Diplomarbeit

Loop Corrections for a Translation-Invariant Renormalizable Non-Commutative ϕ^4 Theory

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Abstract

Today it is widely believed that space-time has to be quantized at very small scales down to the Planck length, which represents a lower bound for distances. All theories of the standard model with the exception of gravity are quantum field theories (QFT), more precisely gauge theories. One mathematical framework to bring together QFT with the quantized space-time is the non-commutative quantum field theory (NCQFT), where the concept of sharply defined points in space-time is abandoned. In the simplest form of this framework products are replaced by the Moyal-Weyl star product, which incorporates all aspects of the deformed space-time. But if this is done in the most basic QFT, the scalar quantum field theory on Euclidean space, one finds that divergences occur in the ultraviolet limit, which is known from commutative QFT, as well as in the infrared limit and those divergences cannot be absorbed with a renormalization procedure. Recently, it was found that a tweaked theory can be renormalized to all orders of perturbation theory [1]. This thesis aims at calculating explicit results for first and higher orders of perturbation for the two and four point function, as well as at exercising a renormalization procedure for the first order. Furthermore, the β functions that indicate the behavior of the parameters for a change of scales of the new theory will be calculated.

Kurzfassung

Eine weit verbreitete Vorstellung heutzutage ist, dass die Raumzeit auf sehr kleinen Skalen bis hinunter zur Plancklänge, die eine untere Schranke für Abstände darstellt, quantisiert sein muss. Alle im Standardmodell enthaltenen Theorien mit Ausnahme der Gravitation sind Quantenfeldtheorien, genauer gesagt Eichtheorien. Ein mathematischer Rahmen, um die Quantenfeldtheorien mit der quantisierten Raumzeit zu vereinen, ist die nichtkommutative Quantenfeldtheorie, kurz NCQFT, bei der das Konzept scharf definierter Punkte in der Raumzeit aufgegeben wird. Im einfachsten Fall wird dabei das Produkt zweier Funktionen durch das sogenannte Moyal-Weyl Sternprodukt ersetzt, welches alle Aspekte der verzerrten Raumzeit beinhaltet. Wird dies allerdings in der einfachsten Form einer Quantenfeldtheorie, nämlich einer skalaren QFT im Euklidischen Raum gemacht, so finden sich Divergenzen im ultravioletten Grenzfall, welche schon von der kommutativen QFT bekannt sind, aber auch im infraroten Grenzfall. Diese Divergenzen können nicht im Zuge einer Renormierungsprozedur zum Verschwinden gebracht werden. Kürzlich wurde jedoch eine verbesserte Theorie gefunden, die für alle Ordnungen einer Störungstheorie renormiert werden kann [1]. Das Ziel dieser Diplomarbeit ist die Berechnung von expliziten Resultaten für die erste und für höhere Ordnungen der Störungsreihe der Zwei- und Vierpunktfunktionen, sowie die Durchführung einer Renormierung für die erste Ordnung. Des Weiteren wird die β Funktion, die Aufschluss über das Verhalten der Parameter bei einer Änderung der Größenordnungen gibt, berechnet.

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Chapter 1

Introduction

1.1 The General Idea

We have come a long way in how much we know of the world we live in from the first observations of our ancestors to the data gained by the sophisticated methods of our time. These days, we know the age and size of our universe quite well as 13.7 billion years and a radius of 4.4×10^{26}m. This estimation is supported by the data of the WMAP spacecraft surveying the microwave background of the universe [2] and, in the future, by the measurements from the Planck spacecraft started in August 2009. On the other side of the scale there are fundamental physical constants like the speed of light c, Planck's constant $\hbar$ and Newton's gravitational constant G, which together, form a smallest length called Planck length $l_P = \sqrt{\hbar G/c^3} = 1.6 \times 10^{-35}$m. We believe that at that scale gravity has the same strength as the other forces and therefore a quantum theory of gravity is needed. This implies that space-time itself has to be quantized at that length. So the scale of our universe spreads over about 61 powers of ten meters. With our advanced telescopes we are able to observe the universe almost to its full extent but on the other end of the scale in the tiny realms we are not that lucky. The best microscopes we have built produced energies of about 1 TeV so far and even our newest gadget, the LHC, will increase this number only by one power of ten. One TeV corresponds to a length of about 10^{-19}m and on the 16 scales which are smaller than that we have to speculate mostly, although we can detect particles with higher energies in cosmic rays. But those particles are to rare to form a good picture of what happens [3]. There are several unconfirmed theories for a quantum gravity from which string theory gets the most attention. In certain effective regimes one can find non-commutative aspects of the string [4]. Other theories are loop quantum gravity and non-commutative geometry itself. A more mathematical point of view for non-commutative geometry laying the basis for non-commutative physics can be found in the work of A. Connes [5, 6]. Another motivation for non-commutative geometry comes from the physics of high external fields and here, especially, from the quantum Hall effect which resists a commutative explanation and therefore it is hoped, that non-commutative methods help solving the problem here [7]. Finally, it should be said that even when one quantizes the Landau problem of a point-like charged particle in an external magnetic field one encounters a non-commutative structure of the spatial operators and the momentum operators respectively. This shows that non-commutativity is more common than may be thought in the first place.

Our current standard model is a quantum theory of fields or quantum field theory (QFT) and is able to explain most of the effects of three of the four fundamental forces in nature with one theory. The lack of gravitational effects and several other problems such as the up to now only postulated Higgs boson or the matter-antimatter asymmetry in the universe show, that this model is not the theory of everything (ToE) we are looking for, nor the grand unified theory (GUT) which should explain all the effects of the strong, weak and electromagnetical interaction and give an explanation for the parameters now only measurable in experiments. Nevertheless, the QFT we use today to describe one of the fundamental interactions at a time work quite well but have some problems too. They predict a lot of different particles and one gets the impression that there have to be some more basic building blocks beneath. Another major problem are the ultraviolet divergences which emerge in the calculations of loop corrections due to the integrations over inner momenta. To overcome this problem, one idea was to introduce a non-commutative space-time. It was hoped that non-commuting coordinates would avoid the UV divergences. But it got even worse because in addition to the UV divergences already present, unexpected divergences occurred at the other end of the spectrum at very small momenta. So from the UV problem in commutative QFT one got to the UV/IR mixing problem in non-commutative (NC)QFT. The theory is not renormalizable because the IR divergence is of a non-renormalizable type. This leads to the question why the whole attempt was not considered a failure and filed and forgotten. As said earlier, the arguments for further investigations where strong because non-commutativity emerged in several different fields and so people where looking for a way to work around the divergences and the non-renormalizability.

Then came the breakthrough of Grosse and Wulkenhaar who added an additional term to the non-commutative action which had the form of a harmonic oscillator. Therefore, they got a different propagator which was essentially the so-called Mehler kernel. They could proof in the matrix base that the new action leads to a renormalizable theory [8]. Now this proof has been confirmed with several other methods [9, 10]. But the renormalizability came with a price, namely that they had to give up translation invariance. Even though the potential is very small on physical grounds (it has not been measured yet) the theory can still not be considered a free theory anymore. Furthermore, in the limit of a commutative theory ($\theta \to 0$) the theory is singular. This model is now called the Grosse-Wulkenhaar (GW) model and it was showed that its β function vanishes to all orders [11].

The idea of adding an extra term to the action to get a renormalizable theory was taken over by Gurau *et al.* [1]. They came up with the solution to implant the IR divergent behavior in the action and added a $\frac{1}{p^2}$ term to the action which led to a propagator which goes to zero for very high and very low momenta. This new theory does not destroy translation invariance like the GW model but it is non-local. With the method of multiscale analysis they were able to prove that their model is renormalizable to all orders of perturbation theory. However, it does not show a Langmann-Szabo symmetry.

The goal of this diploma thesis is to calculate the lowest loop corrections for the $\frac{1}{p^2}$ model with the standard methods using Feynman path integrals to get real results for the perturbation theory. With these results a renormalization to the first order is performed and the β functions of the parameters are calculated. Furthermore, some thoughts regarding higher loop orders and the principal mechanism of how the model produces finite graphs is added. Major parts of this thesis are included in a paper published in the "Journal of High Energy Physics" (JHEP) together with Daniel Blaschke, François Gieres, Erwin Kornberger, Manfred Schweda and René Sedmik [12].

There have been recent attempts, amongst others here at the Vienna University of Technology (VUT), to transfer both theories, the GW and the $\frac{1}{p^2}$ model, to gauge theories. The many fields necessary make this a rather difficult task since there are many vertices and even to one loop order an almost overwhelming number of corrections. Other occurring problems are the vacua which are not zero because the tadpole graphs do not vanish. Many of the calculations were done with the help of a computer algebra system to reduce the workload and avoid errors. See [13, 14, 15, 16, 17, 18] for more informations.

1.2 Conventions

The main subject of this thesis is a four dimensional scalar theory (ϕ^4 theory) in an Euclidean space. Greek indices run from one to four if not specified otherwise explicitly. As usual, the Einstein summation convention is used for the calculations if not stated otherwise. In Euclidean space, there are no co- and contravariant vectors and, therefore, the position of the indices is not important

$$\sum_{\mu=1}^{4} A_\mu B^\mu \equiv A_\mu B_\mu \equiv A^\mu B^\mu \,. \tag{1.1}$$

The differential operator is written in the form

$$\partial_\mu \equiv \frac{\partial}{\partial x_\mu} \,. \tag{1.2}$$

Furthermore, the quabla sign is used for a double differentiation regardless if it is in Euclidean or Minkowski space

$$\Box := \partial_\mu \partial_\mu \,. \tag{1.3}$$

If an integral sign with no interval is given, it is meant to be on an interval from $-\infty$ to $+\infty$. Otherwise the interval is written as usual. In this work a system of natural units is chosen where $c = \hbar = 1$. This leads to a theory where energy and mass have the same units.

1.3 Organization of this Thesis

In the first part of this thesis the framework of NCQFT is explained and the rules for calculations are introduced. The next part deals with the naive model deriving the propagator and vertex as well as showing an example of UV/IR mixing. After that, the new model is introduced and the Feynman rules are calculated. The next chapter brings the one loop corrections and the renormalization to the first order including the β function and renormalization group, followed by a chapter where higher loop orders are considered. Detailed calculations can be found in the appendix.

1.4 Acknowledgment

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Chapter 2

Non-Commutative Quantum Field Theory

2.1 Algebra

In standard quantum field theory it is well known that the operators for coordinates and momenta do not commute. Non-commutative QFT pushes this further and states that different coordinate operators and different momentum operators do not commute, contrary to the commutative QFT where $[\hat{x}^\mu, \hat{x}^\nu] = [\hat{p}^\mu, \hat{p}^\nu] = 0$. As a consequence of this, an object like a point does not exist anymore and one can think of a box or cube as the smallest object in existence, although, of course, the geometrical form is not specified exactly. Hence this non-commutativity deforms space in Euclidean, or space-time in Minkowski space itself. The simplest possible deformation in Euclidean space is described by the algebra

$$\begin{aligned} [\hat{x}^\mu, \hat{x}^\nu] &= \mathrm{i}\theta^{\mu\nu}\,, \\ [\theta^{\mu\nu}, \hat{x}^\rho] &= 0\,. \end{aligned} \tag{2.1}$$

The $\hat{x}$ are the non-commutating coordinate operators and $\theta_{\mu\nu}$ is a constant, antisymmetric matrix of mass dimension -2 that defines the deformation. A common choice in four dimensions is

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix} \quad , \quad \text{with} \quad \theta \in \mathbb{R}\,. \tag{2.2}$$

Others forms of deformation are the so-called Lie-case

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}C^{\mu\nu}_\rho \hat{x}^\rho\,, \quad \text{with} \quad C^{\mu\nu}_\rho \in \mathbb{C}\,, \tag{2.3}$$

or the quantum group space

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\hat{R}^{\mu\nu}_{\rho\sigma} \hat{x}^\rho \hat{x}^\sigma\,, \quad \text{with} \quad \hat{R}^{\mu\nu}_{\rho\sigma} \in \mathbb{C}\,. \tag{2.4}$$

In this work only the simplest form of deformation (2.1) will be used.

2.2 The Moyal-Weyl Star Product

Since it is more convenient to use fields instead of operators, it seems like a good idea to look for a field representation of the commutation relation (2.1). To find this one has to find a substitution for the multiplication of fields. This is the Moyal-Weyl star product with the operator $\star$, which makes it possible to handle the equations more or less in the same way as in commutative QFT. What is needed is a correspondence

$$\hat{\phi}(\hat{x}) \Longleftrightarrow \phi(x)\,. \tag{2.5}$$

The operator valued objects $\hat{\phi}$ can be written as

$$\begin{aligned}\hat{\phi}(\hat{x}) &= \int \mathrm{d}\alpha\ \mathrm{e}^{\mathrm{i}\alpha_\mu \hat{x}^\mu}\phi(\alpha)\,,\\ \phi(\alpha) &= \int \mathrm{d}x\ \mathrm{e}^{-\mathrm{i}\alpha x}\phi(x)\,,\end{aligned} \tag{2.6}$$

with the Fourier integral theorem. Here α and x are real variables. For the product of two operator valued objects one gets

$$\hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x}) = \int \mathrm{d}\alpha \int \mathrm{d}\beta\ \mathrm{e}^{\mathrm{i}\alpha\hat{x}}\mathrm{e}^{\mathrm{i}\beta\hat{x}}\phi_1(\alpha)\phi_2(\beta)\,. \tag{2.7}$$

Since the exponents are operator valued one has to apply the Baker-Campbell-Hausdorff formula to bring them together

$$\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]}\,, \qquad \text{if} \quad [A,[A,B]] = [B,[A,B]] = 0\,. \tag{2.8}$$

The commutator of the spatial operators is in this case a complex constant (2.1) and, therefore, the precondition is always fulfilled. With the Baker-Campbell-Hausdorff formula and inserting Eqn. (2.1) the result is

$$\hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x}) = \int \mathrm{d}\alpha \int \mathrm{d}\beta\ \mathrm{e}^{\mathrm{i}(\alpha+\beta)\hat{x}-\frac{\mathrm{i}}{2}\alpha_\mu\beta_\nu\theta^{\mu\nu}}\phi_1(\alpha)\phi_2(\beta)\ . \tag{2.9}$$

With this the next step is to define the star product of two fields and show that it is the same as the multiplication of two operator valued objects

$$\begin{aligned}\phi_1(x)\star\phi_2(x) :&= \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial^x_\mu\partial^y_\nu}\phi_1(x)\phi_2(y)\Big|_{x=y}\\ &= \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial^x_\mu\partial^y_\nu}\int \mathrm{d}\alpha \int \mathrm{d}\beta\ \mathrm{e}^{\mathrm{i}(\alpha x+\beta y)}\phi_1(\alpha)\phi_2(\beta)\Big|_{x=y}\\ &= \int \mathrm{d}\alpha \int \mathrm{d}\beta \left(1+\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial^x_\mu\partial^y_\nu+\cdots\right)\mathrm{e}^{\mathrm{i}(\alpha+\beta)x}\phi_1(\alpha)\phi_2(\beta)\Big|_{x=y}\\ &= \int \mathrm{d}\alpha \int \mathrm{d}\beta\ \mathrm{e}^{\mathrm{i}(\alpha+\beta)x-\frac{\mathrm{i}}{2}\theta^{\mu\nu}\alpha_\mu\beta_\nu}\phi_1(\alpha)\phi_2(\beta)\,.\end{aligned} \tag{2.10}$$

In comparing this with Eqn. (2.9) one can see the correspondence of those two equations

$$\hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x}) \Longleftrightarrow \phi_1(x) \star \phi_2(x)\,. \tag{2.11}$$

The result is a representation of the commutator (2.1) in normal commutative space-time coordinates but with a deformed product

$$[x^\mu \overset{\star}{,} x^\nu] = x^\mu \star x^\nu - x^\nu \star x^\mu = \mathrm{i}\theta^{\mu\nu}\,. \tag{2.12}$$

For later calculations it is good to know what the star product of two exponentials is.

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}kx} \star \mathrm{e}^{\mathrm{i}k'x} &= \left. \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial^x_\mu\partial^y_\nu}\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{\mathrm{i}k'y}\right|_{x=y} \\
&= \left.\left(1 + \frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial^x_\mu\partial^y_\nu + \cdots\right)\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{\mathrm{i}k'y}\right|_{x=y} \\
&= \mathrm{e}^{\mathrm{i}(k+k')x} - \frac{\mathrm{i}}{2}\theta^{\mu\nu}k_\mu k'_\nu \mathrm{e}^{\mathrm{i}(k+k')x} + \cdots \\
&= \mathrm{e}^{\mathrm{i}(k+k')x}\left(1 - \frac{\mathrm{i}}{2}\theta^{\mu\nu}k_\mu k'_\nu + \cdots\right) \\
&= \mathrm{e}^{\mathrm{i}(k+k')x}\mathrm{e}^{-\frac{\mathrm{i}}{2}k_\mu\theta^{\mu\nu}k'_\nu}\,.
\end{aligned} \tag{2.13}$$

Triple and higher products can be derived in the following way

$$(\phi_1 \star \phi_2 \star \phi_3)(x) = \int \mathrm{d}\alpha \int \mathrm{d}\beta \int \mathrm{d}\gamma \left(\mathrm{e}^{\mathrm{i}\alpha x} \star \mathrm{e}^{\mathrm{i}\beta x} \star \mathrm{e}^{\mathrm{i}\gamma x}\right)\phi_1(\alpha)\phi_2(\beta)\phi_3(\gamma)\,. \tag{2.14}$$

The star product of the exponentials gives[1]

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\alpha x} \star \mathrm{e}^{\mathrm{i}\beta x} \star \mathrm{e}^{\mathrm{i}\gamma x} &= \left(\mathrm{e}^{\mathrm{i}(\alpha+\beta)x}\mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha\theta\beta}\right) \star \mathrm{e}^{\mathrm{i}\gamma x} \\
&= \mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha\theta\beta}\left(\mathrm{e}^{\mathrm{i}(\alpha+\beta)x} \star \mathrm{e}^{\mathrm{i}\gamma x}\right) \\
&= \mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)x}\mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha\theta\beta}\mathrm{e}^{-\frac{\mathrm{i}}{2}(\alpha+\beta)\theta\gamma}\,,
\end{aligned} \tag{2.15}$$

and this leads to

$$(\phi_1 \star \phi_2 \star \phi_3)(x) = \int \mathrm{d}\alpha_1 \int \mathrm{d}\alpha_2 \int \mathrm{d}\alpha_3\; \mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2+\alpha_3)x}\mathrm{e}^{-\frac{\mathrm{i}}{2}\sum\limits_{i<j}^{3}\alpha_i\theta\alpha_j}\phi_1(\alpha_1)\phi_2(\alpha_2)\phi_3(\alpha_3). \tag{2.16}$$

Without proof the formula for higher products is

$$\phi_1(x) \star \cdots \star \phi_n(x) = \int \mathrm{d}\alpha_1 \cdots \int \mathrm{d}\alpha_n\; \mathrm{e}^{\mathrm{i}\sum\limits_i^n \alpha_i x}\mathrm{e}^{-\frac{\mathrm{i}}{2}\sum\limits_{i<j}^{n}\alpha_i\theta\alpha_j}\prod_{i=1}^{n}\phi_i(\alpha_i) \tag{2.17}$$

as can be verified easily.

[1] The short form $\alpha\theta\beta$ stands for $\alpha_\mu\theta_{\mu\nu}\beta_\nu$.

Properties of the Star Product

$k_\mu \theta^{\mu\nu} k'_\nu$ is often written in the short form $k\theta k'$, $k \times k'$ or $k\tilde{k}'$ where $\tilde{k}_\mu = \theta_{\mu\nu} k^\nu$. A very important feature of the star product is that one can replace one star by an ordinary multiplication under an integral. This has the effect that every bilinear expression in the action is exactly the same as in standard commutative QFT. The propagators of NCQFT are the same as the propagators of commutative QFT. The difference between the two theories arises only from the vertices

$$\int \mathrm{d}^n x \; \phi_1(x) \star \phi_2(x) = \int \mathrm{d}^n x \; \phi_1(x) \phi_2(x) \,. \tag{2.18}$$

Proof:

$$\begin{aligned}
\int \mathrm{d}^n x \phi_1(x) \star \phi_2(x) &= \frac{1}{(2\pi)^{2n}} \int \mathrm{d}^n x \int \mathrm{d}^n k_1 \int \mathrm{d}^n k_2 \; \mathrm{e}^{\mathrm{i}(k_1+k_2)x} \tilde{\phi}_1(k_1) \tilde{\phi}_2(k_2) \mathrm{e}^{-\frac{\mathrm{i}}{2} k_1 \theta k_2} \\
&= \frac{1}{(2\pi)^n} \int \mathrm{d}^n k_1 \int \mathrm{d}^n k_2 \; \delta^{(n)}(k_1+k_2) \tilde{\phi}_1(k_1) \tilde{\phi}_2(k_2) \mathrm{e}^{-\frac{\mathrm{i}}{2} k_1 \theta k_2} \\
&= \frac{1}{(2\pi)^n} \int \mathrm{d}^n k_1 \; \tilde{\phi}_1(k_1) \tilde{\phi}_2(-k_1) \mathrm{e}^{\frac{\mathrm{i}}{2} k_1 \theta k_1} \\
&= \frac{1}{(2\pi)^n} \int \mathrm{d}^n k_1 \; \tilde{\phi}_1(k_1) \tilde{\phi}_2(-k_1) \,.
\end{aligned}$$

$k_1 \theta k_1$ vanishes because it represents a multiplication of a totally antisymmetric matrix with a symmetric one which is always zero. Using again Fourier analysis one gets from the last expression

$$\begin{aligned}
\int \mathrm{d}^n x \phi_1(x) \star \phi_2(x) &= \frac{1}{(2\pi)^n} \int \mathrm{d}^n k_1 \int \mathrm{d}^n x \int \mathrm{d}^n x' \; \mathrm{e}^{-\mathrm{i}k_1(x-x')} \phi_1(x) \phi_2(x') \\
&= \int \mathrm{d}^n x \int \mathrm{d}^n x' \; \delta^{(n)}(x-x') \phi_1(x) \phi_2(x') \\
&= \int \mathrm{d}^n x \; \phi_1(x) \phi_2(x) \,,
\end{aligned} \tag{2.19}$$

which finishes the proof.
Furthermore, associativity also holds for the star product

$$[(f \star g) \star h] = [f \star (g \star h)] \,. \tag{2.20}$$

Proof:

$$\begin{aligned}
\text{l.h.s:} & \int \mathrm{d}\alpha \int \mathrm{d}\beta \int \mathrm{d}\gamma \; f(\alpha) g(\beta) h(\gamma) \mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)x} \mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha\theta\beta} \mathrm{e}^{-\frac{\mathrm{i}}{2}(\alpha+\beta)\theta\gamma} \,, \\
\text{r.h.s:} & \int \mathrm{d}\alpha \int \mathrm{d}\beta \int \mathrm{d}\gamma \; f(\alpha) g(\beta) h(\gamma) \mathrm{e}^{\mathrm{i}(\alpha+\beta+\gamma)x} \mathrm{e}^{-\frac{\mathrm{i}}{2}\beta\theta\gamma} \mathrm{e}^{-\frac{\mathrm{i}}{2}(\beta+\gamma)\theta\alpha} \,.
\end{aligned} \tag{2.21}$$

Comparing the left with the right hand side finishes the proof.
Another useful feature is that one can perform cyclic permutations under the integral as can be showed with

$$\int \mathrm{d}x\ (f \star g) = \int \mathrm{d}x\ fg = \int \mathrm{d}x\ gf = \int \mathrm{d}x\ (g \star f)\,. \tag{2.22}$$

From this and with Eqn. (2.20) one can proof that cyclic permutations are allowed.

$$\int \mathrm{d}x\ ((f_1 \star \cdots \star f_{n-1}) \star f_n) = \int \mathrm{d}x\ (f_n \star (f_1 \star \cdots \star f_{n-1}))\,. \tag{2.23}$$

The complex conjugation for the star product is defined as

$$(f \star g)^* = g^* \star f^*\,. \tag{2.24}$$

For the calculations it is necessary to substitute every multiplication between fields in the commutative action with a star product. In bilinear terms under the integral one can leave the multiplication as it is due to Eqn. (2.18).

Chapter 3

The Naive Model

To get a non-commutative scalar quantum field theory in its easiest form one simply takes the most basic commutative quantum field theory and makes the necessary changes to incorporate the non-commutative formalism. This is done by substituting the multiplications of the field operators with the star product (2.10). Therefore, the transition to non-commutativity is easily done. The reason that this simple non-commutative model is here referred to as the naive model is that in this new model some new problems arise which are unknown in the commutative theory. These problems make the theory non-renormalizable in contrary to the standard scalar theory which, for example, is renormalizable. A non-renormalizable theory with its divergences is pretty useless for tackling physical problems and, therefore, one has to modify the scalar theory when one wants to work in a non-commutative environment. In this chapter the naive model is treated to show the principal ways of calculating functions in a non-commutative theory which leads to the reason of non-renormalizability, the notorious UV/IR mixing.

3.1 The Action

The easiest way to get a non-commutative field theory is to simply take the standard commutative action and substitute the multiplications of the fields with the star product (2.10). The action of this theory in coordinate space has the form

$$S[\phi] = \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi \star \partial_\mu\phi + \frac{1}{2}m^2\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi \right) . \qquad (3.1)$$

In the bilinear terms one can drop the star product and use a normal product as shown in Eqn. (2.19). This gives

$$S[\phi] = \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}(\phi\star\phi)(\phi\star\phi) \right) . \qquad (3.2)$$

3.2 General Remarks on Calculating Propagators

For calculating scattering amplitudes of interacting particles in a perturbation theory one uses Feynman diagrams as an intuitive picture of the processes that happen in a certain order. Feynman diagrams are constructed using so-called propagators which represent a moving particle and vertices which stand for interactions between particles. Out of these building blocks one can construct more and more complicated graphs for corrections to the unperturbed propagator or vertex, which generally means that the graphs have a rising number of inner loops. For this scalar theory there are only two building blocks, namely one propagator and one vertex, from which every graph is constructed.

The propagator which is the free two point Green function is defined [19, 20] as the time ordered vacuum expectation value of two free fields,

$$\Delta_{ab}(x,y) = \langle 0|\mathrm{T}\phi_a(x)\phi_b(y)|0\rangle_{(0)}\,. \tag{3.3}$$

Here, ϕ stands for an arbitrary field and in this case even a free field as the subscript (0) suggests. a and b are two generic quantum configurations and T is the time ordering operator which ensures that the field with the later time coordinate is left of the one with the earlier time

$$\mathrm{T}A(t_1)B(t_2) = \begin{cases} A(t_1)B(t_2)\,, & t_1 > t_2\,, \\ B(t_2)A(t_1)\,, & t_2 > t_1\,. \end{cases} \tag{3.4}$$

A and B are arbitrary time dependent operators in this case. The time ordering operator is not necessary in Euclidean space but only in Minkowski space and so it is omitted in the following equations. There exists a generating functional for all Green functions in Euclidean space which is defined as the vacuum to vacuum transition amplitude

$$Z[J] = \langle 0|\mathrm{e}^{-\int \mathrm{d}^4x J_a(x)\phi_a(x)}|0\rangle\,, \tag{3.5}$$

where $J(x)$ is the classical unquantized source of the field operator $\phi(x)$. The sources are Schwartz fast decreasing $\mathcal{C}^\infty$ test functions. This functional can also be written as

$$Z[J] = \frac{\int \mathcal{D}[\phi]\mathrm{e}^{-S-\int \mathrm{d}^4x J_a(x)\phi_a(x)}}{\int \mathcal{D}[\phi]\mathrm{e}^{-S}}\,. \tag{3.6}$$

The denominator in this equation is used to get rid of the unphysical vacuum graphs with no external legs. It is therefore a normalization factor. By varying Eqn. (3.5) twice with respect to the sources and then setting them equal to zero one gets

$$\left.\frac{\delta^2 Z[J]}{\delta J_a(x)\delta J_b(y)}\right|_{J=0} = \langle 0|\phi_a(x)\phi_b(y)|0\rangle\,. \tag{3.7}$$

With the generating functional $Z[J]$ one gets not only results describing propagating and interacting particles but also unphysical, so-called disconnected graphs which are, for example, closed loops. These disconnected graphs would also lead to divergences and to avoid this one introduces the generating functional for the connected Green functions by

$$Z[J] = \mathrm{e}^{-Z^c[J]} = \int \mathcal{D}[\phi] \mathrm{e}^{-S[\phi]+\int \mathrm{d}^4x J_a(x)\phi_a(x)} \,. \tag{3.8}$$

With the generating functional for the connected Green functions one can make a Legendre transformation

$$\Gamma[\phi^{cl}] = \left(Z^c[J] - \int \mathrm{d}^4x J_a(x)\phi^{cl}_a(x) \right)\Bigg|_{J_a=J_a[\phi^{cl}]} \,, \tag{3.9}$$

with the inverse transformation

$$Z^c[J] = \left(\Gamma[\phi^{cl}] + \int \mathrm{d}^4x J_a(x)\phi^{cl}_a(x) \right)\Bigg|_{\phi^{cl}_a=\phi^{cl}_a[J]} \,. \tag{3.10}$$

The classical fields are Schwartz fast decreasing $\mathcal{C}^\infty$ test functions just like the sources. They are the vacuum expectation values of the field operators and defined as

$$\phi^{cl}_a = \frac{\delta Z^c[J]}{\delta J_a(x)} \,, \quad \text{with} \quad \phi^{cl}_a(x) = \langle 0|\phi(x)|0\rangle \,. \tag{3.11}$$

When one makes the functional derivative of the vertex functional $\Gamma[\phi^{cl}]$ with respect to the field one gets

$$\begin{aligned}
\frac{\delta\Gamma[\phi^{cl}]}{\delta\phi^{cl}(y)} &= \int \mathrm{d}^4x \frac{\delta Z^c}{\delta J(x)} \frac{\delta J(x)}{\delta\phi^{cl}(y)} - \int \mathrm{d}^4x \left(\frac{\delta J(x)}{\delta\phi^{cl}(y)} \phi^{cl}(x) + J(x)\delta^4(x-y) \right) \\
&= \int \mathrm{d}^4x \left(\frac{\delta Z^c[J]}{\delta J(x)} - \phi^{cl}(x) \right) \frac{\delta J(x)}{\delta\phi^{cl}(y)} - J(y) \,.
\end{aligned}$$

The argument of the integral is zero due to the definition of the field and so one gets the final result

$$\frac{\delta\Gamma[\phi^{cl}]}{\delta\phi^{cl}(y)} = -J(y) \,. \tag{3.12}$$

Written as a formal power series in $\hbar$ the vertex functional Γ is expanded into the contributions from the different loop orders n

$$\Gamma(\phi) = \sum_{n=0}^{\infty} \hbar^n \Gamma^{(n)}(\phi) \,. \tag{3.13}$$

The tree level (zeroth) order of this expansion $\Gamma^{(0)}$ is equal to the classical action $S[\phi]$ and thus it is possible to calculate the sources from it

$$\frac{\delta S}{\delta\phi^{cl}_a} = -J_a(x) \,. \tag{3.14}$$

For the calculation of the propagators it is possible to show that instead of $Z[J]$ one can use the generating functional of the connected Green functions with a changed sign,

$$\begin{aligned}\Delta_{ab}(x,y) &= \left.\frac{\delta^2 Z[J]}{\delta J_a(x)\delta J_b(y)}\right|_{J=0} = \left.\frac{\delta^2 \mathrm{e}^{-Z^c[J]}}{\delta J_a(x)\delta J_b(y)}\right|_{J=0} \\ &= \left.\frac{\delta}{\delta J_a(x)}\left(-\frac{Z^c[J]}{\delta J_b(y)}\mathrm{e}^{-Z^c[J]}\right)\right|_{J=0} \\ &= -\left.\frac{\delta^2 Z^c}{\delta J_a(x)\delta J_b(y)}\mathrm{e}^{-Z^c[J]}\right|_{J=0} + \left.\frac{\delta Z^c[J]}{\delta J_a(x)}\frac{\delta Z^c[J]}{\delta J_b(y)}\mathrm{e}^{-Z^c[J]}\right|_{J=0} . \end{aligned} \tag{3.15}$$

The second part of this expression leads to unphysical one point functions called tadpoles. The contributions of those are normally set to zero and, therefore, the term is dropped at this point. The normalization of Z^c together with the fact that the sources are set to zero kills the exponential and what is left is the desired result

$$\Delta_{ab}(x,y) = -\frac{\delta^2 Z^c}{\delta J_a(x)\delta J_b(y)} = -\frac{\delta \phi_b^{cl}(y)}{\delta J_a(x)} . \tag{3.16}$$

To calculate the propagator one starts with the bilinear action and varies it with respect to the fields in order to get an equation for the source. This equation is then furthermore varied with respect to the source and gives the propagator. Working with a scalar theory this is pretty simple in this thesis as there is only one type of field and one corresponding source. In gauge theories with many different fields and sources this task can become rather complex.

