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Comparison of Classical and Quantum Universes within the Empirical Logic Approach

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I. INTRODUCTION

"Die Welt ist alles was der Fall ist"

Ludwig Wittgenstein

This master's thesis aims to compare the mathematical models of quantum mechanics and classical mechanics. To this end we draw heavily on a formalism developed by David J. Foulis and Charles H. Randall, termed empirical logic. This formalism is well suited for the comparison of models for the description of physical systems, due to reasons presented shortly. For our comparison we do not single out any specific physical system that quantum or classical mechanics can describe. Rather, in order to grasp intuitively what we compare, imagine the universes that quantum and classical mechanics would describe accurately if they existed. For all we know these are fictional, but for this work they provide an intuitive background.

We will restrict ourselves here to a non-relativistic treatment. Furthermore and possibly by reasonable choices, the universes are conceived to be conservative and not made up from partial systems. Along the way these assumptions simplify our problems a great deal.

Empirical Logic Empirical logic provides abstract formal concepts conceived to be found in any model of any physical system. Moreover it provides at a very basic level an interface that can be used to plug in conceptions, theories, or models, of the considered physical system. More precisely, it poses, and then deliberately leaves open entirely, the question of how the phenomena produced by the considered system are interrelated. Such relations must be derived from an already existing conception of the system. Afterwards these relation can be explicitly incorporated, thereby structuring the phenomena. In the ideal case, all the abstract notions of empirical logic then naturally find their system-specific formalization. It is these specific formalizations of identical abstract concepts that we seek to reveal here. In the cases of quantum and classical universes, that is.

From a certain point of view our work may then also be seen as a translation of these two systems into the language of empirical logic.

Methodology The abstract notions employed here neither cover exhaustively those developed within empirical logic nor are they exclusively taken from there. We have selected concepts that, to us, seemed suitable for the purpose. This means we more or less expected four things from them:

A physical motivation: If possible they should have a clear physical motivation.

Generality: They should be general enough to apply to both considered universes.

Precision: They should be formulated precisely enough to allow a formal derivation of the mathematical objects that represent them.

Compatibility: They should fit into the empirical logic approach.

The concepts we ended up with are largely from the literature of Foulis and Randall, but also from others, notably David W. Cohen, Josef M. Jauch, V.S. Varadarajan and Alexander Wilce. In one case we failed to find a concept general enough to account for both quantum and classical universes.

For lack of a better place we want to stress here, that in case of the quantum universe we have not explicitly ensured, that our assertions hold for unbounded operators and continuous spectra. Where such objects occur, if not stated otherwise, we restrict all our claims to bounded operators. In general we have tried to achieve mathematical rigor where possible, but have not always succeeded.

Overview As an overview we now provide a short roadmap of this work, touching upon the abstract concepts along the way.

We begin with a minimal presentation of classical and quantum mechanics in Sec.II. Then in Sec.III we present exclusively mathematical material that physicists are not necessarily familiar with. In our view this material would have lead too far astray if presented during the main part of this thesis (Sec.IV). Although we frequently refer to it, on a first read, Sec.III might be skipped altogether and used as a reference only.

The main part, Sec.IV, introduces in a bottom up manner the selected concepts and investigates their respective realizations in classical and quantum universes. As empirical logic is conceived to apply to any physical system we motivate the notions in the most general way we are aware of.

We begin Sec.IV.A with the “physical operation”, which is basically any well-defined procedure, as long as it ends by recording some symbols as outcomes. Usually a multitude of physical operations is available for a given system. Choosing the outcome symbols uniquely across all these physical operations results in a long list of outcome sets, called the “primitive test space”. The outcome sets are called “tests”. As outcomes across tests are entirely unrelated, no information about the interdependence of physical operations is contained in this structure. Clearly though many procedures in the “real world” have related outcomes. The power of the empirical logic approach lies in the fact, that by formally disconnecting all outcomes, we have to explicitly (re)introduce relations. This requires deep physical knowledge, and reveals subtle preconceptions about physical systems (e.g. the “counterfactual assumption”). Compare this procedure for example to the (re)construction of a manifold for use as a space-time starting from a mere set. In our case of course we do have excessive physical knowledge as the models are already well-defined. (Re)Connecting the related outcomes is done via the introduction of a “physical equivalence relation” inspired by that knowledge. A procedure is then given to incorporate this relation and obtain a new test space, representing the information contained in the physical equivalence relation. This new test space underlies all following concepts.

In Sec.IV.B a non-physical equivalence relation is used to define the “propositions”. Under the technical assumption of an algebraic test space and the interpretational counterfactual assumption, these propositions which may be confirmed, refuted, or neither, by a physical operation, encode information that is independent of the performed physical operation. Next up, the set of propositions called the “proposition logic”, it is a structure famous in the field of quantum logic, and not surprisingly. The subsequent notions of “statistical states” (Sec.IV.C), “observables”(IV.D.1), and “dynamical systems”(IV.E), are to a high degree determined by this structure in case of quantum mechanics. Somewhat unsatisfying is the failure of an analogous program for classical universes in this thesis. But at least, only our concept of a “dynamical system” will not be applicable. A statistical state attributes probabilities to propositions. The observables capture the ways by which propositions may, somehow unambiguously, be assigned to numerical values. And, a combination of both then provides probabilities for numerical values as well as the definition of an “expectation value”. The dynamical system, as the name suggests describes the time evolution of the universes, here we require it to leave the proposition logic invariant.

II. CONVENTIONAL PHYSICAL FORMALISMS

This chapter will very briefly sketch the conventional formalisms needed for the description of conservative non-relativistic physical systems. We do assume the reader is already familiar with this.

A. Classical Mechanics

1. Mathematical Model

a. Dynamics Conservative classical mechanical systems may be described using phase space Ω and the canonical equations. Phase space coordinates are usually denoted (q,p) where $q=(q_1,...,q_f)$ and $p=(p_1,...,p_f)$ and f is the number of degrees of freedom. Given some initial condition $(q(t_0)=q_0, p(t_0)=p_0)$ the time evolution of a physical system is then described by the integral curve $(q(t),p(t))$ which is a unique[1] solution to the ordinary differential equations (canonical equations):

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{2.1}$$

Here $H=H(q,p)$ is the Hamiltonian of the system which does not depend on time for conservative systems. Note that adding a constant to H does not alter the solutions to the canonical equations. There is an intriguing way to solve these equations using the Hamilton-Jacobi partial differential equations, see Appendix C for a sketch of this method. If H is constructed from kinetic energy T and potential energy V by $H=T+V$, the solutions to Eq.(2.1) correspond to those of Newtons equations of motion. For convenience we can rewrite Eq.(2.1). Let $x=(q_1,...,q_f,p_1,...p_f)$ and define a matrix J by:

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{2.2}$$

Where I is the identity matrix of rank f. Then we can write:

$$\dot{x} = J\nabla H(x) \tag{2.3}$$

The right hand side (RHS) is then a vector field on phase space, in this case called the *Hamiltonian vector field*, and just like H itself constant in time. We can write $\sigma(t,x)$ with $\sigma : \Re\times\Omega\to\Omega$ for the integral curve passing through x at time $t=0$ such that $\sigma(0,x)=x$ is an initial condition. Then σ satisfies

$$\frac{d}{dt}\sigma(t,x) = J\nabla H(\sigma(t,x)). \tag{2.4}$$

[1] By the wellknown existence and uniqueness theorem, the solution exists and is unique at least locally. We assume in the following that it exists for all $t\in\Re$.

Such a σ is called a *phase flow* or just a *flow* generated by the Hamiltonian vector field. As every integral curve of Eq.(2.1) for given initial conditions is unique so is $\sigma(t,x)$ for all $x \in \Omega$. Therefore if we write for a fixed t, $\sigma(t,x) = \sigma_t(x)$, then the maps $\sigma_t : \Omega \to \Omega$ are all uniquely determined by a given Hamiltonian through the canonical equations. It can be shown (Nakahara, 2003, Theorem 5.1) that, for all $t,s \in \Re$, σ satisfies

$$\sigma(t,\sigma(s,x)) = \sigma(t+s,x). \tag{2.5}$$

Each $\sigma_t(x)$ is a diffeomorphism (Nakahara, 2003). From Eq.(2.5) we can see that the σ_t are a one parameter group with respect to function composition as $\sigma_t \circ \sigma_s = \sigma_{t+s}$ and $\sigma_{-t} = \sigma_t^{-1}$. The needed unit element is obviously σ_0. On account of the same three features we can also view $t \mapsto \sigma_t$ as a homomorphism of the additive group of real numbers into the diffeomorphisms of Ω. This will be of importance later on in Sec.IV.E.2.

b. Observables Observables f are usually defined as functions of the phase space coordinates $f : \Omega \to \Re$. Therefore for any given observable, the coordinates determine the value it takes.

It is worthwhile mentioning that the canonical equations may be expressed in a coordinate independent way by means of the Poisson bracket, this is defined for two functions h,g (see e.g. Arnold *et al.*, 1989):

$$\{h,g\}(x) = \frac{d}{dt}|_{t=0}\, h(\sigma_t^g(x)). \tag{2.6}$$

Here $\sigma_t^g(x)$ is the phase flow generated by the Hamiltonian vector field if $H = g$. In a coordinate chart (q,p) this is equal to

$$\{h,g\} = \sum_i \left(\frac{\partial h}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial h}{\partial p_i}\frac{\partial g}{\partial q_i} \right). \tag{2.7}$$

The time evolution of any observable f may then be written:

$$\dot{f} = \{f,H\} \tag{2.8}$$

c. Uncertain States In case we are uncertain of phase space coordinates, we might introduce a probability distribution $\rho(q,p) > 0$ satisfying

$$\int_\Omega \rho(q,p)\, dq\, dp = 1. \tag{2.9}$$

Then we obtain the probability $P(B)$ for the system to be in a subset $B \in \Omega$ through:

$$P(B) = \int_B \rho(q,p)\, dq\, dp. \tag{2.10}$$

The expectation value $\bar{f}$ of an observable f is then calculated by:

$$\bar{f} = \int_\Omega f(q,p)\rho(q,p)\, dq\, dp. \tag{2.11}$$

As the probability distribution is just a special kind of observable, we get

$$\dot{\rho} = \{\rho, H\}. \tag{2.12}$$

2. *Measurement Process*

It is a fundamental assumption in classical mechanics, that the influence of a measurement of any observable on the state of the system can be arbitrarily reduced (given the technological means). Furthermore it is possible to measure (q,p) up to arbitrary precision at any given time (see Jauch, 1968).

B. Quantum Mechanics

1. *Mathematical Model*

Quantum mechanics is usually formulated in complex separable Hilbert space $\mathscr{H}$. The state of a system is represented by a positive operator $\hat{\rho}$ of trace one called the *density operator*. The time evolution of this state $\hat{\rho}(t)$ is the solution to the equation ($\hbar = 1$):

$$-i\frac{d\hat{\rho}}{dt} = [\hat{\rho}, \hat{H}] \tag{2.13}$$

where $\hat{H}$ is now the Hamiltonian operator which is obtained by replacing the coordinates in the classical $H(q,p)$ by their respective operators. For a pure state defined by $\hat{\rho}^2 = \hat{\rho}$ and with $|\psi\rangle$ the normalized eigenvector to the eigenvalue that is one, this is equivalent to the Schrödinger Equation:

$$i\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle. \tag{2.14}$$

A measurement of an observable $\hat{O}$ on a pure state $|\psi\rangle$ causes this state to "collapse" into an eigenspace associated to a (possibly degenerate) eigenvalue λ of the observable. The eigenspace can be written as the span of g orthogonal eigenvectors (to the eigenvalue λ) $\{|\lambda_1\rangle,...,|\lambda_g\rangle\}$ of $\hat{O}$, where g denotes the degeneracy of the eigenvalue λ. The projection operator $\hat{P}_\lambda$ onto this eigenspace can be written as

$$\hat{P}_\lambda = \sum_{i=1}^{g} |\lambda_i\rangle\langle\lambda_i|. \tag{2.15}$$

The probability $P(\lambda)$ for $|\psi\rangle$ to collapse into the eigenspace corresponding to eigenvalue λ is then:

$$P(\lambda) = \langle\psi|\hat{P}_\lambda|\psi\rangle, \tag{2.16}$$

which gives the more famous special case for $g = 1$:

$$P(\lambda) = |\langle\psi|\lambda\rangle|^2. \tag{2.17}$$

After the measurement the system is in a pure state $|\psi'\rangle$, which can be written as a normalized superposition of the $|\lambda_i\rangle$:

$$|\psi'\rangle = \sum_{i=1}^{g} c_i|\lambda_i\rangle, \tag{2.18}$$

where the c_i are complex numbers.

For a general state $\hat{\rho}$ the probability $P(\lambda)$ to collapse into the eigenspace corresponding to eigenvalue λ is:

$$P(\lambda) = Tr(\hat{\rho}\hat{P}_\lambda) \tag{2.19}$$

Sometimes it is interesting to calculate the expectation value $\langle\hat{O}\rangle$ of an operator $\hat{O}$ this is done by:

$$\langle\hat{O}\rangle = Tr(\hat{\rho}\hat{O}) \tag{2.20}$$

Another important quantity is the variance $\Delta\hat{O}$ defined as

$$(\Delta\hat{O})^2 = \langle(\hat{O} - \langle\hat{O}\rangle)^2\rangle \tag{2.21}$$

With this, Heisenberg's principle for two observables $\hat{A}$ and $\hat{B}$ can be formulated:

$$\Delta\hat{A}\cdot\Delta\hat{B} \geq \frac{1}{2} \mid \langle[\hat{A},\hat{B}]\rangle \mid \tag{2.22}$$

We also state here that two observables A, B commute $[A,B] = 0$ if and only if they share a common set of eigenvectors.

III. BASIC MATHEMATICAL DEFINITIONS AND THEOREMS

This section introduces in a formal manner the basic mathematical notions employed in this text.

A. Test Spaces

Test spaces and their tests constitute the basic ideas from which the empirical logic approach is built up. In the following we introduce some properties that endow test spaces with the structure needed in the section on physics. The most important such property for our purpose is algebraicity which allows the definition of the proposition logic, to be described in the next section. Of interest is also the orthocoherence of a test space which makes the proposition logic orthomodular. The presentation follows closely the article of Wilce (2000).

A *test space* $\mathfrak{A}$ is a collection of non-empty sets that satisfies *irredundancy* in the following sense:

$$\forall E,F \in \mathfrak{A},\ E \subseteq F \Rightarrow E = F \tag{3.1}$$

The elements of $\mathfrak{A}$ are called *tests* and elements of a test are called *outcomes*. The set $X = \bigcup \mathfrak{A}$ is the *outcome space* and sets of outcomes that are subsets of (at least) one test are referred to as *events*. The set of events is denoted $\mathscr{E}(\mathfrak{A})$.

Two events A, B are called *compatible* iff they are subsets of a common test E. Furthermore iff A and B are compatible and disjoint as well they are *orthogonal*, denoted $A \perp B$. In case $A \perp B$ and there exists a test E such that $E = A \sqcup B$ (where $A \sqcup B$ denotes a union of orthogonal events) A and B are *orthogonal complements* of each other. This is written A oc B and sometimes the notation $A^{\perp} := B$ is used. Iff two events A and B, not necessarily contained in one test, share a common complement, they are called *operationally perspective* or just *perspective* written $A \sim B$. Formally:

$$A \sim B \Leftrightarrow \exists\, C \subseteq \mathscr{E}(\mathfrak{A}) \text{ such that } A \,\mathrm{oc}\, C \text{ and } C \,\mathrm{oc}\, B \tag{3.2}$$

Recall that a binary relation $R(A,B)$ is an equivalence relation, if it satisfies the conditions:

(i) *Reflexivity*: $R(A,A)$;

(ii) *Symmetry*: $R(A,B) = R(B,A)$;

(iii) *Transitivity*: if $R(A,B)$ and $R(B,C)$ then $R(A,C)$.

Note that perspectivity is a reflexive and symmetric relation, because

$$\forall A \in \mathscr{E}(\mathfrak{A}),\ A \sim A \tag{3.3}$$

and

$$\forall A, B \in \mathscr{E}(\mathfrak{A}),\ A \sim B \Leftrightarrow B \sim A \tag{3.4}$$

hold.

In order to make perspectivity transitive as well, a test space needs to satisfy another requirement, to be defined now. We call a test space *algebraic* if it satisfies *algebraicity*:

$$A \sim B \text{ oc } C \Rightarrow A \text{ oc } C. \tag{3.5}$$

To see that on an algebraic test space perspectivity is transitive, just recall the definition of perspectivity and let $A \sim B$ and $B \sim C$, from $B \sim C$ we know that there exists D such that B oc D, D oc C. Then we have $A \sim B$ oc C which, on an algebraic test space implies A oc D. Finally from A oc D and D oc C we get $A \sim C$.

We therefore have an equivalence relation on algebraic test spaces, in the next section we will look at its equivalence classes. An equivalence class will then be called a *proposition*. First though we look at further consequences of algebraicity.

Let us define then another binary relation between events A, B called *implication* by

$$A \leq B \Leftrightarrow \exists C, C \perp A, (A \sqcup C) \sim B. \tag{3.6}$$

If $A \leq B$ we say that A *implies* B. Note that neither $A \leq B$ nor $B \leq A$ has to hold for arbitrary $A, B \in \mathfrak{A}$.

Now recall further that a binary relation $R(A,B)$ is a preorder, if it satisfies the conditions:

(i) *Reflexivity*: $R(A,A)$;

(ii) *Transitivity*: if $R(A,B)$ and $R(B,C)$ then $R(A,C)$.

We can see immediately that the implication is reflexive as $A \sqcup \emptyset \sim A$, but transitivity again needs algebraicity of the test space. We give a proof inspired by that of Cohen (1989, Lemma 3B.2.) now: Let $A \leq B$ and $B \leq C$, this is to say that there exist events U and L such that $(A \sqcup U) \operatorname{oc} L \operatorname{oc} B$, and events V and K such that $(B \sqcup V) \operatorname{oc} K \operatorname{oc} C$. Then $(V \sqcup K) \operatorname{oc} B \operatorname{oc} L$ gives us $(V \sqcup K) \sim L \operatorname{oc} (A \sqcup U)$ which by algebraicity implies $(V \sqcup K) \operatorname{oc} (A \sqcup U)$. Then we can move the complement around to get $A \sqcup U \sqcup V \operatorname{oc} K$ but we had also $K \operatorname{oc} C$ so $A \sqcup (U \sqcup V) \sim C$ which means $A \leq C$.

For later use we also prove the following property permitted by algebraicity, call it a pre-antisymmetry for the moment:

$$A \leq B \text{ and } B \leq A \Rightarrow A \sim B \tag{3.7}$$

To see this just replace C by A in the previous proof, then we arrive at $A \sqcup U \sqcup V \operatorname{oc} K$ again, but this time $K \operatorname{oc} A$ already, so irredundancy requires that $U \sqcup V = \emptyset$. Then $A \operatorname{oc} L \operatorname{oc} B$ which is $A \sim B$.

A theorem that is not used in this text, but may help to get a clearer picture of algebraic test spaces is the following (Wilce, 2000, Lemma 3.4). It goes by the name of additivity lemma:

If $\mathfrak{A}$ algebraic, $A \sim A'$, $B \sim B'$ in $\mathscr{E}(\mathfrak{A})$, then

$$A \perp B \Rightarrow A' \perp B' \text{ and } A \sqcup B \sim A' \sqcup B'. \tag{3.8}$$

The following property for test spaces has a consequence famous within the field of quantum logic, namely orthomodularity of the poset of propositions. Traditionally orthomodularity is defined directly for the poset of propositions. Here it arises as a consequence of a condition on test spaces.

A test space is called *orthocoherent* if it follows from the pairwise orthogonality of three events, that they are also *jointly orthogonal*. This means their union is an event as well. In other words, they are contained in the same test. Formally:

$$\forall A,B,C \in \mathscr{E}(\mathfrak{A}),\ \text{if } A \perp B,\ B \perp C,\ A \perp C \text{ then } A,B,C \in \mathscr{E}(\mathfrak{A}) \tag{3.9}$$

The consequences of this property will be mentioned in the next section.

B. The Propositions and their Structure

As mentioned above, in algebraic test spaces perspectivity is an equivalence relation. We will now look at the structures admitted by the resulting set of equivalence classes $\Pi(\mathfrak{A})$. Recall that the equivalence classes are called propositions so that $\Pi(\mathfrak{A})$ is the set of propositions. Here we want to mention that within the empirical logic approach there is another important set, the set of *properties* which admits similar structures. Generally these structures do not coincide with those of the set of propositions, but in the cases of classical and quantum mechanics they do (for details see e.g. Foulis *et al.*, 1983). We omit the notion of properties here.

The set of propositions $\Pi(\mathfrak{A})$ is always an orthocomplemented poset, which we will see first. Then we introduce another structure that each set of propositions admits, called an orthoalgebra. Its relation to the posets is not straightforward but the reason we mention orthoalgebras is that there is proof, that every orthoalgebra arises from a test space (Feldman and Wilce, 1993). For posets we have not found an equivalent. Both posets and orthoalgebras are generalizations of orthomodular posets and those are in turn generalizations of Boolean lattices. The claim that every orthomodular poset and thus every Boolean lattice arises from a test space is a sign of the generality of test spaces and a motivation for their study. Orthomodular posets are also known as quantum logics in the literature and have been thoroughly investigated (e.g. Pták and Pulmannová, 1991, and references therein). We will see what condition to impose on a test space to obtain orthomodular posets. What is missing is the conditions that lead to an orthomodular lattice, as we are unaware of these at present. Since we will encounter mainly lattices and often atomistic ones, these concepts are defined at the end as well.

1. Orthocomplemented Posets from Test Spaces

The elements of $\Pi(\mathfrak{A})$ will be called *propositions* and denoted by $p(A)$ for all events $A \in \mathscr{E}(\mathfrak{A})$. Formally they are defined by:

$$p(A) := \{B \in \mathscr{E}(\mathfrak{A}) | A \sim B\}. \tag{3.10}$$

Now we state the defining requirements for a partial order:

(i) *Reflexivity*: $R(A,A)$;

(ii) *Antisymmetry*: if $R(A,B)$ and $R(B,A)$ then $A = B$;

(iii) *Transitivity*: if $R(A,B)$ and $R(B,C)$ then $R(A,C)$.

Then we define a relation on $\Pi(\mathfrak{A})$ by:

$$\forall p(A), p(B) \in \Pi(\mathfrak{A}),\ p(A) \leq p(B) \Leftrightarrow A \leq B \tag{3.11}$$

This relation is a preorder for propositions because it is defined by the preorder for events. Moreover from pre-antisymmetry Eq. (3.7): $A \leq B$ and $B \leq A \Rightarrow A \sim B$ we get:

$$p(A) \leq p(B) \text{ and } p(B) \leq p(A) \Rightarrow A \sim B \Rightarrow p(A) = p(B), \tag{3.12}$$

which is the ordinary antisymmetry. Therefore we have a partial order on $\Pi(\mathfrak{A})$ and say $(\Pi(\mathfrak{A}),\leq)$ is a *partially ordered set* or a *poset*. Sometimes we speak of an arbitrary poset $(L,\leq)$ not necessarily arising from a test space in the above manner.

