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Parking Functions and Generalizations

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unter der Anleitung von Ao.Univ.Prof. Dipl.-Ing. Dr.techn. Alois Panholzer

durch

Georg Seitz

Albertgasse 3A/14
1080 Wien

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Kurzfassung

Betrachte eine Einbahnstraße mit n nummerierten Parkplätzen in einer Reihe. m aufeinanderfolgende Fahrer wollen in dieser Straße parken, wobei jeder einen bevorzugten Parkplatz hat. Jeder Fahrer fährt zu der gewählten Stelle und parkt dort, falls sie frei ist. Falls nicht, nimmt er den nächsten freien Parkplatz, falls vorhanden. Andernfalls verlässt er die Straße.
Die Funktion $p : [m] \to [n]$, die jedem Fahrer i den bevorzugten Parkplatz $p(i)$ zuweist, heißt Parkfunktion (parking function), falls alle m Fahrer erfolgreich parken können.

Parkfunktionen wurden 1966 von Konheim und Weiss in ihrer Analyse von Hashtabellen mit linearem Sondieren eingeführt. Seitdem haben sie das Interesse zahlreicher Mathematiker geweckt und wurden ausgiebig studiert.
Viele Beziehungen zwischen Parkfunktionen und anderen kombinatorischen Objekten wie zum Beispiel markierten Bäumen (labeled trees), azyklischen Funktionen (acyclic functions), Warteschlangen (priority queues), nichtkreuzenden Partitionen (noncrossing partitions) und speziellen Polytopen sind bekannt.
Parkfunktion wurden auf verschiedene Arten verallgemeinert, und sie treten in der Analyse von Hashing-Varianten und Sortier-Algorithmen auf.

Im 1. Kapitel dieser Arbeit werden einige Lösungsmethoden und Hilfsresultate gesammelt.
Parkfunktionen werden in Kapitel 2 definiert und einige grundlegende Eigenschaften hergeleitet. Es wird gezeigt, wo Parkfunktionen bei der Analyse von Hashtabellen auftreten, und ein neues Resultat über Haltepunkte (breakpoints) von Parkfunktionen wird präsentiert.
Im 3. Kapitel werden bekannte Relationen zwischen Parkfunktionen und azyklischen Funktionen bzw. markierten Bäumen sowie eine Beziehung zwischen Parkfunktionen und Warteschlangen behandelt.
Einige Verallgemeinerungen von Parkfunktionen sind in Kapitel 4 gesammelt, unter anderem bucket parking functions und **x**-parking functions.

Das letzte Kapitel befasst sich mit defekten Parkfunktionen (defective parking functions). Nach einer Darstellung bekannter Resultate wird eine erzeugende Funktion für die Anzahl von defekten Parkfunktionen hergeleitet und diese für eine asymptotische Analyse des erwarteten Defekts verwendet. Zum Abschluss werden defective parking functions auf defective bucket parking functions verallgemeinert und eine erzeugende Funktion für deren Anzahl hergeleitet.

Contents

Preface

In 1966, parking functions were introduced by Konheim and Weiss as a by-product in their analysis of hashing with linear probing. Since then they have aroused the interest of numerous mathematicans and the subject has been widely studied.
Many relations between parking functions and other combinatorial objects are now known, such as labeled trees, acyclic functions, priority queues, noncrossing partitions and special polytopes. Parking functions have been generalized in various ways, and they appear in the analysis of hashing variants and sorting algorithms.

We will start our work with a collection of auxiliary results in Chapter 1.
Then we define parking functions in Chapter 2 and derive some basic properties. We will show where they appear in the analysis of hashing with linear probing and then give a new result on breakpoints.
In Chapter 3 we will present known relations between parking functions, acyclic functions and labeled trees and a relation between parking functions and priority queues.
Some known generalizations of parking functions are collected in Chapter 4, such as bucket parking functions and $\mathbf{x}$-parking functions.
Our main point of interest will be defective parking functions in Chapter 5. After presenting some known results we will derive a generating function for the number of defective parking functions and use it to find an asymptotic result on the expected defect. Finally, we will generalize defective parking functions to defective bucket parking functions and find a generating function for their number.

At this point, I want to thank everybody who has contributed to this master thesis in one way or another. I want to thank Prof. Panholzer for suggesting this interesting subject, and for the guidance through the process of writing a master thesis. I want to thank my parents for making my studies possible, and my girlfriend for the moral support.

Notation

We will use the following notation:

$$\mathbb{N} := \{1,\ 2,\ 3, \ldots\},$$

and

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Whenever we use the variables m and n, respectively, we implicitly mean that $m, n \in \mathbb{N}_0$. We further define

$$[n] := \{1, \ldots, n\},$$

with the convention that $[0] = \emptyset$, and

$$[n]_0 := [n] \cup \{0\}.$$

We write $x^{\underline{n}}$ for the *falling factorials* of x,

$$x^{\underline{n}} = x(x-1)\cdots(x-n+1).$$

Let $\bar{x} = (x_1, \ldots, x_n)$ and $cond(\bar{x})$ be a condition on $\bar{x}$. We define the *indicator function* $\mathbf{1}_{\{cond(\bar{x})\}}(\bar{x})$ by

$$\mathbf{1}_{\{cond(\bar{x})\}}(\bar{x}) = \begin{cases} 1, & \text{if } cond(\bar{x}) \text{ evaluates to "true"}, \\ 0, & \text{else.} \end{cases}$$

The symbol $\delta_{i,j}$ denotes the *Kronecker delta*,

$$\delta_{i,j} = \mathbf{1}_{\{i=j\}}(i,j).$$

Whenever it seems useful, we will identify any function $f : [m] \to [n]$ with the tupel $(f(1), \ldots, f(m)) \in [n]^m$.
If X is a random variable, we write $\mathbb{E}\left(X\right)$ for the expected value of X.

When we write an integral of the form

$$\oint F(t)dt,$$

we mean that the integral is taken in positive direction over a suitable contour enclosing the origin. We will only use this symbol if it is obvious what "suitable" means.
If $F(z) = \sum_{k\geq 0} a_k z^k$, or, more generally, $F(z) = \sum_{k\geq -n} a_k z^k$, we write $[z^j]F(z)$ to extract the coefficient of z^j in $F(z)$, i. e.,

$$[z^j]F(z) = a_j.$$

By Cauchy's integral formula, we have

$$[z^j]F(z) = \frac{1}{2\pi \mathrm{i}} \oint \frac{F(z)}{z^{j+1}} dz,$$

which will be used frequently.

Chapter 1

Preliminaries

Before we start with our main sections, we will collect some techniques and auxiliary results which will be useful in order to solve the subsequent problems.

1.1 Lagrange's inversion formula

Consider a power series $f(x) = \sum_{k\geq 0} f_k x^k$. Given a parameter transform $x = x(z) = \sum_{k\geq 0} a_k z^k$, one may be interested in extracting the coefficients of $f(x(z))$. In some cases, the following formula proves useful:

Theorem 1.1.1 (Lagrange's inversion formula [26]). *Let $f(x) = \sum_{k\geq 0} f_k x^k$. Suppose that $x = x(z) = \sum_{k\geq 0} a_k z^k$ satisfies $z = \frac{x}{\phi(x)}$, where $\phi(x) = \sum_{k\geq 0} \phi_k x^k$ with $\phi_0 \neq 0$.*
Then it holds that

$$[z^n]f(x(z)) = \begin{cases} \frac{1}{n}[x^{n-1}]f'(x)(\phi(x))^n, & n \geq 0 \\ [x^0]f(x), & n = 0. \end{cases}$$

Note that, given the relation $z = z(x) = \frac{x}{\phi(x)}$, this theorem can be used to find the coefficients of $x(z)$ (simply set $f(x) = x$).

Proof. We use Cauchy's integral formula and the substitution $x(z) = x$, $dz =$

$(\frac{1}{\phi(x)} - \frac{x\phi'(x)}{(\phi(x))^2})dx$: We obtain

$$\begin{aligned}
[z^n]f(x(z)) &= \frac{1}{2\pi \mathrm{i}} \oint \frac{f(x(z))}{z^{n+1}} dz \\
&= \frac{1}{2\pi \mathrm{i}} \oint \frac{f(x)(\phi(x))^{n+1}}{x^{n+1}} \left(\frac{1}{\phi(x)} - \frac{x\phi'(x)}{(\phi(x))^2} \right) dx \\
&= \frac{1}{2\pi \mathrm{i}} \oint \frac{f(x)(\phi(x))^n}{x^{n+1}} - \frac{1}{2\pi \mathrm{i}} \oint \frac{f(x)\phi(x)^{n-1}\phi'(x)}{x^n} \\
&= [x^n]f(x)(\phi(x))^n - [x^{n-1}]f(x)(\phi(x))^{n-1}\phi'(x).
\end{aligned}$$

This already shows that the claim holds for $n = 0$.
For $n \geq 1$, we use the fact that for an arbitrary power series $h(x)$ the equation

$$[x^n]h(x) = \frac{1}{n}[x^{n-1}]h'(x)$$

holds.
Hence, we have

$$[x^n]f(x)\phi(x))^n = \frac{1}{n}[x^{n-1}]f'(x)(\phi(x))^n + [x^{n-1}]f(x)(\phi(x))^{n-1}\phi'(x),$$

and finally

$$[z^n]f(x(z)) = \frac{1}{n}[x^{n-1}]f'(x)(\phi(x))^n.$$

□

We will now give an example where this method can be used.

1.1.1 Example: Binary trees

Definition 1.1.1. *A binary tree is either empty or consists of a vertex and a "left" and a "right" binary tree.*

Let a_n denote the number of binary trees with n vertices. We will show that the a_n are the well-known *Catalan numbers*, $a_n = \frac{1}{n+1}\binom{2n}{n}$.
We can easily establish the recurrence

$$\begin{aligned}
a_0 &= 1, \\
a_{n+1} &= \sum_{k=0}^{n} a_k a_{n-k}, \quad n \geq 0.
\end{aligned}$$

Now we introduce the generating function

$$A(z) := \sum_{n\geq 0} a_n z^n.$$

When we multiply the above recursion by z^{n+1} and sum up, we obtain

$$\begin{aligned} A(z) - 1 &= \sum_{n\geq 0}\sum_{k=0}^{n} a_k a_{n-k} z^{n+1} \\ &= z\sum_{k=0}^{n} a_k z^k \sum_{n-k\geq 0} a_{n-k} z^{n-k} \\ &= z(A(z))^2. \end{aligned}$$

Note that one of the solutions to this equation, $A(z) = \frac{1+\sqrt{1-4z}}{2z}$, has a pole at $z = 0$ and is therefore not relevant for our problem. Hence, we get the solution

$$A(z) = \frac{1-\sqrt{1-4z}}{2z}.$$

We use the transform $z(x) = \frac{x}{(1+x)^2}$, i. e., $\phi(x) = (1+x)^2$, and $f(x) := A(z(x)) = \frac{1-\sqrt{1-4z(x)}}{2z(x)} = 1 + x$.
An application of Lagrange's inversion formula then shows that

$$\begin{aligned} [z^n]A(z) &= [z^n]f(x(z)) \\ &= \frac{1}{n}[z^{n-1}]f'(x)(\phi(x))^n \\ &= \frac{1}{n}[z^{n-1}](1+x)^{2n} \\ &= \frac{1}{n}\binom{2n}{n-1} \\ &= \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

1.2 The tree function

Definition 1.2.1. *The tree function*

$$T(z) = \sum_{n\geq 0} t_n \frac{z^n}{n!}$$

is implicitely defined by

$$T(z) = z\mathrm{e}^{T(z)}.$$

$T(z)$ is the exponential generating function of the numbers t_n of rooted labeled trees with n vertices [15]. As it turns out (see Section 3.1), parking functions are closely related to labeled trees. Thus, it is natural that generating functions associated with parking functions involve the tree function.
We will now collect some results on $T(z)$.

Proposition 1.2.1.

$$T(z) = \sum_{n\geq 1} n^{n-1}\frac{z^n}{n!},$$

i. e., $t_n = n^{n-1}$

Proof. Since $z = \frac{T(z)}{\mathrm{e}^{T(z)}}$, we can apply Lagrange's inversion formula. We get

$$\frac{t_n}{n!} = [z^n]T(z) = \frac{1}{n}[T^{n-1}]\mathrm{e}^{nT} = \frac{n^{n-1}}{n!}, \quad n \geq 1,$$

and $t_0 = 0$. □

Note that for $n \geq 1$, the *Cayley number* $T_n := \frac{t_n}{n} = n^{n-2}$ is the number of (free) labeled trees with n vertices.

Lemma 1.2.2.

$$[z^n]T(z)^j = \begin{cases} j\frac{n^{n-j-1}}{(n-j)!}, & \textit{if } n \geq j, \\ 0, & \textit{else.} \end{cases}$$

Proof. For $n = 0$ the claim obviously holds. For $n \geq 1$, we can again use Lagrange's inversion formula:

$$[z^n]T(z)^j = \frac{j}{n}[T^{n-1}]\mathrm{e}^{nT}T^{j-1} = \frac{j}{n}[T^{n-j}]\mathrm{e}^{nT} = \begin{cases} j\frac{n^{n-j-1}}{(n-j)!}, & \text{if } n \geq j, \\ 0, & \text{else.} \end{cases}$$

□

We will further need the following expansion of $T(z)$ [10, 11]:

Lemma 1.2.3. *The function $T(z)$ has a dominant singularity at $t = \frac{1}{\mathrm{e}}$, and its singular expansion there is*

$$T(z) = 1 - \sqrt{2}\sqrt{1-ez} + \mathcal{O}(1-ez).$$

1.3 The kernel method

In the following we will have to solve functional equations for multivariate generating functions. In certain cases, the *kernel method* proves useful, which can be described as follows:
Assume that we have a functional equation

$$F(\bar{x}) = \frac{G(F, \bar{x})}{H(\bar{x})},$$

where $\bar{x} = (x_1, \ldots, x_n)$, from which we need to find the generating function F. Now assume that $H(\bar{a}) = 0$ for $\bar{a} = (a_1, \ldots, a_n)$. If we know a priori that a power series expansion for F around $\bar{a}$ exists, we must have $G(F, \bar{a}) = 0$ as well.
In some cases this is sufficient to compute the generating function F at special values, and subsequently in general.
In [22], Prodinger has collected some problems where this method works. We will now present an example.

1.3.1 Example: Knödel walks

Consider the following bin packing model: There are bins of size 1, and items of size $\frac{1}{3}$ and $\frac{2}{3}$, respectively, arriving at random. One tries to complete as many bins as possible.
For each $n \in \mathbb{N}$, we let state n represent the situation that we have n bins filled $\frac{2}{3}$. There is also a special state, which we denote by b, representing a bin filled $\frac{1}{3}$. The possible state changes, depending on the next arriving item, are

- $0 \to 1$ or $0 \to b$,
- $b \to 0$ or $b \to 1$,
- $n \to n-1$ or $n \to n+1$, for $n \geq 1$.

This model can be formulated as a random walk on a special graph (see Figure 1.1). Following the example of Prodinger, we will call the random walks on this graph, starting at state 0, "Knödel walks".
We are now interested in the number $a_{n,i}$ of Knödel walks of length n which end in state i. We introduce the generating functions $f_i(z) = \sum_{n\geq 0} a_{n,i} z^n$, for $i \in \{b,\ 0,\ 1, \ldots\}$.
We have the following recursions:

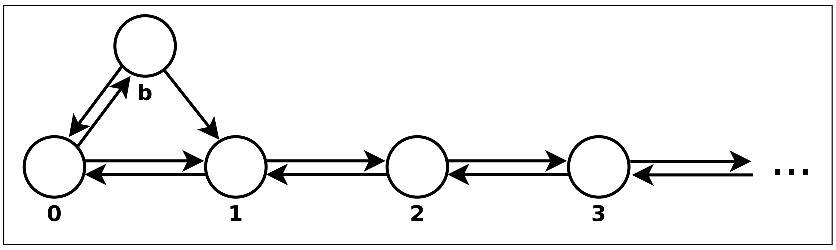

Figure 1.1: The Knödel Graph

$$
\begin{aligned}
f_i(z) &= zf_{i-1}(z) + zf_{i+1}(z), \qquad i \geq 2,\\
f_1(z) &= zf_0(z) + zf_b(z) + zf_2(z),\\
f_0(z) &= 1 + zf_1(z) + zf_b(z),\\
f_b(z) &= zf_0(z).
\end{aligned}
$$

We further introduce the generating function $F(z,x) = \sum_{i\geq 0} f_i(z)x^i$. The above recursions sum up to

$$
\begin{aligned}
F(z,x) &= \sum_{i\geq 2} x^i z f_{i-1}(z) + \sum_{i\geq 2} x^i z f_{i+1}(z)\\
&\quad + xzf_0(z) + xzf_b(z) + xzf_2(z) + 1 + zf_1(z) + zf_b(z)\\
&= xzF(z,x) + \frac{z}{x}F(z,x) + 1 - \frac{z}{x}f_0(z) + (1+x)zf_b(z)\\
&= xzF(z,x) + \frac{z}{x}F(z,x) + 1 - \left(\frac{z}{x} - (1+x)z^2\right)F(z,0),
\end{aligned}
$$

and further

$$F(z,x) = \frac{z(1-x(1+x)z)F(z,0) - x}{x^2z - x + z}.$$

The denominator can be factorized into

$$x^2z - x + z = z(x - r_1(x))(x - r_2(x)),$$

where

$$r_1(x) = \frac{1-\sqrt{1-4z^2}}{2z}, \quad r_2(x) = \frac{1+\sqrt{1-4z^2}}{2z}.$$

Since $x - r_1(z) \sim x - z$ as $x, z \to 0$, the factor $1/(x - r_1(z))$ has no power series expansion around $(0,0)$. But $F(z,x)$ has, so $(x - r_1(z))$ must be a factor of the numerator as well.

From this, we find

$$z(1 - zr_1(z)(1 + r_1(z)))F(z,0) - r_1(z) = 0,$$

and, since

$$r_1^2(z) = \frac{1 - 2\sqrt{1-4z^2} + 1 - 4z^2}{4z^2} = \frac{r_1(z)}{z} - 1,$$

we further get

$$f_0(z) = F(z,0) = \frac{r_1(z)}{z(1+z)(1-r_1(z))}.$$

From this, we find the generating function

$$F(z,x) = \frac{r_1(z)}{z(1+z)(1-r_1(z))} \frac{1 + xzr_1(z)}{1 - xr_1(z)}.$$

By extracting coefficients we get

$$\begin{aligned} f_i(z) &= [x^i]F(z,x) \\ &= \frac{r_1(z)}{z(1+z)(1-r_1(z))} \left([x^i]\frac{1}{1-xr_1(z)} + [x^i]z\frac{xr_1(z)}{1-xr_1(z)} \right) \\ &= \frac{r_1(z)}{z(1+z)(1-r_1(z))}(1+z)r_1^i(z) \\ &= \frac{r_1^{i+1}(z)}{z(1-r_1(z))}, \end{aligned} \tag{1.1}$$

for $i \geq 1$, and we have

$$f_b(z) = zf_0(z) = \frac{r_1(z)}{(1+z)(1-r_1(z))}.$$

In order to extract the coefficients $a_{n,i} = [z^n]f_i(z)$, we use the following lemma:

Lemma 1.3.1. *Let $F(z) = \sum_{n\geq 0} f_n z^n$, and let $z(v) = \frac{v}{1+v^2}$. Then*

$$[z^n]F(z) = [v^n](1-v^2)(1+v^2)^{n-1}F(z(v)).$$

Proof. This follows from an application of Cauchy's integral formula: We have

$$[z^n]F(z) = \frac{1}{2\pi \mathrm{i}} \oint \frac{F(z)}{z^{n+1}} dz,$$

and the substitution $z = z(v) = \frac{v}{1+v^2}$, $dz = dv\frac{1-v^2}{(1+v^2)^2}$, shows that

$$\begin{aligned}
[z^n]F(z) &= \frac{1}{2\pi \mathrm{i}} \oint \frac{(1+v^2)^{n+1}F(z(v))}{v^{n+1}} \frac{1-v^2}{(1+v^2)^2} dv \\
&= [v^n](1-v^2)(1+v^2)^{n-1}F(z(v)).
\end{aligned}$$

□

By applying this lemma to (1.1), we find

$$\begin{aligned}
[z^n]f_i(z) &= [v^n](1-v^2)(1+v^2)^{n-1}\frac{v^i(1+v^2)}{1-v} \\
&= [v^{n-i}](1+v)(1+v^2)^n \\
&= [v^{n-i}]\sum_{k=0}^{n}\binom{n}{k}(v^{2k}+v^{2k+1}) \\
&= \binom{n}{\lfloor \frac{n-i}{2} \rfloor},
\end{aligned}$$

for $i \geq 1$. We further get

$$[z^n](1+z)f_0(z) = [v^n](1-v^2)(1+v^2)^{n-1}\frac{1+v^2}{1-v} = \binom{n}{\lfloor \frac{n}{2} \rfloor},$$

so

$$[z^n]f_0(z) = \sum_{k=0}^{n}\binom{k}{\lfloor \frac{k}{2} \rfloor}(-1)^{n-k}.$$

1.4 Singularity Analysis

Consider a generating function $f(z)$, from which we want to determine the asymptotic order of growth of $[z^n]f(z)$. In certain cases one can apply *singularity analysis* rather than extracting the coefficients exactly.
We will use the following lemma due to Flajolet and Odlyzko [9]:

Lemma 1.4.1. *For fixed $\eta > 0$ and $0 < \phi < \frac{\pi}{2}$, let*

$$\Delta := \{z \mid |z| \leq 1+\eta \text{ and } |\mathrm{Arg}(z-1)| \geq \phi\},$$

where $\mathrm{Arg}(z)$ *denotes the argument of z taken in the inverval $[-\pi, \pi[$. Assume that, with the sole exception of a singularity at $z = 1$, $f(z)$ is analatic in Δ, and that*

$$f(z) = \sum_{j=0}^{m} c_j(1-z)^{\alpha_j} + \mathcal{O}((1-z)^A), \quad \text{for } z \to 1 \text{ in } \Delta,$$

where $\alpha_0 < \alpha_1 < \ldots < \alpha_m < A$.
Then

$$[z^n]f(z) = \sum_{j=0}^{m} \binom{n-\alpha_j-1}{n} + \mathcal{O}(n^{-A-1}), \quad \textit{for } n \to \infty.$$

The asymptotic growth of expressions of the form $\binom{n-\alpha-1}{n}$ is well-known:

Lemma 1.4.2.

$$\binom{n-\alpha-1}{n} \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}$$

Note that if $f(z)$ has a singularity at ζ, $g(z) := f(\zeta z)$ has a singularity at 1. Hence, if Lemma 1.4.1 can be applied to $g(z)$, we can find the asymptotic growth of $[z^n]f(z)$ using the fact that

$$[z^n]f(z) = [z^n]g\left(\frac{z}{\zeta}\right) = \zeta^{-n}[z^n]g(z).$$

1.5 Convergence of random variables

Definition 1.5.1. *Let X be a random variable with distribution function F, and $(X_n)_{n\in\mathbb{N}}$ a sequence of random variables where each X_i is associated to a distribution function F_i. We say that the sequence X_n converges towards X in distribution, if*

$$\lim_{n\to\infty} F_n(a) = F(a)$$

for all $a \in \mathbb{R}$ at which F is continuous. We then write

$$X_n \xrightarrow{d} X.$$

Assume that we want to show that a given sequence X_n of random variables converges in distribution towards a random variable X. If the moments $\mathbb{E}\left(X_n^r\right)$ are known, the following theorem can be useful:

Theorem 1.5.1 (Fréchet and Shohat [18])**.** *If $\lim_{n\to\infty} \mathbb{E}\left(X_n^r\right) = \mathbb{E}\left(X^r\right)$ for all $r \in \mathbb{N}$ and the moments $\mathbb{E}\left(X^r\right)$ uniquely determine the distribution of X, then $X_n \xrightarrow{d} X$.*

It remains to find a criterion for a sequence of moments to uniquely determine the distribution. We will only use the following sufficient condition [6]:

Lemma 1.5.2. *Let X be a random variable and*

$$M_X(t) := \sum_{r \geq 0} \mathbb{E}\left(X^r\right) \frac{t^r}{r!}$$

the moment generating function of X. If $M_X(t)$ exists in a neighbourhood of $t = 0$, then the moments $\mathbb{E}\left(X^r\right)$ uniquely determine the distribution of X.

1.5.1 Example: Rayleigh distribution

Definition 1.5.2. *We say that a random variable X is Rayleigh distributed with parameter s, if X has the density function*

$$f(x) = \frac{x}{s^2} \mathrm{e}^{-\frac{x^2}{2s^2}}, \quad x \geq 0.$$

The moments of a Rayleigh distributed random variable X are given by

$$\mathbb{E}\left(X^r\right) = 2^{\frac{r}{2}} s^r \Gamma\left(\frac{r}{2} + 1\right),$$

see [20].