3.2.1 Propagator

The bilinear part of the action gives the propagator. Due to the feature of the star product that one can drop one star under the integral the propagator in the non-commutative theory is the same as in commutative theory. The starting

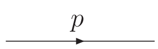

Figure 3.1: The propagator in momentum space.

point is the bilinear part of the action

$$\begin{aligned} S_{bi}[\phi] &= \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi\star\partial_\mu\phi + \frac{1}{2}m^2\phi\star\phi\right) \\ &= \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2\right) . \end{aligned} \tag{3.17}$$

By a Fourier transformation one gets the bilinear action in momentum space,[1]

$$
\begin{aligned}
S_{bi}[\phi] &= \int \mathrm{d}^4x \int \tfrac{\mathrm{d}^4p}{(2\pi)^4} \int \tfrac{\mathrm{d}^4p'}{(2\pi)^4} \left(\tfrac{1}{2}\partial_\mu \left(\phi(p)\mathrm{e}^{\mathrm{i}px}\right) \partial_\mu \left(\phi(p')\mathrm{e}^{\mathrm{i}p'x}\right) + \tfrac{m^2}{2}\phi(p)\phi(p')\mathrm{e}^{\mathrm{i}(p+p')x} \right) \\
&= -\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4} \int \frac{\mathrm{d}^4p'}{(2\pi)^4} \left(\frac{pp'}{2}\phi(p)\phi(p') + \frac{m^2}{2}\phi(p)\phi(p') \right) \mathrm{e}^{\mathrm{i}(p+p')x} \\
&= -\int \frac{\mathrm{d}^4p}{(2\pi)^4} \int \frac{\mathrm{d}^4p'}{(2\pi)^4} \left(\frac{pp'}{2}\phi(p)\phi(p') + \frac{m^2}{2}\phi(p)\phi(p') \right) \delta^{(4)}(p+p') \\
&= \int \frac{\mathrm{d}^4p'}{(2\pi)^4} \left(\frac{p'^2}{2}\phi(p')\phi(-p') + \frac{m^2}{2}\phi(p')\phi(-p') \right) . \qquad (3.18)
\end{aligned}
$$

To get the propagator one has to vary the bilinear action with respect to the fields,

$$
\begin{aligned}
\frac{\delta S_{bi}}{\delta\phi(p)} &= \frac{\delta}{\delta\phi(p)} \int \frac{\mathrm{d}^4p'}{(2\pi)^4} \left(\frac{p'^2}{2}\phi(p')\phi(-p') + \frac{m^2}{2}\phi(p')\phi(-p') \right) \\
&= \int \frac{\mathrm{d}^4p'}{(2\pi)^4} \delta^{(4)}(p-p') \left(p'^2\phi(-p') + m^2\phi(-p') \right) \\
&= \left(p^2 + m^2\right)\phi(-p) = -j(-p) \, . \qquad (3.19)
\end{aligned}
$$

From this one gets $\phi(-p) = -\frac{j(-p)}{p^2+m^2}$ and with a second variation, this time with respect to the source, the propagator,

$$
\Delta(p,p') = -\frac{\delta\phi(-p)}{\delta j(-p')} = \frac{1}{p^2+m^2}\delta^{(4)}(p-p') \, . \qquad (3.20)
$$

The delta functional is dropped at this time as it only represents the momentum conservation, which is understood, and what is left is the propagator

$$
G(p) = \frac{1}{p^2+m^2} \, . \qquad (3.21)
$$

3.3 General Remarks on Calculating Vertices

Given their existence, aside from two point functions one can also calculate n point functions with $n > 2$. When one does that, one gets the unconnected functions, where not all the external points are somehow connected with each other and the connected functions, which are of interest as the so-called vertices. An unconnected four point function would be, for example, built up from two two point functions shown left in Fig. 3.2. Using the generating functional for

[1]Note that here the same symbol is used for the field in momentum as well as in coordinate space.

Figure 3.2: An unconnected and a connected 4-point function.

the connected functions $Z^c[\phi]$ one gets only the connected functions. For a four point function this would be only the vertex shown right in Fig. 3.3. When one is just interested in the vertex itself without the external propagators one has to find a mechanism to amputate them. Following [19], this is done by introducing an expression which can be seen as an inverse of a propagator so that

$$\int \mathrm{d}z \Delta_c(x,z) K(z,y) = \delta(x-y)\,. \tag{3.22}$$

K is called the kernel and is defined as

$$K(x,y) = \frac{\delta^2 \Gamma[\phi^{cl}]}{\delta\phi^{cl}(x)\delta\phi^{cl}(y)}\,. \tag{3.23}$$

Hence, one can proof Eqn. (3.22)

$$\begin{aligned}\int \mathrm{d}^4z \Delta(x,z) K(z,y) &= -\int \mathrm{d}^4z \frac{\delta^2 Z^c[J]}{\delta J(x)\delta J(z)} \frac{\delta^2\Gamma[\phi^{cl}]}{\delta\phi^{cl}(z)\delta\phi^{cl}(y)} \\ &= \int \mathrm{d}^4z \frac{\delta\phi^{cl}(x)}{\delta J(z)} \frac{\delta J(z)}{\delta\phi^{cl}(y)} \\ &= \delta^4(x-y)\,. \end{aligned} \tag{3.24}$$

This equation will now be varied with respect to $J(x'')$ with the rewritten operator

$$\frac{\delta}{\delta J(x'')} = \int \mathrm{d}^4z'' \frac{\delta\phi^{cl}(z'')}{\delta J(x'')} \frac{\delta}{\delta\phi^{cl}(z'')} = -\int \mathrm{d}^4z'' \Delta(x'',z'') \frac{\delta}{\delta\phi^{cl}(z'')}\,, \tag{3.25}$$

to get

$$\begin{aligned}&\int \mathrm{d}^4z \frac{\delta^3 Z^c[J]}{\delta J(x'')\delta J(x)\delta J(z)} K(z,y) \\ &\quad - \int \mathrm{d}^4z \int \mathrm{d}^4z'' \frac{\delta^2 Z^c[J]}{\delta J(x)\delta J(z)} \Delta(x'',z'') \frac{\delta^3\Gamma[\phi^{cl}]}{\delta\phi^{cl}(z'')\delta\phi^{cl}(z)\delta\phi^{cl}(y)} = 0\,. \end{aligned} \tag{3.26}$$

Inserting the propagator gives

$$\begin{aligned}&\int \mathrm{d}^4z \frac{\delta^3 Z^c[J]}{\delta J(x'')\delta J(x)\delta J(z)} K(z,y) \\ &\quad + \int \mathrm{d}^4z \int \mathrm{d}^4z'' \Delta(x,z)\Delta(x'',z'') \frac{\delta^3\Gamma[\phi^{cl}]}{\delta\phi^{cl}(z'')\delta\phi^{cl}(z)\delta\phi^{cl}(y)} = 0\,. \end{aligned} \tag{3.27}$$

When one inserts a propagator $\Delta(x', y)$ on both sides of this equation and integrates over y one gets

$$\int \mathrm{d}^4 y \int \mathrm{d}^4 z \frac{\delta^3 Z^c[J]}{\delta J(x'')\delta J(x)\delta J(z)} \Delta(x', y) K(z, y) + \int \mathrm{d}^4 y \int \mathrm{d}^4 z \int \mathrm{d}^4 z'' \Delta(x', y)\Delta(x, z)\Delta(x'', z'') \frac{\delta^3 \Gamma[\phi^{cl}]}{\delta\phi^{cl}(z'')\delta\phi^{cl}(z)\delta\phi^{cl}(y)} = 0\,, \tag{3.28}$$

and this gives with Eqn. (3.22)

$$\frac{\delta^3 Z^c[J]}{\delta J(x'')\delta J(x)\delta J(x')} = -\int \mathrm{d}^4 y \int \mathrm{d}^4 z \int \mathrm{d}^4 z'' \Delta(x', y)\Delta(x, z)\Delta(x'', z'') \frac{\delta^3 \Gamma[\phi^{cl}]}{\delta\phi^{cl}(z'')\delta\phi^{cl}(z)\delta\phi^{cl}(y)}\,. \tag{3.29}$$

This is the desired result, namely that a connected three point function is the same as a vertex function with external propagators. The inverse of this function has the form

$$\frac{\delta^3 \Gamma[\phi^{cl}]}{\delta\phi^{cl}(z'')\delta\phi^{cl}(z)\delta\phi^{cl}(y)} = -\int \mathrm{d}^4 x \int \mathrm{d}^4 x' \int \mathrm{d}^4 x'' K(x', y) K(x, z) K(x'', z'') \frac{\delta^3 Z^c[J]}{\delta J(x'')\delta J(x)\delta J(x')}\,. \tag{3.30}$$

This result then states that the vertex is a connected three point function without the external propagators. For the scalar model there exist no three point functions but only one connected four point function. It is easy to see that a four vertex can be generated in the same manner as shown by just one additional derivative. Generally, one can create n-point functions by variating enough times with respect to the sources and the general formula for the vertex function is

$$V = -\frac{\delta^n \Gamma[\phi^{cl}]}{\delta\phi^{cl}(x_1)\ldots\delta\phi^{cl}(x_n)}\,. \tag{3.31}$$

As already mentioned in the section of the propagator, Γ can be substituted by the action. Here it is sufficient to use the interaction part of the action as it is the only one which contains enough fields for the number of variations. The result for the vertex in the scalar model is therefore

$$V^{\phi^4} = -\frac{\delta^4 S_{int}}{\delta\phi(x_1)\delta\phi(x_2)\delta\phi(x_3)\delta\phi(x_n)}\,, \tag{3.32}$$

or in momentum space

$$V^{\phi^4}(p_1, p_2, p_3, p_4) = -(2\pi)^{16} \frac{\delta^4 S_{int}}{\delta\tilde{\phi}(-p_1)\delta\tilde{\phi}(-p_2)\delta\tilde{\phi}(-p_3)\delta\tilde{\phi}(-p_4)}\,. \tag{3.33}$$

3.3.1 Vertex

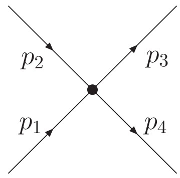

Figure 3.3: The vertex.

The interaction of fields can be calculated from the higher than bilinear parts of the action. Here this is just one term,

$$\begin{aligned}
S_{int}[\phi] &= \int \mathrm{d}^4x \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \\
&= \int \mathrm{d}^4x \frac{\lambda}{4!} (\phi \star \phi)(\phi \star \phi) \\
&= \frac{\lambda}{4!} \int \mathrm{d}^4x \int \frac{\mathrm{d}^4k_{1..4}}{(2\pi)^{16}} \mathrm{e}^{\mathrm{i}(k_1+k_2+k_3+k_4)x} \tilde{\phi}(k_1)\tilde{\phi}(k_2)\mathrm{e}^{-\mathrm{i}\frac{k_1 \times k_2}{2}} \tilde{\phi}(k_3)\tilde{\phi}(k_4)\mathrm{e}^{-\mathrm{i}\frac{k_3 \times k_4}{2}} \\
&= \frac{\lambda}{4!(2\pi)^{12}} \int \mathrm{d}^4k_{1..4}\, \delta(k_1+k_2+k_3+k_4) \\
&\qquad \times \tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4)\mathrm{e}^{-\frac{\mathrm{i}}{2}(k_1 \times k_2 + k_3 \times k_4)} . \qquad (3.34)
\end{aligned}$$

Taking the functional derivatives of this part of the action gives the vertex. It is not necessary to symmetrize the action when one makes every permutation during the derivation. Due to the four functional derivations the calculation is rather lengthy and can be found in the Appendix A.1. The result for the vertex is

$$\begin{aligned}
V^{\phi^4}(p_1,p_2,p_3,p_4) &= -\frac{\delta}{\delta\tilde{\phi}(-p_1)}\frac{\delta}{\delta\tilde{\phi}(-p_2)}\frac{\delta}{\delta\tilde{\phi}(-p_3)}\frac{\delta}{\delta\tilde{\phi}(-p_4)}(2\pi)^{16}S_{int}[\tilde{\phi}] \\
&= -\tfrac{\delta}{\delta\tilde{\phi}(-p_1)}\tfrac{\delta}{\delta\tilde{\phi}(-p_2)}\tfrac{\delta}{\delta\tilde{\phi}(-p_3)}\tfrac{\delta}{\delta\tilde{\phi}(-p_4)}\tfrac{\lambda}{4!}(2\pi)^4 \int \mathrm{d}^4k_{1..4}\, \delta^4(k_1+k_2+k_3+k_4) \\
&\quad \tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4)\mathrm{e}^{-\frac{\mathrm{i}}{2}(k_1 \times k_2 + k_3 \times k_4)} \\
&= -\frac{\lambda}{3}(2\pi)^4\delta^4(p_1+p_2+p_3+p_4) \\
&\quad \left(\cos\tfrac{p_1 \times p_2}{2}\cos\tfrac{p_3 \times p_4}{2} + \cos\tfrac{p_1 \times p_3}{2}\cos\tfrac{p_2 \times p_4}{2} + \cos\tfrac{p_1 \times p_4}{2}\cos\tfrac{p_2 \times p_3}{2}\right) . \qquad (3.35)
\end{aligned}$$

The delta functional in this result ensures the momentum conservation. If one lets $|\theta|$ go to zero the vertex looses its phase factors and one recovers the commutative vertex.

3.4 UV/IR Mixing

The naive non-commutative model has a problem not known in commutative QFT. In standard QFT a loop graph shows a quadratic UV divergent behavior and has to be renormalized. This is not possible in NCQFT because of the fact that not only is there a term with UV divergence as in standard QFT but also a term with an IR divergence which does not appear in the normal theory. A good method to show this is to calculate the one loop correction for the two point function of the model. The graph without external legs is composed from a vertex and a propagator connecting two legs of the vertex (see Fig. 3.4),

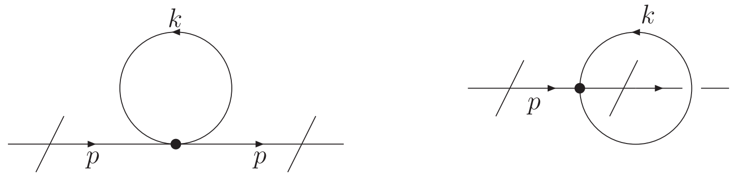

Figure 3.4: The planar and non-planar one loop two point functions.

$$\Pi(p) = -\frac{1}{2}\int \mathrm{d}^4k \frac{\lambda}{3}\frac{1}{(2\pi)^4}\frac{1}{k^2+m^2}\left(\cos\frac{p\times(-k)}{2}\cos\frac{k\times(-p)}{2}\right.$$
$$\left. + \cos\frac{p\times k}{2}\cos\frac{(-k)\times(-p)}{2} + \cos\frac{p\times(-p)}{2}\cos\frac{(-k)\times k}{2}\right).$$

The factor $\frac{1}{2}$ is a symmetry factor necessary because of the mirror symmetry of the graph. The last term gives one because $k \times k$ is a multiplication of a symmetric and an antisymmetric matrix which is zero. The other two terms can be added because the cosine is a symmetric function. This leads to

$$\Pi(p) = -\frac{\lambda}{6(2\pi)^4}\int \mathrm{d}^4k \frac{1}{k^2+m^2}\left(2\cos^2\frac{p\times k}{2} + 1\right).$$

The square of the cosine can be split up in two terms using $\cos^2\frac{p\times k}{2} = \cos^2\frac{k\times p}{2} = \frac{1}{2}\cos k\tilde{p} + \frac{1}{2}$ with $\tilde{p} = \theta_{\mu\nu}p^\nu$ and leads to the result

$$\Pi(p) = -\frac{\lambda}{6(2\pi)^4}\int \mathrm{d}^4k \frac{1}{k^2+m^2}\left(\cos k\tilde{p} + 2\right).$$

Writing the cosine with the help of exponential functions, that means with $\cos k\tilde{p} = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}k\tilde{p}} + \mathrm{e}^{-\mathrm{i}k\tilde{p}}\right)$, one arrives at the final result

$$\Pi(p) = -\frac{\lambda}{12(2\pi)^4}\int \mathrm{d}^4k \sum_{\eta=\pm 1}\frac{\mathrm{e}^{\mathrm{i}\eta k\tilde{p}}}{k^2+m^2} - \frac{\lambda}{3(2\pi)^4}\int \mathrm{d}^4k \frac{1}{k^2+m^2}. \tag{3.36}$$

The second term is just the same as in commutative theory and known to be UV divergent. It is called the *planar* graph. The first term is called the *non-planar* graph because it has an exponential factor which is expected to act as a damping for high momenta because of fast oscillations. This indeed works but it is also responsible for the IR divergent behavior of the theory. Both graphs are depicted in Fig. 3.4. The generic integral which will be solved now has the form

$$I(p) = \int \mathrm{d}^4k \frac{1}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}\eta k\tilde{p}}}{k^2 + m^2} \,. \tag{3.37}$$

The first step is to use Schwinger parameterization to get rid of the fraction and replace it by something which can be integrated more easily. On the downside one gets a second integral to solve instead.

$$\frac{1}{A} = \int_0^\infty \mathrm{d}\alpha \; \mathrm{e}^{-\alpha A} \,, \qquad \forall\, A \neq 0 \,. \tag{3.38}$$

With the Schwinger parameterization the integrand becomes an all exponential expression.

$$\begin{aligned} I(p) &= \frac{1}{(2\pi)^4} \int \mathrm{d}^4k \int_0^\infty \mathrm{d}\alpha \; \mathrm{e}^{-\alpha(k^2+m^2)+\mathrm{i}\eta k\tilde{p}} \\ &= \frac{1}{(2\pi)^4} \int \mathrm{d}^4k \int_0^\infty \mathrm{d}\alpha \; \mathrm{e}^{-\alpha k^2+\mathrm{i}\eta k\tilde{p}-\alpha m^2} \,. \end{aligned}$$

The exponent can be written as a full square with the goal of substituting the integration variable and one gets (with $\eta^2 = 1$)

$$I(p) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4k \int_0^\infty \mathrm{d}\alpha \; \mathrm{e}^{-\alpha\left(k^2 - \frac{\mathrm{i}\eta k\tilde{p}}{\alpha} - \frac{\tilde{p}^2}{4\alpha^2}\right) - \frac{\tilde{p}^2}{4\alpha} - \alpha m^2} \,.$$

Substituting $k' = k - \frac{\mathrm{i}\eta\tilde{p}}{2\alpha}$ leaves the differential unchanged as $\mathrm{d}k' = \mathrm{d}k$ and the resulting Gaussian integral can be solved leaving only the integral over the Schwinger parameter α.

$$\begin{aligned} I(p) &= \frac{1}{(2\pi)^4} \int_0^\infty \mathrm{d}\alpha \int \mathrm{d}^4k' \mathrm{e}^{-\alpha k'^2 - \frac{\tilde{p}^2}{4\alpha} - \alpha m^2} \\ &= \frac{1}{(4\pi)^2} \int_0^\infty \mathrm{d}\alpha \frac{1}{\alpha^2} \mathrm{e}^{-\frac{\tilde{p}^2}{4\alpha} - \alpha m^2} \,. \end{aligned} \tag{3.39}$$

A solution for this integral can be found, for example, in the book of Gradshteyn [21] on p.340 (3.471/9),

$$\int_0^\infty x^{\nu-1}\mathrm{e}^{-\frac{\beta}{x}-\gamma x}\mathrm{d}x = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu\left(2\sqrt{\beta\gamma}\right) \quad \text{with} \quad \mathrm{Re}\,\beta > 0,\ \mathrm{Re}\,\gamma > 0\,. \tag{3.40}$$

The K_ν are the modified Bessel functions of the second kind and given as (Gradshteyn p.961 (8.446) [21])

$$\begin{aligned} K_\nu(z) &= \frac{1}{2}\sum_{k=0}^{\nu-1}(-1)^k\frac{(\nu-k-1)!}{k!\left(\frac{z}{2}\right)^{\nu-2k}} \\ &+ (-1)^{\nu+1}\sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{\nu+2k}}{k!(\nu+k)!}\left[\ln\frac{z}{2}-\frac{1}{2}\psi(k+1)-\frac{1}{2}\psi(\nu+k+1)\right]\,, \end{aligned} \tag{3.41}$$

where the $\psi(\nu+1)$ are Euler-Psi functions given as ([21] p.945 (8.365/4))

$$\psi(\nu+1) = -\gamma_E + \sum_{k=1}^{\nu}\frac{1}{k}\,, \tag{3.42}$$

with $\gamma_E = 0.5772\cdots$ the Euler-Mascheroni constant. The Bessel functions are symmetric which means $K_{-\nu} = K_\nu$. Here ν is -1 and thus K_1 is used,

$$\begin{aligned} K_1(z) &= \frac{1}{z} + \sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2k+1}}{k!(k+1)!}\left[\ln\frac{z}{2}+\gamma_E-\sum_{l=1}^{k}\frac{1}{l}-\frac{1}{2(k+1)}\right] \\ &= \frac{1}{z}+\frac{z}{2}\left(\ln\frac{z}{2}+\gamma_E-\frac{1}{2}\right)+\frac{z^3}{16}\left(\ln\frac{z}{2}+\gamma_E-\frac{5}{4}\right)+\mathcal{O}\left(z^5\right)\,. \end{aligned} \tag{3.43}$$

The result for the generic integral is

$$I(p) = \frac{2}{(4\pi)^2}\left(\frac{4m^2}{\tilde{p}^2}\right)^{\frac{1}{2}} K_1\left(2\sqrt{\frac{\tilde{p}^2}{4}m^2}\right) = \frac{1}{(2\pi)^2}\frac{m}{\tilde{p}}K_1(\tilde{p}m)\,. \tag{3.44}$$

Inserting the expansion for small arguments of the Bessel functions to the first order (3.43) gives

$$\begin{aligned} I(p) &\approx \frac{1}{(2\pi)^2}\frac{m}{\tilde{p}}\left[\frac{1}{\tilde{p}m}+\frac{\tilde{p}m}{2}\left(\ln\left(\tfrac{\tilde{p}m}{2}\right)+\gamma_E-\frac{1}{2}\right)\right] \\ &= \frac{1}{(2\pi)^2}\left[\frac{1}{\tilde{p}^2}+\frac{m^2}{2}\left(\ln\left(\tfrac{\tilde{p}m}{2}\right)+\gamma_E-\frac{1}{2}\right)\right]\,. \end{aligned} \tag{3.45}$$

This expression has a mass dimension of two which normally indicates a quadratic divergence but the exponential phase factor made it finite. However, if one looks at small external momenta (that is $\tilde{p} \to 0$), one finds that the first term diverges. The planar graph, on the other hand, is UV divergent as already mentioned. So the one loop correction for the two point graph is UV as well as IR divergent. In addition to the UV divergence already present in commutative theories, new divergences emerge coming from the non-planar graph. This is the famous UV/IR mixing problem of this theory which makes it not renormalizable. The IR singularity is a problem for higher order calculations because of the integrations of the inner loops which include the value $p = 0$. When one looks at the exponential factor of the non-planar graph responsible for the damping of the UV divergences one can see that these divergences are mapped to the infrared region because when $|\theta|$ or p^2 go to zero the exponential factor vanishes and leaves a planar graph which is UV divergent. A physical interpretation of the whole problem can be found when looking at what non-commutativity means. The non-commutativity relation between spacial directions or between different momenta represents an uncertainty relation which leads to a mixing of long and short distances or high and low momenta because whenever a very long or short distance occurs in one direction, another direction inevitably has to have a distance opposite to the first. The same happens for momenta where a high momentum is accompanied by a low momentum and hence the UV/IR mixing arises.

Chapter 4

The $\frac{1}{p^2}$ Model

4.1 Renormalizable Models

The naive model is not renormalizable and, therefore, some effort was put in the search of an extension of the model to a renormalizable one. Recently, there have been three such models proven to be renormalizable. One was put forward by Grosse and Wulkenhaar [22, 23] who added a term

$$S_{osc}[\phi] = \int \mathrm{d}^4x \frac{\Omega^2}{2} (\tilde{x}_\mu \phi) \star (\tilde{x}_\mu \phi) \ , \quad \text{with} \quad \tilde{x}_\mu = 2\theta^{-1}_{\mu\nu} x_\nu \, , \tag{4.1}$$

to the action (3.2). This additional oscillator potential, with Ω being a dimensionless constant parameter, renders the theory renormalizable. The propagator is essentially the Mehler kernel which is described in [24] and [25] for one dimension and in [26] for more than one dimension. However, the extension has the major problem that it obviously breaks translation invariance, although only by a tiny bit as the oscillator potential is very weak. Another extension has been proposed by Grosse and Vignes-Tourneret [27]. They added a non-local term

$$S_{GV}[\phi] = \frac{\mu}{\theta^4} \int \mathrm{d}^4x \, \phi(x) \int \mathrm{d}^4x' \phi(x') \, , \tag{4.2}$$

with μ being a parameter of mass dimension -2, to the action (3.2). This gives a translation invariant ϕ^4 theory but it is only renormalizable in one direction which makes it not very useful for physical models. Gurau, Magnen, Rivasseau and Tanasa [1] made the third proposal which consists of adding another non-local term

$$S_{1/p^2}[\phi] = -\frac{1}{2} \int \mathrm{d}^4x \ \phi \star \frac{a'^2}{\theta^2 \Box} \phi \, , \tag{4.3}$$

to the action of the naive model (3.2) with the intention that this term acts as a, sort of, counterterm for the IR divergence. This new theory is also translation invariant and will be investigated in more detail in the following chapters.

4.2 The Action

To avoid problems with the UV/IR mixing one adds an additional term to the action. This term is basically a $\frac{1}{\tilde{\Box}}$ with $\tilde{\Box} = \theta^2\Box$. At this point it seems odd to put a derivation in the denominator as this operation seems to be not defined[1]. The Fourier transformation of the expression gives some clarification here as the quabla operator becomes a squared momentum which is known how to handle. The action of the new model is[2] [1]:

$$S[\phi] = \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{1}{2}m^2\phi \star \phi - \frac{1}{2}\phi \star \frac{a'^2}{\theta^2\Box}\phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right) . \quad (4.4)$$

As is known already the star in the bilinear terms can be dropped,

$$S[\phi] = \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 - \frac{1}{2}\phi\frac{a'^2}{\theta^2\Box}\phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right) . \quad (4.5)$$

Here, a' is just a dimensionless parameter and to shorten the notation a' and θ are combined to a new parameter $a = \frac{a'}{\theta}$. When looking at the additional term in momentum space one finds

$$\begin{aligned} S_{1/p^2}[\phi] &= -\frac{1}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4} \int \frac{\mathrm{d}^4p'}{(2\pi)^4} \phi(p)\mathrm{e}^{\mathrm{i}px}\frac{a^2}{\Box}\left(\phi(p')\mathrm{e}^{\mathrm{i}p'x}\right) \\ &= \frac{1}{2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4} \int \frac{\mathrm{d}^4p'}{(2\pi)^4} \phi(p)\frac{a^2}{p'^2}\phi(p')\mathrm{e}^{\mathrm{i}(p+p')x} \\ &= \frac{1}{2}\int \frac{\mathrm{d}^4p}{(2\pi)^4} \int \frac{\mathrm{d}^4p'}{(2\pi)^4} \phi(p)\frac{a^2}{p'^2}\phi(p')\delta^{(4)}(p+p') \\ &= \frac{1}{2}\int \frac{\mathrm{d}^4p}{(2\pi)^4} \phi(p)\frac{a^2}{p^2}\phi(-p) . \end{aligned} \quad (4.6)$$

In Table 4.1 the mass dimensions of the coefficients are listed. The new term

ϕ	m	a	θ	a'
1	1	2	-2	0

Table 4.1: Mass dimensions.

contains $\frac{1}{p^2}$ which happens to be also a result of the one loop correction for the

[1] The derivation in the denominator is seen here as an inverse of the usual derivation which means $\Box_x \frac{1}{\Box_{x'}} \propto \delta^{(4)}(x - x')$.