Posets and thus $(\Pi(\mathfrak{A}),\leq)$ allow the definition of a *greatest lower bound* or *meet* $\bigwedge_i p(A_i)$ and a *least upper bound* or *join* $\bigvee_i p(A_i)$ of a family $p(A_i)$ of their elements.

Formally a join on a poset $(L,\leq)$ with $a_i,c,d \in L$ is defined as follows:

$$\bigvee_i a_i = c \Leftrightarrow \forall i, a_i \leq c \text{ and if } \exists d \text{ such that } \forall i, a_i \leq d \text{ then } c \leq d. \tag{3.13}$$

Accordingly the meet is defined as:

$$\bigwedge_i a_i = c \Leftrightarrow \forall i, c \leq a_i \text{ and if } \exists d \text{ such that } \forall i, d \leq a_i \text{ then } d \leq c. \tag{3.14}$$

Note that neither has to exist. If a largest element, denoted 1, and a smallest element denoted 0 exist the poset is called *bounded* and denoted $(L,\leq,1,0)$. The poset $(\Pi(\mathfrak{A}),\leq)$ is bounded as, for $E \in \mathfrak{A}$, an arbitrary test $p(E)$ implies no other proposition than itself: $p(E) \leq p(A) \Leftrightarrow E \leq A \Rightarrow \exists B \perp E,\ E \cup B = A$ but only $E \perp \emptyset$ such that $B = \emptyset$ and $A = E$. Furthermore it is implied by all propositions as, for all $A \in \mathscr{E}(\mathfrak{A})$, we have $A \subseteq F$ for some $F \in \mathfrak{A}$, then $A \sqcup F \backslash A = F \,\mathrm{oc}\, \emptyset \,\mathrm{oc}\, E$ so $p(A) \leq p(E)$. On the other hand $p(\emptyset)$ is only implied by itself and implies all propositions which is easy to see. Then:

$$p(E) = 1 \text{ and } p(\emptyset) = 0 \tag{3.15}$$

An *orthocomplementation* on a poset $(L,\leq)$ is a unary operation $f : L \to L$ (this is bijective) such that the following conditions are satisfied for all $a,b \in L$:

(i) $a \vee a' = 1$;

(ii) $a \wedge a' = 0$;

(iii) $a \leq b \Rightarrow b' \leq a'$;

(iv) $(a')' = a$.

A poset with orthocomplementation is written $(L,\leq,')$.

Define the *orthogonal complement* of a proposition: For all $p(A) \in \Pi(\mathfrak{A})$, let $A \subseteq E \in \mathfrak{A}$ then :

$$p(A)' := p(E \backslash A). \tag{3.16}$$

This is indeed an orthocomplementation. We proof this now.

ad (i) From the definition of the join we see that this is shown if, for all $A, B \in \mathscr{E}(\mathfrak{A})$ where $A \subseteq E \in \mathfrak{A}$, we have

$$\left.\begin{array}{r} p(A) \leq p(B) \\ p(E \backslash A) \leq p(B) \end{array}\right\} \Rightarrow p(B) = 1 \rightarrow B \in \mathfrak{A}. \tag{3.17}$$

Now

$$\left.\begin{array}{r} p(A) \leq p(B) \Leftrightarrow \exists G, H \subseteq E : A \sqcup G \,\mathrm{oc}\, H \,\mathrm{oc}\, B \\ p(E \backslash A) \leq p(B) \Leftrightarrow \exists C, D \subseteq E : E \backslash A \sqcup C \,\mathrm{oc}\, D \,\mathrm{oc}\, B \end{array}\right\} \Rightarrow A \sqcup G \sim B \,\mathrm{oc}\, D \Rightarrow A \sqcup G \,\mathrm{oc}\, D. \tag{3.18}$$

Note that at the same time $D \subseteq A$ because $E \backslash A \sqcup C \,\mathrm{oc}\, D$ then $D = \emptyset$ and $B \,\mathrm{oc}\, \emptyset$ which means B is a test and $p(B) = 1$.

ad (ii) Similar to before we now need to show:

$$\left.\begin{array}{l} p(B) \leq p(A) \\ p(B) \leq p(E \backslash A) \end{array}\right\} \Rightarrow p(B) = 0 \rightarrow B = \emptyset. \tag{3.19}$$

This time with $F_1, F_2 \in \mathfrak{A}$

$$\left.\begin{array}{ll} p(B) \leq p(A) & \Leftrightarrow \exists G, H \subseteq F_1 : B \sqcup G \,\mathrm{oc}\, H \,\mathrm{oc}\, A \Rightarrow H = E \backslash A \Rightarrow B \sqcup G = A \\ p(B) \leq p(E \backslash A) & \Leftrightarrow \exists C, D \subseteq F_2 : B \sqcup C \,\mathrm{oc}\, D \,\mathrm{oc}\, E \backslash A \Rightarrow D = A \Rightarrow B \sqcup C = E \backslash A \end{array}\right\} \Rightarrow B = \emptyset. \tag{3.20}$$

ad (iii) (Cohen, 1989). We go down to the events for this proof. Let $A \subseteq E$ and $B \subseteq F$, then we need to show that $A \leq B \Leftrightarrow F \backslash B \leq E \backslash A$. Suppose $A \leq B$. Then exists C with $A \sqcup C \sim B \,\mathrm{oc}\, F \backslash B$, algebraicity gives $A \sqcup C \,\mathrm{oc}\, F \backslash B$ this is equivalent to $F \backslash B \sqcup C \,\mathrm{oc}\, A$ while $A \,\mathrm{oc}\, E \backslash A$, but this is $F \backslash B \leq E \backslash A$. The other direction is proven by exchanging A and B for $F \backslash B$ and $E \backslash A$.

ad(iv) This one is obvious.

We conclude that by way of the propositions, every algebraic test space gives rise to a bounded orthocomplemented poset $(\Pi(\mathfrak{A}), \leq, ', 1, 0)$ with

$$p(A) \leq p(B) \;\Leftrightarrow\; A \leq B \tag{3.21a}$$

$$p(A)' \;\coloneqq\; p(E \backslash A) \tag{3.21b}$$

$$p(E) \;=\; 1 \tag{3.21c}$$

$$p(\emptyset) \;=\; 0. \tag{3.21d}$$

Be careful not to assume $p(A) \vee p(B) = p(A \cup B)$ as this restricts the test spaces that would give rise to such a poset, to orthocoherent ones (see section III.B.3.d).

As every test space admits this bounded orthocomplemented poset we will often refer to it simply as the poset of propositions of a test space, or the poset $\Pi(\mathfrak{A})$, the further structure is implied. Sometimes we write $(\Pi(\mathfrak{A}), \leq, ')$ but again boundedness is implied here.

Finally note that orthogonality lifts to the propositions with

$$p(A) \perp p(B) \Leftrightarrow A \perp B \tag{3.22}$$

This definition is equivalent to

$$p(A) \perp p(B) \Leftrightarrow p(A) \leq p(B)' \tag{3.23}$$

as we will see now. Let $A \perp B \Rightarrow A \subseteq E \backslash B$ then $A \sqcup E \backslash (A \sqcup B) \,\mathrm{oc}\, B \,\mathrm{oc}\, E \backslash B$ thus $A \leq E \backslash B$ and $p(A) \leq p(B)'$. For the other direction let $A \leq E \backslash B$ with A not necessarily an event of E. Then exists event C such that $A \sqcup C \sim E \backslash B$ while $E \backslash B \,\mathrm{oc}\, B$ such that algebraicity gives $A \sqcup C \,\mathrm{oc}\, B$ so $A \perp B$.

Equation (3.23) is suited to define orthogonality for arbitrary orthocomplemented posets without reference to an underlying test space. In case of an underlying test space the notions coincide. We therefore define for any orthocomplemented poset L, with $a,\ b \in L$:

$$a \perp b \Leftrightarrow a \leq b'. \tag{3.24}$$

2. *Orthoalgebras from Test Spaces*

An orthoalgebra (Feldman and Wilce, 1993; Wilce, 2000) is a four-tupel $(L, \oplus, 1, 0)$ with L a set and $\oplus$ a partially-defined binary operation $\oplus : L \times L \to L$ such that

(i) the operation $\oplus$ is associative and commutative in a strong sense, meaning e.g. in case of commutativity, that, if for $a, b \in L$, $a \oplus b$ exists, $b \oplus a$ exists as well and $a \oplus b = b \oplus a$;

(ii) there exists an element $0 \in L$ such that $0 \oplus a = a \oplus 0 = a$ for every $a \in L$;

(iii) there exists a *unit element* $1 \in L$ such that, for every $p \in L$, there is a unique element p' with $p \oplus p' = 1$;

(iv) $p \oplus p$ is defined only if $p = 0$.

We obtain the orthoalgebra $(\Pi(\mathfrak{A}), \oplus, 1, 0)$ by using

$$p(A) \perp p(B) \Leftrightarrow A \perp B \tag{3.25}$$

to define $\oplus$ by

$$p(A) \perp p(B) \Rightarrow p(A) \oplus p(B) := p(A \cup B) \tag{3.26}$$

and identifying as before:

$$p(A)' := p(E \backslash A) \tag{3.27a}$$

$$p(E) = 1 \tag{3.27b}$$

$$p(\emptyset) = 0. \tag{3.27c}$$

It is easily verified that this is in fact an orthoalgebra.

It is also possible to construct a test space $\mathfrak{A}_L$ from every orthoalgebra $(L, \oplus, 1, 0)$ such that $(\Pi(\mathfrak{A}_L), \oplus, 1, 0)$ is isomorphic to $(L, \oplus, 1, 0)$, denoted $\Pi(\mathfrak{A}_L) \backsimeq L$. To get this, let $\mathfrak{A}_L$ be the set of all finite subsets $E = \{a_1, ..., a_n\}$ of L with $a_1 \oplus ... \oplus a_n = 1$. A proof of this can be found in Feldman and Wilce (1993).

3. *Orthomodular Posets and Test Spaces*

a. Definition An *orthomodular poset* or *quantum logic* is a five-tupel $(L, \leq, ', 0, 1)$ with $\leq$ a partial ordering and ' a unary operation such that (Pták and Pulmannová, 1991):

(i) 0, 1 are least and greatest elements of L;

(ii) $a \leq b$ implies $b' \leq a'$ for any $a, b \in L$;

(iii) $(a')' = a$;

(iv) if $\{a_i | i \in \mathbb{N}\}$ is a countable subset of L such that $a_i \leq a'_j$ for $i \neq j$, then the join $\bigvee_i a_i$ exists in L;

(v) if $a, b \in L$ and $a \leq b$, then $b = a \vee (b \wedge a')$.

Here (v) is called the *orthomodular law* or *orthomodularity*.

b. Orthomodular Posets as Orthocomplemented Posets To see that every orthomodular poset is an orthocomplemented poset, we need to show that ' is an orthocomplementation, the requirements were:

(1) $a \vee a' = 1$;

(2) $a \wedge a' = 0$;

(3) $a \leq b \Rightarrow b' \leq a'$;

(4) $(a')' = a$.

Then we only need to show (1) and (2). First we prove *De Morgan's law* for orthomodular posets:

$$\text{If } a \vee b \text{ exists in } L \Rightarrow (a \vee b)' = a' \wedge b'. \tag{3.28}$$

Assume $a \vee b$ exists, then consider, using (ii):

$$\left.\begin{array}{ll} a \leq a \vee b & \Rightarrow (a \vee b)' \leq a' \\ b \leq a \vee b & \Rightarrow (a \vee b)' \leq b' \end{array}\right\} (a \vee b)' \leq a' \wedge b' \tag{3.29}$$

The last expression exists because $(a \wedge b)'$ is a candidate for that meet. Now the $(a' \wedge b')'$ exists as well (because ' is a unary operation), so we also have the following:

$$\left.\begin{array}{ll} a' \wedge b' \leq a' & \Rightarrow a \leq (a' \wedge b')' \\ a' \wedge b' \leq b' & \Rightarrow b \leq (a' \wedge b')' \end{array}\right\} a \vee b \leq (a' \wedge b')' \Leftrightarrow a' \wedge b' \leq (a \vee b)' \tag{3.30}$$

Combining the results gives De Morgan's law.

To get (1) consider, using (v), $a \leq 1 \Rightarrow 1 = a \vee (1 \wedge a') = a \vee a'$. Then to get (2) use (iii) and De Morgan's law on (1) to get $(a \vee a')' = 1' = a' \wedge a$ and from (v) and (iii) $0 \leq 1 \Rightarrow 1 = 0 \vee (1 \wedge 0') = 1 \wedge 0' \Rightarrow 0' = 1 \Rightarrow (0')' = 0 = 1'$. Then $a \wedge a' = 0$.

c. Orthomodular Posets as Orthoalgebras Next we show that every orthomodular poset is an orthoalgebra as well. To this end we identify with the operation $\oplus$ the join defined for $a,b \in L$ if $a \leq b'$.

Now, the join is commutative and associative as, if $a \oplus b = a \vee b$ and therefore $a \leq b'$ then because of (ii) $b \leq a'$ so that $b \vee a = b \oplus a$ exists. This generalizes to associativity. For the element 0 (of the orthoalgebra) we take 0 (of the orthomodular poset) and see that $0 \oplus a = 0 \vee a = a \vee 0 = a \oplus 0 = a$. For the unit element we take 1 and see that a' is the unique element such that $a \vee a' = 1$. That $a \oplus a = a \vee a$ is defined only if $a = 0$ can be seen in the following way. On an orthomodular poset $a \vee a$ is only defined if we require $a \leq a'$. From $a \vee a' = 1$ we get that $a \leq 1$ and $a' \leq 1$ and for any $b \in L$ such that $a \leq b$ and $a' \leq b$ implies $b = 1$. Now substitute a' for b into the last equations and we get $a' = 1$. So $a = 0$ from $1' = 0$.

d. Orthomodular Posets from Orthocoherent Test Spaces We now return to the consequence of orthocoherence Eq.(3.9) on the set of propositions. The poset of propositions $(\Pi(\mathfrak{A}), \leq, ', 1, 0)$ is an orthomodular poset iff $\mathfrak{A}$ is orthocoherent (Wilce, 2000). We omit the proof.

There is another important condition for orthocoherence of the test space and thus for orthomodularity of the poset $\Pi(\mathfrak{A})$. It is mentioned frequently in the literature (e.g. Foulis *et al.*, 1983; Wilce, 2003): A test space $\mathfrak{A}$ is orthocoherent (and its poset of propositions orthomodular) iff for all $A, B \in \mathfrak{A}$, we have

$$A \perp B \Rightarrow p(A) \vee p(B) = p(A \cup B). \tag{3.31}$$

We will need this property and therefore orthocoherence to ensure that "weights" induce "statistical states" in section III.C.

4. Boolean Lattices

a. Definition A poset is called a *lattice* if join and meet exist for every pair of elements of a poset, we write $(L, \vee, \wedge)$ for a lattice. The lattice is called *σ-complete* if both operations exist for all countable families of elements and *complete* if they do so for all families of elements.

From any lattice[2] we can recover a poset by setting $a \leq b \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$. All the

[2] A lattice can be defined abstractly as an algebra $(L, \vee, \wedge)$ with the two binary operations being associative, commutative, idempotent, and satisfying the absorption law $a \vee (a \wedge b) = a \wedge (a \vee b) = a$.

laws that hold for join and meet on posets then hold on the lattice as well (Grätzer, 1971).

A *complement* of an element a of a bounded poset $(L, \leq, 1, 0)$ is an element b such that $a \wedge b = 0$ and $a \vee b = 1$. A *complemented lattice* is a lattice where each element has a complement.

A lattice is *distributive* if all elements satisfy the distributive law:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{3.32a}$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{3.32b}$$

A *Boolean lattice* $(B, \vee, \wedge)$ is a complemented distributive lattice.

b. σ-complete Boolean Lattices as Orthomodular Posets Now we show that every Boolean lattice is an orthomodular poset. Let $(B, \vee, \wedge)$ be a σ-complete (bounded) Boolean lattice.

First we show that the complement is unique: Let $a \in B$ and $b, c \in B$ be complements of a. Then

$$b = b \vee 0 = b \vee (a \wedge c) = (b \vee a) \wedge (b \vee c) = b \vee c$$

$$c = c \vee 0 = c \vee (a \wedge b) = (c \vee a) \wedge (c \vee b) = c \vee b.$$

Which gives $b = c$. Next we need property (iii), $a'' := (a')' = a$ with a' the unique complement of a. First we show $a \leq a''$:

$$a'' \vee 0 = a'' \vee (a \wedge a') \tag{3.33}$$

$$= (a'' \vee a) \wedge (a'' \vee a') \tag{3.34}$$

$$= a'' \vee a \tag{3.35}$$

Which gives $a \leq a''$. Now similarly

$$a'' \wedge 1 = a'' \wedge (a \vee a') \tag{3.36}$$

$$= (a'' \wedge a) \vee (a'' \wedge a') \tag{3.37}$$

$$= a'' \wedge a \tag{3.38}$$

Leaving $a'' \leq a$, hence $a'' = a$. Next we show property (ii): $a \leq b$ implies $b' \leq a'$. From $a \leq b$ we know $a \wedge b = a$ then consider:

$$a \wedge b = a \tag{3.39}$$

$$(a \wedge b) \vee a' = a \vee a' \tag{3.40}$$

$$(a \vee a') \wedge (b \vee a') = 1 \tag{3.41}$$

$$(b \vee a') = 1 \tag{3.42}$$

$$b' \wedge (b \vee a') = b' \wedge 1 \tag{3.43}$$

$$(b' \wedge b) \vee (b' \wedge a') = b' \tag{3.44}$$

$$b' \wedge a' = b'. \tag{3.45}$$

Then $b' \leq a'$.

Property (iv) is a direct consequence of σ-completeness, property (i) is boundedness, and property (v) follows from distributivity: let $a \leq b$ then $a \vee (b \wedge a') = (a \vee b) \wedge (a \vee a') = b$. This completes the proof.

5. Further Properties

An *atom* of a bounded poset L is an element $a \neq 0$ such that, for $x \in L$ from $x \leq a$ and $x \neq a$ (also written $x < a$) it follows that $x = 0$. Then a bounded poset L is called *atomic* if for every element $x \in L$, $x \neq 0$, there exists an atom a with $a \leq x$. An *atomistic* lattice is an atomic lattice such that each element $x \neq 0$ is a join of atoms.

For a lattice of propositions $(\Pi(\mathfrak{A}), \leq, ', 1, 0)$ on an orthocoherent test space $\mathfrak{A}$, it is possible to define compatibility in such a way that two propositions $p(A)$ and $p(B)$ are compatible if the events A and B are compatible (see Cohen, 1989, Theorem 3B.17). There is no claim of the converse. The way to define this notion of compatibility is as follows.

Two propositions p, q in an orthomodular lattice $(L, \leq, ', 1, 0)$ are called *compatible* if there exist $u, v, w \in L$ such that:

1. u, v, w is a pairwise orthogonal set in L;

2. $u \vee v = p$ and $v \vee w = q$.

C. Statistical States and Test Spaces

Here we present a possible probability model for test spaces and bounded orthocomplemented posets. As reference we owe tribute to the book by Cohen (1989).

1. Weights on Test Spaces

Let $\mathfrak{A}$ be a test space and X its outcome set, then $\omega : X \to [0,1]$ is called a *weight* on $\mathfrak{A}$ if for every test $E \in \mathfrak{A}$

$$\omega(E) := \sum_{x \in E} \omega(x) = 1. \tag{3.46}$$

The weight of an event A, $\omega(A)$, is the 'sum' of the weights of its outcomes

$$\omega(A) = \sum_{x \in A} \omega(x). \tag{3.47}$$

In case of countably infinite events A, the sum here refers to the supremum of all sums over finite subsets of A.

It is easy to see that on an algebraic test space perspective events have the same weights. Let $A \sim B$, $A \subseteq E \in \mathfrak{A}$ and $B \subseteq F \in \mathfrak{A}$, then there exists C such that $A \operatorname{oc} C \operatorname{oc} B$. From

$$\omega(A \sqcup C) = \omega(A) + \omega(C) = \omega(E) = 1$$

and supposing $\omega(A) = a$, $a \in [0,1]$ we get $\omega(C) = 1 - a$. But at the same time

$$\omega(C \sqcup B) = \omega(C) + \omega(B) = \omega(F) = 1$$

such that $\omega(B) = 1 - \omega(C) = a$.

2. Statistical States

A *state* on a bounded orthomodular poset $(L, \leq, ', 1, 0)$ is a function $s : L \to [0,1]$ such that:

(i) for $a, b \in L$, if $a \perp b$, then $s(a \vee b) = s(a) + s(b)$;

(ii) s(1) = 1.

A σ-additive state is one that satisfies a stronger version of the first condition:

(i)' for $\{a_i\}$, $i \in I$ an orthogonal countable sequence of propositions, $\bigvee_I a_i$ exists, and $s(\bigvee_I a_i) = \sum_I s(a_i)$.

The set of states on L is denoted by $\mathscr{S}(L)$.

In case of an underlying algebraic and orthocoherent test space endowed with a weight ω we immediately get an induced state on $\Pi(\mathfrak{A})$ by setting $s(p(A)) = \omega(A)$. Then from Eq. (3.31) we get for $p(A) \perp p(B) \Leftrightarrow A \perp B$ that

$$s(p(A) \vee p(B)) = s(p(A \sqcup B)) \tag{3.48}$$

$$= \omega(A \sqcup B) \tag{3.49}$$

$$= \omega(A) + \omega(B) \tag{3.50}$$

$$= s(p(A)) + s(p(B)). \tag{3.51}$$

Note that orthocoherence is needed for Eq.(3.48).

3. Mixtures of States

We define a *mixture of states* in the following way.

Let $\{c_j\},\ j \in J$, J countable, be real numbers such that $c_j \in [0,1]$ and $\sum_J c_j = 1$, furthermore let $\{s_j\},\ j \in J$ be statistical states on a logic L, then $s(a) = \sum_J c_j s_j(a)$ is a state on L as well and it is called a *mixture of the states* $\{s_j\}$ or just a *mixed state*. This way to combine states is also known as *convex combination*. To see that mixtures are states, let $\{a_i\}$, $i \in I$ an orthogonal countable sequence of propositions, then

$$s\left(\bigvee_I a_i\right) = \sum_J c_j s_j \left(\bigvee_I a_i\right) \tag{3.52}$$

$$= \sum_J c_j \left(\sum_I s_j(a_i)\right) \tag{3.53}$$

$$= \sum_I \sum_J c_j s_j(a_i) \tag{3.54}$$

$$= \sum_I s(a_i). \tag{3.55}$$

As all sums converge by definition. Also note

$$s(1) = \sum_J c_j s_j(1) \tag{3.56}$$

$$= \sum_J c_j \tag{3.57}$$

$$= 1. \tag{3.58}$$

One fact that we will use later is that all states are mixtures of pure states. A pure state is a state s that cannot be written as a nontrivial mixture. A trivial mixture is the sum over just s itself. The claim follows directly from the definition, as we can split up each "mixing" state s_i of a mixture until only pure states are left.

