in $\mathcal{H}'_{\tilde{\beta}}$ for which $\tilde{\beta}$ lies in the same generalized *knot class* as γ. Two graphs are members of the same knot class if the number of vertices and (nontrivially labeled) edges are equal and they are knotted in the same way.

This construction allows for introducing an inner product on the solution space by linearly extending the map to arbitrary cylindrical functions in the following way: First, decompose an arbitrary function $\Psi_\gamma\in\mathcal{H}_\gamma$

$$\Psi_\gamma = \sum_{\tilde{\gamma}\prec\gamma;\Psi_{\tilde{\gamma}}\in\mathcal{H}'_{\tilde{\gamma}}}\Psi_{\tilde{\gamma}}$$

Then using this decomposition we define the inner product via the action of an element of Cyl* associated with an arbitrary element of Cyl.

$$(\eta(\Psi)|\eta(\Phi)) := (\eta(\Psi)|\eta(\Phi)\rangle := \sum_{\tilde{\gamma}\prec\gamma;\Psi_{\tilde{\gamma}}\in\mathcal{H}'_{\tilde{\gamma}}}\ \sum_{\phi\in\mathrm{Diff}\,/\,\mathrm{Diff}_{\tilde{\gamma}}}\langle\hat{V}(\varphi)\hat{P}_{\mathrm{GS}_{\tilde{\gamma}}}\Psi_{\tilde{\gamma}},\Phi_\beta\rangle \tag{11.6.10}$$

11.7 Represention of the Poisson *-algebra

Now we turn to the represention of $\mathcal{P}$ on $\mathcal{H}$[8]. We represent a cylindrical function $f \in \mathrm{Cyl}(\mathcal{A})$ by a multiplication operator

$$[\pi(f)\Psi](\bar{A}) = f((\bar{A}))\Psi(\bar{A})$$

The action of vector-fields is implemented as a derivative operator

$$\begin{aligned}[\pi(Y^i_{\mathcal{S}})\Psi](\bar{A}) &:= \frac{\hbar}{i} Y^i_{\mathcal{S}}[\Psi(\bar{A})] = -\frac{\hbar}{i}\{E_i(\mathcal{S}), \Psi\}(\bar{A}) = \\ &=: \frac{\ell_P^2\gamma}{4}\sum_{v\in\mathcal{S}}[J^i_{(v,\mathrm{up})} - J^i_{(v,\mathrm{down})}]\Psi(\bar{A}),\end{aligned} \tag{11.7.1}$$

where we defined the "angular momentum" operators

$$\begin{aligned} J^i_{(v,\mathrm{x})} &:= \sum_{e \text{ at } v} J^i_{(e,x)} := \\ &\frac{1}{i}\sum_{e \text{ at } v}(\delta_{e\cap\mathcal{S}=b(e)}R^i_{(e,x)} + \delta_{e\cap\mathcal{S}=f(e)}L^i_{(e,x)})\end{aligned} \tag{11.7.2}$$

The uncountable sum over all points in $\mathcal{S}$ does not cause any troubles, since acting on a cylindrical function over some graph that intersects $\mathcal{S}$, there are only finitely many nonvanishing contributions.

11.8 Spherically symmetric spin-networks and vector fields

Coming from the full theory one would like to understand how to obtain a symmetric state from a generic quantum states, by means of pull backs of functions on the space of generalized G connections to functions on the space of $Z_{[\lambda]}$ connections and Higgs fields. Symmetric states have to be considered as generalized states, which can be understood due to the fact that they have only singular support on invariant connections, i.e. a symmetric state is an element of Cyl^*. [19] provides proof of the fact that the symmetric states on generalized connections modulo generalized gauge transformations can be identified with cylindrical functions on the space on generalized connections restricted the the orbit space B times restricted Higgs fields modulo the generalized gauge transformations restricted to B. This important statement is known as the quantum symmetry reduction theorem.

[8]For the sake of simplicity, we will not take a detour using an abstract quantum *-algebra, which would be necessary for rigorously quantize the Poisson *-algebra.

A graph γ of the reduced theory consists only of the collections of nonoverlapping egdes immersed in the one dimensional basis manifold B, whose orientation agrees with the one of B. Let the orientation of B coincide with the direction ∂_x. Thus, we exclude the down case in the action of momentum operators on cylindrical functions. A spherically symmetric spin network is a labeling of such graphs with irreducible representations of $U(1)$, i.e. by integers as we have seen in chapter 6.3. Since we also discuss Maxwell theory we introduce also a charge network, by assigning a charge quantum number to each edge. At the vertices we insert point holonomies of the Higgs field and the new angular variable η.

The smooth angular variables η are generalized as usual to the set $\hom(V(\gamma), U(1))$ without any continuity requirements. Hence, we also label the vertices with integers. The handling of the Higgs fields is more tricky. The quantum configuration space for Higgs fields which are $\mathbb{R}$ fields is given by the so-called Bohr compactification $\mathbb{R}_B$ of the real line. Hence, we assign also a continuous label to the set of vertices of the reduced graph, which can be considered as subsuming the continuous label - the edge length δ and the discrete $SU(2)$ label of the full theory. This continuous label is problematic and we will postpone its discussion for the time being 12.

We conclude that spherically symmetric spin-charge network functions are given by

$$\begin{aligned} &T_{s,\vec{m},\vec{l},\vec{\mu};\vec{k}} = \\ &= \prod_{e\in E(\gamma)} \prod_{v\in V(\gamma)} e^{\frac{im_e}{2}\int_{\mathcal{I}} A_x(x)dx} e^{i\mu_v\gamma K_t(v)} e^{il_v\eta(v)} \otimes \prod_{e\in E(\gamma)} \prod_{v\in V(\gamma)} e^{ik_e\int_{\mathcal{I}}\phi dx} \beta^n_{e_t}. \end{aligned} \tag{11.8.1}$$

For the action of momentum operators we find

$$\begin{aligned} [\hat{Y}^3_{S^2_{\epsilon^2}} T_{s,\vec{m},\vec{l},\vec{\mu};\vec{k}}] &= \frac{\ell_P^2\gamma}{32\pi}(m_{e^+} + m_{e^-})T_{s,\vec{m},\vec{l},\vec{\mu};\vec{k}} \\ [\hat{Y}^{1/2}_{S_{\theta/\phi}} T_{s,\vec{m},\vec{l},\vec{\mu};\vec{k}}] &= \frac{\ell_P^2\gamma}{16\pi}\sum_{v\in\mathcal{I}} \mu_v T_{s,\vec{m},\vec{l},\vec{\mu};\vec{k}} \\ [\hat{Y}^{\eta}_{\mathcal{V}_\epsilon} T_{s,\vec{m},\vec{l},\vec{\mu};\vec{k}}] &= \frac{\ell_P^2\gamma}{8\pi}\sum_{v\in\mathcal{I}} l_v T_{s,\vec{m},\vec{l},\vec{\mu};\vec{k}} \\ [\hat{\mathcal{Y}}_{S^2_{\epsilon^2}} T_{s,\vec{m},\vec{l},\vec{\mu};\vec{k}}] &= \frac{\alpha_e}{8\pi}(k_{e^+} - k_{e^-})T_{s,\vec{m},\vec{l},\vec{\mu};\vec{k}}, \end{aligned}$$

where α_e denotes the fine structure constant.

11.9 Spherically symmetric solutions of the reduced Gauß and diffeomorphism constraints

Recall the form of the spherically symmetric Gauß constraint given in ch. 8.1.2. In the reduced theory its explicit quantization becomes considerably simple. It is only the function $\int_B \lambda^3 (E^x)' dx$ which turns out to be problematic. Let us examine the action of its quantum analog on cylindrical functions:

$$\begin{aligned}
&\int_B \lambda^3 \frac{d}{dx} \hat{Y}_{S^2_{x,\epsilon^2}} \Psi_\gamma = \frac{\ell_P^2 \gamma}{16\pi} \int_B \lambda^3 \frac{d}{dx} \sum_{e \in E(\gamma)} \int_e dt \dot{e}(t) \delta(x, e(t)) h(e)_{AB} \frac{\partial}{\partial h(e)_{AB}} \Psi_\gamma = \\
&- \frac{\ell_P^2 \gamma}{16\pi} \sum_{e \in E(\gamma)} \int_e dt \int_B (\dot{e}(t) \frac{d}{dx} \lambda^3) \delta(x, e(t)) h(e)_{AB} \frac{\partial}{\partial h(e)_{AB}} \Psi_\gamma = \\
&- \frac{\ell_P^2 \gamma}{16\pi} \sum_{e \in E(\gamma)} \int_e dt \frac{d}{dt} \lambda^3(e(t)) h(e)_{AB} \frac{\partial}{\partial h(e)_{AB}} \Psi_\gamma = \\
&- \frac{\ell_P^2 \gamma}{16\pi} \sum_{e \in E(\gamma)} (\lambda^3(e(1)) R_e - \lambda^3(e(0)) R_e) \Psi_\gamma.
\end{aligned} \tag{11.9.1}$$

Therefore, the complete Gauß constraint acting on a spin network basis function yields the eigenvalues

$$\begin{aligned}
\frac{\ell_P^2 \gamma}{16\pi} \Bigg[- \sum_{e \in E(\gamma)} (\lambda^3(v^+) m_e - \lambda^3(v) m_e) + 2 \sum_{v \in V(\gamma)} \lambda^3(v) l_v \Bigg] &= \\
\frac{\ell_P^2 \gamma}{16\pi} \sum_{v \in V(\gamma)} \lambda^3(v)(m_{e^+} - m_{e^-} + 2 l_v)&,
\end{aligned}$$

which are constrained to vanish.

Hence, we can replace the vertex labels l_v by $-1/2(m_{e^+} - m_{e^-})$. This relation reveals that the difference of two adjacent edges has to be even because l_v is integer. For the matter part, we conclude that the charge labels must be the same on every edge, which is intutively clear, since there are no vertex insertions, i.e. Higgs fields in the $U(1)$ matter sector. Therefore, gauge invariant spin network functions read

$$T^{\text{inv}}_{s,\vec{m},\vec{l},\vec{\mu};\vec{k}} = \prod_{e \in E(\gamma)} \prod_{v \in V(\gamma)} e^{\frac{i m_e}{2} \int_{\mathcal{I}} [A_x(x) + \eta'(x)] dx} e^{i \mu_v \gamma K_t(v)} e^{i k_e \int_{\mathcal{I}} \phi dx} \beta^n_{e_t}. \tag{11.9.2}$$

The remaining diffeomorphisms of the reduced theory shift vertices in the manifold B. Thus, their position is not physically relevant. The solution space is again a subspace of the algebraic dual of the cylindrical functions on generalized $U(1)$ connections and Higgs fields.

11.10 The area operator

The classical area functional is given by the formula

$$\mathrm{Ar}[\mathcal{S}] = \int_{\mathcal{S}} \sqrt{*E_i * E_j \delta^{ij}}, \tag{11.10.1}$$

which we approximate by

$$\sum_I \sqrt{E_i[\mathcal{S}_\mathcal{I}] E_j[\mathcal{S}_\mathcal{I}] \delta^{ij}} \tag{11.10.2}$$

in so far as by sending the number of the partitions $\mathcal{S}_\mathcal{I}$ of $\mathcal{S}$ to infinity their coordinate size shrinks to zero. Then we replace the momenta by the quantum analogs, which yields an operator defined with respect to a graph[9]. The important point here is that we have to refine the partition of the surface only until that very point where every $\mathcal{S}_\mathcal{I}$ is punctured at most once by the graph possibly by a vertex with any valence. Further refinements would not alter the result of the action of the operator:

$$\begin{aligned}
\hat{\mathrm{Ar}}[\mathcal{S}]_\gamma &= \sum_{v\in\mathcal{S}} \sqrt{\hat{[Y]}^i_{\mathcal{S},v} \hat{[Y]}^j_{\mathcal{S},v} \delta_{ij}} = \\
&= \frac{\ell_P^2\gamma}{4} \sum_{v\in\mathcal{S}} \sqrt{(J^i_{v,\mathrm{up}} - J^i_{v,\mathrm{down}})(J^j_{v,\mathrm{up}} - J^j_{v,\mathrm{down}})\delta_{ij}} = \\
&= \frac{\ell_P^2\gamma}{4} \sum_{v\in\mathcal{S}} \sqrt{J^2_{v,\mathrm{up}} + J^2_{v,\mathrm{down}} - J_{v,\mathrm{up}}\cdot J_{v,\mathrm{down}} - J_{v,\mathrm{down}}\cdot J_{v,\mathrm{up}}} = \\
&= \frac{\ell_P^2\gamma}{4} \sum_{v\in\mathcal{S}} \sqrt{2J^2_{v,\mathrm{up}} + 2J^2_{v,\mathrm{down}} - (J_{v,\mathrm{up}} + J_{v,\mathrm{down}})^2}.
\end{aligned} \tag{11.10.3}$$

Recall the definition of the angular momentum operators and of the right invariant vector-fields 11.7.2. In the $(2j+1)$ dimensional representation of $SU(2)$ we have

$$\delta^{ik} \frac{1}{i}\tau_i^{(j)} \frac{1}{i}\tau_k^{(j)} = j(j+1)\mathbb{1}_{2j+1}. \tag{11.10.4}$$

Therefore, the area operator has a discrete spectrum of eigenvalues:

$$\frac{\ell_P^2\gamma}{4} \sum_v \sqrt{2j_{v,\mathrm{up}}(j_{v,\mathrm{up}}+1) + 2j_{v,\mathrm{down}}(j_{v,\mathrm{down}}+1) - j_{v,\mathrm{up+down}}(j_{v,\mathrm{up+down}}+1)}. \tag{11.10.5}$$

[9] One can show that the family of operators is consistent and therefore one obtains a well defined area operator on $\mathcal{H}$.

It is bounded from below and has the lowest eigenvalue, the so-called area gap:

$$a_{\min} = \frac{\ell_P^2 \gamma \sqrt{3}}{8}. \tag{11.10.6}$$

Due to the Clebsch Gordan decomposition the eigenvalues $j_{v,\text{up+down}}$ are constrained to lie in the set

$$\{|j_{v,\text{up}} - j_{v,\text{down}}|, |j_{v,\text{up}} - j_{v,\text{down}}| + 1, \dots, j_{v,\text{up}} + j_{v,\text{down}}\}. \tag{11.10.7}$$

This is one of the most famous results in loop quantum gravity: The quantization of area pointing at the discreteness of space.
Let us briefly look at the analogous expression of the area operator of a sphere in the reduced theory: There we find

$$\begin{aligned} \text{Ar}(S^2) = 4\pi|E^x| &\rightarrow \\ a_{S^2} = \frac{\ell_P^2\gamma}{8}&|(m_{e^+} + m_{e^-})| \end{aligned} \tag{11.10.8}$$

11.11 Quantizing the Hamiltonian constraints

11.11.1 The Thiemann trick

In the expression of the scalar constraint[10]

$$C = \frac{1}{\kappa\omega_h} \tilde{E}_i^a \tilde{E}_j^b \varepsilon^{ij}{}_k (F_{ab}^k - (1+\gamma^2)\varepsilon^k{}_{lm}(K_a^l K_b^m)) \tag{11.11.1}$$

a major difficulty occurs. Quantizing the inverse volume form is particularly problematic if the eigenvalue zero is in the spectrum of the volume operator. In the early LQG literature one absorbed the inverse density weight into the lapse function and used $\gamma^2 = -1$ in order to obtain a polynomial expression. This simple form of the constraint gave hope that the quantization of the dynamics of GR could be successful. As already mentioned the self-dual connection formulation is not favoured nowadays. Furthermore it turned out that the density weight 1 of the constraint is actually crucial in order to obtain a well defined UV-finite operator. In section 11.3 we saw that we can approximate the gauge potential and the field strength with parallel transport and holonomy respectively, while the densitized triads can be approximated with fluxes. Thus, they will not be problematic in the quantization and give well defined operators. Now we have to look for a possibility to express the inverse volume form with the help of the well defined operators and possibly

[10] Here we use abstract and internal index notation again because this renders the following calculation much simpler.

commutators of these. This is indeed possible and was done by Thiemann in the QSD series [36]. The crucial observation is the following: In the classical regime the volume of a region $\mathcal{R}$ is given by

$$V(\mathcal{R}) = \int_{\mathcal{R}} \sqrt{|\frac{1}{3!}\varepsilon_{abc}\varepsilon^{ijk}\tilde{E}_i^a\tilde{E}_j^b\tilde{E}_k^c|}.$$

We have to take the absolute value of the trivector under the squareroot, since we allow the triad to have both orientations. In the classical regime we ignore the possibility that the orientation could change. The variation of the Volume gives

$$\begin{aligned}\frac{\delta V(\mathcal{R})}{\delta \tilde{E}_k^c(x)} &= \frac{2}{\kappa\gamma}\{A_c^k(x), V(\mathcal{R})\} = \\ = \frac{1}{4}\frac{\operatorname{sgn}(E)}{\omega_h}&\varepsilon^{ijk}\tilde{E}_i^a(x)\tilde{E}_j^b(x)\varepsilon_{abc} = \\ &= \frac{1}{2}e_c^k, \end{aligned} \tag{11.11.2}$$

where $x \in \mathcal{R}$. Therefore, we can rewrite the field strength part of the scalar constraint, which now reads (again without indices[11])

$$\mathcal{C}_E = \operatorname{sgn}(E)\frac{2}{\gamma}\left(\frac{2}{\kappa}\right)^2 F \wedge \cdot\{A, V(\mathcal{R})\}. \tag{11.11.3}$$

Here we adopt the usual convention of LQG to order geometrical operators to the right[12]. For the treatment of the second part of the constraint, sometimes called the kinetic part of the scalar constraint, we define a new quantity

$$\begin{aligned}\mathcal{K} &:= \int_{\Sigma} \tilde{E}\cdot\lrcorner K, \\ K &= \frac{2}{\kappa\gamma}\{A, \mathcal{K}\}.\end{aligned}$$

Starting from this we find for the expression of the second part of the scalar constraint

$$-2\operatorname{sgn}(E)\frac{1+\gamma^2}{\gamma}\left(\frac{2}{\kappa}\right)^4\{A,\mathcal{K}\}\wedge\{A,\mathcal{K}\}\wedge\cdot\{A,V(\mathcal{R})\}. \tag{11.11.4}$$

The expression still contains the smeared densitized trace of the extrinsic curvature $\mathcal{K}$, which has to be replaced. To that end, we first compute the variation of the gravitational

[11]Note that in [11] there is a factor 2 missing, which comes from $F \wedge A = 1/2\varepsilon^{abc}F_{ab}A_c$.