Proposition 1.5.3. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables. If*

$$\lim_{n \to \infty} \mathbb{E}\left(X_n^r\right) = 2^{\frac{r}{2}} s^r \Gamma\left(\frac{r}{2} + 1\right), \quad r \in \mathbb{N},$$

then $X_n \xrightarrow{d} X$ where X is Rayleigh distributed with parameter s.

Proof. We only have to show that the moment generating function of X,

$$M_X(t) = \sum_{r \geq 0} \mathbb{E}\left(X^r\right) \frac{t^r}{r!} = \sum_{r \geq 0} 2^{\frac{r}{2}} s^r \Gamma\left(\frac{r}{2} + 1\right) \frac{t^r}{r!},$$

exists in a neighbourhood of $t = 0$.
Since

$$\lim_{r \to \infty} \frac{\Gamma(\frac{r}{2} + 1)}{r!} = \lim_{r \to \infty} \frac{\Gamma(\frac{r}{2} + 1)}{\Gamma(r + 1)} = 0,$$

which can be seen by Stirling's formula, $M_X(t)$ exists at least for $|\sqrt{2}st| < 1$. □

Chapter 2

Parking functions

2.1 Definition

Parking functions have been introduced by Konheim and Weiss in their analysis of hashing with linear probing [16] in the following way:

Consider a one-way street with n numbered parking slots in a line. There are m consecutive drivers who wish to park in this street, each of which has a preferred parking slot in mind. Each driver proceeds to the chosen place and parks there, if it is empty. If not, the driver takes the first available space, if any. If no space is empty, the driver leaves.

Definition 2.1.1. *Let $p : [m] \to [n]$ be a function which associates each driver i with his preferred parking slot $p(i)$. If all drivers are able to park when using the parking strategy described above, then p is called a parking function.*

As a first result, we will prove that the ordering of the elements of a function doesn't affect the property of being a parking function:

Lemma 2.1.1. *A function $p : [m] \to [n]$ is a parking function if and only if $p \circ \pi$ is a parking function for any permutation π on $[m]$.*

Proof. It suffices to show that if p is a parking function, then $p \circ \pi$ is a parking function for any permutation π that only switches two consecutive elements and leaves the other values fixed:
For any $f : [m] \to [n]$, define ιf by

$$\begin{aligned} \iota f(1) &= f(1), \text{ and} \\ \iota f(i) &= \min\{j \in \mathbb{N} \mid j \geq f(i),\ j \notin \{\iota f(1), \dots, \iota f(i-1)\}\}, \end{aligned}$$

for $i \in \{2, \ldots, m\}$. Note that if p is a parking function then $\iota p(i)$ is the final parking slot of driver i. p is not a parking function if and only if $\iota p(i) > n$ for at least one $i \in [m]$.
Let $p : [m] \to [n]$ be a parking function and $p' = p \circ \pi$, where $\pi(k) = k+1$, $\pi(k+1) = k$ and $\pi(x) = x$ for $x \in [m] \backslash \{k, k+1\}$.
We have $\{\iota p(1), \ldots, \iota p(k-1)\} = \{\iota p'(1), \ldots, \iota p'(k-1)\}$. Now we have to consider the following three cases:

- $\iota p(k) < p(k+1) = p'(k)$: In this case, $\iota p'(k) = \iota p(k+1)$ and $\iota p'(k+1) = \iota p(k)$.
- $\iota p(k) \geq p(k+1) = p'(k)$ and $\iota p(k+1) < \iota p(k)$: Then we have $\iota p'(k) = \iota p(k+1)$ and $\iota p'(k+1) = \iota p(k)$ as well.
- $\iota p(k) \geq p(k+1) = p'(k)$ and $\iota p(k+1) > \iota p(k)$: Now $\iota p'(k) = \iota p(k)$ and $\iota p'(k+1) = \iota p(k+1)$.

In all three cases we see that $\{\iota p(1), \ldots, \iota p(k+1)\} = \{\iota p'(1), \ldots, \iota p'(k+1)\}$ and further $\{\iota p(i) \mid i \in [m]\} = \{\iota p'(i) \mid i \in [m]\}$. Hence p' is a parking function. □

By this lemma, it is easy to verify the equivalence of various definitions of parking functions on $[n]$ (i. e., the special case $m = n$):

Lemma 2.1.2. *For $p : [n] \to [n]$, the following statements are equivalent:*

- *p is a parking function in the sense of Definition 2.1.1.*
- *p is a major function, i. e., if $(q_1, \ldots, q_n)$ is the increasing rearrangement of $(p(1), \ldots, p(n))$, then $q_i \leq i$ for all $i \in [n]$.*
- *$|p^{-1}(\{n-i+1, \ldots, n\})| \leq i$ for all $i \in [n]$.*
- *There exists a permutation $\pi : [n] \to [n]$ with $p(i) \leq \pi(i)$ for all $i \in [n]$ (π is then called a certificate for p).*

Example:
$n = m = 5$: $p = (3,\ 5,\ 2,\ 2,\ 1)$ is a parking function and $\pi = (4,\ 5,\ 3,\ 2,\ 1)$ is a certificate for p.
$q = (3,\ 5,\ 4,\ 3,\ 1)$ is not a parking function.

For a parking function on $[n]$, the final parking order gives a permutation which we call the *output* of the parking function. More formally:

Definition 2.1.2. *If $p : [n] \to [n]$ is a parking function, we call $\pi_p := (\iota p)^{-1}$ the output of p.*

Example:
For $p = (3,\ 5,\ 2,\ 2,\ 1)$ we have $\iota p = (3,\ 5,\ 2,\ 4,\ 1)$ and $\pi_p = (5,\ 3,\ 1,\ 4,\ 2)$.

2.2 Basic results

Let $g(n, m)$ denote the number of parking functions $p : [m] \to [n]$. The following result due to Konheim and Weiss [16] shows that the numbers $g(n, n)$ can be expressed in terms of the numbers $T_n = n^{n-2}$ (compare Section 1.2). A short and elegant proof of this fact has been given by Pollak [12] and the proof provided here is an adaptation to the general case which has appeared in [5].

Lemma 2.2.1. *$g(n, m) = (n - m + 1)(n + 1)^{m-1}$, especially $g(n, n) = T_{n+1} = (n + 1)^{n-1}$.*

Proof. Consider m drivers who wish to park in a circular parking lot with $n+1$ slots ($m \leq n$), in which the same rules apply as in the one-way street. Now no driver will have to leave and there will be $n + 1 - m$ empty spaces once all drivers have parked. The driver's choices will be a parking function for the original problem if and only if slot number $n + 1$ is empty. By symmetry, this happens in a fraction $\frac{n+1-m}{n+1}$ of the total number $(n + 1)^m$ of choices. □

A first result on the limiting behaviour of $g(n, m)$ is given in [16] as follows:

Proposition 2.2.2. *For $m, n \in \mathbb{N}$, $m \leq n$, let $f_{n,m}$ be a random variable with values in $[n]^{[m]}$ such that $\mathbb{P}\{[f_{n,m} = h]\} = \frac{1}{n^m}$ for all $h : [m] \to [n]$. Set $P(m, n) := \mathbb{P}\{[f_{n,m}$ is a parking function$]\}$. Then*

$$\lim_{n\to\infty} P(\mu n, n) = (1 - \mu)\mathrm{e}^{\mu}, \qquad 0 < \mu \leq 1.$$

Proof. This follows directly from

$$P(m, n) = \frac{g(n, m)}{n^m} = \left(1 + \frac{1}{n}\right)^m \left(1 - \frac{m}{n + 1}\right).$$

□

We will now prove a useful result on the exponential generating function of the numbers $g(n, n)$.

Proposition 2.2.3. *Let*

$$\theta(z) = \sum_{n\geq 0} g(n,n)\frac{z^n}{n!} = \sum_{n\geq 0}(n+1)^{n-1}\frac{z^n}{n!}. \tag{2.1}$$

Then

$$\theta(z) = \mathrm{e}^{z\theta(z)}. \tag{2.2}$$

Proof. Consider the tree function $T(z)$. We have

$$T(z) = \sum_{k\geq 1} k^{k-1}\frac{z^k}{k!} = \sum_{k\geq 1} k^{k-2}\frac{z^k}{(k-1)!} = \sum_{k\geq 0}(k+1)^{k-1}\frac{z^{k+1}}{k!} = z\theta(z), \tag{2.3}$$

and since $T(z) = z\mathrm{e}^{T(z)}$, this proves the proposition. □

Given a permutation τ of $[n]$, one may be interested in the number of parking functions on $[n]$ with output τ. In [13], Gilbey and and Kalikow have given the following expression for this number:

Proposition 2.2.4. *Let τ be a permutation of $[n]$ and*

$$S(\tau) = \{p : [n] \to [n] \mid \pi_p = \tau\}.$$

Define $\tau(0) := n+1$ and

$$b_\tau(j) := \max\{i \in [j-1]_0 \mid \tau(i) > \tau(j)\}.$$

Then

$$|S(\tau)| = \prod_{j=1}^{n}(j - b_\tau(j)). \tag{2.4}$$

Proof. If parking slot j is occupied by driver $\tau(j)$, this driver can have chosen any slot number $i \leq j$ as long as the spaces $i, \ldots, j-1$ were occupied when he arrived. This happens if and only if all these slots are occupied by cars numbered less than $\tau(j)$. □

2.3 Uses of parking functions

In [16], the numbers $g(n,m)$ appear in the context of a hashing problem using the following occupancy discipline:
Consider m balls $B_1, \ldots, B_m$ which have to be placed in n cells $C_0, \ldots, C_{n-1}$ ($n \geq m$). There are m "fictitious" cell numbers $(j_1, \ldots, j_m)$ with $0 \leq j_k < n$ for all $k \in [m]$, and the actual location l_k of B_k is defined according to the rules

- $l_1 = j_1$
- for $k \geq 2: \quad l_k = j_k + s_k \mod n$,
  where $s_k = \min\{i \mid i \geq 0,\ j_k + i \mod n \notin \{l_1, \dots, l_{k-1}\}\}$.

Let $f(n, m)$ be the number of choices of $(j_1, \dots, j_m)$ such that the last cell C_{n-1} remains empty. Then obviously $f(n, m) = g(n-1, m)$.
The numbers $f(n, m)$ can be used to express the probabilities $\mathbb{P}\{s_k = i\}$, if for m balls the fictitious cell numbers $j = (j_1, \dots, j_m)$ are randomly chosen (with $\mathbb{P}\{j = x\} = n^{-m}$ for all $x \in \{0, \dots, n-1\}^m$). We will now present the results in terms of the numbers $g(n, m,)$:
First observe that, due to symmetry

$$\begin{aligned}
\mathbb{P}\{s_k = i\} &= \sum_{t=0}^{n-1} P\{s_k = i | j_k = t\} \mathbb{P}\{j_k = t\} \\
&= \sum_{t=0}^{n-1} P\{s_k = i | j_k = 0\} \cdot \frac{1}{n} \\
&= P\{s_k = i | j_k = 0\}.
\end{aligned}$$

If $j_k = 0$, then $s_k = 0$ if and only if $0 \notin \{l_1, \dots, l_{k-1}\}$.
For $1 \leq i \leq k-1$ the case $s_k = i$ occurs exactly if there exists a $q \in \{0, \dots, k-i-1\}$ such that

- $\{-q \mod n, \dots,\ 0, \dots,\ i-1 \mod n\} \subseteq \{l_1, \dots, l_{k-1}\}$ and
- $\{-q-1 \mod n,\ i \mod n\} \cap \{l_1, \dots, l_{k-1}\} = \emptyset$.

This shows:

$$\begin{aligned}
\mathbb{P}\{s_k = 0\} &= \frac{g(n-1, k-1)}{n^{k-1}}, \\
\mathbb{P}\{s_k = i\} &= \sum_{q=0}^{k-i-1} \binom{k-1}{q+i} g(q+i, q+i) \frac{g(n-i-q-2, k-i-q-1)}{n^{k-1}},
\end{aligned}$$

for $1 \leq i \leq k-1$. Note that this second formula actually holds for $i = 0$, too.
By using Lemma 2.2.1 this can be simplified to

$$\mathbb{P}\{s_k = i\} = \frac{1}{n^{k-1}} \sum_{q=i}^{k-1} \binom{k-1}{q} (q+1)^{q-1} (n-k)(n-q-1)^{k-q-2}.$$

Furthermore,

$$\begin{aligned}
\mathbb{E}\left(s_k\right) &= \frac{n-k}{n^{k-1}} \sum_{i=1}^{k-1} i \sum_{q=i}^{k-1} \binom{k-1}{q} (q+1)^{q-1} (n-q-1)^{k-q-2} \\
&= \frac{n-k}{n^{k-1}} \sum_{q=1}^{k-1} \binom{k-1}{q} (q+1)^{q-1} (n-q-1)^{k-q-2} \sum_{i=1}^{q} i \\
&= \frac{n-k}{2n^{k-1}} \sum_{q=i}^{k-1} \binom{k-1}{q} (q+1)^{q} q (n-q-1)^{k-q-2}.
\end{aligned}$$

2.4 Breakpoints

The concept of *breakpoints* of a function has been introduced by Gilbey and Kalikow in [13]. We will use the following definition:

Definition 2.4.1. *Let $p : [n] \to [n]$. We say that $b \in [n]$ is a breakpoint of p if and only if $|\{i \mid p(i) \leq b\}| = b$.*

Note that if p is a parking function, this condition is equivalent to saying that $p(i) \leq b$ if and only if $\iota p(i) \leq b$ for all $i \in [n]$. This in turn is equivalent to saying that $\{\pi_p(1), \ldots, \pi_p(b)\} = \{i \mid p(i) \leq b\}$. In a more intuitive way, b is a breakpoint of a parking function p if every driver who wishes to park in one of the first b slots succeeds in doing so.
Obviously, n is a breakpoint of any parking function p on $[n]$. The following lemma from [13] shows that at least one other breakpoint exists in many instances:

Lemma 2.4.1. *Let p be a parking function on $[n]$, and $d = \iota p(n)$. Then d is a breakpoint of p.*

Proof. Since parking slot d is the only empty slot when driver n arrives, we have

$$\begin{aligned}
|\{i \in [n-1] \mid \iota p(i) < d\}| &= d-1, \\
|\{i \in [n-1] \mid \iota p(i) = d\}| &= 0,
\end{aligned}$$

and for all $i \in [n-1]$ holds

$$\iota p(i) < d \text{ if and only if } p(i) < d.$$

Since $p(n) \leq \iota p(n) = d$, this shows that $|\{i \in [n] \mid p(i) \leq d\}| = d$. $\square$

We will now derive a formula for the number of parking functions with exactly k breakpoints:

Proposition 2.4.2. *Let $a_{n,k}$ be the number of parking functions on $[n]$ with exactly k breakpoints, and*

$$F(z,w) = \sum_{n\geq 1}\sum_{k\geq 1} \frac{a_{n,k}}{n!} z^n w^k.$$

Then

$$F(z,w) = \frac{\frac{T(z)-z}{T(z)}w}{1-\frac{T(z)-z}{T(z)}w},$$

where $T(z)$ denotes the tree function.
Furthermore,

$$a_{n,k} = \sum_{j=1}^{k} (-1)^{j+1}\binom{k}{j} j(n-j)^{n-1}.$$

Proof. Let $p : [n] \to [n]$ be a parking function with $k > 1$ breakpoints. If b is the first breakpoint of p, then the function

$$p|_{\{i\in[n]\ \mid\ p(i)\leq b\}} : \{i \in [n] \mid p(i) \leq b\} \to [b]$$

is a parking function with one breakpoint, where the cars are numbered with b elements of $[n]$. The number of choices of these elements is $\binom{n}{b}$. Similarly, the function

$$p|_{\{i\in[n]\ \mid\ p(i)>b\}} : \{i \in [n] \mid p(i) > b\} \to \{b+1,\ldots,n\}$$

is a parking function with $k-1$ breakpoints, where the cars are numbered with the remaining elements of $[n]$ (and the parking slots have been renumbered as well). This gives the recursion

$$a_{n,k} = \sum_{b=1}^{n-1}\binom{n}{b} a_{b,1}a_{n-b,k-1}, \quad n\geq 1,\ k\geq 2.$$

By multiplying the recursion with $\frac{z^n w^k}{n!}$ and summing up, we get

$$\begin{aligned}
F(z,w) - \sum_{n\geq 1} \frac{a_{n.1}}{n!} z^n w &= \sum_{k\geq 2}\sum_{n\geq 1}\sum_{b=1}^{n-1} \frac{a_{b,1}}{b!}\frac{a_{n-b,k-1}}{(n-b)!} z^n w^k \\
&= \sum_{k\geq 2} w^k \sum_{b\geq 1} \frac{a_{b,1}}{b!} z^b \sum_{n\geq 1} \frac{a_{n,k-1}}{n!} z^n \\
&= \sum_{b\geq 1} \frac{a_{b,1}}{b!} z^b w \cdot \sum_{n\geq 1}\sum_{k\geq 2} \frac{a_{n,k-1}}{n!} z^n w^{k-1} \\
&= \sum_{b\geq 1} \frac{a_{b,1}}{b!} z^b w \cdot F(z,w),
\end{aligned}$$

or

$$\sum_{n\geq 1} \frac{a_{n,1}}{n!} z^n w = \frac{F(z,w)}{1+F(z,w)}.$$

Now, consider the function $\theta(z)$ as defined in (2.1).
We must have $F(z,1) = \theta(z) - 1$, and we know that $\theta(z) = \frac{T(z)}{z}$, see (2.3). Hence,

$$\sum_{n\geq 1} \frac{a_{n,1}}{n!} z^n = \frac{T(z)-z}{T(z)},$$

and finally

$$F(z,w) = \frac{\frac{T(z)-z}{T(z)} w}{1 - \frac{T(z)-z}{T(z)} w}.$$

Now, the coefficients $a_{n,k}$ can be extracted:

$$\begin{aligned}
\frac{a_{n,k}}{n!} &= [z^n w^k] \frac{\frac{T(z)-z}{T(z)} w}{1 - \frac{T(z)-z}{T(z)} w} \\
&= [z^n] \left(1 - \frac{z}{T(z)}\right)^k \\
&= \sum_{j=0}^{k} (-1)^j \binom{k}{j} [z^{n-j}] \frac{1}{T(z)^j},
\end{aligned}$$

for $n, k \geq 1$.

We know that $z = \frac{T(z)}{\mathrm{e}^{T(z)}}$. Hence, using Lagrange's inversion formula, we get

$$\begin{aligned}
[z^{n-j}]\frac{1}{T(z)^j} &= -\frac{j}{n-j}[T^{n-j-1}]\frac{\mathrm{e}^{(n-j)T}}{T^{j+1}} \\
&= -\frac{j}{n-j}[T^n]\mathrm{e}^{(n-j)T} \\
&= -\frac{j}{n-j}\frac{(n-j)^n}{n!}.
\end{aligned}$$

This shows that

$$a_{n,k} = \sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}j(n-j)^{n-1}, \quad n,k \geq 1.$$

$\square$

Chapter 3

Relations to other combinatorial objects

Many relations between parking functions and other combinatorial objects are now known, such as labeled trees and acyclic functions [12], priority queues [13], noncrossing partitions [25] and special polytopes [21]. In this section, we will present two of these known relations.

3.1 Parking functions and acyclic functions

We will show that *acyclic functions* on $[n]$ can be represented as free labeled trees with $n+1$ vertices, the total number of which is $T_{n+1} = (n+1)^{n-1} = g(n,n)$ (see Section 1.2). This suggests to find a mapping between acyclic and parking functions. We will then present two different mappings due to Foata and Rioardan [12]. These mappings can be used to translate known properties of acyclic functions and free labeled trees into properties of parking functions.

3.1.1 The representation of acyclic functions as labeled trees

Definition 3.1.1. *Let $f : [n] \to [n]$. We say that f is an acyclic function if and only if all cycles of f are of length* 1*.*

It is easy to see how an acyclic function f on $[n]$ can be represented as a labeled tree with $n+1$ vertices:
First, given f, we define a rooted labeled forest with vertices $x \in [n]$ in the following way: For $x \neq y$, we let (x, y) be an edge of the forest if $f(x) = y$. We further let $x \in [n]$ be a root if and only if $f(x) = x$.

The resulting forest is then mapped to a free tree with $n+1$ labeled vertices by connecting all rooted vertices with a new vertex $n+1$.
An example is shown in Figure 3.1.

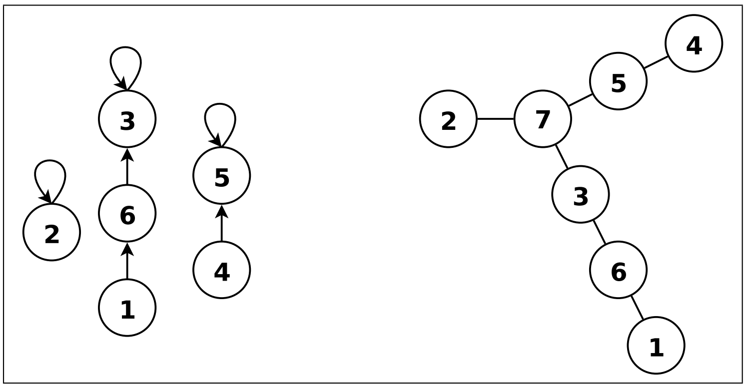

Figure 3.1: Representation of the acyclic function $f = (6,\ 2,\ 3,\ 5,\ 5,\ 3)$ as a rooted labeled forest (roots are indicated by slings) and as a labeled tree

3.1.2 A mapping using tree codes

The first mapping between parking and acyclic functions rests on a mapping of parking functions on codes by Pollak:
The code for a parking function $(p(1), \ldots, p(n))$ is $(c(1), \ldots, c(n-1))$ with

$$c(i) = p(i+1) - p(i) \mod (n+1). \tag{3.1}$$

In order to map the code $c = (c(1), \ldots, c(n-1))$ to an acyclic function, we interpret c as a Prüfer code for a labeled tree with $n+1$ vertices. This correspondence can be described as follows:
Given a labeled tree with n vertices, let $(c_1, \ldots, c_{n-2})$ denote the associated code. c_1 is the vertex adjacent to the leave of the tree with the smallest label. Now this leave and its edge to c_1 are removed. c_2 is then the vertex adjacent to the leave with smallest label of the remaining tree, and so on. The process stops when there are only two vertices left.
For the inverse one can write down two sequences:

$$\begin{array}{l} b_1, \ldots, b_{n-2},\ b_{n-1} \\ c_1, \ldots, c_{n-2},\ n \end{array}$$

The second sequence is the code augmented by n. The first one will be the sequence of vertices adjacent to the code elements in the process described above, which can be determined in the following way: b_1 is the smallest number not contained in $c_1, \ldots, c_{n-2},\ n$. For $i \in \{2, \ldots, n-1\}$, b_i is the smallest number not contained in $b_1, \ldots, b_{i-1},\ c_i, \ldots, c_{n-2},\ n$. The pairs $(b_i,\ c_i)$, $i \in [n-2]$ and (b_{n-1}, n) are $n-1$ lines of the tree, thus the tree is completely determined.

Example:
$n = 6$: The code corresponding to the parking function $p = (4,\ 3,\ 3,\ 1,\ 1,\ 4)$ is $c = (6,\ 0,\ 5,\ 0,\ 3)$. This is the Prüfer code of the labeled tree shown in Figure 3.1, hence the associated acyclic function is $f = (6,\ 2,\ 3,\ 5,\ 5,\ 3)$.

We will now show that the mapping between parking functions and codes is a bijection. If $p(1)$ is known then the inverse of (3.1) is obviously given by

$$p(i) = p(1) + c(1) + \ldots + c(i-1) \mod (n+1), \qquad i = 2, \ldots, n \tag{3.2}$$

Proposition 3.1.1. *For each code* $(c(1), \ldots, c(n-1)) \in \mathbb{Z}_{n+1}^{n-1}$ *there exists a unique* $p(1)$ *such that the function* p *defined by* (3.2) *is a parking function.*

Proof. In [12], this has been proved by providing an algorithm which determines $p(1)$:
Define $(r_1, \ldots, r_n)$ with $r_i = |\{x \mid p(x) = i\}|$ to be the *specification* of a parking function p. From Lemma 2.1.2 it follows that $r_{n-j+1} + \ldots + r_n \le j$ for all $j \in [n-1]$ and it is clear that $r_1 + \ldots + r_n = n$. With $R_j = r_1 + \ldots + r_j - j$ this is equivalent to

$$R_j \ge 0, \quad j \in [n-1], \qquad R_n = 0. \tag{3.3}$$

Now, given a code $(c(1), \ldots, c(n-1))$, consider the function $(h(1), \ldots, h(n))$ with

$$\begin{aligned} h(1) &= n+1 \\ h(i+1) &= c(1) + \ldots + c(i) \mod (n+1), \qquad i \in [n-1]. \end{aligned}$$

Take $r(h) = (r_1, \ldots, r_{n+1})$ as the specification of h and define $R_j(h) = r_1 + \ldots + r_j - j$, for $j \in [n+1]$.
Now let d denote the leftmost position of the minimums of $R_j(h)$, i. e.,

$$\begin{aligned} R_d(h) &< R_j(h),\ j \in \{1, \ldots, d-1\} \\ R_d(h) &\le R_j(h),\ j \in \{d+1, \ldots, n+1\}. \end{aligned}$$

Since $R_{n+1} = -1$, we must have $R_d(h) \le -1$. If $d = 1$, then $-1 \ge R_1(h) = r_1 - 1$, and if $d > 1$, then $r_1 + \ldots + r_{d-1} - d + 1 = R_{d-1}(h) > R_d(h) = r_1 + \ldots + r_d - d$. This shows that $r_d = 0$.