Another type of state is the *dispersion-free state* sometimes also called a *classical state* defined as a state satisfying: for all $a \in L$ either $s(a) = 0$ or $s(a) = 1$.

We then have that every dispersion-free state is pure. The following proof is due to Pták and Pulmannová (1991). Assume s is dispersion-free and we can write $s = cs_1 + (1-c)s_2$ with $c \in (0,1)$ and $s_1 \neq s_2$. Then there exists some $a \in L$ such that $s_1(a) \neq s_2(a)$. But then $cs_1(a) + (1-c)s_2(a)$ can never be equal to one, as it is equal to one if $s_1 = s_2 = 1$ and less otherwise. Neither can it be equal to zero as it is equal to zero for $s_1 = s_2 = 0$ and bigger otherwise. On Boolean lattices the converse holds as well, see Pták and Pulmannová (1991, Proposition 2.1.8) for a proof.

The structure of $\mathscr{S}(L)$ is interesting and there are powerful ways to describe it, but we will content ourselves with this, some further details can be found in the book by Cohen (1989).

D. Observables

Here we state the definitions needed in section IV.D in a very rudimentary form. They are due to Cohen (1989).

Let $\mathbb{I} = \{(a,b] | a,b \in \mathfrak{R}\}$ then the collection $\mathbb{B}$ of *Borel sets* on the real numbers $\mathfrak{R}$ is the smallest collection of subsets of $\mathfrak{R}$ satisfying:

(1) $\mathbb{I} \subseteq \mathbb{B}$;

(2) if $B \in \mathbb{B}$, then $\mathfrak{R} \backslash B \in \mathbb{B}$;

(3) $\mathbb{B}$ is closed under countable unions.

A σ-*algebra* or *Boolean algebra*[3] $\mathbb{A}$ is a collection of subsets of a set X that satisfies:

(i) $X \in \mathbb{A}$;

(ii) if $S \in \mathbb{A}$, then $X \backslash S \in \mathbb{A}$;

(iii) if $\mathbb{S}$ is a countable subset of $\mathbb{A}$, then $\bigcup \mathbb{S} \in \mathbb{A}$.

A *measure on* $\mathbb{A}$ *for set* X is a function μ satisfying

(i) $\mu : \mathbb{A} \to \mathfrak{R}^{\infty}$;

(ii) $\mu(A) \geq 0$ for all $A \in \mathbb{A}$, and $\mu(A) < \infty$ for at least one $A \in \mathbb{A}$;

(iii) if $\mathbb{S} \subseteq \mathbb{A}$ is a pairwise disjoint, countable collection, then $\mu(\bigcup \mathbb{S}) = \sum_{S \in \mathbb{S}} \mu(S)$.

Then the members of $\mathbb{A}$ are called the μ-*measurable sets*. If μ exists $(X, \mathbb{A})$ is called a *measurable space*. If $\mu(X) = 1$ then μ is called a *probability measure*.

If μ is a measure on the σ-algebra $\mathbb{A}$ in X then a function $f : X \to \mathfrak{R}$ is called a μ-*measurable function* iff $f^{-1}(B) := \{x \in X | f(x) \in B\} \in \mathbb{A}$ for every Borel set B. If g is another μ-measurable function then f equals g μ-*almost everywhere* iff

$$\mu\{x \in X | f(x) \neq g(x)\} = 0. \tag{3.59}$$

This is written $f(x) =_{\mu} g(x)$.

The μ-*integral of a* μ-*measurable nonnegative function* f *over a set* $S \in \mathbb{A}$ is then defined by

$$\int_S f d\mu := \sup \left\{ \sum_{i=1}^{n} a_i \, \mu(h^{-1}(a_i) \cap S) \;\middle|\; \begin{array}{l} h \text{ is a nonnegative function with finite image,} \\ \text{image}(h) = \{a_1, \ldots, a_n\}, \text{ and } h(x) \leqq_{\mu} f(x) \text{ for all } x \text{ in } S \end{array} \right\}. \tag{3.60}$$

Suitable generalizations for negative and complex functions can be found in the book by Cohen (1989).

E. Dynamics

Here we state the formal definitions of poset and lattice automorphisms employed in section IV.E. Further concepts related to dynamics will be introduced there.

A $(\Pi, \leq, ')$-automorphism $m : \Pi \to \Pi$ satisfies:

[3] This is at the same time a σ-complete Boolean lattice

(i) let $p \in \Pi$ then $m(p') = m(p)'$;

(ii) let $q,\ p \in \Pi$ then $p \leq q$ iff $m(p) \leq m(q)$.

A $(\Pi, \vee, \wedge, 0, 1)$-automorphism $m : \Pi \to \Pi$ satisfies:

(a) if $p_i \in \Pi$, $i \in I$, I countable, then $m(\bigvee_I p_i) = \bigvee_I m(p_i)$;

(b) if $p_i \in \Pi$, $i \in I$, I countable, then $m(\bigwedge_I p_i) = \bigwedge_I m(p_i)$;

(c) $m(0) = 0$;

(d) $m(1) = 1$.

Let $(\Pi, \leq, ')$ be an orthocomplemented poset such that $(\Pi, \vee, \wedge, 0, 1)$ is a σ-complete bounded lattice. Then a $(\Pi, \leq, ')$-automorphism $m : \Pi \to \Pi$ is also a $(\Pi, \vee, \wedge, 0, 1)$-automorphism.

Note that in this case every property expressible with the symbols $(\leq, \vee, \wedge, ', 0, 1)$ is then preserved as well. For example orthomodularity. We now give a proof of the above assertion. Let m be a $(\Pi, \leq, ')$-automorphism.

ad (a): (Jauch, 1968), from the definition of the join we have $p_i \leq \bigvee_I p_i$ for every $i \in I$, then from (ii) we get $m(p_i) \leq m(\bigvee_I p_i)$. Now again by definition of the join $m(p_i) \leq \bigvee_I m(p_i)$ and $\bigvee_I m(p_i) \leq m(\bigvee_I p_i)$. The inverse $m^{-1}$of any automorphism is again an automorphism such that the former gives $p_i \leq m^{-1}\left(\bigvee_I m(p_i)\right)$, as this is true for every p_i it is also true for their join $\bigvee_I p_i \leq m^{-1}\left(\bigvee_I m(p_i)\right)$ and again by property (ii) $m(\bigvee_I p_i) \leq \bigvee_I m(p_i)$. Then implication in both directions of the equation in (a) is true.

ad (b): Similar to (a): $m(\bigwedge_I p_i) \leq p_i$ then $m(\bigwedge_I p_i) \leq m(p_i)$ and therefore $m(\bigwedge_I p_i) \leq \bigwedge_I m(p_i)$. Also $\bigwedge_I m(p_i) \leq m(p_i)$ thus $m^{-1}(\bigwedge_I m(p_i)) \leq p_i$ then $m^{-1}(\bigwedge_I m(p_i)) \leq \bigwedge_I p_i$ and finally $\bigwedge_I m(p_i) \leq m(\bigwedge_I p_i)$.

ad (c): Consider: $m(0) = m(p \wedge p') = m(p) \wedge m(p)' = 0$.

ad (d): Also: $m(1) = m(p \vee p') = m(p) \vee m(p)' = 1$.

If $(\Pi, \vee, \wedge, 0, 1)$ is atomistic we have two further properties: On the one hand a $(\Pi, \vee, \wedge, 0, 1)$-automorphism maps atoms to atoms.

Proof: (Jauch, 1968) Let a be an atom, then suppose $x < m(a)$ (remember this means $x \neq m(a)$ and $x \leq m(a)$). Then $m^{-1}(x) < a$, but because a is an atom $m^{-1}(x) = 0$. Therefore $x = m(0) = 0$ and $m(a)$ is an atom.

On the other hand a $(\Pi, \leq, ')$-automorphism is completely determined by its restriction to the atoms.

Proof: (Jauch, 1968) Note that because L is atomistic we can write every $x \in L$ as $x = \bigvee_I a_i$ with a_i atoms. Then $m(x) = m(\bigvee_I a_i) = \bigvee_I m(a_i)$. The second equality comes from m also being a $(\Pi, \vee, \wedge, 0, 1)$-automorphism. Knowing $m(a_i)$ is thus equivalent to knowing $m(x)$ for every x. That is what we wanted to proof.

IV. PHYSICS EMPLOYING THE EMPIRICAL LOGIC APPROACH

Preliminaries The previous chapter has only dealt with mathematical definitions and facts. This chapter aims to show how these can be employed within the context of physics.

Our goal in the following sections is first to motivate and define a few concepts sufficiently general as to apply to both classical and quantum mechanics. Second, we try to identify the mathematical representations of these concepts in these special cases.

A. Test Spaces

Overview This section introduces test spaces in a physical context. The idea of test spaces[4] and its application to physics is largely the work of Foulis and Randall, another protagonist in this line of research is Alexander Wilce.

Foulis and Randall (1972, 1974) describe in detail how test spaces arise naturally in physics. Drawing mainly from the latter publication, we extract a way to obtain a test space for a given physical system. This test space will be devoid of physical content and we refer to it as a "primitive test space". By way of "physical equivalence relations" and "refinements of tests" we can then (re-)introduce such content and eliminate superfluous tests. The resulting test space then contains structure of the physical system, and as we will in the course of this text a significant part. Furthermore we mention in this section an assumption about counterfactual events. After summa-

[4] The term test space is used in the more recent publications. The older closely related concept is that of a quasi-manual (see e.g. Foulis *et al.*, 1983). In this context tests are often called experiments or operations.

rizing the concept of a "physical" test space, we apply it to the mathematical models of classical and quantum mechanics. This leads us (in our language) to the test spaces of the classical and the quantum universe. The former will be found to contain a single test equivalent to phase space and the latter will contain as tests orthonormal bases of Hilbert space.

For the reader's convenience we recall here the formal definition of a test space (see Sec. III.A). Mathematically (cf. e.g. Wilce, 2000) a *test space* $\mathfrak{A}$ is a collection of non-empty sets that satisfies the following condition called *irredundancy*:

$$\forall E, F \in \mathfrak{A},\ E \subseteq F \Rightarrow E = F. \tag{4.1}$$

The elements of $\mathfrak{A}$ are called *tests* and elements of a test are called *outcomes*. The set $X = \bigcup \mathfrak{A}$ is the *outcome space* and sets of outcomes that are subsets of at least one test are referred to as *events*. The set of events is denoted $\mathscr{E}(\mathfrak{A})$. Moreover two events A, B are *orthogonal* $A \perp B$ if they are disjoint events of a common test and they are *compatible* if $E \cup F$ is an event. .

1. The Primitive Test Space

a. Physical Operations First let us quote directly from Foulis and Randall (1974) the definition of the main concept underlying test spaces, the *physical operation* (p.81):

> By a *physical operation*, we mean instructions that describe a well-defined, physically realizable, reproducible procedure and furthermore that specify what must be observed and what can be recorded as a consequence of an execution of this procedure. In particular a physical operation must require that, as a consequence of each execution of the instructions, one and only one symbol from a specified set E be recorded as the *outcome* of this realization of the physical operation. We refer to the set E as the *outcome set* for the physical operation. Notice that the outcome of a realization of a physical operation is merely a *symbol*; it is not any real or imagined occurrence in the 'physical world out there'. Also, observe that, if we delete or add details to the instructions for any physical operation, especially if we modify the outcome set R in any way, we thereby define a physical operation.

A physical operation then specifies, a procedure, possible observations and a set of symbols, the outcome set E. Furthermore it must specify which outcome is recorded in case of which observation during the execution of the procedure. This correspondence must be one-to-one. Moreover observations and with them outcomes must be mutually exclusive and exhaustive. Mutually exclusive in the sense that if any outcome is recorded (or any observation is made), no other outcome can be recorded (no other observation can be made) for the same execution of the physical operation. Exhaustive in the sense that nothing could happen such that none of the prespecified outcomes is recorded (none of the observations is made).

Note that it is easy to combine multiple physical operations to new ones by adding instructions, for example to sequentially execute two physical operations and record pairs of outcomes as outcomes.

Also note that in practice the outcome set is always finite, yet we may admit idealized physical operations with infinite, and even continuous outcome sets when appropriate.

b. Obtaining the Primitive Test Space Let us now consider some physical system. In the general case there are certainly multiple physical operations, that may be executed in order to investigate this system. With defining these multiple physical operations, comes the need to specify the associated outcome sets. In addition to previous requirements, these outcome sets are now required to be composed out of unique symbols. That is to say, not only must the symbols be unique within each physical operation, but unique across all physical operations considered. The essential consequence of this is that outcome sets are pairwise disjoint, and there is thus a one-to-one correspondence of physical operations and outcome sets. From this correspondence it is possible to identify the physical operation from the outcome set alone. We then call a collection of physical operations satisfying this condition a *primitive test space*. The outcome sets, from which it is then possible to recover the physical operations, are its *tests*. We see immediately that the primitive test space is irredundant, and therefore a test space. The terminology introduced above for formal test spaces is now applicable. Here is a simple example of a primitive test space with tests E_1, E_2, E_3:

$$\begin{aligned} E_1 &= \{q, w, e, r, t\} \\ E_2 &= \{s, d\} \\ E_1 &= \{6, 4, +, A\}. \end{aligned} \tag{4.2}$$

Above we have not defined which physical operations should be taken into consideration for a primitive test space. We might want to include every physical operation that influences the physical system, but deciding which to exclude is non-trivial. Apriori we cannot be sure that some arbitrary operation does not influence an investigated physical system. Leaving out some possible physical operations then amounts to an idealization of the physical system. The physical operations are then a subjective choice of the physicist and reflect in how far an idealization is accepted.

c. The Primitive Test Space and Physical Content In a primitive test space, physical operations are formally represented as entirely unrelated tests. The tests being disjoint sets, we may not infer any relation of an outcome of one test to an outcome of another test from the primitive test space. As a consequence this test space does not encode information about the associated physical system. Though we may look at cardinality of the primitive test space or at the cardinality of its tests, both of these depend on choices of the physicist, namely which physical operations to consider and how to define them.

It is important to accept, that, within empirical logic, no independent rule defining how physical content should be obtained is given. All such content must be derived from an already existing model of the considered physical system.

Let us look at the physical operations corresponding to the tests. There we might have an intuition, or a theory, telling us that some observation in one physical operation and another observation in another physical operation are "physically equivalent". In other words have the same "physical implications". Then we could try to incorporate this physical equivalence into our primitive test space. Let us cite Foulis and Randall (1974) again (p.83, the original employs a different terminology. Square brackets indicate the identical concept in terms used in this text.):

> We are now in a position to consider just which pairs of outcomes of operations [tests] in the manual $\mathscr{D}$ [primitive test space $\mathscr{D}$] we should construe as being 'physically equivalent'. Such considerations could be based on practically anything from a subjective whim to an elaborate scientific theory, but, often they are based on an appropriate 'world picture' or model. For instance, we often prefer to regard a number of outcomes of distinct physical operations as registering the same 'property' or as representing the same 'measurement'. If a voltage is measured using different instruments - or even different methods - identical numerical results are ordinarily taken to be equivalent.

We are now going to investigate this further.

2. *Incorporating Physical Equivalence Relations into a Test Space*

a. Physical Equivalence Relations In order to encode into the test space the physical equivalence of observations pertaining to different physical operations, an equivalence relation between the associated outcomes seems appropriate. Then for a pair of observations, physically equivalent according to the given model, we can relate the associated outcome pair.

Note that reflexivity, symmetry and transitivity are intuitive requirements for such physical equivalence, as can be easily checked using the example of different voltmeters by Foulis and Randall (1974) in the quote above (Sec.IV.A.1.c).

We will call an equivalence relation derived in the above way from a model, theory or intuition a *physical equivalence relation*. It is defined on the outcome space of a primitive test space. Note that two outcomes of the same test can never be physically equivalent (as they are mutually exclusive).

b. Incorporation Process We will now describe how to incorporate a given physical equivalence relation "$\equiv$" into a primitive test space. The result of the incorporation process will be a new test space encoding the additional structure represented by the physical equivalence relation. The presented procedure is mentioned alongside another equivalent one in Foulis and Randall (1974).

We start with a primitive test space. For tests we write E, F, G, H, outcomes are usually denoted by x_E, x_F, y_E, y_F where the index indicates the test they belong to, $E = \bigcup x_E$. Note that x_E and x_F need not have anything in common.

Through the given equivalence relation "$\equiv$" we obtain an equivalence class

$$[x_E] = \{y_F \in X | x_E \equiv y_F,\ F \text{ a test}\} \tag{4.3}$$

for each outcome $x_E \in X$. For all tests E let $[E]$ denote the set of the equivalence classes of the outcomes of E i.e.

$$[E] = \{[x_E] | x_E \in X\}. \tag{4.4}$$

Observe that elements of $[E]$ may also be elements of the set $[F]$ for F a test, as we may have $x_E \equiv y_F$.

If there is a test F such that $[F] = [E]$ then, by definition, physically it should not matter which of the associated physical operations is executed. Therefore no physical information is lost if we remove F from the test space. In case of $[F] = [E]$ we also refer to the tests E and F as *physically* *equivalent tests*. In case $[F]$ is a proper subset of $[E]$ then we also remove F from the test space[5], as all outcomes of F are physically equivalent to an outcome of E and E even contains further outcomes[6].

The resulting set of tests is a test space, as it is irredundant by construction and there is still a one-to-one correspondence of tests and physical operations. In accordance with the given equivalence relation, in this step, we eliminated superfluous tests from the primitive test space. Later in Sec.IV.A.3 we will see how to remove some more tests, roughly speaking those having counterparts with higher resolution.

c. Revision of the Notation Note that there may still be physically equivalent outcomes, $x_E \equiv y_F$, that belong to multiple tests in case $[E]$ and $[F]$ partly overlap. At the moment such outcomes are still different symbols in the test space. This means that we cannot recognize physically equivalent outcomes directly from the test space. This can be achieved though. Recall that for no two tests $E,\ F$ we still have $[E] = [F]$, because all such $[F]$ have been removed. But then $[E]$ is a unique union of equivalence classes of outcomes and E is uniquely determined by $[E]$. Then, if we denote all outcomes belonging to the same equivalence class by a single symbol, say x, the tests stay in one-to-one correspondence to physical operations and remain irredundant. From a test space thereby obtained we can immediately recognize physically equivalent outcomes as those denoted by the same symbol.

Note the consequences for events. Formally two events, $A,\ B$ with $A \subseteq E, B \subseteq F$ are *physically equivalent* if $[A] = [B]$, where $[A] = \{[x_E] | x_E \in A\}$ and accordingly for $[B]$. In the newly introduced notation this is just denoted $A = B$. Therefore in the new test space events may belong to multiple tests.

d. Standard Terminology With the new notation some further terminology (cf. Foulis *et al.*, 1983) makes sense. Let outcome x be recorded as a consequence of the physical operation associated to

[5] This does not mean we generally remove for example a physical operation that detects through which of two slits a particle went in favor of one with three slits. The condition is that the equivalence relation connects the outcomes, and by default no outcomes were equivalent. In fact irredundancy can be seen as a condition on the equivalence relation (Foulis and Randall, 1974).

[6] The question may be asked whether such situations ever arise in practice. If $[F] \subset [E]$, intuitively we might suspect that the physical operation associated to F cannot be exhaustive as the one associated to E admits the physically equivalent observations and additional ones.

the test E. Moreover let $x \in A$ with $A \subseteq E$ an event. Then the event A is said to have *occurred*, E was *executed* and is called a *test for* A. Hence an event A can only occur as a consequence of the execution of a test for A. Sometimes we just say x occurred for E.

e. The Relation between Primitive and Formal Test Spaces Foulis and Randall (1974) remark that every formal test space, can be obtained by the previous procedure from a primitive test space and a suitable equivalence relation on the outcome space.

3. Coarsenings and Refinements of Tests

a. An Example Next we want to introduce the idea of "coarsening" and "refinement" of a test. To motivate this idea we present a simple example.

Consider two physical operations, one only records whether some particle is in the left or right half of some volume and the other one records whether the particle is in the top-left, top-right, bottom-left or bottom-right part of it. The associated tests may be denoted $E = \{l, r\}$ (l for left and r for right) and $F = \{t-l, t-r, b-l, b-r\}$ ($t-l$ for top-left and b_l for bottom-left and so on).

We can always create a new test $G = \{a, b\}$, a "coarsening" of F, in the following way. Change the instructions of the physical operation associated to F in such a way that, if any of the two observations represented by the symbols $t-l$ and $b-l$ is made, record a as the outcome of the new physical operation. Similarly record b, if any of the two observations represented by the symbols $t-r$ and $b-r$ is made. The test associated to the new physical operation is then G. We may say G is a coarse-grained version or a coarsening of F. In this case we call F a refinement of G.

A physical equivalence relation derived from intuition suggests $l \equiv a$ and $r \equiv b$. We then have $[E] = [G]$ and can remove one of the tests, but as F provides all information and more that G could ever reveal, we might as well remove both E and G from the test space.

b. General Definition Let us generally define coarsenings now. A test G is a *coarsening* of a test F if the following conditions are satisfied:

(i) There is a sequence of orthogonal events $A_1, A_2, A_3, \ldots$ of F such that $\bigcup A_i = F$.

(ii) The physical operation associated to G is identical to the one associated to F with the only difference, that in case any outcome $x \in A_i$ occurred for F, a symbol representing A_i, e.g. i, is recorded as having occurred for G.

We also call F a *refinement* for E.

c. Removing further Tests from the Test Space Given that from a physical point of view a refinement contains equal or more information about the system compared to the original test, we are lead to append a rule to our procedure of incorporating a physical equivalence relation into a test space:

Delete every test physically equivalent to a coarsening of another test. This includes deleting all coarsenings afterwards.

Note that a test may have multiple not necessarily physically equivalent coarsenings. Then by the newly added rule, one test may account for the removal of multiple physically not equivalent tests. All this without losing access to relevant information.

An example of two not physically equivalent coarsenings can be obtained by constructing a further coarsening H of F in the previously mentioned example. This coarsening H concentrates on whether the particle is in the top or the bottom part of the volume. Let $H = \{c, d\}$. Similar to the construction of G before, change the instructions of the physical operation associated to F in such a way that, if any of the two outcomes $t-l$ and $t-r$ occurs, record c as the outcome of the physical operation associated to G. Similarly record d, if any of the two outcomes $b-l$ and $b-r$ occurs. Intuitively, this experiment is not physically equivalent to E. It is equivalent though to other possible tests deciding, whether the particle is in the top or bottom part of the volume.

d. Abuse of Terminology Allow us to abuse the presented terminology on coarsenings and refinements in the following way. We say a test E has a refinement, if there exists a test F whose coarsening is equivalent to E. Conversely we say E is a coarsening of F in this case.