[2] It has to be mentioned at this point that, as terms with $\Box\tilde{\Box}$ have mass dimension zero, it is in principle possible to add such terms with arbitrary powers of the operator to the action. However, they are usually forbidden by power counting or renormalization conditions.

propagator of the naive model (3.2). Thus, it is hoped that, with a proper value for the parameter a, this inconvenient behavior, namely the IR divergence of the propagator, can be suppressed and that the action gives a renormalizable theory. This has been proved by Gurau *et al.* in [1] to arbitrary orders with the so-called *Multiscale Analysis* which looks at different energy scales of the graph and "slices" it in a way that the separate parts are all finite. For each slice an upper bound can be found and with it a general bound for the amplitudes of arbitrary Feynman graphs can be derived. With a recursive scheme starting at the highest energies one can get the field of the next lower energy level at any recursion step and finally arrives at the renormalized effective action. More on the topic of Multiscale Analysis can be found in the books of Rivasseau [3, 28] and, specially for the $\frac{1}{p^2}$ model, in the paper of Gurau *et al.* [1]. Letting θ go to zero to get a commutative limit is possible but not trivial as one has to add several auxiliary terms to the commutative action which vanish for $\theta \neq 0$ [29].

4.3 Feynman Rules

In this work a more traditional form of renormalization than the Multiscale Analysis is used by calculating loop graphs order by order and then renormalizing them. But before one can start with the loops one has to have the Feynman rules of the model first. As it is a scalar model there is only one propagator and one vertex to calculate. Details on deriving propagators and vertices can be found in the Sections 3.2 and 3.3.

4.3.1 Propagator

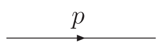

Figure 4.1: The propagator.

The extra term in the action gives a different propagator as in the naive model (3.2) and, therefore, also as in the commutative theory. The bilinear part of the action, which is responsible for the better behavior of the model, reads

$$S_{bi}[\phi] = \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 - \frac{1}{2}\phi\frac{a^2}{\Box}\phi \right) . \tag{4.7}$$

From this the further calculation is done like in the normal case. The source is gained by a variation with respect to ϕ of the action S_{bi} which at tree level is the same as Γ,

$$\begin{aligned}
\frac{\delta S_{bi}}{\delta\phi(p)} &= \frac{\delta}{\delta\phi(p)}\int\frac{\mathrm{d}^4p'}{(2\pi)^4}\left(\frac{p'^2}{2}\phi(p')\phi(-p')+\frac{m^2}{2}\phi(p')\phi(-p')+\frac{1}{2}\phi(p)\frac{a^2}{p^2}\phi(-p)\right)\\
&= \int\frac{\mathrm{d}^4p'}{(2\pi)^4}\delta^{(4)}(p-p')\left(p'^2\phi(-p')+m^2\phi(-p')+\frac{a^2}{p'^2}\phi(-p')\right)\\
&= \left(p^2+m^2+\frac{a^2}{p^2}\right)\phi(-p)=-j(-p)\,. \qquad (4.8)
\end{aligned}$$

This is then solved for the field $\phi(-p)=-\frac{j(-p)}{p^2+m^2+\frac{a^2}{p^2}}$ and the propagator is calculated with a second variation with respect to the source,

$$\Delta(p,p')=-\frac{\delta\phi(-p)}{\delta j(-p')}=\frac{1}{p^2+m^2+\frac{a^2}{p^2}}\delta^{(4)}(p-p')\,. \qquad (4.9)$$

As a last step the delta functional representing the momentum conservation can be dropped giving the propagator

$$G(p)=\frac{1}{p^2+m^2+\frac{a^2}{p^2}}\,. \qquad (4.10)$$

The propagator is the only difference between the new model with the $\frac{1}{\Box}$ term and the naive model. It is therefore the reason for the better UV/IR behavior of the theory and for its renormalizability. One can see that the propagator becomes zero for $p\to 0$ as well as for $p\to\infty$ and it even stays finite for vanishing mass m. For later calculations it will be useful to write the propagator in a different form,

$$\begin{aligned}
\frac{1}{p^2+m^2+\frac{a^2}{p^2}} &= \frac{p^2}{p^4+m^2p^2+\frac{a'^2}{\theta^2}} = \frac{p^2}{\underbrace{p^4+m^2p^2+\frac{m^4}{4}}_{\left(p^2+\frac{m^2}{2}\right)^2}-\left(\underbrace{\frac{m^4}{4}-\frac{a'^2}{\theta^2}}_{M^4}\right)}\\
&= \frac{p^2}{\left(p^2+\frac{m^2}{2}-M^2\right)\left(p^2+\frac{m^2}{2}+M^2\right)}\\
&= \frac{1}{2}\left[\frac{1}{p^2+\frac{m^2}{2}-M^2}+\frac{1}{p^2+\frac{m^2}{2}+M^2}-m^2\frac{1}{\left(p^2+\frac{m^2}{2}-M^2\right)\left(p^2+\frac{m^2}{2}+M^2\right)}\right]\,. \qquad (4.11)
\end{aligned}$$

The third term can be rewritten furthermore as

$$\begin{aligned}
&= \frac{1}{2}\left[\frac{1}{p^2+\frac{m^2}{2}-M^2}+\frac{1}{p^2+\frac{m^2}{2}+M^2}+\frac{m^2}{2M^2}\left(\frac{1}{p^2+\frac{m^2}{2}+M^2}-\frac{1}{p^2+\frac{m^2}{2}-M^2}\right)\right]\\
&= \frac{1}{2}\sum_{\zeta=\pm1}\frac{1+\zeta\frac{m^2}{2M^2}}{p^2+\frac{m^2}{2}+\zeta M^2}\,. \qquad (4.12)
\end{aligned}$$

Here the new mass $M^4 = \frac{m^4}{4} - a^2$ is introduced. M^4 can be either positive or negative depending on the value of a and, therefore, M^2 is either real or purely imaginary. In this form the propagator looks like the propagator from the naive model with a squared momentum and a mass term in the denominator so one can expect the calculation of the generic integrals for loop calculations to be pretty similar to the naive model.

4.3.2 Vertex

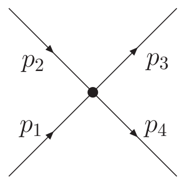

Figure 4.2: The vertex.

The part of the action which gives the vertex is exactly the same as in the naive ϕ^4 theory (3.2) and, therefore, gives the same vertex. The interaction part of the action is

$$S_{int}[\phi] = \int \mathrm{d}^4x \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \,. \tag{4.13}$$

The vertex has already been calculated (A.1) and is

$$\begin{aligned} V^{\phi^4}(p_1, p_2, p_3, p_4) &= -\frac{\lambda}{3}(2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) \\ &\left(\cos\tfrac{p_1 \times p_2}{2} \cos\tfrac{p_3 \times p_4}{2} + \cos\tfrac{p_1 \times p_3}{2} \cos\tfrac{p_2 \times p_4}{2} + \cos\tfrac{p_1 \times p_4}{2} \cos\tfrac{p_2 \times p_3}{2}\right) \,. \end{aligned} \tag{4.14}$$

Chapter 5

One Loop Corrections and Renormalization

5.1 One Loop Correction for the Propagator

The one loop propagator correction depicted in Fig. 5.1 is the easiest correction with mass to calculate. It consists of the planar and the non-planar part. To build it one needs one vertex (4.14), which is the same in the naive and the new model and one propagator (4.10). The external propagators are amputated which is depicted by the crossed out external lines. The left graph in Fig. 5.1 shows

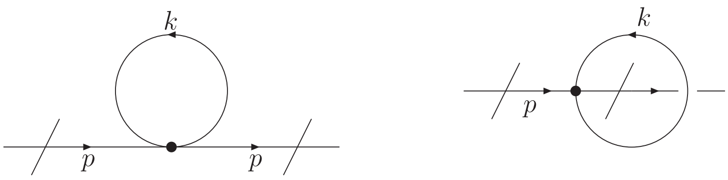

Figure 5.1: The planar and non-planar one loop graphs.

the planar graph and the right graph is the non-planar graph. Without external propagators (external legs) the one loop function has the form

$$\Pi(p) = -\int \mathrm{d}^4k \frac{1}{2}\frac{\lambda}{3(2\pi)^4}\frac{1}{k^2+m^2+\frac{a^2}{k^2}}\left(\cos\frac{p\times(-k)}{2}\cos\frac{k\times(-p)}{2}\right.$$
$$\left.+\cos\frac{p\times k}{2}\cos\frac{(-p)\times(-k)}{2}+\cos\frac{p\times(-p)}{2}\cos\frac{(-k)\times k}{2}\right).$$

λ is the coupling constant and the factor $1/2$ comes from the mirror symmetry of the graph. All the other symmetries are already incorporated in the vertex. A similar calculation as in the naive model (3.36) gives

$$\Pi(p) = -\frac{\lambda}{6(2\pi)^4} \int \mathrm{d}^4k \frac{1}{k^2 + m^2 + \frac{a^2}{k^2}} \left(2\cos^2 \frac{p \times k}{2} + 1 \right) ,$$

and with $\cos^2 \frac{p\times k}{2} = \cos^2 \frac{k\tilde{p}}{2} = \frac{1}{2}\cos k\tilde{p} + \frac{1}{2}$ one gets

$$\Pi(p) = -\frac{\lambda}{6(2\pi)^4} \int \mathrm{d}^4k \frac{1}{k^2 + m^2 + \frac{a^2}{k^2}} \left(\cos k\tilde{p} + 2 \right) .$$

As in the naive model, the term with the cosine is called *non-planar* and is the cause of the IR divergences whereas the other is called *planar* and behaves like in the commutative case. Expanding the cosine in exponential functions gives

$$\Pi(p) = -\frac{\lambda}{12(2\pi)^4} \int \mathrm{d}^4k \sum_{\eta=\pm1} \frac{\mathrm{e}^{\mathrm{i}\eta k\tilde{p}}}{k^2 + m^2 + \frac{a^2}{k^2}} - \frac{\lambda}{3(2\pi)^4} \int \mathrm{d}^4k \frac{1}{k^2 + m^2 + \frac{a^2}{k^2}} . \tag{5.1}$$

The more interesting term of the two is the first one because the second one is just the same as in commutative space and it is well known that it can be handled with an ultraviolet cutoff parameter as is done later. The first term, however, has the additional damping factor in it which should be responsible for the suppression of the mixing problem. The next step is to write the propagator in the form of Eqn. (4.12) because this has the advantage that there is no $\frac{1}{k^2}$ term in the denominator and one can solve the integral in a similar way to the commutative theory.

Non-Planar One Loop Function

The non-planar term looks like

$$\Pi^{n-pl}(p) = -\frac{\lambda}{24(2\pi)^4} \int \mathrm{d}^4k \sum_{\eta,\zeta=\pm1} \frac{1 + \zeta\frac{m^2}{2M^2}}{k^2 + \frac{m^2}{2} + \zeta M^2} \mathrm{e}^{\mathrm{i}\eta k\tilde{p}} . \tag{5.2}$$

The denominator has been expanded like in Eqn. (4.11) in the previous chapter[1]. To shorten the calculations only the generic integral of the non-planar graph is used. It has the form

$$I(p) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{e}^{\eta\mathrm{i}k\tilde{p}}}{k^2 + \frac{m^2}{2} + \zeta M^2} .$$

[1] $M^2 = \sqrt{\frac{m^4}{4} - a^2}$

The first step is to use the Schwinger parameterization (3.38) and complete the exponent to a full square where one can drop the η^2 because it is always one.

$$\begin{aligned}
I(p) &= \int_0^\infty \mathrm{d}\alpha \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{e}^{-\alpha\left(k^2+\frac{m^2}{2}+\zeta M^2\right)+\mathrm{i}\eta k\tilde{p}} \\
&= \int_0^\infty \mathrm{d}\alpha \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{e}^{-\alpha\left(k^2-\frac{\mathrm{i}\eta k\tilde{p}}{\alpha}-\frac{\tilde{p}^2}{4\alpha^2}\right)-\frac{\tilde{p}^2}{4\alpha}-\alpha\left(\frac{m^2}{2}+\zeta M^2\right)} \\
&= \frac{1}{(4\pi)^2} \int_0^\infty \mathrm{d}\alpha \frac{1}{\alpha^2} \mathrm{e}^{-\frac{\tilde{p}^2}{4\alpha}-\alpha\left(\frac{m^2}{2}+\zeta M^2\right)} .
\end{aligned}$$

With the substitution $k' = k - \frac{\mathrm{i}\eta\tilde{p}}{2\alpha}$ and the integration over a Gauss function. This integral can be solved with the tables from Gradshteyn [21] from which the suitable one can be found under Eqn. (3.40) and involves a modified Bessel function,

$$\begin{aligned}
I(p) &= \frac{2}{(4\pi)^2} \left(\frac{\frac{m^2}{2}+\zeta M^2}{\frac{\tilde{p}^2}{4}}\right)^{\frac{1}{2}} K_1\left(\sqrt{\tilde{p}^2\left(\frac{m^2}{2}+\zeta M^2\right)}\right) \\
&= \frac{1}{(2\pi)^2} \sqrt{\frac{1}{\tilde{p}^2}\left(\frac{m^2}{2}+\zeta M^2\right)} K_1\left(\sqrt{\tilde{p}^2\left(\frac{m^2}{2}+\zeta M^2\right)}\right) . \qquad (5.3)
\end{aligned}$$

This result corresponds to the result of the naive model (3.44) for $a \to 0$. To investigate the interesting IR behavior of this graph the Bessel function in it is expanded for small arguments,

$$\begin{aligned}
K_1(z) &= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k+1}}{k!(k+1)!} \left[\ln\frac{z}{2} + \gamma_E - \sum_{l=1}^{k}\frac{1}{l} - \frac{1}{2(k+1)}\right] \\
&= \frac{1}{z} + \frac{z}{2}\left(\ln\frac{z}{2} + \gamma_E - \frac{1}{2}\right) + \frac{z^3}{16}\left(\ln\frac{z}{2} + \gamma_E - \frac{5}{4}\right) + \mathcal{O}\left(z^5\right) , \qquad (5.4)
\end{aligned}$$

where γ_E is the Euler-Mascheroni constant. Only the first two terms of the expansion are of interest and, thus, the rest is neglected. Inserting this in the calculation one gets

$$
\begin{aligned}
I(p) \approx{} & \frac{1}{(2\pi)^2}\sqrt{\frac{1}{\tilde{p}^2}\left(\frac{m^2}{2}+\zeta M^2\right)}\Bigg[\left(\tilde{p}^2\left(\frac{m^2}{2}+\zeta M^2\right)\right)^{-\frac{1}{2}} \\
& +\frac{1}{2}\sqrt{\tilde{p}^2\left(\frac{m^2}{2}+\zeta M^2\right)}\left(\ln\sqrt{\frac{\tilde{p}^2}{4}\left(\frac{m^2}{2}+\zeta M^2\right)}+\gamma_E-\frac{1}{2}\right)\Bigg] \\
={} & \frac{1}{(4\pi)^2}\left[\frac{4}{\tilde{p}^2}+\left(\frac{m^2}{2}+\zeta M^2\right)\left(\ln\left(\frac{\tilde{p}^2}{4}\left(\frac{m^2}{2}+\zeta M^2\right)\right)+2\gamma_E-1\right)\right].
\end{aligned}
\tag{5.5}
$$

With this result it is possible to build the complete non-planar graph but one has to be careful because even though there is no η in the generic result anymore the sum over η has still to be performed and this leads to an additional factor two,

$$
\Pi^{n-pl}(p) = -\frac{\lambda}{48\pi^2}\sum_{\zeta=\pm1}\left(1+\zeta\frac{m^2}{2M^2}\right)\frac{1}{\tilde{p}}\sqrt{\frac{m^2}{2}+\zeta M^2}\,K_1\left(\tilde{p}\sqrt{\frac{m^2}{2}+\zeta M^2}\right).
\tag{5.6}
$$

This result is finite for $\tilde{p}^2 \neq 0$, i.e. if $p \neq 0$ and $\theta \neq 0$. For $\tilde{p}^2 \ll 1$ the Bessel function is expanded and leads to

$$
\begin{aligned}
\Pi^{n-pl}(p) ={} & -\frac{\lambda}{12(4\pi)^2}\sum_{\zeta=\pm1}\left(1+\zeta\frac{m^2}{2M^2}\right) \\
& \times\left[\frac{4}{\tilde{p}^2}+\left(\frac{m^2}{2}+\zeta M^2\right)\left(\ln\left(\frac{\tilde{p}^2}{4}\left(\frac{m^2}{2}+\zeta M^2\right)\right)+2\gamma_E-1\right)\right]+\mathcal{O}(1).
\end{aligned}
$$

The final step is to sum over ζ

$$
\begin{aligned}
={} & -\frac{\lambda}{6(4\pi)^2}\Bigg[\frac{4}{\tilde{p}^2}+m^2\ln\left(\frac{\tilde{p}^2}{4}\sqrt{\frac{m^4}{4}-M^2}\right) \\
& +\left(M^2+\frac{m^4}{4M^2}\right)\ln\sqrt{\frac{m^2+2M^2}{m^2-2M^2}}\Bigg]+\mathcal{O}(1).
\end{aligned}
\tag{5.7}
$$

One can see that this result has a quadratic IR divergence and, furthermore, a logarithmic IR divergence. So, on one loop level, no IR damping takes place. This is only achieved at higher orders as can be seen in Chapter 6 but the result at one loop order is very important for the behavior of higher loop orders since they consist of one loop graphs.

Planar One Loop Function

The planar one loop function has no phase factor. Therefore, it is UV divergent as will be shown here. The equation looks like

$$\Pi^{pl} = -\frac{\lambda}{6(2\pi)^4}\int \mathrm{d}^4k \sum_{\zeta=\pm1} \frac{1+\zeta\frac{m^2}{2M^2}}{k^2+\frac{m^2}{2}+\zeta M^2} . \tag{5.8}$$

To solve the generic integral for the graph it has to be regularized with an ultraviolet cutoff Λ with the limit $\Lambda \to \infty$. With the Schwinger parameterization (3.38) as a first step one can solve the integral over k,

$$\begin{aligned}
I &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2+\frac{m^2}{2}+\zeta M^2} \\
&= \int_0^\infty \mathrm{d}\alpha \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\alpha\left(k^2+\frac{m^2}{2}+\zeta M^2\right)} , \\
I_\Lambda(\Lambda) &= \frac{1}{(4\pi)^2}\int_0^\infty \mathrm{d}\alpha \frac{1}{\alpha^2}\mathrm{e}^{-\frac{1}{4\alpha\Lambda^2}-\alpha\left(\frac{m^2}{2}+\zeta M^2\right)} .
\end{aligned}$$

To solve the integral over α it is necessary to introduce a cutoff term $-\frac{1}{4\alpha\Lambda^2}$ in the exponential as has been done above. The integral can be found in [21] and gives a modified Bessel function like showed in Eqn. (3.40),

$$I_\Lambda(\Lambda) = \frac{2}{(4\pi)^2}\left(\frac{\frac{m^2}{2}+\zeta M^2}{\frac{1}{4\Lambda^2}}\right)^{\frac{1}{2}} K_1\left(\sqrt{\frac{1}{\Lambda^2}\left(\frac{m^2}{2}+\zeta M^2\right)}\right) . \tag{5.9}$$

With the expansion for small arguments like in Eqn. (5.4) the result is

$$\begin{aligned}
I_\Lambda(\Lambda) &\approx \frac{\Lambda}{(2\pi)^2}\sqrt{\frac{m^2}{2}+\zeta M^2} \\
&\quad\times\left[\sqrt{\frac{\Lambda^2}{\frac{m^2}{2}+\zeta M^2}}+\frac{1}{2}\sqrt{\frac{\frac{m^2}{2}+\zeta M^2}{\Lambda^2}}\left(\ln\sqrt{\frac{\frac{m^2}{2}+\zeta M^2}{4\Lambda^2}}+\gamma_E-\frac{1}{2}\right)\right] \\
&= \frac{1}{(4\pi)^2}\left[4\Lambda^2+\left(\frac{m^2}{2}+\zeta M^2\right)\left(\ln\left(\frac{\frac{m^2}{2}+\zeta M^2}{4\Lambda^2}\right)+2\gamma_E-1\right)\right] .
\end{aligned} \tag{5.10}$$

The regularized graph is then

$$\Pi^{pl}(\Lambda) = -\frac{\lambda}{48\pi^2}\sum_{\zeta=\pm1}\left(1+\zeta\frac{m^2}{2M^2}\right)2\Lambda\sqrt{\frac{m^2}{2}+\zeta M^2}K_1\left(\frac{1}{\Lambda}\sqrt{\frac{m^2}{2}+\zeta M^2}\right) . \tag{5.11}$$

For large values of Λ the modified Bessel function can be expanded according to Eqn. (5.4) and gives

$$\Pi^{pl}(\Lambda) = -\frac{\lambda}{6(4\pi)^2}\sum_{\zeta=\pm1}\left(1+\zeta\frac{m^2}{2M^2}\right) \times\left[4\Lambda^2+\left(\frac{m^2}{2}+\zeta M^2\right)\left(\ln\left(\frac{\frac{m^2}{2}+\zeta M^2}{4\Lambda^2}\right)+2\gamma_E-1\right)\right].$$

When one carries out the last summation one finally gets

$$\Pi^{pl}(\Lambda) = -\frac{\lambda}{3(4\pi)^2}\left[4\Lambda^2+m^2\ln\left(\frac{1}{4\Lambda^2}\sqrt{\frac{m^4}{4}-M^2}\right) + \left(M^2+\frac{m^4}{4M^2}\right)\ln\sqrt{\frac{m^2+2M^2}{m^2-2M^2}}\right]+\mathcal{O}(1)\,. \tag{5.12}$$

Performing the limit $\Lambda\to\infty$ the known UV divergences of the commutative theory are encountered again.

5.2 One Loop Function in the Massless Case

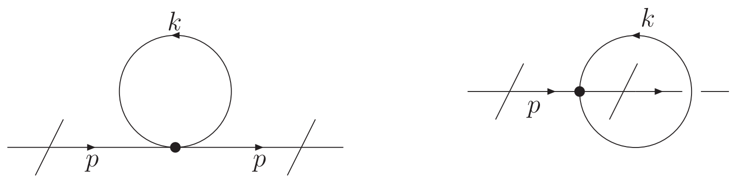

Figure 5.2: The planar and non-planar one loop functions.

For the massless case the action has the form

$$S[\phi] = \int \mathrm{d}^4x\left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi-\frac{1}{2}\phi\frac{a'^2}{\theta^2\Box}\phi+\frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\right). \tag{5.13}$$

The propagator to this action is, with the abbreviation $a=\frac{a'}{\theta}$ and a' being a dimensionless constant like in the massive case,

$$G(p) = \frac{1}{p^2+\frac{a^2}{p^2}}\,. \tag{5.14}$$

It can be achieved by letting the mass m go to zero in the massive propagator (4.10) and, like in the massive case, the denominator is split up for easier integrability. In contrary to the massive case one gets only two terms instead of four

$$\frac{1}{p^2+\frac{a^2}{p^2}} = \frac{p^2}{p^2\left(p^2+\frac{a^2}{p^2}\right)} = \frac{1}{2}\left(\frac{1}{p^2+\mathrm{i}a}+\frac{1}{p^2-\mathrm{i}a}\right). \tag{5.15}$$

The generic integral of the non-planar part of the function is built from a propagator and a vertex like in the massive case and gives with the Schwinger parameterization and a completion to a full square of the exponent the result (see Appendix B.1 for details)

$$\begin{aligned} I(p) &= \int \mathrm{d}^4k \frac{e^{\pm ik\tilde{p}}}{k^2+\frac{a^2}{k^2}} = \frac{1}{2}\int \mathrm{d}^4k\ e^{\pm ik\tilde{p}}\left(\frac{1}{k^2+\mathrm{i}a}+\frac{1}{k^2-\mathrm{i}a}\right) \\ &= 2\pi^2\int_0^\infty \frac{\mathrm{d}\alpha}{\alpha^2}\cos(\alpha a)\, e^{-\frac{\tilde{p}^2}{4\alpha}}\,. \end{aligned} \tag{5.16}$$

The integral over the Schwinger parameter can be solved with the formula (Gradshteyn p.497 (3.957/2) [21])

$$\int_0^\infty x^{\mu-1}e^{-\frac{\beta^2}{4x}}\cos ax\,\mathrm{d}x = \frac{\beta^\mu}{2^\mu}a^{-\frac{\mu}{2}}\left[e^{-i\frac{\mu\pi}{4}}K_\mu\left(\beta e^{i\frac{\pi}{4}}\sqrt{a}\right)+e^{i\frac{\mu\pi}{4}}K_\mu\left(\beta e^{-i\frac{\pi}{4}}\sqrt{a}\right)\right],$$
$$\text{with}\quad \mathrm{Re}\,\beta>0,\ \mathrm{Re}\,\mu<1,\ a>0\,. \tag{5.17}$$

The K_μ are the modified Bessel functions of the second kind already used in the naive and the massive model (3.41). This gives for the generic non-planar one loop function

$$\begin{aligned} I(p) &= \frac{4\pi^2}{\tilde{p}}\sqrt{a}\left[e^{i\frac{\pi}{4}}K_1\left(\tilde{p}e^{i\frac{\pi}{4}}\sqrt{a}\right)+e^{-i\frac{\pi}{4}}K_1\left(\tilde{p}e^{-i\frac{\pi}{4}}\sqrt{a}\right)\right] \\ &= \frac{4\pi^2}{\tilde{p}}\sqrt{a}\left[\frac{1+i}{\sqrt{2}}K_1\left(\frac{1+i}{\sqrt{2}}\tilde{p}\sqrt{a}\right)+\frac{1-i}{\sqrt{2}}K_1\left(\frac{1-i}{\sqrt{2}}\tilde{p}\sqrt{a}\right)\right]. \end{aligned} \tag{5.18}$$

In the last step the exponential function was split up in its trigonometric form which could be solved for the given argument[2]. The logarithmic divergence present in the massive case, coming from the second term in the expansion for small arguments of the Bessel function, does not occur in the massless case since most of the term cancels and only a constant factor remains. The final result is

$$I(p) = \frac{8\pi^2}{\tilde{p}^2}-\pi^3 a-\frac{\pi^2a^2\tilde{p}^2}{2}\left(\ln\frac{\tilde{p}^2a}{4}+2\gamma_E-\frac{5}{2}\right)+\mathcal{O}\left(\tilde{p}^3\right). \tag{5.19}$$

For a detailed calculation see Appendix B.1. The fact that a limit $m\to 0$ is possible is of importance for a later generalization of the action to a gauge theory. This generalization is currently a work in progress at the Institute for Theoretical Physics of the VUT but will not be covered here any further.

[2] $\mathrm{e}^{\pm\mathrm{i}\frac{\pi}{4}} = \cos\frac{\pi}{4}\pm\mathrm{i}\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\pm\mathrm{i}\frac{\sqrt{2}}{2}$

5.3 One Loop Vertex

To construct the one loop correction for the vertex, two vertices and two propagators connecting them in the right way are required. There are three possible ways to do that and every one has a symmetry factor 1/2. Because of momentum conservation the sum of all momenta entering the vertex has to be zero,

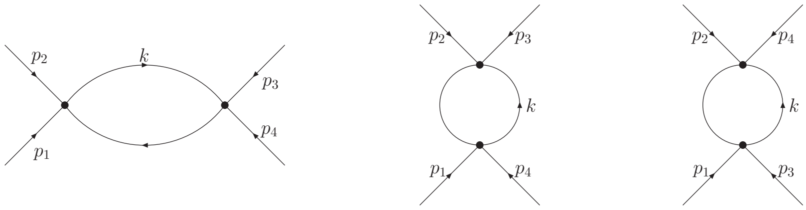

Figure 5.3: The three vertices.

$$p_1 + p_2 + p_3 + p_4 = 0\,. \tag{5.20}$$

The equation for the one loop vertex depicted on the left in Fig. 5.3 is with the coupling constant λ

$$\begin{aligned} V(p) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\lambda^2}{9} & \frac{1}{k^2 + m^2 + \frac{a^2}{k^2}} \frac{1}{(p_1+p_2-k)^2 + m^2 + \frac{a^2}{(p_1+p_2-k)^2}} \\ \times \Big(& \cos\frac{p_1 \times p_2}{2} \cos\frac{k \times (k - p_1 - p_2)}{2} \\ & + \cos\frac{p_1 \times (k - p_1 - p_2)}{2} \cos\frac{p_2 \times k}{2} \\ & \quad + \cos\frac{p_1 \times k}{2} \cos\frac{p_2 \times (k - p_1 - p_2)}{2} \Big) \\ \times \Big(& \cos\frac{k \times (p_1 + p_2 - k)}{2} \cos\frac{(-p_3) \times (-p_4)}{2} \\ & + \cos\frac{(k - p_1 - p_2) \times (-p_3)}{2} \cos\frac{k \times (-p_4)}{2} \\ & \quad + \cos\frac{(k - p_1 - p_2) \times (-p_4)}{2} \cos\frac{k \times (-p_3)}{2} \Big)\,. \end{aligned} \tag{5.21}$$

This can be rewritten without most of the brackets in the cosines because the cosine function is even. With the fact that $a \times a = 0$ and momentum conservation

(5.20) and, furthermore, employing the expansion from Eqn. (4.12) one gets

$$V(p) = \frac{\lambda^2}{36} \sum_{\zeta,\chi=\pm1} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\left(1+\zeta\frac{m^2}{2M^2}\right)}{k^2+\frac{m^2}{2}+\zeta M^2} \frac{\left(1+\chi\frac{m^2}{2M^2}\right)}{(p_1+p_2-k)^2+\frac{m^2}{2}+\chi M^2}$$
$$\left(\cos\tfrac{p_1\times p_2}{2}\cos\tfrac{k\times(p_1+p_2)}{2} + \cos\tfrac{p_1\times(k-p_2)}{2}\cos\tfrac{p_2\times k}{2} + \cos\tfrac{p_1\times k}{2}\cos\tfrac{p_2\times(k-p_1)}{2}\right)$$
$$\left(\cos\tfrac{k\times(p_3+p_4)}{2}\cos\tfrac{p_3\times p_4}{2} + \cos\tfrac{(p_4+k)\times p_3}{2}\cos\tfrac{k\times p_4}{2} + \cos\tfrac{(p_3+k)\times p_4}{2}\cos\tfrac{k\times p_3}{2}\right) . \tag{5.22}$$

The multiplication of the two brackets enveloping the cosines gives for small external momenta (See Appendix B.2.1 for a detailed calculation)

$$\left(\cos\tfrac{p_1\times p_2}{2}\cos\tfrac{k\times(p_1+p_2)}{2} + \cos\tfrac{p_1\times(k-p_2)}{2}\cos\tfrac{p_2\times k}{2} + \cos\tfrac{p_1\times k}{2}\cos\tfrac{p_2\times(k-p_1)}{2}\right)$$
$$\left(\cos\tfrac{k\times(p_3+p_4)}{2}\cos\tfrac{p_3\times p_4}{2} + \cos\tfrac{(p_4+k)\times p_3}{2}\cos\tfrac{k\times p_4}{2} + \cos\tfrac{(p_3+k)\times p_4}{2}\cos\tfrac{k\times p_3}{2}\right)$$
$$= 2 + \frac{1}{2}\sum_{i=2}^{4} \mathrm{e}^{\mathrm{i}k\times(p_1+p_i)} + \sum_{i=1}^{4} \mathrm{e}^{\mathrm{i}k\times p_i} + \frac{3}{2}\mathrm{e}^{\mathrm{i}k\times(p_1+p_2)} . \tag{5.23}$$

