DISSERTATION

Bifurcation and stability of viscous shock waves in viscous conservation laws

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... dedicated to Ines and my family.

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Introduction

A viscous conservation law in one space dimension is a partial differential equation of the form

$$\frac{\partial u}{\partial t}(x,t) + \frac{\partial}{\partial x} f(u(x,t)) = \frac{\partial^2 u}{\partial x^2}(x,t) \tag{1}$$

with variables $t \in \mathbb{R}_+^0$ and $x \in \mathbb{R}$ as well as functions $u : \mathbb{R} \times \mathbb{R}_+^0 \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. Such equations arise frequently in continuum physics and model the effects of nonlinear transport and diffusion. A viscous shock wave $u(x,t)$ is a traveling wave solution of (1),

$$u(x,t) := \bar{u}(\xi) \quad \text{with} \quad \xi := x - s \cdot t,$$

whose (viscous) profile $\bar{u} \in C^1(\mathbb{R}; \mathbb{R}^n)$ is transported with speed $s \in \mathbb{R}$ and approaches constant endstates $u^\pm := \lim_{\xi \to \pm\infty} \bar{u}(\xi)$. The profile $\bar{u}(\xi)$ is governed by an autonomous system of ordinary differential equations,

$$\frac{du}{d\xi}(\xi) = f(u(\xi)) - s \cdot u(\xi) - f(u^-) + s \cdot u^-, \tag{2}$$

and is equivalent to a heteroclinic orbit that connects the distinct stationary points u^- with u^+.

Due to translational invariance of the differential equations, a shifted profile also solves the profile equation (2). Therefore, a viscous shock wave is considered to be nonlinearly stable, if its perturbed profile approaches the manifold of heteroclinic orbits connecting the endstates $u^\pm$ asymptotically in time. It is a natural idea to study the nonlinear stability of viscous shock waves via the spectrum of its linearized evolution operator. A viscous shock wave is called spectrally stable, if the spectrum is confined to the left half-plane and the multiplicity of the eigenvalue zero equals the dimension of the manifold of heteroclinic orbits in the profile equation (2). Although the accumulation of the spectrum at the imaginary axis complicates the analysis, Zumbrun and collaborators proved that spectral stability of a viscous shock wave implies its nonlinear stability [ZH98, MZ04]. This implication holds for a viscous shock wave regardless of the magnitude of its amplitude, which is the distance between its endstates $u^{\pm}$. However, spectral stability of viscous shock waves has been proved only in the small amplitude case [FS02], whereas the large amplitude case remains wide open.

A possible strategy is to consider a viscous shock wave with small amplitude, which is spectrally stable, and to prove that no eigenvalue can move into the right half-plane as a parameter, such as the amplitude, varies. Since the spectrum accumulates at the origin, one has to distinguish between eigenvalues which move through the imaginary axis at the origin and away from the origin, respectively. In this regard, we investigate scenarios for the onset of instability and focus on the first situation.

Next, we give an outline of the thesis and state the main results. In the first chapter, we collect some basic facts about the existence and stability of traveling wave solutions in viscous conservation laws. Additionally, we discuss the Evans function approach to the spectral stability of viscous shock waves. This approach is based on a dynamical system reformulation of the eigenvalue problem, which has found many applications in related contexts [AGJ90, San02]. Briefly speaking, the Evans function is analytic away form the essential spectrum and its zeros correspond to eigenvalues. Moreover, the multiplicity of an isolated eigenvalue equals its order as a root of the Evans function. In the context of viscous shock waves the essential spectrum lies in the left half-plane and touches the imaginary axis at the origin. However, the Gap Lemma [GZ98, KS98] allows to continue the Evans function analytically into a small neighborhood of the origin. We give an alternative proof, where we exploit the slow-fast structure of the eigenvalue equation and use geometric singular perturbation theory [Fen79, Jon95, Szm91] to obtain the result. This idea was put forward by Freistühler and Szmolyan to construct and study the Evans bundles of weak shock waves [FS02].

Zumbrun and Howard based their spectral analysis on the resolvent kernel, rather than the resolvent. The effective spectrum is defined as the set of poles for the meromorphic continuation of the resolvent kernel into the essential spectrum. In particular, the effective spectrum coincides with the zero set of the analytic continuation of the Evans function and the multiplicity of an effective eigenvalue is equal to the order of the roots of the Evans function. Moreover, an effective eigenprojection with respect to a spectral parameter is defined via the residue of the resolvent kernel. The range of an effective eigenprojection is referred to as the effective eigenspace and its elements, the effective eigenfunctions, can be arranged in Jordan chains. In reference to the special position, effective eigenfunctions that decay exponentially in the limits $\xi \to \pm\infty$ are called genuine eigenfunctions [ZH98].

For a viscous shock wave associated to a Lax shock, the simplicity of the effective eigenvalue zero depends on the transversality of the profile and the Liu-Majda condition, which is necessary for dynamical stability of the Lax shock as a solution of the inviscid conservation law [Liu85, Maj83b, Maj83a, Maj84]. An effective eigenvalue can move through the origin only if the effective eigenvalue zero is not simple. Thus two possible scenarios for the onset of instability are the failure of the Liu-Majda condition and the occurrence of a non-transversal profile, which generically signals a bifurcation.

In the second chapter, we consider a viscous shock wave whose profile is non-transversal and associated to a Lax shock. First, we investigate its effective spectrum: The profile is lying in the intersection of invariant manifolds, $W^u(u^-)$ and $W^s(u^+)$, of the profile equation (2). Since the eigenvalue equation for the eigenvalue zero is related to the linearized profile equation, functions in the (two-dimensional) intersection of the tangent spaces, $T_{\bar{u}(\xi)}W^u(u^-) \cap T_{\bar{u}(\xi)}W^s(u^+)$, are genuine eigenfunctions to the eigenvalue zero. Thus the viscous shock wave is not spectrally stable. In accordance with the concept of effective spectrum, we show that the multiplicity of the eigenvalue zero is related to the existence of special bounded solutions of the generalized eigenvalue equation.

Second, we consider a viscous shock wave whose profile is governed by a family of profile equations

$$\frac{du}{d\xi}(\xi, \mu) = F(u(\xi, \mu), \mu),$$

where the smooth dependence of the vector field $F(u, \mu)$ on the parameter μ models the perturbative effects. In this way, the cases of a parameter dependent flux function of the viscous conservation law and dependence on the shock speed are covered.

A non-transversal profile $\bar{u}(\xi, \mu_0)$ may not persist for all parameter values μ close to μ_0. Melnikov theory is used to investigate this situation and we show that the existence of a non-transversal profile associated to a Lax shock indicates generically the occurrence of a saddle-node bifurcation of profiles with respect to the parameter μ. We describe the saddle-node bifurcation in a standard way such that the parameter μ, the family of profiles and the extended Evans function depend smoothly on a new parameter ν. If the Liu-Majda condition holds, then we are able to prove for the Evans function $E(\kappa, \nu)$ that a bifurcation occurs in the equation $E(\kappa, \nu) = 0$. In a neighborhood of the origin, the zero set consists of the line $\kappa = 0$ and a curve of eigenvalues $\kappa = \kappa(\nu)$, which change its sign as the parameter ν is varied. For ν such that the eigenvalue $\kappa(\nu)$ is positive, the associated viscous shock waves are unstable.

In the third chapter, we apply the outlined theory to examples motivated by planar waves in magnetohydrodynamics. Such planar waves are governed by a system of hyperbolic-parabolic conservation laws and the corresponding profile equation has a gradient like vector field, whose stationary points are hyperbolic [Ger59]. Freistühler and Szmolyan investigated the existence and bifurcation of profiles which are associated to intermediate non-degenerate shocks. They found a parameter range such that profiles exist and are generated in a global heteroclinic bifurcation [FS95]. We prove that the conjectured saddle-node bifurcation of profiles occurs and draw first conclusions on the spectral stability of the associated family of viscous shock waves. Subsequently, we consider a simplified model which has, besides reflectional invariance, an additional symmetry. In this example a saddle-node bifurcation occurs, where the associated viscous shock waves are not spectrally stable, since all of them exhibit an eigenvalue zero with multiplicity two. Finally, an appendix contains a short summary of Melnikov theory.

Previously, Kapitula [Kap99] has studied the point spectrum which is associated to traveling wave solutions of semi-linear parabolic equations under perturbations. The Evans function is used to describe the effects of the perturbation on the isolated eigenvalues, whose initial position and multiplicity is given. However, we are interested in the effective spectrum and try to obtain and interpret criteria for the existence and multiplicity of effective eigenvalues.

The close relation between spectral stability of traveling waves and the geometry of the traveling wave problem (transversality and orientation properties of the involved stable and unstable manifolds) goes back to Evans [Eva72, Eva73a, Eva73b, Eva75] and Jones [Jon84]. We are not aware of other work, where the bifurcation in the traveling wave problem is related directly to bifurcations in the equations defining the zero set of the Evans function close to the origin.

Zumbrun and his collaborators investigated transversal profiles and proposed the stability index, which determines the parity of the number of unstable eigenvalues [GZ98, BGSZ01, LZ04a, LZ04b]. It is a necessary, but not sufficient, stability criterion. In contrast, we consider non-transversal viscous shock waves associated to Lax shocks and study the existence of unstable eigenvalues directly.

Chapter 1

Viscous conservation laws

We collect some basic facts about hyperbolic viscous conservation laws and viscous shock waves. A viscous conservation law in one space dimension is a partial differential equation of the form

$$\frac{\partial u}{\partial t}(x,t) + \frac{\partial}{\partial x} f(u(x,t)) = \frac{\partial^2 u}{\partial x^2}(x,t) \tag{1.1}$$

with a spatial variable $x \in \mathbb{R}$ and a time variable $t \in \mathbb{R}^0_+$. The unknown function $u(x,t)$ takes its values in an open convex set $U \subseteq \mathbb{R}^n$ and the given non-linear flux function $f : U \to \mathbb{R}^n$ is smooth. We assume that the inviscid system

$$\frac{\partial u}{\partial t}(x,t) + \frac{\partial}{\partial x} f(u(x,t)) = 0 \tag{1.2}$$

is hyperbolic, i.e. the Jacobian of the flux function, $\frac{df}{du}(u)$, is diagonalizable with real eigenvalues for all $u \in U$. We are interested in a special kind of solutions.

Definition 1.1. A *traveling wave solution* $u(x,t)$ of system (1.1) has the form $u(x,t) := \bar{u}(\xi)$, where the variable ξ is defined as $\xi := x - st$ for some $s \in \mathbb{R}$ and the function $\bar{u} \in C^2(\mathbb{R};\mathbb{R}^n)$ is twice differentiable. A *viscous shock wave* $u(x,t)$ of system (1.1) is a traveling wave solution, whose *viscous profile* $\bar{u}(\xi)$ approaches asymptotically two distinct endstates $u^\pm := \lim_{\xi\to\pm\infty} \bar{u}(\xi)$.

The profile $\bar{u}(\xi)$ associated to a traveling wave solution is governed by the system of ODEs,

$$\frac{d^2u}{d\xi^2}(\xi) = -s\frac{du}{d\xi}(\xi) + \frac{d}{d\xi}f(u(\xi)). \tag{1.3}$$

In case of a viscous profile, integration with respect to ξ yields the *profile equation*

$$\frac{du}{d\xi}(\xi) = f(u(\xi)) - su(\xi) - c =: F(u(\xi)), \tag{1.4}$$

where the constant vector $c \in \mathbb{R}^n$ satisfies the identity

$$c = f(u^+) - su^+ = f(u^-) - su^-. \tag{1.5}$$

The viscous profile associated to a viscous shock wave corresponds to a heteroclinic orbit. It connects the endstates $u^\pm$, which are equilibria of the vector field $F(u)$. Since the profile equation is a system of autonomous ODEs, two orbits are either identical or do not intersect at all. Therefore, a viscous profile $\bar{u}(\xi)$ of system (1.4) is uniquely determined by a point of its orbit. Given a point $u_0 \in \mathbb{R}^n$, we will denote the corresponding viscous profile as $u(\xi; u_0)$, i.e. there exists a $\xi_0 \in \mathbb{R}$ such that $u(\xi_0; u_0) = u_0$.

Remark. [Smo83,Ser99,Daf05] The inviscid system of conservation laws (1.2) is obtained by neglecting the second order derivatives in the system (1.1). Typically, for non-linear flux functions, the associated Cauchy problem with smooth initial data yields classical solutions which exist only for a finite time. Hence, one is forced to consider weak solutions which allow jump discontinuities, i.e. shocks. In the simplest case, these are piecewise constant solutions

$$u(x) := \begin{cases} u^- & \text{for } x < st, \\ u^+ & \text{for } x > st, \end{cases} \tag{1.6}$$

whose parameters $(u^-, u^+; s)$ have to satisfy the Rankine-Hugoniot condition

$$f(u^+) - f(u^-) = s(u^+ - u^-). \tag{1.7}$$

It is apparent from (1.5) and (1.7) that a viscous profile is a smooth regularization of the shock solution (1.6).

A shock solution (1.6) of a system of hyperbolic conservation law (1.2) is called a Lax k-shock, if the real eigenvalues $\tilde{\lambda}_j(u^\pm)$ for $j = 1, \dots, n$ of the Jacobians $\frac{df}{du}(u^\pm)$ are ordered by increasing value and satisfy the inequalities

$$\tilde{\lambda}_{k-1}(u^-) < s < \tilde{\lambda}_k(u^-) \quad \text{and} \quad \tilde{\lambda}_k(u^+) < s < \tilde{\lambda}_{k+1}(u^+).$$

A finer classification is based on the *index* of the shock solution (1.6), which is the number of characteristics that enter the shock discontinuity.

Definition 1.2. A shock solution (1.6) of a system of hyperbolic conservation laws (1.2) is referred to as *undercompressive*, *Lax* or *overcompressive type* if the index of the shock solution is less than, equal to or greater than $n + 1$, where n is the dimension of the state space.

In the following we will assume

(A1) A viscous shock wave $u(x, t) = \bar{u}(\xi)$ of the system of hyperbolic viscous conservation laws (1.1) exists.

(A2) The shock speed s of the viscous shock wave $\bar{u}(\xi)$ is *non-characteristic*, i.e. the shock speed differs from any eigenvalue of the Jacobians $\frac{df}{du}(u^\pm)$.

Remark 1.1. The hyperbolicity of the system (1.1) and the assumption (A2) imply that the endstates $u^\pm$ are hyperbolic stationary points of the vector field $F(u)$. In particular, the Jacobians $\frac{dF}{du}(u^\pm)$ have non-zero real eigenvalues $\lambda_j(u^\pm)$ with associated eigenvectors $r_j(u^\pm)$ for $j = 1, \dots, n$.

In this situation the Hartman-Grobman theorem applies, which states that the flows of the profile equation (1.4) and its linearization about a hyperbolic stationary point,

$$\frac{du}{d\xi}(\xi) = \frac{dF}{du}(u^\pm)u(\xi),$$

are topologically conjugate, i.e. there exists a homeomorphism in a small neighborhood of the hyperbolic stationary point, which maps the trajectories of the profile equation onto trajectories of the linearized system. In addition, smooth stable manifolds

$$W^s(u^\pm) = \left\{u_0 \in \mathbb{R}^n \;\middle|\; \exists \text{ a solution } u(\xi; u_0) \text{ of (1.4)}: \lim_{\xi\to+\infty} u(\xi; u_0) = u^\pm\right\}$$

and smooth unstable manifolds

$$W^u(u^\pm) = \left\{u_0 \in \mathbb{R}^n \;\middle|\; \exists \text{ a solution } u(\xi; u_0) \text{ of (1.4)}: \lim_{\xi\to-\infty} u(\xi; u_0) = u^\pm\right\}$$

of the profile equation exist and are tangent to the respective stable and unstable subspace of the associated linear system. A viscous profile $\bar{u}(\xi)$ with endstates $u^\pm$ corresponds to a non-empty intersection of the stable manifold $W^s(u^+)$ and the unstable manifold $W^u(u^-)$. We recall the general definition of transversality.

Definition 1.3. The intersection of two smooth manifolds M and N, which are embedded in $\mathbb{R}^n$, is *transversal*, if for all points p in the intersection of the manifolds $M \cap N$ the sum of their tangent spaces spans $\mathbb{R}^n$, i.e.

$$\dim\left(T_pM + T_pN\right) = n.$$

Definition 1.4. A viscous profile $\bar{u}(\xi)$ is called *transversal*, if its orbit is a transversal intersection of the stable manifold $W^s(u^+)$ and the unstable manifold $W^u(u^-)$, that means for all points p on the heteroclinic orbit $\{\bar{u}(\xi) \mid \xi \in \mathbb{R}\}$ the identity

$$\dim\left(T_pW^s(u^+) + T_pW^u(u^-)\right) = n$$

holds.

A transversal, heteroclinic orbit will persist under small perturbations of the system. However, viscous profiles associated to an undercompressive shock are necessarily non-transversal.

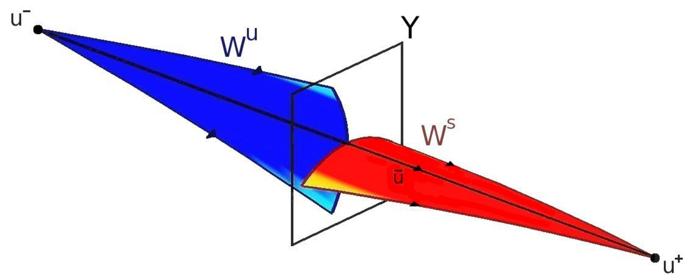

Figure 1.1: A profile which exists by a transversal intersection of the invariant manifolds.

In some examples of systems of viscous conservation laws (1.1) there exists for a pair of endstates (u^-, u^+) a manifold of viscous profiles, whose dimension is greater than one. This property has direct consequences on the (spectral) stability of viscous shock waves associated to such a viscous profile. For this reason, Zumbrun and collaborators introduced a new classification of viscous profiles.

Definition 1.5 ([HZ06]). Let l denote the dimension of the manifold of heteroclinic orbits connecting the endstates $u^\pm$ and the index i be the number of incoming characteristics for the underlying shock solution $(u^-, u^+; s)$. A viscous profile is classified as *pure undercompressive* type if the associated shock solution is undercompressive and $l = 1$, *pure Lax* type if the corresponding shock solution is Lax type and $l = i - n = 1$, and *pure overcompressive* type if the related shock solution is overcompressive and $l = i - n > 1$. Otherwise it is classified as *mixed under-overcompressive* type; see [LZ95, ZH98].

1.1 Stability of viscous shock waves

In the following, we study the stability of viscous shock waves under small perturbations of the viscous profile. We cast the viscous conservation law (1.1) in the moving coordinate frame $(x, t) \mapsto (\xi := x - st, t)$ and obtain an evolutionary system

$$\frac{du}{dt} = s\frac{du}{d\xi} - \frac{d}{d\xi}f(u) + \frac{d^2u}{d\xi^2} =: \mathcal{F}(u). \tag{1.8}$$

Thus the equation for a stationary solution, $0 = \mathcal{F}(u)$, is equivalent to the profile equation (1.3) and the viscous profiles connecting the endstates $u^\pm$ form a smooth manifold of stationary solutions. In order to study their stability, we have to specify an appropriate Banach space $\mathcal{B}$ of solutions and a subspace $\mathcal{A} \subseteq \mathcal{B}$ of admissible perturbations. We will consider classical solutions and choose the Banach space of twice differentiable functions $\mathcal{B} = C^2(\mathbb{R};\mathbb{R}^n)$ and the subspace $\mathcal{A} = C^2_{\exp}(\mathbb{R};\mathbb{R}^n)$ of functions with exponential decay to zero in the limits ξ to $\pm\infty$.

Definition 1.6. A viscous shock wave $u(x,t) = \bar{u}(\xi)$ is *non-linearly stable* with respect to $\mathcal{A}$, if every solution $u(\xi,t)$ of the Cauchy problem (1.8), with initial condition $u(\xi,0) = \bar{u}(\xi) + p(\xi)$ and sufficiently small perturbation $p \in \mathcal{A}$, approaches the manifold of heteroclinic orbits, that connect the endstates $u^\pm$, asymptotically in time.

The function $p(\xi,t) := u(\xi,t) - \bar{u}(\xi)$ describes the evolution of the initial perturbation $p \in \mathcal{A}$. By expanding the non-linear terms in the evolutionary system (1.8), we obtain a differential equation for functions $p(\xi,t)$ as

$$\frac{dp}{dt}(\xi,t) = \underbrace{\mathcal{F}(\bar{u}(\xi))}_{=0} + Lp(\xi,t) + R(p(\xi,t)) \tag{1.9}$$

with a linear operator

$$Lp := \frac{d\mathcal{F}}{du}(\bar{u})p = \frac{d}{d\xi}\left(\frac{dp}{d\xi} - \frac{dF}{du}(\bar{u})p\right) \tag{1.10}$$

and a non-linear function $R(p) = o(\|p\|^2)$. The linear part of system (1.9),

$$\frac{dp}{dt}(\xi,t) = Lp(\xi,t), \tag{1.11}$$

is a good approximation as long as the norm of the perturbation $p(\xi,t)$

remains small. The search for solutions of the linearized problem (1.11) of the form $p(\xi,t) = \exp(\kappa t)p(\xi)$ with $\kappa \in \mathbb{C}$ and p in a complex Banach space $\mathcal{X} \supset \mathcal{A}$ leads to the eigenvalue equation

$$Lp(\xi) = \kappa p(\xi).$$

In accordance with our choice for the Banach space of admissible perturbations, we will restrict the operator L to the space $\mathcal{X} := C^2_{\text{exp}}(\mathbb{R};\mathbb{C}^n)$.

Definition 1.7 ([GK69, Kat95]). A complex number $\kappa \in \mathbb{C}$ is an *eigenvalue* of the linear operator L, if there exists a function $p \neq 0$ in $\mathcal{X}$ such that $(L - \kappa I)p = 0$. We refer to the function p as *eigenfunction*.

An eigenvalue κ is *isolated*, if there exists a small neighborhood of κ, $B(\kappa)$, such that $(L - \tilde{\kappa} I)$ is invertible for all $\tilde{\kappa} \in B(\kappa)/\{\kappa\}$.

Suppose $\kappa \in \mathbb{C}$ is an isolated eigenvalue of the linear operator L, where the kernel of $(L - \kappa I)$ is one-dimensional. The eigenvalue κ has multiplicity $l \in \mathbb{N}$, if there exist functions $p_0 \equiv 0$ and $p_j \in \mathcal{X} \setminus \{p_0\}$ for $j = 1, \dots, l$ such that

$$(L - \kappa I)p_j = p_{j-1},$$

but there is no function $p_* \in \mathcal{X}$ with $(L - \kappa I)p_* = p_l$. The functions p_j for $j = 2, \dots, l$ are referred to as *generalized eigenfunctions*.

The multiplicity of an isolated eigenvalue $\kappa \in \mathbb{C}$, where the kernel of the operator $(L - \kappa I)$ has dimension $m \in \mathbb{N}$, is determined as the sum of the multiplicities of m linearly independent eigenfunctions $p_i \in \ker(L - \kappa I)$ for $i = 1, \dots, m$.

Definition 1.8 ([GK69, Kat95]). Let the linear operator $L : \mathcal{X} \to \mathcal{Y}$ be a map between complex Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. The *resolvent set* of L, $\rho(L)$, is the set of complex numbers κ such that $L - \kappa I$ has a bounded inverse. The resolvent function $R(\kappa) := (L - \kappa I)^{-1}$ is well-defined on $\rho(L)$.

The *spectrum* of L, $\sigma(L)$, is the complement of the resolvent set $\rho(L)$. The *point spectrum* of L, $\sigma_p(L)$, is the set of all isolated eigenvalues of L with finite multiplicity. The *essential spectrum* of L, $\sigma_{ess}(L)$, is the complement of the point spectrum within the spectrum, i.e. $\sigma_{ess}(L) = \sigma(L)/\sigma_p(L)$.

Lemma 1.1. *If the assumptions (A1) and (A2) hold, then the derivative of the viscous profile* $\frac{d\bar{u}}{d\xi}(\xi)$ *is an eigenfunction of the linear operator* L (1.10) *for the eigenvalue* $\kappa = 0$.

Proof. The set $\{\bar{u}(\xi+z) \mid z \in \mathbb{R}\}$ is a smooth manifold of stationary solutions of $\mathcal{F}(u)$ and the identity $\mathcal{F}(\bar{u}(\xi+z)) = 0$ holds for all $z \in \mathbb{R}$. We differentiate the last equation with respect to z at $z = 0$ and obtain

$$L\frac{d\bar{u}}{d\xi}(\xi) = \frac{d\mathcal{F}}{du}(\bar{u}(\xi))\frac{d\bar{u}}{d\xi}(\xi) = 0.$$

Since $\lim_{\xi\to\pm\infty} \frac{d\bar{u}}{d\xi}(\xi) = 0$ exponentially fast, we conclude that $\frac{d\bar{u}}{d\xi}(\xi)$ is an eigenfunction. $\square$

Due to translational invariance, the manifold of viscous profiles connecting the endstates $u^{\pm}$ is at least of dimension one. Each additional invariance implies the existence of another eigenfunction to the eigenvalue zero.

Definition 1.9. Let l denote the dimension of the manifold of viscous profiles that connect the endstates $u^{\pm}$. A viscous shock wave $u(x,t) = \bar{u}(\xi)$ is *spectrally stable*, if the linear operator $L = \frac{d\mathcal{F}}{du}(\bar{u})$ has no spectrum in the closed right half-plane $\overline{\mathbb{C}_+}$ except for an eigenvalue zero with multiplicity l.

Zumbrun and collaborators [ZH98, MZ02, MZ04] proved that a spectrally stable viscous shock wave is indeed non-linearly stable.

Essential spectrum

The linear operator $L = \frac{d\mathcal{F}}{du}(\bar{u})$ depends on the viscous profile $\bar{u}(\xi)$. Hence, it approaches asymptotically operators with constant coefficients as ξ tends to $\pm\infty$. For this reason the essential spectrum can be located by the following theorem.

Theorem 1.1 ([Hen81]). *The essential spectrum of* L *is sharply bounded to the right by* $\sigma_{ess}(L^+) \cup \sigma_{ess}(L^-)$, *where* $L^{\pm} = \frac{d}{d\xi}(\frac{dp}{d\xi} - \frac{dF}{du}(u^{\pm})p)$ *correspond to the operators obtained by linearizing* $\mathcal{F}(u)$ *about the constant solutions* $\bar{u} = u^{\pm}$.

Next, we locate the essential spectrum of the linear operator (1.10).

Theorem 1.2. *If the assumptions (A1) and (A2) hold, then the differential operator* $L = \frac{d\mathcal{F}}{du}(\bar{u})$*, associated to a profile* $\bar{u}(\xi)$*, has no essential spectrum in the punctured, closed right half-plane* $\overline{\mathbb{C}_+}^{\bullet} := \overline{\mathbb{C}_+} \setminus \{0\}$.

Proof. In order to locate the essential spectrum $\sigma_{ess}(L)$, we use the result of Theorem 1.1 and analyze the spectra of the operators

$$L^{\pm}p(\xi) = \frac{d}{d\xi}\left(\frac{dp}{d\xi}(\xi) - \frac{dF}{du}(u^{\pm})p(\xi)\right).$$

A linear operator with constant coefficients has no point spectrum, which implies $\sigma(L^{\pm}) = \sigma_{ess}(L^{\pm})$. An element $\kappa \in \sigma_{ess}(L^{\pm})$ is characterized by the equivalent properties

- The operator $L^{\pm} - \kappa I$ has no bounded inverse.
- The Fourier transform of the operator $L^{\pm} - \kappa I$ is not invertible.

The Fourier transform of the operators $L^{\pm} - \kappa I$ are given by

$$\mathbb{R} \to \mathbb{C}^n, \quad \theta \mapsto \left(-\theta^2 I - i\theta\frac{dF}{du}(u^{\pm}) - \kappa I\right)$$

and we loose invertibility if the right-hand side is singular. Thus a complex number κ is an element of $\sigma_{ess}(L^{\pm})$ if and only if for some $\theta \in \mathbb{R}$ the identity

$$\det\left(-\theta^2 I - i\theta\frac{dF}{du}(u^{\pm}) - \kappa I\right) = 0 \tag{1.12}$$

holds. The Jacobians $\frac{dF}{du}(u^{\pm})$ are diagonalizable with real eigenvalues $\lambda_j(u^{\pm})$ for $j = 1, \ldots, n$ and we obtain the determinant as a finite product

$$\prod_{j=1}^{n}\big(-\theta^2 - i\theta\lambda_j(u^{\pm}) - \kappa\big) = 0. \tag{1.13}$$

The equation (1.13) is satisfied if a single factor vanishes, which happens for spectral parameters $\kappa_j^{\pm}(\theta) := -\theta^2 - i\theta\lambda_j(u^{\pm})$ with $\theta \in \mathbb{R}$ and $j = 1, \ldots, n$.

This defines $2n$ curves

$$\kappa_j^{\pm} := \left\{\kappa \in \mathbb{C} \;\middle|\; \kappa = -\theta^2 - i\theta\lambda_j(u^{\pm}) \text{ for } \theta \in \mathbb{R}\right\}, \tag{1.14}$$

which are parabolas associated to the eigenvalues $\lambda_j(u^{\pm})$ for $j = 1, \ldots, n$. They are contained in the left half-plane and touch the imaginary axis only in the origin, see Figure 1.1. Their unions form the essential spectra $\sigma_{ess}(L^{\pm})$,

$$\sigma_{ess}(L^{\pm}) = \bigcup_{j=1}^{n} \kappa_j^{\pm}.$$

Thus Theorem 1.1 implies that the essential spectrum $\sigma_{ess}(L)$ is bounded to the right by the curves (1.14), which completes the argument. $\square$

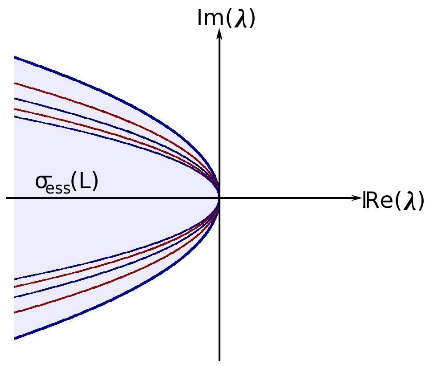

Figure 1.2: The essential spectrum $\sigma_{ess}(L)$ is bounded to the right by the spectrum of $\sigma_{ess}(L^+)$ and $\sigma_{ess}(L^-)$.

1.2 Evans function $E(\kappa)$

In the last section we proved that the essential spectrum does not intersect $\overline{\mathbb{C}_+}^{\bullet}$. Hence, the point spectrum will decide upon spectral stability of a viscous shock wave. Starting with the work of Evans, it became popular to study the spectrum related to a traveling wave solution via a dynamical system approach [Eva72, Eva73a, Eva73b, Eva75]. Soon, the connection between the profile equation and the eigenvalue equation became apparent [Jon84]. Alexander, Gardner and Jones developed a method to locate the point spectrum related to traveling wave solutions in reaction-diffusion equations [AGJ90]. This approach is also applicable to other parabolic equations, notably viscous conservation laws, and is now known as Evans function theory. We refer to the survey of Sandstede [San02] on the stability of traveling waves and references therein.

The point spectrum consists of isolated eigenvalues of finite multiplicity. A pair of an eigenvalue $\kappa \in \mathbb{C}$ and an eigenfunction $p \in C^2_{\exp}(\mathbb{R};\mathbb{C}^n)$ has to satisfy the identity

$$Lp(\xi) = \frac{d}{d\xi}\left(\frac{dp}{d\xi}(\xi) - \frac{dF}{du}(\bar{u}(\xi))p(\xi)\right) = \kappa p(\xi).$$

We consider the variables $(p, q := \frac{dp}{d\xi} - \frac{dF}{du}(\bar{u})p)(\xi)$ and rewrite the equation as a system of first order ODEs

$$\frac{d}{d\xi}\begin{pmatrix} p \\ q \end{pmatrix}(\xi) = \begin{pmatrix} \frac{dF}{du}(\bar{u}(\xi)) & I_n \\ \kappa I_n & 0_n \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}(\xi). \tag{1.15}$$

Thus the eigenvalue problem is to find a complex number $\kappa \in \overline{\mathbb{C}_+}^{\bullet}$ and a non-trivial function $\binom{p}{q} \in C^1_{\exp}(\mathbb{R};\mathbb{C}^{2n})$ such that (1.15) is satisfied. The matrix of the linear ODE,

$$\mathbb{A}(\xi,\kappa) := \begin{pmatrix} \frac{dF}{du}(\bar{u}(\xi)) & I_n \\ \kappa I_n & 0_n \end{pmatrix},$$

is analytic in κ and differentiable in ξ, because $F(u)$ is smooth and the viscous profile $\bar{u}(\xi)$ is differentiable. Since a viscous profile approaches constant endstates $u^{\pm} = \lim_{\xi\to\pm\infty} \bar{u}(\xi)$, the coefficients of the matrix $\mathbb{A}(\xi,\kappa)$ approach constants as ξ tends to $\pm\infty$ and we denote the limits of $\mathbb{A}(\xi,\kappa)$ with

$$\mathbb{A}^{\pm}(\kappa) := \begin{pmatrix} \frac{dF}{du}(u^{\pm}) & I_n \\ \kappa I_n & 0_n \end{pmatrix}.$$

Definition 1.10. A domain $\Omega \subset \mathbb{C}$ has *consistent splitting* if there exists a number $l \in \mathbb{N}$ such that for all $\kappa \in \Omega$ the matrices $\mathbb{A}^{\pm}(\kappa)$ have l eigenvalues with positive real part and $2n - l$ eigenvalues with negative real part.

The matrices $\mathbb{A}^{\pm}(\kappa)$ have pure imaginary eigenvalues precisely for spectral parameters κ which lie on the curves $\kappa_j^{\pm}$ in (1.14). The essential spectrum is contained in the region to the left of the union of these curves and is tangent to the imaginary axis at $\kappa = 0$.

Theorem 1.3. *Suppose the assumptions (A1) and (A2) hold. Then the punctured, closed right half-plane* $\overline{\mathbb{C}_+}^{\bullet} := \overline{\mathbb{C}_+} \setminus \{0\}$ *has consistent splitting with splitting index* $l = n$*. In particular, the matrices* $\mathbb{A}^{\pm}(\kappa)$ *have eigenvalues*

$$\mu_j^{\mp}(u^{\pm}, \kappa) = \frac{\lambda_j(u^{\pm})}{2} \mp \sqrt{\left(\frac{\lambda_j(u^{\pm})}{2}\right)^2 + \kappa}, \qquad \text{for} \quad j = 1, \ldots, n, \tag{1.16}$$

and associated eigenvectors

$$V_j^{\mp}(u^{\pm}, \kappa) = \begin{pmatrix} r_j(u^{\pm}) \\ -\mu_j^{\pm}(u^{\pm}, \kappa) r_j(u^{\pm}) \end{pmatrix}, \qquad \text{for} \quad j = 1, \ldots, n, \tag{1.17}$$

which are analytic in κ *in the domain* $\overline{\mathbb{C}_+}^{\bullet}$*. Moreover, the eigenvalues satisfy for all* $j = 1, \ldots, n$ *the inequality*

$$\mathrm{Re}(\mu_j^{-}(u^{\pm}, \kappa)) < 0 < \mathrm{Re}(\mu_j^{+}(u^{\pm}, \kappa)). \tag{1.18}$$

The projection operators of the associated stable spaces $S^{\pm}(\kappa)$ *and unstable spaces* $U^{\pm}(\kappa)$ *are analytic in* $\kappa \in \overline{\mathbb{C}_+}^{\bullet}$*, too.*

Proof. The eigenvalue equation associated to the matrix $\mathbb{A}^{\pm}(\kappa)$ can be written as

$$\det\left(\mathbb{A}^{\pm}(\kappa) - \mu I_{2n}\right) = \prod_{j=1}^{n} \left(\left(\lambda_j(u^{\pm}) - \mu\right)(-\mu) - \kappa\right) = 0,$$

since the Jacobians $\frac{dF}{du}(u^{\pm})$ are diagonalizable with real eigenvalues $\lambda_j(u^{\pm})$

for $j = 1, \dots, n$. Thus an eigenvalue μ has to fulfill

$$\mu^2 - \lambda_j(u^\pm)\mu - \kappa = 0 \tag{1.19}$$

and we obtain the expressions

$$\mu_j^\mp(u^\pm, \kappa) = \frac{\lambda_j(u^\pm)}{2} \mp \sqrt{\left(\frac{\lambda_j(u^\pm)}{2}\right)^2 + \kappa}$$

for $j = 1, \dots, n$. For a pure imaginary eigenvalue, $\mu = i\theta$ with $\theta \in \mathbb{R}$, the identity (1.19) is equivalent to the defining equation of the curves $\kappa_j^\pm$ in (1.14), which do not intersect with $\overline{\mathbb{C}_+}^\bullet$ by the result of Theorem 1.2. In addition, the eigenvalues $\mu_j^\mp(u^\pm, \kappa)$ of $\mathbb{A}^\pm(\kappa)$ are continuous in κ, which proves that the domain $\overline{\mathbb{C}_+}^\bullet$ exhibits consistent splitting. In order to determine the splitting index, we consider κ to be real and positive. Hence, the eigenvalues $\mu_j^\pm(u^\pm, \kappa)$ are real and their product $\mu_j^-(u^\pm, \kappa)\mu_j^+(u^\pm, \kappa) = -\kappa$ is negative. We infer that the eigenvalues $\mu_j^\pm(u^\pm, \kappa)$ have opposite signs. Consequently, the matrices $\mathbb{A}^\pm(\kappa)$ have n positive eigenvalues and n negative eigenvalues for $\kappa \in \mathbb{R}_+$. Since $\overline{\mathbb{C}_+}^\bullet$ has consistent splitting, the number of eigenvalues with positive and negative real part respectively is constant for $\kappa \in \overline{\mathbb{C}_+}^\bullet$. This proves the identity (1.18).

A direct calculation shows that $V_j^\mp(u^\pm, \kappa)$ are indeed eigenvectors to the eigenvalues $\mu_j^\mp(u^\pm, \kappa)$. The analytic dependence on the spectral parameter of the eigenvalues induces the one of the eigenvectors. The identity (1.18) proves that a spectral gap between $S^\pm(\kappa)$ and $U^\pm(\kappa)$ persists and we conclude from standard matrix perturbation theory, see [Kat95], the analytic dependence of the projections on $\kappa \in \overline{\mathbb{C}_+}^\bullet$. □

In view of the hyperbolicity of $\mathbb{A}^\pm(\kappa)$ for all $\kappa \in \overline{\mathbb{C}_+}^\bullet$, we conclude that an eigenfunction associated to an isolated eigenvalue necessarily decays to zero as ξ tends to $\pm\infty$. Thus the concept of exponential dichotomy for linear systems, see Definition A.2, is a useful tool to study the existence of eigenfunctions.

Lemma 1.2. *Let the assumptions (A1) and (A2) hold. Then the linear system of ODEs*

$$\frac{d}{d\xi}\begin{pmatrix} p \\ q \end{pmatrix}(\xi) = \begin{pmatrix} \frac{dF}{du}(\bar{u}(\xi)) & I_n \\ \kappa I_n & 0_n \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}(\xi), \tag{1.20}$$

exhibits for all $\kappa \in \overline{\mathbb{C}_+}^{\bullet}$ *exponential dichotomies on* $\mathbb{R}_+$ *and* $\mathbb{R}_-$*, respectively. The associated family of projections will be denoted as* $P_+(\xi, \kappa)$ *with* $\xi \in \mathbb{R}_+$ *and* $Q_-(\xi, \kappa)$ *with* $\xi \in \mathbb{R}_-$*, respectively.*

The statement is a consequence of the hyperbolicity of the matrices $\mathbb{A}^{\pm}(\kappa)$ for all $\kappa \in \overline{\mathbb{C}_+}^{\bullet}$ and the roughness property of exponential dichotomies, see [Cop78, chapter 4].

Definition 1.11. Let the assumptions (A1) and (A2) hold. We define for all $\kappa \in \overline{\mathbb{C}_+}^{\bullet}$ the *stable space* $W_0^s(\kappa) := \mathrm{image}(P_+(0, \kappa))$ and the *unstable space* $W_0^u(\kappa) := \mathrm{image}(Q_-(0, \kappa))$ via the projections $P_+(0, \kappa)$ and $Q_-(0, \kappa)$ from Lemma 1.2.

Lemma 1.3. *Let the assumptions (A1) and (A2) hold. Then the stable space* $W_0^s(\kappa)$ *has for* $\kappa \in \overline{\mathbb{C}_+}^{\bullet}$ *the following properties:*

1. *It consists of all initial values* $\binom{p_0}{q_0} \in \mathbb{C}^{2n}$ *such that there exists a solution* $\binom{p}{q}(\xi)$ *of* (1.15)*, that satisfies*

$$\begin{pmatrix} p \\ q \end{pmatrix}(0) = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \qquad \textit{and} \qquad \lim_{\xi \to +\infty}\begin{pmatrix} p \\ q \end{pmatrix}(\xi) = 0.$$

   In addition, $\dim_{\mathbb{C}} W_0^s(\kappa) = \dim_{\mathbb{C}} \mathrm{image}(P_+(0, \kappa)) = n$ *holds.*

2. *It is possible to choose a basis* $\{\eta_j^s(0, \kappa) \mid j = 1, \dots, n\}$ *for the stable space* $W_0^s(\kappa)$*, which is analytic in* κ*. The associated solutions* $\eta_j^s(\xi, \kappa)$ *of system* (1.15) *are analytic in* κ *and satisfy* $\lim_{\xi \to +\infty} \eta_j^s(\xi, \kappa) = 0$.