4. A Counterfactual Assumption

Foulis *et al.* (1983) assume that the occurrence of an event A is independent of the test for A that was actually executed.

Let A be an event and E, F tests for A. Then the assumption states that, if an execution of E results in the occurrence of A, an execution of F would also have resulted in an occurrence of A.

We see here that this is an assumption about counterfactual physical operations. A counterfactual physical operation is one that could have been performed but was not. For lack of a better term we call this assumption, the *counterfactual assumption*.

Recall that A is an event of multiple tests only due to the physical equivalence of the event $A = A_E$ as an event of E and the event $A = A_F$ as an event of F. In this terminology, the assumption states that if an event A_E occurs in a test E, and if a test F contains a physically equivalent event A_F, then A_F would have occurred if F would have been executed.

From this angle, the assumption concerns the meaning of physical equivalence. Physically equivalent events then must occur independent of the test (for them) that was actually executed.

We will not concern ourselves with a critique of this assumption. Instead we accept it for this work. It will be useful for the motivation of propositions in Sec.IV.B.1.

We want to mention though that this assumption is not an empirical concept. By definition it refers to something that can never be tested. Then the only way to justify it is through the validity of the concepts it leads us to develop.

5. *The Test Space of a Physical System*

To sum up the previous sections concerning test spaces, we give the following procedure.

From a chosen set of physical operations on the considered system, generate the primitive test space. Include in it all coarsenings of all tests. From an existing model, theory, or intuition, derive a physical equivalence relation on the outcome space of the primitive test space. Remove all tests physically equivalent to a coarsening of a test and incorporate the equivalence relation into the primitive test space as described in Sec.IV.A.2.b. Finally revise the notation as shown in Sec.IV.A.2.c.

We finally arrive at a test space as formally defined in Sec.III.A, and containing "concentrated" physically relevant information. Sometimes we refer to test spaces generated in this way as *physical test spaces.*

6. *The Test Space of the Classical Universe*

We will analyse the test space of a classical universe, meaning a universe accurately described by classical mechanics. To this end we choose as physical operations to be considered, all physical operations conceivable in such a universe, i.e. all those not in contradiction with the principles of classical mechanics. This leads us to primitive test space which is reduced by removing unnecessary tests according to our conception of classical mechanics.

a. Consequences of the Measurement Process First recall two basic assumption of classical mechanics (see Jauch, 1968):

1. The influence of a measurement on a physical system can be arbitrarily reduced.

2. The precision of a measurement can be arbitrarily increased.

In the language of test spaces, these translate into a requirements for the equivalence relation and the existence of certain tests:

1. The outcomes of each test are physically equivalent to the outcomes of another test, that has less influence on the physical system. In other words, for any given test, there is always a physically equivalent one, that has less influence on the system.

2. Each test is the coarsening of another test. Equivalently, each test has a refinement.

Next, assume a collection of physical operations on a classical system under consideration is given. Construct the primitive test space of this collection. Take any pair of tests E, F, having arbitrarily strong influence on the physical system. Then according to the first basic assumption, there are two tests $\bar{E}, \bar{F}$ such that $[E] = [\bar{E}]$ and $[F] = [\bar{F}]$ with virtually no influence on the system. According to the standard procedure we can delete E and F from the test space. Now we construct a new physical operation from the pair $\bar{E}, \bar{F}$ simply by executing the according physical operations sequentially and recording as outcomes the pairs of outcomes. We thereby obtain a new test, call it "$\bar{E} + \bar{F}$" for the moment. As the first test does not influence the outcome of the second, a coarsening of $\bar{E} + \bar{F}$ recording only the outcome of the first test is physically equivalent to the first test. A coarsening of $\bar{E} + \bar{F}$ recording only the outcomes of the second test is physically equivalent to the second test. Then we observe that $\bar{E} + \bar{F}$ is a refinement of both $\bar{E}$ and $\bar{F}$ which means we can delete those from the test space as well. Repeating the last few steps for each pair of tests we arrive at a single test. Therefore it makes sense to call test spaces consisting of only one single test *classical test spaces*. Of course this hold for the classical universe as well.

b. Consequences of the Mathematical Model Taking into account the phase space representation of the classical universe we might in fact guess what test the above procedure converges to. Namely, an idealized test the outcomes of which determine the phase space coordinates (q, p) of the universe (up to the desired precision). This test certainly refines every other test. See this by noting that any outcome of any test can be deduced from an outcome of such a test, because phase space coordinates determine the whole universe precisely. Following Foulis and Randall (1972) we call this test the "grand canonical test" and denote it by G for the moment.

Note that G is idealized, in "real world" practice, the set of outcomes of any real test is always finite and the precision with respect to any coordinate is not perfect. Then outcomes correspond to volumes in phase space. In the grand canonical test, the outcomes are actually uncountable considering that they encode the continuous coordinates of phase space. An intermediate but already idealized test would have the elements of a partition of phase space as outcomes, these are at most countably infinite (for infinite phase spaces for example).

In the following we will exclusively employ the ideal grand canonical test as the test space of the classical universe. We do this because the physical system we consider here is not the "real world" but instead the "ideal universe of classical mechanics".

Obviously then we could have just started out with the grand canonical test, but we wanted to present the above arguments for its generality.

Let us clarify G a little further. The grand canonical test determines the phase space coordinates. To formalize this we introduce a map $\phi : X \to \Omega$ from the set of outcomes X into phase space Ω. As the outcomes are mutually exclusive, no two outcomes should determine the same phase space coordinates, meaning ϕ is injective. Furthermore as G is exhaustive, every phase space coordinate pair should be the image of some outcome, making ϕ surjective and eventually bijective. If ϕ is bijective, then all operations, relations and functions defined on phase space (e.g. the topology) can be inherited by the previously unstructured outcome set X. In the rest of this text we will therefore say the outcome set of the grand canonical test is phase space. Then we can identify $\Omega = X = G$. For convenience we will sometimes use the shorthand $x := (q, p)$ if it is clear that we are talking about the grand canonical test.

7. The Test Space of the Quantum Universe

In this section we proceed similar to before. We try to reveal features of the test space of a quantum universe. To this end we allow theoretically possible physical operations, namely the measurements of observables, to generate the primitive test space. Employing our model of quantum mechanics we suggest a physical equivalence relation for this test space. We will not achieve an exhaustive treatment of this issue, but hope the arguments are sufficient to be considered. The following sections will eventually reveal more evidence in favor of the result obtained here.

The paradigm of measurements in quantum mechanics leads us to identify each physical operation on a quantum mechanical universe with the measurement of an observable (self-adjoint operators) on a Hilbert space $\mathscr{H}$. Then to obtain a primitive test space we have to label the possible observations as consequences of such physical operations uniquely. As mentioned in Sec.II.B.2 a measurement causes the state of the system to collapse into a subspace of $\mathscr{H}$. Then we say that an observation is always given by a state within this subspace, and we only have to label these uniquely to get the outcomes.

Let us now simplify things a little and only look at measurements on pure states. We could assume for example that every mixed state described by a density operator is actually in a pure state, but we just don't know which one exactly. Then all measurements are really measurements on pure states.

In addition to this we consider only special physical operations, namely those given by observables with non-degenerate eigenvalues. Then the associated eigenvectors provide an orthogonal basis of Hilbert space, which can be normalized. A measurement of such an observable on a system in a pure state then results in a pure state described by one of these (normalized) eigenvectors. To label these observations uniquely we may use this eigenvector written in a prespecified basis. It is then an outcome. We can therefore label all outcomes of an observable by its normalized orthogonal eigenvectors. In order to distinguish between the outcomes of different observables we could note the observable as an index to each eigenvector. The primitive test space is then the set of all sets of uniquely labelled eigenvectors of non-degenerate self-adjoint operators.

Now we should look for a physical equivalence relation in quantum mechanics. It is not a big stretch to consider outcomes denoting the same unit vector in $\mathscr{H}$ as equivalent, but we might argue for more. The "measurable quantities", probability $P(\lambda)$, expectation value and variance are all invariant, if the state vector is multiplied by a complex number of absolute value one. Then we might call two outcomes (associated to different observables) $|\psi\rangle$ and $|\phi\rangle$ physically equivalent if $|\psi\rangle = e^{i\alpha}|\phi\rangle$ for some $\alpha \in \Re$. Formally:

$$|\psi\rangle \equiv |\phi\rangle \Leftrightarrow |\psi\rangle = e^{i\alpha}|\phi\rangle. \tag{4.5}$$

We believe, but do not show, that the coarsenings of the tests in our primitive test space are physically equivalent to all tests given by observables with degenerate eigenvalues. This would justify our previous restriction to operators with non-degenerate eigenvalues.

Now note that for every orthonormal basis of Hilbert space there are multiple observables, whose normalized eigenvectors are physically equivalent to the orthonormal basis. All but one of the tests associated to these observables are then removed from the primitive test space. The physical equivalence relation incorporated into the primitive test space then leaves us with the set of all (physically in-equivalent) orthonormal bases of the Hilbert space. The tests are (by abuse of terminology) the orthonormal bases. The outcomes are (by further abuse of terminology) unit vectors that are, within the linear subspace they specify, unique across the whole outcome space. We call this test space, the frame test space $\mathscr{F}(\mathscr{H})$ of the quantum universe.

Remarks Note that, the construction by which we obtained refinements for any pair of tests in the classical universe (sequential execution) does not work out here, as, in general, a first test influences the outcome of a second. Furthermore we believe that Heisenberg's uncertainty relation prohibits any form of composition of two tests associated to non-commuting observables that results in a refinement for both of these tests. A proof is not given in this text.

B. Propositions and their Logic

Overview In this section we will see how a concept of a proposition arises in the general case of a test space associated to physical system. Note that the mathematics of Sec.III.A, III.B can directly be applied to the physical test spaces we obtain from the procedure in Sec.IV.A.5. To a certain extend the following repeats these mathematics but also tries to motivate the specific choice of objects in a physical context where possible. First we take a look at propositions and what information they may convey in a physical test space. Then we present the structure of the set of propositions, also called the proposition logic and what can be said about it in the general case. Finally we derive the proposition logics of the classical and the quantum universe from their respective test spaces. For the classical universe we will retrieve the power set of phase space, and for the quantum universe the set of (closed linear) subspaces of Hilbert space. In the latter case the projection operators onto the subspaces provide an equivalent structure.

1. The Concept of Propositions

Traditionally propositions are "sharable objects" (read: can be communicated) and bearers of truth-value (can be either true or false) (McGrath, Fall 2008). In the context of the empirical logic approach, intuitively a proposition should be something that is fundamentally testable, hence empirical in a sense. Also it should be clear which test(s) could be performed and which outcomes correspond to confirmation of a proposition and which to refutation. In accordance with these requirements it would be possible to choose events as propositions. In this case confirmation of a proposition would be nothing else than occurrence of the according event in a test for it. Yet Foulis and Randall represent propositions by more abstract objects.

These objects are only defined if a further assumption about the test space is made, namely its algebraicity Eq.(3.5). Algebraicity means that "perspective events share exactly the same complements" (Wilce, 2000, p.87). Why this should generally be required from a test space has not been clearly stated in the reviewed literature. The resulting mathematical structure is a case in point though. In any case the test spaces we are especially interested in are algebraic.

We now turn to perspectivity which has been introduced formally by Eq.(3.2). Observe here that perspectivity has a pretty clear interpretation as long as the counterfactual assumption holds: If an event A occurs in a test E, then its complement $E\backslash A$ of course did not occur. Therefore if another test F for $E\backslash A$ would have been performed, then (by the counterfactual assumption) $F\backslash(E\backslash A)$ would have occurred. This is just what $A \sim F\backslash(E\backslash A)$ means[7]. Algebraicity then says that if $F\backslash(E\backslash A)$ has another complement different from $E\backslash A$ (in any test) then A shares that complement.

As mentioned before, in an algebraic test space perspectivity becomes an equivalence relation[8], but not a physical one in the sense of Sec.IV.A.2.a. The difference is that if two events are perspec-

[7] Note that the counterfactual assumption makes this reasoning very plausible, but of course perspectivity can be defined without it as well. Then two events are perspective if their complements are physically equivalent.

[8] In a general test space, perspectivity is lacking transitivity to be an equivalence relation. Redefining perspectivity to include arbitrary orders of perspectivity, say $A \sim_{new} B$ iff there exists a finite family of events $\{C_i\}, i = \{1, ..., m\}$ such that $A \sim C_1 \sim ... \sim C_m \sim B$, is an alternative that does not need algebraicity. There exists another problem with this definition though, as propositions may be confirmed and refuted by the same outcomes (Foulis, 1998).

tive that does not mean that each outcome contained in the first event has an equivalent outcome in the second event. This was the definition of physically equivalent events in Sec.IV.A.2.c though. The equivalence classes of events under perspectivity are the *propositions*. We will denote an equivalence class under perspectivity or the *proposition of an event* A by $p(A)$. The family of propositions is written $\Pi(\mathfrak{A})$ and referred to as the *proposition logic* of the test space. Prominent examples of perspective events are the tests themselves as they share the empty set as complement, therefore there is a proposition that is the equivalence class of all tests. Of course $p(\emptyset)$ is also a proposition.

With the following definitions the use of the term logic in proposition logic and empirical logic finally gets some justification. A proposition p is *confirmed* if any one of the events belonging to this equivalence class occurs in a test for it. The proposition p is *refuted* in case the orthogonal complement $E \backslash A$ of an event A that belongs to p occurs in a test E for A. In general propositions may neither be confirmed nor refuted as a result of a test. An example of this is given in Sec.IV.B.4.c.

Remarks It should be clear why it makes sense to look at the equivalence classes of perspectivity instead of the events, at least under the counterfactual assumption. Let's say event A occurred, then if we are sure that all the events B with $B \sim A$ would have occurred in other tests (by the counterfactual assumption), then we can equivalently just state that $p(A)$ was confirmed. The information about the system that is independent of what test is executed is then clearly contained in the proposition.

2. *Structure of the Proposition Logic*

We will now try to motivate from a physical viewpoint the structure found on the set of propositions $\Pi(\mathfrak{A})$. Even in the most general case (algebraic test spaces) we quickly arrive at an orthocomplemented poset. There have been attempts to motivate further structure on general proposition logics $\Pi(\mathfrak{A})$ of physical test spaces. Here we omit this challenge and move on swiftly to the explicit examples of classical and quantum mechanics.

A most important structure of the proposition logic is generated by the implication. It is indeed possible to introduce a very natural notion of implication on the logic. By definition, if A, $B \subseteq E \in \mathfrak{A}$ and $A \subseteq B$ then every time the event A occurs, the event B occurs as well. Therefore we might say that event A implies event B. But in situations where the counterfactual assumption holds, it makes sense to extend the implication to events that are perspective to B. That is why the *implication* is defined by

$$\forall A, B \in \mathscr{E}(\mathfrak{A}),\, A \leq B \Leftrightarrow \exists C, C \perp A, (A \sqcup C) \sim B. \tag{4.6}$$

We have seen that this implication induces an implication between the corresponding propositions through: $p(A) \leq p(B) \Leftrightarrow A \leq B$. So with this implication $\Pi(\mathfrak{A})$ becomes a bounded poset where meet, join, unit and zero elements, and an orthocomplementation are defined. The unit element is $p(E)$ for E any test and is always true, as any result of any test confirms it and the zero element is $p(\emptyset)$, which is always refuted. Of course, if the test space is not exhaustive, none of the anticipated results must occur, and both unit and zero element loose their interpretation. Assuming the test space is exhaustive though the poset is clearly bounded.

From the definition of confirmation and refutation in the last section it follows directly, that a proposition $p(A)$ is confirmed ($x \subseteq B \sim A$ recorded) iff $p(A)' = p(E \backslash A)$ is refuted ($y \subseteq C \sim E \backslash (E \backslash A)$ recorded) and vice versa. We have seen in Sec.III.B.1 that ' defines an orthocomplementation. We thus get an interpretation of the orthocomplement as a kind of negation of a proposition. Remember though that propositions neither have to be confirmed nor refuted (see Sec.IV.B.4.c).

The meet $p(A) \wedge p(B)$ of two propositions $p(A)$, $p(B)$, if it exists, is a proposition $p(C)$ that implies both of them and, other than itself, $p(C)$ implies no proposition that also implies $p(A)$ and $p(B)$. Similarly the join $p(A) \vee p(B)$ of two propositions $p(A)$, $p(B)$, if it exists, is a proposition $p(C)$ implied by both of them that implies all propositions that are implied by both $p(A)$ and $p(B)$. In both cases, as propositions are defined as equivalence classes of events, the existence of an underlying event C is necessary. One has to be careful to interpret meet and join as "and" and "or" because they are subject to physical laws (the physical equivalence relation to be precise) here and not to "good reasoning". If the poset of propositions is a Boolean lattice it works well but in other cases our everyday logic fails to predict how the confirmation of composed propositions depend on the confirmation of their parts.

The notion of orthogonality of events can be directly lifted to propositions by $p(A) \perp p(B) \Leftrightarrow A \perp B$.

It would be nice to have a good reason for requiring orthocoherence of a physical test space, but this is not easy to accomplish according to the literature (see e.g. Wilce, 2000). We could in such a case define compatibility (and later, statistical states) in this general setting. Soon we come back to this in the special cases.

Remarks Up to now the only assumption that we have made that influences the structure of the test space is algebraicity. Observe that this is not so much a condition on the primitive test space, but rather on the physical equivalence relation. In fact all condition on the structure of events and therefore on $\Pi(\mathfrak{A})$ are mainly conditions on the physical equivalence relation. As mentioned in Sec.IV.A.2.e every test space can be obtained from a primitive test space and a suitable equivalence relation. The exception is where the amount of physical operations or outcomes in the primitive test space is not sufficient for a certain property of $\Pi(\mathfrak{A})$. A simple example for this occurs if we require incompatible events to exist, then primitive test spaces with a single physical operation are excluded. It is intuitively clear though that once we have a sufficient amount of physical operations we might patch them together by a suitable physical equivalence relation to obtain any possible test space. Then recall that any orthoalgebra, and therefore any orthomodular poset or Boolean lattice arises from a test space (see Sec.III.B). These structures then can in general be seen to depend ultimately on a physical equivalence relation.

3. The Proposition Logic of the Classical Universe

We have seen in Sec.IV.A.6 that the test space of the classical universe Ω contains only one test. It is obviously algebraic. Then all events are only perspective to themselves and therefore the events are exactly the propositions, formally[9] $p(A) = \{A\}$. Let Ω denote the unique test then for the orthocomplement we have $p(A)' = p(\Omega \backslash A) = \{\Omega \backslash A\}$. The implication reduces to standard set inclusion as $p(A) \leq p(B) \Leftrightarrow A \subseteq B$. Meet and join are merely set intersection $\bigwedge_I p(A_i) = p(\bigcap_I A_i) = \{\bigcap_I A_i\}$ and set union $\bigvee_I p(A_i) = p(\bigcup_I A_i) = \{\bigcup_I A_i\}$ and they exist for every family of events $\{A_i\}, i \in I$, making $\Pi(\Omega)$ a complete lattice. As it is well known that set intersection and set union are distributive operations, formally $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, this lattice is also distributive. Bounded distributive complemented[10] lattices are Boolean lattices, which are always orthomodular (see Sec.III.B.4.b). The power set $\mathscr{P}(S)$ of any set S is always a complete Boolean lattice and the set of events of a test S is just the power set $\mathscr{P}(S)$ of the experiment. The proposition logic of classical universes is then $\mathscr{P}(\Omega)$.

[9] The right hand side of the following equation denotes the set, that is the equivalence class.
[10] In case of distributivity a complement is always an orthocomplement (see Sec.III.B.4.b).

What we can take from this is that the test space does not impose any more structure on representatives of propositions than that they are the subsets of a set. When considering the grand canonical test, events and classical propositions are just subsets of phase space. Notice that all events in classical test spaces are compatible and so are the propositions (compatibility is defined, as we are on a orthomodular poset, see Sec.III.B.5). For compatibility of propositions p, q we need according to the definition in Sec.III.B.5, three propositions, u, v, w such that:

1. u, v, w is a pairwise orthogonal set in L;

2. $u \vee v = p$ and $v \vee w = q$.

For two propositions $p(A), p(B)$ just choose $p(A \backslash B), p(A \cap B), p(B \backslash A)$ as the three necessary pairwise orthogonal propositions. Then clearly $p(A \backslash B) \vee p(A \cap B) = p(A)$ and $p(A \cap B) \vee p(B \backslash A) = p(B)$.

4. *The Proposition Logic of the Quantum Universe*

We have argued in Sec.IV.A.7 that the test space of quantum universes is the frame test space $\mathscr{F}(\mathscr{H})$. Now we investigate what kind of proposition logic $\Pi(\mathscr{F}(\mathscr{H}))$ arises from this test space. It should become clear that $\Pi(\mathscr{F}(\mathscr{H}))$ is a σ-complete orthomodular atomistic lattice. We will not give the rigorous proofs as they need only knowledge common to physicists, specifically the theory of closed linear subspaces and orthonormal bases in Hilbert space.

a. The Poset of Propositions First we check the frame test space for algebraicity. Observe that each test E is an orthonormal basis for $\mathscr{H}$, say $E = \{e_i\}$, $i \in I$. Each outcome x is then just a unit vector, e.g. $x = e_k$ and an event A is a set of unit vectors, e.g. $A = \{e_j | j \in J \subseteq I\}$. If e_k is an outcome then $e^{i\alpha} e_k$ with $\alpha \in \Re$ is not an outcome, as we have removed such outcomes due to physical equivalence.

We can immediately see that each event A is part of all tests F that share all unit vectors of A and contain further a set of unit vectors $B = F \backslash A$, such that B's unit vectors are pairwise orthogonal to the unit vectors of A and together with those unit vectors of A are a basis $A \cup B$. Thus, if $A \operatorname{oc} B$, B can be any orthonormal set of vectors spanning the whole subspace orthogonal to the subspace spanned by A. We denote the span (which is a closed linear subspace) of the unit vectors of an event A by $\bigvee A$ and its orthogonal subspace by $(\bigvee A)^{\perp}$. Now we can write

$$A \,\mathrm{oc}\, B \Leftrightarrow \bigvee B = \left(\bigvee A\right)^{\perp}. \tag{4.7}$$

Algebraicity Eq.(3.5) requests:

$$A \sim B \text{ oc } C \Rightarrow A \text{ oc } C. \tag{4.8}$$

For $A \sim B$ we need an event D such that $A \,\mathrm{oc}\, D \,\mathrm{oc}\, B$. Let $A \subseteq E$, then D is uniquely determined by $D = E \backslash A$. From $D \,\mathrm{oc}\, B$ we know that $D \cup B$ is a test, and see that B can be any event with $\bigvee B = \bigvee A$. This means that

$$A \sim B \Leftrightarrow \bigvee A = \bigvee B. \tag{4.9}$$

From this we immediately get algebraicity.