[12]Remark: The field strength part of the Hamiltonian is ususally called the Euclidean constraint, since it would be the only contribution to the scalar constraint in Euclidean GR. Note that the Euclidean constraint differs here from the one defined in equation (10.3.3) of reference [11] with regard to a sign, which is not visible here due to a different Poisson bracket convention.

field F strength w.r.t. A and use $A = \Gamma + \gamma K$ and $Dv = dv + [\Gamma, v]$ for $v \in \Lambda^1(\Sigma) \otimes \mathcal{SU}(2)$

$$\begin{aligned}\delta_A F &= \delta_A(dA + \frac{1}{2}[A, A]) = d\delta A + [A, \delta A] = d\delta A + [\Gamma, \delta A] + \gamma[K, \delta A] \\ &= D\delta A + 2\gamma K \wedge \delta A. \end{aligned} \tag{11.11.5}$$

Taking into account that $\mathcal{K} = \pounds_n \omega_h$ we calculate (again using index notation)

$$\begin{aligned}\{C_E[1], V(\Sigma)\} &= \{\frac{2\,\mathrm{sgn}(E)}{\kappa} F \wedge \cdot e, V(\Sigma)\} \\ &= \frac{\gamma}{2} \int_\Sigma \mathrm{sgn}(E) \gamma K^i_{[a} \delta^d_{b]} \varepsilon_{kij} \delta^j_l e^k_c \varepsilon^{abc} e^l_d = \\ &= \gamma^2 \int_\Sigma K^i_a \frac{\mathrm{sgn}(E)}{2} \varepsilon^{abc} \varepsilon_{ijk} e^j_b e^k_c = \gamma^2 \int_\Sigma K^i_a \tilde{E}^a_i = \gamma^2 \mathcal{K}, \end{aligned} \tag{11.11.6}$$

where the intrinsic covariant derivative in (11.11.5) can be dropped after partial integration – ignoring surface contributions – since it annihilates e^i_a.

This leads to the conclusion that having control over the part $C_E[N]$ of the scalar constraint implies having control over the whole constraint.
For the Yang-Mills sector a similar preparation has to be done. The Yang Mills scalar constraint in the form of (9.0.5) as well as in the form of (9.0.6) involves the metric which has to be taken care of. We will derive an expression valid as a starting point for regularization, which is a kind of admixture of these two forms of the Yang Mills Hamiltonian, namely

$$C_{YM}[N] = \frac{2}{g^2} \int_\Sigma N \left[\frac{e}{2\sqrt{\omega_h}} \wedge \Sigma \cdot \frac{e}{2\sqrt{\omega_h}} \wedge \Sigma \right] + (\Sigma \leftrightarrow F), \tag{11.11.7}$$

where we used

$$e \wedge \Sigma = \tilde{\mathcal{E}} \lrcorner e.$$

We can reexpress the problematic terms in the numerator of (11.11.7) by the same trick shown above, which gives

$$C_{YM}[N] = \frac{32}{\kappa^2 \gamma^2 g^2} \int_\Sigma N \big[\frac{\{A, V(\mathcal{R})\}}{2\sqrt{\omega_h}} \wedge \Sigma \cdot \frac{\{A, V(\mathcal{R})\}}{2\sqrt{\omega_h}} \wedge \Sigma \big] + (\Sigma \leftrightarrow F). \tag{11.11.8}$$

The problematic ω_h in the denominator has to be replaced by well defined expressions in a second step: We approximate the integrals of (11.11.3) and (11.11.8) by a Riemann sum by utilizing a triangulation T^ϵ of Σ.

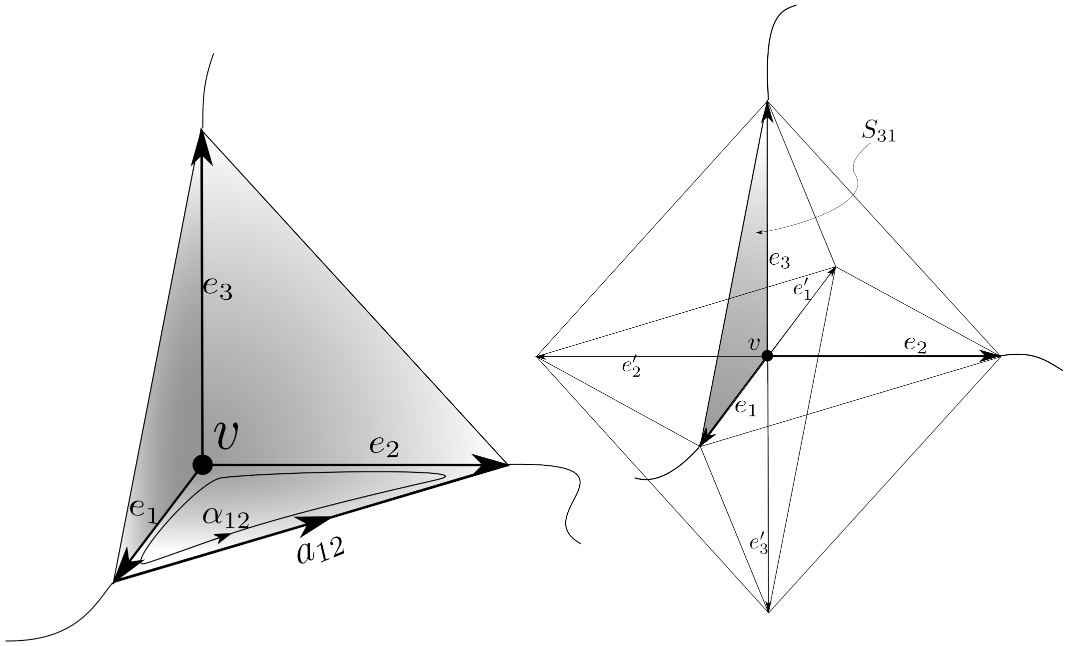


Figure 11.4: *Triangulation of Σ with tetrahedra, here adapted to a graph and its completion to a octahedron.*

11.11.2 Regularization

By triangulating the base manifold Σ we fill Σ with tedrahedra, denoted by Δ, with edge coordinate length ϵ. The intersection point of each ordered triple of edges e_I, $I = \{1, 2, 3\}$, a so-called vertex, is denoted by v[13](Compare with figure 11.4.). The orientation of all edges is chosen to be outgoing from v. The length of the edges of the triangulation is "small"[14]. Furthermore, we introduce loops α_{IJ} starting at v along e_I and ending at v along e_J^{-1}. The arc connecting the endpoints of e_I and e_J is denoted by a_{IJ}. Now, we can use the results of section 11.3, 11.5.1 and the triangulation to approximate the coefficients of the connection and the curvature in order to write the Hamiltonian constraint in terms of the elementary variables. We abbreviate $h(e_I)$ with h_I and, accordingly, the holonomy along the loop $\alpha_{IJ} = e_I \circ a_{IJ} \circ e_J^{-1}$ by h_{IJ}. With

$$\begin{aligned} \frac{1 - h_K^{-1}}{\epsilon} &= (\dot{e}_K \lrcorner A)(v) + \mathcal{O}(\epsilon), \\ \frac{h_{IJ} - h_{IJ}^{-1}}{\epsilon^2} &= \dot{e}_J \lrcorner \dot{e}_I \lrcorner F + \mathcal{O}(\epsilon) \end{aligned} \tag{11.11.9}$$

[13]Of course the whole construction of the triangulation of Σ is analogous to the construction of our probes of space, the graphs. Later on we will adapt the triangulation to the graphs.

[14]Speaking of "small" edges in a background independent theory seems to be without meaning at first glance. Of course, we still have to speak of smallness with respect to a metric. We will remove the regularizing triangulation by perfoming the limit $\epsilon \to 0$ and the vertex number goes to infinity. In performing the limit the length of the edges will become small w.r.t. any metric on Σ.

we replace

$$F \wedge \cdot \{A, V(\mathcal{R})\}$$

by

$$\begin{aligned}
&\operatorname{Tr}\left(\varepsilon^{IJK} \frac{h_{IJ} - h_{IJ}^{-1}}{\epsilon^2} \frac{1}{\epsilon} h_K \{h_K^{-1}, V(\mathcal{R}(\Delta)\}\right) = \\
&= \frac{2}{\epsilon^3} \operatorname{Tr}\left(\varepsilon^{IJK} h_{IJ} h_K \{h_K^{-1}, V(\mathcal{R}(\Delta)\}\right).
\end{aligned} \tag{11.11.10}$$

Finally, we replace the integral by a Riemann sum over $\Delta \in T^\epsilon$ and use $\int_\Sigma F \approx \epsilon^3/6 \sum_{\Delta \in T^\epsilon} F(v)$ and obtain the regulated expression for the Euclidean part of the Hamiltonian constraint

$$C_E^\epsilon[N] = \frac{2}{3} \frac{1}{\gamma} \left(\frac{2}{\kappa}\right)^2 \sum_{\Delta \in T^\epsilon} N(v) \operatorname{Tr}\left(\varepsilon^{IJK} h_{IJ} h_K \{h_K^{-1}, V(\mathcal{R}(\Delta)\}\right). \tag{11.11.11}$$

Let us turn to the regularization of the Yang-Mills Hamiltonian. In equation (11.11.8) we are now able to replace the $1/\sqrt{\omega_h}$. We first introduce a coordinate system. For each of the three regulization edges e_I at v we attach their mirror edges at v such that we obatin an octahedron made of six tetrahedra with center vertex v and choose for the region $\mathcal{R}_v^\epsilon$ in the volume expressions this octahedron. The parameter volume of the octahedron is $4\epsilon^3/3$. Futhermore, we perform a point splitting for the similar contributions in (11.11.8). Therefore, we introduce a so-called characteristic function of the octahedron, denoted by $\chi_{\epsilon,v}(y)$, with the following property

$$\lim_{\varepsilon \to 0} \frac{3}{4\epsilon^3} \chi_{\epsilon,v}(y) = \delta(v, y).$$

Hence, we also have

$$\lim_{\varepsilon \to 0} \frac{3}{4\epsilon^3} V(\mathcal{R}_v^\epsilon) = \sqrt{\det h}. \tag{11.11.12}$$

For any integrable function we define

$$\begin{aligned}
f(v, \epsilon) &:= \int_\Sigma \chi_{\epsilon,v}(y) f(y) \\
\lim_{\varepsilon \to 0} \frac{3}{4\epsilon^3} f(v, \epsilon) &= f(v).
\end{aligned} \tag{11.11.13}$$

This we use in (11.11.8) and obtain

$$C_{YM}[N] = \lim_{\varepsilon\to 0} \frac{3}{4\epsilon^3}\frac{8}{\kappa^2\gamma^2 g^2}\int_\Sigma d^3x N(x)\big[\varepsilon^{IJK}\frac{\{A_K(x),V(x,\epsilon)\}}{2\sqrt[4]{\det h(x)}}\Sigma_{IJ}(x)\cdot \\ \cdot\int_\Sigma d^3y\chi_{\epsilon,v}(y)\varepsilon^{LMN}\frac{\{A_N(y),V(y,\epsilon)\}}{2\sqrt[4]{\det h(y)}}\Sigma_{LM}(y)\big] + (\Sigma\leftrightarrow F). \qquad (11.11.14)$$

Now we use (11.11.12) and with

$$\{A_K(x),\sqrt{V(x,\epsilon)}\} = \frac{\{A_K(x),V(x,\epsilon)\}}{2\sqrt{V(x,\epsilon)}} \qquad (11.11.15)$$

we find

$$C_{YM}[N] = \lim_{\varepsilon\to 0} \frac{8}{\kappa^2\gamma^2 g^2}\int_\Sigma d^3x N(x)\big[\varepsilon^{IJK}\{A_K(x),\sqrt{V(x,\epsilon)}\}\Sigma_{IJ}(x)\cdot \\ \cdot\int_\Sigma d^3y\chi_{\epsilon,v}(y)\varepsilon^{LMN}\{A_K(y),\sqrt{V(y,\epsilon)}\}\Sigma_{LM}(y)\big] + (\Sigma\leftrightarrow F). \qquad (11.11.16)$$

Finally, we replace the integral with a Riemann sum over the tetrahedra (with volume $\epsilon^3/6$) and replace the connections, the Yang-Mills electric and magnetic fields with elementary variables, i.e. holonomies and electric fluxes. To that end, we first observe that via

$$\mathcal{E}[S_{IJ}] := \int_{S_{IJ}}\Sigma = \frac{\epsilon^2}{2}\dot{e}_I\lrcorner\dot{e}_J\lrcorner\Sigma + \mathcal{O}(\epsilon^3) \qquad (11.11.17)$$

we can approximate the dual electric field with electric fluxes. S_{IJ} is the triangular basis of a tetrahedron. For the Yang-Mills magnetic field we write

$$\varepsilon^{IJK}F_{IJ} \approx \varepsilon^{IJK}\frac{\mathcal{B}_{IJ}-\mathcal{B}_{IJ}^{-1}}{\epsilon^2} = \frac{2}{\epsilon^2}\varepsilon^{IJK}\mathcal{B}_{IJ}, \qquad (11.11.18)$$

where $\mathcal{B}_{IJ}$ is a Yang-Mills holonomy along the boundary of the surface S_{IJ}.
As in the gravitational Hamiltonian, regulator volume terms are exactly of the same order as the regulator terms coming from the replacement of phase space coordinates with their elementary variables. The final expression[15], which serves as a starting point

[15] Again the expression here differs the in [34] due to two reasons: First, the Poisson bracket there differs from the one used here by the factor 1/2, which on the one hand is important to reproduce the ADM algebra and on the other to be consistent with the Holst approach. Second, the key identity of the [36] was cited wrong in [34], by a factor 4.

for quantization, reads

$$
\begin{aligned}
C^{\epsilon}_{YM}[N] =& \left(\frac{1}{3}\right)^2 \frac{32}{\gamma^2 g^2} \left(\frac{2}{\kappa}\right)^2 \sum_{\Delta \in T^{\epsilon}} \sum_{\Delta' \in T^{\epsilon}} \chi_{\epsilon,v}(v') N(v) \varepsilon^{IJK} \varepsilon^{L'M'N'} \times \\
& \times \operatorname{Tr}(\tau_i h_K \{h_K^{-1}, \sqrt{V(v,\epsilon)}\}) \operatorname{Tr}(\tau_i h_{N'} \{h_{N'}^{-1}, \sqrt{V(v',\epsilon)}\}) \times \\
& \times \Big[\operatorname{tr}(T^a \mathcal{E}[S_{IJ}]) \operatorname{tr}(T^a \mathcal{E}[S'_{L'M'}]) + \operatorname{tr}(T^a \mathcal{B}_{IJ}) \operatorname{tr}(T^a \mathcal{B}_{L'M'}) \Big]. \qquad (11.11.19)
\end{aligned}
$$

It is important to note that both expressions give rise to well-defined operators on Cyl due to the presence of the volume operator, which only acts on nodes of the underlying graphs and therefore contains only finitely many terms.