For the other two graphs the result is the same with $p_2 \leftrightarrow p_3$ and $p_2 \leftrightarrow p_4$ respectively,

$$2 + \frac{1}{2}\sum_{i=2}^{4} \mathrm{e}^{\mathrm{i}k\times(p_1+p_i)} + \sum_{i=1}^{4} \mathrm{e}^{\mathrm{i}k\times p_i} + \frac{3}{2}\mathrm{e}^{\mathrm{i}k\times(p_1+p_3)} , \tag{5.24}$$

and

$$2 + \frac{1}{2}\sum_{i=2}^{4} \mathrm{e}^{\mathrm{i}k\times(p_1+p_i)} + \sum_{i=1}^{4} \mathrm{e}^{\mathrm{i}k\times p_i} + \frac{3}{2}\mathrm{e}^{\mathrm{i}k\times(p_1+p_4)} . \tag{5.25}$$

These three graphs are then added with a symmetry factor of $\frac{1}{2}$ giving

$$\begin{aligned} V(p) &= V_1(p) + V_2(p) + V_3(p) \\ &= \frac{\lambda^2}{72} \sum_{\zeta,\chi=\pm1} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\left(1+\zeta\frac{m^2}{2M^2}\right)\left(1+\chi\frac{m^2}{2M^2}\right)}{k^2+\frac{m^2}{2}+\zeta M^2} \\ &\times \left[\left(2 + \frac{1}{2}\sum_{i=2}^{4} \mathrm{e}^{\mathrm{i}k\times(p_1+p_i)} + \sum_{i=1}^{4} \mathrm{e}^{\mathrm{i}k\times p_i}\right) \right. \\ &\times \left(\frac{1}{(p_1+p_2-k)^2+\frac{m^2}{2}+\chi M^2} + \frac{1}{(p_1+p_3-k)^2+\frac{m^2}{2}+\chi M^2} + \frac{1}{(p_1+p_4-k)^2+\frac{m^2}{2}+\chi M^2}\right) \\ &\left. + \frac{3}{2}\frac{\mathrm{e}^{\mathrm{i}k\times(p_1+p_2)}}{(p_1+p_2-k)^2+\frac{m^2}{2}+\chi M^2} + \frac{3}{2}\frac{\mathrm{e}^{\mathrm{i}k\times(p_1+p_3)}}{(p_1+p_3-k)^2+\frac{m^2}{2}+\chi M^2} + \frac{3}{2}\frac{\mathrm{e}^{\mathrm{i}k\times(p_1+p_4)}}{(p_1+p_4-k)^2+\frac{m^2}{2}+\chi M^2} \right] . \end{aligned} \tag{5.26}$$

This is the equation to be solved for the one loop vertex correction in the limit of small external momenta. The generic integral of this expression has the form

$$
\begin{aligned}
I(p) &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}k(\tilde{p}+\tilde{q})}}{\left(k^2 + \frac{m^2}{2} + \zeta M^2\right)\left((p-k)^2 + \frac{m^2}{2} + \chi M^2\right)} \\
&= \int\limits_0^\infty \mathrm{d}\alpha \int\limits_0^\infty \mathrm{d}\beta \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{e}^{-\alpha\left(k^2 + \frac{m^2}{2} + \zeta M^2\right) - \beta\left((p-k)^2 + \frac{m^2}{2} + \chi M^2\right) + \mathrm{i}k(\tilde{p}+\tilde{q})} \\
&= \int\limits_0^\infty \mathrm{d}\alpha \int\limits_0^\infty \mathrm{d}\beta \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \Bigg(\mathrm{e}^{-(\alpha+\beta)\left(k^2 - \frac{2\beta k p + \mathrm{i}k(\tilde{p}+\tilde{q})}{\alpha+\beta} + \frac{(2\beta p + \mathrm{i}(\tilde{p}+\tilde{q}))^2}{4(\alpha+\beta)^2}\right) + \frac{(2\beta p + \mathrm{i}(\tilde{p}+\tilde{q}))^2}{4(\alpha+\beta)}} \\
&\quad \mathrm{e}^{-(\alpha+\beta)\frac{m^2}{2} - \alpha\zeta M^2 - \beta\left(p^2 + \chi M^2\right)} \Bigg) .
\end{aligned}
\tag{5.27}
$$

The combination of $\tilde{p}$ and $\tilde{q}$ is capable of mimicking any combination of $\tilde{p}_1 + \tilde{p}_i$ in Eqn. (5.26). To get the planar graph $\tilde{p} = -\tilde{q}$ is inserted and the exponential vanishes. With the substitution $k' = k - \frac{2\beta p + \mathrm{i}(\tilde{p}+\tilde{q})}{2(\alpha+\beta)}$ it is possible to solve the integration over $\mathrm{d}^4 k$

$$
I(p) = \frac{1}{(4\pi)^2} \int\limits_0^\infty \mathrm{d}\alpha \int\limits_0^\infty \mathrm{d}\beta \frac{1}{(\alpha+\beta)^2} \mathrm{e}^{\frac{4\mathrm{i}\beta p(\tilde{p}+\tilde{q}) - (\tilde{p}+\tilde{q})^2 - 4\alpha\beta p^2}{4(\alpha+\beta)} - (\alpha+\beta)\frac{m^2}{2} - \alpha\zeta M^2 - \beta\chi M^2} .
\tag{5.28}
$$

In order to tackle the remaining integrals new variables are introduced:

$$
\begin{aligned}
\alpha &= \lambda \xi , \\
\beta &= (1-\xi)\lambda , \\
\mathrm{d}\alpha\, \mathrm{d}\beta &= \lambda\, \mathrm{d}\lambda\, \mathrm{d}\xi .
\end{aligned}
\tag{5.29}
$$

With the new variables the generic integral has a simpler form. The rule $p \times p = p\tilde{p} = 0$ has to be used for the last term in the exponential and the next step is to introduce a cutoff term $-\frac{1}{4\Lambda^2\lambda}$ which is needed for the calculation of the planar part.

$$
I(p) = \frac{1}{(4\pi)^2} \int\limits_0^1 \mathrm{d}\xi \int\limits_0^\infty \mathrm{d}\lambda \frac{1}{\lambda} \mathrm{e}^{-\frac{(\tilde{p}+\tilde{q})^2}{4\lambda} - \lambda\left(\frac{m^2}{2} + \xi\zeta M^2 + (1-\xi)\chi M^2 + \xi(1-\xi)p^2 - \frac{\mathrm{i}(1-\xi)}{\lambda} p\tilde{q}\right)} ,
\tag{5.30}
$$

$$
I_\Lambda(p) = \frac{1}{(4\pi)^2} \int\limits_0^1 \mathrm{d}\xi \int\limits_0^\infty \mathrm{d}\lambda \frac{1}{\lambda} \mathrm{e}^{-\frac{1}{4\lambda}\left((\tilde{p}+\tilde{q})^2 + \frac{1}{\Lambda^2}\right) - \lambda\left(\frac{m^2}{2} + \xi\zeta M^2 + (1-\xi)\chi M^2 + \xi(1-\xi)p^2\right) + \mathrm{i}(1-\xi)p\tilde{q}}
\tag{5.31}
$$

The integral over λ can be solved with the formula [21]

$$\int_0^\infty x^{\nu-1} \mathrm{e}^{-\frac{\beta}{x}-\gamma x} \mathrm{d}x = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu\left(2\sqrt{\beta\gamma}\right), \quad \text{with} \quad \mathrm{Re}\,\beta > 0,\ \mathrm{Re}\,\gamma > 0\,. \tag{5.32}$$

The K_ν are the modified Bessel functions of the second kind. Here K_0 has to be used and this gives the result

$$I_\Lambda(p) = \frac{2}{(4\pi)^2} \int_0^1 \mathrm{d}\xi$$
$$K_0\left(\left[\left((\tilde{p}+\tilde{q})^2 + \frac{1}{\Lambda^2}\right)\left(\frac{m^2}{2} + M^2(\chi + (\zeta-\chi)\xi) + \xi(1-\xi)p^2\right)\right]^{\frac{1}{2}}\right) \mathrm{e}^{\mathrm{i}(1-\xi)p\tilde{q}}\,. \tag{5.33}$$

The Bessel function can be expanded for small arguments with $K_0(z) \sim -\ln\frac{z}{2} - \gamma_E$ for $z \to 0$ leading to[3]

$$I_\Lambda(p) \approx -\frac{2}{(4\pi)^2} \int_0^1 \mathrm{d}\xi \left[\ln\left(\frac{1}{2}\left[\left((\tilde{p}+\tilde{q})^2 + \frac{1}{\Lambda^2}\right)\right.\right.\right.$$
$$\left.\left.\left.\times\left(\frac{m^2}{2} + M^2(\chi + (\zeta-\chi)\xi) + \xi(1-\xi)p^2\right)\right]^{\frac{1}{2}}\right) + \gamma_E\right] \mathrm{e}^{\mathrm{i}(1-\xi)p\tilde{q}}\,. \tag{5.34}$$

This result contains a planar and a non-planar part. For the planar part $\tilde{p} = -\tilde{q}$ which kills the exponential as $p\tilde{p} = 0$, and this leads to

$$I_\Lambda^{pl} \approx -\frac{1}{(4\pi)^2} \int_0^1 \mathrm{d}\xi$$
$$\times\left[\ln\left(\tfrac{1}{4\Lambda^2}\left[\tfrac{m^2}{2} + M^2(\chi + (\zeta-\chi)\xi) + \xi(1-\xi)p^2\right]\right) + 2\gamma_E\right], \tag{5.35}$$

and for the non-planar part the cutoff factor is not needed but the exponential factor has to be kept

$$I^{n-pl}(p) \approx -\frac{1}{(4\pi)^2} \int_0^1 \mathrm{d}\xi$$
$$\times\left[\ln\left(\tfrac{(\tilde{p}+\tilde{q})^2}{4}\left(\tfrac{m^2}{2} + M^2(\chi + (\zeta-\chi)\xi) + \xi(1-\xi)p^2\right)\right) + 2\gamma_E\right] \mathrm{e}^{\mathrm{i}(1-\xi)p\tilde{q}}\,. \tag{5.36}$$

[3] $\gamma_E = 0.5772\cdots$ is again the Euler-Mascheroni constant.

The integral of the non-planar part is not solvable because of the exponential but with the assumption of small momenta p an approximation of the exponential with 1 is possible. Furthermore, the term with the p^2 can be neglected as it is considered to be small in comparison with the others. When the remaining sums over χ and ζ with the factors $\left(1+\zeta\frac{m^2}{2M^2}\right)\left(1+\chi\frac{m^2}{2M^2}\right)$ are performed (see Appendix B.2.2 for details) the final result is

$$\sum_{\chi,\zeta=\pm1}\left(1+\zeta\tfrac{m^2}{2M^2}\right)\left(1+\chi\tfrac{m^2}{2M^2}\right)I^{n-pl}(p)\approx$$
$$\tfrac{1}{(2\pi)^2}\left(\ln\tfrac{(\tilde{p}+\tilde{q})^2}{4}\sqrt{\tfrac{m^4}{4}-M^4}-\tfrac{1}{2}\left(1-\tfrac{m^4}{4M^4}\right)+2\gamma_E-\tfrac{m^2}{4M^2}\left(3-\tfrac{m^4}{4M^4}\right)\ln\sqrt{\tfrac{m^2-2M^2}{m^2+2M^2}}\right). \tag{5.37}$$

The absolute value of this expression can be seen as an upper limit for the integral of the non-planar part of the vertex. To get the limit for the planar part one has to replace $(\tilde{p}+\tilde{q})^2$ with $\frac{1}{\Lambda^2}$ where Λ is the cutoff.

5.4 Renormalization

As proven by Gurau *et al.* [1] the new model with the additional term $\frac{1}{\tilde{p}^2}$ is renormalizable. For the explicit calculation one starts with the dressed propagator at one loop level

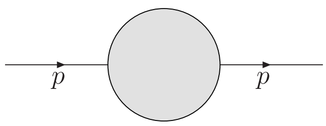

Figure 5.4: The dressed propagator.

$$\Delta'(p)\equiv\frac{1}{A}\;+\;\frac{1}{A}\,\Sigma(\Lambda,p)\,\frac{1}{A}$$
$$\text{with}\quad A=p^2+m_0^2+\frac{a^2}{p^2}\,,\qquad\text{and}\quad\Sigma(\Lambda,p):=\Pi^{pl}(\Lambda)+\Pi^{n-pl}(p)\,. \tag{5.38}$$

A dressed propagator is a propagator with external legs and all quantum corrections. The terms of Σ are given in Eqns. (5.12) and (5.7). Since $A\neq0$, it is possible to use the well known expansion

$$\frac{1}{A+B}=\frac{1}{A}-\frac{1}{A}B\frac{1}{A+B}=\frac{1}{A}-\frac{1}{A}B\frac{1}{A}+\mathcal{O}\left(B^2\right)\,. \tag{5.39}$$

For the dressed propagator this gives

$$\Delta'(p)=\frac{1}{p^2+m_0^2+\frac{a^2}{p^2}-\Sigma(\Lambda,p)}\,. \tag{5.40}$$

$\Sigma(\Lambda, p)$ is expanded to the order p^4 or $\frac{1}{\Lambda^4}$ respectively as the momentum in the denominator is also of quadratical order. For the calculation it is sufficient to write only constant factors in the expansion. One gets

$$
\begin{aligned}
\Sigma(\Lambda, p) = &-\frac{\lambda}{6(4\pi)^2}\left[\frac{4}{\tilde{p}^2} + m_0^2 \ln\left(|a|\tilde{p}^2\right) + \gamma + \alpha\tilde{p}^2 + \beta\tilde{p}^4 + \cdots\right] \\
&-\frac{\lambda}{3(4\pi)^2}\left[4\Lambda^2 + m_0^2 \ln\left(\tfrac{1}{\Lambda^2}\underbrace{\sqrt{\tfrac{m_0^4}{4} - M^4}}_{\sqrt{a^2}=|a|}\right) + \gamma_\Lambda + \alpha_\Lambda\tfrac{1}{\Lambda^2} + \beta_\Lambda\tfrac{1}{\Lambda^4} + \cdots\right] \\
= &-\frac{2\lambda}{3(4\pi)^2\theta^2}\frac{1}{p^2} - \frac{\lambda m_0^2}{6(4\pi)^2}\ln\left(|a|\theta^2 p^2\right) + \lambda\gamma' + \lambda\alpha'\theta^2 p^2 + \lambda\beta'\theta^4 p^4 \\
&-\frac{\lambda}{3(4\pi)^2}\left[4\Lambda^2 + \cdots\right] . \qquad (5.41)
\end{aligned}
$$

Here the γ and γ_Λ are constant terms not depending on p or Λ. α and β as well as α_Λ and β_Λ are the factors for the higher orders in p and $\frac{1}{\Lambda}$ respectively. The primed variables have the form

$$
\gamma' = -\frac{\gamma}{6(4\pi)^2}, \quad \alpha' = -\frac{\alpha}{6(4\pi)^2} \quad \text{and} \quad \beta' = -\frac{\beta}{6(4\pi)^2} . \qquad (5.42)
$$

With this one can regroup the terms and define new variables,

$$
\begin{aligned}
\frac{1}{\Delta'(p)} &= p^2 + m_0^2 + \frac{a^2}{p^2} - \Sigma(\Lambda, p) \\
&= \underbrace{p^2 - \lambda\alpha'\theta^2 p^2}_{p^2(1-\lambda\alpha'\theta^2)} + \underbrace{m_0^2 + \frac{\lambda}{3(4\pi)^2}\left[4\Lambda^2 + \cdots\right] - \lambda\gamma'}_{\tilde{m}_r^2} + \underbrace{\frac{a^2}{p^2} + \frac{2\lambda}{3(4\pi)^2\theta^2}\frac{1}{p^2}}_{\frac{1}{p^2}\tilde{a}_r^2} \\
&\quad + \frac{\lambda m_0^2}{6(4\pi)^2}\ln\left(|a|\theta^2 p^2\right) - \lambda\beta'\theta^4 p^4 - \cdots \\
&= \left(1 - \lambda\alpha'\theta^2\right)\left[p^2 + \underbrace{\frac{\tilde{m}_r^2}{1 - \lambda\alpha'\theta^2}}_{=:m_r^2} + \frac{1}{p^2}\underbrace{\frac{\tilde{a}_r^2}{1 - \lambda\alpha'\theta^2}}_{=:a_r'^2}\right. \\
&\quad \left. + \underbrace{\frac{\lambda m_0^2}{6(4\pi)^2\left(1 - \lambda\alpha'\theta^2\right)}\ln\left(|a|\theta^2 p^2\right) - \frac{\lambda\beta'\theta^4}{1 - \lambda\alpha'\theta^2}p^4 - \cdots}_{=:f(p^2)}\right] . \qquad (5.43)
\end{aligned}
$$

And thus the dressed propagator is

$$
\Delta'(p) = \frac{1}{1 - \lambda\alpha'\theta^2}\,\frac{1}{p^2 + m_r^2 + \frac{a_r'^2}{p^2} + f\left(p^2\right)} . \qquad (5.44)
$$

For small λ an expansion of the first part is performed as $\frac{1}{1-\lambda\alpha'\theta^2} = 1 + \lambda\alpha'\theta^2 + \mathcal{O}\left(\lambda^2\right)$. Only small λ are of interest and, therefore, terms which are of order λ^2 or higher are neglected. This gives

$$\begin{aligned}
m_r^2 &= \left[1 + \lambda\alpha'\theta^2 + \mathcal{O}\left(\lambda^2\right)\right] \tilde{m}_r^2 \\
&= \left[1 + \lambda\alpha'\theta^2 + \mathcal{O}\left(\lambda^2\right)\right] \Bigg[m_0^2 + \frac{\lambda}{3(4\pi)^2}\left(4\Lambda^2 + m_0^2 \ln\frac{|a|}{\Lambda^2}\right) \\
&\qquad\qquad -\lambda\gamma' + \lambda\alpha'_\Lambda\frac{1}{\Lambda^2} + \lambda\beta'_\Lambda\frac{1}{\Lambda^4} + \cdots\Bigg] \\
&= m_0^2 + \frac{\lambda}{3(4\pi)^2}\left(4\Lambda^2 + m_0^2\ln\frac{|a|}{\Lambda^2}\right) \\
&\qquad\qquad + \text{regular term (for } \Lambda \to \infty) + \mathcal{O}\left(\lambda^2\right) . \qquad (5.45)
\end{aligned}$$

The regular term contains all terms which behave nicely for $\Lambda \to \infty$ and which are of order λ.

$$\begin{aligned}
a_r'^2 &= \left[1 + \lambda\alpha'\theta^2 + \mathcal{O}\left(\lambda^2\right)\right] \tilde{a}_r^2 \\
&= \left[1 + \lambda\alpha'\theta^2 + \mathcal{O}\left(\lambda^2\right)\right]\left[a^2 + \frac{2\lambda}{3(4\pi)^2\theta^2}\right] \\
&= a^2 + \frac{2\lambda}{3(4\pi)^2\theta^2} + \lambda\alpha'\theta^2 a^2 + \mathcal{O}(\lambda^2) , \qquad (5.46) \\
f\left(p^2\right) &= \left[1 + \lambda\alpha'\theta^2 + \mathcal{O}\left(\lambda^2\right)\right]\left[\frac{\lambda m_0^2}{6(4\pi)^2}\ln\left(|a|\theta^2 p^2\right) - \lambda\beta'\theta^4 p^4 - \mathcal{O}\left(p^6\right)\right] \\
&= \lambda\left[\frac{m_0^2}{6(4\pi)^2}\ln\left(|a|\theta^2p^2\right) + \mathcal{O}\left(p^4\right)\right] + \mathcal{O}\left(\lambda^2\right) . \qquad (5.47)
\end{aligned}$$

So as a final result for the renormalization of the one loop propagator to order λ and after dropping the prime on the factor α' one gets

$$\Delta'(p) = \frac{Z}{p^2 + m_r^2 + \frac{a_r^2}{p^2}} , \qquad (5.48)$$

with

$$\begin{aligned}
Z &= 1 + \lambda\alpha\theta^2 , \qquad (5.49) \\
m_r^2 &= m_0^2 + \frac{\lambda}{3(4\pi)^2}\left(4\Lambda^2 + m_0^2\ln\frac{|a|}{\Lambda^2}\right) + \text{regular term (for } \Lambda \to \infty) , \qquad (5.50) \\
a_r^2 &= a^2 + \lambda\left(\frac{2}{3(4\pi)^2\theta^2} + \alpha\theta^2 a^2\right) + p^2 f\left(p^2\right) , \qquad (5.51) \\
f\left(p^2\right) &= \lambda\left(\frac{m_0^2}{6(4\pi)^2}\ln\left(|a|\theta^2p^2\right) + \mathcal{O}\left(p^4\right)\right) . \qquad (5.52)
\end{aligned}$$

The numerator Z is the wave function renormalization which is finite because of its Λ-independence and the same is true for a_r^2. Contrary to the commutative scalar theory the wave function renormalization factor is not 1 in this model. $f(p^2)$ shows a logarithmic divergence and it is not quite clear how to handle it. The argument of Gurau *et al.* in [1] is that the logarithmic divergence represents a type of "mild" divergence that has no effect on the physical amplitudes in the limit of small external momenta p because, for vanishing external momenta, the term $f(p^2)$ in the denominator of the dressed propagator is surely smaller than the rest.
To determine what the constant α is one has to look at the expansion of the Bessel function at order z^2 in

$$\frac{1}{z}K_1(z) = \frac{1}{z^2} + \frac{1}{2}\ln z + \frac{1}{2}\left(\gamma_E - \ln 2 - \frac{1}{2}\right) + \frac{z^2}{16}\left(\ln\frac{z}{2} + \gamma_E - \frac{5}{4}\right) + \mathcal{O}\left(z^4\right) , \tag{5.53}$$

and insert this in Eqn. (5.6)

$$\Pi^{n-pl}(p) = -\frac{\lambda}{48\pi^2}\sum_{\zeta=\pm1}\left(1+\zeta\frac{m_0^2}{2M^2}\right)\left(\frac{m_0^2}{2}+\zeta M^2\right) \times \frac{1}{\sqrt{\tilde{p}^2\left(\frac{m_0^2}{2}+\zeta M^2\right)}} K_1\left(\sqrt{\tilde{p}^2\left(\frac{m_0^2}{2}+\zeta M^2\right)}\right) . \tag{5.54}$$

The order $\tilde{p}^2$ without the logarithm is then

$$\begin{aligned}
&-\frac{\lambda}{48\pi^2}\sum_{\zeta=\pm1}\left(1+\zeta\frac{m_0^2}{2M^2}\right)\left(\frac{m_0^2}{2}+\zeta M^2\right)\frac{\tilde{p}^2}{16}\left(\frac{m_0^2}{2}+\zeta M^2\right)\left(\gamma_E - \ln 2 - \frac{5}{4}\right)\\
&= -\frac{\lambda\tilde{p}^2}{3\left(16\pi\right)^2}\left(\gamma_E - \ln 2 - \frac{5}{4}\right)\\
&\qquad \times \sum_{\zeta=\pm1}\left[\frac{m_0^4}{4} + \zeta M^2 m_0^2 + M^4 + \zeta\frac{m_0^6}{8M^2} + \frac{m_0^4}{2} + \zeta\frac{m_0^2 M^2}{2}\right]\\
&= \frac{\lambda\tilde{p}^2}{3\left(16\pi\right)^2}\left(\ln 2 + \frac{5}{4} - \gamma_E\right)\left(\frac{3m_0^4}{2} + 2M^4\right) ,
\end{aligned}$$

and with $M^4 \equiv \frac{m_0^4}{4} - a^2$ one gets

$$\frac{2\lambda\tilde{p}^2}{3\left(16\pi\right)^2}\left(\ln 2 + \frac{5}{4} - \gamma_E\right)\left(m_0^4 - a^2\right) = \lambda\alpha\tilde{p}^2 . \tag{5.55}$$

As a last step one uses $a'^2 = \theta^2 a^2$ and arrives at the final result

$$\alpha\theta^2 = \frac{2}{3\left(16\pi\right)^2}\left(\ln 2 + \frac{5}{4} - \gamma_E\right)\left(\theta^2 m_0^4 - a'^2\right) . \tag{5.56}$$

Hence, α is positive for $\theta^2 m_0^4 > a'^2$. However, it is no problem if $\alpha < 0$ as long as the one loop renormalized parameter $a_r'^2$ is positive. With the abbreviations $A = \frac{2}{3(4\pi)^2}$ and $B = \frac{16}{\ln 2 + \frac{5}{4} - \gamma_E}$ one writes

$$\begin{aligned} a_r'^2 &\equiv a'^2 + \lambda \left(A + \alpha\theta^2 a'^2\right) > 0\,, \\ a'^2 + \lambda \frac{A}{B} &\left(\theta^2 m_0^4 - a'^2\right) a'^2 > -A\lambda\,, \\ a'^4 - &\left(\frac{B}{\lambda A} + \theta^2 m_0^4\right) a'^2 - B < 0\,. \end{aligned} \tag{5.57}$$

This inequality has two bounds between which it is true. As a'^2 should be positive only the upper bound is needed and the lower one is zero because the calculated lower bound is negative which is not in the possible range since no negative values are allowed. The result is

$$a'^2 < \frac{1}{2}\left(\frac{B}{\lambda A} + \theta^2 m^4\right) + \sqrt{\frac{1}{4}\left(\frac{B}{\lambda A} + \theta^2 m_0^4\right)^2 + B}\,. \tag{5.58}$$

$\frac{B}{A} = 2774.6$ and θ is quite small on physical grounds. That makes $\frac{1}{\lambda}$ the dominating factor in the inequality. So, $a_r'^2$ is positive for small values of λ even for $m_0 = 0$. More precisely if $a'^2 \lesssim \frac{10^3}{\lambda}$.

The renormalized coupling constant λ_r at one loop order is obtained by considering the planar part of the one loop vertex correction from Section 5.3. The non-planar part is not needed because the momentum is set to be greater than zero and, therefore, the exponential factor prevents divergences. The integral for the planar part is (planar part from Eqn. (5.26))

$$V^{pl}(p) = \frac{\lambda^2}{36} \sum_{n=2}^{4} \sum_{\zeta,\chi=\pm 1} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\left(1 + \zeta\frac{m_0^2}{2M^2}\right)\left(1 + \chi\frac{m_0^2}{2M^2}\right)}{\left(k^2 + \frac{m_0^2}{2} + \zeta M^2\right)\left[(p_n' - k)^2 + \frac{m_0^2}{2} + \chi M^2\right]}\,. \tag{5.59}$$

The p_n' stands for $p_n' = p_1 + p_m$ with $m = n = 2, 3, 4$ and the result for the planar part is the sum over those three momenta. The integral has the result (5.35)

$$\begin{aligned} V^{pl}(p) = &-\frac{\lambda^2}{36(4\pi)^2} \sum_{n=2}^{4} \sum_{\zeta,\chi=\pm 1} \left(1 + \zeta\frac{m_0^2}{2M^2}\right)\left(1 + \chi\frac{m_0^2}{2M^2}\right) \\ &\times \int_0^1 \mathrm{d}\xi \left[\ln\left[\frac{m_0^2}{8\Lambda^2}\left(1 + \frac{2M^2}{m_0^2}(\chi + (\zeta - \chi)\xi) + \xi(1-\xi)\frac{2p_n'^2}{m_0^2}\right)\right] + 2\gamma_E\right]\,. \end{aligned} \tag{5.60}$$

Expanding the logarithm one can get the Λ dependent term out of the integral and solve the sums. After that the solution has the form

$$V^{pl}(p) = -\frac{\lambda^2}{36(4\pi)^2}\Bigg[12\left(-\ln\frac{8\Lambda^2}{m_0^2}+2\gamma_E\right)+\sum_{n=2}^{4}\sum_{\zeta,\chi=\pm1}\left(1+\zeta\tfrac{m_0^2}{2M^2}\right)\left(1+\chi\tfrac{m_0^2}{2M^2}\right) \times\int_0^1 \mathrm{d}\xi\ln\left(1+\frac{2M^2}{m_0^2}(\chi+(\zeta-\chi)\xi)+\xi(1-\xi)\frac{2p_n'^2}{m_0^2}\right)\Bigg]. \tag{5.61}$$

Performing the sum over ζ and χ in the last term gives

$$\begin{aligned}&\left(1+\frac{m_0^2}{2M^2}\right)^2\int_0^1 \mathrm{d}\xi\ln\left(1+\frac{2M^2}{m_0^2}+\xi(1-\xi)\frac{2p_n'^2}{m_0^2}\right)\\ &\quad+\left(1-\frac{m_0^4}{4M^4}\right)\int_0^1 \mathrm{d}\xi\ln\left(1+\frac{2M^2}{m_0^2}(2\xi-1)+\xi(1-\xi)\frac{2p_n'^2}{m_0^2}\right)\\ &\quad+\left(1-\frac{m_0^4}{4M^4}\right)\int_0^1 \mathrm{d}\xi\ln\left(1+\frac{2M^2}{m_0^2}(1-2\xi)+\xi(1-\xi)\frac{2p_n'^2}{m_0^2}\right)\\ &\quad+\left(1-\frac{m_0^2}{2M^2}\right)^2\int_0^1 \mathrm{d}\xi\ln\left(1-\frac{2M^2}{m_0^2}+\xi(1-\xi)\frac{2p_n'^2}{m_0^2}\right).\end{aligned} \tag{5.62}$$

These integrals can be solved for certain combinations of m_0 and M. This has been done with the help of *Mathematica* and the results contain a term with an arctangent in it which is then expanded in a Taylor series for further calculations. To get the renormalized coupling constant one now substitutes the parameters m_0 and a with their renormalized ones and sorts the result in terms of order of λ. The lowest order is λ^2 and any higher order plays no role at this level. This means that only terms which are independent of λ have to be kept in the result of the integration. For the calculation the renormalized version of M^2 is needed. This is simply done by inserting the renormalized m_r^2 and a_r^2 in the definition of M^2 and gives

$$M_r^2 = \left[\frac{1}{4}\left(m_0^2+\frac{\lambda}{3(4\pi)^2}\left(4\Lambda^2+m_0^2\ln\tfrac{|a|}{\Lambda^2}\right)+reg\right)^2 - a^2-\lambda\left(\frac{2}{3(4\pi)^2\theta^2}+\alpha_r\theta^2a^2\right)\right]^{\frac{1}{2}}.$$