The propositions are then the equivalence classes under perspectivity. These are identified by the last equation as the closed linear (we will not repeat these properties) subspaces of the Hilbert space:

$$p(A) = \bigvee A. \tag{4.10}$$

Note that every subspace of the Hilbert space is a proposition. Next we look at implication Eq.(3.6). The definition was

$$A \leq B \Leftrightarrow \exists C, C \perp A, (A \sqcup C) \sim B. \tag{4.11}$$

Here $C \perp A$ means that all elements of A are pairwise orthogonal to those of C, because then there is indeed a test E such that C and A are events of E. From $(A \sqcup C) \sim B \Leftrightarrow \bigvee(A \sqcup C) = \bigvee B$ we see that

$$A \leq B \Leftrightarrow \bigvee A \subseteq \bigvee B. \tag{4.12}$$

Which lifts directly to the propositions:

$$p(A) \leq p(B) \Leftrightarrow p(A) \subseteq p(B) \Leftrightarrow \bigvee A \leq \bigvee B. \tag{4.13}$$

Another relation that lifts directly to the propositions is orthogonality:

$$p(A) \perp p(B) \Leftrightarrow A \perp B. \tag{4.14}$$

Moreover we identify the orthocomplement from Eq.(3.16):

$$p(A)' := p(E \backslash A), \tag{4.15}$$

such that:

$$p(A)' := \left(\bigvee A\right)^{\perp}. \tag{4.16}$$

In summary, with the easily identified unit and zero elements

$$1 = p(E) = \mathscr{H} \text{ and } 0 = p(\emptyset) = \vec{o}, \tag{4.17}$$

we get for the proposition logic $\Pi(\mathscr{F}(\mathscr{H}))$ of quantum mechanics:

$$p(A) \leq p(B) \;\Leftrightarrow\; \bigvee A \subseteq \bigvee B \tag{4.18a}$$

$$p(A)' \;=\; \left(\bigvee A\right)^{\perp} \tag{4.18b}$$

$$1 \;=\; \mathscr{H} \tag{4.18c}$$

$$0 \;=\; \vec{o}. \tag{4.18d}$$

We have seen in the general case in Sec.III.B.1 that $\Pi(\mathscr{F}(\mathscr{H}))$ is then an orthocomplemented poset.

b. Further Structure of the Proposition Logic The proposition logic $\Pi(\mathscr{F}(\mathscr{H}))$ carries additional structure though. First observe that it is a σ-complete lattice. The join of a sequence of propositions (subspaces) must be by definition, the smallest subspace containing all those propositions, this is the span of the subspaces, or, more precisely the span of the unit vectors spanning the subsets:

$$\bigvee_I p(A_i) = \bigvee_I \left(\bigvee A_i\right) = \bigvee \left(\bigcup_I A_i\right), \tag{4.19}$$

which is again a subspace. Note that $\bigcup_I A_i$ is not necessarily an event, the span exists nonetheless of course and also the join. In the special case of $A_i \perp A_j$ (and therefore $p(A_i) \perp p(A_j)$), for all $i \neq j$, $\bigcup_I A_i$ is an event as well, and we get:

$$\bigvee_I p(A_i) = \bigvee \left(\bigcup_I A_i\right) = p\left(\bigcup_I A_i\right). \tag{4.20}$$

The meet must be the biggest subspace contained in each element of the sequence of subsets, which is given by usual set intersection:

$$\bigwedge_I p(A_i) = \bigcap_I p(A_i). \tag{4.21}$$

Here if all A_i are compatible (part of the same test), we have

$$\bigwedge_I p(A_i) = \bigcap_I p(A_i) = p\left(\bigcap_I A_i\right). \tag{4.22}$$

Furthermore $\Pi(\mathscr{F}(\mathscr{H})$ is an orthomodular poset, consider again the five conditions:

(i) $0,\ 1$ are least and greatest elements of L;

(ii) $a \leq b$ implies $b' \leq a'$ for any $a, b \in L$;

(iii) $(a')' = a$;

(iv) if $\{a_i | i \in \mathbb{N}\}$ is a countable subset of L such that $a_i \leq a'_j$ for $i \neq j$, then the join $\bigvee_i a_i$ exists in L;

(v) if $a, b \in L$ and $a \leq b$, then $b = a \vee (b \wedge a')$.

As the defined orthocomplement is generally an orthocomplementation and (iv) follows directly from Eq.(4.20), we only need to verify (ii) and (v). But (ii) is obviously true as well. Let us look at (v) then.

Let $p(A) \leq p(B)$ then we need $p(B) = p(A) \vee (p(B) \wedge p(A)')$. We show this by choosing from the equivalence classes, which are the propositions, suitable events. From $p(A) \leq p(B) \Leftrightarrow \bigvee A \subseteq \bigvee B$ and Hilbert space basics, we know there are two events D_1, D_2, with $D_1 \subseteq D_2$ and $D_1,\ D_2 \subseteq E$ such that $\bigvee D_1 = p(A) = p(D_1)$ and $\bigvee D_2 = p(B) = p(D_2)$. We also have $p(A)' = p(D_1)' = p(E \backslash D_1)$. Then we need to show $p(D_2) = p(D_1) \vee (p(D_2) \wedge p(E \backslash D_1))$. Which is now easy to evaluate as all events are compatible. We get

$$p(D_2) = p(D_1) \vee (p(D_2) \wedge p(E \backslash D_1)) \tag{4.23}$$

$$= p(D_1) \vee p(D_2 \cap E \backslash D_1) \tag{4.24}$$

$$= p(D_1 \cup (D_2 \backslash D_1)) \tag{4.25}$$

$$= p(D_2). \tag{4.26}$$

So we have a σ-complete orthomodular lattice. Because every subspace of Hilbert space is a proposition, this also means that the subspaces of Hilbert space are the same σ-complete orthomodular lattice. We can furthermore identify the atoms of this lattice as the one dimensional subspaces. That every subspace of Hilbert space has a basis then gives atomicity. Also note that the frame test space must be orthocoherent as $\Pi(\mathscr{F}(\mathscr{H}))$ is orthomodular (see Sec.III.B.3.d).

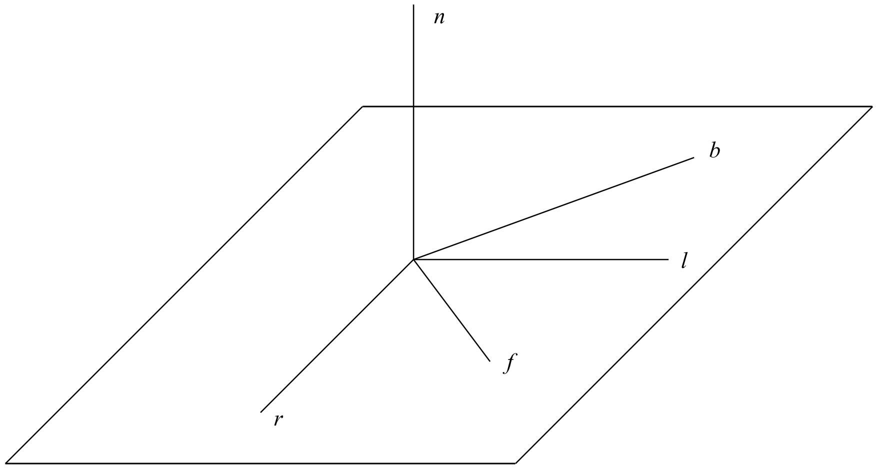

FIG. 1 Sketch of the subspaces representing the atoms of $\Pi(\mathfrak{A}_o)$.

c. A Simple Example We now present a simple test space with two tests, that partly illustrates the previous sections.

Consider the test space:

$$\mathfrak{A} = \{E = \{l, r, n\}, F = \{n, f, b\}\}.$$

A representation in a three dimensional vector space (which is a Hilbert space) is sketched in Fig.1. This is a well known example (Cohen, 1989; Foulis, 1999). As perspective events, we can always identify the tests themselves, and their proposition is the unit element

$$1 = p(E) = p(F) = \{E, F\}.$$

The empty set has

$$p(\emptyset) = 0.$$

Furthermore we observe that the event $\{l, r\}$ is perspective to $\{f, b\}$ and we see

$$p(\{l, r\}) = p(\{f, b\}) = p(E \backslash \{n\}) = p(\{n\})'.$$

Then with unit and zero element, and instead of $p(.)$ denoting directly the equivalence classes, we get

$$\Pi(\mathfrak{A}) = \{0, \{l\}, \{r\}, \{n\}, \{f\}, \{b\}, \{l, n\}, \{r, n\}, \{\{l, r\}, \{f, b\}\}, \{n, f\}, \{n, b\}, 1\}$$

or shorthand, skipping the brackets and using the ortho complements

$$\Pi(\mathfrak{A}) = \{0, l, r, n, f, b, r', l', n', b', f', 1\}.$$

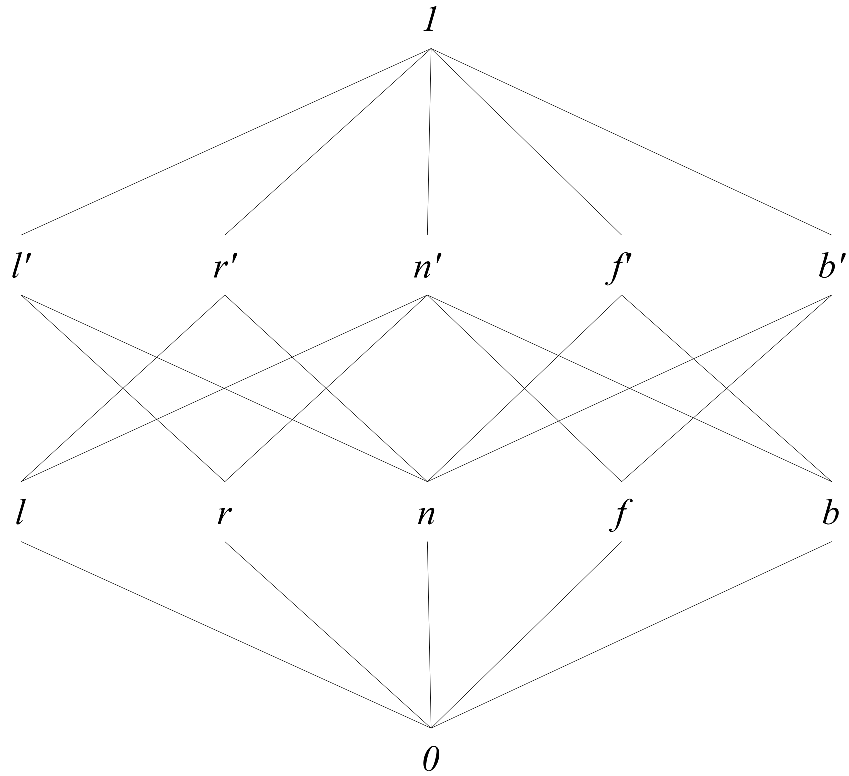

FIG. 2 Hasse diagram of the logic $\Pi(\mathfrak{A}_o)$.

To visualize the whole proposition logic $\Pi(\mathfrak{A})$ and the implications we present its Hasse diagram in Fig.2. There if $a \leq b$, b is written above a and a line connects the two propositions. It is then easy to find joins and meets by tracing the "implication chains" upwards (for joins) and downwards (for meets). In case they are unique and exist, the lowest intersection of the chains starting from two propositions upwards is their join (least upper bound) and the highest intersection downwards their meet (greatest lower bound). In Fig. 2 we can see easily that $\Pi(\mathfrak{A})$ is not distributive by noting for example that $r \wedge (b' \vee f') = r$ but $(r \wedge b') \vee (r \wedge f') = 0$.

Here we can also see that a proposition neither has to be confirmed nor refuted, take for example l and execute the test F, furthermore assume b has been recorded. Then $n' = \{\{l,r\},\{f,b\}\}$ and $f' = \{n,b\}$ are confirmed, but neither l nor l' is confirmed (or refuted).

d. The Proposition Logic as Projection Operators Each subspace of Hilbert space defines a unique projection operator[11] that projects onto it and vice versa. Then we define an the poset relations

[11] A projection operator is a self-adjoint operator P with $P^2 = P$.

and lattice operations for projection operators through their associated subspaces. Let P, Q be projection operators onto the subspaces K and L respectively, and each element P_i of a sequence project onto a subspace K_i then

(i) $P \leq Q \Leftrightarrow K \subseteq L$;

(ii) $P' = I - P$ is the projection onto $K^{\perp}$.

(iii) $\bigvee_I P_i$ is the projection onto $\bigvee_I K_i$;

(iv) $\bigwedge_I P_i$ is the projection onto $\bigcap_I K_i$.

This obviously results in a lattice isomorphic to that of the subspaces of Hilbert space, which we have found to be $\Pi(\mathscr{F}(\mathscr{H}))$. Therefore we will also denote it by $\Pi(\mathscr{F}(\mathscr{H}))$ and choose the more convenient structure as we please. We also note a few more properties of the projection operator lattice:

- $I = 1$, where I is the identity operator;
- $Z = 0$, where Z is the projection onto the null vector $\vec{o}$;
- $P \perp Q \Leftrightarrow K \perp L \Leftrightarrow PQ = 0$;
- if $P_i \perp P_j$ for all $i \neq j$ then $\bigvee_I P_i = \sum_I P_i$.

e. Summary We have established the proposition logic of a quantum universe to be the set of subspaces of a Hilbert space and equivalently the set of projection operators onto these subspaces. From now on we may then use this structure freely to formulate the further concepts relevant to physical systems for quantum universes. Where we are aware of formulations of such concepts directly on the frame test space we will also mention them. We do this in order to support the importance of test space structures in physics.

f. Compatibility on Frame Test Spaces Now let us look at the notion of compatibility on frame test spaces and on their logics. For the frame test space it is clear that there are incompatible events, just choose the events that account for the difference of two tests (in the above example Sec.IV.B.4.c: {f,b} and {l,r,}). Because $\mathscr{F}(\mathscr{H})$ is orthocoherent, the compatibility of events implies the compatibility of their associated propositions (see Sec.III.B.5).

Now in case we represent $\Pi(\mathscr{F}(\mathscr{H}))$ by subspaces, the compatibility of propositions translates into the following requirement for them: Let p,q be two propositions (as subspaces), then p, q are compatible iff both subspaces can be written as the span of their intersection $p \wedge q$ and the subspace orthogonal to the other subspace respectively. Formally p,q are compatible if (Cohen, 1989, Theorem 4.27)

$$p = (p \wedge q) \vee (p \wedge q^{\perp}) \text{ and } q = (p \wedge q) \vee (q \wedge p^{\perp}). \tag{4.27}$$

Note that $p \wedge q$, $p \wedge q^{\perp}$, and $q \wedge p^{\perp}$ are pairwise orthogonal as required. This also means that there is an orthonormal basis which contains as subsets a basis for each of them. In an underlying frame test space we therefore have a test with three orthogonal events A, B, C such that $p(A) = p \wedge q^{\perp}$, $p(B) = p \wedge q$ and $p(C) = q \wedge p^{\perp}$. Then because of orthogonality of the events $p = p(A \cup B)$ and $q = p(C \cup B)$. Hence, there exists a pair of compatible events for each pair of compatible propositions.

In case we represent $\Pi(\mathscr{F}(\mathscr{H}))$ by projection operators, this can be expressed more concisely. Let P,Q be two propositions (as projection operators), then the propositions associated to them are compatible iff (see Cohen, 1989, Theorem 5B.20)

$$PQ - QP = 0. \tag{4.28}$$

Remarks on $\Pi(\mathscr{F}(\mathscr{H}))$ and Hilbert Spaces We want to mention here that the lattice of subspaces of Hilbert space has been thoroughly investigated within the field of quantum logic (for an overview and references see e.g.Coecke *et al.*, 2000). In fact there are known conditions for posets (orthomodularity being one of them) such that they become exactly those representable by this lattice of subspaces of some Hilbert space (first proof by Solèr, 1995; a mathematical review of the proof can be found in Holland, 1995; more physically oriented use is made in Aerts and Steirteghem, 2000; Baltag and Smets, 2005). Even the field over which the Hilbert space is defined (reals, complex numbers or quaternions) can be characterized in this language (Mayet, 1998).

The following three sections IV.C, IV.D, and IV.E will reveal how major parts of the conventional formalism of quantum mechanics arise naturally from the lattice of subspaces of a Hilbert space. That is why the axiomatization of such lattices can provide a crucial part in complete axiomatizations of quantum mechanics (e.g. (Baltag and Smets, 2005)). As one of the used axioms is orthomodularity and every orthomodular poset can be seen to arise from a test space (Feldman and Wilce, 1993; Wilce, 2000), it may also be possible to formulate the axioms exclusively in the language of test spaces. This would make it possible to view the test space as a fundamental structure in quantum theory. We are not aware of such an axiomatization at the moment. But, with this in the back of our head, we mention in the following some candidate notions, defined on test spaces, inducing those defined on the lattice of propositions. For the main purpose of this text, it would of course be sufficient to use only notions defined directly on the proposition logic.

C. Statistical States

Overview This section introduces the notion of a statistical state. The empirical logic approach includes such a notion (Foulis and Randall, 1974), but similar concepts can be found for instance in Cohen (1989); Jauch (1968); Pták and Pulmannová (1991). Statistical states are defined on the proposition logic of the test space. We also mention here the concept of a weight on test spaces, that may be used to induce states on the proposition logic. Trying to apply the concepts, we will, in case of the classical universe encounter some problems with weights. These lead us to propose a slightly different underlying concept for the statistical states. The treatment of the statistical states for a classical universe in this text is not complete, but connects to the conventional description in classical mechanics. For the quantum universe we have Gleason's theorem which directly relates the statistical states of the present approach to the conventional expression.

1. The Concept of States

a. Statistical States of Physical Systems In the words of Foulis (1998) the notion of state refers to the following idea (p.13):

> The state of a system Σ is supposed to encode all available information in regard to the consequences of executing tests on Σ.

The states presented in this section are a special case of this in so far as they also encode the probability that each event occurs with. Such states are sometimes referred to as *probability states* or *statistical states* in contrast to "realistic states" that only carry the possible events.

From the operational point of view the probabilities are in some cases readily accessible. To obtain the probabilities of events we could measure the long term frequency of their occurrence.

This of course is only possible if there is a procedure which ensures that a system is in a certain state.

A function that can carry the information required of a state by Foulis is the weight. Recall the definition from Sec.III.C: Let $\mathfrak{A}$ be a test space and X its outcome set, then $\omega : X \to [0,1]$ is called a weight on $\mathfrak{A}$ if for every test $E \in \mathfrak{A}$, $\omega(E) := \sum_{x\in E} \omega(x) = 1$. The weight of an event A, $\omega(A)$, is the 'sum' of the weights of its outcomes $\omega(A) = \sum_{x\in A} \omega(x)$. In case of countably infinite events A, the sum here refers to the supremum of all sums over finite subsets of A.

Recall as well that the definition implies that perspective events have equal weights. Therefore, in case of an algebraic test space where perspectivity is an equivalence relation, the whole class of equivalent events is assigned the same weight. Hence we can assign the weight of any event to the according proposition. Thereby we define a function s_ω from the set of propositions $\Pi(\mathfrak{A})$ to the interval $[0,1]$, formally $s_\omega(p(A)) := \omega(A)$. Recall that the confirmation and refutation of propositions, under the counterfactual assumption, contain all information about the occurrence or non-occurrence of the contained events. It is then possible to consider the functions s_ω as states in Foulis' sense. This coincides then with the notion of statistical state presented in Sec.III.C.2. We have to require though that $\Pi(\mathfrak{A})$ be orthomodular because, as seen in Sec.III.C.2, we need Eq.(3.31):

$$A \perp B \Rightarrow p(A) \vee p(B) = p(A \cup B).$$

to hold. As mentioned before this is not easy to motivate on general physical test spaces, but the mathematical consequences are desirable and we accept it here.

Orthomodular proposition logics arise from orthocoherent test spaces, so under this condition, s_ω satisfies the formal definition of a statistical state: A *statistical state* on a bounded orthomodular poset $(L, \leq, ', 1, 0)$ is a function $s : L \to [0,1]$ such that:

(i) for $a, b \in L$, if $a \perp b$, then $s(a \vee b) = s(a) + s(b)$;

(ii) s(1) = 1.

A σ-additive statistical state is one that satisfies a stronger version of the first condition:

(i') for $\{a_i\}$, $i \in I$ an orthogonal countable sequence of propositions, then $\bigvee_I a_i$ exists, and $s(\bigvee_I a_i) = \sum_I s(a_i)$.

The set of states on a poset L is denoted $\mathscr{S}(L)$.

b. Special Kinds of States Recall now that sets of statistical (we will not repeat this in this section) states can be linearly combined to give other (statistical) states: If $\{c_i\}$, $i \in I$, I countable, are real numbers such that $c_i \in [0,1]$ and $\sum_I c_i = 1$, and furthermore $\{s_i\}$, $i \in I$ are states on an orthomodular poset L, then $s(a) = \sum_I c_i s_i(a)$ is a mixture of the states $\{s_i\}$, also called a mixed state.

We defined the dispersion-free or classical states as those satisfying: for all $a \in L$ either $s(a) = 0$ or $s(a) = 1$. These are then exactly those states that leave no uncertainty about which propositions will be confirmed and which will be refuted as a consequence of a test.

Then we distinguished two further kinds of states, those that are mixtures of classical states and those that cannot be written as nontrivial mixtures of such states, the latter are called pure states. If a state is a mixture of classical states, it might be argued that this constitutes a lack of knowledge about the state of the system, and that in reality one of the classical states is realized by the system. In other words the uncertainty is not "fundamental" or *ontological* as it is often expressed. Rather such an uncertainty is called *epistemic*. But then in case of a non-dispersion-free and pure state there is still an uncertainty left and we cannot trace it back to some missing knowledge of the system.

2. *The States of the Classical Universe*

In case of the classical universe, where events and propositions coincide, the notion of state and that of weight coincide. If we try to define them naively we run into a problem though. The problem is that weights (just like states) map into the interval $[0,1]$. To describe all possible states of classical mechanics, we propose a probability distribution underlying the states which can take on values in $[0,\infty)$. Such distributions will then induce the statistical states. The formalism we employ is familiar to physicists.

The outcomes of the grand canonical test are the phase space coordinates of a classical system. If the phase space coordinates $x = (q, p)$ are known, no uncertainty is left with regard to the outcomes of any test. This means, knowing these coordinates satisfies the condition by Foulis for knowing the state of the system. Furthermore the influence on the system by the measurement can be reduced arbitrarily such that in principle these coordinates can be obtained from the test without them changing in the course of the measurement. Intuitively then, if the the outcome x_1 is obtained as the consequence of the grand canonical test Ω, it should be given probability one and all others probability zero. We could try to define a weight according to this intuition. But this does not work well. Remember that phase space is continuous, then if we want to attribute a probability to an event or a proposition of Ω, the sum $\omega(A) = \sum_{x \in A} \omega(x)$ defined above has to change into an integral $\omega(A) = \int_A \omega(x)dx$, but this will always return zero if A is just a single outcome or any discrete set of outcomes.