Proof. The first statement is a direct consequence of the properties of an exponential dichotomy on $\mathbb{R}_+$ and its associated projection $P_+(0, \kappa)$. In particular, the dimension of the image of $P_+(0, \kappa)$ equals the number of eigenvalues of $\mathbb{A}^+(\kappa)$ with negative real part, which is n by the results of Theorem 1.3. An analytic basis of $W_0^s(\kappa)$ can be constructed by a standard procedure, see [Kat95, chapter II.4.2.]. Its associated solutions of system (1.15) inherit the analytic dependence. □

Similar results are obtained for the unstable space $W_0^u(\kappa)$.

Lemma 1.4. *Let the assumptions (A1) and (A2) hold. Then the unstable space $W_0^u(\kappa)$ has for $\kappa \in \overline{\mathbb{C}_+}^{\bullet}$ the following properties:*

1. *The unstable space $W_0^u(\kappa)$ consists of all initial values $\binom{p_0}{q_0} \in \mathbb{C}^{2n}$ such that there exists a solution $\binom{p}{q}(\xi)$ of (1.15), that satisfies*

$$\begin{pmatrix} p \\ q \end{pmatrix}(0) = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \qquad \textit{and} \qquad \lim_{\xi \to -\infty} \begin{pmatrix} p \\ q \end{pmatrix}(\xi) = 0.$$

   In addition, $\dim_{\mathbb{C}} W_0^u(\kappa) = \dim_{\mathbb{C}} \operatorname{image}(Q_-(0,\kappa)) = n$ holds.

2. *It is possible to choose a basis $\{\eta_j^u(0,\kappa) \mid j = 1,\ldots,n\}$ for the unstable space $W_0^u(\kappa)$, which is analytic in κ. The associated solutions $\eta_j^u(\xi,\kappa)$ of system (1.15) are analytic in κ and satisfy $\lim_{\xi\to-\infty} \eta_j^u(\xi,\kappa) = 0$.*

The existence of an eigenfunction is equivalent to a non-trivial intersection of the spaces $W_0^s(\kappa)$ and $W_0^u(\kappa)$. In order to detect such an intersection, we will study the Evans function.

Definition 1.12. Let the assumptions (A1) and (A2) hold. The *Evans function*, $E : \mathbb{R} \times \overline{\mathbb{C}_+}^{\bullet} \to \mathbb{C}$, $(\xi,\kappa) \to E(\xi,\kappa)$, is defined as

$$E(\xi,\kappa) := \exp\left(-\int_0^\xi \operatorname{trace}(\mathbb{A}(x,\kappa))dx \right) \det\left(\eta_1^u,\ldots,\eta_n^u,\eta_1^s,\ldots,\eta_n^s\right)(\xi,\kappa),$$

with functions $\eta_j^{s/u}(\xi,\kappa)$ for $j = 1,\ldots,n$ from the Lemmata 1.3 and 1.4.

The Evans function is a Wronskian determinant, which suggests its independence of ξ.

Theorem 1.4 (Abel-Liouville-Jacobi-Ostrogradskii identity [CL55]). *Let A be an n-by-n matrix with continuous elements on an interval $I = [a, b] \subset \mathbb{R}$, and suppose $\Phi(t)$ is a matrix of functions on I satisfying*

$$\frac{d\Phi}{dt}(t) = A(t)\Phi(t), \quad \text{for all} \quad t \in I.$$

Then the determinant of $\Phi(t)$ satisfies on I the first-order equation

$$\frac{d}{dt}(\det \Phi(t)) = \mathrm{trace}(A(t))(\det \Phi(t))$$

and thus for $\tau, t \in I$

$$\det \Phi(t) = \det \Phi(\tau) \exp \int_\tau^t \mathrm{trace}(A(s))ds.$$

Remark 1.2. The functions $\eta_j^{s/u}(\xi, \kappa)$ for $j = 1, \ldots, n$ satisfy the eigenvalue equation and we observe from the result of Theorem 1.4 for any $\xi_0 \in \mathbb{R}$ that

$$\begin{aligned} E(\xi, \kappa) &= \exp\left(- \int_0^\xi \mathrm{trace}(\mathbb{A}(x, \kappa))dx \right) \det\left(\eta_1^u, \ldots, \eta_n^u, \eta_1^s, \ldots, \eta_n^s\right)(\xi, \kappa) \\ &= \exp\left(- \int_0^{\xi_0} \mathrm{trace}(\mathbb{A}(x, \kappa))dx \right) \det\left(\eta_1^u, \ldots, \eta_n^u, \eta_1^s, \ldots, \eta_n^s\right)(\xi_0, \kappa). \end{aligned}$$

Therefore, we consider the Evans function without loss of generality at $\xi = 0$, i.e. $E(\kappa) := \det\left(\eta_1^u, \ldots, \eta_n^u, \eta_1^s, \ldots, \eta_n^s\right)(0, \kappa)$.

Theorem 1.5 ([ZH98]). *Let the assumptions (A1) and (A2) hold. The Evans function in Definition 1.12 has the following properties*

1. *$E(\kappa)$ is analytic in κ for $\kappa \in \overline{\mathbb{C}_+}^{\bullet}$ and independent of ξ.*

2. *$E(\kappa_0) = 0$ if and only if $\kappa_0 \in \sigma_p(L)$.*

3. *The algebraic multiplicity of the eigenvalue $\kappa_0 \in \sigma_p(L)$ equals its order as a root of the Evans function.*

Remark. The Evans function approach was introduced in the setting of reaction -diffusion equations. In this case the properties of the Evans function in a domain of consistent splitting, as stated in Theorem 1.5, have been proved in the article [AGJ90].

1.2.1 Analytic continuation of the Evans function

In the stability analysis of a viscous shock wave, we need to locate the point spectrum within the closed right half-plane $\overline{\mathbb{C}_+}$. However, the Evans function is only well-defined away from the essential spectrum, which lies in the left half-plane and touches the imaginary axis at the origin, see Theorem 1.2. Nonetheless, the Evans function can be analytically continued into a small neighborhood of $\kappa = 0$.

Definition 1.13. Suppose that U and S are complementary $\mathbb{A}$-invariant subspaces for some quadratic matrix $\mathbb{A} \in \mathbb{C}^{n\times n}$. The *spectral gap* of U and S is defined as the difference between the minimum real part of the eigenvalues of $\mathbb{A}$ restricted to U and the maximum real part of the eigenvalues of $\mathbb{A}$ restricted to S.

The unstable space $U^-(\kappa)$ and the stable space $S^-(\kappa)$ of the linear system

$$\frac{dp}{d\xi}(\xi) = \mathbb{A}^-(\kappa)p(\xi)$$

have a positive spectral gap for any κ in the domain $\overline{\mathbb{C}_+}^{\bullet}$. Therefore, the solution manifold of $\frac{dp}{d\xi}(\xi) = \mathbb{A}(\xi,\kappa)p(\xi)$ that approaches the space $U^-(\kappa)$ as ξ tends to $-\infty$ can be uniquely determined. The same reasoning applies to the stable space $S^+(\kappa)$ of the linear system with matrix $\mathbb{A}^+(\kappa)$. Thus the Evans function is well-defined in the domain $\overline{\mathbb{C}_+}^{\bullet}$.

In the present case, the respective spectral gaps become negative as soon as κ enters the essential spectrum and the proper extension of the stable and unstable manifolds is not obvious. However, the differential forms associated to the invariant manifolds distinguish themselves by their maximal rate of convergence to the differential forms related to the respective asymptotic spaces, $U^-(\kappa)$ and $S^+(\kappa)$. This idea was put forward in the Gap Lemma, which has been proved independently by Gardner and Zumbrun [GZ98] in the setting of viscous conservation laws, as well as Kapitula and Sandstede [KS98] for dissipative equations. The results on the existence of an analytic continuation of the Evans function is summarized in following statements.

Theorem 1.6 ([GZ98]). *Let $\mathbb{A}(\xi,\kappa)$ satisfy*

H1 $\overline{\mathbb{C}_+}^{\bullet}$ *has consistent splitting with respect to* $\mathbb{A}^{\pm}(\kappa)$.

H2 exponential convergence of $\mathbb{A}(\xi,\kappa)$ *to* $\mathbb{A}^{\pm}(\kappa)$ *as* $\xi \to \pm\infty$ *with exponential rate* $\alpha > 0$, *uniformly for* κ *in compact sets.*

H3 geometric separation: The eigenvalues $\mu_j(\kappa)$ *of* $\mathbb{A}^{\pm}(\kappa)$ *and the spectral projection operators* $P_S(\kappa)$ *associated to* $S(\kappa)$ *and* $P_U(\kappa)$ *associated to* $U(\kappa)$ *for* $\kappa \in \overline{\mathbb{C}_+}^{\bullet}$ *continue analytically to a simply connected domain* Ω *containing the right half-plane and a small neighborhood of the origin. Furthermore, the associated continuations* $S(\kappa) = P_S(\kappa)\mathbb{C}^{2n}$ *and* $U(\kappa) = P_U(\kappa)\mathbb{C}^{2n}$ *complement each other in* $\mathbb{C}^{2n}$ *for* $\kappa \in \Omega$.

H4 gap condition: $\beta(\kappa) > -\alpha$ *for all* $\kappa \in \Omega$, *where* $\beta(\kappa)$ *is the spectral gap of the pair* $U(\kappa)$ *and* $S(\kappa)$.

Then there is an analytic extension of the Evans function $E(\kappa)$ *to* Ω, *which is unique up to a non-vanishing, analytic factor.*

Lemma 1.5 ([GZ98]). *If the assumptions of Theorem 1.6 hold and in addition* $\overline{\mathbb{A}(\xi,\overline{\kappa})} = \mathbb{A}(\xi,\kappa)$ *is satisfied, where* $\overline{\kappa}$ *denotes complex conjugation, then*

1. *there exist bases* $\{\eta_j^s(\kappa) \mid j = 1,\dots,n\}$ *and* $\{\eta_j^u(\kappa) \mid j = 1,\dots,n\}$ *for the spaces* $S^+(\kappa)$ *and* $U^-(\kappa)$, *respectively, which depend analytically on* κ *for* $\kappa \in \Omega$ *and are real-valued vectors for real* $\kappa \geq 0$.

2. *the Evans function* $E(\kappa)$ *of Theorem 1.6 can be chosen to be real-valued for real* $\kappa \geq 0$.

We are interested in the connection between the multiplicity of the eigenvalue zero and its order as a root of the continued Evans function. Therefore, we need to obtain analytic continuations of the individual vectors, which exist at least locally for κ in a small neighborhood of the origin. This was achieved in the articles [GZ98, ZH98, LZ04a]. We will give an alternative derivation via geometric singular perturbation theory. First, we note preliminary results about the analytic continuation of individual eigenvalues and eigenvectors of the matrices $\mathbb{A}^{\pm}(\kappa)$ with constant coefficients.

Theorem 1.7. *Let the assumptions (A1) and (A2) hold. Then the eigenvalues $\mu_j^{\mp}(u^{\pm},\kappa)$ and the eigenvectors $V_j^{\mp}(u^{\pm},\kappa)$ for $j=1,\dots,n$ in Theorem 1.3 admit an analytic continuation into a small neighborhood of the origin $B_\delta(0) := \{\kappa\in\mathbb{C} \mid |\kappa|<\delta\}$ with radius δ such that*

$$0<\delta<\min\left\{\left(\frac{\lambda_j(u)}{2}\right)^2 \,\middle|\, j=1,\dots,n, \quad u=u^{\pm}\right\}.$$

In this domain the spaces

$$S^+(\kappa) := \oplus_{j=1}^n V_j^-(u^+,\kappa) \text{ and } U^+(\kappa) := \oplus_{j=1}^n V_j^+(u^+,\kappa),$$

as well as the spaces

$$S^-(\kappa) := \oplus_{j=1}^n V_j^-(u^-,\kappa) \text{ and } U^-(\kappa) := \oplus_{j=1}^n V_j^+(u^-,\kappa),$$

complement each other in $\mathbb{C}^{2n}$. The associated projection operators, $P_{S^\pm(\kappa)}$ and $P_{U^\pm(\kappa)}$, are analytic in $\kappa\in B_\delta(0)$, too.

Proof. The expressions for the eigenvalues

$$\mu_j^{\mp}(u^{\pm},\kappa) = \frac{\lambda_j(u^{\pm})}{2} \mp \sqrt{\left(\frac{\lambda_j(u^{\pm})}{2}\right)^2+\kappa}, \qquad \text{for} \quad j=1,\dots,n,$$

and the associated eigenvectors

$$V_j^{\mp}(u^{\pm},\kappa) = \begin{pmatrix} r_j(u^{\pm}) \\ -\mu_j^{\pm}(u^{\pm},\kappa) r_j(u^{\pm}) \end{pmatrix}, \qquad \text{for} \quad j=1,\dots,n,$$

are analytic in κ as long as $|\kappa|<\delta$. Therefore, the stated spaces and their associated projection operators will be analytic in the domain $B_\delta(0)$. Since the eigenvalues $\mu_j^+(u^\pm,\kappa)$ and $\mu_j^-(u^\pm,\kappa)$ are distinct for $\kappa \in B_\delta(0)$, the vectors $V_j^\mp(u^+,\kappa)$ as well as $V_j^\mp(u^-,\kappa)$ will remain linearly independent for $j = 1,\dots,n$. Hence, the spaces $S^+(\kappa)$ and $U^+(\kappa)$, as well as $S^-(\kappa)$ and $U^-(\kappa)$, will complement each other in $\mathbb{C}^{2n}$ for all $\kappa \in B_\delta(0)$. □

In order to understand the dynamics of the eigenvalue equation (1.15) better, we will augment it with the profile equation (1.4). The augmented system is singularly perturbed at $\kappa = 0$ and exhibits a slow-fast structure, which we explore to prove the existence of an extension for the invariant manifolds $W^u(\kappa)$ and $W^s(\kappa)$.

Theorem 1.8. *Suppose the assumptions (A1) and (A2) hold and the indices k^- and k^+ are such that the real eigenvalues $\lambda_j(u^\pm)$ for $j = 1,\dots,n$ of the Jacobians $\frac{dF}{du}(u^\pm)$ are in increasing order of magnitude and satisfy the inequalities*

$$\lambda_{k^-}(u^-) < 0 < \lambda_{k^-+1}(u^-) \quad \text{and} \quad \lambda_{k^+}(u^+) < 0 < \lambda_{k^++1}(u^+),$$

respectively. Then the augmented system

$$\begin{aligned}
\frac{du}{d\xi}(\xi) &= F(u(\xi)),\\
\frac{dp}{d\xi}(\xi) &= \frac{dF}{du}(u(\xi))p(\xi) + q(\xi),\\
\frac{dq}{d\xi}(\xi) &= \kappa p(\xi),
\end{aligned} \tag{1.21}$$

has stationary points $U^\pm = (u^\pm, 0, 0)$. For κ in a small neighborhood of the origin, there exists an invariant manifold $W^s(U^+)$, which is the stable manifold to the stationary point U^+ as long as $\kappa \in \overline{\mathbb{C}_+}^{\bullet}$. The invariant manifold $W^s(U^+)$ has a decomposition into a slow manifold

$$W^{s,slow}(U^+) = \left\{ (u,p,q)^t \in \mathbb{C}^{3n} \;\middle|\; u = u^+, \quad p = -\left(\frac{dF}{du}(u^+)\right)^{-1} q, \right.$$
$$\left. q \in \operatorname{span}\{r_{k^++1}(u^+),\dots,r_n(u^+)\} \right\}$$

and a fast manifold $W^{s,fast}(U^+)$ *with tangent space*

$$T_{U^+}W^{s,fast}(U^+) = \{(u,p,q)^t \in \mathbb{C}^{3n} \mid u,p \in \operatorname{span}\{r_1(u^+),\dots,r_{k^+}(u^+)\},\ q=0\}.$$

Similarly, for κ *in a small neighborhood of the origin, there exists an invariant manifold* $W^u(U^-)$*, which is the unstable manifold to the stationary point* U^- *as long as* $\kappa \in \overline{\mathbb{C}_+}^{\bullet}$*. The invariant manifold* $W^u(U^-)$ *has a decomposition into a slow manifold*

$$W^{u,slow}(U^-) = \Bigg\{(u,p,q)^t \in \mathbb{C}^{3n} \ \Bigg|\ u=u^-, \quad p = -\left(\frac{dF}{du}(u^-)\right)^{-1} q, \\ q \in \operatorname{span}\{r_1(u^-),\dots,r_{k^-}(u^-)\}\Bigg\}$$

and a fast manifold $W^{u,fast}(U^-)$ *with tangent space*

$$T_{U^-}W^{u,fast}(U^-) = \{(u,p,q)^t \in \mathbb{C}^{3n} \mid u,p \in \operatorname{span}\{r_{k^-+1}(u^-),\dots,r_n(u^-)\},\ q=0\}.$$

Proof. We observe that the augmented system (1.21), which is made up of the profile equation (1.4) and the eigenvalue equation (1.15), has stationary points $U^\pm = (u^\pm,0,0)^t$ and is singularly perturbed at $\kappa = 0$. We will construct the invariant manifolds for κ in a small neighborhood of the origin and use the parametrization $\kappa = \rho\exp(i\phi)$ with $\rho \in [0,\delta]$ and $\phi = [0,2\pi[$. The manifold of equilibria for $\kappa = 0$ is given by $M_0 = M_0^- \cup M_0^+$ with

$$M_0^\pm := \left\{(u,p,q)^t \in \mathbb{C}^{3n} \ \middle|\ u=u^\pm, \quad p = -\left(\frac{dF}{du}(u^\pm)\right)^{-1} q, \quad q \in \mathbb{C}^n\right\}.$$

The critical manifolds $M_0^\pm$ are normally hyperbolic, since the linearization of the augmented system at any point $(\tilde{u}, \tilde{p}, \tilde{q})^t \in M_0^{\pm}$ and $\kappa = 0$ yields,

$$\begin{aligned}
\frac{du}{d\xi}(\xi) &= \frac{dF}{du}(\tilde{u})u(\xi),\\
\frac{dp}{d\xi}(\xi) &= \frac{d^2F}{du^2}(\tilde{u})(u, \tilde{p})(\xi) + \frac{dF}{du}(\tilde{u})p(\xi) + q(\xi),\\
\frac{dq}{d\xi}(\xi) &= 0.
\end{aligned}$$

The linearized vector field has exactly $n = \dim(M_0^{\pm})$ eigenvalues with zero real-part. Thus geometric singular perturbation theory [Fen79, Jon95, Szm91] applies. At first, we will construct the invariant manifold $W^s(U^+)$ for $\rho = 0$ and note that it can be decomposed into two invariant manifolds, one within the critical manifold M_0^+ and another one which approaches M_0^+ exponentially fast. The equations on the slow time scale $\tau := \rho\xi$ are

$$\begin{aligned}
\rho\frac{du}{d\tau}(\tau) &= F(u(\tau)),\\
\rho\frac{dp}{d\tau}(\tau) &= \frac{dF}{du}(u(\tau))p(\tau) + q(\tau),\\
\frac{dq}{d\tau}(\tau) &= \exp(i\phi)p(\tau).
\end{aligned}$$

The reduced problem $\rho = 0$ is only defined on M_0 and the slow flow on M_0^+ is governed by

$$\frac{dq}{d\tau}(\tau) = -\exp(i\phi)\left(\frac{dF}{du}(u^+)\right)^{-1} q(\tau).$$

Any subspace spanned by eigenvectors of $\left(\frac{dF}{du}(u^+)\right)^{-1}$ will remain invariant. However, for κ in the domain $\overline{\mathbb{C}_+}^{\bullet}$ the invariant manifold $W^s(U^+)$ should be the stable manifold of the stationary point U^+. By the assumptions, the eigenvalues $-\exp(i\phi)(\lambda_j(u^+))^{-1}$ with associated eigenvectors $r_j(u^+)$ for $j = k^+ + 1, \dots, n$ have negative real part as long as $\phi \in]-\frac{\pi}{2}, \frac{\pi}{2}[$. Thus we obtain the invariant manifold $W^{s,slow}(U^+)$ within M_0^+ in the slow directions as

$$W^{s,slow}(U^+) := \left\{ (u,p,q)^t \in \mathbb{C}^{3n} \;\middle|\; u = u^+, \quad p = -\left(\frac{dF}{du}(u^+)\right)^{-1} q, \right.$$
$$\left. q \in span\{r_{k^++1}(u^+), \dots, r_n(u^+)\} \right\}.$$

The fibers emanating from the slow manifold M_0^+ are described by the equations on the fast time scale ξ. The augmented system reduces for $\rho = 0$ to

$$\begin{aligned}
\frac{du}{d\xi}(\xi) &= F(u(\xi)),\\
\frac{dp}{d\xi}(\xi) &= \frac{dF}{du}(u(\xi))p(\xi) + q(\xi),\\
\frac{dq}{d\xi}(\xi) &= 0.
\end{aligned}$$

We consider without loss of generality the fiber with base point U^+, i.e. solutions satisfying the boundary condition $\lim_{\xi\to\infty}(u,p,q)^t(\xi) = (u^+,0,0)^t$. The constant solution $u(\xi) \equiv u^+$ solves the first equation and the q coordinates are identically zero. Thus the invariant manifold $W^{s,fast}(U^+)$ in the fast directions has at the stationary point U^+ the tangent space

$$T_{U^+}W^{s,fast}(U^+) = \{(u,p,q)^t \in \mathbb{C}^{3n} \mid u,p \in span\{r_1(u^+),\dots,r_{k^+}(u^+)\},\ q = 0\}.$$

In total, the invariant manifold $W^s(U^+)$ can be decomposed into the flow within M_0^+ and the fibration emanating from $W^{s,slow}(U^+) \subset M_0^+$. Since the slow manifold M_0^+ is normally hyperbolic it perturbs smoothly to an invariant manifold M_ρ^+ for $\rho \in [0,\delta]$ small. This implies that the construction of the $W^s(U^+)$ persists for small ρ. In a similar way we are able to construct the stated decomposition of $W^u(U^-)$. □

Corollary 1.1. *Suppose the assumptions (A1) and (A2) hold. Then the solutions $\eta_j^s(\xi,\kappa)$ and $\eta_j^u(\xi,\kappa)$ for $j = 1,\dots,n$ of the eigenvalue equation* (1.15) *in the Lemmata 1.3 and 1.4, respectively, have analytic continuations for κ in a small neighborhood of the origin Ω_0.*

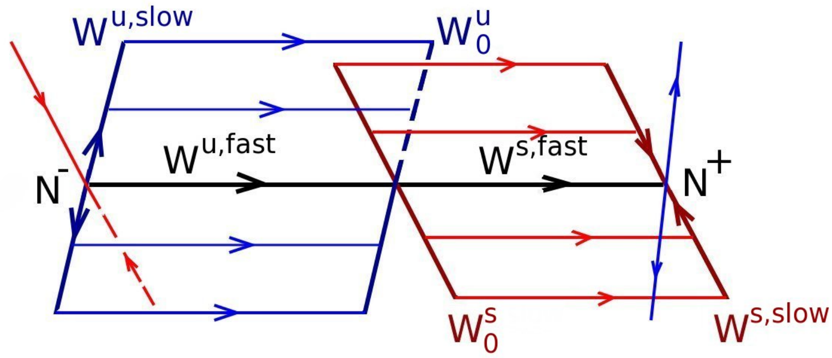

Figure 1.3: Decomposition of the invariant manifolds $W^s(U^+)$ and $W^u(U^-)$.

Proof. By the results of Lemma 1.3, the product space $\{\bar{u}(0)\} \times W_0^s(\kappa)$ is part of the stable manifold $W^s(U^+)$ in Theorem 1.8 for $\kappa \in \overline{\mathbb{C}_+}^{\bullet}$. Hence, the stable space $W_0^s(\kappa)$ has an analytic continuation into a small neighborhood of the origin and a slow-fast decomposition. An analytic basis of $W_0^s(\kappa)$ can be constructed by a standard procedure, see [Kat95, chapter II.4.2.], and the associated solutions $\eta_j^s(\xi, \kappa)$ for $j = 1, \ldots, n$ of the eigenvalue equation (1.15) inherit the analytic dependence on κ. In the same way, the solutions $\eta_j^u(\xi, \kappa)$ for $j = 1, \ldots, n$ associated to the analytic continuation of the unstable space $W_0^u(\kappa)$ are obtained. □

Remark 1.3. The solutions of the eigenvalue equation (1.15) are also denoted as

$$\eta_j^u(\xi, \kappa) = \begin{pmatrix} p_j \\ q_j \end{pmatrix} (\xi, \kappa) \qquad \text{for} \quad j = 1, \ldots, n$$

and

$$\eta_j^s(\xi, \kappa) = \begin{pmatrix} p_{n+j} \\ q_{n+j} \end{pmatrix} (\xi, \kappa) \qquad \text{for} \quad j = 1, \ldots, n$$

with functions $p_j, q_j : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ for $j = 1, \ldots, 2n$.

Theorem 1.9. *Suppose the assumptions (A1) and (A2) hold. For κ in a small neighborhood of the origin and the functions $\eta_j^{s/u}(\xi, \kappa)$ with $j = 1, \ldots, n$ from Corollary 1.1, the analytic continuation of the Evans function is given by*

$$E(\kappa) = \det(\eta_1^u, \ldots, \eta_n^u, \eta_1^s, \ldots, \eta_n^s)(0, \kappa). \tag{1.22}$$

Proof. The solutions in Corollary 1.1 are the analytic continuations of the solutions in the Lemmata 1.3 and 1.4. Hence, the function (1.22) is indeed the analytic continuation of the Evans function in Definition 1.12. □

Remark 1.4. The Evans function is only unique up to a non-vanishing analytic factor.

In the following we will restrict our presentation to viscous shock waves that are related to Lax shocks.

(A3) Let $\lambda_j(u^\pm)$ for $j = 1, \ldots, n$ denote the real eigenvalues of the Jacobians $\frac{dF}{du}(u^\pm)$ in increasing order of magnitude. The viscous profile $\bar{u}(\xi)$ in (A1) is associated to a Lax k-shock, i.e. the inequalities

$$\lambda_{k-1}(u^-) < 0 < \lambda_k(u^-) \quad \text{and} \quad \lambda_k(u^+) < 0 < \lambda_{k+1}(u^+)$$

hold.

Remark. The shock speed s does not show up in the above inequalities, since we consider instead of the flux function $f(u)$ the new vector field $F(u) = f(u) - su - c$. Thus the shock speed is absorbed into the eigenvalues of the Jacobian $\frac{dF}{du}(u^\pm)$.

Corollary 1.2. *Suppose the assumptions (A1), (A2) and (A3) hold. Then the solutions of the eigenvalue equation* (1.15) *in Corollary 1.1 will satisfy for $\kappa = 0$ the reduced system,*

$$\begin{aligned} \frac{dp}{d\xi}(\xi) &= \frac{dF}{du}(\bar{u}(\xi))p(\xi) + q(\xi), \\ \frac{dq}{d\xi}(\xi) &= 0. \end{aligned} \tag{1.23}$$

In addition, the solutions are of the form

$$\eta_j^s(\xi) = \begin{pmatrix} p_{n+j}(\xi) \\ 0 \end{pmatrix}, \qquad \textit{for} \quad j = 1, \ldots, k, \tag{1.24}$$

$$\eta_j^s(\xi) = \begin{pmatrix} p_{n+j}(\xi) \\ r_j(u^+) \end{pmatrix}, \qquad \textit{for} \quad j = k+1, \ldots, n. \tag{1.25}$$

$$\eta_j^u(\xi) = \begin{pmatrix} p_j(\xi) \\ r_j(u^-) \end{pmatrix}, \qquad \textit{for} \quad j = 1, \ldots, k-1, \tag{1.26}$$

$$\eta_j^u(\xi) = \begin{pmatrix} p_j(\xi) \\ 0 \end{pmatrix}, \qquad \textit{for} \quad j = k, \ldots, n. \tag{1.27}$$

Proof. The eigenvalue equation (1.15) reduces for $\kappa = 0$ to the system (1.23). Thus the q-coordinates of the solutions are constant. In Corollary 1.1 we extracted solutions of the eigenvalue equation from the invariant manifolds of the augmented system (1.21). The results on the stable manifold $W^s(U^+)$ and the unstable manifold $W^u(U^-)$ provide boundary conditions for the solutions of (1.15). For example, the solutions $\eta_j^s(\xi, \kappa)$ for $j = 1, \ldots, k$ and $j = k+1, \ldots, n$ are related to the fast manifold $W^{s,fast}(U^+)$ and the slow manifold $W^{s,slow}(U^+)$, respectively. □

Remark 1.5. A new notation for the functions $\eta_j^{s/u}(\xi, \kappa)$ in Corollary 1.1 will emphasize their distinct asymptotic behavior. In the following we will refer to the solutions in the fast manifold as

$$\begin{aligned} S_j^f(\xi, \kappa) &:= \eta_j^s(\xi, \kappa), && \text{for} \quad j = 1, \ldots, k, \\ U_j^f(\xi, \kappa) &:= \eta_{j+k-1}^u(\xi, \kappa), && \text{for} \quad j = 1, \ldots, n-k+1, \end{aligned}$$

and the solutions in the slow manifold as

$$\begin{aligned} S_j^s(\xi, \kappa) &:= \eta_{j+k}^s(\xi, \kappa), && \text{for} \quad j = 1, \ldots, n-k, \\ U_j^s(\xi, \kappa) &:= \eta_j^u(\xi, \kappa), && \text{for} \quad j = 1, \ldots, k-1, \end{aligned}$$

respectively. Additionally, we will denote the matrices spanned by the solutions as

$$\begin{aligned} U^f(\xi,\kappa) &:= (U_1^f,\ldots,U_{n-k+1}^f)(\xi,\kappa), \qquad & U^s(\xi,\kappa) &:= (U_1^s,\ldots,U_{k-1}^s)(\xi,\kappa),\\ S^f(\xi,\kappa) &:= (S_1^f,\ldots,S_k^f)(\xi,\kappa), & S^s(\xi,\kappa) &:= (S_1^s,\ldots,S_{n-k}^s)(\xi,\kappa). \end{aligned}$$

Thus the Evans function in Theorem 1.9 is written as

$$\begin{aligned} E(\kappa) &= \det(\eta_1^u,\ldots,\eta_n^u,\eta_1^s,\ldots,\eta_n^s)(0,\kappa)\\ &= \det\big(U_1^f,\ldots,U_{n-k+1}^f,U_1^s,\ldots,U_{k-1}^s,S_1^f,\ldots,S_k^f,S_1^s,\ldots,S_{n-k}^s\big)(0,\kappa)\\ &= \det\big(U^f,U^s,S^f,S^s\big)(0,\kappa). \end{aligned}$$

1.3 Effective Spectrum

Zumbrun and Howard consider the resolvent kernel, rather than the resolvent, to study the stability of viscous shock waves [ZH98]. The resolvent kernel is the Green's function $G_\kappa(\xi,y)$ associated to the operator $L-\kappa I$ via the identity

$$(L-\kappa I)G_\kappa(.,y) = \delta_y(.)I,$$

where δ_y denotes the Dirac delta distribution centered at y. On the resolvent set $\rho(L)$, the resolvent $(L-\kappa I)^{-1}$ and the Green's function $G_\kappa(\xi,y)$ are meromorphic with poles of finite order. By the result of the Gap Lemma, or alternatively Theorem 1.8 and Corollary 1.1, Zumbrun and Howard are able to construct a representation of the Green's function on the resolvent set and prove the following result.

Lemma 1.6. ([ZH98, Proposition 5.3.]) *Suppose the assumptions (A1) and (A2) hold. Then the Green's function $G_\kappa(\xi,y)$ has a meromorphic continuation into a small neighborhood of the origin, Ω_0, with only poles of finite order, which coincide with zeros (of the analytic continuation) of the Evans function in Theorem 1.9.*

Definition 1.14. The *effective (point) spectrum* is defined as the set of poles of the meromorphic continuation of the Green's function $G_\kappa(x,y)$ in Lemma 1.6.

The space $C^{\infty}_{\exp}(\mathbb{R};\mathbb{C}^n)$ consists of smooth functions that decay exponentially fast to zero. Moreover, the linear operator L maps $C^{\infty}_{\exp}(\mathbb{R};\mathbb{C}^n)$ into itself, since the operator has continuous and bounded coefficients.

Definition 1.15. *([ZH98, Definition 5.1.])* Suppose the assumptions (A1) and (A2) hold. Then for κ_0 in the domain Ω_0 of the meromorphic continuation of the Green's function $G_\kappa(x,y)$ in Lemma 1.6, we define the *effective eigenprojection* $\mathcal{P}_{\kappa_0} : C^{\infty}_{\exp}(\mathbb{R};\mathbb{C}^n) \to C^{\infty}(\mathbb{R};\mathbb{C}^n)$ by

$$\mathcal{P}_{\kappa_0} f(x) := \int_{-\infty}^{+\infty} P_{\kappa_0}(x,y) f(y) dy,$$

where the projection kernel, $P_{\kappa_0}(x,y) := \text{residue}_{\kappa_0} G_\kappa(x,y)$, is defined as the Residue of the Green's function $G_\kappa(x,y)$ at κ_0. Likewise, we define the *effective eigenspace* $\Sigma'_{\kappa_0}(L)$ by

$$\Sigma'_{\kappa_0}(L) := \text{image}(\mathcal{P}_{\kappa_0}).$$

Definition 1.16. *([ZH98, Definition 5.2])* Suppose the assumptions (A1) and (A2) hold. Then for κ_0 in the domain Ω_0 of the meromorphic continuation of the Green's function $G_\kappa(x,y)$ in Lemma 1.6 and k any integer, we define the *effective eigenprojection* $\mathcal{Q}_{\kappa_0,k} : C^{\infty}_{\exp}(\mathbb{R};\mathbb{C}^n) \to C^{\infty}(\mathbb{R};\mathbb{C}^n)$ by

$$\mathcal{Q}_{\kappa_0,k} f(x) := \int_{-\infty}^{+\infty} Q_{\kappa_0,k}(x,y) f(y) dy,$$

where the kernel $Q_{\kappa_0,k}(x,y)$ is defined as

$$Q_{\kappa_0,k}(x,y) := \text{residue}_{\kappa_0} \big((\kappa-\kappa_0)^k G_\kappa(x,y) \big).$$

Additionally, let K be the order of the pole of $G_\kappa(x,y)$ at κ_0 and $k = 0,\dots,K$. Then we define the *effective eigenspace of ascent* k, $\Sigma'_{\kappa_0,k}(L)$, by

$$\Sigma'_{\kappa_0,k}(L) := \text{image}(\mathcal{Q}_{\kappa_0,K-k}).$$

In the following, Zumbrun and Howard prove a modified Fredholm theory.

Lemma 1.7. ([ZH98, Proposition 5.3.]) *Suppose the assumptions (A1) and (A2) hold. Additionally, for κ_0 in the domain Ω_0 of the meromorphic continuation of the Green's function $G_\kappa(x,y)$ in Lemma 1.6, let K be the order of $G_\kappa(x,y)$ at κ_0. Then,*

1. *The operators $\mathcal{P}_{\kappa_0}$, $\mathcal{Q}_{\kappa_0,k}$: $C^\infty_{\exp} \to C^\infty$ are L-invariant,with*

$$\mathcal{Q}_{\kappa_0,k+1} = (L - \kappa_0 I)\mathcal{Q}_{\kappa_0,k} = \mathcal{Q}_{\kappa_0,k}(L - \kappa_0 I)$$

   for all $k \neq -1$, and

$$\mathcal{Q}_{\kappa_0,k} = (L - \kappa_0 I)^k \mathcal{P}_{\kappa_0}$$

   for $k \geq 0$.

2. *The effective eigenspace of ascent k satisfies*

$$\Sigma'_{\kappa_0,k}(L) = (L - \kappa_0 I)\Sigma'_{\kappa_0,k+1}(L)$$

   for all $k = 0, \dots, K$, with

$$\{0\} = \Sigma'_{\kappa_0,0}(L) \subset \Sigma'_{\kappa_0,1}(L) \subset \cdots \subset \Sigma'_{\kappa_0,K}(L) = \Sigma'_{\kappa_0}(L). \tag{1.28}$$

   Moreover, each containment in (1.28) *is strict*

3. *On $\mathcal{P}^{-1}_{\kappa_0}(C^\infty_{\exp})$, $\mathcal{P}_{\kappa_0}$, $\mathcal{Q}_{\kappa_0,k}$ for $k \geq 0$ all commute, and $\mathcal{P}_{\kappa_0}$ is a projection. More generally, $\mathcal{P}_{\kappa_0} f = f$ for any $f \in \Sigma_{\kappa_0}(L : C^\infty_{\exp})$, hence*

$$\Sigma_{\kappa_0,k}(L : C^\infty_{\exp}) \subset \Sigma'_{\kappa_0,k}(L)$$

   for all $k = 0, \dots, K$.

4. *The multiplicity of the effective eigenvalue κ_0, defined as $\dim \Sigma'_{\kappa_0}(L)$, is finite and bounded by $K \cdot n$. Moreover, for all $k = 0, \dots, K$,*

$$\dim \Sigma'_{\kappa_0,k}(L) = \dim \Sigma'_{\kappa_0{}^*,k}(L^*).$$

Further, the projection kernel can be expanded as

$$P_{\kappa_0}(x,y) = \sum_j \varphi_j(x)\pi_j(y),$$

where $\{\varphi_j\}$, $\{\pi_j\}$ *are bases for* $\Sigma'_{\kappa_0}(L)$, $\Sigma'_{\kappa_0{}^*}(L^*)$, *respectively.*

5. *(Restricted Fredholm alternative) For* $g \in C^\infty_{\exp}$,

$$(L - \kappa_0 I)f = g \tag{1.29}$$

is soluble in C^∞ *if, and soluble in* $C^\infty_{\exp}$ *only if,* $\mathcal{Q}_{\kappa_0,K-1}g = 0$, *or equivalently*

$$g \in \Sigma'_{\kappa_0{}^*,1}(L^*)^\perp.$$

Zumbrun and Howard note that for κ_0 away from the essential spectrum, the effective eigenprojection agrees with the standard definition. In that case, the effective eigenspace $\Sigma'_{\kappa_0,k}(L)$ coincides with the usual L^p eigenspace of generalized eigenfunctions of ascent k. However, for $\kappa \in (\sigma_{ess}(L) \cap \Omega_0)$ the operator $\mathcal{P}_\kappa$ is not a projection operator, since its domain does not match its range unless the domain is restricted to $C^\infty_{\exp}$. The special position of the functions in $C^\infty_{\exp}$ in the modified Fredholm theory is emphasized in the following definition.

Definition 1.17. For κ_0 in the domain Ω_0 of the meromorphic continuation of the Green's function $G_\kappa(x,y)$ in Lemma 1.6, a function that lies in the effective eigenspace $\Sigma'_{\kappa_0}(L)$ as well as in the function space $C^\infty_{\exp}$ is referred to as *genuine eigenfunction.*

Lemma 1.8. ([ZH98, Lemma 6.1.]) *Suppose the assumptions (A1) and (A2) hold, and the domain* Ω_0 *as well as the functions* $\eta^u_j(\xi,\kappa)$ *and* $\eta^s_j(\xi,\kappa)$ *for* $j = 1,\dots,n$ *are taken from Corollary 1.1. Then, at any zero* $\kappa \in \Omega_0$ *of the Evans function* $E(\kappa)$ *in Theorem 1.9, there exist analytical choices of bases and indices* $p_1 \geq p_2 \geq \cdots \geq p_J$ *such that for* $j = 1,\dots,J$ *and* $p = 0,\dots,p_j$ *the identities*

$$\frac{\partial^p}{\partial\kappa^p}\eta^u_j(\xi,\kappa) = \frac{\partial^p}{\partial\kappa^p}\eta^s_j(\xi,\kappa) \tag{1.30}$$

and

$$\det\left(\eta_1^u, \ldots, \eta_n^u, \frac{\partial^{p_1+1}}{\partial\kappa^{p_1+1}}(\eta_1^s - \eta_1^u), \ldots, \frac{\partial^{p_J+1}}{\partial\kappa^{p_J+1}}(\eta_J^s - \eta_J^u), \eta_{J+1}^s, \ldots, \eta_n^s\right)(0) \neq 0$$

hold.

Zumbrun and Howard point out that the functions in (1.30) are solutions of the generalized eigenvalue equations

$$\eta_j^{u/s}(\xi, \kappa) = (L - \kappa_0 I)^p \frac{\partial^p}{\partial\kappa^p} \eta_j^{u/s}(\xi, \kappa)$$

with $j = 1, \ldots, J$ and $p = 0, \ldots, p_j$. Thus the functions in (1.30) are, formally, effective eigenfunctions which are arranged in Jordan chains.