Here *reg* stands for the regular term of m_r for $\Lambda \to \infty$ and α_r for the renormalized version of α,[4]

$$
\begin{aligned}
M_r^2 = &\left[\frac{1}{4}\Bigg(\left(m_0^2 + reg\right)^2 + \left(m_0^2 + reg\right) \frac{2\lambda}{3(4\pi)^2} \left(4\Lambda^2 + m_0^2 \ln \tfrac{|a|}{\Lambda^2}\right)\right. \\
&+ \tfrac{\lambda^2}{9(4\pi)^4} \left(4\Lambda^2 + m_0^2 \ln \tfrac{|a|}{\Lambda^2}\right)^2 \Bigg) - a^2 - \tfrac{2\lambda}{3(4\pi)^2} \left[\frac{1}{\theta^2} + \frac{\theta^2 a^2}{16} \left(\ln 2 + \frac{5}{4} - \gamma_E\right)\right. \\
&\left.\left.\left(\left(m_0^2 + \tfrac{\lambda}{3(4\pi)^2} \left(4\Lambda^2 + m_0^2 \ln \tfrac{|a|}{\Lambda^2}\right) + reg\right)^2 - a^2 - \lambda\left(\tfrac{2}{3(4\pi)^2\theta^2} + \alpha\theta^2 a^2\right)\right)\right]\right]^{\frac{1}{2}} \\
= &\left[\left(\frac{m_0^2 + reg}{2}\right)^2 - a^2 + \frac{2\lambda}{3(4\pi)^2} \left[\frac{m_0^2 + reg}{4} \left(4\Lambda^2 + m_0^2 \ln \tfrac{|a|}{\Lambda^2}\right) - \frac{1}{\theta^2}\right.\right. \\
&\left. - \frac{\theta^2 a^2}{16} \left(\ln 2 + \frac{5}{4} - \gamma_E\right) \left(\left(m_0^2 + reg\right)^2 - a^2\right)\right] \\
&+ \frac{\lambda^2}{9(4\pi)^4} \left[\frac{\left(4\Lambda^2 + m_0^2 \ln \tfrac{|a|}{\Lambda^2}\right)^2}{4} - \frac{\theta^2 a^2}{4} \left(m_0^2 + reg\right) \left(\ln 2 + \frac{5}{4} - \gamma_E\right)\right. \\
&\times \left(4\Lambda^2 + m_0^2 \ln \tfrac{|a|}{\Lambda^2}\right) \\
&\left. + \frac{\theta^2 a^2}{4} \left(\ln 2 + \frac{5}{4} - \gamma_E\right) \left(\frac{1}{\theta^2} + \frac{\theta^2 a^2}{16} \left(\ln 2 + \frac{5}{4} - \gamma_E\right) \left(m_0^4 - a^2\right)\right)\right] \\
&\left. - \frac{\lambda^3 \theta^2 a^2}{216(4\pi)^6} \left(\ln 2 + \frac{5}{4} - \gamma_E\right) \left(4\Lambda^2 + m_0^2 \ln \tfrac{|a|}{\Lambda^2}\right)^2\right]^{\frac{1}{2}} .
\end{aligned}
\tag{5.63}
$$

Since the relevant M_r are present in the denominators of the prefactors of the integrals in Eqn. (5.62) one has to use the series expansions

$$
\begin{aligned}
\frac{1}{\sqrt{c+x}} &= \frac{1}{\sqrt{c}} - \frac{x}{2c^{3/2}} + \mathcal{O}\left(x^2\right) , \\
\frac{1}{c+x} &= \frac{1}{c} - \frac{x}{c^2} + \mathcal{O}\left(x^2\right) .
\end{aligned}
\tag{5.64}
$$

The two factors are $\frac{m_r^2}{M_r^2}$ and $\frac{m_r^4}{M_r^4}$ and with the expansion from above they have

[4] From Eqn. (5.56): $\alpha = \frac{2}{3(16\pi)^2} \left(\ln 2 + \frac{5}{4} - \gamma_E\right) \left(m_0^4 - a^2\right)$.

the form

$$
\begin{aligned}
\frac{m_r^2}{M_r^2} &= \left(m_0^2 + \frac{\lambda}{3(4\pi)^2}\left(4\Lambda^2 + m_0^2 \ln\tfrac{|a|}{\Lambda^2}\right) + reg\right) \\
&\quad \times \left(\frac{2}{\sqrt{\left(m_0^2 + reg\right)^2 - 4a^2}} - \frac{4\lambda A}{\left(\left(m_0^2 + reg\right)^2 - 4a^2\right)^{3/2}} + \mathcal{O}\left(\lambda^2\right)\right) \\
&= \frac{2\left(m_0^2 + reg\right)}{\sqrt{\left(m_0^2 + reg\right)^2 - 4a^2}} \\
&\quad + \lambda\left[\frac{8\Lambda^2 + 2m_0^2 \ln\tfrac{|a|}{\Lambda^2}}{3(4\pi)^2\sqrt{\left(m_0^2 + reg\right)^2 - 4a^2}} - \frac{4\left(m_0^2 + reg\right)A}{\left(\left(m_0^2 + reg\right)^2 - 4a^2\right)^{3/2}}\right] + \mathcal{O}\left(\lambda^2\right) ,
\end{aligned}
\tag{5.65}
$$

and

$$
\begin{aligned}
\frac{m_r^4}{M_r^4} &= \left(m_0^2 + \frac{\lambda}{3(4\pi)^2}\left(4\Lambda^2 + m_0^2 \ln\tfrac{|a|}{\Lambda^2}\right) + reg\right)^2 \\
&\quad \times \left(\frac{4}{\left(m_0^2 + reg\right)^2 - 4a^2} - \frac{16\lambda A}{\left(\left(m_0^2 + reg\right)^2 - 4a^2\right)^{2}} + \mathcal{O}\left(\lambda^2\right)\right) \\
&= \frac{4\left(m_0^2 + reg\right)^2}{\left(m_0^2 + reg\right)^2 - 4a^2} \\
&\quad + \lambda\left[\frac{\left(m_0^2 + reg\right)\left(4\Lambda^2 + m_0^2 \ln\tfrac{|a|}{\Lambda^2}\right)}{6\pi^2\left(\left(m_0^2 + reg\right)^2 - 4a^2\right)} - \frac{16\left(m_0^2 + reg\right)A}{\left(\left(m_0^2 + reg\right)^2 - 4a^2\right)^{2}}\right] + \mathcal{O}\left(\lambda^2\right) .
\end{aligned}
\tag{5.66}
$$

Here, A is an abbreviation for the factor of the term of order λ in M_r^2 in Eqn. (5.63)[5]. The zeroth order is independent of Λ and, therefore, behaves just like a factor. The same is true for the results of the integrals with the expanded arctangent. This was checked with *Mathematica*. Therefore, the whole integral is just a finite term in the limit $\Lambda \to \infty$ and, as such, of no significance for the renormalized coupling constant because the behavior is determined by the

[5] $A = \frac{2}{3(4\pi)^2}\left[\frac{m_0^2+reg}{4}\left(4\Lambda^2 + m_0^2 \ln\frac{|a|}{\Lambda^2}\right) - \frac{1}{\theta^2} - \frac{\theta^2 a^2}{16}\left(\ln 2 + \frac{5}{4} - \gamma_E\right)\left(\left(m_0^2 + reg\right)^2 - a^2\right)\right]$.

logarithmic divergence. Thus it is possible to write

$$\begin{aligned} V^{pl}(p) &= \frac{\lambda^2}{36(4\pi)^2}\left[12\left(\ln\frac{8\Lambda^2}{m_0^2} - 2\gamma_E\right) + \text{finite term for } \Lambda\to\infty\right] \\ &= \frac{\lambda^2}{3(4\pi)^2}\left[\ln\frac{\Lambda^2}{m_0^2} + \text{finite term for } \Lambda\to\infty\right]. \end{aligned} \tag{5.67}$$

And the renormalized coupling constant which has to correct this term then has the form

$$\lambda_r = \lambda\left[1 - \frac{\lambda}{3(4\pi)^2}\left(\ln\frac{\Lambda^2}{m_0^2} + \text{finite}\right)\right]. \tag{5.68}$$

5.5 The Renormalization Group and the β Function

A renormalization group is a set of transformations of the renormalization parameter, in this case Λ, which does not change the n-particle function. This means

$$\Lambda\frac{d}{d\Lambda}\Gamma^{(n)} = 0\,. \tag{5.69}$$

Here, $\Gamma^{(n)}$ is the bare n-particle function and Λ the cutoff parameter. In order to get a dimensionless operator the derivative is multiplied by Λ. The renormalized and bare n-point functions are connected by a factor called the wave function renormalization Z_ϕ which depends on the coupling constant and the cutoff,

$$\Gamma_r^{(n)}(p,\lambda_r,m_r,a_r,\Lambda) = Z_\phi^{n/2}(\lambda,\Lambda)\Gamma^{(n)}(p,\lambda,m_0,a)\,. \tag{5.70}$$

Bringing the wave function renormalization on the other side gives

$$\Gamma^{(n)}(p,\lambda,m_0,a) = Z_\phi^{-n/2}(\lambda,\Lambda)\Gamma_r^{(n)}(p,\lambda_r,m_r,a_r,\Lambda)\,. \tag{5.71}$$

When inserted in Eqn. (5.69) this means that the combination of wave function renormalization and renormalized n-point function does not depend on the cutoff. As the renormalized parameters λ_r , m_r and a_r all depend on the cutoff Λ the derivative takes the form

$$\Lambda\frac{d}{d\Lambda}\left[Z_\phi^{-n/2}(\lambda,\Lambda)\Gamma_r^{(n)}(p,\lambda_r,m_r,a_r,\Lambda)\right] = 0\,, \tag{5.72}$$

$$\left[Z^{-n/2}\left(\Lambda\frac{\partial}{\partial\Lambda} + \Lambda\frac{\partial\lambda}{\partial\Lambda}\frac{\partial}{\partial\lambda} + \frac{\Lambda}{m_0^2}\frac{\partial m_0^2}{\partial\Lambda}\frac{\partial}{\partial m_0^2} + \Lambda\frac{\partial a^2}{\partial\Lambda}\frac{\partial}{\partial a^2}\right) + \Lambda\frac{\partial}{\partial\Lambda}Z^{-n/2}\right]\Gamma_r^{(n)} = 0\,.$$

The last term can be written as $\Lambda\frac{\partial}{\partial\Lambda}Z^{-n/2} = -\Lambda\frac{n}{2}\frac{Z^{-n/2}}{Z}\frac{\partial}{\partial\Lambda}Z = -\Lambda\frac{n}{2}Z^{-n/2}\frac{\partial}{\partial\Lambda}\ln Z$ and so the equation is

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \Lambda\frac{\partial\lambda}{\partial\Lambda}\frac{\partial}{\partial\lambda} + \frac{\Lambda}{m_0^2}\frac{\partial m_0^2}{\partial\Lambda}\frac{\partial}{\partial m_0^2} + \Lambda\frac{\partial a^2}{\partial\Lambda}\frac{\partial}{\partial a^2} - \Lambda\frac{n}{2}\frac{\partial}{\partial\Lambda}\ln Z\right]\Gamma_r^{(n)} = 0\,. \tag{5.73}$$

With the abbreviations

$$\beta := \Lambda\frac{\partial\lambda}{\partial\Lambda}\,, \tag{5.74}$$

$$\beta_m := \frac{1}{m_0^2}\Lambda\frac{\partial m_0^2}{\partial\Lambda}\,, \tag{5.75}$$

$$\beta_a := \Lambda\frac{\partial a^2}{\partial\Lambda}\,, \tag{5.76}$$

$$\gamma := \Lambda\frac{\partial}{\partial\Lambda}\ln Z\,, \tag{5.77}$$

this gives the so called renormalization group equation

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta\frac{\partial}{\partial\lambda} + \beta_m\frac{\partial}{\partial m_0^2} + \beta_a\frac{\partial}{\partial a^2} - \frac{n}{2}\gamma\right]\Gamma_r^{(n)} = 0\,. \tag{5.78}$$

The β functions show how the parameters are affected by a change of the cutoff. Of special interest is of course the behavior of the β functions in the limit $\Lambda \to \infty$ because this is what is done after renormalization.

To calculate the β functions one has to express the bare constants in terms of the renormalized constants. In the following this is done for the coupling constant starting from Eqn. (5.68)[6]

$$\lambda_r = \lambda - \frac{\lambda^2}{3(4\pi)^2}\left(\ln\Lambda^2 + \text{finite}\right)\,. \tag{5.79}$$

This can be written as

$$\begin{aligned}\lambda &= \lambda_r + \frac{\lambda^2}{3(4\pi)^2}\left(\ln\Lambda^2 + \text{finite}\right)\\ &= \lambda_r + \frac{\lambda_r^2}{3(4\pi)^2}\left(\ln\Lambda^2 + \text{finite}\right) + \frac{\lambda^4}{9(4\pi)^4}\left(\ln\Lambda^2 + \text{finite}\right)^2 + \mathcal{O}\left(\lambda^6\right)\\ &= \lambda_r + \frac{\lambda_r^2}{3(4\pi)^2}\left(\ln\Lambda^2 + \text{finite}\right) + \mathcal{O}\left(\lambda^4\right)\,. \end{aligned}\tag{5.80}$$

[6]In contrary to Eqn. (5.68) the mass is put in the “finite” term which leaves the remaining logarithm with the dimension afflicted argument Λ^2.

The calculations for the bare mass and the constant a work just the same. For the mass one starts with the renormalized mass from Eqn. (5.50)

$$m_r^2 = m_0^2 + \frac{\lambda}{3(4\pi)^2}\left(4\Lambda^2 + m_0^2 \ln\left(\frac{|a|}{\Lambda^2}\right)\right) . \tag{5.81}$$

This can be written as

$$m_0^2 = m_r^2 - \frac{\lambda_r}{3(4\pi)^2}\left(1 + \frac{\lambda_r}{3(4\pi)^2}\left(\ln\Lambda^2 + \text{finite}\right)\right)\left(4\Lambda^2 + m_0^2 \ln\left(\frac{|a|}{\Lambda^2}\right)\right) .$$

With the expansion of the logarithm $\ln(a+x) = \ln a - \frac{x}{a} - \frac{x^2}{2a^2} + \mathcal{O}\left(x^3\right)$ this gives

$$= m_r^2 - \frac{\lambda_r}{3(4\pi)^2}\left(4\Lambda^2 + m_r^2 \ln\left(\frac{|a_r|}{\Lambda^2}\right)\right) + \mathcal{O}\left(\lambda_r^2\right) . \tag{5.82}$$

For a^2 one has to start from the renormalized constant (5.51)

$$a_r^2 = a^2 + \lambda\left(\frac{2}{3(4\pi)^2\theta^2} + \alpha\theta^2 a^2\right) , \tag{5.83}$$

and gets for a^2

$$\begin{aligned} a^2 = a_r^2 - \lambda_r &\left(1 + \frac{\lambda_r}{3(4\pi)^2}\left(\ln\Lambda^2 + \text{finite}\right)\right) \\ &\times \frac{2}{3(4\pi)^2}\left(\frac{1}{\theta^2} + \theta^2\left(m_0^4 - a^2\right)a^2\left(\ln 2 + \frac{5}{4} - \gamma_E\right)\right) \\ = a_r^2 - \lambda_r &\frac{2}{3(4\pi)^2}\left(\frac{1}{\theta^2} + \theta^2\left(m_r^4 - a_r^2\right)a_r^2\left(\ln 2 + \frac{5}{4} - \gamma_E\right)\right) + \mathcal{O}\left(\lambda_r^2\right) . \end{aligned} \tag{5.84}$$

After these preparations the β functions can be calculated (see also [30]):

$$\beta = \Lambda\frac{\partial\lambda}{\partial\Lambda} = \Lambda\lambda_r^2\frac{2}{3(4\pi)^2\Lambda} = \frac{\lambda_r^2}{24\pi^2} , \tag{5.85}$$

$$\beta_a = \Lambda\frac{\partial a^2}{\partial\Lambda} = 0 , \tag{5.86}$$

$$\begin{aligned} \beta_m &= \frac{1}{m_0^2}\Lambda\frac{\partial m_0^2}{\partial\Lambda} = -\frac{1}{m_0^2}\Lambda\frac{\lambda_r}{3(4\pi)^2}\left(8\Lambda - 2m_r^2\frac{1}{\Lambda}\right) = \frac{2\lambda_r m_r^2}{3(4\pi)^2 m_0^2} - \frac{8\lambda_r\Lambda^2}{3(4\pi)^2 m_0^2} \\ &= \frac{2\lambda_r}{3(4\pi)^2}\frac{m_r^2 - 4\Lambda^2}{m_r^2 - \frac{\lambda_r}{3(4\pi)^2}\left(4\Lambda^2 + m_r^2\ln\frac{|a_r|}{\Lambda^2}\right)} \\ &= \frac{2\lambda_r}{3(4\pi)^2}\left[\frac{m_r^2}{m_r^2 - \frac{\lambda_r}{3(4\pi)^2}\left(4\Lambda^2 + m_r^2\ln\frac{|a_r|}{\Lambda^2}\right)} - \frac{4}{\frac{m_r^2}{\Lambda^2} - \frac{\lambda_r}{3(4\pi)^2}\left(4 + \frac{m_r^2}{\Lambda^2}\ln\frac{|a_r|}{\Lambda^2}\right)}\right] . \end{aligned} \tag{5.87}$$

To get the limit $\Lambda \to \infty$ an expansion of the denominator is made with the second formula from Eqn. (5.64) giving

$$\beta_m = \frac{2\lambda_r}{3(4\pi)^2}\left(1 - \frac{4\Lambda^2}{m_r^2}\right) + \mathcal{O}\left(\lambda_r^2\right) , \tag{5.88}$$

and it can be seen that in the limit $\Lambda \to \infty$, β_m diverges at the lowest order of the renormalized coupling constant. The first term of Eqn. (5.88) is proportional to the one from the commutative model but the divergent second one does not appear in that model. It is possible though, that, when one replaces m_r^2 in the denominator with Eqn. (5.50) and then makes an expansion of the fraction in terms of λ_r, one finds that, in the first order of λ_r, the divergences of the bare mass and the Λ cancel and leave a constant factor.

For the calculation of the γ function one needs the logarithm of the wave function renormalization constant

$$Z = 1 + \lambda\alpha\theta^2 = 1 + \alpha\theta^2\left(\lambda_r + \frac{\lambda_r^2}{3(4\pi)^2}\left(\ln\Lambda^2 + \text{finite}\right)\right) , \tag{5.89}$$

$$\ln Z = \ln\left[1 + \alpha\theta^2\left(\lambda_r + \frac{\lambda_r^2}{3(4\pi)^2}\left(\ln\Lambda^2 + \text{finite}\right)\right)\right] , \tag{5.90}$$

$$\begin{aligned}
\gamma &= \Lambda\frac{\partial}{\partial\Lambda}\ln Z \\
&= -\Lambda\frac{2\theta^2\lambda_r^2}{3(16\pi)^2}\left(\ln 2 + \frac{5}{4} - \gamma_E\right)\frac{\frac{2}{3(4\pi)^2}\left(8m_r^2\Lambda - 2m_r^4\frac{1}{\Lambda}\right) - \frac{2\left(m_r^4 - a_r^2\right)}{3(4\pi)^2\Lambda}}{1 + \alpha\theta^2\left(\lambda_r + \frac{\lambda_r^2}{3(4\pi)^2}\left(\ln\Lambda^2 + \text{finite}\right)\right)} \\
&= -\frac{\theta^2\lambda_r^2}{3^2 2^{10}\pi^4}\left(\ln 2 + \frac{5}{4} - \gamma_E\right)\frac{8m_r^2\Lambda^2 - 3m_r^4 + a_r^2}{1 + \alpha\theta^2\left(\lambda_r + \frac{\lambda_r^2}{3(4\pi)^2}\left(\ln\Lambda^2 + \text{finite}\right)\right)} .
\end{aligned} \tag{5.91}$$

To get this result one has to insert the factor α from Eqn. (5.56):

$$\alpha = \frac{2}{3(16\pi)^2}\left(\ln 2 + \frac{5}{4} - \gamma_E\right)\left(m_0^4 - a^2\right) . \tag{5.92}$$

As can be seen, the essential part of α is the factor $m_0^4 - a^2$. Up to the first order of λ_r this factor is

$$\begin{aligned}
m_0^4 - a^2 = m_r^4 - a_r^2 - \lambda_r\frac{2}{3(4\pi)^2}&\left[4m_r^2\Lambda^2 + m_r^4\ln\left(\frac{|a_r|}{\Lambda^2}\right) - \frac{1}{\theta^2}\right. \\
&\left. - \frac{\theta^2 a_r^2}{16}\left(m_r^4 - a_r^2\right)\left(\ln 2 + \frac{5}{4} - \gamma_E\right)\right] + \mathcal{O}\left(\lambda_r^2\right) .
\end{aligned} \tag{5.93}$$

To get the limit $\Lambda \to \infty$ of Eqn. (5.91) one makes a series expansion with the formula from Eqn. (5.64) giving

$$\gamma = -\frac{\theta^2 \lambda_r^2}{3^2 2^{10} \pi^4} \left(\ln 2 + \frac{5}{4} - \gamma_E \right) \left(8 m_r^2 \Lambda^2 - 3 m_r^4 + a_r^2 \right) + \mathcal{O}\left(\lambda_r^3\right) . \tag{5.94}$$

This diverges quadratically as can be seen easily but it is quite possible that there is another contribution to this order from higher loops which renders it finite. Therefore, the γ function, in agreement with [30], is written as

$$\gamma = 0 + \mathcal{O}\left(\lambda^2\right) . \tag{5.95}$$

Chapter 6

Higher Loop Orders

The goal of this chapter is to show the improvements of the IR behavior of the $\frac{1}{\tilde{p}^2}$ model by looking at non-planar two point correction graphs with higher loop orders. Two of these n-point graphs are shown in Fig. 6.1. Only the IR behavior

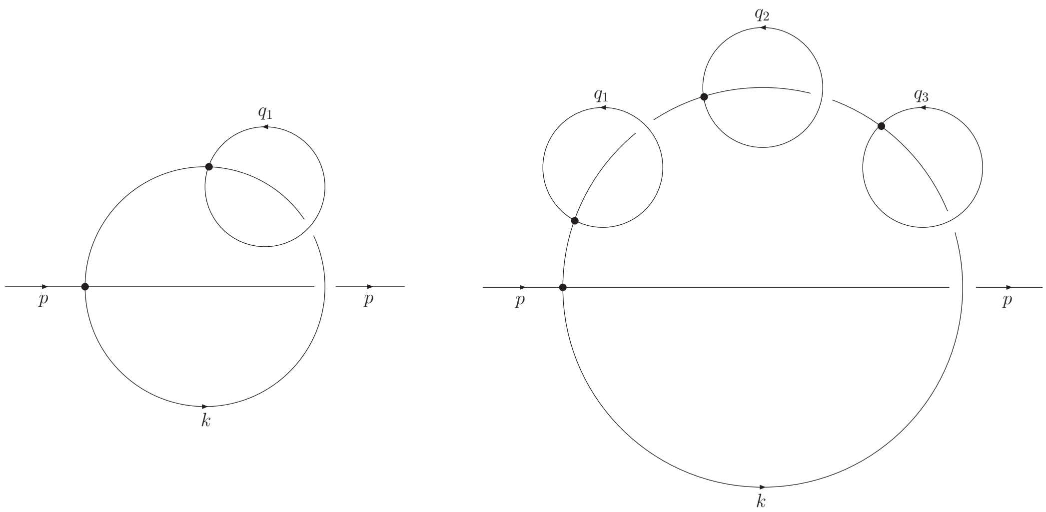

Figure 6.1: A non-planar 2 loop and 4 loop graph.

is part of the investigation here and the first order approximation of the one loop correction for the propagator Π^{n-pl}, which is proportional to $\frac{1}{\tilde{k}^2}$, is sufficient. A graph with n non-planar insertions then has the form

$$\Pi^{n-ins}(p) \equiv \lambda^2 \sum_{\eta=\pm 1} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{\tilde{k}^{2n}} \frac{\mathrm{e}^{\mathrm{i}\eta k\tilde{p}}}{\left(k^2 + m^2 + \frac{a^2}{k^2}\right)^{n+1}} \tag{6.1}$$

$$= \lambda^2 \sum_{\eta=\pm 1} J_n(p)\,. \tag{6.2}$$

The denominator can be written as

$$\frac{1}{\left(k^2+m^2+\frac{a^2}{k^2}\right)^{n+1}} = \left(\frac{1}{2}\sum_{\zeta=\pm 1}\frac{1+\zeta\frac{m^2}{2M^2}}{k^2+\frac{m^2}{2}+\zeta M^2}\right)^{n+1}$$
$$= \frac{1}{2^{n+1}}\sum_{\zeta_1,\dots,\zeta_{n+1}=\pm 1}\prod_{i=1}^{n+1}\left(\frac{1+\zeta_i\frac{m^2}{2M^2}}{k^2+\frac{m^2}{2}+\zeta_i M^2}\right). \tag{6.3}$$

The generic integral for an arbitrary number of insertions is

$$J_n(p) = \frac{1}{2^{n+1}\theta^{2n}}\sum_{\zeta_1\dots\zeta_{n+1}=\pm 1}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}\eta k\tilde{p}}}{k^{2n}}\prod_{i=1}^{n+1}\frac{1+\zeta_i\frac{m^2}{2M^2}}{k^2+\frac{m^2}{2}+\zeta_i M^2}, \tag{6.4}$$

and for the integration of these integrals one needs $n+2$ Schwinger parameters

$$\frac{1}{k^2+\frac{m^2}{2}+\zeta_i M^2} = \int_0^\infty \mathrm{d}\alpha_i\, \mathrm{e}^{-\alpha_i\left(k^2+\frac{m^2}{2}+\zeta_i M^2\right)}, \quad \text{with} \quad i\in\{1,\dots,n+1\}, \tag{6.5}$$

and the formula for higher order Schwinger parameterization

$$\frac{1}{k^{2n}} = \frac{1}{\Gamma(n)}\int_0^\infty \mathrm{d}\alpha_{n+2}\,(\alpha_{n+2})^{n-1}\,\mathrm{e}^{-\alpha_{n+2}k^2}, \quad \text{with} \quad k^2>0. \tag{6.6}$$

With this it is possible to solve the integral. First one starts with the integration over k,

$$J_n(p) = \frac{1}{2^{n+1}\theta^{2n}\Gamma(n)}\sum_{\zeta_1\dots\zeta_{n+1}=\pm 1}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^\infty \mathrm{d}\alpha_{n+2}\,(\alpha_{n+2})^{n-1}\,\mathrm{e}^{-\alpha_{n+2}k^2}$$
$$\times\prod_{i=1}^{n+1}\left(\left(1+\zeta_i\frac{m^2}{2M^2}\right)\int_0^\infty \mathrm{d}\alpha_i\,\mathrm{e}^{-\alpha_i\left(k^2+\frac{m^2}{2}+\zeta_i M^2\right)}\right)\mathrm{e}^{\mathrm{i}\eta k\tilde{p}}$$
$$= \frac{1}{2^{n+1}\theta^{2n}\Gamma(n)}\sum_{\zeta_1\dots\zeta_{n+1}=\pm 1}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\prod_{i=1}^{n+1}\left(\int_0^\infty \mathrm{d}\alpha_i\left(1+\zeta_i\frac{m^2}{2M^2}\right)\right)$$
$$\times\int_0^\infty \mathrm{d}\alpha_{n+2}\,(\alpha_{n+2})^{n-1}\,\mathrm{e}^{-\sum\limits_{i=1}^{n+2}\alpha_i k^2+\mathrm{i}\eta k\tilde{p}-\sum\limits_{i=1}^{n+2}\alpha_i\left(\frac{m^2}{2}+\zeta_i M^2\right)}. \tag{6.7}$$

The integral over k can be solved by completing the square in the exponential

$$\exp\left[-\sum_{i=1}^{n+2}\alpha_i k^2 + \mathrm{i}\eta k\tilde{p} + \frac{\tilde{p}^2}{2\sum\limits_{i=1}^{n+1}\alpha_i} - \frac{\tilde{p}^2}{2\sum\limits_{i=1}^{n+2}\alpha_i} - \sum_{i=1}^{n+2}\alpha_i\left(\frac{m^2}{2}+\zeta_i M^2\right)\right]$$
$$= \exp\left[-\sum_{i=1}^{n+2}\alpha_i\left(k^2 - \frac{\mathrm{i}\eta k\tilde{p}}{\sum\limits_{i=1}^{n+1}\alpha_i} - \frac{\tilde{p}^2}{4\left(\sum\limits_{i=1}^{n+2}\alpha_i\right)^2}\right) - \frac{\tilde{p}^2}{4\sum\limits_{i=1}^{n+2}\alpha_i} - \sum_{i=1}^{n+2}\alpha_i\left(\frac{m^2}{2}+\zeta_i M^2\right)\right]. \tag{6.8}$$

With

$$k' = k - \frac{\mathrm{i}\eta\tilde{p}}{2\sum\limits_{i=1}^{n+2}\alpha_i}, \quad \text{and} \quad \mathrm{d}^4k = \mathrm{d}^4k'. \tag{6.9}$$

one gets

$$J_n(p) = \frac{1}{2^{n+1}\theta^{2n}(4\pi)^2\Gamma(n)} \sum_{\zeta_1\ldots\zeta_{n+1}=\pm1} \prod_{i=1}^{n+1}\left(\int_0^\infty \mathrm{d}\alpha_i\left(1+\zeta_i\frac{m^2}{2M^2}\right)\right)$$
$$\times\int_0^\infty \mathrm{d}\alpha_{n+2}\frac{(\alpha_{n+2})^{n-1}}{\left(\sum\limits_{i=1}^{n+2}\alpha_i\right)^2}\exp\left[-\frac{\tilde{p}^2}{4\sum\limits_{i=1}^{n+2}\alpha_i} - \sum_{i=1}^{n+1}\alpha_i\left(\frac{m^2}{2}+\zeta_i M^2\right)\right]. \tag{6.10}$$

The next step is to change the variables $(\alpha_1,\ldots,\alpha_{n+2}) \to (\xi_1,\ldots\xi_{n+1},\lambda)$. This works like in Eqn. (5.29) for two variables only here more are needed:

$$\begin{aligned}
\alpha_1 &= \lambda\sum_{i=1}^{n+1}\xi_i\,,\\
\alpha_2 &= \lambda\,(1-\xi_1)\prod_{i=2}^{n+1}\xi_i\,,\\
&\vdots\\
\alpha_k &= \lambda\,(1-\xi_{k-1})\prod_{i=k}^{n+1}\xi_i\,,\\
&\vdots\\
\alpha_{n+2} &= \lambda\,(1-\xi_{n+1})\,.
\end{aligned} \tag{6.11}$$