Define a function p as in (3.2) with $p(1) = n + 1 - d$. Then

$$p(i) = h(i) - d \mod (n+1), \qquad i \in [n].$$

and the specification of p is $(r_{d+1}, \ldots, r_{n+1}, r_1, \ldots, r_d)$.
Now, firstly, $R_n(p) = r_1 + \ldots + r_{n+1} - r_d - n = 0$, and secondly

$$R_j(p) = \begin{cases} R_{d+j}(h) - R_d(h), & j \in \{1, \ldots, n+1-d\} \\ R_{d+j-n-1}(h) - R_d(h) + R_{n+1}(h), & j \in \{n+2-d, \ldots, n+1\} \end{cases}$$

These relations together with the inequalities for $R_d(h)$ show that $R_j(p) \geq 0, \ j \in [n-1]$.
Hence p is a parking function. □

Example:
We will show the inverse of the last example:
Take $n = 6$ and $f = (6,\ 2,\ 3,\ 5,\ 5,\ 3)$, so that $c = (6,\ 0,\ 5,\ 0,\ 3)$. We get $h = (7,\ 6,\ 6,\ 4,\ 4,\ 7)$ and $r(h) = (0,\ 0,\ 0,\ 2,\ 0,\ 2,\ 2)$. Then $(R_1(h), \ldots, R_7(h)) = (-1, -2, -3, -2, -3, -2, -1)$, so $d = 3$ and $p = (4,\ 3,\ 3,\ 1,\ 1,\ 4)$.

3.1.3 A mapping using balanced sequences and permutations

For this mapping, a set C_n is introduced which will be put in one-to-one correspondence with both the parking and the acyclic functions on $[n]$ (denoted by A_n and B_n, respectively).
Throughout this section, for each sequence $r = (r_1, \ldots, r_n)$ let the non-zero elements of r be denoted by $(r_{i_1}, \ldots, r_{i_m})$ and set $s_{i_j} = r_{i_1} + \ldots + r_{i_j}$.

Definition 3.1.2. *Let $r = (r_1, \ldots, r_n)$ be a sequence of non-negative integers. We say that r is balanced if, with*

$$R_j = r_1 + \ldots + r_j - j, \quad j \in [n],$$

conditions (3.3) *are fulfilled.*

If r is a balanced sequence, then we clearly have $i_1 = 1$. Furthermore, note that r is balanced if and only if

$$s_{i_j} - (i_{j+1} - 1) \geq 0 \text{ for all } j \in [m-1], \quad \text{and} \quad s_{i_m} = n. \tag{3.4}$$

Definition 3.1.3. *Let π be a permutation of $[n]$ and $r = (r_1, \ldots, r_n)$ a balanced sequence. We say that π is compatible with r if π^{-1} is increasing on $\{1, \ldots, r_{i_1}\}$ and on each interval $\{s_{i_j} + 1, \ldots, s_{i_{j+1}}\},\ j \in [m-1]$.*

We now define C_n as the set of all couples (r, π) with

- $r = r_1, \ldots, r_n$ a balanced sequence,
- π a permutation of $[n]$ which is compatible with r.

A parking function $p \in A_n$ is now mapped to an element $(s(p), \tau_p)$ of C_n in the following way:
$s(p)$ is the specification of p as described earlier.
τ_p is defined by

$$\tau_p(x) = \left|\{y \in [n] \mid p(y) < p(x), \quad \text{or} \quad p(y) = p(x) \text{ and } y \leq x\}\right|.$$

Example:
Take $n = 9$ and $p = (1,\ 6,\ 2,\ 3,\ 5,\ 2,\ 1,\ 5,\ 3)$.
We get $\tau_p = (1,\ 9,\ 3,\ 5,\ 7,\ 4,\ 2,\ 8,\ 6)$ and $s(p) = (2,\ 2,\ 2,\ 0,\ 2,\ 1,\ 0,\ 0,\ 0)$.

Proposition 3.1.2. *The mapping $p \to (s(p), \tau_p)$ maps A_n into C_n.*

Proof. $s(p)$ is balanced due to relation (3.3), and $\tau_p(x)$ is obviously a permutation on $[n]$. □

It is easy to verify that the definition of $(s(p), \tau_p)$ is equivalent to saying that

$$p(x) = \begin{cases} 1, & \text{if} \quad 1 \leq \tau_p(x) \leq r_{i_1}, \\ i_j, & \text{if} \quad s_{i_{j-1}} + 1 \leq \tau_p(x) \leq s_{i_j}, \quad j \in \{2, \ldots, m\}, \end{cases} \tag{3.5}$$

and that τ_p^{-1} is increasing on $\{1, \ldots, r_{i_1}\}$ and on each interval of the form $\{s_{i_j} + 1, \ldots, s_{i_{j+1}}\}$ for $j \in [m-1]$.

Proposition 3.1.3. *The mapping $p \to (s(p), \tau_p)$ is a bijection between A_n and C_n.*

Proof. The mapping is injective, because if p and p' are two distinct parking functions, then either $s(p) \neq s(p')$ or, if $s(p) = s(p')$, then $\tau_p \neq \tau_{p'}$ because of (3.5).
To prove that the mapping $p \to (s(p), \tau_p)$ is surjective and at the same time define its inverse, let (r, π) be an element of C_n. Again, let the non-zero elements of r be denoted by $(r_{i_1}, \ldots, r_{i_m})$ and set $s_{i_j} = r_{i_1} + \ldots + r_{i_j}$.

Define the function p by

$$p(x) = \begin{cases} 1 & \text{if} \quad 1 \leq \pi(x) \leq r_{i_1} \\ i_j & \text{if} \quad s_{i_{j-1}} + 1 \leq \pi(x) \leq s_{i_j}, \quad j \in \{2, \ldots, m\}. \end{cases} \tag{3.6}$$

Then, firstly, r is the specification of p and since r is balanced, p is a parking function. Secondly, π and τ_p are both compatible with r. This together with (3.5) and (3.6) shows that $\pi = \tau_p$.
Hence $p \to (s(p), \tau_p)$ is a bijection between A_n and C_n. □

Now B_n will be mapped to C_n. If f is an acyclic function, then the associated element of C_n will be denoted by $(t(f), \sigma_f)$.
As described in the first mapping, the function f is represented as a forest of rooted trees, labeled with the numbers $1, \ldots, n$.
The usual definition of the *height* $h(x)$ of a vertex will be used: For each vertex x, $h(x)$ is the length of the (unique) path from x to $z(x)$, where $z(x)$ is the root of the tree to which x belongs. Roots are of course of height zero. Furthermore, if y lies on the path from x to $z(x)$, we say that x is at height $h(y) - h(x)$ from y.
We define a total order $<_f$ on the set $[n]$, from which σ_f will be derived: First, let $x <_f y$ if

- $h(x) < h(y)$ or if
- $h(x) = h(y) = 0$ and $x < y$.

Then assume that the order $<_f$ has been defined on the set of all vertices of height $\leq k$. If $h(x) = h(y) = k+1$, then $h(f(x)) = h(f(y)) = k$. By induction, let $x <_f y$ if

- $f(x) <_f f(y)$, or if
- $f(x) = f(y)$ and $x < y$.

We let $\sigma_f^{-1} = (\sigma_f^{-1}(1), \ldots, \sigma_f^{-1}(n))$ be the elements of n in increasing order with respect to $<_f$ and define σ_f to be the inverse permutation of σ_f^{-1}.
Furthermore, we let $t(f) = (r_1, \ldots, r_n)$ be the *forest-specification* of f: r_1 is the number of roots, and for each $i \in \{2, \ldots, n\}$, r_i is the number of vertices at height 1 from $\sigma_f^{-1}(i-1)$.

Example:
Take $n = 9$ and $f = (1,\ 4,\ 1,\ 7,\ 6,\ 1,\ 7,\ 6,\ 7)$.

One can read off the linear order $<_f$ from the rooted labeled forest in Figure 3.2: We have $1 <_f 7 <_f 3 <_f 6 <_f 4 <_f 9 <_f 5 <_f 8 <_f 2$, hence $\sigma_f = (1,\ 9,\ 3,\ 5,\ 7,\ 4,\ 2,\ 8,\ 6)$.
Furthermore, we get $t(f) = (2,\ 2,\ 2,\ 0,\ 2,\ 1,\ 0,\ 0,\ 0)$.

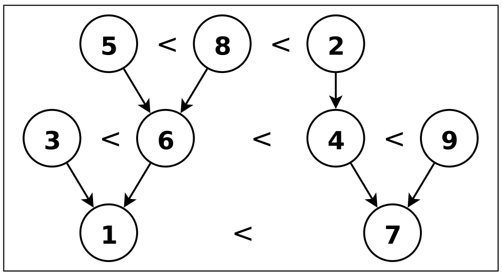

Figure 3.2: The acyclic function $f = (1,\ 4,\ 1,\ 7,\ 6,\ 1,\ 7,\ 6,\ 7)$ represented as a rooted labeled forest. The linear order $<=<_f$ can be read off from bottom to top.

Proposition 3.1.4. *The mapping $f \mapsto (t(f), \sigma_f)$ maps B_n into C_n.*

Proof. As before, let $(r_{i_1}, \ldots, r_{i_m})$ be the sequence of non-zero elements of $t(f)$ and $s_{i_j} = r_{i_1} + \ldots + r_{i_j}$. Obviously

$$s_{i_m} = n \tag{3.7}$$

since the number of roots plus the number of lines in a rooted forest with n vertices is n.
The first $r_1 = r_{i_1}$ elements of σ_f^{-1} are the roots in rising order, thus

$$f(x) = x \quad \text{if} \quad 1 \leq \sigma_f(x) \leq r_{i_1}. \tag{3.8}$$

Furhermore, for $j \in \{2, \ldots, n\}$, the elements $\sigma_f^{-1}(s_{i_{j-1}} + 1), \ldots, \sigma_f^{-1}(s_{i_j})$ are the vertices at height 1 from $\sigma_f^{-1}(i_j - 1)$ written in rising order (compare the definition of $<_f$). Thus

$$f(x) = \sigma_f^{-1}(i_j - 1) \quad \text{if} \quad s_{i_{j-1}} + 1 \leq \sigma_f(x) \leq s_{i_j}, \quad j \in \{2, \ldots, n\}. \tag{3.9}$$

Moreover, the fact that a vertex of smaller height precedes a vertex of greater height in the sequence σ_f^{-1} implies

$$i_j - 1 < s_{i_{j-1}} + 1 \tag{3.10}$$

or equivalenty

$$s_{i_{j-1}} + 1 - (i_j - 1) > 0, \quad j \in \{2, \ldots, n\}. \tag{3.11}$$

Relations (3.7) and (3.11) show that $t(f)$ is balanced (compare relation (3.4)). Also, σ_f^{-1} is increasing on $\{1,\ldots,r_{i_1}\}$ and on each interval $\{s_{i_{j-1}}+1,\ldots,s_{i_j}\}$ for $j\in\{2,\ldots,m\}$, thus σ_f is compatible with $t(f)$. So $f\mapsto(t(f),\sigma_f)$ maps B_n into C_n. $\square$

Proposition 3.1.5. *The mapping $f\mapsto(t(f),\sigma_f)$ is a bijection between B_n and C_n.*

Proof. The mapping is injective, because if f and f' are two distinct acyclic functions on $[n]$, either $\sigma_f\neq\sigma_{f'}$, or, if $\sigma_f=\sigma_{f'}$, then $t(f)\neq t(f')$ due to the relations (3.8) and (3.9).
In order to prove that the mapping $f\mapsto(t(f),\sigma_f)$ is surjective and at the same time define its inverse, let (r,π) be an element of C_n. Again, the non-zero elements of r are denoted by $(r_{i_1},\ldots,r_{i_m})$ and $s_{i_j}=r_{i_1}+\ldots+r_{i_j}$.
Define a function $f:[n]\to[n]$ by

$$f(x)=\begin{cases}x & \text{if}\quad 1\leq\pi(x)\leq r_{i_1},\\ \pi^{-1}(i_j-1) & \text{if}\quad s_{i_{j-1}}+1\leq\pi(x)\leq s_{i_j},\quad j\in\{2,\ldots,m\},\end{cases}\tag{3.12}$$

(note that this is possible because $s_{i_m}=n$).
As r is balanced, relation (3.10) holds for $j\in\{2,\ldots,m\}$, hence f is an acyclic function with roots $\pi^{-1}(1),\ldots,\pi^{-1}(r_{i_1})$. Now let us show that $(t(f),\sigma_f)=(r,\pi)$: π is compatible with r, which, together with the definition of f and the total order $<_f$, implies that $\pi^{-1}(1)<_f\cdots<_f\pi^{-1}(r_{i_1})$ and $\pi^{-1}(s_{i_{j-1}}+1)<_f\cdots<_f\pi^{-1}(s_{i_j})$ for all $j\in\{2,\ldots,m\}$. Clearly $\pi^{-1}(r_{i_1})$ is of height zero and $\pi^{-1}(r_{i_1}+1)$ is of height 1, thus $\pi^{-1}(r_{i_1})<_f\pi^{-1}(r_{i_1}+1)$.
Now assume that $\pi^{-1}(s_{i_{k-1}})<_f\pi^{-1}(s_{i_{k-1}}+1)$ for all k with $2\leq k\leq j<m$. This is equivalent to saying that $\pi^{-1}(1)<_f\cdots<_f\pi^{-1}(s_{i_j})$. The two elements $\pi^{-1}(s_{i_j})$ and $\pi^{-1}(s_{i_j}+1)$ are at height 1 from $\pi^{-1}(i_j-1)$ and $\pi^{-1}(i_{j+1}-1)$, respectively. Relation (3.10) implies that

$$i_j-1<i_{j+1}-1\leq s_{i_j}.$$

By induction, this shows that

$$f(\pi^{-1}(s_{i_j}))=\pi^{-1}(i_j-1)<_f\pi^{-1}(i_{j+1}-1)=f(\pi^{-1}(s_{i_j}+1))$$

and from the definition of $<_f$ follows

$$\pi^{-1}(s_{i_j})<\pi^{-1}(s_{i_j}+1).$$

Hence $\sigma_f=\pi$, and this together with the definition of f implies that $t(f)=r$. Thus $f\mapsto(t(f),\sigma_f)$ is a bijection between B_n and C_n. $\square$

Now the two bijections from C_n to A_n and B_n can be combined in order to get an explicit mapping $f \mapsto p$ from B_n to A_n. This mapping can be expressed as

$$p(x) = \begin{cases} 1 & \text{if} \quad f(x) = x \\ 1 + \sigma_f(f(x)) & \text{if} \quad f(x) \neq x \end{cases}$$

and the inverse is given by

$$f(x) = \begin{cases} x & \text{if} \quad p(x) = 1 \\ \tau_p^{-1}(p(x) - 1) & \text{if} \quad p(x) \neq 1. \end{cases}$$

Example:
Take $n = 9$ and $p = (1,\ 6,\ 2,\ 3,\ 5,\ 2,\ 1,\ 5,\ 3)$.
Then $\tau_p = (1,\ 9,\ 3,\ 5,\ 7,\ 4,\ 2,\ 8,\ 6)$ and $f = (1,\ 4,\ 1,\ 7,\ 6,\ 1,\ 7,\ 6,\ 7)$. The fact that p is mapped to f can also be seen from the previous two examples, where $\tau_p = \sigma_f$ and $s(p) = t(f)$.

3.1.4 Uses of the mappings

In [12], different uses of the mappings described have been given, two of which will now be demonstrated.
We will use the following result about rooted trees, which has been found by Riordan in [23]:

Lemma 3.1.6. *Let $a_{n,k}$ be the number of forests of rooted trees with n labeled vertices which consist of exactly k trees, and define*

$$A_n(x) = \sum_{k \geq 0} a_{n,k} x^k.$$

Then

$$A_n(x) = x(x + n)^{n-1}.$$

Proposition 3.1.7. *Let $b_{n,k}$ be the number of parking functions on $[n]$ with k equal consecutive numbers, and define*

$$B_n(x) = \sum_{k \geq 0} b_{n,k} x^k.$$

Then

$$B_n(x) = x^{-1} A_n(x) = (x + n)^{n-1}.$$

Proof. Using the first mapping, a pair of equal consecutive numbers in the parking funcion corresponds to a zero in the code. Furthermore, a code with k zeros corresponds to an acyclic function with $k+1$ fixed points (which is represented by a forest of $k+1$ rooted trees). Together with Lemma 3.1.6 this proves the proposition. □

Proposition 3.1.8. *Let $c_{n,k}$ be the number of parking functions on $[n]$ with k elements equal to* 1*, and define*

$$C_n(x) = \sum_{k \geq 0} c_{n,k} x^k.$$

Then

$$C_n(x) = A_n(x) = x(x+n)^{n-1}.$$

Proof. In the second mapping, an acyclic function with k fixed points corresponds to a parking function with k elements equal to 1. Again, using Lemma 3.1.6, this proves the proposition. □

3.2 Parking functions and priority queues

The following relation between parking functions on $[n]$ and allowable input-output pairs of permutations of $[n]$ in a priority queue is due to Gilbey and Kalikow [13]. They have described this relation in a more general way by the use of *valet functions*. But for now we will restrict ourselves to the simpler case of ordinary parking functions.

3.2.1 Definition of priority queues and allowable pairs

A *priority queue* is an abstract data type supporting the operations Insert and DeleteMin. There is an input stream $\sigma = (\sigma_1, \ldots, \sigma_n)$ and an output stream $\tau = (\tau_1, \ldots, \tau_n)$ which are tupels of elements of a totally ordered set. Each Insert operation inserts the next element of σ into the queue, whereas each DeleteMin operation removes a minimal element from the queue and places it in the output stream.

Of course, the DeleteMin operation is only allowed if the queue is not empty. An allowable sequence of n Insert's and n DeleteMin's is called a *priority queue computation*. If σ is the input and τ is the output of some priority queue computation, then (σ, τ) is called an *allowable pair*. Throughout this section, we will always assume that $\{\sigma_1, \ldots, \sigma_n\} = \{\tau_1, \ldots, \tau_n\} = [n]$, i. e., σ and τ both are permutations.

The following algorithm from [2] tests whether a given pair (σ, τ) of tupels of length n is an allowable pair. If (σ, τ) is an allowable pair, the priority queue computation executed by this algorithm is called the *natural computation* for (σ, τ).

```
function TestPair((σ, τ), n) {
        Q := empty priority queue
        i := 1
    (*) for j :=1 to n {
                while τ_j ∉ Q {
                        INSERT(σ_i)
                        i := i + 1
                }
                if τ_j ≠ min(Q) {
                        return FALSE
                }
                else {
                        DELETEMIN
                }
        }
        return TRUE
}
```

Let A_n denote the set of parking functions on $[n]$ and B_n the set of allowable pairs (σ, τ) where σ and τ are permutations of $[n]$.
In [3], Atkinson and Thiyagarajah have found that $|B_n| = (n+1)^{n-1} = g(n, n) = g(n, n)$. This suggests to find a mapping between A_n and B_n.

3.2.2 Breakpoints

We have already defined breakpoints of parking functions. We will now define an analogous concept for pairs (σ, τ) of permutations on $[n]$. This will be useful in the subsequent proofs.

Definition 3.2.1. *Let (σ, τ) be a pair of permutations of $[n]$. We say that $b \in [n]$ is a breakpoint of (σ, τ) if and only if $\{\sigma_1, \ldots, \sigma_b\} = \{\tau_1, \ldots, \tau_b\}$.*

If (σ, τ) is an allowable pair, this is equivalent to saying that with the natural computation the queue is empty after outputting τ_b.
Obviously, n is breakpoint of any allowable pair (σ, τ) on $[n]$.
The following lemma shows that at least one other breakpoint exists in many instances:

Lemma 3.2.1. *Let $(\sigma, \tau) \in B_n$ and $\delta = \tau^{-1}(n)$. Then δ is a breakpoint of (σ, τ).*

Proof. With any computation, n is only written to the output stream if there is no element $i < n$ in the queue. Thus the queue is empty after outputting n. □

3.2.3 Bijection between parking functions and allowable pairs

We define functions $\phi_n : A_n \to B_n$ and $\psi_n : B_n \to A_n$ by induction on n. The case $n = 1$ is trivial since $|A_1| = |B_1| = 1$.
Given $p \in A_n$, define $(s, t) = \phi_n(p)$ as follows:

(ϕ1) Set $t := \pi_p$ and $d := \pi_p^{-1}(n) = \iota p(n)$.

(ϕ2) Define $p' \in A_{n-1}$ by

$$p'(i) := \begin{cases} p(i) - 1 & \text{if } p(i) > d \\ p(i) & \text{otherwise} \end{cases}, \quad i \in [n-1].$$

(ϕ3) Set $(s', t') = \phi_{n-1}(p')$.

(ϕ4) Define s by inserting n into the $p(n)$-th position of s'.

Given $(\sigma, \tau) \in B_n$, define $q = \psi_n(\sigma, \tau)$ as follows:

(ψ1) Set $q(n) := \sigma^{-1}(n)$ and $\delta := \tau^{-1}(n)$.

(ψ2) Let σ' and τ' be, respectively, σ and τ with n deleted, so $(\sigma', \tau') \in B_{n-1}$.

(ψ3) Set $q' = \psi_{n-1}(\sigma', \tau')$.

(ψ4) Set

$$q(i) := \begin{cases} q'(i) + 1 & \text{if } q'(i) \geq \delta \\ q'(i) & \text{otherwise} \end{cases}, \quad i \in [n-1].$$

Proposition 3.2.2. *The functions ϕ_n and ψ_n are well-defined, output and breakpoint preserving, mutually inverse bijections between A_n and B_n.*

Following the example of Gilbey and Kalikow, we will split the proof into small parts.

Proposition 3.2.3. *Step (ϕ2) is well-defined and $\pi_{p'}$ is π_p with n deleted (i. e., $\pi_{p'}(i) = \pi_p(i)$, if $i < d$ and $\pi_{p'}(i) = \pi_p(i+1)$, if $d \le i < n$).*

Sketch of proof. For $p : [n] \to [n]$ and $i \in [n]$, set

$$\begin{aligned} E_p(1) &= [n], \\ E_p(i) &= \{j \in [n] \mid j \notin \{\iota p(1), \dots, \iota p(i-1)\}\} \end{aligned}$$

$E_p(i)$ can be interpreted as the set of unoccupied parking slots when driver i arrives. Clearly driver i will choose slot number $\iota p(i) = \min\{j \in E_p(i) \mid j \ge p(i)\}$.
It is easy to verify by induction on i, that

$$E_{p'}(i) = \{j \mid j \in E_p(i) \text{ and } j < d\} \cup \{j-1 \mid j \in E_p(i) \text{ and } d < j \le n\}.$$

It follows that for all $i \in [n]$ holds

$$\iota p'(i) = \iota p(i), \quad \text{if } p(i) < d$$

and

$$\iota p'(i) = \iota p(i) - 1, \quad \text{if } d < p(i) \le n.$$

The fact that $p' \in A_{n-1}$ and the claim about $\pi_{p'}$ follow immediately: For all $i \in [n-1]$, either $\iota p'(i) = \iota p(i) < d \le n$ or $\iota p'(i) = \iota p(i) - 1 \le n-1$, thus $p' \in A_{n-1}$. Furthermore, if $j < d$, then for some $i < n$, $j = \iota p(i) = \iota p'(i)$, thus $\pi_{p'}(j) = \pi_p(j)$. On the other hand, if $n > j \ge d$, then there exists an $i \in [n-1]$ with $j = \iota p(i) - 1 = \iota p'(i)$, hence $\pi_{p'}(j) = \pi_p(j+1)$.
This shows that $\pi_{p'}$ is π_p with n deleted. □

Proposition 3.2.4. *Step (ψ_2) is well-defined.*

Proof. Consider the natural priority queue computation for (σ, τ). Remove the $\sigma^{-1}(n)$-th INSERT and the $\tau^{-1}(n)$-th DELETEMIN to get a priority queue computation, which produces an output of τ' given an input of σ'. Thus $(\sigma', \tau') \in B_{n-1}$. □

Proposition 3.2.5. *ϕ_n preserves breakpoints.*

Proof. Let b be a breakpoint of p.
If $b < d$, then $p(i) \le b$ if and only if $p'(i) \le b$, $i \in [n-1]$ by definition of p', thus b is a breakpoint of p'. By the inductive hypothesis, b is a breakpoint of (s', t'), so $\{s'_1, \dots, s'_b\} = \{t'_1, \dots, t'_b\}$. The fact that $b < d = \iota p(n)$ and b is a breakpoint of p implies that $p(n) > b$. By definitions (ϕ1) and (ϕ4) and Lemma 3.2.3, we have $\{s_1, \dots, s_b\} = \{s'_1, \dots, s'_b\} = \{t'_1, \dots, t'_b\} = \{t_1, \dots, t_b\}$, thus b is a breakpoint of (s, t).