11.11.3 Regularization for the spherically symmetric Einstein-Maxwell Hamiltonian

In this section we will derive the regularized expression for the spherically symmetric gravitational and electromagnetic Hamiltonian constraint. We will follow partly [37], but in some minor issues there will be differences, which are due to the concern to keep as close as possible to the full theory of LQG.
Let us now investigate what the relations examined above imply for the symmetric theory. In the following, we will use the edges and surfaces introduced in section 11.5.1, i.e. we only use radial edges and edges in the homogeneous directions. For that reasons we will not use the triangular loops to approximate the field strength. In that case we have, instead of (11.3.6),

$$h(\alpha^{\epsilon}) \approx 1 + \epsilon^2 v \lrcorner u \lrcorner F. \qquad (11.11.20)$$

The parameter length of edges in homogeneous dirctions is δ, while the parameter for radial edges is denoted by ϵ. First, we check that equation (11.11.2) yields (8.1.11):

$$
\begin{aligned}
e \stackrel{!}{=}& 2\frac{1}{G\gamma}\{A(x), V\} = \\
=& 2\frac{1}{G\gamma} \int_B dz \delta(x,z) \Big[\frac{G\gamma}{2\sqrt{|E^x|}} \operatorname{sgn}(E^x) E^t \Lambda_3 + \frac{G\gamma}{2} \sqrt{|E^x|} (\Lambda_1 d\theta + \Lambda_2 d\phi) \Big] = e. \quad (11.11.21)
\end{aligned}
$$

In this calculation we depart from the discussion given in [37] where the functional $V(\mathcal{R}(\Delta))$ given by equation (8.1.8) is used, which is certainly not the volume of our regularizing box, i.e. $V(\mathcal{R}(\Delta)) = \int_{\mathcal{I} \times S^2_{\epsilon}} \sqrt{|E^x|} \tilde{E}^t$. From the full three dimensional point of view, in the Poisson bracket $\{\cdot,\cdot\}_{\Sigma}$, there appears a $\delta^{(3)}(x,v)$-distribution, setting also the spherical coordinates to those, of the vertex point v on the equator, which, due to our choice, is regular. Thus, in the reduced theory, where one uses the bracket $\{\cdot,\cdot\}_B$ one should therefore use $V = \int_{\mathcal{I}} \sqrt{|E^x|}\bar{E}^t$. Note that only with this form of the volume we arrive at the correct result.

A *small* loop on the symmetry orbit S^2 is given by

$$\begin{aligned}
\alpha_{12} &= h_1 h_2 h_1^{-1} h_2^{-1} = \\
&= \mathbb{1}\left[1 - 2\sin^4\left(\frac{1}{2}\gamma\delta K_t\right)\right] + \sin^2(\gamma\delta K_t)\Lambda_3 + 2\sin(\gamma\delta K_t)\sin^2\left(\frac{1}{2}\gamma\delta K_t\right)(\Lambda_1 - \Lambda_2).
\end{aligned}$$

In this expression any corrections coming from the intrinsic curvature of the 2-sphere have been neglected. For small egdes we assume that a square approximates the loop, which is made of great circles, sufficiently. In order to obtain the loop α_{21} we simply change the sign of the $\mathcal{SU}(2)$-generators. Hence,

$$\begin{aligned}
\frac{\alpha_{12} - \alpha_{21}}{2} &= \sin^2(\gamma\delta K_t)\Lambda_3 + 2\sin(\gamma\delta K_t)\sin^2\left(\frac{1}{2}\gamma\delta K_t\right)(\Lambda_1 - \Lambda_2) = \\
&= \gamma^2\delta^2 K_t^2 \Lambda_3 + \mathcal{O}(\delta^3) \approx F_{\theta\phi} - (\Gamma_t - 1)\Lambda_3,
\end{aligned}$$

where we used

$$\sin(\gamma\delta K_t) \approx \gamma\delta K_t \qquad (11.11.22)$$

to first order in δ.

Next, we consider the loop[16]

$$\begin{aligned}
\alpha_{31} &= h_3 h_1(v^+) h_3^{-1} h_1^{-1}(v) = \\
&= \mathbb{1}\Big[\cos\Big(\int_{\mathcal{I}_\epsilon} A_x dx\Big)\sin\left(\frac{1}{2}\gamma\delta K_t(v^+)\right)\sin\left(\frac{1}{2}\gamma\delta K_t(v)\right) + \\
&\quad + \cos\left(\frac{1}{2}\gamma\delta K_t(v^+)\right)\cos\left(\frac{1}{2}\gamma\delta K_t(v)\right)\Big] + \\
&\quad + \Lambda_3 2\sin\Big(\int_{\mathcal{I}_\epsilon} A_x dx\Big)\sin\left(\frac{1}{2}\gamma\delta K_t(v^+)\right)\sin\left(\frac{1}{2}\gamma\delta K_t(v)\right) +
\end{aligned} \qquad (11.11.24)$$

[16]The following relations might be helpful for understanding this loop calculation:

$$\begin{aligned}
\Lambda^1(v^+)\Lambda^1(v) = \Lambda^2(v^+)\Lambda^2(v) &= -\frac{1}{4}\mathbb{1}\cos(\eta^+ - \eta) - \frac{1}{2}\Lambda^3\sin(\eta^+ - \eta) \\
\Lambda^1(v^+)\Lambda^2(v) = -\Lambda^2(v^+)\Lambda^1(v) &= -\frac{1}{4}\mathbb{1}\sin(\eta^+ - \eta) + \frac{1}{2}\Lambda^3\cos(\eta^+ - \eta)
\end{aligned} \qquad (11.11.23)$$

$$
\begin{aligned}
&+\Lambda_1(v^+)2\cos\Big(\int_{\mathcal{I}_\epsilon} A_x dx\Big)\sin\Big(\frac{1}{2}\gamma\delta K_t(v^+)\Big)\cos\Big(\frac{1}{2}\gamma\delta K_t(v)\Big)\\
&-\Lambda_1(v)2\cos\Big(\frac{1}{2}\gamma\delta K_t(v^+)\Big)\sin\Big(\frac{1}{2}\gamma\delta K_t(v)\Big)+\\
&+2\Lambda_2(v^+)\sin\Big(\int_{\mathcal{I}_\epsilon} A_x dx\Big)\sin\Big(\frac{1}{2}\gamma\delta K_t(v^+)\Big)\cos\Big(\frac{1}{2}\gamma\delta K_t(v)\Big)
\end{aligned}
\tag{11.11.25}
$$

In order to show that this expression indeed approximates $F_{x\theta}(v)$ we first establish some useful results:

$$
\begin{aligned}
&\eta(v^+)-\eta(v) =: \Delta\eta \approx \epsilon\eta'(v)\\
&\Lambda_1(v^+) = \Lambda_1(v)\cos(\Delta\eta)+\Lambda_2(v)\sin(\Delta\eta) = \Lambda_1(v)+\epsilon\eta'(v)\Lambda_2(v)+\mathcal{O}(\epsilon^2)\\
&\Lambda_2(v^+) = \Lambda_2(v)-\epsilon\eta'(v)\Lambda_1(v)+\mathcal{O}(\epsilon^2)\\
&K_t(v^+) \approx K_t(v)+\epsilon K_t'(v)\\
&\int_{\mathcal{I}_\epsilon} A_x dx \approx \epsilon A_x(v)\\
&\cos\Big(\frac{1}{2}\gamma\delta K_t(v)\Big) \approx 1-\frac{1}{8}\gamma^2\delta^2 K_t(v)^2.
\end{aligned}
$$

Only first order terms have to be considered (except for the last line), since in all contributions above there is at least one order of δ involved.
Let us begin with terms proportional to $\mathbb{1}$

$$
\begin{aligned}
&\cos\Big(\int_{\mathcal{I}_\epsilon} A_x dx\Big)\sin\Big(\frac{1}{2}\gamma\delta K_t(v^+)\Big)\sin\Big(\frac{1}{2}\gamma\delta K_t(v)\Big)+\\
&\quad+\cos\Big(\frac{1}{2}\gamma\delta K_t(v^+)\Big)\cos\Big(\frac{1}{2}\gamma\delta K_t(v)\Big) =\\
&\approx 1-\frac{1}{8}\gamma^2\delta^2(K_t^2(v^+)-2K_t(v^+)K_t(v)+K_t^2(v)) \approx 1.
\end{aligned}
$$

The Λ_3 coefficient is of order $\mathcal{O}(\epsilon\delta^2)$ and higher. The $\Lambda_1(v^+)$ together with the $\Lambda_1(v)$ coefficients approximately give

$$
\gamma\delta\epsilon K_t'\Lambda_1+\gamma\delta\epsilon\eta'(v)K_t(v)\Lambda_2.
$$

Finally, let us expand the $\Lambda_2(v^+)$ coefficient

$$
\Lambda_2(v^+)\sin\Big(\int_{\mathcal{I}_\epsilon} A_x dx\Big)\sin\Big(\frac{1}{2}\gamma\delta K_t(v^+)\Big)\cos\Big(\frac{1}{2}\gamma\delta K_t(v)\Big) \approx \gamma\epsilon\delta A_x(v)K_t(v)\Lambda_2.
$$

After synchronizing $\delta = \epsilon$ we hence find in the limit

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{\alpha_{31}-\alpha_{13}}{2\epsilon^2} &= \gamma K_t'\Lambda_1 + \gamma K_t(v)\big(\eta'(v)+A_x(v)\big)\Lambda_2 = \\ &= F_{x\theta} + \Gamma_t'\Lambda_2 - \Gamma_t\big(\eta'(v)+A_x(v)\big).\end{aligned}$$

Now we analyze the terms coming from the Thiemann trick:

$$\begin{aligned}h_3\{h_3^{-1},V\} &\approx -G\Lambda^3\gamma\,\mathrm{sgn}(E^x)\epsilon\frac{E^t}{2\sqrt{|E^x|}} \\ h_2\{h_2^{-1},V\} &\approx -G\Lambda^2\gamma\delta\sqrt{2|E^x|}.\end{aligned}$$

Additionally, we also take radial edges starting at v and going to v^- into account, e.g. the holonomies $h_3(e_v^{v^-})$, which in the classical approximation doubles each constribution to the final expression. Furthermore, we also take into account the loops $\varepsilon^{ij1}\alpha_{ij}$, which doubles the θ-ϕ terms once more. Then, the regularized integrand of gravitational scalar constraint reads

$$\begin{aligned}\mathcal{C}^\epsilon = &-\frac{\mathrm{sgn}(E^x)}{G^2\gamma^3\delta^2\epsilon}N(v)\sum_{\sigma=\pm}\sigma\times \\ &\mathrm{tr}\left\{\left[\varepsilon^{ijk}\alpha_{ij,\sigma} - 2\gamma^2[\delta^2(\Gamma_t^2-1)\Lambda^3 - \delta\int_v^{v^\sigma}\Gamma_t'(\Lambda^1+\Lambda^2)]\right]h_{k,\sigma}\{h_{k,\sigma}^{-1},V\}\right\}.\end{aligned}$$

Note that $\varepsilon^{ij3}\alpha_{ij,\sigma}\equiv\varepsilon^{ij3}\alpha_{ij}$, as well as $\varepsilon^{ij3}h_{i,\sigma}\equiv\varepsilon^{ij3}h_i$.
Now, let us reestablish the 3 dimensional expressions, firstly by observing the correspondences

$$\begin{aligned}\frac{1}{G}\{\cdot,\cdot\}_B &\leftrightarrow \frac{2}{\kappa}\{\cdot,\cdot\}_\Sigma \\ \frac{1}{G}\int_B &\leftrightarrow \frac{2}{\kappa}\int_{B\times S^2}.\end{aligned} \tag{11.11.26}$$

Hence we replace G by $\kappa/2$ in the integrand above. This is because of esthetical reasons as well as for reasons of better agreement with the regulated Hamiltonian of the full theory. If we again replace the integral $\int_{B\times S^2}$ by the Riemann sum $\sum_{\Delta\in\mathcal{T}^{\epsilon,\delta}}\epsilon\delta^2$ we find

$$\begin{aligned}C^\epsilon[N] = &-\frac{4}{\kappa^2\gamma^3}\sum_{\Delta\in\mathcal{T}^{\epsilon,\delta}}N(v)\sum_{\sigma=\pm}\sigma\times \\ &\mathrm{tr}\left\{\left[\varepsilon^{ijk}\alpha_{ij,\sigma} - 2\gamma^2\delta\big(\delta(\Gamma_t^2-1)\Lambda^3 - \int_v^{v^\sigma}\Gamma_t'\Lambda^I\delta_I^k\big)\right]h_{k,\sigma}\{h_{k,\sigma}^{-1},V\}\right\},\end{aligned}$$

where $I = 1,2$ and we absorbed the sign of E^x in the lapse function.
This form of the constraint looks like the Euclidean Hamiltonian of full LQG, except for

the factor 2/3 coming from using tetrahedron above, the factor $1/\gamma^2$, which is due to the choice of phase space variables, and the Γ_t contributions, which we put in by hand. The most important difference is that the expression at hand is already the complete scalar constraint of spherically symmetric Einstein theory, not the Euclidean part. We could even manage to absorb the regularization parameters again. Unfortunately the δ is still present, not surprisingly exactly appearing together with the Γ_t, which we put in by hand. This problem, however, can be circumvented. For this purpose Γ_t has to be expressed via flux variables. This results in the δ parameter appearing exactly to those powers needed to obtain the fluxes $E_3[S^2_\delta]$ and $E[S_{\theta/\phi}]$. We use

$$\Gamma_t = -\frac{(E^x)'}{2E^t} = -\frac{1}{4}\Big(\frac{E^x(v^+) - E^x(v)}{\int_v^{v^+} E^t} - \frac{E^x(v^-) - E^x(v)}{\int_v^{v^-} E^t}\Big) + \mathcal{O}(\epsilon)$$

and we write

$$\Gamma_t' = \Gamma_t(v^\sigma) - \Gamma_t(v).$$

Now, observe that with

$$\delta\Gamma_t = -\frac{1}{8}\sum_{I=1}^{2}\sum_{\sigma=\pm}\frac{E_3[S^2_\delta](v^\sigma) - E_3[S^2_\delta](v)}{E[S^\sigma_I]} + \mathcal{O}(\epsilon), \tag{11.11.27}$$

where $S_{1/2} \equiv S_{\theta/\phi}$, we can get rid of all the regularization parameters. Now, it is only the inverse fluxes are left to be handled. This has to be done analogously to the Thiemann trick.