Therefore we propose to define a probability distribution $\rho : X \to [0,\infty)$ such that $\int_X \rho(x)dx = 1$. This could be seen as a continuous version of a weight, if we require that $\int_E \rho(x)dx = 1$ for all tests E in a test space.

Now in case we know the exact coordinates x_1 we can write, using the Dirac δ-distribution, $\rho(x) = \delta(x - x_1)$. Note that there is then a one-to-one correspondence between such probability distributions and points in phase space.

Using the probability distribution we define a σ-additive statistical state by

$$s(p(A)) = s(A) = \int_A \rho(x)dx. \tag{4.29}$$

In the special case that A is just an outcome x we could use the somewhat ugly expression

$$s(x) = \lim_{\Delta \to 0} \int_{x-\Delta}^{x+\Delta} \rho(x')dx' \tag{4.30}$$

(Δ is a vector just like x). Also note that to get a δ-distribution there is no way to add up two other probability distributions. Then the δ-distributions produce the pure states, in this picture.

For clarity, we show the σ-additivity, let $\{A_i\}$ be a countable sequence of orthogonal propositions which, here, is just a posh expression for disjoint sets, recalling that the join is calculated by set union (see Sec.IV.B.3), and using standard integral manipulation:

$$s\left(\bigvee_I A_i\right) = \int_{\bigvee_I A_i} \rho(x)dx \tag{4.31}$$

$$= \int_{\bigcup_I A_i} \rho(x)dx \tag{4.32}$$

$$= \sum_I \int_{A_i} \rho(x)dx \tag{4.33}$$

$$= \sum_I s(A_i). \tag{4.34}$$

Let us now consider mixtures of states. Because of continuity of phase space, all probability distributions ρ, including continuous ones should be allowed to describe mixed states.

Naively we could try to define mixtures in the same way as before with a countable sequence of coefficients which add up to one. In this case though, we would lose the convenient property that every state is a mixture of pure states. This can be roughly demonstrated in the following way. As we have argued, pure states are represented by δ-distributions. Now it is intuitively clear that a state s_1 induced by a sum over any countable sequence of (weighted) δ-distributions does not converge to a state s_2 induced by a continuous probability distribution. The points (and propositions) x where the states are equal $s_1(x) = s_2(x)$ always remain a null-set. The same problem still pertains if we introduce a continuous function $c : \Re \to [0, \infty)$ such that $\int_R c(t) = 1$ and write a mixed state as $s(A) = \int_\Re c(t)\rho(t)\ dt$. This would only converge on a curve on phase space, which is still a null-set, and so on for $2\text{-}, 3\text{-}, ..., (2n-1)$-parameter functions c (n is the degrees of freedom, so $2n$ is the dimension of phase space). Note that all these constructions produce mixed states, and we allow them, but they cannot produce a continuous ρ (or approximate it arbitrarily). Therefore we will admit as coefficients (in addition to the above cases) functions on phase space itself: $c : \Omega \to [0, \infty)$ such that $\int_\Omega c(y) dy = 1$. Then we can recover any "continuous" state s from an uncountable set of pure states represented by $\rho_y = \delta(x_y - x)$ through:

$$
\begin{aligned}
s(A) &= \int_\Omega c(y) s_y(A)\ dy && (4.35)\\
&= \int_\Omega c(y) \int_A \rho_y(x)\ dx\, dy && (4.36)\\
&= \int_\Omega c(y) \int_A \delta(x_y - x)\ dx\, dy && (4.37)\\
&= \int_A \int_\Omega c(y) \delta(x_y - x)\ dx\, dy && (4.38)\\
&= \int_A \int_\Omega c(x_y) \delta(x_y - x)\ dx\, dx_y && (4.39)\\
&= \int_A c(x)\ dx. && (4.40)
\end{aligned}
$$

Where we have used a substitution $y \mapsto x_y$, which is legal as we integrate over all of Ω and x_y is in one-to-one correspondence with y.

It is easy to see, that each pure state defines a dispersion-free state as

$$
s(A) = \int_A \delta(x_1 - x)\ dx = \begin{cases} 1 & \text{if } x_1 \in A \\ 0 & \text{else.} \end{cases} \tag{4.41}
$$

At the same time, every dispersion-free state must be produced by a single δ-distribution $\rho = \delta(x_1 - x)$, as all other states attribute values different from 1 to at least two (atomic) propositions x_1 and x_2. Recall now the result about states on Boolean lattices that a state is pure if and only if it is dispersion free (see Sec.III.C). This is in accordance with our definitions of pure states and dispersion-free states.

Although the previous development of our proposed notion of state, is neither mathematically rigorous nor completely describes all possible cases, we hope that the general idea has become sufficiently clear.

3. The States of the Quantum Universe

For the quantum universe the situation has been clarified by a celebrated theorem by Andrew Gleason (original paper by Gleason, 1957; for a different proof in more similar terminology see Pták and Pulmannová, 1991). It asserts that for every σ-additive statistical state s on the set of subspaces $L(\mathscr{H})$ of a Hilbert space with dimension greater than two, there is a unique positive operator of trace one $\hat{\rho}$ such that for every proposition $p(M), M \in L(\mathscr{H})$, we have

$$s(p(M)) = Tr(\hat{\rho} P_M). \tag{4.42}$$

Where P_M is the projection operator onto the subspace M. Then all statistical states on the proposition logic $\Pi(\mathscr{F}(\mathscr{H}))$ of the quantum universe can be represented by such operators. The pure statistical states on the logic are exactly those with $\hat{\rho}^2 = \hat{\rho}$ and they are in one-to-one correspondence with the unit vectors of $\mathscr{H}$.

Gleason's theorem also entails that there are no dispersion free states on $L(\mathscr{H})$ if the dimension of Hilbert space is greater than two (see e.g. Svozil (preprint) and references therein).

On the side, note that maybe, Gleason's theorem may be expressed in terms of weights as well. Tests of the frame test space are at most countably infinite, or at least could probably be chosen as such, as the Hilbert space $\mathscr{H}$ is separable. Then weights can be defined as presented. Furthermore, every weight induces a state, and from a state we can recover the weight again through $\omega(x) = p(\omega(x))$. In general though, not every state is induced by a weight (Foulis and Randall, 1974). Unaware of an answer, we leave whether for a quantum universe the later statement would hold.

D. Observables

Overview The term observable is well known from standard physics courses. We will define an abstract notion of an observable on proposition logics that specializes to the well-known expressions of conventional physics. Roughly speaking they provide a connection of numerical values to the propositions. First though we will present the concept of an "elementary observable" (Foulis, 1998). Elementary observables are defined on test spaces and induce observables on their proposition logic.

Then, after the definition of the "normal" observables, we see how these can be used in conjunction with states to provide a probability measure on the real numbers. In other words how a probability for obtaining a certain numerical values from a system in a specified state can be defined. Furthermore we define abstractly the expectation value for an observable if the system is in a certain state.

Then we will see how, applied to the classical and the quantum universe, the notions of elementary observable, observable and expectation value specialize. In case of observables and the expectation value we retrieve the representatives well-known from classical and quantum mechanics.

1. *The Concept of Observables*

a. Elementary Observables and their Pre-images We begin then by defining an *elementary observable* (Foulis, 1998) on a test space $\mathfrak{A}$ as a map $\eta : E \to V$ where $E \in \mathfrak{A}$ and V is some value space in our case it will just be $\mathfrak{R}$. As tests usually have finite (or countable) outcomes, the image of η is usually a finite (countable) subset of $\mathfrak{R}$. The elements of this image will be called *measurable values*. The pre-image η^{-1} of η is defined as usual on the Borel sets of real numbers $\mathbb{B}$ by $\eta^{-1}(B) := \{x \in E | \eta(x) \in B\}$ where $B \in \mathbb{B}$. Then $\eta^{-1}(B)$ defines an event $A \subseteq E$ and, by going over to its equivalence class under perspectivity, a proposition $p(A)$. In this way we can assign propositions to subsets of $\mathfrak{R}$, let us denote this assignment by

$$\mathscr{O}_\eta : \mathbb{B} \to \Pi(\mathfrak{A}) \tag{4.43}$$

$$\mathscr{O}_\eta(B) := p(\eta^{-1}(B)). \tag{4.44}$$

Now we see that because of $\eta^{-1}(\mathfrak{R}) = E$ and $p(E) = 1$ we have $\mathscr{O}_\eta(\mathfrak{R}) = 1$. Furthermore pairwise disjoint sets $B_1, B_2 \in \mathbb{B}$ are mapped to orthogonal events $A_1 \perp A_2$ whose associated propositions are again orthogonal $p(A_1) \perp p(A_2)$ (in the appropriate sense). Finally take a look at $\mathscr{O}_\eta(\bigcup_{i=1}^\infty T_i)$ where the $T_i \in \mathbb{B}$ are a pairwise disjoint sequence. Clearly this is mapped to $p(\bigcup_{i=1}^\infty \{x \in E | \eta^{-1}(x) \in T_i\})$:

$$\mathscr{O}_\eta\left(\bigcup_{i=1}^{\infty} T_i\right) = p\left(\bigcup_{i=1}^{\infty}\{x\in E|\eta^{-1}(x)\in T_i\}\right) \tag{4.45}$$

now if

$$p\left(\bigcup_{i=1}^{\infty}\{x\in E|\eta^{-1}(x)\in T_i\}\right) = \bigvee_{i=1}^{\infty} p\left(\{x\in E|\eta^{-1}(x)\in T_i\}\right) \tag{4.46}$$

$\mathscr{O}_\eta$ satisfies

$$\mathscr{O}_\eta\left(\bigcup_{i=1}^{\infty} T_i\right) = \bigvee_{i=1}^{\infty}\left(\mathscr{O}_\eta\left(T_i\right)\right), \tag{4.47}$$

which is called σ-additivity. For the two cases we are interested in, Eq.(4.46) holds. It is in fact a countably infinite version of Eq.(3.31), which holds for orthocoherent test spaces. We will accept it in this text.

This construction then has the following operational interpretation: If the outcome x is obtained as a result of a test E then if E "measures" the elementary observable η, the measured value $v = \eta(x)$ is recorded. This value is then mapped to the proposition $p(\eta^{-1}(x))$ by the observable $\mathscr{O}_\eta$. Then, $p(\eta^{-1}(x))$ has been confirmed. Note that, the measurable values are always mapped to compatible events and therefore to compatible propositions.

b. Observables For the abstract definition of observables, above properties are required. An *observable* (Cohen, 1989) is then a function $\mathscr{O}:\mathbb{B}\to L$, where $\mathbb{B}$ is the collection of Borel sets of real numbers (see Sec.III.D) and L is a σ-complete lattice of propositions, such that:

1. $\mathscr{O}(\mathfrak{R}) = 1$

2. if $\{T_i\}$, $T_i\in\mathbb{B}$ is a pairwise disjoint sequence, then $\{\mathscr{O}(T_i)\}$ is a pairwise orthogonal sequence, and $\mathscr{O}(\bigcup_{i=1}^{\infty} T_i) = \bigvee_{i=1}^{\infty}(\mathscr{O}(T_i))$.

Later we also need the following: An observable is a *spectral measure* if L is a collection of projection operators on a Hilbert space.

Concerning the question whether every observable $\mathscr{O}$ mapping into the proposition logic $\Pi(\mathfrak{A})$ of a test space $\mathfrak{A}$ can be constructed from an elementary observable in the above way is left open in this text. At least in the general case[12]

[12] The following path may be possible: Show that the range of an observable is a Boolean subalgebra of L. In a Boolean algebra, all propositions are compatible. Then we need that all compatible propositions have compatible

c. Observables and States Together with the notion of a σ-additive state, elementary observables and observables provide a means to attribute probabilities to outcomes of tests depending on the state of the system. To see this note that $s\circ\mathscr{O}$ satisfies the conditions for a probability measure (see Sec.III.D) on $\mathbb{B}$ as

$$
\begin{aligned}
s\circ\mathscr{O}(\mathfrak{R}) &= s(\mathscr{O}(\mathfrak{R})) && (4.48)\\
&= s(1) && (4.49)\\
&= 1, && (4.50)
\end{aligned}
$$

and

$$
\begin{aligned}
s\circ\mathscr{O}\left(\bigcup_{i=1}^{\infty} T_i\right) &= s\left(\mathscr{O}\left(\bigcup_{i=1}^{\infty} T_i\right)\right) && (4.51)\\
&= s\left(\bigvee_{i=1}^{\infty}(\mathscr{O}(T_i)\right) && (4.52)\\
&= \sum_{i=1}^{\infty} s\left(\mathscr{O}\left(T_i\right)\right) && (4.53)\\
&= \sum_{i=1}^{\infty} s\circ\mathscr{O}\left(T_i\right). && (4.54)
\end{aligned}
$$

Then there is a measurable set with finite measure and because s is nonnegative, $s\circ\mathscr{O}$ is as well.

With this measure it is possible to define an *expectation value* $\mathrm{Exp}_{\mathscr{O}}$ *of the observable* $\mathscr{O}$ *if the system is in state* s in a general way, with (Cohen, 1989)

$$\mathrm{Exp}_{\mathscr{O}}(s) := \int_{\mathfrak{R}} I_{\mathfrak{R}} d(s\circ\mathscr{O}). \tag{4.55}$$

Here $I_{\mathfrak{R}}$ is the identity function on $\mathfrak{R}$, $I_{\mathfrak{R}}(x) = x$.

2. *Observables in the Classical Universe*

a. Identification with Functions on Phase Space In the classical universe there is only one test and events and propositions coincide $\Pi(\Omega) = \mathscr{P}(\Omega)$. Varadarajan (1985, Theorem 1.4) has proven that for every observable there exists an elementary observable (a real valued function such that its

underlying events (which is a part we are missing). Compatible events, are all part of the same test and we could use Varadarajan's theorem mentioned in Sec.IV.D.2.a. Note that Pták and Pulmannová (1991) prove the very first part for L an orthomodular poset and observables that satisfy the additional requirement $\mathscr{O}(\mathfrak{R}\backslash T) = \mathscr{O}(T)'$.

pre-image gives the observable). Then the observables are exactly the inverses of the elementary observables. This is in accordance with the conventional terminology. Usually functions $f(x) = f(q,p)$ on phase space are called "observables", though these would be referred to as elementary observables in the terminology presented here, because of the one-to-one correspondence with the observables defined here this does not cause confusion though.

b. Expectation Value Let us now try to obtain the expectation value from its general definition and the notion of state we proposed in Sec.IV.C.2. Again we will neither achieve full mathematical rigor nor an exhaustive treatment. We had

$$\mathrm{Exp}_{\mathscr{O}}(s) := \int_{\mathfrak{R}} I_{\mathfrak{R}} d(s \circ \mathscr{O}) \tag{4.56}$$

for the expectation value and for states, $p(A) = A \in \mathscr{P}(\Omega)$,

$$s(A) = \int_A \rho(x)dx. \tag{4.57}$$

Then clearly for any $B \in \mathbb{B}$:

$$s \circ \mathscr{O}(B) = s(\mathscr{O}(B)) \tag{4.58}$$

$$= \int_{\mathscr{O}(B)} \rho(x)dx. \tag{4.59}$$

We will now argue that the expectation value with this probability measure is indeed the usual expectation value known from conventional classical mechanics. First we evaluate it for a state represented by a finite combination of δ-distributions. Then we use heuristic arguments for the case of a state represented by a continuous probability distribution.

Assume for the moment that $s = \int \sum_J c_j \delta(x_j - x)dx$ with $J = \{1,...,n\}$ finite. Then $s \circ \mathscr{O}(t)$, $t \in \mathfrak{R}$, is zero for $\mathscr{O}(t) \subseteq \Omega \backslash \{x_1,...,x_n\}$. Because disjoint subspaces of $\mathfrak{R}$ are mapped to disjoint subspaces by $\mathscr{O}$ there can be maximally n values of t, say $\{t_1,...,t_k\}$, $k \leq n$ such that at least one element of $\{x_1,...,x_n\}$ is an element of $\mathscr{O}(t)$. Then we can replace $I_{\mathfrak{R}}$ by a function $g : \mathfrak{R} \to \mathfrak{R}$ of finite image, which is $(s \circ \mathscr{O})$-almost everywhere equal to $I_{\mathfrak{R}}$, in this case that is only for the values $\{t_1,...,t_k\}$. Such a function is

$$g(t) = \begin{cases} t_i & \text{if } t = t_i \\ 0 & \text{else.} \end{cases} \tag{4.60}$$

then we can rewrite the expectation value as:

$$\mathrm{Exp}_{\mathscr{O}}(s) = \int_{\Re} I_{\Re} d(s \circ \mathscr{O}) \tag{4.61}$$

$$= \int_{\Re} I_{\Re} d(s \circ \mathscr{O}) \tag{4.62}$$

$$= \sum_{i=1}^{k} t_i \, s(\mathscr{O}_\eta(t_i)) \tag{4.63}$$

$$= \sum_{i=1}^{k} t_i \int_{\mathscr{O}(t_i)} \rho(x) dx \tag{4.64}$$

$$= \sum_{i=1}^{k} t_i \int_{\mathscr{O}(t_i)} \sum_{j=1}^{n} c_j \delta(x_j - x) dx \tag{4.65}$$

$$= \sum_{i=1}^{k} t_i \sum_{j=1}^{n} c_j \int_{\mathscr{O}(t_i)} \delta(x_j - x) dx. \tag{4.66}$$

Then all c_j with $x_j \in \mathscr{O}(t_i)$ get multiplied with t_i. In other words, replacing the observable by its elementary observable counterpart $\mathscr{O}(t_i) = f^{-1}(t_i)$, all c_j with $x_j \in f^{-1}(t_i)$ get multiplied with t_i , but the value of these t_i is just $f(x_j)$ then. We then end up with

$$\mathrm{Exp}_{\mathscr{O}}(s) = \sum_{j=1}^{n} c_j f(x_j). \tag{4.67}$$

This is what we would expect.

Maybe Eq.(4.64) is a good start to anticipate what happens for a continuous distribution. We will get nonzero measure $s \circ \mathscr{O}(t)$ for all t, such that the sum over i becomes an integral over t. Again put $\mathscr{O}(B) = f^{-1}(B)$. We obtain:

$$\mathrm{Exp}_{\mathscr{O}}(s) = \int_{\Re} t \int_{f^{-1}(t)} \rho(x) dx dt. \tag{4.68}$$

Let us look at it for a while. To each t the inner integral is over a subset of phase space. For differing t these subsets are disjoint. For each point x within such a subset the value of $f(x)$ is t, it is multiplied with $\rho(x)$, which may vary within this set. As the outer integral goes over all of $\Re$ all values of t are integrated and as $f^{-1}(\Re) = 1 = \Omega$, the disjoint subsets associated to the t's must add up to Ω. But then we have $f(x)\rho(x)$ at every point x and all "added up" over phase space. Thus we must have:

$$\mathrm{Exp}_{\mathscr{O}}(s) = \int_{\Re} t \int_{f^{-1}(t)} \rho(x) dx dt \tag{4.69}$$

$$= \int_{\Omega} f(x) \rho(x) dx. \tag{4.70}$$

Comparing this to the expectation value in the conventional language Eq.(2.11) we find:

$$\mathrm{Exp}_{f^{-1}} = \bar{f}. \tag{4.71}$$

3. *Observables in the Quantum Universe*

In this section, we will see that in case of a quantum universe the abstract definition of an observable specializes to a spectral measure. By the spectral theorem, these are equivalent to self-adjoint operators, such that once again we recover the conventional description from a more general concept. As a consequence we will also see this happen for the expectation value. On the side we make some conjectures about the connection to elementary observables and their relation to commuting observables. Note that, lacking some mathematical machinery for the treatment of unbounded self-adjoint operators, we will not present the most general statements here.

After not-the-most-general connections have been established, we demonstrate them in case of a finite Hilbert space. This demonstration includes proofs of the conjectures for elementary observables in this case.

a. "General" Statements For quantum universes we have obtained the proposition logic $\Pi(\mathscr{F}(\mathscr{H}))$. The abstract definition of an observable demands a map from the borel sets of the real line $\mathbb{B}$ into this logic. We choose in this case the representation of $\Pi(\mathscr{F}(\mathscr{H}))$ as projection operators (see Sec.IV.B.4.d). Then the observables immediately satisfy the definition of a spectral measure (see Sec.IV.D.1.b). By the famous spectral theorem (for the statement and more references see e.g. Pták and Pulmannová, 1991) each such measure $\mathscr{O}$ uniquely defines a self-adjoint operator $A_{\mathscr{O}}$ and vice versa. Because of this one-to-one correspondence, referring to the self-adjoint operators as "observables" also makes sense. This is exactly what physicists usually do and in this way the notion of observable defined above specializes to the case of conventional quantum physics.

As precise statements of the following would require a lot more mathematical machinery, we will present only the case of finite Hilbert spaces. A more general treatment can be found in Jauch (1968).

The spectral theorem for bounded self-adjoint operators (the general case holds for unbounded ones as well) on infinite dimensional Hilbert spaces can be expressed as follows (Cohen, 1989).

First define a complex measure on $\mathbb{B}$ for every spectral measure $\mathscr{O}$. Let $x, y \in \mathscr{H}$, and set

$$\mu_{\mathscr{O},x,y}(B) = \langle \mathscr{O}(B)x, y \rangle. \tag{4.72}$$

Then the self-adjoint operator $A_{\mathscr{O}}$ corresponding to $\mathscr{O}$ is defined by

$$\langle A_{\mathscr{O}} x, y \rangle = \int_{\mathfrak{R}} I_{\mathfrak{R}} d\mu_{\mathscr{O},x,y}. \tag{4.73}$$

Which holds for all $x, y \in \mathscr{H}$.

Together with Gleason's theorem for representing states this theorem also implies that we can associate a probability measure $s \circ \mathscr{O}_A$ to every self-adjoint operator. Then it is possible to show that the expectation value of an observable $\mathscr{O}$ in state s can be expressed as the well-known trace:

$$\mathrm{Exp}_{\mathscr{O}}(s) = \mathrm{Tr}(\hat{\rho} A_{\mathscr{O}}). \tag{4.74}$$

b. Elementary Observables and Observables In quantum mechanics the test space is the frame test space $\mathscr{F}(\mathscr{H})$, therefore each elementary observable $\eta : E \in \mathscr{F}(\mathscr{H}) \to \mathfrak{R}$ assigns real numbers to the unit vectors of the frame (orthonormal basis) E. Then as the range of the induced observable $\mathscr{O}_\eta$ we use the projection operators onto the subspaces to represent $\Pi(\mathfrak{F}(\mathscr{H}))$.

$$\mathscr{O}_\eta(B) = p(\eta^{-1}(B)), \tag{4.75}$$

with $p(\eta^{-1}(B))$ now a projection operator. These induced observables in turn correspond to self-adjoint operators A_η. So each elementary observable induces such an operator. We conjecture the following theorems but do not prove them for the general case:

A. Every observable (and each self-adjoint operator) is induced by an elementary observable.

B. The eigenvalues of every self-adjoint operator are the measurable values of the elementary observables that induce this operator; and the projection operators associated to the measurable values by the induced observable project onto the eigenspaces associated to these eigenvalues.