On the other hand, if $d \leq b \leq n$, $p(i) \leq b$ if and only if $p'(i) \leq b-1,\ i \in [n-1]$. Since $p(n) \leq \iota p(n) = d \leq b$, we have

$$\begin{aligned} |\{i \in [n-1] \mid p'(i) \leq b-1\}| &= |\{i \in [n-1] \mid p(i) \leq b\}| \\ &= |\{i \in [n] \mid p(i) \leq b\}| - 1 \\ &= b-1, \end{aligned}$$

so $b-1$ is a breakpoint of p'. Now, by induction, $b-1$ is a breakpoint of (s', t'). It follows that

$$\begin{aligned} \{s_1, \ldots, s_b\} &= \{s'_1, \ldots, s'_{b-1}\} \cup \{n\} \\ &= \{t'_1, \ldots, t'_{b-1}\} \cup \{n\} \\ &= \{t_1, \ldots, t_b\}, \end{aligned}$$

thus b is a breakpoint of (s, t). □

Proposition 3.2.6. *ψ_n preserves breakpoints.*

Proof. Let $b < \delta$ be a breakpoint of (σ, τ), then $\{\tau'_1, \ldots, \tau'_b\} = \{\tau_1, \ldots, \tau_b\}$. We also must have $\sigma^{-1}(n) > b$ since b is a breakpoint, thus $\{\sigma'_1, \ldots, \sigma'_b\} = \{\sigma_1, \ldots, \sigma_b\}$. This shows that $\{\tau'_1, \ldots, \tau'_b\} = \{\sigma'_1, \ldots, \sigma'_b\}$, hence b is a breakpoint of (σ', τ'). By the inductive hypothesis, b is a breakpoint of q'.
If, on the other hand, $\delta \leq b \leq n$ is a breakpoint of (σ, τ), then $\{\tau'_1, \ldots, \tau'_b\} = \{\tau_1, \ldots, \tau_{b-1}\} \cup \{n\}$ and $\{\sigma'_1, \ldots, \sigma'_b\} = \{\sigma_1, \ldots, \sigma_{b-1}\} \cup \{n\}$, which implies that $\{\tau'_1, \ldots, \tau'_{b-1}\} = \{\sigma'_1, \ldots, \sigma'_{b-1}\}$, hence $b-1$ is a breakpoint of q'.
Since for $b < \delta$, $q'(i) \leq b$ if and only if $q(i) \leq b,\ i \in [n-1]$, and for $\delta \leq b \leq n$, $q'(i) \leq b-1$ if and only if $q(i) \leq b,\ i \in [n-1]$, one can easily verify that in both cases $|\{i \in [n] \mid q(i) \leq b\}| = b$. Hence b is a breakpoint of q. □

Proposition 3.2.7. *$d-1$ is a breakpoint of p'.*

Proof. Since d is a breakpoint of p and $\iota p(n) = d$, we have $p(i) < d$ if and only if $\iota p(i) < d$ for $i < n$. This implies that

$$\begin{aligned} |\{i \in [n-1] \mid p(i) < d\}| &= |\{i \in [n-1] \mid \iota p(i) < d\}| \\ &= |\{i \in [n] \mid \iota p(i) < d\}| \\ &= d-1, \end{aligned}$$

since p is a parking function. The definition of p' shows that $p'(i) \leq d-1$ if and only if $p(i) < d,\ i \in [n-1]$. Hence, we finally get $|\{i \in [n-1] \mid p'(i) \leq d-1\}| = d-1$. □

Proposition 3.2.8. *$\delta - 1$ is a breakpoint of (σ', τ').*

Proof. Consider the natural computation for (σ, τ). Remove the $\sigma^{-1}(n)$-th INSERT and the $\tau^{-1}(n)$-th DELETEMIN to get a priority queue computation for (σ', τ'). When n is the output in the priority queue computation for (σ, τ), the queue can only contain the element n. Thus, in the priority queue computation for (σ', τ'), the queue will be empty at this point, i. e., after the $(\delta - 1)$-th DELETEMIN. □

Proposition 3.2.9. *ψ_n produces parking functions.*

Proof. We show that given $(\sigma, \tau) \in B_n$, $q = \psi_n(\sigma, \tau)$ is a major function, i. e. $|\{i \in [n] \mid q(i) \leq j\}| \geq j,\ j \in [n]$. By the induction hypothesis, we already have $q' \in B_{n-1}$, so q' is a major function.
Let $j < \delta$, then $q(i) \leq j$ if and only if $q'(i) \leq j,\ i \in [n-1]$, thus

$$|\{i \in [n] \mid q(i) \leq j\}| \geq |\{i \in [n-1] \mid q'(i) \leq j\}| \geq j.$$

On the other hand, if $\delta \leq j \leq n$, $q(i) \leq j$ if and only if $q'(i) \leq j - 1,\ i \in [n-1]$. Since $q(n) = \sigma^{-1}(n) \leq \tau^{-1}(n) = \delta$, we get

$$\begin{aligned} |\{i \in [n] \mid q(i) \leq j\}| &= |\{i \in [n-1] \mid q(i) \leq j\}| + 1 \\ &= |\{i \in [n-1] \mid q'(i) \leq j - 1\}| + 1 \\ &\geq (j-1) + 1 \\ &= j \end{aligned}$$

Hence q is a major function. □

Proposition 3.2.10. *ϕ_n produces allowable pairs.*

Sketch of proof. Given $p \in A_n$, we have to show that $(s, t) = \phi_n(p)$ is an allowable pair. By the inductive hypothesis, we know that (s', t') is an allowable pair, and by $(\phi 4)$ and Lemma 3.2.3 we know that (s, t) is obtained from (s', t') by inserting n into both the $p(n)$-th position of s' and the $\iota p(n)$-th position of t'.
Let $Q(i, j)$ and $Q'(i, j)$ be the content of the queue Q when (*) is reached in the algorithm with values i and j as given, running the algorithm for (s, t) and (s', t') respectively.
The fact that (s, t) is an allowable pair can now be verified by comparing the execution of *TestPair*$((s, t),\ n)$ with the one of *TestPair*$((s', t'),\ n - 1)$. This comparison shows that

$$Q(i, j) = \begin{cases} Q'(i, j) & \text{if } i \leq p(n), \\ Q'(i-1, j) \cup \{n\} & \text{if } p(n) < i \text{ and } j \leq \iota p(n), \\ Q'(i-1, j-1) & \text{if } \iota p(n) < j, \end{cases}$$

and that everytime the test $t_j = \min Q$ is carried out in *TestPair*$((s,t),\, n)$, the test succeeds. □

Proposition 3.2.11. $\psi_n \circ \phi_n = Id_{A_n}$.

Proof. For $n = 1$ this is clearly true, so we proceed by induction for $n \geq 2$. Given $p \in A_n$, we set $(s,t) := \phi_n(p)$, $(\sigma, \tau) := (s,t)$ and $q := \psi_n(\sigma,\tau) = \psi_n \circ \phi_n(p)$. We will show that $p = q$.
As $\tau = t$, we have $\delta = d$. Furthermore, as $p(i) \neq d$ for $i < n$ and $q(n) = p(n)$, it is clear that if $p' = q'$, then $p = q$. But as $p' = \psi_{n-1}(s',t')$ by the inductive hypothesis and $q' = \psi_{n-1}(\sigma',\tau')$ by step (ψ3), it suffices to show that $(s',t') = (\sigma',\tau')$.
By definition in (ψ2), σ' and τ' are σ and τ, respectively, with n deleleted. Furthermore, we showed in Proposition 3.2.3 that t' is t with n deleted, and clearly s' is s with n deleted by step (ϕ4). Since $(s,t) = (\sigma,\tau)$, we see that $(s',t') = (\sigma',\tau')$, so $p = q$ and $\psi_n \circ \phi_n = \mathrm{Id}_{A_n}$. □

Since $|A_n| = |B_n|$ according to [3], we also see that $\phi_n \circ \psi_n = \mathrm{Id}_{B_n}$.
This finishes the proof of Proposition 3.2.2.

3.2.4 Uses of the mapping

Using the presented mapping, we can translate our result on breakpoints of parking functions to a result on breakpoints of allowable pairs:

Proposition 3.2.12. *Let $a_{n,k}$ be the number of allowable pairs $(s,t) \in B_n$ with exactly k breakpoints. Then*

$$a_{n,k} = \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} j(n-j)^{n-1}, \quad n,k \geq 1.$$

Proof. This follows directly from Proposition 2.4.2 and the fact that the functions ϕ_n and ψ_n are breakpoint preserving. □

One can also translate Gilbey's and and Kalikow's result on the number of parking functions with a given output:

Proposition 3.2.13. *Let τ be a permutation of $[n]$ and*

$$S(\tau) = \{(s,t) \in B_n \mid t = \tau\}.$$

Define $\tau(0) := n+1$ and

$$b_\tau(j) := \max\{i \in [j-1]_0 \mid \tau(i) > \tau(j)\}.$$

Then

$$|S(\tau)| = \prod_{j=1}^{n} (j - b_\tau(j)). \tag{3.1}$$

Proof. This follows directly from Proposition 2.2.4 and the fact that the functions ϕ_n and ψ_n are output preserving. $\square$

3.2.5 Non-inductive description of the bijection

Given $(\sigma, \tau) \in B_n$ and $j \in [n]$, we define

$$\begin{aligned} S(\sigma, j) &:= |\{l \in [j] \mid \sigma_l \leq \sigma_j\}|, \\ T(\tau, j) &:= |\{l \in [j-1] \mid \tau_l > \tau_j\}|. \end{aligned}$$

Proposition 3.2.14. *Let $(\sigma, \tau) \in B_n$. Setting*

$$q(i) := S(\sigma, \sigma^{-1}(i)) + T(\tau, \tau^{-1}(i)), \quad i \in [n]$$

gives $q = \psi_n(\sigma, \tau)$.

Proof. We prove this result by induction. The case $n = 1$ is trivial. For $n \geq 2$, we assume that the claim holds for $n - 1$, so we have

$$q'(i) := S(\sigma', \sigma'^{-1}(i)) + T(\tau', \tau'^{-1}(i)),$$

for $i < n$.
Now, consider the relationship between $S(\sigma, \sigma^{-1}(i))$ and $S(\sigma', (\sigma')^{-1}(i))$: We have

$$\begin{aligned} S(\sigma, \sigma^{-1}(i)) &:= \left|\left\{l \in [\sigma^{-1}(i)] \mid \sigma_l \leq i\right\}\right| \text{ and} \\ S(\sigma', (\sigma')^{-1}(i)) &:= \left|\left\{l \in [(\sigma')^{-1}(i)] \mid \sigma'_l \leq i\right\}\right|. \end{aligned}$$

Since σ' is σ with n deleted, the sequences $\sigma_1, \ldots, \sigma_{\sigma^{-1}(i)}$ and $\sigma'_1, \ldots, \sigma'_{(\sigma')^{-1}(i)}$ can only differ in that there may be an n in the first sequence but not in the latter. This shows that $S(\sigma, \sigma^{-1}(i)) = S(\sigma', (\sigma')^{-1}(i)),\ i < n$.
The same argument shows that the sequence $\tau_1, \ldots, \tau_{\tau^{-1}(i)-1}$ can only differ from $\tau'_1, \ldots, \tau'_{(\tau')^{-1}(i)-1}$ by an added n before the $\tau^{-1}(i)$-th position. This implies that

$$T(\tau, \tau^{-1}(i)) = \begin{cases} T(\tau', (\tau')^{-1}(i)), & \text{if } (\tau')^{-1}(i) < \delta, \\ T(\tau', (\tau')^{-1}(i)) + 1, & \text{if } (\tau')^{-1}(i) \geq \delta, \end{cases}$$

Furthermore, we know that $\tau'^{-1} = \iota q'$ and $\iota q'(i) < \delta$ if and only if $q'(i) < \delta$, $i \in [n-1]$, hence

$$q(i) = \begin{cases} q'(i) & \text{if } q'(i) < \delta \\ q'(i) + 1 & \text{if } q'(i) \geq \delta \end{cases}$$

for $i < n$, as required by step ($\psi 4$) of the bijection.
Finally, $S(\sigma, \sigma^{-1}(n)) = |\{l \in [\sigma^{-1}(n)] \mid \sigma_l \leq k\}| = \sigma^{-1}(n)$ and $T(\tau, \tau^{-1}(n)) = 0$, so $q(n) = \sigma^{-1}(n)$ as required by step ($\psi 1$). □

Example:
Take $n = 6$, $\sigma = (3, 1, 6, 4, 5, 2)$ and $\tau = \pi_p = (3, 1, 6, 2, 4, 5)$. Then $q = \psi_n(\sigma, \tau) = (2, 4, 1, 4, 5, 3)$.

Next, given $p \in A_n$, we can calculate $\phi_n(p) = (s, t)$ as follows: We already know that $t = \pi_p$. To find s, we use a modified parking algorithm, which Gilbey and Kalikow call 'Boston parking'. In this algorithm, each driver insists on parking in his preferred space. If it is not empty, he pushes the car currently parking there (and possibly a chain of cars parked in front of this one) one space further.

Example:
Take $n = 6$ and $p = (2, 4, 1, 4, 5, 3)$. Then $\phi_n(p) = (s, t)$ where $t = \pi_p = (3, 1, 6, 2, 4, 5)$ and $s = (3, 1, 6, 4, 5, 2)$.

Chapter 4

Generalizations of parking functions

Parking functions have been generalized in various ways [4, 5, 7, 13, 27, 28]. In this section, we will present some of these generalizations.

4.1 (p,q)-parking functions

The following generalization has been introduced by Cori and Poulalhon in [7]. Throughout this section, we will use *modified parking functions*:

Definition 4.1.1. *A sequence $u = u_1, \ldots, u_n$ of non-negative integers is a modified parking function if there exists a permutation σ on $[n]$ which is strictly larger than u (σ is then called a certificate for u).*

Obviously u is a modified parking function if and only if $p : x \mapsto u(x) + 1$ is a parking function.

4.1.1 Definition

Let p and q be two positive integers, and $n = p + q$.

Definition 4.1.2. *A (p, q)-sequence is a pair (u, v) of sequences of non-negative integers with lengths p and q, respectively, such that*

$$u_i \in [q]_0, \quad i \in [p] \quad \textit{and} \quad v_j \in [p]_0, \quad j \in [q].$$

A partial order $\preccurlyeq$ on the set of all (p, q)-sequences is defined as $(u, v) \preccurlyeq (u', v')$ if for all $i \in [p]$ and $j \in [q]$, $u_i \leq u'_i$ and $v_j \leq v'_j$.

Next, for any permutation σ on $[n]$, define a (p,q)-sequence (x_σ, y_σ) as $x_\sigma = (x_1,\dots,x_p)$ and $y_\sigma = (y_1,\dots,y_q)$ where

$$\begin{aligned} x_i &= |\{1 \le j \le q \mid \sigma_{p+j} < \sigma_i\}| \\ y_j &= |\{1 \le i \le p \mid \sigma_i < \sigma_{p+j}\}| . \end{aligned}$$

Definition 4.1.3. *A (p,q)-sequence (u,v) is a (p,q)-parking function if there exists a permutation σ on $[p+q]$ such that $(u,v) \preccurlyeq (x_\sigma, y_\sigma)$. The permutation σ is then called a certificate for (u,v).*

In a more intuitive way, (p,q)-parking functions can be defined as follows:
Consider a one-way street with n parking slots. p blue cars and q red cars have to park in this street. For all $i \in [p]$, the driver of the i-th blue car wishes to have at least u_i red cars parked before him. Likewise, for all $j \in [q]$, the driver of the j-th red car wants to have at least v_j blue cars parked before him.
The (p,q)-sequence (u,v) is a (p,q)-parking sequence if there exists a parking that satisfies all the wishes.

4.1.2 Relation to modified parking functions

For any integer sequence $u = u_1,\dots,u_p$ define the *rank* function ρ_u of u as

$$\rho_u(i) = |\{1 \le j \le p \mid u_j < u_i\}| + |\{1 \le j < i \mid u_j = u_i\}| .$$

Furthermore, define $\vec{u} = \vec{u}_1,\dots,\vec{u}_p$ by $\vec{u}_i = u_i + \rho_u(i)$. Note that the elements of $\vec{u}$ are all distinct and $\rho_{\vec{u}}(i) = |\{1 \le j \le p \mid \vec{u}_j < \vec{u}_i\}|$.
For two sequences $u = u_1,\dots,u_p$ and $v = v_1,\dots,v_q$, we write $u \cdot v$ for the *concatenation* of u and v, i. e., $u \cdot v = u_1,\dots,u_p,v_1,\dots,v_q$.

Proposition 4.1.1. *A (p,q)-sequence (u,v) is a (p,q)-parking function if and only if $\vec{u} \cdot \vec{v}$ is a modified parking function.*

Proof. Let $w = \vec{u} \cdot \vec{v}$ and $\sigma = (\sigma_1,\dots,\sigma_n)$ be a permutation on $[n]$ which satisfies the following monotonicity condition:

$$w_i < w_j \Rightarrow \sigma_i < \sigma_j, \quad i,j \le n,$$

The following arguments show that σ is a certificate for w if and only if it is a certificate for (u,v):
Consider the (p,q)-sequence (x_σ, y_σ), and let $x_\sigma = x_1,\dots,x_p$ and $y_\sigma = y_1,\dots,y_q$. Then

$$x_i = |\{1 \le j \le q \mid \sigma_{p+j} < \sigma_i\}| ,$$

and

$$\rho_u(i) = \rho_{\vec{u}}(i) = |\{1 \leq k \leq p \mid w_k < w_i\}| = |\{1 \leq k \leq p \mid \sigma_k < \sigma_i\}|,$$

for all $i \in [p]$. Since σ is a permutation, this implies that $x_i + \rho_u(i) = \sigma_i - 1$, or (since $w_i = u_i + \rho_u(i)$) equivalently:

$$x_i - u_i = \sigma_i - w_i - 1, \quad i \in [p].$$

Symmetrically,

$$y_j - v_j = \sigma_{p+j} - w_{p+j} - 1, \quad j \in [q].$$

Hence $w < \sigma$ if and only if $(u, v) \preccurlyeq (x_\sigma, y_\sigma)$.
To end the proof, note that any modified parking function and any (p, q)-parking function has a monotonous certificate, hence this condition on σ is not a restriction. □

As a corollary, we see that the set of modified parking functions on $[n]$ can be interpreted as the diagonal of the set of (n, n)-parking functions:

Proposition 4.1.2. *A sequence $u = u_1, \ldots, u_n$ is a modified parking function if and only if (u, u) is an (n, n)-parking function.*

Proof. If u is a modified parking function, then the bijection $i \mapsto \rho_u(i) + 1$ is a certificate for u. This shows that a sequence u is a modified parking function if and only if $u_i \leq \rho_u(i)$ or equivalenty $\vec{u}_i \leq 2\rho_u(i)$ for all $i \in [n]$.
Let $w = \vec{u}\cdot\vec{u}$. The sequence $\vec{u}$ consists of n distinct elements, and its rank function is $\rho_{\vec{u}} = \rho_u$. Hence the rank function ρ_w of w is given by

$$\rho_w(i) = 2\rho_u(i) \quad \text{and} \quad \rho_w(n+i) = 2\rho_u(i) + 1, \quad i \in [n].$$

According to the above argument, w is a modified parking function if and only if $w_{n+i} = w_i = \vec{u}_i \leq 2\rho_u(i)$ for all $i \in [n]$. The equivalence of $w = \vec{u} \cdot \vec{u}$ being a modified parking function and (u, u) being an (n, n)-parking function has already been proved in Proposition 4.1.1. □

4.1.3 Enumeration of (p, q)-parking functions

Let $K_{p,q,1}$ be the complete tripartite graph with the three subsets $X = \{x_1, \ldots, x_p\}$, $Y = \{y_1, \ldots, y_q\}$, $Z = \{z\}$ of vertices and the set of edges $(X \times Y) \cup (Z \times (X \cup Y))$. In [7], a relation between *recurrent configurations* of $K_{p,q,1}$ and (p, q)-parking functions has been established, which we will use to enumerate (p, q)-parking functions.

A *configuration* of $K_{p,q,1}$ is an assignment of non-negative integers to each vertex in $X \cup Y$. Hence, a configuration is a pair (u, v) of sequences of lengths p and q, respectively. We interpret the integer u_i (resp. v_j) to be the number of grains of sand lying in vertex x_i (resp. y_j).
A *toppling* of vertex $x_i \in X$ occurs if $u_i > q$. In that case we get a new configuration (u', v') where

$$\begin{aligned} u'_i &= u_i - q - 1, \\ u'_k &= u_k, \quad k \in [p] \setminus \{i\}, \\ v'_j &= v_j + 1. \end{aligned}$$

The missing grain of sand is supposed to have fallen in the sink z. For $y_j \in Y$, a toppling is defined similarly.
We say that a configuration (u, v) is *stable* if no vertex can topple, i. e., if and only if (u, v) is a (p, q)-sequence.

Definition 4.1.4. *A stable configuration (u, v) is recurrent if it can be obtained by a sequence of topplings from a configuration (u', v') where for any $i \in [p]$ and $j \in [q]$, $u'_i > q$ and $v'_j > p$.*

Now we show that the recurrent configurations of $K_{p,q,1}$ stand in one-to-one correspondence with (p, q)-parking functions:

Proposition 4.1.3. *A configuration (u,v) is recurrent if and only if the pair (u', v') defined by*

$$\begin{aligned} u'_i &= q - u_i, \quad i \in [p], \\ v'_j &= p - v_j, \quad j \in [q], \end{aligned}$$

is a (p, q)-parking function.

Proof. We use a characterization due to Dhar [8] of recurrent configurations. The criterion claims that (u, v) is a recurrent configuration of $K_{p,q,1}$ if and only if the addition of 1 to each u_i and each v_j leads to a sequence of topplings in which each vertex topples exactly once.
We label the vertices in P with $1, \ldots, p$ and the vertices in Q with $p+1, \ldots, p+q$. Let (u, v) be a recurrent configuration and add 1 to each u_i and each v_j. Now let σ be the permutation of $[n] = [p + q]$, where $\sigma(i) = j$ if vertex i is the j-th one to topple, and consider the (p, q)-sequence (x_σ, y_σ). For $i \in [p]$, exactly x_i vertices of Q topple before vertex $i \in P$ does. Hence we must have $u_i + 1 + x_i > q$ or

equivalently $x_i \geq u'_i$. The same argument shows that $y_j \geq v'_j,\ j \in [q]$. Hence σ is a certificate for (u', v').
Conversely, if (u', v') is a parking function, then any certificate σ of (u', v') will give a possible sequence of topplings when 1 is added to each u_i and each v_j. $\square$

With this result and some known facts about $K_{p,q,1}$ we can easily enumerate (p, q)-parking functions:

Proposition 4.1.4. *The number of (p, q)-parking functions is*

$$(p + q + 1)(p + 1)^{q-1}(q + 1)^{p-1}.$$

Proof. Majumdar and Dhar have shown that recurrent configurations of a graph are in one-to-one correspondence with its spanning trees [19]. In [17], Lewis showed that the number of spanning trees of the complete multipartite graph $K_{n_1,\ldots,n_k}$ is $n^{k-2} \prod_{i=1}^{k} (n - n_i)^{n_i-1}$, where $n = \sum_{i=1}^{k} n_i$. This proves the proposition. $\square$

4.1.4 Increasing parking functions

A (p, q)-sequence is increasing if $u_i \leq u_{i+1}$ and $v_j \leq v_{j+1}$ for all $i \in [p - 1]$ and $j \in [q - 1]$.

Proposition 4.1.5. *The number of increasing (p, q)-parking function is*

$$\frac{p + q + 1}{(p + 1)(q + 1)} \binom{p + q}{p} \binom{p + q}{q}.$$

Proof. Consider a circular parking lot with $p + q + 1$ slots numbered clockwise 0 to $p + q$, where the same parking rules as in the proof of Lemma 2.2.1 apply.
The mapping $(u, v) \mapsto \vec{u} \cdot \vec{v}$ realizes a one-to-one correspondence between the set of increasing (p, q)-parking functions and sequences w of length $p + q$ such that:

- $0 \leq w_i \leq p + q, \quad \text{for all } i \leq p + q$
- $w_i \leq w_{i+1}, \quad \text{for all } i < p + q, i \neq p,$
- the parking process for w leaves slot number $p + q$ empty.

For any preference function satisfying the first two conditions, one slot is left empty. By symmetry, slot $p+q$ is left empty in a fraction $1/(p+q+1)$ of the total number of these preference functions. Thus the number of increasing (p, q)-parking functions is

$$\frac{1}{p + q + 1} \binom{p + q + 1}{p} \binom{p + q + 1}{q},$$

which proves the proposition. $\square$

4.2 Bucket parking functions

Consider a parking lot where cars are parked in n numbered rows, each of which consists of k parking slots. There are $m \leq nk$ consecutive drivers who wish to park there, each of which has a preferred row in mind. Each driver proceeds to the chosen row and parks there, if it is not completely occupied. Otherwise, the driver continues to the next free slot, if any, in one of the successive rows. If no space is empty, the driver leaves.