We proceed with the matter part. This time, we will start right away from the full LQG expression modulo the regularization form factor $(1/3)^2$, since we do not use tetrahedra. Hence, this a opportunity to check our numerical prefactors. Since we have a $U(1)$ gauge group, we have to replace $-2\,\mathrm{tr}$ by -1. Furthermore, we replace, as before, the factor $2/\kappa$ with $1/G$. Again we also want to account for the different orientations of the radial edges, which gives another factor 1/4 due to the point splitting. Thus, collecting all the factors, implies that the prefactor in (11.11.19) gives

$$\frac{2}{G^2\gamma^2}. \tag{11.11.28}$$

Due to the ε-tensors we get four contributions of each summand. Then the regularized Maxwell Hamiltonian of the spherically symmetric theory on B reads

$$\begin{aligned} C_M^{\epsilon,\delta}[N] =& \frac{2}{G^2\gamma^2 e^2\delta^4}\sum_v\sum_{v'}\chi_{\epsilon,v}(v')N(v)\sum_{\sigma=\pm}\sum_{\sigma'=\pm}\sigma\sigma'\times \\ & \times \operatorname{Tr}(\tau_3 h_{3,\sigma}\{h_{3,\sigma}^{-1},\sqrt{V(v)}\})\operatorname{Tr}(\tau_3 h_{3,\sigma'}\{h_{3,\sigma'}^{-1},\sqrt{V(v')}\})\times \\ & \times\left[4\mathcal{E}[S_\delta^2]\mathcal{E}[(S_\delta^2)'] + \varepsilon^{3ij}\varepsilon^{3i'j'}\mathcal{B}_{ij}\mathcal{B}_{i'j'}\right]. \end{aligned} \tag{11.11.29}$$

Let us compute the parts of the summand in (11.11.29):

$$\begin{aligned} &\operatorname{Tr}(\Lambda^3 h_{x,\sigma}\{h_{x,\sigma}^{-1},\sqrt{V}\}) = \frac{\sigma G\gamma\sqrt{\epsilon}\sqrt{E^t}\,\operatorname{sgn}(E^x)}{8\sqrt[4]{|E^x|}\sqrt{|E^x|}} + \mathcal{O}(\epsilon^2) \\ &\mathcal{E}[S_\delta^2] = \delta^2 q^2 + \mathcal{O}(\delta^3) \\ &\mathcal{B}_{12} = 1\cdot e^{i\frac{n}{2}\delta\cos\left(\frac{\pi}{2}+\delta\right)}\cdot 1\cdot 1 = 1 + i\frac{n}{2}\delta^2 + \mathcal{O}(\delta^4) \\ &\frac{\mathcal{B}_{12}-\mathcal{B}_{21}}{2} = i\frac{n}{2}\delta^2 + \mathcal{O}(\delta^4). \end{aligned}$$

Collecting all pieces we find

$$\begin{aligned} &\lim_{\varepsilon\to 0} C_M^{\epsilon,\delta}[N] = \\ &\frac{2}{G^2\gamma^2e^2\delta^4}\frac{1}{\epsilon}\int_{\mathcal{I}}\int_{\mathcal{I}}\delta(v,v')N(v)4\cdot\frac{G^2\gamma^2\epsilon\sqrt{E^t(v)}\sqrt{E^t(v')}}{64\sqrt[4]{|E^x(v)|^3}\sqrt[4]{|E^x(v')|^3}}\cdot 4\cdot\delta^4\left[q(v)q(v') + \frac{n^2}{4}\right] = \\ &= \frac{1}{2e^2}\int_{\mathcal{I}}\frac{E^t}{\sqrt{|E^x|}|E^x|}\left(q^2+\frac{n^2}{4}\right), \end{aligned} \tag{11.11.30}$$

showing that the regularized expression yields the correct classical limit.

11.11.4 The volume operator

The quantization of the volume operator which is obviously essential for the loop quantization of the dynamics of all matter-gravitation coupled systems[17]. The volume functional of some region $\mathcal{R}$ written in terms of traids reads

$$\operatorname{Vol}[\mathcal{R}] = \int_{\mathcal{R}}\sqrt{|{*}\tilde{E}_1\wedge\tilde{E}_2\wedge\tilde{E}_3|}. \tag{11.11.31}$$

We partition this regions adapted to some graph using cubes C of coordinate volume ϵ^3, which we subdive into eight cells using three non-coplanar surfaces $\{\mathcal{S}^I\}$. We choose the three perpendicualar squares that half the cube with coordinate area ϵ^2. It is necessary

[17]The rigorous treatment can be found in [38]

that the if a cube contains a vertex, it should be positioned in the unique intersection point of the three surfaces. The edges of the graph at this vertex must not intersect the surfaces except in v or lie in a surface. Furthermore, if a cube does not contain a vertex, the graph punctures the triplet of surfaces at most in two points.
Now, we can proceed in the meanwhile familiar way to replace the integral with a Riemannian sum and use the electric fluxes instead of the densitized triads.

$$\mathrm{Vol}^{\epsilon}[\mathcal{R}] = \sum_{C\subset\mathcal{R}} \sqrt{|\frac{1}{3!}\varepsilon^{ijk}\varepsilon_{IJK}E_i[\mathcal{S}^I]E_j[\mathcal{S}^J]E_k[\mathcal{S}^K]|}. \tag{11.11.32}$$

After replacing the elementary variables with operators in this expression we obtain

$$\hat{\mathrm{Vol}}[\mathcal{R}] = \Big(\frac{\ell_P^2\gamma}{4}\Big)^{3/2}\sum_{v\in\mathcal{R}}\sum_{e,e',e''\ \text{at}\ v}\sqrt{|\frac{1}{3!}\varepsilon_{ijk}\sigma(e,e',e'')J^i_{(v,e)}J^j_{(v,e')}J^k_{(v,e'')}|}. \tag{11.11.33}$$

The factor $\sigma(e,e',e'')$ is an orientation factor, which vanishes whenever two edges have colinear tangents at v and equals a number $\gtrless 0$ according to their orientation compared to the one chosen classically in Σ. If a cube does not contain any vertex, the eigenvalue of the Volume operator is zero, since we required that the triplet is punctured at most twice. However, this orientation factor inherits a problem: it carries some memory of the regularization we used. Therefore, one averages over the background structures used, basically one uses the orientation preseving group $GL^+ = (3)$ in order to tilt the surfaces and averages over the orientation factors. This yield an orientation factor up to an arbitrary constant

$$\sigma^{\mathrm{av}}(e,e',e'') = \sigma_0\varepsilon(e,e',e'')$$
$$\varepsilon(e,e',e'') = \begin{cases} 1 \\ 0 \\ -1 \end{cases},$$

and we obtain the final expression for the volume operator[18]

$$\hat{\mathrm{Vol}}[\mathcal{R}] = \Big(\frac{\ell_P}{\sqrt{2}}\Big)^3\gamma^{3/2}\sum_{v\in\mathcal{R}}\sum_{e,e',e''\ \text{at}\ v}\sqrt{|\frac{1}{48}\varepsilon_{ijk}\varepsilon(e,e',e'')J^i_{(v,e)}J^j_{(v,e')}J^k_{(v,e'')}|}. \tag{11.11.34}$$

[18]In order to compare the result at hand with the ones given LQG literature e.g. [38], we put some of the numerical factors under the square root. The factor $1/48$ is the familiar one, while the overall factor $1/\sqrt{2}$ comes from a different numerical factor in the definition of the Poisson bracket used here in order to agree with the ADM Poisson bracket, when ADM variables are expressed in terms of Ashtekar variables. For a comparison with the formula given in [11] note that one uses a different convention for the $SU(2)$ generators and consequently for the angular momentum operators there.

Note that unexpectedly the action volume operator on gauge invariant trivalent vertices gives zero due to the constraint $(J^i_{(v,e)} + J^i_{(v,e')} + J^i_{(v,e'')})\Psi^{\text{inv}} = 0$, because

$$\begin{aligned}
&\varepsilon_{ijk} J^i_{(v,e)} J^j_{(v,e')} J^k_{(v,e'')} = -\varepsilon_{ijk} J^i_{(v,e)} J^j_{(v,e')} (J^k_{(v,e)} + J^k_{(v,e')})\Psi^{\text{inv}} = \\
&= i\varepsilon_{ijk}(\varepsilon^{ikl} J^l_{(v,e)} J^j_{(v,e')} + J^i_{(v,e)} \varepsilon^{jkl} J^l_{(v,e')})\Psi^{\text{inv}} = \\
&= 2i(J_{(v,e)} \cdot J_{(v,e')} - J_{(v,e)} \cdot J_{(v,e')})\Psi^{\text{inv}} = 0. \qquad (11.11.35)
\end{aligned}$$

In the symmetry reduced context the volume functional of some region reads

$$\text{Vol}[\mathcal{R}] = \int_{\mathcal{R}} \sqrt{|E^x|} E^t dx d\Omega. \qquad (11.11.36)$$

This time, we choose the partition similarily to the one chosen for regularizing the Hamiltonian constraint and find

$$\text{Vol}[\mathcal{R}] = \sum_{v \in \mathcal{R}} \text{Ar}[S^2_{v,\epsilon^2}] \sqrt{|\hat{Y}_{S^2_{v,\epsilon^2}}|} [\hat{Y}^1_{S_\theta}], \qquad (11.11.37)$$

with eigenvalues

$$V = \left(\frac{\ell_P}{\sqrt{2}}\right)^3 \frac{\gamma^{3/2} \text{Ar}[S^2_{v,\epsilon^2}]}{(8\pi)^{3/2}} \sum_{v \in \mathcal{R}} \sqrt{\frac{1}{2} |m_{e^+} + m_{e^-}| |\mu_v|}. \qquad (11.11.38)$$

11.11.5 The quantized Einstein-Yang-Mills Hamiltonian

What remains to be done is to write down the quantized expressions of the Hamiltonians involved in Einstein-Yang-Mills theory. Therefore, one uses the correspondence

$$\begin{aligned}
\{\cdot,\cdot\} &\to \frac{1}{i\hbar}[\cdot,\cdot], \\
\mathcal{E}_a[\mathcal{S}] &\to \frac{\hbar g^2}{2} \sum_{v \in \mathcal{S}} [\mathcal{J}^a_{(v,\text{up})} - \mathcal{J}^a_{(v,\text{down})}].
\end{aligned}$$

Following Thiemann let us now adapt the regularization to the underlying graph in the following way: Every vertex of the graph coincides with vertices of thetrahedrons. For each triple of edges at a vertex of the graph, introduce one tetrahedron such that the segments of the tetrahedron coincide with the three edges. The arcs of the tetrahedron connecting pairs of edges intersect the edges in points within the range of the edges. Thus, there are no further vertices of the graph between this intersection point and the vertex. Complete each tetrahedron for one triple of edges to an octahedron in the same way described above. The volume of a neighborhood of the vertex of the graph is approximated by these octahedra.

Then the Hamiltonian operators read

$$\boxed{\hat{C}_E[N] = \frac{2}{3}\frac{1}{\gamma}\left(\frac{4m_P}{i\ell_P^3}\right)\sum_{v\in V(\gamma)}\frac{8N(v)}{E(v)}\sum_{v(\Delta)=v}\mathrm{Tr}\left(\varepsilon^{IJK}h_{IJ}h_K[h_K^{-1},\hat{V}(v,\epsilon)]\right)} \tag{11.11.39}$$

$$\boxed{\begin{aligned}\hat{C}_{YM}[N] = -\left(\frac{1}{3}\right)^2\frac{128}{\gamma^2}\frac{\alpha_g m_P}{\ell_P^3}\sum_{v\in V(\gamma)}N(v)\left(\frac{8}{E(v)}\right)^2\sum_{v(\Delta)=v}\sum_{v'(\Delta')=v}\times\\ \times\,\mathrm{Tr}(\tau_i h_K[h_K^{-1},\sqrt{\hat{V}(v,\epsilon)}])\,\mathrm{Tr}(\tau_i h_{N'}[h_{N'}^{-1},\sqrt{\hat{V}(v',\epsilon)}])\times\\ \times\Big[\frac{1}{16}\delta^{IK}\delta^{J'N'}\delta_{ab}\mathcal{J}^a_{e_I}\mathcal{J}^b_{e_{J'}}+\frac{1}{\alpha_g^2}\varepsilon^{IJK}\varepsilon^{L'M'N'}\,\mathrm{tr}(T^a\mathcal{B}_{IJ})\,\mathrm{tr}(T^a\mathcal{B}_{L'M'})\Big]\end{aligned}}$$

$$\tag{11.11.40}$$

We introduced the correction factor

$$E(v) = \binom{n(v)}{3} \tag{11.11.41}$$

where $n(v)$ is the valence of v, giving the number of tetrahedra at v, one for each triple of edges. It prevents from overcounting the volume. The factor 8 accounts for the volume of the octahedron consisting of 8 tetrahedra.
Furthermore, we used the Planck mass defined by

$$m_P := \frac{\ell_P}{\kappa} = \sqrt{\frac{\hbar}{\kappa}}$$

and the fine structure constant defined by

$$\alpha_g = \hbar g^2.$$

Note that the factor $1/\ell_P^3$ cancels the dimensionality of the volume operators, such that the all Hamiltonians have mass dimension 1 (α_g is dimensionless). The action of the Euclidean Hamiltonian can be visualized as follows (see also figure 11.5):
Due to the volume operator the whole operator can only have nontrivial action on vertices where the volume is nonvanishing. Since the volume operator appears in a commutator with a holonomy, the action is also nontrivial on noncoplanar trivalent vertices because the intermediate holonomy breaks gauge invariance. The loop in the Hamiltonian adds (or removes) an edge with color 1/2 connecting two existing edges incident at the vertex. Now, the important point is that although the operator still carries a memory of the

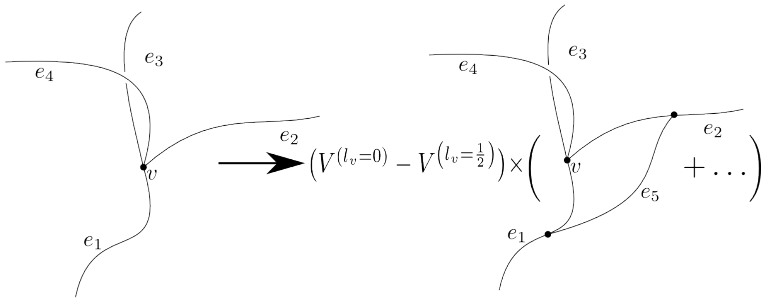


Figure 11.5: *The action of the "Euclidean" part of the Hamiltonian constraint.*

triangulation used, the quantity

$$(\eta(\Psi)|\hat{C}_E[N]\Phi\rangle, \quad \forall\Phi,\Psi \in \mathcal{D} \subseteq \mathcal{H},$$

$\mathcal{D}$ denoting the domain of the operator, is insensitive to the value of the regulator ϵ. At this point, background independence is essential!
In the reduced theory the Hamiltonians read

$$\boxed{\begin{aligned} C_G[N] = &-\frac{m_P}{\gamma^3\ell_P^3\delta^2}\sum_v N(v)\sum_{\sigma=\pm}\sigma\times \\ &\operatorname{tr}\left\{\left[\varepsilon^{ijk}\alpha_{ij,\sigma} - 2\gamma^2[\delta^2(\hat{\Gamma}_t^2 - 1)\Lambda^3 - \delta\int_v^{v^\sigma}\hat{\Gamma}'_t(\Lambda^1+\Lambda^2)]\right]h_{k,\sigma}\{h_{k,\sigma}^{-1},\hat{V}\}\right\}\end{aligned}} \tag{11.11.42}$$

$$\boxed{\begin{aligned} C_M[N] = &\frac{2}{\gamma^2}\frac{\alpha_e m_p}{\ell_P^3\delta^4}\sum_v N(v)\sum_{\sigma=\pm}\sum_{\sigma'=\pm}\sigma\sigma'\times \\ &\times \operatorname{Tr}(\tau_3 h_{3,\sigma}\{h_{3,\sigma}^{-1},\sqrt{V(v)}\})\operatorname{Tr}(\tau_3 h_{3,\sigma'}\{h_{3,\sigma'}^{-1},\sqrt{V(v')}\})\times \\ &\times\left[\mathcal{J}_{v,\sigma}\mathcal{J}_{v,\sigma'} + \frac{1}{\alpha_e^2}\varepsilon^{3ij}\varepsilon^{3i'j'}\mathcal{B}_{ij}\mathcal{B}_{i'j'}\right]\end{aligned}} \tag{11.11.43}$$

Chapter 12

Conclusion

12.1 Summary

Let us summarize the major steps of a loop quantization of GR and symmetry reduction: Starting from the so-called ADM decomposition of spacetime into spacelike hypersurfaces we examined the corresponding initial value formulation of GR. We were able to find evolution equations for the intrinsic metric on the slices and the extrinsic curvature, which described the embedding of the slices. Furthermore, we found two equations constraining the initial data consisting of the intrinsic metric and the extrinsic curvature.
In Chapter 2 we approached the initial value formulation of GR from a somewhat different less (spacetime) geometrical point of view and cast the theory into a symplectic framework. This shed some light on the constraints found before. Actually, we saw that GR is a completely constraint Hamiltonian system. There does not exist any "true" Hamiltonian it is constrained to vanish, which is to be expected since there is no preferred notion of time. Apart from the Hamiltonian constraint, which is also called scalar constraint, we found a second constraint also called the vector constraint.
Both constraints were shown to be first class, which roughly means that the corresponding Poisson algebra closes on shell. This told us that we started with too many degrees of freedom, namely ten instead of two, and that physical solutions are located on the constraint surfaces in phase space. In particular, the corresponding Hamiltonian vector fields generate spatial diffeomorphisms and diffeomorphisms in the direction of the surface normal of the equal time slice (on shell). Therefore, the foliation is completely arbitrary, thereby recovering the full symmetry group of GR.
Then we geared the formulation of GR towards a gauge theory by introducing further "unphysical" degrees of freedom, the vielbeins. Instead of the metric encoding length and angles, we chose the inertial frames, which could be considered as our laboratories in space (-time). Of course, physics should be insensitive to the posture of the laboratory, nevertheless the physicist chooses some reference directions, most conveniently orthogo-

nal ones and thus fixes the gauge in mathematical terms. We showed that we can also formulate GR in terms of densitized triads as momentum variables and their configuration variables, which we found to be specific components of the Lorentz connection. Due to the enlarged degrees of freedom an additional first class constraint appeared called the Gauß constraint, which generates rotations between the different frames. Solving the Gauß constraint forces the components of the configuration variable to be symmetric, which reveals that the configuration variable on the gauge invariant subspace actually is the extrinsic curvature. In addition vielbeins provide a natural means for incoorporating spinors in the theory.