Theorem 1.10. ([ZH98, Theorem 6.3.]) *Suppose the assumptions (A1) and (A2) hold. Then for κ in the domain Ω_0 from Lemma 1.6,*

1. *the functions $\frac{\partial^p}{\partial\kappa^p}\eta_j^u(\xi, \kappa)$ for $j = 1, \ldots, J$ and $p = 0, \ldots, p_j$ in Lemma 1.8, projected onto their first n coordinates are a basis for $\Sigma_\kappa^{'}(L)$. Moreover, the projection of $\frac{\partial^p}{\partial\kappa^p}\eta_j^u(\xi, \kappa)$ is an effective eigenfunction of ascent $p + 1$.*

2. *The dimension of the eigenspace $\Sigma_\kappa^{'}(L)$ is equal to the order of κ as a root of the Evans function in Theorem 1.9.*

Remark 1.6. An effective eigenfunction for an effective eigenvalue κ is lying in the intersection of the spaces $W^u(\kappa) = \mathrm{span}\{\eta_j^u(\xi, \kappa) \mid j = 1, \ldots, n\}$ and $W^s(\kappa) = \mathrm{span}\{\eta_j^s(\xi, \kappa) \mid j = 1, \ldots, n\}$, which are associated to the functions in Corollary 1.1. In particular, the solutions $\eta_j^u(\xi)$ and $\eta_j^s(\xi)$ of the eigenvalue equation for $\kappa = 0$ and $j = 1, \ldots, n$ are bounded functions on $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively. Thus an effective eigenfunction for the effective eigenvalue zero is bounded on $\mathbb{R}$.

1.3.1 Multiplicity of the effective eigenvalue $\kappa = 0$

It turns out that the multiplicity of the effective eigenvalue zero depends on the transversality of the viscous profile and the hyperbolic stability of the associated Lax shock.

Remark 1.7. By the results of Lemma 1.1, the function $\binom{\frac{d\bar{u}}{d\xi}}{0}(\xi)$ is a solution of the eigenvalue equation (1.15) for $\kappa = 0$ and an element of the spaces $S^f(\xi, 0)$ and $U^f(\xi, 0)$ in the Remark 1.5. Thus we assume without loss of generality that for $\kappa = 0$ the identities

$$U_1^f(\xi, 0) = S_1^f(\xi, 0) = \begin{pmatrix} \frac{d\bar{u}}{d\xi} \\ 0 \end{pmatrix}(\xi)$$

hold.

Lemma 1.9. *If the assumptions (A1) and (A2) hold, then the functions* $U_1^f(\xi, \kappa)$ *and* $S_1^f(\xi, \kappa)$ *in Remark 1.7 will satisfy*

$$\frac{\partial U_1^f}{\partial \kappa}(\xi, 0) = \begin{pmatrix} z_1(\xi) \\ \bar{u}(\xi) - u^- \end{pmatrix}$$

and

$$\frac{\partial S_1^f}{\partial \kappa}(\xi, 0) = \begin{pmatrix} z_{n+1}(\xi) \\ \bar{u}(\xi) - u^+ \end{pmatrix},$$

where $z_1(\xi) := \frac{\partial p_1}{\partial \kappa}(\xi, 0)$ *and* $z_{n+1}(\xi) := \frac{\partial p_{n+1}}{\partial \kappa}(\xi, 0)$.

Proof. We differentiate the eigenvalue equation (1.15) with respect to κ and obtain the differential equations,

$$\frac{\partial}{\partial \xi}\begin{pmatrix} \frac{\partial p}{\partial \kappa} \\ \frac{\partial q}{\partial \kappa} \end{pmatrix}(\xi, \kappa) = \begin{pmatrix} \frac{dF}{du}(\bar{u}(\xi)) & I_n \\ \kappa I_n & 0_{n\times n} \end{pmatrix}\begin{pmatrix} \frac{\partial p}{\partial \kappa} \\ \frac{\partial q}{\partial \kappa} \end{pmatrix}(\xi, \kappa) + \begin{pmatrix} 0 \\ p \end{pmatrix}(\xi, \kappa),$$

which govern the vectors $\frac{\partial U_1^f}{\partial \kappa}(\xi, \kappa)$ and $\frac{\partial S_1^f}{\partial \kappa}(\xi, \kappa)$. The q-coordinates for $\kappa = 0$

satisfy the equations

$$\frac{d}{d\xi}\frac{\partial}{\partial\kappa}q(\xi,0) = p(\xi,0),$$

which we integrate from $-\infty$ to ξ. The left hand side equals

$$\int_{-\infty}^{\xi}\frac{\partial}{\partial x}\frac{\partial q_1}{\partial\kappa}(x,0)dx = \frac{\partial q_1}{\partial\kappa}(x,0)\Big|_{-\infty}^{\xi} = \frac{\partial q_1}{\partial\kappa}(\xi) - 0$$

and the right-hand side is obtained as

$$\int_{-\infty}^{\xi}p_1(x,0)dx = \int_{-\infty}^{\xi}\frac{d\bar{u}}{dx}(x,0)dx = \bar{u}(\xi) - u^-,$$

since $p_1(\xi,0) = \frac{d\bar{u}}{d\xi}(\xi)$. This gives

$$\frac{\partial U_1^f}{\partial\kappa}(\xi,0) = \begin{pmatrix} z_1(\xi) \\ \bar{u}(\xi) - u^- \end{pmatrix}$$

with $z_1(\xi) := \frac{\partial p_1}{\partial\kappa}(\xi,0)$. Similarly, we compute

$$\frac{\partial S_1^f}{\partial\kappa}(\xi,0) = \begin{pmatrix} z_{n+1}(\xi) \\ \bar{u}(\xi) - u^+ \end{pmatrix}.$$

□

Theorem 1.11 ([GZ98]). *Suppose the assumptions (A1), (A2) and (A3) hold. Then the first derivative of the Evans function satisfies*

$$\begin{aligned}\frac{dE}{d\kappa}(0) = c\cdot\det(&p_1,\dots,p_{n-k+1},p_{n+2},\dots,p_{n+k})(0)\cdot \\ &\cdot\det(r_1(u^-),\dots,r_{k-1}(u^-),u^+-u^-,r_{k+1}(u^+),\dots,r_n(u^+)),\end{aligned}$$

with a non-zero constant $c\in\mathbb{R}$ and vectors $p_j(0)$ for $j=1,\dots,2n$ in Corollary 1.2.

Proof. We consider the analytic continuation of the Evans function in Theorem 1.9 in the notation of Remark 1.5,

$$E(\kappa) = \det(U_1^f, \dots, U_{n-k+1}^f, U_1^s, \dots, U_{k-1}^s, S_1^f, \dots, S_k^f, S^s)(0, \kappa).$$

In the following, we will restrict our calculations to the real half-line $\kappa \in \mathbb{R}_+$ and note that the involved vectors and the Evans function will be real valued there. By the explanation in Remark 1.7, we assume without loss of generality that the vectors $U_1^f(\xi, \kappa)$ and $S_1^f(\xi, \kappa)$ satisfy for $\kappa = 0$ the identity

$$U_1^f(\xi, 0) = S_1^f(\xi, 0) = \begin{pmatrix} \frac{d\bar{u}}{d\xi} \\ 0 \end{pmatrix}(\xi).$$

Thus the Evans function vanishes, $E(0) = 0$, and the function $\left(\begin{smallmatrix} \frac{d\bar{u}}{d\xi} \\ 0 \end{smallmatrix}\right)(\xi)$ is a genuine eigenfunction for the effective eigenvalue zero. The first derivative of the Evans function with respect to κ is computed by the Leibniz rule,

$$\frac{dE}{d\kappa}(\kappa) = \sum_{i=1}^{n} \det\left(\dots, \begin{pmatrix} p_{i-1} \\ q_{i-1} \end{pmatrix}, \frac{\partial}{\partial\kappa}\begin{pmatrix} p_i \\ q_i \end{pmatrix}, \begin{pmatrix} p_{i+1} \\ q_{i+1} \end{pmatrix}, \dots, \right)(0, \kappa).$$

We evaluate the derivative at $\kappa = 0$ and obtain

$$\begin{aligned} \frac{dE}{d\kappa}(0) &= \det\left(\frac{\partial}{\partial\kappa}U_1^f, U_2^f, \dots, U_{k-1}^f, U^s, S^f, S^s\right)(0) + \\ &\quad + \det\left(U^f, U^s, \frac{\partial}{\partial\kappa}S_1^f, S_2^f, \dots, S_k^f, S^s\right)(0) \\ &= \det\left(U^f, U^s, \frac{\partial}{\partial\kappa}(S_1^f - U_1^f), S_2^f, \dots, S_k^f, S^s\right)(0). \end{aligned}$$

All other summands vanish, since they contain two linearly dependent vectors $S_1^f(0) = U_1^f(0)$. The vectors have been analyzed in Corollary 1.2 and Lemma 1.9 and we obtain the expressions

$$U^f(0) = \begin{pmatrix} p_1 & \cdots & p_{n-k+1} \\ 0 & \cdots & 0 \end{pmatrix}(0),$$

$$
\begin{aligned}
U^s(0) &= \begin{pmatrix} p_{n-k} & \cdots & p_n \\ r_1(u^-) & \cdots & r_{k-1}(u^-) \end{pmatrix}(0), \\
\frac{\partial}{\partial\kappa}\big(S_1^f - U_1^f\big)(\xi) &= \begin{pmatrix} z_{n+1}(\xi) - z_1(\xi) \\ -(u^+ - u^-) \end{pmatrix}, \\
S^{f*}(0) = (S_2^f, \ldots, S_k^f)(0) &= \begin{pmatrix} p_{n+2} & \cdots & p_{n+k} \\ 0 & \cdots & 0 \end{pmatrix}(0)
\end{aligned}
$$

and

$$
S^s(0) = \begin{pmatrix} p_{n+k+1} & \cdots & p_{2n} \\ r_{k+1}(u^+) & \cdots & r_n(u^+) \end{pmatrix}(0).
$$

We change the order of the vectors with an even number of permutations and derive the inner matrix in block diagonal form,

$$
\frac{dE}{d\kappa}(0) = \det\begin{pmatrix} A & B \\ 0_{n\times n} & C \end{pmatrix},
$$

with quadratic matrices

$$
A := (p_1, \ldots, p_{n-k+1}, p_{n+2}, \ldots, p_{n+k})(0) \in \mathbb{R}^{n\times n},
$$

$B \in \mathbb{R}^{n\times n}$, the null matrix $0_{n\times n} \in \mathbb{R}^{n\times n}$ and

$$
C := (r_1(u^-), \ldots, r_{k-1}(u^-), -(u^+ - u^-), r_{k+1}(u^+), \ldots, r_n(u^+)).
$$

Thus, the identity

$$
\det\begin{pmatrix} A & B \\ 0_{n\times n} & C \end{pmatrix} = \det(A)\det(C)
$$

and the assumption (A3) prove the stated result. $\square$

Corollary 1.3 ([GZ98]). *Suppose the assumptions (A1), (A2) and (A3) hold, the viscous profile* $\bar{u}(\xi)$ *is realized by a transversal intersection of the invariant manifolds* $W^u(u^-)$ *and* $W^s(u^+)$ *in* (1.4) *and, additionally, the Liu-Majda criterion*

$$\det\left(r_1(u^-), \dots, r_{k-1}(u^-), [\bar{u}], r_{k+1}(u^+), \dots, r_n(u^+)\right) \neq 0 \tag{1.31}$$

is satisfied. Then the effective eigenvalue $\kappa = 0$ *is simple.*

Proof. The matrix $A = (p_1, \dots, p_{n-k+1}, p_{n+2}, \dots, p_{n+k})(0)$ is spanned by the tangent vectors of the invariant manifolds $W^u(u^-)$ and $W^s(u^+)$ in the profile equation (1.4). The assumption of a transversal intersection along the viscous profile implies that the tangent vectors are linearly independent. Hence the factor $\det(A)$ will be non-zero. Together with the Liu-Majda condition, we obtain that the first derivative of the Evans function at $\kappa = 0$ does not vanish. Thus the order of the root $\kappa = 0$ is one, which implies by the result of Theorem 1.10 that the effective eigenvalue zero is simple. □

In case of a non-transversal viscous profile the first derivative of the Evans function at $\kappa = 0$ vanishes. Hence, the effective eigenvalue zero has multiplicity greater or equal than two, which may signal the onset of instability. We will study this situation in the remainder of this work.

Remark 1.8. The Lax 1-shock and the Lax n-shock are often referred to as *extreme* Lax shocks. The related profiles of (1.4) exist always by a transversal intersection, since for example in case of a Lax 1-shock the unstable manifold $W^u(u^-)$ has dimension n and transversality is trivial.

Chapter 2

Non-transversal profiles

We study the situation of a viscous shock wave whose viscous profile is non-transversal. A non-transversal viscous profile may not persist under small perturbations of the profile equation and indicates a possible bifurcation. We have seen that the multiplicity of the zero eigenvalue depends on the transversality of the viscous profile and the Liu-Majda condition. Thus, the existence and the stability of such a viscous shock wave are sensitive to perturbations. We consider a parametrized family of viscous conservation laws and study the simplest bifurcation scenario: a saddle-node bifurcation of viscous profiles. In particular, we investigate the stability of the associated viscous shock waves.

We consider a family of hyperbolic viscous conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u, \mu) = \frac{\partial^2 u}{\partial x^2}, \tag{2.1}$$

whose flux function $f(u, \mu)$ depends smoothly on the real parameter μ. The associated viscous profile equations are

$$\frac{du}{d\xi}(\xi) = f(u(\xi), \mu) - s(\mu)u(\xi) - c(\mu) =: F(u(\xi), \mu), \tag{2.2}$$

where the vector field $F(u, \mu)$ inherits the smooth dependence on μ from the flux function. A simple example is the case of a parameter independent flux function where the shock speed s becomes the parameter of interest. Next, we adapt the assumptions (A1)–(A3) of the previous chapter.

(B1) For some parameter value μ_0, a viscous shock wave of the system of viscous conservation laws (2.1) exists whose viscous profile $\bar{u}(\xi, \mu_0)$ is non-transversal.

In order to simplify our notation, we will omit for $\mu = \mu_0$ the dependence on the parameter; for example, we will write $\bar{u}(\xi)$ instead of $\bar{u}(\xi, \mu_0)$, $u^{\pm}$ instead of $u^{\pm}(\mu_0)$, s instead of $s(\mu_0)$, etc.

(B2) The shock speed s of the viscous shock wave in assumption (B1) is non-characteristic, that means it differs from any eigenvalue of the Jacobian matrices $\frac{df}{du}(u^{\pm})$.

Again, the assumptions (B1) and (B2) imply that the endstates of the viscous profile $\bar{u}(\xi)$ are hyperbolic fixed points of the vector field $F(u)$. We denote the respective non-zero real eigenvalues of the Jacobians $\frac{dF}{du}(u^{\pm})$ by $\lambda_j(u^{\pm})$ for $j = 1, \ldots, n$ and assume that they are ordered by increasing value. The associated eigenvectors of $\lambda_j(u^{\pm})$ are $r_j(u^{\pm})$ with $j = 1, \ldots, n$. Again, we restrict our presentation to the following kind of viscous shock waves:

(B3) The viscous profile $\bar{u}(\xi)$ in (B1) is related to a Lax k-shock, i.e. the eigenvalues $\lambda_j(u^{\pm})$ satisfy the inequalities

$$\lambda_{k-1}(u^-) < 0 < \lambda_k(u^-) \quad \text{and} \quad \lambda_k(u^+) < 0 < \lambda_{k+1}(u^+). \tag{2.3}$$

Now, we add another assumption that specifies the non-transversal viscous profiles. In general, a viscous profile $\bar{u}(\xi)$ is non-transversal, if its heteroclinic orbit is lying in the intersection of the invariant manifolds, $W^u(u^-)$ and $W^s(u^+)$, whose united tangent spaces fail to cover the state space $\mathbb{R}^n$. That means, for all points p on the orbit $\{\bar{u}(\xi) \mid \xi \in \mathbb{R}\}$ the inequality

$$\dim\left(T_pW^u(u^-) + T_pW^s(u^+)\right) < n$$

holds. The dimensions of the tangent spaces, $T_pW^u(u^-)$ and $T_pW^s(u^+)$, at any point p on the heteroclinic orbit are determined by the assumption (B3) as

$$\dim\big(T_pW^u(u^-)\big) = n - k + 1 \quad \text{and} \quad \dim\big(T_pW^s(u^+)\big) = k,$$

respectively. Therefore, the transversality of the viscous profile can fail in various ways and we give a short list in the Table 2.1. We restrict our attention to the cases where the sum of the tangent spaces has co-dimension one:

(B4) The viscous profile $\bar{u}(\xi)$ in (B1) is non-transversal and for all points p on the orbit $\{\bar{u}(\xi) \mid \xi \in \mathbb{R}\}$ the identity

$$\dim\big(T_pW^u(u^-) + T_pW^s(u^+)\big) = n - 1$$

holds.

Low-dimensional examples are highlighted in red in the Table 2.1.

Remark. For all points p on the orbit $\{\bar{u}(\xi) \mid \xi \in \mathbb{R}\}$, the dimension of the sum of the tangent spaces,

$$\dim\big(T_pW^u(u^-) + T_pW^s(u^+)\big), \tag{2.4}$$

is equal to

$$\dim T_pW^u(u^-) + \dim T_pW^s(u^+) - \dim\big(T_pW^u(u^-) \cap T_pW^s(u^+)\big).$$

Hence, the assumptions (B2) and (B3) imply that the intersection of the tangent spaces is two-dimensional.

2.1 Application of Melnikov theory

We now address the existence of heteroclinic orbits for the family of profile equations (2.2). In (B1), we assumed the existence of a viscous profile of (2.2) for some parameter value μ_0. Due to hyperbolicity, the equilibria $u^\pm$ and their stable and unstable manifolds depend smoothly on the parameter for μ close to μ_0. A viscous profile is said to persist for a parameter close

- n	- k	$\dim T_pW^u(u^-)$ n–k+1	$\dim T_pW^s(u^+)$ k	dimension of the sum (2.4)	transversal
1	1	1	1	1	yes
2	1	2	1	2	yes
2	2	1	2	2	yes
3	1	3	1	3	yes
3	2	2	2	2	no
3	2	2	2	3	yes
3	3	1	3	3	yes
4	1	4	1	4	yes
4	2	3	2	3	no
4	2	3	2	4	yes
4	3	2	3	3	no
4	3	2	3	4	yes
4	4	1	4	4	yes
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
5	3	3	3	3	no
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
6	2	5	2	5	no
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2.1: A list of increasingly degenerate intersection scenarios for the invariant manifolds $W^u(u^-)$ and $W^s(u^+)$.

to μ_0, if there exists a solution of (2.2) that is close to the unperturbed viscous profile. A transversal viscous profile persists for all parameter values in a small neighborhood of μ_0. In case of a non-transversal viscous profile, Melnikov theory is well suited to analyze for which parameter values the profile persists.

We give a short summary on Melnikov theory and refer to the Appendix A for further details. At first, we recall the main hypotheses of Melnikov theory and discuss the connections to our assumptions on the viscous profile.

(M1) For $\mu = \mu_0$, a heteroclinic orbit in the profile equation (2.2) exists, which connects two distinct hyperbolic rest points $u^{\pm}$ of the vector field $F(u)$.

(M2) The heteroclinic orbit $\{\bar{u}(\xi) \mid \xi \in \mathbb{R}\}$ in (M1) is non-transversal, that means the dimension of the sum of the tangent spaces $T_pW^u(u^-)$ and $T_pW^s(u^+)$ is less than the dimension n of the state space. In addition, for some $k \in \mathbb{N}$ and any point p on the orbit $\{\bar{u}(\xi) \mid \xi \in \mathbb{R}\}$ the identity

$$\dim \left(T_pW^u(u^-) \cap T_pW^s(u^+)\right) = k + 1$$

holds.

The hypothesis (M1) follows from our assumptions (B1) and (B2). In particular, (B1) ensures the existence of a heteroclinic orbit and (B2) gives the hyperbolicity of the endstates. The assumptions (B3) and (B4) imply the hypothesis (M2) with index k equal to one.

Remark 2.1. We can relax the assumptions (B3) and (B4) as long as the hypothesis (M2) is met. On the one hand, we can consider viscous profiles which are associated to under- or overcompressive shock solutions. In particular, a viscous profile associated to an undercompressive shock solution is necessarily non-transversal, see also [GZ98]. On the other hand, the sum of the tangent spaces may not be of co-dimension one, for example, the more degenerate cases at the end of the Table 2.1. These changes would influence the index $k \in \mathbb{N}$ and consequently the multiplicity of the zero eigenvalue.

In the following we assume without loss of generality that $\mu_0 = 0$. We study the existence of solutions $\bar{u}(\xi,\mu)$ of the profile equation

$$\frac{\partial u}{\partial \xi}(\xi,\mu) = F(u(\xi,\mu),\mu), \tag{2.5}$$

for parameters μ in a small neighborhood of $\mu_0 = 0$. Any profile $\bar{u}(\xi,\mu)$ of interest can be decomposed into the sum

$$\bar{u}(\xi,\mu) = \bar{u}(\xi) + z(\xi,\mu) \tag{2.6}$$

of the profile $\bar{u}(\xi)$ and a globally bounded function $z(\xi,\mu)$, whose norm $\|z\|_\infty = \sup_{\xi\in\mathbb{R}} \|z(\xi,\mu)\|$ is close to zero. The differential equation for the auxiliary function $z(\xi,\mu)$ is obtained as

$$\frac{dz}{d\xi}(\xi,\mu) = \frac{dF}{du}(\bar{u}(\xi),0)z(\xi,\mu) + g(\xi,z(\xi,\mu),\mu), \tag{2.7}$$

with a function

$$g(\xi,z,\mu) := F(\bar{u}(\xi)+z,\mu) - F(\bar{u}(\xi),0) - \frac{dF}{du}(\bar{u}(\xi),0)z. \tag{2.8}$$

The inhomogeneity satisfies the identities

$$g(\xi,0,0) = 0 \in \mathbb{R}^n \qquad \text{and} \qquad \frac{\partial g}{\partial z}(\xi,0,0) = 0 \in \mathbb{R}^{n\times n}. \tag{2.9}$$

The homogeneous problem

$$\frac{dz}{d\xi}(\xi,\mu) = \frac{dF}{du}(\bar{u}(\xi),0)z(\xi,\mu) \tag{2.10}$$

has exponential dichotomies on $\mathbb{R}^-$ and $\mathbb{R}^+$. This allows to construct the stable manifold

$$W^s(\mu) = \{z \in C^1(\mathbb{R};\mathbb{R}^n) \mid \exists\xi_0 : \|z(\xi,\mu)\| < \infty, \quad \forall\xi_0 \geq \xi\}$$

and the unstable manifold

$$W^u(\mu) = \{z \in C^1(\mathbb{R}; \mathbb{R}^n) \mid \exists \xi_0 : \|z(\xi, \mu)\| < \infty, \quad \forall \xi_0 \leq \xi\}$$

in a small neighborhood of the trivial solution $z_0(\xi) \equiv 0$. A non-empty intersection of these invariant manifolds corresponds to the existence of a globally bounded function $z(\xi, \mu)$. We study this intersection without loss of generality at $\xi = 0$ within a transversal section Y of the profile $\bar{u}(\xi)$. There we define the Melnikov function $M(\nu, \mu)$, which measures the distance between these invariant manifolds along the direction that is orthogonal to the tangent plane $T_{\bar{u}(0)} W^s(u^+) \cap T_{\bar{u}(0)} W^u(u^-)$. The Theorem A.1 reads in our case as follows:

Theorem 2.1. *Under the assumptions (B1)-(B4), there exists a function $z^*(\nu, \mu)(\xi)$ as in Lemma A.8 and a unique (up to a multiplicative factor) globally bounded solution $\psi(\xi)$ of the adjoint equation of* (2.10),

$$\frac{d\psi}{d\xi} = -\left(\frac{dF}{du}(\bar{u}(\xi), 0)\right)^T \psi(\xi), \tag{2.11}$$

The Melnikov function $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has an integral representation

$$M(\nu, \mu) = \int_{-\infty}^{+\infty} < \psi(s), g(s, z^*(\nu, \mu)(s), \mu) > ds, \tag{2.12}$$

which is well-defined and smooth in the domain $B_\delta(0) \times B_\delta(0)$ for a sufficiently small, positive constant δ. Moreover, it satisfies the identities

$$M(0, 0) = 0 \tag{2.13}$$

and

$$\frac{\partial M}{\partial \nu}(0, 0) = 0. \tag{2.14}$$

In order to apply the regular value theorem we have to meet the following hypothesis.

(M3) The dimension of the parameter space is greater than or equal to the number of globally bounded solutions of (2.11).

In our situation the hypothesis holds, since we have a single real parameter μ and a single globally bounded solution of (2.11). Again, (M3) is not a severe restriction, since we always can enlarge the parameter space to ensure the requirement. We use Lemma A.9 to restate the Theorem A.3.

Theorem 2.2. *Suppose the assumptions (B1)–(B4) and*

$$\frac{\partial M}{\partial \mu}(0,0) = \int_{-\infty}^{+\infty} < \psi(s), \frac{\partial F}{\partial \mu}(\bar{u}(s,0),0) > ds \neq 0 \tag{2.15}$$

holds. Then the solution set $B = \{(\nu,\mu) \in \mathbb{R}^2 \mid M(\nu,\mu) = 0\}$ *is a smooth curve in a neighborhood of the origin.*

2.2 Saddle-node bifurcation of profiles

The Melnikov function $M(\nu,\mu)$ in Theorem 2.1 satisfies the identities (2.13) and (2.14). Thus $M(\nu,\mu)$ has a singularity at $(0,0)$, whose nature is determined by its higher order derivatives. We will focus on the least degenerate situation:

(B5) The Melnikov function $M(\nu,\mu)$ in Theorem 2.1 has non-zero derivatives

$$\frac{\partial M}{\partial \mu}(0,0) \neq 0 \tag{2.16}$$

and

$$\frac{\partial^2 M}{\partial \nu^2}(0,0) \neq 0. \tag{2.17}$$

The result of Theorem 2.2 shows that the solution set is a smooth curve which contains the point $(0, 0)$. In addition, the conditions (2.13), (2.14), (2.16) and (2.17) on the Melnikov function imply that a saddle-node bifurcation occurs [GH83]. We follow a standard procedure and obtain a smooth parametrization of the solution curve with respect to the variable ν.

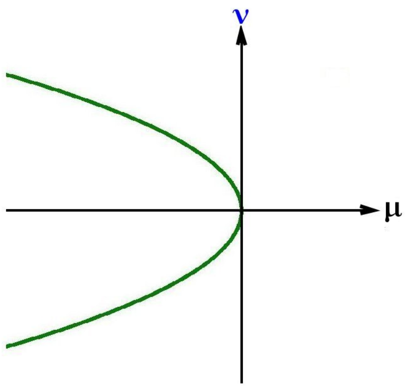

Figure 2.1: Bifurcation diagram of a saddle-node bifurcation.

Theorem 2.3. *If the assumptions (B1)-(B5) hold, then there exists a small neighborhood B of $\nu = 0$ and a function $\mu(\nu) : B \to \mathbb{R}$, $\nu \mapsto \mu(\nu)$, such that $\mu(0) = 0$ and $M(\nu, \mu(\nu)) = 0$ for all $\nu \in B$. In addition, the identities*

$$\frac{d\mu}{d\nu}(0) = 0 \qquad \text{and} \qquad \frac{d^2\mu}{d\nu^2}(0) \neq 0 \tag{2.18}$$

hold.

Proof. We conclude by the implicit function theorem from (2.13) and (2.16) the existence of a unique function $\mu(\nu)$, which satisfies $\mu(0) = 0$ and

$$M(\nu, \mu(\nu)) = 0 \tag{2.19}$$

for sufficiently small ν. We differentiate this identity with respect to ν,

$$0 = \frac{dM}{d\nu}(\nu, \mu(\nu)) = \frac{\partial M}{\partial \nu}(\nu, \mu(\nu)) + \frac{\partial M}{\partial \mu}(\nu, \mu(\nu))\frac{d\mu}{d\nu}(\nu),$$

evaluate the result at $\nu = 0$ and use (2.14) as well as (2.16) to obtain

$$\frac{d\mu}{d\nu}(0) = -\frac{\frac{\partial M}{\partial \nu}(0,0)}{\frac{\partial M}{\partial \mu}(0,0)} = 0. \tag{2.20}$$

Thus the curve $\{(\nu, \mu(\nu)) \in \mathbb{R}^2 \mid \nu \text{ sufficiently small}\}$ is tangent to the ν-axis at $(\nu, \mu) = (0, 0)$. In the next step, we differentiate the identity (2.19) twice with respect to ν,

$$\begin{aligned}
0 &= \frac{d^2 M}{d\nu^2}(\nu, \mu(\nu)) \\
&= \frac{d}{d\nu}\left(\frac{dM}{d\nu}(\nu, \mu(\nu))\right) \\
&= \frac{d}{d\nu}\left(\frac{\partial M}{\partial \nu}(\nu, \mu(\nu)) + \frac{\partial M}{\partial \mu}(\nu, \mu(\nu))\frac{d\mu}{d\nu}(\nu)\right) \\
&= \left(\frac{\partial^2 M}{\partial \nu^2} + \frac{\partial^2 M}{\partial \mu \partial \nu}\frac{d\mu}{d\nu} + \frac{\partial^2 M}{\partial \nu \partial \mu}\frac{d\mu}{d\nu} + \frac{\partial^2 M}{\partial \mu^2}\left(\frac{d\mu}{d\nu}\right)^2 + \frac{\partial M}{\partial \mu}\frac{d^2\mu}{d\nu^2}\right)(\nu, \mu(\nu)),
\end{aligned}$$

evaluate the expression at $\nu = 0$ and use the identity (2.20) to derive

$$0 = \left(\frac{\partial^2 M}{\partial \nu^2} + \frac{\partial M}{\partial \mu}\frac{d^2\mu}{d\nu^2}\right)(0, 0).$$

Hence, we conclude from (2.16) and (2.17) that the second order derivative of the function $\mu(\nu)$ at $\nu = 0$ does not vanish,

$$\frac{d^2\mu}{d\nu^2}(0) = -\frac{\frac{\partial^2 M}{\partial \nu^2}(0,0)}{\frac{\partial M}{\partial \mu}(0,0)} \neq 0. \tag{2.21}$$

□

The assertion of Theorem 2.3 is illustrated in Figure 2.1. The identity $M(\nu, \mu) = 0$ for some (ν, μ) close to $(0, 0)$ is equivalent to the existence of a profile $\bar{u}(\xi, \mu)$ close to $\bar{u}(\xi)$. Hence, on one side of the bifurcation point $\mu = 0$ two profiles exist, which coalesce into a single one as μ reaches zero and cease to exist as the parameter μ moves beyond zero. The profile $\bar{u}(\xi)$ at the bifurcation point $\mu = 0$ exists by a non-transversal intersection of the involved invariant manifolds.

The simplest situation where this bifurcation scenario can be realized is a Lax 2-shock in a system of viscous conservation laws (2.1) in $\mathbb{R}^3$. In this case, the tangent spaces of the invariant manifolds $W^u(u^-)$ and $W^s(u^+)$ are two-dimensional and coincide, see Figure 2.2. The series of pictures in Figure 2.3 shows the invariant manifolds for different values of the parameter μ. Again, any intersection of the invariant manifolds corresponds to a profile.

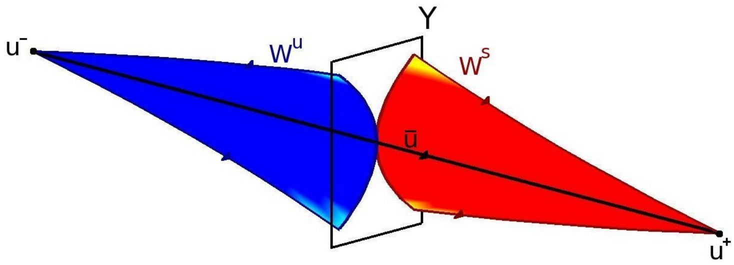

Figure 2.2: A non-transversal profile $\bar{u}(\xi)$ associated to a Lax 2-shock and the invariant manifolds $W^u(u^-)$ and $W^s(u^+)$, which are shown until they reach the transversal cross section Y.

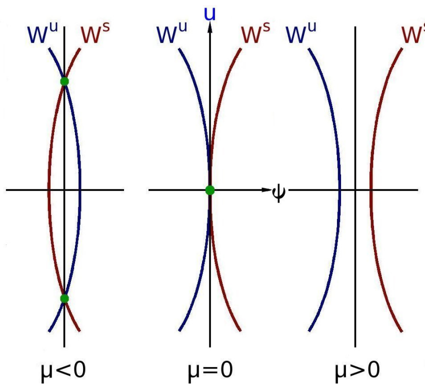

Figure 2.3: The series of pictures indicates how the invariant manifolds move within the cross section Y as the parameter μ changes.

The result of Theorem 2.3 allows us to find expressions for the tangent vectors, which span the two-dimensional intersection of the tangent spaces $T_{\bar{u}(\xi)}W^u(u^-)$ and $T_{\bar{u}(\xi)}W^s(u^+)$.

Lemma 2.1. *If the assumptions (B1)-(B5) hold, then*

1. *The family of profiles* $\bar{u}(\xi,\nu) := \bar{u}(\xi,\mu(\nu))$ *depends smoothly on the parameter* ν.

2. *The function* $\frac{\partial \bar{u}}{\partial \xi}(\xi,\nu)$ *is a solution of*

$$\frac{dp}{d\xi}(\xi) = \frac{dF}{du}(\bar{u}(\xi,\nu),\mu(\nu))p(\xi),$$

and satisfies $\lim_{\xi\to\pm\infty}\frac{\partial \bar{u}}{\partial \xi}(\xi,\nu) = 0$ *for all sufficiently small* ν. *Moreover, the functions* $\frac{\partial \bar{u}}{\partial \xi}(\xi,\nu)$ *are elements of* $C^\infty_{\mathrm{exp}}(\mathbb{R};\mathbb{R}^n)$.

3. *The function* $\frac{\partial \bar{u}}{\partial \nu}(\xi,\nu)\mid_{\nu=0}$ *is a solution of*

$$\frac{dp}{d\xi}(\xi) = \frac{dF}{du}(\bar{u}(\xi))p(\xi),$$

and satisfies $\lim_{\xi\to\pm\infty}\frac{\partial \bar{u}}{\partial \nu}(\xi,0) = 0$. *Moreover, the function* $\frac{\partial \bar{u}}{\partial \nu}(\xi,0)$ *is an element of* $C^\infty_{\mathrm{exp}}(\mathbb{R};\mathbb{R}^n)$.

4. *The vector valued functions* $\frac{\partial \bar{u}}{\partial \xi}(\xi)$ *and* $\frac{\partial \bar{u}}{\partial \nu}(\xi,0)$ *are linearly independent for all* $\xi \in \mathbb{R}$ *and span the intersection of the tangent spaces,*

$$T_{\bar{u}(\xi)}W^u(u^-) \cap T_{\bar{u}(\xi)}W^s(u^+) = \mathrm{span}\left\{\frac{\partial \bar{u}}{\partial \xi}(\xi), \frac{\partial \bar{u}}{\partial \nu}(\xi,0)\right\}. \tag{2.22}$$

Proof. The first two statements are obvious. The family of profiles $\bar{u}(\xi,\nu)$ for sufficiently small ν solves the profile equation (2.2) and has the asymptotic behavior $\lim_{\xi\to\pm\infty}\bar{u}(\xi,\nu) = u^\pm(\mu(\nu))$. We differentiate the profile equation with respect to ν,

$$\frac{\partial}{\partial \xi}\frac{\partial \bar{u}}{\partial \nu}(\xi,\nu) = \frac{dF}{du}(\bar{u}(\xi,\nu),\mu(\nu))\frac{\partial \bar{u}}{\partial \nu}(\xi,\nu) + \frac{\partial F}{\partial \mu}(\bar{u}(\xi,\nu),\mu(\nu))\frac{d\mu}{d\nu}(\nu),$$

evaluate the derivative at $\nu = 0$ and use $\mu(0) = 0$ as well as (2.18) to obtain

$$\frac{\partial}{\partial \xi}\frac{\partial \bar{u}}{\partial \nu}(\xi,0) = \frac{dF}{du}(\bar{u}(\xi))\frac{\partial \bar{u}}{\partial \nu}(\xi,0).$$

Additionally, the function $\frac{\partial \bar{u}}{\partial \nu}(\xi, 0)$ approaches the endstates

$$\lim_{\xi \to \pm\infty} \frac{\partial \bar{u}}{\partial \nu}(\xi, 0) = \frac{\partial u^{\pm}}{\partial \mu} \frac{d\mu}{d\nu}(0),$$

which coincide with the null vector by the identity (2.18). The matrix $\frac{dF}{du}(\bar{u}(\xi))$ in the linearized profile equation approaches hyperbolic matrices $\frac{dF}{du}(u^{\pm})$ with constant coefficients. This fact and the result of Theorem 8.1 in [CL55, chapter 3] imply that the function $\frac{\partial \bar{u}}{\partial \nu}(\xi, 0)$ converges exponentially fast to the null vector as ξ tends to $\pm\infty$.

Finally, we prove the fourth statement. By the result of Lemma A.8, the viscous profiles $\bar{u}(\xi, \nu)$ can be written as the sum of the unperturbed profile $\bar{u}(\xi)$ and the function $z^*(\nu, \mu(\nu))(\xi)$. Moreover, the derivatives $\frac{\partial \bar{u}}{\partial \nu}(0, 0)$ and $\frac{\partial \bar{u}}{\partial \xi}(0, 0)$ are linearly independent and satisfy

$$T_{\bar{u}(0)} W^u(u^-) \cap T_{\bar{u}(0)} W^s(u^+) = \operatorname{span}\left\{ \frac{\partial \bar{u}}{\partial \xi}(0, 0), \frac{\partial \bar{u}}{\partial \nu}(0, 0) \right\}.$$

The associated solutions, $\frac{\partial \bar{u}}{\partial \nu}(\xi, 0)$ and $\frac{\partial \bar{u}}{\partial \xi}(\xi, 0)$, of the linearized profile equation remain linearly independent, which implies that the identity (2.22) holds for all $\xi \in \mathbb{R}$. $\square$

In Theorem 2.1 we observed that a unique (up to a multiplicative factor) globally bounded solution of the adjoint problem (2.11) exists. In the following, we construct this solution. At first, we note a basic fact.

Lemma 2.2. *Suppose $A : \mathbb{R} \to \mathbb{R}^{n\times n}$, $\xi \mapsto A(\xi)$ is a quadratic matrix with continuous coefficients. If $p(\xi)$ is a solution of the associated linear system of ODEs in $\mathbb{R}^n$,*

$$\frac{dp}{d\xi}(\xi) = A(\xi)p(\xi),$$

and $\psi(\xi)$ is a solution of the adjoint problem,

$$\frac{d\psi}{d\xi}(\xi) = -A^T(\xi)\psi(\xi),$$

then their inner product is constant.

Proof. The derivative of the inner product $<\psi, p>(\xi)$ is zero:

$$\begin{aligned}\frac{d}{d\xi} < \psi, p > (\xi) &=< \frac{d\psi}{d\xi}, p > (\xi)+ < \psi, \frac{dp}{d\xi} > (\xi)\\ &=< -A^T\psi, p > (\xi)+ < \psi, Ap > (\xi)\\ &= - < \psi, Ap > (\xi)+ < \psi, Ap > (\xi)\\ &= 0.\end{aligned}$$

Hence, the inner product $<\psi, p>(\xi)$ is constant. $\square$

Any tangent vector of the invariant manifolds $W^u(u^-)$ and $W^s(u^+)$ is a solution of the linearized profile equation (2.10). We will construct a solution of the adjoint system that is orthogonal to these solutions via a generalization of the cross product to higher dimensions.

Definition 2.1 ([Blo79]). Let $n \geq 2$ and e_i with $i = 1, \dots, n$ denote the Euclidean basis vectors of the real vector space $\mathbb{R}^n$. For $n-1$ vectors $p_1, \dots, p_{n-1}$ in $\mathbb{R}^n$, we define the *(generalized) cross product* as the vector

$$p_1 \times \cdots \times p_{n-1} = \sum_{j=1}^{n} \det\big(p_1, \dots, p_{n-1}, e_j\big)e_j.$$

We state some properties of the (generalized) cross product.

Lemma 2.3 ([Blo79]). *Let $n \geq 2$ and w as well as p_i with $i = 1, \dots, n-1$ be vectors in $\mathbb{R}^n$.*

1. *The matrix spanned by the given vectors satisfies the identity*

$$\det(p_1, \dots, p_{n-1}, w) = < p_1 \times \cdots \times p_{n-1}, w > . \tag{2.23}$$

2. *The cross product $p_1 \times \cdots \times p_{n-1}$ is perpendicular to any vector p_i with $i = 1, \dots, n-1$.*

3. *The cross product $p_1 \times \cdots \times p_{n-1}$ is equal to the null vector if and only if the vectors p_i with $i = 1, \dots, n-1$ are linearly dependent.*

4. *In addition, let C be a quadratic matrix whose coefficients c_{ij} are defined by $c_{ij} = \frac{<p_i, p_j>}{\|p_i\| \cdot \|p_j\|}$ for $i, j = 1, \dots, n-1$. Then the length of the cross product satisfies*

$$\|p_1 \times \cdots \times p_{n-1}\| = \|p_1\| \cdot \|p_2\| \cdot \ \cdots \ \cdot \|p_{n-1}\| \cdot (\det(C))^{1/2}.$$

This allows to construct the bounded solution of the adjoint problem from the tangent vectors.