C. Elementary observables on the same test induce commuting self-adjoint operators; and commuting self-adjoint operators can be induced by elementary observables on the same test.

For the finite dimensional case these will become clear in the following demonstration.

c. Treatment of the Finite Case Let $E \in \mathscr{F}(\mathscr{H})$ be a frame of an n-dimensional Hilbert space, η an elementary observable on E. Then η assigns a measurable value to every unit vector $e \in E$, let these real numbers be image$(\eta) = \{m_1, ..., m_k\}$ with $k \leqq n$. Then we can calculate the induced observable $\mathscr{O}_\eta$.

$$\begin{aligned}
\mathscr{O}_\eta(B) &= p\left(\eta^{-1}(B)\right) \\
&= p\left(\eta^{-1}(B \cap image(\eta))\right) \\
&= p\left(\eta^{-1}\left(\bigcup_i \{m_i | m_i \in B\}\right)\right) \\
&= p\left(\bigcup_i \eta^{-1}(\{m_i | m_i \in B\})\right) \\
&= \bigvee_i p\left(\eta^{-1}(\{m_i | m_i \in B\})\right).
\end{aligned} \tag{4.76}$$

From this we can calculate the associated self-adjoint operator, define $A_\eta := A_{\mathscr{O}_\eta}$, then:

$$\langle A_\eta x, y \rangle = \int_{\Re} I_{\Re} d\mu_{\mathscr{O}_\eta, x, y} \tag{4.77}$$

$$\tag{4.78}$$

To calculate the integral we proceed similar to before, replace $I_{\Re}$ by a function g with finite image that is $\mu_{\mathscr{O}_\eta,x,y}$-almost everywhere equal to $I_{\Re}$. To this end we note that $\mu_{\mathscr{O}_\eta,x,y}(B) = \langle \mathscr{O}_\eta(B)x, y \rangle = 0$ for all $x, y \in \mathscr{H}$ iff $\mathscr{O}_\eta(B) = 0$ (here 0 is the projector onto the null vector). And from Eq.(4.76) we see $\mathscr{O}_\eta(B) = 0$ iff $p(\eta^{-1}(\{m_i | m_i \in B\})) = 0$ for all i. This again happens iff all $\eta^{-1}(\{m_i | m_i \in B\}) = \emptyset$ which happens for all $B \subseteq \Re \setminus \text{image}(\eta)$. Then g is arbitrary for t in such B and we choose

$$g(t) = \begin{cases} m_i & \text{if } t = m_i \\ 0 & \text{else.} \end{cases} \tag{4.79}$$

Note that this is μ-almost everywhere the supremum of all functions μ-almost everywhere equal to $I_{\Re}$ as required by Eq.(3.60). The integral on the right side of Eq.(4.77) is then easily evaluated:

$$\langle A_\eta x, y\rangle = \int_{\Re} I_{\Re} d\mu_{\mathscr{O}_\eta,x,y} \tag{4.80}$$

$$= \sum_{i=1}^{k} m_i \, \mu_{\mathscr{O}_\eta,x,y}(g^{-1}(m_i)) \tag{4.81}$$

$$= \sum_{i=1}^{k} m_i \, \langle \mathscr{O}_\eta(m_i)x, y\rangle \tag{4.82}$$

$$= \left\langle \left(\sum_{i=1}^{k} m_i \mathscr{O}_\eta(m_i) \right) x, y \right\rangle. \tag{4.83}$$

This must be true for all x, y such that

$$A_\eta = \sum_{i=1}^{k} m_i \mathscr{O}_\eta(m_i) \tag{4.84}$$

$$= \sum_{i=1}^{k} m_i P^{\eta,m_i}. \tag{4.85}$$

The right hand side is a weighted sum of projectors sometimes called *spectral resolution* for which we introduced a shorthand in the second row. From physics courses on quantum mechanics we know that a self-adjoint operator can be represented by a sum over its eigenvalues multiplied with the projection operators onto their associated eigenspaces. We see then that the measurable values of an elementary observable are the eigenvalues of the self-adjoint operator associated to the induced observable. Moreover the projectors associated to those eigenvalues by the observable project onto the eigenspaces of this operator. For finite Hilbert spaces the above constitutes a proof. This proofs part of conjecture B. above, the existence for every self-adjoint operator is still missing.

We can now evaluate the measure $s \circ \mathscr{O}_\eta$ in case only one measurable value $m = \lambda$ is contained in B :

$$s \circ \mathscr{O}_\eta(B) = s(\mathscr{O}_\eta(B)) \tag{4.86}$$

$$= s(\mathscr{O}_\eta(m)) \tag{4.87}$$

$$= s(P^{\eta,m}) \tag{4.88}$$

$$= \mathrm{Tr}(\hat{\rho} P^{\eta,m}) \tag{4.89}$$

$$= \mathrm{Tr}(\hat{\rho} P_\lambda). \tag{4.90}$$

Which is finite dimensional version of Eq.(2.19).

Now we look at the RHS of Eq.(4.56) and replace A by its spectral resolution:

$$\mathrm{Tr}(\hat{\rho}A_\eta) = \mathrm{Tr}\left(\hat{\rho}\sum_{i=1}^{k} m_i P^{\eta,m_i}\right) \tag{4.91}$$

$$= \sum_{i=1}^{k} m_i\ \mathrm{Tr}(\hat{\rho}P^{\eta,m_i}) \tag{4.92}$$

$$= \sum_{i=1}^{k} m_i\ s(P^{\eta,m_i}). \tag{4.93}$$

The LHS may also be manipulated. Recall that

$$\mathrm{Exp}_{\mathscr{O}_\eta}(s) = \int_{\mathfrak{R}} I_{\mathfrak{R}} d(s\circ\mathscr{O}_\eta). \tag{4.94}$$

Then we proceed to calculate the integral analogous to before. We find that for a general s, the measure $s\circ\mathscr{O}_\eta(B)$ is zero in the same cases as the measure $\mu_{\mathscr{O}_\eta,x,y}$ because[13] $s(0)=0$. Hence we can use the same function g from above to calculate the integral. This gives

$$\mathrm{Exp}_{\mathscr{O}_\eta}(s) = \int_{\mathfrak{R}} I_{\mathfrak{R}} d(s\circ\mathscr{O}_\eta) \tag{4.95}$$

$$= \sum_{i=1}^{k} m_i\ (s\circ\mathscr{O}_\eta)(g^{-1}(m_i)) \tag{4.96}$$

$$= \sum_{i=1}^{k} m_i\ s(\mathscr{O}_\eta(m_i)) \tag{4.97}$$

$$= \sum_{i=1}^{k} m_i\ s(P^{\eta,m_i}) \tag{4.98}$$

$$= \mathrm{Tr}(\hat{\rho}A_\eta), \tag{4.99}$$

as promised.

We will now prove the remaining parts of our conjectures. Let us introduce a second elementary observable κ on E. Let image$(\kappa)=\{n_1,...,n_l\}$ with $l\leqq n$. We want to know whether the operators A,B associated to the induced observables $\mathscr{O}_\eta,\mathscr{O}_\kappa$ commute. The spectral resolution derived above gives us

[13] Remember that the empty subspace is orthogonal to itself $0\perp 0$ and $0\vee 0=0$ therefore $s(0)=s(0)+s(0)$ and we get $s(0)=0$

$$AB - BA = \sum_{i,j=1}^{k,l} m_i P^{\eta,m_i} n_j P^{\kappa,n_j} - n_j P^{\kappa,n_j} m_i P^{\eta,m_i} \tag{4.100}$$

$$= \sum_{i,j=1}^{k,l} m_i n_j \left(P^{\eta,m_i} P^{\kappa,n_j} - P^{\kappa,n_j} P^{\eta,m_i} \right). \tag{4.101}$$

In section IV.B.4.f we mentioned that two projection operators commute iff they are compatible. Furthermore if two events are compatible, then in the present case of the frame test space their associated propositions are compatible as well. Therefore, because all propositions (represented here by the projectors) in the above equation are by definition associated to events of the same test E, they all commute and we get $AB - BA = 0$.

Next we ask, whether all commuting self-adjoint operators on a finite Hilbert space can always be associated to observables induced by elementary observables on the same test. This is the case. From standard text books on quantum mechanics we know that commuting self-adjoint operators A, B have a common set of eigenvectors $\{e_1, ..., e_n\}$ (not necessarily unique). Additionally these eigenvectors can always be chosen orthogonal and then are a basis of the Hilbert space. Choosing such a set of normalized eigenvectors therefore leaves us with a frame $E = \{e_1, ..., e_n\}$ which is nothing else then a test. Remains, to find two elementary observables η, κ on E such that the A, B are the operators associated to the observables induced by η and κ respectively. To this end denote the eigenvalues corresponding to the eigenvectors of A by $\{\lambda_1, ..., \lambda_n\}$ (degenerate eigenvalues are repeated according to their degeneracy) and of B by $\{\gamma_1, ..., \gamma_n\}$. Then as seen above setting $\eta(e_i) = \lambda_i$ and $\kappa(e_i) = \gamma_i$ leads to the spectral resolutions of A and B.

The last statements also imply that for finite Hilbert spaces every observable is induced by at least one elementary observable. All conjectures have been proven then.

E. Dynamics

Overview In this section we will present the selected notion of a dynamical system which is mainly taken from the book by Jauch (1968). First we try to motivate our choice in the general setting of a set of states $\mathscr{S}$ on a proposition logic Π. Then we mention quickly test space symmetries and a connection to dynamical systems. Finally we investigate the dynamical systems of classical and quantum universes. For classical universes our definition of a dynamical system will fail due to our notion of continuity, as we believe. We try to elucidate the problem. Then we go on to the quantum universe, where our definition works well and we recover Schrödinger's equation.

1. The Concept of a Dynamical System

a. General Dynamical Systems As mentioned before, we restrict our treatment to non-relativistic and conservative physical systems. The main advantage of excluding relativity is that we can use a global time parameter (Jauch, 1968).

At the very least a dynamical system (with global time parameter) is one, that if in any state s^t at time t assumes a possibly different state $s^{t+\tau}$ at **any** time $t+\tau$. We say the state s^t *evolves* into the state $s^{t+\tau}$. From a mathematical model of such a system we expect that given the state s^t we can find out about $s^{t+\tau}$. In case we can determine $s^{t+\tau}$ uniquely we speak of a deterministic (model of a) dynamical system. In case this is not possible, e.g. if only probabilities for states at the time $t+\tau$ are obtainable we say the system is indeterministic. We will only treat deterministic systems here, the measurement process (especially in quantum mechanics) which can be seen as an indeterministic dynamical system in the above sense is not taken into account. Note that if the state s^t is a mixed state reflecting uncertainty about the "true" state then this uncertainty prevails at time $t+\tau$ in a deterministic system. The frequencies by which each of the pure states of the mixture is obtained must be preserved under deterministic evolution, and therefore we request for mixtures

$$\left(\sum_I c_i s_i^t\right)^{t+\tau} = \sum_I c_i s_i^{t+\tau}. \tag{4.102}$$

If the correspondence $s^t \mapsto s^{t+\tau}$ satisfies the last equation and is a bijective map, we will call this map a $\mathscr{S}$-automorphism ($\mathscr{S}$ was the set of states) and denote the set of $\mathscr{S}$-automorphisms by $\mathrm{Aut}(\mathscr{S})$. This notion is taken from the book of Varadarajan (1985). For the most general case of differing sets of states $\mathscr{S}_t$ and $\mathscr{S}_{t+\tau}$ let me denote the map that, for given state s^t and times $t, t+\tau$, returns the state $s^{t+\tau}$ by $\phi : \Re \times \Re \times \mathscr{S}_t \to \mathscr{S}_{t+\tau}$. Then $\phi(t, t+\tau, s^t) = s^{t+\tau}$.

As they are deterministic here, we can already derive an important requirement for the maps ϕ. Consider as given arbitrary t_1, t_2, s and $\phi(t_1, t_2, s)$. We know that at any intermediate time $t' \in [t_1, t_2]$ the system is also in a unique state, namely $\phi(t_1, t', s)$. This state evolves further, and at t_2 it is in the state $\phi(t', t_2, \phi(t_1, t', s))$. Effectively this describes an evolution of any s from t_1 to t_2 and since this evolution is unique, we get

$$\phi(t_1, t_2, s) = \phi(t', t_2, \phi(t_1, t', s)). \tag{4.103}$$

Also note that $\phi(t,t,s) = s$ and, if we allow time to move backwards in our model, $\phi(t',t,\phi(t,t',s)) = s$.

As mentioned before we also restrict our treatment to conservative systems. In the present context this is expressed by claiming that the transformation from s^t to $s^{t+\tau}$ only depends on τ not on t (Jauch, 1968). This restriction entails that at any given time t, $\mathscr{S}_t = \mathscr{S}_{t+\tau}$ and hence (together with the deterministic assumption) suggests the use of $\mathscr{S}$-automorphisms to represent the transformations. We can then simplify notation of the map ϕ by $\phi(t,t+\tau,s^t) = \phi(\tau,s)$ with $\phi : \Re \times \mathscr{S} \to \mathscr{S}$. For fixed τ let us write $\phi_\tau : \mathscr{S} \to \mathscr{S}$ so that $\phi_\tau \in \mathrm{Aut}(\mathscr{S})$.

Simplifying Eq. (4.103) for a conservative system results in

$$\phi(t_2 - t_1, s) = \phi(t_2 - t', \phi(t' - t_1, s)). \tag{4.104}$$

With $t_2 - t' = \kappa$ and $t' - t_1 = \tau$ this is equivalent to all of the following:

$$\begin{aligned}
\phi(\tau+\kappa,s) &= \phi(\kappa,\phi(\tau,s)) && (4.105)\\
\phi_{\tau+\kappa}(s) &= \phi_\kappa(\phi_\tau(s)) && (4.106)\\
\phi_{\tau+\kappa}(s) &= (\phi_\kappa \circ \phi_\tau)(s). && (4.107)
\end{aligned}$$

As this is true for every state s we may write

$$\phi_{\tau+\kappa} = \phi_\kappa \circ \phi_\tau. \tag{4.108}$$

Also note that

$$\phi_0(s) = s \tag{4.109}$$

and

$$\phi_\tau \circ \phi_{-\tau} = \phi_0 \tag{4.110}$$

The last two equations show that the empty time interval is mapped to the unit element $e = \phi_0$ and negative time intervals are mapped to the inverse element of their positive counterparts $\phi_\tau^{-1} = \phi_{-\tau}$. In this way Eqs.(4.108),(4.109),(4.110) are equivalent to the definition of a homomorphism $\tau \mapsto \phi_\tau$ of the additive group of real numbers with function composition replacing addition. In this case the homomorphism maps into the automorphisms of the set of states $\mathrm{Aut}(\mathscr{S})$.

A common assumption for the time evolution of states of physical systems is that it is continuous. This is also requested here. More precisely, for any fixed state s and proposition p the function $\tau \mapsto \phi(\tau,s)(p)$ is a continuous real function of τ.

We have then arrived at a continuous homomorphism of the additive group of real numbers $\Re$ into the automorphisms $\mathrm{Aut}(\mathscr{S})$ of the set of states $\mathscr{S}$ of the system. From now on we will call a homomorphism of the additive group of real numbers into another set D, a $[\Re, D]$*-homomorphism*. Then, what we described up to now is a continuous $[\Re, \mathrm{Aut}(\mathscr{S})]$-homomorphism.

Following Jauch (1968) we require one more property. The inspiration for it is, that the proposition logic should be left invariant during time evolution. At every instant of time we expect that the execution of tests leads to the same structure concerning implication, orthocomplementation, join, meet, etc. of propositions. Then the continuous $[\Re, \mathrm{Aut}(\mathscr{S})]$-homomorphisms describing dynamical systems must be those induced by continuous $[\Re, \mathrm{Aut}(\Pi)]$-homomorphisms, where Π is the proposition logic of the system. Their continuity is defined via the continuity of the induced $[\Re, \mathrm{Aut}(\mathscr{S})]$-homomorphism and the rule by which they induce a $[\Re, \mathrm{Aut}(\mathscr{S})]$-homomorphism is the *dynamical law*. Let $m : \Pi \to \Pi$ be a $(\Pi, \leq, ')$-automorphism[14] (see Sec.III.E), p a proposition, then the dynamical law defines an induced $\mathscr{S}$-automorphism through

$$s^m(p) = s(m^{-1}(p)). \tag{4.111}$$

too see this consider a mixture:

$$s^m(p) = \left(\sum c_i s_i\right)^m (p) \tag{4.112}$$

$$= \left(\sum c_i s_i\right)(m^{-1}(p)) \tag{4.113}$$

$$= \sum c_i s_i(m^{-1}(p)) \tag{4.114}$$

$$= \sum c_i s_i^m(p), \tag{4.115}$$

as requested by Eq.(4.102).

Assume now that we have a $[\Re, \mathrm{Aut}(\Pi)]$-homomorphism $m(t, p)$, we write m_t for fixed t again. Then the dynamical law permits us to define the induced $[\Re, \mathrm{Aut}(\mathscr{S})]$-homomorphism $\phi(t, s)$ in

[14] Note that in the two cases we are interested in, $\Pi(\mathfrak{A})$ is a bounded lattice. Then recall that automorphisms of an orthocomplemented poset like $(\Pi, \leq, ')$ that is also a bounded lattice $(\Pi, \vee, \wedge, 0, 1)$ are automatically automorphisms of the latter structure as well (see Sec.III.E). This entails, that all conditions expressible by means of implication, orthocomplementation, join, meet, and the bounds 0, 1 are also preserved by such automorphisms. Examples are compatibility and orthomodularity (see Sec.III.B and Jauch, 1968).

the following way:

$$[\phi_t(s)](p) = s^{m_t}(p) \tag{4.116}$$

$$= s(m_t^{-1}(p)) \tag{4.117}$$

$$= s(m_{-t}(p)). \tag{4.118}$$

It is then straightforward to verify the homomorphism properties from Eqs. (4.108), (4.109), (4.110). For example Eq.(4.108), we need to show that

$$[\phi_{\kappa+\tau}(s)](p) = [(\phi_\kappa \circ \phi_\tau)(s)](p). \tag{4.119}$$

Rewriting the RHS leads directly to the identity:

$$[(\phi_\kappa \circ \phi_\tau)(s)](p)) = \phi_\kappa([\phi_\tau(s)](p)) \tag{4.120}$$

$$= \phi_\kappa(s(m_{-t}(p))) \tag{4.121}$$

$$= s(m_{-\kappa}(m_{-\tau}(p))) \tag{4.122}$$

$$= s(m_{-\kappa-\tau}(p)) \tag{4.123}$$

$$= s(m_{-(\kappa+\tau)}(p)) \tag{4.124}$$

$$= \phi_{\tau+\kappa}(s)(p). \tag{4.125}$$

Because it plays a role later, we look at continuity a little closer. It is defined through continuity of the induced states. A $[\mathfrak{R}, \mathrm{Aut}(\Pi)]$-homomorphism $m(t,p)$ is continuous iff for all $p \in \Pi$, all $s \in \mathscr{S}(\Pi)$, and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|\tau - t| < \delta \Rightarrow |s^{m_\tau}(p) - s^{m_t}(p)| < \varepsilon. \tag{4.126}$$

Finally, we define the *dynamical system of a physical system* as a continuous $[\mathfrak{R}, \mathrm{Aut}(\mathscr{S})]$-homomorphism induced by a continuous $[\mathfrak{R}, \mathrm{Aut}(\Pi)]$-homomorphism.

b. Test Space Symmetries On the side, we now want to mention a connection between test spaces and dynamical systems.

We introduce the notion of a test space symmetry and let it induce a $(\Pi, \leq, ')$-automorphism. Then it is clear from our previous discussion that test space symmetries could also induce $[\mathfrak{R}, \mathrm{Aut}(\mathscr{S})]$-homomorphisms. A *test space symmetry* (Wilce, 2000) is a bijective map $g : X \to X$ preserving tests such that $g(E) = \{g(x) | x \in E \in \mathfrak{A}\}$ is a test. Then orthogonal events A, $B \in \mathscr{E}(\mathfrak{A})$

are mapped to orthogonal events $A \perp B \Leftrightarrow g(A) \perp g(B)$ and perspectivity is preserved: $A \sim B \Leftrightarrow g(A) \sim g(B)$ (Wilce, 2000). It is clear that these maps form a group G, the *symmetry group of the test space*.

If $\mathfrak{A}$ is algebraic, by defining $m_g(p(A)) = p(g(A))$ each test space symmetry induces a morphism $m_g : \Pi(\mathfrak{A}) \to \Pi(\mathfrak{A})$. This morphism m_g has the following properties:

(i) let $p \in \Pi(\mathfrak{A})$ then $m_g(p') = m_g(p)'$;

(ii) let $p, q \in \Pi(\mathfrak{A})$ then $p \leq q$ iff $m_g(p) \leq m_g(q)$.

Therefore m_g is an automorphism of the structure $(\Pi(\mathfrak{A}), \leq, ')$ (see Sec.III.E).

The question whether all automorphisms of $(\Pi(\mathfrak{A}), \leq, ')$ are induced by some symmetry on $\mathfrak{A}$ is again not answered in this text because we have not found it treated in the literature. It could pave the way for further generalizations of dynamical systems. Further investigation of test space symmetries can be found in the articles Foulis (2000) and Foulis and Wilce (2000).

2. *Dynamics of the Classical Universe*

For a classical universe and its grand canonical test we have identified the proposition logic to be the Boolean lattice $\Pi(\Omega) = \mathscr{P}(\Omega)$ of subsets of phase space Ω. Hence to get the dynamical system we should try to identify the possible continuous homomorphisms $m(t, p)$ from $\mathfrak{R}$ into the automorphisms of $\Pi(\Omega)$. We will see that there are none. Nonetheless, for instructive reasons we present how far we get if we assume that the Hamilton formalism is equivalent to the specialization of the dynamical law for the $m(t, p)$. In this case we should be able to derive from a given Hamiltonian a $[\mathfrak{R}, \mathrm{Aut}(\Pi)]$-homomorphism and vice versa. By seeing the failure of this task, maybe the reader quickly sees a solution.

a. Continuous $[\mathfrak{R}, \mathrm{Aut}(\Pi)]$*-homomorphisms from phase flows* First let us how each Hamiltonian could induce a $[\mathfrak{R}, \mathrm{Aut}(\Pi)]$-homomorphisms $m(t, s)$. Recall that by the canonical equations Eq.(2.1) each Hamiltonian gives rise to a phase flow $\sigma(t, x)$ on phase space. For each t, $\sigma_t : \Omega \to \Omega$ is a diffeomorphism and therefore a bijective map. Defining for $A \subseteq \Omega$ that $\sigma_t(A) := \{y \in \Omega | x \in A,\ y = \sigma_t(x)\}$ induces a morphism of $\Pi(\Omega)$: Because propositions $p(A) \in \Pi(\Omega)$ are equal to the events themselves $p(A) = A$ we can define $\sigma_t(p(A)) = \sigma_t(A) = p(\sigma_t(A))$. From this definition and bijectivity the two requirements for a $(\Pi(\Omega), \leq, ')$-automorphism follow immediately.