Next, we made the notion of gauge theory mathematically more precise in terms of fiber bundle theory. Having triads located in space, we were interested in how they are *connected*, i.e. how to parallel transport them. At this point, we developed the basic building blocks of gauge theory: connection and curvature or accordingly - formulated in some local trivialization - gauge potential and field strength.

By using this theory we were then able to implement a symmetry reduction of a gauge theory in a definite way. We started with an action of some symmetry group acting via bundle automorphisms and projected its action to the base space. Then we discussed the decomposition of the base manifold into orbits and concentrated on the orbit space. We restricted the bundle over that space and reduced the structure group. Finally, we could derive an explicit formula for an (gauge fixed) invariant connection on an invariant principal fiber bundle, which gives rise to a connection on the reduced bundle and to scalar fields, the latter being remnants of the full connection.

After these quite general description we made things more concrete and analyzed the structure of the group of rotations. The purpose of this analysis was twofold: On the one hand, the symmetry group generating the transformation under which the connection (or any other field) is said to spherically symmetric is the group of rotations in space which is isomorphic to $SO(3)$. On the other hand, as was shown later, the group $SU(2)$ – the universal cover of $SO(3)$ – is the gauge group from which the most successful aproach of quantum gravity based on the Ashtekar formulation of GR so far, starts. Furthermore, we devoloped some Lie algebra notions. In particular, we discussed the left (right) invariant vector fields and the canonical left invariant Maurer Cartan form on the group $SU(2)$. We proved that $SU(2)$ is the universal (double) cover of $SO(3)$ and showed some properties of the adjoint and fundamental representations of $SU(2)$. Finally, we discussed briefly some notions of representation theory. In particular we derived the explicit form of the Haar measure on $SU(2)$ and $U(1)$. Most important for the development of LQG in this section was the Peter - Weyl theorem, which we could partly prove. It states that we can use the matrix elements of the irreducible (unitary) representations of a compact Lie group to approximate every continuous function on the group and that they provide an orthogonal basis, which we also normalized. This was made explicit in the example of

$U(1)$.

Since the formulation of connection variables due to Ashtekar in terms of spin connections, which was needed in the later construction, we discussed some properties of Clifford algebras in chapter 7, particularly the Pauli algebra. This analysis revealed that the notion of spinors is intimately tied to geometry of space. We rediscovered ideas by Clifford, Grassmann and Hamilton. The algebra we set up naturally incorporates the isometries of the underlying vector space, generated by subsequent reflections. One of the subgroups we found was $Spin(3)$, generating rotations up to phase. Spinors where then found to be the elements of the representation space, which we constructed as ideals of the Clifford algebra. In the matrix representation automatically $SU(2) \sim Spin(3)$ reappeared. Finally we derived the necessary and sufficient conditions for a construction of a spin bundle out of a frame bundle. A spin connection describes parallel transport of spinors, which via the 2-1 covering induces parallel transport of vectors.

Equipped with all necessary tools we introduced the Ashtekar connection. It combines the intrinsic $SU(2)$ connection of the frame bundle over the base manifold Σ and the conjugate configuration variables to the densitized triads multiplied with a parameter called the Immirzi parameter. The classical purely gravitational theory is invariant under the choice of this, in general complex, parameter. By using the Ashtekar connection we were able to recast the Gauß constraint into a form which justifies its naming. We mentioned the attemps to find a action yielding the Ashtekar variables directly that go under the name Holst action. Again its analysis showed that spinors are intimately tied to geometry, i.e. gravitation. Clearly, from the spin bundle approach one could say that spinors configure in the gauge group of gravity. The Holst modification has a nontrivial effect in the spinor sector. In the framework of the Ashtekar connection we performed the spherical symmetry reduction of GR. Chapter 9 is devoted to an analogous description for Yang-Mills theories and electodynamics in particular.

In chapter 10 we applied the connection dynamics of Einstein-Maxwell theory to stationary solutlions. We solved the partial differential evolution equations, i.e. the Hamilton equations of the densitzed triad, the symmetry reduced Ashtekar connection and the scalar fields. The solution was adapted to asymptotic freedom and we found the so-called Reisner-Nordström solution, i.e. the spherically symmetric metric of a charged static black hole, to Einstein field equations. We could then see how the densitized triads behave at the locations of the classical singularity and the event horizons.

Chapter 11 focused on the quantization of Einstein-Yang-Mills connection dynamics. We put emphasis on the parallel transport, again using the general framework of fiber bundle theory. First, we rigorously solved the defining equation of the parallel transport, yielding a path ordered exponential. The notion of parallel transport is connected to the notion of paths, equivalence classes of curves along which the parallel transport takes place under reparametrizations and retracings. Gauge transformations act homogeneously on the parallel transport. This transformation property is a mathematical motivation to choose the parallel transport as elementary variable for the quantization. We also examined a physical motivation for that choice, the Aharonov-Bohm effect and its fiber bundle theoretic interpretation. It is the holonomy (along loops) - an observable phase shift, which incorporates exactly the physical information of a quantum mechanical system interacting with a gauge field. After this short excursus we completed the set of elementary variables with the "electric" fluxes, which are the dualized densitized triads integrated over surfaces, constituting a so-called Poisson *-algebra. The Poisson bracket of elementary variables has very nice properties: A flux acting on holonomies gives a holonomy again, but at the price of having non-commutating fluxes, due to the distributional smearing. We then briefly discussed the construction of the representation space, introducing cylindrical functions over graphs. By using the Haar measure the space of cylindrical functions was eqipped with the diffeomorphism and gauge invariant Ashtekar Lewandowski measure. The completion of the cylindrical functions w.r.t. to this inner product yielded a (non-separable) Hilbert space. The discussion of the Peter-Weyl theorem in 6.3 provided the basic building blocks of an orthonormal basis of this Hilbert space called spin network basis. This construction furnishes the basis for the quantization of the constraints as operators on the kinematical Hilbert space. Also the solutions to the Gauß and diffeomorphism constraint were presented, as well as an inner product on the solution space. It turned out that generically the solutions to the diffeomorphism constraint, except for the constant function, are elements of the algebraic dual to the space of cylindrical functions. After motivating a representation of the Poisson *-algebra we applied the general quantization scheme to spherically reduced Einstein-Maxwell theory and wrote down the explicit expressions for spherically symmetric spin-charge networks. As an example of how LQG supports a discrete picture of space, at least on a kinematical level, we constructed the area operator and its spectrum. The final section was concerned with the quantization of the dynamics of LQG. Apart from several ambiguities, which one always encounters in a quantization of a field theory, one is able to find a version of the quantum Hamiltonian. The extension to Yang-Mills matter is straight forward. In the construction of these operators another geometric operator, namely the volume operator became prominent and we also derived its quantized version. Finally, the finding of the loop quantized versions of the Einstein and Maxwell Hamiltonians concluded the analysis of spherically symmetric loop quantum gravity in this thesis.

12.2 Discussion and outlook

For the major part, this thesis reviewed the approach to symmetry reduction developed by A. Ashtekar, M. Bojowald, H.A. Kastrup, J. Lewandowski and R. Swiderski. As it was shown, the reduction scheme relies heavily on the notion of invariant connections and the definition of symmetric states. The symmetric states have support only on invariant connections and are therefore to be considered as distributions on the space of cylindrical functions. Of course, the spin networks are not symmetric under a continuous symmetry group. Thus, it is not really surprising that difficulties arise. They arose in the form of the continuous parameter labeling the Higgs vertex. Usually this is interpretated as a merging of the continuous edge length and the discrete spin label into one parameter. We lost one of the most intriguing features of loop quantum gravity - discreteness of space: quanta volume and area. The drawback is then circumvented by carrying over results from the full theory to the symmetry reduced one. For example in homogeneous cosmological models the invariant connection is completely determined by the Higgs fields and therefore no discrete label survives. The parameter carries area and from the full theory one knows that there is an area gap. This result has to be imposed by hand in the reduced sector. In the case of spherically symmetry reduced theory the discreteness only in radial directions remains. Higgs fields represent the whole 2-sphere. Furthermore, only diffeomorphisms acting on the radial submanifold remain. We have seen that for the well-posedness of the Hamiltonian constraint diffemorphisms are crucial to obtain an action of the constraint, which is insensitive to a (sufficiently small) regularization parameter. In the reduction process we have also lost these diffeomorphisms acting transversal to the orbit space. Note that it is not the explicit dependence on the parameter length of regularization edges in the homogeneous directions, which is problematic. By reinstating three dimensional Riemannian sums via geodesic cubes as regulators, instead of edges in the radial direction only, we can absorb these parameters.

Let us turn now to another subtlety in the reduction scheme which we blithely ignored up to now. This is related to the specific choice of the phase space coordinates, sometimes referred to as polar coordinates. One makes use of a rotated internal $SU(2)$ basis and introduces the gauge invariant fields $\tilde{E}^t$ and A_t and the angle fields η and β. Unfortunately the fields $\tilde{E}^t$ and A_t do not constitute a conjugate pair. The reason for that we traced back to the internal directions of the intrinsic connection 1-form, which where "orthogonal" to those of the densitized triad field. If one wants to stick to these variables one has to choose between the flux and the connection representation. In view of the specific form of the volume functional the flux representation seems to be more convenient. From the general discussion of the vielbein formalism it is well known that the field $\tilde{E}^a_i$ is canocically conjugate to K^i_a which is part of the Ashtekar connection. This is still true here; $\gamma K_t = A_t - \Gamma_t = A_t \cos\alpha$, where $\alpha = \eta - \beta$ is conjugate to $\tilde{E}^t$. But the argument in [37] goes further. They conclude from the form of the extrinsic curvature of the equal time slices, which has internal directions alligned with those of the densitized triad that $A_t \sin\alpha = \Gamma_t$. This conclusion is false. Despite the intriguing notation used for the object K^i_a appearing in the Ashtekar connection, it is not identically to the extrinsic curvature, this is only true on the Gauß constraint surface in phase space. The Gauß constraint in the vielbein formalism is a symmetry requirement imposed on K_a^i, which in case of spherical symmetry alligns the internal directions with those of the triad. Thus, from that point the reduction process always uses the gauge invariant subspace implicitely. This led to the much simpler form of the Hamiltonian. Making the steps, carried out for the kinetic part of the Hamiltonian, unnecessary. Actually, that choice of variables renders these steps impossible.

Of course, one can say that in the final picture one is only interested in physical, i.e. gauge invariant states. Furthermore, there can always arise ambiguities when defining constraints up to terms proportional to the Gausß constraint, as we have seen in the relation of the vector and the diffeomophism constraint in Yang-Mills theory or in the relation of ADM theory and Ashtekar's connection dynamics. Apart from that, one reduced the full spin-networks to networks immersed in a one dimensional manifold with Higgs vertices. Thus, this procedure is consistent with the general quantum reduction schmeme developed in [19]. The symmetric states were characterized as cylindrical functions on generalized connections restricted to B times equivalence classes of generalized Higgs fields w.r.t. gauge.

On the other hand, in my opinion the above mentioned problems with the continuous label can only be solved, if the full three dimensional picture is reinstated and the definition of symmetric states is changed. Of course, development of symmetry reduced models is of great importance, since these models render calculations more manageable than in the full theory. The theory of gravitation ows its success to such models since Newton, in particular the spherically symmetric ones, which are designed to describe the gravitational field of astrophysical objects, such as planets, stars of all kinds, and blackholes. If the angular momentum of such objects is sufficiently small calculations fit extremely well to observational data. Another example for the success of symmetry reduced models is, of course, the FRW model. It is argueable, however, whether the use of a continuous symmetry group is appropiate in the quanum theory presented here.

What I have in mind here is the following: In the quantum regime the connection looses importance. What is important are the spin-networks, labeling graphs with spin quantum numbers. We could equivalently describe spherical symmetry via measurements. For example, in the vincinity of a point charge measurements of the electric flux through some small reference plaquette will yield the same signal at any point on a sphere centered at the point charge. Applying this idea to a spin network one could require that the area operator on some plaquette “tangential to a sphere” gives the same eigenvalue whenever an edge punctures it. Futhermore, the eigenvalue of the volume operator acting on different vertices on the sphere should be the same. At this point I have to specify the term “sphere”, without using background structure: Consider a subnet of the spinnetwork, consisting of equally valent vertices, and the incident edges should carry the same label.

The number of vertices of this subnet is finite. If there exist edges, which do not link two such equivalent vertices, similar links, i.e. with the same labels, exists at all the vertices connecting them with another subnet characterized in the same way. If the vertices of subnets are linked, the links must not be knotted if the vertices are immersed in some two dimensional surface diffeomorphic to the 2-sphere. One can visualize such an object as stacked fullerenes, but one has to be careful with this picture, since the vertices of the subnets need not to be linked to each other at all. Furthermore, a subnet could consist of one single vertex. In order to get more control over these objects, introduce some background structure given by the rotation group and distribute the vertices of the subnets on the orbits uniformly. One can then classify these networks with graph symmetries, which are discrete subgroups of $SO(3)$. For example, the subnet with the lowest number of vertices togehter with the valence of these vertices determines the degree of symmetry of the graph. Then one would have average over these discrete graph symmetries.
The action of the Hamiltonian constraints break the symmetry of single networks, but yield a linear combinations of networks. Measuring the volume of some region centered at some vertex still yields the same result irrespective of the position on the sphere. Furthermore, one should also apply these constructions to the entropy counting of black holes.
An extension of this framework using spinorial matter leads to an interesting and relatively simple homogeneous toy model. Three valent spin-networks with spinors located at some vertices carry nontrivial volume. Due to the matter spin the gravitational Gauß constraint does in general not vanish, the matter spin gives a source term, in a similar fashion as charges serve as sources for the electric field. As above we start with networks where edges are all labeled equally, e.g. $j = 1/2$ and where matter is located at every vertex. Acting with the Hamiltonian again yields a three valent spinnetwork, only volumes, egde labels and excitations of spinors are changed.
Making these ideas more precise and more mathematically rigorous is a work in progress.