Theorem 2.4. *Suppose the assumptions (B1)-(B4) hold. Let*

$$\{p_i \in C^1(\mathbb{R}; \mathbb{R}^n) \mid i = 1, \dots, n-1\}$$

be a basis for the sum of the tangent spaces $T_{\bar{u}(\xi)} W^u(u^-)$ and $T_{\bar{u}(\xi)} W^s(u^+)$. Then the tangent vectors $p_i(\xi)$ for $i = 1, \dots, n-1$ are solutions of the linearized profile equation,

$$\frac{dp}{d\xi}(\xi) = \frac{dF}{du}(\bar{u}(\xi)) p(\xi), \tag{2.24}$$

which decay to zero in the limit $\xi \to -\infty$ and/or $\xi \to +\infty$. In addition, the function $\psi(\xi)$ defined as

$$\psi(\xi) := \exp\left(- \int_0^\xi \operatorname{trace}\left(\frac{dF}{du}(\bar{u}(x)) \right) dx \right) (p_1 \times \ldots \times p_{n-1})(\xi) \tag{2.25}$$

is the globally bounded solution of the adjoint problem,

$$\frac{d\psi}{d\xi}(\xi) = -\left(\frac{dF}{du}(\bar{u}(\xi))\right)^T \psi(\xi), \tag{2.26}$$

which is unique up to a multiplicative factor and is an element of $C^\infty_{\exp}(\mathbb{R};\mathbb{R}^n)$.

Remark 2.2. The construction of the bounded solution $\psi(\xi)$ in the case of planar [GH83, Pal84] and higher dimensional systems [BL90, Section 4] is well known. However, we use the concept of the generalized cross product to prove the result.

Proof. A solution of the profile equation in the invariant manifolds $W^u(u^-)$ and $W^s(u^+)$ approaches asymptotically a constant endstate as ξ tends to $-\infty$ and $+\infty$, respectively. The associated tangent vector solves the linearized profile equation and decays in the respective limit. By assumption (B4), the sum of the tangent spaces $T_{\bar{u}(\xi)}W^u(u^-)$ and $T_{\bar{u}(\xi)}W^s(u^+)$ has dimension $n-1$. Thus, there are $n-1$ linearly independent solutions of the linearized profile equation that decay in at least one limit.

We will prove that $\psi(\xi)$ is a solution of system (2.26) by a direct computation and use the short hand notation

$$A(\xi) := \frac{dF}{du}(\bar{u}(\xi)) \quad \text{as well as} \quad a(\xi) := \exp\Big(- \int_0^\xi \operatorname{trace}(A(x))dx\Big).$$

First, we obtain the derivative of the cross product

$$(p_1 \times \ldots \times p_{n-1})(\xi) = \sum_{j=1}^n \det\big(p_1(\xi), \ldots, p_{n-1}(\xi), e_j\big)e_j$$

via the Leibniz rule for determinants. Then, we observe from the proof of Theorem 1.4 in [CL55] that for a set of vectors $p_i \in \mathbb{R}^n$ with $i = 1, \ldots, n$ and a quadratic matrix $A \in \mathbb{R}^{n\times n}$ the identity

$$\sum_{i=1}^n \det(p_1, \ldots, p_{i-1}, Ap_i, p_{i+1}, \ldots, p_n) = \operatorname{trace}(A)\det(p_1, \ldots, p_n)$$

holds. In this way, we compute the derivative of the cross product as follows

$$
\begin{aligned}
\frac{d}{d\xi}(p_1 \times \ldots \times p_{n-1})(\xi) &= \frac{d}{d\xi}\sum_{j=1}^{n} \det\big(p_1(\xi), \ldots, p_{n-1}(\xi), e_j\big)e_j \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n-1} \det\left(p_1(\xi), \ldots, p_{i-1}(\xi), \frac{dp_i}{d\xi}(\xi), p_{i+1}(\xi), \ldots, p_{n-1}(\xi), e_j\right)e_j \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n-1} \det(p_1(\xi), \ldots, p_{i-1}(\xi), A(\xi)p_i(\xi), p_{i+1}(\xi), \ldots, p_{n-1}(\xi), e_j)e_j + \\
&\quad + \sum_{j=1}^{n} \det(p_1(\xi), \ldots, p_{n-1}(\xi), A(\xi)e_j)e_j - \\
&\quad - \sum_{j=1}^{n} \det(p_1(\xi), \ldots, p_{n-1}(\xi), A(\xi)e_j)e_j \\
&= \operatorname{trace}(A(\xi))(p_1 \times \ldots \times p_{n-1})(\xi) - \sum_{j=1}^{n} \det(p_1(\xi), \ldots, p_{n-1}(\xi), A(\xi)e_j)e_j.
\end{aligned}
$$

We differentiate the function $\psi(\xi)$ via the product rule and obtain

$$
\begin{aligned}
\frac{d\psi}{d\xi}(\xi) =& \frac{d}{d\xi}\big(a(\xi)(p_1 \times \ldots \times p_{n-1})(\xi)\big) \\
=& \frac{da}{d\xi}(\xi)(p_1 \times \ldots \times p_{n-1})(\xi) + a(\xi)\frac{d}{d\xi}(p_1 \times \ldots \times p_{n-1})(\xi) \\
=& -\operatorname{trace}(A(\xi))\psi(\xi) + \operatorname{trace}(A(\xi))\psi(\xi) - \\
& - a(\xi)\sum_{j=1}^{n} \det(p_1(\xi), \ldots, p_{n-1}(\xi), A(\xi)e_j)e_j \\
=& - a(\xi)\sum_{j=1}^{n} \det(p_1(\xi), \ldots, p_{n-1}(\xi), A(\xi)e_j)e_j.
\end{aligned}
$$

We represent the vectors $A(\xi)e_j$ with respect to the Euclidean basis, $A(\xi)e_j = \sum_{k=1}^{n} < A(\xi)e_j, e_k > e_k$, and obtain after a change in the order of summation the stated result:

$$
\begin{aligned}
\frac{d\psi}{d\xi}(\xi) &= -\, a(\xi)\sum_{j=1}^{n}\det(p_1(\xi),\ldots,p_{n-1}(\xi),A(\xi)e_j)e_j \\
&= -\, a(\xi)\sum_{j=1}^{n}\sum_{k=1}^{n}\det(p_1(\xi),\ldots,p_{n-1}(\xi),<A(\xi)e_j,e_k>e_k)e_j \\
&= -\, a(\xi)\sum_{j=1}^{n}\sum_{k=1}^{n}\det(p_1(\xi),\ldots,p_{n-1}(\xi),e_k)<A(\xi)e_j,e_k>e_j \\
&= -\, a(\xi)\sum_{k=1}^{n}\det(p_1(\xi),\ldots,p_{n-1}(\xi),e_k)\sum_{j=1}^{n}<\big(A(\xi)\big)^T e_k,e_j>e_j \\
&= -\, a(\xi)\sum_{k=1}^{n}\det(p_1(\xi),\ldots,p_{n-1}(\xi),e_k)\big(A(\xi)\big)^T e_k \\
&= -\,\big(A(\xi)\big)^T\left(a(\xi)\sum_{k=1}^{n}\det(p_1(\xi),\ldots,p_{n-1}(\xi),e_k)e_k\right) \\
&= -\,\big(A(\xi)\big)^T\psi(\xi).
\end{aligned}
$$

Next, we prove that the function $\psi(\xi)$ is bounded on $\mathbb{R}$. The functions $p_i(\xi)$ with $i=1,\ldots,n-1$ are linearly independent solutions of (2.24) whose norm decays in at least one limit. In addition, we denote with $p_n(\xi)$ the solution of (2.24) whose norm becomes unbounded in both limits. The solution of the adjoint problem, $\psi(\xi)$, has a constant inner product with any solution of (2.24). In particular, the function $\psi(\xi)$ is orthogonal to $p_i(\xi)$ for $i=1,\ldots,n-1$ by construction and the inner product with the solution $p_n(\xi)$ is non-zero. Hence, the function $\psi(\xi)$ is bounded on $\mathbb{R}$, since it has to compensate for the unbounded growth of $p_n(\xi)$. Moreover, the norm $\|\psi\|(\xi)$ decays in both limits.

Finally, we show that the norm $\|\psi\|(\xi)$ decays exponentially to zero as ξ tends to $\pm\infty$. We conclude from Theorem 8.1 in [CL55, chapter 3] that the solutions $p_i(\xi)$ with $i=1,\ldots,n$ have asymptotic behavior

$$\|p_i\|(\xi)\sim\exp(\lambda_{j_i^{\pm}}(u^{\pm})\xi)$$

as ξ tends to $\pm\infty$ and the indices satisfy $\{j_i^\pm \mid i = 1, \dots, n\} = \{1, \dots, n\}$. Since we assume that $p_n(\xi)$ is the solution of (2.24) whose norm becomes unbounded in both limits, its exponential rates satisfy $\lambda_{j_n^-} < 0$ and $\lambda_{j_n^+} > 0$. The norm of the cross product $(p_1 \times \cdots \times p_{n-1})(\xi)$ has asymptotic behavior

$$\big\|(p_1 \times \cdots \times p_{n-1})\big\|(\xi) \sim \exp\Bigg(\sum_{i=1, i\neq j_n^\pm}^{n} \lambda_i(u^\pm)\xi \Bigg).$$

Therefore, the norm of the function $\psi(\xi)$ satisfies

$$\begin{aligned} \|\psi\|(\xi) &= \exp\Bigg(-\int_0^\xi \operatorname{trace}(A(x))dx \Bigg) \big\|(p_1 \times \cdots \times p_{n-1})\big\|(\xi) \\ &\sim \exp\Bigg(-\sum_{i=1}^{n} \lambda_i(u^\pm)\xi \Bigg) \exp\Bigg(\sum_{i=1, i\neq j_n^\pm}^{n} \lambda_i(u^\pm)\xi \Bigg) \\ &= \exp\big(-\lambda_{j_n^\pm}(u^\pm)\xi \big) \end{aligned}$$

and decays to zero exponentially fast as ξ tends to $\pm\infty$. Hence, the smooth function $\psi(\xi)$ is an element of $C^\infty_{\exp}(\mathbb{R};\mathbb{R}^n)$. $\square$

2.3 Effective eigenvalue $\kappa = 0$

We now investigate the spectral stability of the family of viscous shock waves $u(x,t;\nu) = \bar{u}(\xi, \mu(\nu))$ for sufficiently small ν, that means we study the spectrum of the associated linear operators

$$L(\nu)p(\xi) = \frac{d}{d\xi}\left(\frac{dp}{d\xi}(\xi) - \frac{dF}{du}(\bar{u}(\xi,\nu), \mu(\nu))p(\xi)\right). \tag{2.27}$$

The Evans function theory can be easily extended to incorporate the smooth dependence on the parameter ν. We define the extended Evans function and state its properties.

Theorem 2.5. *Suppose the assumptions (B1)-(B5) hold and δ is a sufficiently small, positive constant. By Corollary 1.1, there exist for sufficiently small ν solutions of the eigenvalue equation,*

$$\frac{d}{d\xi}\begin{pmatrix} p \\ q \end{pmatrix}(\xi) = \begin{pmatrix} \frac{dF}{du}(\bar{u}(\xi,\nu), \mu(\nu)) & I_n \\ \kappa I_n & 0_n \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}(\xi), \tag{2.28}$$

which span the matrices U^f, U^s, S^f and S^s in Remark 1.5. The Evans function $E : B_\delta(0) \times (-\delta, \delta) \subset \mathbb{C} \times \mathbb{R} \to \mathbb{C}$, $(\kappa, \nu) \mapsto E(\kappa, \nu)$, is defined as

$$E(\kappa, \nu) = \det\big(U^f, U^s, S^f, S^s\big)(0, \kappa, \nu). \tag{2.29}$$

In addition, it exhibits the following properties:

1. *The Evans function is analytic in $\kappa \in B_\delta(0)$ and smooth in $\nu \in (-\delta, \delta)$.*
2. *The Evans function can be constructed such that its restriction to a real domain is real valued.*
3. *The zero set of the Evans function coincides with the effective point spectrum and the multiplicity of an effective eigenvalue equals its order as a root of the Evans function.*

Subsequently, we determine the multiplicity of the effective eigenvalue zero via the extended Evans function.

2.3.1 Genuine eigenfunctions

We will identify genuine eigenfunctions for the effective eigenvalue zero.

Theorem 2.6. *Suppose the assumptions (B1)-(B5) hold. Then the functions*

$$\begin{pmatrix} \frac{\partial \bar{u}}{\partial \xi} \\ 0 \end{pmatrix}(\xi) \quad \text{and} \quad \begin{pmatrix} \frac{\partial \bar{u}}{\partial \nu} \\ 0 \end{pmatrix}(\xi, 0) \tag{2.30}$$

are linearly independent, genuine eigenfunctions for the effective eigenvalue zero. Moreover, the Evans function in Theorem 2.5 and its derivatives satisfy the identities

$$E(0,0) = 0, \quad \frac{\partial E}{\partial \kappa}(0,0) = 0 \quad \text{and} \quad \frac{\partial E}{\partial \nu}(0,0) = 0. \tag{2.31}$$

Proof. By the results of Lemma 2.1, the functions (2.30) are solutions of the eigenvalue equation (2.28) for $(\kappa, \nu) = (0,0)$ and decay exponentially fast to zero as ξ tends to $\pm\infty$. Therefore the functions (2.30) are genuine eigenfunctions.

Finally, we verify the identities for the Evans function, which is defined in Theorem 2.5 as

$$E(\kappa, \nu) = \det\left(U^f, U^s, S^f, S^s\right)(0, \kappa, \nu).$$

Since the solutions (2.30) are of the kind proposed in Corollary 1.2 and globally bounded, we can assume without loss of generality that

$$U_1^f(\xi, 0, 0) = S_1^f(\xi, 0, 0) = \begin{pmatrix} \frac{\partial \bar{u}}{\partial \xi} \\ 0 \end{pmatrix}(\xi) \tag{2.32}$$

and

$$U_2^f(\xi, 0, 0) = S_2^f(\xi, 0, 0) = \begin{pmatrix} \frac{\partial \bar{u}}{\partial \nu} \\ 0 \end{pmatrix}(\xi, 0). \tag{2.33}$$

The assumptions (2.32) and (2.33) imply that the Evans function $E(0,0)$ has two pairs of linearly dependent vectors and vanishes at $(\kappa, \nu) = (0,0)$. We transform the Evans function with respect to the linear dependencies,

$$E(\kappa,\nu) = \det\left(U^f, U^s, S_1^f - U_1^f, S_2^f - U_2^f, S_3^f, \dots, S_k^f, S^s\right)(0,\kappa,\nu),$$

and differentiate with respect to κ by the Leibniz rule. The derivative is a sum of determinants, where in each summand a difference vector $(S_1^f - U_1^f)(0,\kappa,\nu)$ and/or $(S_2^f - U_2^f)(0,\kappa,\nu)$ is left. Thus $\frac{\partial E}{\partial \kappa}(0,0)$ equals zero, since the difference vectors coincide with the null vector at $(\kappa,\nu) = (0,0)$. In a similar way, we prove that $\frac{\partial E}{\partial \nu}(0,0)$ vanishes. □

Any tangent vector associated to the invariant manifolds $W^u(u^-)$ and $W^s(u^+)$ satisfies the eigenvalue equation (2.27) for $(\kappa,\nu) = (0,0)$, since it solves the linearized profile equation

$$\frac{dp}{d\xi}(\xi) = \frac{dF}{du}(\bar{u}(\xi))p(\xi).$$

However, only a tangent vector, that is lying in the intersection of the tangent spaces $T_{\bar{u}(\xi)}W^u(u^-)$ and $T_{\bar{u}(\xi)}W^s(u^+)$, is a bounded function. By assumption (B4), the intersection is two-dimensional and, by Lemma 2.1, it is spanned by the functions $\frac{\partial \bar{u}}{\partial \xi}(\xi)$ and $\frac{\partial \bar{u}}{\partial \nu}(\xi,0)$. Thus we conclude that the functions (2.30) are the only genuine eigenfunctions for the effective eigenvalue zero, whose q-coordinates vanish identically.

The result of Lemma 2.1 implies the following result.

Lemma 2.4. *If the assumptions (B1)-(B5) hold, then for all sufficiently small ν the derivative of the viscous profile $\bar{u}(\xi,\nu)$ with respect to ξ is a genuine eigenfunction to the effective eigenvalue zero.*

Corollary 2.1. *If the assumptions (B1)-(B5) hold, then all derivatives of the Evans function with respect to the parameter ν at the point $(\kappa,\nu) = (0,0)$ are zero, i.e.*

$$\frac{d^n E}{d\nu^n}(0,\nu) = 0, \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{2.34}$$

Proof. The Evans function vanishes identically for $\kappa = 0$ and sufficiently small ν, and we conclude the statement. □

Similarly, one can investigate if the eigenvalue equation for $(\kappa, \nu) = (0, 0)$ has solutions, whose q-vector is constant but different from the null vector.

Lemma 2.5. *([Pal84]) Let $A(t)$ be an $n \times n$ matrix function bounded and continuous on $\mathbb{R}$ such that the system*

$$\frac{dx}{dt}(t) = A(t)x(t) \tag{2.35}$$

has an exponential dichotomy on both half lines. Then the linear operator

$$L : C^1(\mathbb{R}, \mathbb{R}^n) \to C^0(\mathbb{R}, \mathbb{R}^n)$$

defined by

$$(Lx)(t) = \frac{dx}{dt}(t) - A(t)x(t)$$

is Fredholm and $f \in \mathrm{image}(L)$ if and only if

$$\int_{-\infty}^{+\infty} < \psi(t), f(t) > dt = 0$$

for all bounded solutions $\psi(t)$ of the adjoint system

$$\frac{d\psi}{dt}(t) = -A^T(t)\psi(t).$$

The index of L is $\dim V + \dim W - n$, where V and W are the stable and unstable subspaces for (2.35).

Later we will need the following technical result, which follows from Palmer's Lemma 2.5.

Lemma 2.6. *Suppose the assumptions (B1)-(B5) hold. The solutions $p_i(\xi)$ for $i = 1, \ldots, n-1$ and $\psi(\xi)$ of the linearized profile equation and the adjoint differential equation, respectively, are taken from Theorem 2.4. If $y(\xi)$ is a bounded solution of the inhomogeneous differential equation*

$$\frac{dy}{d\xi}(\xi) = \frac{dF}{du}(\bar{u}(\xi))y(\xi) + b(\xi),$$

where the inhomogeneity $b : \mathbb{R} \to \mathbb{R}^n$ is a bounded function, then we obtain

$$\begin{aligned}\det(p_1, \ldots, p_{n-1}, y)(0) &= -\int_0^{+\infty} < \psi, b > (\xi) d\xi \\ &= \int_{-\infty}^0 < \psi, b > (\xi) d\xi.\end{aligned}$$

Proof. We use the short hand notation

$$A(\xi) := \frac{dF}{du}(\bar{u}(\xi)) \quad \text{as well as} \quad a(\xi) := \exp\bigg(-\int_0^{\xi} \operatorname{trace}(A(x))dx \bigg).$$

The function $\psi(\xi)$ is defined in Theorem 2.4 as the product of a scalar exponential factor and a cross product of vectors,

$$\psi(\xi) : = a(\xi)(p_1 \times \cdots \times p_{n-1})(\xi).$$

Therefore, the properties of the cross product as stated in Lemma 2.3 imply for a function $f \in C(\mathbb{R}; \mathbb{R}^n)$ and for all $\xi \in \mathbb{R}$ the identity

$$\begin{aligned}< \psi, f > (\xi) &= a(\xi) < (p_1 \times \ldots \times p_{n-1})(\xi), f(\xi) > \\ &= a(\xi) \det(p_1, \ldots, p_{n-1}, f)(\xi). \end{aligned} \tag{2.36}$$

We differentiate the scalar function $< \psi, y > (\xi)$ and obtain

$$\begin{aligned}\frac{\partial}{\partial \xi} < \psi, y > (\xi) &=< \frac{d\psi}{d\xi}, y > (\xi) + < \psi, \frac{dy}{d\xi} > (\xi) \\ &=< -A^T \psi, y > (\xi) + < \psi, Ay + b > (\xi) \\ &= - < \psi, Ay > (\xi) + < \psi, Ay > (\xi) + < \psi, b > (\xi) \\ &=< \psi, b > (\xi).\end{aligned}$$

The scalar function $< \psi, b > (\xi)$ is integrable, since its the inner product of a smooth L^1-integrable function $\psi(\xi)$ and a bounded continuous function $b(\xi) \in C_b(\mathbb{R}; \mathbb{R}^n)$. We integrate the identity $\frac{\partial}{\partial \xi} < \psi, y > (\xi) =< \psi, b > (\xi)$

from 0 to $+\infty$ and obtain from (2.36) for the left hand side

$$\int_0^{+\infty} \frac{\partial}{\partial \xi} < \psi, y > (\xi) d\xi = 0 - < \psi, y > (0) = -\det(p_1, \ldots, p_{n-1}, y)(0).$$

Thus we derive that the first identity

$$\det(p_1, \ldots, p_{n-1}, y)(0) = -\int_0^{+\infty} < \psi, b > (\xi) d\xi$$

holds. In a similar way we obtain the second identity

$$\det(p_1, \ldots, p_{n-1}, y)(0) = \int_{-\infty}^{0} < \psi, b > (\xi) d\xi.$$

□

There exist many bounded solutions of the eigenvalue equation (2.28) for $(\kappa, \nu) = (0, 0)$.

Lemma 2.7. *Suppose the assumptions (B1)-(B5) hold. Then the eigenvalue equation* (2.28) *for* $(\kappa, \nu) = (0, 0)$ *has* $n - 1$ *linearly independent, globally bounded solutions, whose q-vector is constant but different from the null vector.*

Proof. A solution of the proposed form has to satisfy the differential equation

$$\frac{dp}{d\xi}(\xi) = \frac{dF}{du}(\bar{u}(\xi)) p(\xi) + q \tag{2.37}$$

with a constant vector $q \in \mathbb{R}^n \backslash \{0\}$. By the result of Lemma 1.2, the homogeneous system associated to (2.37) has exponential dichotomies on $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively. Thus Palmer's Lemma 2.5 is applicable. By the result of Theorem 2.1, a unique (up to a multiplicative factor) bounded solution $\psi(\xi)$ of the adjoint problem exists. Therefore a bounded solution of (2.37) exists if and only if the identity

$$\int_{-\infty}^{+\infty} < \psi(\xi), q > d\xi = 0$$

holds. For constant vectors q, we obtain a well-defined linear system of equations

$$< \int_{-\infty}^{+\infty} \psi(\xi) d\xi, q >= 0,$$

where the vector $\int_{-\infty}^{+\infty} \psi(\xi) d\xi$ is different from the null vector. Hence, the kernel is $n-1$ dimensional and we conclude the statement. $\square$

2.3.2 Multiplicity of the effective eigenvalue zero - I

By the result of Lemma 2.7, the eigenvalue equation (2.28) for $(\kappa, \nu) = (0, 0)$ has $n-1$ bounded solutions with a constant q-vector different from the null vector, but only solutions in the non-trivial intersection of the spaces

$$\operatorname{span}\left\{ \eta_j^u(\xi) = \begin{pmatrix} p_j(\xi) \\ r_j(u^-) \end{pmatrix} \middle| \, j = 1, \ldots, k-1 \right\} \subset W^u(0) \tag{2.38}$$

and

$$\operatorname{span}\left\{ \eta_j^s(\xi) = \begin{pmatrix} p_{n+j}(\xi) \\ r_j(u^+) \end{pmatrix} \middle| \, j = k+1, \ldots, n \right\} \subset W^s(0) \tag{2.39}$$

are effective eigenfunctions. If the Liu-Majda condition (1.31) holds, i.e.

(B6) $\det(r_1(u^-), \ldots, r_{k-1}(u^-), u^+ - u^-, r_{k+1}(u^+), \ldots, r_n(u^+)) \neq 0,$

then the intersection of the spaces (2.38) and (2.39) is necessarily trivial. In agreement with the modified Fredholm theory in Lemma 1.7, a generalized eigenfunction has to be a solution $p(\xi)$ of the generalized eigenvalue equation $L(Lp) = 0$. The functions $\frac{\partial \bar{u}}{\partial \xi}(\xi)$ and $\frac{\partial \bar{u}}{\partial \nu}(\xi, 0)$ are the only genuine eigenfunctions and are L^1-integrable. Hence, an associated solution $p(\xi)$ of the generalized eigenvalue equation $L(Lp) = 0$ has to satisfy the equation

$$Lp(\xi) = \frac{d}{d\xi}\left(\frac{dp}{d\xi}(\xi) - \frac{dF}{du}(\bar{u}(\xi), 0)p(\xi) \right) = \gamma_1 \frac{\partial \bar{u}}{\partial \nu}(\xi, 0) + \gamma_2 \frac{\partial \bar{u}}{\partial \xi}(\xi)$$

for some real constants γ_1 and γ_2. After integrating the last identity with respect to ξ, we obtain the inhomogeneous linear system of ordinary differential equations

$$\frac{dp}{d\xi}(\xi) = \frac{dF}{du}(\bar{u}(\xi), 0)p(\xi) + \tilde{b}(\xi) \tag{2.40}$$

with a continuous and bounded inhomogeneity

$$\tilde{b}(\xi) := \int_{-\infty}^{\xi} \left(\gamma_1 \frac{\partial \bar{u}}{\partial \nu}(x, 0) + \gamma_2 \frac{\partial \bar{u}}{\partial \xi}(x) \right) dx. \tag{2.41}$$

We will relate the existence of a bounded solution of (2.40) to the vanishing of the second order derivative of the Evans function, $\frac{\partial^2 E}{\partial \kappa^2}(\kappa, \nu)$, at the origin.

The following preliminary result is a direct consequence of the results in Lemma 2.1.

Lemma 2.8. *Suppose the assumptions (B1)-(B5) hold. Then the function*

$$\bar{v}(\xi) := \int_{-\infty}^{\xi} \frac{\partial \bar{u}}{\partial \nu}(x, 0) dx$$

is continuous and bounded on $\mathbb{R}$. *In addition,* $\bar{v}(\xi)$ *approaches constant end-states* $v^{\pm} := \lim_{\xi \to \pm\infty} \bar{v}(\xi)$.

In preparation of Theorem 2.7 we derive an expression for the function $b(\xi)$.

Lemma 2.9. *Suppose the assumptions (B1)-(B6) hold. Then there exist real constants* $\varphi_i \in \mathbb{R}$ *for* $i = 1, \ldots, n$ *such that the identity*

$$-(v^+ - v^-) = \sum_{i=1}^{k-1} \varphi_i r_i(u^-) - \varphi_k(u^+ - u^-) + \sum_{i=k+1}^{n} \varphi_i r_i(u^+) \tag{2.42}$$

holds. The function

$$\begin{aligned} b(\xi) :&= \bar{v}(\xi) - v^+ - \varphi_k(\bar{u}(\xi) - u^+) - \sum_{i=k+1}^{n} \varphi_i r_i(u^+) \\ &= \bar{v}(\xi) - v^- - \varphi_k(\bar{u}(\xi) - u^-) + \sum_{i=1}^{k-1} \varphi_i r_i(u^-) \end{aligned}$$

is continuous and bounded on $\mathbb{R}$.

Proof. The assumption (B6) implies that the set of vectors

$$\{r_i(u^-) \mid i = 1, \dots, k-1\} \cup \{u^+ - u^-\} \cup \{r_i(u^+) \mid i = k, \dots, n\}$$

forms a basis of $\mathbb{R}^n$ and the vector $v^+ - v^-$ has a representation (2.42) with respect to this basis. Thus the function $b(\xi)$ is well-defined as a linear combination of continuous and bounded functions and inherits these properties. □

Theorem 2.7. *Suppose the assumptions (B1)-(B6) hold. Then the second order derivative of the Evans function with respect to the spectral parameter κ satisfies*

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) = c \cdot \int_{-\infty}^{+\infty} < \psi, b > (\xi) d\xi \cdot$$
$$\cdot \det(r_1(u^-), \dots, r_{k-1}(u^-), u^+ - u^-, r_{k+1}(u^+), \dots, r_n(u^+)),$$

with a non-zero, real constant c, the function $\psi(\xi)$ in Theorem 2.4 and the function $b(\xi)$ in Lemma 2.9.

Proof. The assumptions imply by Theorem 2.5 the existence of the Evans function $E(\kappa, \nu)$. We assume without loss of generality that for sufficiently small ν individual solutions of the eigenvalue equation are given by

$$U_1^f(\xi, 0, \nu) = S_1^f(\xi, 0, \nu) = \begin{pmatrix} \frac{\partial \bar{u}}{\partial \xi} \\ 0 \end{pmatrix} (\xi, \nu) \tag{2.43}$$

and

$$U_2^f(\xi, 0, 0) = S_2^f(\xi, 0, 0) = \begin{pmatrix} \frac{\partial \bar{u}}{\partial \nu} \\ 0 \end{pmatrix} (\xi, 0). \tag{2.44}$$

Thus we rewrite the Evans function as in the proof of Theorem 2.6,

$$E(\kappa, \nu) = \det \left(U^f, U^s, S_1^f - U_1^f, S_2^f - U_2^f, S_3^f, \dots, S_k^f, S^s \right) (0, \kappa, \nu),$$

differentiate twice with respect to κ by the Leibniz rule and evaluate the derivative at $(\kappa, \nu) = (0,0)$ to obtain

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) = \det\left(U^f, U^s, \frac{\partial}{\partial \kappa}(S_1^f - U_1^f), \frac{\partial}{\partial \kappa}(S_2^f - U_2^f), S_3^f, \ldots, S_k^f, S^s\right)(0).$$

All other summands vanish at $(\kappa, \nu) = (0,0)$, since they contain a vector $(S_1^f - U_1^f)(0,0,0)$ and/or $(S_2^f - U_2^f)(0,0,0)$ which coincide with the null vector.

We consider the solutions of the eigenvalue equation (2.28) satisfying the identities (2.43) and (2.44). Their derivatives with respect to the spectral parameter κ are governed by the system of differential equations

$$\frac{\partial}{\partial \xi}\begin{pmatrix} \frac{\partial p}{\partial \kappa} \\ \frac{\partial q}{\partial \kappa} \end{pmatrix}(\xi, \kappa, \nu) = \begin{pmatrix} \frac{dF}{du}(\bar{u}(\xi, \nu), \mu(\nu)) & I_n \\ \kappa I_n & 0_n \end{pmatrix}\begin{pmatrix} \frac{\partial p}{\partial \kappa} \\ \frac{\partial q}{\partial \kappa} \end{pmatrix}(\xi, \kappa, \nu) + \begin{pmatrix} 0 \\ p \end{pmatrix}(\xi, \kappa, \nu).$$

In Lemma 1.9, we obtained the expressions

$$\frac{\partial U_1^f}{\partial \kappa}(\xi, 0, 0) = \begin{pmatrix} z_1(\xi) \\ \bar{u}(\xi) - u^- \end{pmatrix} \quad \text{and} \quad \frac{\partial S_1^f}{\partial \kappa}(\xi, 0, 0) = \begin{pmatrix} z_{n+1}(\xi) \\ \bar{u}(\xi) - u^+ \end{pmatrix},$$

where the functions $z_i(\xi)$ are defined as $z_i(\xi) := \frac{\partial p_i}{\partial \kappa}(\xi, 0, 0)$ for $i = 1, n+1$. In a similar way, we derive

$$\frac{\partial U_2^f}{\partial \kappa}(\xi, 0, 0) = \begin{pmatrix} z_2(\xi) \\ \bar{v}(\xi) - v^- \end{pmatrix} \quad \text{and} \quad \frac{\partial S_2^f}{\partial \kappa}(\xi, 0, 0) = \begin{pmatrix} z_{n+2}(\xi) \\ \bar{v}(\xi) - v^+ \end{pmatrix},$$

where the continuous and bounded function $\bar{v}(\xi) := \int_{-\infty}^{\xi} \frac{\partial \bar{u}}{\partial \nu}(x, 0)dx$ with asymptotic endstates $v^{\pm} := \lim_{\xi \to \pm\infty} \bar{v}(\xi)$ is taken from Lemma 2.8 and the functions $z_i(\xi)$ are defined as $z_i(\xi) := \frac{\partial p_i}{\partial \kappa}(\xi, 0, 0)$ for $i = 2, n+2$. We insert these expressions into the derivative of the Evans function $E(\kappa, \nu)$ at $(\kappa, \nu) = (0,0)$ and obtain

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) = \det\begin{pmatrix} U_p^f & U_p^s & z_{n+1} - z_1 & z_{n+2} - z_2 & \tilde{S}_p^f & S_p^s \\ 0 & U_q^s & -[\bar{u}] & -[\bar{v}] & 0 & S_q^s \end{pmatrix}(0) \qquad (2.45)$$

with matrices

$$
\begin{aligned}
U_p^f(0) &:= (p_1, \ldots, p_{n-k+1})(0) \in \mathbb{R}^{n\times(n-k+1)},\\
U_p^s(0) &:= (p_{n-k+2}, \ldots, p_n)(0) \in \mathbb{R}^{n\times(k-1)},\\
U_q^s(0) &:= (r_1(u^-), \ldots, r_{k-1}(u^-)) \in \mathbb{R}^{n\times(k-1)},\\
\tilde{S}_p^f(0) &:= (p_{n+3}, \ldots, p_{n+k})(0) \in \mathbb{R}^{n\times(k-2)},\\
S_p^s(0) &:= (p_{n+k+1}, \ldots, p_{2n})(0) \in \mathbb{R}^{n\times(n-k)}
\end{aligned}
$$

and

$$S_q^s(0) := (r_{k+1}(u^+), \ldots, r_n(u^+)) \in \mathbb{R}^{n\times(n-k)}.$$

In the matrix within the determinant (2.45) the q-coordinates of $n+1$ vectors are different from the null vector. In addition, the assumption (B6) implies that there exist real constants $\varphi_i \in \mathbb{R}$ for $i = 1, \ldots, n$ such that the vector $[\bar{v}] = v^+ - v^-$ has a representation

$$-[\bar{v}] = \sum_{i=1}^{k-1} \varphi_i r_i(u^-) - \varphi_k[\bar{u}] + \sum_{i=k+1}^{n} \varphi_i r_i(u^+). \tag{2.46}$$

We take this linear combination into account and transform the determinant,

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) = \det \begin{pmatrix} U_p^f & U_p^s & z_{n+1} - z_1 & \tilde{z}_{n+2} - \tilde{z}_2 & \tilde{S}_p^f & S_p^s \\ 0 & U_q^s & -[\bar{u}] & 0 & 0 & S_q^s \end{pmatrix}(0),$$

where the auxiliary functions are defined as

$$\tilde{z}_{n+2}(\xi) : = z_{n+2}(\xi) - \varphi_k z_{n+1}(\xi) - \sum_{i=k+1}^{n} \varphi_i p_{n+i}(\xi)$$

and

$$\tilde{z}_2(\xi) : = z_2(\xi) - \varphi_k z_1(\xi) + \sum_{i=1}^{k-1} \varphi_i p_{n-k+1+i}(\xi),$$

respectively. In the next step, we change the order of the vectors by k^2-2 permutations to obtain a matrix in block diagonal form,

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) = (-1)^{k^2-2} \cdot \det \begin{pmatrix} U_p^f & \tilde{S}_p^f & \tilde{z}_{n+2}-\tilde{z}_2 & U_p^s & z_{n+1}-z_1 & S_p^s \\ 0 & 0 & 0 & U_q^s & -[\bar{u}] & S_q^s \end{pmatrix} (0),$$

and factorize the expression into the product

$$\begin{aligned} \frac{\partial^2 E}{\partial \kappa^2}(0,0) = (-1)^{k^2-1} \cdot \det(p_1, \dots, p_{n-k+1}, p_{n+3}, \dots, p_{n+k}, \tilde{z}_{n+2}-\tilde{z}_2)(0) \cdot \\ \cdot \det(r_1(u^-), \dots, r_{k-1}(u^-), u^+ - u^-, r_{k+1}(u^+), \dots, r_n(u^+)). \end{aligned} \tag{2.47}$$

We rewrite the first determinant as a sum of determinants and evaluate each summand in turn. The function $\tilde{z}_{n+2}(\xi)$ is governed by a linear differential equation

$$\begin{aligned} \frac{d\tilde{z}_{n+2}}{d\xi}(\xi) =& \frac{dz_{n+2}}{d\xi}(\xi) - \varphi_k \frac{dz_{n+1}}{d\xi}(\xi) - \sum_{i=k+1}^{n} \varphi_i \frac{dp_{n+i}}{d\xi}(\xi) \\ =& \frac{dF}{du}(\bar{u}(\xi)) z_{n+2}(\xi) + (\bar{v}(\xi) - v^+) \\ & - \varphi_k \left[\frac{dF}{du}(\bar{u}(\xi)) z_{n+1}(\xi) + (\bar{u}(\xi) - u^+) \right] \\ & - \sum_{i=k+1}^{n} \varphi_i \left[\frac{dF}{du}(\bar{u}(\xi)) p_{n+i}(\xi) + r_i(u^+) \right] \\ =& \frac{dF}{du}(\bar{u}(\xi)) \tilde{z}_{n+2}(\xi) + b^+(\xi) \end{aligned} \tag{2.48}$$

with inhomogeneity

$$b^+(\xi) := (\bar{v}(\xi) - v^+) - \varphi_k(\bar{u}(\xi) - u^+) - \sum_{i=k+1}^{n} \varphi_i r_i(u^+). \tag{2.49}$$

The functions $\tilde{z}_{n+2}(\xi)$ and $b^+(\xi)$ are bounded on $\mathbb{R}$, since they are linear combinations of bounded functions. Thus, the requirements of Lemma 2.6

are met and we obtain

$$\det(p_1, \ldots, p_{n-k+1}, p_{n+3}, \ldots, p_{n+k}, \tilde{z}_{n+2})(0) = -\int_0^{+\infty} < \psi, b^+ > (\xi) d\xi.$$

In a similar way, we derive the expression

$$\det(p_1, \ldots, p_{n-k+1}, p_{n+3}, \ldots, p_{n+k}, \tilde{z}_2)(0) = \int_{-\infty}^0 < \psi, b^- > (\xi) d\xi,$$

where the bounded function $b^-(\xi)$ is defined as

$$b^-(\xi) := (\bar{v}(\xi) - v^-) - \varphi_k(\bar{u}(\xi) - u^-) + \sum_{i=1}^{k-1} \varphi_i r_i(u^-).$$

The linear combination (2.46) implies the identity $b^+(\xi) \equiv b^-(\xi)$ and we define $b(\xi) := b^+(\xi)$. Thus we obtain the expression

$$\det(p_1, \ldots, p_{n-k+1}, p_{n+3}, \ldots, p_{n+k}, \tilde{z}_{n+2} - \tilde{z}_2)(0) = -\int_{-\infty}^{+\infty} < \psi, b > (\xi) d\xi$$

and conclude from (2.47) the stated result. □

In the following, we will prove the connection between the existence of a bounded solution of the generalized eigenvalue equation for $(\kappa, \nu) = (0, 0)$ and the second order derivative of the Evans function $\frac{\partial^2 E}{\partial \kappa^2}(0, 0)$.

Theorem 2.8. *Suppose the assumptions (B1)-(B6) hold. Then the second order derivative of the Evans function with respect to the spectral parameter,* $\frac{\partial^2 E}{\partial \kappa^2}(\kappa, \nu)$*, vanishes at* $(\kappa, \nu) = (0, 0)$*, if and only if there exists a generalized eigenfunction for the effective eigenvalue zero that is bounded on* $\mathbb{R}$ *and associated to the genuine eigenfunction* $\frac{\partial \bar{u}}{\partial \nu}(\xi, 0) + \varphi_k \frac{\partial \bar{u}}{\partial \xi}(\xi)$ *with the constant* φ_k *from Lemma 2.9.*

Proof. By the assumptions and the result of Theorem 2.7, the second order derivative of the Evans function, $\frac{d^2 E}{d\kappa^2}(0, 0)$, has a factorization into a product of non-zero factors and the definite integral $\int_{-\infty}^{+\infty} < \psi, b > (\xi) d\xi$, where the function $b(\xi)$ is taken from Lemma 2.9 and $\psi(\xi)$ is the unique (up to a multiplicative factor) bounded solution of the adjoint problem as in Theorem 2.1. By Palmer's Lemma 2.5, the condition

$$\int_{-\infty}^{+\infty} < \psi, b > (\xi) d\xi = 0$$

is equivalent to the existence of a bounded solution of the inhomogeneous linear system of differential equations (2.40). Since the inhomogeneity $b(\xi)$ has the proposed form (2.41) with $\gamma_1 = 1$ and $\gamma_2 = \varphi_k$, the statement follows.

□

2.3.3 Multiplicity of the effective eigenvalue zero - II

Finally, we consider a non-transversal viscous profile, whose associated Liu-Majda determinant vanishes:

(B7) The Liu-Majda condition fails, i.e.

$$\det(r_1(u^-), \dots, r_{k-1}(u^-), u^+ - u^-, r_{k+1}(u^+), \dots, r_n(u^+)) = 0.$$

Remark. The first derivative of the Evans function, $\frac{\partial E}{\partial \kappa}(0,0)$, depends on the Liu-Majda determinant and the transversality of the viscous profile. Hence, the assumption (B7) implies that the first derivative of the Evans function vanishes, $\frac{\partial E}{\partial \kappa}(0,0) = 0$.

Example 2.1. For some non-transversal, intermediate profiles in MHD the Liu-Majda determinant vanishes. We will present this example in the next chapter.