Definition 4.2.1. *Let $p : [m] \to [n]$ be a function which associates each driver i with his preferred parking row $p(i)$. If all drivers are able to park when using the parking strategy described above, then p is called a k-bucket parking function.*

Again, the ordering of the elements of a function doesn't affect the property of being a k-bucket parking function. The proof is a simple generalization of the one given for normal parking functions.

We will now generalize the definition of ι from Section 2:
For any $f : [m] \to [n]$ and $k \geq 1$, we define $\iota_k f$ as

$$\begin{aligned} \iota_k f(1) &= f(1), \text{ and} \\ \iota_k f(i) &= \min \{ j \in \mathbb{N} \mid j \geq f(i), \ |\{t \in [i-1] \mid \iota_k f(t) = j\}| < k \} , \end{aligned}$$

for $i \in \{2, \ldots, m\}$.
Note that if p is a k-bucket parking function then $\iota_k p(i)$ is the final parking row of driver i. Obviously, $\iota p = \iota_1 p$.

We will now give different definitions of k-bucket parking functions $p : [kn] \to [n]$. The fact that they are equivalent is easy to verify.

Lemma 4.2.1. *For $p : [kn] \to [n]$, the following statements are equivalent:*

- *p is a k-bucket parking function in the sense of Definition 4.2.1.*
- *If $(q_1, \ldots, q_{kn})$ is the increasing rearrangement of $(p(1), \ldots, p(kn))$, then $q_i \leq \lceil \frac{i}{k} \rceil$ for all $i \in [kn]$.*
- *$|p^{-1}(\{n-i+1, \ldots, n\})| \leq ki$ for all $i \in [n]$.*

4.2.1 On the number of k-bucket parking functions

The following results on the number of k-bucket parking functions have been given by Blake and Konheim in their analysis of a hashing variant [4]. Let

$$P_{m,n,k} = \{p : [m] \to [n] \mid \text{p is a } k\text{-bucket parking function}\}.$$

We define $T_{k,n,s}$ to be the number of k-bucket parking functions $p : [kn + s] \to [n + 1]$ with the property that in the final parking order the rows $1, \ldots, n$ are fully occupied. More formally,

$$T_{k,n,s} := |\{p \in P_{kn+s,n+1,k} \mid |\{i \in [kn + s] \mid \iota_k p(i) = n + 1\}| = s\}|,$$

for $0 \le s < k$.
Furthermore, we let $f_k(n, m)$ denote the number of k-bucket parking functions $p : [m] \to [n]$ with the property that the last row in the final parking order is not full, i. e.,

$$f_k(n, m) := |\{p \in P_{m,n,k} \mid |\{i \in [m] \mid \iota_k p(i) = n\}| < k\}|.$$

We define the boundary values

$$f_k(n, 0) = 1, \qquad \text{for } n \ge 0,$$

and

$$f_k(n, kn) = 0, \qquad \text{for } n > 0.$$

We already know that $f_1(n, m) = g(n - 1, m) = (n - m)n^{m-1}$, for $m < n$.
Finally, let $g_k(n, m) := |P_{m,n,k}|$ denote the total number of k-bucket parking functions $p : [m] \to [n]$.
Obviously, we have $g_k(n, nk) = T_{k,n,0}$. We will now show that, given the numbers $T_{k,i,0}$ and $f_k(n, m)$, the numbers $g_k(n, m)$ can easily be derived.

Lemma 4.2.2.

$$g_k(n, m) = f_k(n, m) + \sum_{i=1}^{\lfloor \frac{m}{k} \rfloor} \binom{m}{ik} T_{k,i,0} f_k(n - i, m - ik).$$

Proof. The set of final parking orders for k-bucket parking functions $p : [m] \to [n]$ can be decomposed into the disjoint sets E_i of parking orders where

- the rows $n - i + 1, \ldots, n$ are fully occupied, and
- row $n - i$ is not full.

The number of ways of choosing the ki cars which park in the rows $n-i+1,\ldots,n$ is $\binom{m}{ik}$, and $T_{k,i,0}$ is the number of preference functions which place them in these rows. $f_k(n-i, m-ik)$ is the number of preference functions which place the remaining cars in the rows $1,\ldots,n-i$ such that row $n-i$ is not full. $\square$

We will now concentrate on the numbers $T_{k,n,s}$.

Lemma 4.2.3. *The numbers $T_{k,n,s}$ satisfy*

$$T_{k,n,s} = \begin{cases} (n+1)T_{k,n,s-1} + \sum_{i=1}^{n} \binom{kn+s-1}{ki-1} iT_{k,i-1,k-1}T_{k,n-i,s}, & \text{if } 1 \le s < k, \\ \sum_{i=1}^{n} \binom{kn-1}{ki-1} iT_{k,i-1,k-1}T_{k,n-i,0}, & \text{if } s = 0. \end{cases}$$

Proof. Suppose that the final parking row of car $kn+s$ is i, $1 \le i \le n$. Then $ki-1$ cars with smaller number completely occupy the first $i-1$ rows and $k-1$ slots of the i-th row. The rest of the cars occupies the rows $i+1,\ldots,n$ and s slots of row $n+1$. The set of cars with numbers smaller than $kn+s$ can be divided in these two groups in $\binom{kn+s-1}{ki-1}$ different ways. Furthermore, car $kn+s$ can have chosen any of the first i rows in order to finally park in row i.
For $1 \le s < k$, the case $i = n+1$ has to be considered as well. $\square$

We now introduce the generating functions

$$\psi_{k,s}(z) = \sum_{n\ge 0} \frac{T_{k,n,s}}{(kn+s)!} z^n, \quad 0 \le s < k.$$

Then we multiply the above recursion for $1 \le s < k$ with $\frac{z^n}{(kn+s-1)!}$ and sum up. On the left hand side we get

$$\begin{aligned} \sum_{n\ge 0} \frac{T_{k,n,s}}{(kn+s-1)!} z^n &= \sum_{n\ge 0} (kn+s) \frac{T_{k,n,s}}{(kn+s)!} z^n \\ &= k\sum_{n\ge 0}(n+1)\frac{T_{k,n,s}}{(kn+s)!} z^n - k\sum_{n\ge 0} \frac{T_{k,n,s}}{(kn+s)!} z^n \\ &\quad + s\sum_{n\ge 0} \frac{T_{k,n,s}}{(kn+s)!} z^n \\ &= k(z\psi_{k,s}(z))' - k\psi_{k,s}(z) + s\psi_{k,s}(z). \end{aligned}$$

On the right hand side we have

$$\sum_{n\ge 0}(n+1)\frac{T_{k,n,s-1}}{(kn+s-1)!} z^n = (z\psi_{k,s-1}(z))',$$

and

$$\begin{aligned}
&\sum_{n\geq 0}\sum_{i=1}^{n} i\frac{T_{k,i-1,k-1}}{(k(i-1)+k-1)!}\frac{T_{k,n-i,s}}{(k(n-i)+s)!}z^n = \\
&= \sum_{i\geq 1} i\frac{T_{k,i-1,k-1}}{(k(i-1)+k-1)!}z^{i-1}\sum_{n\geq 0}\frac{T_{k,n,s}}{(kn+s)!}z^{n+1} \\
&= (z\psi_{k,k-1}(z))' z\psi_{k,s}(z).
\end{aligned}$$

This leads to the system of differential equations

$$k(z\psi_{k,s}(z))' - (z\psi_{k,s-1}(z))' - z\psi_{k,s}(z)(z\psi_{k,k-1}(z))' = (k-s)\psi_{k,s}(z). \tag{4.1}$$

Equation (4.1) holds for $0 \leq s < k$ if we set $\psi_{k,-1}(z) := 0$.
We will prove that the solution to (4.1) is given by symmetric combinations of the function $\theta(z)$ as defined in Section 2, Equation (2.1). In order to do this, we first collect some results about *elementary symmetric functions*.

Definition 4.2.2. *The r-th elementary symmetric function of the variables $A_k := \{\alpha_j \mid 1 \leq j \leq k\}$ is*

$$\sigma_r(A_k) := [x^{k-r}]\prod_{j=1}^{k}(x+\alpha_j).$$

As an example, we have $\sigma_1(A_k) = \sum_{i=1}^{k}\alpha_k$, and $\sigma_k(A_k) = \prod_{i=1}^{k}\alpha_k$.

Lemma 4.2.4. *Let $\sigma_r := \sigma_r(A_k)$, and $\sigma_{r,i} := \sigma_{r-1}(A_k \setminus \{\alpha_i\})$. Then*

$$r\sigma_r = \sum_{i=1}^{k}\alpha_i\sigma_{r,i}.$$

Proof. We certainly have

$$(k-r)\sigma_r = [x^{k-r}]x\frac{\partial}{\partial x}\prod_{j=1}^{k}(x+\alpha_j) = [x^{k-r}]x\sum_{i=1}^{k}\prod_{\substack{1\leq j\leq k\\ j\neq i}}(x+\alpha_j),$$

hence

$$\begin{aligned}
r\sigma_r &= k\sigma_r - (k-r)\sigma_r \\
&= [x^{k-r}]k\prod_{j=1}^{k}(x+\alpha_j) - [x^{k-r}]x\sum_{i=1}^{k}\prod_{\substack{1\le j\le k\\ j\ne i}}(x+\alpha_j) \\
&= \sum_{i=1}^{k}[x^{k-r}]\alpha_i\prod_{\substack{1\le j\le k\\ j\ne i}}(x+\alpha_j) \\
&= \sum_{i=1}^{k}\alpha_i\sigma_{r,i}.
\end{aligned}$$

□

Lemma 4.2.5. *With σ_r and $\sigma_{r,i}$ defined as above, it holds that*

$$\sigma_r = \sigma_{r+1,i} + \alpha_i\sigma_{r,i},$$

for $1 \le i \le k$.

Proof.

$$\begin{aligned}
\sigma_r &= [x^{k-r}]\prod_{j=1}^{k}(x+\alpha_j) \\
&= [x^{k-r-1}]\prod_{\substack{1\le j\le k\\ j\ne i}}(x+\alpha_j) + [x^{k-r}]\alpha_i\prod_{\substack{1\le j\le k\\ j\ne i}}(x+\alpha_j) \\
&= \sigma_{r+1,i} + \alpha_i\sigma_{r,i}.
\end{aligned}$$

□

Lemma 4.2.6. *Let A_k be a set of functions of a variable z,*

$$A_k = \{\alpha_j(z) \mid 1 \le j \le k\},$$

and write $\sigma_r(z) := \sigma_r(A_k)$ and $\sigma_{r,i}(z) := \sigma_{r,i}(A_k)$. It holds that

$$\sigma_r'(z) = \sum_{i=1}^{k}\sigma_{r,i}(z)\alpha_i'(z).$$

Proof.

$$\begin{aligned}
\sigma_r'(z) &= [x^{k-r}]\frac{\partial}{\partial z}\prod_{j=1}^{k}(x+\alpha_j(z)) \\
&= [x^{k-r}]\sum_{i=1}^{k}\alpha_i'(z)\prod_{\substack{1\le j\le k\\ j\ne i}}(x+\alpha_j(z)) \\
&= \sum_{i=1}^{k}\alpha_i'(z)\sigma_{r,i}(z).
\end{aligned}$$

□

Throughout the rest of this section, let ω denote a primitive k-th root of unity. We let $\sigma_r(z)$ and $\sigma_{r,i}(z)$ be defined as above, with $A_k = \{\omega^j z\theta(\omega^j z) \mid 1 \le j \le k\}$.

Lemma 4.2.7.

$$(kz)^k\psi_{k,s}((kz)^k) = (-1)^{k-s-1}k^{k-s}\sigma_{k-s}(z). \tag{4.2}$$

Proof. By substituting $(kz)^k$ for z in (4.1), we get

$$\begin{aligned}
(k-s)\psi_{k,s}((kz)^k) &= k\rho(z)[(kz)^k\psi_{k,s}((kz)^k)]' \\
&\quad -\rho(z)[(kz)^k\psi_{k,s-1}((kz)^k)]' \\
&\quad -\rho(z)(kz)^k\psi_{k,s}((kz)^k)[(kz)^k\psi_{k,k-1}((kz)^k)]',
\end{aligned}$$

with $\rho(z) = k^{-(k+1)}z^{-(k-1)}$. We will now verify that $\psi_{k,s}(z)$ defined by (4.2) satisfies this equation, i. e., it holds that

$$\begin{aligned}
(k-s)k^{-s}z^{-k}\sigma_{k-s}(z) = {} & k^{-s}z^{1-k}\sigma_{k-s}'(z) \\
& + k^{-s}z^{1-k}\sigma_{k-s+1}'(z) \\
& - k^{-s}z^{1-k}\sigma_{k-s}(z)\sigma_1'(z).
\end{aligned} \tag{4.3}$$

Since $\theta(z)$ satisfies the equation $\theta(z) = \mathrm{e}^{z\theta(z)}$, we have $\theta(\omega^i z) = \mathrm{e}^{\omega^i z\theta(\omega^i z)}$ and hence $(\omega^i z\theta(\omega^i z))' = (\log\theta(\omega^i z))'$. Using Lemma 4.2.6, this allows us to write

$$\sigma_r'(z) = \sum_{i=1}^{k}\sigma_{r,i}(z)\frac{\theta'(\omega^i z)}{\theta(\omega^i z)}.$$

Hence, the right-hand side of (4.3) reduces to

$$k^{-s}z^{1-k}\sum_{i=1}^{k}(\sigma_{k-s,i}(z)+\sigma_{k-s+1,i}(z)-\sigma_{k-s}(z))\frac{\theta'(\omega^i z)}{\theta(\omega^i z)}.$$

Using Lemma 4.2.5, this further reduces to

$$\begin{aligned}
k^{-s}z^{1-k}\sum_{i=1}^{k}(\sigma_{k-s,i}(z)-\omega^i z\theta(\omega^i z)\sigma_{k-s,i}(z))\frac{\theta'(\omega^i z)}{\theta(\omega^i z)} = \\
&= k^{-s}z^{1-k}\sum_{i=1}^{k}\sigma_{k-s,i}(z)(1-\omega^i z\theta(\omega^i z))\frac{\theta'(\omega^i z)}{\theta(\omega^i z)} \\
&= k^{-s}z^{1-k}\sum_{i=1}^{k}\sigma_{k-s,i}(z)((\omega^i z\theta(\omega^i z))'-\omega^i z\theta'(\omega^i z)) \\
&= k^{-s}z^{1-k}\sum_{i=1}^{k}\sigma_{k-s,i}(z)\omega^i\theta(\omega^i z).
\end{aligned}$$

This can easily be identified with the left-hand side of (4.3) using Lemma 4.2.4. □

Using this result, one can at least theoretically compute $T_{k,n,s}$ for special values of k, n and s. For the special case $s = k-1$, i. e., the number of k-bucket parking functions $p : [k(n+1)-1] \to [n+1]$ which leave one slot of the last row empty, one can derive a surprisingly simple formula:

Proposition 4.2.8. $T_{k,n,k-1} = (n+1)^{kn+k-2}$

Proof. According to (4.2), we have

$$\begin{aligned}
(kz)^k\psi_{k,k-1}((kz)^k) &= k\sigma_1(k) \\
&= k\sum_{i=1}^{k}\sum_{n\geq 0}(n+1)^{n-1}\frac{(z\omega^i)^{n+1}}{n!} \\
&= k\sum_{n\geq 0}(n+1)^{n-1}\frac{z^{n+1}}{n!}\sum_{i=1}^{k}\omega^{i(n+1)} \\
&= k^2\sum_{\substack{n\geq 0 \\ n+1=0 \mod k}}(n+1)^{n-1}\frac{z^{n+1}}{n!} \\
&= k^2\sum_{n\geq 1}(kn)^{kn-2}\frac{z^{kn}}{(kn-1)!} \\
&= k^2\sum_{n\geq 0}(k(n+1))^{k(n+1)-2}\frac{z^{k(n+1)}}{(k(n+1)-1)!} \\
&= (kz)^k\sum_{n\geq 0}(n+1)^{kn+k-2}\frac{(kz)^{kn}}{(kn+k-1)!}
\end{aligned}$$

□

This result can also be proved using an adaptation of Pollack's proof for normal parking functions:
Let the $n+1$ parking rows be arranged in a circle, so that each driver finds a free parking slot and one slot is left empty. The number of preference functions for this scenario is $(n+1)^{kn+k-1}$. If a particular preference function leaves one of the slots in the last row empty, then it is a k-bucket parking function $p : [k(n+1)-1] \to [n+1]$. Due to symmetry, this happens in a fraction $\frac{1}{n+1}$ of the total number of preference functions.

We will now find a recurrence formula for the numbers $f_k(n,m)$.

Lemma 4.2.9. *The numbers $f_k(n,m)$ satisfy*

$$f_k(n,m) = \sum_{s=0}^{k-1} \sum_{\substack{0 \le j < n \\ jk \le m-s}} \binom{m}{jk+s} T_{k,j,s} f_k(n-1-j, m-jk-s).$$

Proof. We can decompose the set of final parking orders which do not fill the last row into the disjoint sets $E_{j,s}$ of parking orders where

- s parking slots of row n are occupied,
- rows $n-j, \ldots, n-1$ are fully occupied, and
- row $n-j-1$ is not fully occupied.

The number of ways of choosing the $jk+s$ cars which park in the rows $n-j, \ldots, n$ is $\binom{m}{jk+s}$, and $T_{k,j,s}$ is the number of preference functions which place them in these rows. $f_k(n-1-j, m-jk-s)$ is the number of preference functions which place the remaining cars in the rows $1, \ldots, n-j-1$ such that row $n-j-1$ is not full. □

Using this recurrence formula, we will now derive a generating function for the numbers $f_k(n,m)$. We define

$$\lambda_n(z) = \sum_{m=0}^{kn-1} f_k(n,m) \frac{z^m}{m!},$$

and

$$\Lambda(z,w) = \sum_{n \ge 1} \lambda_n(z) w^{kn}.$$

Proposition 4.2.10.

$$\Lambda(z,w) = \frac{w^k \sum_{s=0}^{k-1} z^s \psi_{k,s}((zw)^k)}{1 - w^k \sum_{s=0}^{k-1} z^s \psi_{k,s}((zw)^k)}.$$

Proof. We have

$$\Lambda(z,w) = \sum_{s=0}^{k-1} \sum_{n\geq 1} w^{kn} \sum_{m=0}^{kn-1} \frac{z^m}{m!} \sum_{\substack{0\leq j<n \\ jk\leq m-s}} \binom{m}{jk+s} T_{k,j,s} f_k(n-1-j, m-jk-s).$$

We know that

$$f_k(n-1-j, m-jk-s) = 0, \text{ for } m > k(n-1)+s,$$

and, for $m = k(n-1)+s$, it holds that

$$f_k(n-1-j, m-jk-s) = 1 \text{ if and only if } j = n-1.$$

Hence, we can write

$$\begin{aligned}
\Lambda(z,w) &= \sum_{s=0}^{k-1}\sum_{n\geq 1} w^{kn} \sum_{m=0}^{k(n-1)+s} z^m \sum_{\substack{0\leq j<n\\ jk\leq m-s}} \frac{T_{k,j,s} f_k(n-1-j, m-jk-s)}{(kj+s)!(m-kj-s)!} \\
&= \sum_{s=0}^{k-1}\sum_{n\geq 1} w^{kn} z^{k(n-1)+s} \frac{T_{k,n-1,s}}{(k(n-1)+s)!} + \sum_{s=0}^{k-1}\sum_{n\geq 1} w^{kn} \\
&\quad \cdot \sum_{m=0}^{k(n-1)+s-1} z^m \sum_{\substack{0\leq j<n\\ jk\leq m-s}} \frac{T_{k,j,s} f_k(n-1-j, m-jk-s)}{(kj+s)!(m-kj-s)!} \\
&= \sum_{s=0}^{k-1} w^k z^s \psi_{k,s}((zw)^k) + \sum_{s=0}^{k-1}\sum_{n\geq 1} w^{kn} \sum_{j=0}^{n-1} \frac{T_{k,j,s}}{(kj+s)!} z^{kj+s} \\
&\quad \cdot \sum_{m=kj+s}^{k(n-1)+s-1} z^{m-(kj+s)} \frac{f_k(n-j-1, m-kj-s)}{(m-kj-s)!} \\
&= \sum_{s=0}^{k-1} w^k z^s \psi_{k,s}((zw)^k) + \sum_{s=0}^{k-1}\sum_{n\geq 1} w^{kn} \sum_{j=0}^{n-1} \frac{T_{k,j,s}}{(kj+s)!} z^{kj+s} \lambda_{n-j-1}(z) \\
&= \sum_{s=0}^{k-1} w^k z^s \psi_{k,s}((zw)^k) + \sum_{s=0}^{k-1}\sum_{j\geq 0} \frac{T_{k,j,s}}{(kj+s)!} z^{kj+s} \sum_{n\geq j+1} w^{kn} \lambda_{n-j-1}(z) \\
&= \sum_{s=0}^{k-1} w^k z^s \psi_{k,s}((zw)^k) + \sum_{s=0}^{k-1}\sum_{j\geq 0} \frac{T_{k,j,s}}{(kj+s)!} z^{kj+s} w^{k(j+1)} \Lambda(z,w) \\
&= \left(\sum_{s=0}^{k-1} w^k z^s \psi_{k,s}((zw)^k)\right)(1+\Lambda(z,w)).
\end{aligned}$$

□

Finally, we can express $\Lambda(z,w)$ in terms of $\theta(z)$:

Proposition 4.2.11.

$$\Lambda(z,w) = \frac{1-\prod_{i=1}^{k}(1-\omega^i w\theta(\frac{\omega^i wz}{k}))}{\prod_{i=1}^{k}(1-\omega^i w\theta(\frac{\omega^i wz}{k}))}.$$

Proof. According to Lemma 4.2.7, we have

$$\begin{aligned} z^k - \prod_{i=1}^{k}(z - k\omega^i v\theta(\omega^i v)) &= \sum_{s=0}^{k-1}(-1)^{k-s-1}k^{k-s}\sigma_{k-s}(v)z^s \\ &= \sum_{s=0}^{k-1}(kv)^k\psi_{k,s}((kv)^k)z^s. \end{aligned}$$

Using the substitution $w = \frac{kv}{z}$, this shows that

$$\sum_{s=0}^{k-1} z^s w^k \psi_{k,s}((zw)^k) = 1 - \prod_{i=1}^{k}(1 - \omega^i w\theta(\frac{\omega^i wz}{k})).$$

□

This finishes this section. We will later consider *defective bucket parking functions* (see Section 5.4), which enables us to find a generating function for the total number $g_k(n, m)$ of k-bucket parking functions $p : [m] \to [n]$.

4.3 x-parking functions

Definition 4.3.1. *Let $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{N}^n$. A sequence $p = (p_1, \dots, p_n)$ of positive integers is called an $\boldsymbol{x}$-parking function if the increasing rearrangement $(q_1, \dots, q_n)$ of p satisfies $q_i \le x_1 + \dots + x_i$.*

By this generalization, many different parking scenarios can be described.

- If $x_i >= 1$ for all $i \in [n]$, the scenario can be described as follows: Consider a street with $\sum_{i=1}^{n} x_i$ parking slots. n drivers want to park there, and for each $k \in [n]$, driver k has parking slot $p(k)$ in mind. But when the drivers arrive, the parking slots i with $i < x_1$ or $x_1 + \dots + x_k < i < x_1 + \dots + x_{k+1}$, $i \in [n-1]$ are already occupied. Again, each driver parks at his preferred space if it is empty, and takes the next empty space (if any) otherwise. If all drivers succeed to park, then the preference function $p : [n] \to [\sum_{i=1}^{n} x_i]$ is an $\mathbf{x}$-parking function for $\mathbf{x} = (x_1, \dots, x_n)$.
- Bucket parking functions $p : [kn] \to [n]$ are $\mathbf{x}$-parking functions for $\mathbf{x} = (x_1, \dots, x_{kn})$ with
$$x_i = \begin{cases} 1, & \text{if } i = 1 \mod k, \\ 0, & \text{else.} \end{cases}$$

- Finally, ordinary parking functions on $[n]$ correspond to the special case $\mathbf{x} = (1, \ldots, 1)$.

Given $\mathbf{x} = (x_1, \ldots, x_n)$, we will denote the number of $\mathbf{x}$-parking functions by $g_n(\mathbf{x})$. In [21], Pitman and Stanley have found a relation between $g_n(\mathbf{x})$ and the volume of some n-dimensional polytope. The following detail of their work will be useful later:

Lemma 4.3.1. *Let $\boldsymbol{x} = (x_1, \ldots, x_n)$. $g_n(\boldsymbol{x})$ is a polynomial in the variables $x_1, \ldots, x_n$:*

$$g_n(\boldsymbol{x}) = \sum_{(p_1,\ldots,p_n)\in P_n} x_{p_1} \cdots x_{p_n},$$

where P_n denotes the set of ordinary parking functions on $[n]$.