This implies

$$\prod_{i=1}^{n+2}\mathrm{d}\alpha_i = \lambda^{n+1}\prod_{l=1}^{n}(\xi_{l+1})^l\,\mathrm{d}\lambda\prod_{j=1}^{n+1}\mathrm{d}\xi_j \quad \text{and} \quad \sum_{i=1}^{n+2}\alpha_i = \lambda\,, \tag{6.12}$$

where

$$\begin{aligned} &\xi_i \in [0,1]\,, \\ &\lambda \in [0,\infty)\,. \end{aligned} \tag{6.13}$$

With the new variables the equation has the form

$$\begin{aligned} J_n(p) = &\frac{1}{2^{n+1}\theta^{2n}(4\pi)^2\Gamma(n)} \sum_{\zeta_1\dots\zeta_{n+1}=\pm1} \prod_{i=1}^{n+1}\left(1+\zeta_i\frac{m^2}{2M^2}\right)\int\limits_0^\infty \mathrm{d}\lambda\,\lambda^{2n-2} \\ &\times \prod_{j=1}^{n+1}\int\limits_0^1 \mathrm{d}\xi_j \prod_{l=1}^{n}\left(\xi_{l+1}\right)^l\left(1-\xi_{n+1}\right)^{n-1}\exp\left(-\frac{\tilde{p}^2}{4\lambda}-\lambda\xi_{n+1}\frac{m^2}{2}-\sum_1^{n+1}\zeta_i\alpha_i M^2\right). \end{aligned}$$

With the sum over ξ_i and the exponential representations of the hyperbolic functions $\sinh x = \frac{1}{2}\left(\mathrm{e}^x - \mathrm{e}^{-x}\right)$ and $\cosh x = \frac{1}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right)$ one gets

$$\begin{aligned} J_n(p) = &\frac{1}{\theta^{2n}(4\pi)^2\Gamma(n)}\int\limits_0^\infty \mathrm{d}\lambda\lambda^{2n-2}\prod_{j=1}^{n+1}\int\limits_0^1 \mathrm{d}\xi_j\prod_{l=1}^{n}\left(\xi_{l+1}\right)^l\left(1-\xi_{n+1}\right)^{n-1}\mathrm{e}^{-\frac{\tilde{p}^2}{4\lambda}-\lambda\xi_{n+1}\frac{m^2}{2}} \\ &\times\prod_{i=1}^{n+1}\left[\cosh\left(\lambda\left(1-\xi_{i-1}\right)\prod_{k=i}^{n+1}\xi_k M^2\right)-\frac{m^2}{2M^2}\sinh\left(\lambda\left(1-\xi_{i-1}\right)\prod_{k=i}^{n+1}\xi_k M^2\right)\right]. \end{aligned} \tag{6.14}$$

For a graph with one non-commutative insertion this integral looks like

$$\begin{aligned} J_1(p) = &\frac{1}{\theta^2(4\pi)^2}\int\limits_0^\infty \mathrm{d}\lambda\int\limits_0^1 \mathrm{d}\xi_1\int\limits_0^1 \mathrm{d}\xi_2\,\xi_2\mathrm{e}^{-\frac{\tilde{p}^2}{4\lambda}-\lambda\xi_2\frac{m^2}{2}} \\ &\times\left[\cosh\left(\lambda\xi_1\xi_2M^2\right)-\frac{m^2}{2M^2}\sinh\left(\lambda\xi_1\xi_2M^2\right)\right] \\ &\times\left[\cosh\left(\lambda\left(1-\xi_1\right)\xi_2M^2\right)-\frac{m^2}{2M^2}\sinh\left(\lambda\left(1-\xi_1\right)\xi_2M^2\right)\right]. \end{aligned} \tag{6.15}$$

As a first step one solves the integral over ξ_1. There are four different types of integrals to solve here. They can be found in Appendix B.3.1 and inserting their results leaves

$$\int_0^1 \mathrm{d}\xi_2\, \xi_2 \mathrm{e}^{-\lambda\xi_2 \frac{m^2}{2}} \left[\frac{1}{2}\cosh\left(\lambda\xi_2 M^2\right) + \frac{\sinh\left(\lambda\xi_2 M^2\right)}{2\lambda\xi_2 M^2} - \frac{m^2}{2M^2}\sinh\left(\lambda\xi_2 M^2\right)\right.$$

$$\left. + \frac{m^4}{4M^4}\left(\frac{1}{2}\cosh\left(\lambda\xi_2 M^2\right) - \frac{\sinh\left(\lambda\xi_2 M^2\right)}{2\lambda\xi_2 M^2}\right)\right]$$

$$= \frac{1}{2}\int_0^1 \mathrm{d}\xi_2\ \mathrm{e}^{-\lambda\xi_2 \frac{m^2}{2}} \left[\xi_2 \cosh\left(\lambda\xi_2 M^2\right)\left(1 + \frac{m^4}{4M^4}\right)\right.$$

$$\left. + \sinh\left(\lambda\xi_2 M^2\right)\left(\frac{1}{\lambda M^2}\left(1 - \frac{m^4}{4M^4}\right) - \frac{\xi_2 m^2}{M^2}\right)\right]$$

$$= \frac{1}{2}\frac{\mathrm{e}^{-\lambda\frac{m^2}{2}}}{\lambda^2}\left[-\lambda\frac{m^2}{2M^4}\cosh\lambda M^2 + \left(\frac{\lambda}{M^2} + \frac{m^2}{2M^6}\right)\sinh\lambda M^2\right] . \tag{6.16}$$

To solve the integral over ξ_2 one has to solve three different types of integrals which is done in Appendix B.3.1. With this result one is now ready for the final integration,

$$J_1(p) = \frac{1}{2\theta^2(4\pi)^2}\int_0^\infty \mathrm{d}\lambda \mathrm{e}^{-\frac{\tilde{p}^2}{4\lambda} - \lambda\frac{m^2}{2}}$$

$$\times\left[\left(\frac{1}{\lambda M^2} + \frac{m^2}{2\lambda^2 M^6}\right)\sinh\lambda M^2 - \frac{m^2}{2\lambda M^4}\cosh\lambda M^2\right]$$

$$= \frac{1}{4\theta^2(4\pi)^2}\int_0^\infty \mathrm{d}\lambda \mathrm{e}^{-\lambda\frac{\tilde{p}^2}{4\lambda} - \lambda\frac{m^2}{2}}$$

$$\times\left[\left(\frac{1}{\lambda M^2} + \frac{m^2}{2\lambda M^6}\right)\left(\mathrm{e}^{\lambda M^2} - \mathrm{e}^{-\lambda M^2}\right) - \frac{m^2}{2\lambda M^4}\left(\mathrm{e}^{\lambda M^2} - \mathrm{e}^{-\lambda M^2}\right)\right]$$

$$= \frac{1}{4\theta^2(4\pi)^2 M^2}\int_0^\infty \mathrm{d}\lambda\frac{1}{\lambda}\mathrm{e}^{-\lambda\frac{\tilde{p}^2}{4\lambda} - \lambda\frac{m^2}{2}}$$

$$\times\left[\left(1 - \frac{m^2}{2M^2} + \frac{m^2}{2\lambda M^4}\right)\mathrm{e}^{\lambda M^2} - \left(1 + \frac{m^2}{2M^2} + \frac{m^2}{2\lambda M^4}\right)\mathrm{e}^{-\lambda M^2}\right] . \tag{6.17}$$

The integrals can be solved with [21] and give

$$\int_0^\infty \mathrm{d}\lambda\frac{1}{\lambda}\mathrm{e}^{-\frac{\tilde{p}^2}{4\lambda} - \lambda\left(\frac{m^2}{2} \pm M^2\right)} = 2K_0\left(\sqrt{\tilde{p}^2\left(\frac{m^2}{2} \pm M^2\right)}\right) , \tag{6.18}$$

and

$$\int_0^\infty \mathrm{d}\lambda \frac{1}{\lambda^2} \mathrm{e}^{-\frac{\tilde{p}^2}{4\lambda}-\lambda\left(\frac{m^2}{2}\pm M^2\right)} = 4\left(\frac{m^2 \pm 2M^2}{2\tilde{p}^2}\right)^{\frac{1}{2}} K_1\left(\sqrt{\tilde{p}^2\left(\frac{m^2}{2}\pm M^2\right)}\right) . \quad (6.19)$$

Hence, one arrives at

$$\begin{aligned} J_1(p) = \frac{1}{2\theta^2(4\pi)^2 M^2}\Bigg[&\left(1-\frac{m^2}{2M^2}\right) K_0\left(\sqrt{\left(\frac{m^2}{2}-M^2\right)\tilde{p}^2}\right) \\ &- \left(1+\frac{m^2}{2M^2}\right) K_0\left(\sqrt{\left(\frac{m^2}{2}+M^2\right)\tilde{p}^2}\right) \\ &+ \frac{m^2}{M^4}\sqrt{\frac{\frac{m^2}{2}-M^2}{\tilde{p}^2}} K_1\left(\sqrt{\left(\frac{m^2}{2}-M^2\right)\tilde{p}^2}\right) \\ &- \frac{m^2}{M^4}\sqrt{\frac{\frac{m^2}{2}+M^2}{\tilde{p}^2}} K_1\left(\sqrt{\left(\frac{m^2}{2}+M^2\right)\tilde{p}^2}\right)\Bigg] , \end{aligned}$$

and with the abbreviation $m_\pm^2 = \frac{m^2}{2} \pm M^2$ this gives

$$\begin{aligned} J_1(p) = -\frac{1}{32\pi^2\theta^2 M^6}\Bigg[&M^2 m_+^2 K_0\left(\sqrt{m_+^2\tilde{p}^2}\right) + M^2 m_-^2 K_0\left(\sqrt{m_-^2\tilde{p}^2}\right) \\ &+ m^2\sqrt{\frac{m_+^2}{\tilde{p}^2}} K_1\left(\sqrt{m_+^2\tilde{p}^2}\right) - m^2\sqrt{\frac{m_-^2}{\tilde{p}^2}} K_1\left(\sqrt{m_-^2\tilde{p}^2}\right)\Bigg] . \end{aligned} \quad (6.20)$$

Finally, one gets the result

$$\begin{aligned} \Pi^{1-ins}(p) &= \lambda^2 \sum_{\eta=\pm1} J_1(p) \\ &= -\frac{\lambda^2}{16\pi^2\theta^2 M^6}\Bigg[M^2 m_+^2 K_0\left(\sqrt{m_+^2\tilde{p}^2}\right) + M^2 m_-^2 K_0\left(\sqrt{m_-^2\tilde{p}^2}\right) \\ &\quad + m^2\sqrt{\frac{m_+^2}{\tilde{p}^2}} K_1\left(\sqrt{m_+^2\tilde{p}^2}\right) - m^2\sqrt{\frac{m_-^2}{\tilde{p}^2}} K_1\left(\sqrt{m_-^2\tilde{p}^2}\right)\Bigg] . \end{aligned} \quad (6.21)$$

Expanding the modified Bessel functions for small external momentum $\tilde{p}^2$ leads to (see Appendix B.3.2)

$$\Pi^{1-ins}(p) = \frac{\lambda^2}{16\pi^2\theta^2 M^6}\left[\underbrace{\left(M^4-\frac{m^4}{4}\right)}_{-a^2}\ln\sqrt{\frac{m_+^2}{m_-^2}} + M^2\frac{m^2}{2}\right] + \mathcal{O}\left(\tilde{p}^2\right) . \quad (6.22)$$

One can see that the result has no IR divergence. This has been a very lengthy calculation already and there was only just one non-commutative insertion involved. With two or more insertions this is soon to long to calculate by hand but one can use the help of a computer to get to the result. The emerging integrals should be solvable for any given order n. In the following a calculation for $n=2$ with *Mathematica* starts with:

$$\begin{aligned}
J_2(p) = {} & \frac{1}{\theta^4(4\pi)^2} \int\limits_0^\infty \mathrm{d}\lambda\,\lambda^2 \int\limits_0^1 \mathrm{d}\xi_1 \int\limits_0^1 \mathrm{d}\xi_2 \int\limits_0^1 \mathrm{d}\xi_3\, \xi_2\xi_3^2\,(1-\xi_3)\,\mathrm{e}^{-\frac{\tilde{p}^2}{4\lambda}-\lambda\xi_3\frac{m^2}{2}} \\
& \times \left[\cosh\left(\lambda\xi_1\xi_2\xi_3M^2\right) - \frac{m^2}{2M^2}\sinh\left(\lambda\xi_1\xi_2\xi_3M^2\right)\right] \\
& \times \left[\cosh\left(\lambda\,(1-\xi_1)\,\xi_2\xi_3M^2\right) - \frac{m^2}{2M^2}\sinh\left(\lambda\,(1-\xi_1)\,\xi_2\xi_3M^2\right)\right] \\
& \times \left[\cosh\left(\lambda\,(1-\xi_2)\,\xi_3M^2\right) - \frac{m^2}{2M^2}\sinh\left(\lambda\,(1-\xi_2)\,\xi_3M^2\right)\right]\,.
\end{aligned} \tag{6.23}$$

The integration is performed with *Mathematica* whereas the order of the integration is ξ_1, ξ_2, ξ_3, λ. A different integration order can result in a much longer calculation time because the subintegrations are getting very complicated. The result is

$$\begin{aligned}
J_2(p) = {} & \frac{1}{16\,\theta^4(4\pi)^2M^{10}} \\
& \times \Bigg[M^2(3m^2+2M^2)K_0\left(\sqrt{m_+^2\tilde{p}^2}\right) + M^2(3m^2-2M^2)K_0\left(\sqrt{m_-^2\tilde{p}^2}\right) \\
& + \sqrt{\tilde{p}^2}\left(\frac{6m^2}{\tilde{p}^2}+M^4\right)\left(\sqrt{m_+^2}K_1\left(\sqrt{m_+^2\tilde{p}^2}\right) - \sqrt{m_-^2}K_1\left(\sqrt{m_-^2\tilde{p}^2}\right)\right)\Bigg]\,.
\end{aligned} \tag{6.24}$$

For small $\tilde{p}^2$ this can be expanded to

$$J_2(p) \approx \frac{1}{512\pi^2\theta^4}\left[\frac{3m^4-4M^4}{M^{10}}\ln\sqrt{\frac{m_+^2}{m_-^2}} - \frac{6m^2}{M^8}\right] + \mathcal{O}\left(\tilde{p}^2\right)\,. \tag{6.25}$$

Thus, for the non-planar tadpole with two insertions one gets

$$\Pi^{2-ins}(p) = \frac{\lambda^2}{256\pi^2\theta^4}\left[\frac{3m^4-4M^4}{M^{10}}\ln\sqrt{\frac{m_+^2}{m_-^2}} - \frac{6m^2}{M^8}\right] + \mathcal{O}\left(\tilde{p}^2\right)\,. \tag{6.26}$$

Like the graph with one insertion this graph also has no IR divergence and it is expected that higher order graphs behave in a similar way. The reason for the

IR-finiteness to every order can be found in the propagators between the non-planar insertions. If one looks at the denominator of the generic integral (6.1) for n non-planar insertions and considers the limit $k^2 \longrightarrow 0$ one sees that because of the $n+1$ propagators in the graph the integrand behaves like

$$\frac{1}{\left(\tilde{k}^2\right)^n \left(\frac{a^2}{k^2}\right)^{n+1}} = \frac{\tilde{k}^2}{(a'^2)^{n+1}} , \tag{6.27}$$

and is, therefore, independent of the loop order. If, by letting a go to zero, one lets the new model go to the naive model, this equation (6.26) diverges since the propagators do not regularize the graph anymore. This holds for every graph with $n \geq 2$.
For a massless field, that is for $m \longrightarrow 0$, the results for one and two insertions from the Eqns. (6.22) and (6.26) become

$$\Pi^{1-ins}(p)\Big|_{m=0} = \frac{\lambda^2}{32\pi\theta^2|a|} + \mathcal{O}\left(\tilde{p}^2\right) , \tag{6.28}$$

$$\Pi^{2-ins}(p)\Big|_{m=0} = \frac{\lambda^2}{128\pi\theta^4|a^3|} + \mathcal{O}\left(\tilde{p}^2\right) . \tag{6.29}$$

So, contrary to the naive model the higher loop graphs do not diverge even for vanishing mass which is important for a generalization to gauge fields.

Chapter 7

Conclusion

7.1 Summary

The model introduced by Gurau *et al.* [1] is a non-commutative scalar ϕ^4 theory with an additional term in the form of an inverse squared momentum in the otherwise from commutative theory straight forward adapted action. This additional term changes the propagator of the theory in a way that the theory becomes renormalizable in contrary to the so-called naive model not containing the term. This was proved to all orders in the above mentioned paper with the powerful tool of multiscale analysis. However there were no explicit calculations for the factors so far. This was a goal of this work. In this work the naive model was revised first in Chapter 3 by calculating Feynman rules and giving an example for the notorious UV/IR mixing which is a problem of most non-commutative theories. In Chapter 4 the new model was presented and its Feynman rules were calculated finding that the vertex function was not changed compared to the naive model, which is quite clear since the terms in the action are identical. The propagator on the other hand gained an additional term in the denominator rendering it convergent for very high as well as very low momenta even when the mass was going to zero. The next Chapter 5 dealed with the one loop corrections for the two and four point function and the renormalization to first order as well as an attempt to calculate the β function of the theory. The result of the one loop correction for the two point function showed that a quadratic infrared divergent term still remained as well as a logarithmic divergence in the next higher order. A reason for this behavior was the splitting of the graph in a *planar* part identical up to a factor with the classical graph and a *non-planar* part bringing the non-commutative effects in the theory. Despite the IR divergent term a renormalization procedure was still possible. The IR divergent term also occurred in the massless case treated in Section 5.2 but the logarithmic divergence was canceled. For the one loop correction of the four point function a distinction between a planar and a non-planar part of the graph could also be made. The non-planar part led to an integral which was not solvable analytically and an approximation for small momenta was made resulting in an expression with a logarithmic divergence as the leading order. Gurau *et al.* did proof the renormalizability but did not calculate explicit coefficients. This was done in Section 5.4 for the one loop order revealing the renormalized mass and the renormalized parameter controlling the non-local insertion as well as a function which is singular for vanishing external momentum. But this divergence is called only a "mild" divergence which has no catastrophic effects on the physical amplitudes as it is smaller than the other terms in the denominator in the limit of small momenta. Another oddity is that a wave function renormalization different from one was found. The renormalized coupling constant was found to have a logarithmic divergence for the regulator parameter. Investigating the β function it was found that the parameter controlling the non-locality does not depend on the regulator whereas the mass parameter shows a divergent behavior for the regulator going to infinity. The coupling constant was found to evince the same characteristics as in commutative theory. Higher order loops where investigated in Chapter 6 particularly by looking at non-planar two point graphs with one or more non-planar insertions as shown in Fig. 6.1. Only the first order approximations where looked at because the interest lied on the IR behavior and also only the first order of the one loop correction was inserted in the higher order graphs due to that reason and to keep the calculation complexity down. Nevertheless the equations became very large for a higher number of insertions and the help of a computer algebra system was needed. The result was that the propagator between the insertions significantly improved the behavior of the whole graph by efficiently damping the IR divergences and thus rendered the whole graph finite for small external momenta. It was also possible to set the mass to zero and get the results for a massless field which were finite as well.

7.2 Outlook

Higher loop graphs were treated exemplarily in this work by picking one type of graphs out (the pearl necklace graph in Fig. 6.1) and perform the calculations on it, but of course there are other types of graphs which deserve attention. For example, the two loop correction for the propagator already consists of five different graphs when one distinguishes between planar and non-planar ones. Also higher order four point functions would be an interesting area for explicit calculations but it is known already that all the higher graphs have to be renormalizable because the proof of Gurau *et al.* [1] covers them all. The calculations for higher than first order graphs in this work where all done with the tree level propagators but actually one should do the renormalization up to one level below first and use the renormalized results for the calculation of the desired level. This model as it is a scalar model on Euclidean space cannot be considered as a physically relevant one and is merely a toy model to investigate the principal aspects of renormalizable non-commutative models on a relatively simple basis. Since the ultimate goal is always to describe real world physics which is mostly done by gauge theories nowadays it is naturally the next step to look for a renormalizable non-commutative gauge theory. By looking at the behavior of the massless limit of the scalar theory in this work a first step in that direction has been made. René Sedmik and Arnold Rofner form the Institute for Theoretical Physics of the VUT both worked on the problem of finding renormalizable non-commutative gauge theories based on an action similar to the one used here on their PhD theses [31, 32]. Their work led to the so-called BRSW model [33] which seems to be a promising candidate for a renormalizable non-commutative gauge theory. Another attempt in this direction is to find a gauge theory based on the scalar GW model which is also currently a work in progress at this department [13, 34]. For a more detailed insight on the topic of non-commutative gauge theories see [17] and references therein. The models mentioned above are all on Euclidean space and a further task will be to transform them to Minkowski space where one has to pay special attention to causality problems that may arise due to the theories non-commutative characters.

Appendix A

The Naive Model

A.1 Vertex

$$V^{\phi^4}(p_1,p_2,p_3,p_4) = -\frac{\delta}{\delta\tilde{\phi}(-p_1)}\frac{\delta}{\delta\tilde{\phi}(-p_2)}\frac{\delta}{\delta\tilde{\phi}(-p_3)}\frac{\delta}{\delta\tilde{\phi}(-p_4)}(2\pi)^{16}S_{int}[\tilde{\phi}]$$

$$= -\tfrac{\delta}{\delta\tilde{\phi}(-p_1)}\tfrac{\delta}{\delta\tilde{\phi}(-p_2)}\tfrac{\delta}{\delta\tilde{\phi}(-p_3)}\tfrac{\delta}{\delta\tilde{\phi}(-p_4)}\tfrac{\lambda(2\pi)^4}{4!}\int \mathrm{d}^4k_{1..4}\,\delta^4(k_1+k_2+k_3+k_4)$$
$$\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4)\mathrm{e}^{-\frac{\mathrm{i}}{2}(k_1\times k_2+k_3\times k_4)}$$

$$= -\tfrac{\lambda(2\pi)^4}{4!}\tfrac{\delta}{\delta\tilde{\phi}(-p_2)}\tfrac{\delta}{\delta\tilde{\phi}(-p_3)}\tfrac{\delta}{\delta\tilde{\phi}(-p_4)}\int \mathrm{d}^4k_{1..4}\,\delta^4(k_1+k_2+k_3+k_4)\mathrm{e}^{-\frac{\mathrm{i}}{2}(k_1\times k_2+k_3\times k_4)}$$
$$\Big(\tilde{\phi}(k_2)\tilde{\phi}(k_3)\tilde{\phi}(k_4)\delta^4(k_1+p_1)+\tilde{\phi}(k_1)\tilde{\phi}(k_3)\tilde{\phi}(k_4)\delta^4(k_2+p_1)$$
$$+\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_4)\delta^4(k_3+p_1)+\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)\delta^4(k_4+p_1)\Big)$$

$$= -\tfrac{\lambda(2\pi)^4}{4!}\tfrac{\delta}{\delta\tilde{\phi}(-p_3)}\tfrac{\delta}{\delta\tilde{\phi}(-p_4)}\int \mathrm{d}^4k_{1..4}\,\delta^4(k_1+k_2+k_3+k_4)\mathrm{e}^{-\frac{\mathrm{i}}{2}(k_1\times k_2+k_3\times k_4)}$$
$$\Big[\Big(\tilde{\phi}(k_3)\tilde{\phi}(k_4)\delta^4(k_2+p_2)+\tilde{\phi}(k_2)\tilde{\phi}(k_4)\delta^4(k_3+p_2)+\tilde{\phi}(k_2)\tilde{\phi}(k_3)\delta^4(k_4+p_2)\Big)$$
$$\times\,\delta^4(k_1+p_1)$$
$$+\Big(\tilde{\phi}(k_3)\tilde{\phi}(k_4)\delta^4(k_1+p_2)+\tilde{\phi}(k_1)\tilde{\phi}(k_4)\delta^4(k_3+p_2)+\tilde{\phi}(k_1)\tilde{\phi}(k_3)\delta^4(k_4+p_2)\Big)$$
$$\times\,\delta^4(k_2+p_1)$$
$$+\Big(\tilde{\phi}(k_2)\tilde{\phi}(k_4)\delta^4(k_1+p_2)+\tilde{\phi}(k_1)\tilde{\phi}(k_4)\delta^4(k_2+p_2)+\tilde{\phi}(k_1)\tilde{\phi}(k_2)\delta^4(k_4+p_2)\Big)$$
$$\times\,\delta^4(k_3+p_1)$$
$$+\Big(\tilde{\phi}(k_2)\tilde{\phi}(k_3)\delta^4(k_1+p_2)+\tilde{\phi}(k_1)\tilde{\phi}(k_3)\delta^4(k_2+p_2)+\tilde{\phi}(k_1)\tilde{\phi}(k_2)\delta^4(k_3+p_2)\Big)$$
$$\times\,\delta^4(k_4+p_1)\Big]$$

$$
\begin{aligned}
V^{\phi^4} = & -\frac{\lambda}{4!}(2\pi)^4 \frac{\delta}{\delta\tilde{\phi}(-p_4)} \int \mathrm{d}^4k_{1..4}\, \delta^4(k_1+k_2+k_3+k_4) \mathrm{e}^{-\frac{i}{2}(k_1\times k_2 + k_3\times k_4)} \\
& \Big\{ \Big[\Big(\tilde{\phi}(k_4)\delta^4(k_3+p_3) + \tilde{\phi}(k_3)\delta^4(k_4+p_3) \Big)\, \delta^4(k_2+p_2) \\
& + \Big(\tilde{\phi}(k_4)\delta^4(k_2+p_3) + \tilde{\phi}(k_2)\delta^4(k_4+p_3) \Big)\, \delta^4(k_3+p_2) \\
& + \Big(\tilde{\phi}(k_2)\delta^4(k_3+p_3) + \tilde{\phi}(k_3)\delta^4(k_2+p_3) \Big)\, \delta^4(k_4+p_2) \Big] \delta^4(k_1+p_1) \\
& + \Big[\Big(\tilde{\phi}(k_4)\delta^4(k_3+p_3) + \tilde{\phi}(k_3)\delta^4(k_4+p_3) \Big)\, \delta^4(k_1+p_2) \\
& + \Big(\tilde{\phi}(k_4)\delta^4(k_1+p_3) + \tilde{\phi}(k_1)\delta^4(k_4+p_3) \Big)\, \delta^4(k_3+p_2) \\
& + \Big(\tilde{\phi}(k_3)\delta^4(k_1+p_3) + \tilde{\phi}(k_1)\delta^4(k_3+p_3) \Big)\, \delta^4(k_4+p_2) \Big] \delta^4(k_2+p_1) \\
& + \Big[\Big(\tilde{\phi}(k_4)\delta^4(k_2+p_3) + \tilde{\phi}(k_2)\delta^4(k_4+p_3) \Big)\, \delta^4(k_1+p_2) \\
& + \Big(\tilde{\phi}(k_4)\delta^4(k_1+p_3) + \tilde{\phi}(k_1)\delta^4(k_4+p_3) \Big)\, \delta^4(k_2+p_2) \\
& + \Big(\tilde{\phi}(k_2)\delta^4(k_1+p_3) + \tilde{\phi}(k_1)\delta^4(k_2+p_3) \Big)\, \delta^4(k_4+p_2) \Big] \delta^4(k_3+p_1) \\
& + \Big[\Big(\tilde{\phi}(k_3)\delta^4(k_2+p_3) + \tilde{\phi}(k_2)\delta^4(k_3+p_3) \Big)\, \delta^4(k_1+p_2) \\
& + \Big(\tilde{\phi}(k_3)\delta^4(k_1+p_3) + \tilde{\phi}(k_1)\delta^4(k_3+p_3) \Big)\, \delta^4(k_2+p_2) \\
& + \Big(\tilde{\phi}(k_2)\delta^4(k_1+p_3) + \tilde{\phi}(k_1)\delta^4(k_2+p_3) \Big)\, \delta^4(k_3+p_2) \Big] \delta^4(k_4+p_1) \Big\} \\
= & -\frac{\lambda}{4!}(2\pi)^4 \int \mathrm{d}^4k_{1..4}\, \delta^4(k_1+k_2+k_3+k_4) \mathrm{e}^{-\frac{\mathrm{i}}{2}(k_1\times k_2 + k_3\times k_4)} \\
& \big\{ \big[\big(\delta^4(k_4+p_4)\delta^4(k_3+p_3) + \delta^4(k_3+p_4)\delta^4(k_4+p_3) \big)\, \delta^4(k_2+p_2) \\
& + \big(\delta^4(k_4+p_4)\delta^4(k_2+p_3) + \delta^4(k_2+p_4)\delta^4(k_4+p_3) \big)\, \delta^4(k_3+p_2) \\
& + \big(\delta^4(k_2+p_4)\delta^4(k_3+p_3) + \delta^4(k_3+p_4)\delta^4(k_2+p_3) \big)\, \delta^4(k_4+p_2) \big] \delta^4(k_1+p_1) \\
& + \big[\big(\delta^4(k_4+p_4)\delta^4(k_3+p_3) + \delta^4(k_3+p_4)\delta^4(k_4+p_3) \big)\, \delta^4(k_1+p_2) \\
& + \big(\delta^4(k_4+p_4)\delta^4(k_1+p_3) + \delta^4(k_1+p_4)\delta^4(k_4+p_3) \big)\, \delta^4(k_3+p_2) \\
& + \big(\delta^4(k_3+p_4)\delta^4(k_1+p_3) + \delta^4(k_1+p_4)\delta^4(k_3+p_3) \big)\, \delta^4(k_4+p_2) \big] \delta^4(k_2+p_1) \\
& + \big[\big(\delta^4(k_4+p_4)\delta^4(k_2+p_3) + \delta^4(k_2+p_4)\delta^4(k_4+p_3) \big)\, \delta^4(k_1+p_2) \\
& + \big(\delta^4(k_4+p_4)\delta^4(k_1+p_3) + \delta^4(k_1+p_4)\delta^4(k_4+p_3) \big)\, \delta^4(k_2+p_2) \\
& + \big(\delta^4(k_2+p_4)\delta^4(k_1+p_3) + \delta^4(k_1+p_4)\delta^4(k_2+p_3) \big)\, \delta^4(k_4+p_2) \big] \delta^4(k_3+p_1) \\
& + \big[\big(\delta^4(k_3+p_4)\delta^4(k_2+p_3) + \delta^4(k_2+p_4)\delta^4(k_3+p_3) \big)\, \delta^4(k_1+p_2) \\
& + \big(\delta^4(k_3+p_4)\delta^4(k_1+p_3) + \delta^4(k_1+p_4)\delta^4(k_3+p_3) \big)\, \delta^4(k_2+p_2) \\
& + \big(\delta^4(k_2+p_4)\delta^4(k_1+p_3) + \delta^4(k_1+p_4)\delta^4(k_2+p_3) \big)\, \delta^4(k_3+p_2) \big] \delta^4(k_4+p_1) \big\}
\end{aligned}
$$

$$
\begin{aligned}
V^{\phi^4} &= -\frac{\lambda}{4!}(2\pi)^4\delta^4(p_1+p_2+p_3+p_4) \\
&\quad \Big(\mathrm{e}^{-\frac{\mathrm{i}}{2}(p_1\times p_2+p_3\times p_4)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_1\times p_2+p_4\times p_3)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_1\times p_3+p_2\times p_4)} \\
&\quad + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_1\times p_4+p_2\times p_3)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_1\times p_3+p_4\times p_2)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_1\times p_4+p_3\times p_2)} \\
&\quad + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_2\times p_1+p_3\times p_4)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_2\times p_1+p_4\times p_3)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_3\times p_1+p_2\times p_4)} \\
&\quad + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_4\times p_1+p_2\times p_3)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_3\times p_1+p_4\times p_2)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_4\times p_1+p_3\times p_2)} \\
&\quad + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_2\times p_3+p_1\times p_4)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_2\times p_4+p_1\times p_3)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_3\times p_2+p_1\times p_4)} \\
&\quad + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_4\times p_2+p_1\times p_3)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_3\times p_4+p_1\times p_2)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_4\times p_3+p_1\times p_2)} \\
&\quad + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_2\times p_3+p_4\times p_1)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_2\times p_4+p_3\times p_1)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_3\times p_2+p_4\times p_1)} \\
&\quad + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_4\times p_2+p_3\times p_1)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_3\times p_4+p_2\times p_1)} + \mathrm{e}^{-\frac{\mathrm{i}}{2}(p_4\times p_3+p_2\times p_1)} \Big) \\
&= -\frac{2\lambda}{4!}(2\pi)^4\delta^4(p_1+p_2+p_3+p_4)\Big(\mathrm{e}^{-\mathrm{i}\frac{p_1\times p_2}{2}}\cos\frac{p_3\times p_4}{2} \\
&\quad + \mathrm{e}^{-\mathrm{i}\frac{p_1\times p_3}{2}}\cos\frac{p_2\times p_4}{2} + \mathrm{e}^{-\mathrm{i}\frac{p_1\times p_4}{2}}\cos\frac{p_2\times p_3}{2} + \mathrm{e}^{-\mathrm{i}\frac{p_2\times p_1}{2}}\cos\frac{p_3\times p_4}{2} \\
&\quad + \mathrm{e}^{-\mathrm{i}\frac{p_3\times p_1}{2}}\cos\frac{p_2\times p_4}{2} + \mathrm{e}^{-\mathrm{i}\frac{p_4\times p_1}{2}}\cos\frac{p_2\times p_3}{2} + \mathrm{e}^{-\mathrm{i}\frac{p_1\times p_4}{2}}\cos\frac{p_2\times p_3}{2} \\
&\quad + \mathrm{e}^{-\mathrm{i}\frac{p_1\times p_3}{2}}\cos\frac{p_2\times p_4}{2} + \mathrm{e}^{-\mathrm{i}\frac{p_1\times p_2}{2}}\cos\frac{p_3\times p_4}{2} + \mathrm{e}^{-\mathrm{i}\frac{p_4\times p_1}{2}}\cos\frac{p_2\times p_3}{2} \\
&\quad + \mathrm{e}^{-\mathrm{i}\frac{p_3\times p_1}{2}}\cos\frac{p_2\times p_4}{2} + \mathrm{e}^{-\mathrm{i}\frac{p_2\times p_1}{2}}\cos\frac{p_3\times p_4}{2} \Big) \\
&= -\frac{\lambda}{3}(2\pi)^4\delta^4(p_1+p_2+p_3+p_4) \\
&\quad \left(\cos\frac{p_1\times p_2}{2}\cos\frac{p_3\times p_4}{2} + \cos\frac{p_1\times p_3}{2}\cos\frac{p_2\times p_4}{2} + \cos\frac{p_1\times p_4}{2}\cos\frac{p_2\times p_3}{2} \right)
\end{aligned}
\tag{A.1}
$$