The multiplicity of an effective eigenvalue equals its order as a root of the Evans function. Whereas the identities $E(0,0) = 0$ and $\frac{\partial E}{\partial \kappa}(0,0) = 0$ still hold, we will have to compute the second order derivative of the Evans function anew.

In this regard, we will refine the assumption (B7) and consider the complementary cases:

(B7a) $\dim\left(\operatorname{span}\left\{r_1(u^-), ..., r_{k-1}(u^-), u^+ - u^-, r_{k+1}(u^+), ..., r_n(u^+)\right\}\right) \leq n-2.$

(B7b) $\dim\left(\operatorname{span}\left\{r_1(u^-), ..., r_{k-1}(u^-), u^+ - u^-, r_{k+1}(u^+), ..., r_n(u^+)\right\}\right) = n-1$ and the vectors $r_1(u^-), \ldots, r_{k-1}(u^-)$ and $r_{k+1}(u^+), \ldots, r_n(u^+)$ are linearly dependent.

(B7c) $\dim\left(\operatorname{span}\left\{r_1(u^-), ..., r_{k-1}(u^-), u^+ - u^-, r_{k+1}(u^+), ..., r_n(u^+)\right\}\right) = n-1$ and the vectors $r_1(u^-), \ldots, r_{k-1}(u^-)$ and $r_{k+1}(u^+), \ldots, r_n(u^+)$ are linearly independent.

Lemma 2.10. *Suppose the assumptions (B1)-(B5) and (B7a) hold. Then the second order derivative of the Evans function,* $\frac{\partial^2 E}{\partial \kappa^2}(\kappa, \nu)$*, vanishes at the origin.*

Proof. We follow the proof of Theorem 2.7 and obtain the second order derivative of the Evans function as

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) = \det \begin{pmatrix} U_p^f & U_p^s & z_{n+1} - z_1 & * & \tilde{S}_p^f & S_p^s \\ 0 & U_q^s & -[\bar{u}] & -[\bar{v}] & 0 & S_q^s \end{pmatrix} (0)$$

with matrices $U_p^f(0) \in \mathbb{R}^{n \times (n-k+1)}$, $U_p^s(0) \in \mathbb{R}^{n \times (k-1)}$, $\tilde{S}_p^f(0) \in \mathbb{R}^{n \times (k-2)}$ and $S_p^s(0) \in \mathbb{R}^{n \times (n-k)}$ as well as

$$U_q^s(0) : = (r_1(u^-), \ldots, r_{k-1}(u^-)) \in \mathbb{R}^{n \times (k-1)}$$

and

$$S_q^s(0) : = (r_{k+1}(u^+), \ldots, r_n(u^+)) \in \mathbb{R}^{n \times (n-k)}.$$

The assumption (B7a) implies that among the $n+1$ vectors in the second row of the matrix there are at most $n-1$ linearly independent ones. Hence, the matrix has not full rank and its determinant vanishes. $\square$

We deduce from the Lemmata 2.5 and 2.10 the following result.

Corollary 2.2. *Suppose the assumptions (B1)-(B5) and (B7a) hold. Then the multiplicity of the effective eigenvalue zero is at least three. Moreover, effective eigenfunctions are given by the bounded solutions of the differential equation*

$$\frac{dp}{d\xi}(\xi) = \frac{dF}{du}(\bar{u}(\xi), 0)p(\xi) + q$$

for any constant vector q in the non-trivial intersection of the spaces

$$\operatorname{span}\{r_1(u^-), \ldots, r_{k-1}(u^-)\} \quad \text{and} \quad \operatorname{span}\{r_{k+1}(u^+), \ldots, r_n(u^+)\}.$$

Theorem 2.9. *Suppose the assumptions (B1)-(B5) as well as (B7b) hold and without loss of generality the vector $r_1(u^-)$ has for real constants φ_i with $i = 1, \ldots, n$ a representation*

$$r_1(u^-) = \sum_{i=2}^{k-1} \varphi_i r_i(u^-) + \sum_{i=k+1}^{n} \varphi_i r_i(u^+).$$

Then the second order derivative of the Evans function satisfies

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) = c \cdot \int_{-\infty}^{+\infty} < \psi(\xi), \sum_{i=k+1}^{n} \varphi_i r_i(u^+) > d\xi \cdot$$
$$\cdot \det(r_2(u^-), \ldots, r_{k-1}(u^-), [\bar{u}], [\bar{v}], r_{k+1}(u^+), \ldots, r_n(u^+))$$

with a non-zero real constant c, the function $\psi(\xi)$ in Theorem 2.4 and the function $\bar{v}(\xi)$ in Lemma 2.8.

Proof. We follow the proof of Theorem 2.7 and obtain the second order derivative of the Evans function as

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) = \det \begin{pmatrix} U_p^f & U_p^s & z_{n+1} - z_1 & * & \tilde{S}_p^f & S_p^s \\ 0 & U_q^s & -[\bar{u}] & -[\bar{v}] & 0 & S_q^s \end{pmatrix} (0)$$

with matrices $U_p^f(0) \in \mathbb{R}^{n\times(n-k+1)}$, $U_p^s(0) \in \mathbb{R}^{n\times(k-1)}$, $\tilde{S}_p^f(0) \in \mathbb{R}^{n\times(k-2)}$ and $S_p^s(0) \in \mathbb{R}^{n\times(n-k)}$ as well as

$$U_q^s(0) := (r_1(u^-), \dots, r_{k-1}(u^-)) \in \mathbb{R}^{n\times(k-1)}$$

and

$$S_q^s(0) := (r_{k+1}(u^+), \dots, r_n(u^+)) \in \mathbb{R}^{n\times(n-k)}.$$

The assumption (B7b) implies that the matrix $(U_q^s, S_q^s)(0) \in \mathbb{R}^{n\times(n-1)}$ has rank $n-2$. We assume without loss of generality that the vector $r_1(u^-)$ has for some real constants φ_i with $i = \{2, \dots, k-1\} \cup \{k+1, \dots, n\}$ a representation

$$r_1(u^-) = \sum_{i=2}^{k-1} \varphi_i r_i(u^-) + \sum_{i=k+1}^{n} \varphi_i r_i(u^+). \tag{2.50}$$

We take this linear combination into account and transform the determinant such that

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) = \det \begin{pmatrix} U_p^f & \tilde{p}_{n-k+2} - \tilde{p}_{n+k+1} & \tilde{U}_p^s & * & * & \tilde{S}_p^f & S_p^s \\ 0 & 0 & \tilde{U}_q^s & -[\bar{u}] & -[\bar{v}] & 0 & S_q^s \end{pmatrix}(0)$$

with matrices

$$\tilde{U}_p^s(0) := (p_{n-k+3}, \dots, p_n)(0) \in \mathbb{R}^{n\times(k-2)},$$

and

$$\tilde{U}_q^s(0) := (r_2(u^-), \dots, r_{k-1}(u^-)) \in \mathbb{R}^{n\times(k-2)}.$$

The auxiliary functions are defined as $\tilde{p}_{n+k+1}(\xi) := \sum_{i=k+1}^{n} \varphi_i p_{n+i}(\xi)$ and $\tilde{p}_{n-k+2}(\xi) := -\sum_{i=1}^{k-1} \varphi_i p_{n-k+1+i}(\xi)$ with $\varphi_1 := -1$. We change the order of the vectors by an even number of permutations to obtain a matrix in block diagonal form,

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) = \det \begin{pmatrix} U_p^f & \tilde{S}_p^f & \tilde{p}_{n-k+2} - \tilde{p}_{n+k+1} & \tilde{U}_p^s & * & * & S_p^s \\ 0 & 0 & 0 & \tilde{U}_q^s & -[\bar{u}] & -[\bar{v}] & S_q^s \end{pmatrix}(0),$$

and factorize the expression into the product

$$\begin{aligned} \frac{\partial^2 E}{\partial \kappa^2}(0,0) = \det(U_p^f, \tilde{S}_p^f, \tilde{p}_{n-k+2} - \tilde{p}_{n+k+1})(0)\cdot \\ \cdot \det(r_2(u^-), \ldots, r_{k-1}(u^-), [\bar{u}], [\bar{v}], r_{k+1}(u^+), \ldots, r_n(u^+)). \end{aligned} \tag{2.51}$$

We write the first determinant as a sum of determinants and evaluate each summand in turn. The function $\tilde{p}_{n-k+2}(\xi)$ is governed by the differential equation

$$\begin{aligned} \frac{d\tilde{p}_{n-k+2}}{d\xi}(\xi) &= -\sum_{i=1}^{k-1} \varphi_i \frac{dp_{n-k+1+i}}{d\xi}(\xi) \\ &= -\sum_{i=1}^{k-1} \varphi_i \left(\frac{dF}{du}(\bar{u}(\xi)) p_{n-k+1+i}(\xi) + r_i(u^-) \right) \\ &= \frac{dF}{du}(\bar{u}(\xi)) \tilde{p}_{n-k+2}(\xi) - \sum_{i=1}^{k-1} \varphi_i r_i(u^-). \end{aligned}$$

The function $\tilde{p}_{n-k+2}(\xi)$ and the inhomogeneity $-\sum_{i=1}^{k-1} \varphi_i r_i(u^+)$ are bounded on $\mathbb{R}$, since they are linear combinations of bounded functions. Thus, the requirements of Lemma 2.6 are met and we obtain

$$\det(U_p^f, \tilde{S}_p^f, \tilde{p}_{n-k+2})(0) = -\int_{-\infty}^{0} < \psi(\xi), \sum_{i=1}^{k-1} \varphi_i r_i(u^-) > d\xi.$$

In a similar way, we derive the expression

$$\det(U_p^f, \tilde{S}_p^f, \tilde{p}_{n+k+1})(0) = -\int_{0}^{+\infty} < \psi(\xi), \sum_{i=k+1}^{n} \varphi_i r_i(u^+) > d\xi.$$

Thus the determinant satisfies

$$\begin{aligned}\det(U_p^f, \tilde{S}_p^f, \tilde{p}_{n-k+2} - \tilde{p}_{n+k+1})(0) &= -\int_{-\infty}^{0} < \psi(\xi), \sum_{i=1}^{k-1} \varphi_i r_i(u^-) > d\xi + \\ &+ \int_{0}^{+\infty} < \psi(\xi), \sum_{i=k+1}^{n} \varphi_i r_i(u^+) > d\xi \\ &= \int_{-\infty}^{+\infty} < \psi(\xi), \sum_{i=k+1}^{n} \varphi_i r_i(u^+) > d\xi.\end{aligned}$$

The last inequality holds by the assumption (B7b) and the linear combination (2.50) with $\varphi_1 = -1$. We combine the last expression and (2.51) to obtain the stated result. □

We infer from Palmer's Lemma 2.5 and Theorem 2.9 the following result.

Corollary 2.3. *Suppose the assumptions from Theorem 2.9 hold. Then the multiplicity of the effective eigenvalue zero is generically two. However a third effective eigenfunction for the effective eigenvalue zero exists, if the differential equation*

$$\frac{dp}{d\xi}(\xi) = \frac{dF}{du}(\bar{u}(\xi), 0)p(\xi) + \sum_{i=k+1}^{n} \varphi_i r_i(u^+)$$

with constants φ_i for $i = k+1, \ldots, n$ from Theorem 2.9 has a bounded solution.

In preparation of the analysis of the case (B7c) we derive an expression for the function $b(\xi)$.

Lemma 2.11. *Suppose the assumptions (B1)-(B5) and (B7c) hold. Then there exist real constants $\varphi_i \in \mathbb{R}$ for $i = 1, \ldots, n$ such that the identity*

$$-[\bar{u}] = \sum_{i=1}^{k-1} \varphi_i r_i(u^-) + \sum_{i=k+1}^{n} \varphi_i r_i(u^+) \tag{2.52}$$

holds. The function

$$\begin{aligned} b(\xi) :&= (\bar{u}(\xi) - u^+) - \sum_{i=k+1}^{n} \varphi_i r_i(u^+) \\ &= (\bar{u}(\xi) - u^-) + \sum_{i=1}^{k-1} \varphi_i r_i(u^-) \end{aligned}$$

is continuous and bounded on $\mathbb{R}$*.*

Proof. The assumption (B7c) implies that the vector $[\bar{u}] = u^+ - u^-$ has a representation with respect to the set of linearly independent vectors

$$\{r_i(u^-) \mid i = 1, \ldots, k-1\} \cup \{r_i(u^+) \mid i = k, \ldots, n\}.$$

Thus the function $b(\xi)$ is well-defined as a linear combination of continuous and bounded functions and inherits these properties. □

Theorem 2.10. *Suppose the assumptions (B1)-(B5) and (B7c) hold. Then the second order derivative of the Evans function with respect to the spectral parameter* κ *satisfies*

$$\begin{aligned} \frac{\partial^2 E}{\partial \kappa^2}(0,0) = c \cdot \int_{-\infty}^{+\infty} &< \psi, b > (\xi) d\xi \cdot \\ &\cdot \det(r_1(u^-), \ldots, r_{k-1}(u^-), [\bar{v}], r_{k+1}(u^+), \ldots, r_n(u^+)), \end{aligned}$$

with a non-zero real constant c, the bounded solution $\psi(\xi)$ *of the adjoint problem in Theorem 2.4 as well as continuous and bounded functions* $\bar{v}(\xi)$ *and* $b(\xi)$ *in the Lemmata 2.8 and 2.11, respectively.*

Proof. We follow the proof of Theorem 2.7 and obtain the second order derivative of the Evans function as

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) = \det \begin{pmatrix} U_p^f & U_p^s & z_{n+1} - z_1 & * & \tilde{S}_p^f & S_p^s \\ 0 & U_q^s & -[\bar{u}] & -[\bar{v}] & 0 & S_q^s \end{pmatrix} (0)$$

with matrices $U_p^f(0) \in \mathbb{R}^{n\times(n-k+1)}$, $U_p^s(0) \in \mathbb{R}^{n\times(k-1)}$, $\tilde{S}_p^f(0) \in \mathbb{R}^{n\times(k-2)}$ and $S_p^s(0) \in \mathbb{R}^{n\times(n-k)}$ as well as

$$U_q^s(0) := (r_1(u^-), \ldots, r_{k-1}(u^-)) \in \mathbb{R}^{n\times(k-1)}$$

and

$$S_q^s(0) := (r_{k+1}(u^+), \ldots, r_n(u^+)) \in \mathbb{R}^{n\times(n-k)}.$$

The assumption (B7c) implies that the vector $[\bar{u}] = u^+ - u^-$ has for some constants $\varphi_i \in \mathbb{R}$ with $i = 1, \ldots, n$ a representation

$$-[\bar{u}] = \sum_{i=1}^{k-1} \varphi_i r_i(u^-) + \sum_{i=k+1}^{n} \varphi_i r_i(u^+). \tag{2.53}$$

We take this linear combination into account and transform the determinant,

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) = \det \begin{pmatrix} U_p^f & U_p^s & \tilde{z}_{n+1} - \tilde{z}_1 & * & \tilde{S}_p^f(0) & S_p^s \\ 0 & U_q^s & 0 & -[\bar{v}] & 0 & S_q^s \end{pmatrix}(0), \tag{2.54}$$

where the auxiliary functions are defined as

$$\tilde{z}_{n+1}(\xi) := z_{n+1}(\xi) - \sum_{i=k+1}^{n} \varphi_i p_{n+i}(\xi) \tag{2.55}$$

and

$$\tilde{z}_1(\xi) := z_1(\xi) + \sum_{i=1}^{k-1} \varphi_i p_{n-k+1+i}(\xi), \tag{2.56}$$

respectively. In the next step, we change the order of the vectors by k^2-3 permutations to obtain a matrix in block diagonal form,

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) =(-1)^{k^2-3}\cdot \det\begin{pmatrix} U_p^f & \tilde{S}_p^f(0) & \tilde{z}_{n+1}-\tilde{z}_1 & U_p^s & * & S_p^s \\ 0 & 0 & 0 & U_q^s & -[\bar{v}] & S_q^s \end{pmatrix}(0),$$

and factorize the expression into the product

$$\begin{aligned}\frac{\partial^2 E}{\partial \kappa^2}(0,0) = (-1)^{k^2-2}\cdot \det(p_1,\dots,p_{n-k+1},p_{n+3},\dots,p_{n+k},\tilde{z}_{n+1}-\tilde{z}_1)(0)\cdot \\ \cdot\det(r_1(u^-),\dots,r_{k-1}(u^-),[\bar{v}],r_{k+1}(u^+),\dots,r_n(u^+)). \quad (2.57)\end{aligned}$$

We write the first determinant as a sum of determinants and evaluate each summand in turn. The function $\tilde{z}_{n+1}(\xi)$ is governed by the differential equation

$$\begin{aligned}\frac{d\tilde{z}_{n+1}}{d\xi}(\xi) &= \frac{dz_{n+1}}{d\xi}(\xi) - \sum_{i=k+1}^{n}\varphi_i\frac{dp_{n+i}}{d\xi}(\xi) \\ &= \frac{dF}{du}(\bar{u}(\xi))z_{n+1}(\xi) + (\bar{u}(\xi)-u^+)- \\ &\quad -\sum_{i=k+1}^{n}\varphi_i\left(\frac{dF}{du}(\bar{u}(\xi))p_{n+i}(\xi)+r_i(u^+)\right) \\ &= \frac{dF}{du}(\bar{u}(\xi))\tilde{z}_{n+1}(\xi) + b^+(\xi), \qquad (2.58)\end{aligned}$$

with inhomogeneity

$$b^+(\xi) := (\bar{u}(\xi)-u^+) - \sum_{i=k+1}^{n}\varphi_i r_i(u^+).$$

The functions $\tilde{z}_{n+1}(\xi)$ and $b^+(\xi)$ are bounded on $\mathbb{R}$, since they are linear combinations of bounded functions. Thus, the requirements of Lemma 2.6 are met and we obtain

$$\det(p_1,\dots,p_{n-k+1},p_{n+3},\dots,p_{n+k},\tilde{z}_{n+1})(0) = -\int_0^{+\infty} <\psi,b^+>(\xi)d\xi.$$

In a similar way, we derive the expression

$$\det(p_1, \ldots, p_{n-k+1}, p_{n+3}, \ldots, p_{n+k}, \tilde{z}_1)(0) = \int_{-\infty}^{0} < \psi, b^- > (\xi) d\xi,$$

where the bounded function $b^-(\xi)$ is defined as

$$b^-(\xi) := (\bar{u}(\xi) - u^-) + \sum_{i=1}^{k-1} \varphi_i r_i(u^-).$$

The linear combination (2.53) implies the identity $b^+(\xi) \equiv b^-(\xi)$ and we define $b(\xi) := b^+(\xi)$. Thus we obtain the expression

$$\det(p_1, \ldots, p_{n-k+1}, p_{n+3}, \ldots, p_{n+k}, \tilde{z}_{n+1} - \tilde{z}_1)(0) = -\int_{-\infty}^{+\infty} < \psi, b > (\xi) d\xi.$$

and conclude from (2.57) the stated result. □

We infer from Palmer's Lemma 2.5 and Theorem 2.10 the following result.

Corollary 2.4. *Suppose the assumptions (B1)-(B5) and (B7c) hold. Then the multiplicity of the effective eigenvalue zero is generically two. However a third generalized eigenfunction for the effective eigenvalue zero exists, if the differential equation*

$$\frac{dp}{d\xi}(\xi) = \frac{dF}{du}(\bar{u}(\xi), 0) p(\xi) + b(\xi)$$

with the function $b(\xi)$ from Lemma 2.11 has a bounded solution. Moreover, such a generalized eigenfunction would be related to the genuine eigenfunction $\frac{\partial \bar{u}}{\partial \xi}(\xi)$.

2.4 Bifurcation analysis of $E(\kappa, \nu) = 0$

The identities $\frac{\partial E}{\partial \kappa}(0,0) = 0$ and $E(0,\nu) = 0$ for all ν sufficiently small indicate a bifurcation in the equation $E(\kappa,\nu) = 0$ defining the zero set of the Evans function. The nature of the singularity of the Evans function at the origin is studied via its higher order derivatives.

First, we establish the connection between (a derivative of) the Evans function and (a derivative of) the Melnikov function.

Theorem 2.11. *If the assumptions (B1)-(B5) hold, then the derivative of the Evans function equals*

$$\frac{\partial^2 E}{\partial \nu \partial \kappa}(0,0) = c \cdot \int_{-\infty}^{+\infty} < \psi(\xi), \frac{d^2 F}{du^2}(\bar{u}(\xi))\left(\frac{\partial \bar{u}}{\partial \nu}, \frac{\partial \bar{u}}{\partial \nu}\right)(\xi, 0) > d\xi \cdot \tag{2.59}$$
$$\cdot \det(r_1(u^-), \ldots, r_{k-1}(u^-), u^+ - u^-, r_{k+1}(u^+), \ldots, r_n(u^+))$$

with a non-zero real constant c and the function $\psi(\xi)$ in Theorem 2.4.

Proof. By Theorem 2.5 the assumptions imply the existence of the Evans function $E(\kappa,\nu)$. We assume without loss of generality that for sufficiently small ν solutions of the eigenvalue equation are given by

$$U_1^f(\xi, 0, \nu) = S_1^f(\xi, 0, \nu) = \begin{pmatrix} \frac{\partial \bar{u}}{\partial \xi} \\ 0 \end{pmatrix}(\xi, \nu) \tag{2.60}$$

and

$$U_2^f(\xi, 0, 0) = S_2^f(\xi, 0, 0) = \begin{pmatrix} \frac{\partial \bar{u}}{\partial \nu} \\ 0 \end{pmatrix}(\xi, 0). \tag{2.61}$$

Thus we rewrite the Evans function as in the proof of Theorem 2.6,

$$E(\kappa, \nu) = \det\left(U^f, U^s, S_1^f - U_1^f, S_2^f - U_2^f, S_3^f, \ldots, S_k^f, S^s\right)(0, \kappa, \nu),$$

differentiate with respect to κ and ν by the Leibniz rule and evaluate the derivative at $(\kappa,\nu)=(0,0)$ to obtain

$$\frac{\partial^2 E}{\partial\nu\partial\kappa}(0,0) = \det\left(U^f, U^s, \frac{\partial(S_1^f-U_1^f)}{\partial\kappa}, \frac{\partial(S_2^f-U_2^f)}{\partial\nu}, S_3^f, \dots, S_k^f, S^s\right)(0) + \\ + \det\left(U^f, U^s, \frac{\partial(S_1^f-U_1^f)}{\partial\nu}, \frac{\partial(S_2^f-U_2^f)}{\partial\kappa}, S_3^f, \dots, S_k^f, S^s\right)(0).$$

All other summands vanish at $(\kappa,\nu)=(0,0)$, since they contain a vector $(S_1^f-U_1^f)(0,0,0)$ and/or $(S_2^f-U_2^f)(0,0,0)$ which coincide with the null vector.

In the proof of Theorem 2.7 we computed the derivatives of the solutions with respect to the spectral parameter κ and obtained the expressions

$$\frac{\partial}{\partial\kappa}\big(S_1^f-U_1^f\big)(\xi,0,0) = \begin{pmatrix} z_{n+1}(\xi)-z_1(\xi) \\ -(u^+-u^-)\end{pmatrix} \tag{2.62}$$

and

$$\frac{\partial}{\partial\kappa}\big(S_2^f-U_2^f\big)(\xi,0,0) = \begin{pmatrix} z_{n+2}(\xi)-z_2(\xi) \\ -(v^+-v^-)\end{pmatrix} \tag{2.63}$$

with functions $z_i(\xi) := \frac{\partial p_i}{\partial\kappa}(\xi,0,0)$ for $i=1,2,n+1,n+2$ and $\bar v(\xi) := \int_{-\infty}^{\xi}\frac{\partial\bar u}{\partial\nu}(x,0)dx$ with asymptotic limits $v^\pm := \lim_{\xi\to\pm\infty}\bar v(\xi)$.

In a similar way, we calculate the derivatives of the solutions of (2.28) with respect to ν, which satisfy the system of differential equations

$$\frac{\partial}{\partial\xi}\begin{pmatrix}\frac{\partial p}{\partial\nu}\\ \frac{\partial q}{\partial\nu}\end{pmatrix}(\xi,\kappa,\nu) = \begin{pmatrix}\frac{dF}{du}(\bar u(\xi,\nu),\mu(\nu)) & I_n \\ \kappa I_n & 0_n\end{pmatrix}\begin{pmatrix}\frac{\partial p}{\partial\nu}\\ \frac{\partial q}{\partial\nu}\end{pmatrix}(\xi,\kappa,\nu) + \\ + \begin{pmatrix}\frac{d^2F}{du^2}(\bar u(\xi,\nu),\mu(\nu))(\frac{\partial\bar u}{\partial\nu},p)(\xi,\kappa,\nu) \\ 0\end{pmatrix} + \\ + \begin{pmatrix}\frac{\partial^2F}{\partial\mu\partial u}(\bar u(\xi,\nu),\mu(\nu))p(\xi,\kappa,\nu)\frac{d\mu}{d\nu}(\nu) \\ 0\end{pmatrix}. \tag{2.64}$$

The functions $U_1^f(\xi,\kappa,\nu)$ and $S_1^f(\xi,\kappa,\nu)$ satisfy the identities (2.60). Hence, the difference vector $(S_1^f-U_1^f)(\xi,\kappa,\nu)$ vanishes identically for $\kappa=0$ and sufficiently small ν, which implies

$$\frac{\partial}{\partial \nu}\big(S_1^f - U_1^f\big)(\xi, 0, \nu) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.65}$$

The solutions $U_2^f(\xi, \kappa, \nu)$ and $S_2^f(\xi, \kappa, \nu)$ are chosen such that the identities (2.61) hold and are part of the fast manifold. By Corollary 1.2, their q-coordinates demonstrate for $\kappa = 0$ and sufficiently small ν the asymptotic behavior

$$\lim_{\xi \to -\infty} q_2(\xi, 0, \nu) = 0 \quad \text{and} \quad \lim_{\xi \to +\infty} q_{n+2}(\xi, 0, \nu) = 0.$$

In addition, the order of taking the limit and the derivative, respectively, can be interchanged for these functions and their derivatives satisfy for $\kappa = 0$ and sufficiently small ν the asymptotic behavior

$$\lim_{\xi \to -\infty} \frac{\partial q_2}{\partial \nu}(\xi, 0, \nu) = 0 \quad \text{and} \quad \lim_{\xi \to +\infty} \frac{\partial q_{n+2}}{\partial \nu}(\xi, 0, \nu) = 0. \tag{2.66}$$

The derivatives of the solutions $U_2^f(\xi, \kappa, \nu)$ and $S_2^f(\xi, \kappa, \nu)$ are governed by the differential equations (2.64). In particular, the q-vectors satisfy for $\kappa = 0$ and sufficiently small ν the equations $\frac{\partial}{\partial \xi}\frac{\partial q_i}{\partial \nu}(\xi, 0, \nu) = 0 \in \mathbb{R}^n$. Thus we conclude that the q-vectors are constant and equal the null vector due to the limits (2.66). Hence, we obtain the expression

$$\frac{\partial}{\partial \nu}\big(S_2^f - U_2^f\big)(\xi, 0, 0) = \begin{pmatrix} y_{n+2} - y_2 \\ 0 \end{pmatrix}(\xi) \tag{2.67}$$

with functions $y_i(\xi)$ defined as $y_i(\xi) := \frac{\partial p_i}{\partial \nu}(\xi, 0, 0)$ for $i = 2, n+2$.

We insert the vectors (2.62), (2.63), (2.65) and (2.67) into the derivative of the Evans function $E(\kappa,\nu)$ at $(\kappa,\nu)=(0,0)$ and obtain

$$\frac{\partial^2 E}{\partial\nu\partial\kappa}(0,0)=\det\begin{pmatrix} U_p^f & U_p^s & * & y_{n+2}-y_2 & \tilde{S}_p^f & S_p^s \\ 0 & U_q^s & -[\bar{u}] & 0 & 0 & S_q^s \end{pmatrix}(0)+$$
$$+\det\begin{pmatrix} U_p^f & U_p^s & 0 & z_{n+2}-z_2 & \tilde{S}_p^f & S_p^s \\ 0 & U_q^s & 0 & -[\bar{v}] & 0 & S_q^s \end{pmatrix}(0)$$

with matrices

$$\begin{aligned}
U_p^f(0) &:= (p_1,\dots,p_{n-k+1})(0)\in\mathbb{R}^{n\times(n-k+1)},\\
U_p^s(0) &:= (p_{n-k+2},\dots,p_n)(0)\in\mathbb{R}^{n\times(k-1)},\\
U_q^s(0) &:= (r_1(u^-),\dots,r_{k-1}(u^-))\in\mathbb{R}^{n\times(k-1)},\\
\tilde{S}_p^f(0) &:= (p_{n+3},\dots,p_{n+k})(0)\in\mathbb{R}^{n\times(k-2)},\\
S_p^s(0) &:= (p_{n+k+1},\dots,p_{2n})(0)\in\mathbb{R}^{n\times(n-k)}
\end{aligned}$$

and

$$S_q^s(0) := (r_{k+1}(u^+),\dots,r_n(u^+))\in\mathbb{R}^{n\times(n-k)}.$$

The second determinant vanishes, since it contains a null vector. However, in the first determinant we change the order of the vectors by k^2-2 permutations to obtain a matrix in block diagonal form,

$$\frac{\partial^2 E}{\partial\nu\partial\kappa}(0,0)=(-1)^{k^2-2}\cdot\det\begin{pmatrix} U_p^f & \tilde{S}_p^f & y_{n+2}-y_2 & U_p^s & * & S_p^s \\ 0 & 0 & 0 & U_q^s & -[\bar{u}] & S_q^s \end{pmatrix},$$

and factorize the expression into the product

$$\begin{aligned}\frac{\partial^2 E}{\partial\nu\partial\kappa}(0,0)=(-1)^{k^2-1}&\cdot\det(U_p^f,\tilde{S}_p^f,y_{n+2}-y_2)(0)\cdot \\ &\cdot\det(r_1(u^-),\dots,r_{k-1}(u^-),[\bar{u}],r_{k+1}(u^+),\dots,r_n(u^+)).\end{aligned}\tag{2.68}$$

We write the first determinant as a sum of determinants and evaluate each summand in turn. The functions that span the matrices $U^f(\xi,0,0)$ and $\tilde{S}^f_p(\xi,0,0)$ are solutions of the linearized profile equation, which decay in at least one limit. In addition, the function $y_{n+2}(\xi) = \frac{\partial p_{n+2}}{\partial \nu}(\xi,0,0)$ is governed by the system of differential equations (2.64), which simplifies for $(\kappa,\nu)=(0,0)$ to

$$\frac{\partial y_{n+2}}{\partial \xi}(\xi) = \frac{dF}{du}(\bar{u}(\xi))y_{n+2}(\xi) + \frac{d^2F}{du^2}(\bar{u}(\xi))\left(\frac{\partial \bar{u}}{\partial \nu},\frac{\partial \bar{u}}{\partial \nu}\right)(\xi,0),$$

since the identities $p_{n+2}(\xi,0,0) = \frac{\partial \bar{u}}{\partial \nu}(\xi,0)$, $\frac{\partial q_{n+2}}{\partial \nu}(\xi,0,0) \equiv 0$, $\mu(0)=0$ and $\frac{d\mu}{d\nu}(0)=0$ hold. The function $y_{n+2}(\xi)$ and the inhomogeneity of its differential equation are bounded on $\mathbb{R}$. Thus we can apply Lemma 2.6 and obtain

$$\det(U^f_p,\tilde{S}^f_p,y_{n+2})(0) = -\int_0^{+\infty} < \psi(\xi), \frac{d^2F}{du^2}(\bar{u}(\xi))\left(\frac{\partial \bar{u}}{\partial \nu},\frac{\partial \bar{u}}{\partial \nu}\right)(\xi,0) > d\xi.$$

In a similar way we derive

$$\det(U^f_p,\tilde{S}^f_p,y_2)(0) = \int_{-\infty}^{0} < \psi(\xi), \frac{d^2F}{du^2}(\bar{u}(\xi))\left(\frac{\partial \bar{u}}{\partial \nu},\frac{\partial \bar{u}}{\partial \nu}\right)(\xi,0) > d\xi.$$

Hence, the first determinant in (2.68) satisfies

$$\det(U^f_p,\tilde{S}^f_p,y_{n+2}-y_2)(0) = -\int_{-\infty}^{+\infty} < \psi(\xi), \frac{d^2F}{du^2}(\bar{u}(\xi))\left(\frac{\partial \bar{u}}{\partial \nu},\frac{\partial \bar{u}}{\partial \nu}\right)(\xi,0) > d\xi.$$

We combine this expression with (2.68) and obtain the stated result, where the constant c is set to $c := (-1)^{k^2}$. □

Corollary 2.5. *If the assumptions (B1)-(B6) hold, then the mixed derivative of the Evans function, $\frac{\partial^2 E}{\partial \nu \partial \kappa}(\kappa,\nu)$, is non-zero at the point $(\kappa,\nu)=(0,0)$.*

Proof. Under the assumptions (B1)-(B5), we obtained in Theorem 2.11 the second order derivative of the Evans function as

$$\frac{\partial^2 E}{\partial \nu \partial \kappa}(0,0) = c\cdot \int_{-\infty}^{+\infty} < \psi(\xi), \frac{d^2F}{du^2}(\bar{u}(\xi))\left(\frac{\partial \bar{u}}{\partial \nu},\frac{\partial \bar{u}}{\partial \nu}\right)(\xi,0) > d\xi \cdot$$
$$\cdot \det(r_1(u^-),\dots,r_{k-1}(u^-),u^+-u^-,r_{k+1}(u^+),\dots,r_n(u^+))$$

with a non-zero constant c. By the results of the Lemmata 2.1 and A.10, the integral expression equals the second order derivative of the Melnikov function at $(\nu, \mu) = (0, 0)$,

$$\int_{-\infty}^{+\infty} < \psi(\xi), \frac{d^2 F}{du^2}(\bar{u}(\xi))\left(\frac{\partial \bar{u}}{\partial \nu}, \frac{\partial \bar{u}}{\partial \nu}\right)(\xi, 0) > d\xi = \frac{\partial^2 M}{\partial \nu^2}(0, 0),$$

which is non-zero by assumption (B5). In addition, the Liu-Majda determinant does not vanish by assumption (B6). Hence, the derivative of the Evans function is the product of non-zero factors, which proves the assertion. □

In the next step, we prove that the Evans function exhibits a bifurcation.

Lemma 2.12. *Suppose the assumptions (B1)-(B6) hold. Then the zero set of the Evans function consists close to the origin of two curves*

$$\{(\kappa, \nu) \in \mathbb{R}^2 \mid \kappa \equiv 0, \nu \in (-\delta, \delta)\} \quad \textit{and} \quad \{(\kappa, \nu) \in \mathbb{R}^2 \mid \nu = \nu(\kappa), \kappa \in (-\delta, \delta)\} \tag{2.69}$$

where δ is a sufficiently small positive constant, and $\nu : (-\delta, \delta) \to \mathbb{R}$, $\kappa \mapsto \nu(\kappa)$ is a differentiable function such that $\nu(0) = 0$ and

$$\frac{d\nu}{d\kappa}(0) = -\frac{1}{2} \frac{\frac{\partial^2 E}{\partial \kappa^2}}{\frac{\partial^2 E}{\partial \nu \partial \kappa}}(0, 0). \tag{2.70}$$

Moreover, the curves intersect transversally at the point $(\kappa, \nu) = (0, 0)$.

Proof. Under the assumptions (B1)-(B5), we conclude from Corollary 2.1 that the Evans function vanishes at $\kappa = 0$ for sufficiently small ν. Thus the curve $\{(\kappa, \nu) \in \mathbb{R}^2 \mid \kappa \equiv 0, \nu \in (-\delta, \delta)\}$ is part of the zero set of the Evans function. In addition, the Evans function is analytic in the spectral parameter κ. Hence, the function

$$\tilde{E}(\kappa, \nu) := \begin{cases} \frac{E(\kappa, \nu)}{\kappa} & \text{if } \kappa \neq 0, \\ \frac{dE}{d\kappa}(\kappa, \nu) & \text{if } \kappa = 0, \end{cases}$$

is well-defined and satisfies

$$\tilde{E}(0, 0) = 0. \tag{2.71}$$

It is differentiable at $(\kappa, \nu) = (0,0)$ and its derivative with respect to ν is non-zero, since the identity

$$\frac{\partial \tilde{E}}{\partial \nu}(0,0) = \frac{\partial^2 E}{\partial \nu \partial \kappa}(0,0) \tag{2.72}$$

holds and the mixed derivative of the Evans function is non-zero by the result of Corollary 2.5. Under the conditions (2.71) and (2.72), we conclude from the implicit function theorem the existence of an interval $(-\delta, \delta)$ for a sufficiently small, positive constant δ and a function $\nu : (-\delta, \delta) \to \mathbb{R}$, $\kappa \mapsto \nu(\kappa)$, such that the identities $\nu(0) = 0$ and

$$\tilde{E}(\kappa, \nu(\kappa)) = 0 \tag{2.73}$$

hold. Thus the zero set of the Evans function $E(\kappa, \nu)$ close to the origin consists of the curves in (2.69), which intersect transversally at $(\kappa, \nu) = (0,0)$.

We differentiate the identity (2.73) with respect to the spectral parameter κ and use L'Hospital's rule as well as the results of Corollary 2.1 to obtain

$$\begin{aligned}
0 &= \frac{d}{d\kappa}\bigg|_{\kappa=0} \tilde{E}(\kappa, \nu(\kappa)) = \frac{d}{d\kappa}\bigg|_{\kappa=0} \frac{E(\kappa, \nu(\kappa))}{\kappa} = \\
&= \lim_{\kappa \to 0} \frac{\left(\frac{\partial E}{\partial \kappa} + \frac{\partial E}{\partial \nu}\frac{d\nu}{d\kappa}\right) \cdot \kappa - E}{\kappa^2}(\kappa, \nu(\kappa)) = \\
&= \lim_{\kappa \to 0} \frac{\left(\frac{\partial^2 E}{\partial \kappa^2} + 2\frac{\partial^2 E}{\partial \nu \partial \kappa}\frac{d\nu}{d\kappa} + \frac{\partial^2 E}{\partial \nu^2}\left(\frac{d\nu}{d\kappa}\right)^2 + \frac{\partial E}{\partial \nu}\frac{d^2\nu}{d\kappa^2}\right) \cdot \kappa + \frac{dE}{d\kappa} - \frac{dE}{d\kappa}}{2\kappa}(\kappa, \nu(\kappa)) = \\
&= \frac{1}{2}\frac{\partial^2 E}{\partial \kappa^2}(0,0) + \frac{\partial^2 E}{\partial \nu \partial \kappa}(0,0)\frac{d\nu}{d\kappa}(0).
\end{aligned}$$

By the result of Corollary 2.5, the mixed derivative of the Evans function is non-zero, $\frac{\partial^2 E}{\partial \nu \partial \kappa}(0,0) \neq 0$. Thus the derivative of the function $\nu(\kappa)$ satisfies

$$\frac{d\nu}{d\kappa}(0) = -\frac{1}{2}\frac{\frac{\partial^2 E}{\partial \kappa^2}}{\frac{\partial^2 E}{\partial \nu \partial \kappa}}(0,0)$$

and is determined by the second order derivative of the Evans function at the bifurcation point. $\square$

Since the curves in (2.69) intersect transversally at the bifurcation point, possible bifurcation scenarios are given in Figure 2.4. The following result guarantees the occurrence of a transcritical bifurcation.

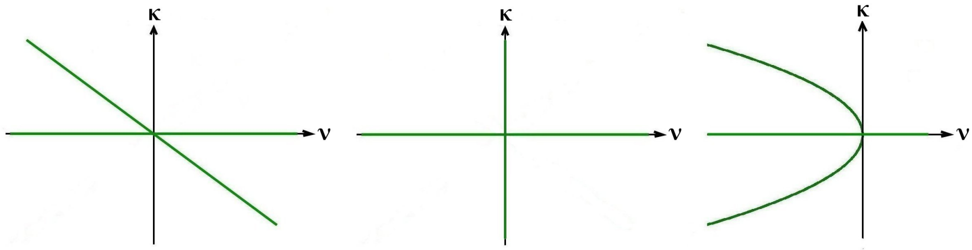

Figure 2.4: Bifurcation diagram of a transcritical, a degenerate and a pitchfork bifurcation.

Theorem 2.12. *Suppose the assumptions (B1)-(B6) hold and the definite integral* $\int_{-\infty}^{+\infty} < \psi, b > (\xi) d\xi$, *with the function* $\psi(\xi)$ *in Theorem 2.4 and the bounded function* $b(\xi)$ *in Theorem 2.7, is non-zero. Then a transcritical bifurcation will occur in the equation* $E(\kappa, \nu) = 0$ *at the bifurcation point* $(\kappa, \nu) = (0, 0)$.