Proof. Given $(p_1, \ldots, p_n) \in P_n$, replace each i by an element of the set $\{\sum_{j=1}^{i-1} x_j + 1, \ldots, \sum_{j=1}^{i} x_j\}$. The number of ways to do this is given by the product $x_{p_1} \cdots x_{p_n}$, and every $\mathbf{x}$-parking function is obtained exactly once in this way. $\square$

4.3.1 Generalization of Section 3.1.3

In this section, some results on the special case $\mathbf{x} = (a, b, b, \ldots, b)$ will be presented.

Proposition 4.3.2. *For $\boldsymbol{x} = (a, \underbrace{b, \ldots, b}_{n-1})$, $g_n(\boldsymbol{x}) = a(a + nb)^{n-1}$.*

This has been proved by Yan in [27] by generalizing the mapping between ordinary parking functions and rooted labeled forests (see Section 3.1.3) in the following way:
Let A_n be the set of all $\mathbf{x}$-parking functions for $\mathbf{x} = (a, \underbrace{b, \ldots, b}_{n-1})$ and B_n the set of all sequences $(S_1, \ldots, S_a)$ of length a such that

- each S_i is a rooted b-forest, i. e., each S_i is a rooted forest in which each edge is colored with one of the colors $0, \ldots, b - 1$,
- $S_i \cap S_j = \emptyset$ for $i \neq j$,
- the union of the vertex sets of $S_1, \ldots, S_a$ is $[n]$.

Like in Section 3.1.3, a third set C_n is introduced, which will be put in one-to-one correspondence with both A_n and B_n.

Let $r = (r_1, \ldots, r_{a+(n-1)b})$ be a sequence of non-negative integers and set $R_j = r_1 + \ldots + r_{a+(j-1)b} - j$. The sequence r is called balanced if

$$\begin{aligned} R_j &\geq 0, \quad i \in [n-1], \\ R_n &= 0. \end{aligned} \tag{4.1}$$

A permutation π of $[n]$ is called compatible with r if π^{-1} is increasing on each interval of the form $\{1 + \sum_{i=1}^{k} r_i, \ldots, \sum_{i=1}^{k+1} r_i\}$ (if $r_{k+1} \neq 0$).
Define C_n to be the set of all couples (r, π) with $r \in \mathbb{N}^{a+(n-1)b}$ balanced and π a permutation of $[n]$ which is compatible with r.

A_n is mapped to C_n in the same way as in Section 3.1.3: For each $a = (a_1, \ldots, a_n) \in A_n$ the associated element of C_n is (r_a, π_a), where r_a is the specification of a, and π_a is defined by

$$\pi_a(i) = |\{j \in [n] \mid a_j < a_i, \quad \text{or} \quad a_j = a_i \text{ and } j \leq i\}|.$$

The proof for the fact that the mapping $a \mapsto (r_a, \pi_a)$ is a bijection doesn't significantly differ from the proof given for Proposition 3.1.5.
Now let us map B_n to C_n: For an element $S = (S_1, \ldots, S_a) \in B_n$, the associated element of C_n will be denoted by $(r(S), \sigma_S)$. The definition of σ_S is just a generalization of the one given in Section 3.1.3. First, a linear order $<_S$ on $[n]$ is defined as follows: Let $x <_S y$ if

- $h(x) < h(y)$, or if
- $h(x) = h(y) = 0$, $x, y \in S_i$ and $x < y$, or if
- $h(x) = h(y) = 0$ and $x \in S_i$, $y \in S_j$ with $i < j$.

Then assume that the order $<_S$ has been defined on the set of all vertices of height $\leq k$. By induction, for two vertices x, y with $h(x) = h(y) = k + 1$, let $x <_S y$ if

- $pre(x) <_S pre(y)$, or if
- $pre(x) = pre(y)$ and color of $edge(x) <$ color of $edge(y)$, or if
- $pre(x) = pre(y)$ and color of $edge(x) =$ color of $edge(y)$ and $x < y$,

where $pre(x)$ denotes the predecessor of x and $edge(x)$ is the edge $(x, pre(x))$.
Let $\sigma_S^{-1} = (\sigma_S^{-1}(1), \ldots, \sigma_S^{-1}(n))$ be the sequence of $\{1, \ldots, n\}$ written in increasing order with respect to $<_S$, and let σ_S be the inverse of σ_S^{-1}.
Now, define the forest specification of S as $r(S) = (r_1, \ldots, r_{a+(n-1)b})$ where

- r_i is the number of roots in S_i, for $i \in [a]$,
- r_{a+k} is the number of children of $\sigma_S^{-1}(1)$ with edge color $k-1$, for $k \in [b]$,
- in general, $r_{a+(i-1)b+k}$ is the number of children of $\sigma_S^{-1}(i)$ with edge color $k-1$, for $k \in [b]$ and $i \in [n-1]$.

Again, the proof for the fact that the mapping $S \mapsto (r(S), \sigma_S)$ is a bijection is a simple generalization of the proof given in Section 3.1.3 (Proposition 3.1.3).

Proof of Proposition 4.3.2. The fact that $|B_n| = a(a+nb)^{n-1}$ [24], and both A_n and B_n are in one-to-one correspondence with C_n, shows that for $\mathbf{x} = (a, \underbrace{b, \ldots, b}_{n-1})$, $g_n(\mathbf{x}) = |A_n| = |B_n| = a(a+nb)^{n-1}$. □

4.4 Valet functions

In [13], the following generalization of parking functions has been presented:
Let $a = (a_1, \ldots, a_k)$ be a sequence of positive integers, $n := \sum_{i=1}^k a_i$, and p a function $p : [k] \to \mathcal{P}([n])$ with $|p(i)| = a_i$.
As before, we have a one-way street with n parking slots and n cars, but in this case there are k types of cars and k valets. Each valet i is responsible for one type of cars and tries to park all a_i cars of this type, having an appropriately sized preferred subset $p(i)$ of parking slots. If all of the cars are able to be parked, then p is called a valet function on a. We let A_a denote the set of valet functions on a. Obviously, parking functions are valet functions on $a = (1, \ldots, 1)$.

Proposition 4.4.1. *The number of valet functions on a is*

$$\frac{1}{n+1} \prod_{i=1}^{k} \binom{n+1}{a_i}.$$

Proof. This can be proved by extending Pollak's proof for the number of parking functions on $[n]$ as follows:
Consider a circular car park with $n+1$ parking slots and allow each valet to choose a subset of size a_i of these slots. The number of possible choices is given by $\prod_{i=1}^k \binom{n+1}{a_i}$. Since the car park is circular, each driver will be able to park, and in the end there will be one empty slot. The choice of subsets will be a valet function for the original problem if slot $n+1$ is empty, which (by symmetry) happens in a fraction $1/(n+1)$ of the total number of possible choices. □

The concept of major functions can be generalized correspondingly: For $k \geq 1$, $a = (a_1, \dots, a_k)$ and $n = \sum_{i=1}^{k} a_i$, we say that $f : [k] \to \mathcal{P}([n])$ with $|f(i)| = a_i,\ i \in [k]$ is a major function if and only if

$$\sum_{i=1}^{k} |f(i) \cap [m]| \geq m, \quad m \in [n].$$

Lemma 4.4.2. *f is a valet function if and only if f is a major function.*

Proof. This can be proved by numbering the $n = \sum_{i=1}^{k} a_i$ cars from 1 to n and using the corresponding result for parking functions. □

If f is a valet function, then the final parking order will give a multiset permutation of $M_a := \{1^{a_1}, \dots, k^{a_k}\}$ which we call the output π_f of f.
The definition of breakpoints can also be generalized: We say that b is a breakpoint of f, if $\sum_{i=1}^{k} |f(i) \cap [b]| = b$.

4.4.1 Generalization of Section 3.2.3

In [13], the bijection between parking functions and allowable pairs has been given in a more general way, using valet functions.
First, one can extend the concept of allowable pairs as follows: We say that (σ, τ) is an allowable pair on the multiset M_a, if σ and τ are multiset permutations of M_a and (σ, τ) is an allowable pair. We let B_a denote the set of these pairs. Furthermore, we say that b is a breakpoint of $(\sigma, \tau) \in B_a$ if and only if $\{\sigma_1, \dots, \sigma_b\} = \{\tau_1, \dots, \tau_b\}$ as multisets.
Now we define two mappings $\phi_a : A_a \to B_a$ and $\psi_a : B_a \to A_a$.
For $k = 1$ the mappings are trivial, since $|A_a| = |B_a| = 1$. For $k \geq 2$ we proceed by induction on k. For this purpose, let $a' = (a_1, \dots, a_{k-1})$.
Given $p \in A_a$, define $(s, t) = \phi_a(p)$ as follows:

(ϕ1) Set $t := \pi_p$ and $D := t^{-1}(k)$. Let $d_0 = 0$, $d_{a_k+1} = n+1$ and $D = \{d_1, \dots, d_{a_k}\}$ where $d_1 < \dots < d_{a_k}$.

(ϕ2) Define $p' \in A_{a'}$ by

$$p'(i) := \bigcup_{w=0}^{a_k} \{l - w \mid l \in p(i) \text{ and } d_w < l < d_{w+1}\}, \quad i \in [k-1].$$

(ϕ3) Set $(s', t') = \phi_{a'}(p')$.

($\phi 4$) Define s by inserting a_k terms labeled k into s' such that $s(j) = k$ if and only if $j \in p(k)$.

Given $(\sigma, \tau) \in B_a$, define $q = \psi_a(\sigma, \tau)$ as follows:

($\psi 1$) Set $q(k) := \sigma^{-1}(k)$ and $\Delta := \tau^{-1}(k)$. Let $\delta_0 = 0$, $\delta_{a_k+1} = n + 1$ and $\Delta = \{\delta_1, \dots, \delta_{a_k}\}$ where $\delta_1 < \cdots < \delta_{a_k}$.

($\psi 2$) Let σ' and τ' be, respectively, σ and τ with all k-s deleted, so $(\sigma', \tau') \in B_{a'}$.

($\psi 3$) Set $q' = \psi_{a'}(\sigma', \tau')$.

($\psi 4$) Set

$$q(i) := \bigcup_{w=0}^{a_k} \{\lambda + w \mid \lambda \in q'(i) \text{ and } \delta_w < \lambda + w < \delta_{w+1}\}, \quad i \in [k-1].$$

Proposition 4.4.3. *The functions ϕ_a and ψ_a are well-defined, output and breakpoint preserving, mutually inverse bijections between A_a and B_a.*

The proof, in detail given in [13], follows the same steps presented in Section 3.2.3. These bijections can be described non-inductively as well:
Let $(\sigma, \tau) \in B_a$. For each $i \in [k]$, let $\hat{\sigma}_i(1), \dots, \hat{\sigma}_i(a_i)$ and $\hat{\tau}_i(1), \dots, \hat{\tau}_i(a_i)$ be the elements of $\sigma^{-1}(i)$ and $\tau^{-1}(i)$, respectively, in increasing order. Setting

$$q(i) := \{S(\sigma, \hat{\sigma}_i(j)) + T(\tau, \hat{\tau}_i(j)) \mid j \in [a_i]\}$$

gives $q = \psi_a(\sigma, \tau)$.
The proof does not significantly differ from the one given for Proposition 3.2.14.
The bijection ϕ_a can be described in the same way as ϕ_n in section 3.2.3, by using the 'Boston parking' algorithm.

Chapter 5

Defective parking functions

As another generalization of parking functions, we will now study preference functions with the property that exactly d drivers do not succeed to park.

Definition 5.0.1. *Let $p : [m] \to [n]$. We say that p is a defective parking function of defect d if and only if*

$$|\{i \in [m] \mid \iota p(i) > n\}| = d.$$

It is easy to verify that $p : [m] \to [n]$ is a defective parking function of defect d if and only if

- for all $i \in [n]$ holds $|f^{-1}(\{n-i+1, \ldots, m\})| \leq d+i$, and
- there exists an $i \in [n]$ such that $|f^{-1}(\{n-i+1, \ldots, m\})| = d+i$.

We let $\mathrm{g}(n, m, d)$ denote the number of defective parking functions of defect d. This is a generalization of the numbers $g(n, m)$: Clearly, ordinary parking functions are of defect 0, hence we have $\mathrm{g}(n, m, 0) = g(n, m) = (n-m+1)(n+1)^{m-1}$.

5.1 Asymptotic results using a Poisson model

In their analysis of the sorting algorithm *linear probing sort*, Gonnet and Munro [14] have found asymptotic results on the expected number of unsuccessful drivers for preference functions $p : [m] \to [n]$. They use a *Poisson filling model*, and then transform their results to our model, which we will refer to as the "exact" model.

5.1.1 The Poisson filling model

As before, we consider a street with n numbered parking slots, and the arriving drivers follow the same rules as in the exact model. But for each $i \in [n]$, we let the number of drivers who wish to park in slot i be Poisson distributed with parameter α, and independent of the number of drivers who wish to park elsewhere. Hence, the total number of drivers itself is a random variable (whose expected value is $n\alpha$). The independency of the parking slots makes this model relatively easy to analyze.

For $i \in \mathbb{N}_0$, we let r_i denote the probability that exactly i cars wish to park in a particular slot. Using a Poisson filling model with parameter α, we have

$$r_i = \mathrm{e}^{-\alpha}\frac{\alpha^i}{i!}.$$

We further let $p_{i,j}$ denote the probability that j cars *overflow* from the first i parking slots, i. e., exactly j of the cars who wish to park in one of the first i parking slots do not succeed to do so. We define the boundary values $p_{0,j} = \delta_{0,j}$, for $j \in \mathbb{N}_0$.

Proposition 5.1.1. *The probabilities $p_{i,j}$ satisfy*

$$p_{i,0} = p_{i-1,0}(r_0 + r_1) + p_{i-1,1}r_0, \quad \textit{for } i \geq 1.$$

and

$$p_{i,j} = \sum_{k=0}^{j+1} p_{i-1,k}r_{j-k+1}, \quad \textit{for } i, j \geq 1.$$

Proof. No cars overflow from the first i parking slots if and only if

- no cars overflow from the first $i-1$ parking slots and at most one driver wishes to park in slot i, or if
- one car overflows from the first $i-1$ parking slots and no driver wishes to park in slot i.

For $j \geq 1$, j cars overflow from the first i parking slots if and only if there exists a $k \in [j+1]_0$ such that

- k cars overflow from the first $i-1$ parking slots, and
- $j-k+1$ drivers wish to park in slot i.

$\square$

We now introduce the generating functions

$$P_i(z) = \sum_{j\geq 0} p_{i,j} z^j.$$

Let W_n denote the random variable which counts the unsuccessful cars for a street with n parking slots. Then W_n is described by $P_n(z)$, and we have $\mathbb{E}(W_n) = P_n'(1)$. Gonnet and Munro have shown that the limit $\lim_{n\to\infty} \mathbb{E}(W_n)$ exists and have found its value. We will only sketch their work.

First, one can translate the recurrence for $p_{i,j}$ into a recurrence for $P_i(z)$:

Proposition 5.1.2. *It holds that*

$$P_0(z) = 1,$$

and, for $i \geq 1$,

$$P_i(z) = \frac{P_{i-1}(z)R(z) + (z-1)p_{i-1,0}r_0}{z}, \tag{5.1}$$

with

$$R(z) := \sum_{j\geq 0} r_j z^j = \sum_{j\geq 0} \mathrm{e}^{-\alpha}\frac{\alpha^j}{j!} z^j = \mathrm{e}^{\alpha(z-1)}.$$

Proof. Since $p_{0,j} = \delta_{0,j}$, we have $P_0(z) = 1$.
For $i \geq 1$, we multiply the above recursions with z^j and sum up for $j \geq 0$. This leads to

$$\begin{aligned}
P_i(z) &= p_{i-1,0}(r_0 + r_1) + p_{i-1,1}r_0 + \sum_{j\geq 1}\sum_{k=0}^{j+1} p_{i-1,k}r_{j-k+1}z^j \\
&= p_{i-1,0}r_0 + \sum_{j\geq 0}\sum_{k=0}^{j+1} p_{i-1,k}r_{j-k+1}z^j \\
&= p_{i-1,0}r_0 + p_{i-1,0}\sum_{j\geq 0} r_{j+1}z^j + \sum_{j\geq 0}\sum_{k=1}^{j+1} p_{i-1,k}r_{j-k+1}z^j \\
&= p_{i-1,0}r_0 + \frac{1}{z}p_{i-1,0}\sum_{j\geq 0} r_{j+1}z^{j+1} + \frac{1}{z}\sum_{k\geq 1} p_{i-1,k}z^k \sum_{j\geq k-1} r_{j-k+1}z^{j-k+1} \\
&= (1-\frac{1}{z})p_{i-1,0}r_0 + \frac{1}{z}p_{i-1,0}\sum_{j\geq 0} r_j z^j + \frac{1}{z}\sum_{k\geq 1} p_{i-1,k}z^k \sum_{j\geq 0} r_j z^j \\
&= (1-\frac{1}{z})p_{i-1,0}r_0 + \frac{1}{z}\sum_{k\geq 0} p_{i-1,k}z^k \sum_{j\geq 0} r_j z^j,
\end{aligned}$$

which proves the proposition. □

If the sequence $(P_i(z))_{i\geq 0}$ converges, then equation (5.1) shows that we must have

$$\lim_{i\to\infty} P_i(z) = \frac{(1-z)\lim_{i\to\infty} p_{i,0} r_0}{R(z)-z}.$$

In order to prove that the limit really exists, one can first derive a formula for $P_i(z)$ in terms of $p_{j,0}$ with $j<i$. This can easily be achieved by solving the linear recurrence relation (5.1).

Lemma 5.1.3.

$$P_i(z) = \gamma(z)^i + \beta(z)\sum_{j=0}^{i-1} p_{i-j-1,0}\gamma(z)^j, \tag{5.2}$$

where $\gamma(z) = \frac{R(z)}{z}$ *and* $\beta(z) = \frac{\mathrm{e}^{-\alpha}(z-1)}{z}$.

By extracting the coefficient of z^0 in this formula, one can derive a recursion for $p_{i,0}$:

Lemma 5.1.4.

$$p_{i,0} = \mathrm{e}^{-i\alpha}\frac{(i\alpha)^i}{i!} + \mathrm{e}^{-\alpha}\sum_{j=0}^{i-1} p_{i-j-1,0}\mathrm{e}^{-j\alpha}\frac{(j\alpha)^j}{j!}\left(1-\frac{j\alpha}{j+1}\right).$$

One can now express $p_{i,0}$ in terms of truncated power series expansions of e^x,

$$e_i(x) := \sum_{j=0}^{i}\frac{x^j}{j!}.$$

Lemma 5.1.5.

$$p_{i,0} = \mathrm{e}^{-i\alpha}(e_{i+1}((i+1)\alpha) - \alpha e_i((i+1)\alpha)). \tag{5.3}$$

Using this representation, one can show that the $p_{i,0}$ converge.

Lemma 5.1.6. *For* $\alpha < 1$ *it holds that*

$$p_{i,0} = (1-\alpha)\mathrm{e}^{\alpha} + \mathcal{O}(i^{-\frac{1}{2}}\mathrm{e}^{(1-\alpha+\ln\alpha)i}),$$

for $i\to\infty$.

Note that $1-\alpha+\ln\alpha<0$, for $\alpha<1$, hence the convergence is exponential. It is interesting to note that $p_{i,0}$, i. e., the probability that all drivers succeed to park, converges towards the same value as in the exact model for $n, m \to \infty$ with $\frac{m}{n}=\alpha$ (compare Proposition 2.2.2).

Using (5.2) and Lemma 5.1.6, one can prove that the $P_i(z)$ converge, i. e.,

$$P(z) := \lim_{i\to\infty} P_i(z) = \frac{(1-z)(1-\alpha)}{R(z)-z},$$

Furthermore, one can show that the limit $\lim_{n\to\infty}\mathbb{E}\left(W_n\right)$ exists and find its value:

Lemma 5.1.7.

$$\lim_{n\to\infty}\mathbb{E}\left(W_n\right) = \frac{\alpha^2}{2(1-\alpha)}.$$

Using (5.2) and (5.3), one can finally write $\mathbb{E}\left(W_n\right)$ as a power series in α, which will be useful when transforming the result to the exact model:

Proposition 5.1.8.

$$\mathbb{E}\left(W_n\right) = \frac{\alpha^2}{2(1-\alpha)} + \mathcal{O}(\alpha^{n+2}). \tag{5.4}$$

5.1.2 Transform to the exact model

The transform which Gonnet and Munro present rests on the following idea: Let $f(n,m)$ be an expected value computed using a model of m objects randomly distributed among n locations. Let $g(n,\alpha)$ be the equivalent expected value using a model with n locations each of which receives a random number of objects which is Poisson distributed with parameter α and independent of all other locations. Then

$$g(n,\alpha) = \sum_{m\geq 0} f(n,m)\mathbb{P}\{X_1+\ldots+X_n=m\},$$

where X_i is a Poisson distributed random variable with parameter α. This is easy to verify since the distribution of the X_i's, under the condition that their sum is m, coincides with the random distribution of m objects in n places.
It is known that the sum of n independent Poisson distributed variables with parameter α is Poisson distributed with parameter $n\alpha$. Hence, we have

$$g(n,\alpha) = \sum_{m\geq 0} f(n,m)\frac{(n\alpha)^m \mathrm{e}^{-n\alpha}}{m!}. \tag{5.5}$$

Note that this is an identity in α.

Lemma 5.1.9. *Let $g(n,\alpha) = \sum_{i\geq 0} a_i(n)\alpha^i$ be an expected value computed using a model with n locations each of which receives a random number of objects which is Poisson distributed with parameter α and independent of all other locations. Let $f(n,m)$ be the equivalent expected value computed using a model of m objects randomly distributed among n locations. Then*

$$f(n,m) = \sum_{k=0}^{m} a_k(n) \frac{m^{\underline{k}}}{n^k}.$$

Proof. We can write (5.5) as

$$e^{n\alpha} g(n,\alpha) = \sum_{m\geq 0} f(n,m) \frac{(n\alpha)^m}{m!}.$$

By extracting coefficients, we get

$$\begin{aligned} f(n,m)\frac{n^m}{m!} &= [\alpha^m]e^{n\alpha} g(n,\alpha) \\ &= \sum_{k=0}^{m} [\alpha^k] g(n,\alpha) [\alpha^{m-k}] e^{n\alpha} \\ &= \sum_{k=0}^{m} a_k(n) \frac{n^{m-k}}{(m-k)!}, \end{aligned}$$

which proves the lemma. □

Example:
Let the expected value in the Poisson model be

$$g(n,\alpha) = \frac{1}{(1-\alpha)^{r+1}}.$$

It is known that

$$\frac{1}{(1-\alpha)^{r+1}} = \sum_{i\geq 0} \binom{r+i}{i} \alpha^i,$$

hence the equivalent expected value in the exact model is

$$f(n,m) = \sum_{k=0}^{m} \binom{r+k}{k} \frac{m^{\underline{k}}}{n^k}.$$

We can now transform our results of the Poisson model to the exact model.