Appendix A

Appendix

A.1 Symmetry of the extrinsic curvature

$$\begin{aligned}
K_{ab} &= -h_a{}^c h_b{}^d \nabla_c (N \nabla_d t) = -h_a{}^c h_b{}^d \nabla_c N \nabla_d t - N h_a{}^c h_b{}^d \nabla_c \nabla_d t = \\
&= h_a{}^c h_b{}^d \nabla_c N \frac{1}{N} n_d - f h_a{}^c h_b{}^d \nabla_c \nabla_d t = -N h_a{}^c h_b{}^d \nabla_d \nabla_c t = \\
&= -h_a{}^c h_b{}^d \nabla_d (N \nabla_c t) + h_a{}^c h_b{}^d \nabla_d N \nabla_c t = h_a{}^c h_b{}^d \nabla_d n_c = \\
&= h_b{}^d \nabla_d n_a = K_{ba}
\end{aligned}$$

A.2 The relation of the extrinsic curvature and the Lie-derivative of the instrinsic metric along the surface normal

$$\begin{aligned}
K_{ab} &= \frac{1}{2}(K_{ab} + K_{ba}) = \frac{1}{2}(h_a{}^c h_{bd} \nabla_c n^d + h_b{}^c h_{ad} \nabla_c n^d) = \\
&= \frac{1}{2}[h_{bd} \nabla_a n^d + h_{bd} n_a (n\nabla) n^d + h_{ad} \nabla_b n^d + h_{ad} n_b (n\nabla) n^d] = \\
&= \frac{1}{2}[h_{db} \nabla_a n^d + n_a (n\nabla) n_b + h_{ad} \nabla_b n^d + n_b (n\nabla) n_a] = \\
&= \frac{1}{2}[h_{db} \nabla_a n^d + h_{ad} \nabla_b n^d + (n\nabla)(n_a n_b)] = \\
&= \frac{1}{2}[h_{db} \nabla_a n^d + h_{ad} \nabla_b n^d + (n\nabla) h_{ab}] = \frac{1}{2} \pounds_n h_{ab}
\end{aligned}$$

A.3 The covariant derivative associated with the spatial metric

$$D_a T_{b_1 \ldots b_n} := h_a{}^c h_{b_1}^{d_1} \ldots h_{b_n}^{d_n} \nabla_c T_{d_1 \ldots d_n}$$

- Linearity:

$$\begin{aligned} &D_a(\alpha T_{b_1 \ldots b_n} + \beta U_{b_1 \ldots b_n}) = \alpha h_a{}^c h_{b_1}^{d_1} \ldots h_{b_n}^{d_n} \nabla_c (\alpha T_{b_1 \ldots b_n} + \beta U_{b_1 \ldots b_n}) = \\ &= \alpha h_a{}^c h_{b_1}^{d_1} \ldots h_{b_n}^{d_n} \nabla_c T_{d_1 \ldots d_n} + \beta h_a{}^c h_{b_1}^{d_1} \ldots h_{b_n}^{d_n} \nabla_c U_{d_1 \ldots d_n} = \\ &= \alpha D_a T_{b_1 \ldots b_n} + \beta D_a \beta U_{b_1 \ldots b_n} \end{aligned}$$

- Leibniz rule

$$\begin{aligned} &D_a(T_{b_1 \ldots b_n} U_{c_1 \ldots c_m}) = h_a{}^c h_{b_1}^{d_1} \ldots h_{b_n}^{d_n} h_{c_1}^{e_1} \ldots h_{c_m}^{e_m} \nabla_c T_{b_1 \ldots b_n} U_{c_1 \ldots c_m} + \\ &+ h_a{}^c h_{b_1}^{d_1} \ldots h_{b_n}^{d_n} h_{c_1}^{e_1} \ldots h_{c_m}^{e_m} T_{b_1 \ldots b_n} \nabla_c U_{c_1 \ldots c_m} = \\ &= D_a T_{b_1 \ldots b_n} U_{c_1 \ldots c_m} + T_{b_1 \ldots b_n} D_a U_{c_1 \ldots c_m} \end{aligned}$$

- Commutativity with contraction

$$\begin{aligned} &D_a(g^{a_i a_j} T_{a_1 \ldots a_i \ldots a_j \ldots a_n}) = h_a{}^c h_{a_1}^{b_1} \ldots h_{a_{i-1}}^{b_{i-1}} h_{a_{i+1}}^{b_{i+1}} \ldots h_{a_{j-1}}^{b_{j-1}} h_{a_{j+1}}^{b_{j+1}} \ldots h_{a_n}^{b_n} g^{a_i a_j} \nabla_c T_{b_1 \ldots a_i \ldots a_j \ldots b_n} = \\ &= g^{a_i a_j} D_a T_{a_1 \ldots a_i \ldots a_j \ldots a_n} \end{aligned}$$

- Symmetry

$$\begin{aligned} &D_a D_b f = h_a{}^c h_b{}^d \nabla_c (h_d{}^e \nabla_e f) = h_a{}^c h_b{}^d \nabla_c (g_d{}^e \nabla_e f + n_d n^e \nabla_e f) = \\ &= h_a{}^c h_b{}^e \nabla_e \nabla_c f + h_a{}^c h_b{}^d \nabla_c n_d n^e \nabla_e f = \\ &= h_a{}^c h_b{}^e \nabla_e (h_c{}^f \nabla_f f) - h_a{}^c h_b{}^e \nabla_e (n_c n^f \nabla_f f) + K_{ab}(n\nabla) f = \\ &= h_a{}^c h_b{}^e \nabla_e (D_c f) - K_{ba}(n\nabla) f + K_{ab}(n\nabla) f = D_b D_a f \\ &\Rightarrow [D_a, D_b] f = 0 \end{aligned}$$

- annihilates the intrinsic metric

$$D_a h_{bc} = h_a{}^d h_b{}^e h_c{}^f (\nabla_d h_{ef}) = h_a{}^d h_b{}^e h_c{}^f [\nabla_d (g_{ef} - n_e n_f)] = 0$$

A.4 Time derivative of the intrinsic metric

$$\begin{aligned}\frac{1}{2}\pounds_n h_{ab} &= \frac{1}{2}[h_{db}\nabla_a n^d + h_{ad}\nabla_b n^d + (n\nabla)h_{ab}] = \\ &= \frac{1}{2}h_a{}^f h_b{}^g[h_{dg}\nabla_f n^d + h_{fd}\nabla_g n^d + (n\nabla)h_{fg}] \end{aligned} \tag{A.4.1}$$

Note that none of the terms of the second line are simply the projected of the corresponding ones in the first line, but that all additional terms cancel. Then we use

$$h_a{}^f h_b{}^g (N\nabla)h_{fg} = h_a{}^f h_b{}^g (N\nabla)(n_f n_g) = 0.$$

A.5 Curvature of the 3-d intrinsic connection

$$\begin{aligned}{}^{(3)}R^d{}_{cab}v^c &:= [D_a, D_b]v^d, \quad v^a \in \Sigma, \quad (\Rightarrow (nv) = 0) \\ D_a D_b v^d &= h_a{}^c h_b{}^e h^d{}_g \nabla_c (h_e{}^f h^g{}_h \nabla_f v^h) = \\ &= h_a{}^c h_b{}^e h^d{}_g \nabla_c[\nabla_e v^g + n^g n_h \nabla_e v^h + n_e n^g n_h (n\nabla)v^h + n_e(n\nabla)v^g)] = \\ &= h_a{}^c h_b{}^e h^d{}_g [\nabla_c \nabla_e v^g + \nabla_c n^g n_h \nabla_e v^h + \nabla_c n_e (n\nabla)v^g] = \\ &= h_a{}^c h_b{}^e h^d{}_g \nabla_c \nabla_e v^g + K_a{}^d h_b{}^e n_h \nabla_e v^h + K_{ab} h^d{}_g (n\nabla)v^g = \\ &= h_a{}^c h_b{}^e h^d{}_g \nabla_c \nabla_e v^g + K_a{}^d h_b{}^e \nabla_e (nv) - K_a{}^d K_{bh} v^h + K_{ab} h^d{}_g (n\nabla)v^g = \\ &= h_a{}^c h_b{}^e h^d{}_g \nabla_c \nabla_e v^g - K_a{}^d K_{bh} v^h + K_{ab} h^d{}_g (n\nabla)v^g\end{aligned}$$

A.6 Time derivative of the extrinsic curvature

$$\dot{K}_{ab} = h_a{}^c h_b{}^d \pounds_t K_{cd} = h_a{}^c h_b{}^d \pounds_{(Nn)} K_{cd} + h_a{}^c h_b{}^d \pounds_N K_{cd}$$

$$\begin{aligned}
&h_a{}^c h_b{}^d \pounds_N K_{cd} = h_a{}^c h_b{}^d [K_{ce} \nabla_d N^e + K_{ed} \nabla_c N^e + N^e \nabla_e K_{cd}] = \\
&= K_{ae} D_b N^e + K_{eb} D_a N^e + N^e D_e K_{ab} = 2K_{(a}{}^e D_{b)} N_e + N^e D_e K_{ab} \\
&h_a{}^c h_b{}^d \pounds_{Nn} K_{cd} = N h_a{}^c h_b{}^d \pounds_n K_{cd} = \\
&= N h_a{}^c h_b{}^d [K_{ce} \nabla_d n^e + K_{ed} \nabla_c n^e + (n\nabla) K_{cd}] = \\
&= N 2 K_a{}^e K_{eb} + N h_a{}^c h_b{}^d (n\nabla) K_{cd}
\end{aligned} \tag{A.6.1}$$

$$\begin{aligned}
&N h_a{}^c h_b{}^d (n\nabla) K_{cd} = N h_a{}^c h_b{}^d (n\nabla)(h_c{}^e \nabla_e n_d) = \\
&= N h_a{}^c h_b{}^d [(n\nabla) h_c{}^e \nabla_e n_d + h_c{}^e (n\nabla) \nabla_e n_d] = \\
&= N h_a{}^c h_b{}^d [(n\nabla)(n_c n^e) \nabla_e n_d + h_c{}^e n^m \nabla_m \nabla_e n_d] = \\
&= N h_a{}^c h_b{}^d [(n\nabla) n_c (n\nabla) n_d - h_c{}^e n^m n_k R^k{}_{dme} + h_c{}^e n^m \nabla_e \nabla_m n_d]
\end{aligned} \tag{A.6.2}$$

The three terms in brackets:

$$\begin{aligned}
&N h_a{}^c (n\nabla) n_c h_b{}^d (n\nabla) n_d = N h_a{}^c (n\nabla)(N \nabla_c t) h_b{}^d (n\nabla)(N \nabla_d t) = \\
&= N^3 h_a{}^c n^m \nabla_c \nabla_m t h_b{}^d n^n \nabla_d \nabla_n t = \frac{1}{N} D_a N D_b N
\end{aligned}$$

$$N h_a{}^e h_b{}^d n^m n_k R^k{}_{dme} = N h_a{}^e h_b{}^d h^m_k R^k{}_{dme} - N h_a{}^e h_b{}^d R_{de}$$

with (1.0.6) $Nh_a^{\ e}h_b^{\ d}h^m_{\ k}R^k_{\ dme} = N(\ ^{(3)}R_{ab} + KK_{ab} - K_b^{\ c}K_{ac})$

with EFE $R_{ab} = 8\pi(T_{ab} - \frac{1}{2}Tg_{ab}) = 8\pi(T_{ab} - \frac{1}{2}T_{cd}h^{cd}g_{ab} + \frac{1}{2}T_{cd}n^c n^d g_{ab}) =$

$= 8\pi(T_{ab} - \frac{1}{2}Jg_{ab} + \frac{1}{2}\rho g_{ab})$

$\Rightarrow Nh_a^{\ e}h_b^{\ d}R_{de} = N8\pi(J_{ab} - \frac{1}{2}Jh_{ab} + \frac{1}{2}\rho h_{ab})$

$$
\begin{aligned}
&Nh_a^{\ e}h_b^{\ d}n^m\nabla_e\nabla_m n_d = Nh_a^{\ e}h_b^{\ d}n^m\nabla_e\nabla_m n_d = \\
&= -Nh_a^{\ e}h_b^{\ d}n^m\nabla_e(h_d^{\ g}\nabla_m(N\nabla_g t)) = \\
&= -Nh_a^{\ e}h_b^{\ d}n^m\nabla_e(h_d^{\ g}N\nabla_g\nabla_m t) = Nh_a^{\ e}h_b^{\ d}n^m\nabla_e(K_{dm} - h_d^{\ g}\frac{n_m}{N}\nabla_g N) = \\
&= Nh_a^{\ e}h_b^{\ d}[\nabla_e(n^m K_{dm}) - \nabla_e n^m K_{dm} + \frac{1}{N}\nabla_e D_d N - n^m D_d N\nabla_e\frac{n_m}{N}] = \\
&= -NK_a^{\ m}K_{mb} + D_aD_bN + ND_bND_a\frac{1}{N} = \\
&= -NK_a^{\ m}K_{mb} + D_aD_bN - \frac{1}{N}D_bND_aN
\end{aligned}
$$

Thus, we find for (A.6.1)

$$
\begin{aligned}
h_a^{\ c}h_b^{\ d}\pounds_{Nn}K_{cd} = &- N\big(\ ^{(3)}R_{ab} + K_{ab}K - 2K_a^{\ c}K_{cb}\big) \\
&+ 8\pi\big(J_{ab} - \frac{1}{2}Jh_{ab} + \frac{1}{2}\rho h_{ab}\big) + D_aD_bN
\end{aligned}
\tag{A.6.3}
$$

Collecting all terms:

$$
\begin{aligned}
\dot{K}_{ab} = &2K_{(a}^{\ \ e}D_{b)}N_e + N^eD_eK_{ab} - N\big(\ ^{(3)}R_{ab} + K_{ab}K - 2K_a^{\ c}K_{cb}\big) + \\
&+ 8N\pi\big(J_{ab} - \frac{1}{2}Jh_{ab} + \frac{1}{2}\rho h_{ab}\big) + D_aD_bN
\end{aligned}
$$

A.7 The Einstein Hilbert Hawking action

The following relations will be used quite often:

$$
\begin{aligned}
&\delta g_{ab} \mid_{\partial\mathcal{R}} = 0 \\
&\delta(g_{ab}) \equiv \delta g_{ab} \\
&\delta(g^{ab}) = \delta(g^{-1})^{ab} = -\delta g^{ab} \\
&\delta\omega_g = \omega_g\frac{1}{2}g^{ab}\delta g_{ab}
\end{aligned}
\tag{A.7.1}
$$

where for the last equation we used (the λ_i are the eigenvalues of g_{ab})

$$\begin{aligned}\delta\sqrt{-g} &= \sqrt{-g}\delta\ln(\sqrt{-g}) = \frac{1}{2}\sqrt{-g}\delta\ln(\prod_i \lambda_i) = \sqrt{-g}\delta\sum_i \ln(\lambda_i) = \\ &= \sqrt{-g}\sum_i \frac{1}{\lambda_i}\delta\lambda_i = \frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ab}.\end{aligned}$$

For the calculation of the curvature quantities we need the the variation of the difference tensor. Let g_{ab} and $\tilde{g}_{ab} = g_{ab} + \delta g_{ab}$ be two "neighboring" metrics and ∇_a and $\tilde{\nabla}_a$ their respective Levi-Civita connections, then the action of $\tilde{\nabla}_a - \nabla_a$ on some vector field is given by a unique tensor-field of valence $[1,2]$, where the lower two indices are symmetric.

$$\begin{aligned}\nabla_a v^b = \tilde{\nabla}_a v^b + C^b_{ac} v^c = \nabla_a v^b \quad \text{v fixed} \\ \nabla_a \tilde{g}_{bc} = \tilde{\nabla}_a \tilde{g}_{bc} - C^k_{ab}\tilde{g}_{kc} - C^k_{ac}\tilde{g}_{bk}\end{aligned} \tag{A.7.2}$$

The same is true for the last equation with cyclic permuted indices. Adding two equations and subtracting the third gives the expression for the difference tensor.

$$C^b_{ac} = \frac{1}{2}\tilde{g}^{bd}(\nabla_c \tilde{g}_{ad} + \nabla_a \tilde{g}_{cd} - \nabla_d \tilde{g}_{ac})$$

Reinserting $\tilde{g}_{ab} = g_{ab} + \delta g_{ab}$ gives to first order in δg_{ab}

$$\delta C^b_{ac} = \frac{1}{2}(\nabla_c \delta g^b_a + \nabla_a \delta g^b_c - \nabla^b \delta g_{ac}).$$

Then for the curvature tensors follows

$$\begin{aligned}&\tilde{R}^a{}_{bcd} = R^a{}_{bcd} + \nabla_c C^a{}_{bd} - \nabla_d C^a{}_{bc} + C^a{}_{mc} C^m{}_{bd} + C^a{}_{md} C^m{}_{bc} \\ &\delta(R^a{}_{bcd}) = \nabla_c \delta C^a{}_{bd} - \nabla_d \delta C^a{}_{bc} \\ &\delta(R_{bd}) = \nabla_a \delta C^a{}_{bd} - \nabla_d \delta C^a{}_{ba} = \nabla_a \delta C^a{}_{bd} - \frac{1}{2}\nabla_d \nabla_b \delta g \\ &g^{bd}\delta(R_{bd}) = \nabla_a \nabla_b \delta g^{ab} - \nabla^2 \delta g\end{aligned} \tag{A.7.3}$$