(i) let $p(A) \in \Pi(\Omega)$ then $\sigma(t,p(A)') = \sigma(t,\Omega \backslash A) = \Omega \backslash \sigma(t,A) = \sigma(t,p(A))'$;

(ii) let $p(A), p(B) \in \Pi(\mathfrak{A})$ then $p(A) \leq p(B)$ iff $\sigma(t,p(A)) \leq \sigma(t,p(B))$.

Now because $\Pi(\Omega)$ is bounded and a σ-complete lattice, the σ_t (and actually any bijective map $\Omega \mapsto \Omega$) become $(\Pi(\Omega),\vee,\wedge,0,1)$-automorphisms as well (see Sec.III.E). Furthermore the points x of phase space are the atoms of the lattice $\Pi(\Omega)$ and because each automorphism is determined by its restriction to the atoms (see Sec.III.E), we know that the automorphisms induced by some σ_t are unique. What is left to show is that these automorphisms are continuous with respect to the induced states. This is not the case. In Appendix A we prove that there are generally no such automorphisms on Boolean lattices. For our notion of state we will see this as well shortly.

The problem is the definition of continuity with respect to induced states in Eq.(4.126). It is obviously not general enough to hold for classical and quantum mechanical systems[15]. The reason it holds for quantum mechanics and not for classical mechanics is interesting. It is a consequence of the fact that there are only dispersion free pure states in classical mechanics while there are (only) dispersive pure states in quantum mechanics. For classical mechanics this results in a finite minimum difference (of value one) between the value assigned by to at least one proposition by the transformed state and the value assigned to it by the original state. One of the two proposition must be assigned the value zero and the other one the value one. In quantum mechanics such a bound does not exist because to the propositions even of pure states there can be assigned any value between zero and one. In other words pure states can be transformed continuously into other pure states in quantum mechanics (on orthomodular lattices) but this is impossible in classical mechanics (on Boolean lattices)

To us it seems the solution has to be found in a more generally applicable notion of continuity or maybe a different notion of state. Clearly, for instance with respect to the metric on phase space, all flows $\sigma(t,x)$ are continuous. If we imagine a pure state as a δ-distribution moving along the

[15] It was actually defined for use in quantum mechanics by Jauch (1968). We had hoped it would hold for classical systems as well.

flow according to $\sigma(t,x)$ ($B \subseteq \Omega$):

$$s^{\sigma_t}(B) = s(\sigma_{-t}(B)) \tag{4.127}$$
$$= \int_{\sigma_{-t}(B)} \rho_s(x)dx \tag{4.128}$$
$$= \int_{\sigma_{-t}(B)} \delta(x_s - x)dx \tag{4.129}$$

we see that

$$s^{\sigma_t}(B) = \begin{cases} 1 & \text{if } x_s \in \sigma_{-t}(B) \\ 0 & \text{else.} \end{cases} \tag{4.130}$$

This is a form of the above problem. For each proposition equivalent to a subset of Ω, once the δ-distribution escapes this subset (formally of course the converse happens) the value attributed to the according proposition by the pure state discontinuously drops from one to zero. Such discontinuous drops are not observed for smooth probability distributions i.e. mixed states. Thus if we want to stick to the previous definition of continuity, it seems we would need Boolean lattices without pure states. We have not investigated this or another version of continuity so far.

b. Phase flows from $[\Re, \mathrm{Aut}(\Pi)]$-homomorphisms Now second we would want to look at whether every continuous homomorphism $m(t,p)$ from $\Re$ into the automorphisms of $\Pi(\Omega)$ has the effect of a flow generated by a Hamiltonian vector field. As there are no such homomorphisms with the present notions of continuity and state, this is futile. Dropping continuity with respect to induced states for a second though we might adopt the following strategy: Look at the effect on the atoms $a \in \Pi(\Omega)$ these are actually the points $x \in \Omega$. Each $m(t,p)$ is determined completely by its restriction $m^R(t,x)$ to these atoms . Then we can go on to investigate $m^R(t,x)$. Because $m_t : \Pi(\Omega) \to \Pi(\Omega)$ is a bijection and atoms are mapped to atoms by $\Pi(\Omega)$-automorphisms, its restriction to the atoms $m_t^R : \Omega \to \Omega$ is also a bijection. At this point we clearly need further requirements for $m(t,p)$ as not every bijection of Ω is a flow. A condition for continuity is needed. Without further investigations this is as far as we get.

3. Dynamics of the Quantum Universe

For the quantum universe we again represent the proposition logic $\Pi(\mathscr{F}(\mathscr{H}))$ by the projection operators. We will sketch how Schrödinger's equation can be seen as an infinitesimal version of a representation of the dynamical law for continuous $[\Re, \mathrm{Aut}(\Pi)]$-homomorphisms. Its solutions are therefore the induced $[\Re, \mathrm{Aut}(\mathscr{S})]$-homomorphisms. In other words, dynamical systems of the quantum universe. The mathematics needed for a rigorous deduction of the following are too involved to be presented in this text. Therefore we only state the main results. To this end we follow closely Jauch (1968, Chapter 10) and in parts the more rigorous treatment of these matters in the book by Varadarajan (1985, Chapter VIII).

Let us write P for a proposition here to stress that it is a projection operator. It can be proven that every continuous $[\Re, \mathrm{Aut}(\Pi)]$-homomorphism $m(t,P)$ is induced by a continuous $[\Re, \mathscr{U}]$-homomorphism into the unitary operators $\mathscr{U}$ on $\mathscr{H}$. We write U_t for the unitary operator associated to the time interval t. It is a property of this construction that U_t acts on a proposition P by $U_t P U_t^{-1}$. For the induced $[\Re, \mathrm{Aut}(\Pi)]$-homomorphism we get:

$$m(t,P) = U_t P U_t^{-1}. \tag{4.131}$$

We also have $U_t + U_r = U_{t+r}$ and $U_t^{-1} = U_{-t}$. This ensures that Eqs.(4.108),(4.109),(4.110) are satisfied for $m(t,P)$. Furthermore all $e^{itc}U_t$ with $c \in \Re$, and only those, induce the same $[\Re, \mathrm{Aut}(\Pi)]$-homomorphism $m(t,P)$ on Π as U_t (Varadarajan, 1985, p.289). That they do is easily seen from Eq. (4.131). The continuity is defined with respect to the induced states, that we will look at next. Note that this continuity is a condition on the above $[\Re, \mathscr{U}]$-homomorphism(s), needed for their definition, and not a consequence of this definition[16].

The induced states are obtained via Eq.(4.111): $s^{m_t}(P) = s(m_t^{-1}(P))$. Gleason's theorem provides the means to represent the states on $\Pi(\mathscr{F}(\mathscr{H}))$ and their induced transformation in Hilbert space. Recall that $s(P) = \mathrm{Tr}(\hat{\rho}P)$, then consider:

$$s^{m_t}(P) = s(m_t^{-1}(P)) \tag{4.132}$$

$$\mathrm{Tr}(\hat{\rho}^{m_t}P) = \mathrm{Tr}(\hat{\rho}\; m_t^{-1}(P)) \tag{4.133}$$

$$\mathrm{Tr}(\hat{\rho}^{m_t}P) = \mathrm{Tr}(\hat{\rho}\; U_t^{-1} P U_t). \tag{4.134}$$

This trace is a continuous function of t. As this is true for all $\hat{\rho}$ and all P it implies that $\langle\psi|U_t\phi\rangle$, for any $|\psi\rangle$, $|\phi\rangle \in \mathscr{H}$, is continuous as well. This continuity is a condition for Stone's theorem (Riesz

[16] At the very least continuity prevents pathological cases of the assignment $t \to U_t$. On separable Hilbert spaces continuity may be expressed as a measurability condition (Riesz and Sz.-Nagy, 1955) as is done by Varadarajan (1985).

and Sz.-Nagy, 1955) which asserts a one-to-one correspondence of $[\Re, \mathscr{U}]$-homomorphisms with self-adjoint operators H by

$$U_t = e^{itH}. \tag{4.135}$$

Here H is defined by

$$H|\psi\rangle = i \lim_{t \to 0} \frac{1}{t} \left(U_t|\psi\rangle - |\psi\rangle \right), \tag{4.136}$$

where the limit exists[17]. For $e^{itc}U_t$ which results in the same $[\Re, \mathrm{Aut}(\Pi)]$-homomorphism $m(t,P)$ we get $H' = H + cI$. Hence each $m(t,P)$ defines a self-adjoint operator H up to an additive constant.

Now let us look at Eq. (4.132) again.

$$s^{m_t}(P) = s(m_t^{-1}(P)) \tag{4.137}$$

$$\mathrm{Tr}(\hat{\rho}^{m_t} P) = \mathrm{Tr}(\hat{\rho}\; U_t^{-1} P U_t) \tag{4.138}$$

$$\mathrm{Tr}(\hat{\rho}^{m_t} P) = \mathrm{Tr}(U_t \hat{\rho} U_t^{-1} P) \tag{4.139}$$

As this must be true for all $\hat{\rho}$, P we can put:

$$\hat{\rho}^{m_t} = U_t \hat{\rho} U_t^{-1} \tag{4.140}$$

Using Stone's theorem and then taking the time derivative we get:

$$\hat{\rho}^{m_t} = e^{itH} \hat{\rho} e^{-itH} \tag{4.141}$$

$$-i \frac{d\hat{\rho}}{dt} = [\hat{\rho}, H]. \tag{4.142}$$

Finally for $\hat{\rho}$ describing a pure state with $|\psi\rangle$ the eigenstate to the eigenvalue that is one

$$i \frac{d}{dt} |\psi\rangle = H|\psi\rangle. \tag{4.143}$$

The last two equations show that H is indeed the Hamilton operator, and therefore that every $m(t,P)$ determines a Hamilton operator up to an additive constant. Conversely, the self-adjoint operators H induce all $[\Re, \mathrm{Aut}(\Pi)]$-homomorphisms, through their corresponding $[\Re, \mathscr{U}]$-homomorphisms. That every $[\Re, \mathscr{U}]$-homomorphisms induces a $[\Re, \mathrm{Aut}(\Pi)]$-homomorphisms is not hard to see, refer to Appendix B.

[17] Which it does on a linear manifold that is dense in $\mathscr{H}$ (Varadarajan, 1985, p.291).

V. CONCLUSION

We have presented a set of formal concepts, definable on the test space emerging from the empirical logic approach. For all concepts except one we were able to identify mathematical representatives in both, the case of a classical and the case of a quantum universe. The exception is the dynamical system for the classical universe.

We now sum up our results in the developed terminology.

The test space of a classical universe is always a single test and isomorphic to phase space Ω. Perspectivity has no effect as it produces subsets of phase space as the propositions, which are just the events again. The proposition logic $\Pi(\Omega)$ obtained from endowing the events with an implication relation is a bounded complete atomistic Boolean lattice, which is always distributive. All events and propositions are thus compatible. We then proposed to obtain statistical states from sums and integrals of pure states represented by δ-distributions on the test space. Pure and dispersion-free states then coincide for classical universes. Observables are the pre-images of functions over Ω and thus inverse to the conventional observables and in one-to-one correspondence. The proposed notion of statistical state is also in one-to-one correspondence with the conventional concept. Our abstract definition of an expectation value leads directly to the usual expression as well. Concerning the dynamical system, our definition of continuity with respect to the statistical states proved to be an impossible requirement. A more general concept needs to be found here.

The test space of a quantum universe was argued to be the frame test space $\mathscr{F}(\mathscr{H})$ of the Hilbert space $\mathscr{H}$ that describes it in the conventional formalism. This has lead to great agreement of the rest of the employed notions, which lends strong support for the correctness of that choice. Nonetheless we believe the arguments presented can be improved upon.

The frame test space $\mathscr{F}(\mathscr{H})$ is the set of orthonormal bases of $\mathscr{H}$. Then perspectivity reveals the subspaces of Hilbert space, which are isomorphic to the projection operators, as the propositions $\Pi(\mathscr{F}(\mathscr{H}))$. Propositions are therfore not identical to events, which are just sets of orthonormal vectors.

The logic $\Pi(\mathscr{F}(\mathscr{H}))$ is a bounded σ-complete orthomodular atomistic lattice. This structure is not distributive and allows for incompatible propositions, which is no surprise as there are already incompatible events in the test space. Furthermore propositions may be neither confirmed nor refuted by a test.

By Gleasons theorem the statistical states on the logic $\Pi(\mathscr{F}(\mathscr{H}))$ are exactly representable by positive operators of trace one which is also the conventional way to describe them.

The observables are spectral measures and reveal themselves as in one-to-one correspondence to self-adjoint operators, which is the usual definition of an observable. Again these are “inverse” concepts. The abstract expectation value also can be represented by its conventional expression.

Finally, although by a nontrivial construction, the dynamical systems of a quantum universe are induced by continuous homomorphisms from the additive group of real numbers into the set of unitary operators, and are in one-to-one correspondence with Hamilton operators up to a constant (operator).

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APPENDIX A: Continuous Automorphisms on Boolean Lattices

Here we show that there is no homomorphism $t \mapsto m_t$ of the additive group of real numbers $\mathfrak{R}$ into the $(B,\vee,\wedge,0,1)$-automorphisms on a Boolean lattice B that is continuous with respect to the induced states[18]. Recall that such a homomorphism is continuous with respect to the induced states iff for all $p \in B$, all $s \in \mathscr{S}(B)$, and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|\tau - t| < \delta \Rightarrow |s^{m_\tau}(p) - s^{m_t}(p)| < \varepsilon. \tag{A1}$$

This fails because the RHS has a finite lower bound. In fact for **every** pair of $(B,\vee,\wedge,0,1)$-automorphisms m^τ,m^t, with $m^\tau \neq m^t$, there is always a state $s_0 \in \mathscr{S}(B)$ and an element $p_0 \in B$ with $|s_0^{m_\tau}(p_0) - s_0^{m_t}(p_0)| = 1$. Choosing e.g. $\varepsilon = 1/2$ then provides a contradiction of this continuity. We go on to proof the existence of the lower bound now.

In short: We show that every nontrivial (not the identity) $\mathscr{S}$-automorphism must transform at least one pure state s_0 into another pure state. Then the value attributed to some element p_0 by the original and the transformed state must be different. Because every pure state is dispersion free on Boolean lattice the only possible difference is equal to one.

[18] Here we use the notion of σ-additive state defined in Sec.IV.C.

In more detail: Let us abbreviate the notation by defining $s := s^{m_t}$, and $s^m := s^{m_\tau}$. Now every state on B is either a pure state or a mixture of pure states. Recall Eq.(4.102), rewritten here for a general map m:

$$\left(\sum_I c_i s_i\right)^m = \sum_I c_i s_i^m. \tag{A2}$$

which is defining for $\mathscr{S}$-automorphisms. Note that mixed states are mapped to mixed states and because the automorphisms are bijective, pure states must be mapped to pure states. According to Eq.(A2) if all pure states are left invariant, all mixed states are left invariant as well. Then for a $\mathscr{S}$-automorphisms $s \mapsto s^m$ to be different from the identity it is necessary that at least one pure state is not mapped to itself. We choose this pure state to be s_0. Then $s_0 \neq s_0^m$ and both states are pure. From this we conclude that there is an element $p_0 \in B$ such that $s_0^m(p_0) \neq s_0(p_0)$. Up to this point the argumentation holds for $\mathscr{S}$-automorphisms on any (not necessary Boolean) bounded σ-complete lattice[19]. The crucial point is that all pure states on a Boolean lattice are dispersion free (see Sec.IV.C.2). Hence to any $p \in B$ they attribute either $s(p) = 0$ or $s(p) = 1$. Therefore $|s_0^m(p_0) - s_0(m)| = 1$. This concludes the proof.

APPENDIX B: Automorphisms of $\Pi(\mathscr{F}(\mathscr{H}))$ from Unitary Operators

I sketch why every $[\mathfrak{R}, \mathscr{U}]$-homomorphism induces a $[\mathfrak{R}, \mathrm{Aut}(\Pi)]$-homomorphism.

First show that every transformation $m(P) = UPU^{-1}$ is a $(\Pi, \leq, ')$-automorphism and then show that if $\langle\psi|U_t\phi\rangle$ is continuous in t for all $|\psi\rangle$, $|\phi\rangle$ which is assured by Stone's theorem, then $s^{m_t}(P) = \mathrm{Tr}(\rho U_t^{-1} P U_t)$ is also continuous in t.

Recall that for projections P, Q implication $P \leq Q$ is defined by set inclusion of the associated subspaces K, L, through $K \subseteq L$. We can write projections as $P = \sum_I |\psi_i\rangle\langle\psi_i|$ with all $\{|\psi_i\rangle\}$ an orthonormal basis of the subspace K associated to P. Then for $P \leq Q$ there is always an orthonormal basis $\{\{|\phi_j\rangle\}, \{|\psi_i\rangle\}\}$ for L such that $Q = \sum_J |\phi_j\rangle\langle\phi_j| + \sum_I |\psi_i\rangle\langle\psi_i|$. Then $UQU^{-1} = \sum_J |U\phi_j\rangle\langle U\phi_j| + \sum_I |U\psi_i\rangle\langle U\psi_i|$ and $UPU^{-1} = \sum_I |U\psi_i\rangle\langle U\psi_i|$. Such that $UPU^{-1} \leq UQU^{-1}$ and implication is conserved. The complement of a projection P is defined by $P' := P^\perp = I - P$ with I the identity. Then $UP^\perp U^{-1} = I - UPU^{-1}$ and $(UPU^{-1})^\perp = I - UPU^{-1}$ so it is conserved as well.

[19] Recall that the states we are talking about are only defined on bounded σ-complete lattices.

For continuity note that:

$$\mathrm{Tr}(\rho U_t^{-1} P U_t) = \mathrm{Tr}(U_t \rho U_t^{-1} P) \tag{B1}$$

$$= \sum_{i,j} \langle \psi_i | U_t \rho | \phi_j \rangle \langle \phi_j | U_t^{-1} P | \psi_i \rangle \tag{B2}$$

$$= \sum_{i,j} \langle \psi_i | U_t \rho | \phi_j \rangle \langle U_t \phi_j | P | \psi_i \rangle. \tag{B3}$$

Which is a sum over products of continuous functions and therefore continuous.

APPENDIX C: Solving the Canonical Equations with Canonical Transformations

The following development draws strongly on the presentation in the book by Goldstein *et al.* (2001). It is possible to transform the coordinates (q,p) to a new set (Q,P) and introduce a new function $K(Q,P)$ that will take the place of the Hamiltonian. If the transformation satisfies the condition:

$$p \cdot \dot{q} - H = P \cdot \dot{Q} - K + \frac{dF}{dt} \tag{C1}$$

where $F(Q,P)$ is an arbitrary function, it is called a *restricted canonical transformation*. Restricted because it is time independent, which is sufficient for conservative systems. After such a transformation a reformulation of the canonical equations Eq.(2.1) is written:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \tag{C2}$$

The advantage of this, is that by choosing (and actually calculating) the right canonical transformation, $K(Q,P)$ may become a very convenient function such that solving Eq.(C2) is a lot easier. Of course by inverting the canonical transformation, the solution of the canonical equations Eq.(2.1) is then available as well.

In short, choosing a specific form for F in Eq.(C1) causes a dependence of the variables on just this F. Further selecting a simple form of K can be exploited by substituting variables dependent on F into that equation. The result is an equation for F and from its solution we can get to the desired integral curves.

I shall sketch this for a special F. Let[20] $F = W(q,P) - Q \cdot P$ and plug it into (C1), expanding

[20] Note that many but not all functions on phase space can be written this way. Thus it is to be expected that this approach does not succeed all the time. For further (similar) approaches see Goldstein *et al.* (2001).

its derivative:

$$p \cdot \dot{q} - H = -Q \cdot \dot{P} - K + \frac{\partial W}{\partial q_i}\dot{q}_i + \frac{\partial W}{\partial P_i}\dot{P}_i \tag{C3}$$

From the independence of $\dot{q}_i$ and P_i (which is a justified assumption in case of a success) follows that their coefficients vanish. This give the equations:

$$p_i = \frac{\partial W}{\partial q_i} \tag{C4}$$

$$Q_i = \frac{\partial W}{\partial P_i} \tag{C5}$$

and leaves

$$K = H \tag{C6}$$

As we want K to be very simple, we can choose

$$K(P) = P_1 \tag{C7}$$

and plug it into Eq.(C6):

$$P_1 = H(q,p) \tag{C8}$$

In order to get an equation for W we replace all p_i by $\partial W/\partial q_i$ (cf. Eq.(C4)):

$$P_1 = H\left(q_1,...,q_n,\frac{\partial W}{\partial q_1},...,\frac{\partial W}{\partial q_i}\right) \tag{C9}$$

This equation is called the *Hamilton-Jacobi partial differential equation* for *Hamilton's characteristic function* W. Solving it gives a function $W = W(q_1,...,q_n,\gamma_1,...,\gamma_n)$ with $\gamma_1,...,\gamma_n$ integration constants. In the new coordinates the solutions to Eq.(C2) are trivial:

$$\dot{Q}_i = \delta_{i1} \;\Rightarrow\; Q_i = t\delta_{i1} + \beta_i \tag{C10}$$

$$\dot{P}_i = 0 \;\Rightarrow\; P_i = \alpha_i \tag{C11}$$

where α_i, β_i are constants. In order to give $W = W(q,P)$ the desired form we are free to set $\gamma_i = P_i = \alpha_i$. Then from Eq.(C4), we have:

$$p_i = \frac{\partial W(q,P)}{\partial q_i} = \frac{\partial W(q,\alpha)}{\partial q_i} \tag{C12}$$

where we plug in the initial conditions $(q_i(t_0),p_i(t_0))$ to evaluate the α_i. The same is done to obtain all β_i with

$$Q_i = \frac{\partial W}{\partial P_i} = \frac{\partial W(q,\alpha)}{\partial \alpha_i} = t\delta_{1i} + \beta_i \tag{C13}$$

After that we finally obtain, first, one half of the canonical transformation, namely $q = q(Q,P) = q(\beta,\alpha,t)$ by resolving Eq.(C13) for q_i, and then, the other half $p = p(Q,P) = p(\beta,\alpha,t)$ by differentiating Eq.(C12) and plugging in the $q_i(\beta,\alpha,t)$.

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