Appendix B

Loop Corrections

B.1 Correction for the Massless Propagator

The first step is to use the propagator in the expanded form (5.15) and perform a Schwinger parameterization,

$$
\begin{aligned}
I(p) &= \int \mathrm{d}^4k \frac{e^{\pm ik\tilde{p}}}{k^2 + \frac{a^2}{k^2}} = \frac{1}{2}\int \mathrm{d}^4k \; e^{\pm ik\tilde{p}} \left(\frac{1}{k^2 + \mathrm{i}a} + \frac{1}{k^2 - \mathrm{i}a}\right) \\
&= \frac{1}{2}\int \mathrm{d}^4k \int\limits_0^\infty \mathrm{d}\alpha \, e^{\pm ik\tilde{p}} \left(e^{-\alpha\left(k^2+\mathrm{i}a\right)} + e^{-\alpha\left(k^2-\mathrm{i}a\right)}\right) .
\end{aligned}
\tag{B.1}
$$

In order to shorten the notation summations over $+1$ and -1 are introduced and the exponent is completed to a full square in order to do a substitution of a variable later on,

$$
\begin{aligned}
I(p) &= \frac{1}{2}\sum_{\eta=\pm1}\sum_{\xi=\pm1}\int \mathrm{d}^4k \int\limits_0^\infty \mathrm{d}\alpha \; e^{i\eta k\tilde{p}} e^{-\alpha\left(k^2+i\xi a\right)} \\
&= \frac{1}{2}\sum_{\eta=\pm1}\sum_{\xi=\pm1}\int \mathrm{d}^4k \int\limits_0^\infty \mathrm{d}\alpha \; e^{-\alpha k^2 + i\eta k\tilde{p} + \frac{\eta^2\tilde{p}^2}{4\alpha} - \frac{\eta^2\tilde{p}^2}{4\alpha} - i\xi\alpha a} .
\end{aligned}
$$

As η has only the value ±1, η^2 is always 1, giving

$$
I(p) = \frac{1}{2}\sum_{\eta=\pm1}\sum_{\xi=\pm1}\int \mathrm{d}^4k \int\limits_0^\infty \mathrm{d}\alpha \; e^{-\alpha\left(k-i\frac{\eta\tilde{p}}{2\alpha}\right)^2} e^{-\frac{\tilde{p}^2}{4\alpha} - i\xi\alpha a} .
$$

With the substitution $k' = k - i\frac{\eta\tilde{p}}{2\alpha}$ one gets a Gaussian integral which can be solved easily

$$\begin{aligned}
I(p) &= \frac{1}{2}\sum_{\eta=\pm1}\sum_{\xi=\pm1}\int\limits_0^\infty \mathrm{d}\alpha \int \mathrm{d}^4k' e^{-\alpha k'^2} e^{-\frac{\tilde{p}^2}{4\alpha}-i\xi\alpha a} \\
&= \pi^2 \sum_{\xi=\pm1}\int\limits_0^\infty \frac{\mathrm{d}\alpha}{\alpha^2} e^{-\frac{\tilde{p}^2}{4\alpha}-i\xi\alpha a}\,.
\end{aligned}$$

Summing over η gives a factor 2 in the equation above,

$$\begin{aligned}
I(p) &= \pi^2 \int\limits_0^\infty \frac{\mathrm{d}\alpha}{\alpha^2} e^{-\frac{\tilde{p}^2}{4\alpha}} \left(e^{i\alpha a} + e^{-i\alpha a}\right) \\
&= 2\pi^2 \int\limits_0^\infty \frac{\mathrm{d}\alpha}{\alpha^2} \cos(\alpha a)\, e^{-\frac{\tilde{p}^2}{4\alpha}}\,.
\end{aligned} \tag{B.2}$$

The expansion for small arguments of the modified Bessel function K_1 gives

$$\begin{aligned}
I(p) = \frac{4\pi^2}{\tilde{p}}\sqrt{a}\Bigg[&\frac{1+i}{\sqrt{2}}\Bigg(\frac{1}{(1+i)\tilde{p}}\left(\frac{2}{a}\right)^{\frac{1}{2}} \\
&+ \frac{(1+i)\tilde{p}}{2}\left(\frac{a}{2}\right)^{\frac{1}{2}}\left(\ln\frac{(1+i)\tilde{p}}{2}\left(\frac{a}{2}\right)^{\frac{1}{2}} + \gamma - \frac{1}{2}\right) \\
&+ \frac{(1+i)^3\tilde{p}^3}{16}\left(\frac{a}{2}\right)^{\frac{3}{2}}\left(\ln\frac{(1+i)\tilde{p}}{2}\left(\frac{a}{2}\right)^{\frac{1}{2}} + \gamma - \frac{5}{4}\right)\Bigg) \\
&+ \frac{1-i}{\sqrt{2}}\Bigg(\frac{1}{(1-i)\tilde{p}}\left(\frac{2}{a}\right)^{\frac{1}{2}} + \frac{(1-i)\tilde{p}}{2}\left(\frac{a}{2}\right)^{\frac{1}{2}}\left(\ln\frac{(1-i)\tilde{p}}{2}\left(\frac{a}{2}\right)^{\frac{1}{2}} + \gamma - \frac{1}{2}\right) \\
&+ \frac{(1-i)^3\tilde{p}^3}{16}\left(\frac{a}{2}\right)^{\frac{3}{2}}\left(\ln\frac{(1-i)\tilde{p}}{2}\left(\frac{a}{2}\right)^{\frac{1}{2}} + \gamma - \frac{5}{4}\right)\Bigg)\Bigg] + \mathcal{O}\left(\tilde{p}^3\right) \\
= \frac{8\pi^2}{\tilde{p}^2} + \pi^2 a\Bigg[&(1+i)^2\left(\ln\frac{(1+i)\tilde{p}}{2}\left(\frac{a}{2}\right)^{\frac{1}{2}} + \gamma - \frac{1}{2}\right) \\
&+ (1-i)^2\left(\ln\frac{(1-i)\tilde{p}}{2}\left(\frac{a}{2}\right)^{\frac{1}{2}} + \gamma - \frac{1}{2}\right)\Bigg] \\
+ \frac{\pi^2 a^2\tilde{p}^2}{16}\Bigg[&(1+i)^4\left(\ln\frac{(1+i)\tilde{p}}{2}\left(\frac{a}{2}\right)^{\frac{1}{2}} + \gamma - \frac{5}{4}\right) \\
&+ (1-i)^4\left(\ln\frac{(1-i)\tilde{p}}{2}\left(\frac{a}{2}\right)^{\frac{1}{2}} + \gamma - \frac{5}{4}\right)\Bigg] + \mathcal{O}\left(\tilde{p}^3\right)
\end{aligned}$$

$$\begin{aligned}
I(p) &= \frac{8\pi^2}{\tilde{p}^2} + 2\mathrm{i}\pi^2 a \left(\ln \frac{(1+i)\tilde{p}}{2} \left(\frac{a}{2}\right)^{\frac{1}{2}} - \ln \frac{(1-i)\tilde{p}}{2} \left(\frac{a}{2}\right)^{\frac{1}{2}} \right) \\
&\quad - \frac{\pi^2 a^2 \tilde{p}^2}{2} \left(\ln \frac{(1+i)\tilde{p}}{2} \left(\frac{a}{2}\right)^{\frac{1}{2}} + \ln \frac{(1-i)\tilde{p}}{2} \left(\frac{a}{2}\right)^{\frac{1}{2}} + 2\gamma - \frac{5}{2} \right) + \mathcal{O}\left(\tilde{p}^3\right) \\
&= \frac{8\pi^2}{\tilde{p}^2} + 2\mathrm{i}\pi^2 a \ln i - \frac{\pi^2 a^2 \tilde{p}^2}{2} \left(\ln \frac{\tilde{p}^2 a}{4} + 2\gamma - \frac{5}{2} \right) + \mathcal{O}\left(\tilde{p}^3\right) \\
&= \frac{8\pi^2}{\tilde{p}^2} - \pi^3 a - \frac{\pi^2 a^2 \tilde{p}^2}{2} \left(\ln \frac{\tilde{p}^2 a}{4} + 2\gamma - \frac{5}{2} \right) + \mathcal{O}\left(\tilde{p}^3\right) \; . \qquad \text{(B.3)}
\end{aligned}$$

B.2 Correction for the One Loop Vertex

B.2.1 Multiplication of the Two Brackets

Multiplication of the two brackets of the vertices in the integral:

$$\begin{aligned}
&\left(\cos\tfrac{p_1\times p_2}{2} \cos\tfrac{k\times(p_1+p_2)}{2} + \cos\tfrac{p_1\times(k-p_2)}{2} \cos\tfrac{p_2\times k}{2} + \cos\tfrac{p_1\times k}{2} \cos\tfrac{p_2\times(k-p_1)}{2} \right) \\
&\left(\cos\tfrac{k\times(p_3+p_4)}{2} \cos\tfrac{p_3\times p_4}{2} + \cos\tfrac{(p_4+k)\times p_3}{2} \cos\tfrac{k\times p_4}{2} + \cos\tfrac{(p_3+k)\times p_4}{2} \cos\tfrac{k\times p_3}{2} \right) \\
&= \cos\frac{p_1\times p_2}{2} \cos^2\frac{k\times(p_1+p_2)}{2} \cos\frac{p_3\times p_4}{2} \\
&\quad + \cos\frac{p_1\times p_2}{2} \cos\frac{k\times(p_1+p_2)}{2} \cos\frac{(p_4+k)\times p_3}{2} \cos\frac{k\times p_4}{2} \\
&\quad + \cos\frac{p_1\times p_2}{2} \cos\frac{k\times(p_1+p_2)}{2} \cos\frac{(p_3+k)\times p_4}{2} \cos\frac{k\times p_3}{2} \\
&\quad + \cos\frac{p_1\times(k-p_2)}{2} \cos\frac{p_2\times k}{2} \cos\frac{k\times(p_3+p_4)}{2} \cos\frac{p_3\times p_4}{2} \\
&\quad + \cos\frac{p_1\times(k-p_2)}{2} \cos\frac{p_2\times k}{2} \cos\frac{(p_4+k)\times p_3}{2} \cos\frac{k\times p_4}{2} \\
&\quad + \cos\frac{p_1\times(k-p_2)}{2} \cos\frac{p_2\times k}{2} \cos\frac{(p_3+k)\times p_4}{2} \cos\frac{k\times p_3}{2} \\
&\quad + \cos\frac{p_1\times k}{2} \cos\frac{p_2\times(k-p_1)}{2} \cos\frac{k\times(p_3+p_4)}{2} \cos\frac{p_3\times p_4}{2} \\
&\quad + \cos\frac{p_1\times k}{2} \cos\frac{p_2\times(k-p_1)}{2} \cos\frac{(p_4+k)\times p_3}{2} \cos\frac{k\times p_4}{2} \\
&\quad + \cos\frac{p_1\times k}{2} \cos\frac{p_2\times(k-p_1)}{2} \cos\frac{(p_3+k)\times p_4}{2} \cos\frac{k\times p_3}{2} \qquad \text{(B.4)}
\end{aligned}$$

$$
\begin{aligned}
(\mathrm{B.4}) = {} & \frac{1}{2}\cos\tfrac{p_1\times p_2}{2}\cos\tfrac{p_3\times p_4}{2}\left[1+\cos(k\times(p_1+p_2))\right] \\
& + \frac{1}{4}\cos\tfrac{p_3\times p_4}{2}\Big[\cos\tfrac{p_1\times(k-p_2)+p_2\times k+k\times(p_1+p_2)}{2}+\cos\tfrac{p_1\times(k-p_2)+p_2\times k-k\times(p_1+p_2)}{2} \\
& \quad + \cos\tfrac{p_1\times(k-p_2)-p_2\times k+k\times(p_1+p_2)}{2}+\cos\tfrac{p_1\times(k-p_2)-p_2\times k-k\times(p_1+p_2)}{2} \\
& \quad + \cos\tfrac{p_1\times k+p_2\times(k-p_1)+k\times(p_1+p_2)}{2}+\cos\tfrac{p_1\times k+p_2\times(k-p_1)-k\times(p_1+p_2)}{2} \\
& \quad + \cos\tfrac{p_1\times k-p_2\times(k-p_1)+k\times(p_1+p_2)}{2}+\cos\tfrac{p_1\times k-p_2\times(k-p_1)-k\times(p_1+p_2)}{2}\Big] \\
& + \frac{1}{4}\cos\tfrac{p_1\times p_2}{2}\Big[\cos\tfrac{-k\times(p_3+p_4)+(p_4+k)\times p_3+k\times p_4}{2}+\cos\tfrac{-k\times(p_3+p_4)+(p_4+k)\times p_3-k\times p_4}{2} \\
& \quad + \cos\tfrac{-k\times(p_3+p_4)-(p_4+k)\times p_3+k\times p_4}{2}+\cos\tfrac{-k\times(p_3+p_4)-(p_4+k)\times p_3-k\times p_4}{2} \\
& \quad + \cos\tfrac{-k\times(p_3+p_4)+(p_3+k)\times p_4+k\times p_3}{2}+\cos\tfrac{-k\times(p_3+p_4)+(p_3+k)\times p_4-k\times p_3}{2} \\
& \quad + \cos\tfrac{-k\times(p_3+p_4)-(p_3+k)\times p_4+k\times p_3}{2}+\cos\tfrac{-k\times(p_3+p_4)-(p_3+k)\times p_4-k\times p_3}{2}\Big] \\
& + \frac{1}{2}\Big[\cos\tfrac{p_1\times(k-p_2)}{2}\cos\tfrac{p_2\times k}{2}+\cos\tfrac{p_2\times(k-p_1)}{2}\cos\tfrac{p_1\times k}{2}\Big] \\
& \quad \Big[\cos\tfrac{(p_4+k)\times p_3+k\times p_4}{2}+\cos\tfrac{(p_4+k)\times p_3-k\times p_4}{2} \\
& \quad + \cos\tfrac{(p_3+k)\times p_4+k\times p_3}{2}+\cos\tfrac{(p_3+k)\times p_4-k\times p_3}{2}\Big]\,. \qquad (\mathrm{B.5})
\end{aligned}
$$

This last term of this expression can be written as

$$
\begin{aligned}
& \frac{1}{4}\Bigg(\cos\frac{-p_3\times p_4+k\times(p_3+p_4)}{2}+\cos\frac{-p_3\times p_4+k\times(p_3-p_4)}{2} \\
& \quad + \cos\frac{p_3\times p_4+k\times(p_3+p_4)}{2}+\cos\frac{p_3\times p_4-k\times(p_3-p_4)}{2}\Bigg) \\
& \quad \Bigg(\cos\frac{p_2\times k-p_1\times(k-p_2)}{2}+\cos\frac{p_2\times k+p_1\times(k-p_2)}{2} \\
& \quad + \cos\frac{p_1\times k-p_2\times(k-p_1)}{2}+\cos\frac{p_1\times k+p_2\times(k-p_1)}{2}\Bigg) \\
& = \frac{1}{2}\left(\cos\frac{p_3\times p_4-k\times(p_3-p_4)}{2}+\cos\frac{p_3\times p_4}{2}\cos\frac{k\times(p_3+p_4)}{2}\right) \\
& \quad \Bigg(\cos\frac{p_1\times p_2+k\times(p_1-p_2)}{2}+\cos\frac{-p_1\times p_2-k\times(p_1+p_2)}{2} \\
& \quad + \cos\frac{-p_1\times p_2-k\times(p_1-p_2)}{2}+\cos\frac{p_1\times p_2-k\times(p_1+p_2)}{2}\Bigg) \\
& = \left(\cos\frac{p_1\times p_2+k\times(p_1-p_2)}{2}+\cos\frac{p_1\times p_2}{2}\cos\frac{k\times(p_1+p_2)}{2}\right) \\
& \quad \times\left(\cos\frac{p_3\times p_4-k\times(p_3-p_4)}{2}+\cos\frac{p_3\times p_4}{2}\cos\frac{k\times(p_3+p_4)}{2}\right)\,. \qquad (\mathrm{B.6})
\end{aligned}
$$

From this result follows for the whole multiplication

$$
\begin{aligned}
(\text{B.5}) = {} & \frac{1}{2}\cos\frac{p_1\times p_2}{2}\cos\frac{p_3\times p_4}{2}\left[1+\cos(k\times(p_1+p_2))\right] \\
& + \frac{1}{4}\cos\frac{p_3\times p_4}{2}\Bigg[2\cos\frac{p_1\times p_2}{2}+\cos\left(\frac{p_1\times p_2}{2}-k\times(p_1+p_2)\right) \\
& + \cos\left(\frac{p_1\times p_2}{2}+k\times(p_1+p_2)\right)+2\cos\left(k\times p_2-\frac{p_1\times p_2}{2}\right) \\
& + 2\cos\left(-k\times p_1-\frac{p_1\times p_2}{2}\right)\Bigg] \\
& + \frac{1}{4}\cos\frac{p_1\times p_2}{2}\Bigg[2\cos\frac{p_3\times p_4}{2}+\cos\left(\frac{p_3\times p_4}{2}-k\times(p_3+p_4)\right) \\
& + \cos\left(\frac{p_3\times p_4}{2}+k\times(p_3+p_4)\right)+2\cos\left(k\times p_3-\frac{p_3\times p_4}{2}\right) \\
& + 2\cos\left(-k\times p_4-\frac{p_3\times p_4}{2}\right)\Bigg] \\
& + \left(\cos\frac{p_1\times p_2+k\times(p_1-p_2)}{2}+\cos\frac{p_1\times p_2}{2}\cos\frac{k\times(p_1+p_2)}{2}\right) \\
& \times\left(\cos\frac{p_3\times p_4-k\times(p_3-p_4)}{2}+\cos\frac{p_3\times p_4}{2}\cos\frac{k\times(p_3+p_4)}{2}\right)\,. \qquad \text{(B.7)}
\end{aligned}
$$

Remembering that $p_1+p_2=-p_3-p_4$ this can be written as

$$
\begin{aligned}
= {} & \frac{1}{2}\cos\frac{p_1\times p_2}{2}\cos\frac{p_3\times p_4}{2}\left[1+\cos(k\times(p_1+p_2))\right] \\
& + \frac{1}{2}\cos\frac{p_3\times p_4}{2}\Bigg[\cos\frac{p_1\times p_2}{2}+\frac{1}{2}\sum_{\pm}\cos\left(\frac{p_1\times p_2}{2}\pm k\times(p_1+p_2)\right) \\
& + \cos\left(k\times p_2-\frac{p_1\times p_2}{2}\right)+\cos\left(-k\times p_1-\frac{p_1\times p_2}{2}\right)\Bigg] \\
& + \frac{1}{2}\cos\frac{p_1\times p_2}{2}\Bigg[\cos\frac{p_3\times p_4}{2}+\frac{1}{2}\sum_{\pm}\cos\left(\frac{p_3\times p_4}{2}\pm k\times(p_3+p_4)\right) \\
& + \cos\left(k\times p_3-\frac{p_3\times p_4}{2}\right)+\cos\left(-k\times p_4-\frac{p_3\times p_4}{2}\right)\Bigg] \\
& + \cos\frac{k\times(p_1+p_2)}{2}\cos\frac{k\times(p_3+p_4)}{2}\cos\frac{p_1\times p_2}{2}\cos\frac{p_3\times p_4}{2} \\
& + \cos\frac{p_1\times p_2}{2}\cos\frac{k\times(p_1+p_2)}{2}\cos\frac{p_3\times p_4-k\times(p_3-p_4)}{2} \\
& + \cos\frac{p_3\times p_4}{2}\cos\frac{k\times(p_3+p_4)}{2}\cos\frac{p_1\times p_2+k\times(p_1-p_2)}{2} \\
& + \cos\frac{p_1\times p_2+k\times(p_1-p_2)}{2}\cos\frac{p_3\times p_4-k\times(p_3-p_4)}{2}\,. \qquad \text{(B.8)}
\end{aligned}
$$

With $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ and $\cos a \cos b = \frac{1}{2}\cos(a+b) + \cos(a-b)$ this gives

$$\begin{aligned}
\text{(B.8)} &= \cos\tfrac{p_1\times p_2}{2}\cos\tfrac{p_3\times p_4}{2}\left[1+\cos(k\times(p_1+p_2))\right] \\
&+ \frac{1}{2}\cos\tfrac{p_3\times p_4}{2}\Big[\cos\tfrac{p_1\times p_2}{2} + \cos\left(k\times p_2 - \tfrac{p_1\times p_2}{2}\right) + \cos\tfrac{p_1\times p_2}{2}\cos(k\times(p_1+p_2)) \\
&+ \cos\left(-k\times p_1 - \tfrac{p_1\times p_2}{2}\right) + 2\cos\tfrac{k\times(p_3+p_4)}{2}\cos\tfrac{p_1\times p_2 + k\times(p_1-p_2)}{2}\Big] \\
&+ \frac{1}{2}\cos\tfrac{p_1\times p_2}{2}\Big[\cos\tfrac{p_3\times p_4}{2} + \cos\left(k\times p_3 - \tfrac{p_3\times p_4}{2}\right) + \cos\left(-k\times p_4 - \tfrac{p_3\times p_4}{2}\right) \\
&+ \cos\tfrac{p_3\times p_4}{2}\cos(k\times(p_3+p_4)) + 2\cos\tfrac{k\times(p_1+p_2)}{2}\cos\tfrac{p_3\times p_4 - k\times(p_3-p_4)}{2}\Big] \\
&+ \frac{1}{2}\left[\cos\tfrac{p_1\times p_2 - p_3\times p_4 + k\times(p_1-p_2+p_3-p_4)}{2} + \cos\tfrac{p_1\times p_2 + p_3\times p_4 + k\times(p_1-p_2-p_3+p_4)}{2}\right] \\
&= 2\cos\tfrac{p_1\times p_2}{2}\cos\tfrac{p_3\times p_4}{2}\left[1+\cos(k\times(p_1+p_2))\right] \\
&+ \frac{1}{2}\cos\tfrac{p_3\times p_4}{2}\Big[\cos\left(k\times p_2 - \tfrac{p_1\times p_2}{2}\right) + \cos\left(k\times p_1 + \tfrac{p_1\times p_2}{2}\right) \\
&+ \cos\tfrac{p_1\times p_2 + k\times(p_1-p_2-p_3-p_4)}{2} + \cos\tfrac{p_1\times p_2 + k\times(p_1-p_2+p_3+p_4)}{2}\Big] \\
&+ \frac{1}{2}\cos\tfrac{p_1\times p_2}{2}\Big[\cos\left(k\times p_4 + \tfrac{p_3\times p_4}{2}\right) + \cos\left(-k\times p_3 + \tfrac{p_3\times p_4}{2}\right) \\
&+ \cos\tfrac{p_3\times p_4 + k\times(-p_3+p_4-p_1-p_2)}{2} + \cos\tfrac{p_3\times p_4 + k\times(-p_3+p_4+p_1+p_2)}{2}\Big] \\
&+ \frac{1}{2}\left[\cos\left(\tfrac{p_1\times p_2 - p_3\times p_4}{2} + k\times(p_1+p_3)\right) + \cos\left(\tfrac{p_1\times p_2 + p_3\times p_4}{2} + k\times(p_1+p_4)\right)\right] \\
&= 2\cos\tfrac{p_1\times p_2}{2}\cos\tfrac{p_3\times p_4}{2}\left[1+\cos(k\times(p_1+p_2))\right] \\
&+ \cos\tfrac{p_1\times p_2}{2}\left[\cos\left(\tfrac{p_3\times p_4}{2} - k\times p_3\right) + \cos\left(\tfrac{p_3\times p_4}{2} + k\times p_4\right)\right] \\
&+ \cos\tfrac{p_3\times p_4}{2}\left[\cos\left(\tfrac{p_1\times p_2}{2} + k\times p_1\right) + \cos\left(\tfrac{p_1\times p_2}{2} - k\times p_2\right)\right] \\
&+ \frac{1}{2}\left[\cos\left(\tfrac{p_1\times p_2 - p_3\times p_4}{2} + k\times(p_1+p_3)\right) + \cos\left(\tfrac{p_1\times p_2 + p_3\times p_4}{2} + k\times(p_1+p_4)\right)\right] .
\end{aligned} \tag{B.9}$$