It should be noted that in general $\partial\mathcal{R}$ is not the union of hypersurfaces Σ bounding $\mathcal{M}$. We compute $h^{ab}\delta(K_{ab})$ for later convenience. h and K are the induced metric and extrinsic curvature on $\partial\mathcal{R}$.

$$\begin{aligned}h^{ab}\delta(K_{ab}) &= h^{ab}\delta(h_a{}^c \nabla_c n_b) = h^{cb}\delta(\nabla_c n_b) = \\ &= -h^{cb} n_d \delta C^d{}_{cb} = \frac{1}{2}h^{cb}(n\nabla)\delta g_{cb}\end{aligned}$$

Now let us turn back to the action:

$$\begin{aligned}
\delta S &= \frac{1}{\kappa}\int_{\mathcal{R}} \omega_g \delta g^{ab} G_{ab} = \frac{1}{\kappa}\int_{\mathcal{R}} \omega_g \delta g^{ab}(R_{ab} - \frac{1}{2}g_{ab}R) = \\
&= \frac{1}{\kappa}\int_{\mathcal{R}} \omega_g(-\delta(g^{ab})R_{ab} - \frac{1}{2}\delta g^{ab} g_{ab} R) = \\
&= \frac{1}{\kappa}\int_{\mathcal{R}} \omega_g(-\delta(g^{ab}R_{ab}) + g^{ab}\delta(R_{ab}) - \frac{1}{2}\delta g^{ab} g_{ab} R) = \\
&= \frac{1}{\kappa}\int_{\mathcal{R}} \omega_g(-\delta(R) + g^{ab}\delta(R_{ab}) - \omega_g^{-1}\delta(\omega_g)R) = \\
&= -\frac{1}{\kappa}\delta\int_{\mathcal{R}} \omega_g R + \frac{1}{\kappa}\int_{M} \omega_g g^{ab}\delta(R_{ab})
\end{aligned}$$

The first term is the well known Einstein-Hilbert action, the second is a contribution of the boundary hypersurface.

$$\begin{aligned}
\int_{\mathcal{R}} \omega_g g^{ab}\delta(R_{ab}) &= \int_{\mathcal{R}} \omega_g \nabla_a\nabla_b \delta g^{ab} - \nabla^2\delta g = \int_{\partial\mathcal{R}} \omega_h n_a(\nabla_b \delta g^{ab} - \nabla^a \delta g) = \\
&= \int_{\partial\mathcal{R}} \omega_h n_a g_{bd}(\nabla^d \delta g^{ab} - \nabla^a \delta g^{bd}) = \int_{\partial\mathcal{R}} \omega_h n_a h_{bd}(\nabla^d \delta g^{ab} - \nabla^a \delta g^{bd})
\end{aligned}$$

$$\int_{\mathcal{R}} \omega_g g^{ab}\delta(R_{ab}) = -\int_{\partial\mathcal{R}} \omega_h h_{bd}(n\nabla)\delta g^{bd} = -2\delta\int_{\partial\mathcal{R}} \omega_h K$$

A.8 The Hamiltonian

$$\begin{aligned}
H &= \int_{\Sigma} \Pi^{ab}\dot{h}_{ab} - \omega_h N[\,^{(3)}R + K^{ab}K_{ab} - K^2] = \\
&= \int_{\Sigma} 2N\omega^{-1}(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi_h^2) + 2\Pi^{ab}D_a N_b - \omega_h N[\,^{(3)}R + \omega^{-2}(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi_h^2)] = \\
&= \int_{\Sigma} N\omega^{-1}(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi_h^2) - \omega_h N\,^{(3)}R - 2N^a h_{ac} D_b \Pi^{cb} + 2D_a(N_b\Pi^{ab}) \qquad \text{(A.8.1)} \\
&= \int_{\Sigma} NC + N^a V_a
\end{aligned}$$

The last term in (A.8.1) gives a boundary term, which vanishes if we assume that N^a has compact support. See [8] for the case of asymptotically flat boundary conditions where as $r \to \infty$ $N^a \to 0$.

A.9 Tensorial densities

The momentum Π^{ab} conjugate to the intrinsic metric h_{ab} is a symmetric tensor of density weight one. Here we will show a useful property of the divergence of such quantities. and their variantions. Consider 2 valent tensor T^{ab} of density weight one

$$D_aT^{ab} = D_a(\omega_h\omega_h^{-1}T^{ab}) = \omega_h D_a(\omega_h^{-1}T^{ab})$$

The term in paranthesis is a tensor of density weight zero, therfor

$$\begin{aligned}\omega_h D_a(\omega_h^{-1}T^{ab}) &= \omega_h \tilde{D}_a\omega_h^{-1}T^{ab} + \tilde{D}_aT^{ab} + C^a_{ac}T^{cb} + C^b_{ac}T^{ac} = \\ &= \tilde{D}_aT^{ab} + C^b_{ac}T^{ac} + C^a_{ac}T^{cb} - \frac{1}{2}h^{cd}\tilde{D}_ah_{cd}T^{ab},\end{aligned}$$

where we have used (A.7.1). Now we use

$$C^a_{ac} = \frac{1}{2}h^{ad}(\tilde{D}_ah_{dc} + \tilde{D}_ch_{ad} - \tilde{D}_dh_{ac}) = \frac{1}{2}h^{ad}\tilde{D}_ch_{ad},$$

to obtain

$$D_aT^{ab} = \tilde{D}_aT^{ab} + C^b_{ac}T^{ac}. \tag{A.9.1}$$

Hence we see that the difference tensor is only involved for the noncontracted indices. In particular for vectorial densities $\mathscr{J}^a$, sometimes referred to as de Rham currents we find that the divergence is completely independent of the covariant derivative, i.e. one could use the partial derivative

$$D_a\mathscr{J}^a = \partial_a\mathscr{J}^a$$

Now we will use the result (A.9.1) to compute the variation of $D_a\Pi^{ab}$ with respect to the intrinsic metric needed in the next section.

$$\begin{aligned}\delta_h\int_\Sigma f_b(D_a\Pi^{ab}) = \delta_h\int_\Sigma f_b\delta_h C^b_{ac}\Pi^{ac} = \delta_h\int_\Sigma f_b\frac{1}{2}(D_a\delta h^b_f + D_f\delta^b_a - D^b\delta h_{af})\Pi^{fa} = \\ = \delta_h\int_\Sigma f_b(D_a\delta h^b_f - \frac{1}{2}D^b\delta h_{af})\Pi^{fa}\end{aligned} \tag{A.9.2}$$

A.10 Constaints as generators of spacetime diffeomorphisms

Befor we start with the variation of the constraints I want to state some important relations of the constraints used in the very elegant derivation of the hypersurface deformation algebra given in [11]. Their form are better suited for a geometric interpretation of them.

After a partial integration of the diffeomorphism constraint we find

$$
\begin{aligned}
V[\vec{N}] =& -2\int_\Sigma N^c h_{cb} D_a \Pi^{ab} = 2\int_\Sigma \Pi^{ab} N^c h_{cb} D_a N_b = \int_\Sigma \Pi^{ab} N^c h_{cb}(D_a N_b + D_a b N_a) = \\
=& \int_\Sigma \Pi^{ab} \pounds_{\vec{N}} h_{ab}
\end{aligned} \tag{A.10.1}
$$

On the other hand we have

$$
\int_\Sigma \Pi^{ab} \pounds_{\vec{N}} h_{ab} = \int_\Sigma [\pounds_{\vec{N}} \Pi - h_{ab} \pounds_{\vec{N}} \Pi^{ab}],
$$

where the trace of the momentum Π is a scalar density, which can be seen as a form of highest degree. The formula for Lie derivatives of forms is given by

$$
\pounds_{\vec{N}} \Pi = d(N \lrcorner \Pi) + N \lrcorner d\Pi = d(N \lrcorner \Pi),
$$

and thus by Stokes theorem, and omiting boundary contributions we end up with

$$
V[\vec{N}] = -\int_\Sigma h_{ab} \pounds_{\vec{N}} \Pi^{ab}. \tag{A.10.2}
$$

From this form of the constraint it becomes clear that $V[\vec{N}]$ is the generator of spatial diffeomophisms and therefore is called diffeomophism constraint. The variations then yield

$$
\delta_h V[\vec{N}] = -\int_\Sigma \delta h_{ab} \pounds_{\vec{N}} \Pi^{ab} \tag{A.10.3}
$$

and

$$
\delta_\Pi V[\vec{N}] = \int_\Sigma \delta \Pi^{ab} \pounds_{\vec{N}} h_{ab}. \tag{A.10.4}
$$

For the variations of the diffeomorphism constraint, we will use its explicit form with the use of (A.9.2):

$$
\begin{aligned}
\delta_h V[\vec{N}] =& -2\delta_h \int_\Sigma h_{cb} D_a \Pi^{ab} N^c = \\
=& -2\int_\Sigma [\delta h_{cb}(D_a \Pi^{ab} N^c) + N_b(D_a \delta h^b_f - \frac{1}{2} D^b \delta h_{af})\Pi^{fa}] = \\
=& \int_\Sigma \delta h_{ac}[2\Pi^{ab} D_b N^c - D_b(\Pi^{ac} N^b)]
\end{aligned} \tag{A.10.5}
$$

$$\delta_\Pi V[\vec{N}] = \int_\Sigma \delta\Pi^{ac}(D_a N_c + D_c N_a) \tag{A.10.6}$$

The necessary compuations for the variation of the Hamiltonian constraint have been already performed in section A.7. We omit all boundary terms:

$$\begin{aligned} \delta_h C[M] = \int_\Sigma \delta h_{ac}\Big\{ -\frac{1}{2}h^{ac}MC + \omega_h^{-1}M(2\Pi^a_b\Pi^{bc} - \Pi^{ac}\Pi) - \\ - \omega_h M(h^{ac}R - R^{ac}) - \omega_h D^a D^c M + \omega_h D^2 M h^{ac}\Big\} \end{aligned} \tag{A.10.7}$$

$$\delta_\Pi C[M] = \int_\Sigma \delta\Pi^{ac}\Big[2M\omega_h^{-1}(\Pi_{ac} - \frac{1}{2}h_{ac}\Pi)\Big] \tag{A.10.8}$$

Again we will write $\delta_\Pi C[M]$ in a form which allows for more geometrical insight. We recognize the term in brackets as $2MK_{ac}$. The intrinsic curvature was the half of Lie derivative of h_{ac} with respect to the hypersurface normal. Note that spatial co-tensors t_{ab} satisfy

$$^{(4)}\pounds_{M\vec{n}}t_{ab} = M^{(4)}\pounds_{\vec{n}}t_{ab}.$$

Therefor we obtain

$$\delta_\Pi C[M] = \int_\Sigma \delta\Pi^{ac(4)}\pounds_{M\vec{n}}h_{ab}. \tag{A.10.9}$$

Hence for functionals on the phase space depending on the spatial metric only, the Hamiltonian constraint generates diffeomorphisms in direction of the hypersurface normal. Note that all indices are projected to Σ, hence the total Hamiltonian indeed generates a time evolution via the Hamiltonian equations

$$\dot{h}_{ab} = \frac{\delta(C[M] + V[\vec{N}])}{\delta\Pi^{ab}} = \perp^{(4)}\pounds_{M\vec{n}}h_{ab} + \pounds_{\vec{N}}h_{ab}, \tag{A.10.10}$$

which is exactly equation (1.0.10) As already mentioned in the main text, this is not the case for functions depending on Π^{ab} in general, but only on the constraint surface and if the vacuum Einstein equations hold. To see this we compute $\pounds_{M\vec{n}}\Pi^{ab}$, using the result (A.6.3), (A.7.1) and

$$\begin{aligned} h^a_c h^b_d \pounds_{M\vec{n}}h^{cd} = h^a_c h^b_d[M(n\nabla)h^{cd} - h^{cg}\nabla_g(Mn^d) - h^{gd}\nabla_g(Mn^b)] = \\ = Mh^a_c h^b_d[-h^{cg}\nabla_g n^d - h^{gd}\nabla_g n^b] = -2MK^{ab} \end{aligned}$$

.

$$\begin{aligned}
\pounds_{M\vec{n}}\Pi^{ab} =& \pounds_{M\vec{n}}\big[\omega_h\big(h^{ac}h^{bd} - h^{ab}h^{cd}\big)K_{cd}\big] = \\
=& M\omega_h\big\{K\big(K^{ab} - h^{ab}K\big) + 2\big(- 2K^{ac}K_c^b + K^{ab}K + h^{ab}K^{cd}K_{cd}\big) + \\
&+ \big(h^{ac}h^{bd} - h^{ab}h^{cd}\big)\big[- \big({}^{(3)}R_{cd} + K_{cd}K - 2K_c{}^eK_{ed}\big) + \\
&+ 8\pi\big(J_{cd} - \frac{1}{2}Jh_{cd} + \frac{1}{2}\rho h_{cd}\big) + \frac{1}{M}D_cD_dM\big]\big\} = \\
=& M\omega_h\big[\big({}^{(3)}Rh^{ab} - {}^{(3)}R^{ab}\big) + D^aD^bM - D^2M - M\big] - \\
&- M\omega_h^{-1}\big(2\Pi^{ac}\Pi_c^b - \Pi_{ab}\Pi\big) + \kappa(J^{ab} - \rho h^{ab}) \qquad \text{(A.10.11)}
\end{aligned}$$

This we insert into (A.10.7) to obtain

$$\delta_hC[M] = \int_\Sigma \delta h_{ac}\Big\{ - \frac{1}{2}h^{ac}MC + \kappa(J^{ab} - \rho h^{ab}) - \pounds_{M\vec{n}}\Pi^{ab}\Big\}, \qquad \text{(A.10.12)}$$

which proves our assertion.

A.11 The hypersurface deformation algebra

Now we are ready to check the Dirac algebra relations.

$$\begin{aligned}
\{V[N], V[\tilde{N}]\} =& \kappa\int_\Sigma \frac{\delta V[N]}{\delta h_{ac}}\frac{\delta V[\tilde{N}]}{\delta\Pi^{ac}} - (N \leftrightarrow \tilde{N}) = \\
=& \kappa\int_\Sigma -2D_b\Pi_c^b(\pounds_{\vec{N}}\tilde{N}^c) - 2\Pi^{ac}(N^dD_aD_b\tilde{N}_c - \tilde{N}^dD_aD_bN_c) + \\
&+ 2\Pi^{ac}(N^dD_bD_a\tilde{N}_c - \tilde{N}^dD_bD_aN_c) = \\
=& \kappa\int_\Sigma V_c(\pounds_{\vec{N}}\tilde{N}^c) + 2\Pi^{ac}(N^dR_{dcba}\tilde{N}^d - \tilde{N}^dR_{dcba}N^d) = \\
=& \kappa\int_\Sigma V_c(\pounds_{\vec{N}}\tilde{N}^c) = \kappa V[\pounds_{\vec{N}}\tilde{N}] \qquad \text{(A.11.1)}
\end{aligned}$$

$$
\begin{aligned}
\{V[N],C[M]\} =&\kappa\int_\Sigma \big[2\Pi^{ab}D_bN^c - D_b(\Pi^{ac}N^b)\big]\big[2M\omega_h^{-1}(\Pi_{ac}-\frac{1}{2}h_{ac}\Pi)\big]-\\
&- 2D_aN_c\Big\{-\frac{1}{2}h^{ac}MC+\omega_h^{-1}M(2\Pi^a_b\Pi^{bc}-\Pi^{ac}\Pi)-\\
&- \omega_hM(h^{ac}R-R^{ac})-\omega_hD^aD^cM+\omega_hD^2Mh^{ac}\Big\} =\\
&\kappa\int_\Sigma -(ND)\Pi^{ac}\big[2M\omega_h^{-1}(\Pi_{ac}-\frac{1}{2}h_{ac}\Pi)\big]-\\
&-(DN)\big[2M\omega_h^{-1}\big(\Pi^{ac}\Pi_{ac}-\frac{1}{2}\Pi^2\big)\big]+\\
&+(DN)MC+\\
&+2\omega_hM((DN)R-D_aN_cR^{ac})+2\omega_hD_aN_cD^cD^aM-2\omega_h(DN)D^2M\Big\} =\\
&\kappa\int_\Sigma -2M\omega_h^{-1}(ND)(\Pi^{ac}-\frac{1}{2}h_{ac})\Pi^{ac}\\
&-(DN)MC\\
&-2\omega_hMD_aN_cR^{ac}-2\omega_hD_cD_aN^cD^aM+2\omega_hD_aD_cN_cD^aM =\\
&\kappa\int_\Sigma -2M\omega_h^{-1}(ND)(\Pi^{ac}-\frac{1}{2}h_{ac})\Pi^{ac}\\
&+(ND)MC+M(ND)2(\Pi^{ac}-\frac{1}{2}h_{ac})\Pi^{ac}-M\omega_h(ND)R\\
&-2\omega_hMD_aN_cR^{ac}-2R_{da}N^dD^aM =\\
&\kappa\int_\Sigma (ND)MC-M\omega_h(ND)R\\
&-2\omega_hMD_aN_cR^{ac}+2D^aN^dR_{da}M+2MD^aR_{da}N^d =\\
&\kappa\int_\Sigma \pounds_{\vec N}MC+2M\omega_hN^dD^a(R_{da}-\frac{1}{2}h_{da}R)
\end{aligned}
$$