Proposition 5.1.10. *Let $W_{n,m}$ denote the defect of a defective parking function $p : [m] \to [n]$. If $m \leq n+1$, then*

$$\mathbb{E}\left(W_{n,m}\right) = \frac{1}{2}\sum_{k=2}^{m} \frac{m^{\underline{k}}}{n^k}. \tag{5.6}$$

Proof. Consider the equivalent expected value $\mathbb{E}\left(W_n\right)$ in the Poisson model. Equation (5.4) shows that

$$g(n,\alpha) = \mathbb{E}\left(W_n\right) = \frac{\alpha^2}{2(1-\alpha)} + \mathcal{O}(\alpha^{n+2}) = \frac{1}{2}\sum_{i=2}^{n+1} \alpha^i + \mathcal{O}(\alpha^{n+2}).$$

Since $m^{\underline{k}} = 0$ for $k > m$, Lemma 5.1.9 proves that

$$\mathbb{E}\left(W_{n,m}\right) = f(n,m) = \frac{1}{2}\sum_{k=2}^{m} \frac{m^{\underline{k}}}{n^k},$$

if $m \leq n+1$. □

We will now show that for $\frac{m}{n} = \alpha < 1$ and $n \to \infty$, $\mathbb{E}\left(W_{n,m}\right)$ converges towards the same value as W_n in the Poisson model:

Proposition 5.1.11. *Let $0 < \alpha < 1$. Then*

$$\lim_{n\to\infty} \mathbb{E}\left(W_{n,\alpha n}\right) = \frac{\alpha^2}{2(1-\alpha)}.$$

We will use the following lemma:

Lemma 5.1.12. *Let $x \in \mathbb{N}$ and $k \in \mathbb{N}_0$. It holds that*

$$x^{\underline{k}} \geq x^k - \binom{k}{2} x^{k-1}.$$

Proof. If $x < k$, then $x^{\underline{k}} = 0$ and the inequality trivially holds.
For $x \geq k$, we will prove the lemma by induction: The claim obviously holds for $k = 0$. If the inequality holds for $k = k_0 < x$, then

$$\begin{aligned}
x^{\underline{k_0+1}} &= x^{\underline{k_0}}(x-k_0) \\
&\geq (x_0^k - \binom{k_0}{2} x^{k_0-1})(x-k_0) \\
&\geq x^{k_0+1} - \binom{k_0}{2} x^{k_0} - k_0 x^{k_0} \\
&= x^{k_0+1} - \binom{k_0+1}{2} x^{k_0},
\end{aligned}$$

hence the claim holds for $k = k_0 + 1$ as well. □

Proof of Proposition 5.1.11. We clearly have

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left(W_{n,\alpha n}\right) &= \lim_{n\to\infty} \frac{1}{2}\sum_{k=2}^{n} \frac{(\alpha n)^{\underline{k}}}{n^k} \\
&\leq \lim_{n\to\infty} \frac{1}{2}\sum_{k=2}^{n} \frac{(\alpha n)^{k}}{n^k} \\
&= \frac{\alpha^2}{2(1-\alpha)}.
\end{aligned}
$$

On the other hand, the above lemma shows that

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left(W_{n,\alpha n}\right) &= \lim_{n\to\infty} \frac{1}{2}\sum_{k=2}^{n} \frac{(\alpha n)^{\underline{k}}}{n^k} \\
&\geq \lim_{n\to\infty} \frac{1}{2}\sum_{k=2}^{n} \frac{(\alpha n)^{k} - \binom{k}{2}(\alpha n)^{k-1}}{n^k} \\
&= \lim_{n\to\infty} \frac{1}{2}\sum_{k=2}^{n} \frac{(\alpha n)^{k}}{n^k} - \lim_{n\to\infty} \frac{1}{2}\sum_{k=2}^{n} \binom{k}{2}\frac{(\alpha n)^{k-1}}{n^k} \\
&= \frac{\alpha^2}{2(1-\alpha)} - \lim_{n\to\infty} \frac{\alpha}{2n}\sum_{k=0}^{n-2} \binom{k+2}{2}\alpha^k \\
&= \frac{\alpha^2}{2(1-\alpha)} - \lim_{n\to\infty} \frac{1}{2n}\frac{\alpha}{(1-\alpha)^3} \\
&= \frac{\alpha^2}{2(1-\alpha)}.
\end{aligned}
$$

□

Now, consider the case $m = n$:

Proposition 5.1.13.

$$
\mathbb{E}\left(W_{n,n}\right) \sim \sqrt{\frac{n\pi}{8}}, \quad \textit{for } n \to \infty.
$$

Proof. Consider *Ramanujan's Q-function*

$$
Q(n) := \sum_{k=1}^{n} \frac{n^{\underline{k}}}{n^k}.
$$

It is well-known (see for example [10]) that

$$Q(n) \sim \sqrt{\frac{n\pi}{2}}, \quad \text{for } n \to \infty,$$

and we obviously have

$$\mathbb{E}\left(W_{n,n}\right) = \frac{1}{2}\sum_{k=2}^{n} \frac{n^{\underline{k}}}{n^k} = \frac{1}{2}(Q(n)-1) \sim \sqrt{\frac{n\pi}{8}}, \quad \text{for } n \to \infty.$$

$\square$

5.2 Exact results

In [5], Cameron, Johannsen, Prellberg and Schweitzer have found an exact formula for $\mathrm{g}(n,m,d)$. We will first present their derivation and then give an alternative. Furthermore, we will show how defective parking functions can be described by **x**-parking functions.

5.2.1 Parameter transform

In order to find a formula for $\mathrm{g}(n,m,d)$, Cameron, Johannsen, Prellberg and Schweitzer first transform the parameters:
For $r,s,d \in \mathbb{N}_0$, let $a(r,s,d)$ be the number of preference functions $p : [d+s] \to [r+s]$ for which

- r parking slots remain empty,
- s slots are occupied in the end, and
- d drivers are unsuccessful.

We clearly have $\mathrm{g}(n,m,d) = a(n-m+d, m-d, d)$, especially $\mathrm{g}(n,n,d) = a(d, n-d, d)$.
It is useful to set $a(r,s,d) = 0$ whenever one of the parameters r, s and d is smaller than 0. We then get the following recursion formula:

Lemma 5.2.1.

$$a(r,s,d) = \begin{cases} 1, & \textit{if } r = s = d = 0, \\ \sum_{i=0}^{d+1} \binom{s+d}{d+1-i} a(r,s-1,i), & \textit{if } d > 0, \\ a(r-1,s,0) + \sum_{i=0}^{d+1} \binom{s+d}{d+1-i} a(r,s-1,i), & \textit{else.} \end{cases}$$

Proof. For $r = s = d = 0$ there exists exactly one assignment.
Next, let $d > 0$. Since d drivers leave in the end, exactly $d+1$ drivers must arrive at parking slot $r + s$ during the parking process. A preference function satisfies this condition if and only if there exists an $i \in [d+1]$ such that $d+1-i$ drivers actually choose the last slot and i drivers arrive at the slot, though they have not chosen it. $\binom{s+d}{d+1-i}$ is the number of ways to choose the drivers who pick the last slot. The number of preference functions for the remaining drivers which satisfy the condition that exactly i drivers arrive at the last space is given by $a(r, s-1, i)$.
Finally, let $d = 0$ but $r > 0$ or $s > 0$. If we only consider the preference functions $p : [d+s] \to [r+s]$ for which the last slot is occupied, then the same recursion as for $d > 0$ holds. On the other hand, the number of preference functions which leave the last slot empty is clearly given by $a(r-1, s, 0)$. □

We can write the above recursion formula as

$$\begin{aligned} a(r,s,d) =& \mathbf{1}_{\{r=s=d=0\}}(r,s,d) + \mathbf{1}_{\{d=0\}} a(r-1,s,0) \\ &+ \sum_{i=0}^{d+1} \binom{s+d}{d+1-i} a(r, s-1, i). \end{aligned} \tag{5.1}$$

Next, we introduce a generating function $A(u, v, t)$ of the numbers $a(r, s, d)$. The above recursion leads to a functional equation which involves evaluations of A at $t = 0$:

Lemma 5.2.2. *Let*

$$A(u,v,t) := \sum_{r,s,d \geq 0} a(r,s,d) u^r \frac{v^s t^d}{(s+d)!}.$$

It holds that

$$A(u,v,t) = \frac{1 + (u - \frac{v}{t}) A(u,v,0)}{1 - \frac{v}{t} \mathrm{e}^t}. \tag{5.2}$$

Proof. Multiplying (5.1) by $u^r \frac{v^s t^d}{(s+d)!}$ and summing up yields

$$\begin{aligned}
A(u,v,t) &= 1 + u \sum_{r,s\geq 0} a(r-1,s,0) u^{r-1} \frac{v^s}{s!} \\
&\quad + \frac{v}{t} \sum_{r,s,d\geq 0} \sum_{i=0}^{d+1} a(r,s-1,i) u^r \frac{v^{s-1} t^{d+1}}{(d+1-i)!(s-1+i)!} \\
&= 1 + uA(u,v,0) - \frac{v}{t} \sum_{r,s\geq 0} a(r,s-1,0) u^r \frac{v^{s-1}}{(s-1)!} \\
&\quad + \frac{v}{t} \sum_{r,s,d\geq 0} \sum_{i=0}^{d} a(r,s-1,i) u^r \frac{v^{s-1} t^d}{(d-i)!(s-1+i)!} \\
&= 1 + (u - \frac{v}{t}) A(u,v,0) \\
&\quad + \frac{v}{t} \sum_{r,s,i\geq 0} a(r,s-1,i) u^r \frac{v^{s-1} t^i}{(s-1+i)!} \sum_{d-i\geq 0} \frac{t^{d-i}}{(d-i)!} \\
&= 1 + (u - \frac{v}{t}) A(u,v,0) + \frac{v}{t} \mathrm{e}^t A(u,v,t),
\end{aligned}$$

which proves the lemma. □

Next, we can obtain an explicit formula for $A(u,v,t)$, which involves the tree function $T(z)$:

Proposition 5.2.3.

$$A(u,v,t) = \frac{1}{1 - \frac{v}{t} e^t} + \frac{u - \frac{v}{t}}{1 - \frac{v}{t} e^t} \cdot \frac{\mathrm{e}^{T(v)}}{1 - u\mathrm{e}^{T(v)}}$$

Proof. In order to find $A(u,v,0)$, we apply the kernel method as described in Section 1.3. We set

$$1 - \frac{v}{t} \mathrm{e}^t = 0,$$

or equivalently

$$t = v\mathrm{e}^t.$$

Clearly, the solution to this equation is

$$t = T(v).$$

From (5.2) we conclude that

$$\left(u - \frac{v}{T(v)}\right) A(u,v,0) + 1 = 0,$$

and further

$$A(u,v,0) = \frac{1}{\frac{v}{T(v)} - u} = \frac{1}{\mathrm{e}^{-T(v)} - u} = \frac{\mathrm{e}^{T(v)}}{1 - u\mathrm{e}^{T(v)}}.$$

Substituting this into (5.2), this finally shows that

$$A(u,v,t) = \frac{1 + (u - \frac{v}{t})A(u,v,0)}{1 - \frac{v}{t}\mathrm{e}^t} = \frac{1}{1 - \frac{v}{t}e^t} + \frac{u - \frac{v}{t}}{1 - \frac{v}{t}e^t} \cdot \frac{\mathrm{e}^{T(v)}}{1 - u\mathrm{e}^{T(v)}}.$$

□

Note that

$$A(u,v,0) = \sum_{r,s \geq 0} a(r,s,0) u^r \frac{v^s}{s!}$$

is a generating function for the number of ordinary parking functions $p : [s] \to [r+s]$. This provides an alternative way to prove Lemma 2.2.1.

Proposition 5.2.4.

$$a(r,s,0) = (r+1)(r+s+1)^{s-1}.$$

Proof. We have

$$\begin{aligned} a(r,s,0) &= s!\ [v^s u^r] A(u,v,0) \\ &= s!\ [v^s u^r] \frac{\mathrm{e}^{T(v)}}{1 - u\mathrm{e}^{T(v)}} \\ &= s!\ [v^s] \mathrm{e}^{(r+1)T(v)}. \end{aligned}$$

Using Lagrange's inversion formula, we obtain

$$\begin{aligned} s!\ [v^s]\mathrm{e}^{(r+1)T(v)} &= (s-1)!\ [T^{s-1}](r+1)\mathrm{e}^{(r+1)T}\mathrm{e}^{sT} \\ &= (r+1)(r+s+1)^{s-1}. \end{aligned}$$

□

From the above result, we get

$$A(u,v,0) = \frac{\mathrm{e}^{T(v)}}{1 - u\mathrm{e}^{T(v)}} = \sum_{r,s \geq 0} (r+1)(r+s+1)^{s-1} u^r \frac{v^s}{s!}, \qquad (5.3)$$

which is proves useful when extracting the coefficients of $A(u,v,t)$.

Proposition 5.2.5. *Let*

$$S(n,m,d) = \sum_{i=0}^{m-d} \binom{m}{i} (n-m+d)(n-m+d+i)^{i-1}(m-d-i)^{m-i}.$$

Then

$$a(r,s,d) = S(r+s, s+d, d) - S(r+s, s+d, d+1), \tag{5.4}$$

or equivalently

$$g(n,m,d) = S(n,m,d) - S(n,m,d+1).$$

Note that this implies that $S(n,m,d)$ is the number of defective parking functions $p : [m] \to [n]$ of defect greater than or equal to d.

Proof. We have

$$\begin{aligned} a(r,s,d) &= (d+s)!\ [u^r v^s t^d] A(u,v,t) \\ &= (d+s)!\ [u^r v^s t^d] \left(\frac{1}{1-\frac{v}{t}e^t} + \frac{u-\frac{v}{t}}{1-\frac{v}{t}e^t} \cdot \frac{\mathrm{e}^{T(v)}}{1-u\mathrm{e}^{T(v)}} \right). \end{aligned}$$

We now read off the coefficients of the summands of $A(u,v,t)$. We let the indicator functions $\mathbf{1}_{\{r=0\}}$, $\mathbf{1}_{\{r\geq 1\}}$ and $\mathbf{1}_{\{s\geq 1\}}$ be defined as usual.
The contribution of the first summand is

$$[u^r v^s t^d] \frac{1}{1-\frac{v}{t}e^t} = \mathbf{1}_{\{r=0\}}(r)[t^d]\frac{\mathrm{e}^{st}}{t^s} = \mathbf{1}_{\{r=0\}}(r) s^{d+s} \frac{1}{(d+s)!}.$$

Using (5.3), we find that the coefficients of the second summand are

$$\begin{aligned} &[u^r v^s t^d] \frac{u}{1-\frac{v}{t}e^t} A(u,v,0) = \\ &= \mathbf{1}_{\{r\geq 1\}}(r)[v^s t^d] \left(\frac{1}{1-\frac{v}{t}e^t} [u^{r-1}] \sum_{i,j\geq 0} (j+1)(j+i+1)^{i-1} u^j \frac{v^i}{i!} \right) \\ &= \mathbf{1}_{\{r\geq 1\}}(r)[v^s t^d] \frac{1}{1-\frac{v}{t}e^t} \sum_{i\geq 0} r(r+i)^{i-1} \frac{v^i}{i!} \\ &= \mathbf{1}_{\{r\geq 1\}}(r)[t^d] \sum_{i=0}^{s} \left(\frac{\mathrm{e}^t}{t} \right)^{s-i} \frac{r(r+i)^{i-1}}{i!} \\ &= \mathbf{1}_{\{r\geq 1\}}(r) \sum_{i=0}^{s} \frac{(s-i)^{s+d-i} r(r+i)^{i-1}}{(s+d-i)!\,i!}. \end{aligned}$$

Finally, the third summand contributes

$$
\begin{aligned}
& [u^r v^s t^d]\frac{\frac{v}{t}}{1-\frac{v}{t}e^t}A(u,v,0) = \\
&= [v^s t^d]\frac{\frac{v}{t}}{1-\frac{v}{t}e^t}\sum_{i\geq 0}(r+1)(r+i+1)^{i-1}\frac{v^i}{i!} \\
&= \mathbf{1}_{\{s\geq 1\}}(s)[t^d]\sum_{i=0}^{s-1}\frac{1}{\mathrm{e}^t}\left(\frac{\mathrm{e}^t}{t}\right)^{s-i}\frac{(r+1)(r+i+1)^{i-1}}{i!} \\
&= \mathbf{1}_{\{s\geq 1\}}(s)\sum_{i=0}^{s-1}\frac{(s-i-1)^{s+d-i}(r+1)(r+i+1)^{i-1}}{(s+d-i)!i!}.
\end{aligned}
$$

Multiplying the obtained expressions by $(s+d)!$, we finally obtain

$$
\begin{aligned}
a(r,s,d) &= \mathbf{1}_{\{r=0\}}(r)s^{s+d} \\
&\quad + \mathbf{1}_{\{r\geq 1\}}(r)\sum_{i=0}^{s}\binom{s+d}{i}r(r+i)^{i-1}(s-i)^{s+d-i} \\
&\quad - \mathbf{1}_{\{s\geq 1\}}(s)\sum_{i=0}^{s-1}\binom{s+d}{i}(r+1)(r+1+i)^{i-1}(s-1-i)^{s+d-i}.
\end{aligned}
$$

Now, the fact that Equation (5.4) holds follows from the fact that

$$
\begin{aligned}
S(s,s+d,d) &= (s+d-d)^{s+d}+\sum_{i=1}^{s}\binom{s+d}{i}\cdot 0\cdot i^{i-1}(s-i)^{s+d-i} \\
&= s^{s+d}.
\end{aligned}
$$

□

Note that $S(n,m,0)$ is the total number of functions $p:[m]\to[n]$, hence we must have

$$
n^m = S(n,m,0) = \sum_{i=0}^{m}\binom{m}{i}(n-m)(n-m+i)^{i-1}(m-i)^{m-i}.
$$

This is a special case of *Abels' Binomial Identity* [1]:

Lemma 5.2.6 (Abels' Binomial Identity)**.** *Let $x,y,z\in\mathbb{R}$ and $m\in\mathbb{N}$. It holds that*

$$
(x+y)^m = \sum_{i=0}^{m}\binom{m}{i}x(x-zi)^{i-1}(x+zi)^{m-i}.
$$

We use this identity with $x = n - m + d$, $y = m - d$ and $z = -1$ to find an alternative expression for $S(n, m, d)$, which is useful for small d:

$$\begin{aligned} S(n,m,d) &= \\ &= \sum_{i=0}^{m-d} \binom{m}{i} (n-m+d)(n-m+d+i)^{i-1}(m-d-i)^{m-i} \\ &= n^m - \sum_{i=m-d+1}^{m} \binom{m}{i} (n-m+d)(n-m+d+i)^{i-1}(m-d-i)^{m-i} \\ &= n^m - \sum_{l=0}^{d-1} \binom{m}{l} (n-m+d)(n+d-l)^{m-l-1}(l-d)^{l}. \end{aligned}$$

This also shows that

$$\bar{S}(n,m,d) := \sum_{l=0}^{d} \binom{m}{l} (n-m+d+1)(n+d+1-l)^{m-l-1}(l-d-1)^{l} \tag{5.5}$$

is the number of defective parking functions $p : [m] \to [n]$ of defect not greater than d.

5.2.2 Block decomposition

We now give an alternative derivation of the numbers $\mathrm{g}(n, m, d)$ due to Panholzer [personal communication, January 2009]. By *block decomposition* of the final parking order, we will find a recursion for $\mathrm{g}(n, m, d)$ and solve this recursion using a generating function approach.

In the following, we will use the notation

$$f(n) := g(n, n, 0).$$

We further let $s(n, d)$ denote the number of defective parking functions $p : [n+d] \to [n]$ of defect d, i. e., the number of preference functions with the property that all n parking slots are occupied in the final parking order and d drivers do not succeed to park.
If $m = n + d$, we clearly have $\mathrm{g}(n, m, d) = s(n, d)$.
For $m < n + d$, we can establish the following recursion:

Proposition 5.2.7. *Let $m < n + d$. It holds that*

$$g(n, m, d) = \sum_{j=1}^{n} \binom{m}{j-1} f(j-1) g(n-j, m-j+1, d). \tag{5.6}$$

Proof. Since $m < n + d$, each preference function $p : [m] \to [n]$ results in a final parking order which leaves at least one slot empty. If the first empty slot is j, then $j - 1$ of the total number of m cars park in the first $j - 1$ slots. The number of ways to choose these cars is given by $\binom{m}{j-1}$, and there are $f(j-1)$ preference functions which place them in the first $j - 1$ slots. The number of assignments of the remaining $m - j + 1$ cars to the slots $\{j + 1, \ldots, n\}$ with the property that exactly d cars overflow is given by $\mathrm{g}(n - j, m - j + 1, d)$. $\square$

We now introduce the generating function

$$G(z, u, v) = \sum_{n \geq 0} \sum_{m \geq 0} \sum_{d \geq 0} \mathrm{g}(n, m, d) \frac{z^m}{m!} u^n v^d.$$

We will further use the auxiliary generating function

$$S(u, v) = \sum_{n \geq 0} \sum_{d \geq 0} s(n, d) \frac{u^n v^d}{(n + d)!},$$

and the tree function

$$T(z) = z\theta(z) = \sum_{m \geq 1} \mathrm{g}(m-1, m-1, 0) \frac{z^m}{(m-1)!} = \sum_{m \geq 1} f(m-1) \frac{z^m}{(m-1)!}.$$

Proposition 5.2.8. *The generating functions defined above satisfy the equation*

$$G(z, u, v) = \frac{S(zu, zv)}{1 - \frac{T(zu)}{z}}. \tag{5.7}$$

Proof. First note that $\mathrm{g}(n - j, m - j + 1, d) = 0$ if $m \geq n + d$. This allows us to write

$$\begin{aligned} \mathrm{g}(n, m, d) &= \mathbf{1}_{\{m=n+d\}}(n, m, d)\ s(n, d) \\ &\quad + \sum_{j=1}^{n} \binom{m}{j-1} g(j-1, j-1, 0) g(n-j, m-j+1, d). \end{aligned}$$

We multiply this equation by $\frac{z^m}{m!}u^n v^d$ and sum up to obtain

$$\begin{aligned}
G(z,u,v) &= \sum_{n\geq 0}\sum_{d\geq 0} s(n,d)\frac{z^{n+d}}{(n+d)!}u^n v^d \\
&\quad + \sum_{n\geq 0}\sum_{m\geq 0}\sum_{d\geq 0}\sum_{j=1}^{n} \frac{f(j-1)}{(j-1)!}\frac{\mathrm{g}(n-j,m-j+1,d)}{(m-j+1)!} z^m u^n v^d \\
&= \sum_{n\geq 0}\sum_{d\geq 0} s(n,d)\frac{(zu)^n(zv)^d}{(n+d)!} + \frac{1}{z}\sum_{j\geq 1}\frac{f(j-1)}{(j-1)!}(zu)^j \\
&\quad \cdot \sum_{n-j\geq 0}\sum_{m\geq 0}\sum_{d\geq 0} \mathrm{g}(n-j,m-j+1,d)\frac{z^{m-j+1}}{(m-j+1)!}u^{n-j}v^d \\
&= S(zu,zv) + \frac{T(zu)}{z}G(z,u,v),
\end{aligned}$$

which proves the proposition. ☐

We can now find an explicit formula for $G(z,u,v)$.

Proposition 5.2.9.

$$G(z,u,v) = \frac{1-\frac{T(zu)}{zv}}{(1-\frac{T(zu)}{z})(1-\frac{u}{v}\mathrm{e}^{zv})}.$$

Proof. It clearly holds that

$$\sum_{d\geq 0} \mathrm{g}(n,m,d) = n^m.$$

Hence, evaluating (5.7) at $v=1$, we obtain

$$\frac{S(zu,z)}{1-\frac{T(zu)}{z}} = G(z,u,1) = \sum_{n\geq 0}\sum_{m\geq 0} n^m \frac{z^m}{m!}u^n = \sum_{n\geq 0}\mathrm{e}^{nz}u^n = \frac{1}{1-u\mathrm{e}^z},$$

or equivalently

$$S(zu,z) = \frac{1-\frac{T(zu)}{z}}{1-u\mathrm{e}^z}.$$

Using the substitution $z \leftarrow zv$, $u \leftarrow \frac{u}{v}$, we find

$$S(zu,zv) = \frac{1-\frac{T(zu)}{zv}}{1-\frac{u}{v}\mathrm{e}^{zv}},$$

and finally

$$G(z,u,v) = \frac{1 - \frac{T(zu)}{zv}}{(1 - \frac{T(zu)}{z})(1 - \frac{u}{v}\mathrm{e}^{zv})}.$$

□

We can now re-prove the result of Section 5.2.1:

Alternative proof of Proposition 5.2.5. It suffices to show that

$$m!\ [u^n z^m v^d] \frac{1}{(1 - \frac{T(zu)}{z})(1 - \frac{u}{v}\mathrm{e}^{zv})} = S(n,m,d), \tag{5.8}$$

and

$$m!\ [u^n z^m v^d] \frac{\frac{T(zu)}{zv}}{(1 - \frac{T(zu)}{z})(1 - \frac{u}{v}\mathrm{e}^{zv})} = S(n,m,d+1). \tag{5.9}$$

Using Lemma 1.2.2 to expand $\frac{1}{1-\frac{T(zu)}{z}}$, we obtain

$$\begin{aligned}
&[u^n z^m v^d] \frac{1}{(1 - \frac{T(zu)}{z})(1 - \frac{u}{v}\mathrm{e}^{zv})} = \\
&= [u^n z^m] \left(\sum_{j\geq 0} \left(\frac{T(zu)}{z} \right)^j [v^d] \sum_{l \geq 0} \left(\frac{u}{v} \mathrm{e}^{zv} \right)^l \right) \\
&= [u^n z^m] \left(\sum_{j\geq 0} \left(\frac{T(zu)}{z} \right)^j \sum_{l \geq 0} [v^{d+l}] (u\mathrm{e}^{zv})^l \right) \\
&= [u^n z^m] \sum_{j\geq 0} \sum_{i \geq j} j \frac{i^{i-j-1}}{(i-j)!} z^{i-j} u^i \sum_{l\geq 0} u^l \frac{(lz)^{d+l}}{(d+l)!} \\
&= [z^m] \sum_{j=0}^{n} \sum_{i=j}^{n} j \frac{i^{i-j-1}}{(i-j)!} z^{i-j} \frac{((n-i)z)^{d+n-i}}{(d+n-i)!} \\
&= [z^m] \sum_{j=0}^{n} z^{d+n-j} \sum_{i=j}^{n} j \frac{i^{i-j-1}}{(i-j)!} \frac{(n-i)^{d+n-i}}{(d+n-i)!} \\
&= \sum_{i=n-m+d}^{n} (n-m+d) \frac{i^{i-(n-m+d)-1}}{(i-(n-m+d))!} \frac{(n-i)^{d+n-i}}{(d+n-i)!} \\
&= \frac{1}{m!} \sum_{l=0}^{m-d} \binom{m}{l} (n-m+d)(n-m+d+l)^{l-1}(m-d-l)^{m-l} \\
&= \frac{1}{m!} S(n,m,d).
\end{aligned}$$

This proves (5.8). The fact that Equation (5.9) holds can be verified in the same manner. $\square$

5.2.3 Defective parking functions as x-parking functions

The following relation between defective parking functions and **x**-parking functions has been mentioned in [5]: Let

$$\mathbf{x}_{n,m,d} := (n-(m-d)+1, \underbrace{1,\dots,1}_{m-d-1}, \underbrace{0,\dots,0}_{d}).$$

Then $p=(p_1,\dots,p_m)$ is an $\mathbf{x}$-parking function for $\mathbf{x}=\mathbf{x}_{n,m,d}$ if and only if p is a defective parking function $p:[m]\to[n]$ of defect not greater than d.
We will use this fact to verify our result of Section 5.2.1, i. e., with $\bar{S}(n,m,d)$ defined as in (5.5) we will show:

Proposition 5.2.10.

$$g_m(\boldsymbol{x}_{n,m,d}) = \bar{S}(n,m,d).$$

Note that for the special case $d=0$, this follows immediately from Proposition 4.3.2, with $a=n-m+1$ and $b=1$:

$$g_m(\mathbf{x}_{n,m,0}) = g_m((n-m+1,1,\dots,1)) = (n-m+1)(n+1)^{m-1}.$$

For $d\ge 1$, we will use an expression for $g_m((a,\underbrace{b,\dots,b}_{m-d-1},c,\underbrace{0,\dots,0}_{d-1}))$ which has been found by Yan in [28]:

Proposition 5.2.11. *For* $\boldsymbol{x}_{m,d,a,b,c} = (a,\underbrace{b,\dots,b}_{m-d-1},c,\underbrace{0,\dots,0}_{d-1})$,

$$g_m(\boldsymbol{x}_{m,d,a,b,c}) = a\sum_{j=0}^{d}\binom{m}{j}(c-(d+1-j)b)^j(a+(m-j)b)^{m-j-1}.$$

Note that with $a=n-(m-d)+1$, $b=1$ and $c=0$, this proves Proposition 5.2.10.