To find a simpler form of this result it is written in its exponential form,

$$
\begin{aligned}
(\mathrm{B.9}) = {}& \frac{1}{4}\left(\mathrm{e}^{\frac{\mathrm{i}}{2}(p_1\times p_2+p_3\times p_4)}+\mathrm{e}^{-\frac{\mathrm{i}}{2}(p_1\times p_2+p_3\times p_4)}+\mathrm{e}^{\frac{\mathrm{i}}{2}(p_1\times p_2-p_3\times p_4)}+\mathrm{e}^{-\frac{\mathrm{i}}{2}(p_1\times p_2-p_3\times p_4)}\right)\\
&\times\left(2+\mathrm{e}^{\mathrm{i}k\times(p_1+p_2)}+\mathrm{e}^{-\mathrm{i}k\times(p_1+p_2)}\right)+\frac{1}{4}\left(\mathrm{e}^{\frac{\mathrm{i}}{2}p_1\times p_2}+\mathrm{e}^{-\frac{\mathrm{i}}{2}p_1\times p_2}\right)\\
&\times\left(\mathrm{e}^{\mathrm{i}\left(\frac{p_3\times p_4}{2}-k\times p_3\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_3\times p_4}{2}-k\times p_3\right)}+\mathrm{e}^{\mathrm{i}\left(\frac{p_3\times p_4}{2}-k\times p_4\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_3\times p_4}{2}-k\times p_4\right)}\right)\\
&+\frac{1}{4}\left(\mathrm{e}^{\frac{\mathrm{i}}{2}p_3\times p_4}+\mathrm{e}^{-\frac{\mathrm{i}}{2}p_3\times p_4}\right)\\
&\times\left(\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2}{2}+k\times p_1\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2}{2}+k\times p_1\right)}+\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2}{2}-k\times p_2\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2}{2}-k\times p_2\right)}\right)\\
&+\frac{1}{4}\Big(\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2-p_3\times p_4}{2}+k\times(p_1+p_3)\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2-p_3\times p_4}{2}+k\times(p_1+p_3)\right)}\\
&+\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2+p_3\times p_4}{2}+k\times(p_1+p_4)\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2+p_3\times p_4}{2}+k\times(p_1+p_4)\right)}\Big)\\
={}&\frac{1}{2}\left(\mathrm{e}^{\frac{\mathrm{i}}{2}(p_1\times p_2+p_3\times p_4)}+\mathrm{e}^{-\frac{\mathrm{i}}{2}(p_1\times p_2+p_3\times p_4)}+\mathrm{e}^{\frac{\mathrm{i}}{2}(p_1\times p_2-p_3\times p_4)}+\mathrm{e}^{-\frac{\mathrm{i}}{2}(p_1\times p_2-p_3\times p_4)}\right)\\
&+\frac{1}{4}\Big(\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2+p_3\times p_4}{2}+k\times(p_1+p_2)\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2+p_3\times p_4}{2}+k\times(p_1+p_2)\right)}\\
&\quad+\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2-p_3\times p_4}{2}+k\times(p_1+p_2)\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2-p_3\times p_4}{2}+k\times(p_1+p_2)\right)}\Big)\\
&+\frac{1}{4}\Big(\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2+p_3\times p_4}{2}-k\times(p_1+p_2)\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2+p_3\times p_4}{2}-k\times(p_1+p_2)\right)}\\
&\quad+\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2-p_3\times p_4}{2}-k\times(p_1+p_2)\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2-p_3\times p_4}{2}-k\times(p_1+p_2)\right)}\Big)\\
&+\frac{1}{4}\Big(\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2+p_3\times p_4}{2}+k\times p_1\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2-p_3\times p_4}{2}+k\times p_1\right)}\\
&\quad+\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2+p_3\times p_4}{2}-k\times p_2\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2-p_3\times p_4}{2}-k\times p_2\right)}\Big)\\
&+\frac{1}{4}\Big(\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2-p_3\times p_4}{2}+k\times p_1\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2+p_3\times p_4}{2}+k\times p_1\right)}\\
&\quad+\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2-p_3\times p_4}{2}-k\times p_2\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2+p_3\times p_4}{2}-k\times p_2\right)}\Big)\\
&+\frac{1}{4}\Big(\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2+p_3\times p_4}{2}-k\times p_3\right)}+\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2-p_3\times p_4}{2}+k\times p_3\right)}\\
&\quad+\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2+p_3\times p_4}{2}-k\times p_4\right)}+\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2-p_3\times p_4}{2}+k\times p_4\right)}\Big)\\
&+\frac{1}{4}\Big(\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2-p_3\times p_4}{2}+k\times p_3\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2+p_3\times p_4}{2}-k\times p_3\right)}\\
&\quad+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2-p_3\times p_4}{2}+k\times p_4\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2+p_3\times p_4}{2}-k\times p_4\right)}\Big)\\
&+\frac{1}{4}\Big(\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2-p_3\times p_4}{2}+k\times(p_1+p_3)\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2-p_3\times p_4}{2}+k\times(p_1+p_3)\right)}\\
&\quad+\mathrm{e}^{\mathrm{i}\left(\frac{p_1\times p_2+p_3\times p_4}{2}+k\times(p_1+p_4)\right)}+\mathrm{e}^{-\mathrm{i}\left(\frac{p_1\times p_2+p_3\times p_4}{2}+k\times(p_1+p_4)\right)}\Big)
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
(\mathrm{B}.10) = {} & \tfrac{1}{4}\mathrm{e}^{\mathrm{i}\frac{p_1\times p_2+p_3\times p_4}{2}}\big(2+\mathrm{e}^{\mathrm{i}k\times(p_1+p_2)}+\mathrm{e}^{-\mathrm{i}k\times(p_1+p_2)}+\mathrm{e}^{\mathrm{i}k\times p_1}+\mathrm{e}^{-\mathrm{i}k\times p_2}+\mathrm{e}^{-\mathrm{i}k\times p_3}\\
&+\mathrm{e}^{-\mathrm{i}k\times p_4}+\mathrm{e}^{\mathrm{i}k\times(p_1+p_4)}\big)+\tfrac{1}{4}\mathrm{e}^{-\mathrm{i}\frac{p_1\times p_2+p_3\times p_4}{2}}\big(2+\mathrm{e}^{-\mathrm{i}k\times(p_1+p_2)}+\mathrm{e}^{\mathrm{i}k\times(p_1+p_2)}\\
&+\mathrm{e}^{-\mathrm{i}k\times p_1}+\mathrm{e}^{\mathrm{i}k\times p_2}+\mathrm{e}^{\mathrm{i}k\times p_3}+\mathrm{e}^{\mathrm{i}k\times p_4}+\mathrm{e}^{-\mathrm{i}k\times(p_1+p_4)}\big)\\
&+\tfrac{1}{4}\mathrm{e}^{\mathrm{i}\frac{p_1\times p_2-p_3\times p_4}{2}}\big(2+\mathrm{e}^{\mathrm{i}k\times(p_1+p_2)}+\mathrm{e}^{-\mathrm{i}k\times(p_1+p_2)}+\mathrm{e}^{\mathrm{i}k\times p_1}+\mathrm{e}^{-\mathrm{i}k\times p_2}+\mathrm{e}^{\mathrm{i}k\times p_3}\\
&+\mathrm{e}^{\mathrm{i}k\times p_4}+\mathrm{e}^{\mathrm{i}k\times(p_1+p_3)}\big)+\tfrac{1}{4}\mathrm{e}^{-\mathrm{i}\frac{p_1\times p_2-p_3\times p_4}{2}}\big(2+\mathrm{e}^{-\mathrm{i}k\times(p_1+p_2)}+\mathrm{e}^{\mathrm{i}k\times(p_1+p_2)}\\
&+\mathrm{e}^{-\mathrm{i}k\times p_1}+\mathrm{e}^{\mathrm{i}k\times p_2}+\mathrm{e}^{-\mathrm{i}k\times p_3}+\mathrm{e}^{-\mathrm{i}k\times p_4}+\mathrm{e}^{-\mathrm{i}k\times(p_1+p_3)}\big)\,. \qquad (\mathrm{B}.11)
\end{aligned}
$$

The interesting result of the graphs is the one for small external momenta meaning $p_i \to 0$. For this case terms with $p_i \times p_j$ will be neglected and this means that it is possible to set the overall exponential of the four terms to one and sum over them. The result is

$$
\begin{aligned}
&2+\frac{1}{4}\sum_{i=2}^{4}\mathrm{e}^{\mathrm{i}k\times(p_1+p_i)}+\frac{1}{4}\sum_{i=2}^{4}\mathrm{e}^{-\mathrm{i}k\times(p_1+p_i)}+\frac{1}{2}\sum_{i=1}^{4}\mathrm{e}^{\mathrm{i}k\times p_i}+\\
&\qquad\qquad\frac{1}{2}\sum_{i=1}^{4}\mathrm{e}^{-\mathrm{i}k\times p_i}+\frac{3}{4}\mathrm{e}^{\mathrm{i}k\times(p_1+p_2)}+\frac{3}{4}\mathrm{e}^{-\mathrm{i}k\times(p_1+p_2)}\,,
\end{aligned}
$$

and as the result of the vertex does not depend on the sign of the exponential one can simply write

$$
2+\frac{1}{2}\sum_{i=2}^{4}\mathrm{e}^{\mathrm{i}k\times(p_1+p_i)}+\sum_{i=1}^{4}\mathrm{e}^{\mathrm{i}k\times p_i}+\frac{3}{2}\mathrm{e}^{\mathrm{i}k\times(p_1+p_2)}\,. \qquad (\mathrm{B}.12)
$$

B.2.2 Approximation for Small Momenta

The integral to solve is Eqn. (5.36) with the factors $\left(1+\zeta\frac{m^2}{2M^2}\right)$ and $\left(1+\chi\frac{m^2}{2M^2}\right)$. That is

$$
\begin{aligned}
&-\frac{\left(1+\zeta\frac{m^2}{2M^2}\right)\left(1+\chi\frac{m^2}{2M^2}\right)}{(4\pi)^2}\int_0^1 \mathrm{d}\xi\ \mathrm{e}^{\mathrm{i}(1-\xi)p\tilde{q}}\\
&\quad\times\left[\ln\left[\frac{(\tilde{p}+\tilde{q})^2}{4}\left(\frac{m^2}{2}+M^2(\chi+(\zeta-\chi)\xi)+\xi(1-\xi)p^2\right)\right]+2\gamma_E\right]\,. \qquad (\mathrm{B}.13)
\end{aligned}
$$

With the assumption of small momenta p one can solve this integral when the exponential is approximated with 1 and the term with the p^2 is neglected as it is considered to be small in comparison with the others. Performing the sum over

χ and ζ finally gives

$$
\begin{aligned}
&-\frac{1}{(4\pi)^2}\int\limits_0^1 \mathrm{d}\xi\, \Bigg[\left(1+\frac{m^2}{2M^2}\right)^2 \left(\ln\left[\frac{(\tilde{p}+\tilde{q})^2}{4}\left(\frac{m^2}{2}+M^2\right)\right]+2\gamma_E\right)\\
&+\left(1-\frac{m^2}{2M^2}\right)^2 \left(\ln\left[\frac{(\tilde{p}+\tilde{q})^2}{4}\left(\frac{m^2}{2}-M^2\right)\right]+2\gamma_E\right)\\
&+\left(1-\frac{m^4}{4M^4}\right)\left(\ln\left[\frac{(\tilde{p}+\tilde{q})^2}{4}\left(\frac{m^2}{2}+(1-2\xi)\,M^2\right)\right]+2\gamma_E\right)\\
&+\left(1-\frac{m^4}{4M^4}\right)\left(\ln\left[\frac{(\tilde{p}+\tilde{q})^2}{4}\left(\frac{m^2}{2}-(1-2\xi)\,M^2\right)\right]+2\gamma_E\right)\Bigg]\\
&=-\frac{1}{(4\pi)^2}\int\limits_0^1 \mathrm{d}\xi\, \Bigg[\left(1+\frac{m^4}{4M^4}\right)\left(\ln\left[\frac{(\tilde{p}+\tilde{q})^2}{4}\left(\frac{m^2}{2}+M^2\right)\right]\right.\\
&\left.+\ln\left[\frac{(\tilde{p}+\tilde{q})^2}{4}\left(\frac{m^2}{2}-M^2\right)\right]+4\gamma_E\right)-\frac{m^2}{M^2}\ln\left(\frac{m^2-2M^2}{m^2+2M^2}\right)\\
&+\left(1-\frac{m^4}{4M^4}\right)\left(\ln\left[\frac{(\tilde{p}+\tilde{q})^2}{4}\left(\frac{m^2}{2}+(1-2\xi)\,M^2\right)\right]\right.\\
&\left.+\ln\left[\frac{(\tilde{p}+\tilde{q})^2}{4}\left(\frac{m^2}{2}-(1-2\xi)\,M^2\right)\right]+4\gamma_E\right)\Bigg]\\
&=-\frac{1}{(4\pi)^2}\int\limits_0^1 \mathrm{d}\xi\, \Bigg[\left(1+\frac{m^4}{4M^4}\right)\left(\ln\left[\frac{(\tilde{p}+\tilde{q})^4}{16}\left(\frac{m^4}{4}-M^4\right)\right]+4\gamma_E\right)\\
&-\frac{m^2}{M^2}\ln\left(\frac{m^2-2M^2}{m^2+2M^2}\right)+\left(1-\frac{m^4}{4M^4}\right)\left(\ln\left[\frac{(\tilde{p}+\tilde{q})^2}{4}\left(\frac{m^2}{2}+(1-2\xi)\,M^2\right)\right]\right.\\
&\left.+\ln\left[\frac{(\tilde{p}+\tilde{q})^2}{4}\left(\frac{m^2}{2}-(1-2\xi)\,M^2\right)\right]+4\gamma_E\right)\Bigg]. \qquad \text{(B.14)}
\end{aligned}
$$

The first two terms of the integral don't contain ξ and are therefore easy to solve. For the third one the formula [21]

$$\int\limits_0^1 \mathrm{d}x\;\ln(a+bx)=\frac{a+b}{b}\ln(a+b)-\frac{a}{b}\ln a-1\,, \qquad \text{(B.15)}$$

is used with $a = \frac{m^2}{2} \mp M^2$ and $b = \pm 2M^2$ and this leads to

$$
\begin{aligned}
&\int_0^1 \mathrm{d}\xi \left[\ln\left[\tfrac{(\tilde{p}+\tilde{q})^2}{4}\left(\tfrac{m^2}{2} + (1-2\xi)\, M^2\right)\right] + \ln\left[\tfrac{(\tilde{p}+\tilde{q})^2}{4}\left(\tfrac{m^2}{2} - (1-2\xi)\, M^2\right)\right]\right] \\
&= \int_0^1 \mathrm{d}\xi \left[\ln\frac{(\tilde{p}+\tilde{q})^4}{16} + \ln\left(\frac{m^2}{2} + M^2 - 2\xi M^2\right) + \ln\left(\frac{m^2}{2} - M^2 + 2\xi M^2\right)\right] \\
&= \ln\frac{(\tilde{p}+\tilde{q})^4}{16} + \frac{\frac{m^2}{2} - M^2 + 2M^2}{2M^2}\ln\left(\frac{m^2}{2} + M^2\right) - \frac{\frac{m^2}{2} - M^2}{2M^2}\ln\left(\frac{m^2}{2} - M^2\right) \\
&\quad - 1 + \frac{\frac{m^2}{2} + M^2 - 2M^2}{-2M^2}\ln\left(\frac{m^2}{2} - M^2\right) - \frac{\frac{m^2}{2} + M^2}{-2M^2}\ln\left(\frac{m^2}{2} + M^2\right) - 1 \\
&= \ln\frac{(\tilde{p}+\tilde{q})^4}{16} + \frac{\left(\frac{m^2}{2} + M^2\right) + \left(\frac{m^2}{2} + M^2\right)}{2M^2}\ln\left(\frac{m^2}{2} + M^2\right) \\
&\quad - \frac{\left(\frac{m^2}{2} - M^2\right) + \left(\frac{m^2}{2} - M^2\right)}{2M^2}\ln\left(\frac{m^2}{2} - M^2\right) - 2 \\
&= \ln\frac{(\tilde{p}+\tilde{q})^4}{16} + \left(\frac{m^2}{2M^2} + 1\right)\ln\left(\frac{m^2}{2} + M^2\right) - \left(\frac{m^2}{2M^2} - 1\right)\ln\left(\frac{m^2}{2} - M^2\right) - 2 \\
&= \ln\left[\frac{(\tilde{p}+\tilde{q})^4}{16}\left(\frac{m^4}{4} - M^4\right)\right] + \frac{m^2}{2M^2}\ln\frac{m^2 + 2M^2}{m^2 - 2M^2} - 2\,. \qquad \text{(B.16)}
\end{aligned}
$$

With this result the complete integral (B.14) gives

$$
\begin{aligned}
&-\frac{2}{(4\pi)^2}\Bigg[\left(1 + \frac{m^4}{4M^4}\right)\ln\frac{(\tilde{p}+\tilde{q})^2}{4}\sqrt{\frac{m^4}{4} - M^4} \\
&\quad + \left(1 - \frac{m^4}{4M^4}\right)\ln\frac{(\tilde{p}+\tilde{q})^2}{4}\sqrt{\frac{m^4}{4} - M^4} + \left(1 - \frac{m^4}{4M^4}\right)\frac{m^2}{2M^2}\ln\sqrt{\frac{m^2 + 2M^2}{m^2 - 2M^2}} \\
&\quad - \left(1 - \frac{m^4}{4M^4}\right) + 4\gamma_E - \frac{m^2}{M^2}\ln\sqrt{\frac{m^2 - 2M^2}{m^2 + 2M^2}}\Bigg] \\
&= \frac{1}{(2\pi)^2}\Bigg(\ln\frac{(\tilde{p}+\tilde{q})^2}{4}\sqrt{\frac{m^4}{4} - M^4} - \frac{1}{2}\left(1 - \frac{m^4}{4M^4}\right) + 2\gamma_E \\
&\qquad\qquad - \frac{m^2}{4M^2}\left(3 - \frac{m^4}{4M^4}\right)\ln\sqrt{\frac{m^2 - 2M^2}{m^2 + 2M^2}}\Bigg)\,. \qquad \text{(B.17)}
\end{aligned}
$$

B.3 Correction for Higher Loop Orders

B.3.1 Integrals

The integral formulas for theses integrals can be found in [21].

$$\int_0^1 \mathrm{d}\xi_1 \cosh\left(\lambda\xi_1\xi_2 M^2\right)\cosh\left(\lambda\xi_1\xi_2 M^2 - \lambda\xi_2 M^2\right)$$
$$= \left[\frac{\xi_1}{2}\cosh\left(\lambda\xi_2 M^2\right) + \frac{1}{4\lambda\xi_2 M^2}\sinh\left(2\lambda\xi_1\xi_2 M^2 - \lambda\xi_2 M^2\right)\right]\Bigg|_0^1$$
$$= \frac{1}{2}\cosh\left(\lambda\xi_2 M^2\right) + \frac{1}{2\lambda\xi_2 M^2}\sinh\left(\lambda\xi_2 M^2\right) \ , \tag{B.18}$$

$$\int_0^1 \mathrm{d}\xi_1 \sinh\left(\lambda\xi_1\xi_2 M^2\right)\cosh\left(\lambda\xi_1\xi_2 M^2 - \lambda\xi_2 M^2\right)$$
$$= \left[\frac{\xi_1}{2}\sinh\left(\lambda\xi_2 M^2\right) + \frac{1}{4\lambda\xi_2 M^2}\cosh\left(2\lambda\xi_1\xi_2 M^2 - \lambda\xi_2 M^2\right)\right]\Bigg|_0^1$$
$$= \frac{1}{2}\sinh\left(\lambda\xi_2 M^2\right) \ , \tag{B.19}$$

$$-\int_0^1 \mathrm{d}\xi_1 \cosh\left(\lambda\xi_1\xi_2 M^2\right)\sinh\left(\lambda\xi_1\xi_2 M^2 - \lambda\xi_2 M^2\right)$$
$$= \left[\frac{\xi_1}{2}\sinh\left(\lambda\xi_2 M^2\right) - \frac{1}{4\lambda\xi_2 M^2}\cosh\left(2\lambda\xi_1\xi_2 M^2 - \lambda\xi_2 M^2\right)\right]\Bigg|_0^1$$
$$= \frac{1}{2}\sinh\left(\lambda\xi_2 M^2\right) \ , \tag{B.20}$$

$$-\int_0^1 \mathrm{d}\xi_1 \sinh\left(\lambda\xi_1\xi_2 M^2\right)\sinh\left(\lambda\xi_1\xi_2 M^2 - \lambda\xi_2 M^2\right)$$
$$= \left[\frac{\xi_1}{2}\cosh\left(\lambda\xi_2 M^2\right) - \frac{1}{4\lambda\xi_2 M^2}\sinh\left(2\lambda\xi_1\xi_2 M^2 - \lambda\xi_2 M^2\right)\right]\Bigg|_0^1$$
$$= \frac{1}{2}\cosh\left(\lambda\xi_2 M^2\right) - \frac{1}{2\lambda\xi_2 M^2}\sinh\left(\lambda\xi_2 M^2\right) \ . \tag{B.21}$$

Integral formulas for the integration over ξ_2 (from [21]) are

$$\int_0^1 \mathrm{d}x\ \mathrm{e}^{ax}\sinh(bx) = \frac{\mathrm{e}^{ax}}{a^2-b^2}\left[a\sinh(bx) - b\cosh(bx)\right]\Big|_0^1$$
$$= \frac{\mathrm{e}^{a}}{a^2-b^2}\left[a\sinh b - b\cosh b\right] + \frac{b}{a^2-b^2}\,, \tag{B.22}$$

$$\int_0^1 \mathrm{d}x\ x\mathrm{e}^{ax}\sinh(bx) = \tfrac{\mathrm{e}^{ax}}{a^2-b^2}\left[\left(ax - \tfrac{a^2+b^2}{a^2-b^2}\right)\sinh(bx) - \left(bx - \tfrac{2ab}{a^2-b^2}\right)\cosh(bx)\right]\Big|_0^1$$
$$= \tfrac{\mathrm{e}^{a}}{a^2-b^2}\left[\left(a - \tfrac{a^2+b^2}{a^2-b^2}\right)\sinh b - \left(b - \tfrac{2ab}{a^2-b^2}\right)\cosh b\right] - \tfrac{2ab}{(a^2-b^2)^2}\,, \tag{B.23}$$

$$\int_0^1 \mathrm{d}x\ x\mathrm{e}^{ax}\cosh(bx) = \tfrac{\mathrm{e}^{ax}}{a^2-b^2}\left[\left(ax - \tfrac{a^2+b^2}{a^2-b^2}\right)\cosh(bx) - \left(bx - \tfrac{2ab}{a^2-b^2}\right)\sinh(bx)\right]\Big|_0^1$$
$$= \tfrac{\mathrm{e}^{a}}{a^2-b^2}\left[\left(a - \tfrac{a^2+b^2}{a^2-b^2}\right)\cosh b - \left(b - \tfrac{2ab}{a^2-b^2}\right)\sinh b\right] + \tfrac{a^2+b^2}{(a^2-b^2)^2}\,. \tag{B.24}$$

Inserting theses three integrals in the calculation gives

$$\frac{1}{2}\int_0^1 \mathrm{d}\xi_2\ \mathrm{e}^{-\lambda\xi_2\frac{m^2}{2}}\Bigg[\xi_2\cosh\left(\lambda\xi_2 M^2\right)\left(1+\frac{m^4}{4M^4}\right)$$
$$+\sinh\left(\lambda\xi_2 M^2\right)\left(\frac{1}{\lambda M^2}\left(1-\frac{m^4}{4M^4}\right)-\frac{\xi_2 m^2}{M^2}\right)\Bigg]$$
$$=\frac{1}{2}\Bigg[\left(1+\frac{m^4}{4M^4}\right)\Bigg[\frac{\mathrm{e}^{-\lambda\frac{m^2}{2}}}{\lambda^2\left(\frac{m^4}{4}-M^4\right)}\Bigg(\left(-\lambda\frac{m^2}{2}-\frac{m^4+4M^4}{m^4-4M^4}\right)\cosh\lambda M^2$$
$$-\left(\lambda M^2+\frac{4m^2M^2}{m^4-4M^4}\right)\sinh\lambda M^2\Bigg)+\frac{\frac{m^4}{4}+M^4}{\lambda^2\left(\frac{m^4}{4}-M^4\right)^2}\Bigg]$$
$$+\frac{1-\frac{m^4}{4M^4}}{M^2}\Bigg[\frac{\mathrm{e}^{-\lambda\frac{m^2}{2}}}{\lambda^2\left(\frac{m^4}{4}-M^4\right)}\left(-\frac{m^2}{2}\sinh\lambda M^2-M^2\cosh\lambda M^2\right)+\frac{M^2}{\lambda^2\left(\frac{m^4}{4}-M^4\right)}\Bigg]$$
$$-\frac{m^2}{M^2}\Bigg[\frac{\mathrm{e}^{-\lambda\frac{m^2}{2}}}{\lambda^2\left(\frac{m^4}{4}-M^4\right)}\Bigg(\left(-\lambda\frac{m^2}{2}-\frac{m^4+4M^4}{m^4-4M^4}\right)\sinh\lambda M^2$$
$$-\left(\lambda M^2+\frac{4m^2M^2}{m^4-4M^4}\right)\cosh\lambda M^2\Bigg)+\frac{m^2M^2}{\lambda^2\left(\frac{m^4}{4}-M^4\right)^2}\Bigg]\Bigg]$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\frac{1}{\lambda^2\left(\frac{m^4}{4}-M^4\right)}\left(\left(1+\frac{m^4}{4M^4}\right)\frac{m^4+4M^4}{m^4-4m^4}+\left(1-\frac{m^4}{4M^4}\right)-\frac{4m^4}{m^4-4M^4}\right)\right.\\
&+\mathrm{e}^{-\lambda\frac{m^2}{2}}\frac{\cosh\lambda M^2}{\lambda^2\left(\frac{m^4}{4}-M^4\right)}\left[\left(1+\frac{m^4}{4M^4}\right)\left(-\lambda\frac{m^2}{2}-\frac{m^4+4M^4}{m^4-4M^4}\right)\right.\\
&\left.-1+\frac{m^4}{4M^4}+\frac{m^2}{M^2}\left(\lambda M^2+\frac{4m^2M^2}{m^4-4M^4}\right)\right]\\
&-\mathrm{e}^{-\lambda\frac{m^2}{2}}\frac{\sinh\lambda M^2}{\lambda^2\left(\frac{m^4}{4}-M^4\right)}\left[\left(1+\frac{m^4}{4M^4}\right)\left(\lambda M^2+\frac{4m^2M^2}{m^4-4M^4}\right)\right.\\
&\left.\left.+\frac{m^2}{2M^2}\left(1-\frac{m^4}{4M^4}\right)-\frac{m^2}{M^2}\left(\lambda\frac{m^2}{2}+\frac{m^4+4M^4}{m^4-4M^4}\right)\right]\right]\\
&= \frac{1}{2}\frac{1}{\lambda^2\left(\frac{m^4}{4}-M^4\right)^2}\left[\frac{m^4}{4}+M^4+\frac{m^8}{16M^4}+\frac{m^4}{4}+\frac{m^4}{4}-M^4-\frac{m^8}{16M^4}+\frac{m^4}{4}-m^4\right.\\
&+\mathrm{e}^{-\lambda\frac{m^2}{2}}\Bigg(\cosh\lambda M^2\Bigg(-\lambda\frac{m^6}{8}+\lambda\frac{m^2M^4}{2}-\frac{m^4}{4}-M^4-\lambda\frac{m^{10}}{32M^4}+\lambda\frac{m^6}{8}\\
&-\frac{m^8}{16M^4}-\frac{m^4}{4}-\frac{m^4}{4}+M^4+\frac{m^8}{16M^4}-\frac{m^4}{4}+\lambda\frac{m^6}{4}-\lambda m^2M^4+m^4\Bigg)\\
&-\sinh\lambda M^2\Bigg(\lambda\frac{m^4M^2}{4}-\lambda M^6+m^2M^2+\lambda\frac{m^8}{16M^2}-\lambda\frac{m^4M^2}{4}+\frac{m^6}{4M^2}\\
&\left.+\frac{m^6}{8M^2}-\frac{m^2M^2}{2}-\frac{m^{10}}{32M^6}+\frac{m^6}{8M^2}-\lambda\frac{m^8}{8M^2}+\lambda\frac{m^4M^2}{2}-\frac{m^6}{4M^2}-m^2M^2\Bigg)\Bigg)\right]\\
&= \frac{1}{2}\frac{\mathrm{e}^{-\lambda\frac{m^2}{2}}}{\lambda^2\left(\frac{m^4}{4}-M^4\right)^2}\left[\lambda\left(\frac{m^6}{4}-\frac{m^2M^4}{2}-\frac{m^{10}}{32M^4}\right)\cosh\lambda M^2\right.\\
&\left.-\left(\lambda\left(-\frac{m^8}{16M^2}-M^6+\frac{m^4M^2}{2}\right)-\frac{m^2M^2}{2}-\frac{m^{10}}{32M^6}+\frac{m^6}{4M^2}\right)\sinh\lambda M^2\right]\\
&= \frac{1}{2}\frac{\mathrm{e}^{-\lambda\frac{m^2}{2}}}{\lambda^2}\left[-\lambda\frac{m^2}{2M^4}\cosh\lambda M^2+\left(\frac{\lambda}{M^2}+\frac{m^2}{2M^6}\right)\sinh\lambda M^2\right]. \qquad \text{(B.25)}
\end{aligned}
$$

B.3.2 Expansion of the Bessel Function

The integration over λ gives

$$
\begin{aligned}
\Pi^{1-ins}(p) = -\frac{\lambda^2}{16\pi^2\theta^2M^6}&\left[M^2m_+^2K_0\left(\sqrt{m_+^2\tilde{p}^2}\right)+M^2m_-^2K_0\left(\sqrt{m_-^2\tilde{p}^2}\right)\right.\\
&\left.+m^2\sqrt{\frac{m_+^2}{\tilde{p}^2}}K_1\left(\sqrt{m_+^2\tilde{p}^2}\right)-m^2\sqrt{\frac{m_-^2}{\tilde{p}^2}}K_1\left(\sqrt{m_-^2\tilde{p}^2}\right)\right], \qquad \text{(B.26)}
\end{aligned}
$$

and expanding the modified Bessel functions for small external momentum $\tilde{p}^2$ leads to

$$\begin{aligned}
\Pi^{1-ins}(p) = -\frac{\lambda^2}{16\pi^2\theta^2 M^6}\Bigg[& -M^2 m_+^2 \ln\frac{\sqrt{m_+^2\tilde{p}^2}}{2} - M^2 m_-^2 \ln\frac{\sqrt{m_-^2\tilde{p}^2}}{2} - m^2 M^2 \gamma_E \\
& + m^2\sqrt{\frac{m_+^2}{\tilde{p}^2}}\left(\frac{1}{\sqrt{m_+^2\tilde{p}^2}} + \frac{\sqrt{m_+^2\tilde{p}^2}}{2}\left(\ln\frac{\sqrt{m_+^2\tilde{p}^2}}{2} + \gamma_E - \frac{1}{2}\right)\right) \\
& - m^2\sqrt{\frac{m_-^2}{\tilde{p}^2}}\left(\frac{1}{\sqrt{m_-^2\tilde{p}^2}} + \frac{\sqrt{m_-^2\tilde{p}^2}}{2}\left(\ln\frac{\sqrt{m_-^2\tilde{p}^2}}{2} + \gamma_E - \frac{1}{2}\right)\right)\Bigg] \\
= -\frac{\lambda^2}{16\pi^2\theta^2 M^6}\Bigg[& -M^2\frac{m^2}{2}\ln\frac{\sqrt{m_+^2\tilde{p}^2}\sqrt{m_-^2\tilde{p}^2}}{4} - M^4 \ln\sqrt{\frac{m_+^2}{m_-^2}} \\
& + \frac{m^4}{4}\ln\frac{\sqrt{m_+^2\tilde{p}^2}}{2} + \frac{m^2}{2}M^2\ln\frac{\sqrt{m_+^2\tilde{p}^2}}{2} - \frac{m^4}{4}\ln\frac{\sqrt{m_-^2\tilde{p}^2}}{2} \\
& + \frac{m^2}{2}\ln\frac{\sqrt{m_-^2\tilde{p}^2}}{2} - m^2 M^2 \gamma_E + \frac{m^4}{4}\left(\gamma_E - \frac{1}{2}\right) + \frac{m^2}{2}M^2\left(\gamma_E - \frac{1}{2}\right) \\
& - \frac{m^4}{4}\left(\gamma_E - \frac{1}{2}\right) + \frac{m^2}{2}M^2\left(\gamma_E - \frac{1}{2}\right)\Bigg] \\
= \frac{\lambda^2}{16\pi^2\theta^2 M^6} & \left[\left(M^4 - \frac{m^4}{4}\right)\ln\sqrt{\frac{m_+^2}{m_-^2}} + M^2\frac{m^2}{2}\right] + \mathcal{O}\left(\tilde{p}^2\right) . \qquad \text{(B.27)}
\end{aligned}$$

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