Proof. A transcritical bifurcation in the equation $E(\kappa, \nu) = 0$ is characterized by the following conditions on the Evans function $E(\kappa, \nu)$ and its partial derivatives at $(\kappa, \nu) = (0, 0)$:

$$E(0,0) = 0, \tag{2.74}$$

$$\frac{\partial E}{\partial \kappa}(0,0) = 0, \tag{2.75}$$

and

$$\frac{\partial E}{\partial \nu}(0,0) = 0, \tag{2.76}$$

$$\frac{\partial^2 E}{\partial \kappa \partial \nu}(0,0) \neq 0, \tag{2.77}$$

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) \neq 0. \tag{2.78}$$

The four conditions (2.74)-(2.77) have been established in Theorem 2.6 and Corollary 2.5. Additionally, the second order derivative of the Evans function, $\frac{\partial^2 E}{\partial \kappa^2}(0,0)$, has by Theorem 2.7 a representation

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) = c \cdot \int_{-\infty}^{+\infty} < \psi, b > (\xi) d\xi \cdot$$
$$\cdot \det(r_1(u^-), \dots, r_{k-1}(u^-), u^+ - u^-, r_{k+1}(u^+), \dots, r_n(u^+)).$$

The assumptions imply that the factors are non-zero and the identity (2.78) holds. Hence, the conditions (2.74)-(2.78) are satisfied and a transcritical bifurcation occurs. □

In the next step, we identify viscous shock waves that are not spectrally stable.

Corollary 2.6. *Suppose the assumptions of Theorem 2.12 hold. Then the viscous shock waves with viscous profiles $\bar{u}(\xi, \nu)$ are not spectrally stable*

1. *for sufficiently small positive parameters ν if the factor* (2.70) *is positive.*

2. *for sufficiently small negative parameters ν if the factor* (2.70) *is negative.*

3. *for $\nu = 0$.*

Proof. By the result of Lemma 2.12, the zero set of the Evans function close to the origin consists of two curves (2.69), which represent effective eigenvalues. Since the derivative of a viscous profile is always a genuine eigenfunction associated to the effective eigenvalue zero, the curve $\{(\kappa, \nu) \in \mathbb{R}^2 \mid \kappa \equiv 0, \nu \in (-\delta, \delta)\}$ is present.

The other curve, $\{(\kappa, \nu) \in \mathbb{R}^2 \mid \nu = \nu(\kappa), \kappa \in (-\delta, \delta)\}$ has a representation with respect to κ. Moreover, the function $\nu(\kappa)$ satisfies the identity (2.70). Hence, in the proposed parameter regimes there exist positive real eigenvalues κ, which imply the instability of the associated viscous shock wave. In contrast, the viscous shock wave with viscous profile $\bar{u}(\xi)$ is not spectrally stable, since the multiplicity of the effective eigenvalue is two and exceeds the dimension (one) of the manifold of heteroclinic orbits connecting the endstates $u^{\pm}$ in the profile equation. □

2.4.1 Marginal case

We consider again the case of a non-transversal profile, whose associated Liu-Majda determinant vanishes:

(B7) The Liu-Majda condition fails, i.e.

$$\det(r_1(u^-), \ldots, r_{k-1}(u^-), u^+ - u^-, r_{k+1}(u^+), \ldots, r_n(u^+)) = 0.$$

Corollary 2.7. *Suppose the assumptions (B1)-(B5) and (B7) hold. Then the second order mixed derivative of the Evans function,* $\frac{\partial^2 E}{\partial\nu\partial\kappa}(\kappa, \nu)$*, vanishes at the origin.*

Proof. In Theorem 2.11 we computed the mixed derivative of the Evans function as

$$\frac{\partial^2 E}{\partial\nu\partial\kappa}(0,0) = c \cdot \int_{-\infty}^{+\infty} < \psi(\xi), \frac{d^2 F}{du^2}(\bar{u}(\xi))\left(\frac{\partial\bar{u}}{\partial\nu}, \frac{\partial\bar{u}}{\partial\nu}\right)(\xi, 0) > d\xi \cdot$$
$$\cdot \det(r_1(u^-), \ldots, r_{k-1}(u^-), u^+ - u^-, r_{k+1}(u^+), \ldots, r_n(u^+))$$

with a non-zero constant c and the function $\psi(\xi)$ in Theorem 2.4. The expression vanishes, since the Liu-Majda determinant is zero by assumption (B7). □

Thus the bifurcation analysis in the previous section does not apply. The nature of the singularity of the Evans function at the origin is determined by higher order derivatives.

Chapter 3

Applications

We will apply the theory to selected model problems.

3.1 Viscous shock waves in MHD

Planar waves in magnetohydrodynamics (MHD) are governed by a system of hyperbolic -parabolic conservation laws. Freistühler and Szmolyan proved that all magnetohydrodynamic shocks have viscous profiles in a certain range of the dissipation coefficients [FS95, Theorem 1.1]. Moreover, they show that the viscous profiles with the same relative flux are generated in a global bifurcation [FS95, Theorem 1.3]. After presenting their results, we will prove via Melnikov theory that a saddle-node bifurcation of viscous profiles occurs and discuss the stability of these viscous profiles.

In the parameter regime of interest, where the dissipative effects due to electrical resistivity ν and longitudinal viscosity λ dominate those of transversal viscosity and heat conductivity, a slow-fast structure in the profile equation is evident. An application of geometric singular perturbation theory [Fen79, Szm91] leads to the study of the reduced system,

$$\left.\begin{aligned} \nu\frac{d\mathbf{b}}{d\xi} &=(\tau - d^2)\mathbf{b} - \mathbf{c}, \\ \lambda\frac{d\tau}{d\xi} &=\frac{1}{2}\|\mathbf{b}\|^2 + \tau - j + \frac{1}{k\tau}\Big(- \frac{\tau^2}{2} - \frac{d^2}{2}\|\mathbf{b}\|^2 - < \mathbf{b}, \mathbf{c} > + e\Big), \end{aligned}\right\} \tag{3.1}$$

where the unknowns are the transversal magnetic field $\mathbf{b} : \mathbb{R} \to \mathbb{R}^2$ and the specific volume $\tau : \mathbb{R} \to \mathbb{R}$. The constants d, e, j and k are real numbers, whereas the constant k is bigger than one and the constant vector $\mathbf{c} \in \mathbb{R}^2$ is different from the null vector. The physical domain U^3 is given by

$$U^3 := \left\{ \begin{pmatrix} \mathbf{b} \\ \tau \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, \tau > 0,\, \theta(\mathbf{b}, \tau) > 0 \right\},$$

where the temperature θ is defined as

$$\theta(\mathbf{b}, \tau) := \frac{1}{c_v} \left(\frac{1}{2}(\tau - d^2)\|\mathbf{b}\|^2 - < \mathbf{b}, \mathbf{c} > + \frac{\tau^2}{2} - j\tau + e \right)$$

with a positive real constant c_v.

Definition 3.1. The reduced system (3.1) restricted to the domains U^3 and the half-space $\overline{U}^3 := \mathbb{R}^2 \times \mathbb{R}_+$ is referred to as Σ^3 and $\overline{\Sigma}^3$, respectively.

Systems Σ^3 and $\overline{\Sigma}^3$ are gradient-like and the physical domain U^3 is positively invariant under the flow of $\overline{\Sigma}^3$. The well known symmetry property of the full MHD equations imply a reflectional invariance of the reduced system.

Lemma 3.1. *([FS95, Lemma 4.3]) System $\overline{\Sigma}^3$ is invariant under the reflection across the plane*

$$E := \left\{ \begin{pmatrix} \mathbf{b} \\ \tau \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, \mathbf{b} \in \mathrm{span}\{\mathbf{c}\} \right\}.$$

In particular, E is invariant under the flow of $\overline{\Sigma}^3$.

For non-degenerate intermediate shocks the constant vector $\mathbf{c}$ is different from the null vector [FS95, Lemma 2.3], which implies that all stationary points of (3.1) are lying in the invariant plane E. Moreover, we can choose a vector $\mathbf{d}$ that is orthogonal to the vector $\mathbf{c}$. In the new coordinates

$$b(\xi) :=< \mathbf{b}(\xi), \frac{\mathbf{c}}{\|\mathbf{c}\|} > \quad \text{and} \quad b_*(\xi) :=< \mathbf{b}(\xi), \frac{\mathbf{d}}{\|\mathbf{d}\|} >,$$

the system (3.1) is obtained as

$$\nu\frac{db}{d\xi} = (\tau - d^2)b - c, \tag{3.2}$$

$$\nu\frac{db_*}{d\xi} = (\tau - d^2)b_*, \tag{3.3}$$

$$\lambda\frac{d\tau}{d\xi} = \left(\frac{1}{2}\|\mathbf{b}\|^2 + \tau - j + \frac{1}{k\tau}\left(-\frac{\tau^2}{2} - \frac{d^2}{2}\|\mathbf{b}\|^2 - bc + e\right)\right), \tag{3.4}$$

with a positive constant $c := \|\mathbf{c}\|$ and the invariant plane E has the representation $E := \{(b, b_*, \tau)^t \in \mathbb{R}^3 \,|\, b_* = 0\}$.

Definition 3.2. The restriction of system $\overline{\Sigma}^3$ to the plane E is referred to as $\overline{\Sigma}^2$. The differential equations of system $\overline{\Sigma}^2$ are given by the equations (3.2) and (3.4).

The nullclines of system $\overline{\Sigma}^2$ are given by

$$G := \left\{ \begin{pmatrix} b \\ \tau \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, g(b,\tau) := b^2(k\tau - d^2) - 2bc + (2k-1)\tau^2 - 2kj\tau + 2e = 0 \right\}$$

and

$$H := \left\{ \begin{pmatrix} b \\ \tau \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, h(b,\tau) := (\tau - d^2)b - c = 0 \right\}.$$

Thus any stationary point of system $\overline{\Sigma}^2$ lies in $G \cap H \cap (\mathbb{R} \times \mathbb{R}_+)$.

Lemma 3.2. *([FS95, Lemma 4.4])*
The nullclines G and H intersect transversally in exactly four points. With appropriate numbering, these points $u_i = (b_i, \tau_i)^t$ for $i = 0, 1, 2, 3$, satisfy $H^+ \cap G = \{u_0, u_1\}$, $H^- \cap G = \{u_2, u_3\}$, more precisely,

$$\tau_0 > \tau_1 > d^2 > \tau_2 > \tau_3 > 0,$$
$$b_1 > b_0 > 0 > b_3 > b_2.$$

At least the fixed points u_1 and u_3 are lying in the physical range

$$U^2 := \left\{ \begin{pmatrix} b \\ \tau \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, \tau > 0,\, \theta\Big(b\frac{\mathbf{c}}{\|\mathbf{c}\|}, \tau\Big) > 0 \right\}.$$

We repeat the discussion of the geometry of the nullclines in [KL61,FS95]: The nullcline H has asymptotes at $\tau = d^2$ as b tends to $\pm\infty$ and consists of two hyperbolas

$$H^- := H \cap \{(b,\tau)^t \in \mathbb{R}^2 \,|\, \tau < d^2,\, b < 0\}$$

and

$$H^+ := H \cap \{(b,\tau)^t \in \mathbb{R}^2 \,|\, \tau > d^2,\, b > 0\}.$$

The nullcline G has asymptotes at $\tau = d^2/k$, which G approaches from above as b tends to $+\infty$ and from below as b tends to $-\infty$. Additionally, the asymptote of H, $\tau = d^2$, lies above the asymptote of G, $\tau = d^2/k$, since the constant k is bigger than one.

The function $g(b,\tau)$ can be regarded as a quadratic polynomial in either b or τ. Hence horizontal or vertical lines will intersect G at most in two points. For given τ, the identity $g(b,\tau) = 0$ has solutions

$$b^\pm(\tau) = \frac{c \pm \sqrt{\pi(\tau)}}{k\tau - d^2}, \qquad \pi(\tau) := c^2 - \big((2k-1)\tau^2 - 2kj\tau + 2e\big)(k\tau - d^2),$$

which are real valued as long as the discriminant $\pi(\tau)$ is non-negative. Thus the geometry of the nullcline G will depend on the number and location of zeros τ_i^* of the polynomial $\pi(\tau)$. Since, the leading coefficient of $\pi(\tau)$ is negative and $\pi(d^2/k) = c^2$ is positive, one has to distinguish three cases:

(C1) there exists one zero $\tau_1^* > \frac{d^2}{k}$,

(C2) there exist three zeros $\tau_1^* > \frac{d^2}{k} > \tau_2^* > \tau_3^*$,

(C3) there exist three zeros $\tau_1^* > \tau_2^* > \tau_3^* > \frac{d^2}{k}$.

The set G consists in cases (C1) and (C2) of two connected components, G_1 and G_2, and in case (C3) of three connected components, G_1, G_2 and G_3. The components G_i are labeled due to their order of appearance with respect to decreasing τ. In all three cases the connected components are separated by either vertical or horizontal strips, see Figure 3.1.

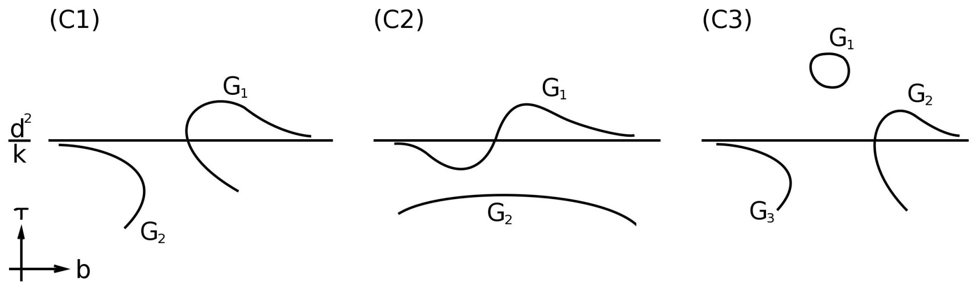

Figure 3.1: [FS95, Figure 1] Nullclines $g(b, \tau) = 0$ in the cases (C1), (C2) and (C3).

Freistühler and Szmolyan continue to discuss the relative position of the nullclines G and H.

Lemma 3.3. *([FS95, Lemma 4.5])*
In case (C1) the nullclines H^- and H^+ both intersect G_1, whereas in case (C2) H^+ intersects G_1 twice and H^- intersects G_1 and G_2 each once. In case (C3) there are two possible scenarios: Either H^- and H^+ both intersect G_1 or both intersect G_2.

Lemma 3.4. *([FS95, Lemma 4.6])*
The set $G \cap ([b_1, b_2] \times \mathbb{R})$ consists of two smooth graphs $G^\pm$ of functions $g^\pm : [b_1, b_2] \to \mathbb{R}$, distinguished by $g^-(b) < g^+(b)$ for all $b \in [b_1, b_2]$. u_0 belongs to G^+, u_3 belongs to G^-. u_1 and u_2 each lie on G^+ or G^- or both. (At least) in (b_1, b_2), both functions g^- and g^- are smooth, and are stationary in at most one point.

By the results of Lemma 3.4, the vector field of system $\overline{\Sigma}^2$ looks generically like in Figure 3.2.

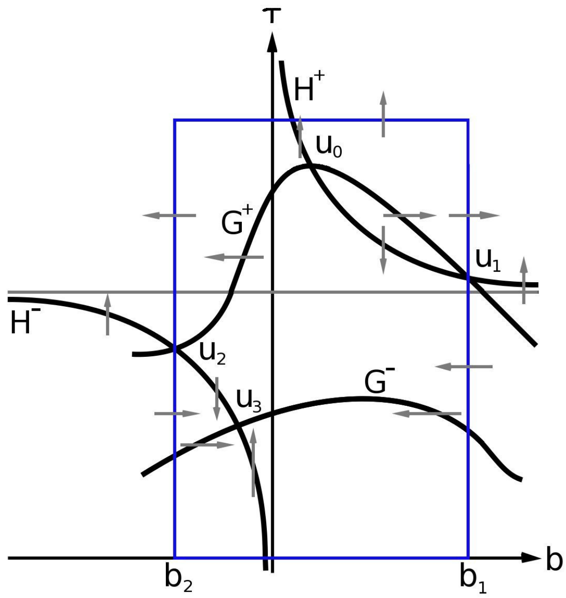

Figure 3.2: [FS95, Figure 2] Phase Portrait of $\overline{\Sigma}^2$ in the domain $[b_2, b_1] \times [0, \tau_*]$.

Lemma 3.5. *([FS95, Lemma 4.7])*
The stationary points of system $\overline{\Sigma}^2$ have the following properties:

1. *u_0, u_1, u_2, u_3 are hyperbolic fixed points for the flow of $\overline{\Sigma}^2$, u_0 is an unstable node, u_3 is a stable node, u_1 and u_2 are saddles.*

2. *Interpreted, via the suspension*

$$\mathbb{R}^2 \to \mathbb{R}^3, \quad \begin{pmatrix} b \\ \tau \end{pmatrix} \mapsto \begin{pmatrix} b\frac{\mathbf{c}}{\|\mathbf{c}\|} \\ \tau \end{pmatrix}, \tag{3.5}$$

   as points in $\mathbb{R}^3$, u_i for $i = 0, 1, 2, 3$ are hyperbolic fixed points in $\overline{\Sigma}^3$. As such the u_i have stable (unstable) manifolds of dimension i $(3 - i)$ for $i = 0, 1, 2, 3$.

Freistühler and Szmolyan established the following heteroclinic bifurcation scenario for system $\overline{\Sigma}^2$.

Lemma 3.6. *([FS95, Lemma 5.1])*

1. *With a certain fixed μ_0, the two dimensional system $\overline{\Sigma}^2$, depending on $\nu, \lambda > 0$, has heteroclinic orbits of the following types and no others:*
   - *(a) $u_0 \to u_1$, $u_0 \to u_2$, $u_0 \to u_3$, $u_1 \to u_3$, $u_2 \to u_3$, for $\frac{\nu}{\lambda} > \mu_0$,*
   - *(b) $u_0 \to u_1$, $u_1 \to u_2$, $u_2 \to u_3$, for $\frac{\nu}{\lambda} = \mu_0$,*
   - *(c) $u_0 \to u_1$, $u_2 \to u_3$, for $\frac{\nu}{\lambda} < \mu_0$.*
2. *At the bifurcation ratio $\frac{\nu}{\lambda} = \mu_0$, the unstable manifold of $\{u_1\} \times (0, \infty)^2$ and the stable manifold of $\{u_2\} \times (0, \infty)^2$, with respect to the extension of $\overline{\Sigma}^2$ by the equations $\frac{d\nu}{d\xi}(\xi) = 0$, $\frac{d\lambda}{d\xi}(\xi) = 0$, intersect transversally.*
3. *All orbits of types $u_0 \to u_1$, $u_0 \to u_2$, $u_1 \to u_2$, $u_1 \to u_3$, $u_2 \to u_3$ are unique, while the orbits of type $u_0 \to u_3$ occur in a one-parameter family. In all cases there exist also orbits with α-limit u_0 (ω-limit u_3) which have no ω-limit (α-limit) in the physical range U^2.*
4. *The fixed points which lie in the physical range are ordered according to increasing values of the entropy S, i.e., $i < j$ implies $S(u_i) < S(u_j)$.*
5. *The fixed points u_1, u_2 and u_3 always lie in the physical range U^2.*

The heteroclinic orbits which are described in the previous lemma become solutions of system Σ^3 via the suspension (3.5). The authors note that the stationary point u_0 may not be physically admissible, depending on its (potentially negative) temperature $\theta\big(b_0 \frac{\mathbf{c}}{\|\mathbf{c}\|}, \tau_0\big)$. Hence they prove two distinct scenarios:

Lemma 3.7. *([FS95, Lemma 5.2])*

1. *Assume that Σ^3 has four fixed points and that an orbit $u_0 \to u_3$ exists in E. Then unique orbits $u_0 \to u_1$, $u_2 \to u_3$, a pair of orbits $u_1 \to u_2$, and one-parameter families of orbits $u_0 \to u_2$, $u_1 \to u_3$ exist for Σ^3. The union of these orbits and the fixed points is the boundary of a two-parameter family of orbits $u_0 \to u_3$.*

2. *Assume that Σ^3 has the three fixed points u_1, u_2, and u_3 and that an orbit $u_1 \to u_3$ exists. Then, a unique orbit $u_2 \to u_3$, a pair of orbits $u_1 \to u_2$, and a one-parameter family of orbits $u_1 \to u_3$ exist for Σ^3.*

Remark 3.1. Freistühler and Rohde investigated the parameter space in the MHD equations and locate the subset of parameters such that either 3 or 4 physically admissible stationary points of the original profile equation exist [FR03].

In a remark, Freistühler and Szmolyan note that one can deduce the result of Lemma 5.2 in [FS95] from the λ-Lemma, see [GH83], and the condition that the invariant manifolds $W^u(u_1)$ and $W^s(u_2)$ in system Σ^3 are in sufficiently general position at the bifurcation value μ_0: By the reflectional symmetry of system (3.1), the intersection of the invariant manifolds $W^u(u_1)$ and $W^s(u_2)$ is non-transversal at the parameter value μ_0. However, it remained an open problem to prove the (expected) quadratic contact of the invariant manifolds $W^u(u_1)$ and $W^s(u_2)$.

We will use Melnikov theory to study this heteroclinic bifurcation in more detail. By introducing the new time scale $\xi = \nu t$ in the equations (3.2)–(3.4), we obtain the system

$$\left.\begin{aligned}
\frac{db}{dt} &= (\tau - d^2)b - c,\\
\frac{db_*}{dt} &= (\tau - d^2)b_*,\\
\frac{d\tau}{dt} &= \mu\Big(\frac{1}{2}\|\mathbf{b}\|^2 + \tau - j + \frac{1}{k\tau}\Big(-\frac{\tau^2}{2} - \frac{d^2}{2}\|\mathbf{b}\|^2 - bc + e\Big)\Big),
\end{aligned}\right\} \tag{3.6}$$

where the dissipation ratio $\mu = \frac{\nu}{\lambda}$ is the parameter of interest and all others are constant. Moreover, we will write the profile equation (3.6) as

$$\frac{du}{dt}(t) = F(u(t), \mu),$$

where $u(t) := (b, b_*, \tau)^t(t)$ and $F(u, \mu) := (F_1, F_2, F_3)^t(u, \mu)$ denote the solution and the right hand side of system (3.6), respectively. In order to apply the analysis of Chapter 2, we will verify the conditions (B1)-(B4):

Lemma 3.8. *Suppose μ_0 is the parameter value from Lemma 3.6*

1. *For $\mu = \mu_0$, there exists a viscous profile $\bar{u}(t)$ that connects the endstates u_1 with u_2. Moreover, the viscous profile has the form*

$$\bar{u}(t) = (\bar{b}, 0, \bar{\tau})^t(t) \tag{3.7}$$

for some scalar functions $\bar{b}, \bar{\tau} : \mathbb{R} \to \mathbb{R}$, which are strict monotonically decreasing with respect to t.

2. *The viscous profile $\bar{u}(t)$ is associated to a Lax 2-shock.*

3. *The viscous profile $\bar{u}(t)$ exists by a non-transversal intersection of the invariant manifolds $W^u(u_1)$ and $W^s(u_2)$. In particular, for any point p on the orbit $\{\bar{u}(t) \,|\, t \in \mathbb{R}\}$ the identity*

$$T_pW^u(u_1) = T_pW^s(u_2) = \operatorname{span}\left\{\frac{\partial \bar{u}}{\partial t}(t), v(t)\right\} \tag{3.8}$$

holds, where the derivative of the viscous profile $\frac{\partial \bar{u}}{\partial t}(t)$ and the function

$$v(t) = \begin{pmatrix} 0 \\ \exp\left(\int_0^t (\bar{\tau}(s) - d^2) ds\right) \\ 0 \end{pmatrix} \tag{3.9}$$

are two linearly independent, bounded solutions of the linearized profile equation

$$\frac{dp}{dt}(t) = \frac{dF}{du}(\bar{u}(t), \mu_0) p(t). \tag{3.10}$$

Proof. 1. By the result of Lemma 3.6, such a viscous profile $\bar{u}(t)$ exists and lies in the plane E. Thus the viscous profile has the form (3.7) for some scalar functions $\bar{b}, \bar{\tau} : \mathbb{R} \to \mathbb{R}$. In the proof of Lemma 5.1 in [FS95] it is observed that the functions $g(b, \tau)$ and $h(b, \tau)$, whence $F_1(b, 0, \tau)$ and $F_3(b, 0, \tau)$, are negative along the viscous profile $\bar{u}(t)$ and vanish only in the stationary points u_1 and u_2. Hence the scalar functions $\bar{b}(t)$ and $\bar{\tau}(t)$ decrease strict monotonically with respect to t.

2. By the results of Lemma 3.5, the hyperbolic stationary points u_1 and u_2 are saddle points whose associated eigenvalues satisfy (with appropriate numbering) the inequalities $\lambda_1(u_1,\mu_0) < 0 < \lambda_2(u_1,\mu_0) < \lambda_3(u_1,\mu_0)$ and $\lambda_1(u_2,\mu_0) < \lambda_2(u_2,\mu_0) < 0 < \lambda_3(u_2,\mu_0)$, respectively.

3. The intersection of the invariant manifolds $W^u(u_1)$ and $W^s(u_2)$ is non-transversal at the parameter value μ_0, due to the reflectional symmetry of system (3.1). Moreover, the second ODE of the linearized profile equation,

$$\begin{aligned}
\frac{dp_1}{dt} &= (\overline{\tau} - d^2)p_1 + \bar{b}p_3,\\
\frac{dp_2}{dt} &= (\overline{\tau} - d^2)p_2,\\
\frac{dp_3}{dt} &= \frac{\partial F_3}{\partial b}(\bar{u},\mu_0)p_1 + \frac{\partial F_3}{\partial \tau}(\bar{u},\mu_0)p_3,
\end{aligned}$$

is independent. Hence the function (3.9) and the derivative of the viscous profile are solutions of the linearized profile equation and linearly independent. In addition, the viscous profile $\bar{u}(t)$ tends to endstates, which satisfy by the results of Lemma 3.2 the inequalities

$$\lim_{t\to-\infty} \overline{\tau}(t) = \tau_1 > d^2 \quad \text{and} \quad \lim_{t\to+\infty} \overline{\tau}(t) = \tau_2 < d^2.$$

Thus the integral

$$\int_0^t (\overline{\tau}(s) - d^2)ds \to -\infty$$

diverges in both limits $t \to \pm\infty$ to $-\infty$ and we conclude that $v(t)$ is globally bounded on $\mathbb{R}$. Since the invariant manifolds $W^u(u_1)$ and $W^s(u_2)$ are two-dimensional, we obtain that for any point p on the orbit $\{\bar{u}(t)\,|\,t \in \mathbb{R}\}$ the identity (3.8) holds. □

Thus the conditions (B1)-(B4) of Chapter 2 hold and we conclude from Theorem 2.1 the following result.

Lemma 3.9. *The Melnikov function $M : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $(\nu,\mu) \mapsto M(\nu,\mu)$, is well-defined and smooth in a small neighborhood of the point $(\nu,\mu) = (0,\mu_0)$.*

Moreover, it satisfies the identities

$$M(0,\mu_0) = 0 \quad \textit{and} \quad \frac{\partial M}{\partial \nu}(0,\mu_0) = 0.$$

We have to compute additional derivatives of the Melnikov function.

Lemma 3.10. *The Melnikov function satisfies*

$$\frac{\partial M}{\partial \mu}(0,\mu_0) = \int_{-\infty}^{+\infty} a(t)w(t)F_1(\bar{u}(t),\mu_0)\frac{\partial F_3}{\partial \mu}(\bar{u}(t),\mu_0)dt, \tag{3.11}$$

with functions

$$a(t) := \exp\Big(- \int_0^t \operatorname{trace}\Big(\frac{dF}{du}(\bar{u}(s),\mu_0)\Big)ds\Big)$$

and $v(t) = (0, w(t), 0)^t$ *from Lemma 3.8. Moreover, the derivative of the Melnikov function* (3.11) *is non-zero at the point* $(0,\mu_0)$.

Proof. By the results of Lemma A.9, the Melnikov function satisfies

$$\frac{\partial M}{\partial \mu}(0,\mu_0) = \int_{-\infty}^{+\infty} < \psi(s), \frac{\partial F}{\partial \mu}(\bar{u}(s),\mu_0) > ds,$$

where $\psi(s)$ is the unique (up to a multiplicative factor) bounded solution of the adjoint differential equation of (3.10). We derive from the results of Theorem 2.4 and Lemma 3.8 the expression

$$\psi(t) := \exp\Big(- \int_0^t \operatorname{trace}\Big(\frac{dF}{du}(\bar{u}(s),\mu_0)\Big)ds\Big)\Big(\frac{\partial \bar{u}}{\partial t}(t) \times v(t)\Big).$$

In addition, the derivative of the vector field $F(u,\mu)$ with respect to μ satisfies

$$\frac{\partial F}{\partial \mu}(\bar{u}(t),\mu_0) = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial F_3}{\partial \mu}(\bar{u}(t),\mu_0) \end{pmatrix}.$$

Hence, the first order derivative of the Melnikov function is obtained as

$$\begin{aligned}
\frac{\partial M}{\partial \mu}(0,\mu_0) &= \int_{-\infty}^{+\infty} < \psi(s), \frac{\partial F}{\partial \mu}(\bar{u}(s),\mu_0) > ds = \\
&= \int_{-\infty}^{+\infty} a(t) < \Big(\frac{\partial \bar{u}}{\partial t} \times v\Big)(t), \frac{\partial F}{\partial \mu}(\bar{u}(t),\mu_0) > dt \\
&= \int_{-\infty}^{+\infty} a(t) \det \begin{pmatrix} F_1(\bar{u},\mu_0) & 0 & 0 \\ 0 & w & 0 \\ F_3(\bar{u},\mu_0) & 0 & \frac{\partial F_3}{\partial \mu}(\bar{u},\mu_0) \end{pmatrix} (t) dt \\
&= \int_{-\infty}^{+\infty} a(t) w(t) F_1(\bar{u}(t),\mu_0) \frac{\partial F_3}{\partial \mu}(\bar{u}(t),\mu_0) dt
\end{aligned}$$

where $a(t) := \exp\big(-\int_0^t \operatorname{trace}\big(\frac{dF}{du}(\bar{u}(s),\mu_0)\big)ds\big)$. The third equality holds by the results of Lemma 2.3. The integrand

$$a(t)w(t)F_1(\bar{u}(t),\mu_0)\frac{\partial F_3}{\partial \mu}(\bar{u}(t),\mu_0)$$

is the product of scalar and continuous functions, which do not change sign by the equation $\frac{\partial F_3}{\partial \mu}(\bar{u}(t),\mu_0) = \frac{g(\bar{b},\bar{\tau})}{2k\bar{\tau}}$ and the results of Lemma 3.8. Thus the integrand has a common sign and is integrable, which implies that $\frac{\partial M}{\partial \mu}(0,\mu_0)$ does not vanish. □

In addition, we need to compute the second order derivative of the Melnikov function with respect to ν.

Lemma 3.11. *The Melnikov function satisfies*

$$\frac{\partial^2 M}{\partial \nu^2}(0,\mu_0) = \int_{-\infty}^{+\infty} a(t) w^3(t) F_1(\bar{u}(t),\mu_0) \frac{\partial^2 F_3}{\partial b_*^2}(\bar{u}(t),\mu_0) dt.$$

with functions

$$a(t) := \exp\Big(-\int_0^t \operatorname{trace}\Big(\frac{dF}{du}(\bar{u}(s),\mu_0)\Big)ds\Big)$$

and $v(t) = (0, w(t), 0)^t$ *from Lemma 3.8. Each of the following conditions implies that* $\frac{\partial^2 M}{\partial \nu^2}(0,\mu_0)$ *is non-zero:*

1. *The function* $(k\overline{\tau}(t) - d^2)$ *has a common sign for all* $t \in \mathbb{R}$.

2. *The expression* $k\tau_2 - d^2$ *is positive.*

Proof. By the results of Lemma A.10, the Melnikov function satisfies

$$\frac{\partial^2 M}{\partial \nu^2}(0, \mu_0) = \int_{-\infty}^{+\infty} < \psi(s), \frac{\partial^2 F}{\partial u^2}(\bar{u}(s), \mu_0)(v(s), v(s)) > ds,$$

where $\psi(s)$ is the unique (up to a multiplicative factor) bounded solution of the adjoint differential equation of (3.10). We derive from the results of Theorem 2.4 and Lemma 3.8 the expression

$$\psi(t) := \exp\Big(- \int_0^t \operatorname{trace}\Big(\frac{dF}{du}(\bar{u}(s), \mu_0)\Big) ds\Big)\Big(\frac{\partial \bar{u}}{\partial t}(t) \times v(t)\Big).$$

In addition, the special form of the solution $v(t) = (0, w(t), 0)^t$ implies that

$$\begin{aligned} \frac{\partial^2 F}{\partial u^2}(\bar{u}(t), \mu_0)(v(t), v(t)) &= \Big(0, 0, \frac{\partial^2 F_3}{\partial b_*^2}(\bar{u}(t), \mu_0) w^2(t)\Big)^t \\ &= \Big(0, 0, \frac{k\overline{\tau}(t) - d^2}{k\overline{\tau}(t)} w^2(t)\Big)^t \end{aligned}$$

where k is bigger than one and $\overline{\tau}(t)$ is positive for all $t \in \mathbb{R}$. We use these expressions to obtain

$$\begin{aligned} \frac{\partial^2 M}{\partial \nu^2}(0, \mu_0) &= \int_{-\infty}^{+\infty} < \psi(s), \frac{\partial^2 F}{\partial u^2}(\bar{u}(s), \mu_0)(v(s), v(s)) > ds \\ &= \int_{-\infty}^{+\infty} a(t) < \Big(\frac{\partial \bar{u}}{\partial t} \times v\Big), \frac{\partial^2 F}{\partial u^2}(\bar{u}, \mu_0)(v, v) > (t) dt \\ &= \int_{-\infty}^{+\infty} a(t) \det \begin{pmatrix} F_1(\bar{u}, \mu_0) & 0 & 0 \\ 0 & w & 0 \\ F_3(\bar{u}, \mu_0) & 0 & \frac{\partial^2 F_3}{\partial b_*^2}(\bar{u}, \mu_0) w^2 \end{pmatrix} (t) dt \\ &= \int_{-\infty}^{+\infty} a(t) w^3(t) F_1(\bar{u}, \mu_0) \frac{\partial^2 F_3}{\partial b_*^2}(\bar{u}, \mu_0)(t) dt, \end{aligned}$$

where the third equality holds by the results of Lemma 2.3. The integrand

$$a(t)w^3(t)F_1(\bar{u}(t),\mu_0)\frac{\partial^2 F_3}{\partial b_*^2}(\bar{u}(t),\mu_0)$$

is the product of scalar factors. The functions $a(t)$, $w(t)$ and $F_1(\bar{u}(t),\mu_0)$ do not change sign by the results of Lemma 3.8. Additionally, the continuous function $\frac{\partial^2 F_3}{\partial b_*^2}(\bar{u}(t),\mu_0) = \frac{k\bar{\tau}-d^2}{k\bar{\tau}}(t)$ does not vanish by the first assumption. Thus the integrand has a common sign and is integrable, which implies that $\frac{\partial^2 M}{\partial \nu^2}(0,\mu_0)$ is non-zero.

By the results of Lemma 3.8, the coordinate functions of the viscous profile $\bar{u}(t) = (\bar{b},0,\bar{\tau})^t(t)$ decrease strict monotonically to $(b_2,0,\tau_2)^t$. Thus the second assumption, $k\tau_2 - d^2 > 0$, implies that for all $t \in \mathbb{R}$ the inequality $k\bar{\tau}(t) - d^2 > 0$ holds and we obtain the statement from the previous result. □

Indeed we can identify a parameter regime such that a saddle-node bifurcation of the type studied in Chapter 2 occurs.

Theorem 3.1. *In case (C3) where H^- and H^+ both intersect G_1 a saddle-node bifurcation will occur.*

Proof. In case (C3), all stationary points are elements of the intersection $G_1 \cap H$ and the component G_1 lies entirely above the line $\tau = \frac{d^2}{k}$. Thus the τ coordinates of all four stationary points are greater than $\frac{d^2}{k}$, which implies by Lemma 3.11 that the second order derivative of the Melnikov function $\frac{\partial^2 M}{\partial \nu^2}(0,\mu_0)$ is not zero. We conclude from the Lemmata 3.9 and 3.10 that the assumptions of Theorem 2.3 hold, which implies the occurrence of a saddle-node bifurcation. □

Hence, the results from Chapter 2 on the spectral stability of the involved viscous profiles are applicable.

Lemma 3.12. *Any viscous profile $\bar{u}(t,\mu)$ that connects the endstates u_1 with u_2 is associated to a Lax 2-shock. Moreover, the Liu-Majda determinant,* $\det\big(r_1(u_1,\mu), u_2 - u_1, r_3(u_2,\mu)\big)$, *vanishes.*

Proof. All stationary points u_j for $j = 0, \ldots, 3$ of the vector field $F(u, \mu)$ lie in the plane E and are independent of the parameter μ. By the results of Lemma 3.5, the related eigenvalues of the vector field satisfy (with appropriate numbering) the inequalities $\lambda_1(u_1, \mu) < 0 < \lambda_2(u_1, \mu) < \lambda_3(u_1, \mu)$ and $\lambda_1(u_2, \mu) < \lambda_2(u_2, \mu) < 0 < \lambda_3(u_2, \mu)$. Thus any viscous profile that connects the endstates u_1 with u_2 is associated to a Lax 2-shock. Moreover, the points u_1 and u_2 are saddle points of the system (3.6) restricted to the plane E. Hence, the vectors $r_1(u_1, \mu)$, $u_2 - u_1$ and $r_3(u_2, \mu)$ are confined to the plane E and the Liu-Majda determinant vanishes. □

We conclude from Theorem 1.11 and Lemma 3.12 that for any viscous shock wave whose viscous profile connects the endstates u_1 with u_2 the related linear operator has an effective eigenvalue zero with multiplicity at least two. If, in addition, the parameter values are specified, it is possible to determine the type (B7a)-(B7c) of the viscous profile and to apply the results of Subsection 2.3.3 on the spectral stability of the associated viscous shock waves. However, we are not able to give a general classification for all viscous profiles.

In the following, we will consider a model problem which resembles the case (C3) and exhibits, besides the reflectional invariance, an additional symmetry. This will allow to verify the occurrence of a saddle-node bifurcation and to obtain explicit expressions for the profiles and the bifurcation value.

3.2 A model problem

We consider the model problem, due to Freistühler,

$$\left.\begin{aligned} \frac{\partial b_1}{\partial t} + \frac{\partial}{\partial x}(v b_1) &= \frac{\partial^2 b_1}{\partial x^2}, \\ \frac{\partial b_2}{\partial t} + \frac{\partial}{\partial x}(v b_2) &= \frac{\partial^2 b_2}{\partial x^2}, \\ \frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(v^2 + b_1^2 + b_2^2) &= \mu \frac{\partial^2 v}{\partial x^2}, \end{aligned}\right\} \tag{3.12}$$

where b_1, b_2 and v are functions of $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ and μ is a small positive parameter. In reference to the physical motivation $(b_1, b_2)^t$ corresponds to a transversal magnetic field, v to a specific volume and μ to a ratio of dissipation constants. The system (3.12) is a system of viscous conservation laws of the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = \frac{\partial}{\partial x}\left(B \frac{\partial u}{\partial x} \right), \tag{3.13}$$

where the functions are given by

$$u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^3, \quad (x,t) \mapsto u(x,t) := (b_1, b_2, v)^t(x,t)$$

and

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad (b_1, b_2, v)^t \mapsto (b_1 v, b_2 v, (b_1^2 + b_2^2 + v^2)/2)^t.$$

The introduction of a regular viscosity matrix $B := \operatorname{diag}(1, 1, \mu)$ does not affect the previous analysis. We will investigate the existence and stability of viscous shock waves of (3.12), whose associated viscous profile, $\bar{u}(\xi)$ with $\xi := x - st$, approaches asymptotic endstates. The associated profile equation is given by

$$\frac{du}{d\xi} = B^{-1}\big(f(u) - s \cdot u - c\big) =: F(u, \mu) \tag{3.14}$$

where the constant vector c satisfies $c = f(u^-) - su^- = f(u^+) - su^+$. We assume without loss of generality that the shock speed s is zero, otherwise we switch to a moving coordinate frame $(x,t) \to (\xi = x - st, t)$. Additionally, we choose an appropriate basis of the state space such that the relative flux satisfies $f(u^-) = f(u^+) = (0, \alpha_1, \alpha_2)^t$. Thus the profile equation (3.14) is obtained as

$$\left.\begin{aligned} \frac{db_1}{d\xi} &= v b_1, \\ \frac{db_2}{d\xi} &= v b_2 - \alpha_1, \\ \mu \frac{dv}{d\xi} &= \frac{1}{2}(v^2 + b_1^2 + b_2^2) - \alpha_2. \end{aligned}\right\} \tag{3.15}$$

The profile equation exhibits two symmetries

Lemma 3.13. *The profile equation* (3.15) *is invariant with respect to*

1. *reflections about the plane* $E := \{(b_1, b_2, v)^t \in \mathbb{R}^3 \,|\, b_1 \equiv 0\}$*:*

$$b_1 \mapsto -b_1, \quad b_2 \mapsto b_2, \quad v \mapsto v. \tag{3.16}$$

2. *time reversal and a reflection of* $(b_2, v)^t$ *at the origin:*

$$\xi \mapsto -\xi, \quad b_1 \mapsto b_1, \quad b_2 \mapsto -b_2, \quad v \mapsto -v. \tag{3.17}$$

Lemma 3.14. *Suppose* μ *is positive and the parameters* α_1 *and* α_2 *satisfy the inequality* $0 < \alpha_1 < \alpha_2$.