Proof. Due to Lemma 4.3.1, we know that for fixed a and b, $g_m(\mathbf{x})$ is a polynomial in c. Hence, it suffices to prove the claim under the assumption that $c>(d+1)b$. Every $\mathbf{x}_{m,d,a,b,c}$-parking function $p=(p_1,\dots,p_m)$ satisfies the conditions

$$1\le p_i\le a+(m-d-1)b+c, \quad \text{for } i\in[m],$$

and

$$|\{p_i \mid p_i \leq a + (l-1)b\}| \geq l, \quad \text{for } l \in [m-d]. \tag{5.10}$$

We can decompose every $\mathbf{x}_{m,d,a,b,c}$-parking function into two subsequences β, γ by the following rule:
Let t be the largest integer such that the following condition holds:

$$|\{p_i \mid p_i \leq a + (m-d-1+l)b\}| \geq m-d+l, \quad \text{for } l \in [t]. \tag{5.11}$$

If no such t exists, we say that $t = 0$. Let β be the subsequence of p which consists of all terms p_i which are less than or equal to $a + (m-d-1+t)b$, and γ be the subsequence of the remaining terms.
Since t is the largest integer for which (5.11) holds, β must be a sequence of length $m-d+t$. Furthermore, from (5.10) and (5.11) follows that β is an $\mathbf{x}'$-parking function for $\mathbf{x}' = (a, b, \ldots, b)$. Let $f_i = a(a+bi)^{i-1}$, then we know from Proposition 4.3.2 that there are f_{m-d+t} such sequences.
On the other hand, γ is a sequence of length $d-t$ in which every term is an element of the set $\{a + (m-d-1)b + (t+1)b + 1, \ldots, a + (m-d-1)b + c\}$. There are $(c-(t+1)b)^{d-t}$ such sequences. Finally, the terms of γ can take any $d-t$ positions in p, hence we conclude

$$\begin{aligned}
g_m(\mathbf{x}) &= \sum_{t=0}^{d} \binom{m}{d-t} (c-(t+1)b)^{d-t} f_{m-d+t} \\
&= \sum_{j=0}^{d} \binom{m}{j} (c-(d+1-j)b)^j f_{m-j} \\
&= a \sum_{j=0}^{d} \binom{m}{j} (c-(d+1-j)b)^j (a+(m-j)b)^{m-j-1}.
\end{aligned}$$

□

5.3 Asymptotic results derived from the exact results

We will now give a result on the asymptotics of $\mathrm{g}(n,m,d)$ for the special case $n = m$ which has been derived by Panholzer [personal communication, January 2009].
We will first determine the generating function of the numbers $\mathrm{g}(n,m,d)$,

$$Q(z,v) := \sum_{n\geq 0}\sum_{d\geq 0} \mathrm{g}(n,n,d)\frac{z^n}{n!}v^d.$$

Proposition 5.3.1.

$$Q(z,v) = \frac{(v-1)T(z)}{vT(z) - z\mathrm{e}^{vT(z)}}.$$

Proof. Since

$$G(z,u,v) = \sum_{n\geq 0}\sum_{m\geq 0}\sum_{d\geq 0} \mathrm{g}(n,m,d)\frac{z^m}{m!}u^n v^d,$$

we can obtain $Q(z,v)$ by

$$\begin{aligned} Q(z,v) &= \frac{1}{2\pi\mathrm{i}}\oint \frac{G(t,\frac{z}{t},v)}{t}dt \\ &= \frac{1}{2\pi\mathrm{i}}\oint \frac{1-\frac{T(z)}{tv}}{t(1-\frac{T(z)}{t})(1-\frac{z}{tv}\mathrm{e}^{tv})}dt \\ &= \frac{1}{2\pi\mathrm{i}}\oint \frac{t-\frac{T(z)}{v}}{(t-T(z))(t-\frac{z}{v}\mathrm{e}^{tv})}dt. \end{aligned}$$

We can now determine $Q(z,v)$ using the residue theorem. The integrand has a singularity where $t = \frac{z}{v}\mathrm{e}^{tv}$, i. e., $tv = z\mathrm{e}^{tv}$ or equivalently $tv = T(z)$, but one can easily check that this is a removable singularity. Hence, we only have to consider the residue at $t = T(z)$, which is a simple pole. This gives

$$\begin{aligned} Q(z,v) &= \operatorname*{Res}_{t=T(z)} \frac{t-\frac{T(z)}{v}}{(t-T(z))(t-\frac{z}{v}\mathrm{e}^{tv})} \\ &= \frac{T(z)-\frac{T(z)}{v}}{T(z)-\frac{z}{v}\mathrm{e}^{T(z)v}} \\ &= \frac{(v-1)T(z)}{vT(z)-z\mathrm{e}^{vT(z)}}. \end{aligned}$$

□

For a random defective parking function $p : [n] \to [n]$, we let X_n denote the defect of p. By setting $w := v-1$ and

$$\tilde{Q}(z,w) := Q(z,w+1) = Q(z,v),$$

we clearly have

$$\mathbb{E}\left(X_n^r\right) = \frac{n!}{n^n}\,[z^n]\left(\frac{\partial^r}{\partial v^r}Q(z,v)\right)\bigg|_{v=1} = \frac{n!}{n^n}r!\,[z^n w^r]\tilde{Q}(z,w).$$

Proposition 5.3.2.

$$[w^r]\tilde{Q}(z,w) = \frac{1}{(1-T(z))^{r+1}}\left(\frac{T(z)^2}{2}\right)^r + \mathcal{O}\left(\frac{1}{(1-T(z))^r}\right),$$

for $T(z) \to 1$, *i. e.,* $z \to \frac{1}{\mathrm{e}}$.

Proof. We have

$$\begin{aligned}
\tilde{Q}(z,w) &= \frac{wT(z)}{(w+1)T(z) - z\mathrm{e}^{(w+1)T(z)}} \\
&= \frac{wT(z)}{(w+1)T(z) - z\mathrm{e}^{T(z)}\sum_{k\geq 0}\frac{w^kT(z)^k}{k!}} \\
&= \frac{wT(z)}{wT(z) - T(z)\sum_{k\geq 1}\frac{w^kT(z)^k}{k!}} \\
&= \frac{1}{1-\sum_{k\geq 0}\frac{w^kT(z)^{k+1}}{(k+1)!}} \\
&= \frac{1}{1-T(z)-\sum_{k\geq 1}\frac{w^kT(z)^{k+1}}{(k+1)!}} \\
&= \frac{1}{(1-T(z))(1-\frac{1}{1-T(z)}\sum_{k\geq 1}\frac{w^kT(z)^{k+1}}{(k+1)!}} \\
&= \frac{1}{1-T(z)}\left(1+\sum_{q\geq 1}\frac{1}{(1-T(z))^q}\left(\sum_{k\geq 1}\frac{T(z)^{k+1}}{(k+1)!}w^k\right)^q\right)
\end{aligned}$$

Hence it holds that

$$[w^0]\tilde{Q}(z,w) = \frac{1}{1-T(z)},$$

and, if $r \geq 1$,

$$[w^r]\tilde{Q}(z,w) = \frac{1}{(1-T(z))^{r+1}}\left(\frac{T(z)^2}{2}\right)^r + \mathcal{O}\left(\frac{1}{(1-T(z))^r}\right),$$

for $T(z) \to 1$. □

By singularity analysis, we can now asymptotically determine $\mathbb{E}\left(X_n^r\right)$.

Proposition 5.3.3.

$$\mathbb{E}\left(X_n^r\right) = \frac{\sqrt{\pi}r!n^{\frac{r}{2}}}{2^{\frac{3r}{2}}\Gamma(\frac{r+1}{2})}(1+\mathcal{O}(n^{-\frac{1}{2}})).$$

Proof. We use the singular expansion of $T(z)$ at $t = \frac{1}{\mathrm{e}}$, see Lemma 1.2.3.

$$\begin{aligned}
&[w^r]\tilde{Q}(z,w) = \\
&= \frac{1}{(1-T(z))^{r+1}}\left(\frac{T(z)^2}{2}\right)^r + \mathcal{O}\left(\frac{1}{(1-T(z))^r}\right) \\
&= \frac{1}{(\sqrt{2}\sqrt{1-ez}+\mathcal{O}(1-ez))^{r+1}}\left(\frac{(1-\sqrt{2}\sqrt{1-ez}+\mathcal{O}(1-ez))^2}{2}\right)^r \\
&\quad + \mathcal{O}\left(\frac{1}{(\sqrt{2}\sqrt{1-ez}+\mathcal{O}(1-ez))^r}\right) \\
&= \frac{1+\mathcal{O}(\sqrt{1-ez})}{2^{\frac{3r+1}{2}}(1-ez)^{\frac{r+1}{2}}(1+\mathcal{O}(\sqrt{1-ez}))} + \mathcal{O}\left(\frac{1}{(1-ez)^{\frac{r}{2}}}\right) \\
&= \frac{1}{2^{\frac{3r+1}{2}}(1-ez)^{\frac{r+1}{2}}} + \mathcal{O}\left(\frac{1}{(1-ez)^{\frac{r}{2}}}\right).
\end{aligned}$$

So, using Lemma 1.4.1 and approximating $n!$ by Stirling's formula, we obtain

$$\begin{aligned}
\mathbb{E}\left(X_n^{\underline{r}}\right) &= \frac{r!n!}{n^n}[z^n w^r]\tilde{Q}(z,w) \\
&= \frac{r!n!}{n^n}[z^n]\frac{1}{2^{\frac{3r+1}{2}}(1-ez)^{\frac{r+1}{2}}} + \mathcal{O}\left(\frac{1}{(1-ez)^{\frac{r}{2}}}\right) \\
&= \frac{r!n!}{n^n}\frac{1}{2^{\frac{3r+1}{2}}}\frac{\mathrm{e}^n n^{\frac{r-1}{2}}}{\Gamma(\frac{r+1}{2})}(1+\mathcal{O}(n^{-\frac{1}{2}})) \\
&= \frac{r!n^n\sqrt{2\pi n}}{n^n e^n}(1+\mathcal{O}(n^{-1}))\frac{1}{2^{\frac{3r+1}{2}}}\frac{\mathrm{e}^n n^{\frac{r-1}{2}}}{\Gamma(\frac{r+1}{2})}(1+\mathcal{O}(n^{-\frac{1}{2}})) \\
&= \frac{\sqrt{\pi}r!n^{\frac{r}{2}}}{2^{\frac{3r}{2}}\Gamma(\frac{r+1}{2})}(1+\mathcal{O}(n^{-\frac{1}{2}})).
\end{aligned}$$

Now, observe that $\mathbb{E}\left(X_n^r\right)$ is given by a sum of the form

$$\mathbb{E}\left(X_n^r\right) = \mathbb{E}\left(X_n^{\underline{r}}\right) + \sum_{i=0}^{r-1} a_i \mathbb{E}\left(X_n^{\underline{i}}\right).$$

Hence, we finally obtain

$$\mathbb{E}\left(X_n^r\right) = \frac{\sqrt{\pi}r!n^{\frac{r}{2}}}{2^{\frac{3r}{2}}\Gamma(\frac{r+1}{2})}(1+\mathcal{O}(n^{-\frac{1}{2}})).$$

$\square$

Proposition 5.3.4.

$$\lim_{n\to\infty} \mathbb{E}\left(\left(\frac{X_n}{\sqrt{n}}\right)^r\right) = 2^{-\frac{r}{2}}\Gamma\left(\frac{r}{2}+1\right).$$

Proof. It suffices to show that

$$\sqrt{\pi}\frac{r!}{2^r\Gamma(\frac{r+1}{2})} = \Gamma\left(\frac{r}{2}+1\right),$$

or equivalently

$$\Gamma\left(\frac{r}{2}+\frac{1}{2}\right)\Gamma\left(\frac{r}{2}+1\right) = \sqrt{\pi}\frac{r!}{2^r}. \tag{5.1}$$

We will prove this equation by induction. It is well-known that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, hence the claim clearly holds for $r = 0$. Now assume that (5.1) holds for $r = r_0 \in \mathbb{N}_0$. By the reflection formula $\Gamma(x+1) = x\Gamma(x)$, we have

$$\begin{aligned} \Gamma\left(\frac{r_0+1}{2}+\frac{1}{2}\right)\Gamma\left(\frac{r_0+1}{2}+1\right) &= \Gamma\left(\frac{r_0}{2}+1\right)\Gamma\left(\frac{r_0}{2}+\frac{1}{2}\right)\frac{r_0+1}{2} \\ &= \sqrt{\pi}\frac{r_0!}{2^{r_0}}\frac{r_0+1}{2} \\ &= \sqrt{\pi}\frac{(r_0+1)!}{2^{r_0+1}}, \end{aligned}$$

hence (5.1) holds for $r = r_0 + 1$ as well. □

Proposition 5.3.5. $\frac{X_n}{\sqrt{n}}$ *is asymptotically Rayleigh distributed with parameter* $s = \frac{1}{2}$, *i. e.,*

$$\frac{X_n}{\sqrt{n}} \xrightarrow{d} X,$$

where X *is a random variable with density*

$$f(x) = 4x\mathrm{e}^{-2x^2}, \quad \textit{for } x \geq 0.$$

and moments

$$\mathbb{E}\left(X^r\right) = 2^{-\frac{r}{2}}\Gamma\left(\frac{r}{2}+1\right).$$

Proof. This follows directly from Proposition 1.5.3 and Proposition 5.3.4. □

5.4 Defective bucket parking functions

We will now generalize the concept of defective parking functions to bucket parking functions.

Definition 5.4.1. *Let $p : [m] \to [n]$. We say that p is a defective k-bucket parking function of defect d if and only if $|\{i \in [m] \mid \iota_k p(i) > n\}| = d$.*

We let $\mathrm{g}_k(n,m,d)$ denote the number of defective k-bucket parking functions of defect d. This is a generalization of the numbers $g_k(n,m)$ of k-bucket parking functions $p : [m] \to [n]$, $g_k(n,m) = \mathrm{g}_k(n,m,0)$, and of the numbers $\mathrm{g}(n,m,d)$ of defective parking functions $p : [m] \to [n]$ of defect d, $\mathrm{g}(n,m,d) = \mathrm{g}_1(n,m,d)$.

In the following, we will use the numbers $f_k(n,m)$ from Section 4.2.1, i. e., $f_k(n,m)$ is the number of k-bucket parking functions $p : [m] \to [n]$ with the property that the last row in the final parking order is not full.
Furthermore, we let $s_k(n,d)$ denote the number of defective parking functions $p : [kn+d] \to [n]$ of defect d, i. e., the number of preference functions with the property that all n parking rows are fully occupied in the final parking order and d drivers do not succeed to park.

If $m = kn+d$, we clearly have $\mathrm{g}_k(n,m,d) = s_k(n,d)$.
For $m < kn+d$, we can establish the following recursion:

Proposition 5.4.1. *Let $m < kn+d$. It holds that*

$$g_k(n,m,d) = \sum_{j=0}^{n-1} \binom{m}{kj+d} s_k(j,d) f_k(n-j, m-kj-d).$$

Proof. Since $m < kn+d$, each preference function $p : [m] \to [n]$ of defect d results in a final parking order in which at least one row is not full. If the last not fully occupied row is at position $n-j$, then $kj+d$ of the m drivers have chosen one of the last j rows. The number of ways to choose these cars is given by $\binom{m}{kj+d}$, and there are $s_k(j,d)$ preference functions which place them in the last j rows such that exactly d drivers are not successful. The number of assignments of the remaining $m-kj-d$ cars to the first $n-j$ rows with the property that row $n-j$ is not full is given by $f_k(n-j, m-kj-d)$. □

We now introduce the generating function

$$G_k(z,u,v) = \sum_{n\geq 0}\sum_{m\geq 0}\sum_{d\geq 0} \mathrm{g}_k(n,m,d)\frac{z^m}{m!}u^n v^d.$$

For fixed k, we will further use the auxiliary generating function

$$S(u,v) = \sum_{n\geq 0}\sum_{d\geq 0} s_k(n,d)\frac{u^n v^d}{(kn+d)!},$$

and the generating function of the numbers $f_k(n,r)$ from Section 4.2.1,

$$\Lambda(z,u) = \sum_{n\geq 1}\sum_{r\geq 0} f_k(n,r)u^{kn}\frac{z^r}{r!} = \frac{1-\prod_{i=1}^{k}(1-\omega_k^i u\theta(\frac{\omega_k^i uz}{k}))}{\prod_{i=1}^{k}(1-\omega_k^i u\theta(\frac{\omega_k^i uz}{k}))},$$

where ω_k denotes a primitive k-th root of unity.

Proposition 5.4.2. *The generating functions defined above satisfy the equation*

$$G_k(z,u,v) = S(z^k u, zv)\big(1+\Lambda\big(z,u^{\frac{1}{k}}\big)\big). \tag{5.1}$$

Proof. We set $f_k(0,r) := 0$ for all $r \in \mathbb{N}_0$. This allows us to write

$$\begin{aligned}
\mathrm{g}_k(n,m,d) \;&=\; \mathbf{1}_{\{m=kn+d\}}(n,m,d)\ s_k(n,d) \\
&\quad + \sum_{j=0}^{n}\binom{m}{kj+d} s_k(j,d) f_k(n-j,m-kj-d).
\end{aligned}$$

We multiply this equation by $\frac{z^m}{m!}u^n v^d$ and sum up to obtain

$$\begin{aligned}
G_k(z,u,v) \;&=\; \sum_{n\geq 0}\sum_{d\geq 0} s(n,d)\frac{z^{kn+d}}{(kn+d)!}u^n v^d \\
&\quad + \sum_{n\geq 0}\sum_{m\geq 0}\sum_{d\geq 0}\sum_{j=0}^{n}\frac{s_k(j,d)}{(kj+d)!}\frac{f_k(n-j,m-kj-d)}{(m-kj-d)!}z^m u^n v^d \\
&= \sum_{n\geq 0}\sum_{d\geq 0} s(n,d)\frac{(z^k u)^n (zv)^d}{(kn+d)!} + \sum_{d\geq 0}\sum_{j\geq 0}\frac{s_k(j,d)}{(kj+d)!}z^{kj+d}u^j \\
&\quad \cdot \sum_{n-j\geq 0}\sum_{m\geq 0}\frac{f_k(n-j,m-kj-d)}{(m-kj-d)!}z^{m-kj-d}u^{n-j} \\
&= S(z^k u, zv) \\
&\quad + \sum_{d\geq 0}\sum_{j\geq 0} s(n,d)\frac{(z^k u)^j (zv)^d}{(kn+d)!}\sum_{n\geq 0}\sum_{m\geq 0} f_k(n,m)(u^{\frac{1}{k}})^{kn}\frac{z^m}{m!} \\
&= S(z^k u, zv) + S(z^k u, zv)\Lambda\big(z,u^{\frac{1}{k}}\big).
\end{aligned}$$

□

From this, we can find an explicit formula for $G_k(z,u,v)$:

Proposition 5.4.3. *Let ω_k denote a primitve k-th root of unity and $T(z)$ the tree function. Then*

$$G_k(z,u,v) = \frac{1}{1-\frac{u}{v^k}\mathrm{e}^{zv}} \cdot \frac{\prod_{i=1}^{k}\left[1-\frac{k}{zv}T\left(\frac{\omega_k^i u^{\frac{1}{k}} z}{k}\right)\right]}{\prod_{i=1}^{k}\left[1-\frac{k}{z}T\left(\frac{\omega_k^i u^{\frac{1}{k}} z}{k}\right)\right]}.$$

Proof. It clearly holds that

$$\sum_{d\geq 0} \mathrm{g}_k(n,m,d) = n^m.$$

Hence, evaluating (5.1) at $v=1$, we obtain

$$S(z^k u, z)\left(1+\Lambda\left(z,u^{\frac{1}{k}}\right)\right) = G_k(z,u,1) = \sum_{n\geq 0}\sum_{m\geq 0} n^m \frac{z^m}{m!}u^n = \frac{1}{1-u\mathrm{e}^z},$$

or equivalently

$$S(z^k u, z) = \frac{1}{1-u\mathrm{e}^z}\frac{1}{1+\Lambda\left(z,u^{\frac{1}{k}}\right)}.$$

Using the substitution $z \leftarrow zv$, $u \leftarrow \frac{u}{v^k}$, we find

$$S(z^k u, zv) = \frac{1}{1-\frac{u}{v^k}\mathrm{e}^{zv}}\frac{1}{1+\Lambda\left(zv,\frac{u^{\frac{1}{k}}}{v}\right)},$$

and finally

$$\begin{aligned}
G(z,u,v) &= S(z^k u, zv)\big(1+\Lambda\big(z,u^{\frac{1}{k}}\big)\big)\\
&= \frac{1}{1-\frac{u}{v^k}\mathrm{e}^{zv}} \cdot \frac{1+\Lambda\big(z,u^{\frac{1}{k}}\big)}{1+\Lambda\big(zv,\frac{u^{\frac{1}{k}}}{v}\big)}\\
&= \frac{1}{1-\frac{u}{v^k}\mathrm{e}^{zv}} \cdot \frac{\prod_{i=1}^{k}\left[1-\omega_k^i\frac{u^{\frac{1}{k}}}{v}\theta\Big(\frac{\omega_k^i u^{\frac{1}{k}} z}{k}\Big)\right]}{\prod_{i=1}^{k}\left[1-\omega_k^i u^{\frac{1}{k}}\theta\Big(\frac{\omega_k^i u^{\frac{1}{k}} z}{k}\Big)\right]}\\
&= \frac{1}{1-\frac{u}{v^k}\mathrm{e}^{zv}} \cdot \frac{\prod_{i=1}^{k}\left[1-\frac{k}{zv}T\Big(\frac{\omega_k^i u^{\frac{1}{k}} z}{k}\Big)\right]}{\prod_{i=1}^{k}\left[1-\frac{k}{z}T\Big(\frac{\omega_k^i u^{\frac{1}{k}} z}{k}\Big)\right]}.
\end{aligned}$$

□

As a direct consequence, we can complete our considerations on the numbers $g_k(n,m)$ of k-bucket parking functions $p:[m]\to[n]$ from Section 4.2.1:

Proposition 5.4.4. *Let*

$$H_k(z,u) = \sum_{n\ge 0}\sum_{m\ge 0} g_k(n,m)\frac{z^m}{m!}u^n.$$

It holds that

$$H_k(z,u) = \frac{(-1)^{k+1}}{u} \cdot \frac{\prod_{i=1}^{k}\left[\frac{k}{z}T\Big(\frac{\omega_k^i u^{\frac{1}{k}} z}{k}\Big)\right]}{\prod_{i=1}^{k}\left[1-\frac{k}{z}T\Big(\frac{\omega_k^i u^{\frac{1}{k}} z}{k}\Big)\right]}.$$

Proof. This follows directly from

$$H_k(z,u) = G_k(z,u,0).$$

□

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