The last term in the last line is the contracted Bianchi identity

$$
\begin{aligned}
D^aR_{da}-\frac{1}{2}D^dR^{abce}h_{ac}h_{be} &= D^aR_{da}-\frac{1}{2}(-D^aR^{bdce}--D^bR^{dace})h_{ac}h_{be}\\
&= D^aR_{da}-D^aR_{da}=0, \qquad \text{(A.11.2)}
\end{aligned}
$$

thus we obtain

$$
\{V[N],C[M]\}=\kappa C[\pounds_{\vec N}M] \qquad \text{(A.11.3)}
$$

Finally we need the Poisson bracket of two scalar constraints. We can make use of the fact that due to the commutator, all terms where no derivative acts on the the lapse functions, cancel:

$$
\begin{aligned}
\{C[N], C[M]\} =& \kappa \int_\Sigma 2(\Pi_{ac} - \frac{1}{2} h_{ac}\Pi)[N(D_a D_c - h_a c D^2)M - (N \leftrightarrow \tilde{N})] = \\
& \kappa \int_\Sigma 2\Pi_{ac} N D_a D_c M - (N \leftrightarrow \tilde{N}) \\
& \kappa \int_\Sigma -2D_a \Pi_{ad} h_{de} h^{ec}(N D_c M - M D_c N) = \\
& V[h^{-1}(N dM - M dN)]
\end{aligned} \tag{A.11.4}
$$

A.12 Notes on the Reissner Nordström solution

From the equations of motion we obtain:

$$
\begin{aligned}
K_t &= \frac{E^t \dot{E}^x}{2E^x} \\
K_x = \frac{E^t \dot{E}^t}{E^x} - \frac{(E^t)^2 \dot{E}^x}{2(E^x)^2} &= \left(\frac{(E^t)^2}{2E^x}\right)^{\cdot}
\end{aligned}
$$

and thus their time derivatives read

$$
\begin{aligned}
\dot{K}_t &= \frac{\dot{E}^t \dot{E}^x}{2E^x} + \frac{E^t \ddot{E}^x}{2E^x} - \frac{E^t (\dot{E}^x)^2}{2(E^x)^2} \\
\dot{K}_x = \frac{(\dot{E}^t)^2}{E^x} + \frac{E^t \ddot{E}^t}{E^x} - \frac{2E^t \dot{E}^t \dot{E}^x}{(E^x)^2} &- \frac{(E^t)^2 \ddot{E}^x}{2(E^x)^2} + \frac{(E^t)^2 (\dot{E}^x)^2}{(E^x)^3}
\end{aligned}
$$

inserted in the vector constraint:

$$
E^x (\dot{E}^x E^t)' - (E^x)' (E^x E^t)^{\cdot} = 0
$$

and in the scalar constraint:

$$
\frac{\dot{E}^x E^t \dot{E}^t}{E^x} - \frac{(\dot{E}^x)^2 (E^t)^2}{4(E^x)^2} + 1 - \frac{GQ^2}{|E^x|} - \frac{((E^x)')^2}{4(E^t)^2} - \frac{E^x (E^x)''}{(E^t)^2} + \frac{E^x (E^x)' (E^t)'}{(E^t)^3} = 0
$$

The equation of motion for K_t then becomes

$$
\frac{(E^t)^4}{(E^x)^2}(2E^x \ddot{E}^x - (\dot{E}^x)^2) = -(2E^x (E^x)'' - ((E^x)')^2)
$$

The equation of motion for K_x then becomes

$$\frac{(\dot{E}^t)^2}{E^x} + \frac{E^t\ddot{E}^t}{E^x} - \frac{\dot{E}^x\dot{E}^tE^t}{(E^x)^2} - \frac{(E^t)^2\ddot{E}^x}{2(E^x)^2} + \frac{(E^t)^2(\dot{E}^x)^2}{2(E^x)^3} =$$
$$= \frac{3(E^x)''}{2(E^t)^2} - \frac{3(E^x)'(E^t)'}{(E^t)^3} + \frac{3E^x((E^t)')^2}{(E^t)^4} - \frac{E^x(E^t)''}{(E^t)^3} - \frac{GQ^2}{|E^x|E^x} \tag{A.12.1}$$

We use the scalar constraint to rewrite this equation in the form:

$$(\dot{E}^t)^2 + E^t\ddot{E}^t - \frac{(E^t)^2\ddot{E}^x}{2(E^x)} + \frac{(E^t)^2(\dot{E}^x)^2}{2(E^x)^2} =$$
$$= \frac{5E^x(E^x)''}{2(E^t)^2} - \frac{4E^x(E^x)'(E^t)'}{(E^t)^3} + \frac{3(E^x)^2((E^t)')^2}{(E^t)^4} -$$
$$- \frac{(E^x)^2(E^t)''}{(E^t)^3} - 1 + \frac{((E^x)')^2}{4(E^t)^2}$$

This we can rewrite again as using the equation of motion for K_t

$$\frac{1}{2}((E^t)^2)^{\cdot} = \frac{2E^x(E^x)''}{(E^t)^2} - \frac{4E^x(E^x)'(E^t)'}{(E^t)^3} + \frac{3(E^x)^2((E^t)')^2}{(E^t)^4} -$$
$$- \frac{(E^x)^2(E^t)''}{(E^t)^3} - 1 + \frac{((E^x)')^2}{2(E^t)^2}$$

The equation of motion for K_t suggests a variable substitution:
$E^x = \mathrm{sgn}(R)R^2(t,r)$, for which we find

$$\ddot{R} = -\frac{R^4}{(E^t)^4}R'' \tag{A.12.2}$$

We rewrite the r.h.s. (A.12.1) with the substitution $E^t = \sqrt{E^x/F}$

$$\frac{F''}{2} + \frac{(E^x)'F'}{2E^x} + \frac{F[2E^x(E^x)'' - ((E^x)')^2]}{2(E^x)^2} - \frac{GQ^2}{|E^x|E^x} =$$
$$\frac{(F'E^x)'}{2E^x} + \frac{F[2E^x(E^x)'' - ((E^x)')^2]}{2(E^x)^2} - \frac{GQ^2}{|E^x|E^x}$$

and the l.h.s.

$$\frac{\dot{F}^2}{F^3} - \frac{\ddot{F}}{2F^2} - \frac{\dot{E}^x\dot{F}}{2E^xF^2}$$

thus

$$\frac{\dot{F}^2}{F^3} - \frac{\ddot{F}}{2F^2} - \frac{\dot{R}\dot{F}}{RF^2} = \frac{F''}{2} + \frac{R'F'}{R} + \frac{2FR''}{R} - \frac{GQ^2}{\mathrm{sgn}(R)R^4}$$

while for (A.12.2) we find

$$\ddot{R} = -F^2 R'' \tag{A.12.3}$$

A.13 The Poisson bracket of the holonomy with the gravito-electric flux:

For this calculation we introduce a local coordinate system x^α in a neighborhood $\mathcal{U} \subset \Sigma$ of x, u^1, u^2 on $\mathcal{S}$ and u^3 on the curve c. The surface $\mathcal{S}$ is then defined by $\mathcal{S} : (u^1, u^2) \to x^\alpha(u^1, u^2, 0) = x^\alpha(u^1, u^2)$ and the curve by $c : t \to x^\alpha(0,0,t) = x^\alpha(t)$. Actually we constructed a congruence (2-parameter family) of curves $c_{u^1,u^2}(t)$ filling $\mathcal{U}$ and a foliation $\mathcal{S}_t$ of $\mathcal{U}$. The integral in this coordinate formulation reads

$$\int_0^1 \int_{\mathcal{S}} du^1 du^2 dt \varepsilon_{\alpha\beta\gamma} \frac{\partial x^\alpha}{\partial u^1} \frac{\partial x^\beta}{\partial u^2} \frac{\partial x^\gamma}{\partial t} f_i(t) \delta^3(x(u^1, u^2), x(t))$$

Using analycity we expand $x^\alpha(u^1, u^2, t) = x^\alpha(u^1, u^2) + x^\alpha(t)$ around the single intersection point $\mathcal{S} \cap c$ with coordinates $x\alpha(0,0,0) = 0$

$$\begin{aligned} x^\alpha(u^1, u^2, t) &= 0 + u^i \frac{\partial x^\alpha}{\partial u^i} + \mathcal{O}(u^i) + \frac{t^n}{n!} \frac{d^n x^\alpha}{dt^n}(0) + \mathcal{O}(t^n) = \\ &=: M_i^\alpha u^i + \mathcal{O}(u^i) + u^3 M_3^\alpha + \mathcal{O}((u^3)) \end{aligned} \tag{A.13.1}$$

$-M_3^\alpha$ is the first vector appearing in the expansion of $x^\alpha(t)$ that is not tangential to $\mathcal{S}$. We introduced the coordinate transformation

$$\frac{t^n}{n!} =: u^3, \qquad \frac{t^{n-1}}{(n-1)!} dt = du^3 \tag{A.13.2}$$

The components vectorial 1-form $\dot{x}^a(t)dt$ expanded (omiting the part tangential to $\mathcal{S}$) are

$$\begin{aligned} \frac{\partial x^\alpha}{\partial t} dt &= (-M_3^\alpha \frac{t^{n-1}}{(n-1)!} + \mathcal{O}(u^3))dt = \\ &= (-M_3^\alpha + \mathcal{O}((u^3)^{\frac{1}{n}}))du^3 \end{aligned} \tag{A.13.3}$$

Then we can rewrite the integral

$$\int_0^1 \int_{\mathcal{S}} du^1 du^2 du^3 (-\det M + \mathcal{O}((u)^{\frac{1}{n}})) f_i(\sqrt[n]{n! u^3}) \delta^3(u^\nu M_\nu^\alpha + \mathcal{O}(u)) \tag{A.13.4}$$

with

$$\delta^3(u^\nu M^\alpha_\nu + \mathcal{O}((u)^{\frac{1}{n}})) = \frac{1}{|\det M|}\delta^3(u^\nu) \tag{A.13.5}$$

we end up with

$$\int_0^1 \int_{\mathcal{S}} du^1 du^2 du^3 \frac{(-\det M + \mathcal{O}((u)^{\frac{1}{n}}))}{|\det M|} f_i(\sqrt[n]{n! u^3})\delta^3(u^\nu) = \frac{f_i(0)}{2} \tag{A.13.6}$$

A.14 Differential forms

Differential forms on an n-dimensional manifold with metric h_{ab}: An m-form expanded in a coordinate basis reads

$$\omega = \frac{1}{m!}\omega_{\mu_1\ldots\mu_m} dx^{\mu_1}\ldots dx^{\mu_m} \tag{A.14.1}$$

where $dx^{\mu_1}\ldots dx^{\mu_m}$ denotes the wedge product

$$dx^\mu \wedge dx^\nu \equiv dx^\mu dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu \tag{A.14.2}$$

which we will suppress in the following.
The volume form:

$$\omega_h = \frac{1}{n!}\varepsilon_{\mu_1\ldots\mu_n} dx^{\mu_1}\ldots dx^{\mu_n} = \sqrt{|h|}dx^1\ldots dx^n \tag{A.14.3}$$

Hodge dual for differential forms

$$*\omega = \frac{1}{m!}\frac{1}{(n-m)!}\varepsilon^{\mu_1\ldots\mu_m}{}_{\mu_{m+1}\ldots\mu_n}\omega_{\mu_1\ldots\mu_m} dx^{\mu_{m+1}}\ldots dx^{\mu_n}. \tag{A.14.4}$$

The components of the (metric) volume-form in some basis are normalized by the condition

$$\epsilon^{I_1\ldots.I_n}\epsilon_{I_1\ldots.I_n} = (-1)^s n!$$

whith $s = 1$ for Lorentzian manifolds. In a orthonormal basis we have

$$\varepsilon_{0123} = 1$$

Using the Hodge star we can define the inner product of two p-forms as

$$\langle\omega, \psi\rangle = \int \omega * \psi, \tag{A.14.5}$$

which for example could be used to write the Yang-Mills action in four dimensions as follows

$$
\begin{aligned}
S = -\frac{1}{2}\int_{\mathcal{M}} \mathcal{F}\star\mathcal{F} = \\
= -\frac{1}{2}\int_{\mathcal{M}} \frac{1}{2}\mathcal{F}_{\mu\nu}e^{\mu}e^{\nu}\frac{1}{2}\mathcal{F}_{\rho\sigma}\frac{1}{2}\varepsilon^{\rho\sigma}{}_{\tau\lambda}e^{\tau}e^{\lambda} = \\
= -\frac{1}{16}\int_{\mathcal{M}} \mathcal{F}_{\mu\nu}\mathcal{F}^{\rho\sigma}\varepsilon_{\rho\sigma\tau\lambda}e^{\mu}e^{\nu}e^{\tau}e^{\lambda} = \\
= -\frac{1}{16}\int_{\mathcal{M}} \mathcal{F}_{\mu\nu}\mathcal{F}^{\rho\sigma}\varepsilon_{\rho\sigma\tau\lambda}(-1)\varepsilon^{\mu\nu\tau\lambda}e^{0}e^{1}e^{2}e^{3} = \\
= \frac{1}{16}\int_{\mathcal{M}} \mathcal{F}_{\mu\nu}\mathcal{F}^{\rho\sigma}\omega_g(-2)\delta^{\mu\nu}_{\rho\sigma} = \\
= -\frac{1}{4}\int_{\mathcal{M}} \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\omega_g
\end{aligned}
$$

m-currents (m-vector valued density):

$$\tilde{v} = \frac{1}{m!}v^{\mu_1\ldots\mu_m}\partial_{\mu_1}\ldots\partial_{\mu_n}\omega_h \tag{A.14.6}$$

We define the $*$-operation on m-currents as follows

$$*\tilde{v} = \frac{1}{m!}v^{\mu_1\ldots\mu_m}\frac{1}{m!}(\partial_{\mu_1}\ldots\partial_{\mu_n})\lrcorner\omega_h \tag{A.14.7}$$

The hodge dual for the $m+1$ split: For a form $G \in \Lambda^p(\mathcal{M})$ with $dim(\mathcal{M}) = n$, $p < n$, $dt \wedge G \neq 0$ we find

$$dtG \star dtG = dtG \star \frac{1}{N}e^0 G = -dt\frac{1}{N}G * G \tag{A.14.8}$$

and for a form $F \in \Lambda^p(\mathcal{M})$ with $dim(\mathcal{M}) = n$, $p = 2m < n$, $dt \wedge F = 0$ we find

$$F \star F = NdtF * F \tag{A.14.9}$$

In the main text one encounters Lie algebra valued objects several times. Here I define some useful abbreviations. Consider the Lie algebra valued differential form $\varphi = \varphi^a T_a$, where T^a denotes a Lie algebra basis element. Instead of writing $-2\,\mathrm{Tr}(\varphi\wedge\psi)$ I denote the contraction with $\cdot$, i.e.

$$\varphi \wedge \cdot\psi := -2\,\mathrm{Tr}(\varphi\wedge\psi) = \varphi^a \wedge \psi^b k_{ab},$$

where k denmotes the Cartan-Killing metric.
The commutator of two Lie algebra valued forms is defined by

$$[\varphi, \psi] := [T_a, T_b]\varphi^a \wedge \psi^b = f_{ab}^c T_c \varphi^a \wedge \psi^b \tag{A.14.10}$$

If all indices of Lie algebra valued p-forms are contracted I simply write

$$\varphi * \psi \tag{A.14.11}$$

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