1. *Then the vector field* $F(u, \mu)$ *of the profile equation* (3.15) *has four hyperbolic fixed points*

$$u_0 = \begin{pmatrix} 0 \\ \beta_0 \\ \beta_1 \end{pmatrix}, \quad u_1 = \begin{pmatrix} 0 \\ \beta_1 \\ \beta_0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ -\beta_1 \\ -\beta_0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ -\beta_0 \\ -\beta_1 \end{pmatrix}. \tag{3.18}$$

where the positive constants β_0 *and* β_1 *are defined as*

$$\beta_0 := \frac{1}{\sqrt{2}}\big(\sqrt{\alpha_2 + \alpha_1} - \sqrt{\alpha_2 - \alpha_1}\big) \tag{3.19}$$

and

$$\beta_1 := \frac{1}{\sqrt{2}}\big(\sqrt{\alpha_2 + \alpha_1} + \sqrt{\alpha_2 - \alpha_1}\big). \tag{3.20}$$

2. *The Jacobian matrix of the vector field* $F(u, \mu)$ *at a stationary point* $u_j = (0, b_2, v)$ *for* $j = 0, \ldots, 3$ *satisfies*

$$\frac{dF}{du}(u_j, \mu) = \begin{pmatrix} v & 0 & 0 \\ 0 & v & b_2 \\ 0 & b_2/\mu & v/\mu \end{pmatrix} \tag{3.21}$$

and has real eigenvalues

$$\lambda_1(u_j,\mu) = \frac{1}{2}\left(\left(1+\frac{1}{\mu}\right)v - \sqrt{\left(1+\frac{1}{\mu}\right)^2 v^2 + \frac{4}{\mu}\left(b_2^2 - v^2\right)}\right), \quad (3.22)$$

$$\lambda_2(u_j,\mu) = v, \quad (3.23)$$

$$\lambda_3(u_j,\mu) = \frac{1}{2}\left(\left(1+\frac{1}{\mu}\right)v + \sqrt{\left(1+\frac{1}{\mu}\right)^2 v^2 + \frac{4}{\mu}\left(b_2^2 - v^2\right)}\right), \quad (3.24)$$

with associated eigenvectors

$$r_1(u_j,\mu) = \begin{pmatrix} 0 \\ b_2 \\ \lambda_1(u_j,\mu) - v \end{pmatrix}, \quad (3.25)$$

$$r_2(u_j,\mu) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad (3.26)$$

$$r_3(u_j,\mu) = \begin{pmatrix} 0 \\ b_2 \\ \lambda_3(u_j,\mu) - v \end{pmatrix}. \quad (3.27)$$

In addition, the eigenvalues (3.22)- (3.24) *are real valued and satisfy for all positive* μ *the order*

$$\lambda_1(u_j,\mu) < \lambda_2(u_j,\mu) < \lambda_3(u_j,\mu).$$

3. *The stationary point* u_0 *is a source,* u_1 *and* u_2 *are saddle points, and* u_3 *is a sink. The number of positive and negative eigenvalues of the Jacobian* $\frac{dF}{du}(u_j,\mu)$ *for* $j = 0,\ldots,3$ *are* $(3-j)$ *and* j*, respectively.*

Remark 3.2. If the parameters α_1 and α_2 satisfy $0 < -\alpha_1 < \alpha_2$, then a similar result holds.

Proof. A stationary point $u = (b_1, b_2, v)^t$ of the vector field $F(u, \mu)$ of (3.15) has to satisfy the identity

$$\begin{aligned} vb_1 &= 0, \\ vb_2 &= \alpha_1, \\ v^2 + b_1^2 + b_2^2 &= 2\alpha_2, \end{aligned}$$

which is equivalent to

$$b_1 = 0, \tag{3.28}$$
$$(b_2 + v)^2 = 2(\alpha_2 + \alpha_1), \tag{3.29}$$
$$(b_2 - v)^2 = 2(\alpha_2 - \alpha_1). \tag{3.30}$$

By the assumption $0 < \alpha_1 < \alpha_2$, the right hand sides of the equations (3.29) and (3.30) are positive. We take the square root and solve the linear system

$$\begin{aligned} b_2 + v &= \pm\sqrt{2}\sqrt{\alpha_2 + \alpha_1}, \\ b_2 - v &= \pm\sqrt{2}\sqrt{\alpha_2 - \alpha_1}. \end{aligned}$$

Thus we obtain four stationary points

$$\begin{pmatrix} b_1 \\ b_2 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \pm\frac{1}{\sqrt{2}}\left(\sqrt{\alpha_2 + \alpha_1} \mp \sqrt{\alpha_2 - \alpha_1}\right) \\ \pm\frac{1}{\sqrt{2}}\left(\sqrt{\alpha_2 + \alpha_1} \pm \sqrt{\alpha_2 - \alpha_1}\right) \end{pmatrix}, \tag{3.31}$$

which lie in the plane E. We define the constants β_0 and β_1 like in (3.19) and (3.20), respectively, and obtain the stationary points (3.31) in the proposed form (3.18). Moreover, the assumption $0 < \alpha_1 < \alpha_2$ implies that the constants β_0 and β_1 are real and positive.

The b_1-coordinate of all stationary points is identically zero. Hence the expression for the Jacobian matrix, the eigenvalues and the eigenvectors follow from a direct computation. The eigenvalues are real, since for a positive constant μ the discriminant,

$$\left(1+\frac{1}{\mu}\right)^2 v^2 + \frac{4}{\mu}(b_2^2 - v^2) = \left(1-\frac{1}{\mu}\right)^2 v^2 + \frac{4}{\mu}b_2^2,$$

is non-negative. The order of the eigenvalues is proved in two steps: First, we show that for positive μ the inequality $\lambda_1(u_j,\mu) < \lambda_2(u_j,\mu)$ holds, that means

$$\frac{1}{2}\left(\left(1+\frac{1}{\mu}\right)v - \sqrt{\left(1+\frac{1}{\mu}\right)^2 v^2 + \frac{4}{\mu}\left(b_2^2 - v^2\right)}\right) < v.$$

This is equivalent to

$$-\sqrt{\left(1-\frac{1}{\mu}\right)^2 v^2 + \frac{4}{\mu}b_2^2} < \left(1-\frac{1}{\mu}\right)v,$$

which holds since v- and b_2-coordinate of a stationary point u_j are non-zero. In the same way, for positive μ the inequality $\lambda_2(u_j,\mu) < \lambda_3(u_j,\mu)$ is proved.

The assumption $0 < \alpha_1 < \alpha_2$ implies that $0 < \beta_0 < \beta_1$. Thus we obtain from the definition of the eigenvalues (3.22)- (3.23) the classification of the stationary points (3.18). □

Remark 3.3. The assumption $0 < \alpha_1 < \alpha_2$ implies that the constants β_0 and β_1 in Lemma 3.14 are real and positive. Moreover, the dependence on the parameters α_1 and α_2 is invertible and we obtain

$$\alpha_2 = \frac{\beta_0^2 + \beta_1^2}{2} \quad \text{and} \quad \alpha_1 = \beta_0\beta_1. \tag{3.32}$$

Thus the profile equation can be written as

$$\left.\begin{aligned} \frac{db_1}{d\xi} &= vb_1, \\ \frac{db_2}{d\xi} &= vb_2 - \beta_0\beta_1, \\ \mu\frac{dv}{d\xi} &= \frac{1}{2}(v^2 + b_1^2 + b_2^2) - \frac{\beta_0^2 + \beta_1^2}{2}. \end{aligned}\right\} \tag{3.33}$$

Remark 3.4. In Lemma 3.13, we recorded two symmetries of the profile equation. Due to the second one, the stationary point $u_1 = (0, \beta_1, \beta_0)^t$ is mapped onto $u_2 = (0, -\beta_1, -\beta_0)^t$ and the identities

$$\begin{aligned} \lambda_1(u_1,\mu) &= -\lambda_3(u_2,\mu), & r_1(u_1,\mu) &= r_3(u_2,\mu),\\ \lambda_2(u_1,\mu) &= -\lambda_2(u_2,\mu), & r_2(u_1,\mu) &= r_2(u_2,\mu),\\ \lambda_3(u_1,\mu) &= -\lambda_1(u_2,\mu), & r_3(u_1,\mu) &= r_1(u_2,\mu), \end{aligned}$$

hold. In the same way, u_0 is associated to u_3.

We study the existence of viscous profiles in the profile equation (3.33).

Theorem 3.2. *Suppose the inequalities $0 < \beta_0 < \beta_1$ hold. Then the profile equation* (3.33) *has viscous profiles of the following kind:*

1. *For non-negative μ, there exist a transversal heteroclinic orbit connecting u_0 with u_1, and a transversal heteroclinic orbit connecting u_2 with u_3.*

2. *For all $0 < \mu \le \mu_0 := \frac{\beta_0^2+\beta_1^2}{2\beta_0^2}$, there exist viscous profiles of the form*

$$\bar{u}_{12}(\xi,\mu) := \begin{pmatrix} \pm\frac{\sqrt{\beta_0^2+\beta_1^2-2\mu\beta_0^2}}{\cosh(\beta_0\xi)} \\ -\beta_1\tanh(\beta_0\xi) \\ -\beta_0\tanh(\beta_0\xi) \end{pmatrix}. \tag{3.34}$$

   The associated heteroclinic orbits connect u_1 with u_2.

3. *For $0 < \mu \le \mu_1 := \frac{\beta_0^2+\beta_1^2}{2\beta_1^2}$, there exist viscous profiles of the form*

$$\bar{u}_{03}(\xi,\mu) := \begin{pmatrix} \pm\frac{\sqrt{\beta_0^2+\beta_1^2-2\mu\beta_1^2}}{\cosh(\beta_1\xi)} \\ -\beta_0\tanh(\beta_1\xi) \\ -\beta_1\tanh(\beta_1\xi) \end{pmatrix}. \tag{3.35}$$

   The associated heteroclinic orbits connect u_0 with u_3 and exist by a transversal intersection of the invariant manifolds $W^u(u_0)$ and $W^s(u_3)$.

Proof. 1. We will show that the proposed heteroclinic orbits exist and lie entirely in the plane $E = \{(b_1, b_2, v)^t \in \mathbb{R}^3 \,|\, b_1 \equiv 0\}$: An associated solution has a b_1-coordinate which vanishes identically and the profile equation (3.33) reduces to the system

$$\left.\begin{aligned} \frac{db_2}{d\xi} &= vb_2 - \beta_0\beta_1, \\ \mu\frac{dv}{d\xi} &= \frac{1}{2}(v^2 + b_2^2) - \frac{\beta_0^2 + \beta_1^2}{2}. \end{aligned}\right\} \tag{3.36}$$

The stationary points $\tilde{P}_0 = (\beta_0, \beta_1)$ and $\tilde{P}_1 = (\beta_1, \beta_0)$ are a source and a saddle point of the system (3.36), respectively. The nullclines of (3.36) describe a hyperbola, $vb_2 = \beta_0\beta_1$, and a circle, $v^2 + b_2^2 = \beta_0^2 + \beta_1^2$, in the plane E, see Figure 3.3.

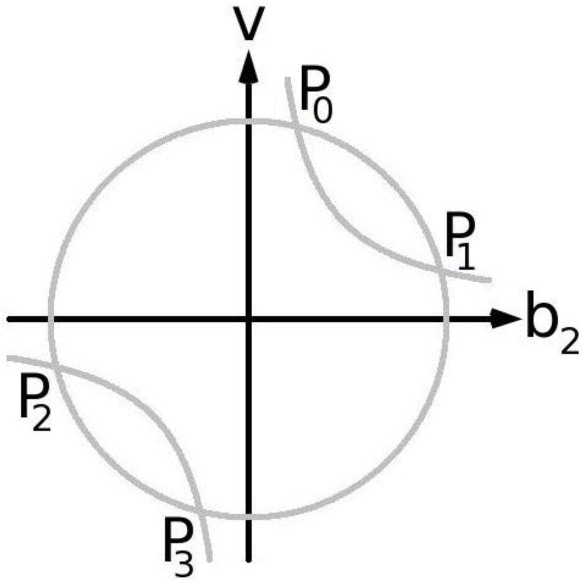

Figure 3.3: The stationary points are the intersection points of the nullclines.

We consider the domain

$$D := \left\{ \begin{pmatrix} b_2 \\ v \end{pmatrix} \in \mathbb{R}_+^2 \,\middle|\, vb_2 \geq \beta_0\beta_1, \quad v^2 + b_2^2 \leq \beta_0^2 + \beta_1^2 \right\}, \tag{3.37}$$

which is negatively invariant with respect to the flow (3.36), see Figure 3.4. Moreover, the stable manifold $W^s(u_1)$ points into the domain D as long as μ

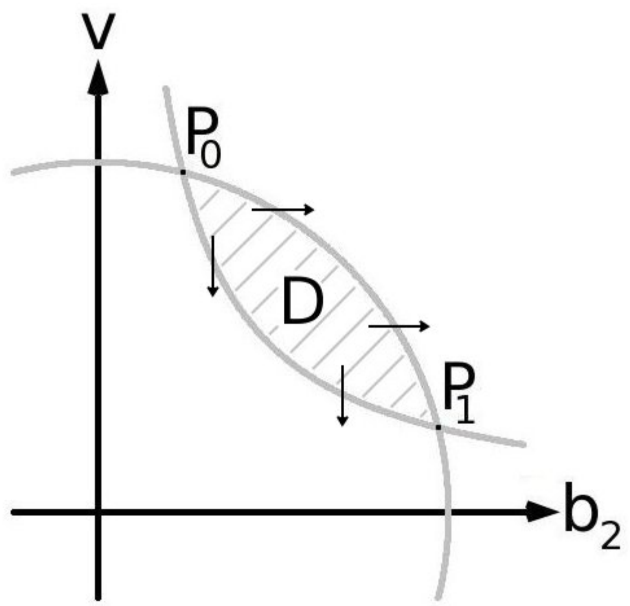

Figure 3.4: The domain D is negatively invariant.

is positive. Since u_0 is a source, there exists a transversal heteroclinic orbit connecting u_0 with u_1.

In case $\mu = 0$, the profile equation (3.36) reduces to the system

$$\left.\begin{aligned} \frac{db_2}{d\xi} &= vb_2 - \beta_0\beta_1, \\ 0 &= \frac{1}{2}(v^2 + b_2^2) - \frac{\beta_0^2 + \beta_1^2}{2}. \end{aligned}\right\} \tag{3.38}$$

Thus solutions of the reduced system (3.38) are restricted to the circle, $v^2 + b_2^2 = \beta_0^2 + \beta_1^2$. The stationary point u_0 is a source and u_1 is a sink of the reduced system (3.38) and lie on the solution manifold. Hence, the arc of the circle connecting u_0 with u_1 is a heteroclinic orbit, since there are no additional stationary points along this arc. Again, the heteroclinic orbit exists by a transversal intersection of the invariant manifolds $W^u(u_0)$ and $W^s(u_1)$, since u_0 is a source of the original profile equation (3.33).

By the symmetry of the profile equation (3.36), see Lemma 3.13, a viscous profile $\bar{u}(\xi) = (b_1, b_2, v)^t(\xi)$ induces the existence of another solution of the form $\hat{u}(\xi) = (b_1, -b_2, -v)^t(-\xi)$. Moreover, the symmetry implies that the identities $u_3 = -u_0$ and $u_2 = -u_1$ hold. Thus the existence of a heteroclinic orbit between u_0 and u_1 implies the existence of a heteroclinic orbit between u_2 and u_3, which is transversal since u_3 is a sink.

2. Since the stationary points $u_1 = (0, \beta_1, \beta_0)^t$ and $u_2 = (0, -\beta_1, -\beta_0)^t$ lie on the straight line $\text{span}\{(0, \beta_1, \beta_0)^t\}$, we make an ansatz for a viscous profile of the form

$$\bar{u}(\xi, \mu) = \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix} (\xi, \mu) + \begin{pmatrix} 0 \\ \beta_1 \\ \beta_0 \end{pmatrix} w(\xi, \mu) \tag{3.39}$$

with functions $b_1, w : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$. Such a solution has to satisfy the profile equation (3.33), which becomes the overdetermined system

$$\frac{db_1}{d\xi} = \beta_0 b_1 w, \tag{3.40}$$

$$\beta_1 \frac{dw}{d\xi} = \beta_0 \beta_1 (w^2 - 1), \tag{3.41}$$

$$\mu \beta_0 \frac{dw}{d\xi} = \frac{\beta_0^2 + \beta_1^2}{2}(w^2 - 1) + \frac{b_1^2}{2}. \tag{3.42}$$

Specifically we obtain from the equations (3.41) and (3.42) an implicit definition of the solution manifold

$$0 = \left(\frac{\beta_0^2 + \beta_1^2}{2} - \mu \beta_0^2 \right)(w^2 - 1) + \frac{b_1^2}{2}. \tag{3.43}$$

The defining equation (3.43) of the solution manifold is consistent with the differential equations (3.40) and (3.41), if the derivative of the right hand side of (3.43),

$$\frac{d}{d\xi} \left(\left(\frac{\beta_0^2 + \beta_1^2}{2} - \mu \beta_0^2 \right)(w^2 - 1) + \frac{b_1^2}{2} \right) \tag{3.44}$$

vanishes identically for all $\xi \in \mathbb{R}$. This is true, since the expression (3.44) simplifies to

$$2\beta_0 w\left(\left(\frac{\beta_0^2+\beta_1^2}{2}-\mu\beta_0^2\right)(w^2-1)+\frac{b_1^2}{2}\right),$$

with (3.40) and (3.41), which vanishes identically by equation (3.43). Since β_1 is non-zero, the differential equation (3.41) has a solution

$$w(\xi,\mu) = -\tanh(\beta_0\xi + C)$$

with a constant $C \in \mathbb{R}$. We use the identity (3.43) to obtain the expression

$$b_1(\xi,\mu) = \pm\sqrt{\left(\beta_0^2+\beta_1^2-2\mu\beta_0^2\right)(1-w^2(\xi,\mu))},$$

which is real-valued as long as

$$(\beta_0^2+\beta_1^2-2\mu\beta_0^2)(1-w^2(\xi,\mu))$$

is non-negative. Since the expression $(1-w^2(\xi,\mu)) = (1-\tanh^2(\beta_0\xi))$ is non-negative for all $\xi \in \mathbb{R}$, viscous profiles of the proposed form (3.34) will exist as long as $0 \leq \mu \leq \frac{\beta_0^2+\beta_1^2}{2\beta_0^2}$. Moreover, the function $w(\xi,\mu)$ has the asymptotic behavior $\lim_{\xi\to\pm\infty} w(\xi,\mu) = \mp 1$ and the associated heteroclinic orbits connect the endstates u_1 with u_2.

3. In the same way, we prove the existence of heteroclinic orbits connecting the endstates $u_0 = (0,\beta_0,\beta_1)^t$ with $u_3 = (0,-\beta_0,-\beta_1)^t$. Since the endstates lie on the straight line $\operatorname{span}\{(0,\beta_0,\beta_1)^t\}$, we make the ansatz

$$\bar{u}(\xi,\mu) = \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix}(\xi,\mu) + \begin{pmatrix} 0 \\ \beta_0 \\ \beta_1 \end{pmatrix} w(\xi,\mu)$$

with scalar functions $b_1, w : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$. Thus we obtain for $0 \leq \mu \leq \frac{\beta_0^2+\beta_1^2}{2\beta_1^2}$ the existence of viscous profiles of the proposed form (3.35). In addition, the associated heteroclinic orbits are transversal, since u_0 is a source and u_3 is a sink. □

Remark 3.5. The assumption $0 < \beta_0 < \beta_1$ implies the inequality

$$\mu_1 := \frac{\beta_0^2 + \beta_1^2}{2\beta_1^2} < \frac{\beta_0^2 + \beta_1^2}{2\beta_0^2} =: \mu_0.$$

Consequently, for all $0 \leq \mu \leq \mu_1$, there exist viscous profiles of the form (3.34) and (3.35) which connect u_1 with u_2 and u_0 with u_3, respectively. In the range $\mu_1 < \mu \leq \mu_0$ only the viscous profiles from u_1 to u_2 remain, which cease to exist for $\mu > \mu_0$.

We will investigate the obtained families of viscous profiles (3.34), which satisfies the conditions (B1)-(B3) of the previous chapter. In particular, the viscous profiles (3.34) are associated to a Lax 2-shock.

Lemma 3.15. *The family of viscous profiles $\bar{u}_{12}(\xi, \mu)$ in Theorem 3.2, which connect the stationary points u_1 and u_2, exhibit a saddle-node bifurcation at the parameter value $\mu_0 = \frac{\beta_0^2+\beta_1^2}{2\beta_0^2}$.*

Proof. In order to prove the occurrence of a saddle-node bifurcation, we will verify that the associated heteroclinic orbit exists for $\mu = \mu_0$ by a non-transversal intersection of the invariant manifolds $W^u(u_1)$ and $W^s(u_2)$. This will allow us to construct the Melnikov function and to check the necessary conditions (B5) on its derivatives.

By the result of Lemma 3.14, the stationary points u_1 and u_2 are hyperbolic and have a two-dimensional unstable manifold $W^u(u_1)$ and a two-dimensional stable manifold $W^s(u_2)$, respectively. In accordance with the result of Theorem 2.3, we define a new parameter ν via

$$\mu(\nu) := \mu_0 - \nu^2 = \frac{\beta_0^2 + \beta_1^2}{2\beta_0^2} - \nu^2$$

and obtain a smooth parameterization of the family of viscous profiles (3.34):

$$\bar{u}_{12}(\xi, \nu) := \begin{pmatrix} \frac{\sqrt{2}\beta_0 \nu}{\cosh(\beta_0 \xi)} \\ -\beta_1 \tanh(\beta_0 \xi) \\ -\beta_0 \tanh(\beta_0 \xi) \end{pmatrix}. \tag{3.45}$$

Moreover, the partial derivatives of $\bar{u}_{12}(\xi,\nu)$ at $(\xi,0)$,

$$\frac{\partial \bar{u}_{12}}{\partial \xi}(\xi,0) = \frac{\beta_0}{\cosh^2(\beta_0\xi)}\begin{pmatrix}0\\-\beta_1\\-\beta_0\end{pmatrix} \quad \text{and} \quad \frac{\partial \bar{u}_{12}}{\partial \nu}(\xi,0) = \frac{\sqrt{2}\beta_0}{\cosh(\beta_0\xi)}\begin{pmatrix}1\\0\\0\end{pmatrix},$$

are solutions of the linearized profile equation

$$\frac{dp}{d\xi}(\xi) = \frac{dF}{du}(\bar{u}_{12}(\xi,0),\mu_0)p(\xi) \tag{3.46}$$

and decay in both limits $\xi \to \pm\infty$. Hence, the linearly independent functions $\frac{\partial \bar{u}_{12}}{\partial \xi}(\xi,0)$ and $\frac{\partial \bar{u}_{12}}{\partial \nu}(\xi,0)$ span a two-dimensional intersection of tangent spaces associated to the invariant manifolds $W^u(u_1)$ and $W^s(u_2)$. Consequently, the viscous profile is non-transversal and the assumption (B4) holds with $k=1$ and $n=3$. By the result of Theorem 2.1, we are able to construct a Melnikov function $M(\nu,\mu)$ which satisfies

$$M(0,\mu_0) = 0 \quad \text{and} \quad \frac{\partial M}{\partial \nu}(0,\mu_0) = 0.$$

Moreover, Theorem 2.4 implies that

$$\psi(\xi) := \sqrt{2}\beta_0^2\big(\cosh(\beta_0\xi)\big)^{\frac{\beta_0^2-\beta_1^2}{\beta_0^2+\beta_1^2}}\begin{pmatrix}0\\-\beta_0\\\beta_1\end{pmatrix}, \tag{3.47}$$

is the unique (up to a multiplicative factor) bounded solution of the adjoint differential equation of (3.46). Thus the derivatives of the Melnikov function are determined by the results of Lemmata A.9 and A.10 as

$$\begin{aligned}\frac{\partial M}{\partial \mu}(0,\mu_0) &= \int_{-\infty}^{+\infty} < \psi(\xi), \frac{\partial F}{\partial \mu}(\bar{u}_{12}(\xi,0),\mu_0) > d\xi \\ &= \frac{\sqrt{2}}{\mu_0}\beta_0^4\beta_1\int_{-\infty}^{+\infty}\big(\cosh(\beta_0\xi)\big)^{\frac{\beta_0^2-\beta_1^2}{\beta_0^2+\beta_1^2}-2}d\xi \end{aligned} \tag{3.48}$$

and

$$\begin{aligned}\frac{\partial^2 M}{\partial \nu^2}(0,\mu_0) &= \int_{-\infty}^{+\infty} < \psi(\xi), \frac{\partial^2 F}{\partial u^2}(\bar{u}_{12},\mu_0)\Big(\frac{\partial \bar{u}_{12}}{\partial \nu}, \frac{\partial \bar{u}_{12}}{\partial \nu}\Big)(\xi,0) > d\xi \\ &= \frac{2\sqrt{2}}{\mu_0}\beta_0^4\beta_1 \int_{-\infty}^{+\infty} \big(\cosh(\beta_0\xi)\big)^{\frac{\beta_0^2-\beta_1^2}{\beta_0^2+\beta_1^2}-2} d\xi, \qquad (3.49)\end{aligned}$$

respectively. The positive-valued function $\big(\cosh(\beta_0\xi)\big)^{\frac{\beta_0^2-\beta_1^2}{\beta_0^2+\beta_1^2}-2}$ can be shown to be integrable, since the exponent is negative by the assumption $0 < \beta_0 < \beta_1$. Thus the integral

$$\int_{-\infty}^{+\infty} \big(\cosh(\beta_0\xi)\big)^{\frac{\beta_0^2-\beta_1^2}{\beta_0^2+\beta_1^2}-2} d\xi$$

is definite and does not vanish, which implies that (3.48) and (3.49) are also non-zero. Hence, the assumptions of Theorem 2.3 hold and we conclude the occurrence of a saddle-node bifurcation of viscous profiles (3.34) with respect to μ. □

Remark 3.6. Heteroclinic orbits from u_0 to u_3 will persist for all parameter values μ close to μ_1, since they are transversal. The special family of solutions in Theorem 3.2 cease to exist for $\mu > \mu_1$, but other solutions nearby will connect the two endstates.

3.2.1 Stability of the viscous shock waves

The family of viscous profiles in Theorem 3.2, which connect the endstates u_1 with u_2, have a smooth parameterization $\bar{u}_{12}(\xi,\mu(\nu))$ with $\mu(\nu) := \mu_0 - \nu^2$. In the proof of Lemma 3.15, we verified for these viscous profiles the conditions (B1)-(B5). Therefore, we can use the Evans function $E(\kappa,\nu)$ in Theorem 2.5 to locate the (effective) eigenvalues of the associated linear operator.

Lemma 3.16. *The viscous profiles $\bar{u}_{12}(\xi,\mu(\nu))$ with $\mu(\nu) := \mu_0 - \nu^2$ in Theorem 3.2 are associated to a Lax 2-shock. Moreover, the associated Liu-*

Majda determinant,

$$\det(r_1(u_1, \mu(\nu)), u_2 - u_1, r_3(u_2, \mu(\nu))), \tag{3.50}$$

vanishes, since the constant vectors $r_1(u_1, \mu(\nu))$ and $r_2(u_2, \mu(\nu))$ are linearly dependent.

Proof. Due to the nature of the saddle points u_1 and u_2, see Lemma 3.14, the viscous profiles are associated to a Lax 2-shock and the Liu-Majda determinant equals (3.50). However, by the results of Lemma 3.14 and Remark 3.4, the constant vectors $r_1(u_1, \mu)$ and $r_3(u_2, \mu)$ are linearly dependent. □

Since the determinant (3.50) vanishes, the Liu-Majda condition (B6) fails. In particular, the case (B7b) occurs and we prove the following result.

Theorem 3.3. *Consider the viscous shock waves whose viscous profiles are given by $\bar{u}_{12}(\xi, \mu(\nu))$ with $\mu(\nu) := \mu_0 - \nu^2$ in Theorem 3.2. For sufficiently small ν, the associated linear operator has an effective eigenvalue zero of multiplicity two, which exceeds the dimension (one) of the manifold of heteroclinic orbits connecting u_1 with u_2 for fixed $\mu(\nu)$.*

Proof. By the result of Theorem 1.9, we obtain an analytic continuation of the Evans function,

$$E(\kappa, \nu) = \det\left(U_1^f, U_2^f, U_1^s, S_1^f, S_2^f, S_1^s\right)(0, \kappa, \nu), \tag{3.51}$$

into a small neighborhood of the origin. The result of Lemma 3.16 implies that the case (B7b) occurs. Thus the first derivative of the Evans function, $\frac{\partial E}{\partial \kappa}(0, \nu)$, vanishes for all $\nu \in \mathbb{R}$. By the result of Theorem 2.9, the second order derivative of the Evans function with respect to κ satisfies

$$\frac{\partial^2 E}{\partial \kappa^2}(0,0) = \int_{-\infty}^{+\infty} < \psi(\xi), r_3(u_2, \mu_0) > d\xi \cdot \det([\bar{u}], [\bar{v}], r_3(u_2, \mu_0)) \tag{3.52}$$

with functions

$$\psi(\xi) := \sqrt{2}\beta_0^2\big(\cosh(\beta_0\xi)\big)^{\frac{\beta_0^2-\beta_1^2}{\beta_0^2+\beta_1^2}}\begin{pmatrix}0\\-\beta_0\\\beta_1\end{pmatrix},$$

and constant vectors

$$[\bar{u}] = u_2 - u_1 = \begin{pmatrix}0\\-2\beta_1\\-2\beta_0\end{pmatrix}$$

$$[\bar{v}] = \int_{-\infty}^{+\infty}\frac{\partial\bar{u}_{12}}{\partial\nu}(x,0)dx = \begin{pmatrix}\pi\sqrt{2}\\0\\0\end{pmatrix}$$

and

$$r_3(u_2,\mu_0) = -r_1(u_1,\mu_0) = \begin{pmatrix}0\\-\beta_1\\2\frac{\beta_1^2\beta_0}{\beta_1^2+\beta_0^2}\end{pmatrix}.$$

Thus the expression (3.52) simplifies to

$$\frac{\partial^2 E}{\partial\kappa^2}(0,0) = 4\pi\beta_1^2\beta_0^4\left(\frac{\beta_0^2+3\beta_1^2}{\beta_0^2+\beta_1^2}\right)^2\int_{-\infty}^{+\infty}\big(\cosh(\beta_0\xi)\big)^{\frac{\beta_0^2-\beta_1^2}{\beta_0^2+\beta_1^2}}d\xi. \qquad (3.53)$$

By the assumption $0<\beta_0<\beta_1$, the constant factors do not vanish and the exponent $\frac{\beta_0^2-\beta_1^2}{\beta_0^2+\beta_1^2}$ is negative. Thus the positive-valued function

$$\big(\cosh(\beta_0\xi)\big)^{\frac{\beta_0^2-\beta_1^2}{\beta_0^2+\beta_1^2}}$$

can be shown to be integrable and the integral

$$\int_{-\infty}^{+\infty}\big(\cosh(\beta_0\xi)\big)^{\frac{\beta_0^2-\beta_1^2}{\beta_0^2+\beta_1^2}}d\xi$$

is definite and does not vanish. Consequently, the second order derivative of the Evans function (3.53) is non-zero at the origin. The smoothness of the Evans function implies that also $\frac{\partial^2 E}{\partial \kappa^2}(0, \nu)$ does not vanish for sufficiently small $\nu \in \mathbb{R}$. Hence, for all sufficiently small ν, the linear operator associated to viscous shock waves with viscous profile $\bar{u}_{12}(\xi, \mu(\nu))$ has an effective eigenvalue zero of multiplicity two. For each $\mu(\nu)$, the viscous shock wave with viscous profile $\bar{u}_{12}(\xi, \mu(\nu))$ is not spectrally stable, since the manifold of heteroclinic orbits connecting u_1 with u_2 is only one-dimensional. □

The model problem resembles the case (C3) in magnetohydrodynamics, where all stationary points lie on the closed nullcline G_1. Due to the additional symmetry, we are able to obtain the family of viscous profile involved in the saddle-node bifurcation and classify the failure of the Liu-Majda condition (B7b). Thus we obtain the general result of Theorem 3.3 on spectral stability of the involved viscous profiles.

Appendix A

Melnikov theory

We will use Melnikov theory to study the persistence of heteroclinic orbits in a parameter dependent family of autonomous differential equations. In particular, we obtain a system of bifurcation equations for the parameters, whose solution set ensures the existence of heteroclinic orbits that are close to the original one. In this way, we observe that transversal heteroclinic orbits persist for small variations of the parameters. Whereas, non-transversal heteroclinic orbits persist only for a proper subset of a small neighborhood of the original parameters. In this account on Melnikov theory we follow the references [Van92, Kok88, Wig03].

A.1 Persistence of heteroclinic orbits

We consider a family of autonomous differential equations

$$\frac{du}{dt}(t,\mu) = F(u(t,\mu),\mu) \tag{A.1}$$

with $t \in \mathbb{R}$, state variable $u \in \mathbb{R}^n$, parameter $\mu \in \mathbb{R}^m$ and a smooth function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.

Definition A.1. The system (A.1) has for some μ_0 a heteroclinic orbit γ, if there exist two distinct hyperbolic fixed points $u^\pm$ of the vector field $F(u,\mu_0)$ and a solution $\bar{u}(t)$ of (A.1) that satisfies $\lim_{t\to\pm\infty} \bar{u}(t) = u^\pm$.

In other words, a heteroclinic orbit exists if the intersection of the stable manifold $W^s(u^+)$ and the unstable manifold $W^u(u^-)$ is not empty.

The hyperbolicity of the fixed points $u^\pm$ implies by the implicit function theorem the existence of a small neighborhood of μ_0, $B(\mu_0)$, and functions $u^\pm : B(\mu_0) \to \mathbb{R}^n$, $\mu \mapsto u^\pm(\mu)$, such that $u^\pm(\mu)$ are hyperbolic fixed points for $F(u, \mu)$ and $u^\pm(\mu_0) = u^\pm$. The heteroclinic orbit $\gamma(\mu_0)$ persists for μ, if a heteroclinic orbit $\gamma(\mu)$ exists that connects the fixed points $u^\pm(\mu)$ and is close to the original orbit $\gamma(\mu_0)$. We make the following assumptions:

(M1) For $\mu = \mu_0$, a heteroclinic orbit γ in the profile equation (A.1) exists that connects two distinct hyperbolic fixed points $u^\pm$ of the vector field $F(u, \mu_0)$.

In order to study the persistence of the heteroclinic orbit γ in (M1), we fix a solution $\bar{u}(t)$ of $(\text{A.1})_{\mu_0}$ that parametrizes the orbit $\gamma = \{\bar{u}(t) \mid t \in \mathbb{R}\}$. We are interested in solutions $u(t, \mu)$ of (A.1) whose orbits remain close to γ and consider the ansatz

$$u(t, \mu) := \bar{u}(t) + z(t). \tag{A.2}$$

Hence, the auxiliary function $z : \mathbb{R} \to \mathbb{R}^n$ is globally bounded and its norm $\|z\|_\infty := \sup_{t\in\mathbb{R}} \|z(t)\|$ is small. We insert the ansatz into (A.1) and obtain a differential equation for the auxiliary function as

$$\frac{dz}{dt}(t) = \frac{dF}{du}(\bar{u}(t), \mu_0) z(t) + g(t, z(t), \mu). \tag{A.3}$$

The smooth inhomogeneity is given by

$$g(t, z, \mu) := F(\bar{u}(t) + z, \mu) - F(\bar{u}(t), \mu_0) - \frac{dF}{du}(\bar{u}(t), \mu_0) z \tag{A.4}$$

and satisfies the identities

$$g(t, 0, \mu_0) = 0 \in \mathbb{R}^n \quad \text{and} \quad \frac{\partial g}{\partial z}(t, 0, \mu_0) = 0 \in \mathbb{R}^{n\times n}. \tag{A.5}$$

We note that the existence of a pair (z, μ) is equivalent to the existence of a heteroclinic orbit with $\bar{u}(t,\mu) := \bar{u}(t) + z(t)$ for μ close to μ_0. In the following we will draw upon properties of the homogeneous part of (A.3), which represents a linear system of ODEs. We recall for general linear systems with associated evolution operator $\phi(t,s)$ the definition of exponential dichotomies.

Definition A.2. Let $I = \mathbb{R}^-, \mathbb{R}^+$ or $\mathbb{R}$. A linear system of ODEs has an *exponential dichotomy* on I if constants $K > 1$ and $\kappa > 0$ exist as well as a family of projections $P(t)$, defined and continuous for all $t \in I$, such that the following holds true for all $s, t \in I$:

1. The projections commute with the evolution operator

$$\phi(t,s)P(s) = P(t)\phi(t,s),$$

   such that for all points $z_0 \in \mathbb{R}^n$ and the associated family of projections $Q(t) := I - P(t)$ the following identities hold:

$$\begin{aligned} \phi(t,s)P(s)z_0 &\in \text{image}(P(t)), \quad t \geq s, \\ \phi(t,s)Q(s)z_0 &\in \text{kernel}(P(t)), \quad t \leq s. \end{aligned}$$

2. $\left|\phi(t,s)P(s)\right| \leq K \cdot e^{-\kappa(t-s)}, \quad t \geq s,$

3. $\left|\phi(t,s)Q(s)\right| \leq K \cdot e^{-\kappa(s-t)}, \quad t \leq s.$

Lemma A.1. *Let the assumption (M1) hold. The linear system of ODEs*

$$\frac{dz}{dt}(t) = \frac{dF}{du}(\bar{u}(t), \mu_0)z(t) \tag{A.6}$$

has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$ with associated families of projections $\{P_+(t) \mid t \in \mathbb{R}^+\}$ and $\{P_-(t) \mid t \in \mathbb{R}^-\}$, respectively.

The statement follows from the hyperbolicity of the matrices $\frac{dF}{du}(u^\pm, \mu_0)$ and the roughness property of exponential dichotomies [Cop78, chapter 4]. The linear system (A.6) is the linearization of the nonlinear system (A.1) about the solution $\bar{u}(t)$. Hence, the stable subspace of (A.6) corresponds to the tangent space of the stable manifold $W^s(u^+)$ of (A.1), that means for all $t \geq 0$ the identity

$$\text{image}\big(P_+(t)\big) = T_{\bar{u}(t)}W^s(u^+) \tag{A.7}$$

holds. Similarly, for all $t \leq 0$ the unstable subspace of (A.6) satisfies

$$\text{image}\big(Q_-(t)\big) = T_{\bar{u}(t)}W^u(u^-). \tag{A.8}$$

The projections P_+ and Q_- are not unique. At a later stage, we fix the kernels of these projections such that subsequent expressions simplify.

We will restrict our attention to heteroclinic orbits $\gamma(\mu)$ for μ in a small neighborhood of μ_0, which are close to $\gamma(\mu_0)$ and intersect each transversal section of $\gamma(\mu_0)$ once. First, we find for the associated solution $\bar{u}(t)$ an appropriate description of the transversal section at $t = 0$:

Definition A.3. Suppose the solution $\bar{u}(t)$ of $(\text{A.1})_{\mu_0}$ parametrizes the heteroclinic orbit in (M1). We define the transversal section Y with respect to $\bar{u}(t)$ at $t = 0$ as the linear space that is orthogonal to the tangent vector $\frac{d\bar{u}}{dt}(0) = F(\bar{u}(0), \mu_0)$ such that $\mathbb{R}^n = \text{span}\{F(\bar{u}(0), \mu_0)\} \oplus Y$.

We decompose the transversal section Y further. The derivative of the solution, $\frac{d\bar{u}}{dt}(t) = F(\bar{u}(t), \mu_0)$, lies in the intersection of the stable and unstable subspace of (A.6) and we observe that

$$\frac{d\bar{u}}{dt}(0) = F(\bar{u}(0), \mu_0) \in T_{\bar{u}(0)}W^u(u^-) \cap T_{\bar{u}(0)}W^s(u^+).$$

We will assume that there exist, additionally, $k \in \mathbb{N} \cup \{0\}$ linearly independent vectors u_i for $i = 1, \dots, k$ in the intersection of the tangent spaces, such that

$$T_{\bar{u}(0)}W^u(u^-) \cap T_{\bar{u}(0)}W^s(u^+) = \text{span}\{F(\bar{u}(0), \mu_0)\} \oplus \text{span}\{u_1, \dots, u_k\}$$

and define the space $U := \text{span}\{u_1, \dots, u_k\}$. The spaces V_s and V_u are determined via the identities

$$T_{\bar{u}(0)}W^s(u^+) = \big(T_{\bar{u}(0)}W^u(u^-) \cap T_{\bar{u}(0)}W^s(u^+)\big) \oplus V_s$$

and

$$T_{\bar{u}(0)}W^u(u^-) = \big(T_{\bar{u}(0)}W^u(u^-) \cap T_{\bar{u}(0)}W^s(u^+)\big) \oplus V_u,$$

respectively. Moreover, we define W as the complement of the sum of the tangent spaces $T_{\bar{u}(0)}W^u(u^-)$ and $T_{\bar{u}(0)}W^s(u^+)$ in $\mathbb{R}^n$. In this way, we obtain a splitting of the transversal section Y into orthogonal subspaces

$$Y = U \oplus V_s \oplus V_u \oplus W.$$

Since we are interested in solutions $u(t,\mu) = \bar{u}(t) + z(t)$ of (A.1) whose orbits remain close to the orbit of $\bar{u}(t)$, the auxiliary function $z(t)$ of (A.3) has to be globally bounded and lies in the intersection of the stable manifold

$$W^s(\mu) := \{z \in C_b^1(\mathbb{R}_+;\mathbb{R}^n) \mid z(t) \text{ solves (A.3) }\}$$

and the unstable manifold

$$W^u(\mu) := \{z \in C_b^1(\mathbb{R}_-;\mathbb{R}^n) \mid z(t) \text{ solves (A.3) }\}.$$

We are able to characterize these manifolds via the exponential dichotomies of the homogeneous system (A.6) in Lemma A.1.

Lemma A.2. *Suppose the assumption (M1) holds and δ is a sufficiently small, positive constant. Then a bounded solution $z(t)$ of* (A.3) *has to satisfy for $t \in \mathbb{R}_+$ the identity*

$$z(t) = \phi(t,0)P_+(0)\xi + \int_0^t \phi(t,s)P_+(s)g(s,z(s),\mu)ds - \\ - \int_t^\infty \phi(t,s)Q_+(s)g(s,z(s),\mu)ds, \qquad \text{(A.9)}$$

where $P_+(t)$ *for* $t \in \mathbb{R}_+$ *is the family of projections in Lemma A.1 and* $Q_+(t) = I - P_+(t)$. *For parameter values* μ *close to* μ_0 *and starting values* ξ *in a small neighborhood of the origin,* $\omega_s := \operatorname{image}(P_+(0)) \cap B_\delta(0)$, *there exists a parametrization* z_+ *for solutions of* (A.3) *that are bounded on* $\mathbb{R}_+$. *The function*

$$z_+ : \omega_s \times \mathbb{R}^m \to C_b^1(\mathbb{R}_+; \mathbb{R}^n), \quad (\xi_s, \mu) \mapsto z_+(\xi_s, \mu),$$

is smooth and satisfies the identities (A.9) *as well as*

$$z_+(0, \mu_0)(t) \equiv 0 \in \mathbb{R}^n \quad \textit{and} \quad \frac{\partial z_+}{\partial \xi}(0, \mu_0)(t) = \phi(t, 0) \in \mathbb{R}^{n \times n}. \tag{A.10}$$

The stable manifold $W^s(\mu)$ *has in a small neighborhood of the trivial solution,* $z_0(t) \equiv 0 \in \mathbb{R}^n$, *and for all parameter values* $\mu \in B_\delta(\mu_0)$ *a representation*

$$W^s(\mu) \cap B_\delta(z_0) = \{z_+(\xi_s, \mu) \in C_b^1(\mathbb{R}_+; \mathbb{R}^n) \mid \xi_s \in \omega_s\}.$$

Proof. By the variation of constants formula, a solution of (A.3) has to satisfy for all $t \in \mathbb{R}$ the identity

$$z(t) = \phi(t, 0)\xi + \int_0^t \phi(t, s) g(s, z(s), \mu) ds.$$

The homogeneous part of (A.3) has an exponential dichotomy on $\mathbb{R}_+$ and we rewrite the identity for $t \in \mathbb{R}_+$ as

$$\begin{aligned} z(t) = &\phi(t, 0) P_+(0)\xi + \int_0^t \phi(t, s) P_+(s) g(s, z(s), \mu) ds - \\ &- \int_t^\infty \phi(t, s) Q_+(s) g(s, z(s), \mu) ds + \\ &+ \phi(t, 0) Q_+(0)\xi + \int_0^\infty \phi(t, s) Q_+(s) g(s, z(s), \mu) ds. \end{aligned}$$

We conclude that a solution of (A.3) is bounded on $\mathbb{R}_+$ if and only if it satisfies

$$0 = \phi(t,0)Q_+(0)\xi + \int_0^\infty \phi(t,s)Q_+(s)g(s,z(s),\mu)ds$$

or equivalently (A.9). Thus the function

$$H : \mathrm{image}(P_+(0)) \otimes C_b^1(\mathbb{R}_+;\mathbb{R}^n) \otimes \mathbb{R}^m \to C_b^1(\mathbb{R}_+;\mathbb{R}^n), \quad (\xi,z,\mu) \mapsto H(\xi,z,\mu),$$

with

$$\begin{aligned} H(\xi,z,\mu) :=& z(t) - \phi(t,0)\xi - \int_0^t \phi(t,s)P_+(s)g(s,z(s),\mu)ds + \\ &+ \int_t^\infty \phi(t,s)Q_+(s)g(s,z(s),\mu)ds \end{aligned}$$

is well-defined and smooth. Since $H(0,z_0,\mu_0) = 0$ and $\frac{\partial H}{\partial z}(0,z_0,\mu_0) = I$, the implicit function theorem implies the existence of a small neighborhood of $(\xi,z,\mu) = (0,z_0,\mu_0)$, $\omega_s \otimes B_\delta(z_0) \otimes B_\delta(\mu_0)$, and a smooth function

$$z_+ : \omega_s \otimes B_\delta(\mu_0) \to C_b^1(\mathbb{R}_+;\mathbb{R}^n), \quad (\xi,\mu) \mapsto z_+(\xi,\mu),$$

such that $z_+(0,\mu_0) = z_0(t) \equiv 0$ and $H(\xi, z_+(\xi,\mu),\mu) = 0$ holds. We differentiate the identity, $H(\xi,z_+(\xi,\mu),\mu) = 0$, with respect to ξ,

$$\begin{aligned} \frac{\partial z_+}{\partial \xi}(\xi,\mu) = \phi(t,0) + \int_0^t \phi(t,s)P_+(s)\frac{\partial g}{\partial z}(s,z_+(\xi,\mu)(s),\mu)\frac{\partial z_+}{\partial \xi}(\xi,\mu)(s)ds - \\ - \int_t^\infty \phi(t,s)Q_+(s)\frac{\partial g}{\partial z}(s,z_+(\xi,\mu)(s),\mu)\frac{\partial z_+}{\partial \xi}(\xi,\mu)(s)ds, \end{aligned}$$

evaluate at $(\xi,\mu) = (0,\mu_0)$ and obtain from $z_+(0,\mu_0) \equiv 0$ and $\frac{\partial g}{\partial z}(t,0,\mu_0) = 0$ the second identity in (A.10). Finally, the parametrization for bounded solutions allows to define the stated representation of the stable manifold $W^s(\mu)$. □

In a similar way, we prove the existence of a representation for the unstable manifold.

Lemma A.3. *Suppose the assumption (M1) holds and δ is a sufficiently small, positive constant. Then a bounded solution $z(t)$ of* (A.3) *has to satisfy for $t \in \mathbb{R}_-$ the identity*

$$z(t) = \phi(t,0)Q_-(0)\xi + \int_{-\infty}^{t} \phi(t,s)P_-(s)g(s,z(s),\mu)ds - \\ - \int_{t}^{0} \phi(t,s)Q_-(s)g(s,z(s),\mu)ds, \qquad \text{(A.11)}$$

where $P_-(t)$ for $t \in \mathbb{R}_-$ is the family of projections in Lemma A.1 and $Q_-(t) = I - P_-(t)$. For parameter values μ close to μ_0 and starting values ξ in a small neighborhood of the origin, $\omega_u := \operatorname{image}(Q_-(0)) \cap B_\delta(0)$, there exists a parametrization z_- for solutions of (A.3) *that are bounded on $\mathbb{R}_-$. The function*

$$z_- : \omega_u \times \mathbb{R}^m \to C_b^1(\mathbb{R}_-; \mathbb{R}^n), \quad (\xi_u, \mu) \mapsto z_-(\xi_u, \mu),$$

is smooth and satisfies the identities (A.11) *as well as*

$$z_-(0,\mu_0)(t) \equiv 0 \in \mathbb{R}^n \quad \text{and} \quad \frac{\partial z_-}{\partial \xi}(0,\mu_0)(t) = \phi(t,0) \in \mathbb{R}^{n\times n}. \qquad \text{(A.12)}$$

The unstable manifold $W^u(\mu)$ has in a small neighborhood of the trivial solution, $z_0(t) \equiv 0 \in \mathbb{R}^n$, and for all parameter values $\mu \in B_\delta(\mu_0)$ a representation

$$W^u(\mu) \cap B_\delta(z_0) = \{z_-(\xi_u, \mu) \in C_b^1(\mathbb{R}_-; \mathbb{R}^n) \mid \xi_u \in \omega_u\}.$$

Instead of studying the intersection of the manifolds $W^s(\mu)$ and $W^u(\mu)$ directly, we consider the associated stable space

$$W_0^s(\mu) := \{y \in Y \mid \exists z \in W^s(\mu) : \quad z(0) = y\}$$

and unstable space

$$W_0^u(\mu) := \{y \in Y \mid \exists z \in W^u(\mu) : \quad z(0) = y\}.$$

These spaces consist of all points y in the transversal section Y such that a solution $z(t)$ of (A.3) that satisfies the initial condition, $z(0) = y$, remains bounded on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. We obtain for μ close to μ_0 a parametrization of the spaces $W_0^s(\mu)$ and $W_0^u(\mu)$ with respect to a coordinate system which is related to the decomposition of Y.

Lemma A.4. *Suppose the assumption (M1) holds. Then the family of projections, $P_+(t)$ for $t \in \mathbb{R}_+$, in Lemma A.1 and the associated family of projections, $Q_+(t) = I - P_+(t)$ for $t \in \mathbb{R}_+$, are uniquely determined by the condition*

$$\mathrm{kernel}(P_+(0)) = V_u \oplus W. \tag{A.13}$$

In particular, for a sufficiently small, positive constant δ and parameter values $\mu \in B_\delta(\mu_0)$ there exists for vectors in the stable space $W_0^s(\mu)$ a parametrization

$$y_+ : \mathbb{R}^k \times V_s \times \mathbb{R}^m \to Y \cap W_0^s(\mu), \quad (\nu, v_s, \mu) \mapsto y_+(\nu, v_s, \mu),$$

with

$$y_+(\nu, v_s, \mu) = \sum_{i=1}^{k} \nu_i u_i + v_s + \tilde{v}_u(\nu, v_s, \mu) + \tilde{w}_+(\nu, v_s, \mu). \tag{A.14}$$

The functions $\tilde{v}_u : \mathbb{R}^k \times V_s \times \mathbb{R}^m \to V_u$ and $\tilde{w}_+ : \mathbb{R}^k \times V_s \times \mathbb{R}^m \to W$ are smooth and satisfy the identities

$$\tilde{v}_u(0,0,\mu_0) = 0 \quad \textit{as well as} \quad \frac{\partial \tilde{v}_u}{\partial(\nu, v_s)}(0,0,\mu_0) = 0 \tag{A.15}$$

and

$$\tilde{w}_+(0,0,\mu_0) = 0 \quad \textit{as well as} \quad \frac{\partial \tilde{w}_+}{\partial(\nu, v_s)}(0,0,\mu_0) = 0, \tag{A.16}$$

respectively.

Proof. The transversal section Y at $t = 0$ has a decomposition into orthogonal subspaces,

$$Y = U \oplus V_s \oplus V_u \oplus W.$$

In Lemma A.2 a representation of the stable manifold $W^s(\mu)$ via the function $z_+(\xi_s, \mu)$ was obtained, which satisfies at $t = 0$ the identity

$$z_+(\xi_s, \mu)(0) = \xi_s - Q_+(0) \int_0^\infty \phi(0, s) g(s, z_+(\xi_s, \mu)(s), \mu) ds.$$

The family of projections $P_+(t)$ for $t \in \mathbb{R}_+$ satisfies (A.7) and especially

$$\text{image}(P_+(0)) = T_{\bar{u}(0)} W^s(u^+) = \text{span}\{F(u, \mu_0)\} \oplus U \oplus V_s,$$

but the kernel is not specified. If we set

$$\text{kernel}(P_+(0)) = \text{image}(Q_+(0)) := V_u \oplus W,$$

then the families of projections are uniquely determined by the properties of exponential dichotomies in Lemma A.1. Moreover, the vector $\xi_s \in U \oplus V_s$ and the integral term,

$$h_+(\xi_s, \mu) := -Q_+(0) \int_0^\infty \phi(0, s) g(s, z_+(\xi_s, \mu)(s), \mu) ds \in V_u \oplus W,$$

now take values in orthogonal spaces. Thus we can find $\nu \in \mathbb{R}^k$ and $v_s \in V_s$, such that

$$\xi_s(\nu, v_s) := \sum_{i=1}^k \nu_i u_i + v_s.$$

Whereas, the auxiliary function $h_+ : \omega_s \times \mathbb{R}^m \to V_u \oplus W$ is smooth and takes values in two orthogonal spaces. We decompose it into

$$h_+(\xi_s(\nu, v_s), \mu) = \tilde{v}_u(\nu, v_s, \mu) + \tilde{w}_+(\nu, v_s, \mu)$$

with smooth functions $\tilde{v}_u : \mathbb{R}^k \times V_s \times \mathbb{R}^m \to V_u$ and $\tilde{w}_+ : \mathbb{R}^k \times V_s \times \mathbb{R}^m \to W$. Their stated properties follow from direct computation and the properties of the function $g(s, z, \mu)$ in (A.5). □

In a similar way, we prove the existence of a representation for the unstable space.

Lemma A.5. *Suppose the assumption (M1) holds. Then the family of projections, $P_-(t)$ for $t \in \mathbb{R}_-$, in Lemma A.1 and the associated family of projections, $Q_-(t) = I - P_-(t)$ for $t \in \mathbb{R}_-$, are uniquely determined by the condition*

$$\mathrm{kernel}(Q_-(0)) = V_s \oplus W. \tag{A.17}$$

In particular, for a sufficiently small, positive constant δ and parameter values $\mu \in B_\delta(\mu_0)$ there exists for vectors in the unstable space $W_0^u(\mu)$ a parametrization

$$y_- : \mathbb{R}^k \times V_u \times \mathbb{R}^m \to Y \cap W_0^u(\mu), \quad (\nu, v_u, \mu) \mapsto y_-(\nu, v_u, \mu),$$

with

$$y_-(\nu, v_u, \mu) = \sum_{i=1}^{k} \nu_i u_i + v_u + \tilde{v}_s(\nu, v_u, \mu) + \tilde{w}_-(\nu, v_u, \mu). \tag{A.18}$$

The functions $\tilde{v}_s : \mathbb{R}^k \times V_u \times \mathbb{R}^m \to V_s$ and $\tilde{w}_- : \mathbb{R}^k \times V_u \times \mathbb{R}^m \to W$ are smooth and satisfy the identities

$$\tilde{v}_s(0,0,\mu_0) = 0 \quad \textit{as well as} \quad \frac{\partial \tilde{v}_s}{\partial(\nu, v_u)}(0,0,\mu_0) = 0 \tag{A.19}$$

and

$$\tilde{w}_-(0,0,\mu_0) = 0 \quad \textit{as well as} \quad \frac{\partial \tilde{w}_-}{\partial(\nu, v_u)}(0,0,\mu_0) = 0, \tag{A.20}$$

respectively.

Finally, we investigate the intersection of the spaces $W_0^s(\mu)$ and $W_0^u(\mu)$. We observe from the Lemmata A.4 and A.5 that for $\mu \in B_\delta(\mu_0)$ a point $y = \sum_{i=1}^k \nu_i u_i + v_s + v_u + w \in Y$ will belong to $W_0^s(\mu) \cap W_0^u(\mu)$ if and only if the system of equations

$$\begin{cases} v_s & = \tilde{v}_s(\nu, v_u, \mu), \\ v_u & = \tilde{v}_u(\nu, v_s, \mu), \\ \tilde{w}_+(\nu, v_s, \mu) & = \tilde{w}_-(\nu, v_u, \mu). \end{cases} \tag{A.21}$$

is satisfied.

Lemma A.6. *Suppose the assumption (M1) holds and δ is a sufficiently small, positive constant. Then there exist unique, smooth functions*

$$v_s^* : B_\delta(0) \times B_\delta(\mu_0) \to V_s, \quad (\nu, \mu) \longmapsto v_s^*(\nu, \mu),$$

and

$$v_u^* : B_\delta(0) \times B_\delta(\mu_0) \to V_u, \quad (\nu, \mu) \longmapsto v_u^*(\nu, \mu),$$

that satisfy the equations

$$\begin{cases} v_s^*(\nu, \mu) & = \tilde{v}_s(\nu, v_u^*(\nu, \mu), \mu), \\ v_u^*(\nu, \mu) & = \tilde{v}_u(\nu, v_s^*(\nu, \mu), \mu). \end{cases} \tag{A.22}$$

In addition, the functions satisfy the identities

$$v_s^*(0, \mu_0) = 0 \quad \textit{as well as} \quad \frac{\partial v_s^*}{\partial \nu}(0, \mu_0) = 0 \tag{A.23}$$

and

$$v_u^*(0, \mu_0) = 0 \quad \textit{as well as} \quad \frac{\partial v_u^*}{\partial \nu}(0, \mu_0) = 0, \tag{A.24}$$

respectively.

Proof. We consider the function

$$\begin{aligned} V : \mathbb{R}^k \times V_s \times V_u \times \mathbb{R}^m &\longrightarrow V_s \times V_u, \\ (\nu, v_s, v_u, \mu) &\longmapsto \begin{pmatrix} v_s - \tilde{v}_s(\nu, v_u, \mu) \\ v_u - \tilde{v}_u(\nu, v_s, \mu) \end{pmatrix}, \end{aligned}$$

which satisfies the identities $V(0,0,0,\mu_0) = 0$ and $\frac{\partial V}{\partial(v_s, v_u)}(0,0,0,\mu_0) = I$, due to the properties of $\tilde{v}_u(\nu, v_s, \mu)$ and $\tilde{v}_s(\nu, v_u, \mu)$ in the Lemmata A.4 and A.5, respectively. Hence, we conclude from the implicit function theorem the existence of unique, smooth functions $v_s^*(\nu, \mu)$ and $v_u^*(\nu, \mu)$ such that $v_s^*(0, \mu_0) = 0$ and $v_u^*(0, \mu_0) = 0$, respectively. In addition, for $(\nu, \mu) \in B_\delta(0) \times B_\delta(\mu_0)$ the identity $V(\nu, v_s^*(\nu, \mu), v_u^*(\nu, \mu), \mu) = 0$ holds, which is equivalent to system (A.22). We differentiate system (A.22) with respect to ν,

$$\begin{cases} \frac{\partial v_s^*}{\partial \nu}(\nu, \mu) & = \frac{\partial \tilde{v}_s}{\partial v}(\nu, v_u^*(\nu, \mu), \mu) \frac{\partial v_u^*}{\partial \nu}(\nu, \mu), \\ \frac{\partial v_u^*}{\partial \nu}(\nu, \mu) & = \frac{\partial \tilde{v}_u}{\partial v}(\nu, v_s^*(\nu, \mu), \mu) \frac{\partial v_s^*}{\partial \nu}(\nu, \mu), \end{cases}$$

evaluate it at $(\nu, \mu) = (0, \mu_0)$ and use the identities (A.15) and (A.19) to obtain the stated result,

$$\frac{\partial v_s^*}{\partial \nu}(0, \mu_0) = 0 \quad \text{and} \quad \frac{\partial v_u^*}{\partial \nu}(0, \mu_0) = 0.$$

□

Thus, the system of equations (A.21) reduces to the bifurcation equation

$$\tilde{w}_-(\nu, v_u^*(\nu, \mu), \mu) = \tilde{w}_+(\nu, v_s^*(\nu, \mu), \mu). \tag{A.25}$$

For μ close to μ_0 a heteroclinic orbit will exist, if there is some $\nu \in \mathbb{R}^k$ such that (ν, μ) satisfies the bifurcation equation (A.25).

Definition A.4. A heteroclinic orbit γ of (A.1) is transversal, if for any point p on the orbit γ the tangent spaces of the invariant manifolds $W^s(u^+)$ and $W^u(u^-)$ satisfy $\dim \big(T_p W^u(u^-) + T_p W^s(u^+)\big) = n$.

Lemma A.7. *A transversal, heteroclinic orbit of* (A.1) *for* $\mu = \mu_0$ *persists for all* μ *in a small neighborhood of* μ_0.

Proof. For a transversal, heteroclinic orbit, the sum of the tangent spaces $T_{\bar{u}(0)}W^u(u^-)$ and $T_{\bar{u}(0)}W^s(u^+)$ has dimension n. Consequently, the space W is empty and the bifurcation equation (A.25) is trivially satisfied. □

A.2 Melnikov function

In the following, we consider a non-transversal, heteroclinic orbit and want to describe the parameter set, such that the heteroclinic orbit persists.

(M2) The heteroclinic orbit γ in (M1) is non-transversal, that means for all points p on the orbit γ the dimension of the sum of the tangent spaces $T_pW^u(u^-)$ and $T_pW^s(u^+)$ is less than the dimension n of the state space. In addition, we assume that for some $k \in \mathbb{N}$ and any point p on the orbit γ the identity $\dim\big(T_pW^u(u^-) \cap T_pW^s(u^+)\big) = k+1$ holds.

The *Melnikov function* $M : \mathbb{R}^k \times \mathbb{R}^m \to W$ with

$$M(\nu,\mu) := \tilde{w}_-(\nu, v_u^*(\nu,\mu),\mu) - \tilde{w}_+(\nu, v_s^*(\nu,\mu),\mu) \tag{A.26}$$

is well-defined and smooth in $B_\delta(0) \times B_\delta(\mu_0)$ and satisfies $M(0,\mu_0) = 0$ and $\frac{\partial M}{\partial \nu}(0,\mu_0) = 0$. We will derive an integral representation of the Melnikov function (A.25), which makes use of the following auxiliary functions.

Lemma A.8. *Let the assumptions (M1) and (M2) hold. Then the functions* $v_s^*(\nu,\mu)$ *and* $v_u^*(\nu,\mu)$ *in Lemma A.6 as well as* $z_+(\xi_s,\mu)$ *and* $z_-(\xi_u,\mu)$ *in the Lemmata A.2 and A.3, respectively, are well-defined. The functions*

$$\begin{aligned} \xi_s^* : B_\delta(0) \times B_\delta(\mu_0) &\to U \oplus V_s, \\ (\nu,\mu) \mapsto \xi_s^*(\nu,\mu) &:= \sum_{i=1}^{k} \nu_i u_i + v_s^*(\nu,\mu), \end{aligned} \tag{A.27}$$

and

$$\xi_u^* : B_\delta(0) \times B_\delta(\mu_0) \to U \oplus V_u,$$
$$(\nu, \mu) \mapsto \xi_u^*(\nu, \mu) := \sum_{i=1}^{k} \nu_i u_i + v_u^*(\nu, \mu), \qquad \text{(A.28)}$$

as well as

$$z^* : B_\delta(0) \times B_\delta(\mu_0) \times \mathbb{R} \to \mathbb{R}^n,$$
$$(\nu, \mu, t) \mapsto z^*(\nu, \mu)(t),$$

with

$$z^*(\nu, \mu)(t) := \begin{cases} z_-^*(\nu, \mu)(t) = z_-(\xi_u^*(\nu, \mu), \mu)(t), & \textit{for } t \leq 0, \\ z_+^*(\nu, \mu)(t) = z_+(\xi_s^*(\nu, \mu), \mu)(t), & \textit{for } t \geq 0, \end{cases} \qquad \text{(A.29)}$$

are well-defined, too. In particular, the function $z^*(\nu, \mu)(t)$ has the following properties:

1. *The function $z^*(\nu, \mu)$ is continuous in $t \in \mathbb{R}$, except for a possible discontinuity at $t = 0$.*

2. *The function $z^*(\nu, \mu)$ is differentiable with respect to ν_i for $i = 1, \ldots, k$ and satisfies $\frac{\partial z^*}{\partial \nu_i}(0, \mu_0)(t) = \phi(t, 0) u_i$ for $i = 1, \ldots, k$.*

Proof. We investigate each assertion in turn:

1. The function $z^*(\nu, \mu)$ is on each half line, $\mathbb{R}_-$ as well as $\mathbb{R}_+$, defined as composition of continuous functions. It is continuous at $t = 0$ if and only if there exists a globally bounded solution for $(\nu, \mu) \in \mathbb{R}^k \times \mathbb{R}^m$.

2. The function $z_+^*(\nu,\mu)$ satisfies for $t \in \mathbb{R}_+$

$$\begin{aligned} z_+^*(\nu,\mu)(t) =& \phi(t,0)\xi_s^*(\nu,\mu)+ \\ &+\int_0^t \phi(t,s)P_+(s)g(s,z_+^*(\nu,\mu)(s),\mu)ds- \\ &-\int_t^\infty \phi(t,s)Q_+(s)g(s,z_+^*(\nu,\mu)(s),\mu)ds. \end{aligned}$$

We differentiate with respect to ν_i for $i = 1,\ldots,k$,

$$\begin{aligned} \frac{\partial z_+^*}{\partial \nu_i}(\nu,\mu)(t) =& \phi(t,0)\Big(u_i + \frac{\partial v_s^*}{\partial \nu_i}(\nu,\mu)\Big)+ \\ &+\int_0^t \phi(t,s)P_+(s)\frac{\partial g}{\partial z}(s,z_+^*(\nu,\mu)(s),\mu)\frac{\partial z_+^*}{\partial \nu_i}(\nu,\mu)(s)ds- \\ &-\int_t^\infty \phi(t,s)Q_+(s)\frac{\partial g}{\partial z}(s,z_+^*(\nu,\mu)(s),\mu)\frac{\partial z_+^*}{\partial \nu_i}(\nu,\mu)(s)ds, \end{aligned}$$

evaluate at $(\nu,\mu) = (0,\mu_0)$ and use the identities (A.23) as well as (A.5) to obtain

$$\frac{\partial z_+^*}{\partial \nu_i}(0,\mu_0)(t) = \phi(t,0)u_i, \quad \text{for all} \quad t \in \mathbb{R}_+.$$

Similarly we show that

$$\frac{\partial z_-^*}{\partial \nu_i}(0,\mu_0)(t) = \phi(t,0)u_i, \quad \text{for all} \quad t \in \mathbb{R}_-.$$

Thus, the derivative is well-defined and satisfies

$$\frac{\partial z^*}{\partial \nu_i}(0,\mu_0)(t) = \phi(t,0)u_i, \quad \text{for all} \quad t \in \mathbb{R}.$$

□

Theorem A.1. *Under the assumptions (M1) and (M2), there exist a function $z^*(\nu,\mu)(t)$ as in Lemma A.8 and a family of linearly independent, globally bounded solutions $\psi_i(t)$, $i = 1,\dots,l = \dim(W)$ of the adjoint equation of* (A.6),

$$\frac{d\psi}{dt} = -\left(\frac{dF}{du}(\bar{u}(t),\mu_0)\right)^T \psi(t). \tag{A.30}$$

The Melnikov function $M : \mathbb{R}^k \times \mathbb{R}^m \to W$ has an integral representation

$$M(\nu,\mu) = \sum_{i=1}^{l} \psi_i(0) \int_{-\infty}^{+\infty} < \psi_i(s), g(s, z^*(\nu,\mu)(s),\mu) > ds, \tag{A.31}$$

which is well-defined and smooth in the domain $B_\delta(0) \times B_\delta(\mu_0)$ for a sufficiently small, positive constant δ. Moreover, it satisfies the identities

$$M(0,\mu_0) = 0 \quad \textit{and} \quad \frac{\partial M}{\partial \nu}(0,\mu_0) = 0. \tag{A.32}$$

Proof. The space W is defined as the complement of the sum of tangent spaces $T_{\bar{u}(0)}W^u(u^-)$ and $T_{\bar{u}(0)}W^s(u^+)$, which is by the assumption of a non-transversal orbit not empty. We denote the dimension of the space W with $l = \dim(W)$ and consider an orthonormal basis ψ_i^0 for $i = 1,\dots,l$. In addition, the space is related to the exponential dichotomies of (A.6) via the identity

$$W = \mathrm{image}(P_-^T(0)) \cap \mathrm{image}(Q_+^T(0)). \tag{A.33}$$

The functions $\psi_i(t)$ for $i = 1,\dots,l$ defined as,

$$\psi_i(t) := \phi(0,t)^T \psi_i^0, \tag{A.34}$$

constitute a basis for the subspace of globally bounded solutions of (A.30).

The Melnikov function was defined as

$$M(\nu,\mu) = \tilde{w}_-(\nu, v_u^*(\nu,\mu),\mu) - \tilde{w}_+(\nu, v_s^*(\nu,\mu),\mu),$$

which can be rewritten by (A.14) and (A.18) as

$$= \sum_{i=1}^{l} \psi_i^0 < \psi_i^0, y_-(\nu, v_u^*(\nu,\mu),\mu) - y_+(\nu, v_s^*(\nu,\mu),\mu) > .$$

We use the functions $z_\pm^*(\nu,\mu)(t)$ from Lemma A.8 and the identities

$$\begin{aligned} y_-(\nu, v_u^*(\nu,\mu),\mu) &= z_-(\xi_u^*(\nu,\mu),\mu)(0) = z_-^*(\nu,\mu)(0) \\ &= \xi_u^*(\nu,\mu) + P_-(0)\int_{-\infty}^{0} \phi(0,s) g(s, z_-^*(\nu,\mu)(s),\mu) ds \end{aligned}$$

as well as

$$\begin{aligned} y_+(\nu, v_s^*(\nu,\mu),\mu) &= z_+(\xi_s^*(\nu,\mu),\mu)(0) = z_+^*(\nu,\mu)(0) \\ &= \xi_s^*(\nu,\mu) - Q_+(0)\int_{0}^{+\infty} \phi(0,s) g(s, z_+^*(\nu,\mu)(s),\mu) ds \end{aligned}$$

to obtain

$$\begin{aligned} M(\nu,\mu) = \sum_{i=1}^{l} \psi_i^0 <\psi_i^0, \Big[\xi_u^*(\nu,\mu) + P_-(0)\int_{-\infty}^{0} \phi(0,s) g(s, z_-^*(\nu,\mu)(s),\mu) ds - \\ - \xi_s^*(\nu,\mu) + Q_+(0)\int_{0}^{+\infty} \phi(0,s) g(s, z_+^*(\nu,\mu)(s),\mu) ds\Big] >, \end{aligned}$$

The vectors ψ_i^0 for $i = 1, \dots, l$ are by definition orthogonal to the space $span\{F(\bar{u}(0),\mu_0)\} \oplus U \oplus V_s \oplus V_u$. Since $\xi_u^*(\nu,\mu) \in U \oplus V_u$ as well as $\xi_s^*(\nu,\mu) \in U \oplus V_s$, the expression simplifies to

$$\begin{aligned} M(\nu,\mu) = \sum_{i=1}^{l} \psi_i^0 < \psi_i^0, \Big[P_-(0)\int_{-\infty}^{0} \phi(0,s) g(s, z_-^*(\nu,\mu)(s),\mu) ds + \\ + Q_+(0)\int_{0}^{+\infty} \phi(0,s) g(s, z_+^*(\nu,\mu)(s),\mu) ds\Big] > . \end{aligned}$$

We split the summands and observe that the vectors satisfy $(P_-(0))^T\psi_i^0 = \psi_i^0$ as well as $(Q_+(0))^T\psi_i^0 = \psi_i^0$ for $i = 1, \dots, l$ by (A.33). Then we use the definition (A.34) of the functions $\psi_i(t) := \phi(0,t)^T\psi_i^0$ for $i = 1, \dots, l$ and the auxiliary function $z^*(\nu, \mu)$ in Lemma A.8 to obtain the stated expression for the Melnikov function:

$$M(\nu, \mu) = \sum_{i=1}^{l} \psi_i(0) \int_{-\infty}^{+\infty} < \psi_i(s), g(s, z^*(\nu, \mu)(s), \mu) > ds.$$

The identities (A.32) follow from direct computations. □

We are interested in the set of parameter values μ close to μ_0 for which heteroclinic orbits exist. The identities (A.32) prevent an application of the implicit function theorem. Therefore, we want to make use of the regular value theorem.

Theorem A.2 (regular value theorem)**.** *Let M, N be smooth manifolds. If $q \in M$ is a regular value of a smooth map $f : M \to N$, then its preimage $f^{-1}(q) \subset M$ is a submanifold of co-dimension* $\dim(N)$.

The Melnikov function $M : \mathbb{R}^k \times \mathbb{R}^m \to W \subset \mathbb{R}^n$ can possess regular values, only if the following hypothesis is fulfilled.

(M3) The dimension of the parameter space, m, is greater than or equal to the number of globally bounded solutions of (A.30), $l = \dim(W)$.

From the mathematical point of view, (M3) is not a severe restriction since we always can enlarge the parameter space to ensure $m \geq l = \dim(W) \geq 1$.

Theorem A.3. *Suppose the assumptions (M1), (M2) and (M3) hold. The Melnikov function with respect to the orthogonal coordinate system of W, $\{\psi_i(0) \mid i = 1, \ldots, l\}$ is defined as $M : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^l$, $(\nu, \mu) \mapsto M(\nu, \mu) = (M_1, \ldots, M_l)^T(\nu, \mu)$ with coordinate functions*

$$M_i(\nu, \mu) := \int_{-\infty}^{+\infty} < \psi_i(s), g(s, z^*(\nu, \mu)(s), \mu) > ds. \tag{A.35}$$

If the vectors

$$\frac{\partial M_i}{\partial \mu}(0, \mu_0), \quad \textit{for} \quad i = 1, \ldots, l = \dim(W), \tag{A.36}$$

are linearly independent, then the solution set $B = \{(\nu, \mu) \mid M(\nu, \mu) = 0\}$ *forms a submanifold of co-dimension* l *in a neighborhood of* $(0, \mu_0)$*.*

Proof. We consider the Melnikov function $M(\nu, \mu) = (M_1, \ldots, M_l)^T(\nu, \mu)$ which satisfies $M(0, \mu_0) = 0 \in \mathbb{R}^n$. We compute the Jacobian of $M(\nu, \mu)$ at $(0, \mu_0)$ and obtain

$$\frac{\partial M}{\partial(\nu, \mu)}(0, \mu_0) = \left(0, \cdots, 0, \frac{\partial M}{\partial \mu_1}, \cdots, \frac{\partial M}{\partial \mu_m}\right)(0, \mu_0).$$

We note that it has rank l, as we recover the linearly independent vectors (A.36) in the row vectors of the Jacobian. This holds true in a small neighborhood of $(0, \mu_0)$ and we can apply the regular value theorem at least locally and obtain the stated results. □

Lemma A.9. *Suppose the assumptions (M1) and (M2) hold. Then the derivatives* $\frac{\partial M_i}{\partial \mu_j}(0, \mu_0)$ *are computed as*

$$\frac{\partial M_i}{\partial \mu_j}(0, \mu_0) = \int_{-\infty}^{+\infty} < \psi_i(s), \frac{\partial F}{\partial \mu_j}(\bar{u}(s, \mu_0), \mu_0) > ds, \tag{A.37}$$

with $j = 1, \ldots, m$ *and* $i = 1, \ldots, l = \dim(W)$*.*

Proof. The coordinate functions of the Melnikov function in Theorem A.3 are defined as

$$M_i(\nu, \mu) := \int_{-\infty}^{+\infty} < \psi_i(s), g(s, z^*(\nu, \mu)(s), \mu) > ds.$$

The differentiation can be interchanged with the integration and we obtain

$$\begin{aligned}
\frac{\partial M_i}{\partial \mu_j}(0, \mu_0) &= \frac{\partial}{\partial \mu_j}\bigg|_{(\nu,\mu)=(0,\mu_0)} \int_{-\infty}^{+\infty} < \psi_i(s), g(s, z^*(\nu, \mu)(s), \mu) > ds \\
&= \int_{-\infty}^{+\infty} < \psi_i(s), \frac{\partial}{\partial \mu_j}\bigg|_{(\nu,\mu)=(0,\mu_0)} g(s, z^*(\nu, \mu)(s), \mu) > ds \\
&= \int_{-\infty}^{+\infty} < \psi_i(s), \frac{\partial F}{\partial \mu_j}(\bar{u}(s, \mu_0), \mu_0) ds.
\end{aligned}$$

The last equality holds, since the derivative of the function,

$$g(t,z,\mu)=F(\bar{u}(t)+z,\mu)-F(\bar{u}(t),\mu_0)-\frac{dF}{du}(\bar{u}(t),\mu_0)z,$$

with respect to μ_j satisfies

$$\begin{aligned}\frac{\partial g}{\partial \mu_j}(s,z^*(\nu,\mu),\mu)=&\frac{\partial F}{\partial u}(\bar{u}(s,\mu_0)+z^*(\nu,\mu),\mu)\frac{\partial z^*}{\partial \mu_j}(\nu,\mu)+\\&+\frac{\partial F}{\partial \mu_j}(\bar{u}(s,\mu_0)+z^*(\nu,\mu),\mu)-\\&-\frac{\partial F}{\partial u}(\bar{u}(s,\mu_0),\mu_0)\frac{\partial z^*}{\partial \mu_j}(\nu,\mu).\end{aligned}$$

We evaluate the derivative at $(\nu,\mu)=(0,\mu_0)$ and obtain

$$\frac{\partial g}{\partial \mu_j}(s,z^*(0,\mu_0),\mu_0)=\frac{\partial F}{\partial \mu_j}(\bar{u}(s,\mu_0),\mu_0),$$

since $z^*(0,\mu_0)=0$. □

We will also need higher order derivatives of the Melnikov function.

Lemma A.10. *Suppose the assumptions (M1) and (M2) hold. Then for* $j=1,\dots,k=\dim(U)$ *and* $i=1,\dots,l=\dim(W)$*, the derivatives* $\frac{\partial^2 M_i}{\partial \nu_j^2}(0,\mu_0)$ *are computed as*

$$\frac{\partial^2 M_i}{\partial \nu_j^2}(0,\mu_0)=\int_{-\infty}^{+\infty}<\psi_i(s),\frac{\partial^2 F}{\partial u^2}(\bar{u},\mu_0)(u_j,u_j)(s,\mu_0)>ds, \tag{A.38}$$

where $u_j(s):=\phi(s,0)u_j$.

Proof. The coordinate functions of the Melnikov function in Theorem A.3 are defined as

$$M_i(\nu,\mu):=\int_{-\infty}^{+\infty}<\psi_i(s),g(s,z^*(\nu,\mu)(s),\mu)>ds.$$

The differentiation can be interchanged with the integration and we obtain

$$\begin{aligned}
\frac{\partial^2 M_i}{\partial \nu_j^2}(0,\mu_0) &= \frac{\partial^2}{\partial \nu_j^2}\bigg|_{(\nu,\mu)=(0,\mu_0)} \int_{-\infty}^{+\infty} < \psi_i(s), g(s, z^*(\nu,\mu)(s), \mu) > ds \\
&= \int_{-\infty}^{+\infty} < \psi_i(s), \frac{\partial^2}{\partial \nu_j^2}\bigg|_{(\nu,\mu)=(0,\mu_0)} g(s, z^*(\nu,\mu)(s), \mu) > ds \\
&= \int_{-\infty}^{+\infty} < \psi_i(s), \frac{\partial^2 F}{\partial u^2}(\bar{u}(s,\mu_0),\mu_0)(u_j(s), u_j(s)) > ds.
\end{aligned}$$

The last equality holds, since the derivative of the function,

$$g(t,z,\mu) = F(\bar{u}(t)+z,\mu) - F(\bar{u}(t),\mu_0) - \frac{dF}{du}(\bar{u}(t),\mu_0)z,$$

with respect to ν_j satisfies

$$\begin{aligned}
\frac{\partial^2 g}{\partial \nu_j^2}(s, z^*(\nu,\mu),\mu) &= \frac{\partial}{\partial \nu_j}\Bigg(\frac{\partial F}{\partial u}(\bar{u}(s,\mu_0) + z^*(\nu,\mu),\mu)\frac{\partial z^*}{\partial \nu_j}(\nu,\mu) - \\
&\qquad - \frac{\partial F}{\partial u}(\bar{u}(s,\mu_0),\mu_0)\frac{\partial z^*}{\partial \nu_j}(\nu,\mu)\Bigg) \\
&= \frac{\partial^2 F}{\partial u^2}(\bar{u}(s,\mu_0) + z^*(\nu,\mu),\mu)\left(\frac{\partial z^*}{\partial \nu_j}, \frac{\partial z^*}{\partial \nu_j}\right)(\nu,\mu) + \\
&\quad + \frac{\partial F}{\partial u}(\bar{u}(s,\mu_0) + z^*(\nu,\mu),\mu)\frac{\partial^2 z^*}{\partial \nu_j^2}(\nu,\mu) - \\
&\quad - \frac{\partial F}{\partial u}(\bar{u}(s,\mu_0),\mu_0)\frac{\partial^2 z^*}{\partial \nu_j^2}(\nu,\mu).
\end{aligned}$$

We evaluate the derivative at $(\nu,\mu) = (0,\mu_0)$ and use the results of Lemma A.8, $z^*(0,\mu_0) = 0$ and $\frac{\partial z^*}{\partial \nu_j}(0,\mu_0)(s) = \phi(s,0)u_j = u_j(s)$, to obtain

$$\begin{aligned}
\frac{\partial^2 g}{\partial \nu_j^2}(s,0,\mu_0) &= \frac{\partial^2 F}{\partial u^2}(\bar{u}(s,\mu_0),\mu)\left(\frac{\partial z^*}{\partial \nu_j}, \frac{\partial z^*}{\partial \nu_j}\right)(0,\mu_0) \\
&= \frac{\partial^2 F}{\partial u^2}(\bar{u}(s,\mu_0),\mu)(u_j(s), u_j(s)).
\end{aligned}$$

□

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